

# Random Number Generation

732A90

Computational Statistics

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- A computer is a deterministic machine
- *Congruential generators*
- Functions of time
- Be careful with respect to application

# First step: Generating Unif[0, 1]

## Linear congruential generator

Define a sequence of integers according to

$$x_{k+1} = (a \cdot x_k + c) \mod m, \quad k \geq 0$$

$x_0$  is **seed**, e.g. based on time

$\mod m$ : remainder after division by  $m$

- $x_k \in \{0, \dots, m-1\}$  and integer
- $x_k/m \sim \text{Unif}[0, 1]$
- $a, c \in [0, m)$  need to be carefully selected

## First step: Generating Unif[0, 1]

Generated numbers will get into a loop with a certain **period**

$$x_{k+1} = (a \cdot x_k + c) \mod m, \quad k \geq 0$$

$$x_0 = a = c = 7, \quad m = 10$$

❶  $x_1 = (7 \cdot 7 + 7) \mod 10 = 56 \mod 10 = 6$

❷  $x_1 = (7 \cdot 6 + 7) \mod 10 = 49 \mod 10 = 9$

❸  $x_1 = (7 \cdot 9 + 7) \mod 10 = 70 \mod 10 = 0$

❹  $x_1 = (7 \cdot 0 + 7) \mod 10 = 7 \mod 10 = 7$

❺  $x_1 = (7 \cdot 7 + 7) \mod 10 = 56 \mod 10 = 6$

❻ ...

## First step: Generating Unif[0, 1]

```
fthreetbits<-function(k,s,L,N){  
  X0<-4*s+1;a<-8*k+5;m<-2^L;X<-X0  
  for (i in 1:N){  
    print(c(X,rev(intToBits(X)[1:5])))  
    X<-(a*X)%m ##c=0  
  }  
}
```

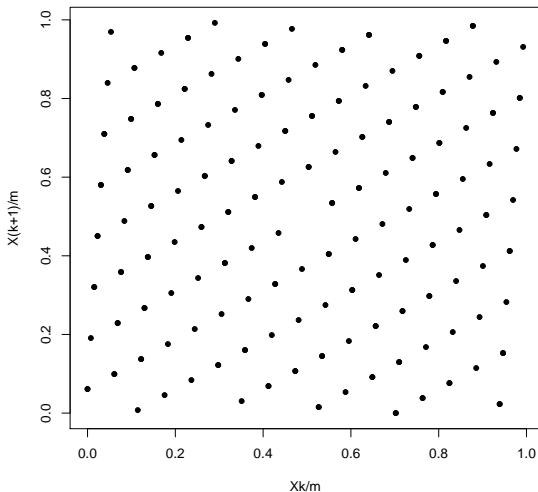
```
> source("CongGen.R");fthreetbits(k=2,s=3,L=8,N=10)  
[1] 13  0  1  1  0  1  
[1] 17  1  0  0  0  1  
[1] 101  0  0  1  0  1  
[1] 73  0  1  0  0  1  
[1] 253  1  1  1  0  1  
[1] 193  0  0  0  0  1  
[1] 213  1  0  1  0  1  
[1] 121  1  1  0  0  1  
[1] 237  0  1  1  0  1  
[1] 113  1  0  0  0  1
```

Last three bits change between 001 and 101

**Discard less significant bits**

# First step: Generating Unif[0, 1]

See also D. E. Knuth (1998). The Art of Computer Programming, Volume 2, Addison-Wesley. Ch. 3.3.4



# First step: Generating Unif[0, 1]

- Period is  $\leq m$  by definition
- $a, c, m$  (**large**) have to be chosen carefully
  - ①  $c$  and  $m$  have to be relatively prime (no common divisors bar 1)
  - ②  $a = 1 \pmod p$  for every prime divisor  $p$  of  $m$
  - ③  $a = 1 \pmod 4$  if 4 divides  $m$
  - ④ Then full period  $m$  reached (**what about**  $a = c = 1$  ?)
- Seed defines the random sequence — same seed, same sequence

**Be careful when re-opening an R workspace**
- Other methods (not in this course)

## Second step: Generating $\text{Unif}[a, b]$

- $U \sim \text{Unif}[0, 1]$  can be transformed into  $X \sim \text{Unif}[a, b]$  as

$$X = a + U \cdot (b - a)$$

- $U$  can also be transformed into **discrete** uniform distribution on integers  $\in \{1, \dots, n\}$  as  $([\cdot], \text{integer part})$

$$X = [nU] + 1$$

### Questions

- 1 Why  $+1$ ?
- 2 How can  $U$  be transformed into  $Y$ , where  $Y$  is discrete uniform on integers  $(50, 55, 60)$ ?

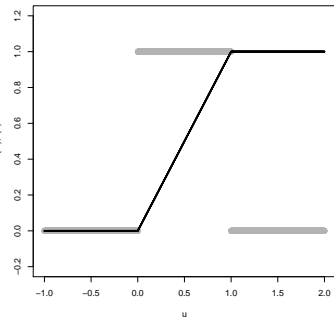


## Second step: Generating nonuniform random numbers

- $U \sim \text{Unif}(0, 1)$
- Let  $F_U$  be the *cumulative distribution function* (CDF) of  $U$

$$F_U(u) = P(U \leq u) = \begin{cases} 0 & u \leq 0 \\ u & 0 < u \leq 1 \\ 1 & 1 < u \end{cases}$$

- The *probability distribution function* (PDF) of  $U$



$$f_U(u) = \begin{cases} 1 & 0 < u < 1 \\ 0 & u \notin (0, 1) \end{cases}$$

Let  $X$  be a random variable with CDF  $X \sim F_X$   
( $F_X$  strictly increasing)

Consider  $Y = F_X^{-1}(U)$ , where  $U \sim \text{Unif}(0, 1)$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(F_X^{-1}(U) \leq y) \\ &= P(F_X(F_X^{-1}(U)) \leq F_X(y)) \\ &= P(U \leq F_X(y)) = F_U(F_X(y)) = F_X(y) \end{aligned}$$

**$Y$  has same probability distribution as  $X$**

If we can generate  $U \sim \text{Unif}(0, 1)$ , then

we can generate  $X \sim F_X$  as

$$X = F_X^{-1}(U)$$

Provided we can calculate  $F_X^{-1} \dots$

# Inverse CDF method: Example

Let  $X \sim \exp(\lambda)$ , i.e. with pdf

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

implying (**SHOW THIS**)

$$F_X(x) = \int_{-\infty}^x f_X(s) ds = 1 - e^{-\lambda x}, \quad x \geq 0$$

## QUESTIONS:

What is  $F_X(x)$  for  $x < 0$ ?

What is  $E[X]$ ?

## Inverse CDF method: Example

Find  $F_X^{-1}$

$$y = 1 - e^{-\lambda x}$$

$$e^{-\lambda x} = 1 - y$$

$$x = -\frac{1}{\lambda} \ln(1 - y)$$

$$F_X^{-1}(y) = -\frac{1}{\lambda} \ln(1 - y)$$

Hence, if  $U \sim U(0, 1)$ , then

$$-\frac{1}{\lambda} \ln(1 - U) = X \sim \exp(\lambda)$$

① When  $F_X^{-1}$  can be derived: **EASY**

② When **NOT**: numerical solution

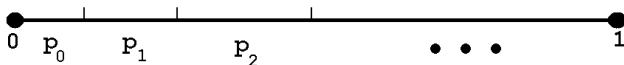
time-consuming

numerical errors ?

Situation 2 is common ... e.g.  $\mathcal{N}(0, 1)$

# Generating discrete RVs

- 1 Define distribution  $P(X = x_i) = p_i$
- 2 Generate  $U \sim \text{Unif}(0, 1)$
- 3 If  $U \leq p_0$ , set  $X = x_0$
- 4 Else if  $U \leq p_0 + p_1$ , set  $X = x_1$
- 5 ...



