6.3-Higher-Order Linear Systems

#UCLA #Y1Q3 #Math33B

Higher-Order Linear Systems

Key Definitions

Limited to homogenous, constant coefficient, linear higher order differentials

Determinant by Laplace (Cofactor) Expansion:

$$\det(A) = \sum_{i}^{n} a_{ij}(-1)^{i+j}\det(cof(A_{ij}))$$

Steps

1. Convert nth order to nxn matrix
2. Solve linear system
3. Convert to linear differential equation

Solution

Given nth order diff. eq.

Auxiliary Functions

$$x_1(t) := y(t)$$

S.t.

$$x_1' = x_2$$

and so on.

Then, create a nxn matrix of aux. funcs.:

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= x_3(t) \\ x_3'(t) &= x_4(t) \end{aligned}$$

What about $x_4'(t) = y^{(4)}(t)$? The original differential equation itself tells us how to relate this to the lower derivatives:

$$x_4'(t) \;=\; 13y''(t) - 36y(t) \;=\; 13x_3(t) - 36x_1(t)$$

Combining these four equations yields the system:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 0 & 13 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

General Solution

$$\vec{x}(t; C_i) = \sum_{i}^{N} C_i e^{\lambda_i t} \vec{v}_i$$

Such that, we can find the original diff. eq.

$$y(t; C_i) = x_1(t) = \sum_{i}^{N} C_i e^{\lambda_i t} \vec{v}_{i,1}$$

E.g.

$$\mathbf{x}(t;C_1,C_2,C_3,C_4) = e^{2t}\begin{bmatrix}1\\2\\4\\8\end{bmatrix} + C_2e^{-2t}\begin{bmatrix}-1\\2\\-4\\8\end{bmatrix} + C_3e^{3t}\begin{bmatrix}1\\3\\9\\27\end{bmatrix} + C_4e^{-3t}\begin{bmatrix}-1\\3\\-9\\27\end{bmatrix}$$

$$= \begin{bmatrix} C_1e^{2t} - C_2e^{-2t} + C_3e^{3t} - C_4e^{-3t} \\ 2C_1e^{2t}2C_2e^{-2t} + 3C_3e^{3t} + 3C_4e^{-3t} \\ 4C_1e^{2t} - 4C_2e^{-2t} + 9C_3e^{3t} - 9C_4e^{-3t} \\ 8C_1e^{2t} + 8C_2e^{-2t} + 27C_3e^{3t} + 27C_4e^{-3t} \end{bmatrix}$$

In particular, the general solution to $y^{(4)} - 13y'' + 36y = 0$ is

$$y(t;C_1,C_2,C_3,C_4) = x_1(t) = C_1e^{2t} - C_2e^{-2t} + C_3e^{3t} - C_4e^{-3t}$$

which we might as well instead write as

$$y(t) = C_1e^{2t} + C_2e^{-2t} + C_3e^{3t} + C_4e^{-3t} \qquad \square$$

General Solution

We can find a solution of form:

$$\vec{x}' = A\vec{x}$$

where A is the **companion matrix** and if:

$$\mathbf{x}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix}$$

So we get the equation in matrix form:

We then form the linear system:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_{n-1}'(t) \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \vdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}$$

Then, if $y_1, \ldots, y_n$ are solutions to the nth order differential equation, we ca get the vector valued functions:

$$\begin{bmatrix} y_1(t) \\ y_1'(t) \\ y_1''(t) \\ \vdots \\ y_1^{(n-1)}(t) \end{bmatrix}, \ldots, \begin{bmatrix} y_n(t) \\ y_n'(t) \\ y_n''(t) \\ \vdots \\ y_n^{(n-1)}(t) \end{bmatrix}$$

For which, the 4.1-2nd Order Linear Differentials > Wronskian $\neq$ 0

$$W(t) = \det \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{n-1}(t) & \cdots & y_n^{(n-1)}(t) \end{bmatrix}$$

Thus the matrix has linearly independent column vectors $\vec{y}_1, \ldots, \vec{y}_n$

Then finally, we get the **general solution**:

$$Y(T) = \sum_{I}^{N} C_I \vec{Y}_I(T)$$