Math 170E: Winter 2023

Lecture 23, Fri 10th Mar

Functions of a random variable and several random variables

$X \rightsquigarrow Y = u(X)$.

$\hookrightarrow$ What is the PMF/PDF of Y?

Example 1:

- Let $X \sim \text{Uniform}([-1, 1])$
- Let $Y = X^2$.
- What is the PDF of Y?

Key: Compute CDF $F_Y(y)$ & then set $f_Y(y) = F_Y'(y)$.

- Y takes values in $[0,1]$.
- If $y \le 0$, $F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = 0$.
- If $y \ge 1$, $F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(X \in [-1,1]) = 1$.
- If $0 < y < 1$: $F_Y(y) = \boxed{\mathbb{P}(X^2 \le y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y})}$.

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\,dx = \sqrt{y}.$$

$$\hookrightarrow F_Y(y) = \begin{cases} 0 & \text{if } y \le 0 \\ \sqrt{y} & \text{if } 0 < y < 1 \\ 1 & \text{if } y \ge 1 \end{cases}$$

$$\hookrightarrow f_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{2\sqrt{y}} & \text{if } 0 < y < 1 \\ 0 & \text{if } y > 1 \end{cases} = \begin{cases} \frac{1}{2\sqrt{y}} & \text{if } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

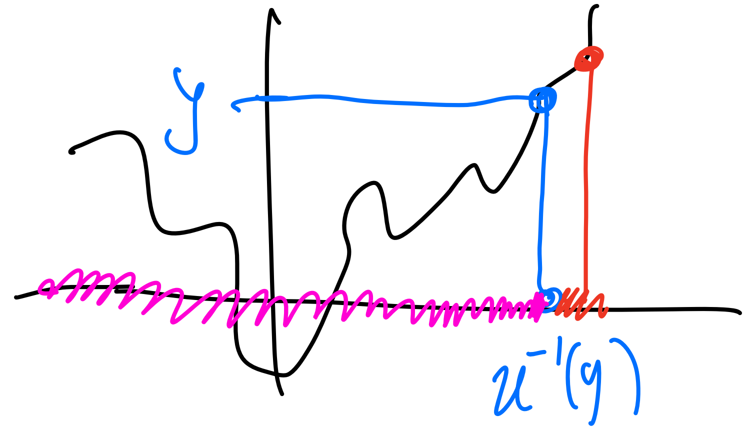
$$F_Y(y) = \mathbb{P}(u(X) \le y) \text{ “}= \mathbb{P}(X \le u^{-1}(y))\text{”}$$

$x \longmapsto x^2 = u(x) \rightarrow$ is not one-to-one.

there are pts $X_1 \neq X_2$ for which $u(X_1) = u(X_2)$.

$$\mathbb{P}(X \in A)$$

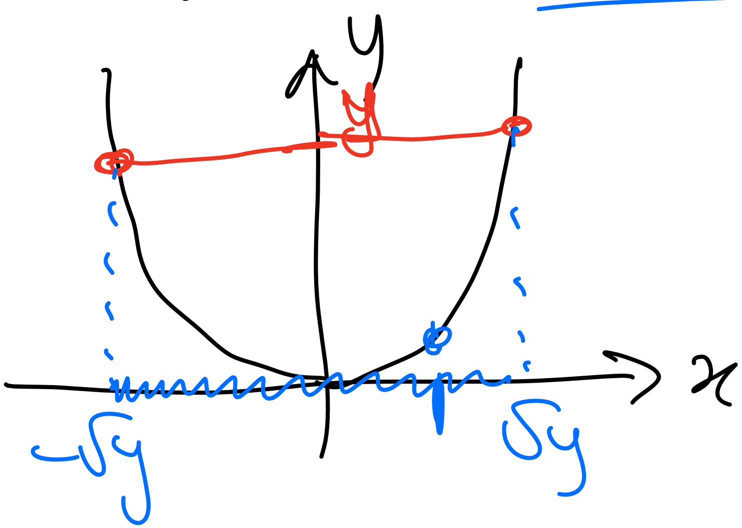
$$A = \{x : u(x) \le y\},$$
$$= u^{-1}((-\infty, y]).$$



