$\boxed{7.2}$

When we have a recurrence relation that expresses a_n in terms of $a_0, a_1, \ldots$, we want to find a closed formula for a_n.

We can do this using
1) Iteration, and
2) (in certain cases) linear homogenous rec. rel with constant coefficients.

We have seen option 1 before:

$\boxed{Ex}$ Let $a_n = a_{n-1} + 3$ where $a_1 = 2$.
Find a closed formula for $a_n, n \geq 1$

$$a_n = a_{n-1} + 3 = (a_{n-2} + 3) + 3 = a_{n-2} + 2 \cdot 3$$
$$= (a_{n-3} + 3) + 2 \cdot 3 = a_{n-3} + 3 \cdot 3$$
$$= \cdots$$
$$= a_{n-k} + 3k$$

For $k = n-1$,
$$a_n = a_1 + 3(n-1) \quad \text{where } n \geq 1$$
is our closed formula

$\boxed{Ex}$ Tower of Hanoi,
We found $c_n = 2c_{n-1} + 1$ where $c_1 = 1$.

Then
$$c_n = 2c_{n-1} + 1$$
$$= 2(2c_{n-2} + 1) + 1 = 2^2 c_{n-2} + 2^1 + 1$$
$$= 2^2(2c_{n-3} + 1) + 2^1 + 1 = 2^3 c_{n-3} + 2^2 + 2^1 + 1$$
$$= \cdots$$
$$= 2^k c_{n-k} + 2^{k-1} + \cdots + 2^1 + 1$$

$\Rightarrow$ for $k = n-1$,
$$c_n = 2^{n-1} c_0 + 2^{n-2} + \cdots + 2^1 + 1$$
$$= 2^{n-1} + 2^{n-2} + \cdots + 2^1 + 1 = \sum_{k=0}^{n-1} 2^k$$
$$= \frac{2^n - 1}{2-1} = 2^n - 1 \quad \text{by geometric sum formula}$$

Def A linear homogeneous rec. rel. of order k (LHRC) w/ constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \quad \text{where } c_k \neq 0.$$

Note: If we know $\{a_{n-1}, a_{n-2}, \ldots, a_{n-k}\}$ this defines the sequence

Ex ① Fibonacci numbers $f_n = f_{n-1} + f_{n-2}$ is a linear homog. rec rel of order 2.

② The recurrence $S_n = 2S_{n-1}$ is a LHRC of order 1.

(non) Ex ① The recurrence relation $a_n = 3a_{n-1}a_{n-2}$ is not a LHRC since there can be no terms $a_i a_j$ in the recurrence. If a recurrence has such terms, we say it is **nonlinear**.

② The rec. rel. $a_n = 3n a_{n-1}$ does not have constant coeff. so is not a LHRR

How to solve LHRC?
We will only discuss the solutions for LHRC of order ≤ 2.

Theorem Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ ($n \geq 2$) be LHRC where $a_0 = C_0$, $a_1 = C_1$.
Let $r_1 \neq r_2$ be the roots of the equation $t^2 - c_1 t - c_2$.
Then there exist constants b, d such that
$$a_n = b r_1^n + d r_2^n \quad n \geq 0,$$
where $b + d = C_0$ and $b r_1 + d r_2 = C_1$.

Ex) Suppose $d_n = 3d_{n-1} - 2d_{n-2}$ $\quad n \geq 2$
where $d_0 = 200$, $d_1 = 220$.

First we find r_1, r_2:
$\Rightarrow$ need to solve
$$t^2 - 3t + 2 = 0 \Rightarrow r_1 = 1, \ r_2 = 2$$
$\Rightarrow d_n = b \cdot 1^n + c \cdot 2^n = b + c \cdot 2^n$
$\Rightarrow d_0 = b + c2^0 = 200$ and $d_1 = b + c2^1 = 220$

$\Rightarrow b = 200 - c$
$\Rightarrow b = 200 - 20$
$= 180$

$\Rightarrow c = \dfrac{220 - b}{2}$
$\Rightarrow 2c = 220 - (200 - c)$
$\Rightarrow c = 20$

$\Rightarrow d_n = 180 + 20 \cdot 2^n$ for $n \geq 0$.

