Math 170E: Winter 2023

Lecture 7, Wed 25th Jan

Discrete random variables

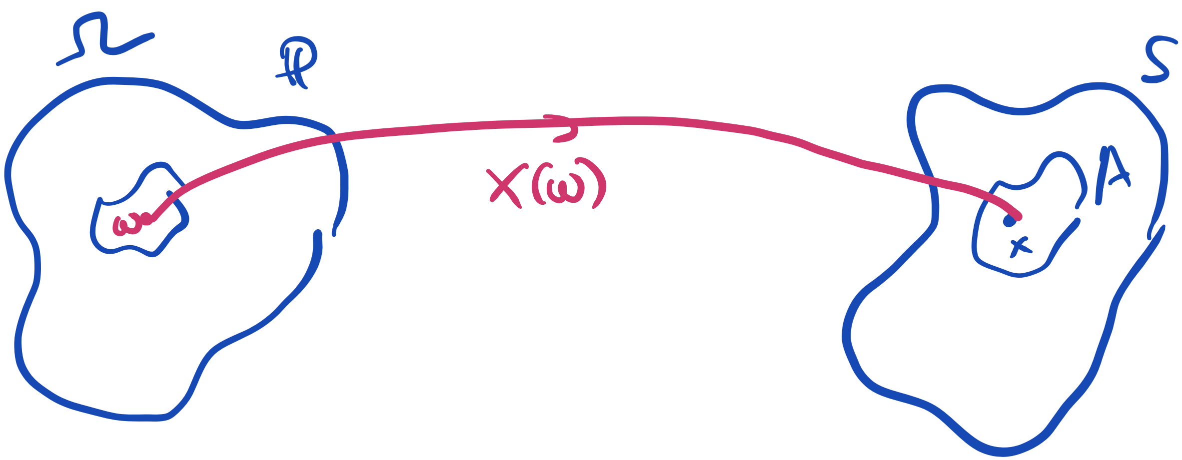
Definition 2.1: (**Random variable**)

Given a set S and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a **random variable** (r.v.) is a function

$$X : \Omega \to S$$

<u>Notation:</u> If $x \in S$, and $A \subseteq S$, we write

$$\mathbb{P}(X = x) := \mathbb{P}(\{\omega \in \Omega \,:\, X(\omega) = x\})$$
$$\mathbb{P}(X \in A) := \mathbb{P}(\{\omega \in \Omega \,:\, X(\omega) \in A\})$$



Example 1: You flip two fair coins.

- The state space is $\Omega = \{HH, HT, TH, TT\}$
- We define the random variable X as the number of heads that appear i.e.

$$X(\{HH\}) = 2, \quad X(\{HT\}) = X(\{TH\}) = 1, \quad X(\{TT\}) = 0.$$

- $X : \Omega \to \{0, \boxed{1, 2}\} \subseteq \mathbb{N}$

What is $\mathbb{P}(X \geq 1)$?

$$\mathbb{P}(X \geq 1) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \geq 1\})$$

$$= \mathbb{P}(\boxed{\{\omega \in \Omega : X(\omega) = 1\}} \cup \underbrace{\{\omega \in \Omega : X(\omega) = 2\}}_{\mathbb{P}(X=2)})$$

$\mathbb{P}(X=1)$

$$= \mathbb{P}(\{HT, TH\}) + \mathbb{P}(\{HH\})$$

$$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

Example 2: You flip two fair coins.

