# Math 170E: Fall 2020

Lecture 5, Fri 20th Jan

Conditional probability and Bayes' theorem

## Last time:

Last time, we were interested in taking $r$ samples from $n$ objects

- we can do this with or without replacement
- we can seek ordered or unordered samples
- the Binomial theorem

## Today:

We'll discuss today:

- the notion of *conditional probability*
- Bayes' theorem

Suppose $A, B \subseteq \Omega$ are events.

If we know that event $B$ has occurred, how does this affect the probability of event $A$ occuring?

**Example 13:** You roll a fair six-sided die twice. You know one of the die rolls is a 6. What is the probability that the sum of the two rolls is a 7?

$\Omega = \{(1,1), (1,2), \ldots, (6,6)\}.$

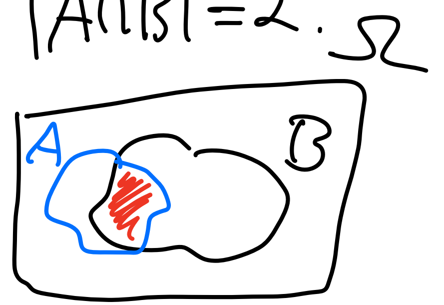
$A = \{\text{sum is a } 7\} = \{(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)\}.$

$\hookrightarrow \mathbb{P}(A) = \dfrac{|A|}{|\Omega|} = \dfrac{6}{36} = 1/6. \rightsquigarrow \mathbb{P}(A \text{ given that } B \text{ happened})$

$B = \{\text{one of the rolls is a } 6\} = \{(1,6), (2,6), \ldots, (6,6), (6,1), \ldots, (6,5)\}$

$\hookrightarrow |B| = 11.$

$\hookrightarrow A \cap B = \{\text{elements in } A \text{ also in } B\} = \{(1,6), (6,1)\}. \rightarrow |A \cap B| = 2.$




$\mathbb{P}(A \text{ given that } B \text{ happened}) = \dfrac{|A \cap B|}{|B|} = \dfrac{2}{11}$

$\dfrac{|A \cap B| / |\Omega|}{|B| / |\Omega|} = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

**Definition 1.13:** (**Conditional probability**)

Let $B \subseteq \Omega$ be an event so that $\mathbb{P}(B) \neq 0$. The probability of an event $A \subseteq \Omega$ conditioned on the event B is given by

$$\underbrace{\mathbb{P}(A|B)}_{\text{A given B}} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

A given B

↪ Because we know that B has occurred our state space (all possible outcomes) has shrunk!

"probability that A occurs given that B has occurred."

**Theorem 1.14:** (**Properties of the conditional probability**)

$(\Omega, \mathcal{F}, \mathbb{P})$

$B$, $\mathbb{P}(B) \neq 0$

$(B, \mathcal{F}, \mathbb{P}(\cdot|B))$

is a prob. space

If $B \subseteq \Omega$ such that $\mathbb{P}(B) \neq 0$, the $\mathbb{P}(\cdot|B)$ is a probability measure.
i.e. $\mathbb{P}(\cdot|B) : \mathcal{F} \mapsto [0,1]$ satisfies:

1. $\mathbb{P}(\Omega|B) = 1$

2. (Countable additivity) If $\{A_j\}_{j=1}^{k}$ are mutually exclusive events, then

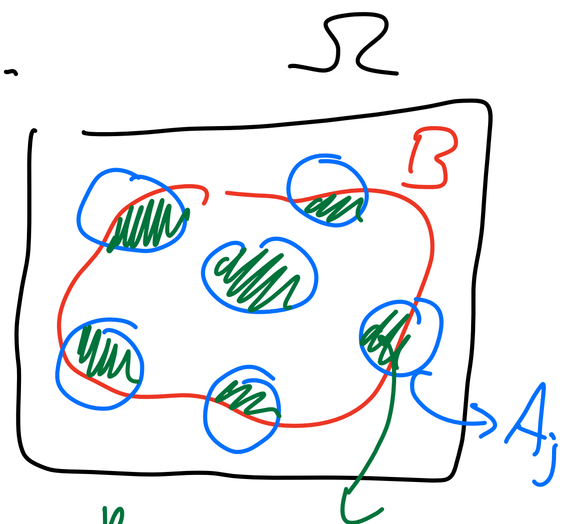
$$\mathbb{P}\left(\bigcup_{j=1}^{k} A_j \middle| B\right) = \sum_{j=1}^{k} \mathbb{P}(A_j|B),$$

and (when "$k = +\infty$"),

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} A_j \middle| B\right) = \sum_{j=1}^{\infty} \mathbb{P}(A_j|B),$$