2D

$u(x) = x^2$:

If $0 < y < 1$, $u^{-1}((-\infty, y]) = \{x : u(x) \le y\}$
$$= \{x : x^2 \le y\}.$$
$$= \{x : -\sqrt{y} \le x \le \sqrt{y}\}.$$
$$= [-\sqrt{y}, \sqrt{y}]$$



Discrete case: Sps X is discrete & $Y=u(X)$.

$$p_Y(y) = \mathbb{P}(Y=y) = \mathbb{P}(u(X)=y).$$
$$= \mathbb{P}(X \in \underbrace{u^{-1}(\{y\})}_{=\{x:\, u(x)=y\}})$$

$$= \sum_{x \in S \cap u^{-1}(\{y\})} p_X(x).$$

e.g. if $Y = X^2$, if $0<y<1$

$$\mathbb{P}(X \in u^{-1}(\{y\})) = \mathbb{P}(X=-\sqrt{y}, X=\sqrt{y}).$$

$$u^{-1}(\{y\}) = \{x : u(x)=y\} = \{x : x^2=y\} = \{-\sqrt{y}, \sqrt{y}\}$$

Continuous case:

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(u(X) \le y)$$
$$= \mathbb{P}(X \in u^{-1}((-\infty, y])).$$

$$= \int_{u^{-1}((-\infty,y])} f_X(x)\,dx.$$

In general, we can't say anything about $u^{-1}((-\infty, y])$.

If we make some assumptions on u, then we can say something:

Proposition 5.1:

Let X be continuous random variable with PDF $f_X(x)$

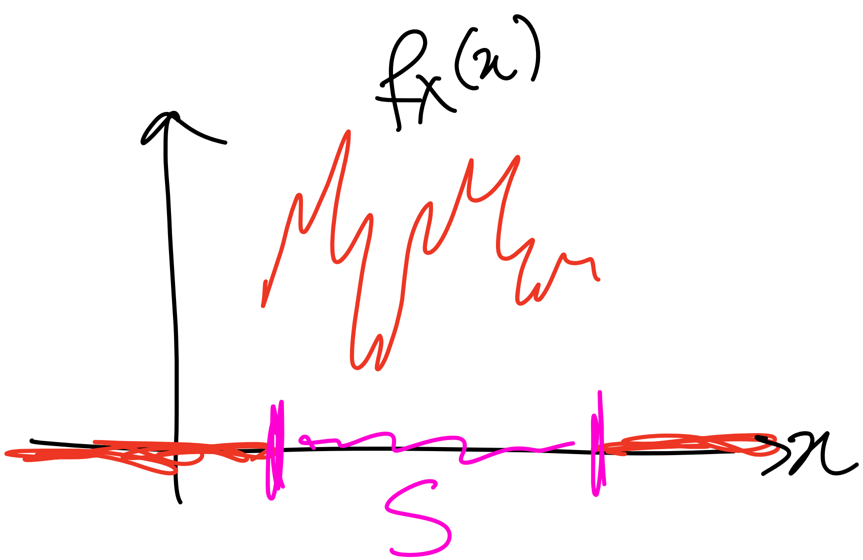
u is decreasing on S.

