Math 170E: Winter 2023

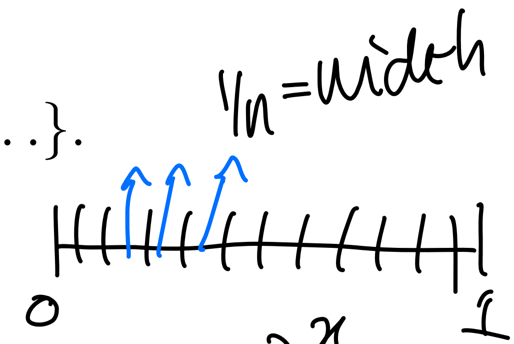
Lecture 13, Fri 10th Feb

The Poisson distribution and random variables of the continuous type

Last time:

- On average, $\lambda > 0$ customers arrive at a Walmart every hour
- Let X denote the number of customers arriving in 1 hour
- X takes values in $S = \{0, 1, \ldots\}$. (We assume the population is ∞)
- Under some further assumptions, $X \sim \text{Poisson}(\cancel{p}\ \lambda)$ and has PMF

$$p_X(\cancel{X}\ x) = e^{-\lambda} \frac{\lambda^x}{x!} \quad \text{if} \quad x \in \{0, 1, 2, \ldots\}.$$



Binomial$(n, \lambda/n)$

$$\hookrightarrow p_X(x) = \lim_{n \to \infty} \left\{ \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \right\} = e^{-\lambda} \frac{\lambda^x}{x!}$$

Binomial$(n, p) \Leftarrow$ approx by Poisson(np).

$\lambda/n = p$

Today:

Good approx: when n is very big & p is very small.

We'll discuss today:

- how to compute the MGF, mean and variance of a Poisson r.v.
- what it means for a random variable to be continuous
- the definition and properties of the probability density function and cumulative distribution function for a continuous r.v.

Assumptions:

- We make the following assumptions about the arrivals:

  1. If the time intervals $(t_1, t_2], (t_2, t_3], \ldots, (t_n, t_{n+1}]$ are *disjoint*, then the number of arrivals in each time interval are *independent*
  2. If $h = t_2 - t_1 > 0$ is sufficiently small, then the probability of exactly one arrival in the time interval $(t_1, t_2]$ is λh
  3. If $h = t_2 - t_1 > 0$ is sufficiently small, then the probability of two or more arrivals in the time interval $(t_1, t_2]$ converges rapidly to zero as $h \to 0$

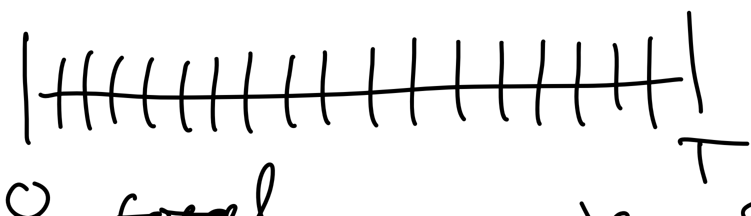
In our derivation of the Poisson process, we assumed that the time interval was of unit length i.e. $[0, 1]$.
How does the Poisson process change if we measure for time T i.e. on $[0, T]$?

- If $T < 1$, fewer arrivals
- If $T > 1$, more arrivals

e.g. $\lambda = \#$ per min
$X = \#$ per hour

Proposition 2.30: Consider an approximate Poisson process with rate $\lambda > 0$ per unit time. Let X be the number of arrivals in a time interval of length $T > 0$ units. Then $X \sim \text{Poisson}(\lambda T)$.

Proof:



n subintervals of length T/n.

$X_n = \#$ of ~~total~~ arrivals $\Rightarrow X_n \sim \text{Bin}(n, \lambda \cdot T/n)$.

$$P_X(x) = \lim_{n\to\infty} \left[\binom{n}{x} \left(\frac{\lambda T}{n}\right)^x \left(1 - \frac{\lambda T}{n}\right)^{n-x} \right].$$

$$= e^{-\lambda T} \cdot \frac{(\lambda T)^x}{x!} \text{ for any } x \in \{0, 1, \dots\} \Rightarrow X \sim \text{Poisson}(\lambda T)$$

Proposition 2.31: If $\lambda > 0$ and $X \sim \text{Poisson}(\lambda)$, then its MGF is

$$M_X(t) = e^{\lambda(e^t - 1)}. \quad \text{for any } t \in \mathbb{R}$$

Proof:

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \cdot e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} \boxed{\sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!}}$$

Taylor series for $\exp(y)$

$e^t \lambda \in \mathbb{R}$

$\hookrightarrow t \in \mathbb{R}$.

