Math 170E: Winter 2023

Lecture 20, Wed 1st Mar

Conditional distributions

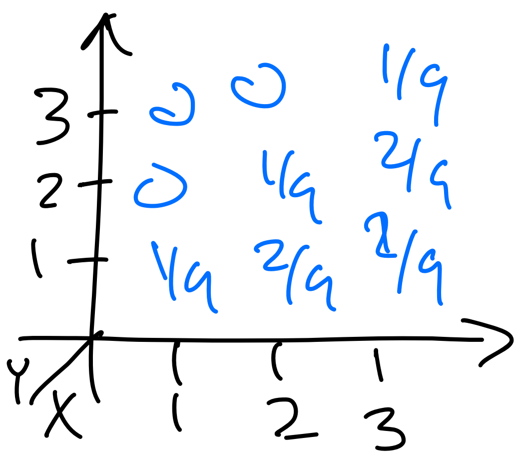
Today:

We'll discuss today:

- how to define and compute the conditional PMF of one discrete random variable conditioned on another
- how to define and compute the conditional expectation and variance
- how to prove and apply the law of iterated expectation and the law of total variance

Example 9:

- You choose two numbers at random from the set $\{1, 2, 3\}$
- Let X be the larger and Y be the smaller of these two numbers
- What is $P(X = 3|Y = 1)$?



Fix $Y=y$, let x vary we get a distribution

$$\mathbb{P}(X=x \mid Y=y).$$

A) $\frac{2}{5}$

B) $\frac{2}{9}$

C) $\frac{5}{9}$

D) $\frac{3}{5}$

$$\mathbb{P}(X=3 \mid Y=1)$$

$$= \frac{\mathbb{P}(X=3, Y=1)}{\mathbb{P}(Y=1)}$$

$$= \frac{p_{X,Y}(3,1)}{p_Y(1)}$$

$$= \frac{2/9}{5/9} = 2/5$$

Definition 4.14: Let X, Y be a pair of discrete random variables taking values in $S_X, S_Y \subseteq \mathbb{R}$, respectively.

- For each fixed $y \in S_Y$, we define the random variable $X|y$ with PMF

$$p_{X|Y}(x|y) = \mathbb{P}(X = x | Y = y) \quad \text{for } x \in S_X$$

$$= \frac{p_{X,Y}(x,y)}{p_Y(y)}.$$

$X|y =$ "$X|Y=y$"

- For each fixed $x \in S_X$, we define the random variable $Y|x$ with PMF

$$p_{Y|X}(y|x) = \mathbb{P}(Y = y | X = x) \quad \text{for } y \in S_Y$$

Proposition 4.15: Let X, Y be a pair of discrete random variables taking values in $S_X, S_Y \subseteq \mathbb{R}$, respectively.

- For each fixed $y \in S_Y$, we have

$$\sum_{x \in S_X} \boxed{p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}} = 1$$

well-defined if and only if $p_Y(y) \neq 0$.

- For each fixed $x \in S_X$, we have

$$\sum_{y \in S_Y} p_{Y|X}(y|x) = 1$$

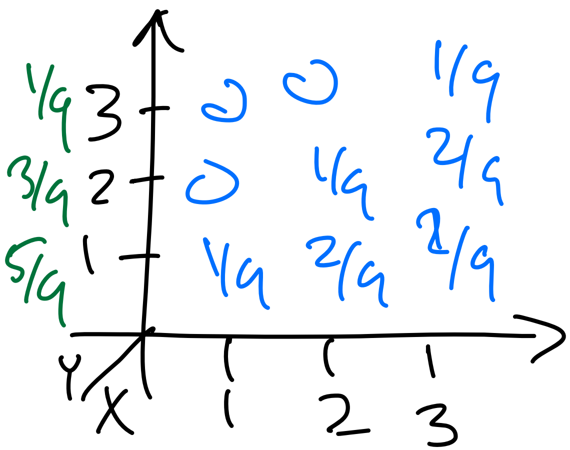
Proof: For the first one:

$$\sum_{x \in S_X} p_{X|Y}(x|y) = \sum_{x \in S_X} \frac{p_{X,Y}(x,y)}{p_Y(y)} = \frac{1}{p_Y(y)} \sum_{x \in S_X} p_{X,Y}(x,y)$$

$$= \frac{1}{p_Y(y)} \cdot p_Y(y) = 1.$$

Example 10:

- You choose two numbers at random from the set $\{1, 2, 3\}$
- Let X be the larger and Y be the smaller of these two numbers
- What is $p_{X|Y}(x|1)$? $\rightsquigarrow$ PMF of the r.v. $X|1$, $x \in \{1,2,3\}$.



$$p_{X|Y}(1|1) = \frac{p_{X,Y}(1,1)}{p_Y(1)} = \frac{1/9}{5/9} = 1/5.$$

$$p_{X|Y}(2|1) = \frac{p_{X,Y}(2,1)}{p_Y(1)} = \frac{2/9}{5/9} = 2/5.$$

$$p_{X|Y}(3|1) = \frac{p_{X,Y}(3,1)}{p_Y(1)} = \frac{2/9}{5/9} = 2/5.$$

$$p_{X|Y}(1|2) = 0$$

$$p_{X|Y}(x|1) = \begin{cases} 1/5 & \text{if } x=1 \\ 2/5 & \text{if } x=2,3. \end{cases}$$

Example 11: $\mathbb{E}[X|y] = g(y). \rightsquigarrow \mathbb{E}[X|Y]$

- You choose two numbers at random from the set $\{1, 2, 3\}$
- Let X be the larger and Y be the smaller of these two numbers
- What is $\mathbb{E}[X|1]$?

