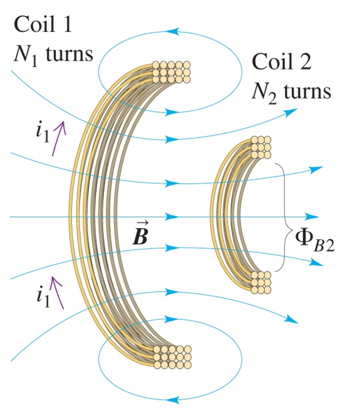
Chapter 30: Inductance

Mutual Inductance (1 of 3)

- Suppose we have two coils of wire placed next to each other, and there is a current in coil 1 that changes with time.
- The current in coil 1 produces a magnetic field, which then induces a current in coil 2 by Faraday's law.
- If i_1 is the current in coil 1, N_2 is the number of turns in coil 2, and Φ_{B2} is the magnetic flux through each turn of wire in coil 2, then the emf in coil 2 is

$$\mathcal{E}_2 = -N_2 \frac{\mathrm{d}\Phi_{B2}}{\mathrm{d}t}.$$

Mutual inductance: If the current in coil 1 is changing, the changing flux through coil 2 induces an emf in coil 2.



- The flux Φ_{B2} is proportional to i_1, so we define a constant of proportionality M_{21} called the **mutual inductance:**

$$N_2 \Phi_{B2} = M_{21} i_1.$$

Mutual Inductance (2 of 3)

- Using the mutual inductance, we can rewrite the emf for coil 2:

$$N_2 \frac{\mathrm{d}\Phi_{B2}}{\mathrm{d}t} = M_{21} \frac{\mathrm{d}i_1}{\mathrm{d}t} \quad \rightarrow \quad \mathcal{E}_2 = -M_{21} \frac{\mathrm{d}i_i}{\mathrm{d}t}.$$

- For the case where i_1 in coil 1 generates a magnetic field that induces an emf in coil 2, the mutual indutance M_{21} is

$$M_{21} = \frac{N_2 \Phi_{B2}}{i_1}.$$

- We can repeat this analysis for the case where there is a current i_2 in coil 2 that generates a magnetic field, inducing an emf in coil 1, with a mutual inductance M_{12}:

$$\mathcal{E}_1 = -M_{12} \frac{\mathrm{d}i_2}{\mathrm{d}t} \quad \rightarrow \quad M_{12} = \frac{N_1 \Phi_{B1}}{i_2}.$$

Mutual Inductance (3 of 3)

- The mutual inductance M_{12} turns out to be the same as M_{21}, so we instead just call it M:

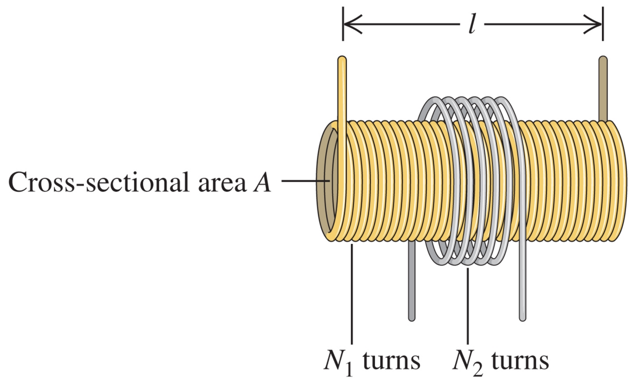
$$M = \frac{N_2 \Phi_{B2}}{i_1} = \frac{N_1 \Phi_{B1}}{i_2}.$$

- Mutual inductance is purely defined by the geometry of coils 1 and 2, and actually can be determined without reference to the currents or the flux (assuming the relative permeability of the material K_m is constant).
- The unit of inductance is the **henry:**

$$1 \text{ H} = 1 \text{ Wb/A} = 1 \text{ V} \cdot \text{s/A} = 1 \ \Omega \cdot \text{s}$$

Example 30.1: Calculating Mutual Inductance

In one form of Tesla coil (a high-voltage generator popular in science museums), a long solenoid with length l and cross-sectional area A is closely wound with N_1 turns of wire. A coil with N_2 turns surrounds it at its center. Find the mutual inductance M.



