No class Monday (20th Feb).

Math 170E: Winter 2023

Lecture 16, Fri 17th Feb

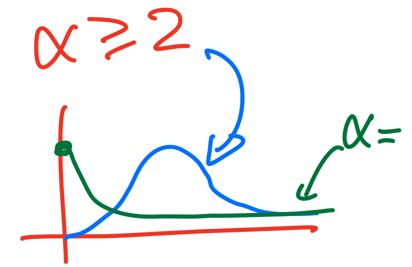
The Gamma and Normal distributions

Example 11: The Gamma distribution

- Consider an approximate Poisson process with rate $\lambda > 0$ per unit time
- Let $\alpha \geq 1$ be an integer
- Let X be the time of the αth arrival
- We say that X is gamma distributed with mean parameters α and $\theta = \frac{1}{\lambda}$ and write $X \sim \text{Gamma}(\alpha, \theta)$
- If $X \sim \text{Gamma}(1, \theta)$, then $X \sim \text{Exponential}(\theta)$

1 Find CDF $\mathbb{P}(X \leq x)$, $\longrightarrow$ 2 Compute $F_X'(x) = f_X(x)$

Proposition 3.14: If $\alpha \geq 1$ is an integer, $\theta > 0$ and $X \sim \text{Gamma}(\alpha, \theta)$, then it has PDF



$$f_X(x) = \frac{1}{\theta^{\alpha}(\alpha-1)!} x^{\alpha-1} e^{-\frac{x}{\theta}} \quad \text{if } x > 0$$

Why?

Let $x > 0$. Let $N = \#$ of arrivals in the time $[0, x]$.

Then $N \sim \text{Poisson}(x\lambda) = \text{Poisson}(x/\theta)$.

The CDF of X is:

$$F_X(x) = \mathbb{P}(X \leq x) = 1 - \underbrace{\mathbb{P}(X > x)}$$

x^{th} arrival happen after time x.

$$= 1 - \mathbb{P}(N \leq \alpha - 1).$$

$\{X > x\} = \{N \leq \alpha - 1\}$.

$$= 1 - \sum_{n=0}^{\alpha-1} e^{-x/\theta} \frac{x^n}{\theta^n n!}$$

$x^0 + x^1 + x^2 + \cdots$
$= 1 + x^1 + x^2 + \cdots$

So

$$f_X(x) = F_X'(x) = -\frac{d}{dx}\left(\sum_{n=0}^{\alpha-1} e^{-x/\theta} \frac{x^n}{\theta^n n!}\right)$$

$$= -\sum_{n=0}^{\alpha-1} \frac{d}{dx}\left(e^{-x/\theta}\frac{x^n}{\theta^n n!}\right).$$

$$= -\sum_{n=0}^{\alpha-1}\left(-\frac{1}{\theta}\right)e^{-x/\theta}\frac{x^n}{\theta^n n!} - \sum_{n=1}^{\alpha-1} e^{-x/\theta}\frac{n x^{n-1}}{\theta^n n!}$$

$$= \frac{1}{\theta}e^{-x/\theta}\sum_{n=0}^{\alpha-1}\frac{x^n}{\theta^n n!} - e^{-x/\theta}\sum_{n=1}^{\alpha-1}\frac{x^{n-1}}{\theta^n (n-1)!}$$

Relabel:
$m = n-1$
$n = m+1$

$$= \frac{1}{\theta}e^{-x/\theta}\sum_{m=0}^{\alpha-1}\frac{x^m}{\theta^m m!} - \frac{e^{-x/\theta}}{\theta}\sum_{m=0}^{\alpha-2}\frac{x^m}{\theta^m m!}$$

$$= \frac{1}{\theta}e^{-x/\theta}\frac{x^{\alpha-1}}{\theta^{\alpha-1}(\alpha-1)!} = \frac{1}{\theta^\alpha(\alpha-1)!}x^{\alpha-1}e^{-x/\theta} \quad //$$

- For $\alpha > 0$, we define the *Gamma function*:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$$

- $\Gamma(1) = 1$

- If $\alpha > 1$, then $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \overset{\text{IBP}}{=} \underbrace{\left[-x^{\alpha-1} e^{-x}\right]_0^\infty}_{0} + (\alpha-1)\int_0^\infty x^{\alpha-2} e^{-x} dx$$

$$= (\alpha-1)\int_0^\infty x^{(\alpha-1)-1} e^{-x} dx$$

$$= (\alpha-1)\Gamma(\alpha-1).$$

- If $\alpha \geq 1$ is an integer, then

$$\int_0^\infty x^{\alpha-1} e^{-x} dx$$

$$\Gamma(\alpha) = (\alpha - 1)!$$

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) = (\alpha-1)(\alpha-2)\Gamma(\alpha-2) = \cdots = (\alpha-1)!\,\Gamma(1)$$
$$= (\alpha-1)!$$

