Homework 8

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1

a

We can check that the PDF is well defined over the relevant interval by taking the double integral over the interval space to check the normalization condition:

$$\int_0^1 \int_0^1 x + y \, dx dy = \int_0^1 \frac{1}{2} + y \, dy = 1$$

As we can see, the PDF is well defined over the relevant interval.

b

We can integrate over the interval of the other variables to find the marginal PDF of each variable:

$$f_X(x) = \int_0^1 x + y \, dy = \frac{1}{2} + x$$

$$f_Y(y) = \int_0^1 x + y \, dx = \frac{1}{2} + y$$

c

The variables are independent if they follow that:

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \text{but} \\ \left(\frac{1}{2} + x\right)\left(\frac{1}{2} + y\right) \neq x + y$$

So, the variables are not independent.

d

We can use a function g of the variables to find the means and variances

$$g(X,Y)=X \implies E[X]=\int_0^1\int_0^1 x(x+y)\,dxdy=\frac{7}{12}$$

$$g(X,Y)=Y \implies E[Y]=\int_0^1\int_0^1 y(x+y)\,dxdy=\frac{7}{12}$$

We can also use the formulas for variance in terms of the expectation:

$$\begin{aligned}\mathrm{var}(X)=E[X^2]-E[X]^2&=\int_0^1\int_0^1 x^2(x+y)\,dxdy-\left(\frac{7}{12}\right)^2\\&=\frac{5}{12}-\left(\frac{7}{12}\right)^2=\frac{11}{144}\end{aligned}$$

$$\mathrm{var}(Y)=\int_0^1\int_0^1 y^2(x+y)\,dxdy-\left(\frac{7}{12}\right)^2=\frac{5}{12}-\left(\frac{7}{12}\right)^2=\frac{11}{144}$$

e

The correlation coefficient is given as

$$\rho(X,Y)=\frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}}$$

Then, we can find the covariance using (d):

$$\begin{aligned}\mathrm{cov}(X,Y)=E[XY]-E[X]E[Y]&=\int_0^1\int_0^1 xy(x+y)\,dxdy-\left(\frac{7}{12}\right)^2\\&=\frac{1}{3}-\left(\frac{7}{12}\right)^2=-\frac{1}{144}\end{aligned}$$

So using (d) we can find the covariance:

$$\rho=\frac{-\frac{1}{144}}{\sqrt{\left(\frac{11}{144}\right)^2}}=-\frac{1}{11}$$

2

We can first find the marginal PDFs using the integral of the joint PDF over the intervals described: $0\le x\le y$ and $x\le y<\infty$, so:

$$f_X(x)=\int_0^y 2e^{-x-y}dy=2e^{-2y}\quad f_Y(y)=\int_x^\infty 2e^{-x-y}dx=-2e^{-x-y}$$

By the independence theorem, the joint PDF is independent if $f_{X,Y}=f_Xf_Y$:

$$f_X(x)f_Y(y)=-4e^{-x-3y}\neq f_{X,Y}(x,y)$$

So, X,Y are not independent.

▼ 3

We can check the normalization condition for each function over its defined interval space to check that it is a true joint PDF

▼ a

$$\int_0^1 \int_{x^2}^{x} cxy \, dydx = \frac{c}{24} = 1 \implies c = 24$$

▼ b

$$\int_0^1 \int_0^{y^2} cye^x \, dxdy = \frac{(e-2)c}{2} = 1 \implies c = \frac{2}{e-2}$$

▼ c

$$\int_0^{\pi/2} \int_0^{\pi/2} c\sin(x+y) \, dxdy = 2c = 1 \implies c = \frac{1}{2}$$

▼ 4

We can model this probability using the expression:

$$P(-0.1 \leq X - Y \leq 0.1) = P((X,Y) \in A) \implies A = \{x, y : y - 0.1 \leq x \leq y + 0.1, 2 < x < 2.5, 2 < y < 2.3\}$$

