# Math 170E: Winter 2023

Lecture 4, Wed 18th Jan

Methods of enumeration

# Last time:

We introduced the notion of *independence* and discussed:

- what it means for two events to be independent
- what it means for a collection of sets to be (mutually) independent

We also discussed the *multiplication principle* which tells us how to compute the number of outcomes of $r$ independent composite experiments

# Today:

We'll discuss today:

- ordered samples of $r$ objects from a set of $n$ with replacement
- permutations of $n$ objects taken $r$ at a time
- combinations of $n$ objects taken $r$ at a time
- unordered samples of $r$ objects from a set of $n$ with replacement
- the binomial theorem

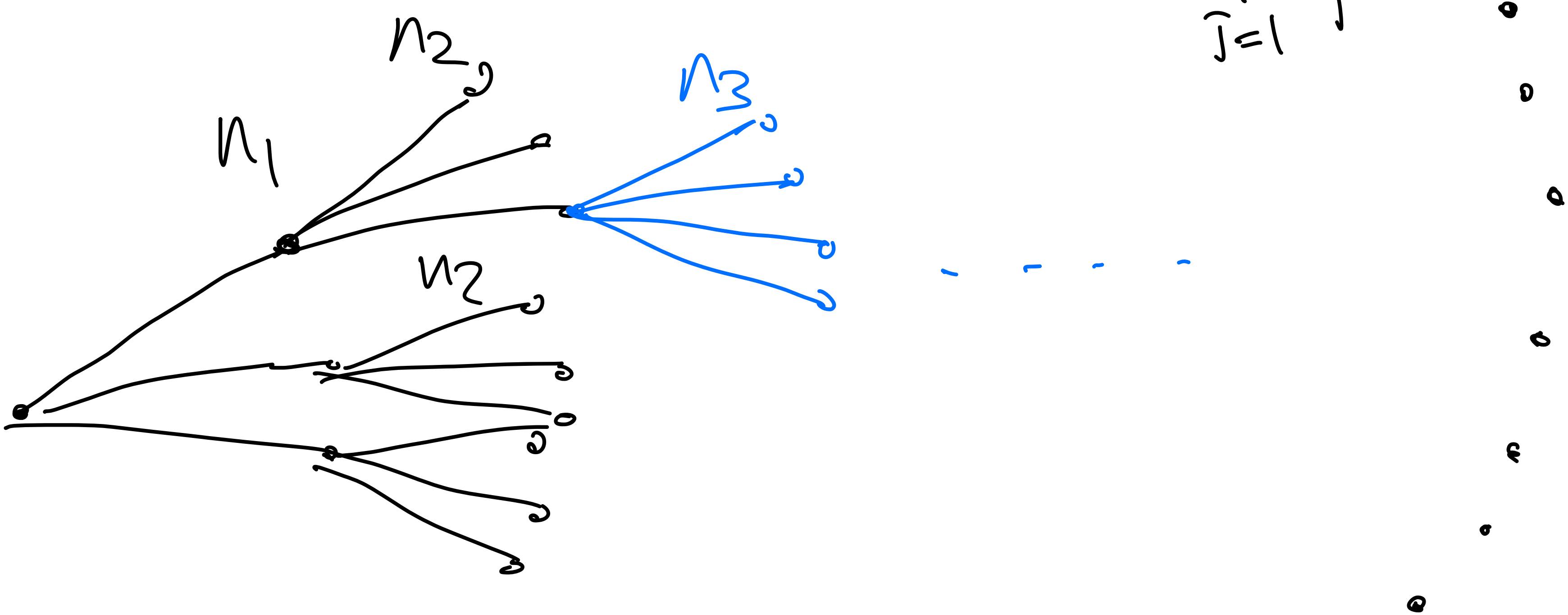
## The multiplication principle:

Let $r \in \{1, 2, 3, \ldots\}$. Suppose that we run $r$ independent experiments and that:

- the 1st experiment has $n_1$ possible outcomes
- the 2nd experiment has $n_2$ possible outcomes
- ...
- the $r$th experiment has $n_r$ possible outcomes

Then, the number of possible outcomes the composite experiment has is

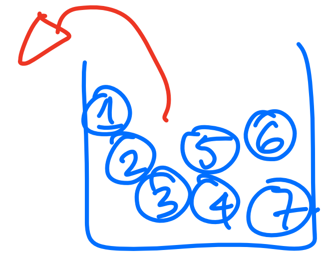
$$n_1 \times n_2 \times n_3 \times \cdots \times n_r = \prod_{j=1}^{r} n_j$$



