Math 170E: Winter 2023

Lecture 6, Mon 23rd Jan

Bayes' Theorem and Discrete random variables

Last time:

- how to compute the probability of an event, given another event has happened (conditional probability)
- properties of the conditional probability
- the law of total probability and applications

Today:

We'll discuss today:

- Bayes' theorem

Recall:

Definition 1.13: (Conditional probability)

Let $B \subseteq \Omega$ be an event so that $\mathbb{P}(B) \neq 0$. The probability of an event $A \subseteq \Omega$ conditioned on the event B is given by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

$(\Omega, \mathcal{F}, \mathbb{P}(\cdot|B))$ is a probability space!

Theorem 1.16: (The law of total probability)

Let $A \subseteq \Omega$ be an event and $\{B_j\}_{j=1}^k \subseteq \Omega$ be mutually exclusive, satisfying $\mathbb{P}(B_j) \neq 0$, for every $j \in \{1, \ldots, k\}$, and

$$A \subseteq \bigcup_{j=1}^{k} B_j.$$

Then

$$\mathbb{P}(A) = \mathbb{P}(A|B_1)\mathbb{P}(B_1) + \cdots + \mathbb{P}(A|B_k)\mathbb{P}(B_k) = \sum_{j=1}^{k} \mathbb{P}(A|B_j)\mathbb{P}(B_j).$$

Theorem 1.17: **(Bayes' Theorem)**
If $A, B \subseteq \Omega$ are events so that $\mathbb{P}(A), \mathbb{P}(B) \neq 0$, then

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A)}$$

Proof:

$$\mathbb{P}(B|A) \overset{\text{def.}}{=} \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A)}$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Theorem 1.18: (**Bayes' Theorem v2**)
If $A \subseteq \Omega$ is an event, $\{B_j\}_{j=1}^k \subseteq \Omega$ are mutually exclusive events so that $\mathbb{P}(A), \mathbb{P}(B_j) \neq 0$, and

$$A \subseteq \bigcup_{j=1}^{k} B_j$$

Then for any $1 \leq \ell \leq k$, we have

$$\mathbb{P}(B_\ell|A) = \frac{\mathbb{P}(A|B_\ell)\mathbb{P}(B_\ell)}{\sum_{j=1}^{n} \mathbb{P}(A|B_j)\mathbb{P}(B_j)}$$

Proof:

$$\mathbb{P}(B_\ell|A) \overset{\text{Bayes thm.}}{=} \frac{\mathbb{P}(A|B_\ell)\,\mathbb{P}(B_\ell)}{\mathbb{P}(A)}$$

+ law of total probability

Example 16:

1 - A test for a virus is 95% effective at detecting the virus when it is present.
2 - If a healthy person is tested, there's a 1% chance of a positive result (*false positive*)
3 - 0.5% of the population have the virus

What is the probability that a person is infected if they tested positive? 4

$+ = \{\text{test positive}\}$, $I = \{\text{infected}\}$

1 $P(+ \mid I) = 0.95$

2 $P(+ \mid I') = 0.01$

3 $P(I) = 0.005$

4 $P(I \mid +) = ?$

$\boxed{\text{Bayes thm}}$

$$P(I \mid +) = \frac{P(+ \mid I)\, P(I)}{P(+)}$$

$$= \frac{P(+ \mid I)\, P(I)}{P(+ \cap I) + P(+ \cap I')}$$

$$= \frac{\mathbb{P}(+ \mid I)\,\mathbb{P}(I)}{\mathbb{P}(+ \mid I)\,\mathbb{P}(I) + \mathbb{P}(+ \mid I')\,\mathbb{P}(I')}$$

$$= \frac{0.95 \times 0.005}{0.95 \times 0.005 + 0.01 \times 0.995} \approx 0.323$$

What?:

- 0.5% have the disease $\implies$ 1 out of every 200 people tests are infected
- the test is 95% accurate if infected $\implies$ out of 200 people, on average, the test correctly identifies 0.95 of them are infected

Meanwhile,

- 199 people are healthy $\implies$ the test says incorrectly that $(199) \times 0.01 = 1.99$ of them are infected
- for every 0.95 infected people who tested positive, there are 1.99 that are incorrectly tested as infected

The proportion of time the test is correct when it says someone is infected is

$$\frac{0.95}{0.95 + 1.99} = \frac{93}{294} \sim 0.323.$$

You might think this means that the test is **useless**. **It isn't!**

- Taking the test, your odds of being infected given a +ve test shot up from 0.5% to 32%!

Take the test multiple times:

$$\mathbb{P}(I \mid +_1, +_2) = \frac{\mathbb{P}(+_1, +_2 \mid I)\,\mathbb{P}(I)}{\mathbb{P}(+_1, +_2)}$$

$$= \frac{\mathbb{P}(+_1 \mid I)\,\mathbb{P}(+_2 \mid I)\,\mathbb{P}(I)}{\mathbb{P}(+_1, +_2)}$$

$$= \frac{\mathbb{P}(+_1 \mid I)\,\mathbb{P}(+_2 \mid I)\,\mathbb{P}(I)}{\mathbb{P}(+_1, +_2 \mid I)\,\mathbb{P}(I) + \mathbb{P}(+_1, +_2 \mid I')\,\mathbb{P}(I')}$$

$$= \frac{\mathbb{P}(+ \mid I)^2\,\mathbb{P}(I)}{\mathbb{P}(+ \mid I)^2\,\mathbb{P}(I) + \mathbb{P}(+ \mid I')^2\,\mathbb{P}(I')}$$

$$= \frac{0.95^2 \times 0.005}{0.95^2 \times 0.005 + 0.01^2 \times 0.995} \approx 0.9784$$

$$\mathbb{P}(+_1, +_2 \mid I) = \mathbb{P}(+_1 \mid I)\,\mathbb{P}(+_2 \mid I)$$

$$\Updownarrow$$

$$\mathbb{P}(+_2 \mid +_1, I) = \mathbb{P}(+_2 \mid I)$$

Example: Let's play Chuck-a-Luck!

- I roll 3 fair six-sided dies
- You pick a number in $\{1, 2, 3, 4, 5, 6\}$
- If your number comes up:
  - once, you win \$10
  - twice, you win \$20
  - thrice, you win \$30
- If your number doesn't come up, you owe me \$10
- Should you play?

$\Omega = \{ (d_1, d_2, d_3) : d_1, d_2, d_3 \in \{1, \dots, 6\} \}$

$\omega \in \Omega \longrightarrow W(\omega) =$ winnings if ω happens

$\in \{-10, 10, 20, 30\}$

$\omega \in \Omega \longrightarrow X(\omega) =$ number of times that my number shows up in ω

$\in \{0, 1, 2, 3\}$

$$W(\omega) = \begin{cases} -10 & \text{if } X(\omega) = 0 \\ 10 & \text{if } X(\omega) = 1 \\ 20 & \text{if } X(\omega) = 2 \\ 30 & \text{if } X(\omega) = 3 \end{cases}$$

<u>Q</u>: What is the probability that I do <u>NOT</u> lose money?

$$\mathbb{P}(\{\omega \in \Omega : W(\omega) \geq 10\}) = \mathbb{P}(\{\omega \in \Omega : W(\omega) \in \{10, 20, 30\}\})$$

$$= 1 - \mathbb{P}(\{\omega \in \Omega : W(\omega) = -10\})$$

$$= 1 - \mathbb{P}(\{\omega \in \Omega : X(\omega) = 0\})$$

$$= 1 - \left(\frac{5}{6}\right)^3$$

$$< 0.5$$

Definition 2.1: (Random variable)

Given a set S and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a **random variable** (r.v.) is a function

$$X : \Omega \to S$$

<u>Notation:</u> If $x \in S$, and $A \subseteq S$, we write

$$\mathbb{P}(X = x) := \mathbb{P}(\{\omega \in \Omega \, : \, X(\omega) = x\})$$
$$\mathbb{P}(X \in A) := \mathbb{P}(\{\omega \in \Omega \, : \, X(\omega) \in A\})$$