13.3 Relations (let X, Y be sets)

Def A (binary) <u>relation</u> R from X to Y is a subset of $X \times Y$. If $(x,y) \in R$, we write xRy. If $X = Y$, we say R is a (binary) relation on X.

Ex) $X = \{x \mid x \text{ is a MATH 61 student}\}$
$Y = \{y \mid y \text{ is a UCLA building}\}$

We can define $R \subseteq X \times Y$ such that xRy if x has a class in building y.

Ex) Suppose Sarah is in our class + she has no classes in Royce.
Then $(\text{Sarah}, \text{MS}) \in R$, but $(\text{Sarah}, \text{Royce}) \notin R$.

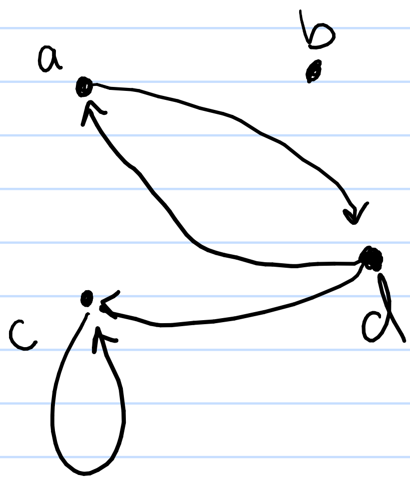
Ex) $X = \{13, 7\}$, $Y = \mathbb{Z}$
Define $R \subseteq X \times Y$, where xRy if x divides y.
Then $(7, 21), (13, 26) \in R$ but $(7, 26) \notin R$.

We can visualize relations on X using <u>digraphs</u>:

① Draw a dot, called a <u>vertex</u> for each $x \in X$.

② Draw an arrow from vertex x_1 to vertex x_2 if $x_1 R x_2$. We call this a <u>directed edge</u>. If $x_1 R x_1$, we call this a <u>loop</u>.

Ex) Consider $X = \{a, b, c, d\}$

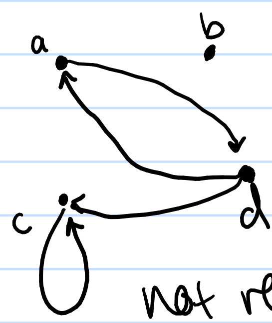


let $R = \{(a,d), (c,c), (d,a), (d,c)\}$

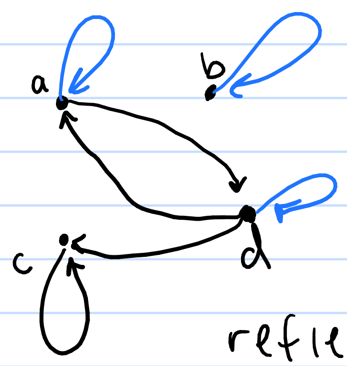
Then,

Def) A relation on X is reflexive if $(x,x) \in R$ for every $x \in X$.

Ex)



not reflexive



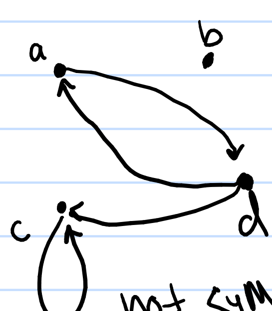
reflexive

Ex) Let $R \subseteq \mathbb{Z} \times \mathbb{Z}$ be defined by $R = \{(x,y) \mid x \le y\}$.

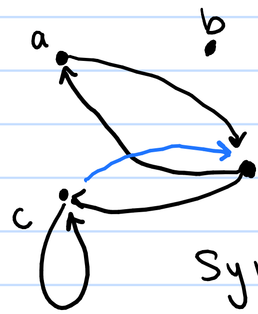
Show R is reflexive.

Def) A relation on X is symmetric if for all $x, y \in X$, if xRy then yRx.

Ex)



not symmetric



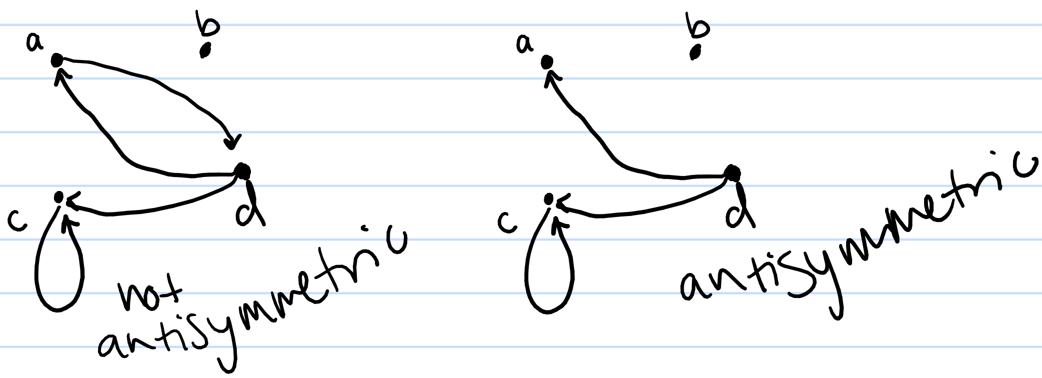
symmetric

Ex Let $R \subseteq \mathbb{Z} \times \mathbb{Z}$ be defined by
$R = \{(x,y) \mid x \leq y\}$.

