Math 170E: Winter 2023

Lecture 12, Wed 8th Feb

The Negative binomial and Poisson distributions

Last time:

- $X \sim \text{Binomial}(n, p)$ has PMF

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad \text{if } x \in \{0, 1, 2, \ldots, n\}$$

- $X \sim \text{Geometric}(p)$ has PMF

$$p_X(x) = (1-p)^{x-1} p \quad \text{if } x \in \{1, 2, 3, \ldots\}$$

\# of trials until we first achieve r successes,

- $X \sim \text{Negative Binomial}(r, p)$ has PMF

$$p_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \quad \text{if } x \in \{r, r+1, r+2, \ldots\}$$

Today:

We'll discuss today:

- how to compute the MGF, mean and variance of a negative binomial r.v.
- the formal derivation of Poisson distribution as a limit of the Binomial distribution
- the definition of an approximate Poisson process
- how to compute the MGF, mean and variance of a Poisson r.v.

Lemma 2.26:
If $r \geq 1$ is an integer and $0 < s < 1$, then

$$\left(\frac{1}{1-s}\right)^r = \sum_{x=r}^{\infty} \binom{x-1}{r-1} s^{x-r}$$

$1 + cs + \tilde{c}s^2 + \ldots$

Proof:

Idea: $\sum_{x=r}^{\infty} s^{x-r} = \frac{1}{1-s}$.

Take derivatives of both sides.

Taylor's Th$^{\underline{m}}$: $g(s) = (1-s)^{-r}$

Then, $g(s) = \sum_{l=0}^{\infty} \frac{d^l g}{ds^l}(0) \frac{1}{l!} s^l$

$g'(s) = (-1)(-r)(1-s)^{-r-1} = r(1-s)^{-r+1}$

$g''(s) = r(r+1)(1-s)^{-(r+2)}$

$\vdots$

$$g^{(l)}(s) = r(r+1)\cdots(r+l-1)(1-s)^{-(r+l)}$$

Hence $$g^{(l)}(0) = \frac{(r+l-1)!}{(r-1)!}$$

Therefore,

$$g(s) = \sum_{l=0}^{\infty} \frac{(r+l-1)!}{(r-1)!\,l!} s^l$$

$$= \sum_{l=0}^{\infty} \binom{r+l-1}{r-1} s^l \leftarrow$$

reindex:

put $l = x-r$

$\rightarrow x = l+r$

$$= \sum_{x=r}^{\infty} \binom{x-1}{r-1} s^{x-r}$$

Proposition 2.27: If $X \sim$ Negative Binomial(r, p), then its MGF is

$$M_X(t) = \left(\frac{pe^t}{1-(1-p)e^t}\right)^r \quad \text{if} \quad t < -\log(1-p)$$

Proof:

$$M_X(t) = E[e^{tX}] = \sum_{x=r}^{\infty} e^{tx} \binom{x-1}{r-1} p^r (1-p)^{x-r}.$$

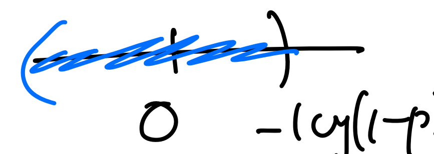
$$\left(\frac{1}{1-s}\right)^r = \sum_{x=r}^{\infty} \binom{x-1}{r-1} s^{x-r}$$

$$= p^r \sum_{x=r}^{\infty} \binom{x-1}{r-1} e^{tr} e^{t(x-r)} (1-p)^{x-r}$$

$$= p^r e^{tr} \sum_{x=r}^{\infty} \binom{x-1}{r-1} \underbrace{\left(e^t(1-p)\right)}_{0 < s = e^t(1-p) < 1.}{}^{x-r}.$$

$$= p^r e^{tr} \left(\frac{1}{1-e^t(1-p)}\right)^r$$

$$= \left(\frac{pe^t}{1-e^t(1-p)}\right)^r.$$



Proposition 2.28: If $X \sim$ Negative Binomial(r, p), then

$$\mathbb{E}[X] = \frac{r}{p}$$

$$\mathrm{var}(X) = \frac{r(1-p)}{p^2}.$$

Proof: $\log M_X(t) = r \log M_Y(t)$, where $Y \sim \mathrm{Geom}(p)$.

$$\mathbb{E}[X] = \frac{d}{dt} \log M_X \Big|_{t=0} = r \frac{d}{dt} \log M_Y \Big|_{t=0} = r/p.$$

$$\mathrm{Var}(X) = \frac{d^2}{dt^2} \log M_X \Big|_{t=0} = r \frac{d^2}{dt^2} \log M_Y \Big|_{t=0} = r \cdot \frac{1-p}{p^2}.$$

Example 15:

- You roll a fair six sided die over and over again.
- On average, how many rolls do you need in order to see a 6 three times?

