1 Introduction

In this homework we look at three different methods to remove projective and affine distortions from images,

1. Remove projective distortion with Point-to-Point Correspondence and then remove affine distortion by using the $cos\theta$ expression with $\theta = 90\,\text{deg}$
2. Remove projective distortion with the Vanishing line method and then remove affine distortion by using the $cos\theta$ expression with $\theta = 90\,\text{deg}$
3. Remove projective distortion and affine distortion using a One-Step method with a dual degenerate conic

2 Description of Methods

1. ***Point-to-Point Correspondence Method to Remove Projective Distortion***

   In this method let H_p be the homography that linearly transforms a point in the homogeneous representation space in the distorted image to a point in the world image (undistorted image). H_p in general can be represented as,

$$H_p = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & \nu \end{bmatrix}. \tag{1}$$

   We can set $\nu = 1$, since in the representational space, it is not the absolute values that matter but the ratios. If

$$\vec{X}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \vec{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix},$$

   where (x, y) and (x', y') are the points in the physical space in the distorted image and world image then the transformation can be represented by,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}. \tag{2}$$

   Expanding Eq. 2 we get,

$$\begin{aligned} x' &= a_{11}x + a_{12}y + t_x \\ y' &= a_{21}x + a_{22}y + t_y \\ 1 &= v_1x + v_2y + 1, \end{aligned} \tag{3}$$

   or,

$$\begin{aligned} x' &= \frac{a_{11}x + a_{12}y + t_x}{v_1x + v_2y + 1} \\ y' &= \frac{a_{21}x + a_{22}y + t_y}{v_1x + v_2y + 1}. \end{aligned} \tag{4}$$

   Given the pair of points, (x, y) and (x', y'), using Eq. 3 we can determine the unknown parameters of H namely, $a_{11}, a_{12}, a_{21}, a_{22}, t_x, t_y, v_1, v_2$. Since there are eight unknown parameters we need at least 8 equations. Each pair of (x, y), (x', y') points gives two equations.

Hence for eight equations we will need four pairs of (x_i, y_i), (x'_i, y'_i) points where $i = 1, 2, 3, 4$. Substituting the values of (x_i, y_i), (x'_i, y'_i) in Eq. 3 we get a set of equations which can be written as,

$$\begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ x'_3 \\ y'_3 \\ x'_4 \\ y'_4 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x'_1 & -y_1x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y'_1 & -y_1y'_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2x'_2 & -y_2x'_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2y'_2 & -y_2y'_2 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3x'_3 & -y_3x'_3 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & -x_3y'_3 & -y_3y'_3 \\ x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4x'_4 & -y_4x'_4 \\ 0 & 0 & 0 & x_4 & y_4 & 1 & -x_4y'_4 & -y_4y'_4 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ t_x \\ a_{21} \\ a_{22} \\ t_y \\ v_x \\ v_y \end{bmatrix} \tag{5}$$

or,

$$C' = AB, \tag{6}$$

where,

$$C' = \begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ x'_3 \\ y'_3 \\ x'_4 \\ y'_4 \end{bmatrix} A = \begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x'_1 & -y_1x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y'_1 & -y_1y'_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2x'_2 & -y_2x'_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2y'_2 & -y_2y'_2 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3x'_3 & -y_3x'_3 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & -x_3y'_3 & -y_3y'_3 \\ x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4x'_4 & -y_4x'_4 \\ 0 & 0 & 0 & x_4 & y_4 & 1 & -x_4y'_4 & -y_4y'_4 \end{bmatrix} B = \begin{bmatrix} a_{11} \\ a_{12} \\ t_x \\ a_{21} \\ a_{22} \\ t_y \\ v_x \\ v_y \end{bmatrix}.$$

We also define C as,

$$C = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix}, \tag{7}$$

for coding purposes.

2. ***Vanishing Line Method to Remove Projective Distortion***

The second method considered in this study to remove the projective distortion in the image is by using the Vanishing Line method. In the Vanishing line method we first identify two pairs of parallel lines in the physical world. Let the images of those two pairs of lines be l_1, m_1 and l_2, m_2. The intersection of a pair of physically parallel lines in an image is called a vanishing point and is given by,

$$VP_1 = l_1 \times m_1, \tag{8}$$

$$VP_2 = l_2 \times m_2. \tag{9}$$

where VP_1 and VP_2 are the two vanishing points for the two pairs of parallel lines. The line obtained by joining these two vanishing points VP_1 and VP_2 is called the vanishing line VL and its equation is obtained using,

$$VL = VP_1 \times VP_2. \tag{10}$$

If the homogeneous representation of VL is $VL = [vl_1, vl_2, vl_3]$ then homography that removes projective distortion from an image is given by,

$$H_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ vl_1 & vl_2 & vl_3 \end{bmatrix}. \tag{11}$$

3. *Cosθ* ***Expression Method to Remove Affine Distortion***

Once the projective distortion is removed either by using the Point-to-Point correspondence method or the Vanishing Line method discussed above, we remove the affine distortion using the $cos\theta$ expression method. If $l = l_1x + l_2y + l_3$ and $m = m_1x + m_2y + m_3$ represent two lines in the physical world then the angle between these two lines is given by,

$$cos\theta = \frac{l_1m_1 + l_2m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}. \tag{12}$$

This can be expressed in terms of the dual degenerate conic C_∞^*,

$$cos\theta = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* m)(m^T C_\infty^* l)}}, \tag{13}$$

where,

$$C_\infty^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

Using the homography H, numerator of Eq. 13 can be written in terms of images of l and m which are l' and m' respectively as,

$$l'^T H C_\infty^* H^T m' = l'^T \begin{bmatrix} A & \vec{t} \\ \vec{0} & 1 \end{bmatrix} \begin{bmatrix} I & \vec{0} \\ \vec{0} & 1 \end{bmatrix} \begin{bmatrix} A^T & \vec{0} \\ \vec{t}^T & 1 \end{bmatrix} m' = l'^T \begin{bmatrix} AA^T & \vec{0} \\ \vec{0} & 1 \end{bmatrix} m'. \tag{15}$$

Here, the affine homography H_a is given by,

$$H_a = \begin{bmatrix} A & \vec{0} \\ \vec{0} & 1 \end{bmatrix}. \tag{16}$$

If the two lines l and m are orthogonal in the physical world then $cos\theta = 0$ so that $l'^T H C_\infty^* H^T m' = 0$. If we use the notation $S = AA^T$, then for a pair of orthogonal lines Eq. 15 collapses to,

$$\begin{bmatrix} l_1' & l_2' \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} m_1' \\ m_2' \end{bmatrix} = 0. \tag{17}$$

Since S is symmetric $s_{12} = s_{21}$ we essentially have three unknowns, s_{11}, s_{22}, s_{12}. But in the homogeneous representational space the information is only in the ratios so essentially we have only two unknowns s_{11}, s_{22} by setting $s_{12} = 1$. We can get these two unknowns from Eq. 17 by considering two pairs of orthogonal lines.

Once we get S, we can use singular value decomposition (SVD) to get A required in the affine homography H_a in Eq. 16 by using the fact that $S = AA^T$ and $A = VDV^T$ so that $S = VD^2V^T$.

4. ***One-Step Method to Remove Projective and Affine Distortion***

   The third method used to remove the projective and affine distortion in this study is a One-Step Method. Let H_{pa} be the homography that removes the projective and affine distortion and it can be represented as,

$$H_{pa} = \begin{bmatrix} A & 0 \\ v^T & 1 \end{bmatrix}. \tag{18}$$

   In this method we use the image of the Dual Degenerate Conic C^*_∞ represented by the 3×3 matrix in Eq. 14. The image of C^*_∞ in the image plane is $C^{*\prime}_\infty$ and in general form can be represented as,

$$C^{*\prime}_\infty = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}. \tag{19}$$

   $C^{*\prime}_\infty$ has five unknown parameters (one of a, b, c, d, e, f can be set to 1). The five unknown parameters of can be obtained using five pairs of orthogonal lines in the physical world and,

$$l'^T C^{*\prime}_\infty m' = 0. \tag{20}$$

   Once we get $C^{*\prime}_\infty$, we can get the A matrix in Eq. 18 using SVD of S where,

$$S = AA^T = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}, \tag{21}$$

   and then use A to get the vector v in Eq. 18 using,

$$v^T A^T = \begin{bmatrix} d/2 & e/2 \end{bmatrix}. \tag{22}$$

   In the following sections we discuss the implementation of the Two-Step Method and the One-Step Method.

3 Description of the Two-Step Method Implementation

The Two-Step Method has been implemented in the Jupyter notebook **TwoStepMethod_Tejas_Pant.ipynb**

1. In the first step, the image is loaded using **mpimg.imread**.

2. In the next step, we identify the set of points required to calculate the homography matrices H_p and H_a

3. Next we calculate the projective distortion matrix H_p which maps the distorted input image to a world image without any distortion. The matrix H_p can be obtained using either the Point-to-Point Correspondence Method or the Vanishing Line Method. If the Point-to-Point Correspondence Method is selected then H_p is calculated using the method **calcHomographyPointCorrespond** and if the Vanishing Line Method is selected then H_p is calculated using the method **calcHomographyVanishLine**.

4. We now have to calculate the homography H_a that removes the affine distortion. This calculation is done in the method **calcHomographyAffine** using two pairs of orthogonal lines. Note that the end points of the orthogonal lines should be in an image without projective distortion so we apply the homography H_p to the end points of the orthogonal lines to get their images in the image plane without projective distortion. This transformation of the end points is done using the method **transformEndPointsPerpendLinesAffine**.

5. Since H_a is from the world image without any projective or affine distortion to an image without projective distortion but with affine distortion, the total homography H_{pa} that removes projective and affine distortion is given by $H_{pa} = H_a^{-1} H_p$.

6. Finally we apply this homography H_{pa} to the distorted image to get the world image without any projective and affine distortion. While calculating the world image it is absolutely necessary to define a scaling factor to control the size of the world image. The calculation of the world image is done in the two methods **getUndisortedImgData** and **getImage**.

4 Description of the One-Step Method Implementation

The One-Step Method has been implemented in the Jupyter notebook **OneStepMethod_Tejas_Pant.ipynb**

1. In the first step, the image is loaded using **mpimg.imread**.

2. In the next step, we identify the endpoints of the five pairs of orthogonal lines that are required to calculate the homography H_{pa}^{-1} that maps the distorted image to a world image without projective and affine distortion. Note the inverse operation. In the current implementation we try to identify four points which form a square so that the five pairs of orthogonal lines in are a combination of four sides of the square and and the final pair of orthogonal lines is the intersection of the two diagonals of the square.

3. The homography H_{pa} is calculated using the method **calHomographyOneStep**.

4. Finally we apply this homography H_{pa} to the distorted image to get the world image without any projective and affine distortion. While calculating the world image it is absolutely necessary to define a scaling factor to control the size of the world image. The calculation of the world image is done in the two methods **getUndisortedImgData** and **getImage**.

5 Conclusions

1. In general it is observed that the Two-Step methods are more robust than the One-Step Method. However it appears that the One-Step method can also be made more robust by constructing the five pairs of orthogonal lines using the four sides of a square and the diagonals of the square.

2. Among the Two-Step Methods, for the Point-to-Point Correspondence it is observed that if the same set of points that are used for calculating H_p are used to calculate H_a then the difference between the images without projective distortion and without projective and affine distortion is very small.

3. For the Point-to-Point Correspondence, the sequence of the points is important otherwise the world image without distortion can be inverted or a mirror image of the distorted original image.

4. Among the two Two-Step Methods, the Vanishing Line Method is observed to be the most robust.

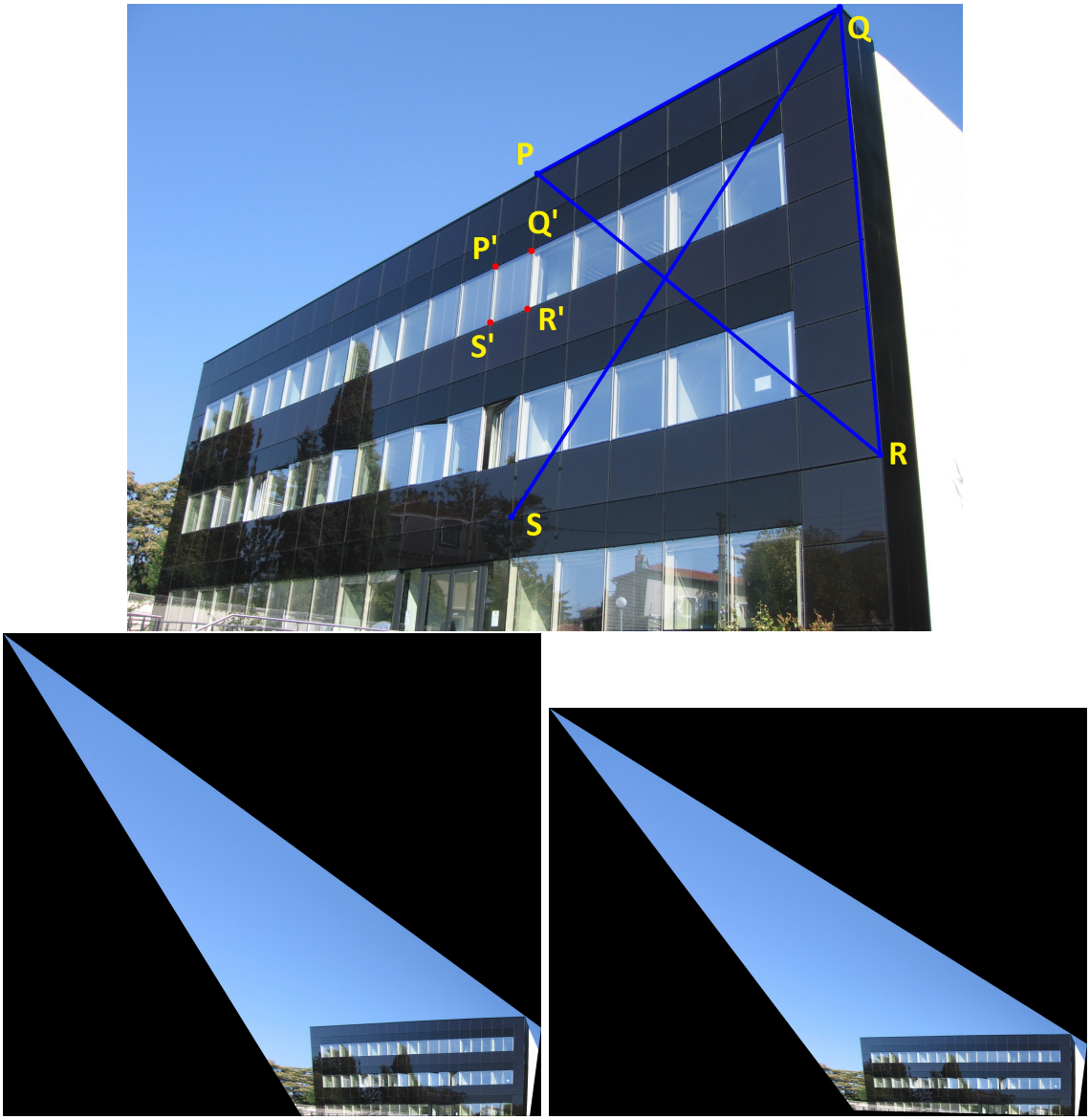

Figure 1: Removing **projective** (bottom left) and **affine distortion** (bottom right) using the **Point-to-Point Correspondence**. For removing the projective distortion we use the points P', Q', R' and S'. For removing the affine distortion the two pairs of orthogonal lines used are PQ-QR and PR-QS.

