Comput Econ (2012) 40:49–62
DOI 10.1007/s10614-011-9268-9

A Second-Order Difference Scheme for the Penalized Black–Scholes Equation Governing American Put Option Pricing

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Accepted: 7 May 2011 / Published online: 18 May 2011

Abstract In this paper we present a stable finite difference scheme on a piecewise uniform mesh along with a power penalty method for solving the American put option problem. By adding a power penalty term the linear complementarity problem arising from pricing American put options is transformed into a nonlinear parabolic partial differential equation. Then a finite difference scheme is proposed to solve the penalized nonlinear PDE, which combines a central difference scheme on a piecewise uniform mesh with respect to the spatial variable with an implicit time stepping technique. It is proved that the scheme is stable for arbitrary volatility and arbitrary interest rate without any extra conditions and is second-order convergent with respect to the spatial variable. Furthermore, our method can efficiently treats the singularities of the non-smooth payoff function. Numerical results support the theoretical results.

Keywords Black–Scholes equation · Option valuation · Power penalty method · Central difference scheme · Piecewise uniform mesh

Mathematics Subject Classification (2000) 65M06 · 65M12 · 65M15

1 Introduction

The pricing and hedging of derivative securities, also known as contingent claims, is a subject of much practical importance. One basic type of derivative is an option. The owner of a call option has the right but not the obligation to purchase an underlying asset (such as a stock) for a specified price (called the exercise price or strike price)

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on or before a specified expiry date. A put option is similar except the owner of such a contract has the right but not the obligation to sell. Options which can be exercised only on the expiry date are called European, whereas options which can be exercised any time up to and including the expiry date are classified as American. It was shown by Black and Scholes (1973) that the value of an European option is governed by a second order parabolic partial differential equation with respect to time and the price of its underlying stock. The determination of the value of an American option is more complicated. It has been shown (cf., for example, Wilmott et al. 1993) that American option pricing is governed by a linear complementarity problem involving the Black–Scholes differential operator and a constraint (or obstacle) on the value of the option. This gives rise to a free boundary in the solution, and thus it is also called a free boundary problem. To determine the value of an American option, a linear complementary problem needs to be solved. Since analytical solutions can rarely be found for practical problems, numerical methods need to be developed for solving these problems.

There are several methods in the open literature for the valuation of European and American options. The lattice technique was proposed in Cox et al. (1979) to solve the Black-Schoels equations and was improved in Hull and White (1988). That approach is equivalent to an explicit time-stepping scheme. Classical finite difference methods applied to constant-coefficient heat equations have also been developed (cf. Rogers and Talay 1997; Schwartz 1977; Courtadon 1982; Wilmott et al. 1993). Vazquez (1998) studied an upwind numerical approach for an American option pricing model. Ikonen and Toivanen (2004) proposed a operator splitting method to solve the linear complementarity problems arising from the pricing of American options. Wu and Kong (2005) applied a linearized method based on differential quadrature method to the problem of pricing American better-of options on two assets. Zvan et al. (1998a) suggested a general finite element approach for PDE of option pricing models. In Forsyth and Vetzal (2002) and Zvan et al. (1998b) numerical methods of option pricing models are studied by applying a standard finite volume method to obtain a difference scheme. The numerical schemes use central difference for a given mesh, but switch to upstream weighting for a small number of nodes, which are second order spatial convergent. In this paper we change the grid spacing to a piecewise uniform mesh which is constructed so that central difference is used everywhere.

It is well known that when using the standard finite difference method to solve those problems involving the convection–diffusion operators, such as the Black–Scholes partial differential operator, numerical difficulty can be caused. The main reason is that when the volatility or the asset price is small, the Black–Scholes partial differential operator becomes a convection-dominated operator. Hence, the implicit Euler scheme with central spatial difference method may lead to nonphysical oscillations in the computed solution. The implicit Euler scheme with upwind spatial difference method does not have this disadvantage, but this difference scheme is only first-order convergent. Recently, a stable fitted finite volume method (cf., for example, Angermann and Wang 2007) is employed for the discretization of the Black–Scholes equation. But it is also first-order convergent.

The power penalty method was introduced by Zvan et al. (1998b) for American options with stochastic volatility by adding a source term to the discrete equation. Nielsen et al. (2002) presented a refinement of their work by adding a penalty term to the continuous equation and illustrated the performance of various numerical schemes. By adding a power penalty term the linear complementarity problem for pricing American options can be transformed into a nonlinear parabolic partial differential equation. As the solution approaches the payoff function at expiry, the penalty term forces the solution to stay above it. When the solution is far from the barrier, the term is small and thus the Black–Scholes equation is approximatively satisfied in this region.

