Geometry

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Chapter 1

Quadratic Curves

1.1 Conic Sections

Definition 1.1. A conic section, or a conic, is a curve obtained from intersection of a plane with the surface of a cone.

Germinal Pierre Dandelin, a 19th century French-Belgian Mathematician, discovered this elegant proof to demonstrate that any plane that cuts through a right circular cone produces a quadratic curve.

Theorem 1.1. *When a plane intersects a right circular cone, the curve produced will either be an ellipse, a parabola or a hyperbola.*

Proof. Place a sphere tangent to the intersecting plane π and the cone such that it touches the plane at F, and the cone in a circle C with centre O, that lies on a horizontal plane ϵ.

Take an aribtrary point P on the curve Q produced by the intersection of the plane π and the cone, and extend the line VP from the vertex V of the cone such that it meets the circle C at point L. Let D be the point on the intersection on the planes π and ϵ such that PD is perpendicular to the line of intersection.

Drop a perpendicular PM on OL such that $\triangle PML$ and $\triangle PMD$ are both right angled. Denote $\angle PLM$ as α, and $\angle PDM$ as β. Consider the triangles $\triangle PML$ and $\triangle PMD$:

$$\begin{aligned} && \sin\alpha &= \frac{PM}{PD} \\ \text{and} && \sin\beta &= \frac{PM}{PL} \\ \implies && \frac{PL}{PD} &= \frac{\sin\alpha}{\sin\beta} \end{aligned}$$

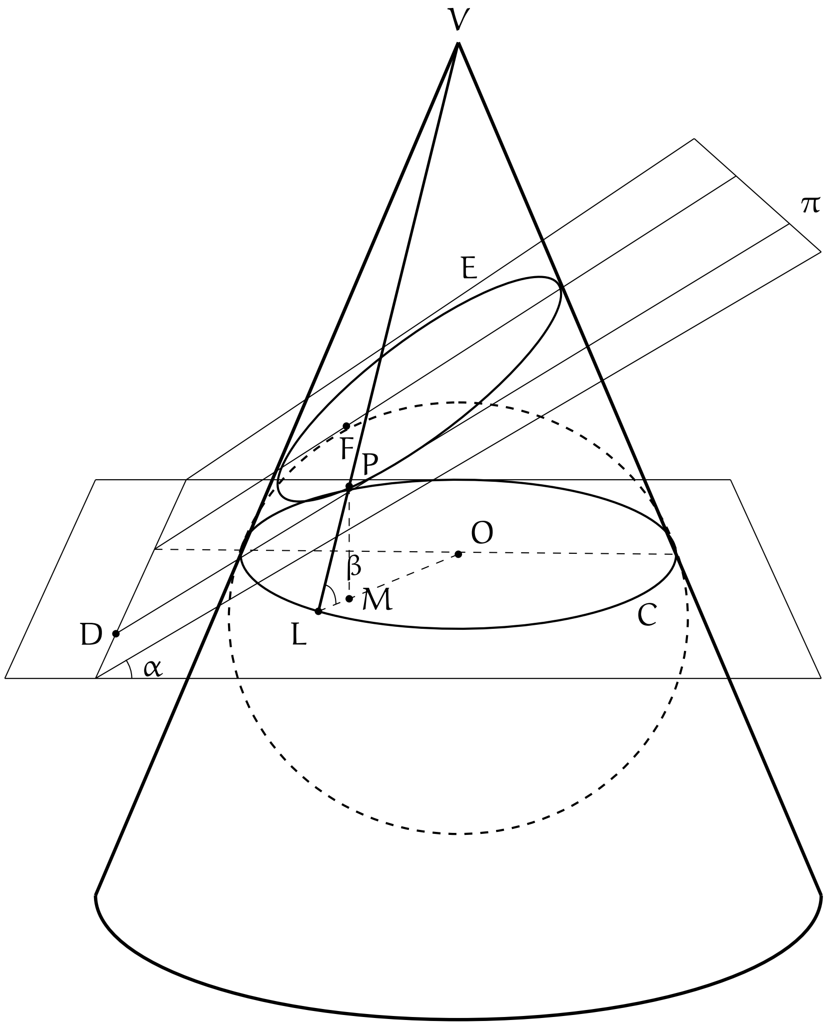

Figure 1.1: When $0 < \alpha < \beta < \frac{\pi}{2}$.

Since PL and PF are both tangents from P to the sphere, $PF = PL$. Therfore,

$$\frac{PF}{PD} = \frac{\sin\alpha}{\sin\beta}$$
$$\implies PF = e \cdot PD$$

where $e = \sin\alpha / \sin\beta$
It follows from the focus-directrix definition that Q will be an ellipse if $\alpha < \beta$, a parabola if $\alpha = \beta$, or a hyperbola if $\alpha > \beta$. ■

1.2 Group Laws on Conics

Consider a conic section

$$C = \{(x, y) \in \mathbb{F}^2 : f(x, y) = 0, f \in \mathbb{F}[x, y]\}$$

where f is squarefree, $\deg(f) = 2$, and $\mathrm{ch}(\mathbb{F}) \neq 2$. Given a fixed point $O \in C$, for any $P, Q \in C$, define a binary operation $\oplus : C \times C \to C$ by $P \oplus Q = R$, where R is such that l_{PQ} is parellel to l_{OR}.

Theorem 1.2. *Set of points of C forms a group under the binary operation $\oplus$.*

Proof. Under $\oplus$, the point O serves as the identity, and when Q is such that the line parellel to l_{PQ} that passes through O is tangent to the conic, that is when $R = O$, we get $P \oplus Q = O$. Such Q is the inverse for any $P \in C$.

