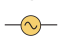
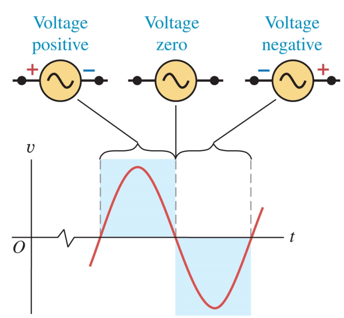
Chapter 31: Alternating Current

Phasors and Alternating Currents

- Most household and industrial power systems operate on alternating current (ac).
- An **ac source** supplies a sinusoidally varying voltage v or current i. Such a voltage source is represented in a circuit by the following symbol: 
- A sinusoidal voltage with amplitude V and angular frequency $\omega = 2\pi f$ is described by the following function:



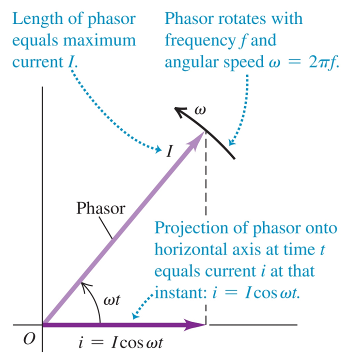
$$v = V \cos \omega t.$$

- In the US and Canada, commercial electric-power distribution systems use a frequency of $f = 60$ Hz. Other parts of the world use $f = 50$ Hz.
- A sinusoidal alternating current with amplitude I may correspondingly be described by

$$i = I \cos \omega t.$$

Phasor Diagrams

- **Phasors** are vectors that rotate counterclockwise with constant angular speed ω, which represent sinusoidally varying voltages and currents.
- They are geometric representations that help with describing sinusoidally varying quantities.
- A phasor sweeps out an angle ωt as a function of time t.
- The projection of a phasor onto the horizontal axis gives us the instantaneous value of the voltage or current.
  - The horizontal component is proportional to $\cos \omega t$.
  - The length of the phasor corresponds to the amplitude for voltage or current.



Root-Mean-Square Values

- It is not useful to describe sinusoidal functions by their average value because the average over a full period is zero.
- We instead use **root-mean-square (rms)** values for voltage and current.
  - Start by squaring the function:

$$i^2 = I^2 \cos^2 \omega t = I^2 \frac{1}{2}(1 + \cos 2\omega t).$$

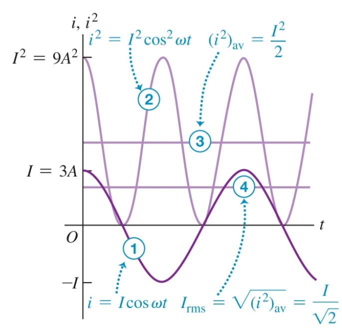
  - Compute the average:

$$\left(i^2\right)_{\text{avg}} = \frac{1}{2} I^2.$$

  - Take the square-root of the result:

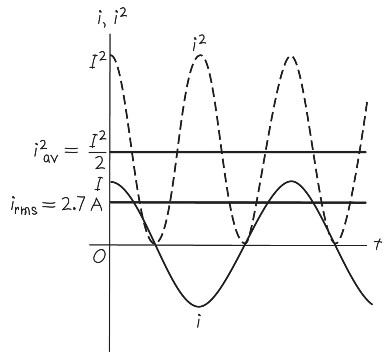
$$I_{\text{rms}} = \sqrt{\left(i^2\right)_{\text{avg}}} = \sqrt{\frac{1}{2} I^2} = \frac{I}{\sqrt{2}} \quad \rightarrow \quad V_{\text{rms}} = \frac{V}{\sqrt{2}}.$$

- Rms values are typically quoted for voltage and current in ac circuits.



Example 31.1: Current in a Desktop Computer

The plate on the back of a desktop computer says it draws 2.7 A from a 120 V, 60 Hz line. For this computer, what are (a) the average current, (b) the average square of the current, and (c) the current amplitude?



(a) The average of any sinusoidal function over any whole number of cycles is zero.

