CS163: Deep Learning for Computer Vision

Lecture 15: Generative Models: Introduction, PixelRNN, VAE

Courtesy of Prof. Justin Johnson at Umich and the team of Stanford CS231n for the slides

Supervised vs Unsupervised Learning

Supervised Learning

Data: (x, y)

x is data, y is label

Goal: Learn a *function* to map x -> y

Examples: Classification, regression, object detection, semantic segmentation, image captioning, etc.

Classification

Cat

Supervised vs Unsupervised Learning

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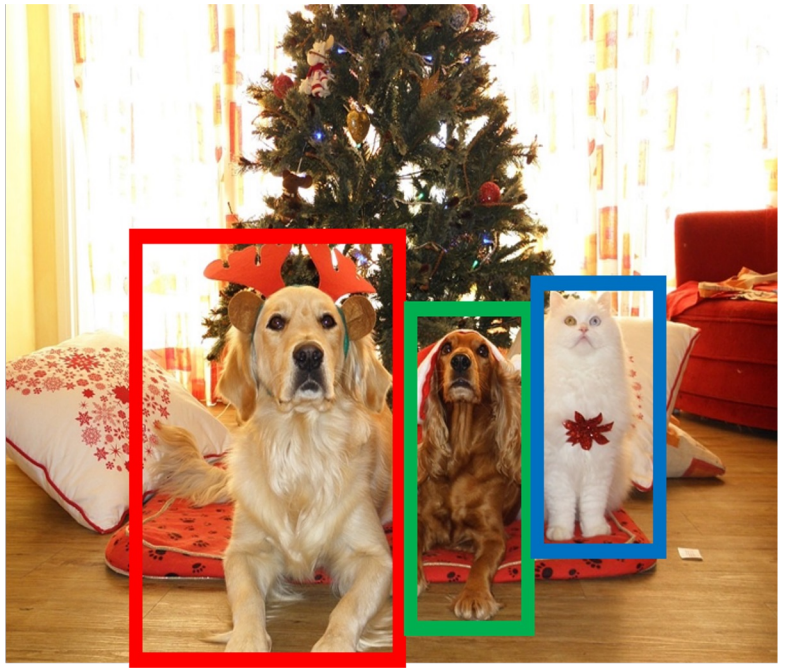
Object Detection



DOG, **DOG**, **CAT**

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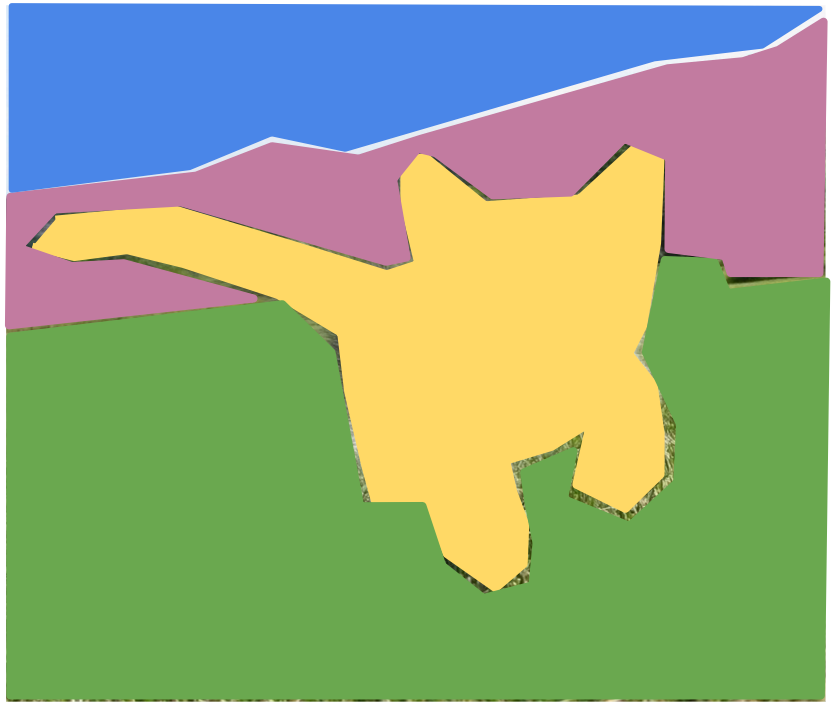
Semantic Segmentation



GRASS, CAT, TREE, SKY

Supervised vs Unsupervised Learning

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Unsupervised Learning

Data: x

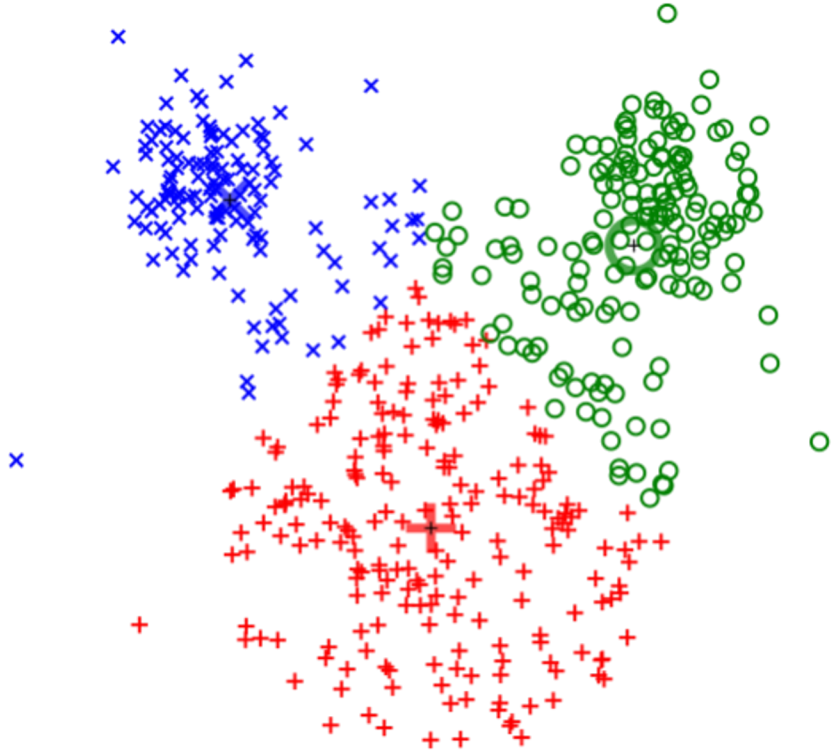
Just data, no labels!

Goal: Learn some underlying hidden *structure* of the data

Examples: Clustering, dimensionality reduction, feature learning, density estimation, etc.

Supervised vs Unsupervised Learning

Clustering
(e.g. K-Means)



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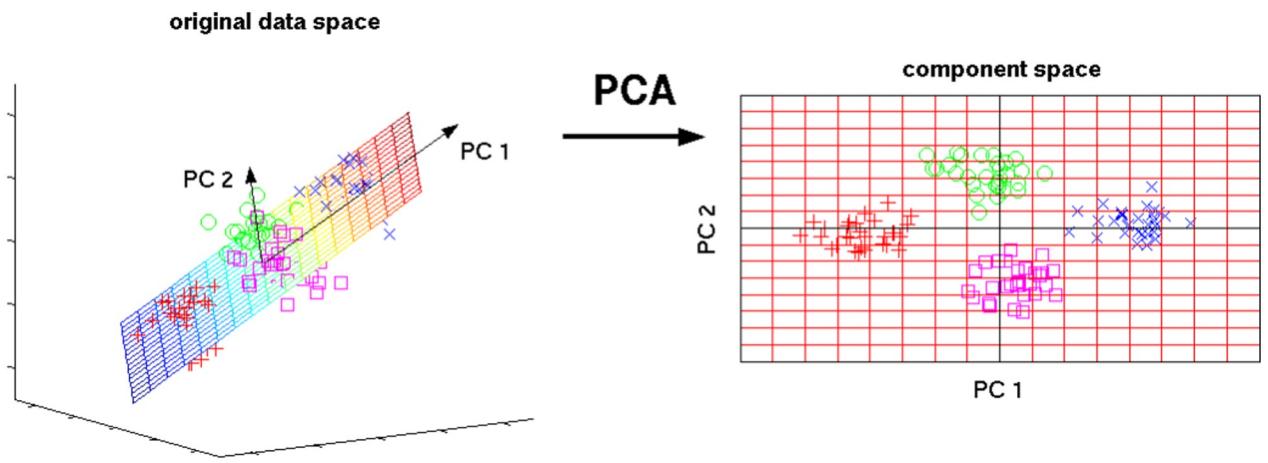
Supervised vs Unsupervised Learning

Dimensionality Reduction (e.g. Principal Components Analysis)



3D → 2D

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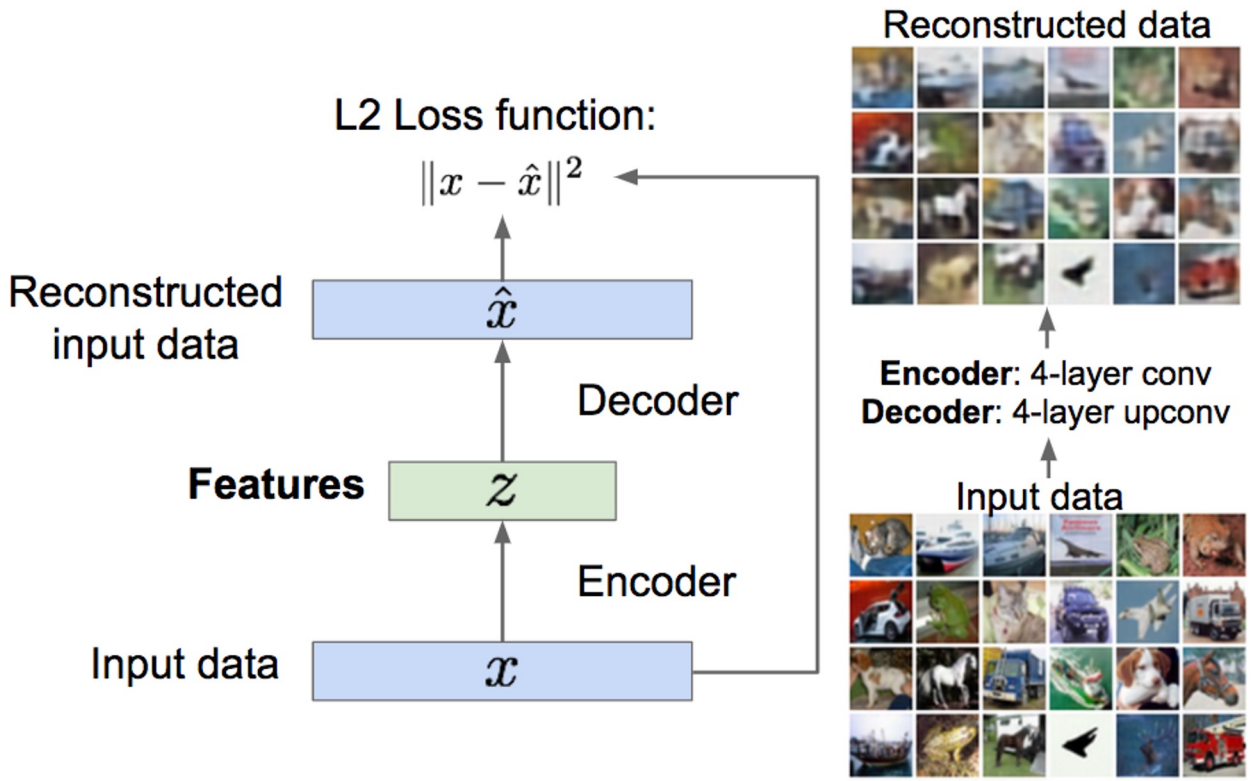
Supervised vs Unsupervised Learning

Feature Learning (e.g. autoencoders)



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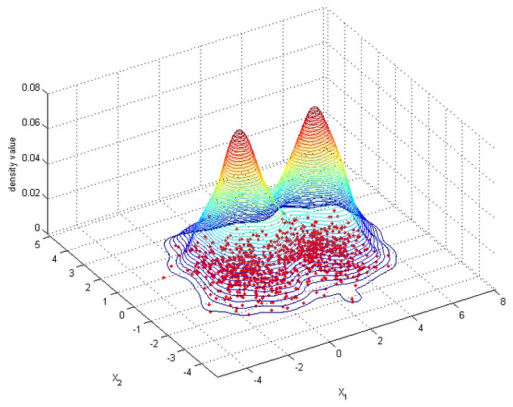
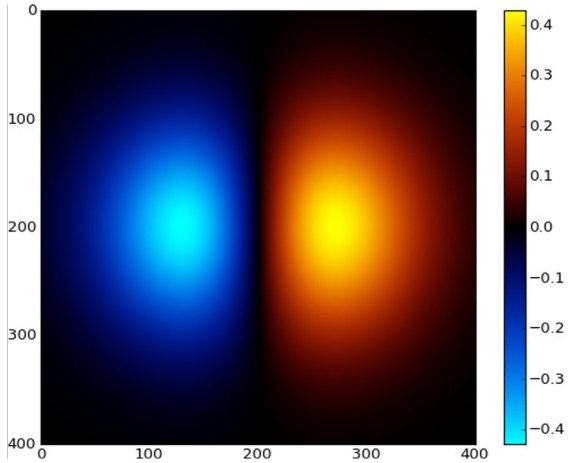
Supervised vs Unsupervised Learning

Density Estimation





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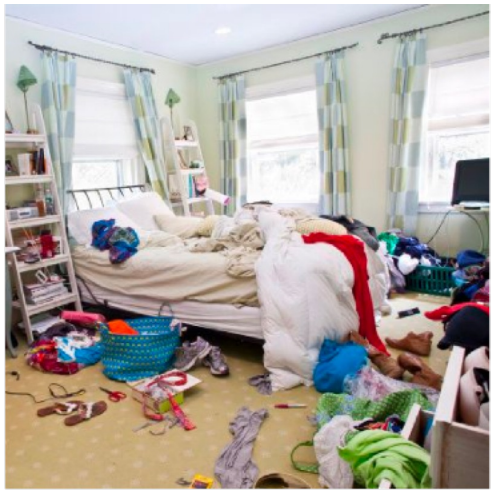
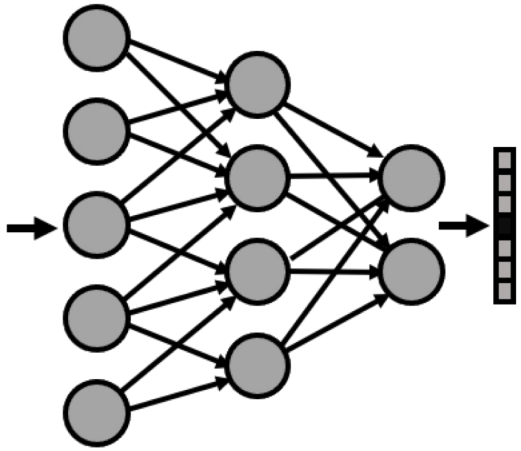
Discriminative vs Generative Models in CV

Discriminative models are everywhere for visual recognition





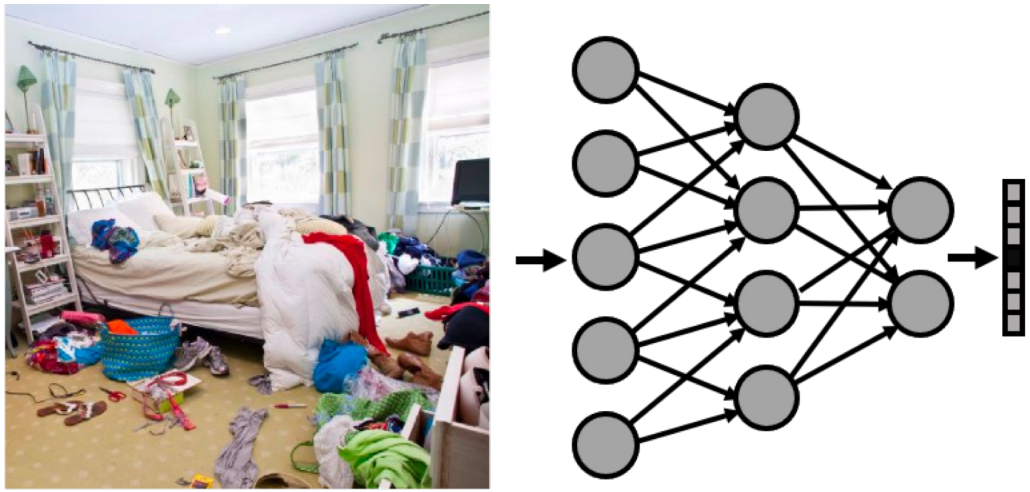
Scene categorization
Bedroom

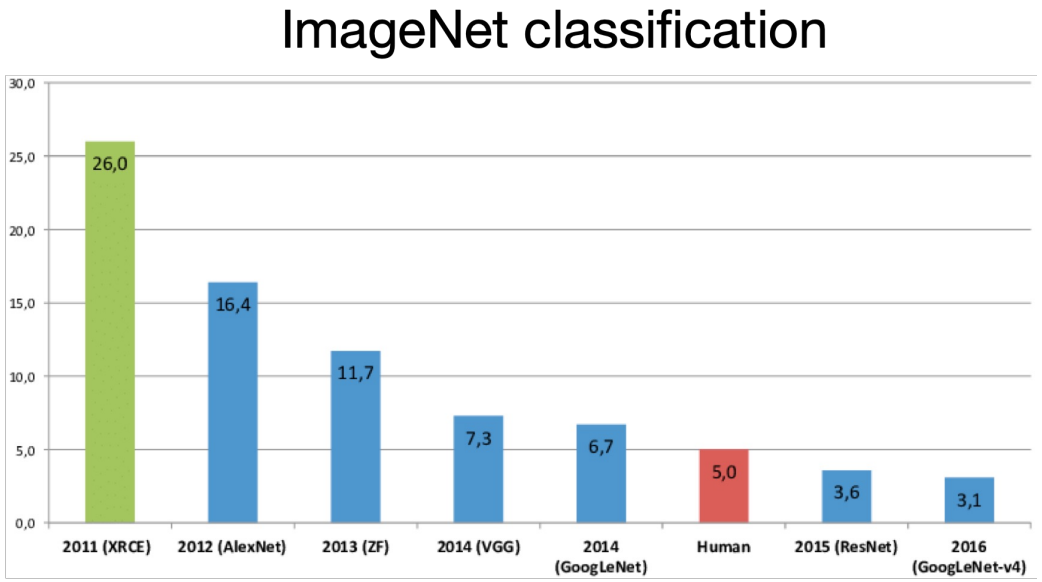
Attribute prediction
Messy, natural-lighting

Semantic segmentation

Successful Progress for Discriminative Models

Image classification, object detection, attribute prediction, semantic segmentation, few shot recognition, weakly supervised localization, etc.



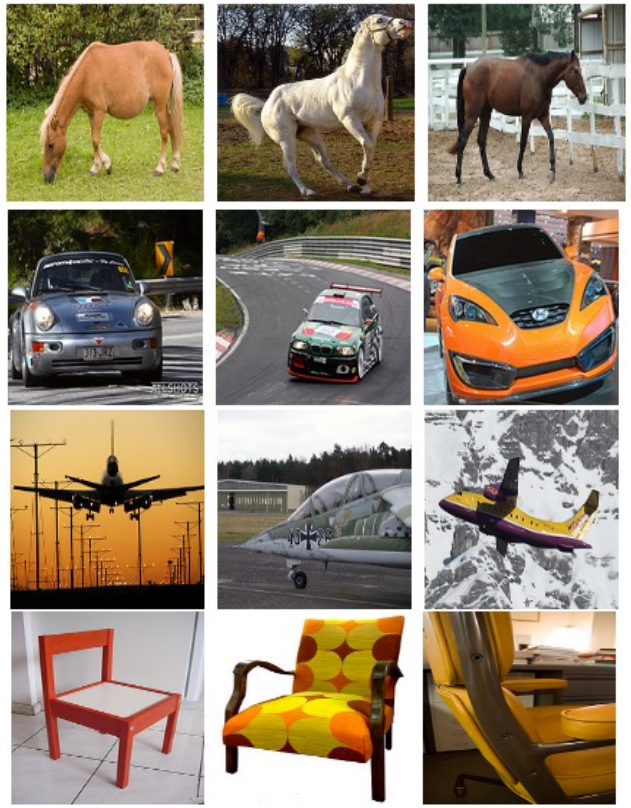


Successful Progress for Discriminative Models

2016: ResNet
2017:DenseNet
>100 layers

IMAGENET

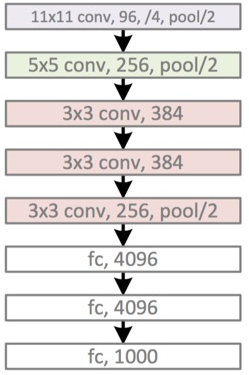
2009: 1.2 million images



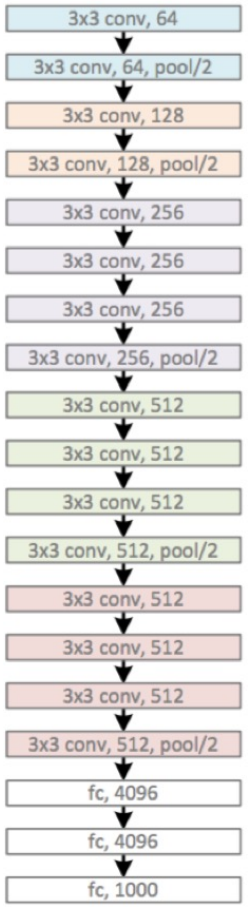
COCO
Common Objects in Context

2014: 120k annotated images

2012: AlexNet
5 layers



2014: VGG
16 layers

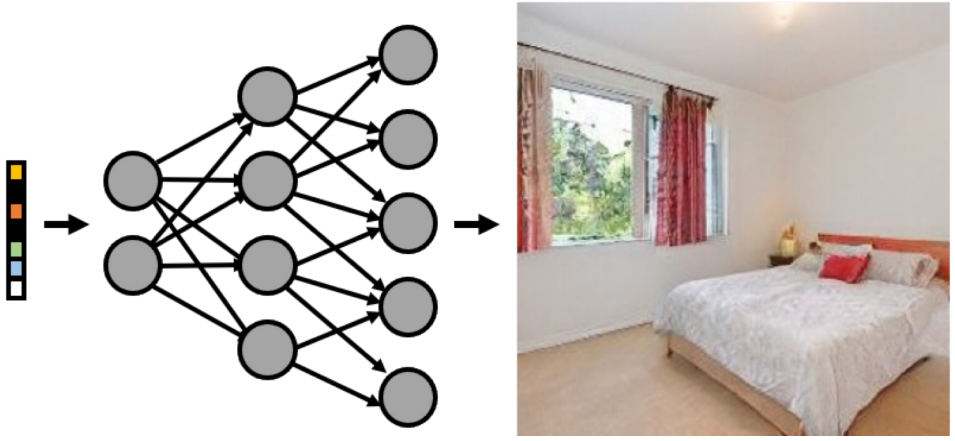


2015: GoogLeNet
22 layers

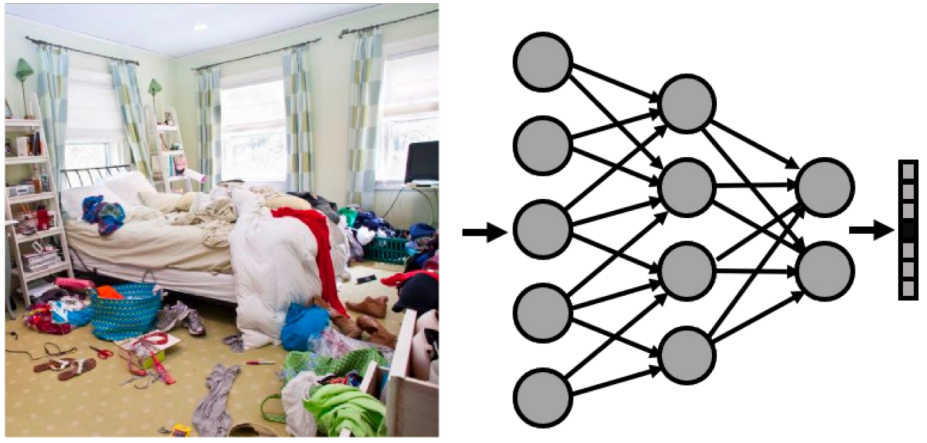
Paradigm Shifting to Generative Models

Nowadays…

Generative Model

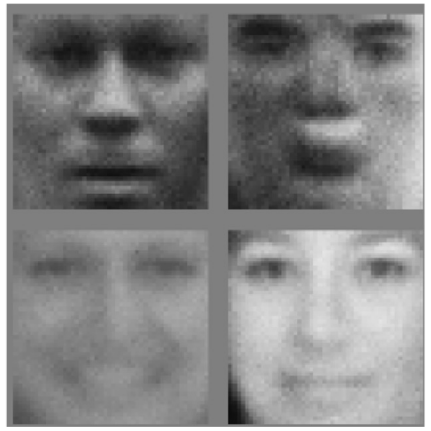


Discriminative Model



Recent Advances in Generative Models

2014



GAN

2015

DCGAN

2017

PG-GAN

2018

StyleGAN

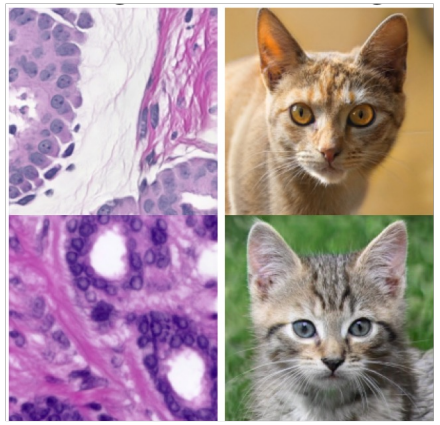
2018

BigGAN

2019

StyleGANv2

2020



StyleGAN-ADA

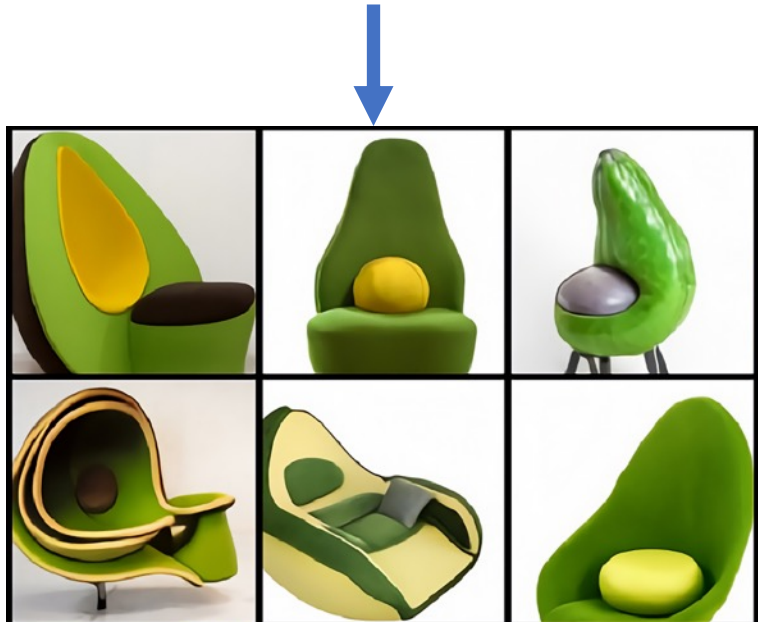
2021

Alias-Free GAN

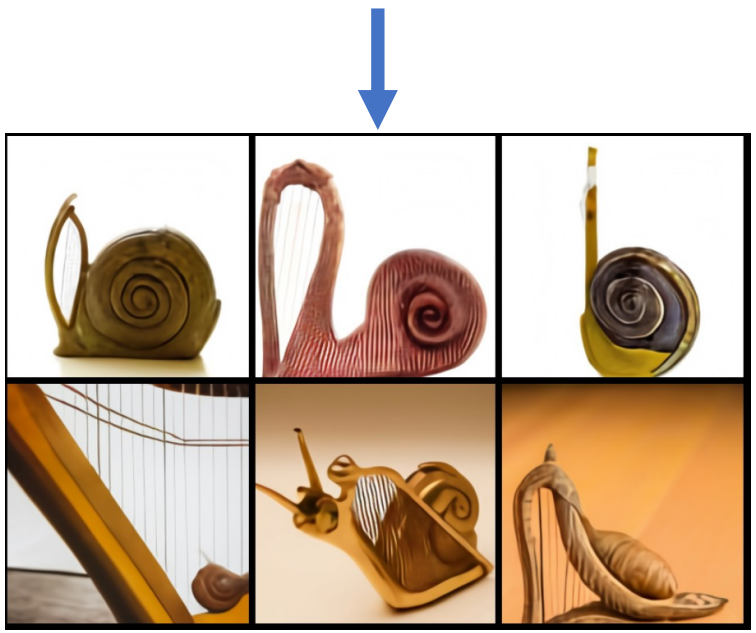
2021: DALLE: Text2Image Model from OpenAI

