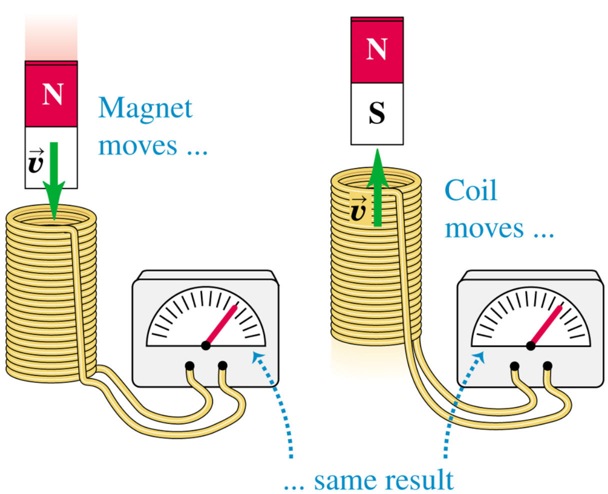
Chapter 37: Relativity

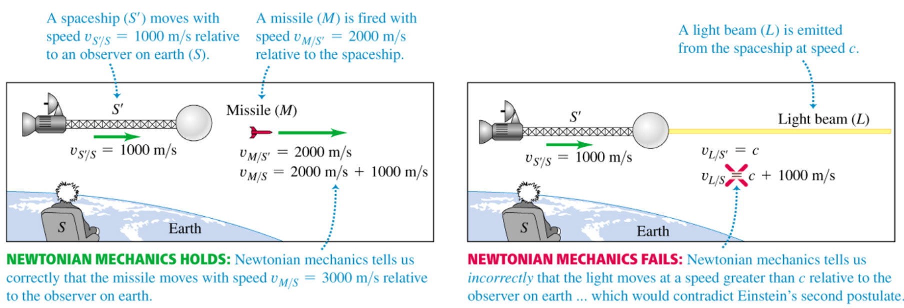
Einstein's First Postulate

- Einstein's first postulate, known as the **principle of relativity,** states that **the laws of physics are the same in every inertial reference frame.**
- For example, the same emf is induced in the coil whether the magnet moves relative to the coil, or the coil moves relative to the magnet.



Einstein's Second Postulate

- Einstein's second postulate is that **the speed of light in vacuum is the same in all inertial frames of reference and is independent of the motion of the source.**
- Suppose two observers measure the speed of light in vacuum. One is at rest with respect to the light source, and the other is moving away from it.
- According to the principle of relativity, the two observers must obtain the same result, despite the fact that one is moving with respect to the other.
- The second postulate immediately implies that it is impossible for an inertial observer to travel at the speed of light in vacuum.



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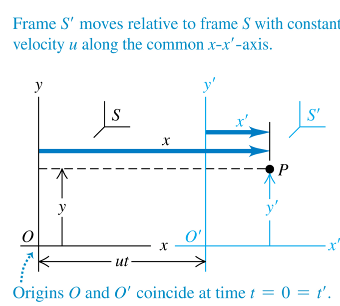
The Galilean Transformation (1 of 2)

- The **Galilean transformation** is a coordinate transformation between two inertial frames of reference.
- Suppose we have two frames S and S', with S' moving at velocity u in the x-direction relative to S.
- In the figure, and the equations below, the position of particle P is described in two frames of reference:



$$x = x' + ut, \quad y = y', \quad z = z', \quad \text{(Galilean coordinate transformation)}.$$

- If particle P moves in the x-direction, then the velocities v_x and v'_x in S and S' are related by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x'}{\mathrm{d}t} + u \quad \rightarrow \quad v_x = v'_x + u.$$

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The Galilean Transformation (2 of 2)

- In relativity this is not exactly correct, and velocities do not add in a linear manner.
- The Galilean transformation rule for adding velocities implies that if $v_x = c$, then
$$c = c' + u.$$
- But Einstein's second postulate says that $c = c'$ since the speed of light must be c in all inertial frames of reference. That means that the relationship $v_x = v'_x + u$ *cannot* be correct.

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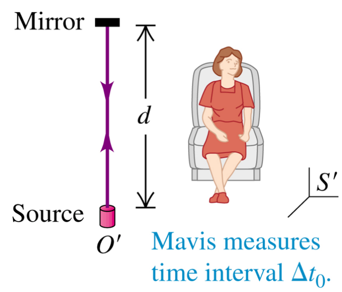
Relativity of Time Intervals (1 of 5)

- In special relativity, the time interval between two events Δt, as measured in one frame of reference S, will be different from the time interval Δt_0 for the same two events as measured in another frame of reference S' that is moving with respect to S.
- Consider a thought experiment, where Mavis, in S', measures the time interval between two events.



- Event 1 is when a flash of light from a light source leaves O'. Event 2 is when the flash returns to O', having been reflected from a mirror a distance d away.
- The flash of light moves a total distance $2d$, so the time interval is
$$\Delta t_0 = \frac{2d}{c}.$$

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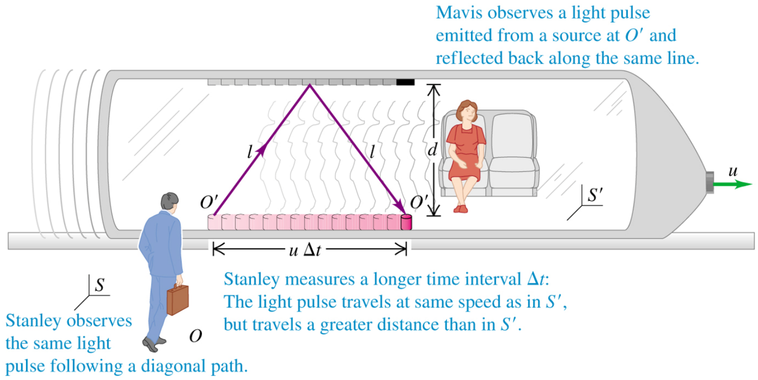
Relativity of Time Intervals (2 of 5)