# Generating $\mathcal{N}(0, 1)$

Assume

- $\theta \in \text{Unif}(0, 2\pi)$
- $D \in \text{Unif}(0, 1)$

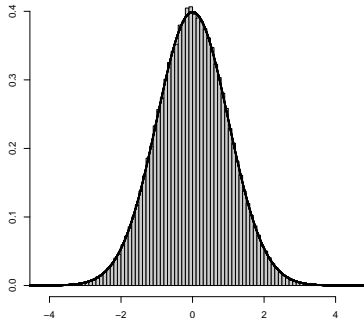
- 1: Generate  $\theta, D$
- 2: Generate  $X_1$  and  $X_2$  as

$$X_1 = \sqrt{-2 \ln D} \cos \theta$$

$$X_2 = \sqrt{-2 \ln D} \sin \theta$$

$X_1$  and  $X_2$  are independent and normally distributed

But finding such transformations is not easy





# Acceptance/rejection methods

- IDEA: generate  $Y \sim f_Y$  similar to some known PDF  $f_X$
- IDEA:  $f_Y$  is easy to generate from
- REQUIREMENT: there exists a constant  $c$

$$\forall_x c f_Y(x) \geq f_X(x)$$

- $f_Y$ : majorizing density, proposal density
- $f_X$ : target density
- $c$ : majorizing constant

# Acceptance/rejection methods

```
1: while  $X$  not generated do  
2:   Generate  $Y \sim f_Y$   
3:   Generate  $U \sim \text{Unif}(0, 1)$   
4:   if  $U \leq f_X(Y)/(cf_Y(Y))$  then  
5:      $X = Y$   
6:     Set  $X$  is generated  
7:   end if  
8: end while
```

- $X \sim f_X$  **CHECK THIS**
- Larger  $c$ : larger rejection rates— $c$  as small as possible  
number of draws  $\sim \text{Geometric}(1/c)$  mean:  $c$
- Can work in higher dimensions—**but** high rejection rate

# Acceptance/rejection methods: Example

Generate  $\text{beta}(2,7)$

```
y<-dbeta(seq(0,2,0.0001),2,7)
```

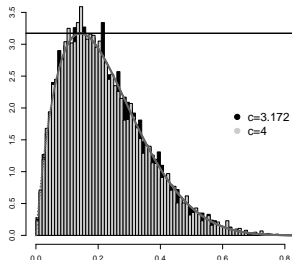
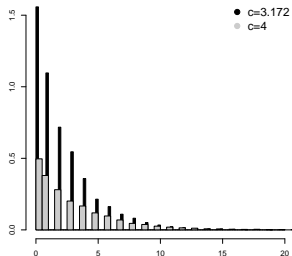
```
c<-max(y);c
```

```
[1] 3.172554
```

- 1: **while**  $X$  not generated **do**
- 2:   Generate  $Y \sim \text{Unif}(0, 1)$
- 3:   Generate  $U \sim \text{Unif}(0, 1)$
- 4:   **if**  $U \leq \text{dbeta}(Y, 2, 7)/(c \cdot 1)$  **then**
- 5:      $X = Y$
- 6:     Set  $X$  is generated
- 7:   **end if**
- 8: **end while**

**QUESTION:**

Compare acceptance and rejection regions (of  $Y$ ) for different  $c$ .



# Acceptance/rejection methods:

- Acceptance/rejection is difficult to apply
- Difficult to find majorizing density
  - can always take  $\sup(f_X) \cdot \text{Unif}(0, 1)$
  - but what is the problem?

# Generating multivariate normal

Generate  $\mathcal{N}(\vec{\mu}, \Sigma) \in \mathbb{R}^n$

- 1: Generate  $n$  i.i.d.  $\mathcal{N}(0, 1)$  r.vs.  $\vec{X} = (X_1, \dots, X_n)$   
{We know how to do this, see slide 16}
- 2: Compute Cholesky decomposition (a.k.a. matrix square root) of  $\Sigma$ , i.e. find  $\mathbf{A}$ , lower triangular s.t.  $\mathbf{A}\mathbf{A}^T = \Sigma$ ,  
{in R: `chol()` }
- 3:  $\vec{Y} = \mu + \mathbf{A}\vec{X}$

## QUESTION:

what is the expectation and variance-covariance of  $\vec{Y}$ ?

- ① `ddistribution name()`: density of distribution
- ② `pdistribution name()`: CDF of distribution
- ③ `qdistribution name()`: quantiles of distribution
- ④ `rdistribution name()`: simulate from distribution

- Computers generate pseudo-random numbers
- We draw from pseudo-uniform and transform to desired distribution
- Analytical methods for transforming exist but are distribution specific