→ No class on Monday.

Math 170E: Winter 2023

Lecture 3, Fri 13th Jan

Independence and methods of enumeration

Last time:

We proved some general properties of a probability measure:

- $\mathbb{P}(A') = 1 - \mathbb{P}(A)$
- If $A \subset B$ are events, then $\mathbb{P}(A) \leq \mathbb{P}(B)$
- (inclusion-exclusion principle) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$A \cup B = \{A \text{ or } B \text{ occurs}\}$

$\left\{\begin{array}{l} A \text{ but not } B \\ B \text{ but not } A \\ \text{Both } A \text{ \& } B \end{array}\right\}$

Today:

We'll discuss today:

- some applications and further directions for the inclusion-exclusion principle
- What it means for two events to be independent
- How to compute probabilities of successive independent trials

Theorem 1.6: (The inclusion-exclusion principle) For any events A and B,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Proof:

Example 4: A survey of the Scottish population found:

- 40% of Scots disliked haggis $\rightarrow \mathbb{P}(A') = 0.4$
- 49% of Scots liked black pudding $\rightarrow \mathbb{P}(B) = 0.49$
- 80% of Scots who liked haggis also liked black pudding

Q: What is the probability that a randomly chosen Scot likes haggis ~~and~~ OR black pudding?

A) 0.59

~~B) 0.95~~

~~C) 0.03~~

D) 0.61


Let $\Omega = \{$ Scottish population in the survey $\}$.

$\mathcal{F} =$ all subsets of Ω. $A \in \mathcal{F}$.

collection of people.

Let $A = \{$ people who like haggis $\}$

$B = \{$ ______ BP $\}$.

WANT: $\mathbb{P}(A \cup B)$. , $\mathbb{P}(A) = 1 - \mathbb{P}(A') = 0.6$,

$\mathbb{P}(B) = 0.49$

$|A \cap B| = (0.8)|A|$

$\hookrightarrow \mathbb{P}(A \cap B) = \dfrac{|A \cap B|}{|\Omega|} = \dfrac{(0.8)|A|}{|\Omega|} = (0.8)\mathbb{P}(A)$

$= (0.8) \times (0.6)$

$= 0.48$.

By inclusion-exclusion,

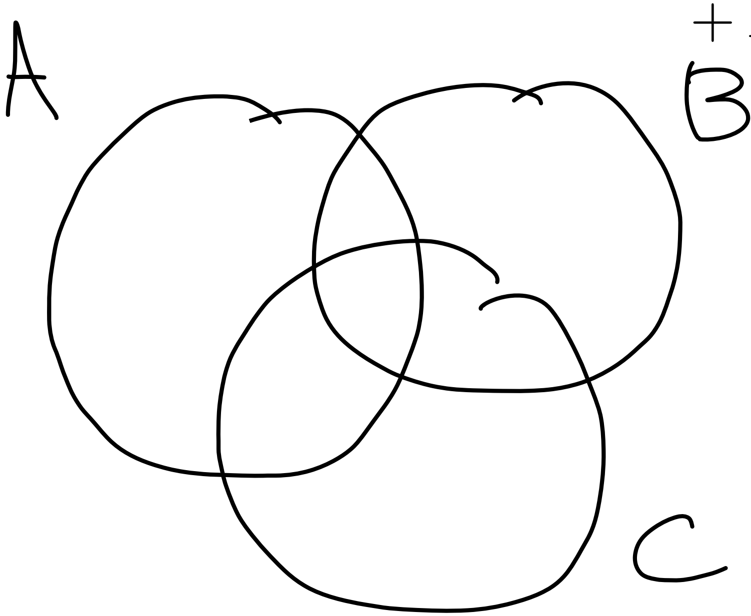
$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$= 0.6 + 0.49 - 0.48 = 0.61$ //

Theorem 1.7: If A, B and C are events, then

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) =& \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\ &- \mathbb{P}(A \cap B) - \mathbb{P}(\cancel{A \cap B}) - \mathbb{P}(B \cap C) \\ &+ \mathbb{P}(A \cap B \cap C)\end{aligned}$$

$A \cap C$.



↪ See Homework.

Inclusion-Exclusion generalises to any finite number of sets:

$$\mathbb{P}\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=k}} \mathbb{P}\left(\bigcap_{j \in I} A_j\right)$$

$\mathbb{P}(A_1 \cap A_n)$.

$\mathbb{P}(A_2 \cap A_3)$

$\mathbb{P}(A_1 \cap A_2)$

Definition 1.9: We say that two events $A, B \subseteq \Omega$ are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

↳ don't influence each other.

If two events are not independent, we say they are dependent.

$A \cap B$ = event that both A & B occur.

$\mathbb{P}(A \cap B)$ = (proportion of outcomes for which A & B occur)

$$= \frac{|A \cap B|}{|\Omega|}$$

if A, B independent

$$= \frac{|A|}{|\Omega|} \cdot \frac{|B|}{|\Omega|} = \mathbb{P}(A)\mathbb{P}(B).$$

Example 1: You flip a fair coin twice.
Define the events

- $A = \{\text{First flip is a head}\}$
- $B = \{\text{Second is a tail}\}$

Are the events A and B independent?

$\Omega = \{HH, HT, TH, TT\}$.

$A = \{HH, HT\} \rightarrow \mathbb{P}(A) = 2/4 = 1/2$

$B = \{HT, TT\} \rightarrow \mathbb{P}(B) = 1/2$

$A \cap B = \{HT\} \rightarrow \mathbb{P}(A \cap B) = 1/4$.

Check: $\mathbb{P}(A \cap B) = 1/4 = 1/2 \times 1/2 = \mathbb{P}(A)\,\mathbb{P}(B)$

$\hookrightarrow$ Thus, A and B are independent.

Example 2: You flip a fair coin twice.

Define the events

- $A = \{\text{First flip is a head}\}$
- $C = \{\text{Both are tails}\}$

Are the events A and C independent?

$\Omega = \{HH, HT, TH, TT\}$.

$\mathbb{P}(A) = 1/2$

$\hookrightarrow C = \{TT\}$, $\mathbb{P}(C) = 1/4$.

$\hookrightarrow A \cap C = \emptyset$ (if first is a H, then C can't have happened).

$\hookrightarrow \mathbb{P}(\emptyset) = 0$.

$$\mathbb{P}(A \cap C) = 0 \neq \frac{1}{2} \times 1/4 = \mathbb{P}(A)\,\mathbb{P}(C).$$

So A & C are dependent.

⚠: Don't confuse independence with disjoint

$A \cap B = \emptyset$

$0 = \mathbb{P}(A \cap B)$, $\mathbb{P}(A) \cdot \mathbb{P}(B)$

Proposition 1.10: If A and B are independent events, then so are

$\mathbb{P}(A\cap B)=\mathbb{P}(A)\,\mathbb{P}(B)$

(i) A and B'

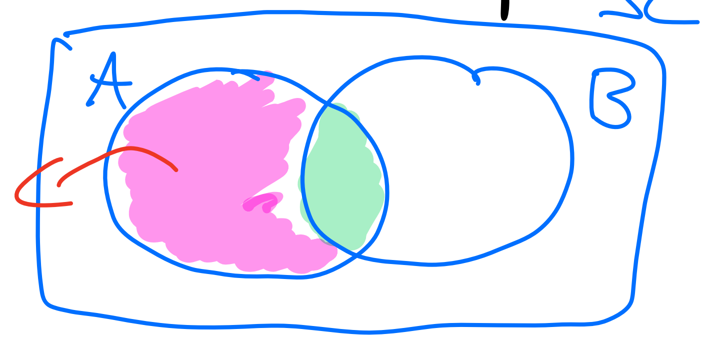
(ii) A' and B' $\longrightarrow$ See HW

Proof:

(i) <u>WANT</u>: $\boxed{\mathbb{P}(A\cap B')=\mathbb{P}(A)\,\mathbb{P}(B')}$ $\longrightarrow$ A & B' are indep.

