Math 170E: Winter 2023

Lecture 22, Wed 8th Mar

Bivariate distributions of the continuous type

Last time:

- Let X, Y be continuous random variables with joint PDF $f_{X,Y}(x,y)$. Then if $A \subseteq \mathbb{R}^2$,

$$\mathbb{P}((X,Y) \in A) = \iint_A f_{X,Y}(x,y)dxdy$$

- We define the marginals by integrating out one of the variables:
  - the marginal PDF of X to be

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

  - the marginal PDF of Y to be

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

Proposition 4.22:

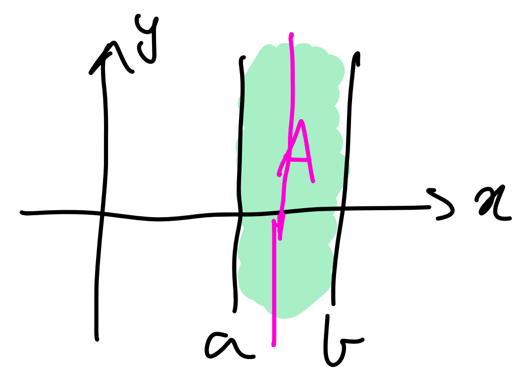
Let X, Y be continuous random variables and f_X be the marginal PDF of X. If $a < b$, then

$$\mathbb{P}(a < X \leq b) = \int_a^b f_X(x)dx$$

Proof:

$$A = \{(x,y) \in \mathbb{R}^2 : a < x \leq b, y \in \mathbb{R}\}.$$

$$\mathbb{P}(a < X \leq b) = \mathbb{P}((X,Y) \in A)$$



$$= \iint_A f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_a^b \underbrace{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy}_{f_X(x)}\,dx = \int_a^b f_X(x)\,dx.$$

Definition 4.23:

Let X, Y be continuous random variable with joint PDF $f_{X,Y}(x, y)$.

Given a function $g : \mathbb{R}^2 \to \mathbb{R}$, we define the expected value of $g(X, Y)$ to be

$$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

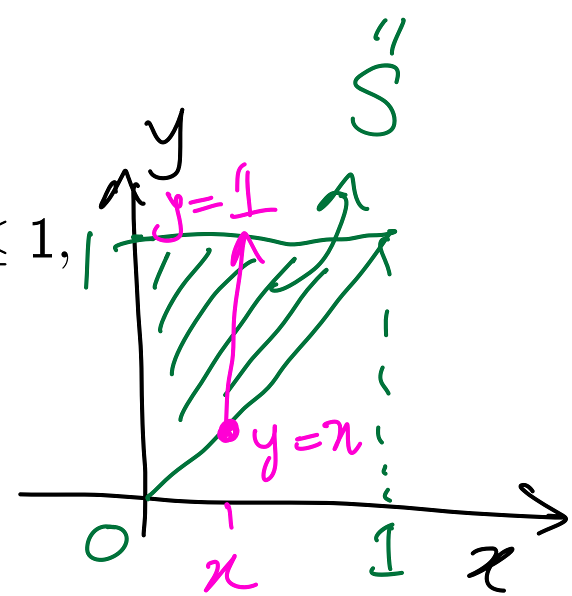
1-d case: $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

$\{0 \le x \le y \le 1\}$

Example 21:

- Let X, Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } 0 \le x \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$



- What is $\mathbb{E}[XY]$?