Ex) For Fibonacci, $f_n = f_{n-1} + f_{n-2}$ $\quad n \geq 3$, $f_1 = f_2 = 1$.

To find r_1, r_2: solve $t^2 - t - 1 = 0$
by quadratic formula $\Rightarrow r_1, r_2 = \dfrac{1 \pm \sqrt{5}}{2}$

$$\Rightarrow f_n = b\left(\frac{1+\sqrt{5}}{2}\right)^n + d\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Then we solve
$$b\left(\frac{1+\sqrt{5}}{2}\right) + d\left(\frac{1-\sqrt{5}}{2}\right) = 1$$
and $$b\left(\frac{1+\sqrt{5}}{2}\right)^2 + d\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1$$
to find $b = \frac{1}{\sqrt{5}}$, $d = -\frac{1}{\sqrt{5}}$

$$\Rightarrow f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \quad n \geq 1.$$

Theorem Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ be LHRC where $a_0 = C_0$, $a_1 = C_1$.
Let r be a repeated root of the equation
$$t^2 - c_1 t - c_2.$$
Then there exist constants b, d such that
$$a_n = b r^n + d n r^n \quad n \geq 0.$$

Ex) Let $d_n = 4d_{n-1} - 4d_{n-2}$ $n \geq 2$ where $d_0 = d_1 = 1$.
Then we solve
$$t^2 - 4t - 4 = 0 \Rightarrow r_1 = r_2 = 2 \text{ is a repeated root}$$
$$\Rightarrow d_n = b 2^n + d n 2^n$$
Then we solve
$$d_0 = 1 = b 2^0 + d \cdot 0 = b$$
$$d_1 = 1 = b 2^1 + d \cdot 1 \cdot 2^1 = 1 \cdot 2 + 2d$$
$$\Rightarrow 1 = 2 + 2d \Rightarrow d = -1/2$$
$$\Rightarrow d_n = 2^n - n/2\, 2^n \text{ for } n \geq 0.$$
$$= 2^n - n 2^{n-1}$$

$\boxed{\text{Extra}}$ (7.2 - Exercises 42-48)

How to solve second order, linear inhomogeneous recurrence relation with constant coeff:

$(\varnothing)$ $a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n)$, where $f(n)$ is not identically 0.

$p(t) = t^2 - c_1 t - c_2$ is the auxillary polynomial.

We say $g(n)$ is a solution of $(\varnothing)$ if $\forall n \geq 2$,

$$g(n) = c_1 g(n-1) + c_2 g(n-2) + f(n)$$

How to find a solution based on roots of $\underbrace{t^2 - c_1 t - c_2 = 0}_{\text{homogeneous eq.}}$ + $f(n)$

$f(n)$	1 not a root	1 is single root	1 is double root
C	A	An	An^2
Cn	$An+B$	An^2+Bn	An^3+Bn^2
Cn^2	An^2+Bn+C	An^3+Bn^2+Cn	$An^4+Bn^3+Cn^2$

$f(n)$	C not a root	C is single root	C is double root
C^n	AC^n	AnC^n	An^2C^n

We use this to find the particular solution $g(n)$

$\boxed{\text{Thm}}$ Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n)$ be SOLIRRCC & $g(n)$ be a particular solution. Then any solution of $(\varnothing)$ is of the form

$$U_n = V_n + g(n), \text{ where}$$

$$V_n = \begin{cases} b r_1^n + d r_2^n & \text{if } p(t) \text{ has roots } r_1 \neq r_2 \\ b r^n + d n r^n & \text{if } p(t) \text{ has double root } r \end{cases}$$

Steps:

1. Find $p(t)$ + compute its roots
2. Find / Guess particular solution $g(n)$
3. Write out a_n in terms of roots + particular solution
4. Use initial conditions to determine constants in $g(n)$

Ex) $a_n = 6a_{n-1} - 9_{n-2} + n \quad n \geq 2 \quad (\otimes)$ where $a_0 = a_1 = 1$

