## CS 181 PRACTICE MIDTERM 2B

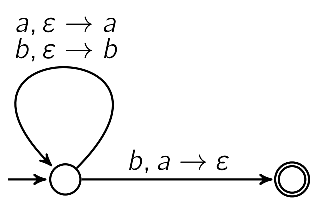
**You may state without proof any fact taught in lecture.**

**1** Describe the languages corresponding to the following CFGs/PDAs with alphabet $\Sigma = \{a, b\}$. You may provide a verbal description or a regular expression.

**a.** $S \to aSa \mid aSb \mid bSa \mid bSb \mid a$

**b.** $S \to Sa \mid abB$
$B \to bB \mid \varepsilon$

**c.**



**2** Draw a pushdown automaton for the language of strings over the alphabet $\{a, b\}$ in which the number of $a$'s is not equal to the number of $b$'s.

**3** Give context-free grammars for the following languages over the alphabet $\{a, b\}$:

**a.** strings that contain a pair of $a$'s separated by an even number of symbols;

**b.** strings that contain exactly two more $a$'s than $b$'s.

**4** Consider the context-free grammar $S \to SaS \mid aS \mid Sa \mid b$, over the alphabet $\{a, b\}$.

- **a.** Describe the language generated by this grammar.
- **b.** Prove that this grammar is ambiguous.
- **c.** Give an equivalent unambiguous grammar.

**5** Prove or disprove: if $L$ is not context-free and $F$ is finite, then $L \setminus F$ is not context-free.

**6** Given a language $L$, define $L^\diamond = \{v^k : v \in L \text{ and } k \geqslant 0\}$. Prove or disprove: if $L$ is context-free, so is $L^\diamond$.

**7** For a language $L$, let $\mathrm{Drop}(L)$ denote the set of all strings that can be obtained by taking a nonempty string in $L$ and deleting a single character from it. Given a context-free grammar $G$ for $L$, explain in detail how to modify $G$ to obtain a context-free grammar for $\mathrm{Drop}(L)$. Your solution must not use pushdown automata in any way.

**8** For each of the following languages $L$, determine whether it is context-free and prove your answer:

**a.** strings of the form $ss^Rs$, where $s \in \{0, 1\}^*$;

**b.** palindromes over the decimal alphabet that represent even integers, namely, $\{0, 2, 4, 6, 8, 22, 44, \ldots\}$.

# SOLUTIONS

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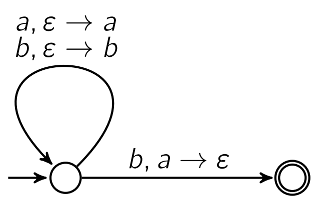
**You may state without proof any fact taught in lecture.**

**1** Describe the languages corresponding to the following CFGs/PDAs with alphabet $\Sigma = \{a, b\}$. You may provide a verbal description or a regular expression.

**a.** $S \to aSa \mid aSb \mid bSa \mid bSb \mid a$

**b.** $S \to Sa \mid abB$
$B \to bB \mid \varepsilon$

**c.**




**Solution:**

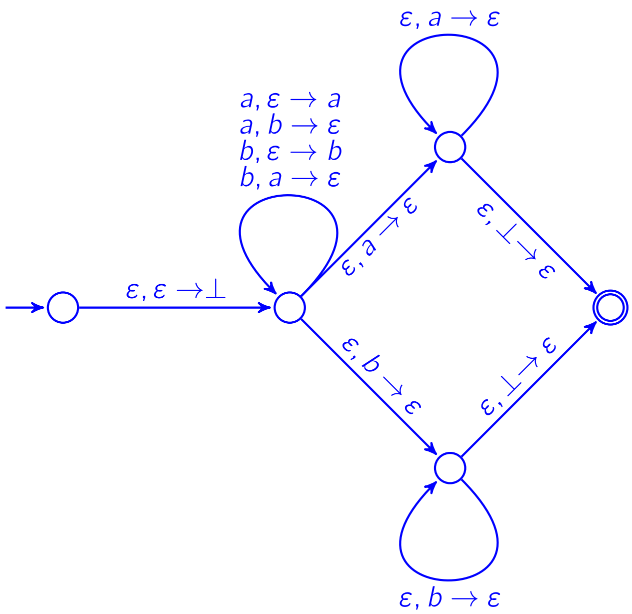
**a.** odd-length strings with middle symbol $a$;

**b.** $ab^+a^*$;

**c.** $\Sigma^*ab$.

**2** Draw a pushdown automaton for the language of strings over the alphabet $\{a, b\}$ in which the number of $a$'s is not equal to the number of $b$'s.