$\Longrightarrow$ all properties of prob. measures we proved in Lectures 1 & 2 hold for $\mathbb{P}(\cdot|B)$:
e.g. $\mathbb{P}(A'|B) = 1 - \mathbb{P}(A|B)$

**Proof:** 1) $\mathbb{P}(\Omega \mid B) = \dfrac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = \dfrac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1.$




2) Suppose $\{A_j\}_{j=1}^{k}$ are mutually exclusive.

$$\mathbb{P}\left(\bigcup_{j=1}^{k} A_j \,\Big|\, B\right) = \frac{\mathbb{P}\left(\left(\bigcup_{j=1}^{k} A_j\right) \cap B\right)}{\mathbb{P}(B)}$$

$$\left(\bigcup_{j=1}^{k} A_j\right) \cap B = \bigcup_{j=1}^{k} (A_j \cap B)$$

$$= \frac{\mathbb{P}\left(\bigcup_{j=1}^{k} (A_j \cap B)\right)}{\mathbb{P}(B)}$$

by countable add of $\mathbb{P}$

$$= \frac{\sum_{j=1}^{k} \mathbb{P}(A_j \cap B)}{\mathbb{P}(B)}$$

$\left[\begin{array}{c} \{A_j\} \text{ are m-e.} \\ \Downarrow \\ \{A_j \cap B\} \text{ are m-e.} \end{array}\right].$

$$= \sum_{j=1}^{k} \frac{\mathbb{P}(A_j \cap B)}{\mathbb{P}(B)} = \sum_{j=1}^{K} \mathbb{P}(A_j \mid B).$$

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## Example 14:

- You have a regular pack of 52 cards
- You draw 3 cards at random
- Given that the first card you draw is not an ace, what is the probability that you draw at least one ace?

4 aces in 52 card deck.

$\Omega = \{\text{all possible 3 card hands}\}$.

$A = \{\text{drew at least 1 ace}\}$

$B = \{\text{first is not an ace}\}$

Want:

$\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$.

A) $\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50}$

B) $1 - \frac{48}{52} \times \frac{47}{51} \times \frac{46}{50}$

C) $\frac{1}{52} \times \frac{48}{51}$

D) $1 - \frac{47}{51} \times \frac{46}{50}$

$\mathbb{P}(B) = \dfrac{48}{52}$.

$\mathbb{P}(A\cap B) = ??$

3rd not

$A\cap B = \{1^{st}\text{ not an ace}, 2^{nd}\text{ is an ace}\}$

$\cup \{\text{———}, 3^{rd}\text{ ace}, 2^{nd}\text{ is not}\}$

$\cup \{\text{———}, 2^{nd}\text{ ace}, 3^{rd}\text{ ace}\}$

$\hookrightarrow A' = \{\text{drew no aces}\}$.

$\hookrightarrow A' \subseteq B \hookrightarrow A'\cap B = A'$.

$$\mathbb{P}(A|B) = 1 - \mathbb{P}(A'|B).$$

$$= 1 - \frac{\mathbb{P}(A' \cap B)}{\mathbb{P}(B)}$$

$$\mathbb{P}(A' \cap B) = \mathbb{P}(A')$$

$$= \frac{48}{52} \times \frac{47}{51} \times \frac{46}{50}$$

$$= 1 - \frac{\cancel{\frac{48}{52}} \times \frac{47}{51} \times \frac{46}{50}}{\cancel{48/52}} = 1 - \frac{47}{51} \times \frac{46}{50}. \quad //$$

**Proposition 1.15:** If $A, B \subseteq \Omega$ are *independent* and $\mathbb{P}(B) \neq 0$, then

$$\underbrace{\mathbb{P}(A|B) = \mathbb{P}(A)}$$

If A & B are indep, knowing B tells me nothing about A.