Ω

$A\cap B'$



We can write

$$A\cap B' = A\setminus(A\cap B).$$

and since $A\cap B\subseteq A$, then

$$\mathbb{P}(A\cap B') = \mathbb{P}(A\setminus(A\cap B))$$

$$= \mathbb{P}(A)-\mathbb{P}(A\cap B),$$

A, B indep.

$$= \mathbb{P}(A)-\mathbb{P}(A)\,\mathbb{P}(B) = \mathbb{P}(A)(1-\mathbb{P}(B)).$$

$$= \mathbb{P}(A)\,\mathbb{P}(B')$$

Definition 1.11: We say events $A_1, A_2, \ldots, A_n \subseteq \Omega$ are (mutually) independent if, given $1 \leq k \leq n$ and $1 \leq j_1 < j_2 < \ldots < j_n \leq n$, we have

$$\mathbb{P}\left(\bigcap_{\ell=1}^{k} A_{j_\ell}\right) = \prod_{\ell=1}^{k} \mathbb{P}(A_{j_\ell})$$

$n=10$:

Put $k=1$: $1 \leq j_1 \leq 10$, $\mathbb{P}(A_{j_1}) = \mathbb{P}(A_{j_1})$

Put $k=2$: $1 \leq j_1 < j_2 \leq 10$, $\mathbb{P}(A_{j_1} \cap A_{j_2}) = \mathbb{P}(A_{j_1})\mathbb{P}(A_{j_2})$,

Put $k=3$: $1 \leq j_1 < j_2 < j_3 \leq 10$, $\mathbb{P}(A_{j_1} \cap A_{j_2} \cap A_{j_3})$

$= \mathbb{P}(A_{j_1})\mathbb{P}(A_{j_2})\mathbb{P}(A_{j_3})$, In the special case $n = 3$, this says the events $A, B, C \subseteq \Omega$ are mutually independent if and only if **all** the following are satisfied:

$$\begin{aligned} \mathbb{P}(A \cap B) &= \mathbb{P}(A)\mathbb{P}(B) \\ \mathbb{P}(A \cap C) &= \mathbb{P}(A)\mathbb{P}(C) \\ \mathbb{P}(B \cap C) &= \mathbb{P}(B)\mathbb{P}(C) \\ \mathbb{P}(A \cap B \cap C) &= \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C). \end{aligned}$$

Example 3: You flip a fair coin twice.

Define the events

- $A = \{\text{First flip is a head}\} = \{HH, HT\}$
- $B = \{\text{Second flip is a head}\} = \{HH, TH\}$
- $C = \{\text{Both are the same}\} = \{HH, TT\}$.

$\left.\right\} \mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = 1/2$.

Are the events A, B and C independent?

$A \cap B = \{HH\} \rightarrow \mathbb{P}(A \cap B) = 1/4 = \mathbb{P}(A)\mathbb{P}(B)$

$A \cap C = \{HH\} \rightarrow \mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$

$B \cap C = \{HH\} \rightarrow \mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C)$.

$\hookrightarrow A \cap B \cap C = \{HH\} \rightarrow$

$$\mathbb{P}(A \cap B \cap C) = 1/4 \neq 1/2 \times 1/2 \times 1/2 = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$$

$\hookrightarrow$ So $A, B, \&\ C$ are not mutually indep.