- The round-trip time measured by Stanley in frame S is a longer interval Δt.
- In his frame of reference, the two events occur at different points in space.
- Instead, the time he measures is

$$\Delta t = \frac{2l}{c} = \frac{2}{c}\sqrt{d^2 + \left(\frac{u\Delta t}{2}\right)^2} = \frac{2}{c}\sqrt{\left(\frac{c\Delta t_0}{2}\right)^2 + \left(\frac{u\Delta t}{2}\right)^2}.$$



Relativity of Time Intervals (3 of 5)

- We can use the fact that $\Delta t_0 = 2d/c$ in order to get Δt in terms of Δt_0:

$$\Delta t = \frac{2}{c}\sqrt{\left(\frac{c\Delta t_0}{2}\right)^2 + \left(\frac{u\Delta t}{2}\right)^2}$$

$$(\Delta t)^2 = (\Delta t_0)^2 + \frac{u^2}{c^2}(\Delta t)^2$$

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}}.$$

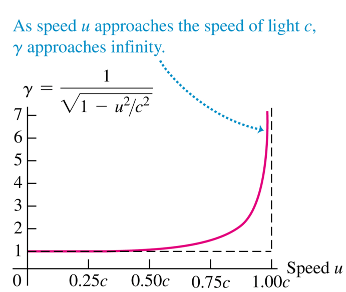
- Since the quantity $\sqrt{1-u^2/c^2}$ is less than 1, Δt is greater than Δt_0. So Stanley measures a longer round-trip time for the light pulse than Mavis does.
- This result generalizes for any two frames S and S', with S' moving with velocity u relative to S.

Relativity of Time Intervals (4 of 5)

- The **proper time** Δt_0 is defined as the time interval between two events that occur at the same point.
- Suppose Δt_0 is measured in frame S'. An observer in frame S will measure the time interval to be Δt:
$$\Delta t = \gamma \Delta t_0.$$
- The factor γ is called the **Lorentz factor:**
$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}.$$
- This effect is known as **time dilation.** If observers in S' measure the ticks on a clock in their frame to be $\Delta t_0 = 1$ s, observers in S will measure a time between the ticks on a clock in S' to be $\Delta t > \Delta t_0$.
- The result is that **observers measure any clock to run slow if it moves relative to them.**

Relativity of Time Intervals (5 of 5)

- When u is very small compared to c, γ is very nearly equal to 1, and the effects of time dilation are not noticeable. In this limit, Newtonian mechanics is a very good approximation.
- If the relative speed u is great enough that γ is appreciably greater than 1, the speed is said to be relativistic.



- The Lorentz factor γ shows up in many other expressions in special relativity, such as in energy and momentum.

Example 37.1: Time Dilation at $0.990c$

High-energy subatomic particles coming from space interact with atoms in Earth's upper atmosphere, in some cases producing unstable particles called *muons.* A muon decays into other particles with a mean lifetime of $\tau = 2.20\ \mu\text{s} = 2.2 \times 10^{-6}$ s, as measured in the rest frame of the muon. If a muon is moving at $0.990c$ relative to the Earth, what will an observer on Earth measure its mean lifetime to be?

In the frame of the Earth, the mean lifetime of a muon moving at $0.990c$ is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}} = \frac{2.20\ \mu\text{s}}{\sqrt{1-(0.990)^2}} = 15.6\ \mu\text{s}.$$

This is about seven times longer than the mean lifetime in the rest frame of a muon.

Example 37.2: Time Dilation at Airliner Speeds

An airplane flies from San Francisco to New York, which is about $d = 4800$ km away, at a steady speed of $u = 300$ m/s. How much time does the trip take, as measured by an observer on the ground? By an observer in the plane?

As measured in a frame S on the ground, the time interval is

$$\Delta t = \frac{d}{u} = \frac{4.80 \times 10^6\ \text{m}}{300\ \text{m/s}} = 1.60 \times 10^4\ \text{s},$$

which is about 4.5 hours.

In the plane's frame S', San Francisco and New York pass under the plane at the same point, so the time interval in the plane is the proper time Δt_0. We have

$$\frac{u^2}{c^2} = \frac{(300\ \text{m/s})^2}{(3.00 \times 10^8\ \text{m/s})^2} = 1.00 \times 10^{-12}.$$

Therefore, the proper time is

$$\Delta t_0 = (1.60 \times 10^4\ \text{s})\sqrt{1 - 1.00 \times 10^{-12}}$$

The factor inside of the square root is very nearly 1, to the point where we need to use a binomial expansion of the square root to find the first-order deviation from 1:

$$\sqrt{1 - 1.00 \times 10^{-12}} = 1 - \frac{1}{2}(1.00 \times 10^{-12}) + \cdots .$$

Then to first-order, the proper time is

$$\Delta t_0 = (1.60 \times 10^4\ \text{s})(1 - 0.50 \times 10^{-12}),$$

so the difference between the time measured on the ground and the time measured on the plane is

$$\Delta t - \Delta t_0 = (1.60 \times 10^4\ \text{s})(0.50 \times 10^{-12}) = 8.0 \times 10^{-9}\ \text{s} = 8\ \text{ns}.$$

This difference is so small that it is not at all noticeable in everyday life. But this effect has actually been verified with atomic clocks that can attain a precision of one part in 10^{13}!

Example 37.3: Just When is it Proper?

Mavis boards a spaceship and then zips past Stanley on Earth at a relative speed of $0.600c$. At the instant she passes him, they both start timers. (a) A short time later Stanley measures that Mavis has traveled 9.00×10^7 m beyond him and is passing a space station. What does Stanley's timer read as she passes the space station? (b) Stanley starts to blink just as Mavis flies past him, and Mavis measures that the blink takes 0.400 s from beginning to end. According to Stanley, what is the duration of his blink?