**Proof:**

$$\hookrightarrow \mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} \overset{\text{indep}}{=} \frac{\mathbb{P}(A)\cancel{\mathbb{P}(B)}}{\cancel{\mathbb{P}(B)}} = \mathbb{P}(A).$$

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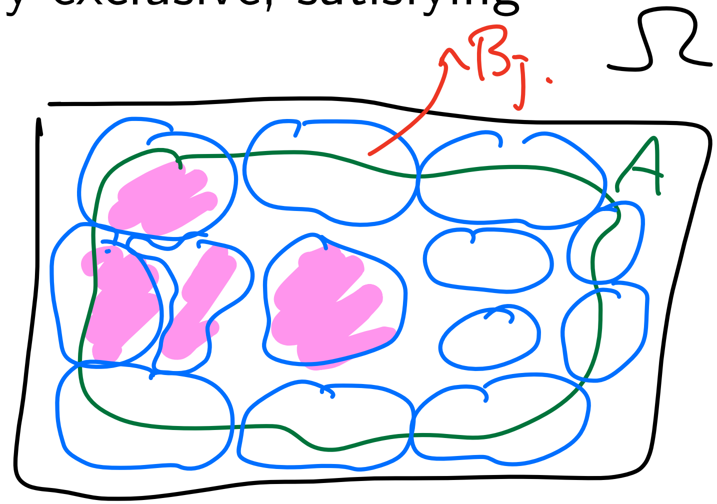
**Theorem 1.16:** (**The law of total probability**)

Let $A \subseteq \Omega$ be an event and $\{B_j\}_{j=1}^k \subseteq \Omega$ be mutually exclusive, satisfying $\mathbb{P}(B_j) \neq 0$, for every $j \in \{1, \ldots, k\}$, and




$$\underbrace{A \subseteq \bigcup_{j=1}^{k} B_j.}_{\{B_j\} \text{ cover } A.}$$

Then

$$\mathbb{P}(A) = \mathbb{P}(A|B_1)\mathbb{P}(B_1) + \cdots + \mathbb{P}(A|B_k)\mathbb{P}(B_k) = \sum_{j=1}^{k} \mathbb{P}(A|B_j)\mathbb{P}(B_j).$$

$$\mathbb{P}(A|B_j) = \frac{\mathbb{P}(A \cap B_j)}{\mathbb{P}(B_j)}$$

**Proof:** $A \subseteq \bigcup_{j=1}^{k} B_j$. $\hookrightarrow$ $A = A \cap \left(\bigcup_{j=1}^{k} B_j\right) = \bigcup_{j=1}^{k} (A \cap B_j)$

$\{B_j\}$ m.e. $\Rightarrow$ $\{A \cap B_j\}$ are m.e.

By countable additivity,

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{j=1}^{k} (A \cap B_j)\right)$$

$$= \sum_{j=1}^{k} \mathbb{P}(A \cap B_j) = \sum_{j=1}^{k} \mathbb{P}(A \mid B_j)\, \mathbb{P}(B_j).$$

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**Example 15:** A bin has three types of disposable flashlights in it.

- The probability a flashlight of type 1 lasts more than 100 hours of use is 70%
- for types 2 and 3, the probability is 40% and 30% respectively
- suppose 20% in the bin are type 1, 30% are type 2 and 50% are type 3

What is the probability that a randomly selected flashlight has more than 100 hours of use?

$\Omega = \{\text{all flashlights in the bin}\}$

$A = \{\text{chosen has} \geq 100 \text{ hrs}\}$. → WANT: $\mathbb{P}(A)$.

$B_j = \{\text{chosen is of type } j\}, \ j = 1, 2, 3.$ ← mutually exclusive

$\mathbb{P}(B_1) = 0.2, \ \mathbb{P}(B_2) = 0.3, \ \mathbb{P}(B_3) = 0.5.$

$\mathbb{P}(A \mid B_j)$: $\mathbb{P}(A \mid B_1) = 0.7, \ \mathbb{P}(A \mid B_2) = 0.4, \ \mathbb{P}(A \mid B_3) = 0.3.$

By the law of total. Prob,

$$\mathbb{P}(A) = \sum_{j=1}^{3} \mathbb{P}(A \mid B_j)\,\mathbb{P}(B_j)$$

$$= (0.7) \times (0.2) + (0.4) \times (0.3) + (0.3) \times (0.5)$$

$$= 0.41.$$

↳ 41% chance that a randomly selected flashlight lasts for $\geq 100$ hrs.

$$\frac{7}{10} \times \frac{1}{5} + \frac{2}{5} \times \frac{3}{10} + \frac{3}{10} \cdot \frac{1}{2}.$$