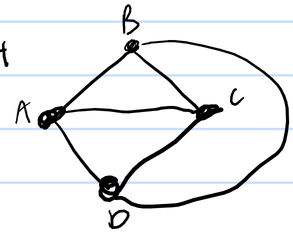
8.7 Planar Graphs

Def A graph is <u>planar</u> if it can be drawn on the plane without any edges crossing.

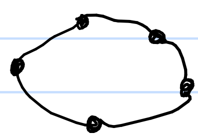
Ex $G = K_4$



is planar

Def The <u>cycle graph</u> C_n has $|V(C_n)| = n$ is the simple, connected graph consisting of a single cycle where each vertex has degree 2.

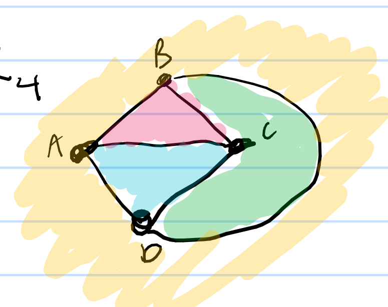
Ex C_5



is planar

Def For G a connected, planar graph, the edges $E($ divide the plane into regions called faces. Each face is defined by the cycle that forms its boundary

Ex $G = K_4$



F_1 has boundary (A, B, D, A)
F_2 has boundary (B, C, D, B)
F_3 has boundary (A, B, C, A)
F_4 has boundary (A, C, D, A)

Then K_4 has $f = 4$ faces, $v = 4$ vertices, & $e = 6$ edges.

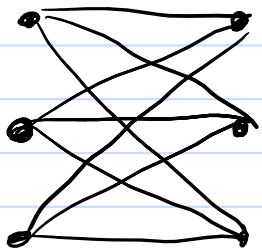
We see $f = e - v + 2$

Theorem (Euler) For G connected, planar graph with f faces, $|V(G)| = v$, and $|E(G)| = e$,

$$f = e - v + 2.$$

Therefore if connected G does not have f, e, v satisfying $f = e - v + 2$, we know G is not planar.

Ex $K_{3,3}$



is clearly connected.

we know $e=9$, $v=6$.

How many faces? we know each cycle has at least 4 edges since $K_{3,3}$ has no cycles of length 3 (why?). ($\therefore$ # edges bounding faces is $\geq 4f$)

Each edge is a part of ≤ 2 faces

$\Rightarrow 2e \geq 4f$

If $K_{3,3}$ were planar, $2e \geq 4f = 4(e-v+2)$

$\Rightarrow 18 \geq 20$ $\rightarrow\leftarrow$

Ex We can show K_5 is not planar through a similar argument.

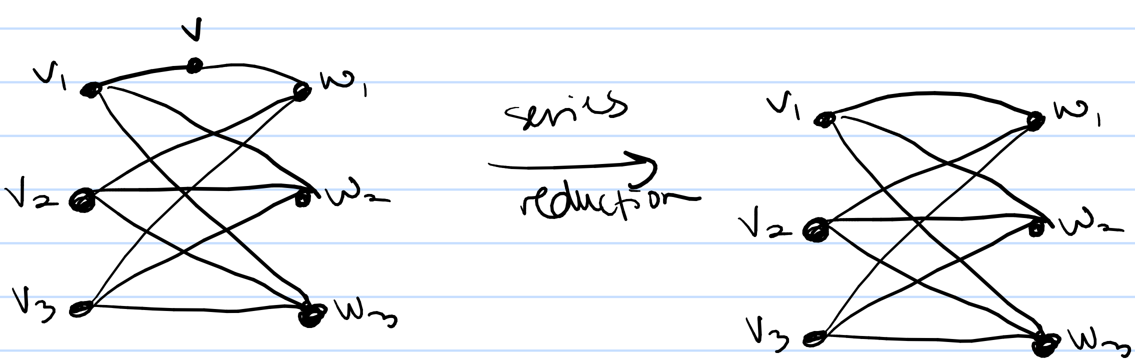
Def Suppose G has $v \in V(G)$ where $\delta(v)=2$ with $v_1 \neq v_2$ vertices in $V(G)$ such that $(v_1,v),(v_2,v) \in E(G)$. Then we say the edges $(v_1,v),(v_2,v)$ are in <u>series</u>. A <u>series reduction</u> forms a graph G', where

$V(G') = V(G) - \{v\}$

$E(G') = E(G) - \{(v_1,v),(v_2,v)\} \cup \{(v_1,v_2)\}$.