↳ $g(x,y) = xy$.

$$\mathbb{E}[XY] = \mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy.$$

$$= \iint_S xy \cdot 2\,dx\,dy$$

$$= 2\iint_S xy\,dx\,dy.$$

$$= 2\int_0^1\int_x^1 xy\,dy\,dx,$$

$$= 2\int_0^1 \left[xy^2/2\right]_{y=x}^{1} dx$$

$$= \cancel{2}\int_0^1 \left[\frac{x}{\cancel{2}} - \frac{x^3}{\cancel{2}}\right] dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{x=0}^{1}$$

$$= 1/4.$$

Proposition 4.24:

Let X, Y be continuous random variable with joint PDF $f_{X,Y}(x, y)$. Then

- If $a, b \in \mathbb{R}$ and $g, h : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\mathbb{E}[ag(X, Y) + bh(X, Y)] = a\mathbb{E}[g(X, Y)] + b\mathbb{E}[h(X, Y)]$$

- If $g, h : \mathbb{R}^2 \to \mathbb{R}$ and $g(x, y) \le h(x, y)$ for all $(x, y) \in \mathbb{R}^2$ we have

$$\mathbb{E}[g(X, Y)] \le \mathbb{E}[h(X, Y)]$$

Proof: Exactly the same as 1-variable case.

If X, Y discrete, indep if $P_{X,Y}(x,y) = P_X(x) P_Y(y)$

$\boxed{\{X=x\},\ \{Y=y\} \text{ are indep events}}$

Definition 4.25:

Let X, Y be continuous random variables with joint PDF $f_{X,Y}(x,y)$ and marginal PDFs $f_X(x), f_Y(y)$.

We say that X, Y are independent if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$

$\{X \leq x\}, \{Y \leq y\}$ are independent $\Leftarrow$ definition for X, Y indep.

$\hookrightarrow$ $F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$
$= \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y) = F_X(x) F_Y(y).$

Then
$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y}\left[F_X(x) F_Y(y)\right]$$
$$= F_X'(x) F_Y'(y) = f_X(x) f_Y(y).$$

Proposition 4.25:

Let X, Y be independent continuous random variables. Then if $g, h : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$$

Proof: Exercise,

Definition 4.26:

Let X, Y be continuous random variables.

- We define the covariance of X and Y to be

$$\text{cov}(X, Y) = \mathbb{E}\Big[\big(X - \mathbb{E}[X]\big)\big(Y - \mathbb{E}[Y]\big)\Big] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

- We define the correlation coefficient of X and Y to be

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}}$$

- The Cauchy-Schwarz inequality also holds in the continuous setting

- $\implies -1 \leq \rho(X, Y) \leq 1$

$|\rho(X,Y)| = 1 \iff Y = aX + b$, for some $a, b \in \mathbb{R}$.

Example 21:

We need: $\mathrm{cov}(X,Y) \to \mathbb{E}[X], \mathbb{E}[Y]$, $\mathbb{E}[XY]$

$\mathrm{Var}(X), \mathrm{Var}(Y) \to \mathbb{E}[X], \mathbb{E}[Y]$, $\mathbb{E}[X^2], \mathbb{E}(Y^2)$.

- Let X, Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } 0 \le x \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

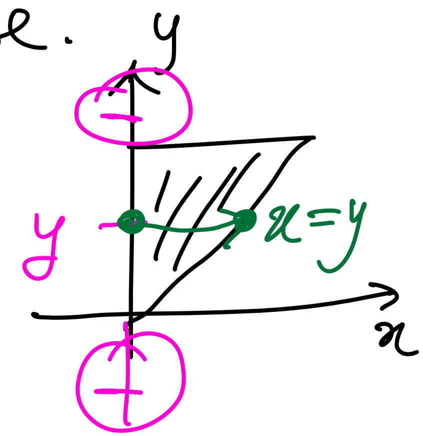
- What is $\rho(X,Y)$?

$$f_X(x) = \begin{cases} 2-2x, & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Last time:

$$f_Y(y) = \begin{cases} \int_0^y 2\,dx & \text{if } 0 \le y \le 1 \\ 0 & \text{if } y > 1 \text{ or } y < 0 \end{cases}$$

$$= \begin{cases} 2y & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}.$$



$$\mathbb{E}[X] = \int_0^1 x(2-2x)\,dx = 2\int_0^1 x - x^2\,dx$$

$$= 2\left[\tfrac{1}{2} - \tfrac{1}{3}\right] = \tfrac{1}{3}.$$

$$\mathbb{E}[Y] = \int_0^1 y \cdot 2y \, dy = 2/3.$$

$\mathbb{E}[XY] = 1/4$ (from earlier).