In this paper we present a stable finite difference scheme on a piecewise uniform mesh along with a power penalty method for solving the American put option problem. By adding a power penalty term the linear complementarity problem arising from pricing American put options is transformed into a nonlinear parabolic partial differential equation. Then a finite difference scheme is proposed to solve the penalized nonlinear PDE, which combines a central difference scheme on a piecewise uniform mesh with respect to the spatial variable with an implicit time stepping technique. We show that the scheme is stable for arbitrary volatility and arbitrary interest rate without any extra conditions and is second-order convergent with respect to the spatial variable. Furthermore, our method can efficiently treats the singularities of the non-smooth payoff function.

The rest of the paper is organized as follows. In the next section we describe some theoretical results on the continuous complementarity problem for the American put option pricing model. The discretization method is described in Sect. 3. In Sect. 4, we present a stability and error analysis for the finite difference scheme. It is shown that the finite difference solution converges to the exact solution at the rate of $O(h^2 + \tau)$. Finally, numerical experiments are provided to support these theoretical results in Sect. 5.

2 The Continuous Problem

Let v denote the value of an American put option with strike price E and expiry date T; Let x denote the price of the underlying asset. It is known that v satisfies the following strong form of the linear complementarity problem (Jaillet et al. 1990, Theorem 3.6):

$$Lv(x,t) \geq 0, \quad x > 0 \quad \text{and} \quad t \in [0, T), \tag{2.1}$$

$$v(x,t) - V^*(x) \geq 0, \quad x > 0 \quad \text{and} \quad t \in [0, T], \tag{2.2}$$

$$Lv(x,t) \cdot \left[v(x,t) - V^*(x)\right] = 0, \quad x > 0 \quad \text{and} \quad t \in [0, T), \tag{2.3}$$

$$v(x,T) = V^*(x), \quad x \geq 0 \quad \text{and} \quad t = T, \tag{2.4}$$

$$v(0,t) = E, \quad x = 0 \quad \text{and} \quad t \in [0, T], \tag{2.5}$$

$$v(x,t) \to 0, \quad x \to +\infty \quad \text{and} \quad t \in [0, T], \tag{2.6}$$

where L denotes the Black–Scholes operator defined by

$$Lv(x,t) \equiv -\frac{\partial v}{\partial t} - \frac{1}{2}\sigma^2(t)x^2\frac{\partial^2 v}{\partial x^2} - r(t)x\frac{\partial v}{\partial x} + r(t)v,$$

and $V^*(x)$ is the final (payoff) condition defined by

$$V^*(x) = \max\{E - x, 0\}. \tag{2.7}$$

We can see that

(i) above the optimal stopping boundary x_f, $Lv = 0$;
(ii) below the optimal stopping boundary x_f, $Lv = rE > 0$, thus $v(x,t) = V^*(x)$.

In this paper we assume that the volatility of the asset $\sigma(t)$ and the interest rate $r(t)$ satisfy

$$\alpha^* \geq \sigma^2(t) \geq \alpha > 0, \qquad \beta^* \geq r(t) \geq \beta > 0$$

for some positive constants α^*, α, β^* and β.

For applying the numerical method we truncate the domain $(0, +\infty)$ into $(0, S_{max})$. In this paper we choose $S_{max} = 4E$. The boundary condition at $x = S_{max}$ is chosen to be $v(S_{max}, t) = 0$. Normally, this truncation of the domain leads to a negligible error in the value of the option (Kangro and Nicolaides 2000).

Now we consider the following linear complementary problem:

$$\begin{aligned}
Lv(x,t) &\geq 0, && (x,t) \in (0, S_{max}) \times (0, T), && (2.8)\\
v(x,t) - V^*(x) &\geq 0, && (x,t) \in (0, S_{max}) \times [0, T], && (2.9)\\
Lv(x,t) \cdot [v(x,t) - V^*(x)] &= 0, && (x,t) \in (0, S_{max}) \times [0, T], && (2.10)\\
v(x,T) &= V^*(x), && x \in [0, S_{max}], && (2.11)\\
v(0,t) = E,\ v(S_{max}, t) &= 0, && t \in [0, T]. && (2.12)
\end{aligned}$$

Because the function $v(x, T) = \max\{E - x, 0\}$ has a discontinuous first-order partial derivative at $x = E$, $v(x,t)$ is not very smooth in the region where x near E and t near T. The numerical solution will have large truncation error at the region near $x = E$ and $t = T$. This is the place where the second-order derivative goes to infinity. Here we use a smoothing technique to treat the singularities of the non-smooth payoff function.