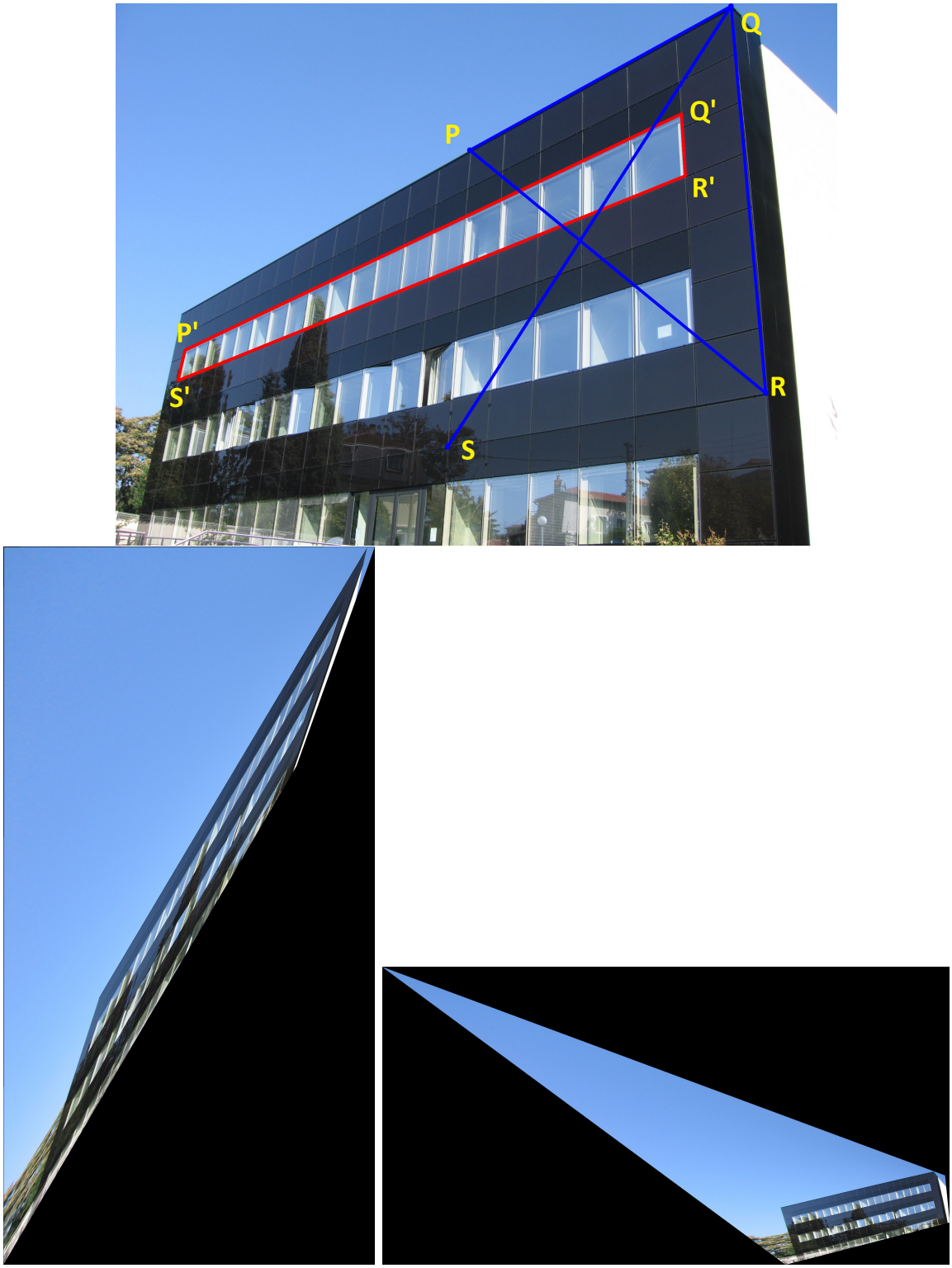

Figure 2: Removing **projective** (bottom left) and **affine distortion** (bottom right) using the **Vanishing Points Method**. For removing the projective distortion we use the vanishing points obtained by the P'Q'-S'R' and P'S'-Q'R'. This is the **first pair** of vanishing lines. For removing the affine distortion the two pairs of orthogonal lines used are PQ-QR and PR-QS.