Let X = # of trials until 3^{rd} 6

Then $X \sim \text{Neg.Bin}(3, 1/6)$

$\hookrightarrow E[X] = 3/(1/6) = 18.$

A) 20

B) 12

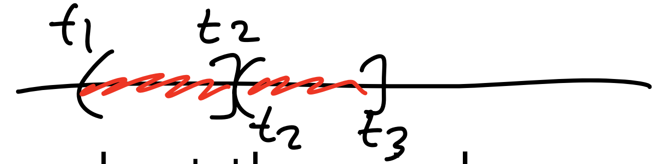
C) 32

D) 18

Problem:

- On average, $\lambda > 0$ customers arrive at a Walmart every hour
- Let X denote the number of customers arriving in 1 hour
- X takes values in $S = \{0, 1, \ldots\}$. (We assume the population is ∞)
- Can we describe the PMF of X?

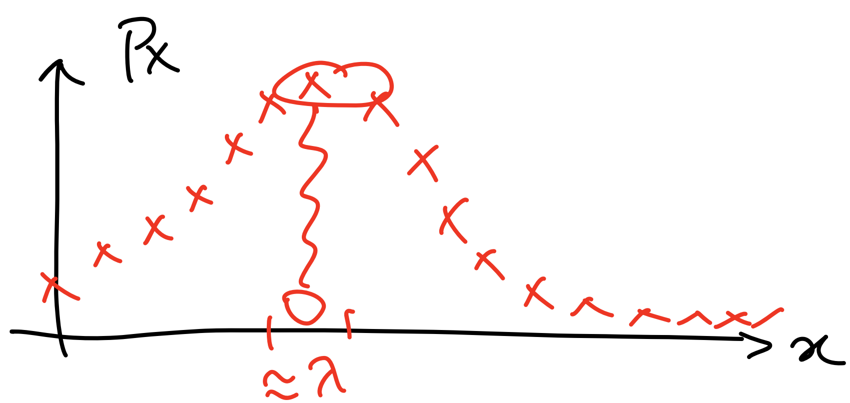
Assumptions:



- We make the following assumptions about the arrivals:
    1. If the time intervals $(t_1, t_2], (t_2, t_3], \ldots, (t_n, t_{n+1}]$ are *disjoint*, then the number of arrivals in each time interval are *independent*
    2. If $h = t_2 - t_1 > 0$ is sufficiently small, then the probability of exactly one arrival in the time interval $(t_1, t_2]$ is λh
    3. If $h = t_2 - t_1 > 0$ is sufficiently small, then the probability of two or more arrivals in the time interval $(t_1, t_2]$ converges rapidly to zero as $h \to 0$

- An arrival process satisfying these assumptions is called an approximate Poisson process

- The random variable X is called a Poisson random variable and we write $X \sim \text{Poisson}(\lambda)$

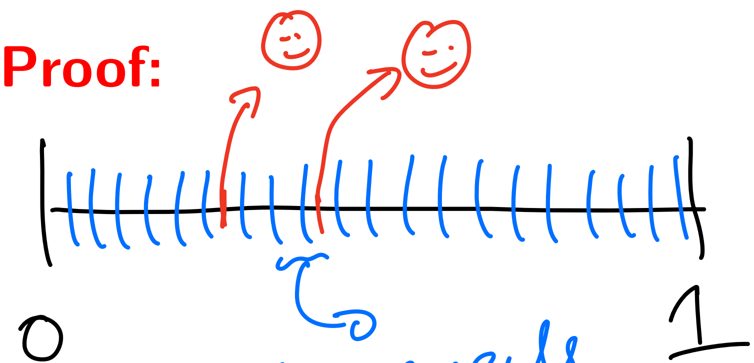


Proposition 2.29: If $X \sim \text{Poisson}(\cancel{p}\ \lambda)$, then its PMF is

$$p_X(\cancel{x}) = e^{-\lambda}\frac{\lambda^x}{x!} \quad \text{if} \quad x \in \{0, 1, 2, \ldots\}.$$

If x is very large, $x! \approx x^x$.
$P_X(x) \sim \dfrac{\lambda^x}{x^x}$
$= \exp(x\log\lambda - x\log x) \longrightarrow 0$ as $x \to +\infty$

Proof:



n subintervals of width $1/n$.

prob of one arrival ② $= \lambda \cdot 1/n$

- If n is suff. large, so that $1/n$ is suff. small, ③ then the # of arrivals in each subinterval is ① Bernoulli(λ/n) r.v. indep!