Step 1: Find homog. eq + roots

Compute roots of $p(t) = t^2 - 6t + 9 = 0 \Rightarrow t = -3$ double root

Step 2: make guess for $g(n)$

we'll guess $g(n) = An + B$ + substitute to compute constants

$$\Rightarrow g(n-1) = A(n-1) + B$$
$$g(n-2) = A(n-2) + B$$

$\Rightarrow$ plugging into $(\otimes)$, we get

$$An + B = 6(An + (B-A)) - 9(An + (B-2A)) + n$$

$$\Rightarrow \underbrace{4An}_{} + \underbrace{(4B - 12A)}_{} = \underbrace{n}_{} + \underbrace{0}_{}$$

$$\Rightarrow 4A = 1, \quad 4B - 12A = 0$$
$$\Rightarrow A = 1/4 \quad B = 3/4$$

$$\Rightarrow g(n) = An + B = \frac{n+3}{4}$$

By Thm, $u_n = v_n + g(n) = (br^n + dnr^n) + \frac{n+3}{4}$

$$= br^n + dnr^n + \frac{n+3}{4}$$

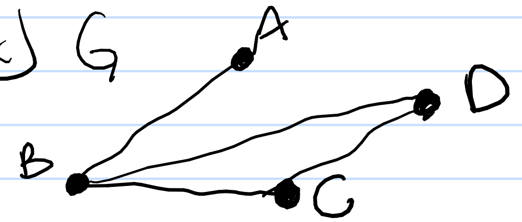
So $a_0 = 1 = b$, $a_1 = br + dr + \frac{1+3}{4} = b + d + 1 \Rightarrow d = -1$

$$\Rightarrow a_n = r^n - nr^n + \frac{n+3}{4}$$

$\boxed{8.1}$ Graphs

Def A <u>graph</u> G is a set of <u>vertices</u> $V(G)$ and <u>edges</u> $E(G) \subseteq \{(u,v) \mid u,v \in V\}$. Here $(u,v) = e \in E(G)$ is **not** an ordered pair, so we view $(u,v) = (v,u)$.

ex) G



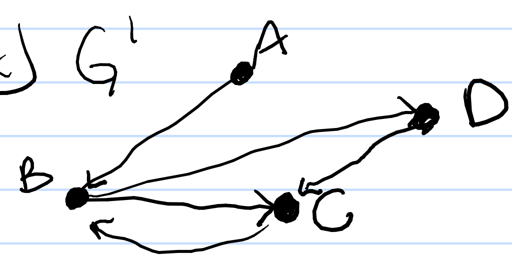
$V(G) = \{A, B, C, D\}$
$E(G) = \{(A,B), (B,D), (C,D), (B,C)\}$

Def A <u>directed graph</u> G is a set of vertices $V(G)$ and edges $E(G) \subseteq \{(u,v) \mid u,v \in V\} \subseteq V \times X$. Here $(u,v) = e \in E(G)$ **is** an ordered pair.

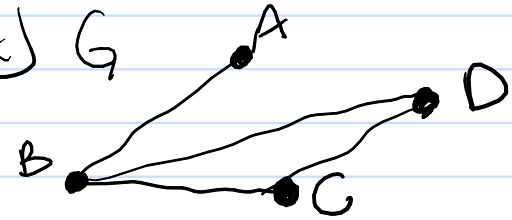
Def An edge $e \in E(G)$ is <u>incident to</u> vertices v + w if $e = (v,w)$. In this case we call v, w <u>adjacent vertices</u>

We often write $G = (V, E)$.

ex) G'



is a directed graph (digraph)
Here $V(G) = V(G')$
and
$E(G) = \{(A,B), (B,D), (D,C), (C,B)\}$

Ex) 

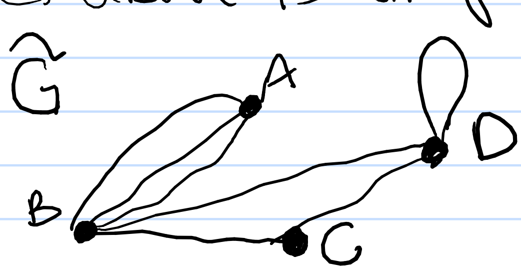
A is adjacent to B

A is not adjacent to D

Def Multiple edges associated to the pair $\{v_1, v_2\} \subseteq V$ are called <u>parallel</u>. An edge (v, v) is called a <u>loop</u>, where $v \in V$. If v_1 is not adjacent to any $v_2 \in V$, we say v_1 is <u>isolated</u>.