Show R is not symmetric.

Def A relation R on X is <u>antisymmetric</u> if for all $x, y \in X$, if xRy and yRx, then $x = y$.

Ex



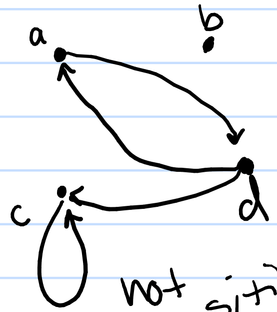
Ex Let $R \subseteq \mathbb{Z} \times \mathbb{Z}$ be defined by
$R = \{(x,y) \mid x \leq y\}$.

Show R is antisymmetric.

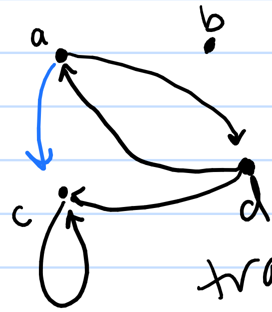
Note: another way to define antisymmetric:
if $x \neq y$ then either
$(x,y) \notin R$ or $(y,x) \notin R$

Def A relation R on X is <u>transitive</u> if for all $x, y, z \in X$, if $(x,y), (y,z) \in R$, then $(x,z) \in R$

Ex



not transitive



transitive

Ex Let $R \subseteq \mathbb{Z} \times \mathbb{Z}$ be defined by
$R = \{(x,y) \mid x \leq y\}$.

Show R is transitive.

Def If R is reflexive, antisymmetric, + transitive, we call R a <u>partial order</u>.

Ex Let $R \subseteq \mathbb{Z} \times \mathbb{Z}$ be defined by
$R = \{(x,y) \mid x \leq y\}$.
This is a partial order.

Def Suppose R is a partial order on X. If $x, y \in X$ and xRy or yRx, we say x and y are <u>comparable</u>. Otherwise we say x and y are <u>incomparable</u>. If each pair $x, y \in X$ are comparable, we say R is a <u>total order</u>.

Ex $R = \{(x, y) \mid x \text{ divides } y\}$
is a partial order, not a total order

$R = \{(x,y) \mid x \leq y\}$.
is a total order

$\boxed{\text{Def}}$ Let $R \subseteq X \times Y$. Define the <u>inverse</u> of R be the relation

$$R^{-1} = \{(y,x) \mid x R y\} \subseteq Y \times X$$

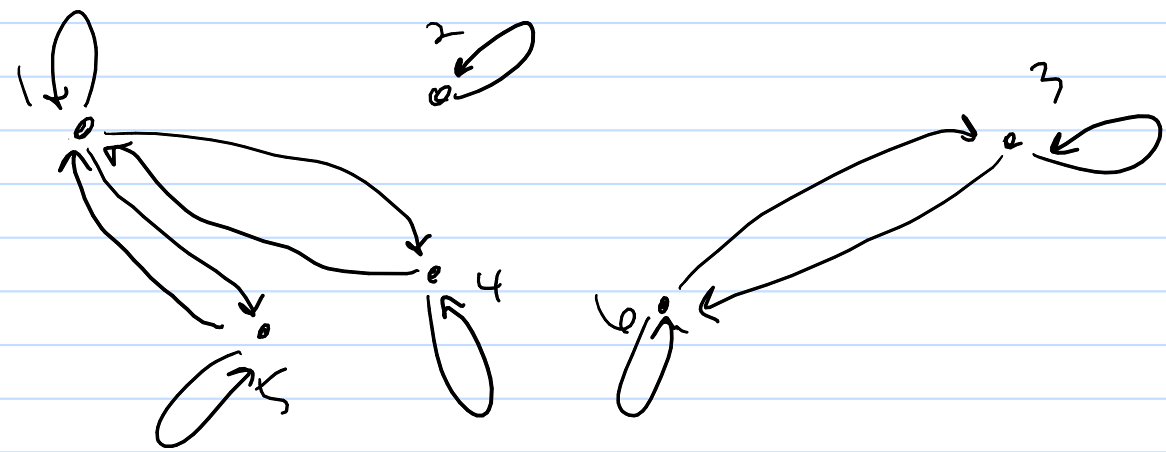
$\boxed{\text{Def}}$ Let $R_1 \subseteq X \times Y$, $R_2 \subseteq Y \times Z$. The <u>composition</u> of R_1 and R_2 is the relation

$$R_2 \circ R_1 = \{(x,z) \mid x R_1 y,\ y R_2 z \text{ for some } y \in Y\}$$

3.4 Equivalence Relations (let X be a set)

Theorem Let S be a partition of X. Define xRy to mean that for some $A \in S$, $x, y \in A$.
Then R is reflexive, symmetric, and transitive.

