Math 170E: Winter 2023

Lecture 2, Wed 11th Jan

Properties of probability spaces

Last time:

We reviewed some concepts in set theory:

- union $\cup$
- intersection $\cap$
- complement $\Omega \setminus A$, A'
- De Morgan's laws

Today:

Today, we'll begin discussing **probability**:

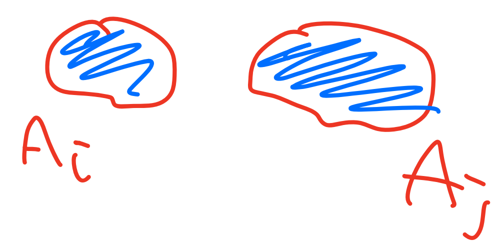
- Definition of a probability space
- Properties of a probability measure
- How to compute probabilities using the *inclusion-exclusion* principle

Definition 1.1: A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where:

- Ω is a nonempty set called the state space, $\rightsquigarrow$ finite or infinite.
- $\mathcal{F}$ is a collection of subsets of Ω (note: $\emptyset, \Omega \in \mathcal{F}$)
  - An element $A \in \mathcal{F}$ is called an event and $A \subseteq \Omega$
  - (In this course, $\mathcal{F} = \{\text{all subsets of } \Omega\}$)
- A function $\mathbb{P} : \mathcal{F} \mapsto [0, 1]$ is called a probability measure if it satisfies:

1. $\mathbb{P}(A) \geq 0$ for any $A \in \mathcal{F}$
2. $\mathbb{P}(\Omega) = 1$
3. (Countable additivity) If $\{A_j\}_{j=1}^{k}$ are events, such that $\overbrace{A_i \cap A_j = \emptyset}^{\text{disjoint}}$ for $i \neq j$ (mutually exclusive), then

$$\mathbb{P}\left(\bigcup_{j=1}^{k} A_j\right) = \sum_{j=1}^{k} \mathbb{P}(A_j),$$



and (when "$k = +\infty$"),

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(A_j),$$

Comments:

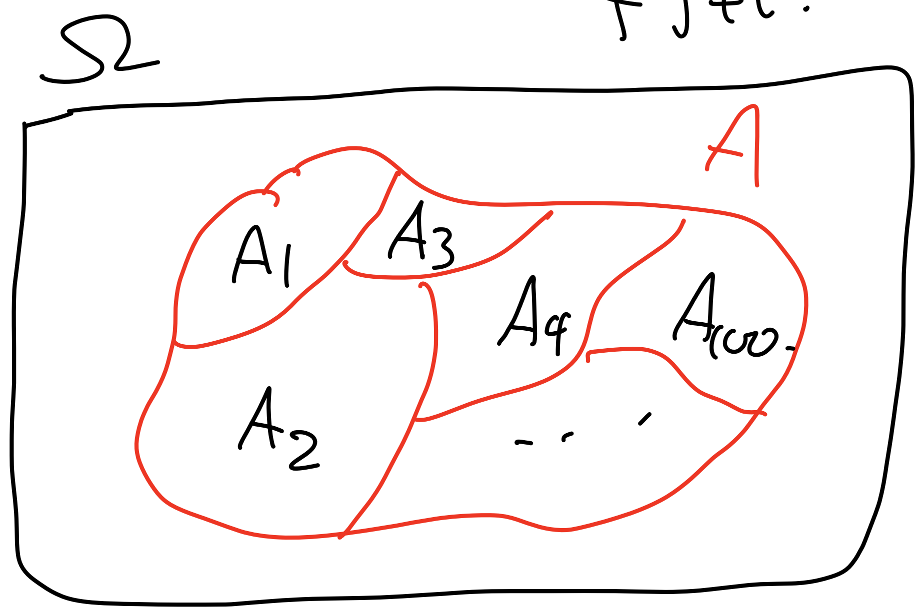
Ω = the set of all possible outcomes of some random experiment.

- Given $A \in \mathcal{F}$, $\mathbb{P}(A)$ represents the "probability of A occurring"

- Countable additivity $\{A_j\}_{j=1}^{100} \to$ mutually exclusive: $A_j \cap A_i = \phi$ if $j \neq i$.

$\hookrightarrow$ event: $A = \bigcup_{j=1}^{100} A_j$

$$\mathbb{P}(A) = \sum_{j=1}^{100} \mathbb{P}(A_j)$$



Example 1.1: You flip a fair coin twice. What is the probability that you see at least one head? Probability of HEADS or TAILS equal.

Enumerate all the outcomes: $\underbrace{HH, HT, TH}_{\text{3 possible outcomes at least 1H}}, \cancel{TT}$ ← 4 total outcomes.