In some experiments, we are interested in taking $r$ samples from $n$ objects

↳ distinct

- we can do this with or without replacement
- we can seek ordered or unordered samples




| | ordered | unordered |
|---|---|---|
| with replacement | $n^r$ | $\binom{n+r-1}{r}$. |
| without rep. | ${}_nP_r = \frac{n!}{(n-r)!}$ | $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$ |

→ Suppose, $n=6$, $r=4$. $\rightsquigarrow \{1, \ldots, 6\}$.

e.g. $\{1,1,6,2\}$

## Ordered with replacement:

- We take ordered samples of size $r$ from a set of $n$ objects with replacement.
- Then, the number of samples is $n^r$.

$\{1,1,6,2\} \neq \{6,2,1,1\}$.

**Proof:** Trial-by-trial approach $\hookrightarrow \{a_1, \ldots, a_n\}$, $a_j \neq a_k$ if $j \neq k$.

Pick r out with replacement.

1st choice: $n$ choices

2nd choice: $n$ choices

$\vdots$

$r$th choice: $n$ choices

By the multip. principle, there are

$$\underbrace{n \times n \times \cdots \times n}_{r \text{ many}} = n^r.$$

possible outcomes.

**Ordered without replacement:**

Recall: $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$

$0! = 1$.

- We take ordered samples of size $r$ from a set of $n$ objects without replacement.

$\cancel{\{1,1,6,2\}}$

$\{1,6,2,3\} \neq \{6,3,2,1\}$

different.

- Then, the number of samples is

$$_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! \qquad {}_nP_r := \frac{n!}{(n-r)!}$$

- Ordered samples without replacement are called permutations
- Important case: If $r = n$, we have number of ways of rearranging $n$ *distinct* objects, which is $n!$

**Argument 1:** (Trial-by-trial) $\to \{a_1, \ldots, a_n\}$.

$1^{st}$ choice: $n$ choices

$2^{nd}$ choice: $n-1$ choices.

$3^{rd}$ choice: $n-2$ choices.

$\vdots$

$r^{th}$ choice: $n-(r-1) = n-r+1$ choices.

By the multip. principle, we have

$$n(n-1)(n-2)\times\cdots\times(n-r+1).$$

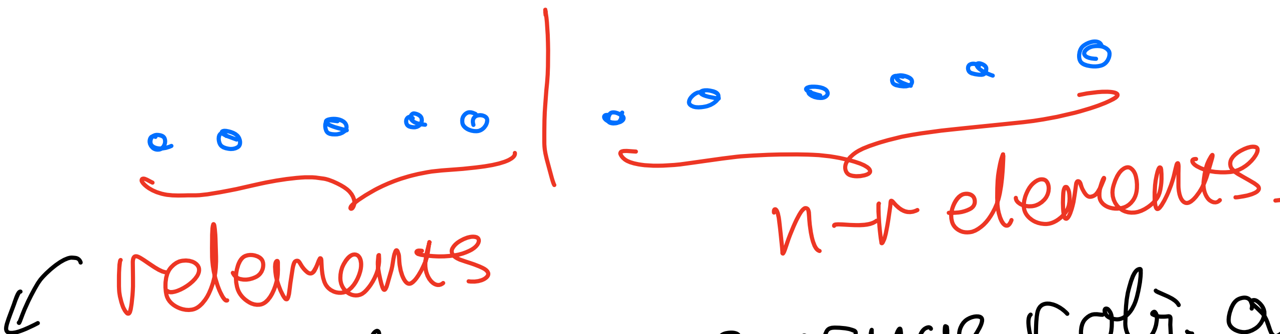
$$= n(n-1)(n-2)\times\cdots\times(n-r+1)\times\left(\frac{(n-r)(n-r-1)\times\cdots\times2\times1}{(n-r)(n-r-1)\times\cdots\times2\times1}\right)$$

$$= \frac{n!}{(n-r)!} \quad //$$

**Argument 2:** (Top-down approach):

Given $n$ distinct objects, we have $n!$ total permutations (ways to arrange them).

Suppose we fix <u>one</u> such arrangement:




Gives us one possible way to arrange $r$ obj. out of $n$.

But for this arrangement, there are $(n-r)!$ many ways the remaining $n-r$ objects could have been arranged. For each permutation, we <u>over</u> counted by $(n-r)!$ : $\longrightarrow$

$$ {}^{n}P_{r} = \frac{n!}{(n-r)!} $$

//

distinct/no clones.