(a) The two events of Mavis passing the earth and Mavis passing the space station occur at different positions in Stanley's frame, but at the same position in Mavis's frame. Stanley Measures time interval Δt, while Mavis measures the proper time Δt_0. As measured by Stanley, Mavis moves at

$$u = 0.600c = 0.600(3.00 \times 10^8 \text{ m/s}) = 1.80 \times 10^8 \text{ m/s},$$

and travels 9.00×10^7 m in time

$$\Delta t = \frac{9.00 \times 10^7 \text{ m}}{1.80 \times 10^8 \text{ m/s}} = 0.500 \text{ s}.$$

Meanwhile, Mavis's timer reads an elapsed time of

$$\Delta t_0 = \Delta t\sqrt{1 - u^2/c^2} = (0.500 \text{ s})\sqrt{1 - (0.600)^2} = 0.400 \text{ s}.$$

Since Stanley measures 0.500 s on his timer, but only 0.400 s have passed on Mavis's timer, Stanley concludes that her timer runs slow.

(b) Since 0.400 s corresponds to the time it takes for Stanley to blink in Mavis's frame, and Stanley is moving with respect to Mavis in the opposite direction at $0.600c$, the proper time for Stanley blinking is

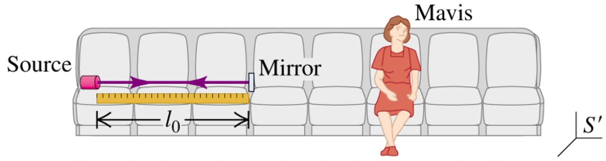
$$\Delta t_0 = \Delta t\sqrt{1 - u^2/c^2} = (0.400 \text{ s})\sqrt{1 - (0.600)^2} = 0.320 \text{ s}.$$

Since Mavis measures 0.400 s on her timer for the duration of Stanley's blink, but Stanley measures 0.320 s on his timer, Mavis concludes that Stanley's timer runs slow.

Relativity of Length (1 of 5)

- In addition to time intervals changing between frames, lengths also change in relativity.
- Consider the same thought experiment as before, but now we attach a light source to one end of a ruler and a mirror to the other end.
- The ruler is at rest in reference frame S', and its length in this frame is l_0.
- The time required for a light pulse to make the round trip from the source to mirror and back is

$$\Delta t_0 = \frac{2l_0}{c}.$$

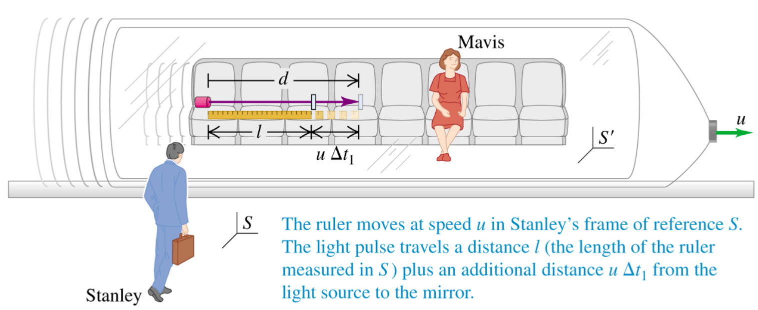


The ruler is stationary in Mavis's frame of reference S'. The light pulse travels a distance l_0 from the light source to the mirror.

Relativity of Length (2 of 5)

- In reference frame S, Stanley sees the ruler is moving to the right with speed u.
- The length of the ruler in S is l, and the time of travel is measured to be Δt_1. The total length of path d from source to mirror is not l, but is instead

$$d = l + u\Delta t_1.$$



The ruler moves at speed u in Stanley's frame of reference S. The light pulse travels a distance l (the length of the ruler measured in S) plus an additional distance $u\,\Delta t_1$ from the light source to the mirror.

Relativity of Length (3 of 5)

- The light pulse travels at speed c, so we also have

$$d = c\Delta t_1 \quad \rightarrow \quad c\Delta t_1 = l + u\Delta t_1 \quad \rightarrow \quad \Delta t_1 = \frac{l}{c-u}.$$

- On the return trip, the time Δt_2 it takes for the light to be reflected back to the source is

$$\Delta t_2 = \frac{l}{c+u}.$$

- Thus, the total time in Stanley's frame is $\Delta t = \Delta t_1 + \Delta t_2$, giving us

$$\Delta t = \frac{l}{c-u} + \frac{l}{c+u} = \frac{2l}{c(1-u^2/c^2)}.$$

- Since $\Delta t = \gamma \Delta t_0$, we also have

$$\Delta t = \frac{2l_0}{c\sqrt{1-u^2/c^2}}.$$

Relativity of Length (4 of 5)

- Setting the two expressions for Δt equal to each other gives us

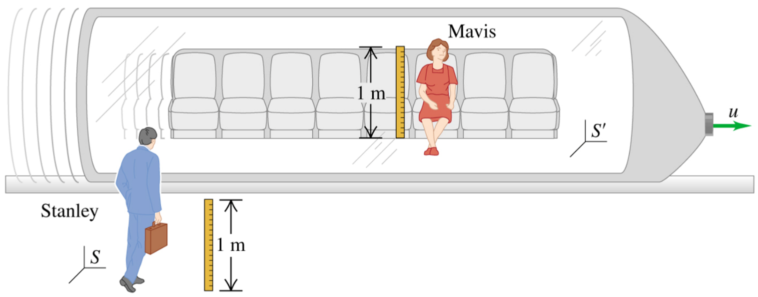
$$\begin{aligned} \frac{2l}{c(1-u^2/c^2)} &= \frac{2l_0}{c\sqrt{1-u^2/c^2}} \\ l &= l_0 \frac{1-u^2/c^2}{\sqrt{1-u^2/c^2}} \\ l &= l_0\sqrt{1-u^2/c^2}. \end{aligned}$$

- A length measured in the frame in which the object is at rest (the rest frame of the object) is called a **proper length.** Thus, l_0 is a proper length in S', and the length measured in any other frame moving relative to S is less than l_0.
- This effect is called **length contraction:**

$$l = l_0\sqrt{1-u^2/c^2} = \frac{l_0}{\gamma}.$$

Relativity of Length (5 of 5)

- Length contraction applies only to lengths that are parallel to the direction of u. Lengths perpendicular to u remain the same between S and S'.
- If Mavis were to measure a height perpendicular to u in her frame S', Stanley would measure that same height in his frame S.



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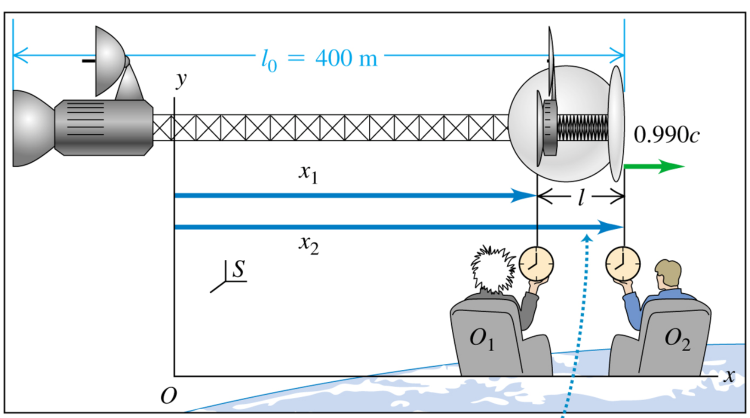
Example 37.4: How Long is the Spaceship?

A spaceship flies past Earth at a speed of $0.990c$. A crewmember on the ship measures its length, obtaining the value of 400 m. What length do observers measure on Earth?



The two observers on earth (S) must measure x_2 and x_1 simultaneously to obtain the correct length $l = x_2 - x_1$ in their frame of reference.

The length of the ship in the Earth's frame is

$$l = l_0\sqrt{1 - u^2/c^2} = (400 \text{ m})\sqrt{1 - (0.990)^2} = 56.4 \text{ m},$$

which, as expected, is shorter than the length as measured in the ship's frame of reference.

Example 37.5: How Far Apart are the Observers?

Observers O_1 and O_2 in the previous figure are 56.4 m apart on the Earth. How far apart does the spaceship crew measure them to be?

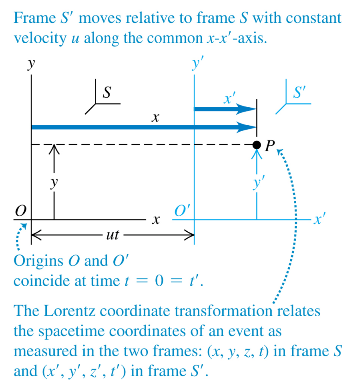
Again we apply the formula for length contraction to get

$$l = l_0\sqrt{1 - u^2/c^2} = (56.4 \text{ m})\sqrt{1 - (0.990)^2} = 7.96 \text{ m}.$$

So the crew on the ship sees the observers to be closer than in the Earth's reference frame.

The Lorentz Transformations (1 of 3)

- The Galilean transformation, as we have seen, is valid only in the limit when u approaches zero.
- The more general relationships are called the **Lorentz transformations.**
- These transformations relate the coordinates (x, y, z) at time t and (x', y', z') at time t' in two inertial frames S and S'. They are more general than the Galilean transformations and are consistent with the principle of relativity.



- To derive the transformations, we assume that the origins coincide at initial time $t = 0 = t'$. In frame S, the distance from the origin O to the origin O' of S' at time t is ut.

The Lorentz Transformations (2 of 3)

- Since the coordinate x' is a proper length in S', it is contracted by the factor $1/\gamma$. Then the coordinate x' in terms of x is

$$x = ut + \frac{x'}{\gamma} = ut + x'\sqrt{1 - u^2/c^2} \quad \rightarrow \quad x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}}.$$

- Similarly, since x is a proper length in S, we also have that

$$x' = -ut' + x\sqrt{1 - u^2/c^2}.$$

- We can use this to obtain the corresponding transformation for t':

$$\frac{x - ut}{\sqrt{1 - u^2/c^2}} = -ut' + x\sqrt{1 - u^2/c^2} \quad \rightarrow \quad t' = \frac{t - xu/c^2}{\sqrt{1 - u^2/c^2}}.$$

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The Lorentz Transformations (3 of 3)

- Since lengths perpendicular to the direction of relative motion are not affected by length contraction, $y' = y$ and $z' = z$. The full set of the coordinate transformations is

$$\begin{aligned} x' &= \frac{x - ut}{\sqrt{1 - u^2/c^2}} = \gamma(x - ut), \\ y' &= y, \\ z' &= z, \\ t' &= \frac{t - xu/c^2}{\sqrt{1 - u^2/c^2}} = \gamma(t - ux/c^2). \end{aligned}$$

- This generalizes the results of the Galilean transformation, and we can obtain the Galilean results in the limit that $u \ll c$, and hence $\gamma \to 1$.
- The Lorentz transformations are fundamentally different from the Galilean transformation and tell us that space and time are no longer separate, and we now refer to time and the three spatial dimensions as a four-dimensional entity called spacetime, with the coordinates of an event denoted by (x, y, z, t).