- Recall that if $\alpha \geq 1$ is an integer, then $X \sim \text{Gamma}(\alpha, \theta)$ if it has PDF

$$f_X(x) = \frac{1}{\theta^\alpha (\alpha - 1)!} x^{\alpha-1} e^{-\frac{x}{\theta}} \quad \text{if } x > 0$$

- We generalise this to any $\alpha, \theta > 0$: $X \sim \text{Gamma}(\alpha, \theta)$ if it has PDF

$$f_X(x) = \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\theta}} \quad \text{if } x > 0$$

- $\Gamma(1/2) = \sqrt{\pi}$

Proposition 3.15: If $\alpha, \theta > 0$ and $X \sim \text{Gamma}(\alpha, \theta)$, then it has MGF, mean and variance

$$M_X(t) = \frac{1}{(1-\theta t)^{\alpha}} \quad \text{if } t < \frac{1}{\theta}$$

$$\mathbb{E}[X] = \alpha\theta$$

$$\text{var}(X) = \alpha\theta^2$$

Proof: see HW 5

	Bernoulli	Poisson
# of arrivals / success in a fixed # of trials.	Binomial	Poisson.
Time/trial to get a first success	Geometric	EXP.
Time/trial to get a # of successes	Neg. Bin.	Gamma.

Motivation:

- Let X be the number of HEADS from n weighted coin flips
- Then $X \sim \text{Binomial}(n, p)$, then

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x \in \{0, 1, 2, \ldots, n\}.$$

- For large n, this becomes difficult to calculate
- $\Longrightarrow$ approximate the probability distribution!
- If n is *large* and p is *small*, then $X \sim \text{Poisson}(np)$, then it has PMF

$$p_X(x) = e^{-np} \frac{(np)^x}{x!} \quad \text{if } x \in \{0, 1, 2, \ldots\}$$

- $\Longrightarrow$ this is not so good when $p \sim \frac{1}{2}$

- We can write $X = X_1 + \cdots + X_n$, where $X_j \sim \text{Bernoulli}(p)$
- (baby) Central Limit Theorem:

$$\mathbb{P}\left(a \leq \frac{\overbrace{X_1 + \cdots + X_n}^{X} - \overset{\mathbb{E}[X]}{\overset{\parallel}{np}}}{\underbrace{\sqrt{np(1-p)}}_{\sigma_X}} \leq b\right) \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}x^2}\, dx$$

as $n \to \infty$

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx = 1}$$

- the limit object is independent of p!

If n is really big,

$$\mathbb{P}(\mu_X - a\sigma_X \leq X \leq \mu_X + b\sigma_X) \approx \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}x^2}\, dx.$$

Example: What is the probability of getting 20 HEADS in 40 flips? → fair.

$X = \#$ of Heads in 40 flips

$X \sim \text{Binomial}(40, \tfrac{1}{2})$

$\hookrightarrow \mathbb{P}(X=20) = \binom{40}{20}\frac{1}{2^{40}} \approx 0.1254.$

Approximate using CLT → $n = 40$, $p = 1/2$ ← kinda large.

$\mu_X = 40 \cdot \tfrac{1}{2} = 20$

$\sigma_X = \sqrt{40 \cdot \tfrac{1}{4}} = \sqrt{10}.$

$\mathbb{P}(X=20) = \mathbb{P}(19.5 \leq X \leq 20.5).$

$= \mathbb{P}\left(\frac{19.5-20}{\sqrt{10}} \leq \frac{X-20}{\sqrt{10}} \leq \frac{20.5-20}{\sqrt{10}}\right).$

$= \mathbb{P}\left(-0.16 \leq \frac{X-20}{\sqrt{10}} \leq 0.16\right)$

$\overset{CLT}{\approx} \frac{1}{\sqrt{2\pi}}\int_{-0.16}^{0.16} e^{-\frac{1}{2}x^2}\,dx \approx 0.1272.$

If we tried to use Poisson$(40 \cdot \frac{1}{2})$ = Poisson(20).