Define $\pi_\varepsilon(y)$ as

$$\pi_\varepsilon(y) = \begin{cases} y, & y \geq \varepsilon, \\ c_0 + c_1 y + \cdots + c_9 y^9, & -\varepsilon < y < \varepsilon, \\ 0, & y \leq -\varepsilon, \end{cases} \tag{2.13}$$

where $0 < \varepsilon \ll 1$ is a transition parameter and $\pi_\varepsilon(y)$ is a function which smooths out the original $\max(y, 0)$ around $y = 0$. This requires that $\pi_\varepsilon(y)$ satisfies

$$\pi_\varepsilon(-\varepsilon) = \pi'_\varepsilon(-\varepsilon) = \pi''_\varepsilon(-\varepsilon) = \pi'''_\varepsilon(-\varepsilon) = \pi^{(4)}_\varepsilon(-\varepsilon) = 0,$$
$$\pi_\varepsilon(\varepsilon) = \varepsilon,\ \pi'_\varepsilon(\varepsilon) = 1,\ \pi''_\varepsilon(\varepsilon) = \pi'''_\varepsilon(\varepsilon) = \pi^{(4)}_\varepsilon(\varepsilon) = 0.$$

Using these ten conditions we can easily find that

$$c_0 = \frac{35}{256}\varepsilon,\ c_1 = \frac{1}{2},\ c_2 = \frac{35}{64\varepsilon},\ c_4 = -\frac{35}{128\varepsilon^3},$$
$$c_6 = \frac{7}{64\varepsilon^5},\ c_8 = -\frac{5}{256\varepsilon^7},\ c_3 = c_5 = c_7 = c_9 = 0.$$

Replacing $\max\{E-x, 0\}$ in the terminal condition (2.7) by the fourth-order smooth function $\pi_\varepsilon(E-x)$, the linear complementarity problem (2.8)–(2.12) has a smooth solution. When smoothing function $\pi_\varepsilon(E-x)$ approaches to $\max\{E-x, 0\}$, the smooth solution approaches to the original solution. Therefore, in the remaining of this paper we will consider the following linear complementarity problem:

$$Lv(x,t) \geq 0, \quad (x,t) \in (0, S_{max}) \times (0, T), \tag{2.14}$$
$$v(x,t) - \pi_\varepsilon(E-x) \geq 0, \quad (x,t) \in (0, S_{max}) \times [0, T], \tag{2.15}$$
$$Lv(x,t) \cdot [v(x,t) - \pi_\varepsilon(E-x)] = 0, \quad (x,t) \in (0, S_{max}) \times [0, T], \tag{2.16}$$
$$v(x,T) = \pi_\varepsilon(E-x), \quad x \in [0, S_{max}], \tag{2.17}$$
$$v(0,t) = E,\ v(S_{max}, t) = 0, \quad t \in [0, T]. \tag{2.18}$$

The above linear complementarity problem can be solved by a power penalty approach. Let $0 < \mu \ll 1$ be a small regularization parameter and consider the following initial-boundary value problem

$$L_\mu v(x,t) = 0, \quad (x,t) \in (0, S_{max}) \times (0, T), \tag{2.19}$$
$$v(x,T) = \pi_\varepsilon(E-x), \quad x \in [0, S_{max}], \tag{2.20}$$
$$v(0,t) = E,\ \ v(S_{max}, t) = 0, \quad t \in [0, T], \tag{2.21}$$

where

$$L_\mu v(x,t) \equiv Lv(x,t) - \frac{C\mu}{v(x,t) + \mu - q(x)},$$

$C \geq rE$ is a positive constant and $q(x) = E - x$.

Nielsen et al. (2002) motivated the choice of the penalty term

$$\frac{C\mu}{v(x,t) + \mu - q(x)}.$$

Essentially, it is of order μ in regions where $v(x,t) \gg q(x)$, and hence the Black–Scholes equation is approximately satisfied. When $v(x,t)$ approaches $q(x)$ this term is approximately equal to C assuring that the early exercise constraint is not violated. Thus the penalty term is chosen so that the solution stays above the payoff function as the solution approaches expiry. Moreover, far from the barrier, $q(x)$, the penalty term is chosen small enough so that the PDE still resembles the Black–Scholes equation very closely. For detailed discussions and analysis of the penalty methods, we refer to the aforementioned references. In this paper we consider a second-order finite difference scheme to discretize the semilinear partial differential equation (2.19)–(2.21) and present an error analysis for the full discretization method.