We say G' is obtained from G by a <u>series reduction</u>.

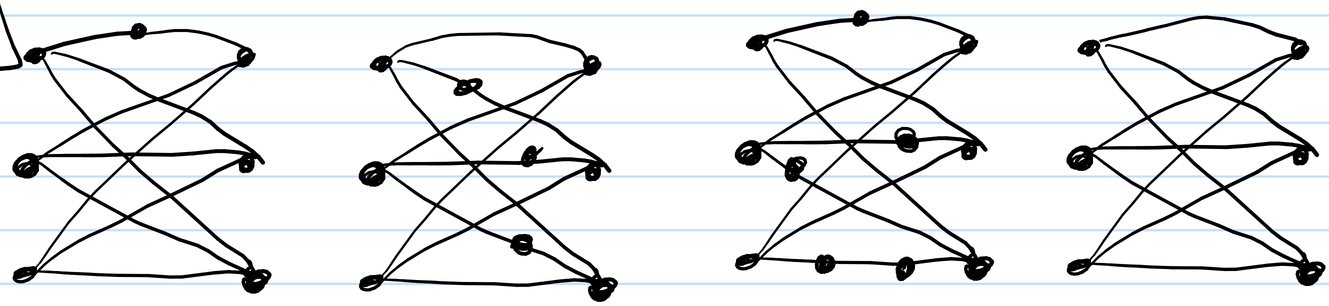
Ex G:



$\Rightarrow K_{3,3}$ is obtained from G by series reduction

Def Graphs G_1, G_2 are <u>homeomorphic</u> if G_1 + G_2 can be reduced to isomorphic graphs by applying a sequence of series reductions.

Ex

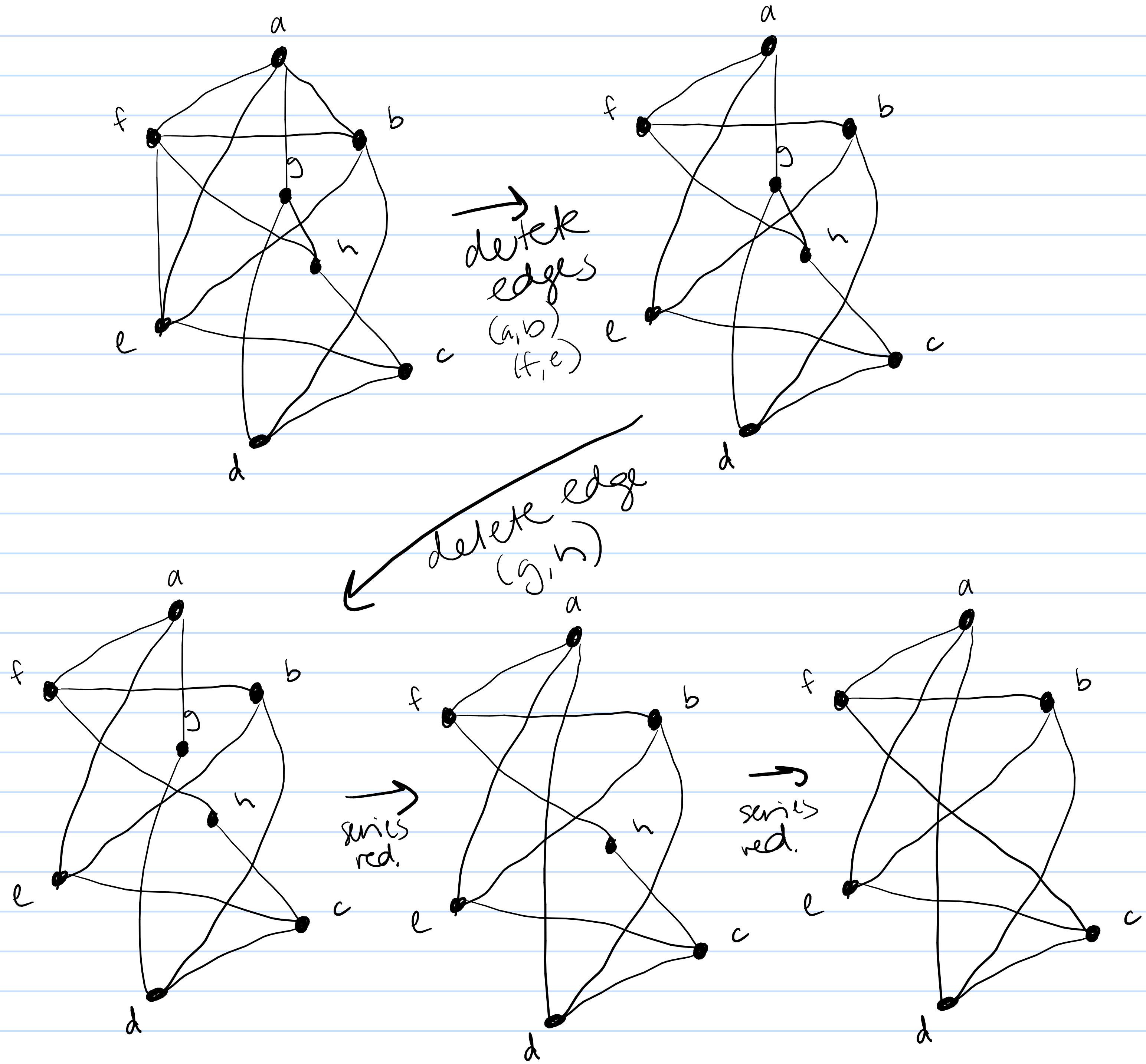


these graphs above are all homeomorphic to $K_{3,3}$.

Ex We can define a relation R on the set of graphs where $G_1 R G_2$ when G_1 + G_2 are homeomorphic. Then R is an equivalence relation.

Theorem (Kuratowski) A graph G is <u>planar</u> $\iff$ G does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.

Ex) Let's apply this theorem to prove G below is not planar. (we'll try to find $K_{3,3}$)



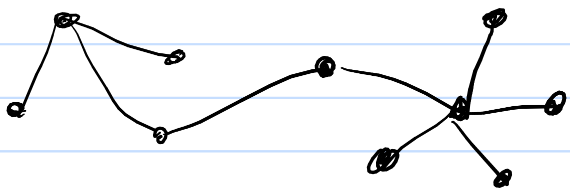
$K_{3,3}$

9.1 Trees

Def A *tree* T is a simple graph with the property that if $v, w \in V(T)$, there is a unique simple path from v to w

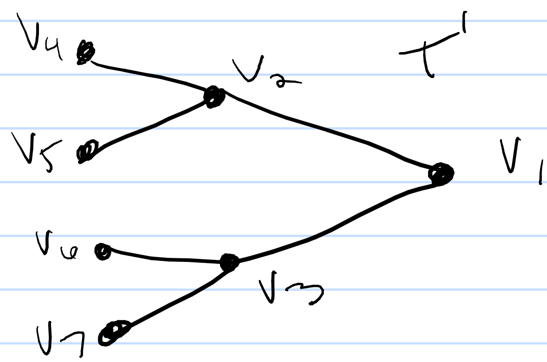
A *rooted tree* is a tree where a certain vertex is defined as the root

Ex T



a tree

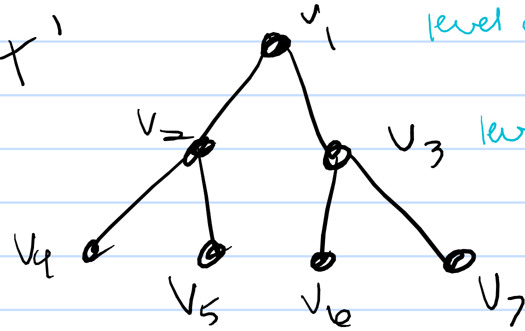
Ex Can visualize tournaments:



this is a rooted tree, where we will define v_1 to be the root

We will usually draw rooted trees w/ the root at the top where vertices that are paths of length ℓ from the root are all drawn at level ℓ

