Math 170E: Winter 2023

Lecture 14, Fri 13th Feb

Random variables of the continuous type

Definition 3.1: Let $S \subseteq \mathbb{R}$, and $X : \Omega \to S$ is a random variable

- we define the cumulative distribution function of X, $F_X : \mathbb{R} \to [0, 1]$ by

$$F_X(x) := \mathbb{P}(X \leq x).$$

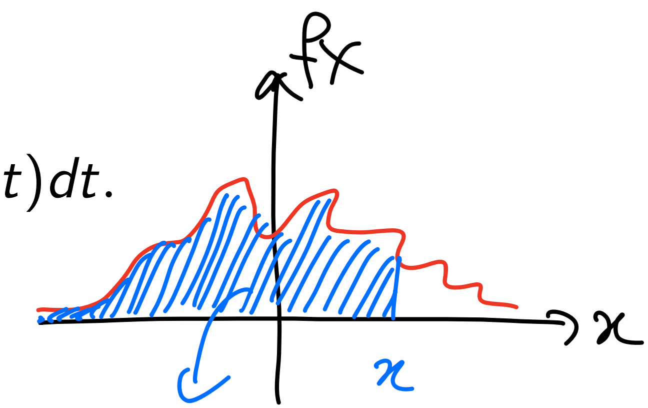
  We have

$$\lim_{x \to -\infty} F_X(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} F_X(x) = 1.$$

- we say that X is a continuous random variable if there exists a *non-negative integrable* function $f_X : \mathbb{R} \to [0, \infty)$ such that

$\int_{-\infty}^{\infty} f_X(t)dt < +\infty.$

$$\mathbb{P}(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)dt.$$



(PDF).

$F_X(x) = \mathbb{P}(X \leq x).$

- we call f_X a probability density function for X.

Proposition 3.2: If X is a continuous random variable with PDF $f_X : \mathbb{R} \to [0, \infty)$, then

$$\int_{-\infty}^{\infty} f_X(x)dx = 1.$$

(normalisation condition)

Proof:

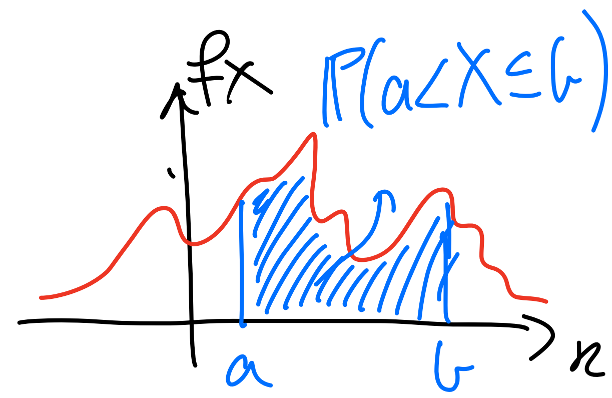
$$1 = \mathbb{P}(X \in \mathbb{R}) = \lim_{x \to +\infty} F_X(x) = \lim_{x \to +\infty} \int_{-\infty}^{x} f_X(t)dt,$$

X cts r.v.

$$= \int_{-\infty}^{\infty} f_X(t)dt$$

Proposition 3.3: If X is a continuous random variable with PDF $f_X : \mathbb{R} \to [0, \infty)$ and $a < b$, then

$$\mathbb{P}(X \in (a, b]) \;\; || \;\; \mathbb{P}(a < X \leq b) = \int_a^b f_X(x)dx.$$



Proof:

$$\mathbb{P}(a<X\leq b) = \mathbb{P}(\{X \leq b\} \setminus \{X \leq a\}).$$

$$= \mathbb{P}(X \leq b) - \mathbb{P}(X \leq a)$$

$$= F_X(b) - F_X(a).$$

X cts r.v.