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The Lorentz Velocity Transformation (1 of 2)

- We can use the results of the Lorentz transformations to derive the generalization of the Galilean velocity transformation by taking the differentials of x' and t':

$$\begin{aligned} \mathrm{d}x' &= \gamma(\mathrm{d}x - u\,\mathrm{d}t) \\ \mathrm{d}t' &= \gamma(\mathrm{d}t - u\,\mathrm{d}x\,/c^2) \end{aligned} \quad \rightarrow \quad \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\frac{\mathrm{d}x}{\mathrm{d}t} - u}{1 - \frac{u}{c^2}\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

- The velocity v'_x in S' is therefore

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2}.$$

- This reduces to the expression for the Galilean transformation of velocities in the appropriate limit. When $v_x \ll c$ and $u \ll c$, we get $v'_x = v_x - u$.

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The Lorentz Velocity Transformation (2 of 2)

- On the other hand, if $v_x = c$, we obtain

$$v'_x = \frac{c - u}{1 - uc/c^2} = \frac{c(1 - u/c)}{1 - u/c} = c,$$

which says that anything moving with velocity $v = c$ as measured in S also has velocity $v'_x = c$ in S', consistent with Einstein's second postulate.

- Since there is no fundamental distinction between the frames S and S', we can simply reverse the roles of v_x and v'_x and switch the sign on u to get v_x in terms of v'_x:

$$v_x = \frac{v'_x + u}{1 + uv'_x/c^2}.$$

- These results tell us that an object moving with a speed less than c in one frame of reference therefore always has a speed less than c in **every other** frame of reference.

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Example 37.6: Was It Received Before It Was Sent?

Winning an interstellar race, Mavis pilots her spaceship across a finish line in space at a speed of $0.600c$ relative to that line. A "hooray" message is sent from the back of her ship (event 2) at the instant (in her frame of reference) that the front of her ship crosses the line (event 1). She measures the length of her ship to be 300 m. Stanley is at the finish line and is at rest relative to it. When and where does he measure events 1 and 2 to occur?

To set this up, we will take the origin in Mavis's frame S' to be the front of the spaceship, and for Stanley's frame S to be the finish line. We will also set $t = 0 = t'$ to be the time at which Mavis crosses the finish line, so that $x = 0 = x'$ at $t = 0 = t'$. Then for Mavis and Stanley, event 1 takes place at $(x = 0, t = 0)$ in S and $(x' = 0, t' = 0)$ in S'. Since event 2 takes place at the back of Mavis's ship, the spacetime coordinates for event 2 in S' are $(x' = -300 \text{ m}, t' = 0)$.

The Lorentz transformations can be used to determine the spacetime coordinates of event 2 in Stanley's frame S, as we have

$$x = \gamma(x' + ut'), \quad t = \gamma(t' + ux'/c^2).$$

For $u = 0.600c = 1.80 \times 10^8$ m/s, the Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1 - (0.600)^2}} = 1.25.$$

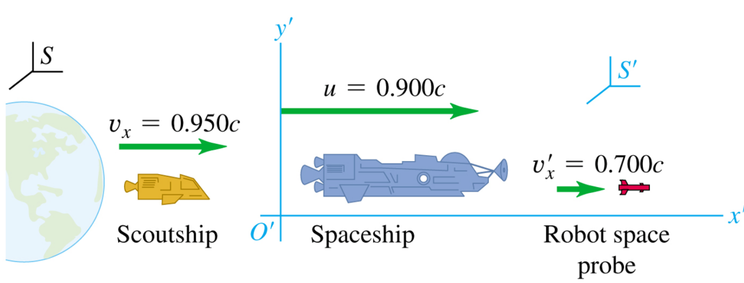
Therefore,

$$x = (1.25)(-300 \text{ m}) = -375 \text{ m}, \quad t = (1.25)\frac{(1.80 \times 10^8 \text{ m/s})(-300 \text{ m})}{(3.00 \times 10^8 \text{ m/s})^2} = -7.50 \times 10^{-7} \text{ s} = -0.750 \text{ μs}.$$

Thus, the spacetime coordinates for event 2 in S are $(x = -375 \text{ m}, t = -0.750 \text{ μs})$.

Example 37.7: Relative Velocities

(a) A spaceship moving away from the Earth at $0.900c$ fires a robot space probe in the same direction as its motion at $0.700c$ relative to the space ship. What is the probe's velocity relative to the Earth? (b) A scoutship is sent to catch up with the space ship by traveling at $0.950c$ relative to the Earth. What is the velocity of the scoutship relative to the spaceship?



(a) The probe's velocity relative to the Earth is

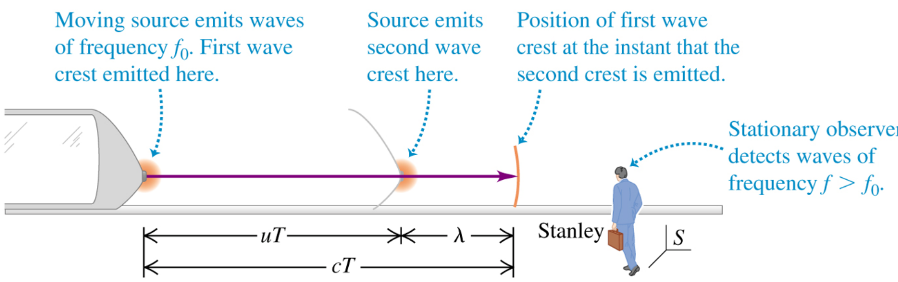
$$v_x = \frac{v'_x + u}{1 + uv'_x/c^2} = \frac{0.700c + 0.900c}{1 + (0.900c)(0.700c)/c^2} = 0.982c.$$

(b) Relative to the spaceship, the scoutship's velocity is

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} = \frac{0.950c - 0.900c}{1 - (0.900c)(0.950c)/c^2} = 0.345c.$$

The Doppler Effect for Electromagnetic Waves (1 of 3)

- When a source moves toward the observer, the observed frequency f is greater than the emitted frequency f_0.
- A source of light is moving with constant speed u toward Stanley, who is stationary in an inertial frame. As measured in its rest frame, the source emits light waves with frequency f_0 and period $T_0 = 1/f_0$.