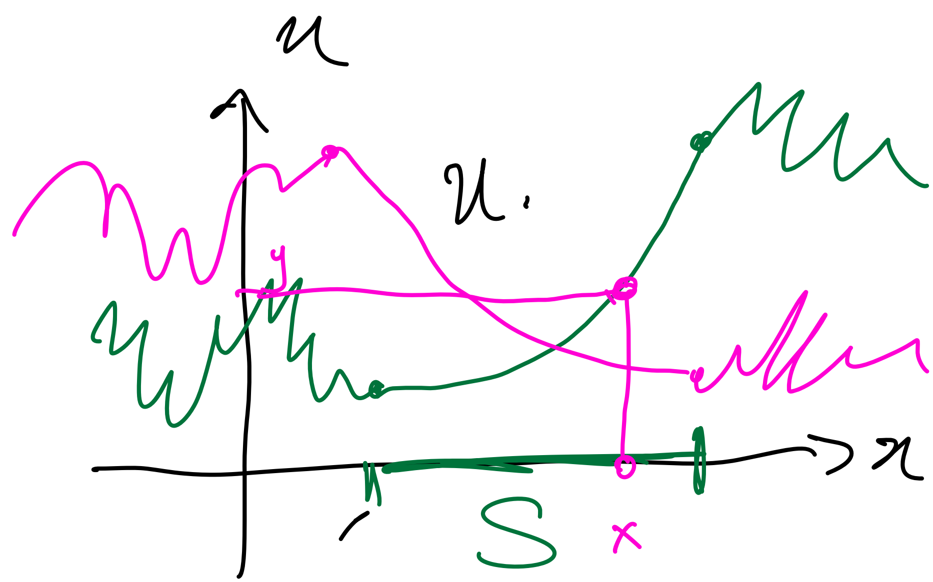
- Let $S \subseteq \mathbb{R}$ so that $f_X(x) = 0$ for all $x \in \mathbb{R} \setminus S$
- Let $u : \mathbb{R} \to \mathbb{R}$ be smooth and satisfy $u'(x) > 0$ or $u'(x) < 0$ for all $x \in S$.
- Then $Y = u(X)$ has PDF

u is increasing on S.

$$f_Y(y) = \left| \frac{d}{dy} u^{-1}(y) \right| \cdot f_X(u^{-1}(y))$$

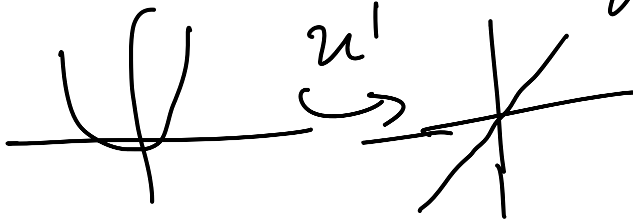
Proof:





Previous example: $S=[-1,1]$, $u(x)=x^2$, $u'(x)=2x>0$ if $x>0$

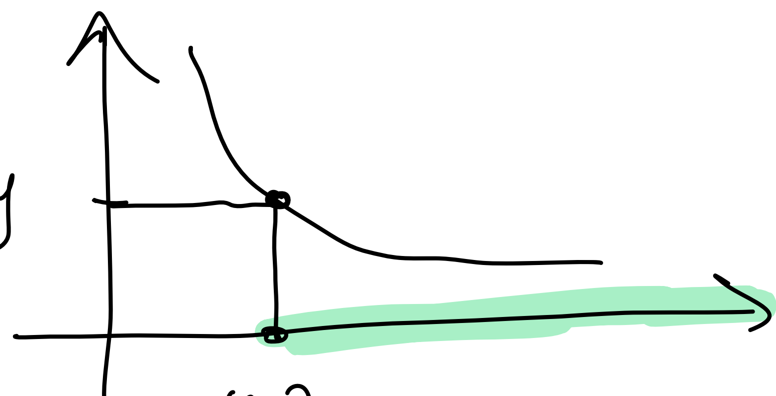
but $u'(x)<0$ if $x<0$



Why? $\mathbb{P}(Y\le y)=\mathbb{P}(u(X)\le y)$

$= \mathbb{P}(X\in \underbrace{u^{-1}((-\infty,y])}_{\{x:\, u(x)\le y\}})$

Suppose u is strictly decreasing on S ($u'(x)<0$).



As u strictly decreasing,

$$u^{-1}((-\infty, y]) = [u^{-1}(y), +\infty).$$

$$\mathbb{P}(Y \le y) = \mathbb{P}(X \in [u^{-1}(y), \infty)) = \mathbb{P}(X \ge u^{-1}(y))$$

$$= \int_{u^{-1}(y)}^{\infty} f_X(x)\,dx.$$

Several random variables

Definition 5.4:

- Let $X_1, X_2, \ldots, X_n$ be discrete random variables taking values in sets $S_1, S_2, \ldots, S_n \subseteq \mathbb{R}$ and let $S = S_1 \times S_2 \times \cdots \times S_n \subseteq \mathbb{R}^n$.