The field due to a current i_1 in the solenoid is

$$B_1 = \frac{\mu_0 N_1 i_1}{l}.$$

The mutual inductance is therefore

$$M = \frac{N_2 \Phi_{B2}}{i_1} = \frac{N_2 B_1 A}{i_1} = \frac{\mu_0 A N_1 N_2}{l}.$$

Notice how the current i_1 cancels out in the final expression, leaving M only dependent on the geometry of the coils.

Now suppose that $l = 0.50$ m, $A = 10$ cm^2, $N_1 = 1000$, and $N_2 = 10$. Then the mutual inductance is

$$M = \frac{(4\pi \times 10^{-7} \text{ Wb} \cdot \text{m/A})(1.0 \times 10^{-3} \text{ m}^2)(1000)(10)}{0.5 \text{ m}} = 25 \text{ μH}.$$

Example 30.2: Emf Due to Mutual Inductance

In the previous example, suppose the current i_2 in the outer coil is given by $i_2 = (2.0 \times 10^6 \text{ A/s})t$. (Currents in wires can indeed increase this rapidly for brief periods.) (a) At $t = 3.0$ μs, what is the average magnetic flux through each turn of the solenoid (coil 1) due to the current in the outer coil? (b) What is the induced emf in the solenoid?

(a) At $t = 3.0$ μs, the current in the outer coil is $i_2 = (2.0 \times 10^6 \text{ A/s})(3.0 \times 10^{-6} \text{ s}) = 6.0$ A. The flux through each turn of coil 1 is then

$$\Phi_{B1} = \frac{M i_2}{N_1} = \frac{(25 \times 10^{-6} \text{ H})(6.0 \text{ A})}{1000} = 1.5 \times 10^{-7} \text{ Wb}.$$

(b) Since $i_2 = (2.0 \times 10^6 \text{ A/s})t$, we have that

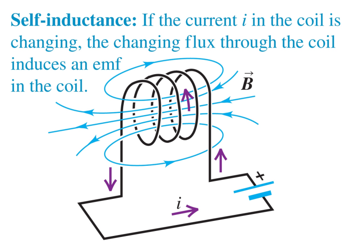
$$\frac{\mathrm{d}i_2}{\mathrm{d}t} = 2.0 \times 10^6 \text{ A/s}.$$

Then the induced emf in the solenoid is

$$\mathcal{E}_1 = -M\frac{\mathrm{d}i_2}{\mathrm{d}t} = -(25 \times 10^{-6}\ \mathsf{H})(2.0 \times 10^{6}\ \mathsf{A/s}) = -50\ \mathsf{V}.$$

Self-Inductance and Inductors

- Any circuit with a coil that carries a varying current can produce a **self-induced** emf.
- The self-inductance L for a coil with N turns, current i running through it, and magnetic flux Φ_B running through the coil is defined as



$$L = \frac{N\Phi_B}{i}.$$

- If the current i changes, then so does Φ_B. We can rearrange the expression for L and take the time derivative to get

$$N\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = L\frac{\mathrm{d}i}{\mathrm{d}t}.$$

- From Faraday's law, the self-induced emf in a coil with N turns is

$$\mathcal{E} = -N\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -L\frac{\mathrm{d}i}{\mathrm{d}t}.$$

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Inductors as Circuit Elements (1 of 2)

- In circuits, inductors are used to oppose changes in current, and are represented by the following symbol: 
- If an inductor is present in a circuit with a changing current, the total electric field in the coil contains a conservative and non-conservative electric field $\mathbf{E}_c + \mathbf{E}_n$, which must be zero if the resistance in the inductor is negligible.
- Faraday's law says that the line integral of $\mathbf{E}_n$ in the coil is equal to the emf:

$$\int \mathbf{E}_n \cdot \mathrm{d}\mathbf{l} = -L\frac{\mathrm{d}i}{\mathrm{d}t}.$$

- Only the coil has non-zero $\mathbf{E}_n$, so we may replace the closed line integral with an integral across the terminals a and b of the coil to get

$$\int_a^b \mathbf{E}_n \cdot \mathrm{d}\mathbf{l} = -L\frac{\mathrm{d}i}{\mathrm{d}t}.$$

Inductors as Circuit Elements (2 of 2)