To prove that $\oplus$ is associative, we'll derive expressions for $P \oplus Q$ by paremeterization for standard conics: for the circle $x^2 + y^2 = 1$, for the parabloa $y = x^2$, and for the hyperbola $xy = 1$. In the next Chapter, we will prove that all ellipses, hyperbolas, and parabolas are affine congruent to their respective standard forms. It will generalize our results to all conics.

Let the point P be (p_1, p_2), Q be (q_1, q_2), O be (o_1, o_2), and R be (r_1, r_2). The slope of the line l_{PQ} will be $\lambda = \frac{q_2 - p_2}{q_1 - p_1}$, assuming $P \neq Q$ (associativity would be trivial then). Let ℓ be the line through O with slope λ. (r_1, r_2) will satisfy:

$$\begin{aligned} \lambda &= \frac{r_2 - o_2}{r_1 - o_2} = \frac{q_2 - p_2}{q_1 - p_1} \\ \implies r_2 &= o_2 + \mu(q_2 - p_2) \text{ and} \\ r_1 &= o_1 + \mu(q_1 - p_1) \end{aligned}$$

for some $\mu \in \mathbb{F}$.

(i) **Circle**
Without loss of generality, let $O = (1, 0)$. Since R also lies on C, $r_1^2 + r_2^2 = 1$. i.e.

$$\begin{aligned} & (1 + \mu(q_1 - p_1))^2 + (0 + \mu(q_2 - p_2))^2 = 1 \\ \implies \quad & \mu(\mu(q_1 - p_1)^2 + \mu(q_2 - p_2)^2 + 2(q_1 - p_1)) = 0 \\ \implies \quad & \mu = 0 \text{ or } \mu = -\frac{2(q_1 - p_1)}{(q_1 - p_1)^2 + (q_2 - p_2)^2} \end{aligned}$$

We assume that $(q_1 - p_1)^2 + (q_2 - p_2)^2 \neq 0$, leaving out the case when $P = Q$.

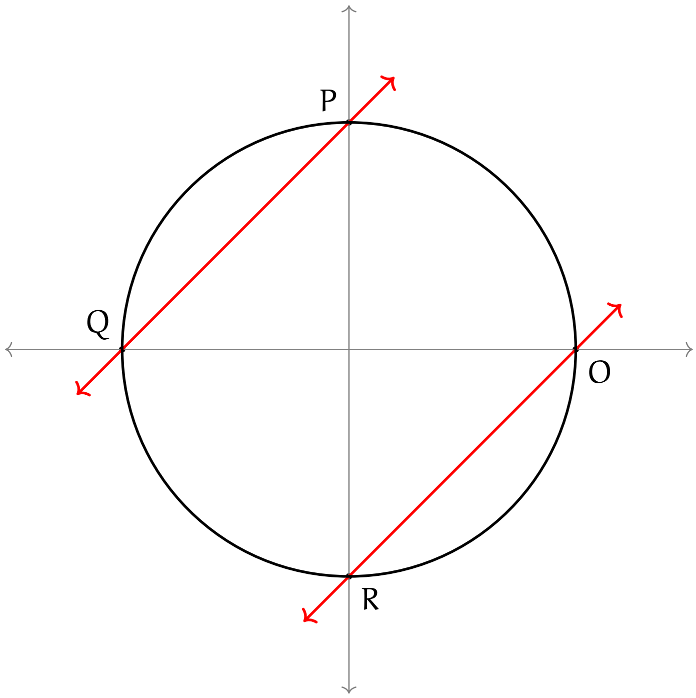

Figure 1.2: $R = P \oplus Q$ when C is a circle.

The $\mu = 0$ solution corresponds to $O = R$. Considering the other solution,

$$\begin{aligned}
r_1 &= 1 - \frac{2(q_1 - p_1)^2}{(q_1 - p_1)^2 + (q_2 - p_2)^2} \\
&= \frac{(q_2 - p_2)^2 - (q_1 - p_1)^2}{(q_1 - p_1)^2 + (q_2 - p_2)^2} \\
&= \frac{1 - p_1^2 - q_1^2 + p_1 q_1 - p_2 q_2}{1 - p_1 q_1 - p_2 q_2} \\
&= \frac{(p_1 q_1 - p_2 q_2)(1 - p_1 q_1 - p_2 q_2)}{1 - p_1 q_1 - p_2 q_2} \\
&= p_1 q_1 - p_2 q_2
\end{aligned}$$

and

$$\begin{aligned}
r_2 &= -\frac{2(q_1-p_1)(q_2-p_2)}{(q_1-p_1)^2+(q_2-p_2)^2} \\
&= \frac{p_2q_2+p_2q_1-p_1p_2-q_1q_2}{1-p_1q_1-p_2q_2} \\
&= \frac{(p_1q_2+p_2q_1)(1-p_1q_1-p_2q_2)}{1-p_1q_1-p_2q_2} \\
&= p_1q_2+p_2q_1
\end{aligned}$$

$$\implies P \oplus Q = (p_1q_1-p_2q_2, p_1q_2+p_2q_1)$$

(ii) **Parabola**
Without loss of generality, let $O = (0,0)$. The points of the standard parabloa can be parameterized as (t, t^2). Let $P = (p, p^2)$, $Q = (q, q^2)$, and $R = (r, r^2)$. Substituting them in λ:

$$\begin{aligned}
\lambda &= \frac{r^2}{r} = \frac{q^2-p^2}{q-p} \\
&\implies r = p+q \\
&\implies P \oplus Q = (p+q, (p+q)^2)
\end{aligned}$$

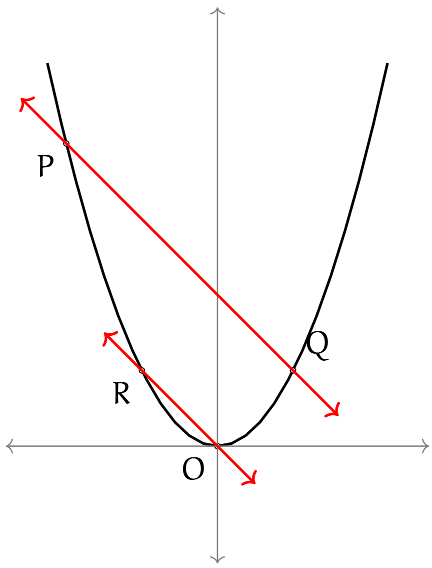

Figure 1.3: $R = P \oplus Q$ when C is the standard parabola.