3 Discretization

We now consider the approximation of the solution to the semilinear partial differential equation (2.19)–(2.21) by a second-order finite difference scheme.

The use of central difference scheme on a uniform mesh may produces nonphysical oscillations in the computed solution. To overcome this oscillation we use a piecewise uniform mesh Ω^N on the space interval $[0, S_{max}]$:

$$x_i = \begin{cases} h & i = 1, \\ h[1 + \frac{\alpha}{\beta^*}(i-1)] & i = 2, \dots, N/4 - 1, \\ E & i = N/4, \\ E + \varepsilon & i = N/4 + 1, \\ E + \varepsilon + \frac{S_{max} - E - \varepsilon}{3N/4 - 1}(I - N/4 - 1) & i = N/4 + 2, \dots, N, \end{cases}$$

where

$$h = \frac{E - \varepsilon}{1 + \frac{\alpha}{\beta^*}(N/4 - 2)}.$$

Here we have used a refined mesh at the region near $x = E$ for treating the singularities of the non-smooth payoff function. For the time discretization, we use a uniform mesh Ω^K on $[0, T]$ with K mesh elements. Then the piecewise uniform mesh $\Omega^{N\times K}$ on $\Omega = (0, S_{max}) \times (0, T)$ is defined to be the tensor product $\Omega^{N\times K} = \Omega^N \times \Omega^K$. It is easy to see that the mesh sizes $h_i = x_i - x_{i-1}$ and $\tau_j = t_j - t_{j-1}$ satisfy

$$h_i = \begin{cases} h & i = 1, \\ \frac{\alpha}{\beta^*} h & i = 2, \dots, N/4 - 1, \\ \varepsilon & i = N/4, N/4 + 1, \\ \frac{S_{max} - E - \varepsilon}{3N/4 - 1} & i = N/4 + 2, \dots, N \end{cases}$$

and

$$\tau = \tau_j = T/K, \quad j = 1, \dots, K,$$

respectively.

We discretize the semilinear partial differential equation (2.19)–(2.21) using a central difference scheme on the above piecewise-uniform mesh:

$$
\begin{aligned}
L_\mu^{N,K} V_i^j \equiv & -\frac{V_i^{j+1} - V_i^j}{\tau_{j+1}} - \frac{\left(\sigma^j\right)^2 x_i^2}{h_i + h_{i+1}} \left(\frac{V_{i+1}^j - V_i^j}{h_{i+1}} - \frac{V_i^j - V_{i-1}^j}{h_i} \right) \\
& - r^j x_i \frac{V_{i+1}^j - V_{i-1}^j}{h_i + h_{i+1}} + r^j V_i^j - \frac{C\mu}{V_i^j + \mu - q_i} = 0, \quad 1 \le i < N,\ 0 \le j < K,
\end{aligned} \tag{3.1}
$$

$$
V_i^K = \pi_\varepsilon \left(E - x_i \right), \qquad 0 \le i \le N, \tag{3.2}
$$

$$
V_0^j = E,\ V_N^j = 0, \qquad 0 \le j < K. \tag{3.3}
$$

Then from the above solution V_i^j we can obtain the optimal stopping price which is the maximum stock price such that $V_i^j = V_i^*$ for each t_j.

4 Analysis of the Method

To investigate the convergence of the method, note that the error functions $z_i^j = V_i^j - v_i^j$ $(0 \le i \le N, 0 \le j \le K)$ are the solutions of the discrete problem

$$
\begin{aligned}
& -\frac{z_i^{j+1} - z_i^j}{\tau_{j+1}} - \frac{\left(\sigma^j\right)^2 x_i^2}{h_i + h_{i+1}} \left(\frac{z_{i+1}^j - z_i^j}{h_{i+1}} - \frac{z_i^j - z_{i-1}^j}{h_i} \right) \\
& - r^j x_i \frac{z_{i+1}^j - z_{i-1}^j}{h_i + h_{i+1}} + r^j z_i^j + \frac{C\mu}{\left(\xi_i^j + \mu - q_i\right)^2} z_i^j \\
& = \left(\frac{v_i^{j+1} - v_i^j}{\tau_{j+1}} - \frac{\partial v}{\partial t} \left(x_i, t_j \right) \right) + \frac{1}{2} \left(\sigma^j\right)^2 x_i^2 \\
& \quad \times \left[\frac{2}{h_i + h_{i+1}} \left(\frac{v_{i+1}^j - v_i^j}{h_{i+1}} - \frac{v_i^j - v_{i-1}^j}{h_i} \right) - \frac{\partial^2 v}{\partial x^2} (x_i, t_j) \right] \\
& \quad + r^j x_i \left(\frac{v_{i+1}^j - v_{i-1}^j}{h_i + h_{i+1}} - \frac{\partial v}{\partial x} \left(x_i, t_j \right) \right), \qquad 1 \le i < N, 0 \le j < K,
\end{aligned} \tag{4.1}
$$

$$
z_i^K = 0, \qquad 0 \le i \le N, \tag{4.2}
$$

$$
z_0^j = z_N^j = 0, \qquad 0 \le j < K, \tag{4.3}
$$

where $\xi_i^j = v_i^j + \lambda z_i^j, 0 < \lambda < 1$.