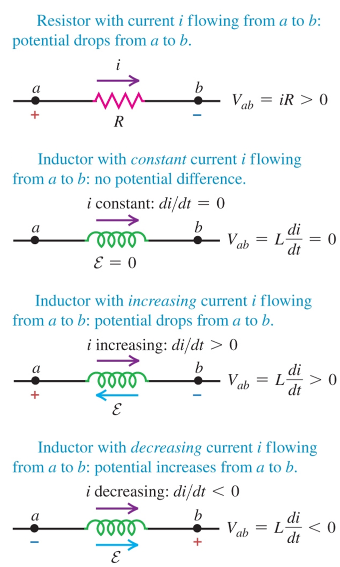
- Because $\mathbf{E}_c + \mathbf{E}_n = \mathbf{0}$, we know $\mathbf{E}_c = -\mathbf{E}_n$, so we can rewrite the integral as

$$\int_a^b \mathbf{E}_c \cdot \mathrm{d}\mathbf{l} = L\frac{\mathrm{d}i}{\mathrm{d}t}.$$

- An inductor therefore has a potential difference V_{ab} across its terminals given by

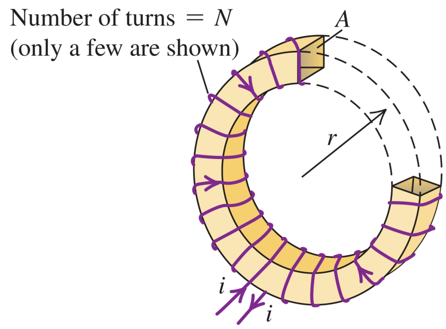
$$V_{ab} = V_a - V_b = L\frac{\mathrm{d}i}{\mathrm{d}t}.$$

- A constant current i flowing through an inductor will result in no voltage drop: $V_{ab} = 0$.
- An increasing current i flowing through an inductor results in a potential drop: $V_{ab} > 0$.
- A decreasing current i flowing through an inductor results in a potential gain: $V_{ab} < 0$.



Example 30.3: Calculating Self-Inductance

Determine the self-inductance of a toroidal solenoid with cross sectional area A and mean radius r, closely wound with N turns of wire on a nonmagnetic core. Assume that B is uniform across a cross section (that is, neglect the variation of B with distance from the toroid axis).



The flux through the solenoid is

$$\Phi_B = BA = \frac{\mu_0 NiA}{2\pi r},$$

So the self-inductance is

$$L = \frac{N\Phi_B}{i} = \frac{\mu_0 N^2 A}{2\pi r}.$$

Suppose $N = 200$, $A = 5.0$ cm^2, and $r = 0.10$ m. This would give us an inductance of

$$L = \frac{(4\pi \times 10^{-7}\ \mathsf{T \cdot m/A})(200)^2(5.0 \times 10^{-4}\ \mathsf{m}^2)}{2\pi(0.10\ \mathsf{m})} = 40 \times 10^{-6}\ \mathsf{H}.$$

Example 30.4: Calculating Self-Induced Emf

If the current in the toroidal solenoid from the previous example increases uniformly from 0 to 6.0 A in 3.0 μs, find the magnitude and direction of the self-induced emf.

We have that

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{6.0\ \mathsf{A}}{3.0 \times 10^{-6}\ \mathsf{s}} = 2.0 \times 10^6\ \mathsf{A/s}.$$

Therefore, the magnitude of the induced emf is

$$|\mathcal{E}| = L\left|\frac{\mathrm{d}i}{\mathrm{d}t}\right| = (40 \times 10^{-6}\ \mathsf{H})(2.0 \times 10^6\ \mathsf{A/s}) = 80\ \mathsf{V}.$$

Since the current is increasing, Lenz's law tells us that the direction of the emf is opposite to that of the current.