- We define the random variable Y as the number of *tails* that appear i.e.

$$Y(\{HH\}) = 0, \quad Y(\{HT\}) = Y(\{TH\}) = 1, \quad Y(\{TT\}) = 2.$$

- $Y : \Omega \to \{0, 1, 2\} \subseteq \mathbb{N}$

What is $\mathbb{P}(Y \geq 1)$? Is $X = Y$?

$$\mathbb{P}(Y \geq 1) = \mathbb{P}(\{\omega \in \Omega : Y(\omega) \geq 1\})$$
$$= 1 - \mathbb{P}(Y < 1)$$
$$= 1 - \mathbb{P}(\{\omega \in \Omega : Y(\omega) = 0\})$$
$$= 1 - \mathbb{P}(\{HH\})$$
$$= 1 - \frac{1}{4} = \frac{3}{4}$$

"identically distributed"?

$$\boxed{\begin{aligned} \mathbb{P}(X=1) &= \mathbb{P}(Y=1) = \tfrac{1}{2} \\ \mathbb{P}(X=0) &= \mathbb{P}(Y=0) = \tfrac{1}{4} \\ \mathbb{P}(X=2) &= \mathbb{P}(Y=2) = \tfrac{1}{4} \end{aligned}}$$

$X(\{HH\}) = 2$
$\neq$
$Y(\{HH\}) = 0$

$X = Y$ if and only if $X(\omega) = Y(\omega)$ for all $\omega \in \Omega$

Definition 2.2: (**Discrete random variable**)

A random variable $X : \Omega \to S$ is **discrete** if $S \subseteq \mathbb{R}$ is finite or countable (i.e. in one-to-one correspondence with $\mathbb{N}$).

We can think of discrete r.v.s as random numbers

$$\begin{array}{ccccc} 1, & 2, & 3, & 4, & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ (a_1, & a_2, & a_3, & a_4, & a_5, \ldots) \end{array}$$

Definition 2.3: Given a discrete r.v. X taking values in $S \subseteq \mathbb{R}$, we define:

- the **probability mass function** (PMF) of X as the function $p_X : S \to [0,1]$ defined by

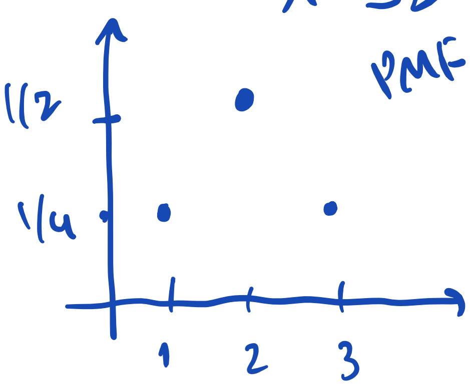
$$p_X(x) = \mathbb{P}(X = x).$$

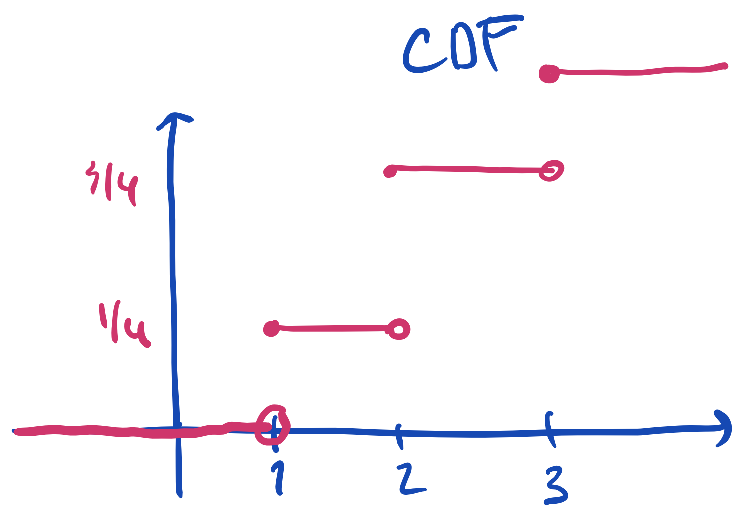
- the **cumulative distribution function** (CDF) of X as the function $F_X : \mathbb{R} \to [0,1]$ defined by

$$F_X(x) = \mathbb{P}(X \leq x).$$

- we say that two r.v.s X and Y are **identically distributed** if they have same CDF and we write $X \sim Y$

$X : \Omega \to \{1,2,3\}$





Example 3: Uniform random variables

Let $m \geq 1$. A discrete r.v. X is uniformly distributed on $\{1, 2, \ldots, m\}$ and we write

$X: \Omega \to \{1, 2, \ldots, m\}$

$$X \sim \text{Uniform}(\{1, 2, \ldots, m\}),$$

if it has PMF

$$p_X(x) = \frac{1}{m}, \quad \text{for } x \in \{1, 2, \ldots, m\}.$$

Which of the following could you model with this random variable?