If G has no loops or parallel edges, we say G is a <u>simple graph</u>.

Ex) G above is simple



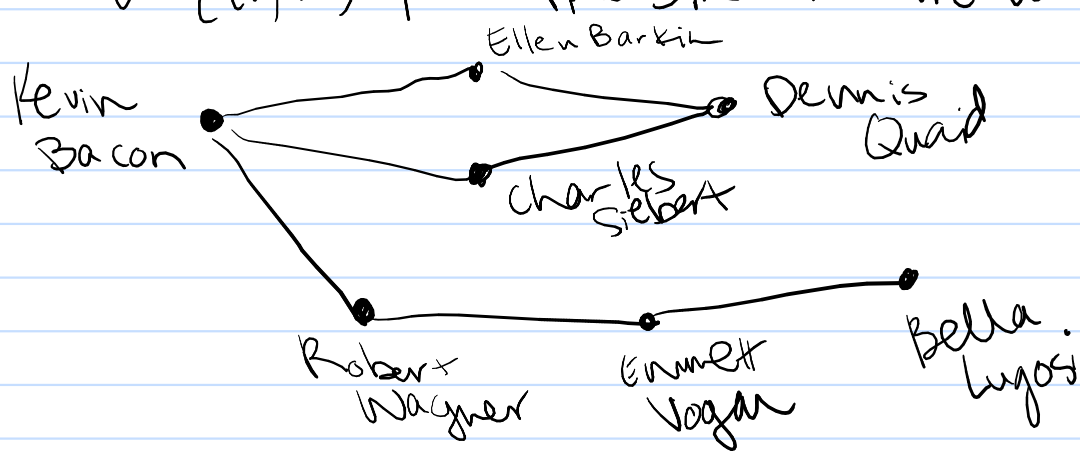
is not

Def A graph with numbers on the edges is a <u>weighted graph</u>. If $e \in E(G)$ has label k, we say the <u>weight of edge e</u> is k. For a path P in a weighted graph, we say the length of P is the sum of weights of the edges in the path.

Ex) Bacon numbers

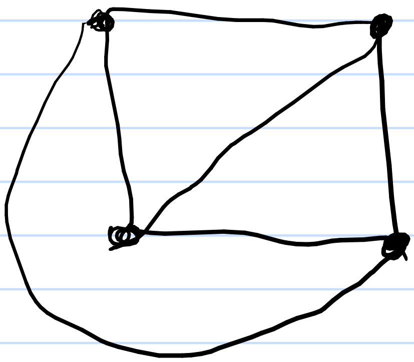
let $V \subseteq \{\text{Hollywood actors}\}$

$E = \{(v, w) \mid v \text{ appears in} \geq 1 \text{ movie with } w\}$



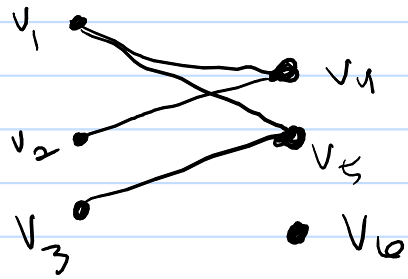
Def The *complete graph on n vertices*, denoted K_n, is the simple graph with $|V(K_n)| = n$ and an edge between each pair of vertices.

Ex K_4



Def $G = (V, E)$ is *bipartite* if there exist $V_1, V_2 \subseteq V$ (may be empty) where $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V$ and each $e \in E$ is such that $e = (v_1, v_2)$ where $v_1 \in V_1$, $v_2 \in V_2$.