Over the n subintervals, we have
X_n = # of arrivals in the n subintervals

$\hookrightarrow$ $X_n \sim \text{Binomial}(n, \lambda/n)$.

Goal: If $\lambda > 0$ and $X_n \sim \text{Bin}(n, \lambda/n)$, we show that

$$p_{X_n}(x) = \mathbb{P}(X_n = x) \longrightarrow e^{-\lambda} \frac{\lambda^x}{x!} \quad \text{as } n \to \infty$$

for every $x \in \{0, 1, \ldots\}$.

Morally: If $X \sim \text{Poisson}(\lambda)$, then

$X \approx X_n \sim \text{Bin}(n, \lambda/n)$

for very big n.

Fix $x \in \{0, 1, \ldots\}$ and $\lambda > 0$.

$$\mathbb{P}(X_n = x) = \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}.$$

$$= \frac{\lambda^x}{x!} \underbrace{\frac{n!}{n^x (n-x)!}}_{1} \cdot \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{2} \cdot \underbrace{\left(1-\frac{\lambda}{n}\right)^{-x}}_{3} .$$

3 $\quad \left(1-\frac{\lambda}{n}\right)^{-x} \to 1^{-x} = 1 \text{ as } n \to \infty .$

1 $\quad \frac{n!}{n^x (n-x)!} = \frac{n(n-1) \cdots (n-(x+1))}{n^x}$

$$= \frac{n}{n} \cdot \frac{n-1}{n} \cdots\cdots \frac{n-(x+1)}{n} .$$

$$= 1 \cdot \left(1-1/n\right) \cdots \left(1 - \frac{x+1}{n}\right).$$

$$\to 1 \text{ as } n \to \infty .$$

② $$\left(1-\frac{\lambda}{n}\right)^n = \sum_{j=0}^{n} \binom{n}{j} \frac{(-\lambda)^j}{n^j}$$

Binomial theorem

$$= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \frac{(-\lambda)^j}{n^j}$$

$$= \sum_{j=0}^{n} \boxed{\frac{n!}{n^j(n-j)!}} \cdot \frac{(-\lambda)^j}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{n!}{n^j(n-j)!} \frac{(-\lambda)^j}{j!} \mathbb{1}_{[0,n]}(j)$$

$$\mathbb{1}_{[0,n]}(j) = \begin{cases} 1 & \text{if } j \in [0,n] \\ 0 & \text{if } j \notin [0,n] \end{cases} = \begin{cases} 1 & \text{if } 0 \le j \le n \\ 0 & \text{if } j > n. \end{cases}$$

$$\lim_{n\to\infty} \left(1-\lambda/n\right)^n = \lim_{n\to\infty} \sum_{j=0}^{\infty} \frac{n!}{n^j(n-j)!} \frac{(-\lambda)^j}{j!} \mathbb{1}_{[0,n]}(j)$$

assuming we can (?) need a Th

$$= \sum_{j=0}^{\infty} \underbrace{\lim_{n\to\infty}\left(\frac{n!}{n^j(n-j)!}\right)}_{=1.} \cdot \frac{(-\lambda)^j}{j!} \underbrace{\lim_{n\to\infty} \mathbb{1}_{[0,n]}(j)}_{= \mathbb{1}_{[0,\infty)}(j) = 1.}$$

$$= \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} = e^{-\lambda}.$$

$$\hookrightarrow \lim_{n\to\infty} \mathbb{P}(X_n = x) = e^{-\lambda} \frac{\lambda^x}{x!} = \text{PMF for Poisson}(\lambda).$$

Example 16:

- You have one gram of Uranium-235.
- You count the number of α-particles it emits in a 1-second time interval
- You observe that on average it emits 3 α particles per second
- What is a good approximation to the probability that no more than 2 α particles appear?

n = # of atoms in 1-gram of U-235, $n \gg 1$

$\hookrightarrow$ Each individual atom has a prob of $3/n \ll 1$ of emitting an α-particle in 1-second

X = # of particles emitted in 1-second.

$\hookrightarrow X \sim \text{Poisson}(3)$

$$\mathbb{P}(X \leq 2) = p_X(0) + p_X(1) + p_X(2)$$
$$= e^{-3} + 3e^{-3} + \frac{3^2}{2!}e^{-3} = \frac{17}{2e^3} \simeq 0.41$$