https://openai.com/blog/dall-e/

An armchair in the shape of an avocado



A snail made of harp

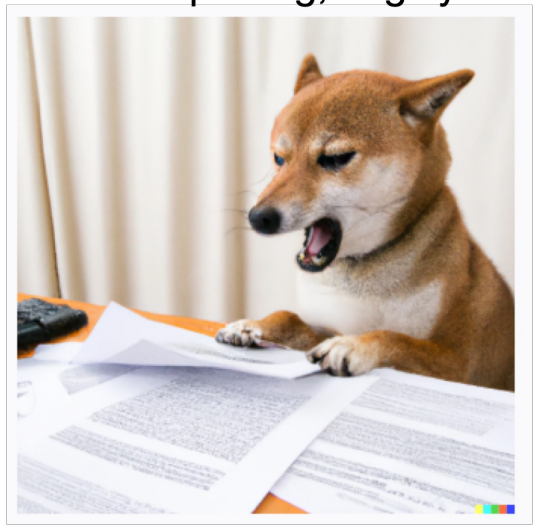


2022: DALLE-2 for AICG

Whitney, the good boi
ig: @whitneythegoodboi

shiba inu overlooking Grand Canyon in starry night

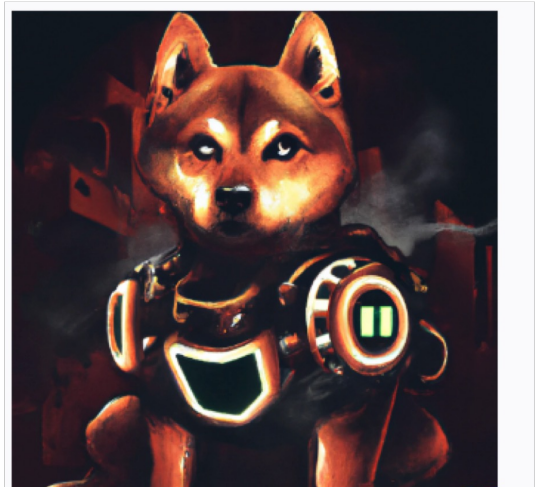
a shiba inu is doing tax reporting, angrily



shiba inu is playing heavy metal music

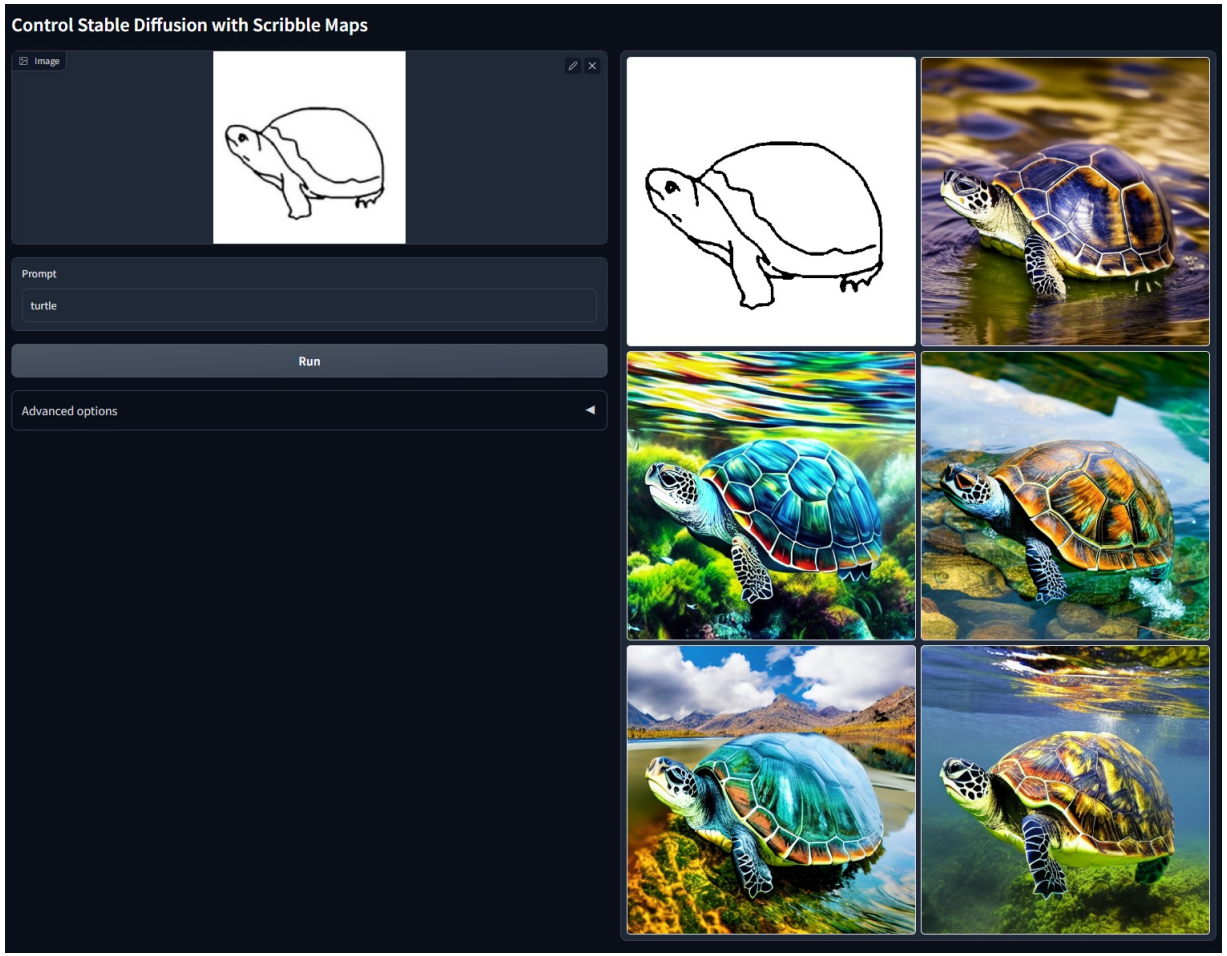
a superhero shiba inu fighting an evil corgi, watercolor

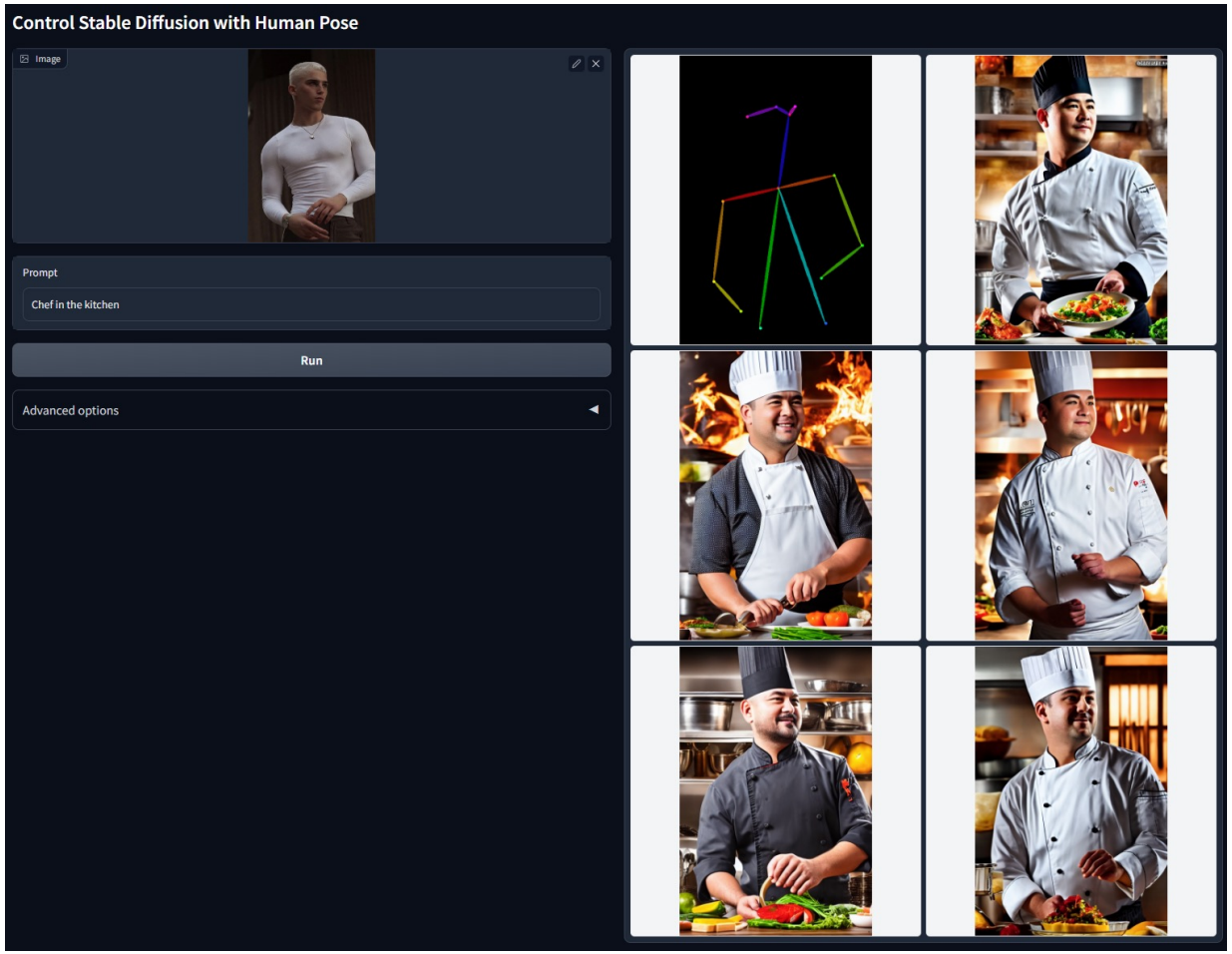
a robot shiba inu, cyberpunk style



A shiba inu is doing rock climbing, pixel art

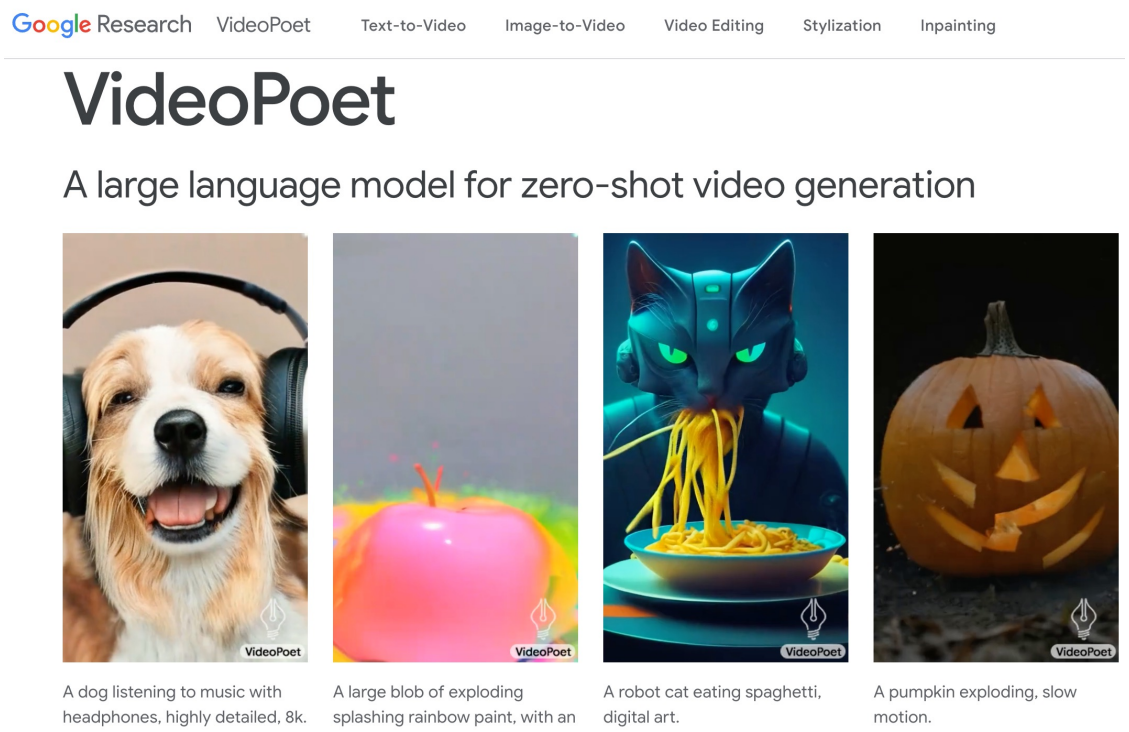
2023: ControlNet + Stable Diffusion (open-source)





https://github.com/lllyasviel/ControlNet

2024: Video generation and 3D generation



https://sites.research.google/videopoet/

https://www.youtube.com/watch?v=HK6y8DAPN_0

Discriminative vs Generative Models

Discriminative Model:
Learn a probability distribution p(y|x)

Generative Model:
Learn a probability distribution p(x)

Data: x

Label: y

Cat

Discriminative vs Generative Models

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Cat

Probability Recap:

Density Function
p(x) assigns a positive number to each possible x; higher numbers mean x is more likely

Density functions are **normalized**:

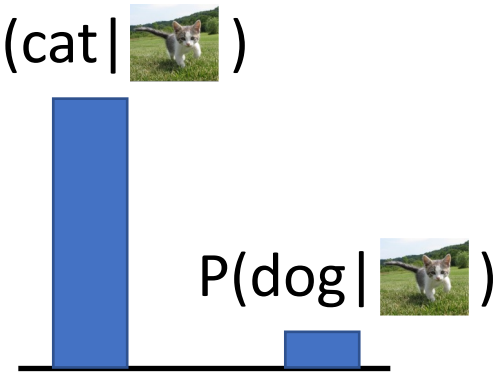
$$\int p(x)dx = 1$$

Different values of x **compete** for density

Discriminative vs Generative Models

Discriminative Model: Learn a probability distribution p(y|x)

Data: x



Generative Model: Learn a probability distribution p(x)

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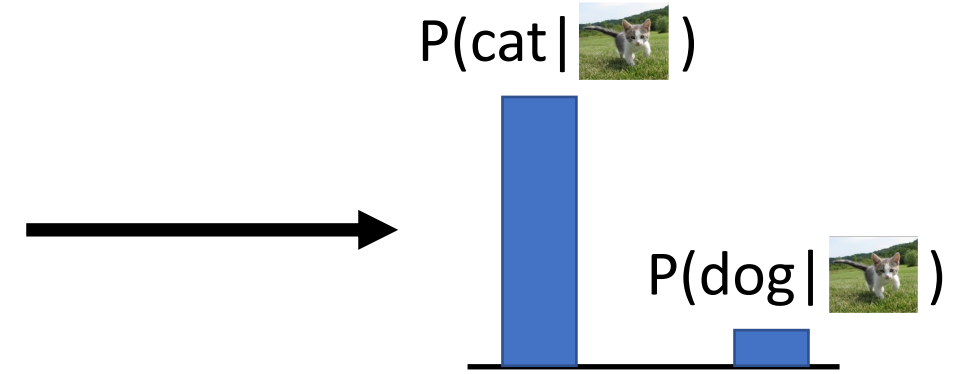
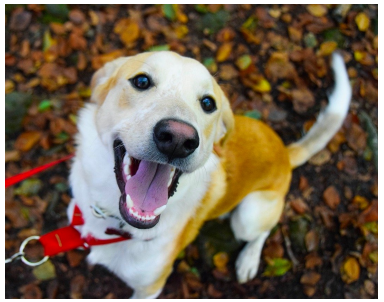
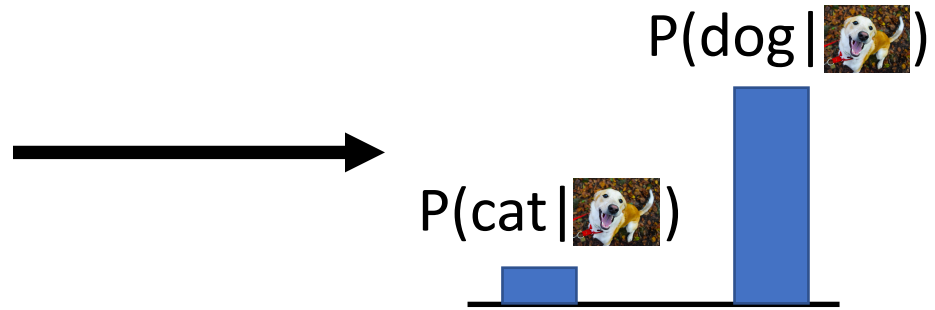


Generative Model: Learn a probability distribution p(x)

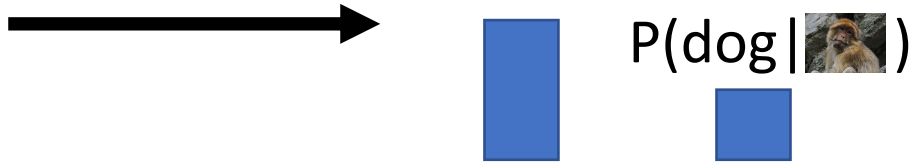




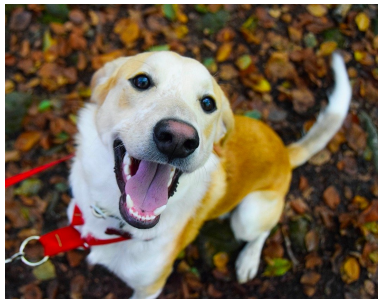
Discriminative model: the possible labels for each input ”compete” for probability mass. But no competition between **images**

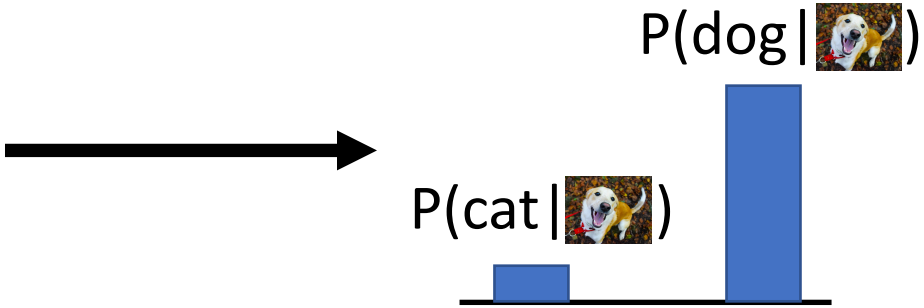
Discriminative vs Generative Models

Discriminative Model: Learn a probability distribution p(y|x)



Generative Model: Learn a probability distribution p(x)



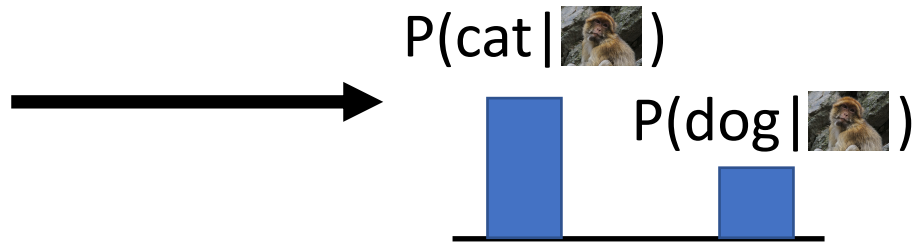


Discriminative model: No way for the model to handle unreasonable inputs; it must give label distributions for all images

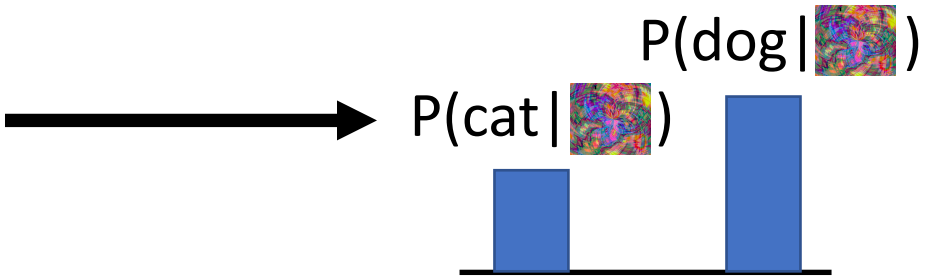
Monkey image is CC0 Public Domain

Discriminative vs Generative Models

Discriminative Model: Learn a probability distribution p(y|x)



Generative Model: Learn a probability distribution p(x)



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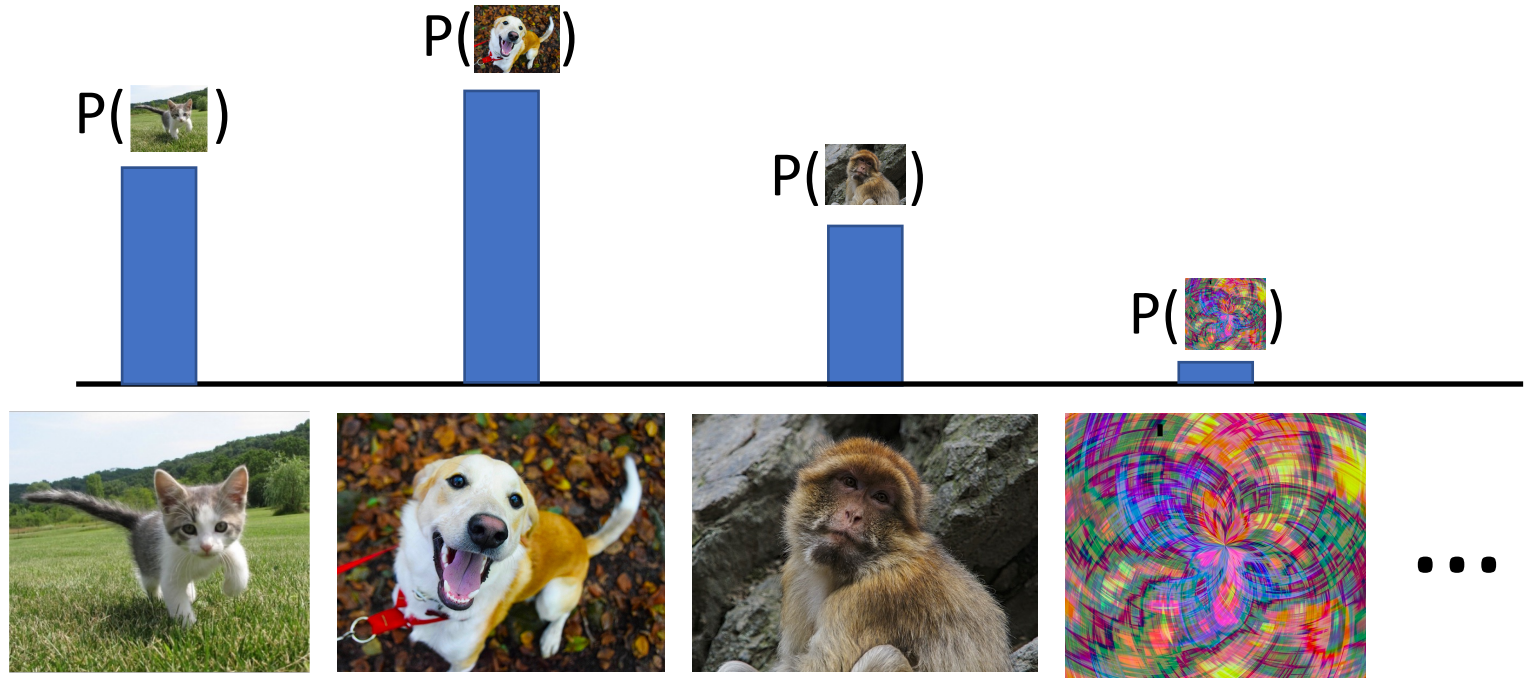
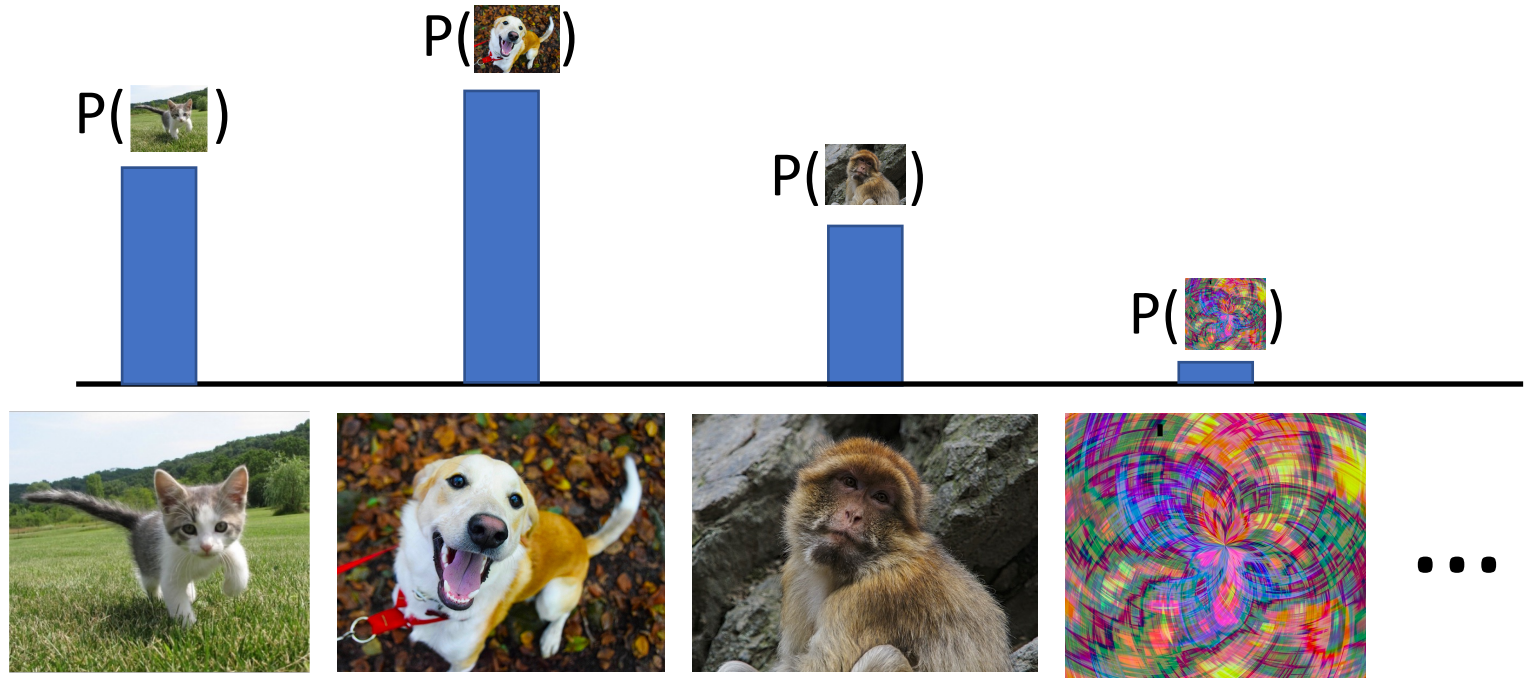


Generative model: All possible images compete with each other for probability mass

Discriminative vs Generative Models

Discriminative Model: Learn a probability distribution p(y|x)

Generative Model: Learn a probability distribution p(x)



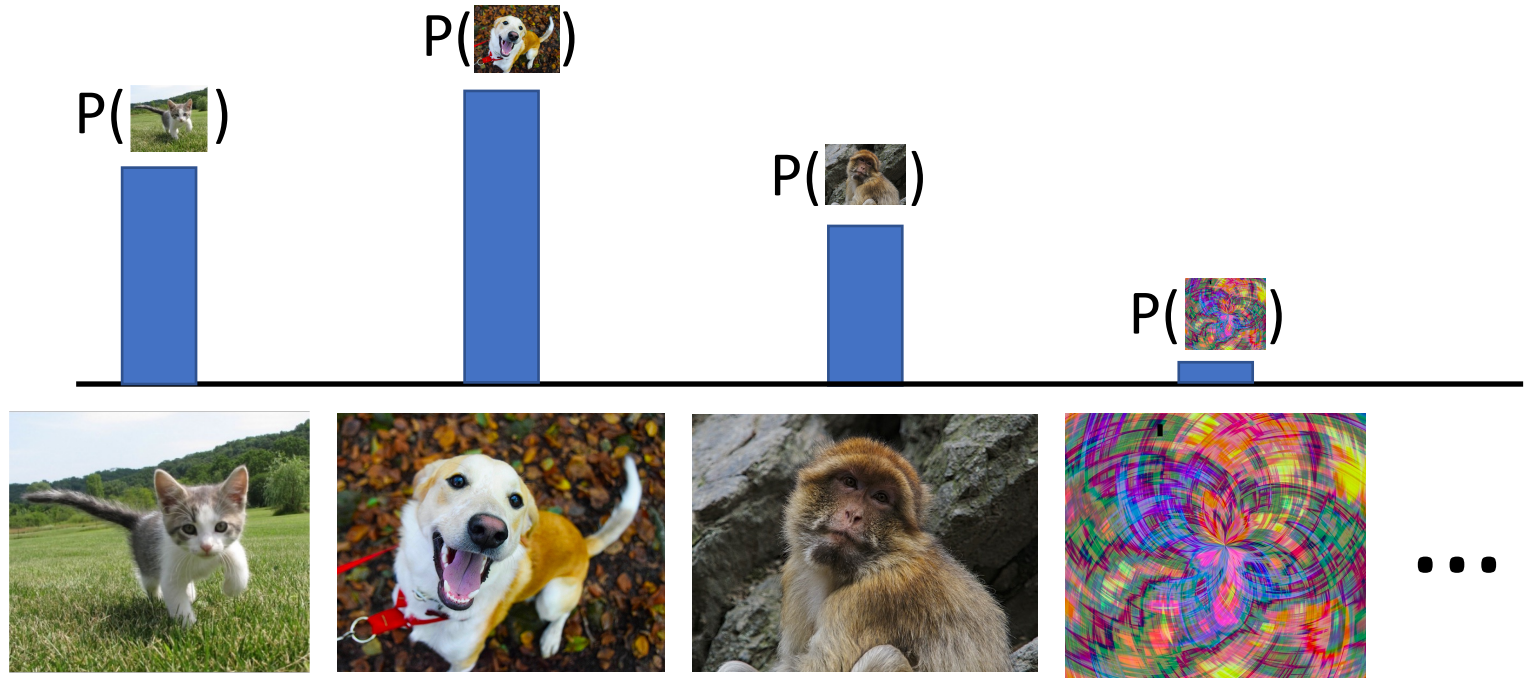
Generative model: All possible images compete with each other for probability mass

Requires deep image understanding! Is a dog more likely to sit or stand? How about 3-legged dog vs 3-armed monkey?

