Math 170E: Winter 2023

Lecture 11, Mon 6th Feb

The Geometric and Negative binomial distributions

Last time:

- A Bernoulli trial is an experiment that has a probability $p \in (0, 1)$ of success and probability $(1 - p)$ of failure
- Let X be the number of successes out of $n \geq 1$ *independent, identical* Bernoulli trials
- Then, $X \sim \text{Binomial}(n, p)$

Today:

We'll discuss today:

- the Geometric distribution and computing its PMF, MGF, mean, variance
- the negative binomial distribution and its PMF

Example 13:

- You have an unfair coin which flips HEADS with probability $\frac{1}{4}$ and TAILS with probability $\frac{3}{4}$
- You flip the coin many times
- What is the probability that you first see HEADS on the 5th flip?

A) $\left(\frac{3}{4}\right)^5$

B) $\left(\frac{3}{4}\right)^4 \times \frac{1}{4}$

C) $\left(\frac{1}{4}\right)^5$

D) $\left(\frac{1}{4}\right)^4 \times \frac{3}{4}$

Definition 2.21:

- Suppose we run independent, identical Bernoulli trials with probability $p \in (0, 1)$ of success
- Let X be the trial on which we first achieve success
- Then X is a discrete random variable taking values in the set $S = \{1, 2, 3, \ldots\}$
- X takes values in $S = \{0, 1, \ldots, n\}$.
- We say that X is a Geometric random variable with parameter p and write $X \sim \text{Geometric}(p)$

Proposition 2.22: If $X \sim \text{Geometric}(p)$, then its PMF is

$$p_X(x) = (1-p)^{x-1}p \quad \text{if} \quad x \in \{1, 2, 3, \ldots\}$$

Proof:

Example 13:

- You roll fair six sided die over and over again.
- How many rolls do you need to so that the probability of getting a 6 with this number of rolls is at least 50%?

Proposition 2.22: If $X \sim \text{Geometric}(p)$, then its MGF is

$$M_X(t) = \frac{pe^t}{1 - (1-p)e^t} \quad \text{if} \quad t < -\log(1-p)$$

Proof:

Proposition 2.23: If $X \sim \text{Geometric}(p)$, then its mean is

$$\mathbb{E}[X] = \frac{1}{p}$$

Proof:

Proposition 2.24: If $X \sim \text{Geometric}(p)$, then its variance is

$$\text{var}(X) = \frac{1-p}{p^2}$$

Proof:

Example 14:

- Consider the die roll example again.
- On average, how many rolls do I need in order to see a 6?

A) 6

B) 11

C) 4

D) 36

Example 15:

- You have an unfair coin which flips HEADS with probability $\frac{1}{4}$ and TAILS with probability $\frac{3}{4}$
- You flip until you see 3 HEADS
- What is the probability that you get your 3rd HEAD on the 6th flip?

Definition 2.25:

- Suppose we run independent, identical Bernoulli trials with probability $p \in (0, 1)$ of success
- Let $r \geq 1$ and X be the trial on which we first achieve the rth success
- Then X is a discrete random variable taking values in the set $S = \{r, r + 1, r+\}$
- X takes values in $S = \{r, r + 1, r + 2, \ldots\}$.
- We say that X is a negative binomial random variable with parameter r, p and write $X \sim \text{Negative Binomial}(r, p)$
- If $r = 1$, then $X \sim \text{Negative Binomial}(1, p) \sim \text{Geometric}(p)$

Proposition 2.26: If $X \sim$ Negative Binomial(r, p), then its PMF is

$$p_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \quad \text{if} \quad x \in \{r, r+1, \ldots\}$$

Proof: