Math 170E: Winter 2023

Lecture 15, Wed 15th Feb

Last time:

- Let $S \subseteq \mathbb{R}$ and $X : \Omega \to S$ be a random variable with CDF $F_X(x)$
- We say X is a continuous random variable if there exists a function $f_X : \mathbb{R} \to [0, \infty)$ so that

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

- We call f_X a probability density function for X
- If X is a continuous random variable with PDF $f_X(x)$, we define its expected value to be

$$\mu_X = \mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx.$$

More generally, if $g : \mathbb{R} \to \mathbb{R}$ is any function, then

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx.$$

Definition/Proposition 3.10: If X is a continuous random variable we define its moment generating function to be

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx.$$

for all $t \in \mathbb{R}$ for which this makes sense.

If $M_X(t)$ is smooth on some interval $(-\delta, \delta)$, $\delta > 0$, then for all $n \geq 0$,

$$\left.\frac{d^n}{dt^n} M_X\right|_{t=0} = \mathbb{E}[X^n],$$

$$\left.\frac{d}{dt} \log M_X\right|_{t=0} = \mathbb{E}[X],$$

$$\left.\frac{d^2}{dt^2} \log M_X\right|_{t=0} = \mathrm{var}(X)$$

Proof: same as the discrete case

$$\int_{a_n}^{b_n} \qquad b_n \to +\infty$$

$$a_n \to -\infty \qquad \int_{-\infty}^{\infty} f_X(x)\,dx = 1.$$

Example 6: Let X have PDF $f_X(x) = \frac{1}{\pi(1+x^2)}$ for $x \in \mathbb{R}$. What is the MGF, mean and variance of X?

odd function

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \boxed{\frac{x}{\pi(1+x^2)}} dx. \quad g(x) = \frac{x}{1+x^2}, \; g(-x) = -g(x).$$

$$\int_{-M}^{M} \frac{x}{\pi(1+x^2)} dx = 0 \text{ for every } M>0 \Rightarrow \lim_{M\to+\infty} \int_{-M}^{M} \frac{x}{\pi(1+x^2)} dx = 0$$

$$\int_{-M}^{2M} \frac{x}{\pi(1+x^2)} dx = \int_{M}^{2M} \frac{x}{\pi(1+x^2)} dx = \frac{1}{2\pi} \log\left(\frac{4M^2+1}{M^2+1}\right).$$

$$\frac{4+1/M^2}{1+1/M^2} \to 4.$$

$$\lim_{M\to+\infty} \int_{-M}^{2M} \frac{x}{\pi(1+x^2)} dx = \frac{1}{2\pi} \log(4) \neq 0$$

$\hookrightarrow$ So $\mathbb{E}[X]$ is <u>undefined</u> (the integral only converges conditionally but <u>not</u> absolutely).

MGF is also not defined

$$M_X(t) = \int_{-\infty}^{\infty} \frac{e^{tx}}{\pi(1+x^2)} dx \leftarrow \text{does not converge if } t \neq 0.$$

$\hookrightarrow$ f_X is called the <u>Cauchy distribution</u>.

Example 7:

- Customers arrive at a Coffee shop according to an approximate Poisson process with rate 1 $\underbrace{\text{customer per minute}}_{\lambda}$ $\rightarrow X \in [0, +\infty)$, cts r.v.
- Let X be the arrival time (in minutes) of the *first* customer
- What is $\mathbb{P}(X \leq \frac{1}{2})$? → "$\mathbb{P}$(first customer arrives in 30 seconds)." $\frac{1}{2}$ minute

Let N be the number of customers arriving in first 30 seconds.

$$\Rightarrow N \sim \text{Poisson}\left(\frac{\lambda}{2}\right) = \text{Poisson}\left(\frac{1}{2}\right)$$

. If $X > 1/2$, then $N = 0$.
Also if $N = 0$, then $X > 1/2$.

$$\{\omega : X > 1/2\} = \{\omega : N = 0\}.$$

$$\mathbb{P}(X \leq 1/2) = 1 - \mathbb{P}(X > 1/2)$$
$$= 1 - \mathbb{P}(N = 0).$$
$$= 1 - e^{-1/2} \frac{(1/2)^0}{0!} = 1 - e^{-1/2} \approx 0.39.$$

Example 8: The Exponential distribution

$[\lambda] = \frac{1}{\text{time}}$, $[\theta] = \frac{1}{[\lambda]} = \text{time}$.

- Consider an approximate Poisson process with rate $\lambda > 0$ per unit time
- Let X be the time of the first arrival
- We say that X is exponentially distributed with mean waiting time $\theta = \frac{1}{\lambda}$ and write $X \sim \text{Exponential}(\theta) = \text{Exp}(\theta)$. $\hookrightarrow \mathbb{E}[X] = \theta$.

