Math 170E: Homework 5

Due: Fri. 17th February by 11:59pm PDT via Gradescope

Submit answers to all problems via Gradescope. The reader will grade three problems each out of five points. Up to five further points will be awarded based on the proportion of the remaining problems that are completed.

Please make sure that your submission is readable. If your pencil is too faint, get a thicker one. If your handwriting is cramped and small, write bigger and use more paper. Please use simple plain paper or lined paper (e.g. please avoid graph paper etc.). It is your responsibility to ensure that your submission is readable. If we cannot read a solution, we may refuse to grade it. Thank you!

I encourage you to discuss and work on problems with other students in the class. Nevertheless, the solutions you present have to be your own. In particular, if the solution you present is identical to someone else's, or it is identical to some other resource (book, online, etc.), this will be considered cheating.

1. Customers arrive randomly at a bank teller's window. Given that one customer arrived during a particular 10-minute period, let X equal the time within the 10 minutes that the customer arrived. If $X \sim \text{Uniform}([0, 10])$, find:
   (a) The PDF of X
   (b) $\mathbb{P}(X \geq 8)$
   (c) $\mathbb{P}(2 < X \leq 8)$
   (d) $\mathbb{E}[X]$
   (e) $\text{var}(X)$

2. Let $\{X_j\}_{j=1}^k$ be continuous random variables and $\{f_{X_j}\}_{j=1}^k$ be their PDFs, respectively, each with sample space $\mathbb{R}$. Let $\{a_j\}_{j=1}^k$ be constants.
   (a) Give sufficient conditions for the constants $\{a_j\}_{j=1}^k$ so that $\sum_{j=1}^k a_j f_{X_j}(x)$ is a PDF.
   (b) If X is a continuous random variable with PDF $\sum_{j=1}^k a_j f_j(x)$ and $\mathbb{E}[X_j] = \mu_j$, $\text{var}(X_j) = \sigma_j^2$ for every $j = 1, \ldots, k$, find the mean and variance of X.

3. Let X be a continuous random variable. Show that:
   (a) If $a \in \mathbb{R}$ is a constant, then $\mathbb{E}[a] = a$
   (b) If $a, b \in \mathbb{R}$ are constants and $g, h : \mathbb{R} \to \mathbb{R}$ are functions, then

$$\mathbb{E}[ag(X) + bh(X)] = a\mathbb{E}[g(X)] + b\mathbb{E}[h(X)].$$

(c) If $g(x) \leq h(x)$ for all $x \in \mathbb{R}$, then $\mathbb{E}[g(X)] \leq \mathbb{E}[h(X)]$

4. Let $I \subset \mathbb{R}$ be an interval and X be a continuous random variable with state space $\mathbb{R}$. Define the function

$$\mathbf{1}_I(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } x \notin I. \end{cases}$$

This is called the *indicator function* of the set I. Show that $\mathbb{E}[\mathbf{1}_I(X)] = \mathbb{P}(X \in I)$ and $\mathbb{E}[\mathbf{1}_{\mathbb{R}\setminus I}(X)] = 1 - \mathbb{P}(X \in I)$.

5. A grocery store has n watermelons to sell and makes \$1.00 on each sale. Let X be the number of consumers of these watermelons which is a random variable with a distribution that can be approximated by

$$f_X(x) = \frac{1}{200} \quad \text{if } 0 < x < 200,$$

and 0 otherwise; i.e. X is a continuous random variable with PDF f_X. Suppose each customer only buys one watermelon. If the grocer does not have enough watermelons to sell to all consumers, she figures that she loses \$5.00 in goodwill from each unhappy customer. But if she has surplus watermelons, she loses 50 cents on each extra watermelon.

   (a) If $X \leq n$, what is her profit in terms of X and n?

   (b) If $X > n$, what is her profit in terms of X and n?

   (c) How many watermelons should she stock in order to maximise her profit?

6. Let X be a continuous random variable with PDF $f_X(x) = Ce^{-|x|}$ for $x \in \mathbb{R}$.

   (a) Argue why f_X can be made into a well-defined PDF.

   (b) Find the MGF of X.

   (c) Find the mean and variance of X.

   (d) Show that this distribution satisfies a $76 - 94 - 99$ rule; that is, 76% of values lie within one standard deviation of the mean, 94% within two standard deviations and 99% within three standard deviations.

7. Let $\alpha, \theta > 0$ and $X \sim \text{Gamma}(\alpha, \theta)$.

   (a) Find the MGF of X

   (b) Find the mean of X

   (c) Find the variance of X

8. Given an integer $r \geq 1$, we say that X has *chi-square* distribution with r degrees of freedom and write $X \sim \chi^2(r)$ if it has PDF

$$f_X(x) = \frac{1}{2^{\frac{r}{2}}\Gamma(\frac{r}{2})} x^{\frac{r}{2}-1} e^{-\frac{x}{2}}, \quad \text{if } x > 0.$$

By relating the Chi-square distribution with r degrees of freedom to a suitable Gamma distribution, find the MGF, mean and variance of X.

9. Let the random variable X be equal to the number of days that it takes a high-risk driver to have an accident. Assume that X has an exponential distribution. If $\mathbb{P}(X < 50) = 0.25$, compute $\mathbb{P}(X > 100 \,|\, X > 50)$.

10. Cars arrive at a tollbooth at a mean rate of 5 cars every 10 minutes according to a Poisson process. Find the probability that the toll collector will have to wait longer than 26.30 minutes before collecting the eighth toll.