(iii) **Hyperbola**
Without loss of generality, let $O = (1, 1)$.The points of the standard hyperbola can be parameterized as $(t, \frac{1}{t})$. Let $P = (p, \frac{w}{p})$, $Q = (q, \frac{1}{q})$, and $R = (r, \frac{1}{r})$. Substituting them in λ:

$$\begin{aligned}\lambda &= \frac{\frac{1}{r} - 1}{r - 1} = \frac{\frac{1}{q} - \frac{1}{p}}{p - q} \\ &\implies r = pq \\ &\implies P \oplus Q = (pq, \frac{1}{pq})\end{aligned}$$

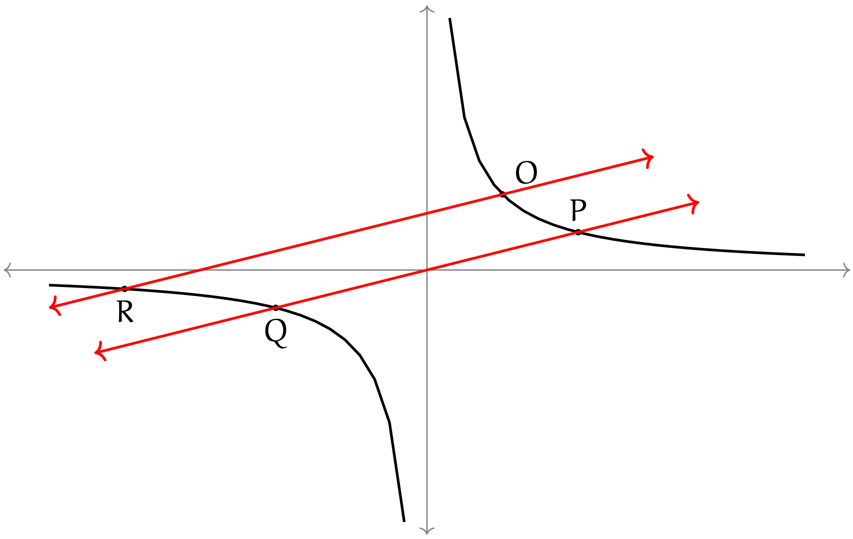

Figure 1.4: $R = P \oplus Q$ when C is the standard hyperbola.

From the expressions obtained for $P \oplus Q$, in all three cases it can be easily proved that $P \oplus (Q \oplus R) = (P \oplus Q) \oplus R, \forall P, Q, R \in C$. ■

Remark. It can also be shown that the group $\langle C, \oplus \rangle$ is isomorphic to some other well known groups in each case:

- When C is an ellipse, $\langle C, \oplus \rangle \cong \langle S^1, \cdot \rangle$
  where $S^1 = \{e^{i\theta} \in \mathbb{C} : \theta \in [0, 2\pi)\}$.
- When C is a parabola, $\langle C, \oplus \rangle \cong \langle \mathbb{R}, + \rangle$.
- When C is a hyperbola, $\langle C, \oplus \rangle \cong \langle \mathbb{R}^\times, \cdot \rangle$.

1.3 Solutions for Diophantine Equations

Consider the conic $C = \{(x, y) \in \mathbb{Q} : x^2 + y^2 = 1\}$, and $P = (1, 0) \in C$. Let l_m be the line with slope $m \in Q$, passing through P and another point $Q = (x, y) \in C$. The coorrdinates of Q can be found by substituting $y = m(x - 1)$.

$$x^2 + m^2(x-1)^2 - 1 = (1+m^2)x^2 - 2m^2x - (1-m^2) = 0$$

using the quadratic formula,

$$x = \frac{m^2 \pm 1}{1 + m^2}$$

from the non-trivial solution, we get $x = \frac{m^2-1}{m^2+1}$ and $y = \frac{-2m}{m^2+1}$. substituting these values in the equation for the conic,

$$\begin{aligned} \left(\frac{m^2-1}{m^2+1}\right)^2 + \left(\frac{-2m}{m^2+1}\right)^2 &= 1 \\ \implies (m^2-1)^2 + (2m)^2 &= (m^2+1)^2 \end{aligned}$$

This equation will produce integer solutions for $x^2 + y^2 = 1$, though not all of them. Rational or integer solutions for any equations of the form $ax^2 + by^2 = c$, where $a, b, c \in Q$ can be similarly generated.

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Chapter 2

Affine Geometry

2.1 Affine Space

Definition 2.1. A set $\mathcal{E}$ is endowed with the structure of an affine space by a vector space E and a mapping Θ that associates a vector of E with any ordered pair of points in $\mathcal{E}$,

$$\begin{array}{ccc} \mathcal{E} \times \mathcal{E} & \longrightarrow & E \\ (A, B) & \longmapsto & \overrightarrow{AB} \end{array}$$

such that:

- for any point A of $\mathcal{E}$, the partial map $\Theta_A : B \mapsto \overrightarrow{AB}$ is a bijection from $\mathcal{E}$ to E;
- for any points A, B, and C in $\mathcal{E}$, we have $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$.