T'



level 0

level 1

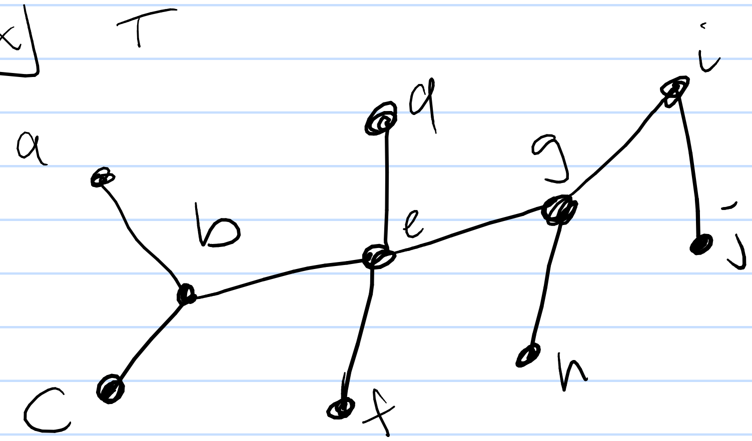
level 2

has height 2

Def The *level of a vertex* v is the length of the simple path from the root to v in a rooted tree.

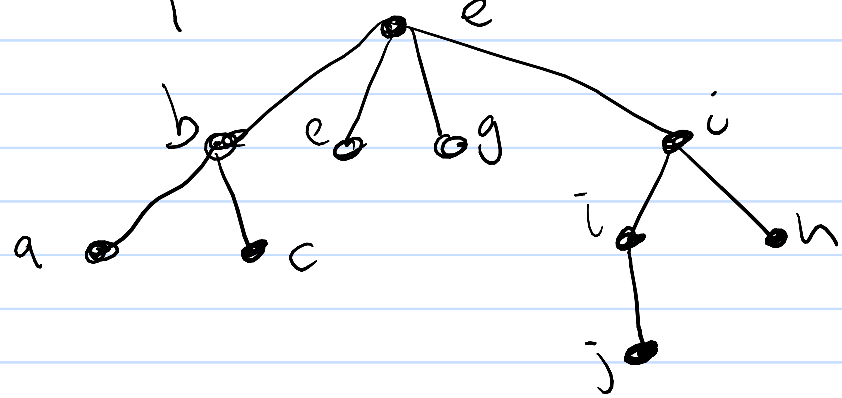
The *height* of a rooted tree T is the maximal level that can appear as the level of some $v \in V(T)$.

Ex T



$\downarrow$ root e

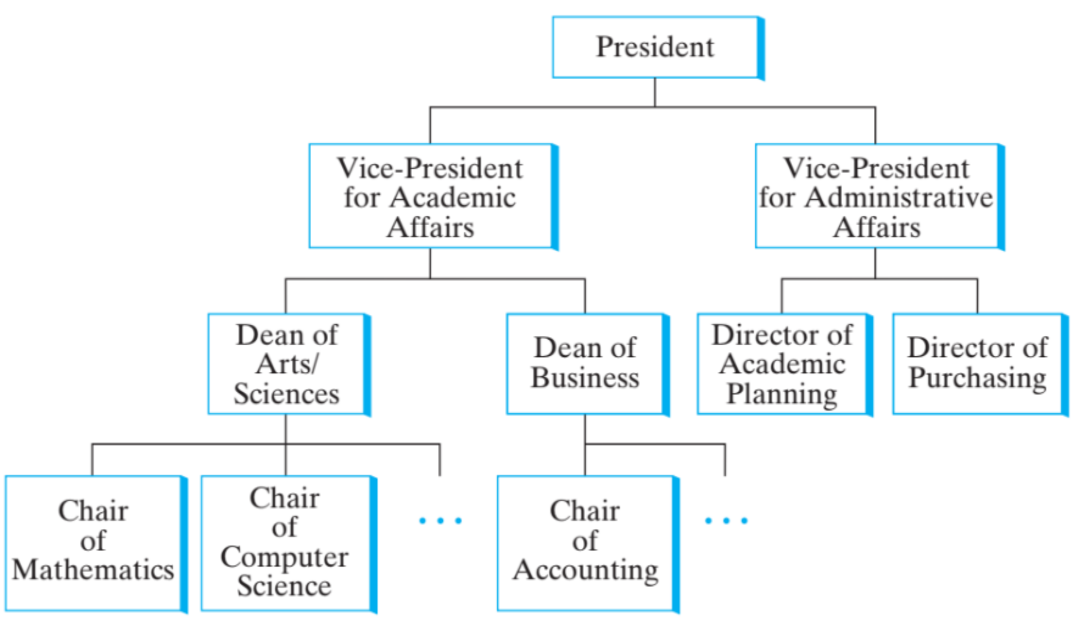
T'