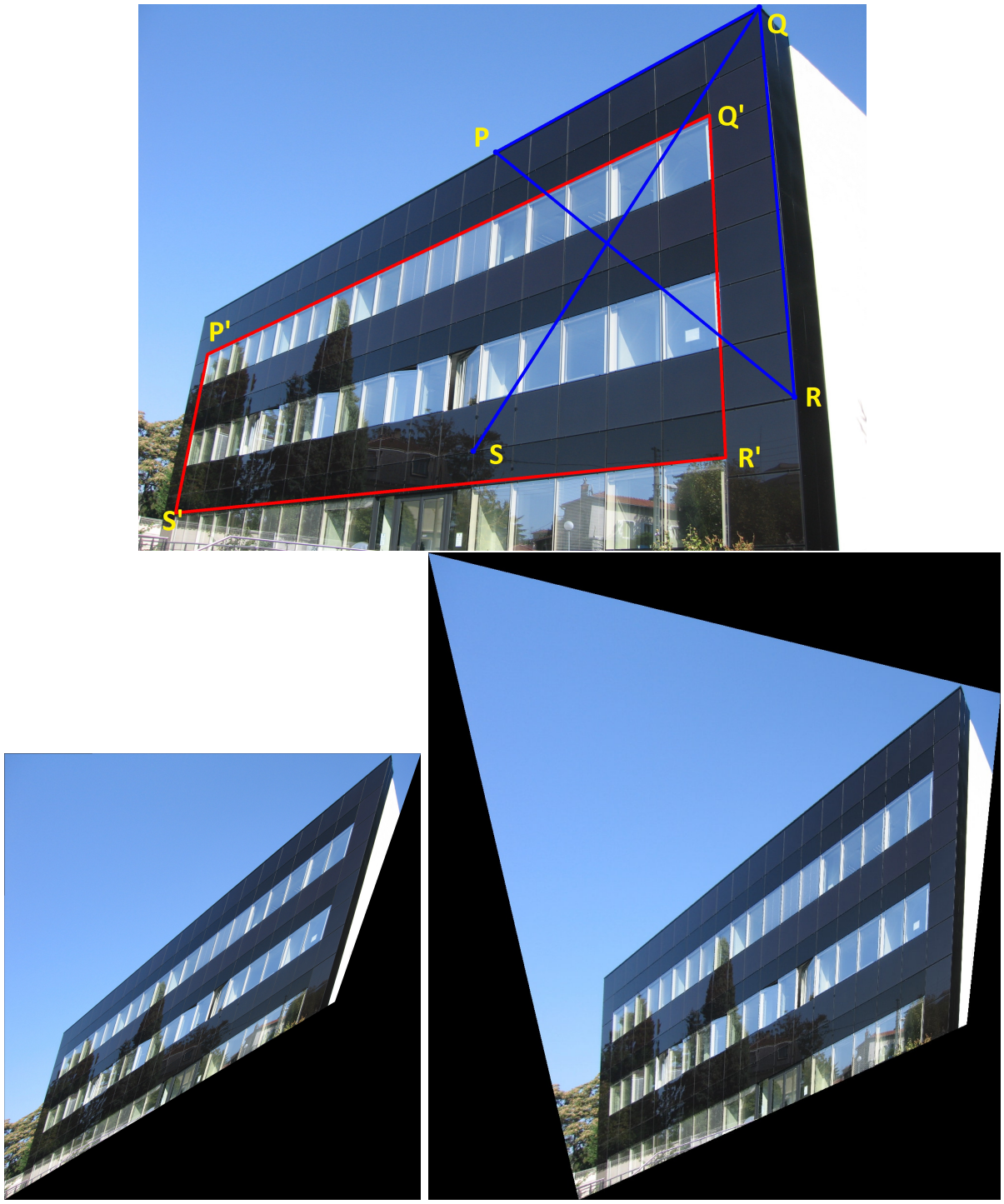

Figure 3: Removing **projective** (bottom left) and **affine distortion** (bottom right) using the **Vanishing Points Method**. For removing the projective distortion we use the vanishing points obtained by the P'Q'-S'R' and P'S'-Q'R'. This is the **second pair** of vanishing lines. For removing the affine distortion the two pairs of orthogonal lines used are PQ-QR and PR-QS.

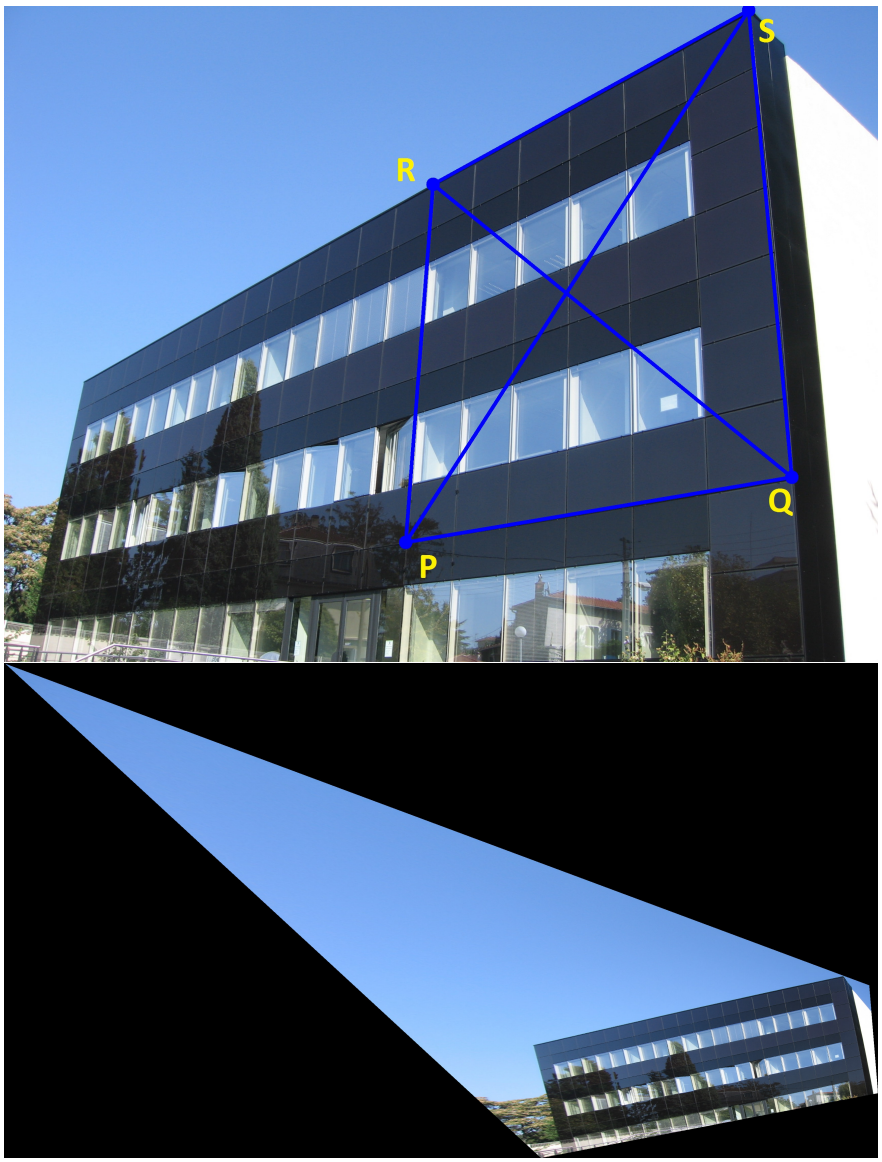

Figure 4: Removing **projective** and **affine distortion** using the **One-Step Method**. The five pairs of orthogonal lines used in this method are PQ-QR, PR-RS, RS-QS, QS-PQ and PS-QR.