The vector space E is called the direction of $\mathcal{E}$, or its underlying vector space. The elements of $\mathcal{E}$ are called points, and the dimension of $\mathcal{E}$ is defined to be the dimension of the vector space E.

A subset $\mathcal{F}$ of $\mathcal{E}$ is an affine subspace if it is empty, or if it contains a point A such that $\Theta_A(\mathcal{F})$ is a vector subspace of E.

2.1.1 Affine Transfromations

Definition 2.2. Let $\mathcal{E}$ and $\mathcal{F}$ be two affine spaces directed respectively by the vector spaces E and F. A transformation $\varphi : \mathcal{E} \to \mathcal{F}$ is said to be affine if there exists a point $O \in \mathcal{E}$ and a linear transformation $f : E \to F$ such that $\forall M \in E\ f(\overrightarrow{OM}) = \overrightarrow{\varphi(O)\varphi(M)}$.

When $E = F = \mathbb{R}^2$, it can be proved that the map f is of the form $f(x) = Ax + b$, where $b \in \mathbb{R}^2$, and A is a 2×2 invertible matrix. The set of all affine transformations of $\mathbb{R}^2$ is denoted by $A(2)$. If A is orthagonal, the map f would produce a Euclidean transformation, where all distances and angles are preserved. From the definition, all affine transformations:

1. Map straight lines into straight lines;
2. Map parellel straight lines into parellel straight lines;
3. Preserve ratios of lengths along a given straight line.

Translations and Rotations are examples of affine transformations in 2 dimensional affine spaces.

Remark. For the affine transformations $t : \mathbb{R}^2 \to \mathbb{R}^2$, $t(x) = Ax + b$, the inverse is given by $t^{-1}(x) = A^{-1}x - A^{-1}b$

2.2 Fundamental Theorem of Affine Geometry

Lemma 2.1. *The points $(0,0)$, $(0,1)$, and $(1,0)$ can be mapped into any three non-collinear points p, q, and r by a unique affine transformation.*

Proof. Any affine transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$ has the form $t(x) = Ax + b$. Where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } b = \begin{pmatrix} e \\ f \end{pmatrix}$$

The images of the points $(1,0)$ and $(0,1)$ are $(a+e, c+f)$ and $(b+e, d+f)$ respectively. For any p, q, and r, we can choose t to be such that $(e, f) = p$, $(a, e) = q - p$, and $(b, d) = r - p$. For some t' such that $t'((0,0)) = p$, $t'((1,0)) = q$, and $t'((0,1)) = r$, we would have $t(x) - t'(x) = 0\ \forall x$. Hence, such transformation t is unique. ■

Definition 2.3. Two subsets of an affine space are said to be affine congruent if there exists an affine transformation which takes one to the other.

Using the above lemma, we prove the two-dimensional analog of the so called Fundamental theorem of Affine Geometry.

Theorem 2.1 (Fundamental theorem of Affine Geometry)**.** *For any two sets of three non-collinear points, $\{p, q, r\}$ and $\{p', q', r'\}$ in $\mathbb{R}^2$, there exists a unique affine transformation t which maps p, q, and r to p', q', and r', respectively.*

Proof. Let t_1 be the affine transformation which maps $(0,0)$, $(0,1)$, and $(1,0)$ to the points p, q, and r respectively, and let t_2 be the affine transformation which maps $(0,0)$, $(0,1)$, and $(1,0)$ to the points p', q', and r' respectively. Then, the map $t = t_2 \circ t_1^{-1}$ is an affine transformation, and it maps p, q, and r to p', q', and r' respectively.

Suppose t and s are both affine transformations which map p, q, and r to p', q' and r' respectively. Then the composites $t \circ t_1$, and $s \circ t_1$ both map the points $(0,0)$, $(0,1)$, and $(1,0)$ to the points p', q', and r' respectively. Since such map is unique, $t \circ t_1 = s \circ t_1$. Composing both of them with t_1^{-1} to the right, it follows that $t = s$. ■

Corollary. *All triangles are affine congruent.*

2.3 Affine Congruence of Conics

Lemma 2.2. *Affine transformations map ellipses to ellipses, parabolas to parabolas, and hyperbolas to hyperbolas.*

Proof. Consider the non-degenerate conic with the equation

$$Ax^2 + Bxy + Cy^2 + Fx + Gy + H = 0,$$

and its image under the transformation t given by $t(m) = Jm + k$, for some invertible matrix J, and $k \in \mathbb{R}^2$. The inverse transformation given by $t^{-1}(n) = J^{-1}n - J^{-1}k$ can be written in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}$$

where $m = (x, y)$, and $n = (x', y')$. It follows that $x = px' + qy' + u$, and $y = rx' + sy' + v$. If these expressions are substituted into the equaiton of the conic, we get back a second degree equation. So the image of a conic must be a conic. It cannot be degenerate since affine transformations map lines into lines, and if the resultant conic is a line, then the inverse would map it into a degenrate conic. But this cannot happen since our original conic is non-degenerate.