↳ $\mathbb{P}(\text{at least } 1H) = 3/4$, ↳ $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$ ↳ $|A|$ = # of elements in A.

Construct a probability space: ↳ $(\Omega, \mathcal{F}, \mathbb{P})$.

$\Omega = \{HH, HT, TH, TT\}$.

$\mathcal{F}$ = set of all subsets of $\Omega = \{\phi, \{HH\}, \{TT\}, \{HT\}, \{TH\}, \{HH, HT\}, \{TH, TT\}, \ldots, \Omega\}$

$2^4 = 16$ non-empty events.

Define $\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{4}$ for $A \in \mathcal{F}$.

Check: {$\mathbb{P}$ satisfies properties (1), (2), (3)

$|A_1 \cup A_2| = |A_1| + |A_2|$.

disjoint

$\hookrightarrow \mathbb{P}(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$.

WANT: $\mathbb{P}(\text{at least } 1H) = \mathbb{P}(A)$

where A is the event we get at least 1H

$\mathbb{P}(A) = \mathbb{P}(\{HH, HT, TH\})$

$\{HH, HT\} \cup \{TH\}$ $= \frac{|\{HH, HT, TH\}|}{|\Omega|} = 3/4.$

$A = A_1 \cup A_2 \cup A_3$

$= \underbrace{\{HH\} \cup \{HT\} \cup \{TH\}}_{\text{mutually exclusive.}}$

Countable additivity:

$\mathbb{P}(A) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3)$

$= \frac{|A_1|}{4} + \frac{|A_2|}{4} + \frac{|A_3|}{4} = 3/4.$

Properties of $\mathbb{P}$:

Proposition 1.2: $\mathbb{P}(\emptyset) = 0$

Proof: Since $\Omega = \Omega \cup \phi$, $\Omega \cap \phi = \phi$

The sets Ω & ϕ are mutually exclusive $(\Omega, \phi \in F)$

So by countable additivity:

$$1 \overset{2}{=} \mathbb{P}(\Omega) = \mathbb{P}(\Omega \cup \phi)$$

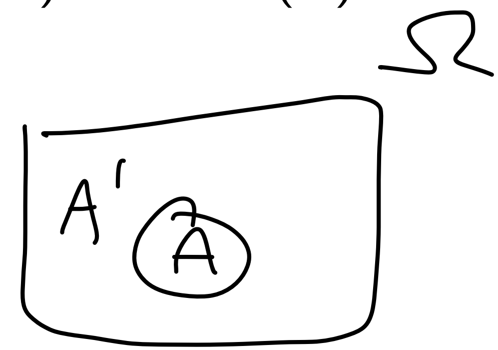
$$\overset{3}{=} \mathbb{P}(\Omega) + \mathbb{P}(\phi)$$

$$\overset{2}{=} 1 + \mathbb{P}(\phi)$$

$$\Rightarrow \mathbb{P}(\phi) = 0.$$

Proposition 1.3: If A is an event, then A' is an event and $\mathbb{P}(A') = 1 - \mathbb{P}(A)$

Proof:



We write $\Omega = A \cup A'$, $A \cap A' = \phi$

So A & A' are mutually exclusive.

So

$$1 \overset{(2)}{=} \mathbb{P}(\Omega) = \mathbb{P}(A \cup A') \overset{(3)}{=} \mathbb{P}(A) + \mathbb{P}(A')$$

$$\hookrightarrow \quad \mathbb{P}(A') = 1 - \mathbb{P}(A).$$

Example 2: I flip two fair coins. What is the probability of getting no heads?

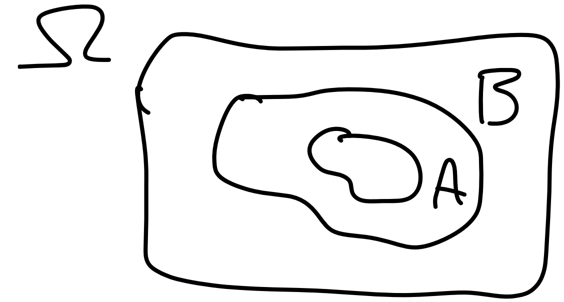
Let $A = \{\text{at least one HEAD}\} = \{HH, HT, TH\}$

$B = \{\text{no HEADS}\} = \{TT\}$. $\hookrightarrow \mathbb{P}(B) = \frac{|B|}{4} = 1/4$.

$\hookrightarrow A' = B$. $\hookrightarrow \mathbb{P}(B) = 1 - \mathbb{P}(A)$

$= 1 - 3/4 = 1/4$.

$\hookrightarrow \mathbb{P}(A) = 1 - \mathbb{P}(B) = 1 - 1/4 = 3/4$.



Proposition 1.4: If $A \subseteq B$, then $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$

Proof: • $B = (B \setminus A) \cup A \Rightarrow$ mutually exclusive.

$$(B \setminus A) \cap A = \phi$$

So
$$\mathbb{P}(B) = \mathbb{P}\big((B \setminus A) \cup A\big) \overset{③}{=} \mathbb{P}(B \setminus A) + \mathbb{P}(A)$$

Rearrange: $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$.

Corollary 1.5: If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$ $\rightarrow$ $\mathbb{P}$ is "increasing"

Proof: $\mathbb{P}(B) = \mathbb{P}(B \setminus A) + \mathbb{P}(A)$
$\geq 0 + \mathbb{P}(A) = \mathbb{P}(A)$. //

If we put $B = \Omega$, $\Rightarrow$ $\mathbb{P}(A) \leq 1$ for all events $A \in \mathcal{F}$.

- Cor. 1.5 implies $\mathbb{P}(A) \leq \mathbb{P}(\Omega) = 1$

⚠: if you calculate a probability and it's > 1, it's wrong! (also if < 0)

Example 3: I roll a fair 20 sided die. Which of the following is true?

$$\mathbb{P}(\text{rolled a} \# \geq 5) \leq \mathbb{P}(\text{rolled a} \# \geq 11)$$

$$\mathbb{P}(\underbrace{\text{rolled a} \# \geq 5}_{A}) \geq \mathbb{P}(\underbrace{\text{rolled a} \# \geq 11}_{B})$$

$\Omega = \{1, 2, 3, \ldots, 20\}$.

$A = \{\text{rolled} \geq 5\} = \{5, 6, \ldots, 20\}$.

$B = \{\text{rolled} \geq 11\} = \{11, 12, \ldots, 20\}$.

So $B \subseteq A$ so Corollary 1.5,

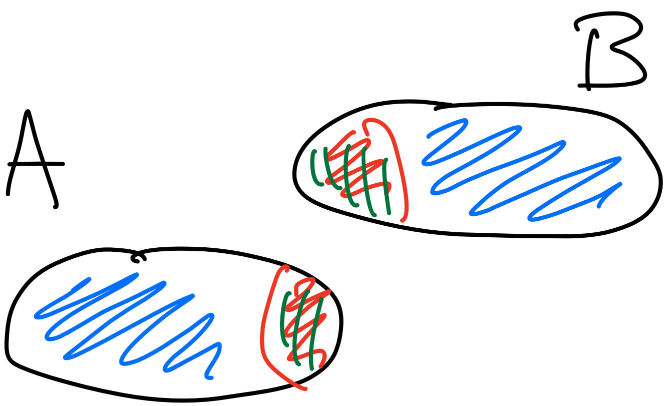
$\mathbb{P}(A) \geq \mathbb{P}(B)$.

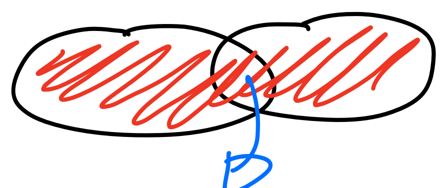
Example 3 (cont.): I now roll a weighted 20 sided die. Which of the following is true?

$$\mathbb{P}(\text{rolled a } \# \geq 5) \leq \mathbb{P}(\text{rolled a } \# \geq 11)$$

$$\underbrace{\mathbb{P}(\text{rolled a } \# \geq 5)}_{\mathbb{P}(A) = \sum_{j=5}^{20} \mathbb{P}(\{j\})} \geq \underbrace{\mathbb{P}(\text{rolled a } \# \geq 11)}_{\mathbb{P}(B) = \sum_{j=11}^{20} \mathbb{P}(\{j\})} \leftarrow \text{TRUE}$$

$$\mathbb{P}(A) = \sum_{j=5}^{20} \mathbb{P}(\{j\}) \geq \mathbb{P}(B) = \sum_{j=11}^{20} \mathbb{P}(\{j\}).$$





$$\text{Area}(A \cup B)$$
$$= \text{Area}(A) + \text{Area}(B) - \text{Area}(A \cap B),$$
$$= \text{Area}(A \setminus (A \cap B)) + \text{Area}(B \setminus (A \cap B)) + \text{Area}(A \cap B)$$

Theorem 1.6: (The inclusion-exclusion principle) For any events A and B,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Proof: $A \cup B = \underbrace{[A \setminus (A \cap B)] \cup [B \setminus (A \cap B)] \cup [A \cap B]}_{\text{mutually exclusive}}$.

So countable additivity,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A \setminus (A \cap B)) + \mathbb{P}(B \setminus (A \cap B)) + \mathbb{P}(A \cap B).$$

Prop. 1.4

$$= \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B)$$

$$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$