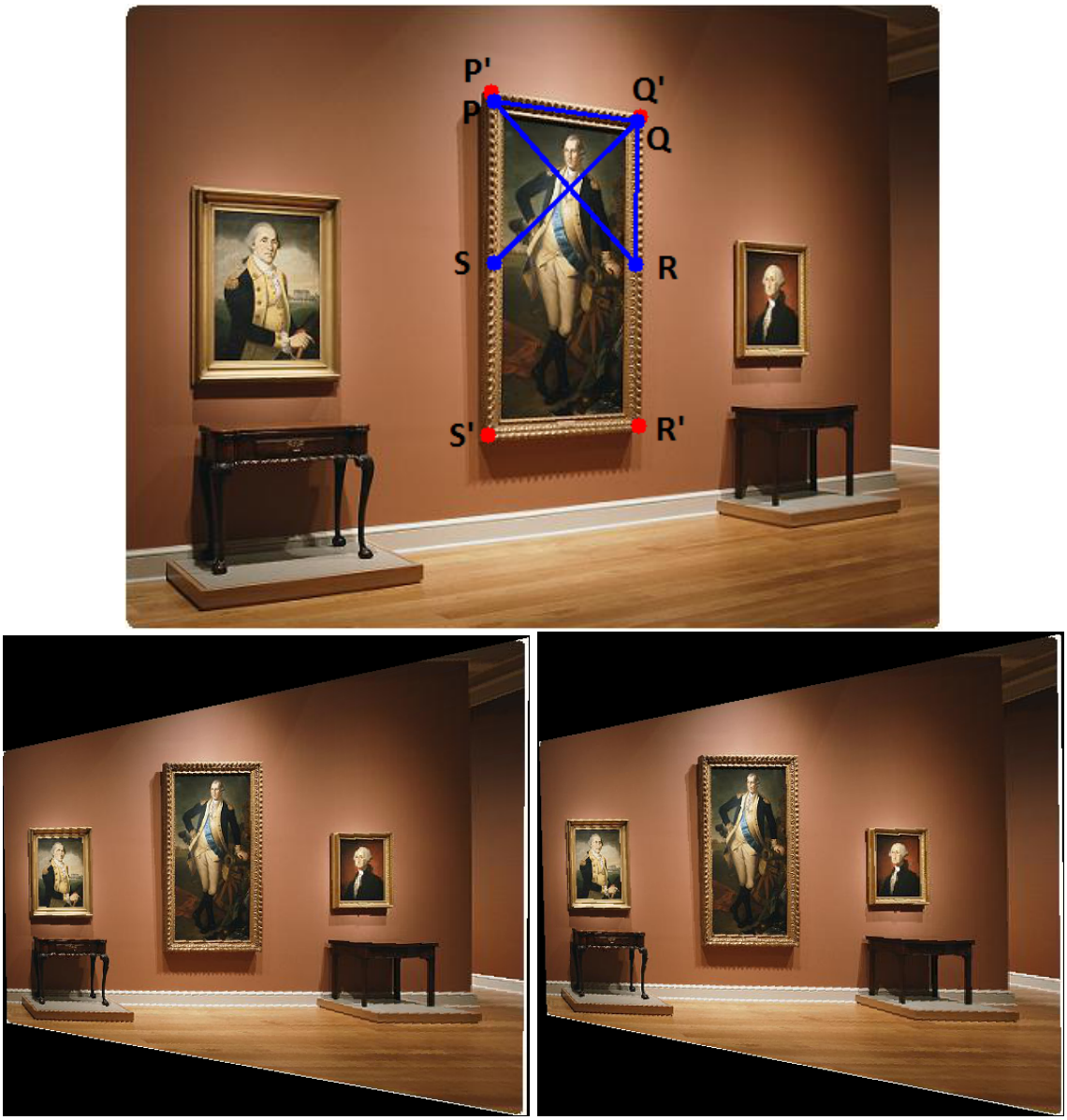

Figure 5: Removing **projective** (bottom left) and **affine distortion** (bottom right) using the **Point-to-Point Correspondence**. For removing the projective distortion we use the points P', Q', R' and S'. For removing the affine distortion the two pairs of orthogonal lines used are PQ-QR and PR-QS.

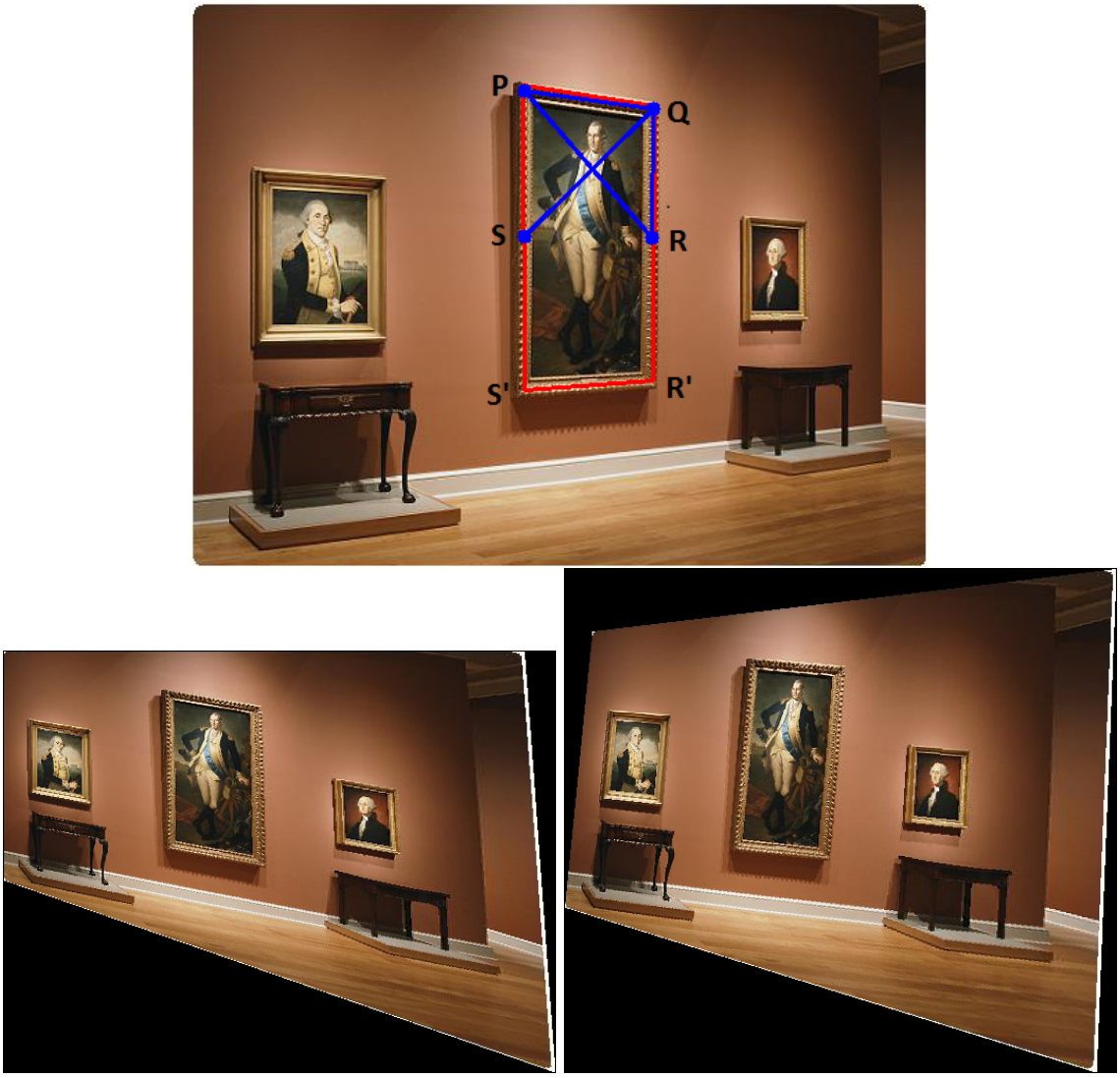

Figure 6: Removing **projective** (bottom left) and **affine distortion** (bottom right) using the **Vanishing Points Method**. For removing the projective distortion we use the vanishing points obtained by the PQ-S'R' and PS'-QR'. This is the **first pair** of vanishing lines. For removing the affine distortion the two pairs of orthogonal lines used are PQ-QR and PR-QS.