If we substitute x and y to the equation of the original conic, it turns out that the discriminant obtained is just

$$(ps - rq)^2(B^2 - 4AC)$$

Where $B^2 - 4AC$ is the discriminant of the original conic.
Since $(ps - rq)^2 > 0$, the sign of the discriminant does not change. Hence the type of conic is also unchanged. ■

Theorem 2.2. *All ellipses are congruent to each other.*

Proof. Consider a general elliipse centered at the origin, with aligned axes,
$E_1 : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
With a translation to move the center to the origin and a rotation to align the axes, any ellipse can be transformed into this form. If we apply the affine transformation $t_1 : (x, y) \mapsto (x', y')$, where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Then the equation becomes $(x')^2 + (y')^2 = 1$. The map $t = t_2 \circ t_1^{-1}$ takes the ellipse E_1 to E_2. ■

Theorem 2.3. *All hyperbolas are congruent to each other.*

Proof. Consider a general hyperbola centerd at the origin with aligned axes,
$H_1 : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
With a translation to move the center to the origin and a rotation to align the axes, any hyperbola can be transformed into this form. If we apply the affine transformation $t_1 : (x, y) \mapsto (x', y')$, where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{1}{a} \\ \frac{1}{b} & \frac{1}{b} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Then the equation becomes $x'y' = 1$. The map $t = t_2 \circ t_1^{-1}$ takes the hyperbola H_1 to H_2. ■

Theorem 2.4. *All parabolas are congruent to each other.*

Proof. Consider a general parabola centered at the origin with aligned axes, $P_1 : y^2 = ax$. With a translation to move the center to the origin and a rotation to align the axes, any parabola can be transformed into this form. If we apply the affine transformation $t_1 : (x, y) \mapsto (x', y')$, where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Then the equation becomes $(y')^2 = x'$. The map $t = t_2 \circ t_1^{-1}$ takes the parabola P_1 to P_2. ■

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Chapter 3

Projective Geometry

3.1 The Projective Space

Definition 3.1. Let E be a finite dimensional vector space. The *projective space* $P(E)$ deduced from E is the set of all 1 dimensional linear subspaces of E.

Remark. The dimension of $P(E)$ is $\mathrm{dim}\, E-1$. If E consists only of the point 0, it does not contain any lines, and $P(E)$ is empty. Thus it shall be implicitly assumed that dim $E \geqslant 1$. If dim $E = 1$, E itself is a line, and thus the set of lines contains a unique element, hence $P(E)$ is a point.

A subset V of $P(E)$ is a projective subspace if it is an image of a nonzero vector subspace F of E.

Remark. Usually, $P(\mathbb{R}^{n+1})$ is denoted as $\mathbb{RP}^n$.

Proposition 3.1. *Let V and W be two projective subspaces of $P(E)$.*

- *If dim V + dim $W \geqslant$ dim $P(E)$, then $V \cap U$ is not empty.*
- *Let H be a hyperplane of $P(E)$, and let m be a point not in H. Every line through m intersects H at a unique point.*

Proof. Let F and G be the vector subspaces of E from which V and W were deduced, i.e. $V = P(F)$, and $W = P(G)$. The statement can be translated into vector subspaces as

$$\begin{aligned} & (\dim F - 1) + (\dim G - 1) \geqslant (\dim E - 1) \\ \Longrightarrow \quad & \dim F + \dim G \geqslant \dim E + 1 \end{aligned}$$

We can use the linear algebraic properties to further deduce that:

$$\dim F + \dim G = \dim (F + G) + \dim (F \cap G) \leqslant \dim E + \dim (F \cap G)$$

Therefore,

$$\dim\,(F \cap G) \geqslant 1$$

This can be translated back into projective geometry to conclude that $V \cap W$ is not empty.

Now, to prove the second proposition, let J be the vector hyperplane of which H is image of. The point m is the image of a line l in E, not contained in the hyperplane J. The assertion, translated in terms of linear algebra, is that any plane P containing l meets J along a unique line. Since l is not in J, we have $P + J = E$. Hence,

$$\begin{aligned} \dim\,(P \cap J) &= \dim P + \dim J - \dim\,(P + J) \\ &= 2 + \dim E - 1 - \dim E = 1 \end{aligned}$$

■

3.1.1 Projective Transformations

Definition 3.2. Let E and E' be two vector subspaces, and $p : E - \{0\} \to P(E)$ and $p' : E' - \{0\} \to P(E')$ be the two projections. A *projective transformation* $g : P(E) \to P(E')$ is a mapping such that there exists a linear isomorphism $f : E \to E'$ with $p' \circ f = g \circ p$.

Proposition 3.2. *The set of projective transformations from* $P(E)$ *to itself,* $PGL(E)$*, is a group under composition.*

Proof. From the definitions, the projective transformation that descends from identity map of E forms the identity of the group. For any projective tranformation g that descends from a linear isomorphism f, the transformation g' that descends from f^{-1} will act as its inverse. Since functional composition obeys associativity, $PGL(E)$ is a group. ■

3.2 Projective Frames

Given a basis of vector space E, the vectors in E can be descirbed by their coordinates with respect to the basis.

Definition 3.3. A point m in $P(E)$ can be described by the nonzero vector that generates the line m. In a n-dimensional projective space $P(E)$, the $(n + 1)$ tuples $(x_1, \ldots, x_{n+1})$ and $(x'_1, \ldots, x'_{n+1})$ represent the same point iff there exists a nonzero scalar λ such that $x_i = \lambda x'_i$ for all i.

Remark. Usually, the coordinates for projective spaces are represented as $[x_1 : x_2 : \ldots : x_{n+1}]$.

In a projective space $P(E)$ with dimension n, we actually need $n+2$ points to uniquely determine the basis of the underlying space E, which will be proved in the next lemma. It will also justify the next definition.

Definition 3.4. If E is a vector space of dimension $n+1$, a *projective frame* of $P(E)$ is a set of $n+2$ points $(m_0, \ldots, m_{n+1})$ such that $m_1, \ldots, m_{n+1}$ are the images of the vectors $e_1, \ldots, e_{n+1}$ in a basis of E, and m_0 is the image of $e_1 + \cdots + e_{n+1}$.