$$P_X(20) = e^{-20} \frac{20^{20}}{20!}$$

$$\simeq \boxed{0.0889}.$$

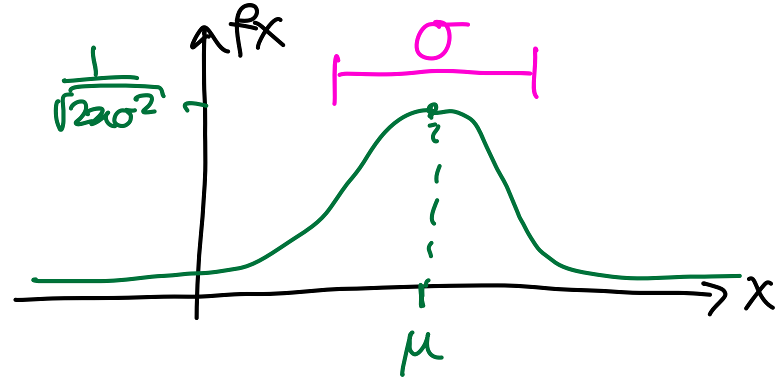
Example 11: The Normal distribution

$\in \mathbb{R}$

- We say a continuous random variable X is normally distributed with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ if it has PDF

(if $|x| \gg 1$, $\sim e^{-x^2}$

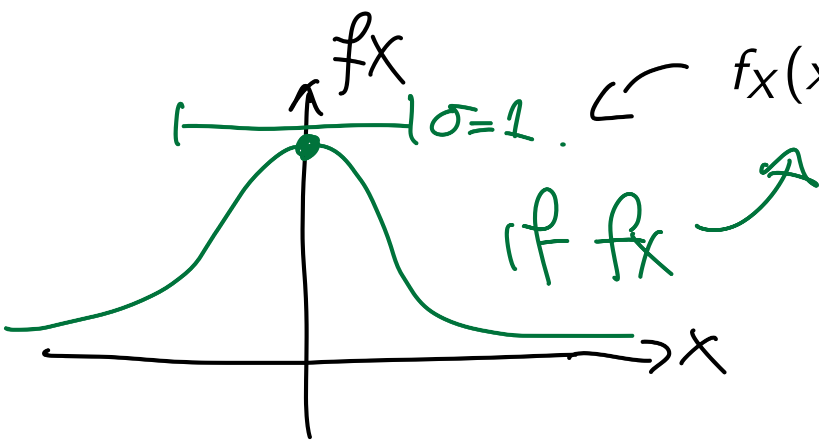
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for } x \in \mathbb{R}$$

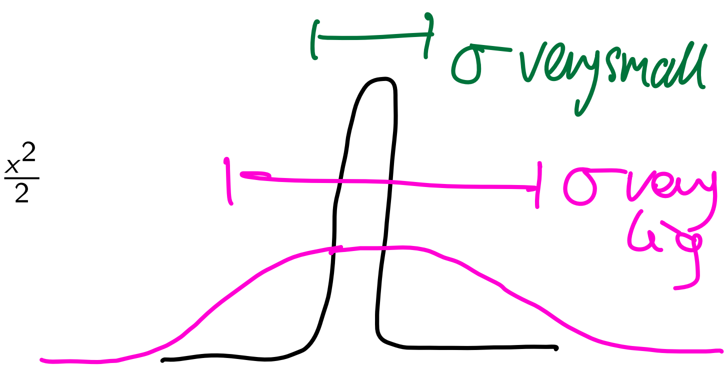


- We write $X \sim \mathcal{N}(\mu, \sigma^2)$

- If $\mu = 0$ and $\sigma^2 = 1$, we say that X is a standard normal random variable; i.e. $X \sim \mathcal{N}(0, 1)$, where

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$





Proposition 3.16 : We have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt = 1.$$

Proof: Change variables: $x = \frac{t-\mu}{\sigma}$, $dt = \sigma dx$.

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\cancel{\sigma}^2}} e^{-x^2/2} \cancel{\sigma} dx.$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \qquad (x,y) \to (r,\theta)$$

Use polar coordinates!

$x = r\cos\theta$, $y = r\sin\theta$.

$x^2+y^2 = r^2$.

$dxdy = r dr d\theta$.

$$I^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right)^2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dxdy.$$

$$= \int_0^{\infty} \int_0^{2\pi} \frac{1}{2\pi} e^{-r^2/2} r dr d\theta.$$

$$= \int_0^\infty r e^{-r^2/2}\,dr. \qquad \frac{d}{dr}\left(e^{-r^2/2}\right) = -r e^{-r^2/2}$$

$$= \int_0^\infty \frac{d}{dr}\left(-e^{-r^2/2}\right) dr.$$

$$= \left[-e^{-r^2/2}\right]_{r=0}^{\infty} = -\left(-e^{-0}\right) = 1.$$

$\Rightarrow I^2 = 1$ but $I \geq 0$ so $\underline{\underline{I = 1}}$.