Homework 7

course	MATH 170E
area	ucla
created	@March 7, 2023 12:03 PM
updated	@March 8, 2023 11:56 PM
✅ tasks	
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tags	math

1

a) $\mathbb{E}[Y|X] = \mathbb{E}[Y]$

The statement implies that the variables are independent as knowing anything about X tells us nothin about Y. This means

$$p_{Y|X}(y|x) = \mathbb{P}(Y = y|X = x) = \frac{\mathbb{P}(Y \cap X)}{\mathbb{P}(X)} = \mathbb{P}(Y) = p_Y(Y)$$

Then we can find that for any $X = x$

$$\mathbb{E}[Y|X = x] = \sum_{y \in S_Y} y \cdot p_{Y|X}(y|x) = \sum_{y \in S_Y} y \cdot p_Y(y)$$

The RHS is specifically the definition of the expected value for Y so:

$$\mathbb{E}[Y|X] = \sum_{y \in S_Y} y \cdot p_Y(y) = \mathbb{E}[Y] \iff \mathbb{P}(Y \cap X) = \mathbb{P}(Y)\mathbb{P}(X)$$

b)

We can observe that the rolling of the 6-sided die (1 time) is

$$Y \sim \text{Uniform}([1, 6]) \implies \mathbb{E}[Y] = \frac{6+1}{2} = 3.5$$

Thus, using the proof in (a) since rolling a die and flipping a coin are independent events:

$$\mathbb{E}[Y|X] = \mathbb{E}[Y] = 3.5$$

c)

We can use the definition of the conditional variance as:

$$\text{var}(Y|X) = \mathbb{E}\left[(Y - \mathbb{E}[Y|X])^2 | X\right]$$

Now, we can use the proof from part (a) since X, Y are independent

$$\mathbb{E}\left[(Y - \mathbb{E}[Y|X])^2 | X\right] = \mathbb{E}\left[(Y - \mathbb{E}[Y|X])^2\right] = \mathbb{E}\left[(Y - \mathbb{E}[Y])^2\right]$$

This is precisely the definition of the variance of Y so:

$$\text{var}(Y|X) = \mathbb{E}\big[(Y - \mathbb{E}[Y])^2\big] = \text{var}(Y)$$

▼ 2

▼ a)

We can first define the random number N which describes the outcome of a die (assuming it is fair and is 6-sided) as

$$N \sim \text{Uniform}([1, 6])$$

Then, we can describe the coin flips (assuming it is fair) X as a binomial distribution using N

$$X|n \sim \text{Binom}(N = n, 1/2)$$

We can use the Law of Iterated Expectation:

$$\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|N]\big] = \sum_{n \in S_N} \mathbb{E}[X|n] \cdot p_N(n) = \frac{1}{6}\sum_{n=1}^{6} \mathbb{E}[X|n]$$

Now, we can find the mean using the expected value of the binomial distribution defined above:

$$\mathbb{E}[X] = \frac{1}{6}\sum_{n=1}^{6} \mathbb{E}[X|n] = \frac{1}{6}\sum_{n=1}^{6} \frac{n}{2} = \frac{1}{12} \cdot 21 = \frac{21}{12}$$

▼ b)

We can use the definition of the variance to describe in terms of the expected value using the law of iterated expectation

$$\text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}\big[\mathbb{E}[X^2|N]\big] - \left(\frac{21}{12}\right)^2$$

We can use the expansion method using the proof for the law of iterated expectation and the definition of the expected value of a conditional distribution and the probability defined in (a):

$$= \frac{1}{6}\sum_{n=1}^{6} \mathbb{E}[X^2|n] \; - \frac{49}{16} = \frac{1}{6}\sum_{n=1}^{6} \left(\sum_{x \in S_X} x^2 \cdot p_{X|N}(x|n) \right) \; - \frac{49}{16}$$

$$= \frac{1}{6}\sum_{n=1}^{6} \frac{1}{2} \left(\sum_{x \in S_X} x^2 \right) \; - \frac{49}{16} = \frac{1}{2} \left(\sum_{x \in S_X} x^2 \right) \; - \frac{49}{16}$$

Now, we know X is defined as the number of heads given $N \in [1, 6]$ flips, so $X \in [0, 6]$:

$$\text{var}(X) = \frac{1}{2}\sum_{x=0}^{6} x^2 \; - \frac{49}{16} = \frac{91}{2} - \frac{49}{16} = \frac{679}{16}$$