Lemma 3.1. *Let* $(m_0, \ldots, m_{n+1})$ *be a projective frame of* $P(E)$. *If the two bases of* E $(e_1, \ldots, e_{n+1})$ *and* $(e'_1, \ldots, e'_{n+1})$ *are such that* $p(e_i) = p(e'_i) = m_i$ *and* $p(e_1 + \cdots + e_{n+1}) = p(e'_1 + \cdots + e'_{n+1}) = m_0$, *then they are proportional.*

Proof. Consider the points m_i of $P(E)$. Since the vectors e_i and e'_i both generate the line m_i, $e_i = \lambda_i e'_i$ for some nonzero λ_i. Using the $(n+2)^{th}$ point, we can conclude that

$$(e_1 + \cdots + e_{n+1}) = \lambda(e'_1 + \cdots + e'_{n+1})$$

Thus,

$$\lambda_1 e_1 + \cdots + \lambda_{n+1} e_{n+1} = \lambda(e_1 + \cdots + e_{n+1})$$

As we are dealing with a basis, $\lambda_i = \lambda$. Thus two bases are proportional. ■

3.3 Some Results in Projective Geometry

Theorem 3.1 (Fundamental Theorem of Projective Geometry)**.** *Let* $(a_1, \ldots, a_{n+2})$ *and* $(b_1, \ldots, b_{n+2})$ *be two sets of points in* $\mathbb{RP}^n$ *such that none of the* a_i *and* b_i *belong to the projective subspace defined by* n *of the others in their respective sets. Then there exists a unique projective transformation* $f : \mathbb{RP}^n \to \mathbb{RP}^n$ *such that,* $f(a_i) = b_i$ *for all* $i = 1, 2, \ldots, n+2$.

Proof. The set of points (a_i) and (b_i) are both projective frames of $\mathbb{RP}^n$. Let $(e_1, \ldots, e_{n+1}), (e'_1, \ldots, e'_{n+1}) \in \mathbb{R}^{n+1}$ be the basis that generate the frames (a_i) and (b_i) respectively. We know that there exists a unique isomorphism $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $f(e_i) = e'_i$. The projective transformation g that descends from f will map the first frame to the second.

To prove the uniqueness: let f and f' two such projcetive transformations. The projective transormation $g^{-1} \circ g'$ from $\mathbb{RP}^n$ into itself keeps the frame invariant. Thus it is the identity transformation. ■

Theorem 3.2 (Desargues's Theorem)**.** *Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles in $\mathbb{R}^2$ such that the lines AA', BB', and CC' meet at point U. Let BC and $B'C'$ meet at P, CA and $C'A'$ at Q, and AB and $A'B'$ at R. Then P, Q, and R are colinear.*

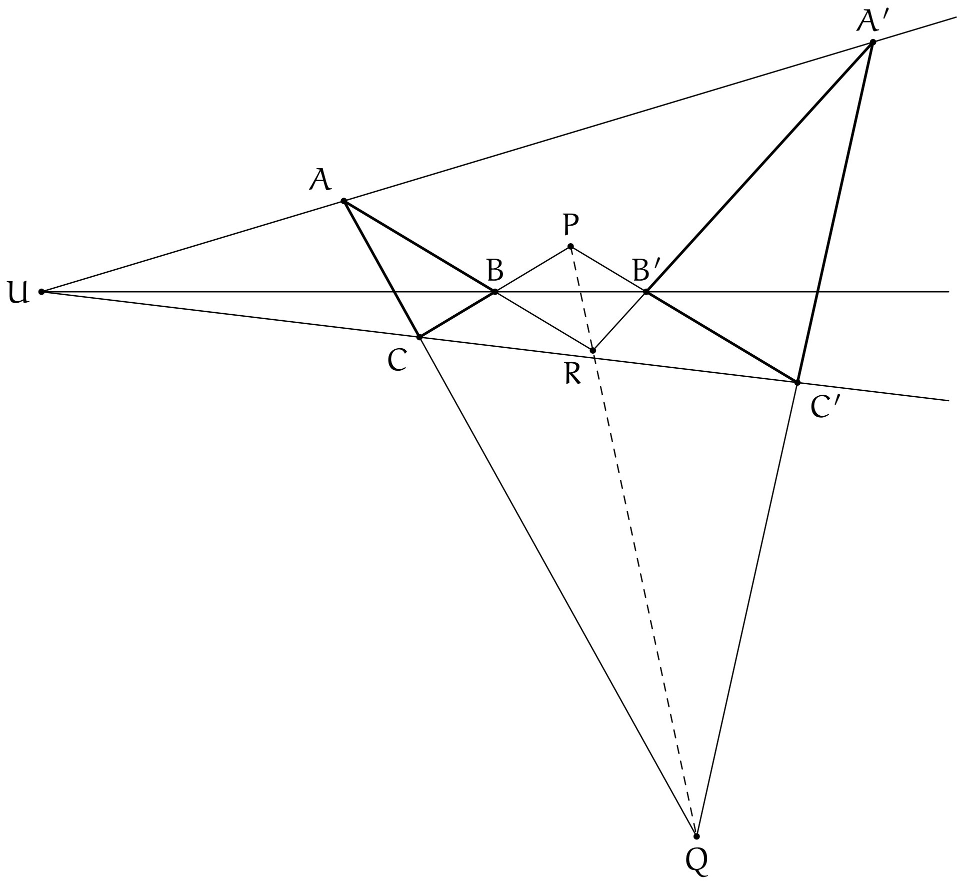

Figure 3.1: P, Q, and R are collinear.

Proof. We will prove the theorem for the special case where $A = [1 : 0 : 0]$, $B = [0 : 1 : 0]$, $C = [0 : 0 : 1]$, and $U = [1 : 1 : 1]$. From the fundamental theorem of projective geometry, we know that it will be congruent to any other configuration. We can use the fact that projective congruence preserves projective properties, to deduce that the theorem holds in general.