Let

$$
\begin{aligned}
\bar{L}_\mu^{N,K} u_i^j \equiv & -\frac{u_i^{j+1} - u_i^j}{\tau_{j+1}} - \frac{\left(\sigma^j\right)^2 x_i^2}{h_i + h_{i+1}} \left(\frac{u_{i+1}^j - u_i^j}{h_{i+1}} - \frac{u_i^j - u_{i-1}^j}{h_i} \right) \\
& - r^j x_i \frac{u_{i+1}^j - u_{i-1}^j}{h_i + h_{i+1}} + r^j u_i^j + \frac{C\mu}{\left(\xi_i^j + \mu - q_i\right)^2} u_i^j .
\end{aligned} \tag{4.4}
$$

$$R_i^j \equiv \left(\frac{v_i^{j+1}-v_i^j}{\tau_{j+1}} - \frac{\partial v}{\partial t}\left(x_i,t_j\right)\right) + \frac{1}{2}(\sigma^j)^2 x_i^2\left[\frac{2}{h_i+h_{i+1}}\left(\frac{v_{i+1}^j-v_i^j}{h_{i+1}} - \frac{v_i^j-v_{i-1}^j}{h_i}\right) - \frac{\partial^2 v}{\partial x^2}\left(x_i,t_j\right)\right] + r^j x_i\left(\frac{v_{i+1}^j-v_{i-1}^j}{h_i+h_{i+1}} - \frac{\partial v}{\partial x}\left(x_i,t_j\right)\right). \tag{4.5}$$

The operator $\bar{L}_\mu^{N,K}$ satisfies the following discrete maximum principle.

Lemma 1 (Discrete maximum principle) *The operator $\bar{L}_\mu^{N,K}$ defined by (4.4) on the piecewise uniform mesh $\Omega^{N\times K}$ satisfies a discrete maximum principle, i.e. if u_i^j and w_i^j are mesh functions that satisfy $u_0^j \ge w_0^j$, $u_N^j \ge w_N^j$ $(0 \le j < K)$, $u_i^K \ge w_i^K$ $(0 \le i \le N)$ and $L^{N,K}u_i^j \ge L^{N,K}w_i^j$ $(1 \le i < N,\ 0 \le j < K)$, then $u_i^j \ge w_i^j$ for all i, j.*

Proof Let

$$a_i = -\frac{\left(\sigma^j\right)^2 x_i^2}{\left(h_i+h_{i+1}\right)h_i} + \frac{r^j x_i}{h_i+h_{i+1}}, \quad b_i = \frac{\left(\sigma^j\right)^2 x_i^2}{h_i h_{i+1}} + r^j + \frac{C\mu}{\left(\xi_i^j+\mu-q_i\right)^2},$$

$$c_i = -\frac{\left(\sigma^j\right)^2 x_i^2}{\left(h_i+h_{i+1}\right)h_{i+1}} - \frac{r^j x_i}{h_i+h_{i+1}}, \qquad i=1,\ldots,N-1.$$

Then

$$\bar{L}_\mu^{N,K}u_i^j = -\frac{u_i^{j+1}-u_i^j}{\tau_{j+1}} + a_i u_{i-1}^j + b_i u_i^j + c_i u_{i+1}^j.$$

We can obtain

$$a_i < -\frac{\left(\sigma^j\right)^2 x_1 x_i}{\left(h_i+h_{i+1}\right)h_i} + \frac{r^j x_i}{h_i+h_{i+1}} \le \frac{\left(-\alpha x_1+\beta^* h_i\right)x_i}{\left(h_i+h_{i+1}\right)h_i} = \frac{\left(-\alpha h+\beta^*\frac{\alpha}{\beta^*}h\right)x_i}{\left(h_i+h_{i+1}\right)h_i} = 0, \qquad 2 \le i < N/4$$

and

$$a_i \le \frac{\left(-\alpha x_i+\beta^* h_i\right)x_i}{\left(h_i+h_{i+1}\right)h_i} < 0, \qquad N/4 \le i \le N-1$$

for sufficiently large N. Clearly,

$$b_i > 0 \quad \text{for } 1 \le i \le N-1, \qquad c_i < 0 \quad \text{for } 1 \le i \le N-2$$

and

$$\begin{aligned} &b_1 + c_1 > 0, \\ &a_i + b_i + c_i > 0, \qquad 2 \le i \le N-2, \\ &a_{N-1} + b_{N-1} > 0. \end{aligned}$$

Hence we verify that the matrix associated with $\bar{L}_\mu^{N,K}$ is an M-matrix. □

Now we can get the main result for our difference scheme.

Theorem 1 *Let v be the solution of (2.19)–(2.21) and V be the solution of the finite difference scheme (3.1)–(3.3). Then we have the following error estimates*

$$\left|V_i^j - v(x_i, t_j)\right| \le C(\tau + h^2), \qquad 0 \le i \le N,\ 0 \le j \le K,$$

where C is a constant independent of τ and h.

Proof We use Taylor expansion to obtain

$$\begin{aligned} |R_i^j| &\le C_1 \int_{t_{j-1}}^{t_{j+1}} \left|\frac{\partial^2 v}{\partial t^2}(x_i, t)\right| \mathrm{d}t + C_2 h \int_{x_{i-1}}^{x_{i+1}} \left[x_i^2 \left|\frac{\partial^4 v}{\partial x^4}\left(x, t_j\right)\right| + x_i \left|\frac{\partial^3 v}{\partial x^3}\left(x, t_j\right)\right|\right] \mathrm{d}x \\ &\le C_3 \left(\tau + h^2\right) \end{aligned}$$

for $0 < i < N,\ 0 < j < K$, where C_i $(i = 1, 2, 3)$ are positive constants independent of τ and h. Hence, using the barrier function $W_i^j = C(\tau + h^2)$ (with the constant C sufficiently large), Lemma 1 implies that

$$\left|V_i^j - v(x_i, t_j)\right| \le C(\tau + h^2), \qquad 0 \le i \le N, \quad 0 \le j \le K,$$

which completes the proof. □

5 Numerical Experiments

In this section we verify experimentally the theoretical results obtained in the preceding section. Errors and convergence rates for the second-order finite difference scheme are presented for two test problems.

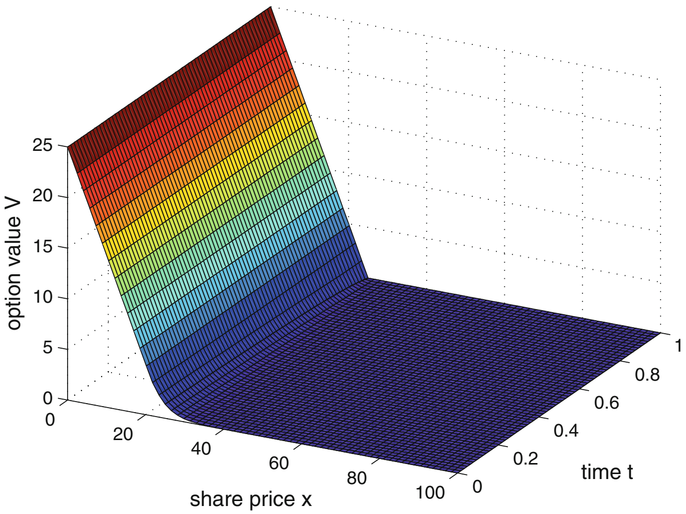

Fig. 1 Computed option value V for Test 1

Test 1 American put option with parameters: $S_{max} = 100$, $T = 1$, $r = 0.06$, $\sigma = 0.2$ and $E = 25$.

Test 2 American put option with parameters: $S_{max} = 100$, $T = 1$, $r = 0.06$, $\sigma = 0.4$ and $E = 25$.

To solve the nonlinear problem (3.1)–(3.3) we use Newton iterative method. The initial guesses for Tests 1 and 2 are taken as $\left[V_i^j\right]^{(0)} = V_i^{j+1}$ for time mesh point j and the stoping criterion is

$$\max_i \left| \left[V_i^j\right]^{(m)} - \left[V_i^j\right]^{(m-1)} \right| \leq 10^{-5}.$$

For Test 1 we choose $N = K = 64$ and $\varepsilon = 0.0001$, $\mu = 0.0001$, $C = rE$. The computed option value V and the constraint $V - V^*$ are depicted in Figs. 1 and 2, respectively.