Magnetic-Field Energy

- The energy in an inductor is stored in the magnetic field of the coil, in the same way that the energy of a capacitor is stored in the electric field between the capacitor plates.
- We can compute the energy stored in an inductor by considering the rate at which energy is delivered to the inductor when the current flowing through it goes from $i = 0$ to its final value of $i = I$.
  - The power delivered to the inductor is

$$P = V_{ab} i = L i \frac{\mathrm{d}i}{\mathrm{d}t}.$$

  - The energy $\mathrm{d}U$ supplied during an interval $\mathrm{d}t$ is $\mathrm{d}U = P\,\mathrm{d}t$, so

$$\mathrm{d}U = L i\,\mathrm{d}i\,.$$

  - The total energy supplied to the inductor when $i = 0$ to $i = I$ is therefore

$$U = L \int_0^I i\,\mathrm{d}i = \frac{1}{2} L I^2.$$

Magnetic Energy Density (1 of 2)

- To see that the energy of an inductor is contained within the magnetic field it produces, consider an ideal toroidal solenoid.
- The volume of a solenoid with cross-sectional area A and circumference $2\pi r$ is $V = 2\pi r A$. This gives us a self-inductance of

$$L = \frac{N\Phi_B}{I} = \frac{N}{I}\frac{\mu_0 N I}{2\pi r} A = \frac{\mu_0 N^2 A}{2\pi r}.$$

- The corresponding energy is

$$U = \frac{1}{2} L I^2 = \frac{1}{2}\frac{\mu_0 N^2 A}{2\pi r} I^2.$$

- The energy per unit volume, or *energy density*, is the energy U divided by the volume $V = 2\pi r A$:

$$u = \frac{U}{2\pi r A} = \frac{1}{2}\mu_0 \frac{N^2 I^2}{(2\pi r)^2}.$$

Magnetic Energy Density (2 of 2)

- The energy density can instead be written in terms of B:

$$B = \frac{\mu_0 N I}{2\pi r} \quad \rightarrow \quad \frac{N^2 I^2}{(2\pi r)^2} = \frac{B^2}{\mu_0^2}.$$

- The magnetic energy density in a vacuum is then

$$u = \frac{B^2}{2\mu_0}.$$

- If the material inside the toroid is not vacuum but a material with magnetic permeability $\mu = K_m \mu_0$, we replace μ_0 by μ to get

$$u = \frac{B^2}{2\mu}.$$

- This is the correct expression for *any* magnetic field configuration in a material with constant permeability.

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Example 30.5: Storing Energy in an Inductor

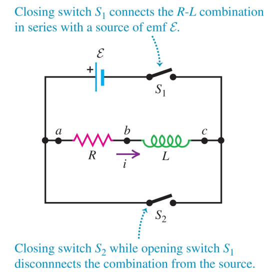
The electric-power industry would like to find efficient ways to store electrical energy generated during low-demand hours to help meet customer requirements during high-demand hours. Could a large inductor be used? What inductance would be needed to store 1.00 kW · h of energy in a coil carrying a 200 A current?

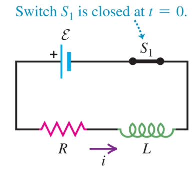
Solving for L from the expression for U, we get

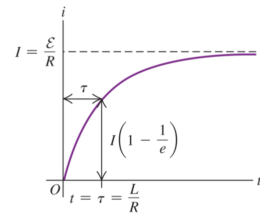
$$L = \frac{2U}{I^2} = \frac{2(3.60 \times 10^6 \text{ J})}{(200 \text{ A})^2} = 180 \text{ H}.$$

The R-L Circuit

- An **R-L circuit** contains both an inductor and a resistor, and possibly an emf source.
- Suppose we have an R-L circuit with an emf source $\mathcal{E}$, and we close a switch at $t = 0$ to allow flowing current.
  - Using Kirchhoff's loop rule, the equation for the circuit is
$$\mathcal{E} - iR - L\frac{\mathrm{d}i}{\mathrm{d}t} = 0 \quad \rightarrow \quad \frac{\mathrm{d}i}{\mathrm{d}t} = \frac{\mathcal{E} - iR}{L}.$$
  - The current initially starts at $i = 0$, before it reaches some steady-state value I.
  - The solution to the differential equation satisfying the initial and final conditions is
$$i = \frac{\mathcal{E}}{R}\left[1 - e^{-(R/L)t}\right].$$
  - The steady-state value of the current is $I = \mathcal{E}/R$, and the **time constant** for the circuit is $\tau = L/R$.