- We define their joint PMF to be

$$p_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = \mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

- We define the marginal PMF of X_j to be

$$\begin{aligned} p_{X_j}(x_j) &= \mathbb{P}(X_j = x_j) \\ &= \sum_{x_1 \in S_1} \cdots \sum_{x_{j-1} \in S_{j-1}} \sum_{x_{j+1} \in S_{j+1}} \cdots \sum_{x_n \in S_n} p_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) \end{aligned}$$

- We say that $X_1, X_2, \cdots, X_n$ are independent if

$$p_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = p_{X_1}(x_1) p_{X_2}(x_2) \cdots p_{X_n}(x_n)$$

  for all $(x_1, x_2, \ldots, x_n)$

- We say that $X_1, X_2, \cdots, X_n$ are independent and identically distributed (i.i.d) if they are independent and $p_{X_j} = p_{X_1}$ for every $j = 2, \ldots n$

Example 4:

- Let $X_1, X_2, X_3 \sim \text{Bernoulli}(\frac{1}{4})$ be independent
- What is $\mathbb{P}(X_1 + X_2 + X_3 \geq 1)$?

i.i.d.

$$P_{X_j}(x) = \begin{cases} 1/4 & \text{if } x = 1 \\ 3/4 & \text{if } x = 0 \end{cases}$$

What is the joint PMF P_{X_1,X_2,X_3}?

Since X_1, X_2, X_3 indep., $P_{X_1,X_2,X_3}(x_1,x_2,x_3) = P_{X_1}(x_1)P_{X_2}(x_2)P_{X_3}(x_3)$.

$$= P_{X_1}(x_1)\,P_{X_1}(x_2)\,P_{X_1}(x_3)$$

$$\mathbb{P}(X_1+X_2+X_3 \geq 1) = 1 - \mathbb{P}(\underbrace{X_1+X_2+X_3 < 1}_{\Leftrightarrow X_1 = X_2 = X_3 = 0}),$$

$$= 1 - P_{X_1,X_2,X_3}(0,0,0) = 1 - P_{X_1}(0)^3$$

$$= 1 - (3/4)^3.$$

$$= \frac{37}{64}$$

Definition 5.5:

- Let $X_1, X_2, \ldots, X_n$ be continuous random variables
- We define their joint PDF

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$$

so that for any $A \subseteq \mathbb{R}^n$, we have

$$\mathbb{P}\big((X_1, X_2, \ldots, X_n) \in A\big) = \int_A f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) dx_1 \ldots dx_n$$

- We define the marginal PMF of X_j to be

$$f_{X_j}(x_j) = \int_{\mathbb{R}^{n-1}} f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_n$$

- We say that $X_1, X_2, \cdots, X_n$ are independent if

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

for all $(x_1, x_2, \ldots, x_n)$

- We say that $X_1, X_2, \cdots, X_n$ are independent and identically distributed (i.i.d) if they are independent and $f_{X_j} = f_{X_1}$ for every $j = 2, \ldots n$

Example 5:

- Let $X_1, X_2, X_3 \sim$ Exponential(1) be independent

$$f_{X_1}(x) = \begin{cases} e^{-x} & \text{if } x>0 \\ 0 & \text{otherwise.} \end{cases}$$

- What is $\mathbb{P}(\min(X_1, X_2, X_3) > 1)$?

Joint pdf: $f_{X_1,X_2,X_3}(x_1,x_2,x_3) \overset{\text{indep}}{=} \prod_{j=1}^{3} f_{X_j}(x_j)$