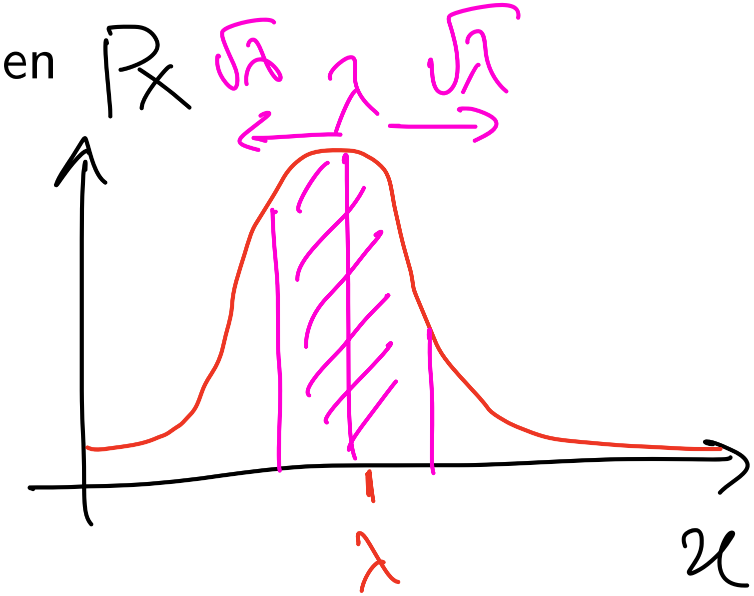
$$= e^{-\lambda} e^{e^t \lambda} = e^{\lambda(e^t - 1)}$$

$$\hookrightarrow \log M_X(t) = \lambda(e^t - 1).$$

$$\mathbb{E}[X] = \frac{d}{dt} \log M_X(t) \Big|_{t=0} = \lambda = \frac{d^2}{dt^2} \log M_X \Big|_{t=0} = \text{Var}(X).$$

Proposition 2.31: If $\lambda > 0$ and $X \sim \text{Poisson}(\lambda)$, then

$$\mathbb{E}[X] = \lambda$$
$$\text{var}(X) = \lambda.$$



Proof:

Chapter 3: Continuous random variables

Many practical situations are better described using a random variable taking on a *continuum* of values, rather than a *discrete* number of values.

Examples:

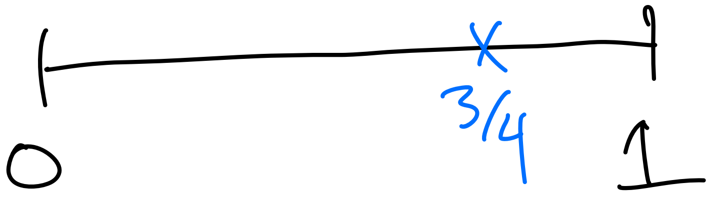
- the *time* until the occurrence of the next earthquake
- the *lifetime* of a battery
- the annual *rainfall* in Edinburgh
- the *heights* of people

A fundamental difference:

$\mathbb{P}(X=x)$

- discrete r.v.s take on a *specific* value with probability > 0
- the probability of a continuous random variable taking on a *specific* value is 0!
- all that has meaning is the probability they take on a value within a given interval

$\mathbb{P}(X=x)=0$!



→ Pick a # in $(0,1)$ at "random"

X " $X \in (0,1)$.

↳ $\mathbb{P}(X = 3/4) = ?$

Suppose that $X = x$ for some $x \in (0,1)$.

How to ascribe a prob?

$$\mathbb{P}(X = x) = \frac{\#\{x\}}{\#(0,1)} = \frac{1}{\infty} = 0.$$

$$\int_0^{\varepsilon} 1\,dx = \varepsilon.$$

Use length of the events?

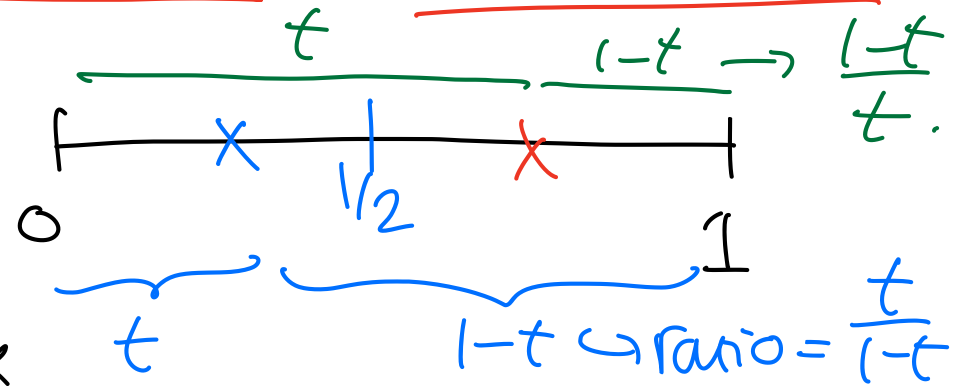
$$\mathbb{P}(X = x) = \frac{\text{length}(\{x\})}{\text{length}((0,1))} = \frac{0}{1} = 0.$$

↳ All that has meaning in the cts world is

$$\mathbb{P}(X \in (a,b)), \quad a < b.$$

↳ $F_X(x) = \mathbb{P}(X \leq x)$

Example 1: A stick of unit length is broken into two pieces at random. What is the probability that the ratio of the length of the shorter side to that of the longer side is $\leq x$, for $0 < x < 1$?