Discriminative vs Generative Models

Discriminative Model: Learn a probability distribution p(y|x)

Generative Model: Learn a probability distribution p(x)



Generative model: All possible images compete with each other for probability mass

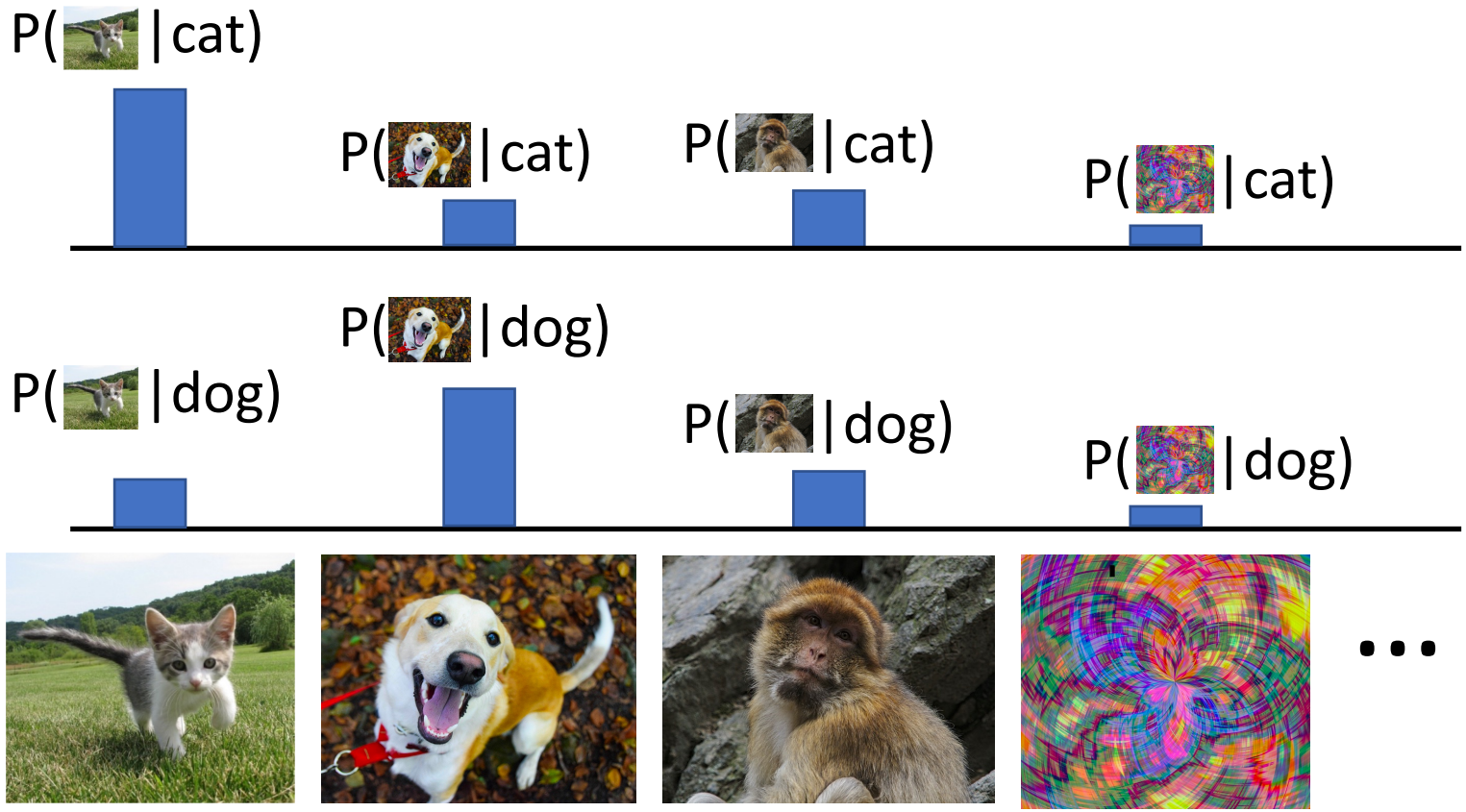
Model can “reject” unreasonable inputs by assigning them small values

Discriminative vs Generative Models

Discriminative Model: Learn a probability distribution p(y|x)

Generative Model: Learn a probability distribution p(x)

Conditional Generative Model: Learn p(x|y)



Conditional Generative Model: Each possible label induces a competition among all images

What can we do with a discriminative model?

Discriminative Model: Learn a probability distribution p(y|x)

Assign labels to data

Feature learning (with labels)

Generative Model: Learn a probability distribution p(x)

Conditional Generative Model: Learn p(x|y)

What can we do with a generative model?

Discriminative Model:
Learn a probability distribution p(y|x)

→ Assign labels to data
Feature learning (with labels)

Generative Model:
Learn a probability distribution p(x)

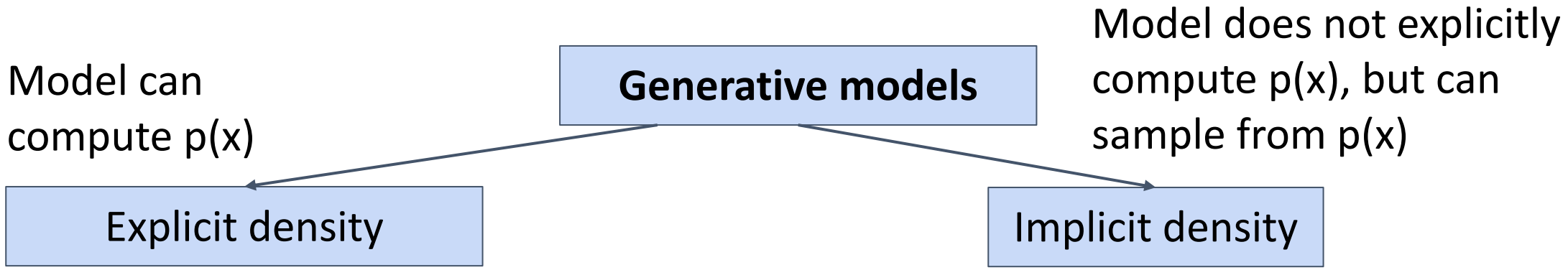
→ Detect outliers
Feature learning (without labels)
Sample to **generate** new data

Taxonomy of Generative Models

Generative models

Figure adapted from Ian Goodfellow, Tutorial on Generative Adversarial Networks, 2017.

Taxonomy of Generative Models



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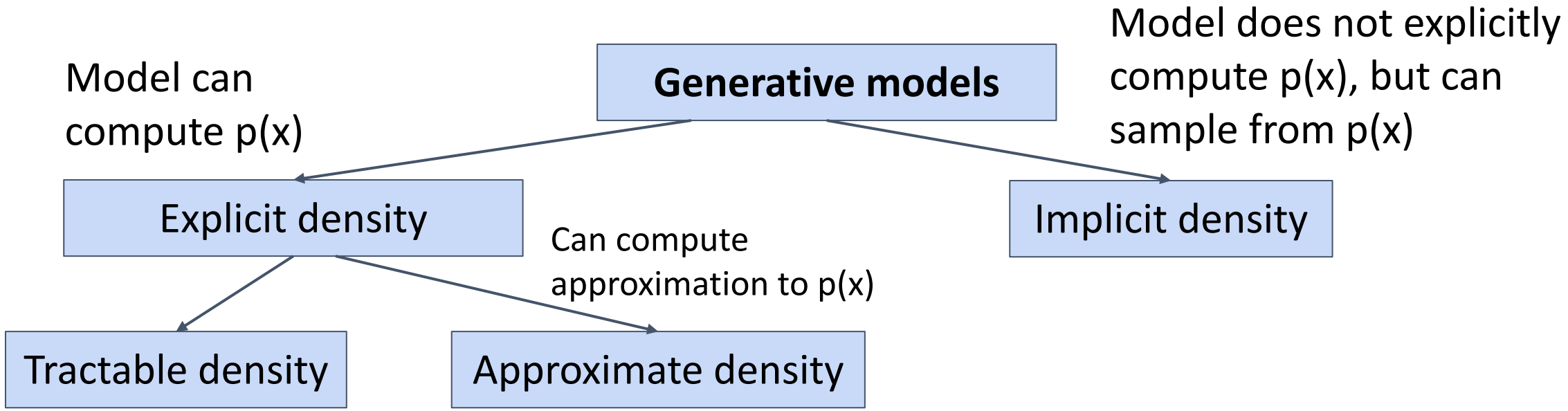
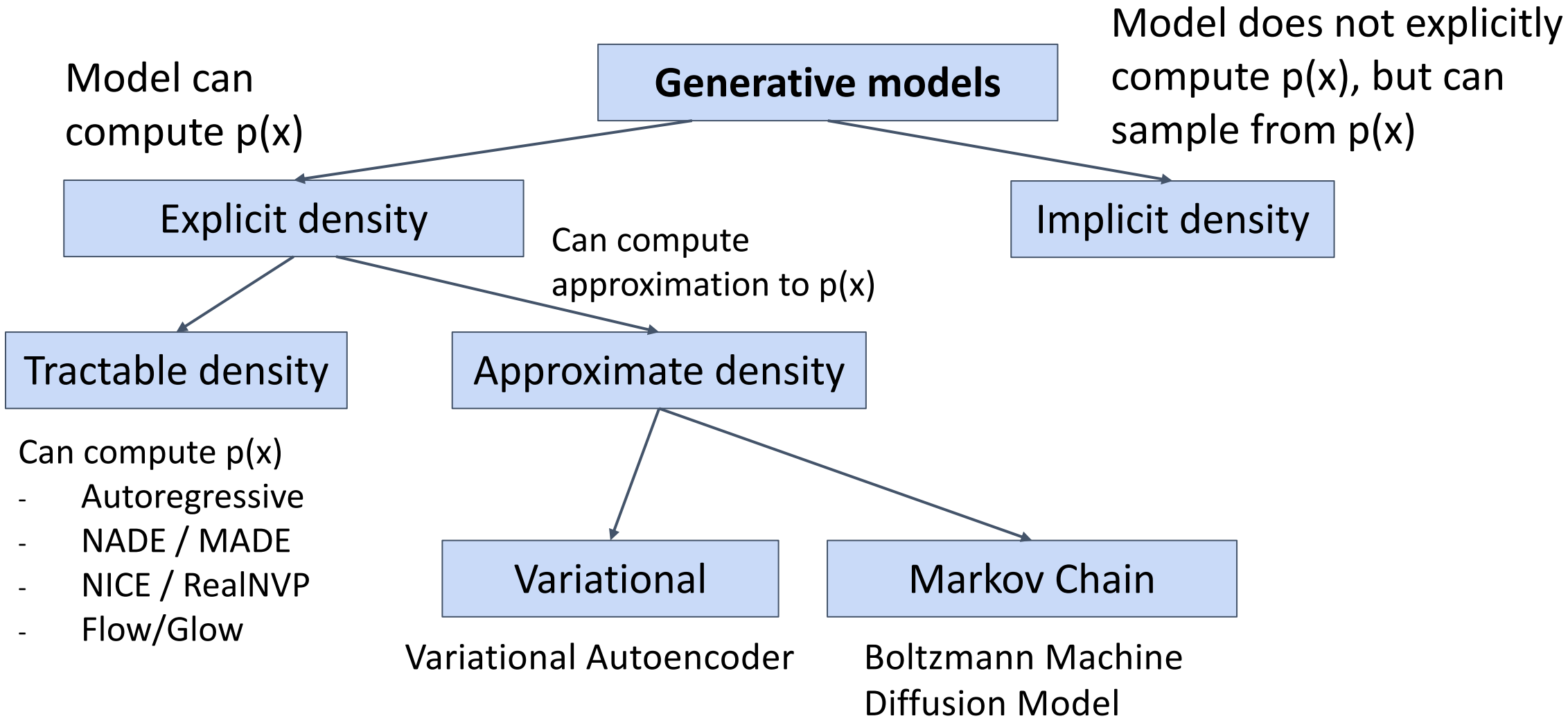


Can compute p(x)

- Autoregressive
- NADE / MADE
- NICE / RealNVP
- Flow/Glow

Figure adapted from Ian Goodfellow, Tutorial on Generative Adversarial Networks, 2017.

Taxonomy of Generative Models



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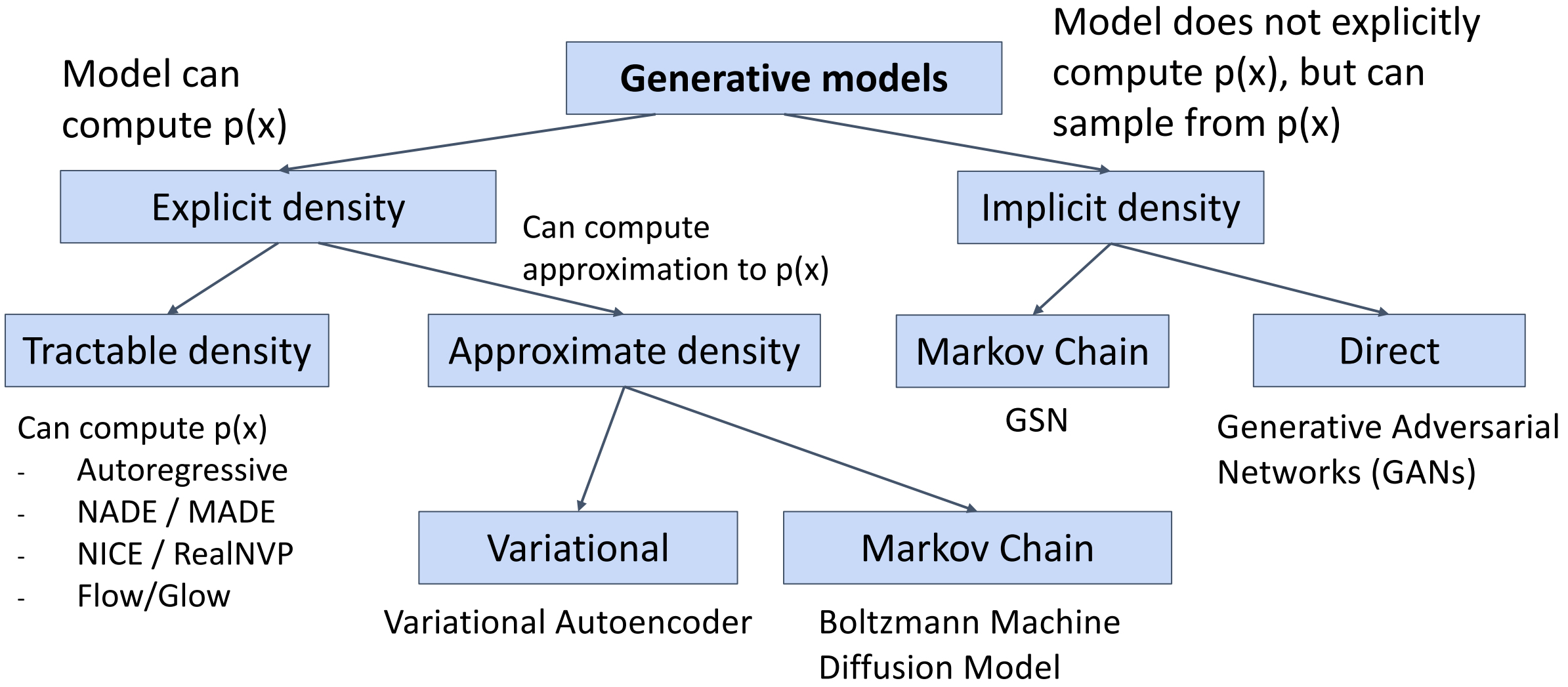

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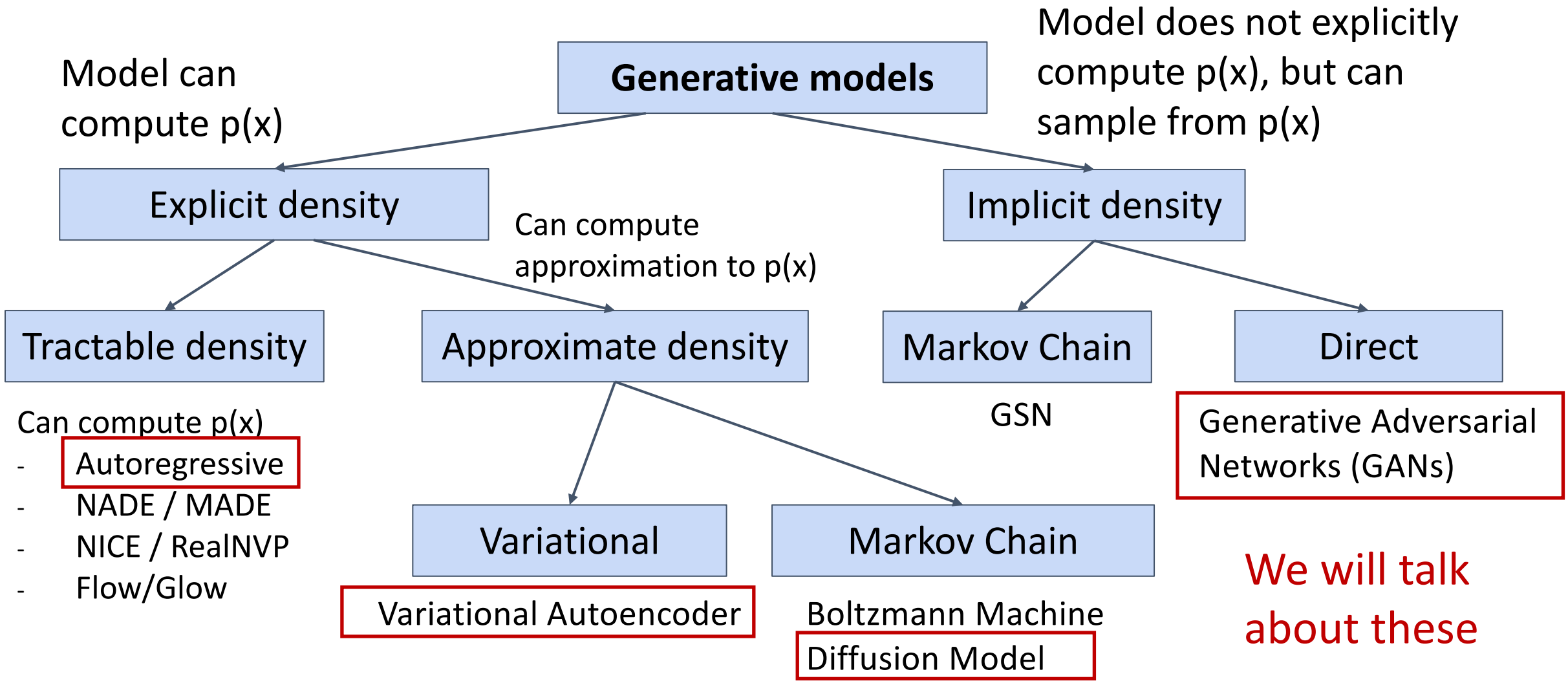

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Autoregressive models

Explicit Density Estimation

Goal: Write down an explicit function for $p(x) = f(x, W)$

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Given dataset $x^{(1)}, x^{(2)}, \ldots x^{(N)}$, train the model by solving:

$$W^* = \arg\max_W \prod_i p(x^{(i)})$$

Maximize probability of training data
(Maximum likelihood estimation)

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Log trick to exchange product for sum

$$= \arg\max_W \sum_i \log f(x^{(i)}, W)$$

This will be our loss function!
Train with gradient descent

Explicit Density: Autoregressive Models

Goal: Write down an explicit function for $p(x) = f(x, W)$

Assume x consists of multiple subparts:

$$x = (x_1, x_2, x_3, \ldots, x_T)$$

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Break down probability using the chain rule:

$$\begin{aligned} p(x) &= p(x_1, x_2, x_3, \ldots, x_T) \\ &= p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1, x_2) \ldots \end{aligned}$$

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Probability of the next subpart given all the previous subparts

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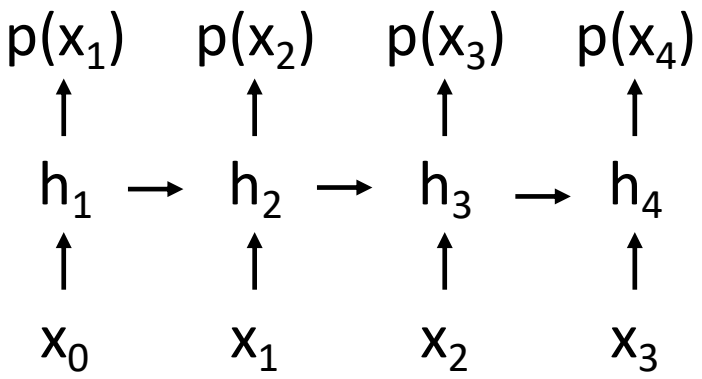


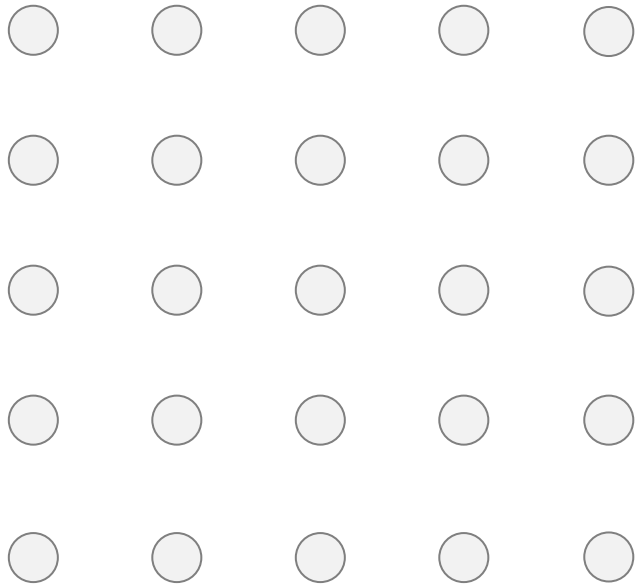
PixelRNN

Generate image pixels one at a time, starting at the upper left corner

Compute a hidden state for each pixel that depends on hidden states and RGB values from the left and from above (LSTM recurrence)

$$h_{x,y} = f(h_{x-1,y}, h_{x,y-1}, W)$$

At each pixel, predict red, then blue, then green: softmax over [0, 1, …, 255]



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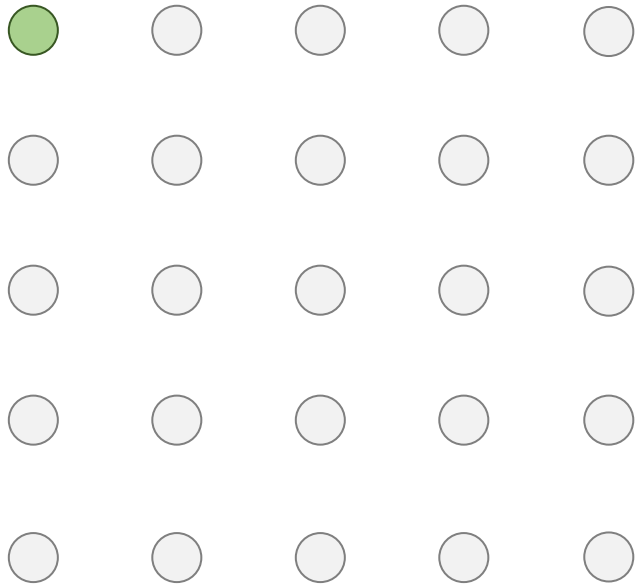

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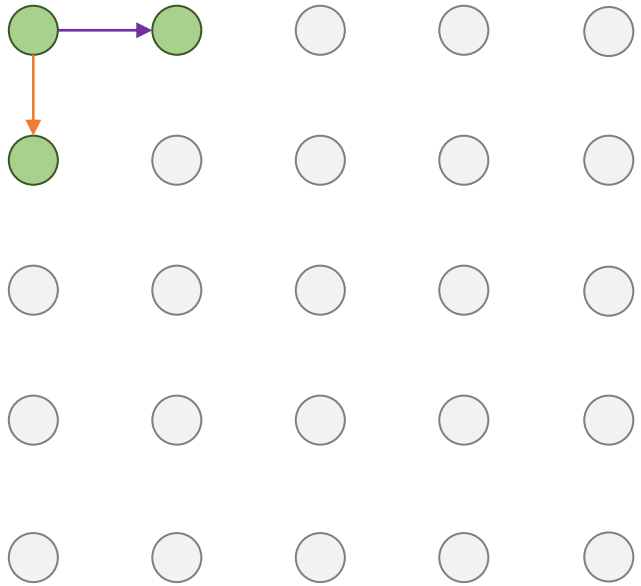

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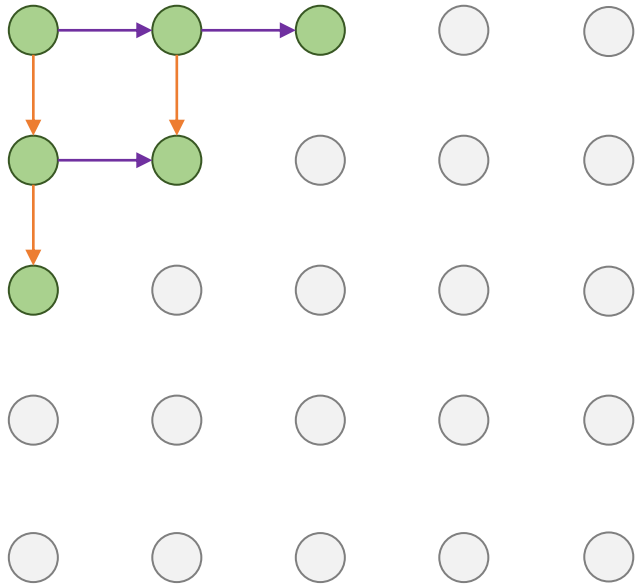

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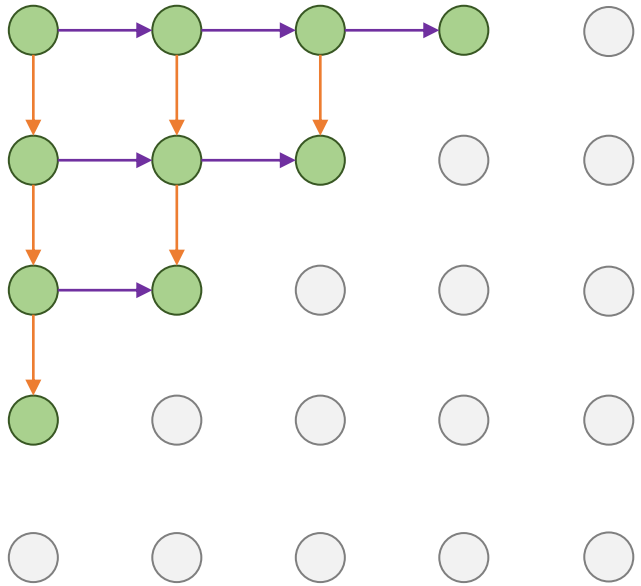

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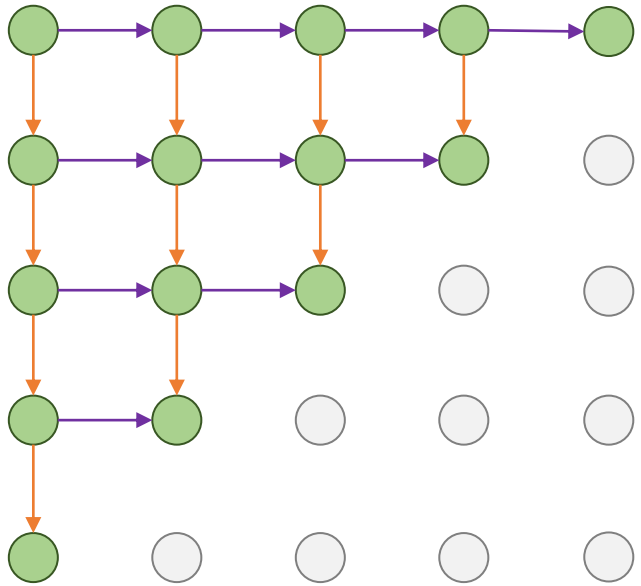