$\Rightarrow$ T' has height 3.

We see e is on level 0, h is on level 2, j on level 3.

Ex Rooted trees can also visualize chains of command



Ex) Rooted trees also visualize organization of files/folders.

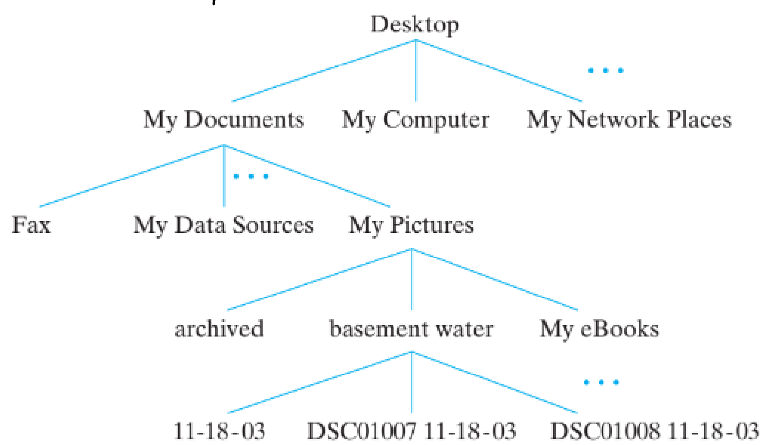
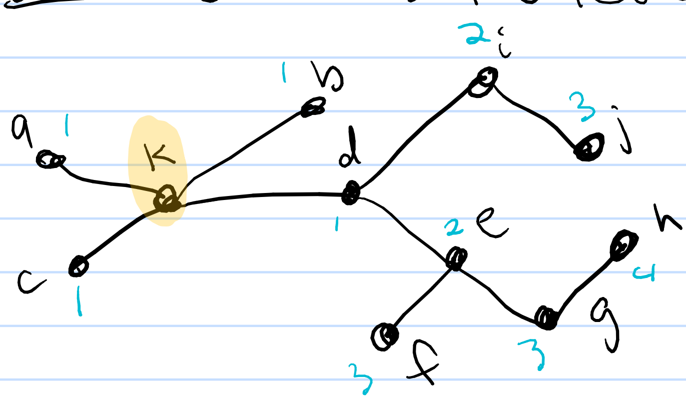
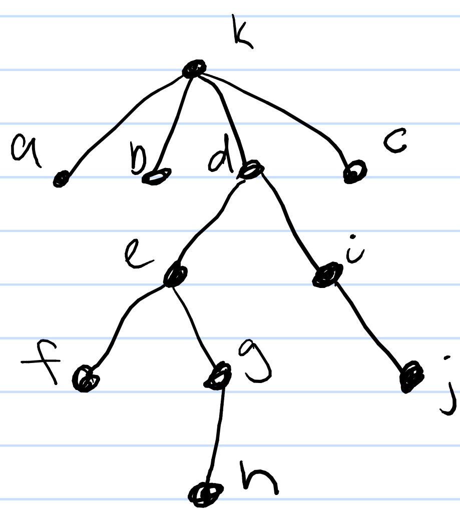


Figure 9.1.8 The structure of Figure 9.1.7 shown as a rooted tree.