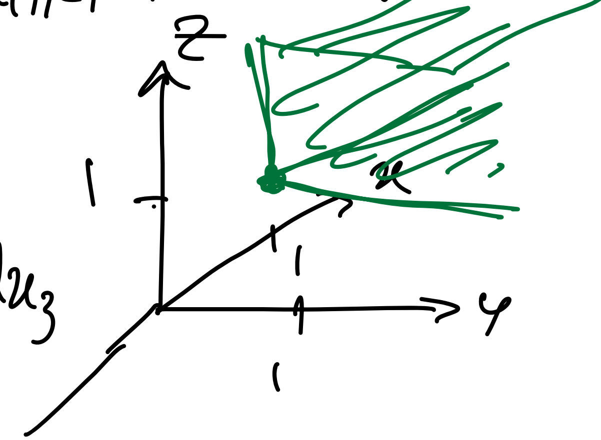
$$\{\min(X_1,X_2,X_3)>1\} = \{X_1>1, X_2>1, X_3>1\}.$$

$$\mathbb{P}(\quad) = \mathbb{P}(X_1>1, X_2>1, X_3>1).$$

$$A = \{(x,y,z)\in\mathbb{R}^3 : x>1, y>1, z>1\}$$

$$= \mathbb{P}((X_1,X_2,X_3)\in A)$$

$$= \int_1^\infty\int_1^\infty\int_1^\infty f_{X_1,X_2,X_3}(x_1,x_2,x_3)\,dx_1\,dx_2\,dx_3$$



$$= \int_1^\infty\int_1^\infty\int_1^\infty f_{X_1}(x_1)\,f_{X_1}(x_2)\,f_{X_1}(x_3)\,dx_1\,dx_2\,dx_3.$$

$$= \underbrace{\left(\int_1^\infty f_{X_1}(x)\,dx\right)^3}_{\mathbb{P}(X_1>1)^3} = \left(\int_1^\infty e^{-x}\,dx\right)^3 = e^{-3}.$$

If not ident. distributed but still indep,

$$\mathbb{P}(X_1>1, X_2>1, X_3>1) = \prod_{j=1}^{3} \mathbb{P}(X_j>1).$$

Definition 5.6:

- Let $u : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$

- If $X_1, \ldots, X_n$ are discrete random variables, we define

$$\mathbb{E}[u(X_1, \ldots, X_n)] = \sum_{(x_1, \ldots, x_n) \in S} u(x_1, \ldots, x_n) p_{X_1, \ldots, X_n}(x_1, \ldots, x_n)$$

- If $X_1, \ldots, X_n$ are discrete random variables, we have

$$\mathbb{E}[g(X_j)] = \sum_{x_j \in S_j} g(x_j) p_{X_j}(x_j)$$

Definition 5.7:

- Let $u : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$
- If $X_1, \ldots, X_n$ are continuous random variables, we define

$$\mathbb{E}[u(X_1, \ldots, X_n)] = \int_{\mathbb{R}^n} u(x_1, \ldots, x_n) f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

- If $X_1, \ldots, X_n$ are continuous random variables, we have

$$\mathbb{E}[g(X_j)] = \int_{-\infty}^{\infty} g(x_j) f_{X_j}(x_j) dx_j$$

Proposition 5.8:

- Let $X_1, \ldots, X_n$ be discrete or continuous random variables

- If $u, v : \mathbb{R}^n \to \mathbb{R}$ and $a, b \in \mathbb{R}$

$$\begin{aligned}&\mathbb{E}[au(X_1, \ldots, X_n) + bv(X_1, \ldots, X_n)] \\ &\quad = a\mathbb{E}[u(X_1, \ldots, X_n)] + b\mathbb{E}[v(X_1, \ldots, X_n)]\end{aligned}$$

- If $u(x_1, \ldots, x_n) \leq v(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n)$, then

$$\mathbb{E}[u(X_1, \ldots, X_n)] \leq \mathbb{E}[v(X_1, \ldots, X_n)]$$

Proof:

Proposition 5.8:

Let $X_1, \ldots, X_n$ be discrete or continuous random variables. Let $a_1, \ldots, a_n \in \mathbb{R}$ and let

$$Y = a_1 X_1 + \ldots + a_n X_n$$

→ linear combination of $\{X_j\}_{j=1}^{n}$.

Then

$$\mathbb{E}[Y] = a_1 \mathbb{E}[X_1] + \ldots + a_n \mathbb{E}[X_n]$$

$$= \sum_{j=1}^{n} a_j \mathbb{E}[X_j].$$

Proof: