Math 170E: Winter 2023

Lecture 21, Mon 6th Mar

Conditional distributions and Bivariate distributions of the continuous type

Today:

We'll discuss today:

- how to define and compute the conditional variance
- how to prove and apply the law of total variance
- how to compute two-dimensional integrals
- the definition of a continuous joint probability distribution

Let X, Y be a pair of discrete random variables taking values in $S_X, S_Y \subseteq \mathbb{R}$, respectively.

- For each fixed $y \in S_Y$, we define the random variable $X|y$ with PMF

$$p_{X|Y}(x|y) = \mathbb{P}(X = x|Y = y) \quad \text{for } x \in S_X$$

$$= \frac{p_{X,Y}(x,y)}{p_Y(y)} \hookrightarrow \neq 0.$$

Recap

- Similar for $Y|x$
- Define the function $g : S_X \to \mathbb{R}$ by

$$g(x) = \mathbb{E}[Y|x]$$

- We define the conditional expectation of Y conditioned on X to be the random variable

$$\mathbb{E}[Y|X] = g(X) = g(X(\omega)).$$

The Law of Iterated Expectation

Let X, Y be discrete random variables. Then

$$\mathbb{E}\big[\mathbb{E}[Y|X]\big] = \mathbb{E}[Y]$$

Definition 4.18: Let X, Y be a pair of discrete random variables taking values in $S_X, S_Y \subseteq \mathbb{R}$, respectively.

- Define the function $h : S_X \to \mathbb{R}$ by

$$h(x) = \mathrm{var}(Y|x)$$

- We define the conditional variance of Y conditioned on X to be the random variable

$$\mathrm{var}(Y|X) = h(X)$$

- We can similarly define $\mathrm{var}(X|Y)$

Theorem 4.19: **The Law of Total Variance**

$\rightarrow \mathrm{var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2.$

Let X, Y be discrete random variables. Then

$$\mathrm{var}(Y) = \mathbb{E}[\mathrm{var}(Y|X)] + \mathrm{var}(\mathbb{E}[Y|X])$$

Proof: $g(x) = \mathbb{E}[Y|x] = \sum_{y \in S_Y} y\, P_{Y|X}(y|x).$

$\hookrightarrow g(X) = \mathbb{E}[Y|X]$

$$h(x) = \mathrm{var}(Y|x) = \sum_{y \in S_Y} y^2 P_{Y|X}(y|x) - g(x)^2 \quad \text{--- (1)}$$

Now $\mathrm{var}(Y|X) = h(X)$, so

$$\mathbb{E}[\mathrm{var}(Y|X)] = \mathbb{E}[h(X)] = \sum_{x \in S_X} h(x) P_X(x)$$

$$\text{by (1)} = \sum_{x \in S_X} \sum_{y \in S_Y} y^2 \underbrace{P_{Y|X}(y|x) P_X(x)}_{= P_{X,Y}(x,y)} - \sum_{x \in S_X} g(x)^2 P_X(x)$$

$$= \sum_{x \in S_X} \sum_{y \in S_Y} y^2 P_{X,Y}(x,y) - \mathbb{E}[g(X)^2].$$

$$= \mathbb{E}[Y^2] - \mathbb{E}[g(X)^2] \quad \ldots (2)$$

Now

$$\mathrm{Var}(\mathbb{E}[Y|X]) = \mathrm{Var}(g(X))$$

$$= \mathbb{E}[g(X)^2] - \mathbb{E}[g(X)]^2$$

$$= \mathbb{E}[g(X)^2] - (\mathbb{E}[Y])^2 \quad \ldots (3)$$

by law of iterated expectation

Adding (2) & (3) we get

$$\mathrm{Var}(\mathbb{E}[Y|X]) + \mathbb{E}[\mathrm{Var}(Y|X)] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$$

$$= \mathrm{Var}(Y).$$

Example 13:

- Let $X \sim \text{Geometric}(\frac{1}{4})$ and $Y|x \sim \text{Uniform}(\{1, 2, \ldots, x\})$.
- What is the $\text{var}(Y)$?

By the law of ~~total~~ variance,

$$\text{var}(Y) = \mathbb{E}[\text{var}(Y|X)] + \text{var}(\mathbb{E}[Y|X]).$$

$$g(x) = \mathbb{E}[Y|x] = \frac{x+1}{2}, \quad h(x) = \text{var}(Y|x) = \frac{x^2-1}{12}.$$

$$g(X) = \mathbb{E}[Y|X] = \frac{X+1}{2}. \quad h(X) = \text{var}(Y|X) = \frac{X^2-1}{12}.$$

- $$\mathbb{E}[\text{var}(Y|X)] = \mathbb{E}\left[\frac{X^2-1}{12}\right]$$
$$= \frac{1}{12}\left\{\mathbb{E}[X^2] - 1\right\}.$$
$$= \frac{1}{12}\{28-1\} = \frac{27}{12}.$$

$X \sim \text{Geom}(1/4)$

$\mathbb{E}[X] = 4.$

$$\text{var}X = \frac{1-p}{p^2} = \frac{1-1/4}{(1/4)^2} = \frac{3}{4} \times 4^2 = 12.$$

$\mathbb{E}[X^2] = \text{var}(X) + \mathbb{E}[X]^2$
$= 12 + 16 = 28$

- $\text{var}(\mathbb{E}[Y|X]) = \text{var}\left(\frac{X+1}{2}\right)$

$$= \frac{1}{4}\text{var}(X+1)$$
$$= \frac{1}{4}\text{var}(X) = \frac{12}{4} = 3.$$

So therefore,
$$\text{Var}(Y) = \frac{27}{12} + 3 = \frac{63}{12}$$

$\exists$ r.v. cts: $\mathbb{P}(X \in A) = \int_A f_X(x)\,dx$. $\rightarrow$ one integral

$\hookrightarrow \subseteq \mathbb{R}$

Discrete bivariate: $\mathbb{P}((X,Y) \in A) = \sum_{(x,y)\in A\cap S} p_{X,Y}(x,y)$

$\uparrow$ two sums.

CTS Givariate:

X, Y CTS and they have some joint PDF:

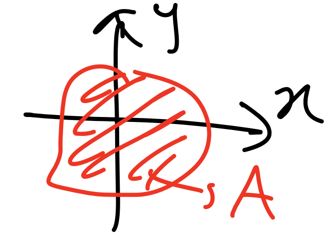
$$\mathbb{P}((X,Y)\in A) = \iint_A f_{X,Y}(x,y)\,dx\,dy.$$

$A \subseteq \mathbb{R}^2$

two integrals!

"Double integral"