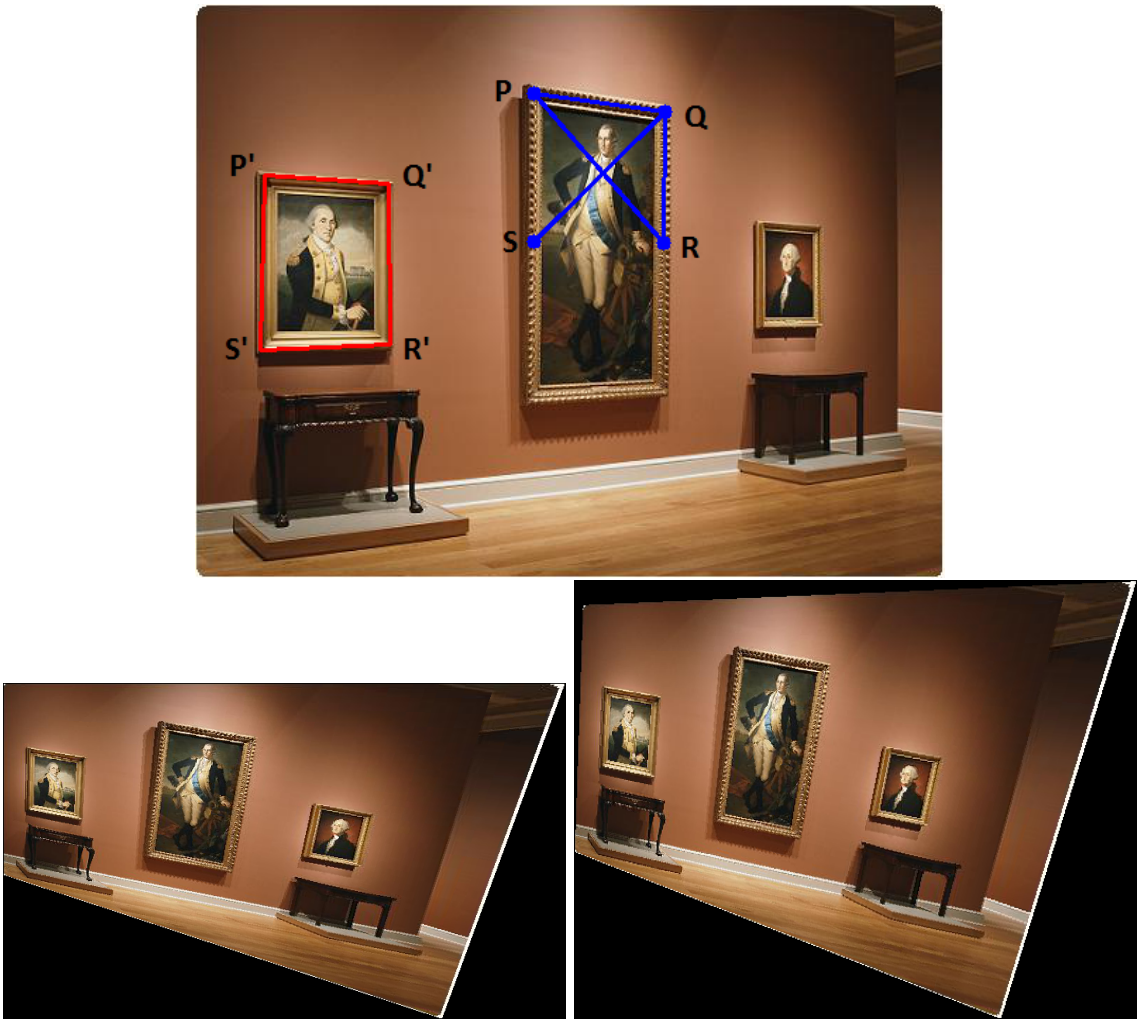

Figure 7: Removing **projective** (bottom left) and **affine distortion** (bottom right) using the **Vanishing Points Method**. For removing the projective distortion we use the vanishing points obtained by the P'Q'-S'R' and P'S'-Q'R'. This is the **second pair** of vanishing lines. For removing the affine distortion the two pairs of orthogonal lines used are PQ-QR and PR-QS.

Figure 8: Removing **projective** and **affine distortion** using the **One-Step Method**. The five pairs of orthogonal lines used in this method are PQ-QR, PR-RS, RS-QS, QS-PQ and PS-QR

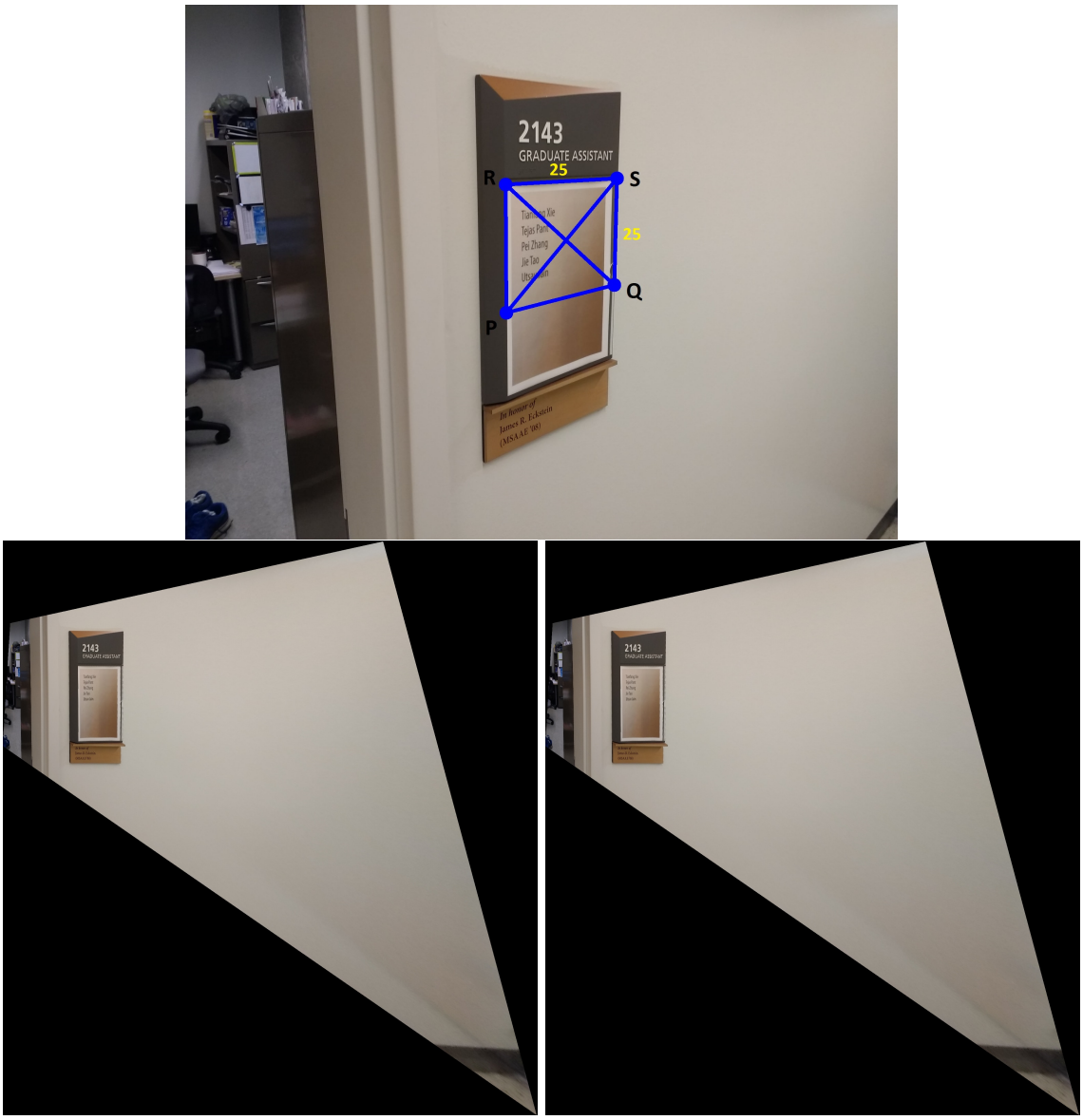

Figure 9: Removing **projective** (bottom left) and **affine distortion** (bottom right) using the **Point-to-Point Correspondence Method**. For removing the projective distortion we use the points P, Q, R and S. The length of PR and RS in world co-ordinates is 25 cm each. For removing the affine distortion the two pairs of orthogonal lines used are RS-QS and RQ-PS.

Figure 10: Removing **projective** (bottom left) and **affine distortion** (bottom right) using the **Vanishing Points Method**. For removing the projective distortion we use the vanishing points obtained by the PQ-SR and PR-QS. For removing the affine distortion the two pairs of orthogonal lines used are RS-QS and RQ-PS.