The Doppler Effect for Electromagnetic Waves (2 of 3)

- In Stanley's frame, T is the time interval between the emission of successive wave crests from the source. The crests ahead of the source move a distance cT, and the source moves a distance uT in the same direction.
- The distance between successive crests—the wavelength—is thus $\lambda = (c - u)T$, as measured in Stanley's frame. He measures the frequency c/λ.
- Therefore, the frequency he observes is

$$f = \frac{c}{(c-u)T}.$$

- Because T_0 is a proper time in the rest frame of the source, T_0 and T are related by

$$T = \frac{T_0}{\sqrt{1 - u^2/c^2}} = \frac{cT_0}{\sqrt{c^2 - u^2}}.$$

The Doppler Effect for Electromagnetic Waves (3 of 3)

- Substituting T into our expression for f, we get

$$f = \frac{c}{c-u}\frac{\sqrt{c^2-u^2}}{c}f_0 = \sqrt{\frac{c+u}{c-u}}f_0, \quad \text{(source approaching observer)}.$$

- This shows that when the source moves *toward* the observer, the observed frequency f is *greater* than the emitted frequency f_0.
- The difference $f - f_0 = \Delta f$ is called the Doppler frequency shift. When u/c is much smaller than 1, the fractional shift $\Delta f/f$ is small and is approximately equal to u/c:

$$\frac{\Delta f}{f} = \frac{u}{c}.$$

- When the source moves *away* from the observer, we change the sign of u in the expression for f to get

$$f = \sqrt{\frac{c-u}{c+u}}f_0, \quad \text{(source moving away from observer)}.$$

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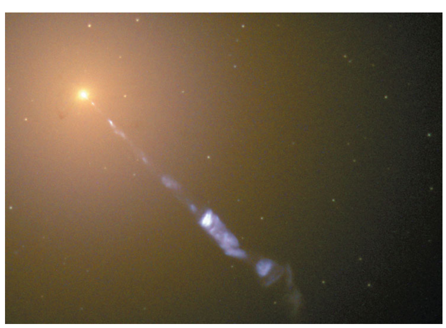
Example 37.8: A Jet from a Black Hole

Many galaxies have supermassive black holes at their centers. As material swirls around such a black hole, it is heated, becomes ionized, and generates strong magnetic fields. The resulting magnetic forces steer some of the material into high-speed jets that blast out of the galaxy and into intergalactic space. The light we observe from the jet in the figure has a frequency of 6.66×10^{14} Hz (in the far ultraviolet), but in the reference frame of the jet material the light has a frequency of 5.55×10^{13} Hz (in the infrared). What is the speed of the jet material with respect to us?



The frequency we observe is $f = 6.66 \times 10^{14}$ Hz, while the frequency in the frame of the sources is $f_0 = 5.55 \times 10^{13}$ Hz. Since $f > f_0$, the jet is approaching us, so we must use the expression

$$f = \sqrt{\frac{c+u}{c-u}}f_0.$$

We can solve for u by squaring both sides to get

$$\left(\frac{f}{f_0}\right)^2 = \frac{c+u}{c-u} \quad \rightarrow \quad u[(f/f_0)^2+1] = [(f/f_0)^2-1]c \quad \rightarrow \quad u = \frac{(f/f_0)^2-1}{(f/f_0)^2+1}c.$$

Since $f/f_0 = (6.66 \times 10^{14}$ Hz$)/(5.55 \times 10^{13}$ Hz$) = 12.0$, we obtain

$$u = \frac{(12.0)^2-1}{(12.0)^2+1}c = 0.986c,$$

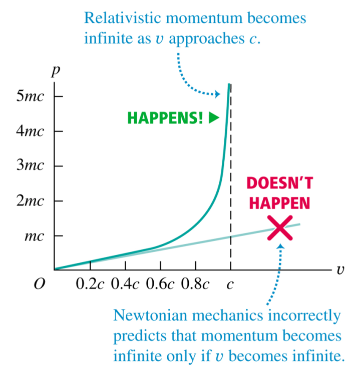
so 98.6% the speed of light.

Relativistic Momentum

- Conservation of momentum between frames S and S' still holds in relativity, but we have to modify the definition of momentum. The relativistic momentum is defined as

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}} = \gamma m\mathbf{v}.$$

- Notice that in the limit $v \ll c$, $\gamma \rightarrow 1$ and we retain $\mathbf{p} = m\mathbf{v}$ as in Newtonian mechanics.



- But we also have that as $v \rightarrow c$, the momentum approaches infinity, whereas the Newtonian definition of momentum is unbounded. This means that any object with mass m cannot reach the speed of light.

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Newton's Second Law in Relativity (1 of 3)

- In Newtonian mechanics, the second law is

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}.$$

- This still holds in relativity, provided that we use the relativistic form of momentum:

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}).$$

- If the net force $\mathbf{F}$ and $\mathbf{v}$ are both parallel to each other, then we have

$$\begin{aligned} F &= \gamma m\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mathrm{d}\gamma}{\mathrm{d}t}mv \\ &= \gamma m\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{v}{c^2(1 - v^2/c^2)^{3/2}}\frac{\mathrm{d}v}{\mathrm{d}t}mv \\ &= \gamma ma\left(\frac{1}{1 - v^2/c^2}\right) \\ &= \gamma^3 ma. \end{aligned}$$

Newton's Second Law in Relativity (2 of 3)

- Solving for the acceleration gives us

$$a = \frac{F}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2}.$$

- This again tells us that it is not possible for a massive object to reach the speed of light. As $v \to c$, the acceleration, regardless of the force F, approaches zero.
- For the case where $\mathbf{F}$ is perpendicular to $\mathbf{v}$, the speed remains constant since no work is done by $\mathbf{F}$, which means γ is also constant. Therefore,

$$F = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma mv) = \gamma m\frac{\mathrm{d}v}{\mathrm{d}t} = \gamma ma.$$