Proposition 3.11: If $\theta > 0$ and $X \sim \text{Exponential}(\theta)$, then it has PDF

$$f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \quad \text{if } x > 0$$

Proof: ① Want CDF of X: $\mathbb{P}(X \le x)$. $= 1 - \mathbb{P}(X > x)$.
Fix $x > 0$. Let N be the number of arrivals on the interval $(0, x]$.

Then, $N \sim \text{Poisson}(\underbrace{\lambda x}_{= x/\theta})$ $\rightarrow \theta = \frac{1}{\lambda}$.

Then the CDF is

$$F_X(x) = \mathbb{P}(X \le x) = 1 - \mathbb{P}(X > x) \rightarrow \{X > x\} = \{N = 0\}.$$

$$= 1 - \mathbb{P}(N = 0)$$

$$= 1 - e^{-x/\theta} \frac{\left(\frac{x}{\theta}\right)^0}{0!} = 1 - e^{-x/\theta}.$$

② Take deriv. of CDF:

$$\hookrightarrow f_X(x) = F_X'(x) = \frac{1}{\theta} e^{-x/\theta}. \rightarrow$$

Check:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$$

$f_X(x) = \frac{1}{\theta} e^{-x/\theta}$, $f_X(0) = \frac{1}{\theta} = \lambda$.

$\lambda_1 > \lambda_2$

$\Updownarrow$

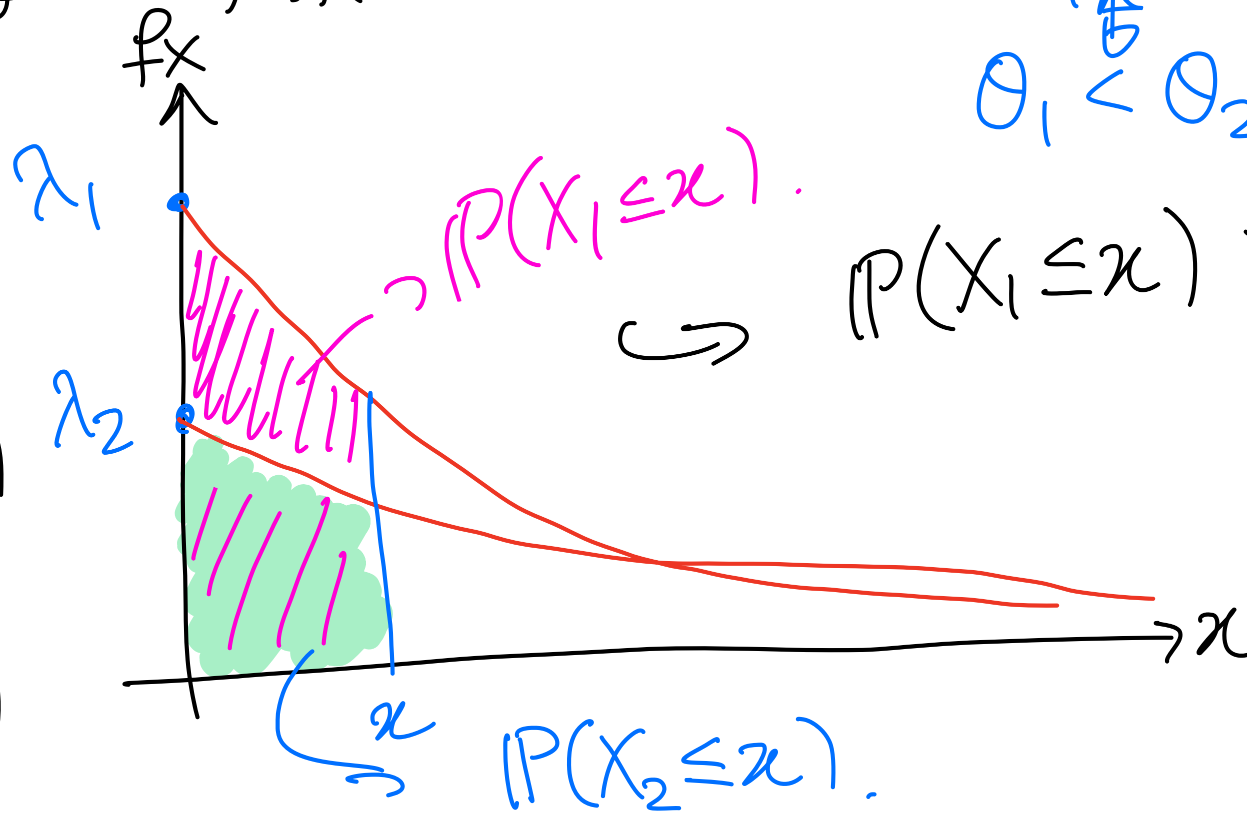
$\theta_1 < \theta_2$.

f_{X_1}, f_{X_2}

$\downarrow$

$X_1 \sim Exp(\theta_1) = Exp(\frac{1}{\lambda_1})$

$X_2 \sim Exp(\theta_2) = Exp(\frac{1}{\lambda_2})$



$\hookrightarrow \mathbb{P}(X_1 \le x) \ge \mathbb{P}(X_2 \le x)$.

