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A robust and accurate finite difference method for a generalized Black–Scholes equation

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ABSTRACT

In this paper we present a numerical method for a generalized Black–Scholes equation, which is used for option pricing. The method is based on a central difference spatial discretization on a piecewise uniform mesh and an implicit time stepping technique. Our scheme is stable for arbitrary volatility and arbitrary interest rate, and is second-order convergent with respect to the spatial variable. Furthermore, the present paper efficiently treats the singularities of the non-smooth payoff function. Numerical results support the theoretical results.

1. Introduction

An option is a financial contract that gives its owner the right to buy or sell a specified amount of a particular asset at a fixed price, called the exercise price, on or before a specified date, called the maturity date. Options that can be exercised at any time up to the maturity are called American, while options that can only be exercised on the maturity data are European. Options which provide the right to buy the underlying asset are known as calls, whereas options conferring the right to sell the underlying asset are referred to as puts. It was shown in [1] that these option prices satisfy a second-order partial differential equation with respect to the time horizon t and the underlying asset price x. This equation is now known as the Black–Scholes equation, and can be solved exactly when the coefficients are constant or space-independent. However, in many practical situations, numerical solutions are normally sought. Therefore, efficient and accurate numerical algorithms are essential for solving this problem accurately.

There are several methods in the open literature for the valuation of European and American options. The lattice technique was proposed in [2] to solve the Black–Scholes equations and was improved in [3]. That approach is equivalent to an explicit time-stepping scheme. Classical finite difference methods applied to constant-coefficient heat equations have also been developed (cf. [4–7]). The reason for this is that when the coefficients of the Black–Scholes equation are constant or space-independent, the equation can be transformed into a diffusion equation. However, when a problem is space-dependent, this transformation is impossible, and thus the Black–Scholes equation in the original form needs to be solved.

Other numerical schemes based on finite difference methods for the valuation of European and American options have also been developed. Vazquez [8] studied an upwind numerical approach for an American and European option pricing model. Ikonen and Toivanen [9] proposed a operator splitting method to solve the linear complementarity problems arising from the pricing of American options. Zvan et al. [10] suggested a general finite element approach for PDE option pricing models. In [11,12] numerical methods of option pricing models are studied by applying a standard finite volume method

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to obtain a difference scheme. The numerical schemes use central difference for a given mesh, but switch to upstream weighting for a small number of nodes, which are second order spatial convergent. In this paper we change the grid spacing to a piecewise uniform mesh which is constructed so that central difference is used everywhere.

It is well known that when using the standard finite difference method to solve those problems involving the convection-diffusion operator, such as the Black–Scholes partial differential operator, numerical difficulties can be caused. The main reason is that when the volatility or the asset price is small, the Black–Scholes partial differential operator becomes a convection-dominated operator. Hence, the implicit Euler scheme with central spatial difference method may lead to nonphysical oscillations in the computed solution. The implicit Euler scheme with upwind spatial difference method does not have this disadvantage, but this difference scheme is only first-order convergent. Recently, a stable fitted finite volume method (cf., for example, [13,14]) is employed for the discretization of the Black–Scholes equation. But it is also first-order convergent.

In this paper we present a numerical method for a generalized Black–Scholes equation, which is used for option pricing. The method is based on a central difference spatial discretization on a piecewise uniform mesh and an implicit time stepping technique. The matrix associated with discrete operator is an M-matrix, which ensure that the scheme is stable for arbitrary volatility and arbitrary interest rate without any extra conditions. The scheme is second-order convergent with respect to the spatial variable. Furthermore, the present paper efficiently treats the singularities of the non-smooth payoff function. Without loss of generality, we shall discuss the method using the model for European options in our paper. Naturally, the method is applicable to the American option if it is used together with a technique for free boundary problems.

The rest of the paper is organized as follows. In the next section we discuss the continuous model of the Black–Scholes equations. The discretization method is described in Section 3. In Section 4 we present a stability and error analysis for the finite difference scheme. It is shown that the finite difference solution converges to the exact solution at the rate of $O(h^2+\tau)$. Numerical examples are presented in Section 5. Finally a discussion on results is indicated in Section 6.