Set of outcomes $= \Omega = (0,1)$

$\omega \in \Omega \to \omega = t$, means the pt at which we broke the stick (as measured from 0).

Let X denote the ratio of shorter to longer side.

Want: $\mathbb{P}(X \leq x)$ for $0 < x < 1$ (CDF). $\boxed{X, T \in (0,1)}$

Let T be the place where we broke the stick.

$$\mathbb{P}(X \leq x) = \mathbb{P}(X \leq x, T \leq 1/2) + \mathbb{P}(X \leq x, T > 1/2).$$

$$= \mathbb{P}\left(\frac{T}{1-T} \leq x, T \leq 1/2\right) + \mathbb{P}\left(\frac{1-T}{T} \leq x, T > 1/2\right)$$

$$= \mathbb{P}\left(T \leq \frac{x}{1+x}, T \leq 1/2\right) + \mathbb{P}\left(T \geq \frac{1}{1+x}, T > 1/2\right).$$

$\frac{T}{1-T} \leq x$

$T \leq x - Tx$

$1-T \le Tx$
$T \ge \frac{1}{1+x}$

$$= \mathbb{P}\left(T \le \frac{x}{1+x}\right) + \mathbb{P}\left(T \ge \frac{1}{1+x}\right).$$

$$\text{“} = \mathbb{P}\left(\text{pick a random \# in } (0,1) \text{ \& it lies in } \left(0, \frac{x}{1+x}\right)\right) + \mathbb{P}\left(\overline{\qquad\qquad} \text{ \& it lies } \left(\frac{1}{1+x}, 1\right)\right)\text{”}.$$

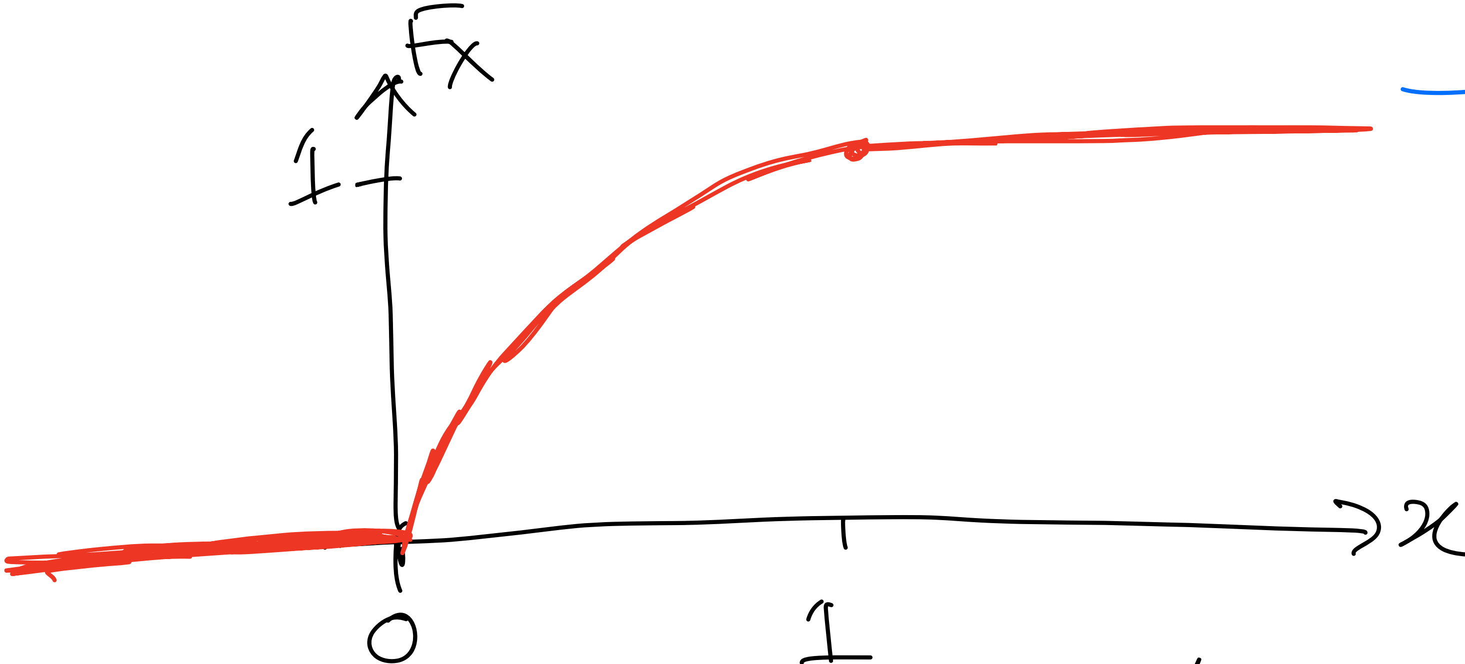
$$= \frac{\text{length}\left(\left(0, \frac{x}{1+x}\right)\right)}{\text{length}((0,1))} + \frac{\text{length}\left(\left(\frac{1}{1+x}, 1\right)\right)}{\text{length}((0,1))}$$

$$= \frac{x}{1+x} + 1 - \frac{1}{1+x} = \frac{2x}{1+x}, \quad 0 < x < 1.$$

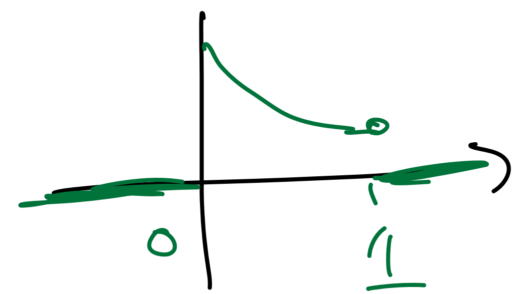
$$\hookrightarrow \quad F_X(x) = \begin{cases} \dfrac{2x}{1+x} & \text{if } 0 < x < 1. \\ 0 & \text{if } x \le 0 \\ 1 & \text{if } x \ge 1. \end{cases}$$

$\longrightarrow \mathbb{P}(X \le x)$
$= \mathbb{P}(X \le 1)$
$= \mathbb{P}(\Omega)$

$\mathbb{P}(a \le X \le b)$
“$= F_X(b) - F_X(a)$”



→ CDF is a cts function of x.



If $x \neq 0, 1$, F_X is differentiable & $F_X'(x) = f_X(x)$.

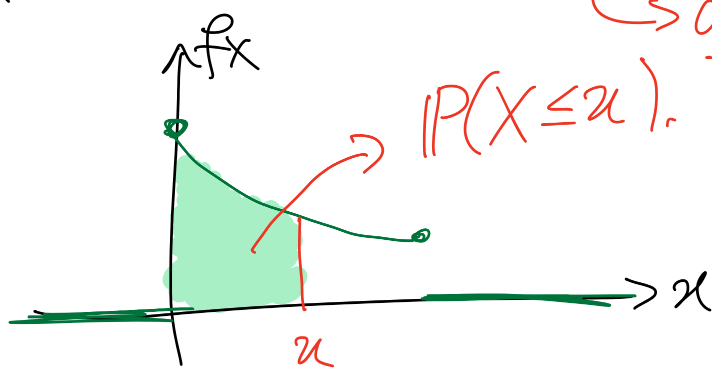
$$f_X(x) = \begin{cases} \dfrac{2}{(1+x)^2} & \text{if } 0 < x < 1 \\ 0 & \text{if } x < 0 \\ 0 & \text{if } x > 1 \end{cases}$$

By the Fundamental theorem of calculus, for any $x \in \mathbb{R}$,

$$F_X(x) = F_X(x) - \underbrace{\lim_{x \to -\infty} F_X(x)}_{=0} = \int_{-\infty}^{x} f_X(y)\,dy.$$

$$\hookrightarrow \quad \mathbb{P}(X \le x) = \int_{-\infty}^{x} \boxed{f_X(y)}\,dy.$$

probability density



$$\mathbb{P}(X \in (-\infty, x)) = \int_{(-\infty, x)} f_X(y)\,dy.$$

$$\hookrightarrow \quad \mathbb{P}(X \in (a,b)) = \int_a^b f_X(y)\,dy.$$

If X discrete, $\mathbb{P}(X \in (a,b)) = \sum_{x \in (a,b) \cap S_X} p_X(x)$.

Definition 3.1: Let $S \subseteq \mathbb{R}$, and $X : \Omega \to S$ is a random variable

- we define the cumulative distribution function of X, $F_X : \mathbb{R} \to [0,1]$ by

$$F_X(x) := \mathbb{P}(X \leq x).$$

  We have

$$\lim_{x \to -\infty} F_X(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} F_X(x) = 1.$$

- we say that X is a continuous random variable if there exists a non-negative integrable function $f_X : \mathbb{R} \to [0, \infty)$ such that

$$\mathbb{P}(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)dt.$$

- we call f_X a probability density function for X.