Flip n fair coins.

$$\Omega = \{ \{ \underbrace{||||||||}_{n} \}, \ldots \}$$

$1 \leq k \leq n$

$$\mathbb{P}(\text{exactly } k \text{ heads}) = \frac{|\text{strings which have } k\,H \;\&\; (n-k)\,T|}{|\Omega|}.$$

$n = 100$

$k = 52.$

Methods of enumeration:

The multiplication principle:

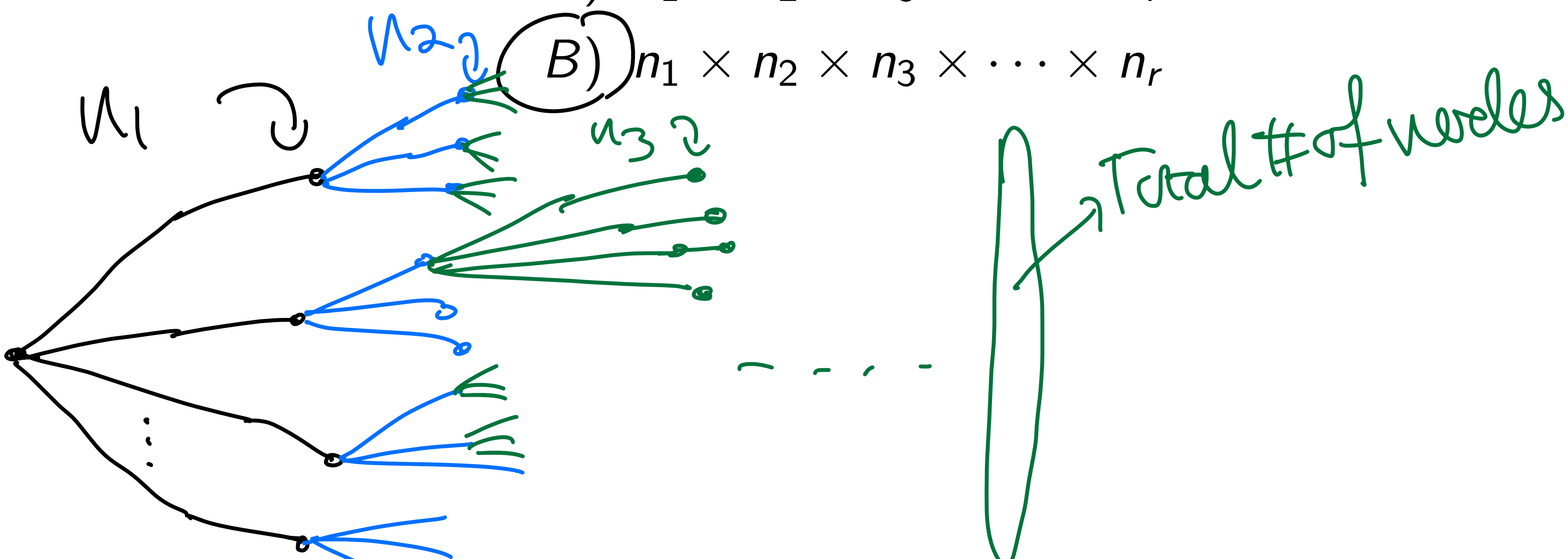
Let $r \in \{1, 2, 3, \ldots\}$. Suppose that we run r independent experiments and that:

- the 1st experiment has n_1 possible outcomes
- the 2nd experiment has n_2 possible outcomes
- $\ldots$
- the rth experiment has n_r possible outcomes

How many possible outcomes does the composite experiment have?

$$A)\ \ n_1 + n_2 + n_3 + \cdots + n_r$$

$$B)\ \ n_1 \times n_2 \times n_3 \times \cdots \times n_r$$



Example 4: You roll a fair six-sided die and then flip a fair coin. What is the probability that you rolled a 6 and you flipped a head?

$n_1 = 6$

$n_2 = 2$ outcomes.

A) $\frac{1}{24}$

B) $\frac{2}{3}$

C) $\frac{1}{12}$

D) $\frac{1}{6}$

$\Omega = \{ \{1H\}, \{2H\}, \ldots, \{6H\}, \{1T\}, \ldots, \{6T\} \}$

$|\Omega| = 12.$

$= n_1 \times n_2 = 6 \times 2.$

multip. principle.

$$\mathbb{P}(\{6H\}) = \frac{|\{6H\}|}{|\Omega|} = \frac{1}{12}$$