When specifying a path of a file, this is precisely specifying a path to a file from the root in this tree

Ex:- What is the level of each vertex if we take k as the root?





$\boxed{\text{Thm}}$ Every tree with 2 or more vertices has a vertex of degree 1.

Pf

Suppose T is a tree where $|V(T)| \geq 2$.

Pick some vertex $v_1 \in V(T)$. Since T is a tree + $|V(T)| \geq 2$, there is some $v_2 \in V(T)$ such that $(v_1, v_2) \in E(T)$.

If $\delta(v_1) = 1$ we are done

Otherwise move to v_2. If $\delta(v_2) > 1$, there is some $v_3 \in V(T)$ such that $(v_2, v_3) \in E(T)$

If $\delta(v_2) = 2$ we are done.

Continue in this manner:

If $\delta(v_i) = 1$ at any point, we are done.

Otherwise $\delta(v_i) \geq 2$ so there is some $v_{i+1} \in V(T)$ such that $(v_i, v_{i+1}) \in E(T)$

Build a path as we go:

$$(v_1, v_2, \ldots, v_i, v_{i+1})$$

We will never re-use a vertex v_i or else this would create a cycle.

T has only finitely many vertices, so this process must conclude.

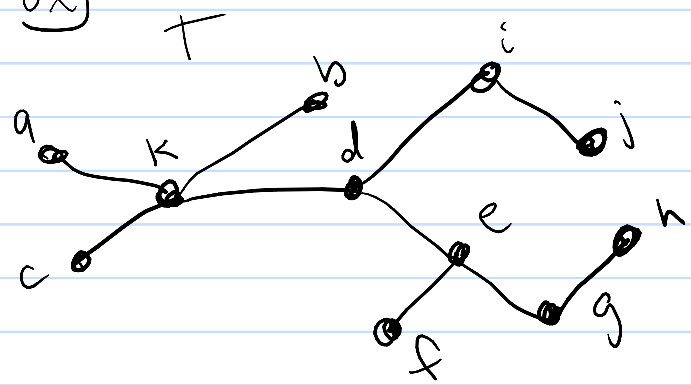
$\therefore$ there is some v_j s.t. $\delta(v_j) = 1$.

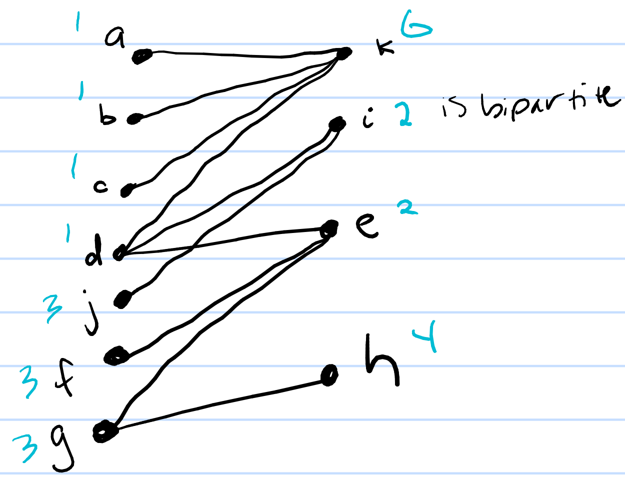
Ex) Claim: A tree T is a bipartite graph.

How to do this?

try an example

Ex) T





How to generalize this?

We see that the partite sets of $V(T)$ are exactly based on the parity of the level of v in the rooted tree

Pf

Pick any vertex $v \in T$ to be the root. Compute the level of each vertex in the rooted tree T'.

Make $V_1 = \{v' \in V(T') \mid v' \text{ has even level}\}$

$V_2 = \{v' \in V(T') \mid v' \text{ has odd level}\}$.

Clearly V_1, V_2 partition $V(T')$.

To show: if $v_1, v_2 \in V_1 \Rightarrow (v_1, v_2) \notin E(T)$ (same for V_2)

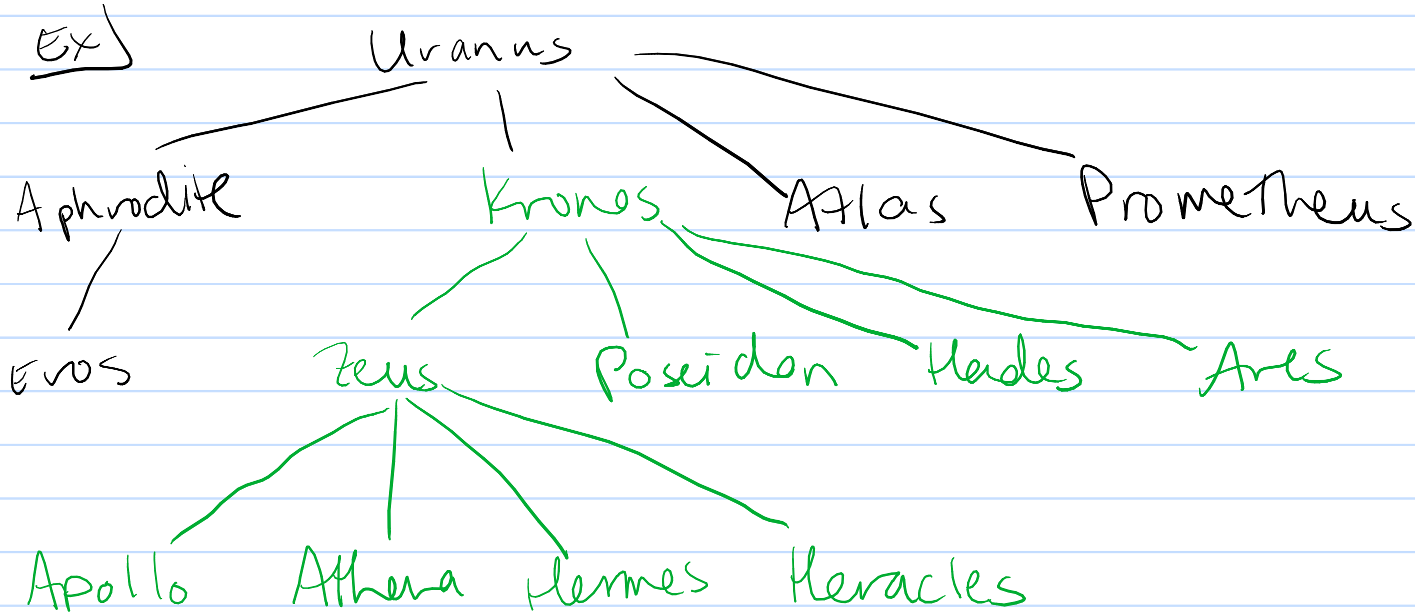
Suppose not. Then we have a path $(v, \dots, v_1)$ in T' + path $(v, \dots, v_2, v_1)$ in T' distinct paths, but this contradicts that T' is a tree $\rightarrow\leftarrow$

$\therefore (v_1, v_2) \notin E(T)$.

9.2 More trees

We can view a family tree as a rooted tree:

Ex)



Def Let T be a rooted tree with root v_0. Suppose $x, y, z \in V(T)$ and $(v_0, v_1, \ldots, v_n)$ is a simple path in T. Then:

a) v_{n-1} is the <u>parent</u> of v_n

b) $v_0, \ldots, v_{n-1}$ are the <u>ancestors</u> of v_n

c) v_n is a <u>child</u> of v_{n-1}

d) If x is an ancestor of y, y is a <u>descendant</u> of x.

e) If $x + y$ are children of z, $x + y$ are <u>siblings</u>.

f) If x has no children, x is a <u>leaf</u>

g) If x is not a leaf, x is an <u>internal vertex</u>.

h) A <u>subtree of T rooted at x</u> is the graph T' where $V(T') = \{x\} \cup \{v \in V(T) \mid v \text{ a descendant of } x\}$ and $E(T') = \{e \in E(T) \mid e \text{ is a simple path from } x \text{ to some } v \in V(T')\}$

Ex) The subtree rooted at Kronos is highlighted in green above

Def A graph with no cycles is called <u>acyclic</u>.

