Math 170E: Homework 2

Due: Fri. 27th January by 11:59pm PDT via Gradescope

Submit answers to all problems via Gradescope. The reader will grade three problems each out of five points. Up to five further points will be awarded based on the proportion of the remaining problems that are completed.

Please make sure that your submission is readable. If your pencil is too faint, get a thicker one. If your handwriting is cramped and small, write bigger and use more paper. Please use simple plain paper or lined paper (e.g. please avoid graph paper etc.). It is your responsibility to ensure that your submission is readable. If we cannot read a solution, we may refuse to grade it. Thank you!

I encourage you to discuss and work on problems with other students in the class. Nevertheless, the solutions you present have to be your own. In particular, if the solution you present is identical to someone else's, or it is identical to some other resource (book, online, etc.), this will be considered cheating.

1. Let A be a set with n elements.
   (a) How many non-empty subsets of A are there?
   (b) How many non-empty subsets of size k are there? Here $1 \leq k \leq n$.

2. Use the binomial theorem to show that:
   (a) $\displaystyle\sum_{r=0}^{n} \binom{n}{r} = 2^n$
   (b) $\displaystyle\sum_{r=0}^{n} \binom{n}{r} (-1)^r = 0$

3. Argue as in the proof of the binomial theorem to prove the *multinomial theorem*:
$$(x_1 + x_2 + \cdots + x_r)^n = \sum \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r},$$
   where the sum is taken over all non-negative integers $n_1, n_2, \ldots, n_r$ such that $n_1 + n_2 + \ldots + n_r = n$.

4. In a state lottery, four digits are drawn at random one at a time with replacement from 0 to 9. Suppose that you win if any permutation of your selected integers is drawn. Give the probability of winning if you select:
   (a) $6, 7, 8, 9$

(b) $6, 7, 8, 8$

(c) $7, 7, 8, 8$

(d) $7, 8, 8, 8$

5. How many ways can you rearrange the letters in ABRACADABRA?

6. How many ways can you give 10 cookies to 4 friends if each friend gets at least 1 cookie?

7. A poker hand is defined as drawing 5 cards at random without replacement from a deck of 52 playing cards. Find the probability of each of the following poker hands:
   (a) Four of a kind (four cards of equal face value and one card of a different value).
   (b) Full house (one pair and one triple of cards with equal face value).
   (c) Three of a kind (three equal face values plus two cards of different values).
   (d) Two pairs (two pairs of equal face value plus one card of a different value).
   (e) One pair (one pair of equal face value plus three cards of different values).

8. Suppose that the alleles for eye color for a certain male fruit fly are (R, W) and the alleles for eye color for the mating female fruit fly are (R,W), where R and W represent red and white, respectively. Their offspring receive one allele for eye color from each parent. Assume that each of the four possible outcomes has equal probability. If an offspring ends up with either two white alleles or one red and one white allele for eye color, its eyes will look white. Given that an offspring's eyes look white, what is the probability that it has two white alleles for eye color?

9. An urn contains four colored balls: two orange and two blue. Two balls are selected at random without replacement, and you are told that at least one of them is orange. What is the probability that the other ball is also orange?

10. An urn contains four balls numbered 1 through 4. The balls are selected one at a time without replacement. A match occurs if the ball numbered m is the mth ball selected. Let the event A_i denote a match on the ith draw, $i = 1, 2, 3, 4$.
   (a) Show that $\mathbb{P}(A_i) = \frac{3!}{4!}$ for every $i = 1, 2, 3, 4$.
   (b) Show that $\mathbb{P}(A_i \cap A_j) = \frac{2!}{4!}$ for $i \neq j$.
   (c) By computing $\mathbb{P}(A_i \cap A_j \cap A_k)$ for $i \neq j \neq k$, show that the probability of at least one match is

   $$\mathbb{P}(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}.$$

   (d) Extend this exercise so that there are n balls in the urn. Show that the probability of at least one match is

   $$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^{n+1}}{n!}.$$

   What is the limit of this probability as $n \to \infty$?

11. An urn contains eight red and seven blue balls. A second urn contains an unknown number of red balls and nine blue balls. A ball is drawn from each urn at random, and the probability of getting two balls of the same color is 151/300. How many red balls are in the second urn?

12. Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement.

    (a) If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

    (b) Now suppose that the balls have different weights, with each red ball having weight r and each white ball having weight w. Suppose that the probability that a given ball in the urn is the next one selected is its weight divided by the sum of the weights of all balls currently in the urn. Now what is the probability that both balls are red? (You can test your result by checking some extreme cases: e.g. the red balls are suddenly weightless!! ($r = 0$), or the red balls are infinitely heavy compared to the white balls!! ($r \to \infty$ while $w < \infty$).

13. At a certain stage of a criminal investigation, the inspector in charge is 60% convinced of the guilt of a certain suspect. Suppose, however, that a new piece of evidence which shows that the criminal has a certain characteristic (such as left-handedness, baldness, or brown hair) is uncovered. If 20% of the population possesses this characteristic, how certain of the guilt of the suspect should the inspector now be if it turns out that the suspect has the characteristic? You may suppose that the probability of the suspect having the characteristic if they are, in fact, innocent is equal to 0.2, the proportion of the population possessing the characteristic.

14. There are two bags. Bag A contains 1 red ball and 9 blue balls. Bag B contains 10 blue balls. You choose a ball at random from bag A and place it into bag B. You then choose a ball at random from bag B and place it into bag A. If after this process the red ball is in bag A, what is the probability that it was moved?