The line AU has the equation $y = z$. Since A' lies on AU, it must have coordinates $[a : b : b]$, where $b \neq 0$, since $A' \neq A$. We can also wirte $A' = [p : 1 : 1]$, where $p = a/b$. Similary, $B' = [1 : q : 1]$, and $C' = [1 : 1 : r]$.

Now to find the point P, we find the equation of the line BC.

$$\begin{vmatrix} x & y & z \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0 \Longrightarrow (qr-1)x - (r-1)y + (1-q) = 0$$

Substituting $x = 0$ in the equation for the line $B'C'$, we get $(r-1)y = (1-q)z$, which immplies $P = [0 : 1-q : r-1]$. Similarly we find that $Q = [1-p : 0 : r-1]$, and $R = [1-p : q-1 : 0]$.

To check the colinearity of P, Q, and R:

$$\begin{vmatrix} 0 & 1-q & r-1 \\ 1-p & 0 & r-1 \\ 1-p & q-1 & 0 \end{vmatrix}$$

$$\begin{aligned} &= -(1-q)(1-p)(1-r) + (r-1)(1-p)(q-1) \\ &= 0 \end{aligned}$$

Therefore P, Q, and R are colinear. ■

Proposition 3.3. *There is a unique projective conic through any given set of five points, no three of which are colinear.*

Proof. By the fundamental theorem of projective geometry, there exists a projective transformation t which maps the four out of given five points to the points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$ and $[1 : 1 : 1]$. Let $[a : b : c]$ be the image of the fifth point under t. Since projective transformations preserve colinearity, no three of the five points are colinear, and also it can be deduced that a, b and c are nonzero, since if any of them were zero, the image of the fifth point would be collinear with other two points.

The conic that passes through these 5 points will be of the general form

$$Ax^2 + Bxy + Cy^2 + Fxz + Gyz + Hz^2 = 0$$

By substituting the points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$, the equation can be reduced to the form

$$Bxy + Fxz + Gyz = 0$$

Since the porjective concic also passes through $[1 : 1 : 1]$ and $[a : b : c]$, we get the equations

$$B + F + G = 0$$

and

$$Bab + Fac + Gbc = 0$$

Solving these simultaneous equations, we get

$$F = -G\left(\frac{ab - bc}{ab - ac}\right)$$

and

$$B = -G\left(\frac{ac - bc}{ac - ab}\right)$$

It follows that the conic is of the form

$$-G\left(\frac{ac - bc}{ac - ab}\right)xy - G\left(\frac{ab - bc}{ab - ac}\right)xz + Gyz = 0$$

that is,

$$c(a - b)xy + b(c - a)xz + a(b - c)yz = 0$$

Since the conic is uniquely determined by the fifth point, it follows that it is unique.

■

Remark. The projective conic $E = \{[x : y : z] : xy + yz + zx = 0\}$ is called the standard projective conic. It passes through the traingle of reference formed by the points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. This fact can be used to simplify calculations involving projcetive conics.

All the points on the conic except than $[1 : 0 : 0]$ can be parameterized as $[t^2 + t : t + 1 : -t]$, where $t \in \mathbb{R}$. All points on E satisfy $xy + yz + zx = 0$. Suppose $y \neq 0$, let $t = x/y$. Substituting $x = ty$ to the equatin of the conic:

$$\begin{aligned} & (ty)y + yz + z(ty) = 0 \\ \Longrightarrow \quad & ty + (t + 1)z = 0 \\ \Longrightarrow \quad & y = -\tfrac{t+1}{t}z, x = -(t + 1)z \end{aligned}$$

Thus the point has homogeneous coordinates $\left[-(t + 1)z : -\frac{t+1}{t}z : z\right]$, which can be rewritten in the form $[t(t + 1) : t + 1 : -t]$. This also happens to hold for the point $[0 : 0 : 1]$, where $y = 0$.

Proposition 3.4. *Let E_1 and E_2 be non-degenrate conics that pass through the points P_1, Q_1, R_1 and P_2, Q_2, R_2 respectively. Then there exists a projective transformation t which maps E_1 to E_2 such that $t(P_1) = P_2$, $t(Q_1) = Q_2$, and $t(R_1) = R_2$.*

Proof. We prove this result by proving that for any conic E_1, there exists a transormation t_1 which maps it to the standard conic $xy+yz+zx=0$ such that $t_1(P_1) = [1:0:0]$, $t_2(Q_1) = [0:1:0]$, and $t_3(R_1) = [0:0:1]$ for any $P_1, Q_1, R_1 \in E$.

Let f be a transformation that maps P_1 to $[1:0:0]$, Q_1 to $[0:1:0]$, and R_1 to $[0:0:1]$. It will map the conic E_1 into a conic E' of the form

$$Fxy + Gyz + Hzy = 0$$

for some $F, G, H \in \mathbb{R}$. Divide the equation by FGH to rewrite E' in the form

$$\frac{xy}{GH} + \frac{yz}{FH} + \frac{zx}{FG} = 0$$

Now, let g be the transformation of the form $g([x:y:z]) = A[x:y:z] \forall [x:y:z] \in \mathbb{R}\,\mathrm{Pr}^3$ where A is a 3×3 matrix such that

$$A = \begin{pmatrix} \frac{1}{H} & 0 & 0 \\ 0 & \frac{1}{G} & 0 \\ 0 & 0 & \frac{1}{F} \end{pmatrix}$$

Then, g maps E' to the standrd conic $xy+yz+zx=0$, leaving P, Q, and R unchanged. Let $t_1 = g \circ f$. Similarly, let t_2 be the function that maps the conic E_2 to the standard conic such that $t_2(P_2) = [1:0:0]$, $t_2(Q_2) = [0:1:0]$, and $t_2(R_2) = [0:0:1]$ for any $P_2, Q_2, R_2 \in E_2$.