$$\mathbb{E}[X^2] = \int_0^1 x^2(2-2x)\,dx = 2\int_0^1 x^2 - x^3 \, dx$$
$$= 2\left[\frac{1}{3} - 1/4\right] = 1/6.$$

$$\mathbb{E}[Y^2] = \int_0^1 y^2 \cdot 2y \, dy = 2\int_0^1 y^3 \, dy = 1/2.$$

$$\hookrightarrow \operatorname{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{1}{4} - \frac{2}{9}$$
$$= \frac{9-8}{36} = \frac{1}{36}.$$

$$\operatorname{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$
$$= \frac{1}{6} - \frac{1}{9} = \frac{9-6}{54} = \frac{3}{54} = \frac{1}{18}.$$

$$\operatorname{var}(Y) = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}.$$

$$\Rightarrow \rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}} = \frac{1/36}{\sqrt{\frac{1}{18}\times\frac{1}{18}}}$$

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1.$$

$$= \frac{1/36}{1/18} = \frac{18}{36} = 1/2$$

Conditional distributions of the continuous type

$Y \mid \boxed{X=x}$ → zero probab. always.

$$p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}.$$

$$\mathbb{P}(Y\in A \mid X=x) = \frac{\mathbb{P}(X=x, Y\in A)}{\mathbb{P}(X=x)} = 0 \text{ if } X \text{ is cts!}$$

$$\mathbb{P}(Y\le y \mid x-\delta \le X \le x+\delta) = \frac{\mathbb{P}(Y\le y,\ x-\delta\le X\le x+\delta)}{\mathbb{P}(x-\delta\le X\le x+\delta)}.$$

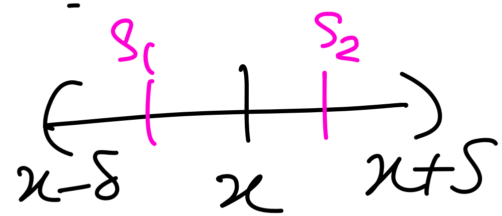
Mean value th$^{=}$ for integrals

$$= \frac{\dfrac{1}{2\delta}\displaystyle\int_{x-\delta}^{x+\delta}\left[\int_{-\infty}^{y} f_{X,Y}(s,t)\,dt\right]ds.}{\dfrac{1}{2\delta}.\displaystyle\int_{x-\delta}^{x+\delta}[f_X(s)]\,ds.}$$

$$\frac{1}{b-a}\int_a^b f(s)\,ds = f(s_0)$$

$s_0 \in (a,b)$

$$= \frac{\displaystyle\int_{-\infty}^{y} f_{X,Y}(s_1,t)\,dt}{f_X(s_2)}, \text{ for some } s_1, s_2 \in (x-\delta, x+\delta).$$



$$\delta \to 0^+ \quad \longrightarrow \quad \frac{\displaystyle\int_{-\infty}^{y} f_{X,Y}(x,t)\,dt}{f_X(x).}$$

$$\underbrace{\lim_{\delta\to 0^+} \mathbb{P}(Y \le y \mid x-\delta \le X \le x+\delta)} = \frac{1}{f_X(x)}\int_{-\infty}^{y} f_{X,Y}(x,t)\,dt.$$

" $\mathbb{P}(Y \le y \mid X = x)$ $\longrightarrow$ $Y \mid X = x$

$$\hookrightarrow f_{Y|X}(y|x) = \frac{d}{dy}\mathbb{P}(Y \le y \mid X = x)$$

$$= \frac{d}{dy}\left(\frac{1}{f_X(x)}\int_{-\infty}^{y} f_{X,Y}(x,t)\,dt\right)$$

$$= \frac{f_{X,Y}(x,y)}{f_X(x)}.$$

Definition 4.27:

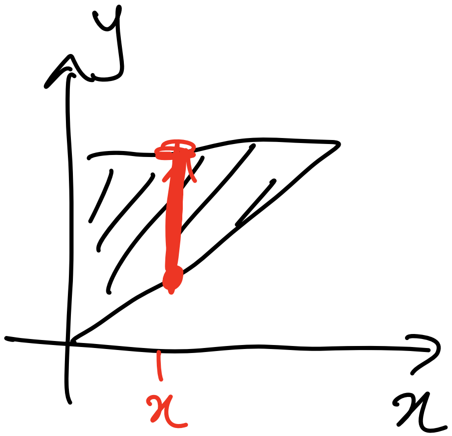
Let X, Y be continuous random variable with joint PDF $f_{X,Y}(x, y)$.
Given $x \in \mathbb{R}$ such that $f_X(x) > 0$, we define the continuous random variable $Y|x$ with PDF

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

Example 23:

- Let X, Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } 0 \le x \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$



- Given $0 < x < 1$, what is $f_{Y|X}(y|x)$?

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_{X,Y}(x,y)}{2(1-x)}.$$

$$= \begin{cases} \frac{1}{1-x} & \text{if } x \le y \le 1. \\ 0 & \text{otherwise.} \end{cases}$$

$\rightsquigarrow Y|x \sim \text{Uniform}((x,1)).$

Proposition 4.28:

Let X, Y be continuous random variable and $x \in \mathbb{R}$ such that $f_X(x) > 0$. Then,

$$\int_{-\infty}^{\infty} f_{Y|X}(y|x)dy = 1.$$

Proof: exercise.

Example 24:

- Let X, Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } 0 \le x \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

- If $0 \le x < 1$, what is $\mathbb{E}[Y|x]$?

$Y|x \sim \text{Uniform}((x,1))$

$$\mathbb{E}[Y|x] = \frac{x+1}{2}$$

$$= \int_x^1 y \cdot f_{Y|X}(y|x)\,dy$$

$$= \int_x^1 y \cdot \frac{1}{1-x}\,dy = \frac{1}{1-x} \cdot \left[\frac{y^2}{2}\right]_{y=x}^1$$

$$= \frac{1}{1-x} \cdot \frac{1-x^2}{2} = \frac{x+1}{2}.$$

$g(x) = \mathbb{E}[Y \mid x] = \frac{x+1}{2}$.

$\hookrightarrow \mathbb{E}[Y \mid X] = g(X) = \frac{X+1}{2}$.

Uniform([a,b])

$\text{Var}(\checkmark) = \frac{b^2 - a^2}{12}$.

$h(x) = \text{var}(Y \mid x) = \frac{1 - x^2}{12}$

$\hookrightarrow$ Unif([x,1])

$\hookrightarrow \text{var}(Y \mid X) = h(X) = \frac{1 - X^2}{12}$.

Law of iterated expectancy

$$\begin{cases} \mathbb{E}[\mathbb{E}[Y \mid X]] = \mathbb{E}[Y] \\ \text{var}(Y) = \text{var}(\mathbb{E}[Y \mid X]) + \mathbb{E}[\text{var}(Y \mid X)] \end{cases}$$

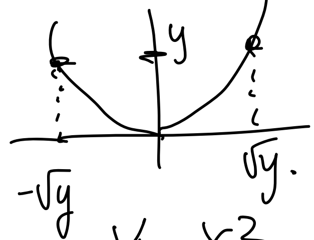
$\hookrightarrow$ these hold true if X, Y are crs with a joint pdf.

$\hookrightarrow X \rightsquigarrow Y = u(X).$

$\hookrightarrow$ what is the distribution of X?

$$\mathbb{P}(Y=y) = \mathbb{P}(u(X)=y).$$
$$= \mathbb{P}(\cancel{X = u^{-1}(y)}).$$
$$= \mathbb{P}(X \in u^{-1}(\{y\}))$$

$u(x) = x^2$



$Y = X^2$

$u^{-1}(\{y\}) = \{-\sqrt{y}, \sqrt{y}\}$