2. The continuous problem

In this paper we consider the following generalized Black–Scholes equation

$$-\frac{\partial v}{\partial t}-\frac{1}{2}\sigma^2(x,t)x^2\frac{\partial^2 v}{\partial x^2}-r(t)x\frac{\partial v}{\partial x}+r(t)v=0,\quad (x,t)\in\mathbb{R}^+\times(0,T) \tag{2.1}$$

equipped with the terminal and boundary conditions

$$v(x,T)=\max(x-K,0),\quad x\in\mathbb{R}^+, \tag{2.2}$$

$$v(0,t)=0,\quad t\in[0,T]. \tag{2.3}$$

Here $v(x,t)$ is the European call option price at asset price x and at time t, K is the exercise price, T is the maturity, $r(t)$ is the risk-free interest rate, and $\sigma(x,t)$ represents the volatility function of the underlying asset. Here we assume that $\sigma^2\geq\alpha>0$, $\beta^*\geq r\geq\beta>0$. When σ and r are constant functions, it becomes the classical Black–Scholes model.

Note that (2.1) degenerates when x goes to zero. Nevertheless, the existence and uniqueness of a classical solution of (2.1)–(2.3) is well known. In fact, it can be transformed to a Cauchy problem for a uniformly parabolic operator.

Now, we see that the above model is described in a infinite domain $\mathbb{R}^+\times(0,T)$, which makes difficulties in constructing numerical solutions. This motivates the consideration of the following model defined on a truncated domain $\Omega=(0,S_{\max})\times(0,T)$, where $S_{\max}$ is suitably chosen positive number:

$$-\frac{\partial w}{\partial t}-\frac{1}{2}\sigma^2(x,t)x^2\frac{\partial^2 w}{\partial x^2}-r(t)x\frac{\partial w}{\partial x}+r(t)w=0,\quad (x,t)\in\Omega, \tag{2.4}$$

$$w(x,T)=\max(x-K,0),\quad x\in[0,S_{\max}], \tag{2.5}$$

$$w(0,t)=0,\quad t\in[0,T], \tag{2.6}$$

$$w(S_{\max},t)=S_{\max}-Ke^{-\int_t^T r(s)\mathrm{d}s},\quad t\in[0,T]. \tag{2.7}$$

The existence and uniqueness of a classical solution of (2.4)–(2.7) can be found in [15,16]. It is proved in [17] that if v and w are solutions of (2.1)–(2.3) and (2.4)–(2.7), respectively, then

$$|v(x,t)-w(x,t)|\leq K\exp\left(-\frac{\left[\ln\frac{S_{\max}}{x}\right]^2}{2\left[\min\limits_{\bar\Omega}\sigma^2\right](T-t)}\right),\quad (x,t)\in\Omega \tag{2.8}$$

holds.

Since the final condition is not smooth, the resulting solution is not smooth enough for the convergence of finite difference approximations. Hence we need to modify the model (2.4)–(2.7).

Define $\pi_\varepsilon(y)$ as

$$\pi_\varepsilon(y) = \begin{cases} y, & y \geq \varepsilon, \\ c_0 + c_1 y + \cdots + c_9 y^9, & -\varepsilon < y < \varepsilon, \\ 0, & y \leq -\varepsilon, \end{cases}$$

where $0 < \varepsilon \ll 1$ is a transition parameter and $\pi_\varepsilon(y)$ is a function which smooths out the original $\max(y, 0)$ around $y = 0$. This requires that $\pi_\varepsilon(y)$ satisfies

$$\pi_\varepsilon(-\varepsilon) = \pi'_\varepsilon(-\varepsilon) = \pi''_\varepsilon(-\varepsilon) = \pi'''_\varepsilon(-\varepsilon) = \pi^{(4)}_\varepsilon(-\varepsilon) = 0,$$
$$\pi_\varepsilon(\varepsilon) = \varepsilon, \qquad \pi'_\varepsilon(\varepsilon) = 1, \qquad \pi''_\varepsilon(\varepsilon) = \pi'''_\varepsilon(\varepsilon) = \pi^{(4)}_\varepsilon(\varepsilon) = 0.$$

Using these ten conditions we can easily find that

$$c_0 = \frac{35}{256}\varepsilon, \qquad c_1 = \frac{1}{2}, \qquad c_2 = \frac{35}{64\varepsilon}, \qquad c_4 = -\frac{35}{128\varepsilon^3},$$
$$c_6 = \frac{7}{64\varepsilon^5}, \qquad c_8 = -\frac{5}{256\varepsilon^7}, \qquad c_3 = c_5 = c_7 = c_9 = 0.$$

