$\boxed{8.4}$ Shortest-Path Algorithm

Let $w(i,j)$ denote the weight of edge (i,j) in a weighted graph G.

In this section G will always be a connected, weighted graph.

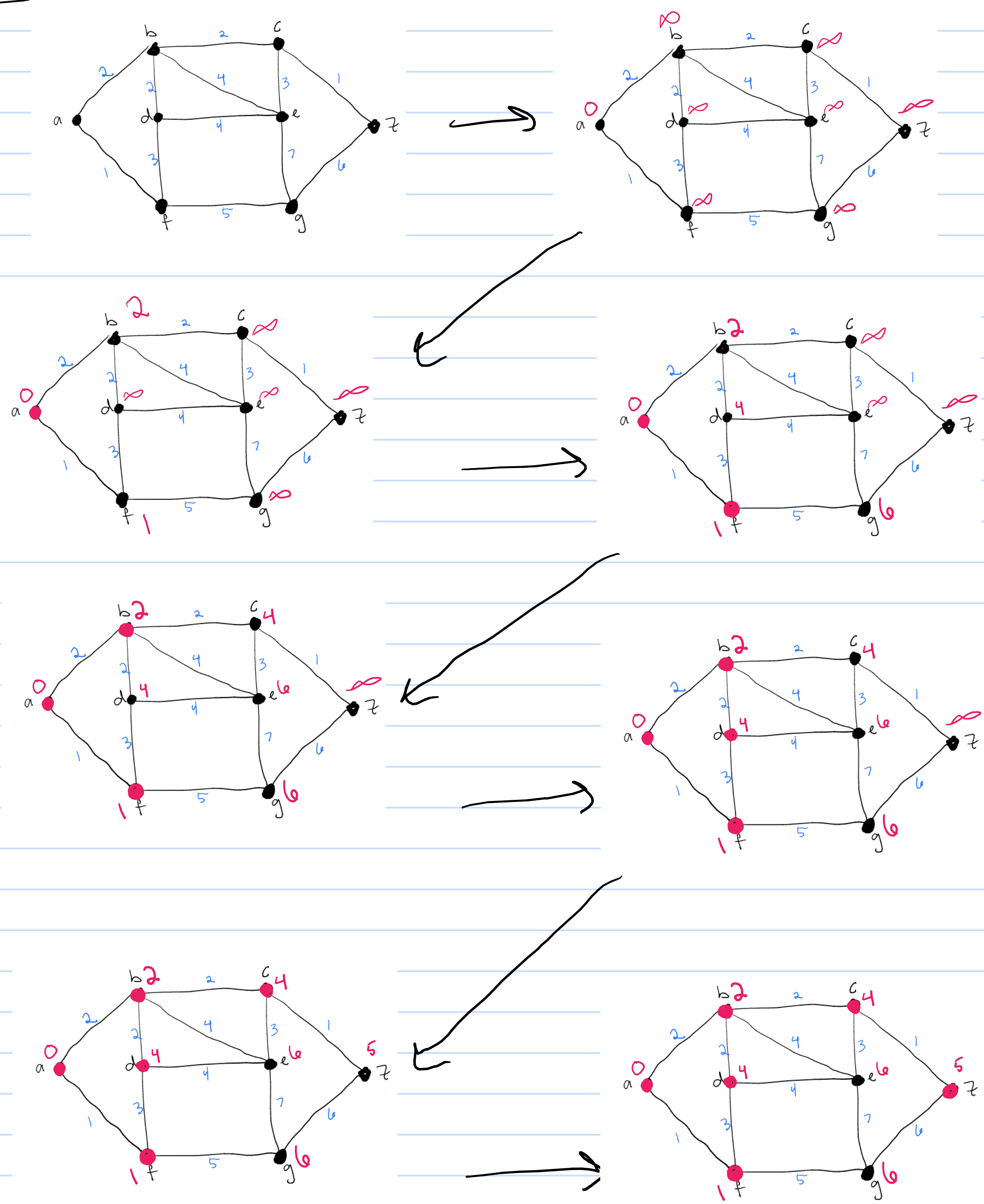
$\boxed{\text{Algorithm}}$ (Dijkstra) The following finds the length $L(z)$ of the shortest path from vertex a to z in G. The weight of edge $e(i,j)$ is $w(i,j) > 0$ & the label of $x \in V(G)$ is $L(x)$.