For Test 2 we also choose $N = K = 64$ and $\varepsilon = 0.0001$, $\mu = 0.0001$, $C = rE$. The computed option value V and the constraint $V - V^*$ are depicted in Figs. 3 and 4, respectively.

To demonstrate the theoretical rates of convergence numerically, we take $K = 1024$ which is a sufficiently large choice to bring out second-order convergence in space. Here we choose $\varepsilon = 0.0001$, $\mu = 0.0001$ and $C = rE$. The exact solutions of our test problems are not available. We use the approximated solution of $N = 2048$, $K = 1024$ as the exact solution. We present the error estimates for different N. Because we only know "the exact solution" on mesh points, we use the linear interpolation to get solutions at other points. In this paper $\bar{V}(x, t)$ denotes "the exact solution" which is a linear

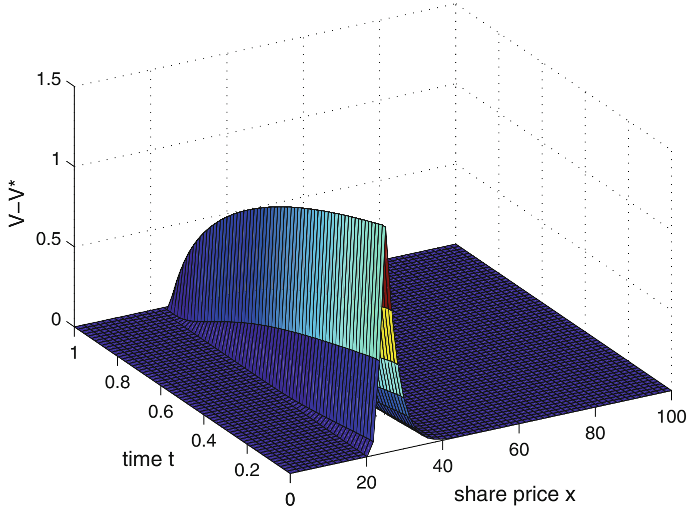

Fig. 2 The constraint $V - V^*$ for Test 1

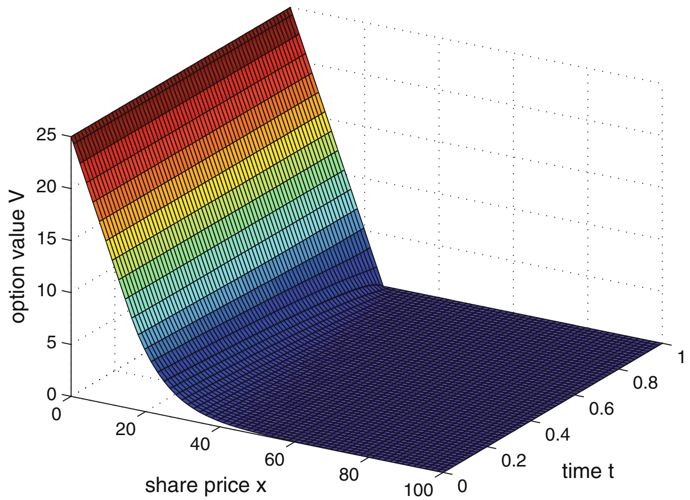

Fig. 3 Computed option value V for Test 2

interpolation of the approximated solution $V^{2048,1024}$. We measure the accuracy in the discrete maximum norm

$$e^{N,K} = \max_{i,j} |V_{ij}^{N,K} - \bar{V}(x_i, t_j)|,$$

and the convergence rate

$$\bar{r}^{N,K} = \log_2 \left(\frac{e^{N,K}}{e^{2N,K}} \right).$$

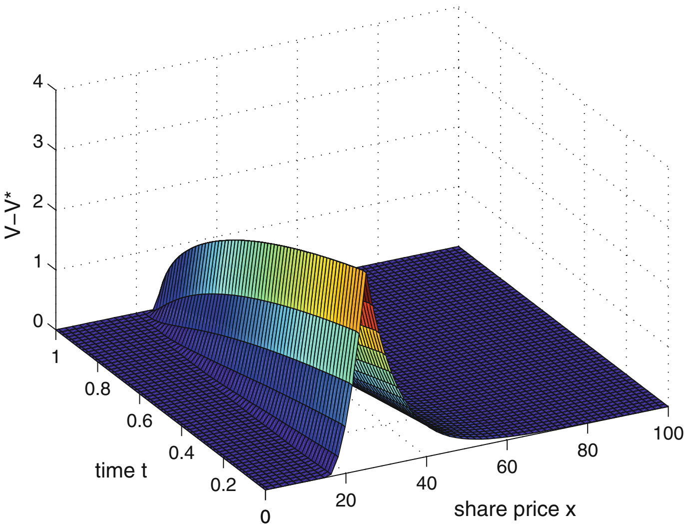

Fig. 4 The constraint $V - V^*$ for Test 2

Table 1 Numerical results for Test 1

K	N	Error	Rate
1024	128	6.8029e−2	1.122
	256	3.1260e−2	1.312
	512	1.2587e−2	1.881
	1024	3.4184e−3	–