**Example 6:** At a competition of 100 athletes, only the order of the first 10 are recorded. How many different outcomes does the competition have?

$n=100$ · ordered

$r=10$ · without replacement.

$\# \text{outcomes} = {}_{100}P_{10} = \frac{100!}{(100-10)!} = \frac{100!}{90!}$

$= 100 \times 99 \times 98 \times \cdots \times 91.$

**Unordered without replacement:** "n choose r"

- We take unordered samples of size $r$ from a set of $n$ objects without replacement.
- Then, the number of samples is

Binomial coefficients.

$$\binom{n}{r} := {}_nC_r := \frac{n!}{(n-r)!r!} \quad "\frac{{}_nP_r}{r!}"$$

$\cancel{\{1,1,6,2\}}$

$\{1,6,2,3\} = \{6,2,1,3\}$ Same one.

- Unrdered samples without replacement are called combinations
- the number of subsets of size $r$ from a set of $n$ objects
  ↳ non-empty.

**Argument:** (Top-down approach)

↳ we have $\frac{n!}{(n-r)!}$ many ordered lists of size $r$ from $n$ objects.

To "un-order", we have overcounted by $r!$. So total number of outcomes is:

$$\frac{\frac{n!}{(n-r)!}}{r!} = \binom{n}{r}. \quad //$$

**Example 7:** I have a deck of 52 cards and draw 5 of them out. How many possible hands do I have?

- without replacement.
- unordered.

$n = 52$

$r = 5$

$$\binom{52}{5} = \frac{52!}{47!\,5!}$$ many possible outcomes.

**Example 8:**

- A quiz has 10 TRUE/FALSE questions
- You choose answers at random
- What is the probability that you get at least 80% of them correct?

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}.$$

$\Omega = \{TT\cdots T, TT\cdots TF, \cdots, FF\cdots F\}$, $|\Omega| = 2^{10}$
(since ordered, with rep $n=2, r=10$).

$A = \{\geq 80\% \text{ correct}\}$.
$= \{\text{at least 8 answers right}\} = A_1 \cup A_2 \cup A_3$.
$= \{\text{exactly 8 right}\} \cup \{=9\} \cup \{=10\}$.

By mutual exclusivity,

$$\mathbb{P}(A) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) = \frac{|A_1| + |A_2| + |A_3|}{2^{10}}$$

$|A_3| = |\{=10 \text{ right}\}| = 1.$

$|A_2| = |\{=9 \text{ right}\}| = {}_{10}C_9 = \binom{10}{9} = \frac{10!}{9!\,1!} = 10.$

$$|A_1| = |\{8 \text{ right}\}| = {}_{10}C_8 = \frac{10!}{8!\,2!} = \frac{10\times 9}{2} = 45.$$

$$\Rightarrow |A| = 1+10+45 = 56$$

$$\hookrightarrow |\mathbb{P}(A)| = \frac{56}{2^{10}}.$$

$$|A_1| = |\{8 \text{ right}\}| = |\{2 \text{ wrong}\}| = {}_{10}C_2 = \frac{10!}{2!\,8!} = {}_{10}C_8.$$

Note: $\boxed{{}_nC_r = {}_nC_{n-r}, \quad \binom{n}{r} = \binom{n}{n-r}}$

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \binom{n}{n-r}.$$

## Unordered with replacement:

with replacement: no restriction on $n$ & $r$.

without: $r \le n$.

- We take unordered samples of size $r$ from a set of $n$ objects with replacement.

$\{1,1,6,2\} = \{1,6,2,1\}$.

- Then, the number of samples is

$$_{n+r-1}C_r = \binom{n+r-1}{r}$$

**Argument:** Stars & bars method

For simplicity, sps that $n=4$, $r=5$. $\rightarrow \{A,B,C,D\}$.

Some samples are: $\{ABBBC\} = \{ABCBB\}$, $\{DABCD\}$.

Use $*$ to denote which object we picked, use bars $|$ to denote how many $A$'s, ..., $D$'s:

e.g.

$$\underbrace{*}_{1A}\,|\,\underbrace{*\;*\;*}_{3B}\,|\,\underbrace{*}_{1C}\,|\,\underbrace{\;}_{0D} \longrightarrow \{ABBBC\}$$

$$* \mid * \; * \; * \mid * \mid$$

1A 3B 1C 0D

# of unordered samples without replacement is the # of ways we can place, say, 5 stars out of 8 total symbols.

$$= \binom{8}{5}$$

Alternatively, we can place 3 bars out of 8 symbols

$$= \binom{8}{3} = \binom{8}{5}.$$

<u>In general</u>: $r$ stars, $n-1$ bars $\longrightarrow$ $n+r-1$ total symbols.