$$= \int_{-\infty}^{b} f_X(t)dt - \int_{-\infty}^{a} f_X(t)dt.$$

$$= \int_a^b f_X(t)dt.$$

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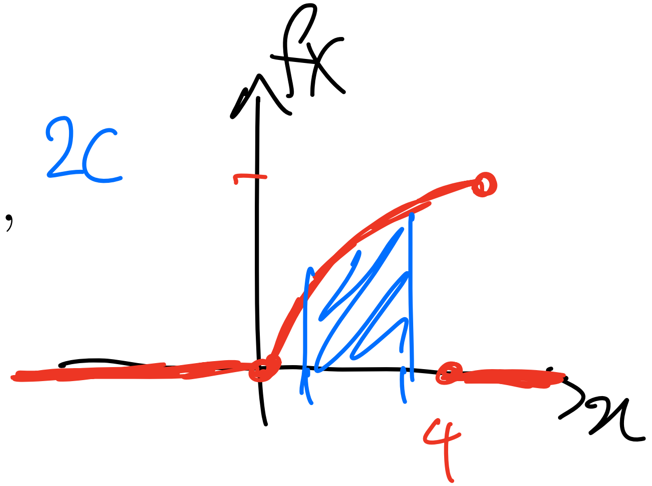
Example 2: A continuous r.v. X has PDF

$$f_X(x) = \begin{cases} C\sqrt{x} & \text{if } x \in [0,4], \\ 0 & \text{otherwise} \end{cases}$$



What is $\mathbb{P}(1 < X \leq 3)$?

① Find $C \in \mathbb{R}$:

- f_X is non-neg. provided $C \geq 0$.

$$4^{3/2} = (2^2)^{3/2} = 2^3$$

- Normalisation:

$$1 = \int_{-\infty}^{\infty} f_X(t)\,dt = \int_0^4 C\sqrt{x}\,dx = C\left[\frac{2}{3}x^{3/2}\right]_0^4$$

$$= C\left[\frac{2}{3}\cdot 2^3\right] = \frac{16}{3}C.$$

$$\Rightarrow C = \frac{3}{16}.$$

② Find $\mathbb{P}(1 < X \leq 3)$

So by our previous result,

$$\mathbb{P}(1 < X \leq 3) = \int_1^3 f_X(t)\,dt$$

$$= \int_1^3 \frac{3}{16}\sqrt{x}\,dx = \frac{1}{8}\left[3^{3/2} - 1\right].$$

Proposition 3.4: If X is a continuous random variable with PDF $f_X : \mathbb{R} \to [0, \infty)$, then for any $x \in \mathbb{R}$

$$\mathbb{P}(X = x) = 0.$$

$$\text{"} \int_x^x f_X(t)\,dt = 0 \text{"}$$

In particular, if $a < b$, then

$$\mathbb{P}(a \leq X \leq b) = \mathbb{P}(a < X \leq b) = \mathbb{P}(a \leq X < b) = \mathbb{P}(a < X < b)$$

Proof: $\mathbb{P}(X \in [a,b]) = \underbrace{\mathbb{P}(X=a)}_{=0} + \mathbb{P}(X \in (a,b]) = \mathbb{P}(X \in (a,b])$

$\hookrightarrow \mathbb{P}(X \in [a,b]) = \underbrace{\mathbb{P}(X=a)}_{=0} + \underbrace{\mathbb{P}(X=b)}_{=0} + \mathbb{P}(X \in (a,b))$

$0 \leq \mathbb{P}(X=x) \leq \mathbb{P}(x-\delta < X \leq x+\delta)$ for $\delta > 0$.

$= \int_{x-\delta}^{x+\delta} f_X(t)\,dt \to 0$ as $\delta \to 0^+$.

By the squeeze Theorem, $\mathbb{P}(X=x) = 0$ for any $x \in \mathbb{R}$.

"X is as if $x \mapsto F_X(x)$ is a cts function on $\mathbb{R}$"

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt.$$

$$F_X(x+\varepsilon) - F_X(x) \longrightarrow 0 \text{ as } \varepsilon \to 0^+.$$

by cty of F_X.