Ex) $S = \{\{1,4,5\}, \{2\}, \{3,6\}\}$ partitions $X = \{1,2,..,6\}$



Def A relation that is reflexive, symmetric, & transitive on X is called an <u>equivalence relation</u> on X.

Ex) Previous theorem tells us partitions of X are equivalence relations on X.

Ex) $R = \{(x,y) \mid x \leq y\} \subseteq \mathbb{Z} \times \mathbb{Z}$

is not an equiv. relation.

Theorem Let R be an equivalence relation on X. For each $a \in X$, let
$$[a] = \{x \in X \mid xRa\}.$$
Then $S = \{[a] \mid a \in X\}$ is a partition of X

Def We call these sets $[a]$ above <u>equivalence classes</u> on X given by R.

Ex) $S = \{\{1,4,5\}, \{2\}, \{3,6\}\}$

Then $[1] = [4] = [5] = \{1,4,5\}$

$[2] = \{2\}$

$[3] = [6] = \{1,3,6\}$

Ex) let $X = \{1, 2, \dots, 10\}$. Define

$R = \{(x,y) \mid x,y \text{ such that } 3 \text{ divides } x-y\} \subseteq X \times X$

Show this is an equivalence relation, Find the equivalence classes.

$[1] = \{x \in X \mid 3 \text{ divides } x-1\} = \{1,4,7,10\}$

$[2] = \{x \in X \mid 3 \text{ divides } x-2\} = \{2,5,8\}$

$[3] = \{3,6,9\}$

we see this relation is also

$R = \{(x,y) \mid x \bmod 3 = y \bmod 3\}$

$\boxed{3.5}$ We can also visualize relations using matrices.

$\boxed{\text{Def}}$ the <u>matrix of relation</u> $R^{\subseteq X \times Y}$ is the matrix formed by labelling the rows by $x \in X$, columns by $y \in Y$. In the matrix entry in row x and column y, we place a 1 if xRy and 0 otherwise.

$\boxed{\text{Ex}}$ $R = \{(1,b),(1,d),(2,c),(3,c),(3,a),(4,c)\}$

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

<u>Note</u>: this matrix depends on the choice of ordering on elements of X, Y.

$\boxed{\text{Theorem}}$ Let $R_1 \subseteq X \times Y$, $R_2 \subseteq Y \times Z$. Choose orderings of X, Y, Z. Let A_1 be the matrix of R_1 and A_2 be the matrix of R_2. Let A be the matrix formed from $A_1 A_2$ by replacing any entries $r \neq 0$ with 1. Then A is the matrix of $R_2 \circ R_1$.

$\boxed{6.1}$ Counting principles

Ex) Suppose at LSU, STEM majors have to take
≥ 1 out of Bio, Chem, Physics, Geology lectures
and
≥ 1 out of Bio, Chem, + Physics lab
How many ways to minimally satisfy this requirement?

Claim: 12 ways

Pf

Can choose Bio + any lab — 3 ways
Chem + any lab — 3 ways
Phys + any lab — 3 ways
or Geol + any lab — 3 ways

12 ways

$\boxed{\text{Thm}}$ (Multiplication Principle) If an event can be broken down into t independent steps where there are n_1 ways to do step 1, n_2 for step 2, ..., and n_t for step t,
there are $n_1 \cdot n_2 \cdot \ldots \cdot n_t$ possible events.

Ex) What if above we also have a requirement for ≥ 1 of swimming, juggling, or archery?

Then $4 \cdot 3 \cdot 3 = 36$ options

Ex) a) What is # strings of length 4 from ABCDE with no repetition?

Pf

$$\left(\begin{array}{c}\text{\#ways to choose} \\ \text{1st letter}\end{array}\right)\left(\begin{array}{c}\text{\#ways to choose} \\ \text{2nd letter}\end{array}\right)\left(\begin{array}{c}\text{\#ways to choose} \\ \text{3rd letter}\end{array}\right)\left(\begin{array}{c}\text{\#ways to choose} \\ \text{4th letter}\end{array}\right)$$

$= 5 \cdot 4 \cdot 3 \cdot 2$ ways

Ex) b) How many of those start w/ B?