# choosing $r$ stars out of $n+r-1$ symbols $= \binom{n+r-1}{r}$.

**Example 8:** How many ways are there to buy 8 fruits if your options are APPLES, BANNAS, PEARS, ORANGES?

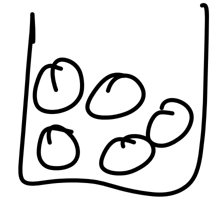
$\{ABPOABPO\} \rightsquigarrow n=4, r=8,$ unordered with replacement.

$$\# \text{ways is } {}_{4+8-1}C_8 = \binom{11}{8} = \frac{11!}{8!\,3!}$$

$$= \frac{11 \cdot 10 \cdot 9}{3 \cdot 2}$$

$$= 165.$$

## Example 9:



- You have 4 RED balls and 2 BLUE balls
- You take them all out of a bag one at a time
- How many distinguishable permutations (combinations) are there?

R R B B R R

B R R R B R

# ways of choosing 4 red balls from 6 total balls $= \binom{6}{4}.$

↪ # ways of choosing 2 blue out of 2 total $= \binom{2}{2}.$

# combinations $= \binom{6}{4}\binom{2}{2} = 15.$

R G R G G B B R R

## Example 10:

- You have 4 RED balls, 2 BLUE and 3 GREEN balls
- You take them all out of a bag one at a time
- How many distinguishable permutations are there now?

choose 4 red from 9 slots $= \binom{9}{4}$

choose 3 green from 5 slots $= \binom{5}{3}$

choose 2 blue — 2 slots $= \binom{2}{2}$.

Multip. princ $\Rightarrow$ $\binom{9}{4}\binom{5}{3}\binom{2}{2} = \dfrac{9!}{\cancel{5!}\,4!} \cdot \dfrac{\cancel{5!}}{3!\,\cancel{2!}} \cdot \dfrac{\cancel{2!}}{2!\,0!}$

# of combinations

$= \dfrac{9!}{4!\,3!\,2!}$

If all different, 9! ways.

overcounted by:

4! ways of ordering R

3! — G

2! — B.

In general, suppose we have $n$ objects, of which:

- $n_1$ are of type 1
- $n_2$ are of type 2
- ...
- $n_r$ are of type $r$

for $1 \leq r \leq n$ and $n_1 + \cdots + n_r = n$. Then, there are

multinomial coefficient "n choose $n_1, n_2, \ldots, n_r$".

$$\boxed{\binom{n}{n_1, n_2, \ldots, n_r}} := \frac{n!}{n_1! n_2! \cdots n_r!}$$

distinguishable permutations

Exercise: Show that this is correct by doing each of the two arguments in Example 10.

**Example 11:** You have a 52 card deck. How many ways are there to deal 5 cards to each of 6 players?

$n = 52$

30 cards.

$n_1, \ldots, n_6$ : $n_j = 5$ cards for each $j = 1, \ldots, 6$.

7th player

$n_7$ = cards remaining in deck $= 52 - 5 \times 6 = 22$.

$$\text{Total number of ways} = \binom{52}{5,5,5,5,5,5,22} = \frac{52!}{(5!)^6\, 22!}$$

**Theorem 1.12:** **(The Binomial Theorem)**

generalisation
$(x_1 + \cdots + x_k)^n = \sum$ —

If $n \in \mathbb{N} \cup \{0\}$ and $x, y \in \mathbb{R}$, then

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

See HW2.

$$(x+y)^2 = (x+y)(x+y) = xx + xy + yx + yy = x^2 + 2xy + y^2$$

$$= \binom{2}{2}x^2 + \binom{2}{1}xy + \binom{2}{0}y^2.$$

$$\hookrightarrow (x+y)^n = \underbrace{x \cdots x}_{n\text{'s}} + \underbrace{xyx \cdots x}_{(n-1)\,x\text{'s}} + \cdots + \underbrace{yy \cdots y}_{0\,x\text{'s}}$$

Coefficient of $x^r y^{n-r}$ = # of ways of choosing $r$ $x$'s out of $n$ symbols

$$= \binom{n}{r}.$$