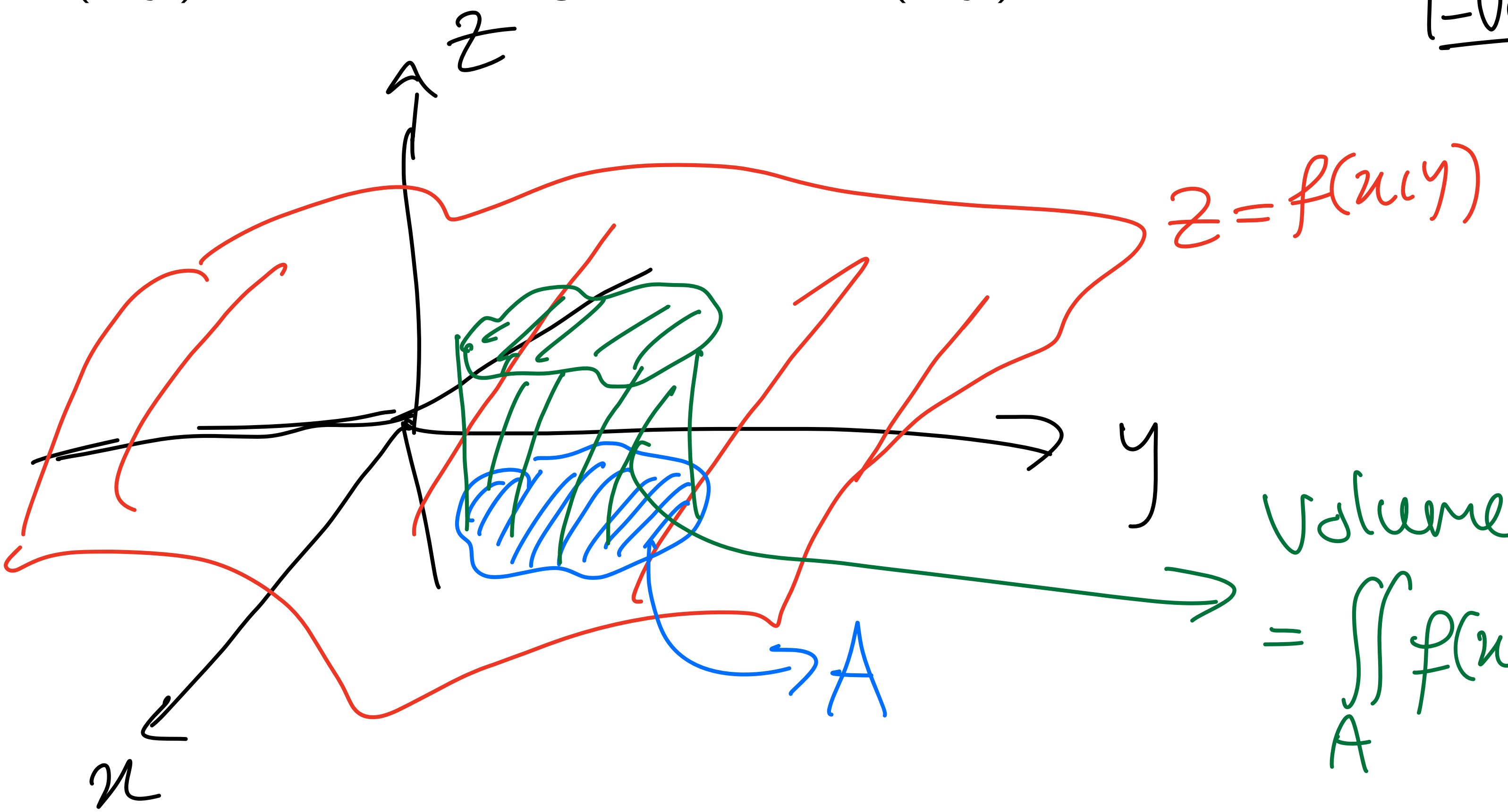
A primer on double integrals:



- Given $A \subseteq \mathbb{R}^2$ and $f : A \to \mathbb{R}$, we will define the *double integral*:

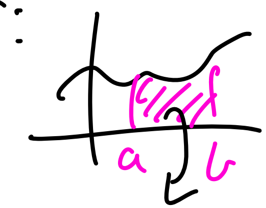
$$\iint_A f(x,y)\,dxdy$$

- If f is non-negative, this computes the volume between the region A in the (x,y)-plane and the graph of $z = f(x,y)$



Volume $= \iint_A f(x,y)\,dxdy$

1-Var:



$\int_a^b f(x)dx$.

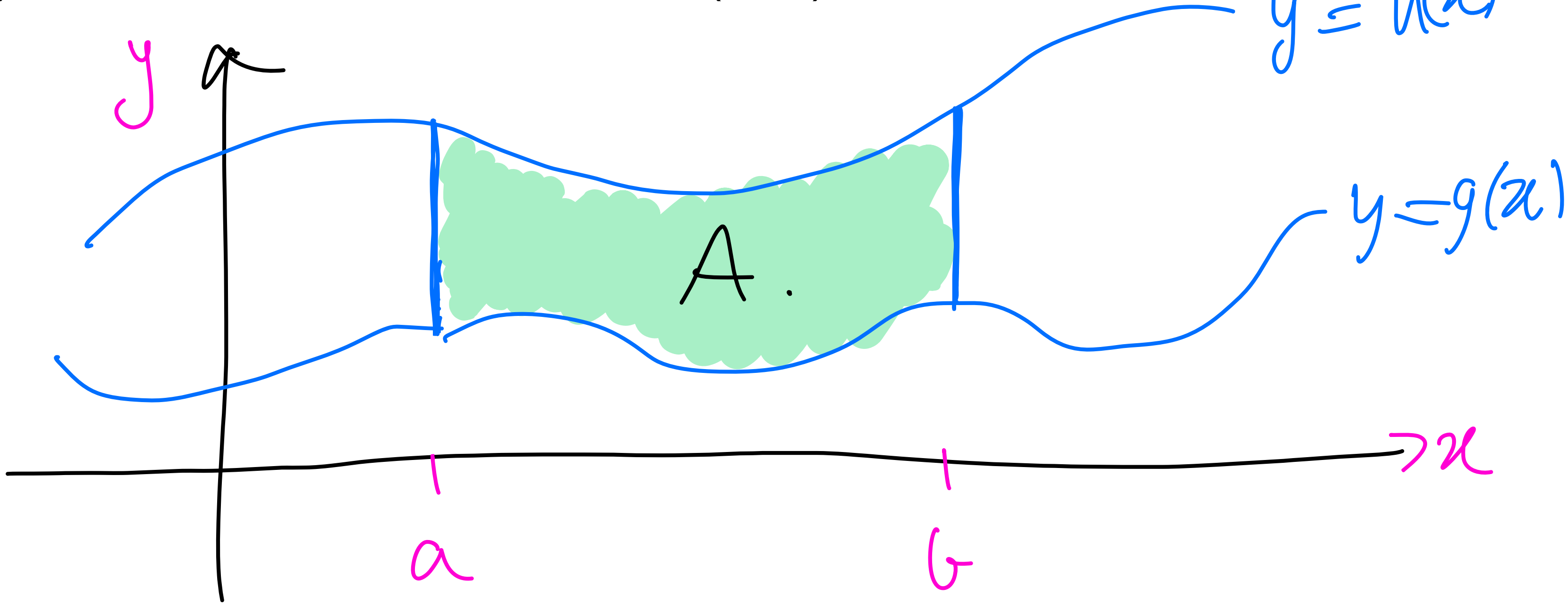
- If there exist constants $a < b$ and functions $g, h : [a, b] \to \mathbb{R}$ so that