Table 2 Numerical results for Test 2

K	N	Error	Rate
1024	128	6.5298e−2	1.244
	256	2.7577e−2	1.635
	512	8.8779e−3	2.187
	1024	1.9491e−3	–

The error estimates and convergence rates in our computed solutions of Tests 1 and 2 are listed in Tables 1 and 2, respectively.

From the figures it is seen that the numerical solutions by our method are non-oscillatory. From Tables 1 and 2 we see that $e^{N,K}/e^{2N,K}$ is close to 4 for sufficiently large K, which supports the convergence estimate of Theorem 1. They indicate that the theoretical results are fairly sharp.

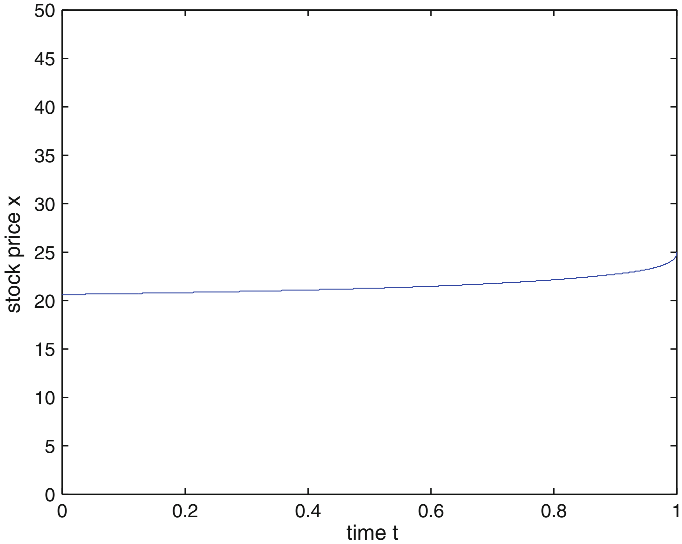

Fig. 5 The optimal stopping boundary for Test 1

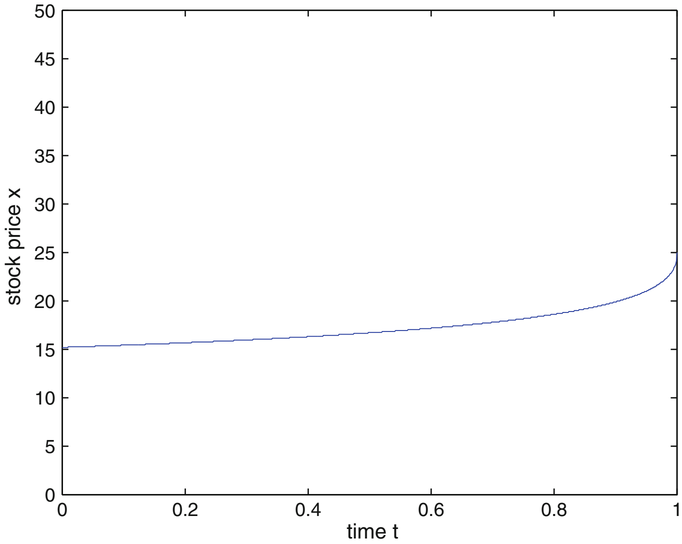

Fig. 6 The optimal stopping boundary for Test 2

As discussed above, the optimal stopping boundary is obtained as the maximum stock price such that $V_i^j = V_i^*$ for each t_j. Note that in order to find the maximum stock price in the numerical computation, we regard that $V_i = V_i^*$ if $|V_i^j - V_i^*| < \frac{1}{N}$ since the numerical error is $O(N^{-1}+K^{-1})$. We sketch the optimal stopping boundary for the two test problems with $N = K = 1024$, see Figs. 5 and 6.

Acknowledgment We would like to thank the anonymous referees for several suggestions for the improvement of this paper. The work was supported by Zhejiang Province Natural Science Foundation (Grant No. Y6100021, Y6110310) of China and Ningbo Municipal Natural Science Foundation (Grant Nos. 2009A610082, 2010A610099) of China.

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