▼ 3

▼ a)

We can first simplify the expression

$$K(m) = \mathbb{E}\left[(Y - \mu_Y)^2 - 2m(X - \mu_X)(Y - \mu_Y) + m^2 X^2\right]$$

Now, we can use the linear addition of expectance to show:

$$K(m) = \mathbb{E}\big[(Y-\mu_Y)^2\big] - 2m\cdot\mathbb{E}\big[(X-\mu_X)(Y-\mu_Y)\big] + m^2\cdot\mathbb{E}\big[(X-\mu_X)^2\big]$$

These include some known expectancies, which we can replace s.t.

$$K(m) = \sigma_Y^2 - 2m\cdot\operatorname{cov}(X,Y) + m^2\sigma_X^2$$

We can now derive the function with respect to m:

$$K'(m) = -2\operatorname{cov}(X,Y) + 2m\sigma_X^2$$

Now, we can express the covariance in terms of the correlation coefficient:

$$\operatorname{cov}(X,Y) = \rho\sqrt{\sigma_X^2\sigma_Y^2} = \rho\sigma_X\sigma_Y$$

Thus

$$K'(m) = -2\rho\sigma_X\sigma_Y + 2m\sigma_X$$

b)

We can see from taking a simple derivative test, to minimize $K(m)$

$$m = \rho\sigma_Y$$

The second derivative (trivial to calculate) identifies that $K(m)$ is concave up so $m = \rho\sigma_Y$ is indeed the minimum.

4

We can first observe that the MGFs can be expressed using the expected value:

$$M_{X+Y}(t) = \mathbb{E}[e^{t(X+Y)}] = \mathbb{E}[e^{tX}e^{tY}]$$
$$\&$$
$$M_X^{1/2}(2t)M_Y^{1/2}(2t) = \sqrt{\mathbb{E}[e^{2tX}]\mathbb{E}[e^{2tY}]} = \sqrt{\mathbb{E}\big[(e^{tX})^2\big]\mathbb{E}\big[(e^{tY})^2\big]}$$

From here, we can observe that the inequality is then precisely the Cauchy-Schwarz Inequality:

$$\mathbb{E}[AB] \le \sqrt{\mathbb{E}[A^2]\mathbb{E}[B^2]} \quad \text{s.t.} \quad A = e^{tX},\ B = e^{tY}$$

Therefore,

$$E[e^{tX}e^{tY}] \le \sqrt{\mathbb{E}\big[(e^{tX})^2\big]\mathbb{E}\big[(e^{tY})^2\big]} \quad \equiv \quad M_{X+Y}(t) \le M_X^{1/2}(2t)M_Y^{1/2}(2t)$$

(by the Cauchy-Schwarz Inequality)

5

a)

We can first find the marginal PMFs as

$$p_X(x) = \sum_{y\in S_Y} p_{X,Y}(x,y) = \frac{4x+10}{32} \qquad p_Y(y) = \sum_{x\in S_X} p_{X,Y}(x,y) = \frac{2y+3}{32}$$

Now, we can find the expectations as

$$E[X] = \sum_{x \in S_X} x p_X(x) = \frac{25}{16} \qquad E[Y] = \sum_{y \in S_Y} y p_Y(y) = \frac{47}{16}$$
$$\&$$
$$E[XY] = \sum_{(x,y) \in S} xy p_{X,Y}(x,y) = \frac{35}{8}$$

We can also find the individual variances:

$$\operatorname{var}(X) = \sum_{x \in S_X} x^2 p_X(x) - \left(\frac{25}{16}\right)^2 = \frac{63}{256}$$
$$\&$$
$$\operatorname{var}(Y) = \sum_{y \in S_Y} y^2 p_Y(y) - \left(\frac{47}{16}\right)^2 = \frac{111}{256}$$

Now, we can find the covariance as

$$\operatorname{cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{35}{8} - \frac{25 \cdot 47}{16 \cdot 16} = -\frac{55}{256}$$

Finally the correlation coefficient:

$$\rho = \frac{-\frac{55}{256}}{\sqrt{\frac{63}{256} \cdot \frac{111}{256}}} = -\frac{55\sqrt{777}}{2331}$$

▼ b)