Replacing $\max(x - K, 0)$ in the terminal condition (2.5) by the fourth-order smooth function $\pi_\varepsilon(x - K)$ we obtain

$$-\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2(x, t)x^2\frac{\partial^2 u}{\partial x^2} - r(t)x\frac{\partial u}{\partial x} + r(t)u = 0, \quad (x, t) \in \Omega, \tag{2.9}$$
$$u(x, T) = \pi_\varepsilon(x - K), \quad x \in [0, S_{\max}], \tag{2.10}$$
$$u(0, t) = 0, \quad t \in [0, T], \tag{2.11}$$
$$u(S_{\max}, t) = S_{\max} - Ke^{-\int_t^T r(s)\mathrm{d}s}, \quad t \in [0, T]. \tag{2.12}$$

The existence and uniqueness of a classical solution of (2.9)–(2.12) can be found in [15,16], which also contain the proof of the following estimate: There exists a positive constant C independent of $\pi_\varepsilon(x - K)$ such that

$$|u(x, t) - w(x, t)| \leq C\|\pi_\varepsilon(x - K) - \max(x - K, 0)\|_{L^\infty} \tag{2.13}$$

for $(x, t) \in \bar{\Omega}$.

It follows from (2.8) and (2.13) that we can make the solution of our modified model (2.9)–(2.12) become close to that of the original model (2.1)–(2.3) by choosing sufficiently large $S_{\max}$ and sufficiently small ε. In the remaining of this paper we will consider the model (2.9)–(2.12).

3. Discretization

We now consider the approximation of the solution to the Black–Scholes equation by a central difference scheme on a piecewise uniform mesh.

The use of central difference scheme on a uniform mesh may produces nonphysical oscillations in the computed solution. To overcome this oscillation we use a piecewise uniform mesh Ω^N on the space interval $[0, S_{\max}]$:

$$x_i = \begin{cases} h & i = 1, \\ h\left[1 + \frac{\alpha}{\beta^*}(i - 1)\right] & i = 2, \ldots, N/4 - 1, \\ K & i = N/4, \\ K + \varepsilon & i = N/4 + 1, \\ K + \varepsilon + \frac{S_{\max} - K - \varepsilon}{3N/4 - 1}(I - N/4 - 1) & i = N/4 + 2, \ldots, N, \end{cases}$$

where

$$h = \frac{K - \varepsilon}{1 + \frac{\alpha}{\beta^*}(N/4 - 2)}.$$

Here we have used a gridded mesh at the region near $x = K$ for treating the singularities of the non-smooth payoff function. For the time discretization, we use a uniform mesh Ω^M on $[0, T]$ with M mesh elements. Then the piecewise uniform mesh $\Omega^{N\times M}$ on $\Omega = (0, S_{\max}) \times (0, T)$ is defined to be the tensor product $\Omega^{N\times M} = \Omega^N \times \Omega^M$. It is easy to see that the mesh sizes $h_i = x_i - x_{i-1}$ and $\tau_j = t_j - t_{j-1}$ satisfy

$$h_i = \begin{cases} h & i = 1, \\ \frac{\alpha}{\beta^*}h & i = 2, \ldots, N/4 - 1, \\ \varepsilon & i = N/4, N/4 + 1, \\ \frac{S_{\max} - K - \varepsilon}{3N/4 - 1} & i = N/4 + 2, \ldots, N \end{cases}$$

and

$$\tau = \tau_j = T/M, \quad j = 1, \ldots, M,$$

respectively.

We discretize the Black–Scholes operator using the central difference scheme

$$L^{N,M} z_i^j = -\frac{z_i^{j+1} - z_i^j}{\tau_{j+1}} - \frac{(\sigma_i^j)^2 x_i^2}{h_i + h_{i+1}} \left(\frac{z_{i+1}^j - z_i^j}{h_{i+1}} - \frac{z_i^j - z_{i-1}^j}{h_i} \right) - r^j x_i \frac{z_{i+1}^j - z_{i-1}^j}{h_i + h_{i+1}} + r^j z_i^j \tag{3.1}$$

on the above piecewise uniform mesh, which ensures that the matrix associated with the discrete operator is an M-matrix. Hence the scheme is stable for arbitrary volatility and arbitrary interest rate.

Then our scheme reads: Find $U_i^j \in R^{N+1} \times R^{M+1}$ with

$$L^{N,M} U_i^j = 0, \quad 1 \le i < N,\ 0 \le j < M, \tag{3.2}$$

$$U_i^M = \pi_\varepsilon (x_i - K), \quad 1 \le i < N, \tag{3.3}$$

$$U_0^j = 0, \qquad U_N^j = S_{\max} - K e^{-\int_{t_j}^T r(s) \mathrm{d}s}, \quad 0 \le j \le M. \tag{3.4}$$

4. Analysis of the method

Our analysis is based on discrete maximum principle, truncation error analysis and barrier function techniques.