(b) We have $I_{\mathrm{rms}} = 2.7$ A, hence

$$\left(i^2\right)_{\mathrm{avg}} = I_{\mathrm{rms}}^2 = (2.7 \text{ A})^2 = 7.3 \text{ A}^2.$$

(c) The amplitude of the current is

$$I = \sqrt{2} I_{\mathrm{rms}} = \sqrt{2}(2.7 \text{ A}) = 3.8 \text{ A}.$$

Resistor in an Ac Circuit

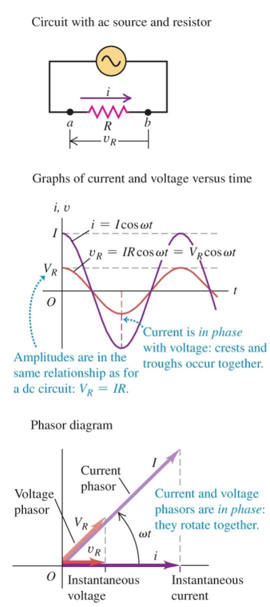
- When a resistor is connected to an ac voltage source, the voltage across the resistor is

$$v_R = iR = (IR)\cos\omega t.$$

- We can define $V_R = IR$ to be the voltage amplitude across the resistor, so that

$$v_R = V_R \cos\omega t.$$

- Resistors are in phase with the voltage because both v_R and v are proportional to $\cos\omega t$. Their phasors therefore rotate together.



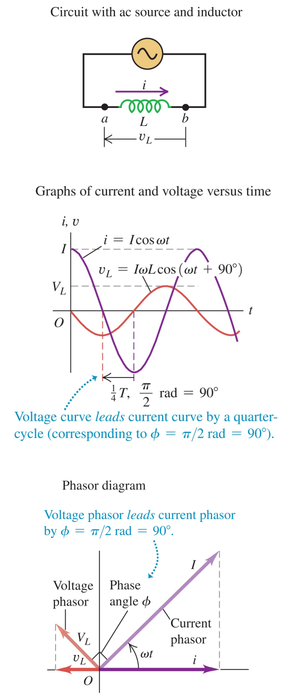
Inductor in an Ac Circuit

- Inductors behave differently compared to resistors because the voltage across an inductor is proportional to the rate of change of the current:

$$v_L = L\frac{\mathrm{d}i}{\mathrm{d}t} = -I\omega L \sin\omega t = I\omega L\cos(\omega t + \phi).$$

- As a result, inductors are out of phase with the current by $\phi = 90°$, which is the **phase angle.** The phasor for the inductor is ahead of the voltage by ϕ.
- The amplitude of the inductor voltage is $V_L = I\omega L$. We define the **inductive reactance** X_L as $X_L = \omega L$.
- This allows us to write V_L as

$$V_L = IX_L.$$



Example 31.2: An Inductor in an Ac Circuit

The current amplitude in a pure inductor in a radio receiver needs to be 250 μA when the voltage amplitude is 3.60 V at a frequency of 1.60 MHz (at the upper end of the AM broadcast band). (a) What inductive reactance is needed? What inductance? (b) If the voltage amplitude is kept constant, what will be the current amplitude through the inductor at 16.0 MHz? At 160 kHz?

(a) The inductive reactance needs to be

$$X_L = \frac{V_L}{I} = \frac{3.60 \text{ V}}{250 \times 10^{-6} \text{ A}} = 1.44 \times 10^4 \ \Omega,$$

which corresponds to an inductance of

$$L = \frac{X_L}{2\pi f} = \frac{1.44 \times 10^4 \ \Omega}{2\pi(1.60 \times 10^6 \text{ Hz})} = 1.43 \times 10^{-3} \text{ H}.$$

(b) The current amplitude is $I = V_L/X_L = V_L/2\pi f L$, so at 16.0 MHz and 160 kHz, we get

$$I_1 = \frac{3.60 \text{ V}}{2\pi(1.43 \times 10^{-3} \text{ H})(16.0 \text{ MHz})} = 25 \text{ μA},$$

along with

$$I_2 = \frac{3.60 \text{ V}}{2\pi(1.43 \times 10^{-3} \text{ H})(160 \text{ kHz})} = 2.5 \text{ mA}.$$

Capacitor in an Ac Circuit

- Capacitors also behave differently compared to resistors since its voltage is proportional to the charge:

$$i = \frac{\mathrm{d}q}{\mathrm{d}t} = I\cos\omega t \quad \rightarrow \quad v_C = \frac{q}{C} = \frac{I}{\omega C}\sin\omega t.$$

- The phase angle in this case is $\phi = -90°$:

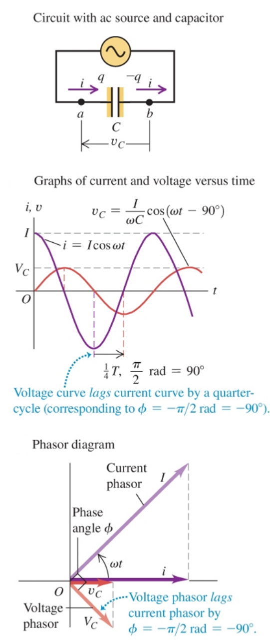
$$v_C = \frac{I}{\omega C}\cos(\omega t + \phi).$$

- The voltage amplitude is

$$V_C = \frac{I}{\omega C}.$$

- We can therefore define the capacitive reactance as $X_C = 1/\omega C$, which gives us a voltage amplitude of

$$V_C = IX_C.$$



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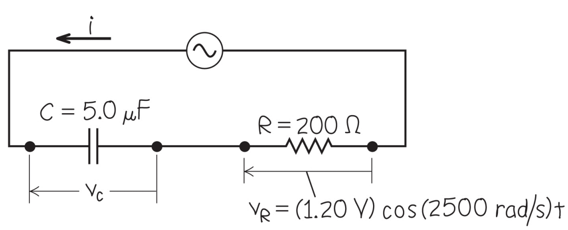
Example 31.3: A Resistor and a Capacitor in an Ac Circuit

A $200\ \Omega$ resistor is connected in series with a $5.0\ \mu$F capacitor. The voltage across the resistor is $v_R = (1.20\ \text{V})\cos[(2500\ \text{rad/s})t]$. (a) Derive an expression for the circuit current. (b) Determine the capacitive reactance of the capacitor. (c) Derive an expression for the voltage across the capacitor.



(a) The current is

$$i = \frac{v_R}{R} = \frac{(1.20\ \text{V})\cos[(2500\ \text{rad/s})t]}{200\ \Omega} = (6\times 10^{-3}\ \text{A})\cos[(2500\ \text{rad/s})t].$$

(b) The capacitive reactance is given by

$$X_C = \frac{1}{\omega C} = \frac{1}{(2500\ \text{rad/s})(5.0\times 10^{-6}\ \text{F})} = 80\ \Omega.$$

(c) First, we must find the voltage amplitude V_C:

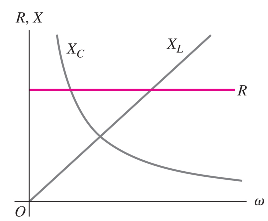
$$V_C = IX_C = (6.0 \times 10^{-3}\ \text{A})(80\ \Omega) = 0.48\ \text{V}$$

Then the voltage across the capacitor is

$$v_C = V_C \cos(\omega t - 90°) = (0.48\ \text{V}) \cos[(2500\ \text{rad/s})t - \pi/2\ \text{rad}].$$

Comparing Ac Circuit Elements

- The reactances X_L and X_C both depend on ω in different ways, while R remains constant regardless of ω.
- If $\omega = 0$ corresponding to a dc circuit, there is no current through a capacitor since $X_C \to \infty$.
- If $\omega \to \infty$, the current through an inductor becomes vanishingly small because $X_L \to \infty$.



- To summarize the relationships of voltage and current amplitudes for the three circuit elements:

Circuit Element	Amplitude Relationship	Circuit Quantity	Phase of v
Resistor	$V_R = IR$	R	In phase with i
Inductor	$V_L = IX_L$	$X_L = \omega L$	Leads i by $90°$
Capacitor	$V_C = IX_C$	$X_C = 1/\omega C$	Lags i by $90°$

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The L-R-C Series Circuit (1 of 2)

- When a resistor, inductor, and capacitor are in series with an ac source, the voltage across the circuit is equal to the sum of the instantaneous voltages across each of the elements.
  - Suppose the current in the circuit is $i = I\cos\omega t$.
  - The amplitudes for the voltages across each circuit element are

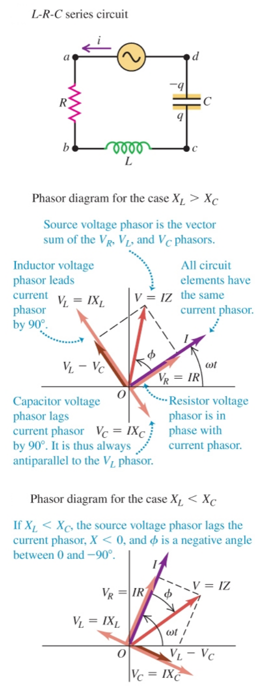
$$V_R = IR, \quad V_L = IX_L, \quad V_C = IX_C.$$