$\begin{pmatrix}\#\text{ways to choose} \\ 1^{st}\text{ letter}\end{pmatrix}\begin{pmatrix}\#\text{ways to choose} \\ 2^{nd}\text{ letter}\end{pmatrix}\begin{pmatrix}\#\text{ways to choose} \\ 3^{rd}\text{ letter}\end{pmatrix}\begin{pmatrix}\#\text{ways to choose} \\ 4^{th}\text{ letter}\end{pmatrix}$

$= 1 \cdot 4 \cdot 3 \cdot 2$

c) How many of these do not start with B?

We know there are $5 \cdot 4 \cdot 3 \cdot 2$ total and $1 \cdot 4 \cdot 3 \cdot 2$ that do start with B

$\Rightarrow$ there are $5 \cdot 4 \cdot 3 \cdot 2 - 1 \cdot 4 \cdot 3 \cdot 2 = 4 \cdot 4 \cdot 3 \cdot 2$ not starting with B

Ex) Prove if $|X| = n$, then $|P(X)| = 2^n$.

We will think about # ways to build $S \in P(X)$. For each $y \in X$, we decide to either
1) add y to S
2) not add y to S

$\Rightarrow$ 2 choice for each $y \in X$

$\Rightarrow \underbrace{2}_{\text{decide for }1^{st}\text{ elt}} \cdot 2 \cdot \ldots \cdot \underbrace{2}_{\text{decide for }n^{th}\text{ elt}} = 2^n$ ways

Ex) What is # of reflexive relations on X where $|X| = n$?

If R is reflexive then xRx for each $x \in X$. Therefore in relation matrix, we have all 1's on the diagonal. This accounts for n matrix entries. There n^2 total entries. The others are not constrained, so have 2 choices for each of the $n^2 - n$ entries

$\Rightarrow 2^{n^2-n}$ choices $\Rightarrow 2^{n^2-n}$ refl. rel.

Ex] How many 8-bit strings begin w/ either 101 or 111?

We just have to choose last 5 in both cases.
There are 2^5 ways for this
$\Rightarrow$

$$\underbrace{2^5}_{\text{start w/ 101}} + \underbrace{2^5}_{\text{start w/ 111}} = 2^6 \text{ total ways}$$

$\boxed{\text{Theorem}}$ (Addition Principle) Suppose $X_1, \dots, X_t$ are sets where $|X_i| = n_i$. If each X_i, X_j are disjoint when $i \neq j$, the # elements that can be chosen from

X_1 or X_2 or X_3 ... or X_t

is $n_1 + n_2 + \dots + n_t$

Ex] Suppose you only had to choose 2 movies from different genres to watch on the plane. There are 3 romances, 4 comedies, and 5 children's animated movies.
How many ways to choose.

$X_1 = \{\text{choices in romance + comedy}\}$
$X_2 = \{\text{choices in rom. + kid's}\}$
$X_3 = \{\text{choices in comedy + kid's}\}$

Then $|X_1| = 3 \cdot 4$, $|X_2| = 3 \cdot 5$, $|X_3| = 4 \cdot 5$

$\Rightarrow$ total # ways $= 3 \cdot 4 + 3 \cdot 5 + 4 \cdot 5$

Ex) You are have a person committee of Alice, Ben, Connie, Don, Eddie, & Fran. From these, choose a president, secretary, & treasurer.

1) What is total # ways?

(choose pres)(choose sec)(choose treas)

$6 \cdot 5 \cdot 4$ ways

2) What if A or B must be pres?

(A is pres)(choose sec)(choose treas)

1 5 4

+

(B is pres)(choose sec)(choose treas)

1 5 4

$2 \cdot 5 \cdot 4$ ways

3) What if D & E must have an office

(choose office for D)(choose office for E)(choose last officer)

$3 \cdot 2 \cdot 4$ ways

$\boxed{\text{Theorem}}$ (Inclusion-Exclusion) If X, Y finite,

$$|X \cup Y| = |X| + |Y| - |X \cap Y|$$

Ex) What if above we want A or D or both to be officers?

Let $X = \{\text{choices with A on board}\}$

$Y = \{\text{choices with D on board}\}$

Then we need to find $|X \cup Y|$

By Inclusion-Exclusion

$$\underbrace{|X \cup Y| = |X|}_{\text{A on board}} + \underbrace{|Y|}_{\text{D on board}} - \underbrace{|X \cap Y|}_{\text{both on board}} = 3 \cdot 5 \cdot 4 + 3 \cdot 5 \cdot 4 - 3 \cdot 2 \cdot 4$$

$$= 2 \cdot 3 \cdot 4 \cdot 5 - 2 \cdot 3 \cdot 4$$

$$= 2 \cdot 3 \cdot 4^2$$