Lemma 1 (*Maximum Principle*)**.** *The operator $L^{N,M}$ defined by (3.1) satisfies a discrete maximum principle, i.e. if $\{v_i^j\}$ and $\{w_i^j\}$ are mesh functions that satisfy $v_0^j \le w_0^j$, $v_N^j \le w_N^j$ $(j = 0, 1, \ldots, M)$, $v_i^M \le w_i^M$ $(i = 0, 1, \ldots, N)$ and $L^{N,M} v_i^j \le L^{N,M} w_i^j$ $(i = 1, \ldots, N-1, j = M-1, \ldots, 1, 0)$, then $v_i^j \le w_i^j$ for all i, j.*

Proof. It is easy to verify that the matrix associated with $L^{N,M}$ is an M-matrix, as in the proof of [18, Lemma 3.1]. □

The next lemma gives us a useful formula for the truncation error.

Lemma 2. *Let $v(x, t)$ be a smooth function defined on $\Omega^{N,M}$. Then the following estimate for the truncation error holds true:*

$$|L^{N,M} v_i^j - (Lv)_i^j| \le C_1 \int_{t_{j-1}}^{t_{j+1}} \left| \frac{\partial^2 v}{\partial t^2}(x_i, t) \right| \mathrm{d}t + C_1 h \int_{x_{i-1}}^{x_{i+1}} \left[x_i^2 \left| \frac{\partial^4 v}{\partial x^4}(x, t_j) \right| + x_i \left| \frac{\partial^3 v}{\partial x^3}(x, t_j) \right| \right] \mathrm{d}x,$$
$$1 \le i < N,\ 0 < j < M,$$

where C_1 is a positive constant independent of the mesh.

Proof. It can be easily obtained by using Taylor's formula with the integral form of the remainder. □

Now we can get the main result for our difference scheme.

Theorem 1. *Let u be the solution of (2.9)–(2.12) and U be the solution of the finite difference scheme (3.2)–(3.4). Assume that u satisfies $x^2 \frac{\partial^4 u}{\partial x^4} \in C(\bar{\Omega})$ and $x \frac{\partial^3 u}{\partial x^3} \in C(\bar{\Omega})$. Then we have*

$$|u(x_i, t_j) - U_i^j| \le C(h^2 + \tau), \quad 0 \le i \le N,\ 0 \le j \le M,$$

where C is a positive constant independent of h and τ.

Proof. Applying Lemma 2 we have

$$\begin{aligned} |L^{N,M}(u_i^j - U_i^j)| &= |L^{N,M} u_i^j - (Lu)_i^j| \le C_1 \int_{t_{j-1}}^{t_{j+1}} \left| \frac{\partial^2 u}{\partial t^2}(x_i, t) \right| \mathrm{d}t + C_1 h \int_{x_{i-1}}^{x_{i+1}} \left[x_i^2 \left| \frac{\partial^4 u}{\partial x^4}(x, t_j) \right| + x_i \left| \frac{\partial^3 u}{\partial x^3}(x, t_j) \right| \right] \mathrm{d}x \\ &\le C_2(\tau + h^2), \quad 1 \le i < N,\ 0 < j < M, \end{aligned}$$

where we have used the assumptions $x^2 \frac{\partial^4 u}{\partial x^4} \in C(\bar{\Omega})$ and $x \frac{\partial^3 u}{\partial x^3} \in C(\bar{\Omega})$, C_2 is also a positive constant independent of the mesh. Hence, using the barrier function $W_i^j = C(\tau + h^2)(1 + T - t_j)$ (with the constant C sufficient large), Lemma 1 implies that for all i, j,

$$|u(x_i, t_j) - U_i^j| \le C(h^2 + \tau),$$

which completes the proof. □

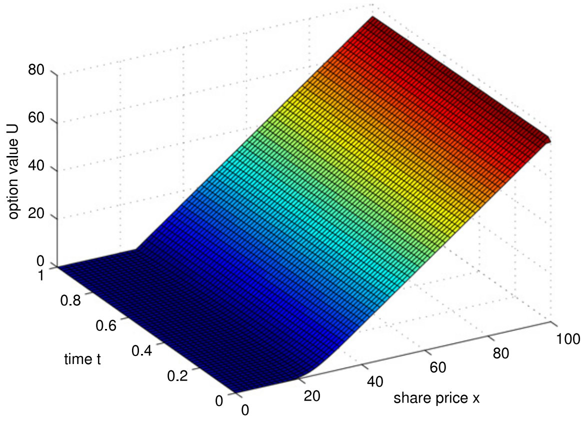

Fig. 1. Computed option value U for Test 1.