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Example 30.6: Analyzing an R-L Circuit

A sensitive electronic device of resistance $R = 175\ \Omega$ is to be connected to a source of emf (of negligible internal resistance) by a switch. The device is designed to operate with a 36 mA current, but to avoid damage to the device, the current can rise to no more than 4.9 mA in the first 58 μs after the switch is closed. An inductor is therefore connected in series with the device, as in the figure with the R-L circuit; the switch in question is S_1. (a) What is the required source emf $\mathcal{E}$? (b) What is the required inductance L? (c) What is the R-L time constant τ?

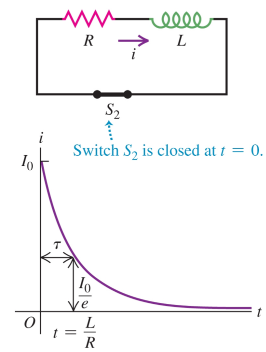
(a) The emf is
$$\mathcal{E} = IR = (0.036 \text{ A})(175\ \Omega) = 6.3 \text{ V}.$$

(b) We can solve for the inductance by considering the expression for i:
$$1 - \frac{iR}{\mathcal{E}} = e^{-(R/L)t} \quad \rightarrow \quad L = \frac{-Rt}{\ln(1 - iR/\mathcal{E})}$$
$$= \frac{-(175\ \Omega)(58 \times 10^{-6} \text{ s})}{\ln[1 - (4.9 \times 10^{-3} \text{ A})(175\ \Omega)/(6.3 \text{ V})]}$$
$$= 69 \text{ mH}.$$

(c) From the definition of the time constant, we get
$$\tau = \frac{L}{R} = \frac{69 \times 10^{-3} \text{ H}}{175\ \Omega} = 390 \text{ μs}.$$

Current Decay in an R-L Circuit

- Consider the same R-L circuit as before, but after the current has built up, a switch is flipped so that the emf $\mathcal{E}$ is no longer present.
  - Now the differential equation for the circuit is
  $$-iR - L\frac{\mathrm{d}i}{\mathrm{d}t} = 0 \quad \rightarrow \quad \frac{\mathrm{d}i}{\mathrm{d}t} = -\frac{R}{L}i.$$
  - For an initial current I_0 at $t = 0$, the solution to the differential equation is
  $$i = I_0 e^{-(R/L)t}.$$
  - The current in the circuit now exponentially decays, with the same time constant $\tau = L/R$ as in the previous case.



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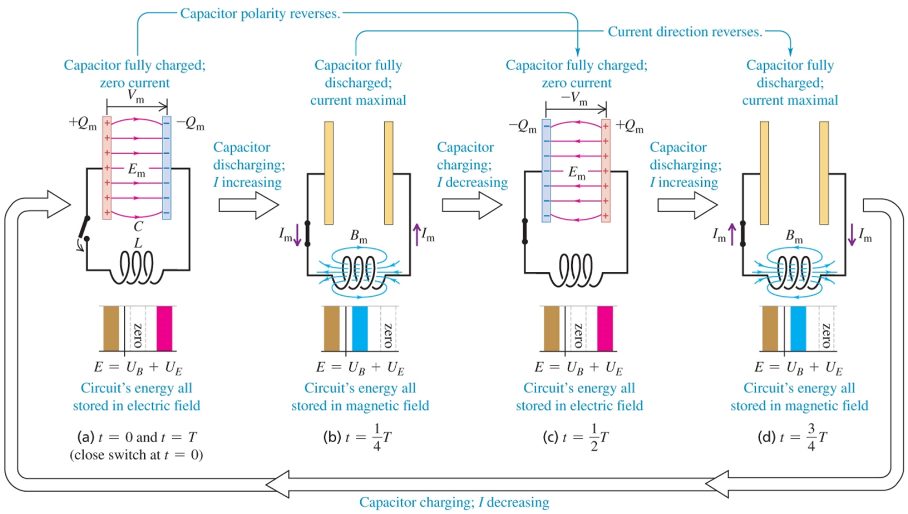
Example 30.7: Energy in an R-L Circuit

When the current in an R-L circuit is decaying, what fraction of the original energy stored in the inductor has been dissipated after 2.3 time constants?