INPUT: connected, weighted graph w/ all positive weights, vertices a, z

OUTPUT: $L(z)$

```
dijkstra(w, a, z, L) {
    L(a) = 0
    for all vertices x≠a
        L(x) = ∞
    T = set of all vertices      } vert. whose shortest dist
    while (z ∈ T) {                from a hasn't been
        choose v∈T with min L(v)   computed
        T = T - {v}
        for each x∈T adjacent to v
            L(x) = min{L(x), L(v) + w(v,x)}
    }
}
```

Ex



$\Rightarrow L(z) = 5$ using the path (a, b, c, z)

$\boxed{\text{Thm}}$ Dijkstra's Algorithm correctly finds a path from a to z of minimal length

<u>Pf</u> by induction on i

We will prove that the during the i^{th} time entering the while loop, $L(v)$ is the shortest path from a to v.

<u>Base case</u>: $i=1$.
Then in this case $L(a)=0$ and all other values are ∞.
Thus in the 1^{st} loop, a is the chosen vertex + is the length of the shortest path from a to a.

<u>Ind Assume</u>: Assume for all $k<i$, the k^{th} time we arrive in while loop, $L(v)$ is length of shortest path a to v.

Suppose we are now entered loop for i^{th} time.
Choose $v \in T$ with minimal $L(v)$.

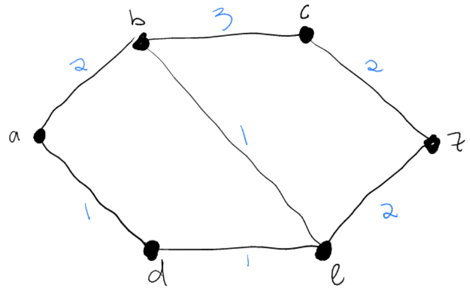
Suppose there is a path P from a to w less than $L(v)$ where $w \in T$. (arguing towards a contradiction).
Let P be a shortest path from a to w. Let $x \in T$ be the nearest vertex to a on P. let u precede x on P. Then $u \notin T \Rightarrow$ u was chosen at $i-1$ step. By ind. assump,
$L(u)$ is length of shortest path a to u. Then

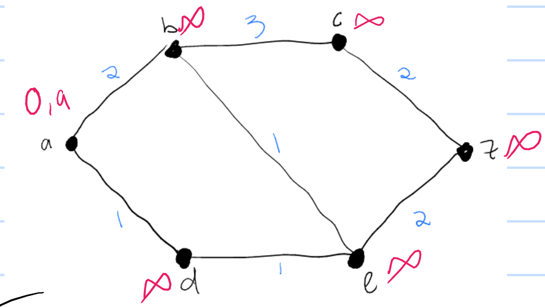
$$L(x) \leq L(u) + (u,x) \leq \text{length of } P < L(v)$$

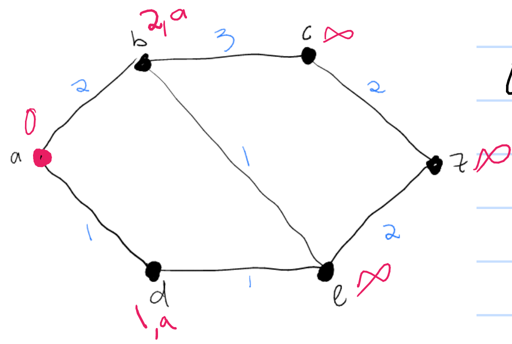
but then v was not a vertex in T with $L(v)$ minimal $\rightarrow\leftarrow$
($L(x)$ was a smaller choice)
Therefore it must be that $w \notin T$.
$\Rightarrow$ if there is a path from a to v of length less than $L(v)$, v would have been already selected + removed from T prior.
$\therefore$ every path has length $\leq L(v)$
Since we have a path of len. $L(v)$ it must be min.

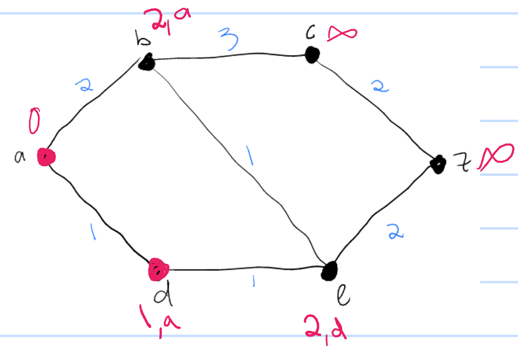
Ex) Find shortest path a to z + find its length

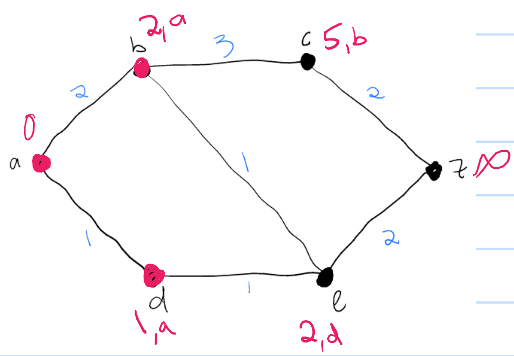
We will label the neighbors of the newly added v as we go.