$\hookrightarrow$ Show that: $\mathbb{P}(X_1 \le x) \ge \mathbb{P}(X_2 \le x)$ if and only if $\lambda_1 \ge \lambda_2$. $\|$

Proposition 3.12: If $\theta > 0$ and $X \sim \text{Exponential}(\theta)$, then its MGF is

$$M_X(t) = \frac{1}{1-\theta t} \quad \text{if } t < \frac{1}{\theta}.$$

Proof: $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$

$$= \frac{1}{\theta}\int_0^{\infty} e^{tx} e^{-x/\theta}\,dx.$$

$$= \frac{1}{\theta}\int_0^{\infty} e^{(t-1/\theta)x}\,dx.$$

We need $t - 1/\theta < 0$
$\Leftrightarrow t < 1/\theta$.

$$= \frac{1}{\theta}\left[\frac{1}{t-1/\theta} e^{(t-1/\theta)x}\right]_0^{\infty}$$

$$= \frac{1}{\theta}\left[-\frac{1}{t-1/\theta}\right] = -\frac{1}{\theta t - 1} = \frac{1}{1-\theta t}.$$ provided that $t < 1/\theta$.

Proposition 3.13: If $\theta > 0$ and $X \sim \text{Exponential}(\theta)$, then it has mean and variance

$$\mathbb{E}[X] = \theta$$

mean waiting time.

$$\text{var}(X) = \theta^2$$

Proof:

$$\mathbb{E}[X] = \frac{d}{dt} M_X(t)\Big|_{t=0}$$

$$= \frac{d}{dt}\left((1-\theta t)^{-1}\right)\Big|_{t=0} = -(-\theta)(1-\theta t)^{-2}\Big|_{t=0}$$

$$= \theta.$$

Example 9:

$\lambda = 20$ (in hours)

- Customers arrive at a coffee shop at a rate of 20 customers per hour

  1 customer every $\frac{1}{20}$ hrs

- The shop opens at 9 am

  $\frac{60}{20}$ mins $= 3$ min.

- At what time does the shop owner expect their first customer?

X = arrival time of first customer.

$\hookrightarrow X \sim Exp\left(\frac{1}{20}\right)$.

$\Downarrow$

$E[X] = \frac{1}{20}$ hrs

$= 3$ mins.

A) 9 : 03am

B) 10 : 00am

C) 9 : 10am

D) 9 : 05am

Example 10:

- Customers arrive at a Coffee shop according to an approximate Poisson process with rate 1 customer per minute
- Let X be the arrival time (in minutes) of the *third* customer
- What is $\mathbb{P}(X \leq \frac{1}{2})$? $\longrightarrow \mathbb{P}(X \leq 1/2) = 1 - \mathbb{P}(X > 1/2).$

third customer arrives after 1/2-minute.

Let N = # of arrivals in first 30 seconds

Then $N \sim \text{Poisson}(1/2)$.

$\hookrightarrow \{X > 1/2\} = \{N=0\} \cup \{N=1\} \cup \{N=2\} = \{N \leq 2\}.$

Hence

$$\begin{aligned}\mathbb{P}(X \leq 1/2) &= 1 - \mathbb{P}(X > 1/2) \\ &= 1 - \mathbb{P}(N \leq 2). \\ &= 1 - e^{-1/2} - e^{-1/2}\frac{(\frac{1}{2})^1}{1!} - e^{-1/2}\frac{(\frac{1}{2})^2}{2!}.\end{aligned}$$

Example 11: The Gamma distribution

- Consider an approximate Poisson process with rate $\lambda > 0$ per unit time
- Let $\alpha \geq 1$ be an integer
- Let X be the time of the αth arrival
- We say that X is gamma distributed with mean parameters α and $\theta = \frac{1}{\lambda}$ and write $X \sim \text{Gamma}(\alpha, \theta)$
- If $X \sim \text{Gamma}(1, \theta)$, then $X \sim \text{Exponential}(\theta)$

Proposition 3.14: If $\alpha \geq 1$ is an integer, $\theta > 0$ and $X \sim \mathrm{Gamma}(\alpha, \theta)$, then it has PDF

$$f_X(x) = \frac{1}{\theta^\alpha (\alpha-1)!} x^{\alpha-1} e^{-\frac{x}{\theta}} \quad \text{if } x > 0$$

- If $\alpha \geq 2$, $f_X(0) = 0$.

$\hookrightarrow$ Maximum at $x_* = \theta(\alpha-1)$.

$$f_X(x_*) = \frac{1}{\theta} e^{-(\alpha-1)} \frac{(\alpha-1)^{\alpha-1}}{\boxed{(\alpha-1)!}} \sim \frac{1}{\theta} \cdot \frac{1}{\sqrt{\alpha}} \quad \text{for } \alpha \to +\infty$$

Stirling formula