$$A = \{(x, y) \in \mathbb{R}^2 \, : \, a \leq x \leq b, \quad g(x) \leq y \leq h(x)\},$$

then "Vertically simple regions"

$$\iint_A f(x, y)dxdy = \overbrace{\int_a^b \underbrace{\left(\int_{g(x)}^{h(x)} f(x, y)dy \right)}_{\text{a function of } x \text{ only}} dx}^{\text{iterated integral}}$$

- If f is non-negative, this computes the volume between the region A in the (x, y)-plane and the graph of $z = f(x, y)$



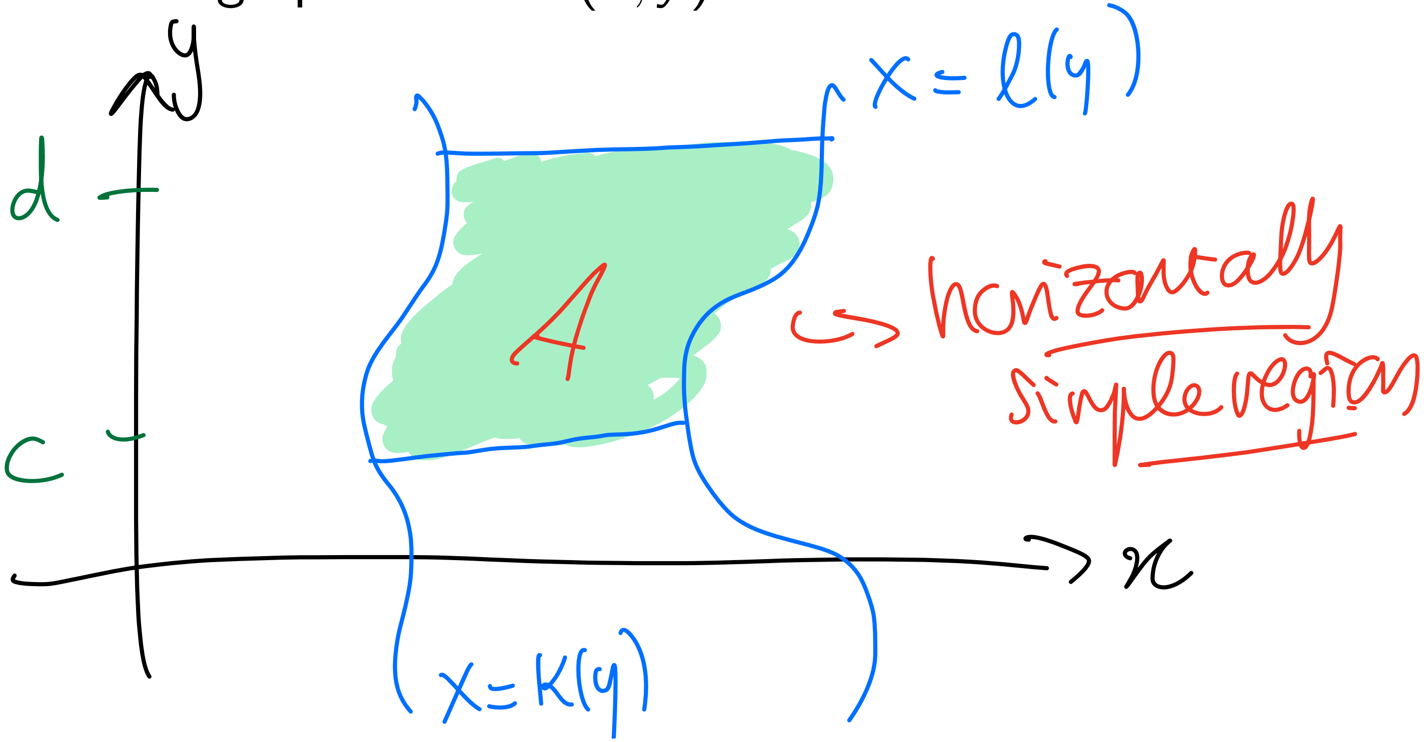
- If there exist constants $c < d$ and functions $k, \ell : [a, b] \to \mathbb{R}$ so that