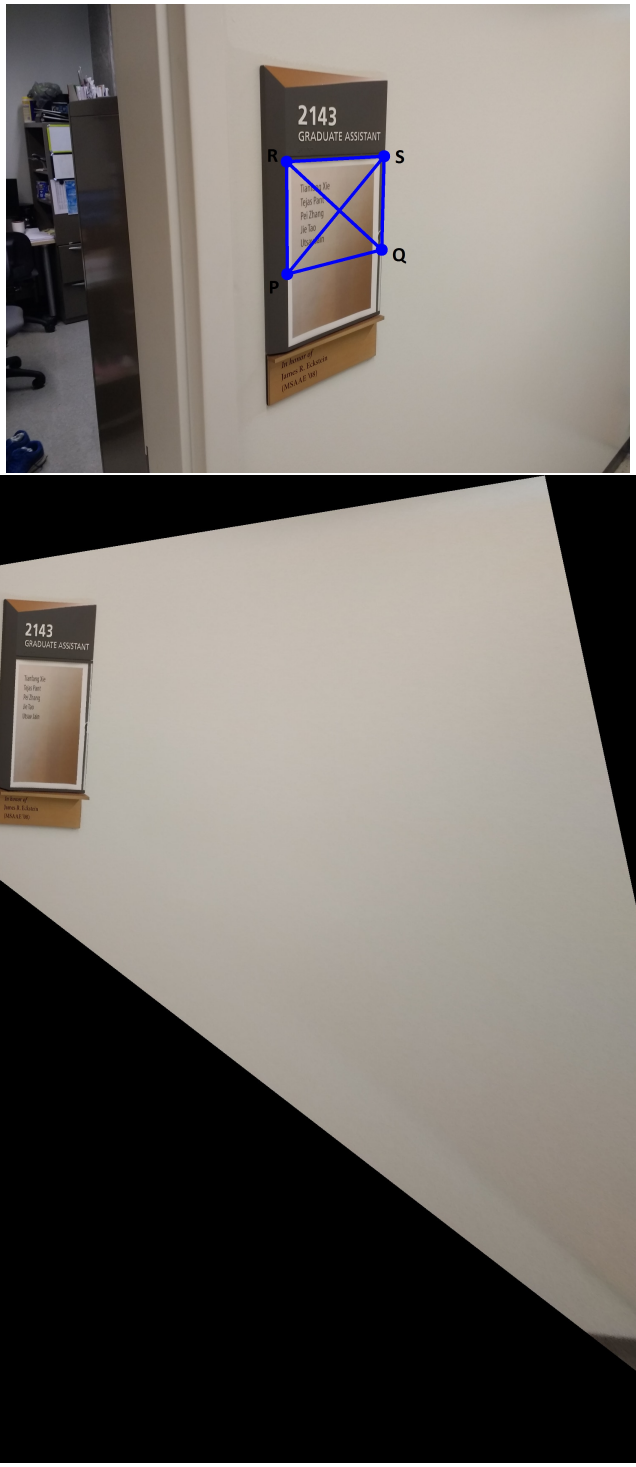

Figure 11: Removing **projective** and **affine distortion** using the **One-Step Method**. The five pairs of orthogonal lines used in this method are PQ-QR, PR-RS, RS-QS, QS-PQ and PS-QR

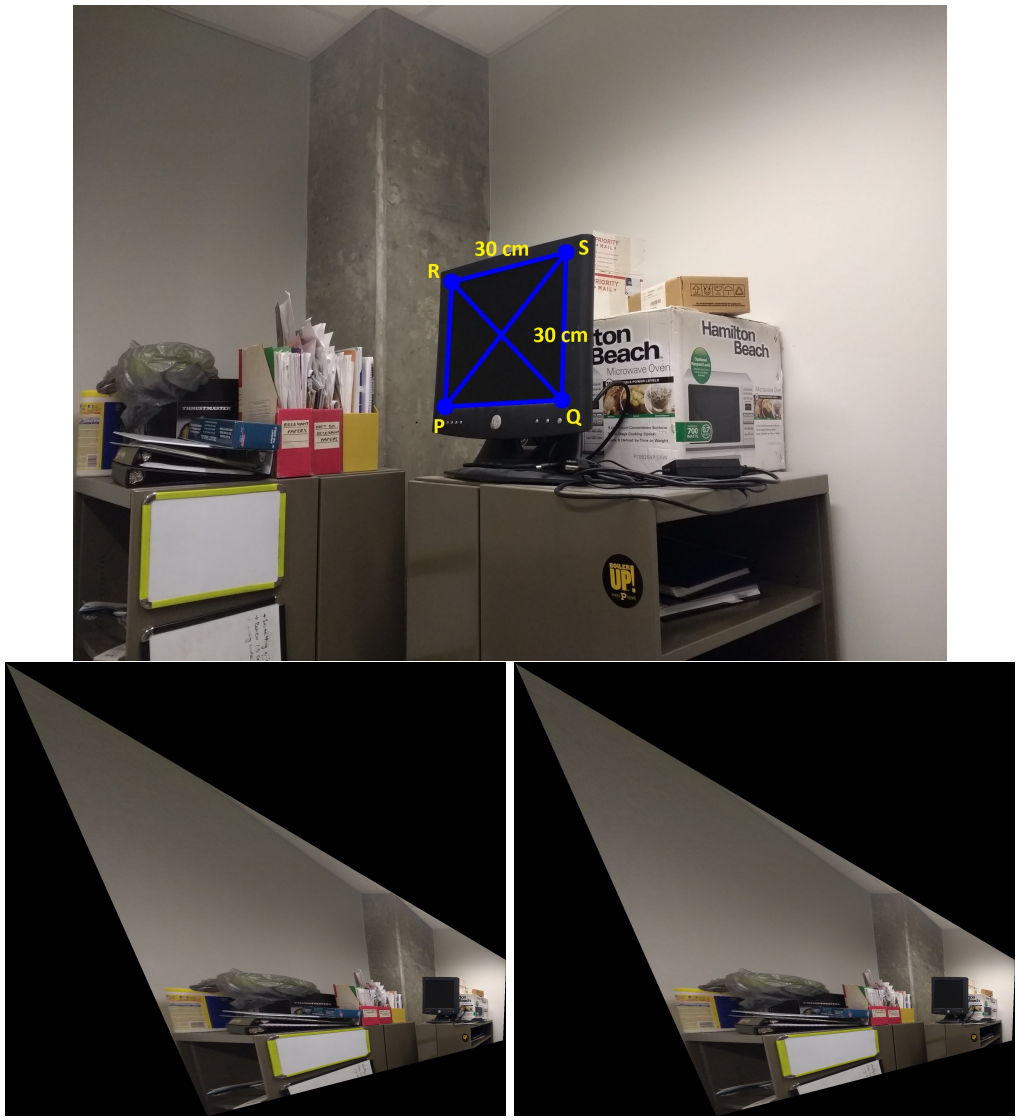

Figure 12: Removing **projective** (bottom left) and **affine distortion** (bottom right) using the **Point-to-Point Correspondence Method**. For removing the projective distortion we use the points P, Q, R and S. The length of PR and RS in world co-ordinates is 30 cm each. For removing the affine distortion the two pairs of orthogonal lines used are RS-QS and RQ-PS.