We can use our expression for i and insert it into our expression for the energy U stored in an inductor, which gives us
$$U = \frac{1}{2}Li^2 = \frac{1}{2}LI_0^2 e^{-2(R/L)t} = U_0 e^{-2(R/L)t},$$
where $U_0 = LI_0^2/2$. After $t = 2.3\tau = 2.3L/R$, we get
$$U = U_0 e^{-2(2.3)} = U_0 e^{-4.6} = 0.010U_0.$$
So only 1% of the energy remains, which means 99% of the original energy U_0 has been dissipated.

The L-C Circuit

- An **L-C circuit** contains both an inductor and a capacitor.
- L-C circuits are characterized by oscillating behavior of current and charge:

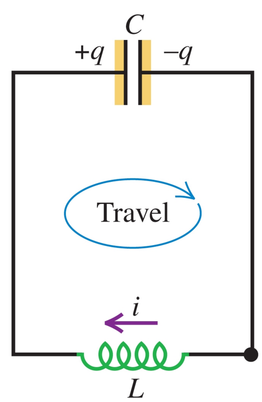


Electrical Oscillations in an L-C Circuit

- The differential equation describing an L-C circut is

$$-L\frac{\mathrm{d}i}{\mathrm{d}t} - \frac{q}{C} = 0 \quad \rightarrow \quad \frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{LC}q = 0.$$

- This is the equation of simple harmonic motion!
- The solution for q on the capacitor given a total charge Q is therefore exactly the same as for the position of a harmonic oscillator:



$$q = Q\cos(\omega t + \phi), \quad \omega = \sqrt{\frac{1}{LC}}.$$

- The corresponding equation for the current i in the inductor is analogous to the velocity of a harmonic oscillator:

$$i = \frac{\mathrm{d}q}{\mathrm{d}t} = -\omega Q\sin(\omega t + \phi).$$

Electrical and Mechanical Oscillations: Analogies

- All of the relationships derived for the harmonic oscillator apply for an L-C circuit:

Mass-Spring System	**Inductor-Capacitor Circuit**
Kinetic energy $= \frac{1}{2}mv_x^2$	Magnetic energy $= \frac{1}{2}Li^2$
Potential energy $= \frac{1}{2}kx^2$	Electrical energy $= q^2/2C$
$\frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$	$\frac{1}{2}Li^2 + q^2/2C = Q^2/2C$
$v_x = \pm\sqrt{k/m}\sqrt{A^2 - x^2}$	$i = \pm\sqrt{1/LC}\sqrt{Q^2 - q^2}$
$v_x = \mathrm{d}x/\mathrm{d}t$	$i = \mathrm{d}q/\mathrm{d}t$
$\omega = \sqrt{\frac{k}{m}}$	$\omega = \sqrt{\frac{1}{LC}}$
$x = A\cos(\omega t + \phi)$	$q = Q\cos(\omega t + \phi)$

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Example 30.8: An Oscillating Circuit

A 300 V dc power supply is used to charge a 25 μF capacitor. After the capacitor is fully charged, it is disconnected from the power supply and connected across a 10 mH inductor. The resistance in the circuit is negligible. (a) Find the frequency and period of oscillation of the circuit. (b) Find the capacitor charge and circuit current 1.2 ms after the inductor and capacitor are connected.

(a) From the expression for ω, the angular frequency is

$$\omega = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(10\times10^{-3}\text{ H})(25\times10^{-6}\text{ F})}} = 2.0\times10^{3}\text{ rad/s}.$$

Therefore, the frequency f and period T are

$$f = \frac{\omega}{2\pi} = \frac{2.0\times10^{3}\text{ rad/s}}{2\pi\text{ rad}} = 320\text{ Hz} \quad\rightarrow\quad T = \frac{1}{f} = \frac{1}{320\text{ Hz}} = 3.1\text{ ms}.$$

(b) First, we must find the total charge on the capacitor. We have

$$Q = C\mathcal{E} = (25\times10^{-6}\text{ F})(300\text{ V}) = 7.5\times10^{-3}\text{ C}.$$

We also have that at $t = 0$, the capacitor is fully charged, so the phase angle is $\phi = 0$. Therefore, at $t = 1.2$ ms, the charge is

$$q = Q\cos\omega t = (7.5\times10^{-3}\text{ C})\cos\left[(2.0\times10^{3}\text{ rad/s})(1.2\times10^{-3}\text{ s})\right] = -5.5\times10^{-3}\text{ C}.$$

The current is then

$$i = -\omega Q\sin\omega t = -(2.0\times10^{3}\text{ rad/s})(7.5\times10^{-3}\text{ C})\sin\left[(2.0\times10^{3}\text{ rad/s})(1.2\times10^{-3}\text{ s})\right] = -10\text{ A}.$$

Example 30.9: Energy in an Oscillator Circuit

For the L-C circuit of the previous example, find the magnetic and electrical energies (a) at $t = 0$ and (b) at $t = 1.2$ ms.