Newton's Second Law in Relativity (3 of 3)

- We can also obtain a more general expression for $\mathbf{F}$ for the case where $\mathbf{v}$ is in some arbitrary direction:

$$\begin{aligned}\mathbf{F} &= \gamma m \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \frac{\mathrm{d}\gamma}{\mathrm{d}t} m\mathbf{v} \\ &= \gamma m \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \left[\frac{\mathbf{v}}{c^2(1-v^2/c^2)^{3/2}} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right] m\mathbf{v} \\ &= \gamma m\mathbf{a} + \frac{\gamma^3 m}{c^2}\mathbf{v}(\mathbf{v}\cdot\mathbf{a}) \\ &= \gamma m\left[\mathbf{a} + \frac{\gamma^2}{c^2}\mathbf{v}(\mathbf{v}\cdot\mathbf{a})\right].\end{aligned}$$

- In the limit $v \ll c$, this reduces to the familiar Newtonian expression $\mathbf{F} = m\mathbf{a}$.

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Example 37.9: Relativistic Dynamics of an Electron

An electron (rest mass 9.11×10^{-31} kg, charge -1.60×10^{-19} C) is moving opposite to an electric field of magnitude $E = 5.00 \times 10^5$ N/C. All other forces are negligible in comparison to the electric-field force. (a) Find the magnitudes of momentum and of acceleration at the instants when $v = 0.010c$, $0.90c$, and $0.99c$. (b) Find the corresponding accelerations if a net force of the same magnitude is perpendicular to the velocity.

(a) For $v = 0.010c$, $0.90c$, and $0.99c$, we have the following values of γ:

$$\gamma(v = 0.010c) = 1.00, \quad \gamma(v = 0.90c) = 2.29, \quad \gamma(v = 0.99c) = 7.09$$

Then the values of the magnitude of the momentum $p = \gamma m v$ are

$$\begin{aligned}p_1 &= (1.00)(9.11 \times 10^{-31}\ \text{kg})(0.010)(3.00 \times 10^8\ \text{m/s}) = 2.7 \times 10^{-24}\ \text{kg}\cdot\text{m/s}, \\ p_2 &= (2.29)(9.11 \times 10^{-31}\ \text{kg})(0.90)(3.00 \times 10^8\ \text{m/s}) = 5.6 \times 10^{-22}\ \text{kg}\cdot\text{m/s}, \\ p_3 &= (7.09)(9.11 \times 10^{-31}\ \text{kg})(0.99)(3.00 \times 10^8\ \text{m/s}) = 1.9 \times 10^{-21}\ \text{kg}\cdot\text{m/s}.\end{aligned}$$

The magnitude of the force on the electron is

$$F = |q|E = (1.60 \times 10^{-19}\ \text{C})(5.00 \times 10^5\ \text{N/C}) = 8.00 \times 10^{-14}\ \text{N},$$

and since $a = F/\gamma^3 m$, then we have

$$\begin{aligned}a_1 &= \frac{8.00 \times 10^{-14}\ \text{N}}{(1.00)^3(9.11 \times 10^{-31}\ \text{kg})} = 8.8 \times 10^{16}\ \text{m/s}^2, \\ a_2 &= \frac{8.00 \times 10^{-14}\ \text{N}}{(2.29)^3(9.11 \times 10^{-31}\ \text{kg})} = 7.3 \times 10^{15}\ \text{m/s}^2,\end{aligned}$$

$$a_3 = \frac{8.00 \times 10^{-14}\ \text{N}}{(7.09)^3(9.11 \times 10^{-31}\ \text{kg})} = 2.5 \times 10^{14}\ \text{m/s}^2.$$

(b) If the force is perpendicular to the velocity, then $a = F/\gamma m$, and we instead get

$$a_1 = \frac{8.00 \times 10^{-14}\ \text{N}}{(1.00)(9.11 \times 10^{-31}\ \text{kg})} = 8.8 \times 10^{16}\ \text{m/s}^2,$$
$$a_2 = \frac{8.00 \times 10^{-14}\ \text{N}}{(2.29)(9.11 \times 10^{-31}\ \text{kg})} = 3.8 \times 10^{16}\ \text{m/s}^2,$$
$$a_3 = \frac{8.00 \times 10^{-14}\ \text{N}}{(7.09)(9.11 \times 10^{-31}\ \text{kg})} = 1.2 \times 10^{16}\ \text{m/s}^2.$$

Relativistic Work and Energy (1 of 4)

- We can use the work-energy theorem to derive an expression for the kinetic energy of a massive particle in special relativity. The work done on a particle of mass m under a force F from point x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F\,\mathrm{d}x = \int_{x_1}^{x_2} \frac{ma\,\mathrm{d}x}{(1 - v_x^2/c^2)^{3/2}}.$$

- To evaluate this expression, we rewrite it in terms of the speed v_x of the particle at positions x_1 and x_2. The work done is equal to the kinetic energy K of the particle if it is accelerated from rest. At x_1 the speed is zero, and at x_2 the speed is v.
- We can convert the expression for W into an integral over v_x since

$$a\,\mathrm{d}x = \frac{\mathrm{d}v_x}{\mathrm{d}t}\,\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}v_x = v_x\,\mathrm{d}v_x\,.$$

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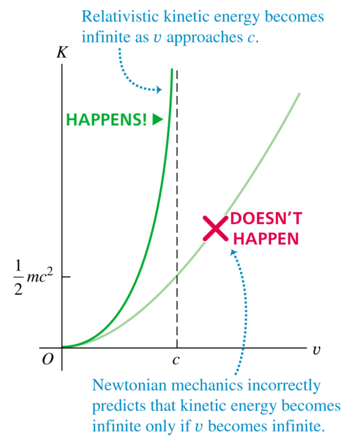
Relativistic Work and Energy (2 of 4)

- Therefore, the kinetic energy is

$$
\begin{aligned}
K &= \int_0^v \frac{m v_x \, \mathrm{d}v_x}{(1 - v_x^2/c^2)^{3/2}} \\
&= \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 \\
&= (\gamma - 1) mc^2.
\end{aligned}
$$

- This expression is quite different from the Newtonian expression for kinetic energy. As $v \to c$, $(\gamma - 1)$ approaches infinity, and the relativistic kinetic energy also approaches infinity.