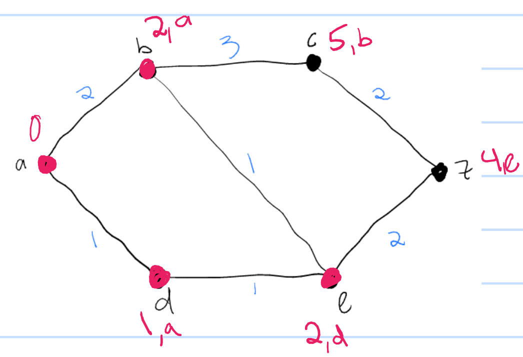


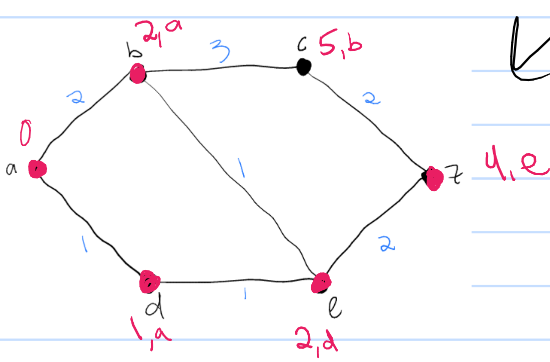










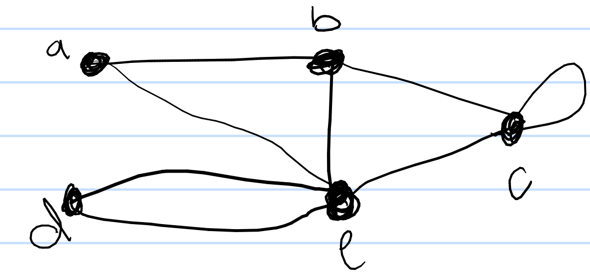


(a, d, e, z) length 4

8.5 Representations of Graphs

We want to use matrices called <u>adjacency matrices</u>.

Ex

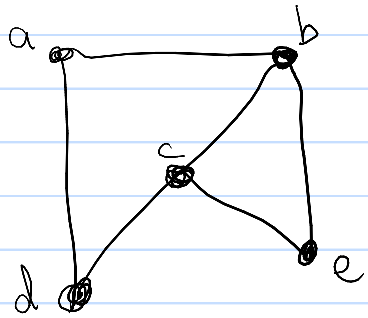


$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 0 & 0 & 1 \\ b & 1 & 0 & 1 & 0 & 1 \\ c & 0 & 1 & 2 & 0 & 1 \\ d & 0 & 0 & 0 & 0 & 2 \\ e & 1 & 1 & 1 & 2 & 0 \end{array}$$

Def The <u>adjacency matrix</u> of G is an $n\times n$ matrix, where $|V(G)|=n$ with rows + columns labeled by $V(G)$ (after fixing an order). The entry in position i,j records the number of edges between the i^{th} + j^{th} vertices. (We count a loop as 2 edges)

Because adjacency is a symmetric condition, this matrix will be symmetric.

Ex



$$A = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 0 & 1 & 0 \\ b & 1 & 0 & 1 & 0 & 1 \\ c & 0 & 1 & 0 & 1 & 1 \\ d & 1 & 0 & 1 & 0 & 0 \\ e & 0 & 1 & 1 & 0 & 0 \end{array}$$

What will powers of A tell us?

Ex

$$A^2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Idea:

$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 0 & 1 & 0 \end{array} \quad \begin{array}{c|c} & c \\ \hline a & 0 \\ b & 1 \\ c & 0 \\ d & 1 \\ e & 1 \end{array} = 0\cdot 0 + 1\cdot 1 + 0\cdot 0 + 1\cdot 1 + 0\cdot 1 = 2$$

Theorem If A is the adjacency matrix of a simple graph, the ij^{th} entry of A^n is equal to the number of paths from vertex i to vertex j of length n, for $n \in \mathbb{Z}_{>0}$

Ex

$$A^2 = \begin{pmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

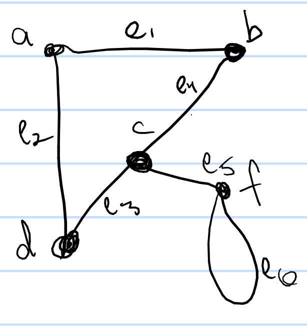
$$A^4 = A^2 \cdot A^2 = \begin{pmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 9 & 3 & 11 & 1 & 6 \\ 3 & 15 & 7 & 11 & 8 \\ 11 & 7 & 15 & 3 & 8 \\ 1 & 11 & 3 & 9 & 6 \\ 6 & 8 & 8 & 6 & 8 \end{pmatrix} \end{array}$$

$\Rightarrow$ there are 6 paths of length 4 from d to e.

Def The <u>incidence matrix</u> of G has its rows labeled by $V(G)$ + cols by $E(G)$. We store a 1 in row v and col e if e is incident to v and 0 otherwise.

Ex

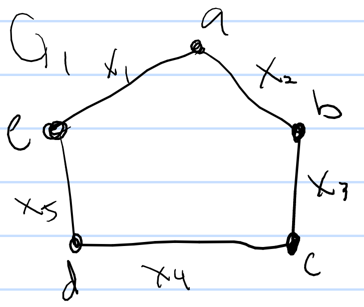


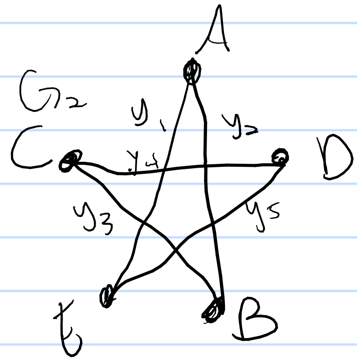
$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

8.6 Isomorphisms of Graphs

Idea: We can draw the graph G defined by $V(G) = \{a, b, c, d, e\}$ & $E(G) = \{(a,b), (b,c), (c,d), (d,e), (a,e)\}$ many ways

Ex)





we want to think of these as the "same" graph

Def Graphs G_1 and G_2 are <u>isomorphic</u> if there is a one-to-one, onto function $f: V(G_1) \to V(G_2)$ and a one-to-one & onto function $g: E(G_1) \to E(G_2)$ such that

$$e = (v, w) \text{ for } v, w \in V(G_1) \iff g(e) = (f(v), f(w))$$

We call this pair of functions f, g an <u>isomorphism</u> of G_1 onto G_2.

Ex) For the above G_1, G_2, for f, g defined by $f(a) = A$, $f(b) = B$, $\ldots$, $f(e) = E$ and

$$g(x_i) = y_i \quad \text{for } i \in \{1, 2, \ldots, 5\}$$

give an isomorphism of G_1 onto G_2.

Note: We can define a relation R on the set of graphs where $G_1 R G_2$ when G_1 & G_2 are isomorphic.
This is an equivalence relation.

$\boxed{\text{Thm}}$ Graphs G_1 + G_2 are isomorphic $\iff$

for some ordering of their vertices, their adjacency matrices are equal.

$\boxed{\text{Corollary}}$ Let G_1, G_2 be simple graphs. The following are equivalent:

1) G_1 + G_2 are isomorphic
2) There is a 1-1 and onto function $f: V(G_1) \to V(G_2)$ such that v, w are adjacent in $G_1 \iff f(v), f(w)$ are adjacent in G_2.

EX) For running examples,

$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 0 & 0 & 1 \\ b & 1 & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 0 & 1 & 0 \\ d & 0 & 0 & 1 & 0 & 1 \\ e & 1 & 0 & 0 & 1 & 0 \end{array} \qquad \begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 0 & 1 & 0 & 0 & 1 \\ B & 1 & 0 & 1 & 0 & 0 \\ C & 0 & 1 & 0 & 1 & 0 \\ D & 0 & 0 & 1 & 0 & 1 \\ E & 1 & 0 & 0 & 1 & 0 \end{array}$$

This again shows G_1 + G_2 are isomorphic.

Q: How can we show simple graphs G_1 + G_2 are <u>not</u> isomorphic?

