Homework 9

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2

a

We can first find the joint PDF by finding the individual PDFs for the exponential r.v.s. knowing they are independent

$$f_{X_j} = \frac{1}{1000}e^{-x/1000} \therefore f_{X_1,X_2} = f_{X_1}f_{X_2} = \frac{e^{-x_1/1000}e^{-x_2/1000}}{1,000,000}$$

We can note that this probability can be expressed as:

$$P(Y_1 \leq y_1, Y_2 \leq y_2) = P(X_1 \leq y_1, X_1 \leq X_2 \leq y_2)$$

Now we can find the probability as

$$G(y_1, y_2) = \frac{1}{1000^2} \int_0^{y_1} \int_{x_1}^{y_2} e^{-x_1/1000} e^{-x_2/1000} dx_2 dx_1$$

$$= \frac{1}{1000} \int_0^{y_1} \mathrm{e}^{-\frac{x1}{500}} - \mathrm{e}^{-\frac{y2}{1000}-\frac{x1}{1000}} dx_1$$

$$G(y_1, y_2) = \frac{1}{2} \mathrm{e}^{-\frac{y2+3y1}{1000}} \cdot \left(\mathrm{e}^{\frac{y2+3y1}{1000}} - \mathrm{e}^{\frac{y2+y1}{1000}} - 2\mathrm{e}^{\frac{3y1}{1000}} + 2\mathrm{e}^{\frac{y1}{500}} \right)$$

▼ b

We can see that the probability is

$$P(Y_2 > 1200) = 1 - P(Y_2 \leq 1200) = 1 - P(X_1, X_2 \leq 1200)$$

Because these variables are independent, we can calculate:

$$= 1 - \int_0^{1200} \frac{1}{1000} e^{-x_1/1000} dx_1 \int_0^{1200} \frac{1}{1000} e^{-x_2/1000} dx_2$$

So

$$P(Y_2 > 1200) \approx 1 - 0.6988^2 \approx 0.5117$$

▼ 3

We can first find the joint PDF since the variables re independent as

$$f_{X,Y}(x, y) = \frac{x}{125} e^{-x/5} e^{-y/5}$$

Now, we can find the CDF of Z as $P(Z \leq z) = P(X/Ylez) = P(X \leq zY)$

$$\frac{1}{125} \int_0^{\infty} \int_0^{yz} x e^{-x/5} e^{-y/5} dx dy = \frac{z^2}{(z+1)^2}$$

Now, we can find the PDF by taking the derivative with respect to z

$$f_Z(z) = \frac{2z}{(z+1)^3}$$

▼ 4

▼ a

Since each variable is independent, we can use their PMFs to find the probability

$$P(X_1 = 2, X_2 = 2, X_3 = 5) = \binom{4}{2} 0.5^2 0.5^2 \cdot \binom{6}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4 \cdot \binom{12}{5} \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^7$$

$$P(X_1 = 2, X_2 = 2, X_3 = 5) \approx 0.007896$$

▼ b

The independence theorem for expectation allow us to show the following

$$E[X_1 X_2 X_3] = E[X_1]E[X_2]E[X_3] = \frac{4}{2} * \frac{6}{3} * \frac{12}{6} = 8$$

▼ c

The r.v. Y is a liner combination, so we can use the linear property of expectation to find the mean:

$$E[Y] = E[X_1] + E[X_2] + E[X_3] = 6$$

The variance can be found as

$$\text{var}(Y) = E[Y^2] - E[Y]^2 = E[(X_1 + X_2 + X_3)^2] - 36$$

For which

$$E[Y^2] = E[X_1^2] + E[X_2^2] + E[X_3^2] + 2E[X_1]E[X_2] + 2E[X_2]E[X_3] + 2E[X_1]E[X_3]$$

The 2nd moment for the binomial is known to be $n(n-1)p^2$

$$E[Y^2] = 3 + \frac{10}{3} + \frac{11}{3} + 8 \cdot 3 = 34$$

So:

$$\text{var}(Y) = 34 - 36 = -2$$

▼ 5

We can express:

$$E[Z] = E[2Y_1 + Y_2]$$

Since the function Y_1, Y_2 depend on independent variable X_1, X_2 and we know that one i the minimum and one i the maximum of the X variable, the following must be true

$$Y_1 + Y_2 = \min(X_1, X_2) + \max(X_1, X_2) = X_1 + X_2 \implies Y_2 = X_1 + X_2 - Y_1$$

So $E[Z] = E[Y_1 + X_1 + X_2]$ and using the linearity from problem (2):

$$E[Z] = E[\min(X_1, X_2)] + 2 + 2$$

We also know for an exponentially distributed minimum function the following is true:

$$Y_1 \sim \text{Exp}\left(\frac{1}{\frac{1}{X_1} + \frac{1}{X_2}}\right)$$

For which, $E[Y_1] = 1$ so

$$E[Z] = 1 + 2 + 2 = 5$$

▼ 6

▼ a

We can first rewrite the probability to match Chebyshev's format

$$P(23 < X < 43) = P(-10 < X - \mu < 10)$$
$$= P(|X - \mu| < 10) = 1 - P(|X - \mu| \geq 10)$$

This is now in Chebyshev's format:

$$P(|X - \mu| \geq 10) \leq \frac{\sigma^2}{\lambda^2} = \frac{16}{100}$$

So, the lower bound is

$$1 - \frac{16}{100} = 0.84$$

▼ b

This is already in the proper format so the upper bound can be found as

$$P(|X - 33| \geq 14) \leq \frac{16}{14^2} = \frac{4}{49}$$

▼ 7

We must first convert this to the proper format as

$$P(|Y/n - 0.25| < 0.05) = 1 - P(|Y - 0.25n| \geq 0.05n) \leq 1 - \frac{\sigma^2}{.0025n^2}$$

Because the distribution is binomial, $\sigma^2 = n(0.25)(0.75) = \frac{3}{16}n$

So for $n = 100$

$$P(|Y/100 - 0.25| < 0.05) \leq 1 - \frac{18.75}{25} = 0.25$$

For $n = 1000$

$$P(|Y/1000 - 0.25| < 0.05) \leq 1 - \frac{187.5}{2500} = 0.925$$

For $n = 10,000$

$$P(|Y/10,000 - 0.25| < 0.05) \leq 1 - \frac{1875}{250000} = 0.9925$$