**Solution.** It is helpful to view this language as the union of two simpler languages: strings with an excess of $a$'s and strings with an excess of $b$'s. One way to implement this idea is shown below.



**3** Give context-free grammars for the following languages over the alphabet $\{a, b\}$:

**a.** strings that contain a pair of $a$'s separated by an even number of symbols;

**b.** strings that contain exactly two more $a$'s than $b$'s.

**Solution:**

**a.** $S \to XaYaX$
$X \to aX \mid bX \mid \varepsilon$
$Y \to aaY \mid abY \mid baY \mid bbY \mid \varepsilon$

**b.** $S \to EaEaE$
$E \to aEbE \mid bEaE \mid \varepsilon$ ($E$ generates strings with equally many $a$'s and $b$'s.)

**4** Consider the context-free grammar $S \to SaS \mid aS \mid Sa \mid b$, over the alphabet $\{a, b\}$.

**a.** Describe the language generated by this grammar.

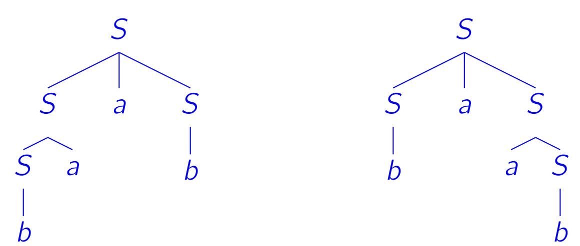
**b.** Prove that this grammar is ambiguous.

**c.** Give an equivalent unambiguous grammar.

**Solution.**

**a.** Strings that contain $b$ but not $bb$.

**b.** The string $baab$ has at least two parse trees:




**c.** $S \to ATA$
$A \to Aa \mid \varepsilon$
$T \to b \mid TAab$ $\qquad$ ($T$ generates $b(a^+b)^*$.)

**5** Prove or disprove: if $L$ is not context-free and $F$ is finite, then $L \setminus F$ is not context-free.

**Solution.** The claim is true. We will prove the contrapositive: if $F$ is finite and $L \setminus F$ context-free, then $L$ is context-free. For this, write

$$L = (L \setminus F) \cup (L \cap F).$$

For any finite $F$, the language $L \cap F$ is also finite, hence regular, hence context-free. We conclude that, with $F$ finite and $L \setminus F$ context-free, $L$ is the union of two context-free languages and is therefore itself context-free (by the closure of context-free languages under union).

**6** Given a language $L$, define $L^\diamond = \{v^k : v \in L \text{ and } k \geqslant 0\}$. Prove or disprove: if $L$ is context-free, so is $L^\diamond$.

**Solution.** False. Consider the regular (hence context-free!) language $L = 0^+1^+$. We will use the pumping lemma to show that $L^\diamond$ is not context-free. For this, take an arbitrary integer $p \geqslant 1$ and consider the string $w = 0^{p+1}1^{p+1}0^{p+1}1^{p+1} \in L^\diamond$. Fix any decomposition $w = uvxyz$ for some strings $u, v, x, y, z$ with $|v| + |y| \neq 0$ and $|vxy| \leqslant p$. Then pumping down must reduce the number of 0s or 1s, or both.

Case 1: Pumping down reduces the number of 0s. In this case, the length restriction $|vxy| \leqslant p$ implies that $uxz \in 0^{p_1}1^+0^{p_2}1^+$ for some positive integers $p_1, p_2$ one of which equals $p+1$ and the other is less than $p+1$. Thus, $uxz \notin L^\diamond$.

Case 2: Pumping down reduces the number of 1s. In this case, the length restriction $|vxy| \leqslant p$ implies that $uxz \in 0^+1^{p_1}0^+1^{p_2}$ for some positive integers $p_1, p_2$ one of which equals $p+1$ and the other is less than $p+1$. Again, $uxz \notin L^\diamond$.

By the pumping lemma, $L^\diamond$ is not context-free.

**7** For a language $L$, let $\mathrm{Drop}(L)$ denote the set of all strings that can be obtained by taking a nonempty string in $L$ and deleting a single character from it. Given a context-free grammar $G$ for $L$, explain in detail how to modify $G$ to obtain a context-free grammar for $\mathrm{Drop}(L)$. Your solution must not use pushdown automata in any way.

**Solution.** Let $G = (V, \Sigma, R, S)$ be given.

- *Variables.* The grammar for $\mathrm{Drop}(L)$ includes all the variables in $V$. In addition, for each $X \in V \cup \Sigma$, we create a new variable $\overline{X}$ meant to generate the Drop of the strings generated by $X$.
- *Start variable.* The start variable for $\mathrm{Drop}(L)$ is $\overline{S}$.
- *Rules.* The rules for $\mathrm{Drop}(L)$ include all of $R$. In addition, for every rule in $R$ of the form $X \to Y_1Y_2 \dots Y_k$ where $k \geqslant 1$ and $Y_1, Y_2, \dots, Y_k \in V \cup \Sigma$, we add the rules $\overline{X} \to Y_1 \dots Y_{i-1}\overline{Y_i}Y_{i+1} \dots Y_k$ for every $i = 1, 2, \dots, k$. Lastly, we add the rule $\overline{\sigma} \to \varepsilon$ for every $\sigma \in \Sigma$.

**8** For each of the following languages $L$, determine whether it is context-free and prove your answer:

**a.** strings of the form $ss^Rs$, where $s \in \{0,1\}^*$;

**b.** palindromes over the decimal alphabet that represent even integers, namely, $\{0, 2, 4, 6, 8, 22, 44, \ldots\}$.

**Solution.**

**a.** Not context-free. Take an arbitrary integer $p \geqslant 1$ and consider the string $w = 0^p1^{2p}0^{2p}1^p = (0^p1^p)(0^p1^p)^R(0^p1^p) \in L$. Fix any decomposition $w = uvxyz$ for some strings $u, v, x, y, z$ with $|v| + |y| \neq 0$ and $|vxy| \leqslant p$. We will show that $uxz \notin L$. For the sake of contradiction, suppose that $uxz = ss^Rs$ for some string $s$. The length restriction $|vxy| \leqslant p$ implies that pumping down removes at most $p$ symbols and hence $s = 0^{p'}1^{p''}$ for some integers $p', p''$ no greater than $p$. As a result, $uxz = 0^{p'}1^{2p''}0^{2p'}1^{p''}$. Comparing that with the original string $w = 0^p1^{2p}0^{2p}1^p$, we see that $p' = p$ because the restriction $|vxy| \leqslant p$ ensures that pumping down cannot simultaneously affect $w$'s leading run of zeroes and inner run of zeroes. Analogously, $p'' = p$ because pumping down cannot simultaneously affect $w$'s trailing run of ones and inner run of ones. In conclusion, $uxz = 0^p1^{2p}0^{2p}1^p$ and therefore $v = y = \varepsilon$. We have arrived at the promised contradiction since $|v| + |y| \neq 0$. Therefore, $uxz \notin L$ and $L$ is not context-free by the pumping lemma.

**b.** Context-free, with grammar

$$
\begin{aligned}
S &\to 0 \mid 2 \mid 4 \mid 6 \mid 8 \mid 2T2 \mid 4T4 \mid 6T6 \mid 8T8\\
T &\to 0T0 \mid 1T1 \mid 2T2 \mid 3T3 \mid 4T4 \mid 5T5 \mid 6T6 \mid 7T7 \mid 8T8 \mid 9T9 \mid \Sigma \mid \varepsilon\\
\Sigma &\to 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9.
\end{aligned}
$$

Note that integers other than 0 have no leading zeroes.