  - To find the total amplitude of voltage across the circuit, we must consider the phasor diagram for the circuit and take the vector sum of V_R, V_L, and V_C:

$$V = \sqrt{V_R^2 + (V_L - V_C)^2} = I\sqrt{R^2 + (X_L - X_C)^2}.$$

  - We can then define the **impedance** Z as

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad \rightarrow \quad V = IZ.$$



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The L-R-C Series Circuit (2 of 2)

- The impedance plays a similar role to resistance in a dc circuit, but is instead a function of R, L, C, and ω:

$$Z = \sqrt{R^2 + [\omega L - (1/\omega C)]^2}.$$

- Using the phasor diagrams, we can determine the phase angle ϕ between the current and the voltage using the fact that

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{X_L - X_C}{R} = \frac{\omega L - 1/\omega C}{R}.$$

- The source voltage can therefore be written as

$$v = V\cos(\omega t + \phi).$$

- The rms voltage V_{rms} also obeys a similar relation to $V = IZ$:

$$V_{\text{rms}} = I_{\text{rms}} Z.$$

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Example 31.4: An L-R-C Series Circuit I

In the series circuit of the figure on the previous page, suppose $R = 300\ \Omega$, $L = 60$ mH, $C = 0.50$ μF, $V = 50$ V, and $\omega = 10,000$ rad/s. Find the reactances X_L and X_C, the impedance Z, the current amplitude I, the phase angle ϕ, and the voltage amplitude across each circuit element.

For the reactances, we have

$$X_L = \omega L = (10,000 \text{ rad/s})(60 \text{ mH}) = 600\ \Omega, \qquad X_C = \frac{1}{\omega C} = \frac{1}{(10,000 \text{ rad/s})(0.5 \times 10^{-6} \text{ F})} = 200\ \Omega.$$

The total impedance Z is therefore

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(300\ \Omega)^2 + (600\ \Omega - 200\ \Omega)^2} = 500\ \Omega.$$

Meanwhile, the current amplitude I and phase angle ϕ are

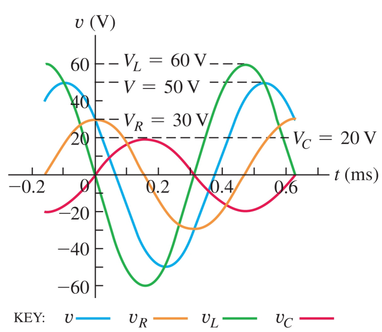
$$I = \frac{V}{Z} = \frac{50 \text{ V}}{500\ \Omega} = 0.10 \text{ A}, \qquad \phi = \arctan\left(\frac{X_L - X_C}{R}\right) = \arctan\left(\frac{400\ \Omega}{300\ \Omega}\right) = 53°.$$

Finally, the voltage amplitudes are

$$\begin{aligned} V_R &= IR = (0.10 \text{ A})(300\ \Omega) = 30 \text{ V}, \\ V_C &= IX_C = (0.10 \text{ A})(200\ \Omega) = 20 \text{ V}, \\ V_L &= IX_L = (0.10 \text{ A})(600\ \Omega) = 60 \text{ V}. \end{aligned}$$

Example 31.5: An L-R-C Series Circuit II

For the L-R-C series circuit of the previous example, find expressions for the time dependence of the instantaneous current i and the instantaneous voltages across the resistor (v_R), inductor (v_L), capacitor (v_C), and ac source (v).



The current and voltages all oscillate with the same frequency $\omega = 10,000$ rad/s. Therefore, the current is

$$i = I\cos\omega t = (0.10 \text{ A})\cos[(10,000 \text{ rad/s})t].$$

The resistor voltage is in phase with the current, so its voltage is

$$v_R = V_R\cos\omega t = (30 \text{ V})\cos[(10,000 \text{ rad/s})t].$$

The inductor voltage leads the current by a phase factor of $\phi = 90°$, so we get

$$v_L = V_L \cos(\omega t + 90°) = -V_L \sin \omega t = -(60 \text{ V}) \sin[(10,000 \text{ rad/s})t].$$

On the other hand, the capacitor voltage lags behind the current by $90°$, so it is instead