(a) At $t = 0$, there is no current and $q = Q$, so all of the energy is stored in the capacitor, and we have

$$U_B = 0, \qquad U_E = \frac{Q^2}{2C} = \frac{(7.5 \times 10^{-3}\ \text{C})^2}{2(25 \times 10^{-6}\ \text{F})} = 1.1\ \text{J}.$$

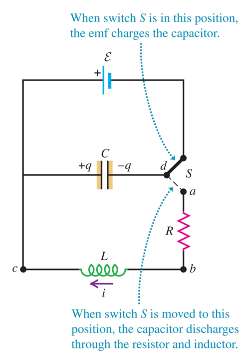
(b) From the previous example, at $t = 1.2$ ms, we have $i = -10$ A and $q = -5.5 \times 10^{-3}$ C. Therefore,

$$U_B = \frac{1}{2}Li^2 = \frac{1}{2}(10 \times 10^{-3}\ \text{H})(-10\ \text{A})^2 = 0.5\ \text{J}, \qquad U_E = \frac{q^2}{2C} = \frac{(-5.5 \times 10^{-3}\ \text{C})^2}{2(25 \times 10^{-6}\ \text{F})} = 0.6\ \text{J}.$$

The L-R-C Series Circuit (1 of 2)

- An **L-R-C series circuit** is a circuit with an inductor, a resistor, and a capacitor all in series with each other.
- Suppose we initially charge a capacitor with an emf source $\mathcal{E}$, and then flip a switch so that the capacitor is now in series with an inductor and a resistor.
- The differential equation for the circuit is

$$-iR - L\frac{\mathrm{d}i}{\mathrm{d}t} - \frac{q}{C} = 0 \quad \rightarrow \quad \frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{LC}q = 0.$$



- This is exactly analogous to a damped harmonic oscillator, with the following solution:

$$q = Ae^{-(R/2L)t}\cos(\omega' t + \phi), \quad \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

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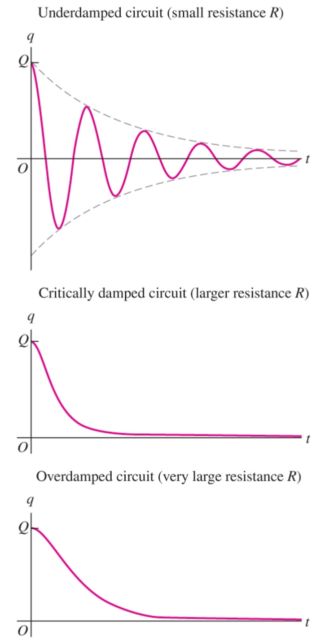
The L-R-C Series Circuit (2 of 2)

- As with a damped harmonic oscillator, an L-R-C series circuit can exhibit underdamped, critically damped, and overdamped motion of charge.
- In the underdamped case, $R^2 < 4L/C$, and the system oscillates while having an exponentially decaying envelope that reduces the amplitude of oscillation.
- For the critically damped case, $R^2 = 4L/C$, which results in $\omega' = 0$, thereby removing the oscillatory behavior entirely and causing the system to return to equilibrium in the shortest amount of time.
- The overdamped case occurs when $R^2 > 4L/C$, which causes ω' to become imaginary, and q becomes the sum of two decreasing exponential functions.



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Example 30.10: An Underdamped L-R-C Series Circuit

What resistance R is required (in terms of L and C) to give an L-R-C series circuit a frequency that is one-half the undamped frequency?

Setting ω' equal to $\omega/2$, we get

$$\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{1}{2}\sqrt{\frac{1}{LC}} \quad \rightarrow \quad \frac{1}{LC} - \frac{R^2}{4L^2} = \frac{1}{4LC} \quad \rightarrow \quad R = \sqrt{\frac{3L}{C}}.$$