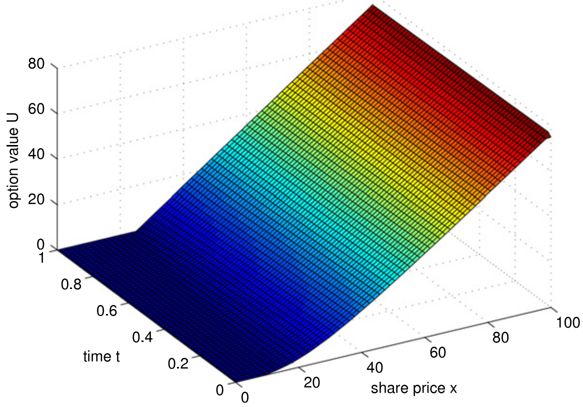

Fig. 2. Computed option value U for Test 2.

5. Numerical experiments

In this section we verify experimentally the theoretical results obtained in the preceding section. Errors and convergence rates for the difference scheme are presented for two test problems.

Test 1. European call option with parameters: $T = 1$, $r = 0.06$, $\sigma = 0.2(1 + te^{-x})$, $K = 25$ and $S_{\max} = 100$.

Test 2. European call option with parameters: $T = 1$, $r = 0.06$, $\sigma = 0.4(2 + \sin x)$, $K = 25$ and $S_{\max} = 100$.

For Tests 1 and 2 we choose $N = M = 64$ and $\varepsilon = 10^{-4}$. The computed option value U are depicted in Figs. 1 and 2, respectively.

To demonstrate the theoretical rates of convergence numerically, we take $M = 1024$ which is a sufficiently large choice to bring out second-order convergence in space. The exact solutions of our test problems are not available. We use the approximated solution of $N = 2048$, $M = 1024$ as the exact solution. We present the error estimates for different N. Because we only know "the exact solution" on mesh points, we use the linear interpolation to get solutions at other points. In this paper $\bar{U}(x, t)$ denotes "the exact solution" which is a linear interpolation of the approximated solution $U^{2048,1024}$. We measure the accuracy in the discrete maximum norm

$$e^{N,M} = \max_{i,j} |U_{ij}^{N,M} - \bar{U}(x_i, t_j)|,$$

and the convergence rate

$$\bar{r}^{N,M} = \log_2 \left(\frac{e^{N,M}}{e^{2N,M}} \right).$$

The error estimates and convergence rates in our computed solutions of Tests 1 and 2 are listed in Tables 1 and 2, respectively.

From the figures it is seen that the numerical solutions by our method are non-oscillatory. From Tables 1 and 2 we see that $e^{N,M}/e^{2N,M}$ is close to 4 for sufficiently large M, which supports the convergence estimate of Theorem 1. They indicate that the theoretical results are fairly sharp.

Table 1
Numerical results for Test 1.

M	N	Error $e^{N,M}$	Rate $r^{N,M}$
1024	128	6.8027e−2	1.122
	256	3.1258e−2	1.312
	512	1.2588e−2	1.880
	1024	3.4189e−3	–

Table 2
Numerical results for Test 2.

M	N	Error $e^{N,M}$	Rate $r^{N,M}$
1024	128	5.7124e−2	1.551
	256	1.9498e−2	1.919
	512	5.1550e−3	2.284
	1024	1.0586e−3	–

6. Conclusion

In this paper we present a numerical method for a generalized Black–Scholes equation, which is used for option pricing. The method is based on a central difference spatial discretization on a piecewise uniform mesh and an implicit time stepping technique. The matrix associated with discrete operator is an M-matrix, which ensure that the scheme is stable for arbitrary volatility and arbitrary interest rate without any extra conditions. The scheme is second-order convergent with respect to the spatial variable. Furthermore, the present paper efficiently treats the singularities of the non-smooth payoff function. Numerical experiments, performed to demonstrate the effectiveness of the method, showed that the approach is stable and accurate. This method can be applied to American option pricing model along with a technique for free boundary problems.

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