$(X|1)(\omega).$

$$P_{X|Y}(x|1) = \begin{cases} 1/5 & \text{if } x=1 \\ 2/5 & \text{if } x=2,3. \end{cases}$$

$$\mathbb{E}[X|1] = \sum_{x \in S_X} x \, P_{X|Y}(x|1)$$

$$= 1 \times P_{X|Y}(1|1) + 2 \times P_{X|Y}(2|1) + 3 \times P_{X|Y}(3|1)$$

$$= 1/5 + 4/5 + 6/5 = 11/5.$$

Example 11:

- You choose two numbers at random from the set $\{1, 2, 3\}$
- Let X be the larger and Y be the smaller of these two numbers
- What is var$(X|1)$?

Definition 4.16: Let X, Y be a pair of discrete random variables taking values in $S_X, S_Y \subseteq \mathbb{R}$, respectively.

fix $x \in S_X$

$\hookrightarrow Y|x = Y|X=x.$

- Define the function $g : S_X \to \mathbb{R}$ by

$$g(x) = \mathbb{E}[Y|x]$$

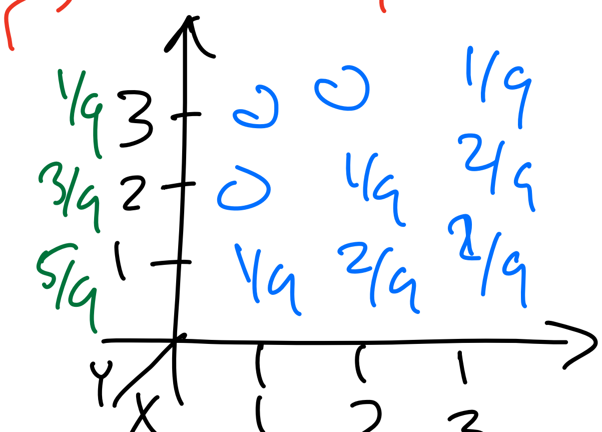
- We define the conditional expectation of Y conditioned on X to be the random variable

$$\mathbb{E}[Y|X] = g(X)$$

- We can similarly define $\mathbb{E}[X|Y]$

$$\mathbb{E}[Y|X](\omega) \overset{\text{def}^n}{=} g(X(\omega)).$$

$\mathbb{E}[Y] = \frac{14}{9}$



$\Omega = \{(1,1),(1,2),(1,3),\dots,(3,3)\}$

$\hookrightarrow |\Omega| = 9.$

Suppose that we know that $X(\omega) = 2$.

What is my best guess for Y given

that $\underline{X(\omega)=2}$?

- On the event, $\{\omega : X(\omega)=2\} = \{(1,2),(2,1),(2,2)\}$.

$\hookrightarrow$ The PMF of Y should change if we know $\{X=2\}$

$\hookrightarrow$ We let $Y|2$ be a new r.v. ?

$$P_{Y|2}(y|2) = \begin{cases} 2/3, & y=1 \\ 1/3, & y=2 \\ 0, & y=3 \end{cases}$$

$\hookrightarrow$ $$\mathbb{E}[Y|2] = \sum_{y=1}^{3} y\, P_{Y|2}(y|2) = 1 \times 2/3 + 2 \times 1/3$$
$$= 4/3 < \frac{14}{9} = \mathbb{E}[Y].$$

All the info related to X is contained in the union of the events:

$$\{X=1\}, \quad \{X=2\}, \{X=3\}.$$

↪ We can also define r.v. $Y|1$ & $Y|3$.

So for each $x \in \{1,2,3\}$, we get some numbers

$\mathbb{E}[Y|x] \rightsquigarrow \mathbb{E}[Y|X=x]$.

To compute $\mathbb{E}(Y|1)$ we needed the info $\{X=1\}$.

$\{\omega : X(\omega)=1\} = \{(1,1)\}$.

If $\omega = (1,1) \xrightarrow{X=1} \mathbb{E}[Y|X] = \mathbb{E}[Y|1]$.

If $\omega \in \{(1,2),(2,1),(2,2)\} \xrightarrow{X=2} \mathbb{E}[Y|X] = \mathbb{E}[Y|2] = 4/3$

If $\omega \in \Omega \setminus \{(1,2),(2,1),(2,2),(1,1)\} \xrightarrow{X=3} \mathbb{E}[Y|X] = \mathbb{E}[Y|3]$

So we have a map : $\omega \in \Omega \longrightarrow \mathbb{E}[Y|X](\omega) \in \mathbb{R}$.

$\mathbb{E}[Y|X]$ is a random variable

(depends on outcomes!).

We view this as a composition:

$$\omega \longmapsto \{X(\omega) = x\} \longmapsto g(X(\omega)) \in \mathbb{R}$$

$$\omega \in \Omega.$$

$$g(x) = \mathbb{E}[Y|x].$$

Example 12:

$$P_X(x) = \frac{1}{4}\left(\frac{3}{4}\right)^{x-1}, \quad x = 1, 2, \dots$$

- Let $X \sim \text{Geometric}(\frac{1}{4})$ and $Y|x \sim \text{Uniform}(\{1, 2, \dots, x\})$.
- What is the PMF of $\mathbb{E}[Y|X]$? $\mathbb{E}[Y|x] = g(x)$.

$$Y|x \sim \text{Unif}(\{1, \dots, x\}) \Rightarrow \mathbb{E}[Y|x] = \frac{x+1}{2} = g(x).$$

So $\mathbb{E}[Y|X] = g(X) = \frac{X+1}{2}$

$$\mathbb{P}(\mathbb{E}[Y|X] = z) = \mathbb{P}\left(\frac{X+1}{2} = z\right) = \mathbb{P}(X = 2z-1).$$

$$= P_X(\boxed{2z-1}).$$

$$= \frac{1}{4}\left(\frac{3}{4}\right)^{2z-2}.$$

for $z \in \{1, 3/2, 2, 5/2, \dots\}$.

This makes sense only if $2z-1 \in \{1, 2, \dots\}$.

$\Leftrightarrow 2z \in \{2, 3, \dots\}$

$z \in \{1, 3/2, 2, 5/2, \dots\}$

Theorem 4.17: **The Law of Iterated Expectation**

Let X, Y be discrete random variables. Then

$$\mathbb{E}\big[\mathbb{E}[Y|X]\big] = \mathbb{E}[Y]$$

Proof: Let $g(x) = \mathbb{E}[Y|x] = \sum_{y \in S_Y} y \, P_{Y|X}(y|x)$.

Then $\mathbb{E}[Y|X] = g(X)$.

So

$$\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[g(X)] = \sum_{x \in S_X} g(x) P_X(x).$$

$$P_{X,Y}(x,y) / P_X(x).$$

$$= \sum_{x \in S_X} \Big(\sum_{y \in S_Y} y \, P_{Y|X}(y|x) \Big) P_X(x).$$

$$= \sum_{x \in S_X} \sum_{y \in S_Y} y \, P_{Y|X}(y|x) P_X(x)$$

$$= \sum_{x \in S_X} \sum_{y \in S_Y} y \, P_{X,Y}(x,y)$$

$$= \sum_{y \in S_Y} y \sum_{x \in S_X} P_{X,Y}(x,y) = \underline{\sum_{y \in S_Y} y \, P_Y(y)} = \mathbb{E}[Y]$$

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Example 13:

- Let $X \sim \text{Geometric}(\frac{1}{4})$ and $Y|x \sim \text{Uniform}(\{1, 2, \ldots, x\})$.
- What is $\mathbb{E}[Y]$?

$X \sim Geom(1/4)$

$\mathbb{E}[X] = 4.$

$\mathbb{E}[Y|X] = \frac{X+1}{2}$

By the law of iterated expectation,

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}\left[\frac{X+1}{2}\right]$$

$$= \frac{1}{2}\{\mathbb{E}[X]+1\}$$

$$= \frac{1}{2}(4+1) = 5/2.$$

$P_{Y|X}(y|x) = \frac{1}{x}$ for each $x = 1, 2, \ldots$

$$P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)} \Rightarrow P_{X,Y}(x,y) = P_X(x) P_{Y|X}(y|x) = \frac{P_X(x)}{x}.$$

if $y = 1, 2, \ldots x.$

$1 \le y \le x$

If we fix y, then

$$P_Y(y) = \sum_{x=y}^{\infty} \frac{P_X(x)}{x}.$$

$\{y \le x < +\infty\}$

$1 \le y < +\infty$

$$\mathbb{E}[Y] = \sum_{y=1}^{\infty} y P_Y(y) = \sum_{y=1}^{\infty} \sum_{x=y}^{\infty} \frac{P_X(x)}{x} y.$$

$\{1 \le y \le x\}$

$\{1 \le x \le +\infty\}$

$$= \sum_{x=1}^{\infty} \sum_{y=1}^{x} \frac{P_X(x)}{x} y$$

$$= \sum_{x=1}^{\infty} \frac{P_X(x)}{x} \sum_{y=1}^{x} y \longrightarrow \frac{x(x+1)}{2}$$

$$= \sum_{x=1}^{\infty} \frac{P_X(x)}{\cancel{x}} \cdot \frac{\cancel{x}(x+1)}{2}$$

$$= \frac{1}{2} \left\{ \underbrace{\sum_{x=1}^{\infty} x P_X(x)}_{E[X]} + \underbrace{\sum_{x=1}^{\infty} P_X(x)}_{=1} \right\}$$

$$= \frac{1}{2} \left\{ E[X] + 1 \right\}.$$