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Relativistic Work and Energy (3 of 4)

- It is not immediately apparent that we recover the Newtonian form of kinetic energy in the limit that $v \ll c$. To verify this, we must make use of a binomial expansion of the form

$$(1 + x)^n = 1 + nx + n(n-1)\frac{x^2}{2} + \cdots .$$

- In the case of the expression for γ, $n = -1/2$ and $x = -v^2/c^2$:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots .$$

- Inserting this into our expression for K then yields

$$K = \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots - 1\right) mc^2 = \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^4}{c^4} + \cdots .$$

- When $v \ll c$, the higher order terms become negligibly small, and we recover the Newtonian expression for kinetic energy.

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Relativistic Work and Energy (4 of 4)

- The constant mc^2 term in the expression for K suggests that the total energy of a particle is the sum of the kinetic energy and the mc^2, which is known as the rest energy:

$$E = K + mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \gamma mc^2.$$

- For a particle at rest, $v = 0$, and $E = mc^2$. This explains why a particle at rest can decay into two or more particles that have kinetic energy.
- We can also write the total energy E of a particle in terms of its momentum:

$$\left(\frac{E}{mc^2}\right)^2 - \left(\frac{p}{mc}\right)^2 = \frac{1}{1 - v^2/c^2} - \frac{v^2/c^2}{1 - v^2/c^2} \quad \rightarrow \quad E^2 = (mc^2)^2 + (pc)^2.$$

- If we have a particle with no mass, then $m = 0$, but it still has energy given by $E = pc$. This also means that its momentum is given by $p = E/c$. Any such particle must also travel at the speed of light at all times.

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Example 37.10: Energetic Electrons

(a) Find the rest energy of an electron ($m = 9.109 \times 10^{-31}$ kg, $q = -e = -1.602 \times 10^{-19}$ C) in joules and electron volts. (b) Find the speed of an electron that has been accelerated by an electric field, from rest, through a potential increase of 20.0 kV or of 5.00 MV.

(a) The rest energy is

$$mc^2 = (9.109 \times 10^{-31} \text{ kg})(2.998 \times 10^8 \text{ m/s})^2 = 8.187 \times 10^{-14} \text{ J}.$$

An electron volt is defined as 1 eV $= 1.602 \times 10^{-19}$ J, so we can also write the energy as

$$mc^2 = (8.187 \times 10^{-14} \text{ J})\frac{1 \text{ eV}}{1.602 \times 10^{-19} \text{ J}} = 5.11 \times 10^5 \text{ eV} = 0.511 \text{ MeV}.$$

(b) The total energy E of an accelerated electron is the sum of its rest energy mc^2 and the kinetic energy eV_{ba} that it gains from the work done on it by the electric field in moving from point a to point b. Thus,

$$E = \gamma mc^2 = mc^2 + eV_{ba}.$$

We can use this to solve for the Lorentz factor γ, which we can then use to find the velocity v. Solving for γ, we obtain

$$\gamma = 1 + \frac{eV_{ba}}{mc^2}.$$

For a potential increase of $V_{ba} = 20.0$ kV, the electron gains 20.0 keV of energy, and hence

$$\gamma = 1 + \frac{20.0 \times 10^3 \text{ eV}}{0.511 \times 10^6 \text{ eV}} = 1.039.$$

The speed of the electron is therefore

$$v = c\sqrt{1-(1/\gamma)^2} = c\sqrt{1-(1/1.039)^2} = 0.272c = 8.15\times 10^7 \text{ m/s}.$$

We can repeat this process for the case where $V_{ba} = 5.00$ MV, in which case we get

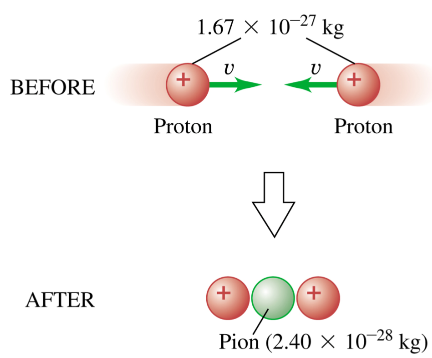
$$\gamma = 1 + \frac{5.00\times 10^6 \text{ eV}}{0.511\times 10^6 \text{ eV}} = 10.78,$$

and thus

$$v = c\sqrt{1-(1/10.78)^2} = 0.996c = 2.99\times 10^8 \text{ m/s}.$$

Example 37.11: A Relativistic Collision

Two protons (each with mass $m_\mathrm{p} = 1.67\times 10^{27}$ kg) are initially moving with equal speeds in opposite directions. They continue to exist after a head-on collision that also produces a neutral pion of mass $m_\pi = 2.40\times 10^{-28}$ kg. If all three particles are at rest after the collision, find the initial speed of the protons.



The total energy of each proton before the collision is $\gamma m_\mathrm{p} c^2$. Therefore, by conservation of energy,

$$2(\gamma m_\mathrm{p} c^2) = 2(m_\mathrm{p} c^2) + m_\pi c^2,$$

which gives us a Lorentz factor of

$$\gamma = 1 + \frac{m_\pi}{2m_\mathrm{p}} = 1 + \frac{2.40\times 10^{-28} \text{ kg}}{2(1.67\times 10^{-27} \text{ kg})} = 1.072.$$

The initial speed of the protons is then

$$v = c\sqrt{1-(1/\gamma)^2} = 0.360c.$$