Math 170E: Homework 8

Due: Wed 15th March by 11:59pm PDT via Gradescope

Submit answers to all problems via Gradescope. The reader will grade three problems each out of five points. Up to five further points will be awarded based on the proportion of the remaining problems that are completed.

Please make sure that your submission is readable. If your pencil is too faint, get a thicker one. If your handwriting is cramped and small, write bigger and use more paper. Please use simple plain paper or lined paper (e.g. please avoid graph paper etc.). It is your responsibility to ensure that your submission is readable. If we cannot read a solution, we may refuse to grade it. Thank you!

I encourage you to discuss and work on problems with other students in the class. Nevertheless, the solutions you present have to be your own. In particular, if the solution you present is identical to someone else's, or it is identical to some other resource (book, online, etc.), this will be considered cheating.

1. Let X and Y have the joint PDF $f_{X,Y}(x,y) = x + y$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$.
   (a) Show that $f_{X,Y}$ is a well-defined joint probability density function.
   (b) Find the marginal PDFs of X and Y.
   (c) Are X and Y independent?
   (d) Compute the means $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ and the variances $\mathrm{var}(X)$ and $\mathrm{var}(Y)$.
   (e) Compute the correlation coefficient of X and Y.

2. Let X and Y have joint PDF

$$f_{X,Y}(x,y) = 2e^{-x-y}, \quad 0 \leq x \leq y < \infty.$$

   Are X and Y independent?

3. Find the value of c such that the following functions are the joint PDF of two continuous random variables X and Y:
   (a) $f_{X,Y}(x,y) = cxy$ on $0 \leq x \leq 1$ and $x^2 \leq y \leq x$.
   (b) $f_{X,Y}(x,y) = cye^x$ on $0 \leq x \leq y^2$ and $0 \leq y \leq 1$.
   (c) $f_{X,Y}(x,y) = c\sin(x+y)$ on $0 \leq x, y \leq \frac{\pi}{2}$.

4. Two construction companies make bids of X and Y (in \$100,000's) on a remodeling project. The joint pdf of X and Y is uniform on the space $2 < x < 2.5, 2 < y < 2.3$. If X and Y are within 0.1 of each other, the companies will be asked to rebid; otherwise, the low bidder will be awarded the contract. What is the probability that they will be asked to rebid?

5. Let T_1 and T_2 be random times for a company to complete two steps in a certain process. Say T_1 and T_2 are measured in days and they have the joint pdf that is uniform over the space $1 < t_1 < 10, 2 < t_2 < 6, t_1 + 2t_2 < 14$. What is $\mathbb{P}(T_1 + T_2 > 10)$?

6. An automobile repair shop makes an initial estimate X (in thousands of dollars) of the amount of money needed to fix a car after an accident. Say X has the pdf

$$f_X(x) = 2e^{-2(x-0.2)}, \quad 0.2 < x < \infty.$$

   Given that $X = x$, the final payment Y has a uniform distribution between $x - 0.1$ and $x + 0.1$. What is the expected value of Y?

7. Let $X \sim \text{Uniform}((0, 2))$, and let the conditional distribution of Y, given that $X = x$, be $\text{Uniform}((0, x^2))$.
   (a) Determine $f_{X,Y}(x, y)$ the joint PDF of X and Y.
   (b) Calculate the marginal $f_Y(y)$.
   (c) Find the conditional mean $\mathbb{E}[X|y]$.

8. Let $X \sim \text{Uniform}((0, 1))$ and let $Y = e^X$. Find the PDF of Y.

9. Let $X \sim \text{Exponential}(1)$. Find, for $a > 0$, the PDF of $Y = e^{-\frac{X}{a}}$.