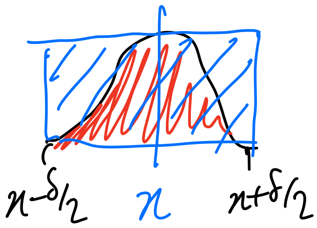
$$0 = \lim_{\varepsilon \to 0^+} \left(F_X(x+\varepsilon) - F_X(x)\right) = \mathbb{P}(X=x).$$

- $\underbrace{f_X(x)}_{\text{density}} \neq \mathbb{P}(X=x) = 0.$

If $\delta > 0$ small, then

$$\mathbb{P}\left(X \in (x-\delta/2,\, x+\delta/2]\right) = \int_{x-\delta/2}^{x+\delta/2} f_X(t)\,dt$$

"length.



$$\approx f_X(x)\cdot\delta.$$

$$\frac{\text{prob}}{\text{length}} \approx f_X(x).$$

$f_X(x) \approx$ probability per unit length.

$X \rightsquigarrow$ discrete approximations $\{X_n\}_{n\in\mathbb{N}}$

"$X_n \to X$ as $n \to +\infty$".

$$\mathbb{E}[X] = \lim_{n\to\infty} \mathbb{E}[X_n]$$
$$= \lim_{n\to\infty}\left(\sum_{x\in S_n} x\, p_{X_n}(x)\right).$$

$f_X(x)dx \approx$ probability in dx interval.

Definition 3.6: If X is a continuous random variable with PDF $f_X(x)$, we define its expected value to be

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx.$$

$\to \mathbb{E}[X] = \sum_{x \in S} x \, P_X(x),$

We will use the notation $\mu_X = \mathbb{E}[X]$.
More generally, if $g : \mathbb{R} \to \mathbb{R}$ is any function, then

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

Example 3 : Uniform random variable

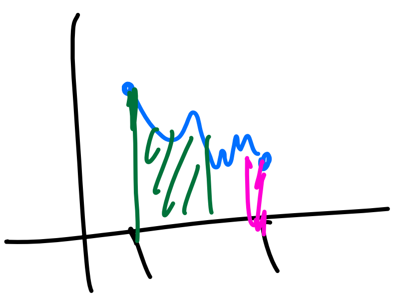
- Let $a < b$
- Pick a point X at random from the interval $[a, b]$
- If we have an equal probability of picking every point in $[a, b]$, we say X is uniformly distributed on the interval $[a, b]$
- We say $X \sim \text{Uniform}([a, b])$

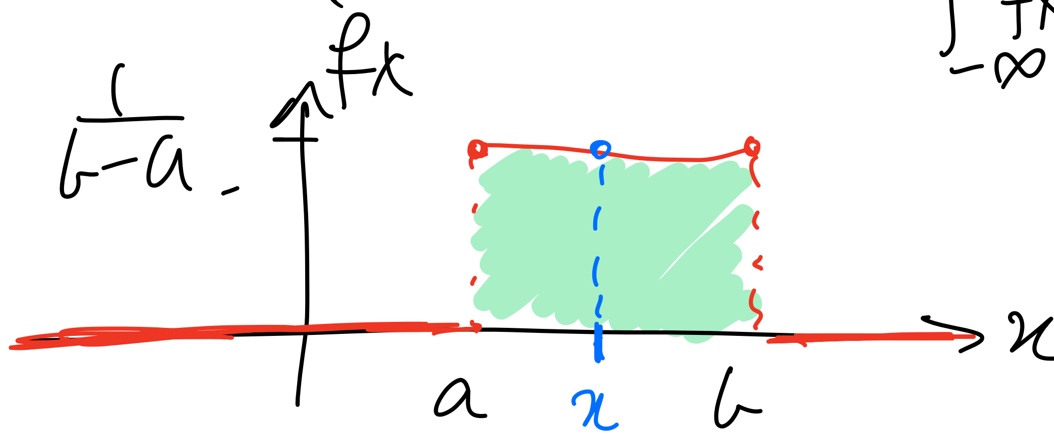
Proposition 3.7: If $a < b$ and $X \sim \text{Uniform}([a, b])$, then it has PDF

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in (a, b) \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f_X(t)\,dt = \text{Area of box} = \frac{b-a}{b-a} = 1.$$

Proof:

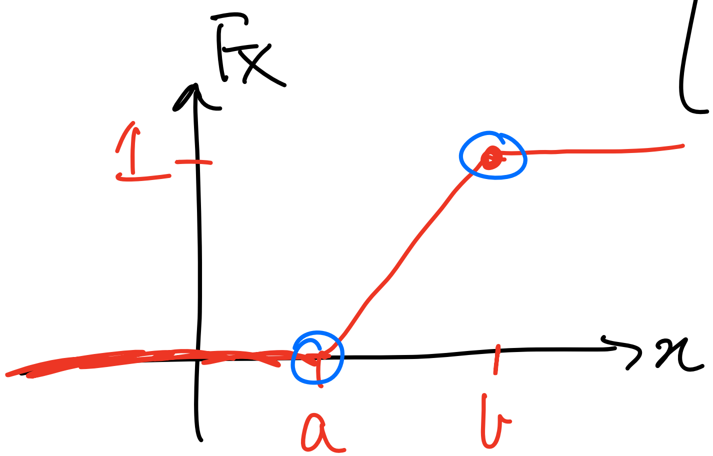




Let $x \in (a,b)$. Then $\mathbb{P}(a < X \leq x) = \frac{|[a,x]|}{|[a,b]|} = \frac{x-a}{b-a}$.

Some CDF:

$$F_X(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } x \in (a,b) \\ 0 & \text{if } x \leq a \\ 1 & \text{if } x \geq b \end{cases}$$



$\hookrightarrow F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

$F_X'(x) = f_X(x)$.

F_X is differentiable everywhere except at $x = a, b$!

$$f_X(x) = F_X'(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in (a,b) \\ 0 & \text{otherwise.} \end{cases}$$

Example 4:

- Let $a < b$ and $X \sim \text{Uniform}([a, b])$ $\quad \mathbb{E}[X] = \frac{a+b}{2}.$
- What is $\mathbb{E}[X]$?

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{a} x \underbrace{f_X(x)}_{=0}\,dx + \int_a^b x f_X(x)\,dx + \int_b^{+\infty} x \underbrace{f_X(x)}_{=0}\,dx$$

$$= \int_a^b x f_X(x)\,dx$$

$$= \frac{1}{b-a}\int_a^b x\,dx.$$

$$= \frac{1}{b-a}\left[\frac{1}{2}x^2\right]_a^b$$

$$= \frac{1}{b-a}\,\frac{b^2-a^2}{2} = \frac{1}{\cancel{b-a}}\cdot\frac{\cancel{(b-a)}(b+a)}{2}$$

$$= \frac{b+a}{2}.$$

midpoint of $[a,b]$.

Prop$^{\underline{n}}$ 5.8: Let X be a cts r.v. Then:

(i) If $a \in \mathbb{R}$, $\mathbb{E}[a] = a$ $\rightarrow$ $\int_{-\infty}^{\infty} a f_X(x)\,dx = a$

(ii) If $a, b \in \mathbb{R}$, & $g, h : \mathbb{R} \to \mathbb{R}$ are functions then

$$\mathbb{E}[a g(X) + b h(X)] = a\,\mathbb{E}[g(X)] + b\,\mathbb{E}[h(X)].$$

(iii) If $g(x) \le h(x)$ for all $x \in \mathbb{R}$, then

$$\mathbb{E}[g(X)] \le \mathbb{E}[h(X)].$$

$\hookrightarrow$ Why?: Do the same arguments as in discrete case.

Def^n: If X is a cts r.v., we define its variance by

$$\sigma_X^2 = Var(X) = \mathbb{E}\left[(X-\mathbb{E}[X])^2\right]$$
$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Standard deviation: $\sigma_X = \sqrt{Var(X)}$.

Same as discrete case.

Example: Suppose that $a<b$ & $X \sim Unif([a,b])$.

$\hookrightarrow Var(X) = \frac{(b-a)^2}{12}$.

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \frac{1}{b-a}\int_a^b x^2\,dx$$
$$= \frac{1}{b-a}\,\frac{1}{3}(b^3-a^3)$$
$$= \cancel{\frac{1}{b-a}}\,\frac{1}{3}\cancel{(b-a)}(b^2+ab+a^2)$$
$$= \frac{b^2+ab+a^2}{3}.$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4(b^2+ab+a^2) - 3(a+b)^2}{12}$$

$$= \frac{4b^2+4ab+4a^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12} \quad //$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum f(x_n)(x_n - x_{n-1}).$$