6.1-2-Planar Systems

#UCLA #Y1Q3 #Math33B

Planar Systems

Key Definitions

Characteristic polynomial - the determinant of the matrix, A, minus the identity, I, multiplied by λ

Planar Systems - usually homogenous linear systems with constant coefficients solved using linear algebra, specifically in 2×2 matrices below

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Problem

Given a linear system:

$$\vec{x}' = A\vec{x} \quad \text{AND} \quad A = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

With possible IVP:

$$\vec{x}(T_0) = \begin{bmatrix} A \\ B \end{bmatrix}$$

Steps

1. Find Eigenvalues using characteristic polynomial
2. Find Eigenvectors using $null(A - \lambda I_n)$
3. General Solution:

$$x(t; c_1, c_2) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

4. Plug in IVP and solve augmented matrix using general solution

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General Solutions

Distinct Real Roots

Different real Eigenvalues

$$x(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

Complex Conjugate Roots

Complex Eigenvalues

$$\lambda = a + bi$$

$$\vec{w} = \vec{v}_1 + \vec{v}_2 i$$

$$\bar{\lambda} = a - bi$$

$$\bar{\vec{w}} = \vec{v}_1 - \vec{v}_2 i$$

Complex Version

$$x(t) = c_1 e^{\lambda t}\vec{w} + c_2 e^{\bar{\lambda} t}\bar{\vec{w}}$$

Real Version

$$x(t) = c_1 e^{at}(\vec{v}_1 \cos bt - \vec{v}_2 \sin bt) + c_2 e^{at}(\vec{v}_1 \sin bt + \vec{v}_2 \cos bt)$$

Double Real Roots

One Eigenvalue

Easy Case

2 linearly independent Eigenvectors
Same as Distinct Real Roots case:

$$x(t) = C_1 e^{\lambda t}\vec{v}_1 + C_2 e^{\lambda t}\vec{v}_2$$

Hard Case

1 Eigenvector
Find $\vec{v}_2$ by setting up augmented matrix:

$$\vec{v}_2 = [A - \lambda I \quad | \quad \vec{v}_1]$$

Then solution is given by:

$$x(t) = C_1 e^{\lambda t}\vec{v}_1 + C_2 e^{\lambda t}(\vec{v}_2 + t\vec{v}_1)$$