We can use the definition of conditional expectation to find

$$E[Y|X] = \sum_{y \in S_Y} y \cdot p_{Y|X}(y|x) = \sum_y y \cdot \frac{p_{X,Y}(x,y)}{p_X(x)} = \frac{5x+15}{16} = g(X)$$

▼ c)

We can use conditional probability and the joint and marginal PMFs we defined in (a):

$$P(1 \le Y \le 3 | X = 1) = \frac{P(Y \cap X)}{P(X=1)} = \frac{\sum_{y=1}^{3} p_{X,Y}(x=1,y)}{p_X(x=1)}$$

Now, we can calculate the probability

$$P(1 \le Y \le 3 | X = 1) = \frac{\sum_{y=1}^{3} \frac{1+y}{32}}{\frac{4(1)+10}{32}} = \frac{32}{14} \cdot \frac{1+1+1+2+1+3}{32} = \frac{9}{14}$$

▼ d)

We can us the alternate definition of conditional variance:

$$\operatorname{var}(Y|X) = E[Y^2|X] - E[Y|X]^2 = E[Y^2|X] - \left(\frac{5x+15}{16}\right)^2$$

Now we can solve for the remaining conditional expectation

$$E[Y^2|X] = \sum_y y^2 \cdot \frac{p_{X,Y}(x,y)}{p_X(X)} = \sum_y \frac{y^2(x+y)}{4x+10} = \frac{15x+50}{2x+5}$$

So, the conditional variance is

$$\operatorname{var}(Y|X) = \frac{15x+50}{2x+5} - \left(\frac{5x+15}{16}\right)^2 = -\frac{5\left(10x^3+85x^2-528x-2335\right)}{256(2x+5)}$$

▼ 6

We can first define the distributions of the random variables

$$X \sim \text{Uniform}([1,4]) \qquad Y \sim \text{Uniform}([2,8])$$

We can construct the joint PMF using the table:

X	1/16	2/16	3/16	4/16	3/16	2/16	1/16
4	0	0	0	1/16	1/16	1/16	1/16
3	0	0	1/16	1/16	1/16	1/16	0
2	0	1/16	1/16	1/16	1/16	0	0
1	1/16	1/16	1/16	1/16	0	0	0
$p_{X,Y}$	2	3	4	5	6	7	8

Now we can find expectations:

$$E[X] = \frac{4+1}{2} = \frac{5}{2} \qquad E[Y] = \frac{8+2}{2} = 5$$
$$\&$$
$$E[XY] = \sum_{x,y} xy \cdot p_{X,Y} = \frac{55}{4}$$

We can also find the variances as

$$\operatorname{var}(X) = E[X^2] - E[X]^2 = \sum_x x^2 p_X(x) - \frac{25}{4} = \frac{30}{4} - \frac{25}{4} = \frac{5}{4}$$
$$\&$$
$$\operatorname{var}(Y) = E[Y^2] - E[Y]^2 = \sum_y y^2 p_Y(y) - 25 = \frac{55}{2} - 25 = \frac{5}{2}$$

Now we can find the covariance

$$\operatorname{cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{55}{4} - \frac{25}{2} = \frac{5}{4}$$

So, the correlation coefficient is

$$\rho = \frac{\frac{5}{4}}{\sqrt{\frac{5}{4}\cdot\frac{5}{2}}} = \frac{\sqrt{2}}{2}$$

▼ 7

We will assume the variables are uniformly distributed (chosen fairly) such that

$$X \sim \text{Uniform}([1,10] \implies E[X] = \frac{11}{2}, \operatorname{var}(X) = \frac{33}{4}$$
$$\&$$
$$Y|x \sim \text{Uniform}([1,x]) \implies E[Y|x] = \frac{x+1}{2}$$

Then, using what we learned in problem (2) and the given identity:

$$E[Y] = \frac{1}{10}\sum_{x=1}^{10} E[Y|x] = \frac{1}{20}\sum_{x=1}^{10} x+1 = \frac{1}{20}\left(\frac{10(10+1)}{2}+10\right) = \frac{13}{4}$$

The variances are found using the given identities:

$$\begin{gathered}\operatorname{var}(X) = \frac{10}{12} \\ \operatorname{var}(Y) = \frac{1}{10}\sum_{x=1}^{10}\left(\frac{x+1}{2}\right)^2 - \frac{169}{16} = \frac{1}{40}\left(\sum_{1}^{10} x^2 + 2\sum_{1}^{10} x + 10\right) \\ = \frac{1}{40}\left(\frac{10(10+1)(20+1)}{6} + 10(10+1) + 10\right) = \frac{101}{8}\end{gathered}$$

