Math 170E: Homework 3

Due: Sat. 4th February by 11:59pm PDT via Gradescope

Submit answers to all problems via Gradescope. The TA will grade three problems each out of five points. Up to five further points will be awarded based on the proportion of the remaining problems that are completed.

Please make sure that your submission is readable. If your pencil is too faint, get a thicker one. If your handwriting is cramped and small, write bigger and use more paper. Please use simple plain paper or lined paper (e.g. please avoid graph paper etc.). It is your responsibility to ensure that your submission is readable. If we cannot read a solution, we may refuse to grade it. Thank you!

I encourage you to discuss and work on problems with other students in the class. Nevertheless, the solutions you present have to be your own. In particular, if the solution you present is identical to someone else's, or it is identical to some other resource (book, online, etc.), this will be considered cheating.

1. You have an ordinary deck of 52 well-shuffled cards. You remove the cards one at a time until you get an ace. Let the random variable X denote the number of cards removed. What is the probability mass function of X? Make sure to specify where X can take values. Recall that there are four aces in a standard 52-card deck.

2. Let X be a discrete random variable with PMF

$$p_X(x) = \frac{c}{(1+x)(2+x)}, \quad x \in \{0, 1, 2, 3, \ldots\},$$

   where $c \in \mathbb{R}$.

   (a) Find $\mathbb{P}(a \leq X \leq b)$ for integers $0 \leq a < b$. You may assume for now that $p_X(x)$ is a genuine PMF.

   (b) Use your result from part (a) to find a c which makes p_X a genuine PMF.

   (c) Find the probability that $X \geq 4$ given that $X \geq 1$.

   (d) What is the expected value of X?

3. Let X be a discrete random variable with PMF

$$p_X(x) = C2^{-|x|},$$

   where x is an integer and $C > 0$ is a constant. Find $\mathbb{P}(-10 \leq X \leq 10)$. You may use that for $0 \leq a \leq b$, $\sum_{x=a}^{b} 2^{-x} = 2^{1-a} - 2^{-b}$. In particular, $\sum_{x=1}^{\infty} 2^{-x} = 1$.

4. Consider a collection of N objects of which N_1 are of type 1 and N_2 are of type 2. We select n objects from the collection of N at random and without replacement. Then the probability that exactly x (where $x \leq n$, $x \leq N_1$ and $n - x \leq N_2$) of these n objects are of type 1 and $n - x$ are of type 2 is

$$\frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}},$$

where $N = N_1 + N_2$

   (a) Provide a brief argument for the above formula of the probability.

   (b) We say a discrete random variable is *hypergeometrically distributed* if it has PMF

$$p_X(x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}},$$

   where $x \leq n$, $x \leq N_1$ and $n - x \leq N_2$. Use the Vandermonde identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

   to justify that p_X is a PMF.
   Remark: One key difference between the hypergeometric and the Binomial distribution is how they count objects: the former is without replacement, while the latter is with replacement.

5. Suppose there are 3 defective items in a collection of 50 items. A sample of size 10 is taken at random and without replacement. Let X denote the number of defective items in the sample. Find the probability that the sample contains:

   (a) Exactly one defective item.

   (b) At most one defective item.

6. Let X be a discrete random variable with MGF $M_X(t)$, which is well-defined and smooth for $t \in (-\delta, \delta)$, for some $\delta > 0$. Define $f(t) = \log M_X(t)$ and show that:

   (a) $f'(0) = \mathbb{E}[X]$

   (b) $f''(0) = \mathrm{var}(X)$

7. Let X equal an integer selected at random from the first m positive integers, $\{1, 2, ..., m\}$. Find the value of m for which $\mathbb{E}[X] = \mathrm{var}(X)$.

8. Let $m \geq 1$ and $X_j \sim \mathrm{Uniform}(\{1, 2, \ldots, m\})$ for $j = 1, 2$ be independent (i.e. for any sets A and B, the events $\{X_1 \in A\}$ and $\{X_2 \in B\}$ are independent). Define the random variable $X = \max(X_1, X_2)$.

   (a) Find the PMF of X.

(b) Find the mean and variance of X. You may use that

$$\sum_{x=1}^{m} x^3 = \frac{m^2(m+1)^2}{4}.$$

(c) Suppose you roll two fair six-sided die. What is the expected value of the larger outcome? What is its standard deviation?