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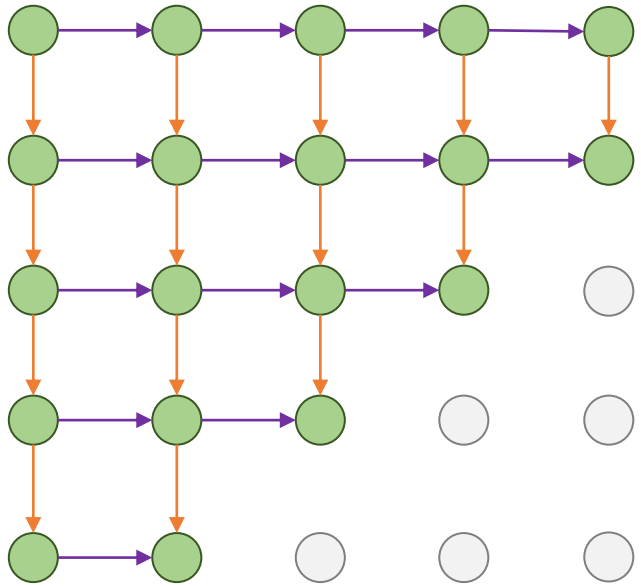

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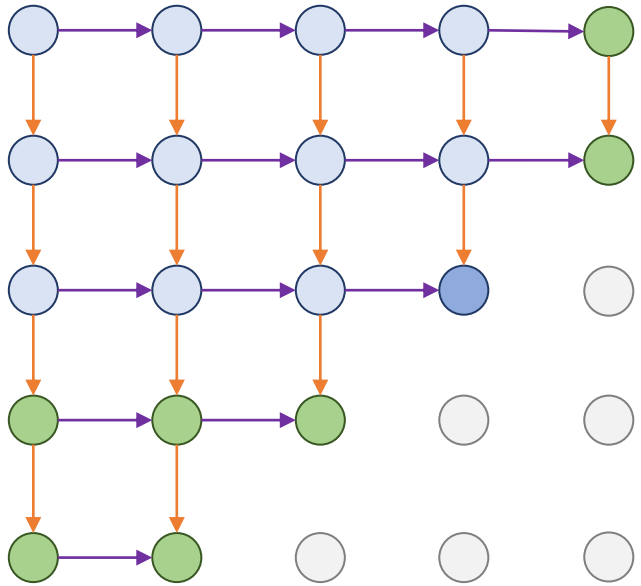
Each pixel depends **implicitly** on all pixels above and to the left:



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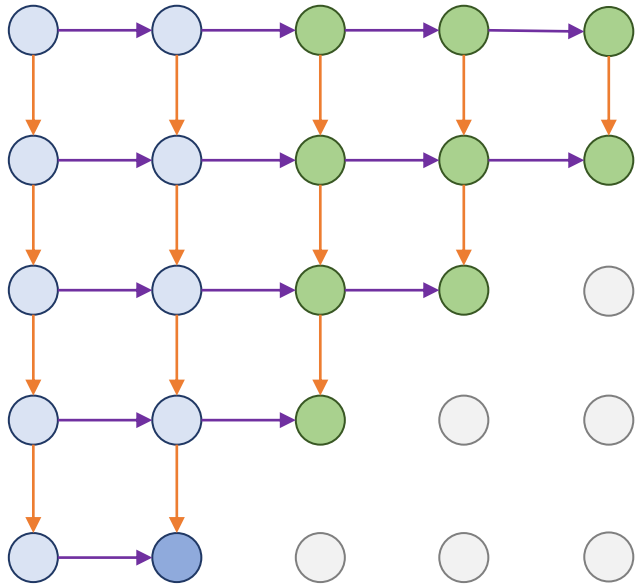

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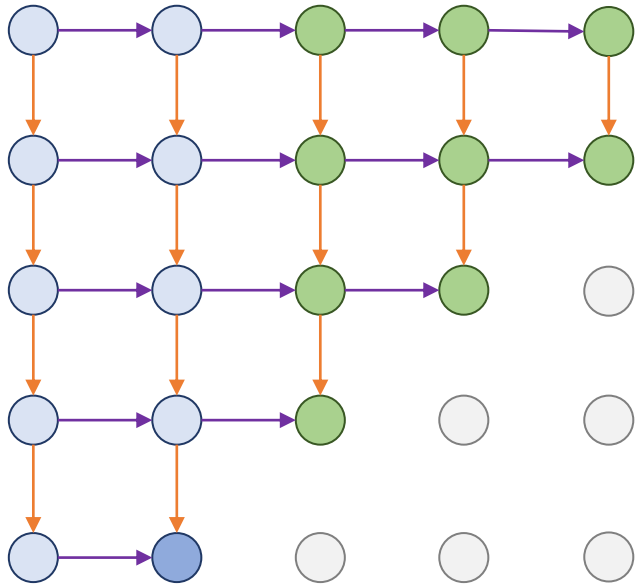
Problem: Very slow during both training and testing; N x N image requires 2N-1 sequential steps



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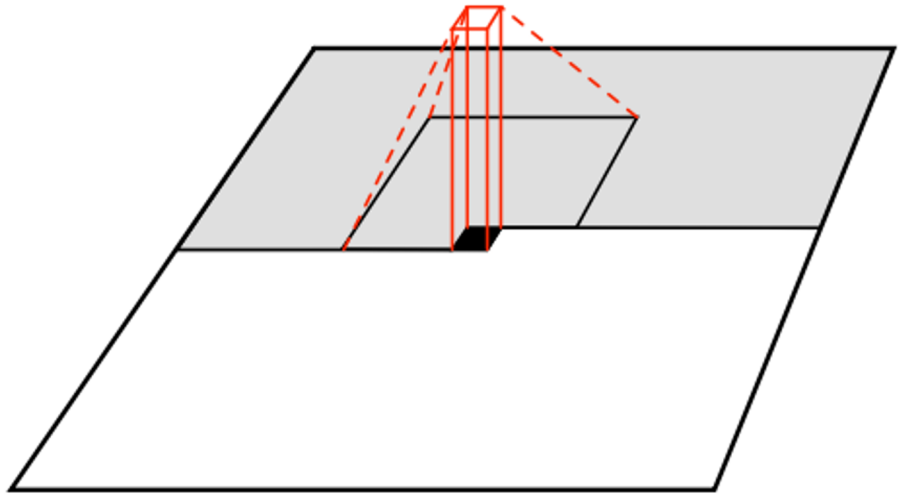
PixelCNN

Still generate image pixels starting from corner

Dependency on previous pixels now modeled using a CNN over context region



Van den Oord et al, “Conditional Image Generation with PixelCNN Decoders”, NeurIPS 2016

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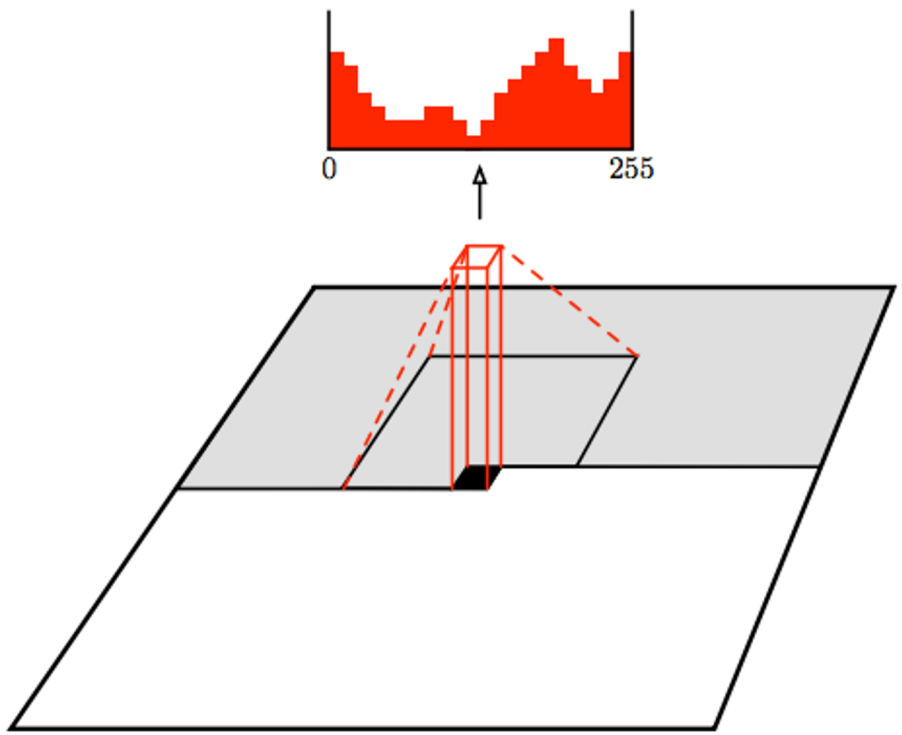
Training: maximize likelihood of training images

$$p(x) = \prod_{i=1}^{n} p(x_i | x_1, \ldots, x_{i-1})$$



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PixelCNN

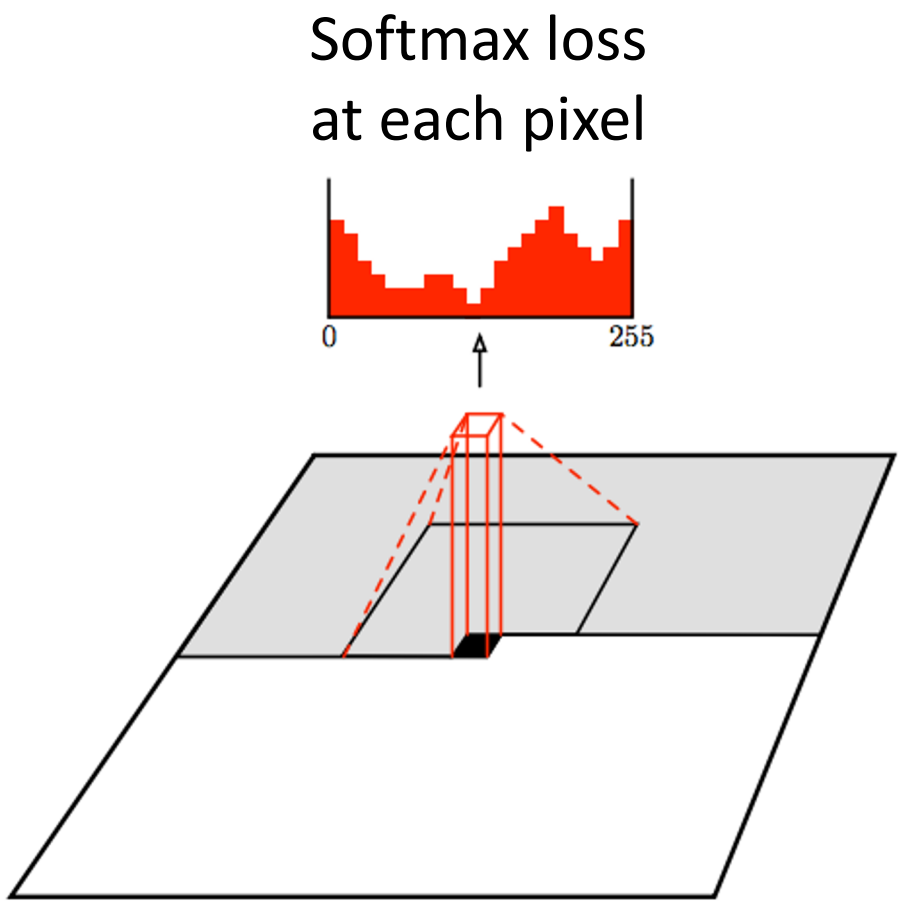
Still generate image pixels starting from corner

Dependency on previous pixels now modeled using a CNN over context region

Training: maximize likelihood of training images

Training is faster than PixelRNN
(can parallelize convolutions since context region values known from training images)

Generation must still proceed sequentially
=> still slow



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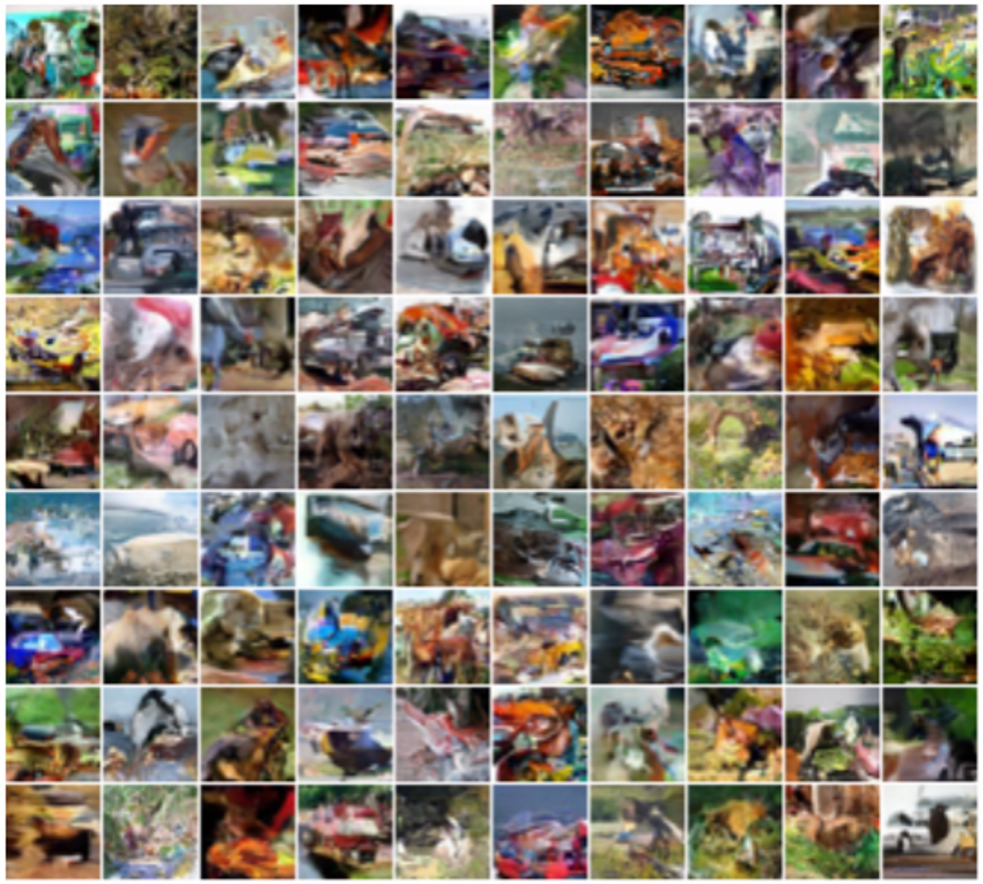
PixelRNN: Generated Samples



32x32 CIFAR-10

32x32 ImageNet

Van den Oord et al, “Pixel Recurrent Neural Networks”, ICML 2016

Autoregressive Models: PixelRNN and PixelCNN

Pros:

- Can explicitly compute likelihood p(x)
- Explicit likelihood of training data gives good evaluation metric
- Good samples

Con:

- Sequential generation => slow

Improving PixelCNN performance

- Gated convolutional layers
- Short-cut connections
- Discretized logistic loss
- Multi-scale
- Training tricks
- Etc…

See

- Van der Oord et al. NIPS 2016
- Salimans et al. 2017 (PixelCNN++)

Variational Autoencoders

Variational Autoencoders

PixelRNN / PixelCNN explicitly parameterizes density function with a neural network, so we can train to maximize likelihood of training data:

$$p_\theta(x) = \prod_{i=1}^{n} p_\theta(x_i | x_1, ..., x_{i-1})$$

Variational Autoencoders (VAE) define an **intractable density** that we cannot explicitly compute or optimize

But we will be able to directly optimize a **lower bound** on the density

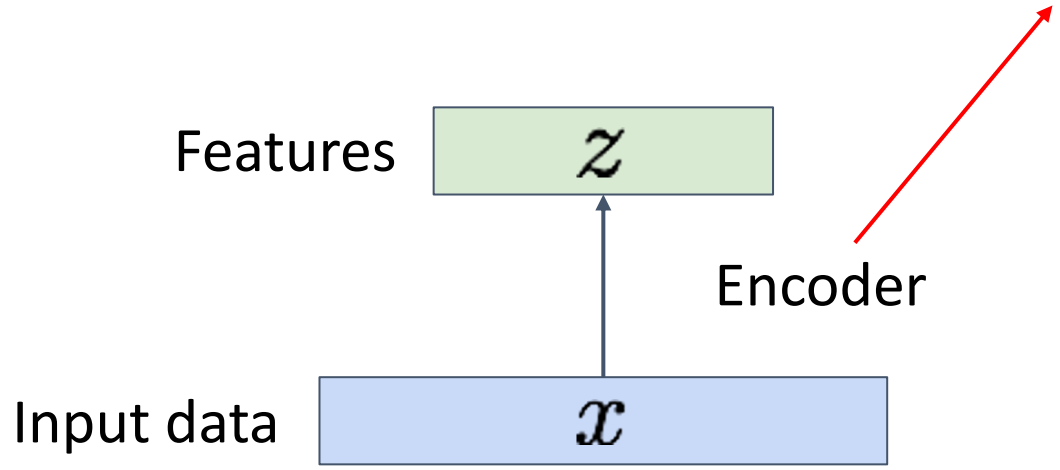
Variational Autoencoders

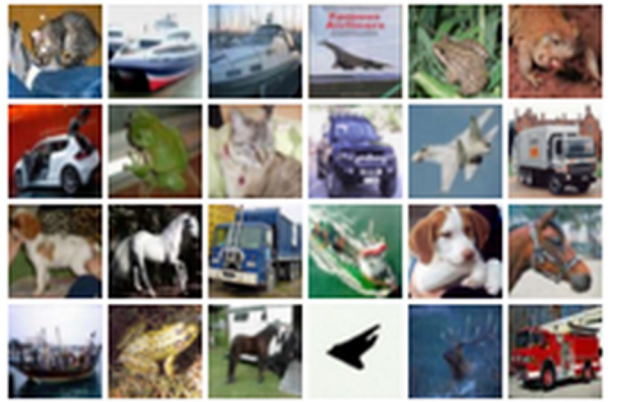
(Regular, non-variational) Autoencoders

Unsupervised method for learning feature vectors from raw data x, without any labels

Features should extract useful information (maybe object identities, properties, scene type, etc) that we can use for downstream tasks

Originally: Linear + nonlinearity (sigmoid)
Later: Deep, fully-connected
Later: ReLU CNN





Input Data

Hinton et al. Reducing the Dimensionality of Data with Neural Networks. Science, 2006
https://www.cs.toronto.edu/~hinton/science.pdf

(Regular, non-variational) Autoencoders

Problem: How can we learn this feature transform from raw data?

Features should extract useful information (maybe object identities, properties, scene type, etc) that we can use for downstream tasks
But we can't observe features!

Originally: Linear + nonlinearity (sigmoid)
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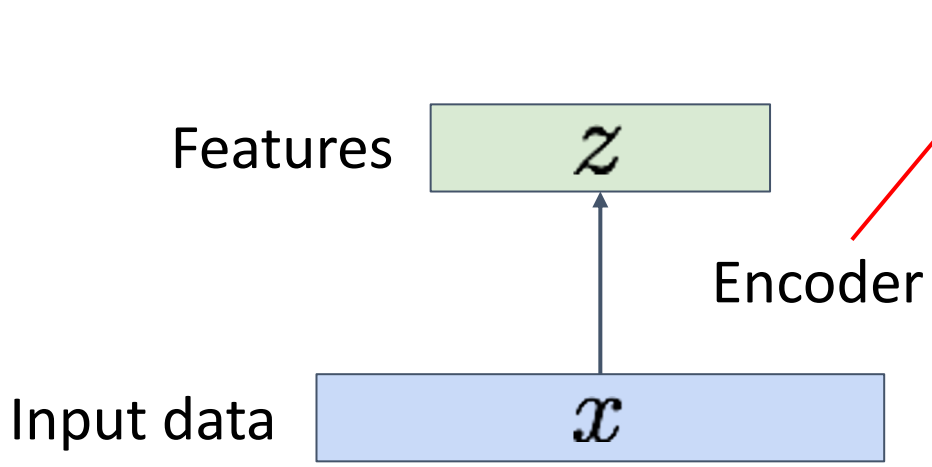
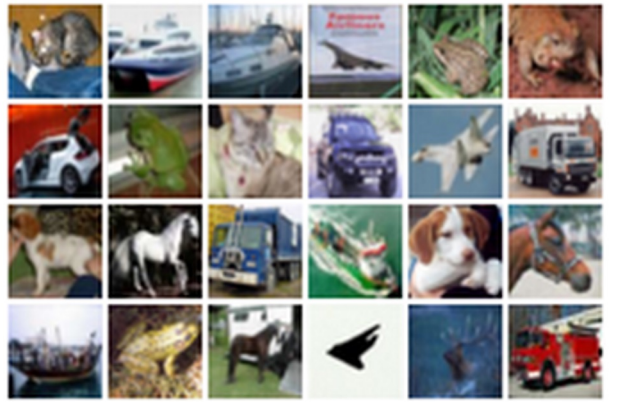




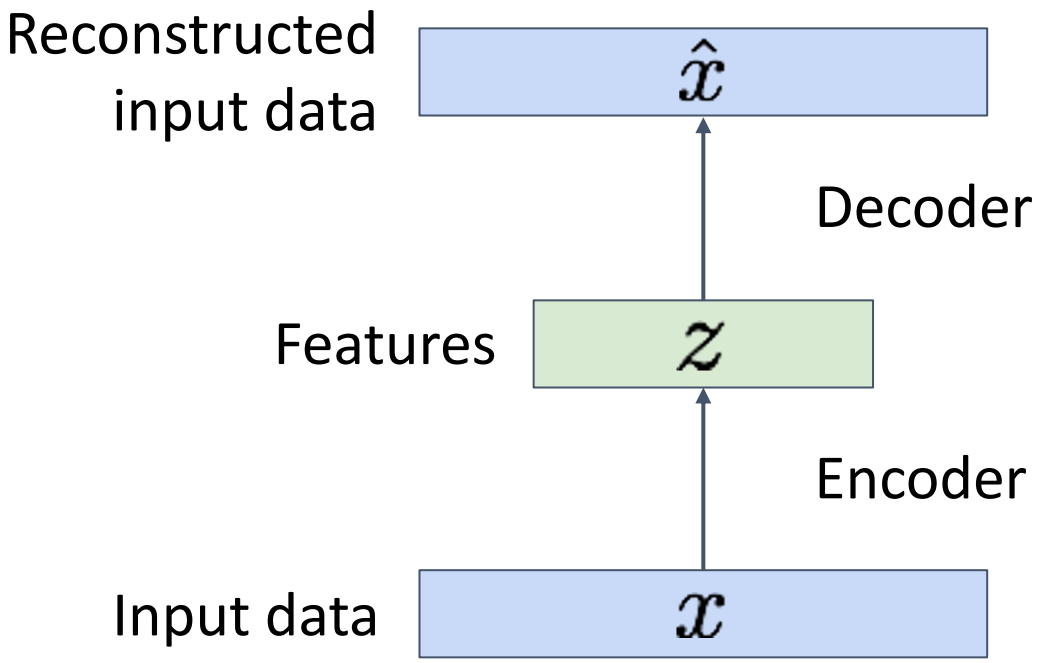
Input Data

(Regular, non-variational) Autoencoders

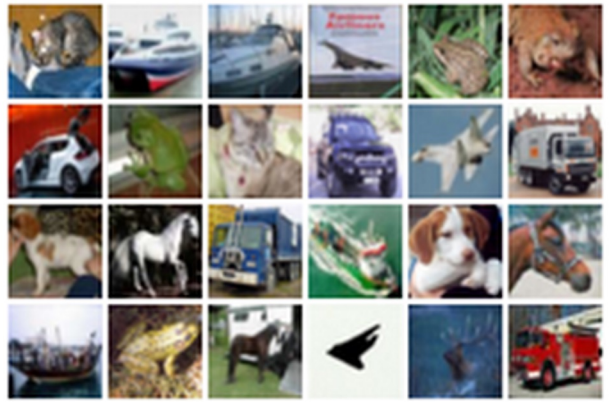
Problem: How can we learn this feature transform from raw data?

Idea: Use the features to reconstruct the input data with a **decoder**

"Autoencoding" = encoding itself



Originally: Linear + nonlinearity (sigmoid)
Later: Deep, fully-connected
Later: ReLU CNN (upconv)

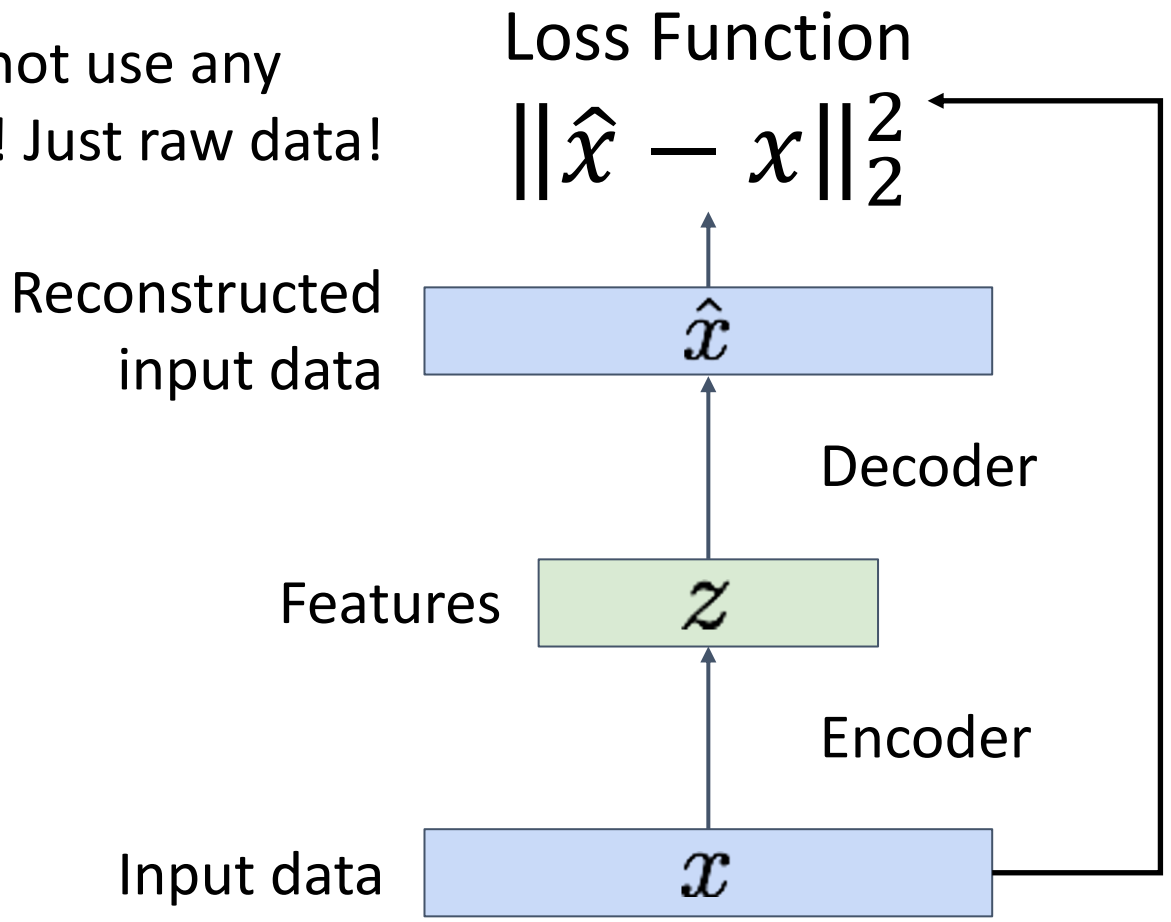


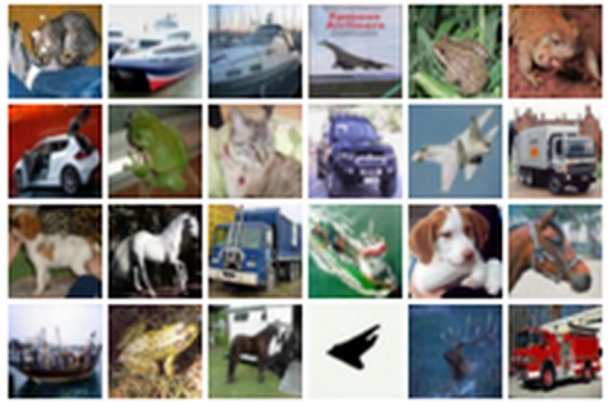
Input Data

(Regular, non-variational) Autoencoders

Loss: L2 distance between input and reconstructed data.

Does not use any labels! Just raw data!