Wee can also visualize the sample space for which the PDF is defined as the rectangle formed by the bounds given because each variable is uniformly distributed. This gives us a total area of the rectangle of $(2.5-2)(2.3-2) = 0.15$. This means we can find the probability as fraction of the overall sample space. The area the probability covers is defined in the expression above which we can find as:

$$A = \int_2^{2.3} \int_{y-0.1}^{y+0.1} dxdy + \int_{1.9}^{2} \int_2^{x+0.1} dydx = 0.055$$

So, the probability is:

$$P(|X - Y| \leq 0.1) = \frac{0.055}{0.15} = \frac{11}{30}$$

▼ 5

We can use. similar method as question (4) by first defining the sample space as

$$\Omega = \int_1^{10} \int_2^{(14-t_1)/2} dt_2 dt_1 - \int_1^2 \int_6^{(14-t_1)/2} dt_2 dt_1 = 20$$

Then, we can also express the space the probability is defined over as an area integral of the bounds:

$$A = \int_6^{10} \int_2^{(14-t_1)/2} dt_2 dt_1 - \int_6^8 \int_2^{10-t_1} dt_2 dt_1 = 2$$

So, the probability is found as:

$$P(T_1 + T_2 > 10) = \frac{2}{20}$$

6

We know the conditional variable is uniformly distributed over the bounds, so:

$$E[Y|x] = \frac{x + 0.1 + x - 0.1}{2} = x$$

Now, we can use iterated expectation:

$$E[Y] = E[E[Y|X]] = E[X] = \int_{0.2}^{\infty} x(2e^{-2(x-0.2)})\, dx = 0.7$$

7

a

By conditional probability (given that the conditional variable is uniformly distributed:

$$f_{X,Y}(x, y) = f_{Y|X}(y|x) f_X(x) = \frac{1}{2} \cdot \frac{1}{x^2} = \frac{1}{2x^2} \quad (0 < y < x^2)$$

b

The marginal PDF of Y can be found using the bounds $x > \sqrt{y}$ and $x < 2$

$$f_Y(y) = \int_{\sqrt{y}}^{2} \frac{1}{2x^2}\, dx = -\frac{1}{4} + \frac{1}{2\sqrt{y}}$$

c

First we can find the conditional PDF for $X|Y$ using the conditional probability using (a),(b)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{\frac{1}{2x^2}}{-\frac{1}{4} + \frac{1}{2\sqrt{y}}}$$

Then, we can find the conditional expectation as:

$$E[X|y] = \int_{\sqrt{y}}^{2} x f_{X|Y}(x|y)\, dx = \frac{\ln(2) - \ln(\sqrt{y})}{2\left(\frac{1}{2\sqrt{y}} - \frac{1}{4}\right)}$$

8

We first know the PDF of X to be:

$$f_X(x) = 1$$

We can use the proposition of functions of random variables s.t.

$$u(X) = e^X \implies u'(X) > 0 \quad \forall x \in [0,1]$$
$$u^{-1}(y) = \{x : e^x = y\} = \ln(y)$$

And, since we know the function $u(X) = e^X$ is smooth and increasing for all values of x in the domain defined by the uniform distribution, we can use the proposition that:

$$f_Y(y) = \left|\frac{d}{dy}u^{-1}(y)\right| \cdot f_X(u^{-1}(y)) = \left|\frac{1}{y}\right| \cdot 1 = \frac{1}{|y|}$$

▼ 9

We know the PDF of the exponential distribution to be

$$f_X(x) = e^{-x}$$

Now, we can find a function of X over the domain of the distribution given $a > 0$ s.t.

$$u(X) = e^{-X/a} \implies u'(X) = -\frac{1}{5}e^{-X/5} < 0 \quad \forall X$$
$$u^{-1}(y) = \{x : e^{-x/a} = y\} = -a\ln(y)$$

We can use the proposition for functions of random variables to determine the PDF of Y:

$$f_Y(y) = \left|\frac{d}{dy}u^{-1}(y)\right| \cdot f_X(u^{-1}(y)) = \left|-\frac{a}{y}\right| \cdot e^{a\ln(y)} = \left|\frac{a}{y}\right| \cdot y^a$$

Finally, since we know the domains s.t. $y > 0$ and $a > 0$ we can simplify to:

$$f_Y(y) = ay^{a-1}$$

📌 **SUMMARY**