A) the outcome of a fair die roll $\sim \text{Uniform}(\{1, \ldots, 6\})$

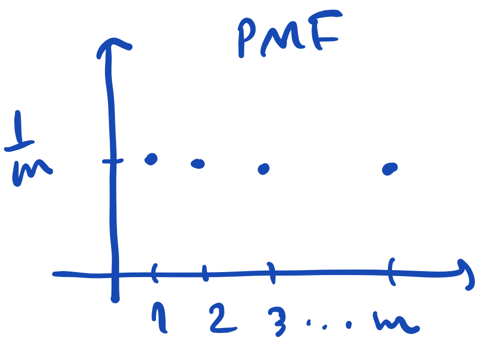
B) the sum of two die rolls $\times$

C) the outcome of a weighted coin flip $\times$

D) the outcome of a fair coin flip $m = 2$

If $X \sim \text{Uniform}(\{1, 2, \dots, m\})$, then it has CDF

$$F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{k}{m} & \text{if } k \leq x \leq k+1, \quad k \in \{1, \dots, m-1\} \\ 1 & \text{if } x \geq m \end{cases}$$



$\cdot\ x < 1, \quad F_X(x) = \mathbb{P}(X \leq x) = 0$

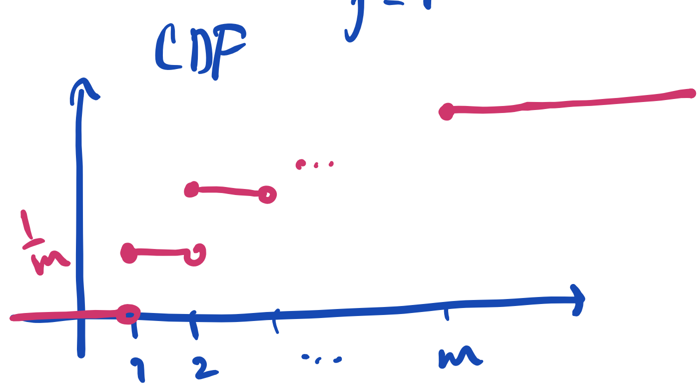
$\cdot\ F_X(1) = \mathbb{P}(X \leq 1) = \mathbb{P}(X = 1) = \frac{1}{m}$

$\cdot\ F_X(2) = \mathbb{P}(X \leq 2) = \mathbb{P}(\{X=1\} \cup \{X=2\})$

$= \mathbb{P}(X=1) + \mathbb{P}(X=2) = \frac{1}{m} + \frac{1}{m} = \frac{2}{m}$

$\cdot\ k \in \{1, \dots, m\}, \quad F_X(k) = \mathbb{P}(X \leq k) = \sum_{j=1}^{k} \mathbb{P}(X = j) = \sum_{j=1}^{k} \frac{1}{m} = \frac{k}{m}$