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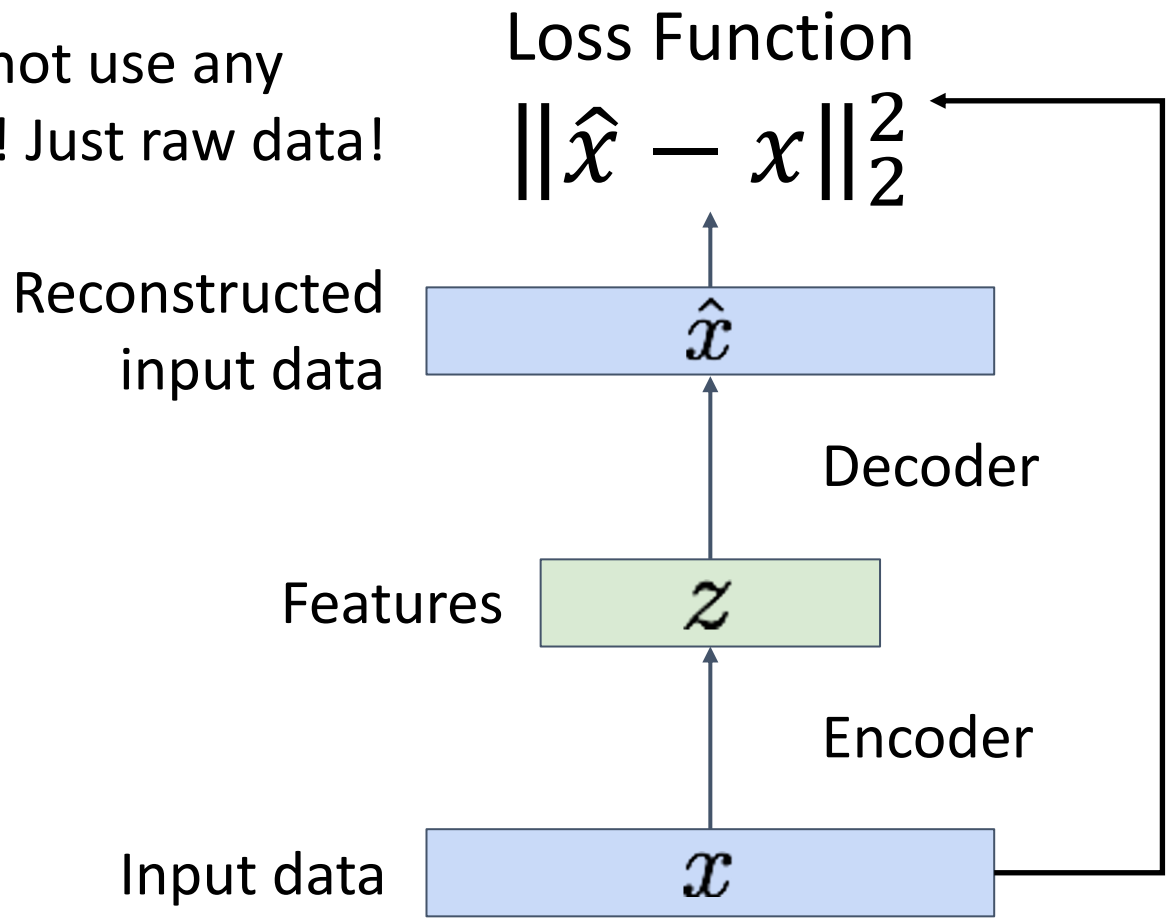
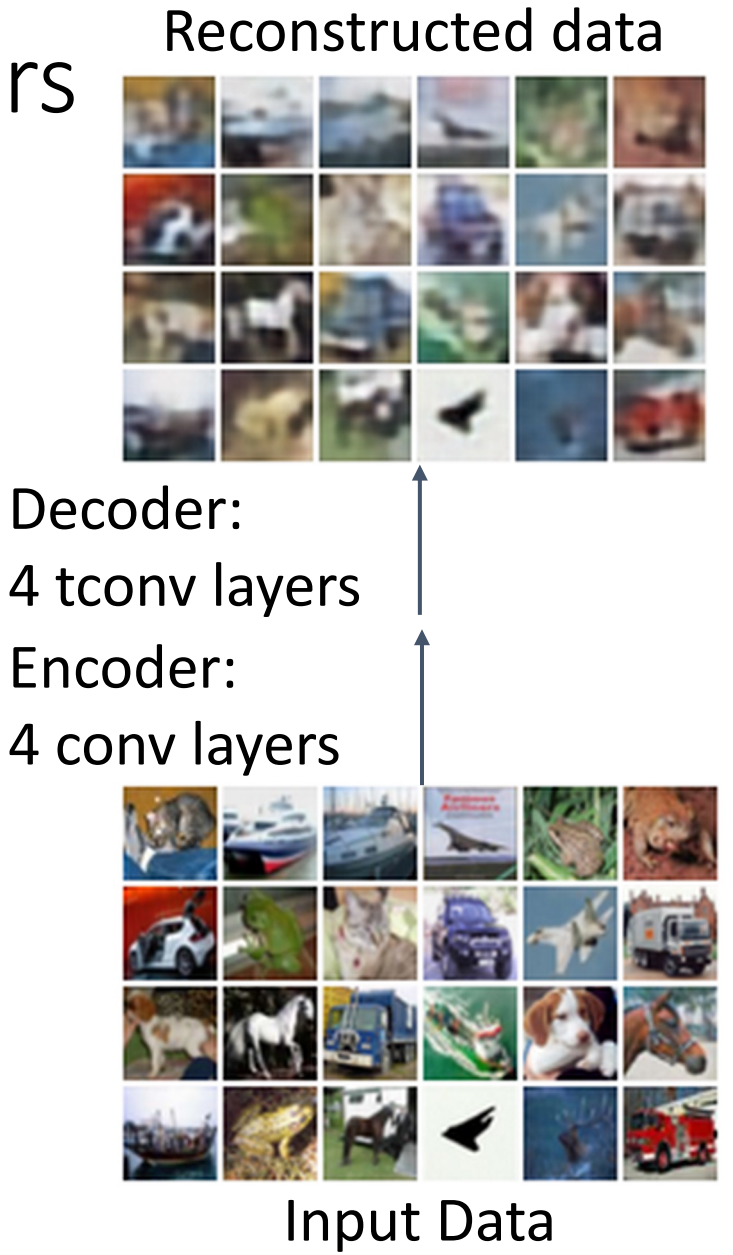




(Regular, non-variational) Autoencoders

Loss: L2 distance between input and reconstructed data.

Does not use any labels! Just raw data!

Loss Function

$$\|\hat{x} - x\|_2^2$$

Reconstructed input data

$\hat{x}$

Decoder

Features need to be **lower dimensional** than the data

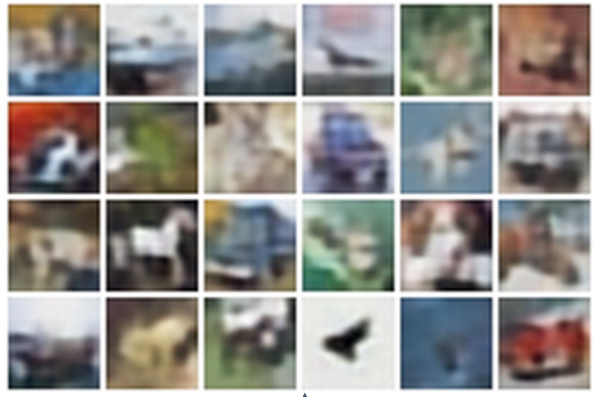
Features

z

Encoder

Input data

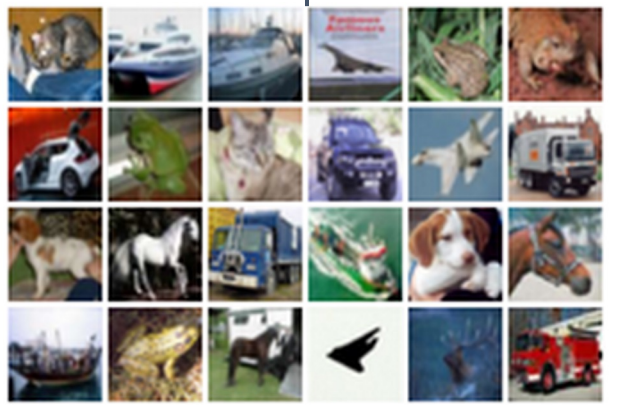
x



Reconstructed data

Decoder:
4 tconv layers

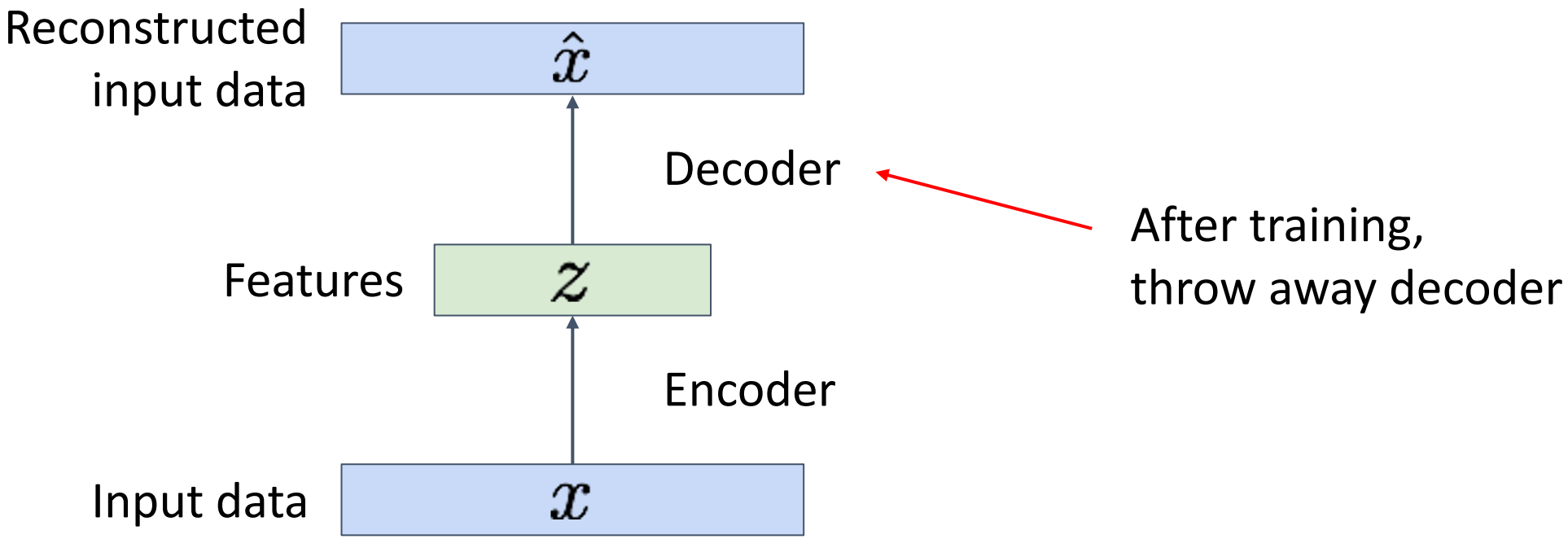
Encoder:
4 conv layers



Input Data

(Regular, non-variational) Autoencoders

After training, **throw away decoder** and use encoder for a downstream task



(Regular, non-variational) Autoencoders

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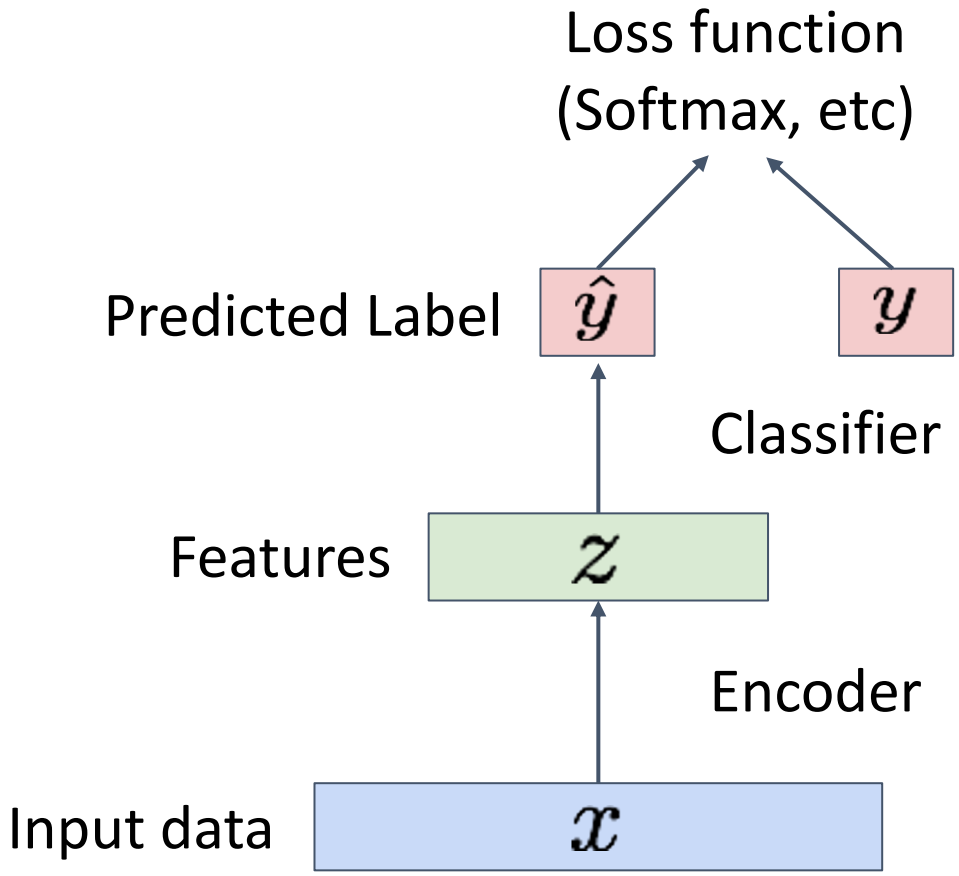


Fine-tune encoder jointly with classifier

Encoder can be used to initialize a **supervised** model

Train for final task (sometimes with small data)

Dimensionality reduction with unsupervised learning

MNIST dataset

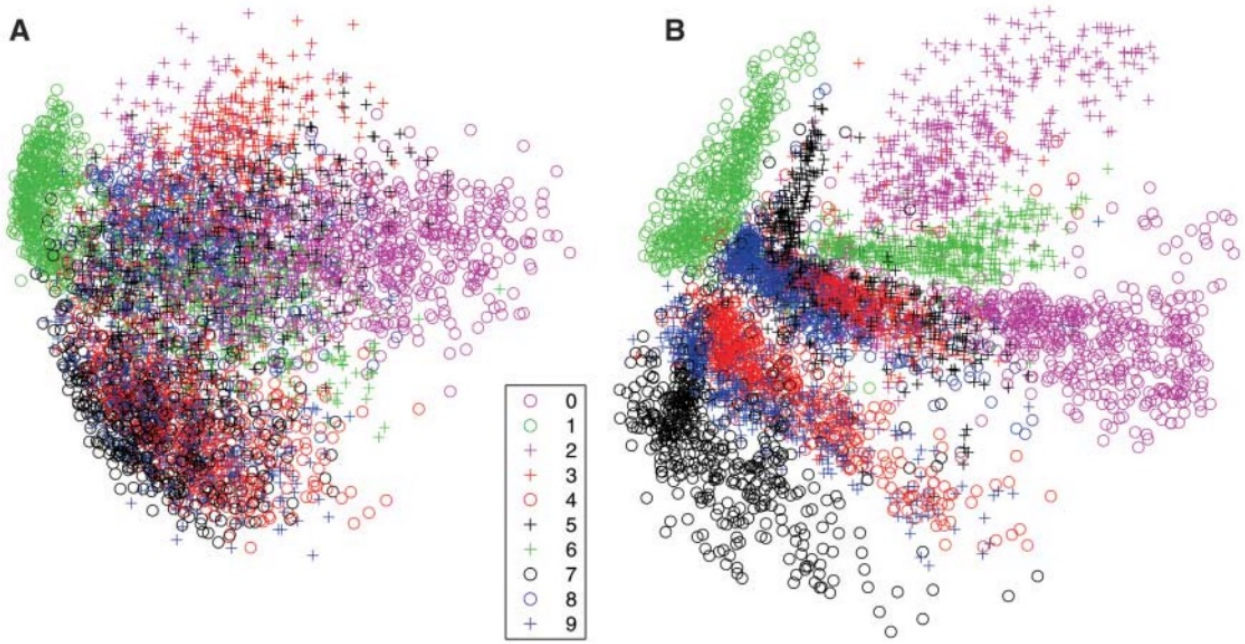
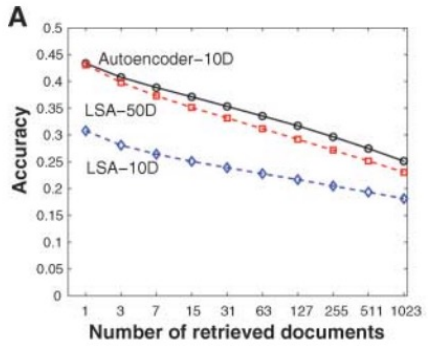
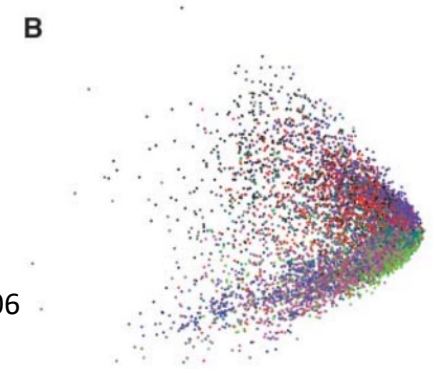


Fig. 3. (**A**) The two-dimensional codes for 500 digits of each class produced by taking the first two principal components of all 60,000 training images. (**B**) The two-dimensional codes found by a 784-1000-500-250-2 autoencoder. For an alternative visualization, see (*8*).

Document dataset





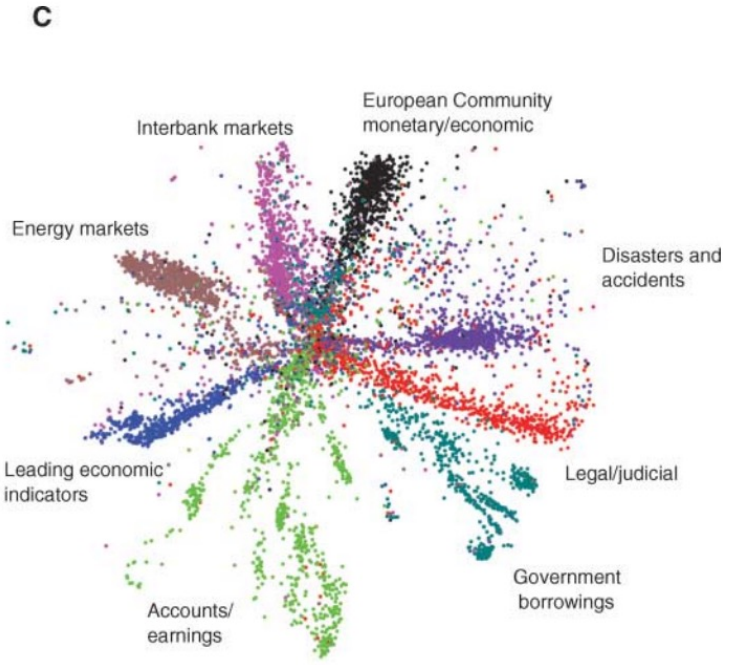
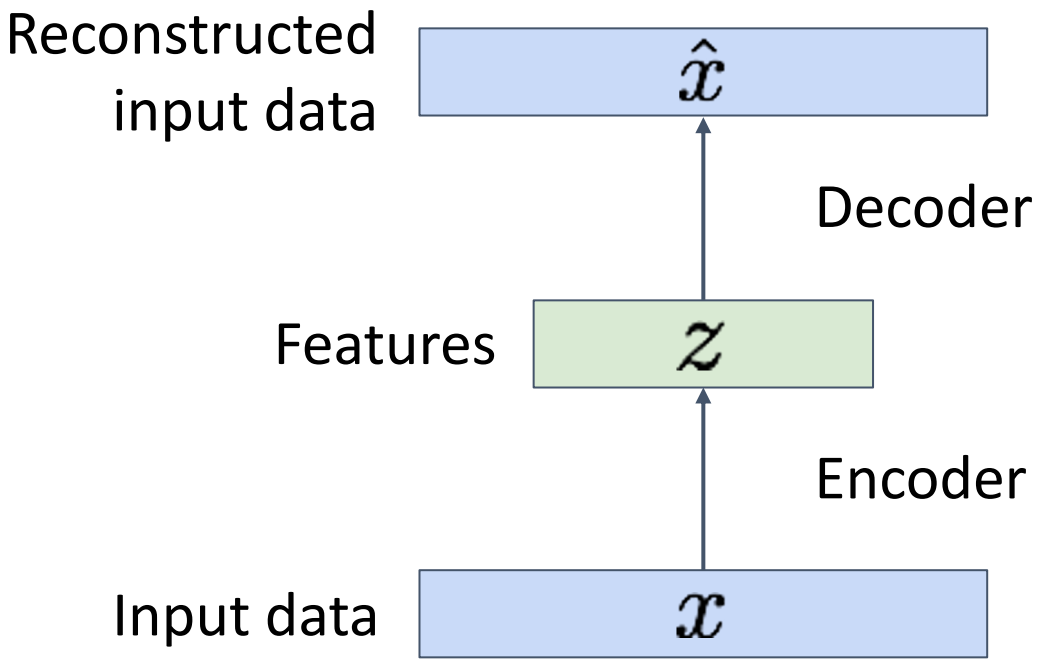


Fig. 4. (**A**) The fraction of retrieved documents in the same class as the query when a query document from the test set is used to retrieve other test set documents, averaged over all 402,207 possible queries. (**B**) The codes produced by two-dimensional LSA. (**C**) The codes produced by a 2000-500-250-125-2 autoencoder.

Hinton et al. Reducing the Dimensionality of Data with Neural Networks. Science, 2006
https://www.cs.toronto.edu/~hinton/science.pdf

(Regular, non-variational) Autoencoders

Autoencoders learn **latent features** for data without any labels!
Can use features to initialize a **supervised** model
Not probabilistic: No way to sample new data from learned model



Variational Autoencoders

Kingma and Welling, Auto-Encoding Variational Beyes, ICLR 2014

Variational Autoencoders

Probabilistic spin on autoencoders:

1. Learn latent features z from raw data
2. Sample from the model to generate new data

Variational Autoencoders

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1. Learn latent features z from raw data
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Assume training data $\{x^{(i)}\}_{i=1}^{N}$ is generated from unobserved (latent) representation **z**

Intuition: x is an image, **z** is latent factors used to generate **x:** attributes, orientation, etc.

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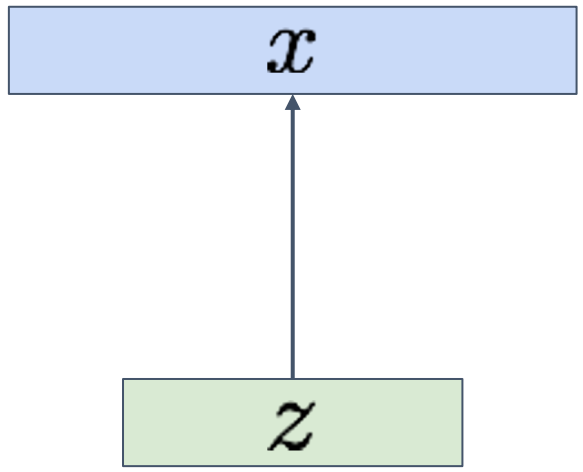
After training, sample new data like this:

Sample from conditional $p_{\theta^*}(x \mid z^{(i)})$

Sample z from prior $p_{\theta^*}(z)$



Variational Autoencoders

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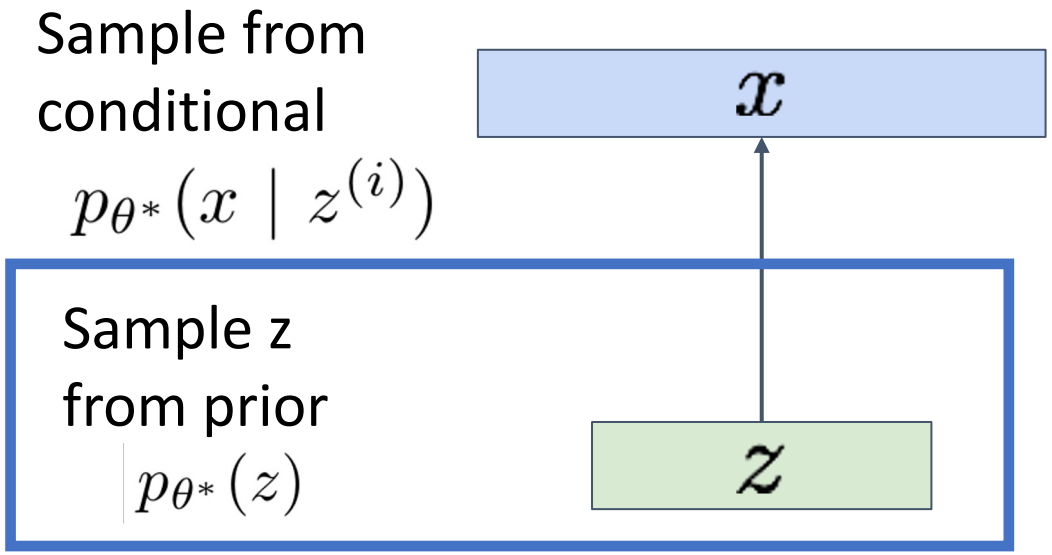

Assume training data $\{x^{(i)}\}_{i=1}^{N}$ is generated from unobserved (latent) representation **z**

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Assume simple prior p(z), e.g. Gaussian

Variational Autoencoders

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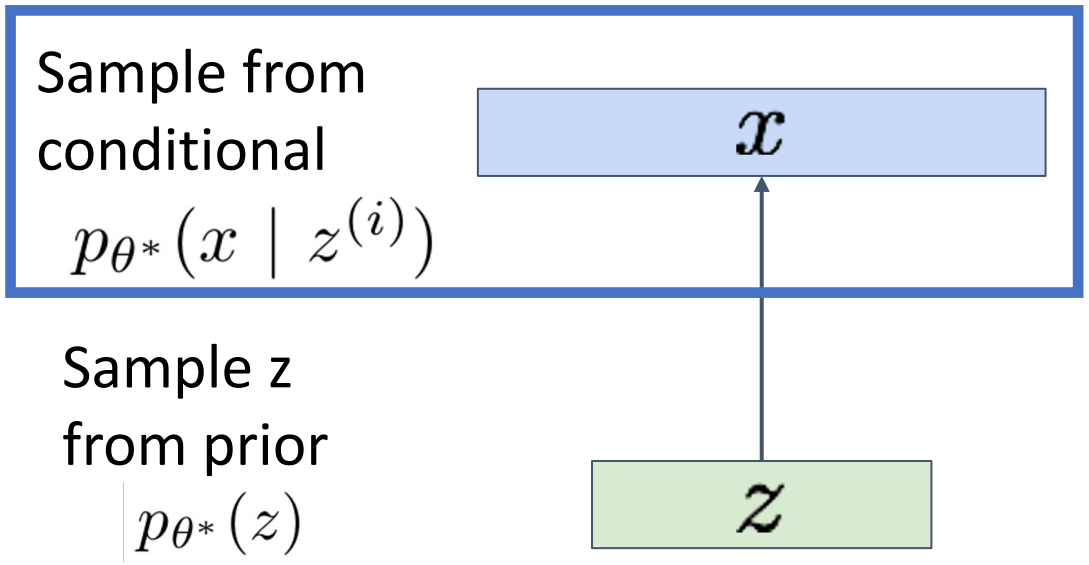

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Represent p(x|z) with a neural network (Similar to **decoder** from autoencoder)

Variational Autoencoders

Decoder must be **probabilistic**:
Decoder inputs z, outputs mean $\mu_{x|z}$ and (diagonal) covariance $\sum_{x|z}$

Sample x from Gaussian with mean $\mu_{x|z}$ and (diagonal) covariance $\sum_{x|z}$

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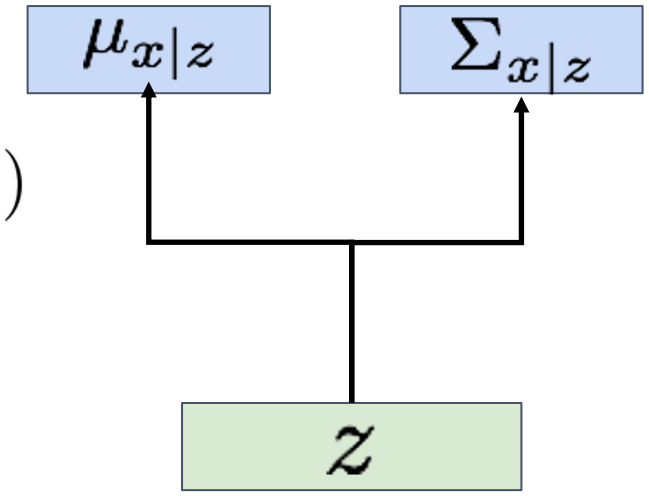
Sample z from prior $p_{\theta^*}(z)$



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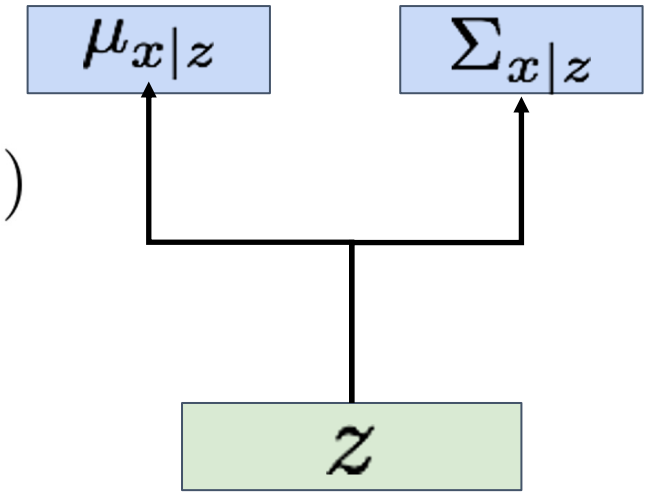
Sample from conditional

$$p_{\theta^*}(x \mid z^{(i)})$$

Sample z from prior

$$p_{\theta^*}(z)$$



Assume training data $\{x^{(i)}\}_{i=1}^{N}$ is generated from unobserved (latent) representation **z**

How to train this model?

Basic idea: **maximize likelihood of data**

If we could observe the z for each x, then could train a *conditional generative model* p(x|z)

Variational Autoencoders

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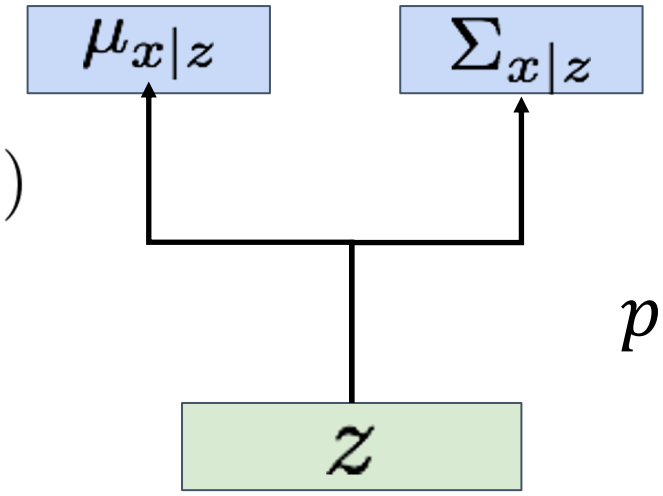
Sample x from Gaussian with mean $\mu_{x|z}$ and (diagonal) covariance $\sum_{x|z}$

Sample from conditional $p_{\theta^*}(x \mid z^{(i)})$



Sample z from prior $p_{\theta^*}(z)$

Assume training data $\{x^{(i)}\}_{i=1}^{N}$ is generated from unobserved (latent) representation **z**

How to train this model?

Basic idea: **maximize likelihood of data**

We don't observe z, so need to marginalize:

$$p_\theta(x) = \int p_\theta(x,z)dz = \int p_\theta(x|z)p_\theta(z)dz$$

Variational Autoencoders

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Sample x from Gaussian with mean $\mu_{x|z}$ and (diagonal) covariance $\sum_{x|z}$

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How to train this model?

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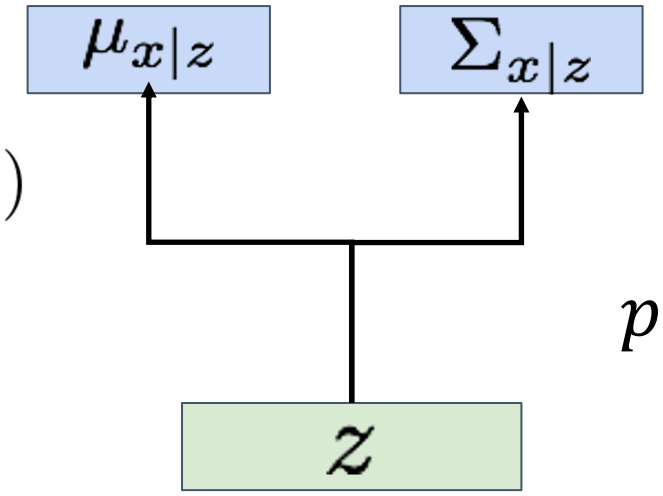

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We don't observe z, so need to marginalize:

$$p_\theta(x) = \int p_\theta(x,z)dz = \int p_\theta(x|z)p_\theta(z)dz$$

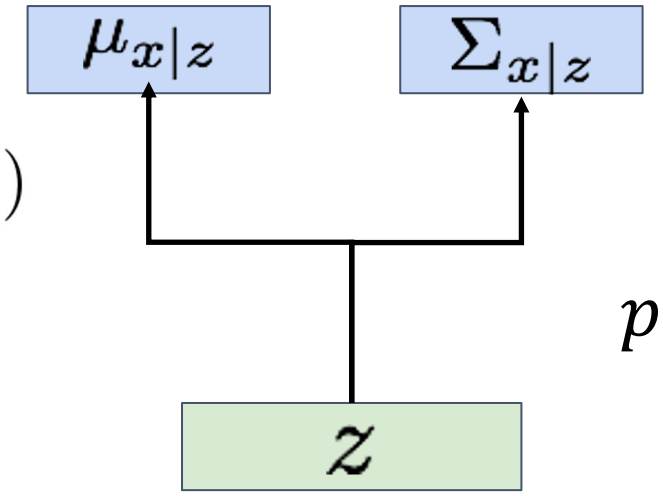
Ok, can compute this with decoder network

Variational Autoencoders

Decoder must be **probabilistic**:
Decoder inputs z, outputs mean $\mu_{x|z}$ and (diagonal) covariance $\sum_{x|z}$

Sample x from Gaussian with mean $\mu_{x|z}$ and (diagonal) covariance $\sum_{x|z}$

Sample from conditional $p_{\theta^*}(x \mid z^{(i)})$



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Ok, we assumed Gaussian prior for z

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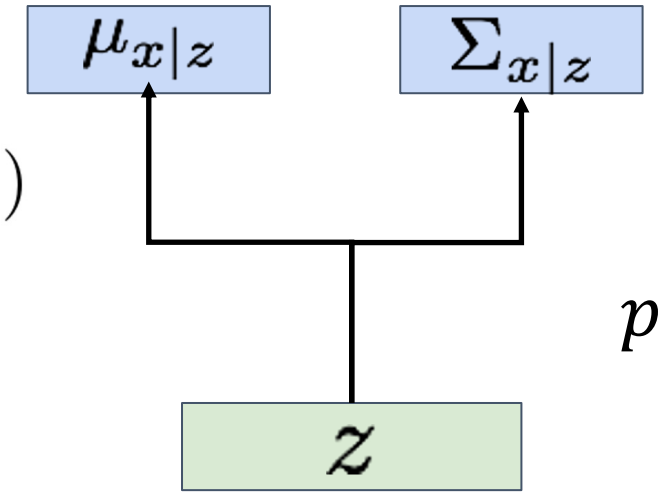

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$$p_\theta(x) = \int p_\theta(x,z)dz = \int p_\theta(x|z)p_\theta(z)dz$$

Problem: Impossible to integrate over all z!

Recall $p(x, z) = p(x \mid z)p(z) = p(z \mid x)p(x)$

Variational Autoencoders

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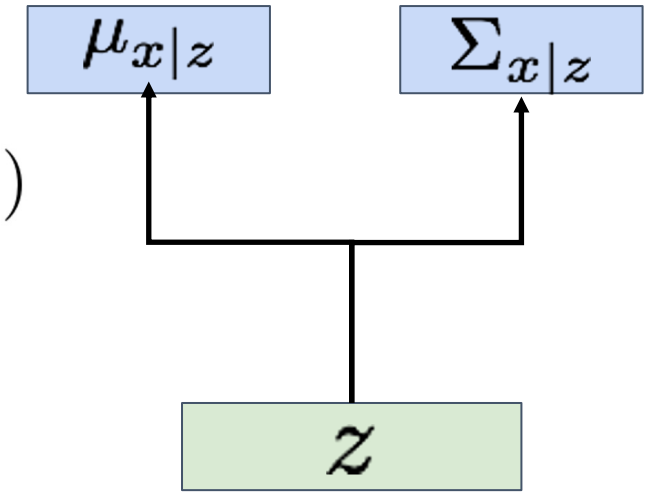
Sample z from prior $p_{\theta^*}(z)$



Basic idea: **maximize likelihood of data**

Another idea: Try Bayes' Rule:

$$p_\theta(x) = \frac{p_\theta(x \mid z)p_\theta(z)}{p_\theta(z \mid x)}$$

Recall $p(x, z) = p(x \mid z)p(z) = p(z \mid x)p(x)$

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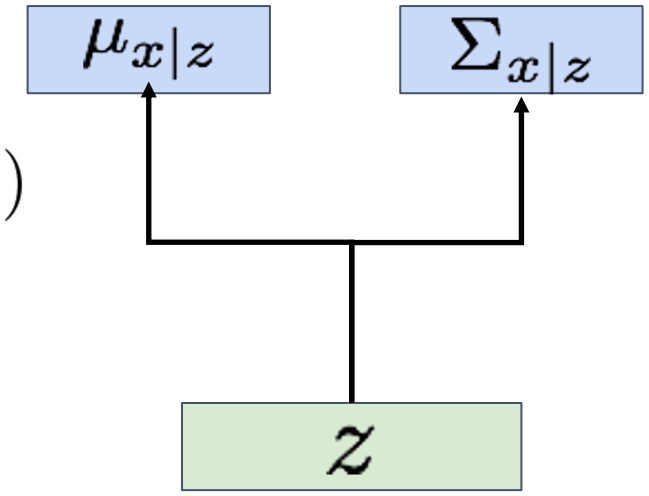

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Ok, compute with decoder network

Recall $p(x, z) = p(x \mid z)p(z) = p(z \mid x)p(x)$

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Sample x from Gaussian with mean $\mu_{x|z}$ and (diagonal) covariance $\sum_{x|z}$

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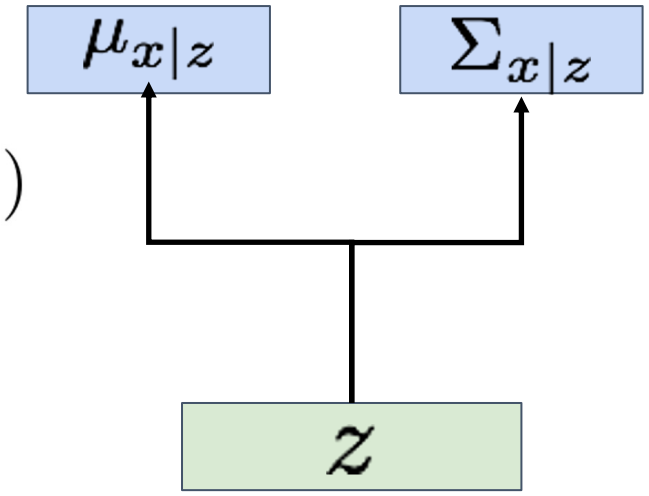

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Another idea: Try Bayes' Rule:

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Ok, we assumed Gaussian prior

Recall $p(x, z) = p(x \mid z)p(z) = p(z \mid x)p(x)$

Variational Autoencoders

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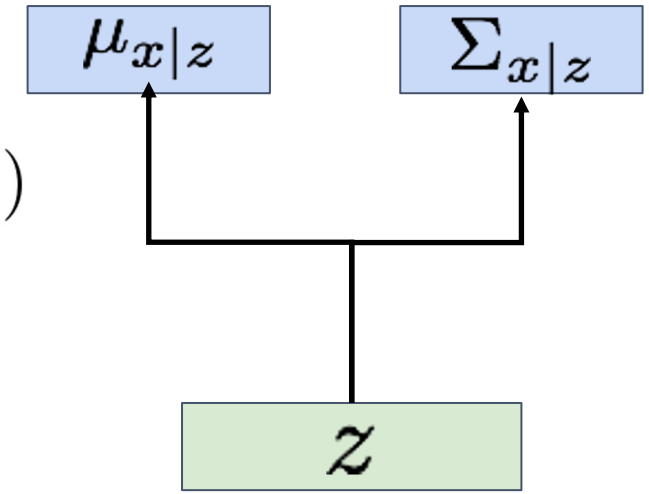

Basic idea: **maximize likelihood of data**

Another idea: Try Bayes' Rule:

$$p_\theta(x) = \frac{p_\theta(x \mid z)p_\theta(z)}{p_\theta(z \mid x)}$$

Problem: No way to compute this!

Recall $p(x, z) = p(x \mid z)p(z) = p(z \mid x)p(x)$

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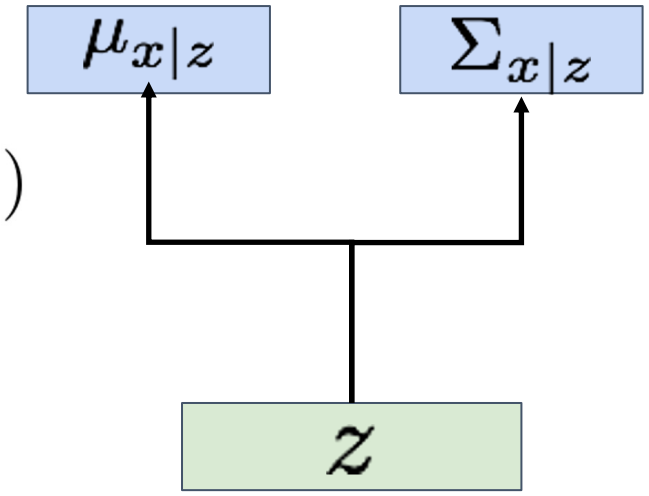

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Another idea: Try Bayes' Rule:

$$p_\theta(x) = \frac{p_\theta(x \mid z)p_\theta(z)}{p_\theta(z \mid x)}$$

Solution: Train another network **(encoder)** that learns $q_\phi(z \mid x) \approx p_\theta(z \mid x)$

Recall $p(x, z) = p(x \mid z)p(z) = p(z \mid x)p(x)$

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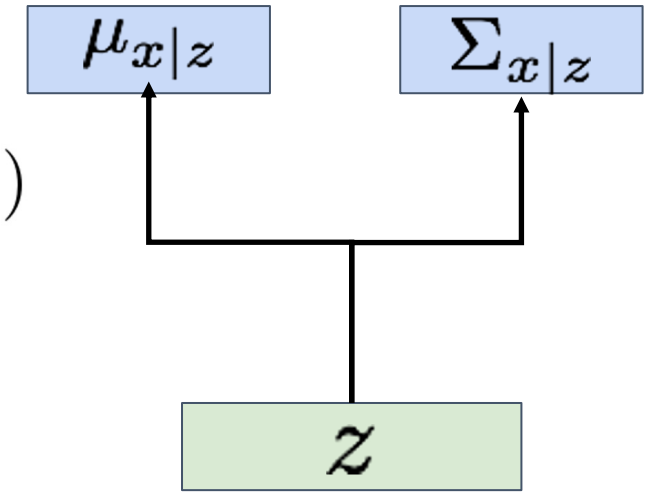


Basic idea: **maximize likelihood of data**

Another idea: Try Bayes' Rule:

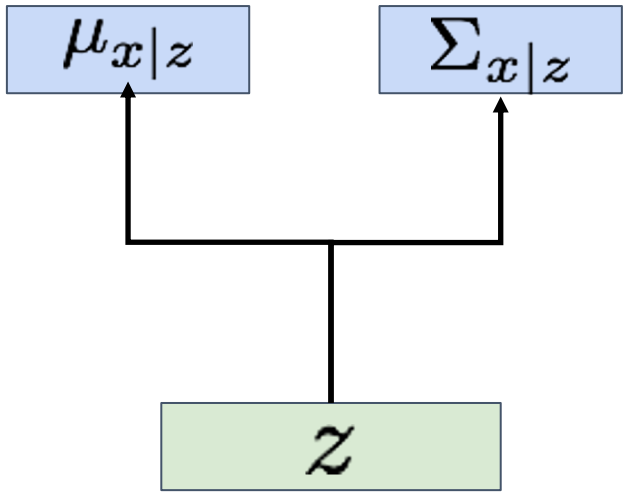
$$p_\theta(x) = \frac{p_\theta(x \mid z)p_\theta(z)}{p_\theta(z \mid x)} \approx \frac{p_\theta(x \mid z)p_\theta(z)}{q_\phi(z \mid x)}$$

Use **encoder** to compute $q_\phi(z \mid x) \approx p_\theta(z \mid x)$

Variational Autoencoders

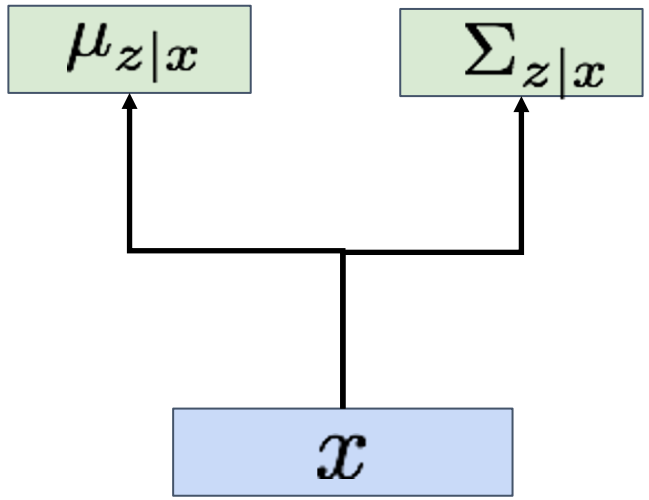
Decoder network inputs latent code z, gives distribution over data x

$$p_\theta(x \mid z) = N(\mu_{x|z}, \Sigma_{x|z})$$



Encoder network inputs data x, gives distribution over latent codes z

$$q_\phi(z \mid x) = N(\mu_{z|x}, \Sigma_{z|x})$$



If we can ensure that $q_\phi(z \mid x) \approx p_\theta(z \mid x)$,

then we can approximate

$$p_\theta(x) \approx \frac{p_\theta(x \mid z)p(z)}{q_\phi(z \mid x)}$$

Idea: Jointly train both encoder and decoder

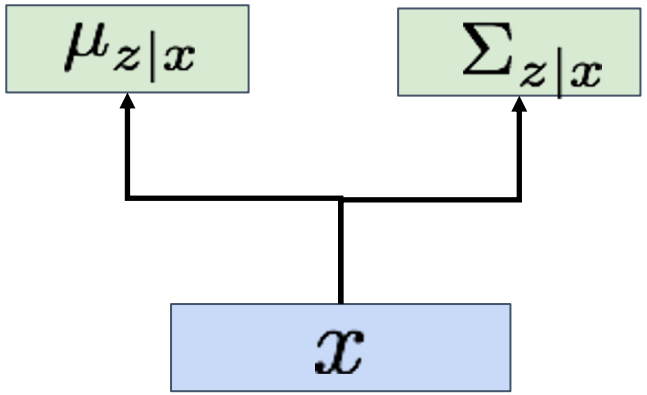
Variational Autoencoders

Jointly train **encoder** q and **decoder** p to maximize the **variational lower bound** on the data likelihood

$$\log p_\theta(x) \geq E_{z\sim q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

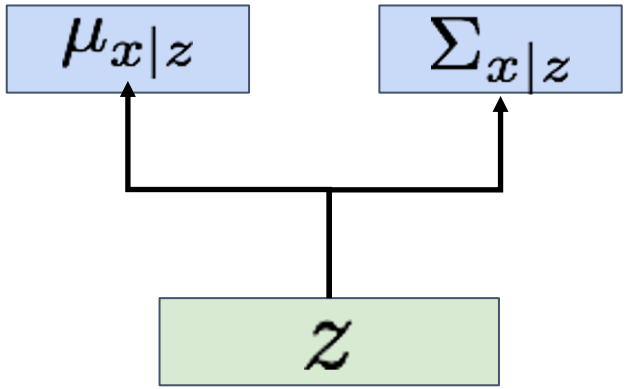
Encoder Network

$$q_\phi(z \mid x) = N(\mu_{z|x}, \Sigma_{z|x})$$



Decoder Network

$$p_\theta(x \mid z) = N(\mu_{x|z}, \Sigma_{x|z})$$



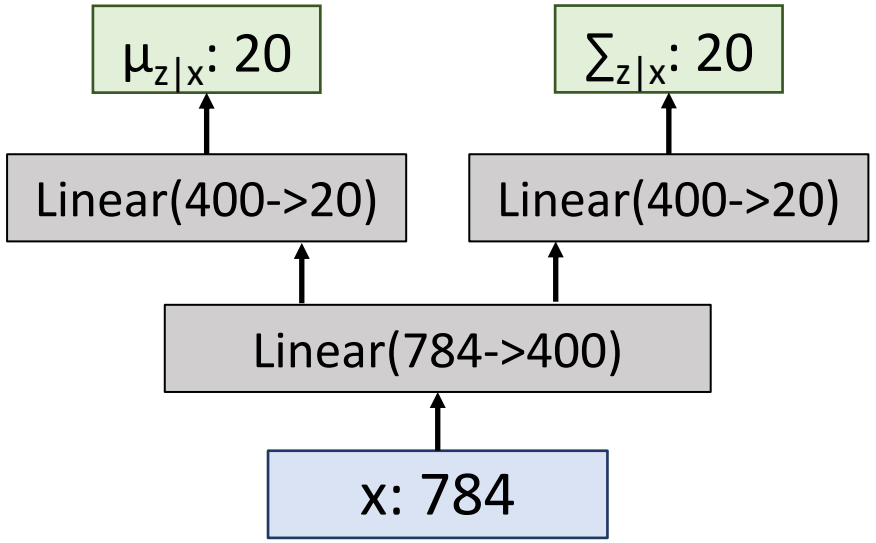
Example: Fully-Connected VAE

x: 28x28 image, flattened to 784-dim vector

z: 20-dim vector

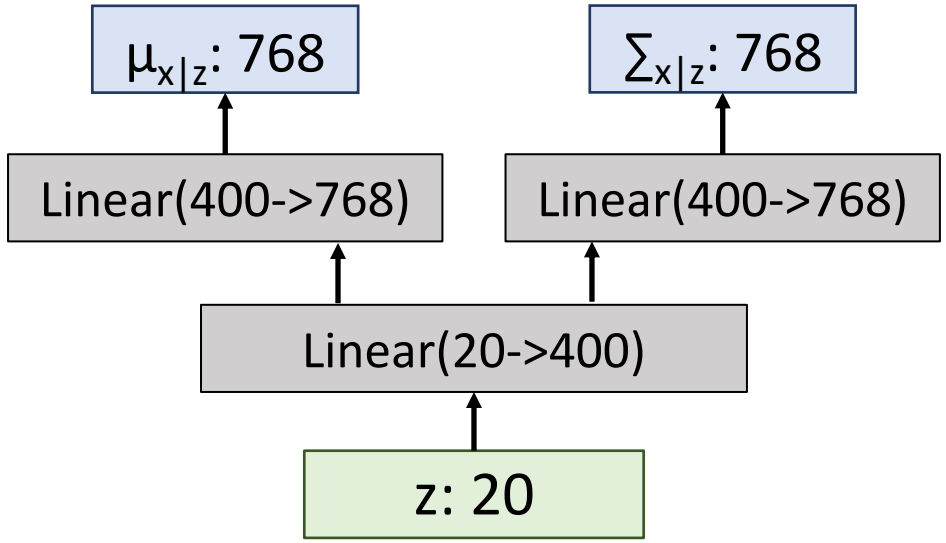
Encoder Network

$$q_\phi(z \mid x) = N(\mu_{z|x}, \Sigma_{z|x})$$



Decoder Network

$$p_\theta(x \mid z) = N(\mu_{x|z}, \Sigma_{x|z})$$



Code helps understand! PyTorch official example

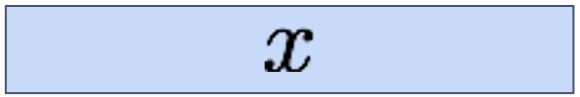
https://github.com/pytorch/examples/tree/master/vae

Variational Autoencoders

Train by maximizing the
variational lower bound

$$E_{z\sim q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

Input Data



Variational Autoencoders

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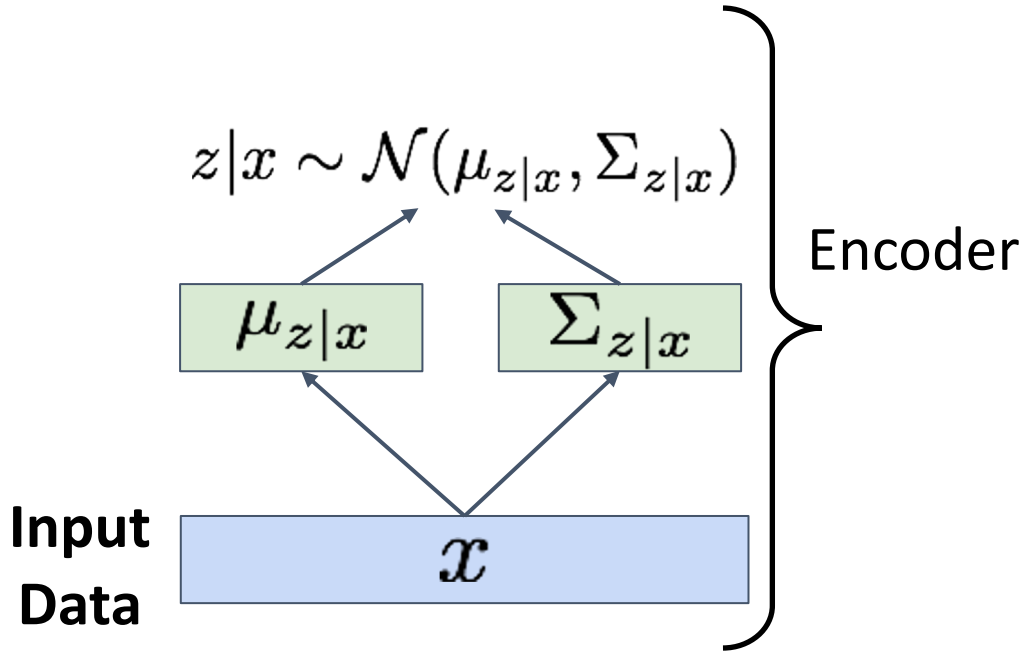
1. Run input data through **encoder** to get a distribution over latent codes



Variational Autoencoders

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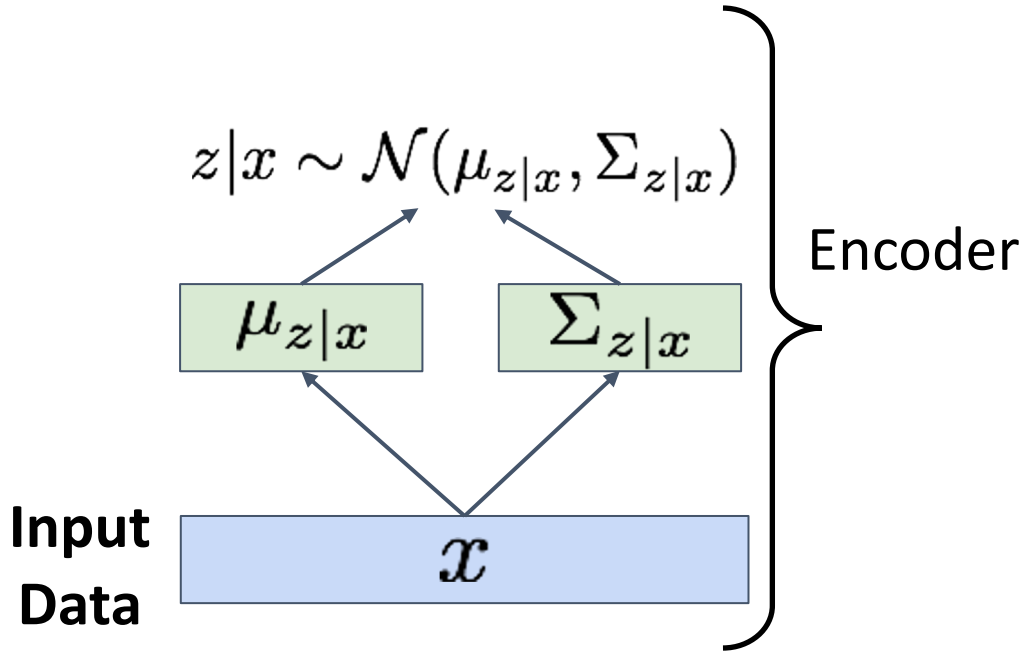
$$E_{z\sim q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

1. Run input data through **encoder** to get a distribution over latent codes
2. **Encoder output should match the prior p(z)!**



https://github.com/pytorch/examples/tree/master/vae

```
# Reconstruction + KL divergence losses summed over all elements and batch
def loss_function(recon_x, x, mu, logvar):
    BCE = F.binary_cross_entropy(recon_x, x.view(-1, 784), reduction='sum')

    # see Appendix B from VAE paper:
    # Kingma and Welling. Auto-Encoding Variational Bayes. ICLR, 2014
    # https://arxiv.org/abs/1312.6114
    # 0.5 * sum(1 + log(sigma^2) - mu^2 - sigma^2)
    KLD = -0.5 * torch.sum(1 + logvar - mu.pow(2) - logvar.exp())

    return BCE + KLD
```

Variational Autoencoders

Train by maximizing the **variational lower bound**

$$E_{z\sim q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

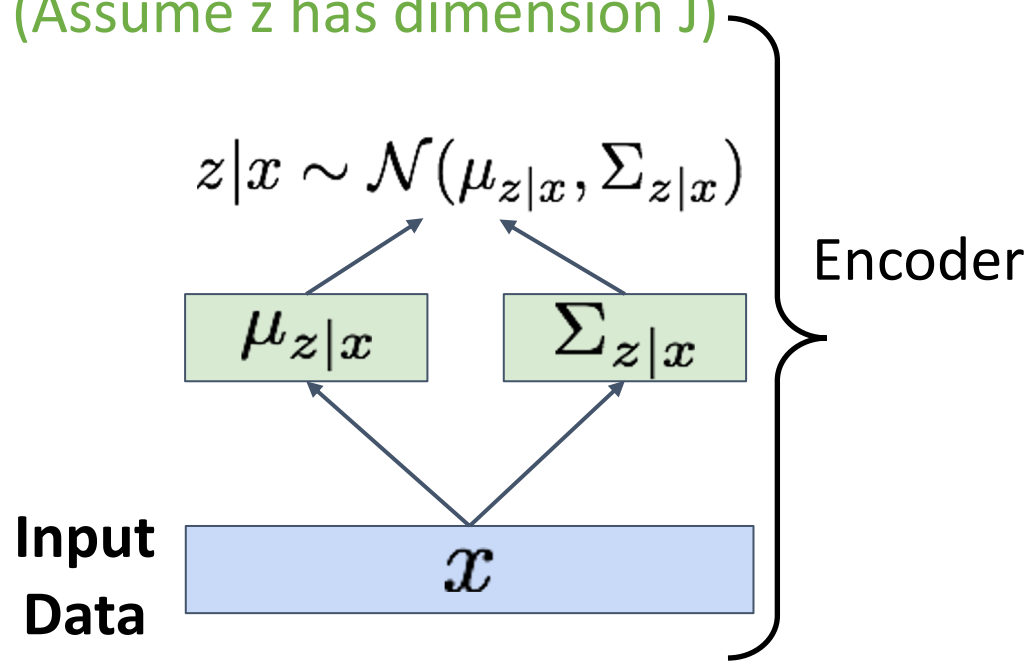
Closed form solution when q_ϕ is diagonal Gaussian and p is unit Gaussian!
(Assume z has dimension J)

1. Run input data through **encoder** to get a distribution over latent codes
2. **Encoder output should match the prior p(z)!**

$$-D_{KL}\left(q_\phi(z|x), p(z)\right) = \int_Z q_\phi(z|x)\log\frac{p(z)}{q_\phi(z|x)}dz$$

$$= \int_Z N\left(z;\mu_{z|x},\Sigma_{z|x}\right)\log\frac{N(z;0,I)}{N\left(z;\mu_{z|x},\Sigma_{z|x}\right)}dz$$

$$= \frac{1}{2}\sum_{j=1}^{J}\left(1+\log\left(\left(\Sigma_{z|x}\right)_j^2\right)-\left(\mu_{z|x}\right)_j^2-\left(\Sigma_{z|x}\right)_j^2\right)$$



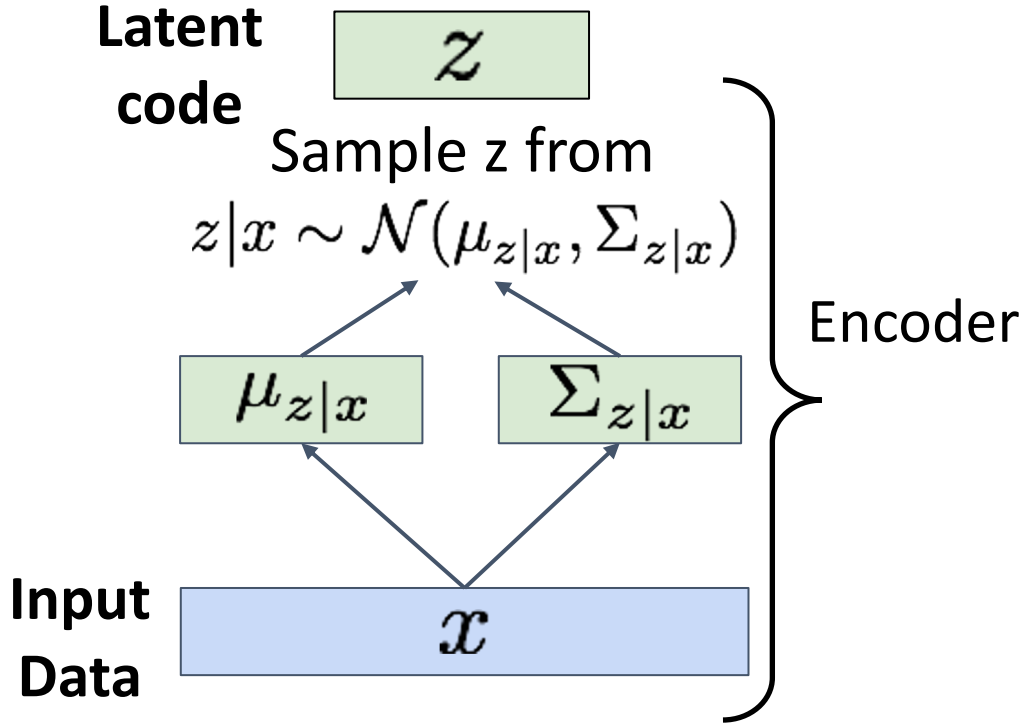
p(z) is *N(0,I)*

Variational Autoencoders

Train by maximizing the
variational lower bound

$$E_{z\sim q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

1. Run input data through **encoder** to get a distribution over latent codes
2. **Encoder output should match the prior p(z)!**
3. Sample code z from encoder output



Variational Autoencoders

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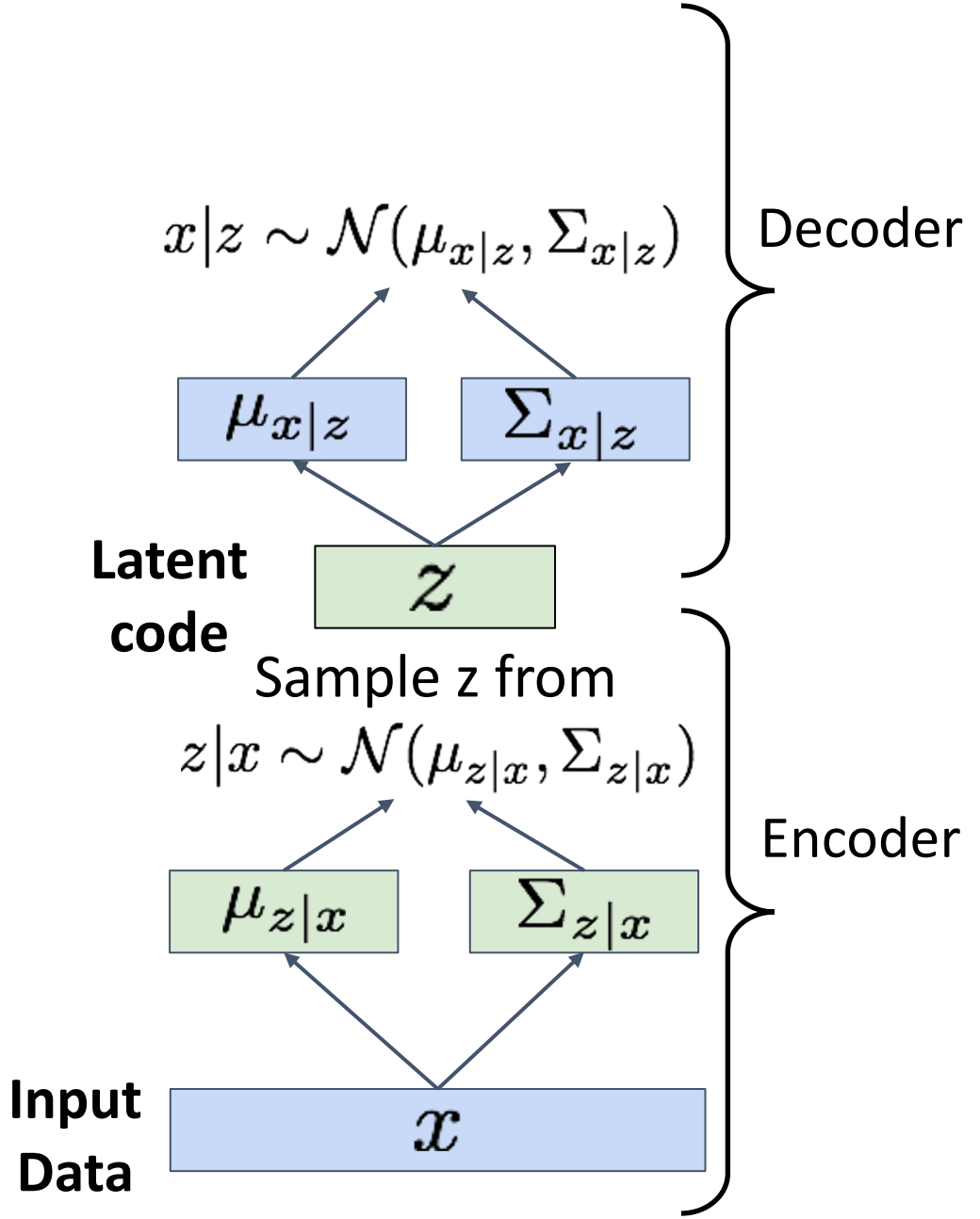
1. Run input data through **encoder** to get a distribution over latent codes
2. **Encoder output should match the prior p(z)!**
3. Sample code z from encoder output
4. Run sampled code through **decoder** to get a distribution over data samples



Variational Autoencoders

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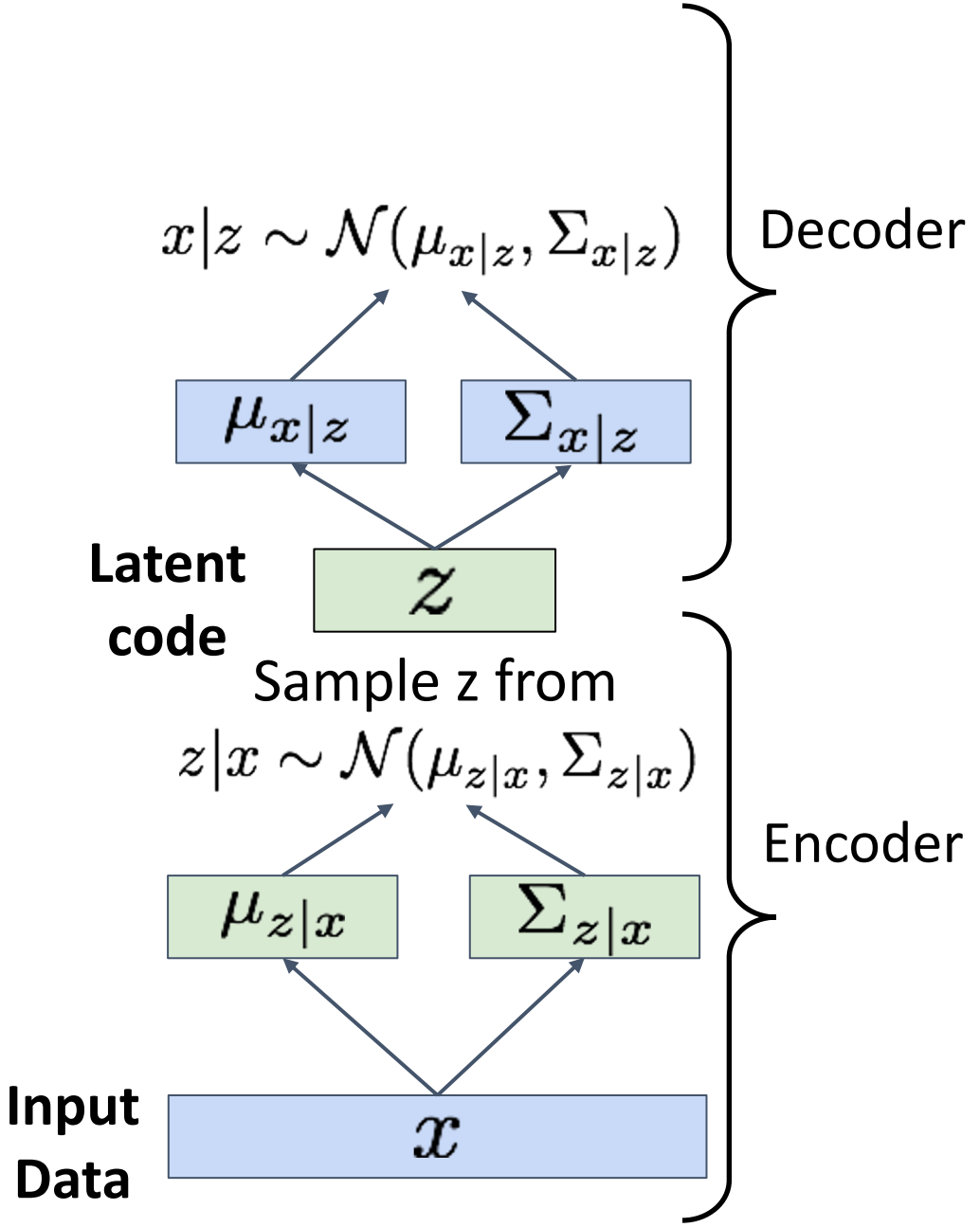
$$E_{z\sim q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

1. Run input data through **encoder** to get a distribution over latent codes
2. **Encoder output should match the prior p(z)!**
3. Sample code z from encoder output
4. Run sampled code through **decoder** to get a distribution over data samples
5. **Original input data should be likely under the distribution output from (4)!**



Variational Autoencoders

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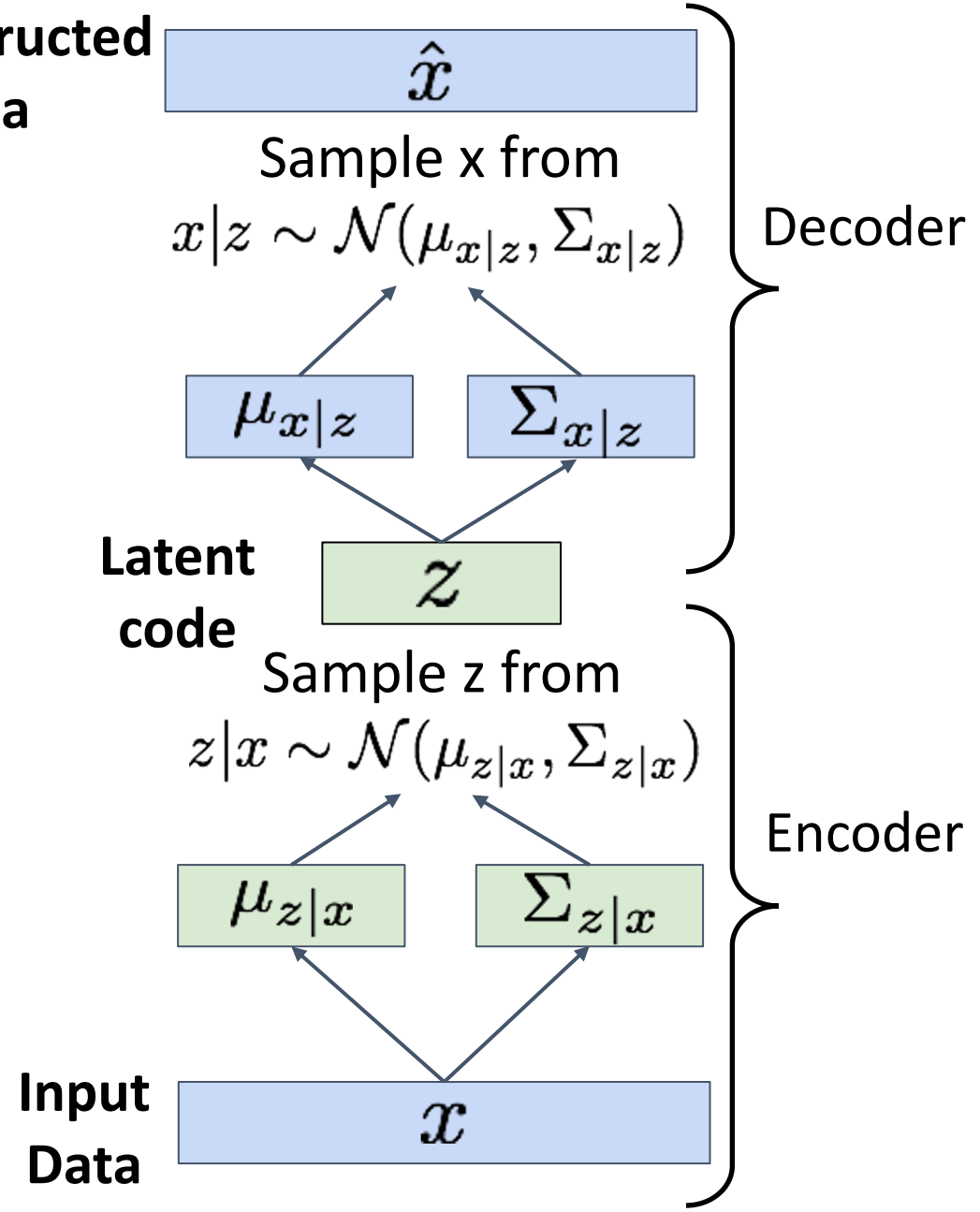
2. **Encoder output should match the prior p(z)!**
3. Sample code z from encoder output
4. Run sampled code through **decoder** to get a distribution over data samples
5. **Original input data should be likely under the distribution output from (4)!**
6. Can sample a reconstruction from (4)

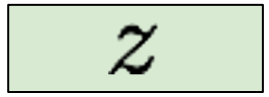


Variational Autoencoders: Generating Data

After training we can generate new data!

1. Sample z from prior p(z)

Latent code

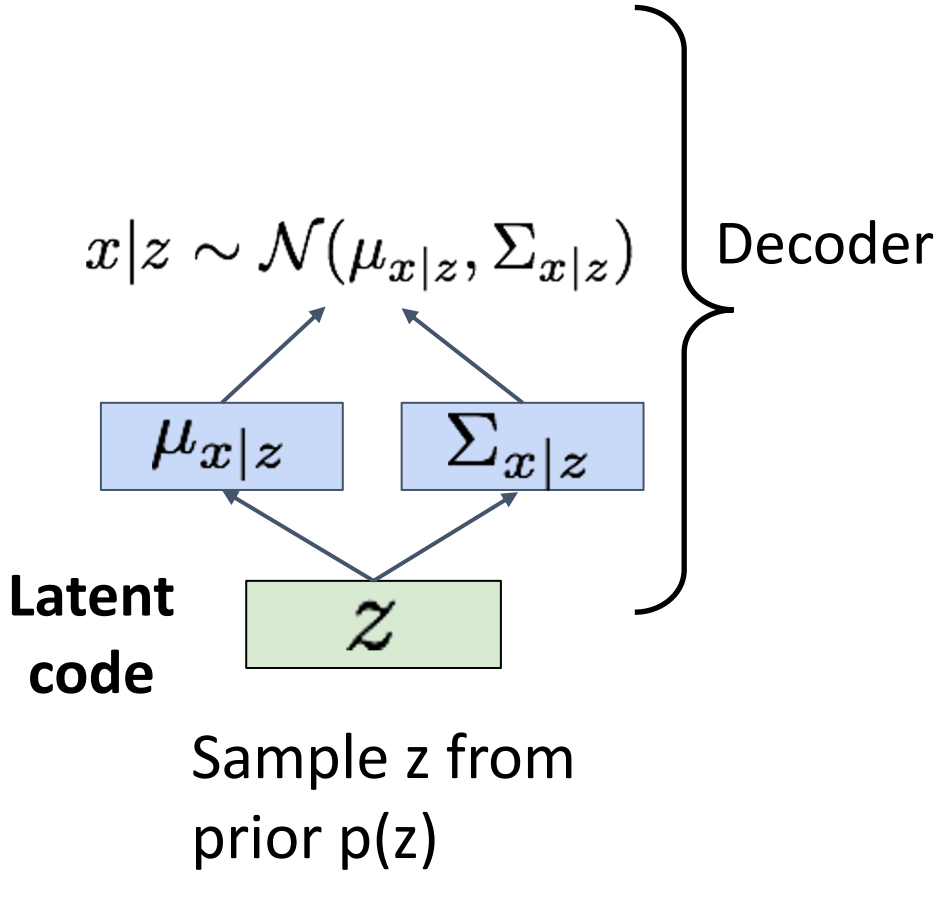


Sample z from prior p(z)

Variational Autoencoders: Generating Data

After training we can generate new data!

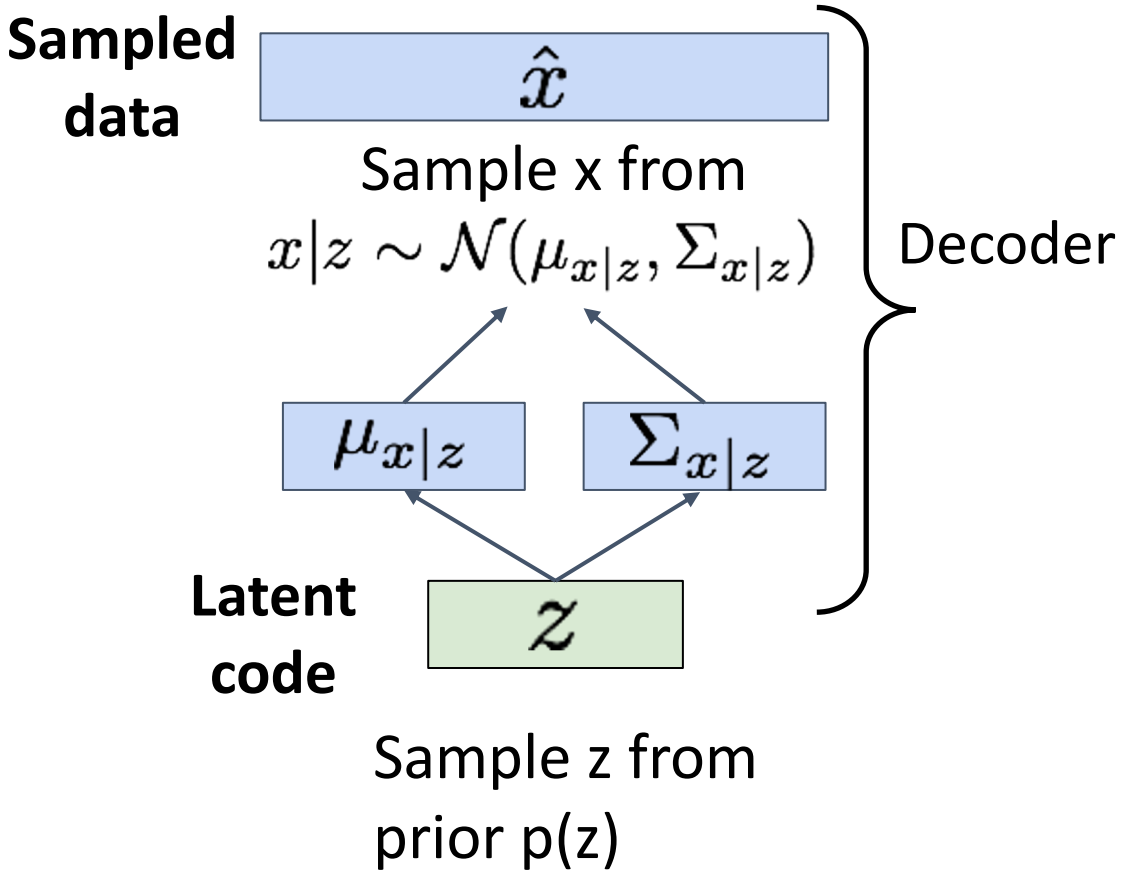
1. Sample z from prior p(z)
2. Run sampled z through decoder to get distribution over data x



Variational Autoencoders: Generating Data

After training we can generate new data!

1. Sample z from prior p(z)
2. Run sampled z through decoder to get distribution over data x
3. Sample from distribution in (2) to generate data



Variational Autoencoders: Generating Data

32x32 CIFAR-10

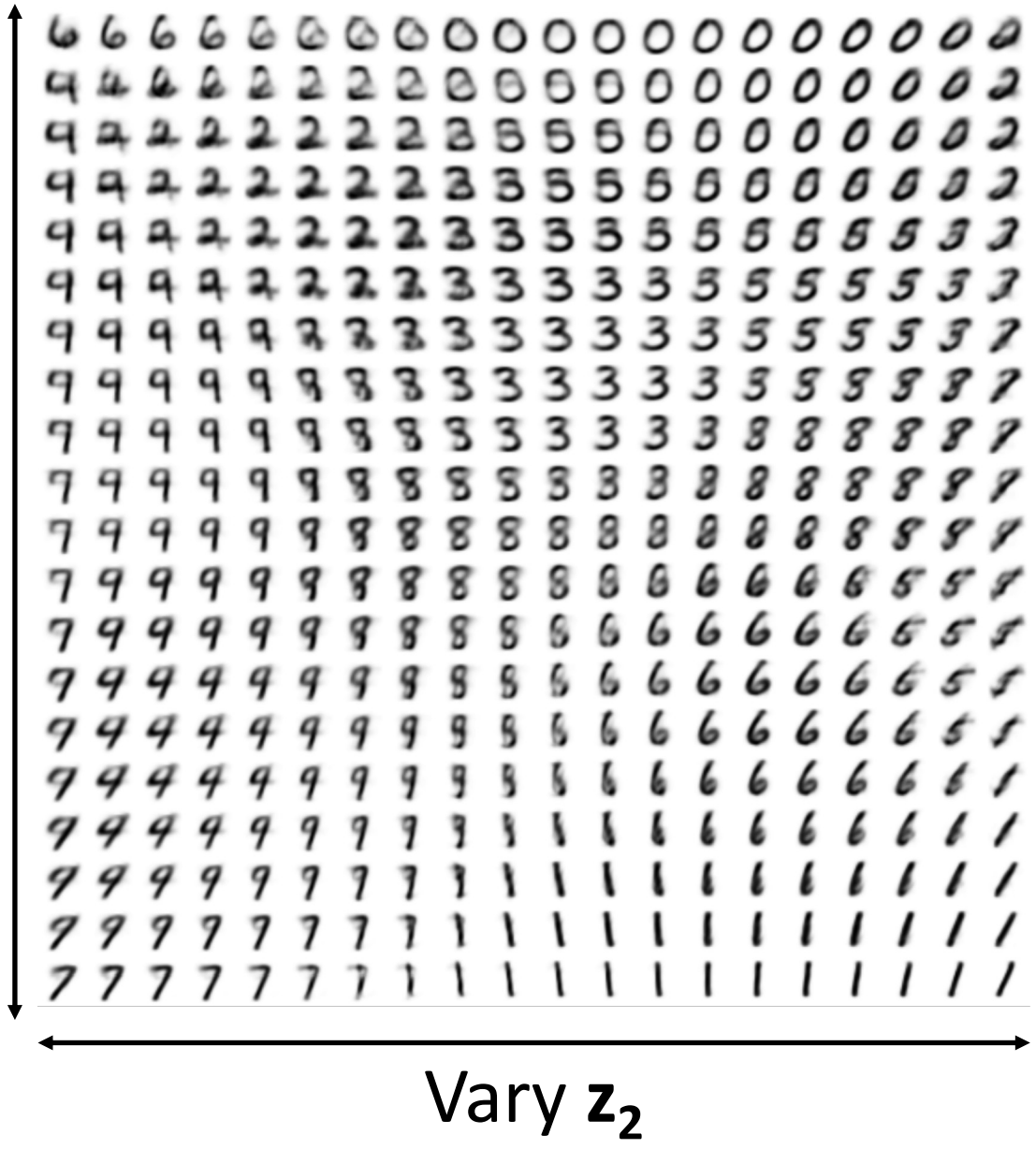
Labeled Faces in the Wild

Figures from (L) Dirk Kingma et al. 2016; (R) Anders Larsen et al. 2017.

Variational Autoencoders

The diagonal prior on p(z) causes dimensions of z to be independent

“Disentangling factors of variation”

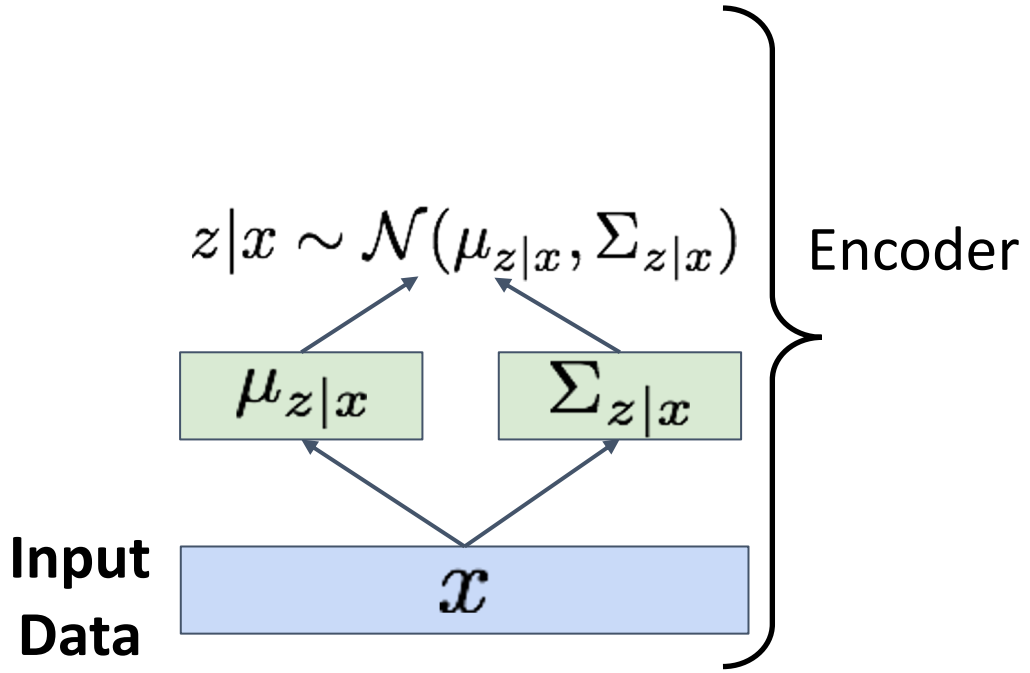


Kingma and Welling, Auto-Encoding Variational Beyes, ICLR 2014

Variational Autoencoders

After training we can **edit images**

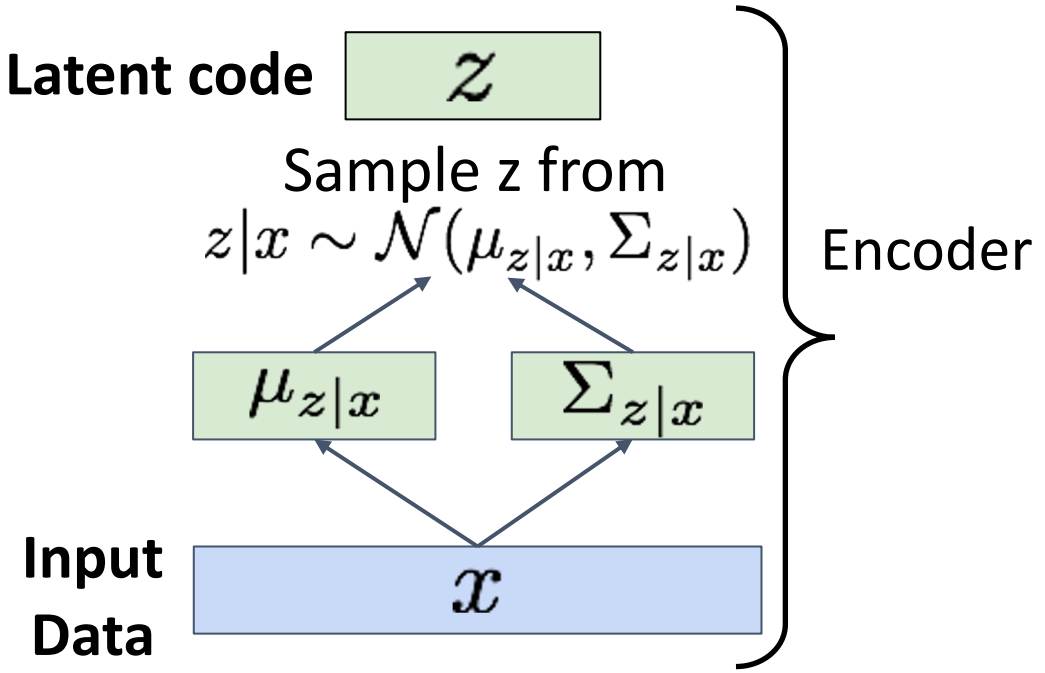
1. Run input data through **encoder** to get a distribution over latent codes



Variational Autoencoders

After training we can **edit images**

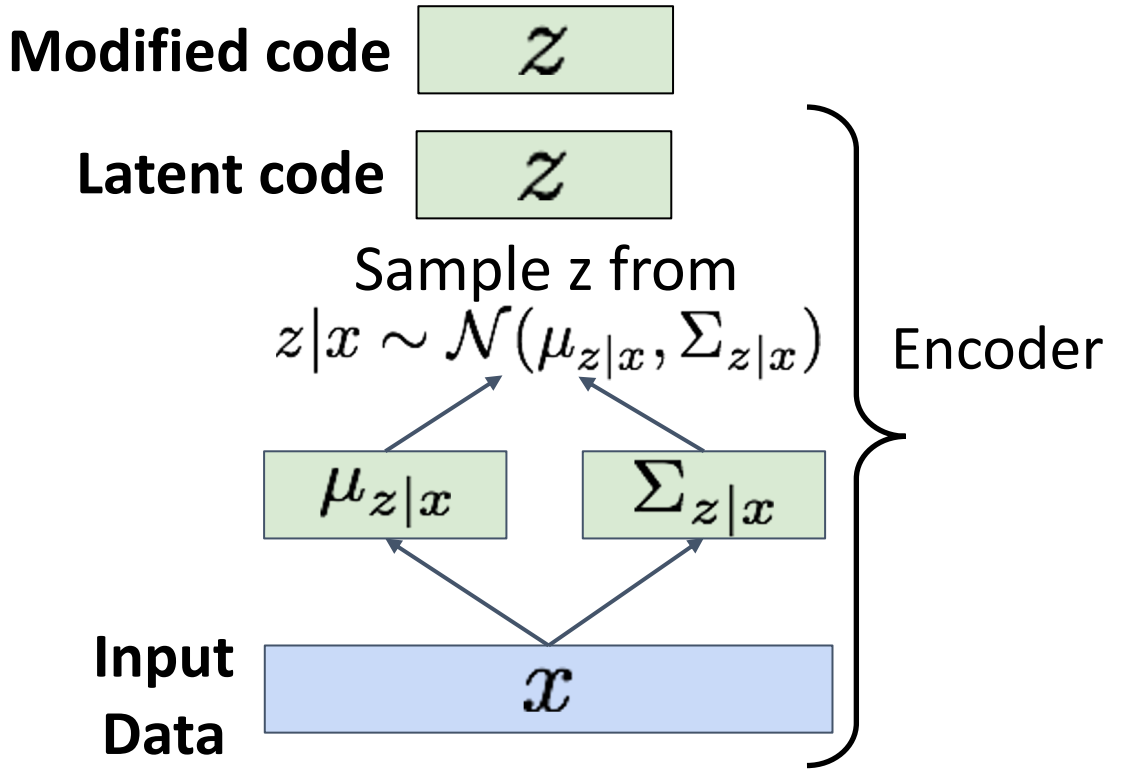
1. Run input data through **encoder** to get a distribution over latent codes
2. Sample code z from encoder output



Variational Autoencoders

After training we can **edit images**

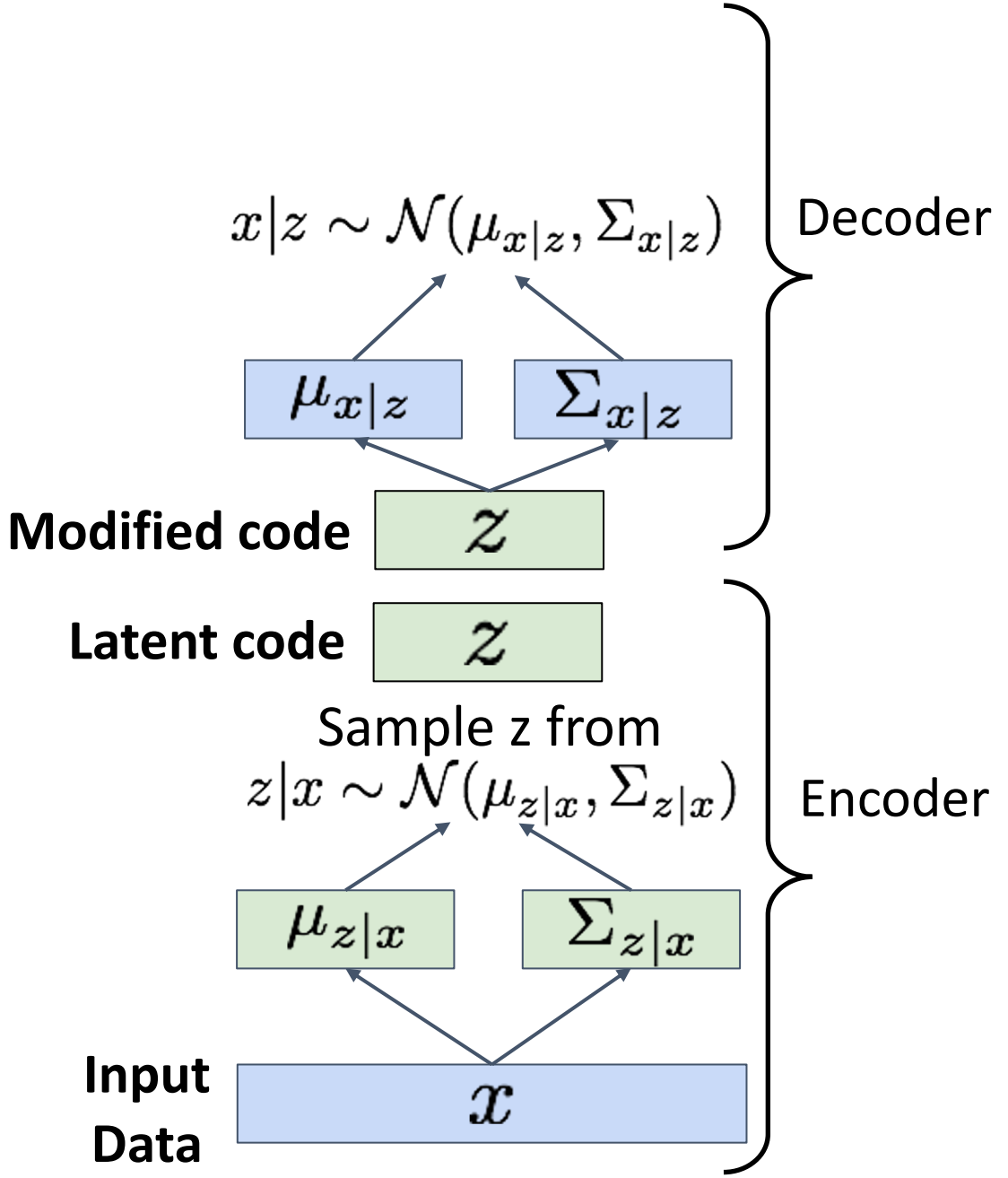
1. Run input data through **encoder** to get a distribution over latent codes
2. Sample code z from encoder output
3. Modify some dimensions of sampled code



Variational Autoencoders

After training we can **edit images**

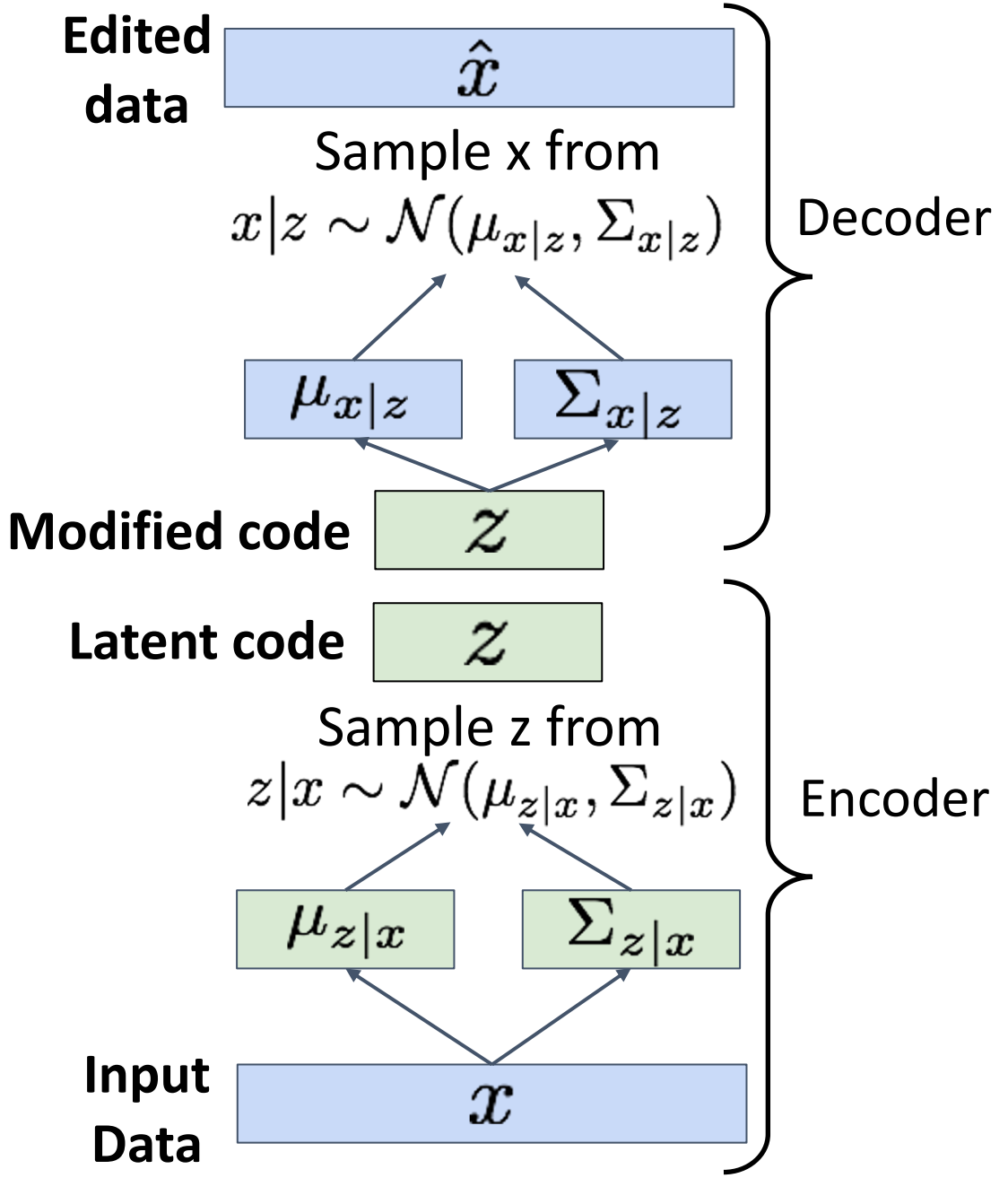
1. Run input data through **encoder** to get a distribution over latent codes
2. Sample code z from encoder output
3. Modify some dimensions of sampled code
4. Run modified z through **decoder** to get a distribution over data sample



Variational Autoencoders

After training we can **edit images**

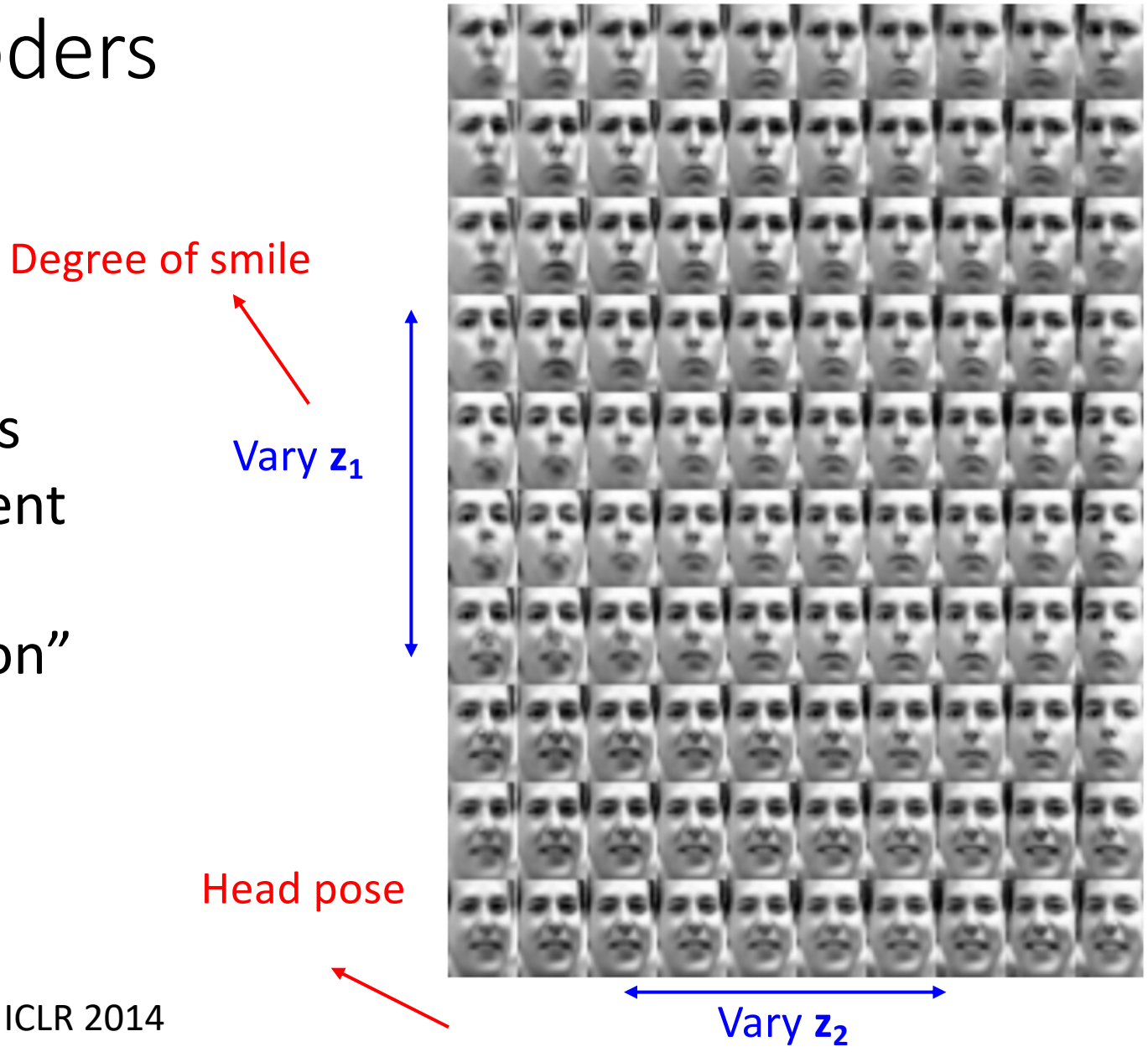
1. Run input data through **encoder** to get a distribution over latent codes
2. Sample code z from encoder output
3. Modify some dimensions of sampled code
4. Run modified z through **decoder** to get a distribution over data samples
5. Sample new data from (4)



Variational Autoencoders

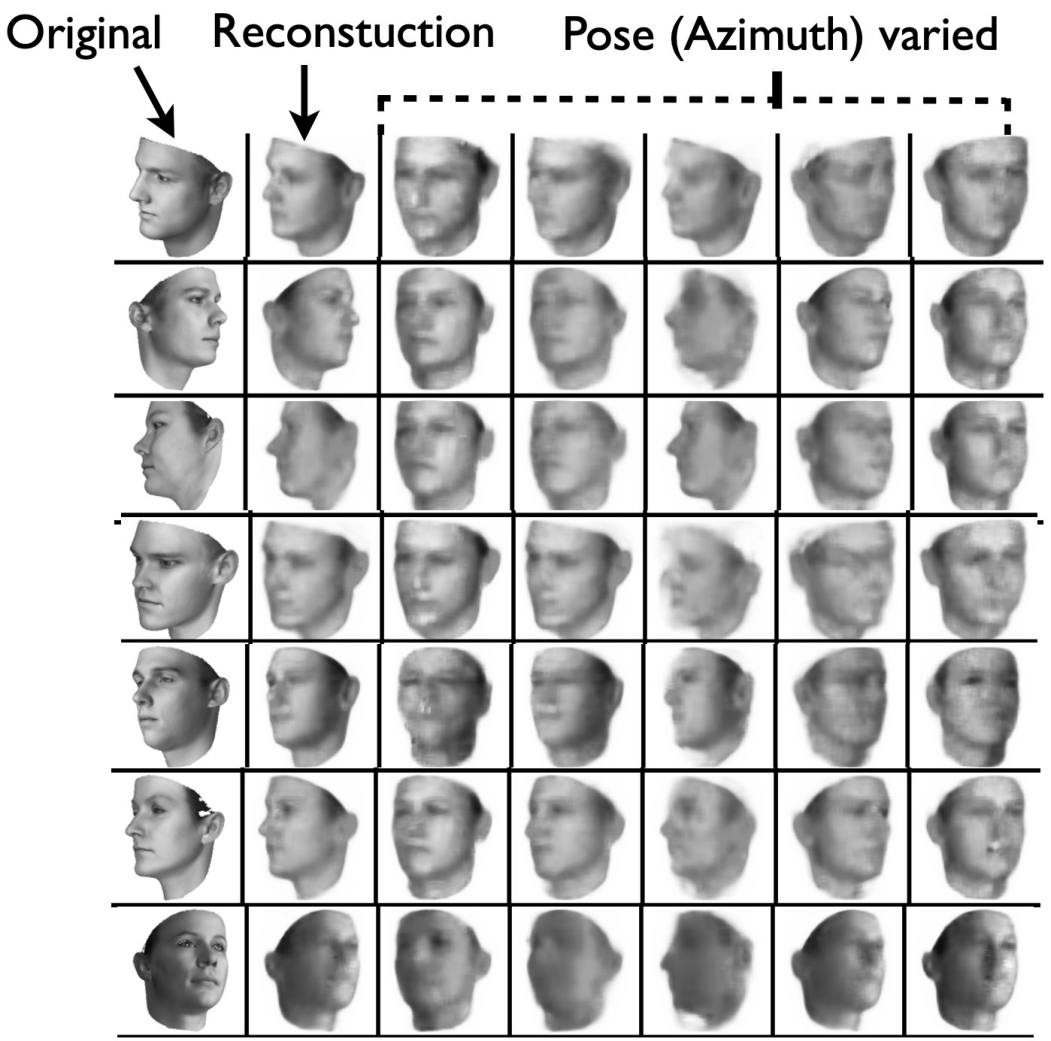
The diagonal prior on p(z) causes dimensions of z to be independent

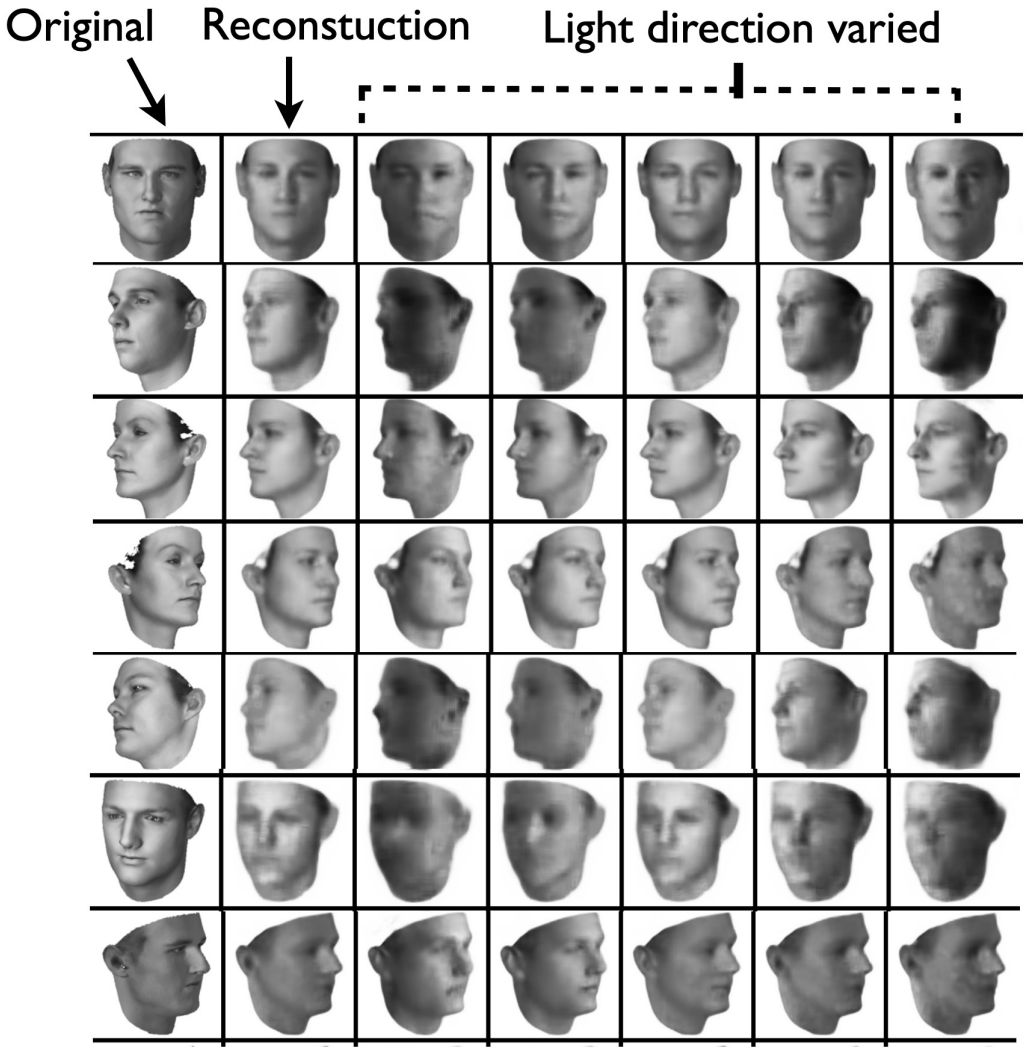
"Disentangling factors of variation"



Kingma and Welling, Auto-Encoding Variational Beyes, ICLR 2014

Variational Autoencoders: Image Editing





Kulkarni et al, "Deep Convolutional Inverse Graphics Networks", NeurIPS 2014

Variational Autoencoder: Summary

Probabilistic spin to traditional autoencoders => allows generating data
Defines an intractable density => derive and optimize a (variational) lower bound

Pros:

- Principled approach to generative models
- Allows inference of $q(z|x)$, can be useful feature representation for other tasks

Cons:

- Maximizes lower bound of likelihood: okay, but not as good evaluation as PixelRNN/PixelCNN
- Samples blurrier and lower quality compared to state-of-the-art (GANs)

Active areas of research:

- More flexible approximations, e.g. richer approximate posterior instead of diagonal Gaussian, e.g., Gaussian Mixture Models (GMMs)
- Incorporating structure in latent variables, e.g., Categorical Distributions

Next Time:
Generative Models, part 2

VQ-VAE, GAN, and Diffusion

Derivation of the Variational Autoencoders (Appendix)

Variational Autoencoders

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)}$$

Bayes' Rule

Variational Autoencoders

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

Multiply top and bottom by $q_\Phi(z|x)$

Variational Autoencoders

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

$$= \log p_\theta(x|z) - \log \frac{q_\phi(z|x)}{p(z)} + \log \frac{q_\phi(z|x)}{p_\theta(z|x)}$$

Split up using rules for logarithms

Variational Autoencoders

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

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Split up using rules for logarithms

Variational Autoencoders

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

$$= \log p_\theta(x|z) - \log \frac{q_\phi(z|x)}{p(z)} + \log \frac{q_\phi(z|x)}{p_\theta(z|x)}$$

$$\log p_\theta(x) = E_{z \sim q_\phi(z|x)}[\log p_\theta(x)]$$

We can wrap in an expectation since it doesn't depend on z

Variational Autoencoders

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

$$= E_z[\log p_\theta(x|z)] - E_z\left[\log \frac{q_\phi(z|x)}{p(z)}\right] + E_z\left[\log \frac{q_\phi(z|x)}{p_\theta(z|x)}\right]$$

$$\log p_\theta(x) = E_{z\sim q_\phi(z|x)}[\log p_\theta(x)]$$

We can wrap in an expectation since it doesn't depend on z

Variational Autoencoders

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

$$= E_z[\log p_\theta(x|z)] - E_z\left[\log \frac{q_\phi(z|x)}{p(z)}\right] + E_z\left[\log \frac{q_\phi(z|x)}{p_\theta(z|x)}\right]$$

$$= E_{z\sim q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right) + D_{KL}(q_\phi(z|x), p_\theta(z|x))$$

Data reconstruction

Variational Autoencoders

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

$$= E_z[\log p_\theta(x|z)] - E_z\left[\log \frac{q_\phi(z|x)}{p(z)}\right] + E_z\left[\log \frac{q_\phi(z|x)}{p_\theta(z|x)}\right]$$

$$= E_{z\sim q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right) + D_{KL}(q_\phi(z|x), p_\theta(z|x))$$

KL divergence between prior, and
samples from the encoder network

Variational Autoencoders

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

$$= E_z[\log p_\theta(x|z)] - E_z\left[\log \frac{q_\phi(z|x)}{p(z)}\right] + E_z\left[\log \frac{q_\phi(z|x)}{p_\theta(z|x)}\right]$$

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KL divergence between encoder and posterior of decoder

p(z) is *N(0,1)*

Variational Autoencoders

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

$$= E_z[\log p_\theta(x|z)] - E_z\left[\log \frac{q_\phi(z|x)}{p(z)}\right] + E_z\left[\log \frac{q_\phi(z|x)}{p_\theta(z|x)}\right]$$

$$= E_{z\sim q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right) + D_{KL}(q_\phi(z|x), p_\theta(z|x))$$

KL is >= 0, so dropping this term gives a **lower bound** on the data likelihood:

Variational Autoencoders

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

$$= E_z[\log p_\theta(x|z)] - E_z\left[\log \frac{q_\phi(z|x)}{p(z)}\right] + E_z\left[\log \frac{q_\phi(z|x)}{p_\theta(z|x)}\right]$$

$$= E_{z\sim q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right) + D_{KL}(q_\phi(z|x), p_\theta(z|x))$$

$$\log p_\theta(x) \geq E_{z\sim q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

Evidence Lower Bound (ELBO): $\log P(\mathbf{X}) \geq \mathbb{E}_{\mathbf{Z}\sim Q}\left[\log \frac{P(\mathbf{X},\mathbf{Z})}{Q(\mathbf{Z})}\right].$