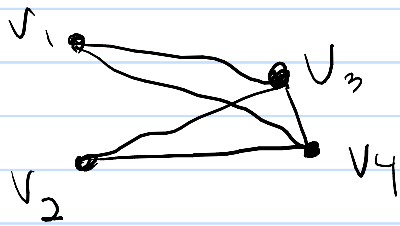
Ex



is bipartite. We could take $V_1 = \{v_1, v_2, v_3\}$

$V_2 = \{v_4, v_5, v_6\}$

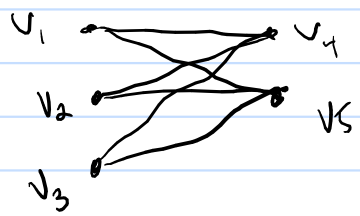
Ex



is not bipartite

Def The *complete bipartite graph on m and n vertices*, denoted $K_{m,n}$ is the simple graph whose vertex set is partitioned by V_1, V_2 where $|V_1| = m$, $|V_2| = n$, and $E(K_{m,n}) = \{(v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2\}$

Ex



is $K_{3,2}$

8.2 Paths + Cycles

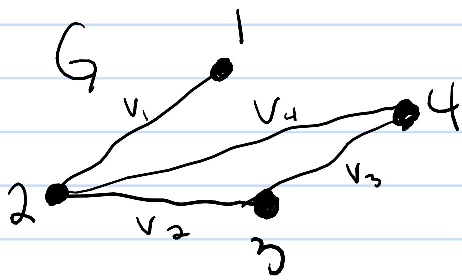
Def Let $v_0, v_n \in V(G)$. A path from v_0 to v_n of length n is an alternating sequence of $n+1$ vertices + n edges, beginning w/ v_0 + ending w/ v_n,

$$(v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n)$$

where each $e_i = (v_{i-1}, v_i) \in E(G)$ for $i \in [1, n]$.

(for simple graphs, we just list vertices)

Ex)

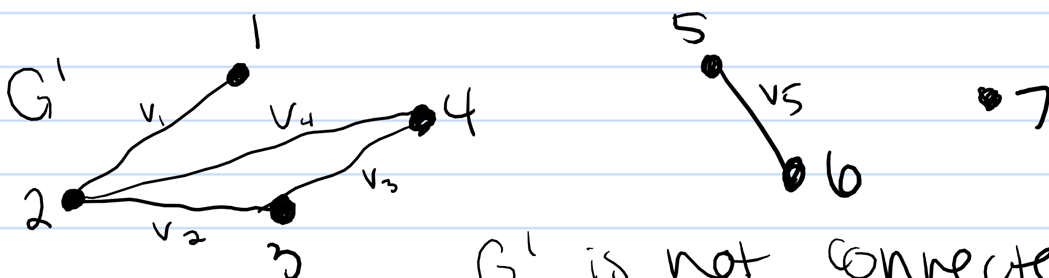


$(1, v_1, 2, v_4, 4, v_3, 3)$ is a path of length 3.

(3) is a path of length 0

Def A graph G is connected if for any $v, w \in V(G)$ there exists a path starting at v and ending at w in G.

Ex) G above is connected



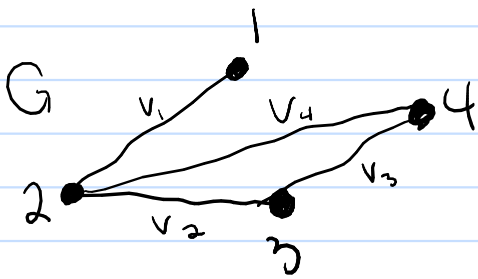
G' is not connected since there is no path between 1 and 5.

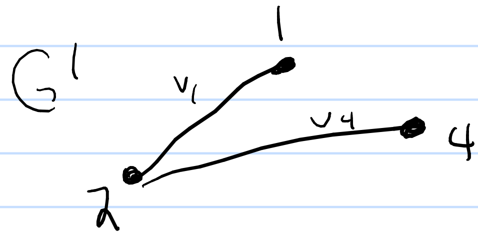
Def Let $G = (V, E)$ be a graph. $G' = (V', E')$ is a subgraph of G if

a) $V' \subseteq V$ and $E' \subseteq E$, and

b) If $e = (v, w) \in E'$, then $v, w \in V'$.