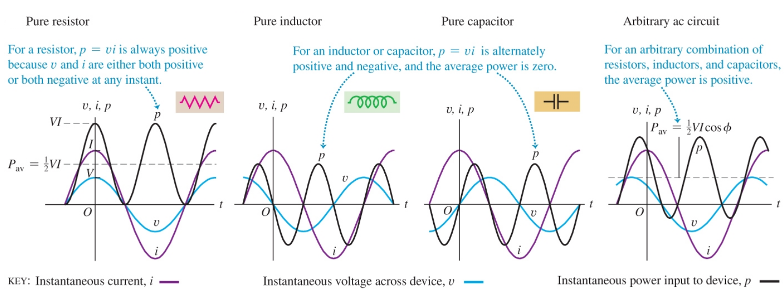
$$v_C = V_C \cos(\omega t - 90°) = V_C \sin \omega t = (20 \text{ V}) \sin[(10,000 \text{ rad/s})t].$$

For the source voltage, we found in the previous example that the phase angle of the entire circuit is $\phi = 53° = 0.93$ rad, which gives us

$$v = V \cos(\omega t + \phi) = (50 \text{ V}) \cos[(10,000 \text{ rad/s})t + 0.93 \text{ rad}].$$

Power in Alternating-Current Circuits

- The instantaneous power delivered to a circuit element with an instantaneous potential difference v and with current i running through it is $p = vi$. Suppose we have a circuit with $i = I \cos \omega t$:
  - For a resistor: $p_R = v_R i = I^2 R \cos^2 \omega t$.
  - For an inductor: $p_L = v_L i = -I^2 \omega L \sin \omega t \cos \omega t$.
  - For a capacitor: $p_C = v_C i = (I^2/\omega C) \sin \omega t \cos \omega t$.
- Resistors always have positive p_R, and the average power delivered is $P_{\text{avg}} = I_{\text{rms}}^2 R = V_{\text{rms}} I_{\text{rms}}$.
- Capacitors and inductors can have energy deposited and discharged, so p_L and p_C can be negative.



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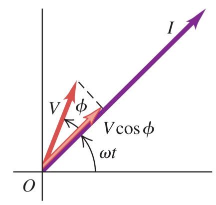
Power in a General Ac Circuit

- In general, the instantaneous power in a circuit is given by

$$p = vi = [V\cos(\omega t + \phi)](I\cos\omega t).$$

Average power $= \frac{1}{2}I(V\cos\phi)$, where $V\cos\phi$ is the component of V in phase with I.



- Using a trig identity, we can rewrite p as

$$\begin{aligned} p &= [V(\cos\omega t\cos\phi - \sin\omega t\sin\phi)](I\cos\omega t) \\ &= VI\cos\phi\cos^2\omega t - VI\sin\phi\cos\omega t\sin\omega t. \end{aligned}$$

- The average power is then

$$P_{\text{avg}} = \frac{1}{2}VI\cos\phi = V_{\text{rms}}I_{\text{rms}}\cos\phi.$$

- The factor $\cos\phi$ is called the **power factor**, and when $\phi = 0$, $P_{\text{avg}} = V_{\text{rms}}I_{\text{rms}}$, which only occurs for a pure resistance. On the other hand, for a pure inductor or capacitor, $\phi = \pm 90^\circ$, and $P_{\text{avg}} = 0$.

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Example 31.6: Power in a Hair Dryer

An electric hair dryer is rated at 1500 W (the *average* power) at 120 V (the *rms* voltage). Calculate (a) the resistance, (b) the rms current, and (c) the maximum instantaneous power. Assume that the dryer is a pure resistor. (The heating element acts as a resistor.)

(a) The resistance is

$$R = \frac{V_{\text{rms}}^2}{P_{\text{avg}}} = \frac{(120\text{ V})^2}{1500\text{ W}} = 9.6\ \Omega.$$

(b) The current is

$$I_{\text{rms}} = \frac{P_{\text{avg}}}{V_{\text{rms}}} = \frac{1500\text{ W}}{120\text{ V}} = 12.5\text{ A}.$$

(c) Since the hair dryer only has pure resistance, the maximum instantaneous power is

$$p_{\max} = VI = 2P_{\text{avg}} = 2(1500\text{ W}) = 3000\text{ W}.$$

Example 31.7: Power in an L-R-C Series Circuit

For the L-R-C series circuit of the second-to-last example, (a) calculate the power factor and (b) calculate the average power delivered to the entire circuit and to each circuit element.

(a) The power factor is

$$\cos\phi = \cos(53°) = 0.60.$$

(b) The average power is

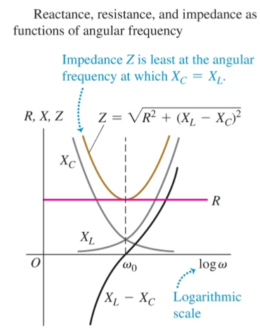
$$P_{\text{avg}} = \frac{1}{2}VI\cos\phi = \frac{1}{2}(50\text{ V})(0.10\text{ A})(0.60) = 1.5\text{ W}.$$

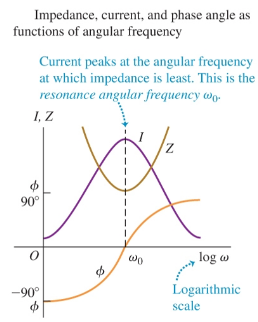
Resonance in Alternating-Current Circuits (1 of 2)

- As the angular frequency of the source ω is varied, the maximum value of I occurs at the frequency for which the impedance Z is minimized.
  - This peaking of the current amplitude at a certain frequency is called **resonance.**
  - The angular frequency ω_0 at which the resonance peak occurs is called the **resonance angular frequency.**
- At $\omega = \omega_0$, the inductive and capacitive reactances are equal, so $Z = R$ and $\omega_0 L = 1/\omega_0 C$

$$\omega_0 L = \frac{1}{\omega_0 C} \quad \rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}}.$$

- The resonance frequency is equal to the natural frequency of oscillation for an L-C circuit.





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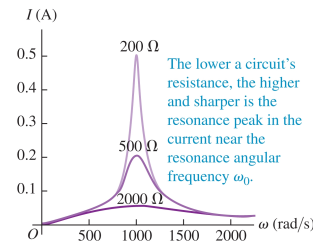
Resonance in Alternating-Current Circuits (2 of 2)

- An L-R-C series circuit with an ac current can be tailored to result in the maximum current amplitude.
- This behavior is analogous to a damped harmonic oscillator with a driving force.



- At the resonance frequency ω_0, the impedance in the circuit is at a minimum, and the current amplitude is maximized, which results in a peak around ω_0 in the current amplitude as a function of ω.
  - The resulting curve will have a sharper peak around ω_0 for lower values of resistance.
  - At resonance, $Z = R$ and $I = V/R$, so the height of the resonance curve is inversely proportional to R.

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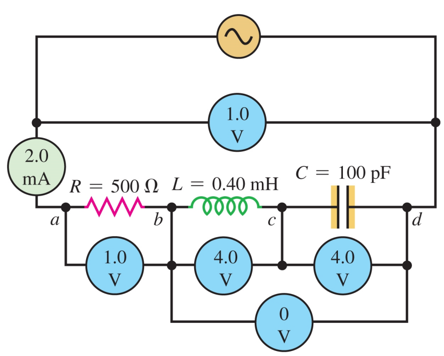
Example 31.8: Tuning a Radio

The circuit shown below is similar to a radio tuning circuit, with an rms terminal voltage of 1.0 V. (a) Find the resonance frequency ω_0. At ω_0, find (b) the inductive reactance X_L, the capacitive reactance X_C, and the impedance Z; (c) the rms current I_{rms}; (d) the rms voltage across each circuit element.



(a) For ω_0, we have

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(0.40 \times 10^{-3}\ \text{H})(100 \times 10^{-12}\ \text{F})}} = 5.0 \times 10^6\ \text{rad/s}.$$

(b) At ω_0, the reactances are

$$X_L = \omega_0 L = 2000\ \Omega, \qquad X_C = \frac{1}{\omega_0 C} = 2000\ \Omega \quad \rightarrow \quad X_L = X_C \quad \rightarrow \quad Z = R = 500\ \Omega.$$

(c) The rms current at resonance is

$$I_{\rm rms} = \frac{V_{\rm rms}}{R} = \frac{1.0\ \text{V}}{500\ \Omega} = 2.0\ \text{mA}.$$