$\cdot\ k = m, \quad F_X(m) = \frac{m}{m} = 1$



Proposition 2.4: If X is a discrete r.v. and $A \subseteq \mathbb{R}$ is any set, then

$$\mathbb{P}(X \in A) = \sum_{x \in A \cap S} p_X(x). \qquad X : \Omega \to S$$

Proof:

$$\begin{aligned}
\mathbb{P}(X \in A) &= \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}) \\
&= \mathbb{P}(\{\omega \in \Omega : X(\omega) \in \boxed{A \cap S}\}) \\
&= \mathbb{P}\Big(\bigcup_{j=1}^{\infty} \{\omega \in \Omega : X(\omega) = a_j\}\Big) \\
&\overset{\text{countable addit.}}{=} \sum_{j=1}^{\infty} \mathbb{P}(\{\omega \in \Omega : X(\omega) = a_j\}) \\
&= \sum_{j=1}^{\infty} \mathbb{P}(X = a_j) = \sum_{j=1}^{\infty} p_X(a_j) = \sum_{x \in A \cap S} p_X(x). \quad \square
\end{aligned}$$

$A \cap S \subseteq S$

$\downarrow$

$(a_1, a_2, a_3, \dots)$

Proposition 2.5: If X is a discrete r.v. and $\cancel{A \subseteq \mathbb{R}}$ is any set, then

$$F_X(x) = \sum_{\substack{y \in S \\ y \le x}} p_X(y).$$

Proof:

$$F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(X \in \overset{= A}{(-\infty, x]})$$

$$= \sum_{y \in \boxed{A \cap S}} p_X(y) \longrightarrow y \in S \text{ and } y \le x$$

$$= \sum_{\substack{y \in S \\ y \le x}} p_X(y)$$

$y \in A \cap S$

$\underbrace{y \in A}_{y \in (-\infty, x]}$ and $y \in S$

$y \le x$

Proposition 2.6: If X is a discrete r.v. and $a < b$, then

$$\mathbb{P}(a < X \leq b) = F_X(b) - F_X(a).$$

⚠ Now, we know how to calculate $\mathbb{P}(X \in I)$ for any interval $I \subseteq \mathbb{R}$

Proof:

$$\mathbb{P}(a < X \leq b) = \mathbb{P}(X \in \underbrace{(a,b]}_{=A})$$

$$\overset{\text{Prop. 2.4}}{=} \sum_{x \in A \cap S} p_X(x)$$

$x \in S$ and $\underbrace{x \in A}_{a < x \leq b}$

$$= \sum_{\substack{x \in S \\ a < x \leq b}} p_X(x)$$

$$= \sum_{\underbrace{\substack{x \in S \\ x \leq b}}_{(-\infty, b] \cap S}} p_X(x) - \sum_{\underbrace{\substack{x \in S \\ x \leq a}}_{(-\infty, a] \cap S}} p_X(x)$$

Example 5: A gambler plays a game on the flip of a weighted coin. They

- win x_1 dollars on a HEAD
- lose x_2 dollars on a TAIL

The coin is weighted so that $\mathbb{P}(\{H\}) = p$ and $\mathbb{P}(\{T\}) = 1 - p$ for some $0 < p < 1$. The gambler flips the coin N times.
What is the theoretical average winnings of the gambler in any given flip?

average times that I see H $\rightarrow$ Np $\rightsquigarrow$ x_1
" " " " " T $\rightarrow$ $N(1-p)$ $\rightsquigarrow$ $-x_2$

$$\frac{1}{N}\left[(\mathbb{P}(\{H\})N)x_1 + (\mathbb{P}(\{T\})N)(-x_2)\right]$$

$$= \mathbb{P}(\{H\})x_1 + \mathbb{P}(\{T\})(-x_2) = x_1\,\mathbb{P}(X = x_1) + (-x_2)\,\mathbb{P}(X = -x_2)$$

$$X : \Omega \longrightarrow \{x_1, -x_2\}$$

Definition 2.7: (**Expected value**)

If X is a discrete random variable taking values in a countable set $S \subseteq \mathbb{R}$, its expected value is defined to be

$$\mathbb{E}[X] = \sum_{x \in S} x \underbrace{p_X(x)}_{\mathbb{P}(X=x)},$$

provided the sum converges.

Notation: we also write $\mu_X = E[X]$.

"μ_X = the best guess for the value of X".