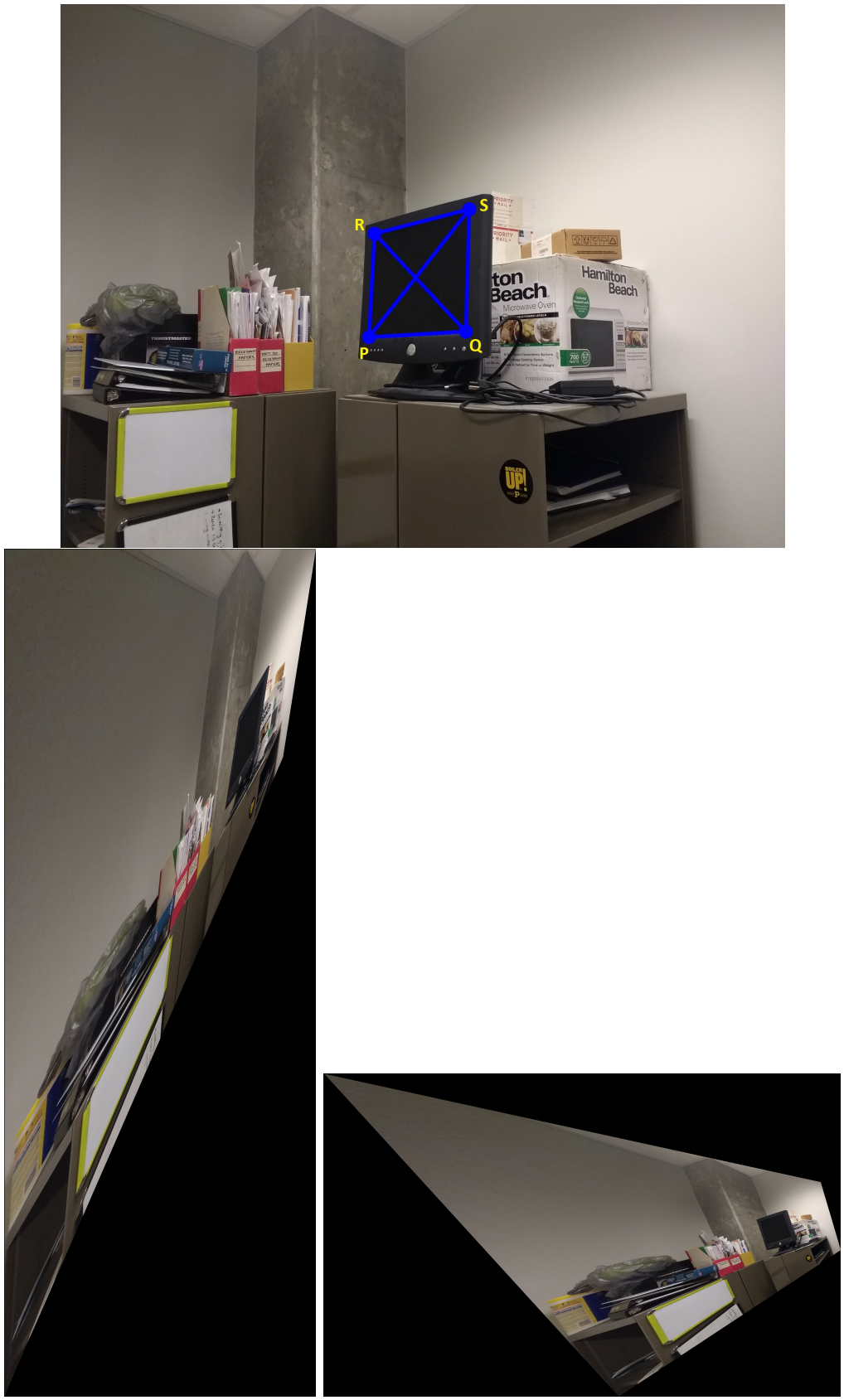

Figure 13: Removing **projective** (bottom left) and **affine distortion** (bottom right) using the **Vanishing Points Method**. For removing the projective distortion we use the vanishing points obtained by the PQ-SR and PR-QS. For removing the affine distortion the two pairs of orthogonal lines used are RS-QS and RQ-PS.

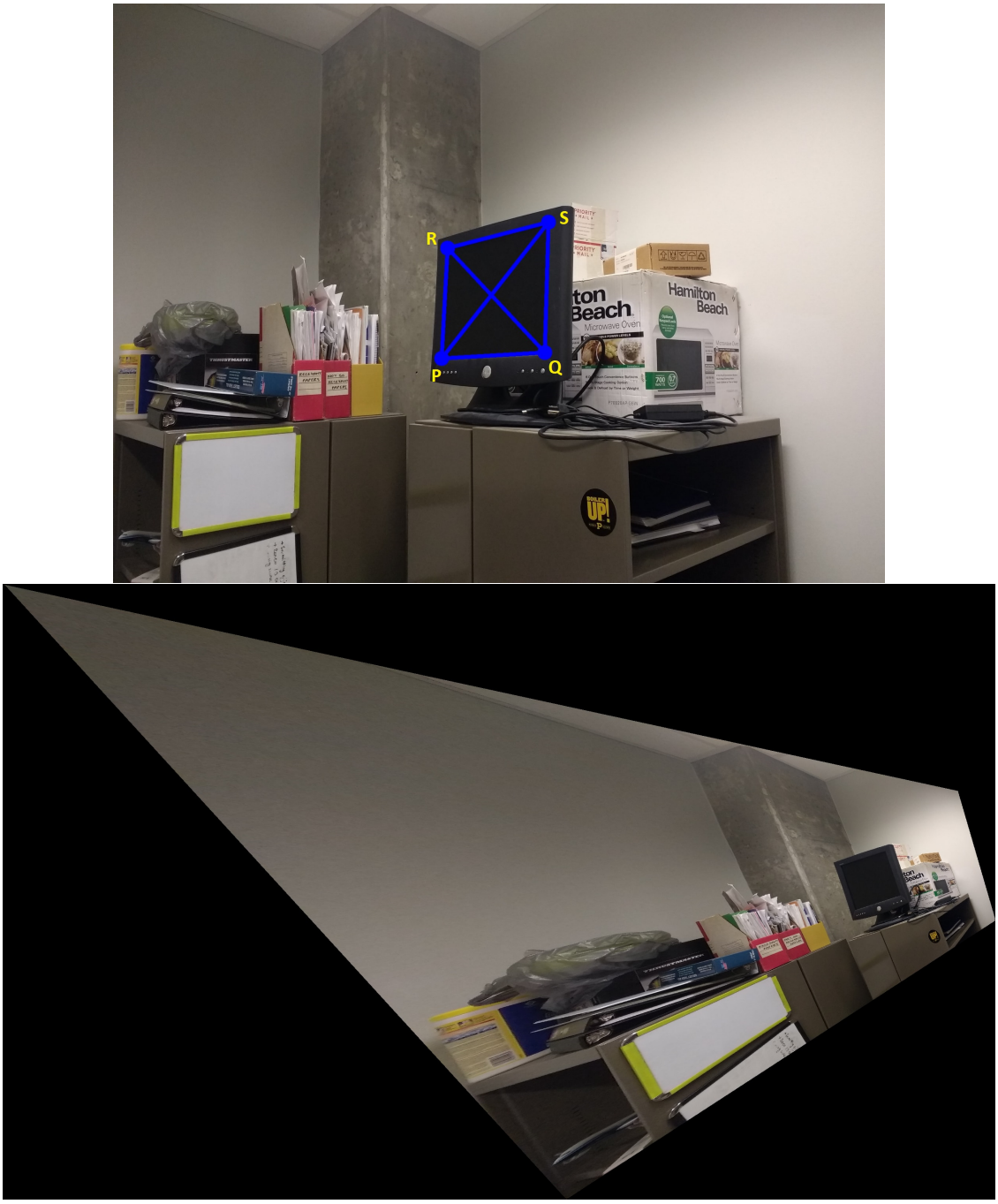

Figure 14: Removing **projective** and **affine distortion** using the **One-Step Method**. The five pairs of orthogonal lines used in this method are PQ-QR, PR-RS, RS-QS, QS-PQ and PS-QR