$$A = \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d, \quad k(y) \leq x \leq \ell(y)\},$$

then

$$\iint_A f(x, y)dxdy = \int_c^d \overbrace{\left(\int_{k(y)}^{\ell(y)} f(x, y)dx\right)}^{\text{function of } y \text{ only.}} dy$$

- If f is non-negative, this computes the volume between the region A in the (x, y)-plane and the graph of $z = f(x, y)$



Example:

$y = 1 - x$.

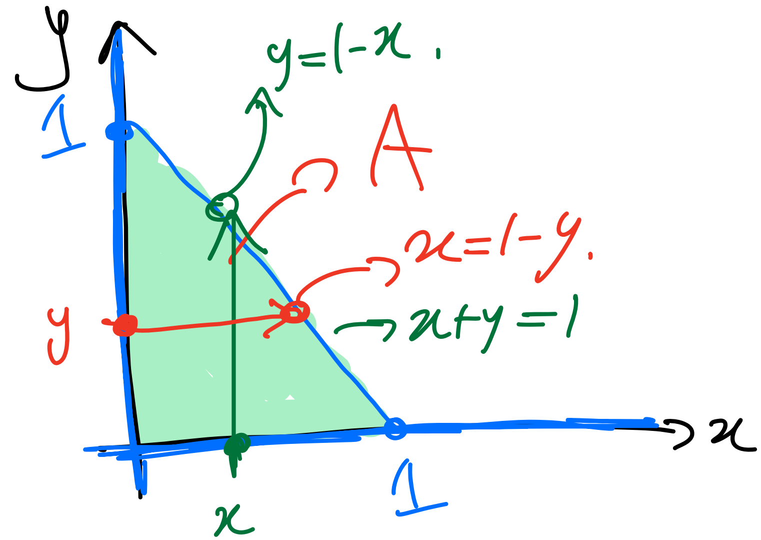
- Let $f(x, y) = x + y$ and A be the region bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$.

$A = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le 1-x\}$

↪ vertically simple

→ horizontally simple

$= \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1,\ 0 \le x \le 1-y\}$.



choose whichever is simplest to compute.

$$\iint_A (x+y)\,dx\,dy = \int_0^1 \int_0^{1-x} (x+y)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-y} (x+y)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + xy\right]_{x=0}^{1-y} dy = \int_0^1 \left[\frac{(1-y)^2}{2} + (1-y)y\right] dy$$

$$= \int_0^1 \frac{1}{2} - \cancel{y} + \frac{y^2}{2} + \cancel{y} - y^2\,dy$$

$$= \int_0^1 \frac{1}{2} - \frac{y^2}{2}\,dy = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

Bivariate distributions of the continuous type:

Given two continuous random variables X, Y we may define a joint probability density function $f_{X,Y} : \mathbb{R}^2 \to [0, \infty)$ so that for any $A \subseteq \mathbb{R}^2$, we have

$$\mathbb{P}((X, Y) \in A) = \iint_A f_{X,Y}(x, y)\,dx\,dy.$$

↪ $F_{X,Y}(x,y) = \mathbb{P}(X \le x, Y \le y)$

↪ $f_{X,Y}(x,y) = \dfrac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y).$

Example 19:

- Let X, Y have joint PDF

$S = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le y \le 1\}$

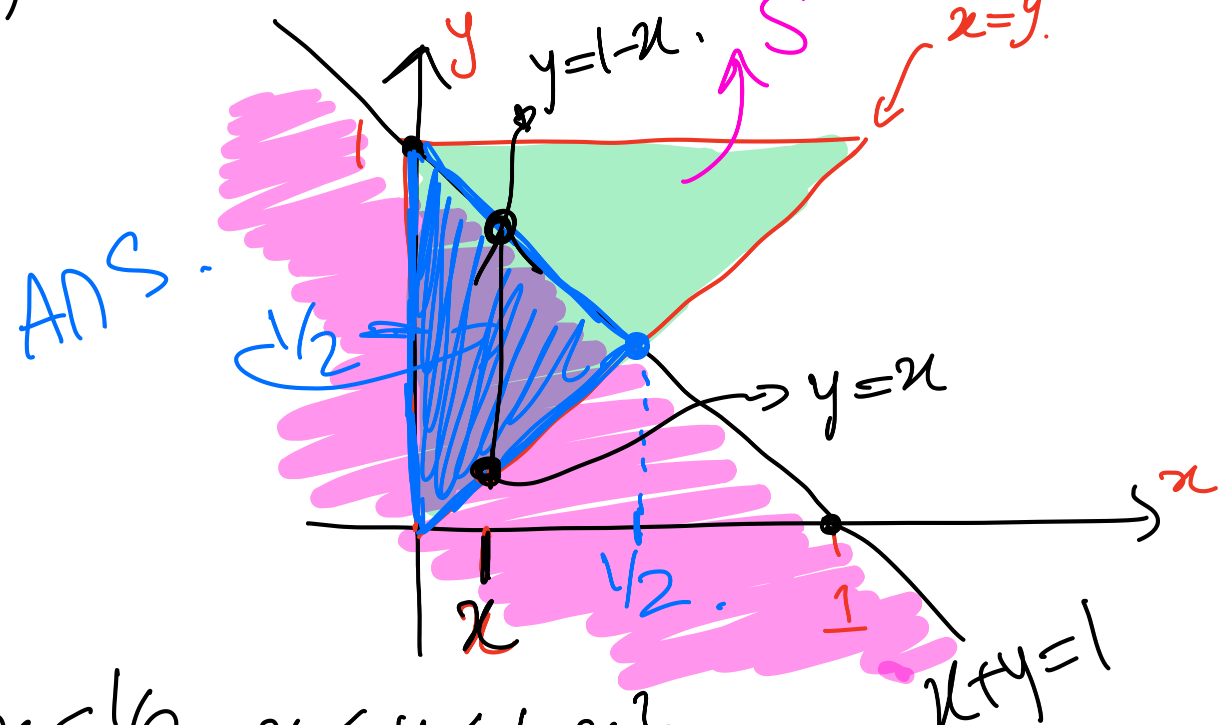
$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } 0 \le x \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

$\text{Area}(S) = 1/2.$

- What is $\mathbb{P}(X + Y \le 1)$?

$\mathbb{P}(X+Y \le 1) = \mathbb{P}((X,Y) \in A) \rightsquigarrow A = \{(x,y) \in \mathbb{R}^2 : x+y \le 1\},$

$$= \iint_A f_{X,Y}(x,y)\,dx\,dy = \iint_{A \cap S} f_{X,Y}(x,y)\,dx\,dy.$$



$A \cap S = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1/2,\ x \le y \le 1-x\}$

↪ vertically simple.

$$\iint_{A\cap S} f_{X,Y}(x,y)\,dx\,dy = \int_0^{1/2}\int_x^{1-x} 2\,dy\,dx.$$

$$= \int_0^{1/2} 2\left[(1-x)-x\right]dx$$

$$= \int_0^{1/2} 2\left[1-2x\right]dx$$

$$= 2\left[x-x^2\right]_{x=0}^{1/2} = 2\left[\frac{1}{2}-\left(\frac{1}{2}\right)^2\right]$$

$$= 1-1/2 = 1/2.$$

$$\iint_{A\cap S} f_{X,Y}(x,y)\,dx\,dy = 2\iint_{A\cap S} 1\,dx\,dy$$

$$= 2\,\text{Area}(A\cap S).$$

Proposition 4.20:

If X, Y are continuous random variable with joint PDF $f_{X,Y}(x,y)$ then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$$

Proof:

$$1 = \mathbb{P}(\Omega) = \mathbb{P}((X,Y) \in \mathbb{R}^2).$$

$$= \iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy$$

$-\infty < x < +\infty$

$-\infty < y < +\infty$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy.$$

Definition 4.21:

If X, Y are continuous random variable with joint PDF $f_{X,Y}(x, y)$ then we define

- the marginal PDF of X to be

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$$

- the marginal PDF of Y to be

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx$$

Example 20:

- Let X, Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } 0 \le x \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

- What is the marginal PDF of X?

$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$.

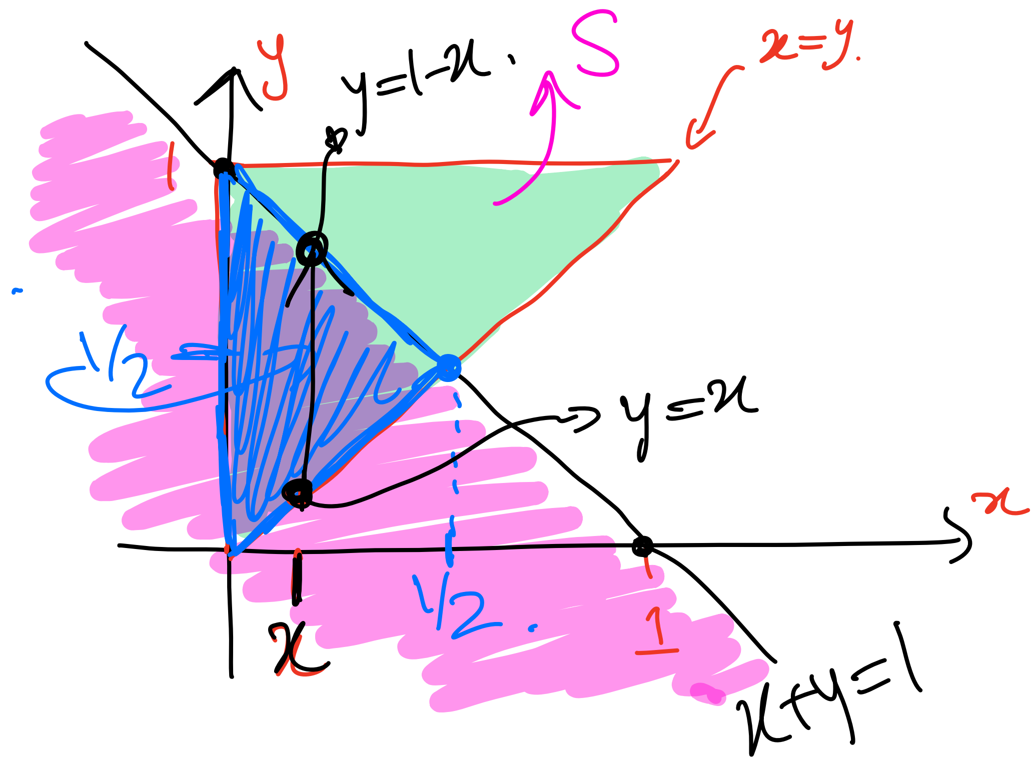
If $x<0$, $f_{X,Y}(x,y)=0$ for all $y \in \mathbb{R}$.

$\Rightarrow f_X(x)=0$

Similarly if $x>1$.

If $0<x<1$:

$\Rightarrow \quad x \le y \le 1.$



$$\hookrightarrow f_X(x) = \int_x^1 2\, dy = 2(1-x).$$

$$\hookrightarrow f_X(x) = \begin{cases} 2(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$g(x) = E[Y|x].$$

$$g : S_X \longrightarrow \mathbb{R}.$$

$$X : \Omega \longrightarrow S_X \subseteq \mathbb{R}.$$

$$g \circ X = g(X) : \Omega \longrightarrow \mathbb{R}.$$