Proposition If T is a tree, T is acyclic + connected.

Pf

By definition of T, there is exactly one simple path between $v, w \in V(T)$
$\Rightarrow T$ is connected.

Suppose there is a cycle $(v_1, v_2, \ldots, v_k, v_1)$ in T. Then the path
(v_1, v_2) is one path from v_1 to v_2 +

$(v_1, v_k, v_{k-1}, \ldots, v_2)$ is another $\rightarrow\leftarrow$

$\therefore T$ is acyclic

Theorem Let T be a graph where $|V(T)| = n$. The following are equivalent:

a) T is a tree
b) T is connected + acyclic
c) T is connected + $|E(T)| = n-1$
d) T is acyclic + has $|E(T)| = n-1$

Pf

We will show $a \Rightarrow b$, $b \Rightarrow c$, $c \Rightarrow d$, + $d \Rightarrow a$. Then we will be done

$(a \Rightarrow b)$ This is given by the proposition above.

$(b \Rightarrow c)$ Assume T is connected + acyclic
<u>want to show</u>: $|E(T)| = n-1$

We proceed by Induction on n.

<u>Base Case</u>: $n=1 \Rightarrow T$ is • $\Rightarrow$ result holds

($b \Rightarrow c$ ctd)

Ind Assump: Suppose $b \Rightarrow c$ for the n case.

Let T be connected + acyclic where $|V(T)| = n+1$.
Then it follows there must be some $v \in V(T)$ where $\delta(v) = 1$. (otherwise either T is disconn or there is a cycle)
Then let T' be T w/ v and the edge incident to v removed.
$\Rightarrow$ by Ind. Assump $|E(T')| = n-1$ since $|V(T')| = n$ + T' is connected, acyclic
$\Rightarrow |E(T)| = |E(T')| + 1 = n \Rightarrow$ result follows

($c \Rightarrow d$) Suppose T is connected + $|E(T)| = n-1$.
Claim: T is acyclic.
Suppose not. Then T has some cycle. Obtain T' from T by deleting as many edges as possible, while retaining connectivity.
Then T' is connected + acyclic, so by prev. proof $\Rightarrow |E(T')| = n-1$
but $|E(T')| < |E(T)|$ $\rightarrow\leftarrow$
$\therefore$ there must be no cycles in T.

($d \Rightarrow a$) Suppose T acyclic + $|E(T)| = n-1$.
Claim: T is a tree
($\Rightarrow$ need to show T is simple, where there is a unique path between each v, w)
If T had any loops, or multi-edges, these would form a cycle $\Rightarrow T$ is simple.
Suppose T were not connected
$\Rightarrow T$ has connected components $T_1, \ldots, T_k$, where $|V(T_i)| = n_i$ and $k > 1$
$\Rightarrow$ by ($b \Rightarrow c$) for each connected component,
$$n-1 = (n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1)$$
$$= (\textstyle\sum n_i) - k = n - k < n-1 \rightarrow\leftarrow$$
$\therefore T$ is connected

($d \Rightarrow a$ ctd) Suppose T had distinct simple paths P_1, P_2 from v to w.

Then let a be the 1st vertex on P_1 that is not on P_2. Let b be the vertex immed. before a on P_1.

Let c be the next vertex after b that is on P_1 + P_2. Then

$$P_1 = (v, \ldots, a, b, v_1, v_2, \ldots, v_k, c, \ldots, w)$$

$$P_2 = (v, \ldots, a, b', w_1, w_2, \ldots, w_\ell, c, \ldots, w)$$

Then

$$(a, b, v_1, \ldots, v_k, c, w_\ell, \ldots, w_2, w_1, b', a)$$

is a cycle in T $\rightarrow\leftarrow$

This is simple by our definition of c as being the earliest choice.

$\therefore$ T is a tree