Homework 3

The problems below are from the Sipser textbook.

1 Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA that recognizes a language L. Does the NFA $(Q, \Sigma, \delta, q_0, Q \setminus F)$, which is result of swapping the accept and reject states in M, necessarily recognize the complement of L? Prove or give a counterexample.

1.32 Let

$$\Sigma_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Σ_3 contains all size 3 columns of 0s and 1s. A string of symbols in Σ_3 gives three rows of 0s and 1s. Consider each row to be a binary number and let

$$B = \{w \in \Sigma_3^* \mid \text{the bottom row of } w \text{ is the sum of the top two rows}\}.$$

For example,

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in B, \quad \text{but} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \notin B.$$

Prove that B is regular. Hint: Prove the regularity of B^R and then apply a closure property.

1.40 Recall that string x is a ***prefix*** of string y if a string z exists where $xz = y$, and that x is a ***proper prefix*** of y if in addition $x \neq y$. In each of the following parts, we define an operation on a language A. Show that the class of regular languages is closed under that operation.

- [A]**a.** $NOPREFIX(A) = \{w \in A \mid \text{no proper prefix of } w \text{ is a member of } A\}$.
- **b.** $NOEXTEND(A) = \{w \in A \mid w \text{ is not the proper prefix of any string in } A\}$.

1.41 For languages A and B, let the ***perfect shuffle*** of A and B be the language

$$\{w \mid w = a_1 b_1 \cdots a_k b_k, \text{ where } a_1 \cdots a_k \in A \text{ and } b_1 \cdots b_k \in B, \text{ each } a_i, b_i \in \Sigma\}.$$

Show that the class of regular languages is closed under perfect shuffle.

1.62 Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. For each $k \geq 1$, let D_k be the language consisting of all strings that have at least one $\mathtt{a}$ among the last k symbols. Thus $D_k = \Sigma^* \mathtt{a} (\Sigma \cup \varepsilon)^{k-1}$. Describe a DFA with at most $k+1$ states that recognizes D_k in terms of both a state diagram and a formal description.

1.66 A ***homomorphism*** is a function $f \colon \Sigma \longrightarrow \Gamma^*$ from one alphabet to strings over another alphabet. We can extend f to operate on strings by defining $f(w) = f(w_1) f(w_2) \cdots f(w_n)$, where $w = w_1 w_2 \cdots w_n$ and each $w_i \in \Sigma$. We further extend f to operate on languages by defining $f(A) = \{f(w) \mid w \in A\}$, for any language A.

a. Show that the class of regular languages is closed under homomorphism. In other words, given a DFA M that recognizes B and a homomorphism f, construct a finite automaton M' that recognizes $f(B)$. Consider the machine M' that you constructed. Is it a DFA in every case?

1.69 Let $\Sigma = \{\mathtt{0},\mathtt{1}\}$. Let $WW_k = \{ww \mid w \in \Sigma^* \text{ and } w \text{ is of length } k\}$.

- **a.** Show that for each k, no DFA can recognize WW_k with fewer than 2^k states.
- **b.** Describe a much smaller NFA for $\overline{WW_k}$, the complement of WW_k.