CS 181: May 15th Notes - CFL Closure #9

9.1 Union & Concat: Thm: L', L'' context-free $\Rightarrow L' \cup L''$ & $L' \oplus L''$ CFLs (Closed)

Pf.

- $CFG(L') = (V', \Sigma, R', S')$, $CFG(L'') = (V'', \Sigma, R'', S'')$: $V' \cap V'' = \emptyset$
- $S \notin V' \cup V''$
- $L' \cup L''$: $S \to S' | S''$ $\Rightarrow$ union CFL
- i.e. $CFG(L' \cup L'')$: $(V' \cup V'' \cup \{S\}, \Sigma, R' \cup R'' \cup \{S \to S' | S''\})$
- $CFG(L' \oplus L'') = (V' \cup V'' \cup \{S\}, \Sigma, R' \cup R'' \cup \{S \to S'S''\})$

9.2 Kleene Star: Thm: L is CFL $\Rightarrow L^*$ is CFL (Closed)

- $CFG(L) = (V, \Sigma, R, S)$ s.t. $S' \notin V$
- $CFG(L^*) = (V \cup \{S'\}, \Sigma, R \cup \{S' \to \varepsilon | SS'\}, S')$ $\therefore L^*$ is CFL

Cor: Every ~~Reg.~~ Reg. Lang. is a CFL

Pf: (1) ε, (2) $\emptyset$, (3) $\sigma \in \Sigma$ $\forall \sigma$ are context-free & reg.

- all reg langs arise from the above w/ union, concat, or kleene star (reg. ops)
  $\therefore$ resulting lang must be context-free $\because$ CFLs are closed under reg. ops.
- pf by Kleene's Thm.

9.3 Intersection: Thm: CFLs not closed under intersection (NOT Closed)

Pf:

$$\underbrace{L_1 = \{a^n b^n c^m\}}_{CFL} \cap \underbrace{L_2 = \{a^m b^n c^n\}}_{CFL} = \boxed{\{a^n b^n c^n\}} \text{ not CFL}$$

Cor: Complement not closed (NOT closed)

- $L = \{a^n b^n c^n\}$ not CFL but $\overline{L} = \{a^n b^m c^k\}$ is CFL

- De Morgan's: spse. L_1, L_2 are closed under complement, then $\overline{L_1} \cup \overline{L_2}$ should be a CFL, but $\underbrace{\overline{L_1} \cup \overline{L_2}}_{\text{should be CFL}} = \underbrace{\overline{L_1 \cap L_2}}_{\text{not CFL}}$ $\Rightarrow$ Contradiction to intersection

Cor: Set difference not closed (NOT closed)

Pf:

$\overline{L} = \Sigma^* \setminus L \Rightarrow L$ is CFL but $\overline{L}$ not CFL $\Rightarrow$ not closed

Thm: if L' reg. & L'' is CFL, $\Rightarrow L' \cap L''$ is CFL

Pf:

$\text{DFA}(L') = (Q', \Sigma, \delta', q_0', F')$ &

$\text{PDA}(L'') = (Q'', \Sigma, \delta'', \Gamma, q_0'', F'')$

then,

$$\text{PDA}(L' \cap L'') = (Q' \times Q'', \Sigma, \Gamma, \delta, (q_0', q_0''), F' \times F'')$$

s.t. $\delta((q', q''), \sigma, \gamma) = \{\underbrace{\{\delta'(q', \sigma)\}}_{\subseteq Q'} \times \underbrace{\delta''(q'', \sigma, \gamma)}_{\subseteq Q'' \times \Gamma}\} \in Q' \times Q'' \times \Gamma$

Cartesian product

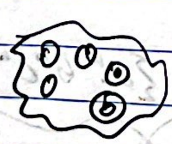
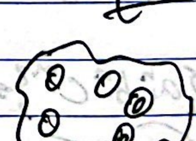
if $\sigma \in \Sigma$, $\subseteq Q' \times Q'' \times \Gamma$

elif $\sigma = \varepsilon$:

$$\delta((q', q''), \varepsilon, \gamma) = \{q'\} \times \delta''(q'', \sigma, \gamma)$$

9.4 Prefix & Suffix: Thm: L is CFL $\Rightarrow$ prefix(L) is CFL (closed)

Pf: (prefix)

given PD(A) L =  , create a copy s.t.  } change transition labels to $\varepsilon, \gamma \to \gamma'$

now map each state in original to its copy in the copy. $\Rightarrow$ PDA(prefix(L))

Pf. (suffix): given CFL L, suffix(L) is CFL (closed)

given PDA(L) =  , we make 2 copies:

PDA(suffix(L))  } change transitions to $\varepsilon, \gamma \to \gamma'$ from σ, γ, γ'

(Closed)

9.5 Substring! Def: $w = w_1 \ldots w_n$, substring$(w) = w_i w_{i+1} \cdots w_j$ s.t. $j \geq i$ $\forall i \in |w|$

i.e. substring$(L) = \{a : a$ is a substring of $w \in L\}$

Thm: ~~~~ L is CFL $\Rightarrow$ substring(L) is CFL

Pf.:

given CFL L, substring(L) = ~~~~ Suffix(prefix(L)) is CFL

9.6. Reverse (Thm. If L is CFL $\Rightarrow$ L^R is CFL (Closed)

Pf: given CFL L, L has CFG $G = (V, \Sigma, R, S)$ then

CFG(~~~~reverse(L)) $= (V, \Sigma, R', S)$ s.t. R' reverses all RHS of R.

~~eg~~ $S \to XY \in R \Rightarrow S \to YX \in R'$ and so on.