Ex

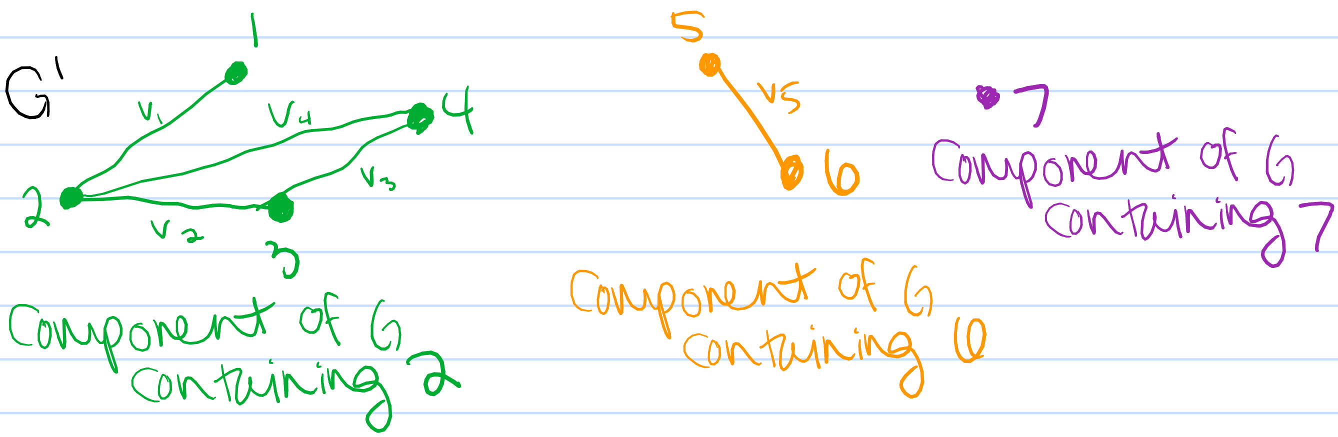




is a subgraph of G.

Def Let G be a graph and $v \in V(G)$. The subgraph G' of G containing all edges + vertices contained in some path beginning at v is called the <u>component</u> of G containing v.

Ex



Ex) G is connected if G has only one component

Def For $v, w \in V(G)$: a <u>simple path</u> from v to w is a path from v to w w/ no repeated vertices

A <u>cycle</u> at v is a path of nonzero length from v to v w/ no repeated edges.

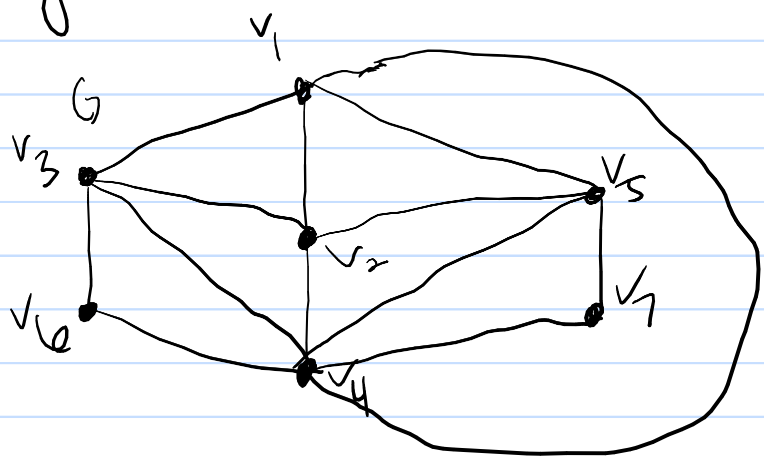
A <u>simple cycle</u> at v is a cycle from v to v in which there are no repeated vertices, other than the first + last.

Def A cycle in G that includes each edge & vertex in G is called an <u>Eulerian cycle</u>.

The <u>degree</u> of $v \in V(G)$, denoted $\delta(v)$, is the number of edges incident to v.
(Note: we say a loop at v gives +2 to degree of v)

Thm G has an Eulerian cycle $\iff$ G is connected & every vertex has even degree.

Ex



G is connected & each vertex has even degree $\Rightarrow$ G has Euler cycle.

$(6, 4, 7, 5, 1, 3, 4, 1, 2, 5, 4, 2, 3, 6)$

Thm If G has m edges & $V(G) = \{v_1, \dots, v_n\}$ then

$$\sum_{i=1}^{n} \delta(v_i) = 2m$$

Cor For any G, there are always an even number of $v \in V(G)$ such that $\delta(v)$ is odd

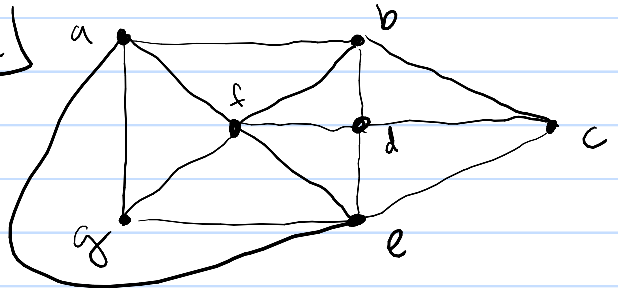
Thm G has a path w/ no repeated edges from v to $w \neq v$ containing all vertices & edges $\iff$ G is connected & v, w are the only vertices in G w/ odd degree

Thm If G contains a cycle from v to v, G contains a simple cycle from v to v

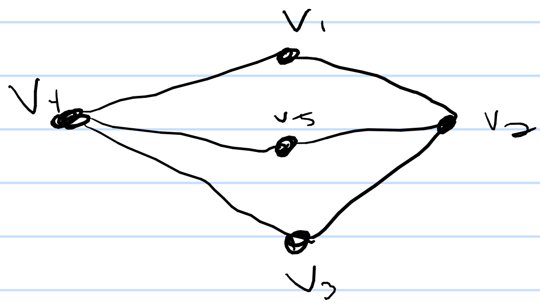
8.3 Hamiltonian cycles

Def A cycle in G that contains each $v \in V(G)$ exactly once (except for the start + end) is a <u>Hamiltonian cycle</u>.

Ex



has a Hamiltonian cycle (a, e, g, f, d, c, d, a)



does not (see next page for argument why)

(The traveling salesman problem is the following:

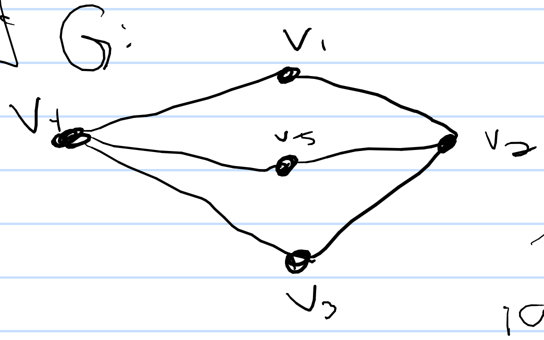
Given a weighted graph G, find a min length Hamiltonian cycle in G.)

<u>Q</u>: How to determine G has no Hamiltonian cycle?

very hard in general.

In some cases, once we analyze G, we can reason why one will not exist.

Ex) G:

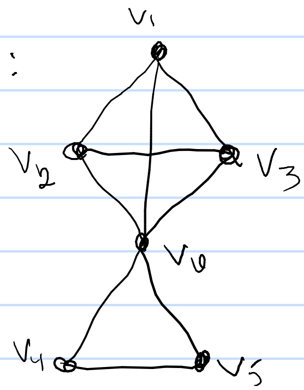


v_2, v_4 have no edges between them

To draw a Ham cycle, we would ignore all but 2 edges from each vertex. Then for G, we would delete 1 edge from v_2 + 1 from v_4. But then this would leave 4 edgets total, but we need 5 for a Ham. cycle $\rightarrow\leftarrow$

$\Rightarrow$ no Ham cycle

Ex) G:



Any Ham cycle must contain $(v_4, v_5), (v_5, v_6), (v_4, v_6)$, but this is already a cycle that doesn't contain v_1, v_2, v_3 $\rightarrow\leftarrow$

$\Rightarrow$ no Ham cycle.