Math 170E: Homework 6

Due: Fri. 24th Feb by 11:59pm PDT via Gradescope

Submit answers to all problems via Gradescope. The reader will grade three problems each out of five points. Up to five further points will be awarded based on the proportion of the remaining problems that are completed.

Please make sure that your submission is readable. If your pencil is too faint, get a thicker one. If your handwriting is cramped and small, write bigger and use more paper. Please use simple plain paper or lined paper (e.g. please avoid graph paper etc.). It is your responsibility to ensure that your submission is readable. If we cannot read a solution, we may refuse to grade it. Thank you!

I encourage you to discuss and work on problems with other students in the class. Nevertheless, the solutions you present have to be your own. In particular, if the solution you present is identical to someone else's, or it is identical to some other resource (book, online, etc.), this will be considered cheating.

1. Given $X \sim \mathcal{N}(6, 25)$, find
   (a) $\mathbb{P}(6 \leq X \leq 12)$
   (b) $\mathbb{P}(0 \leq X \leq 8)$
   (c) $\mathbb{P}(|X - 6| \leq 10)$

2. Let $X \sim \mathcal{N}(\mu, \sigma^2)$.
   (a) For any $k > 0$, show that $\mathbb{P}(|X - \mu| > k\sigma) = 2 - 2\Phi(k)$.
   (b) Using part (a), verify the $68 - 95 - 99.7\%$ rule for the normal distribution.

3. Let $Z \sim \mathcal{N}(0, 1)$ be a standard normal random variables and let $Y = Z^2$.
   (a) Find the CDF of Y.
   (b) Using your result to part (a), find the PDF of Y.
   (c) By comparing your answer to part (a) with the PDF of the $\chi^2(1)$ distribution, find the value of $\Gamma(\frac{1}{2})$.

4. A candy maker produces mints that have a label weight of 20.4 grams. Assume that the distribution of the weights of these mints is $\mathcal{N}(21.37, 0.16)$.
   (a) Let X denote the weight of a single mint selected at random from the production line. Find $\mathbb{P}(X > 22.07)$.

(b) Suppose that 15 mints are selected independently and weighed. Let Y equal the number of these mints that weigh less than 20.857 grams. Find $\mathbb{P}(Y \leq 2)$.

5. Roll a pair of four-sided dice, one red and one black, each of which has possible outcomes $1, 2, 3, 4$ that have equal probabilities. Let X equal the outcome on the red die, and let Y equal the outcome on the black die.
   (a) Find the joint PMF of X and Y.
   (b) What are the marginal PMFs of X and Y?
   (c) Are X and Y independent?

6. Let the joint PMF of X and Y be

$$p_{X,Y}(x, y) = \frac{x + y}{32}, \quad x = 1, 2, \quad y = 1, 2, 3, 4.$$

   (a) Find the marginal PMF of X.
   (b) Find the marginal PMF of Y.
   (c) Find $\mathbb{P}(X > Y)$.
   (d) Find $\mathbb{P}(Y = 2X)$.
   (e) Find $\mathbb{P}(X \leq 3 - Y)$.
   (f) Are X and Y independent?

7. Roll a pair of four-sided die, one red and one black. Let X equal the outcome on the red die and let Y the sum of the two die.
   (a) What values do X and Y take?
   (b) What is the joint PMF of X and Y?
   (c) What are the marginals of X and Y?
   (d) Are X and Y independent?

8. You repeatedly draw a number at random from the numbers $1, 2, ..., 10$. Let X be the number of draws until the number 1 appears and Y the number of draws until the number 10 appears.
   (a) What is the joint probability mass function of X and Y?
   (b) Verify your answer to part (a) by checking your result is a genuine joint PMF. You may use the Geometric series formula:

$$\sum_{x=N}^{\infty} a^x = \frac{a^N}{1 - a}, \quad |a| < 1.$$