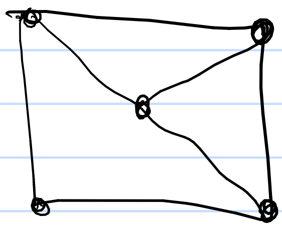
$\boxed{\text{Def}}$ A property of graphs that is preserved under isomorphism is called an <u>invariant</u>.
This means a property P is an invariant when G_1 has property $P \Rightarrow G \in [G_1]$ has property P

<u>Idea</u>: use invariants to detect when graphs are not isomorphic

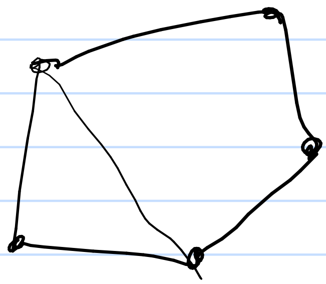
EX G has n vertices,

G has m edges are invariants.

EX



G_1



G_2

$|E(G_1)| = 7$

$|E(G_2)| = 6$ so

G_1 can not be isomorphic to G_2

EX Suppose $k \in \mathbb{Z}_{\geq 0}$. Then the property of having a vertex of degree k is an invariant.

Pf

Suppose G_1, G_2 are isomorphic in terms of $f: V(G_1) \to V(G_2)$ + $g: E(G_1) \to E(G_2)$.

Suppose $x \in V(G_1)$ s.t. $\delta(x) = k$. Let $\{e_1, e_2, \dots, e_k\} \subseteq E(G_1)$ be the edges incident to x.

Then $f(x)$ is incident to $\{g(e_1), g(e_2), \dots, g(e_k)\} \Rightarrow \delta(f(x)) \geq k$

These are distinct since g is injective.

We will show this is all edges incident to $f(x)$.

Consider $e \in E(G_2)$ incident to $f(x)$.

Then by surjectivity of g, there is some $e' \in E(G_1)$ where $g(e') = e$.

By the def of f and g, we know, e' must be incident to x. Therefore $e' \in \{e_1, \dots, e_k\}$.

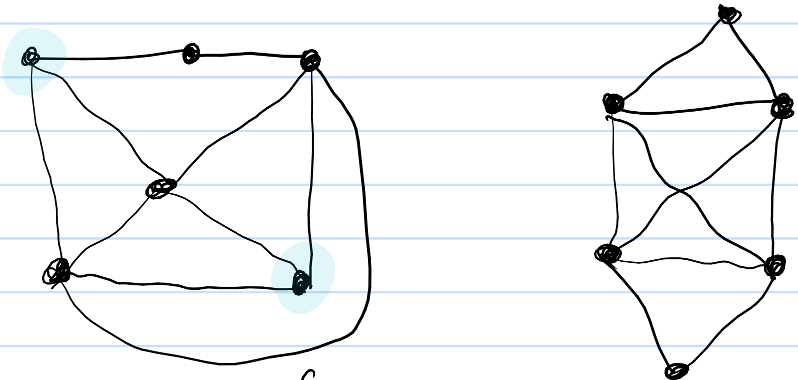
$\Rightarrow e \in \{f(e_1), \dots, f(e_k)\}$

$\Rightarrow \delta(f(x)) = k$, so we are done

Ex) Using this theorem,

$|V(G_1)| = |V(G_2)|$

$|E(G_1)| = |E(G_2)|$



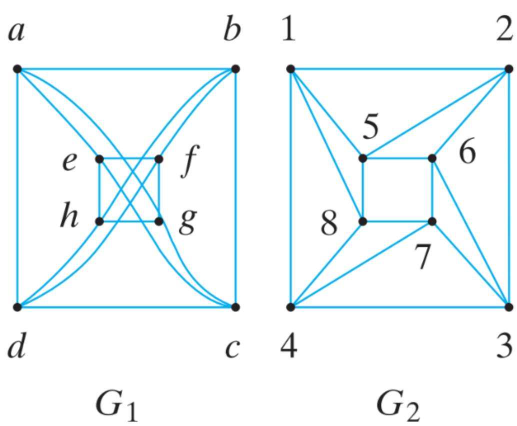
G_1 G_2

G_1 has 2 vertices of degree 3 + G_2 has none. This tells us G_1, G_2 are not isomorphic

Another property:

Proposition: The property of having a simple cycle of length k is an invariant.

Ex)



G_2 has simple cycles of length 3, but G_1 does not.

Therefore G_1 + G_2 are not isomorphic.