Math 170E: Homework 7

Due: Wed. 8th March by 11:59pm PDT via Gradescope

Submit answers to all problems via Gradescope. The reader will grade three problems each out of five points. Up to five further points will be awarded based on the proportion of the remaining problems that are completed.

Please make sure that your submission is readable. If your pencil is too faint, get a thicker one. If your handwriting is cramped and small, write bigger and use more paper. Please use simple plain paper or lined paper (e.g. please avoid graph paper etc.). It is your responsibility to ensure that your submission is readable. If we cannot read a solution, we may refuse to grade it. Thank you!

I encourage you to discuss and work on problems with other students in the class. Nevertheless, the solutions you present have to be your own. In particular, if the solution you present is identical to someone else's, or it is identical to some other resource (book, online, etc.), this will be considered cheating.

1. Let X, Y be independent discrete random variables.
   (a) Show that $\mathbb{E}[Y|X] = \mathbb{E}[Y]$.
   You can interpret this to say that knowing X tells me absolutely nothing about Y, so my best guess for Y given X is just my best guess for X
   (b) Suppose that I flip a weighted coin and roll a six-sided die. Let X be the outcome on the coin and Y be the outcome on the die. What is $\mathbb{E}[Y|X]$?
   (c) Show that $\mathrm{var}(Y|X) = \mathrm{var}(Y)$.
2. Suppose I roll a fair die to get a number $N \in \{1, 2, \ldots, 6\}$ and then flip N fair coins. Let X be the number of heads I obtain.
   (a) What is $\mathbb{E}[X]$?
   (b) What is $\mathrm{var}(X)$?
3. Let X, Y be discrete random variables with mean μ_X, μ_Y, respectively. Define the function

$$K(m) = \mathbb{E}\left[\left((Y - \mu_Y) - m(X - \mu_X)\right)^2\right], \quad m \in \mathbb{R}.$$

   (a) Show that $K'(m) = -2\rho\sigma_X\sigma_Y + 2m\sigma_X^2$, where ρ is the correlation coefficient of X, Y and σ_X^2, σ_Y^2 are the variances of X, Y.

(b) Find the value of m which minimises $K(m)$. You may assume that $\sigma_X > 0$. The line $y = mx + (\mu_Y - m\mu_X)$ is known as the least squares regression line.

4. Let X be a random variable with MGF $M_X(t)$ defined for $t \in \mathbb{R}$ and Y a random variable with MF $M_Y(t)$ also defined for all $t \in \mathbb{R}$. Show that

$$M_{X+Y}(t) \leq M_X^{\frac{1}{2}}(2t) M_Y^{\frac{1}{2}}(2t).$$

5. Let the random variables X and Y have joint PMF

$$p_{X,Y}(x, y) = \frac{x+y}{32},$$

for $x = 1, 2$ and $y = 1, 2, 3, 4$.

(a) Find the correlation coefficient of X and Y.

(b) Find $\mathbb{E}[Y|X]$

(c) Compute $\mathbb{P}(1 \leq Y \leq 3|X = 1)$.

(d) Find $\mathrm{var}(Y|X)$.

6. Roll a fair four-sided die twice. Let X equal the outcome on the first roll and let Y equal the sum of the two rolls. What is the correlation coefficient of X, Y?

7. Let X be a randomly chosen integer from $1, ..., 10$ and Y a randomly chosen integer from $1, ..., X$. What is the correlation coefficient of X, Y?
You may use the identities from lectures:

$$\sum_{x=1}^{n} x = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}.$$

8. Let X and Y be two (discrete) random variables with finite and non-zero second moments.

(a) Show that if there are $a, b \in \mathbb{R}$, $a \neq 0$, such that $Y = aX + b$, then $|\rho(X, Y)| = 1$. This says that the Pearson correlation coefficient is maximised (± 1) when X and Y are linear combination of each other.

Now we will show the reverse statement: if the correlation coefficient is maximised, then X and Y must be linearly related to each other. For this, we need to revisit the Cauchy-Schwarz inequality.

(b) Show that

$$uv + \frac{(u-v)^2}{2} = \frac{u^2+v^2}{2}$$

for any $u, v \in \mathbb{R}$.

(c) Let $U = \frac{X}{\sqrt{\mathbb{E}[X^2]}}$ and $V = \frac{Y}{\sqrt{\mathbb{E}[Y^2]}}$. Using part (b), prove the Cauchy-Schwarz inequality

$$|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$$

with equality in the above if and only if $U = V$.

(d) Now, apply Cauchy-Schwarz as we did in lecture to show that $|\rho(X, Y)| \leq 1$, but keep track of the case of equality. You will find that X and Y must be linearly related.

(e) Consider the situation when X is perturbed by some random noise Y. Suppose that both X and Y have finite, non-zero variance. Compute $\rho(X, X + Y)$.