To find the joint expectation, we must first find the joint PMF which we can do so using the conditional probability theorem:

$$\begin{gathered}p_{X,Y}(x,y) = P(X \cap Y) = P(X)P(Y|X) \\ = p_X(x)p_{Y|X}(y|x) = \frac{1}{10}\cdot\frac{1}{x} = \frac{1}{10x} \quad 1 \le y \le x\end{gathered}$$

Then, using this probability we can find the joint expctation:

$$\begin{gathered}E[XY] = \sum_{x,y} xy \cdot p_{X,Y}(x,y) = \sum_x\sum_y \frac{xy}{10x} = \frac{1}{10}\sum_{x=1}^{10}\sum_{y=1}^{x} y \\ = \frac{1}{10}\sum_{x=1}^{10}\frac{x(x+1)}{2} = \frac{1}{20}\left(\sum_{x=1}^{10} x^2 + \sum_{x=1}^{10} x\right) \\ = \frac{1}{20}\left(\frac{10(10+1)(2(10)+1)}{6} + \frac{10(10+1)}{2}\right) = 22\end{gathered}$$

Now, we can find the covariance

$$\operatorname{cov}(X,Y) = E[XY] - E[X]E[Y] = 22 - \frac{11}{2}\cdot\frac{13}{4} = \frac{33}{8}$$

So, finally, the correlation coefficient is

$$\rho = \frac{\frac{33}{8}}{\sqrt{\frac{33}{4}\cdot\frac{101}{8}}} = \frac{\sqrt{6666}}{202}$$

8

a)

The correlation coefficient for these random variables is defined as

$$\rho = \frac{\operatorname{cov}(X, Y = aX+b)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y = aX+b)}}$$

We can first simplify these terms using their respective transformation rules

$$\begin{gathered}\operatorname{cov}(X, Y = aX+b) = a\cdot\operatorname{cov}(X,X) = a\cdot\operatorname{var}(X) \\ \operatorname{var}(Y = aX+b) = a^2\cdot\operatorname{var}(X)\end{gathered}$$

So, we can simplify the correlation coefficient to

$$\rho = \frac{a\cdot\operatorname{var}(X)}{\sqrt{a^2\cdot\operatorname{var}^2(X)}} = \frac{a}{|a|} = \pm 1$$

Thus, the correlation coefficient is maximized (by definition) when the random variable are linear combinations of each other.

▼ b)

We can start by expanding the numerator of the LHS and simplifying:

$$uv + \frac{(u-v)^2}{2} = uv + \frac{u^2 - 2uv + v^2}{2} = \frac{u^2 + v^2}{2}$$

Now, the LHS = RHS, QED.

▼ c)

We can first suppose $U = V$, then plugging in the variables we get:

$$\frac{XY}{\sqrt{E[X^2]E[Y^2]}} + 0 = \frac{\frac{X^2}{E[X^2]} + \frac{Y^2}{E[Y^2]}}{2}$$

We also know $U = V \implies X = Y$, so

$$\frac{X^2}{E[X^2]} = \frac{2\frac{X^2}{E[X^2]}}{2} \implies 1 = 1$$

Since this is true, we can apply the property $U = V \implies X = Y$ to the inequality:

$$E[X^2] \leq \sqrt{E[X^2]^2} \implies E[X^2] = E[X^2]$$

I.e. this is the only “equality” case in the inequality when $U = V$

▼ d)

We can begin with the inequality and dividing over the root so that:

$$\frac{E[XY]}{\sqrt{E[X^2]E[Y^2]}} \leq 1 \quad : \quad \rho \leq 1$$

We can observe this is precisely correlation coefficient we found in (a) IF and ONLY IF X, Y are linear combinations of each other

▼ e)

We use the definition of the coefficient and transformation properties of the covariance

$$\rho(X, X+Y) = \frac{\text{cov}(X, X+Y)}{\sigma_X \sigma_{X+Y}} = \frac{\text{var}(X) + \text{cov}(X, Y)}{\sigma_X \sigma_{X+Y}}$$

📌 **SUMMARY**