(d) The rms potential difference across the resistor is

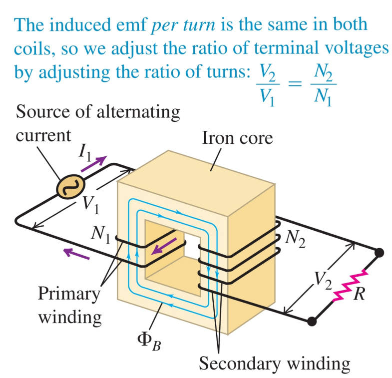
$$V_{R,\rm rms} = I_{\rm rms} R = (0.0020\ \text{A})(500\ \Omega) = 1.0\ \text{V}.$$

For the rms potential differences across the inductor and capacitor, we get

$$V_{L,\rm rms} = I_{\rm rms} X_L = (0.0020\ \text{A})(2000\ \Omega) = 4.0\ \text{V},$$
$$V_{C,\rm rms} = I_{\rm rms} X_C = (0.0020\ \text{A})(2000\ \Omega) = 4.0\ \text{V}.$$

Transformers (1 of 3)

- One reason why we use ac power instead of dc power is because it is much easier to step voltage levels up or down with ac than with dc.
- This is accomplished using **transformers,** which are represented by the following symbol:



- A transformer has power supplied to a **primary** coil wrapped around a core with high relative permeability K_m (typically iron), which is connected to a **secondary** coil that then delivers the power to a resistor.
- The purpose of a transformer like the example shown is to increase the delivered voltage relative to the supplied voltage.

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Transformers (2 of 3)

- A transformer works by having the ac source set up an alternating flux in the core, which produces an emf in each winding of the coils.
- The emfs in the cores are

$$\mathcal{E}_1 = -N_1 \frac{\mathrm{d}\Phi_B}{\mathrm{d}t}, \quad \mathcal{E}_2 = -N_2 \frac{\mathrm{d}\Phi_B}{\mathrm{d}t}.$$

- The flux per turn Φ_B is the same in both coils, so the ratio of the emfs is

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}.$$

- If the windings of the coils have zero resistance, then $\mathcal{E}_1$ and $\mathcal{E}_2$ are equal to the terminal voltages across the primary and secondary, and thus

$$\frac{V_2}{V_1} = \frac{N_2}{N_1}.$$

- If $N_2 > N_1$, then $V_2 > V_1$, and we have a **step-up** transformer.
- If $N_2 < N_1$, then $V_2 < V_1$, and we have a **step-down** transformer.

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Transformers (3 of 3)

- If the secondary circuit is completed by a resistance R, then the amplitude of the current through it is $I_2 = V_2/R$. Since the power delivered to the primary equals that taken out of the secondary, then we have

$$V_1 I_1 = V_2 I_2.$$

- Using the fact that $V_2/V_1 = N_2/N_1$,

$$\frac{V_1}{I_1} = \frac{R}{(N_2/N_1)^2}.$$

- This shows that transformers can transform not only voltages and currents, but resistances as well. In fact, transformers can be thought of as transforming the impedance of the network to which the secondary circuit is completed.

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Example 31.9: Transformer for a Coffeemaker

A friend returns to the US from Europe with a 960 W coffeemaker, designed to operate from a 240 V line. (a) What can she do to operate it at the USA-standard 120 V? (b) What current will the coffeemaker draw from the 120 V line? (c) What is the resistance of the coffeemaker? (The voltages are rms values.)

(a) She can use a step-up transformer to convert 120 V ac to the 240 V ac that the coffeemaker requires. Since $N_2/N_1 = V_2/V_1$, she requires a ratio of $N_2/N_1 = (240 \text{ V})/(120 \text{ V}) = 2$.

(b) The average power is $P_{\text{avg}} = V_1 I_1$, so we have

$$I_1 = \frac{P_{\text{avg}}}{V_1} = \frac{960 \text{ W}}{120 \text{ V}} = 8.0 \text{ A},$$

while the secondary current is

$$I_2 = \frac{P_{\text{avg}}}{V_2} = \frac{960 \text{ W}}{240 \text{ V}} = 4.0 \text{ A}.$$

(c) We have $V_1 = 120$ V, $I_1 = 8.0$ A, and $N_2/N_1 = 2$, so

$$\frac{V_1}{I_1} = \frac{120 \text{ V}}{8.0 \text{ A}} = 15 \ \Omega \quad \rightarrow \quad R = \left(\frac{N_2}{N_1}\right)^2 \frac{V_1}{I_1} = (2)^2(15 \ \Omega) = 60 \ \Omega.$$