The composite function $t = t_2^{-1} \circ t_1$ maps E_1 to E_2 such that $t(P_1) = P_2$, $t(Q_1) = Q_2$, and $t(R_1) = R_2$, as required. ■

Theorem 3.3 (Pascal's Theorem)**.** *Let A, B, C, A', B', and C' be six distinct points on a non-degenrate projective conic. Let BC and $B'C$ intersect at P, CA' and $C'A$ at Q, and AB' at R. The points P, Q, and R are colinear.*

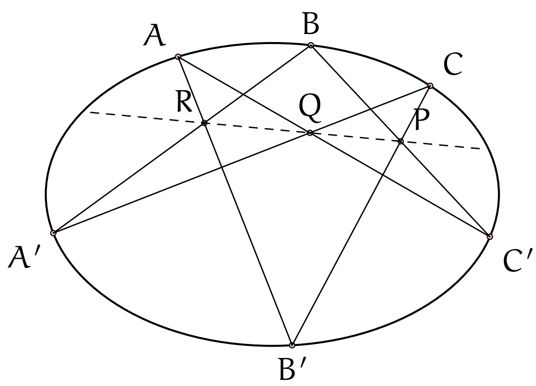

Figure 3.2: When the E is an ellipse.

Proof. We know that any non-degenerate conic can pe transformed to the standard conic. Let the conic be in the standard form $xy + yz + zx = 0$, with $A = [1 : 0 : 0]$, $B = [0 : 1 : 0]$, and $C = [0 : 0 : 1]$. Let the point $A' = [a^2 + a : a + 1 : -a]$, $B' = [b^2 + b : b + 1 : -b]$, and $C' = [c^2 + c : c + 1 : -c]$, for some $a, b, c \in \mathbb{R}$

The line BC' has the equation $x = -(c + 1)z$, and the line $B'C$ has the equation $x = by$. The point P lies on both of these lines. Hence it has the homogeneous coordinates $[b(c + 1) : c + 1 : -b]$. Similarly, $Q = [a(c + 1) : c + 1 : -c]$, and $R = [b(a + 1) : b + 1 : -b]$.

To check their colinearity, evaluate the determinant:

$$\begin{vmatrix} b(c+1) & c+1 & -b \\ a(c+1) & c+1 & -c \\ b(a+1) & b+1 & -b \end{vmatrix}$$

Which, after some row operations, simplifies to be equal to 0. Hence, the points P, Q, and R are colinear. ■

3.4 The Cross-Ratio

Definition 3.5. Let a, b, c and d be four points on a projective line D. There exists a unique map $g : D \to K \cup \{\infty\}$ that maps a to ∞, b to 0, and c to 1. The image of d under this projective mapping is called the *cross-ratio* of the points (a, b, c, d), and denoted $[a, b, c, d]$.

Proposition 3.5. *Let* a_1, a_2, a_3, *and* a_4 *be four points on the line* D (*the first three being distinct*) *and* a'_1, a'_2, a'_3, *and* a'_4 *be four points on another line* D' (*satisfying the same assumption*). *There exists a projective transformation* $f : D \to D'$ *such that* $f(a_i) = a'_i \Leftrightarrow [a_1, a_2, a_3, a_4] = [a'_1, a'_2, a'_3, a'_4]$.

Proof. Assume f is a projective mapping that sends a_i to a'_i. Let g and g' be functions such that $[a_1, a_2, a_3, a_4] = g(a_4)$, and $[a'_1, a'_2, a'_3, a'_4] = g'(a'_4)$. $g' \circ f$ is a function which maps a_1 to ∞, a_2 to 0, and a_3 to 1. But such function is unique. Hence, $g = g' \circ f$, which implies that $g(a_4) = g'(a'_4)$. That is,

$$[a_1, a_2, a_3, a_4] = [a'_1, a'_2, a'_3, a'_4]$$

■

Remark. **Formulas for cross-ratio**
Let a, b, and c be four points on the affine line, the first three being distinct. Then

$$[a, b, c, d] = \frac{(d - b)(c - a)}{(d - a)(c - b)}$$

Also, since the points a and b are distinct, c and d can be written as

$$\begin{aligned} c &= \alpha a + \beta b \\ d &= \gamma a + \delta b \end{aligned}$$

Then the cross-ratio

$$[a, b, c, d] = \frac{\gamma\beta}{\alpha\delta}$$

Proposition 3.6. *If* a*,* b*,* c*, and* d *are four colinear distinct points, then the following equalities hold*

$$\begin{aligned} [a, b, c, d] + [a, c, b, d] &= 1 \\ [b, a, c, d] &= [a, b, c, d]^{-1} \\ [a, b, d, c] &= [a, b, c, d]^{-1} \end{aligned}$$

Proof. Let f be the function that defines the cross-ratio, such that $[a, b, c, d] = f(d)$, and let f' be a function such that $f'(x) = 1 - f(x) \forall x$. The composite function $f' \circ f$ maps a to ∞, b to 0, and c to 1. But the function that defines the cross-ratio $[a, c, b, d]$ also maps a to ∞, b to 0, and c to 1. Since such function is unique,

$$\begin{aligned} [a, c, b, d] &= f' \circ f(d) \\ \implies [a, c, b, d] &= 1 - [a, b, c, d] \end{aligned}$$

Let g be a function such that $g(x) = \frac{1}{f(x)} \forall x$. The composite function $g \circ f$ maps a to 0, b to ∞, and c to 1. Thus it is the function that defines the cross-ratio $[b, a, c, d]$. That is,

$$\begin{aligned} [b, a, c, d] &= g \circ f(d) \\ \implies [b, a, c, d] &= \frac{1}{f(d)} \\ &= [a, b, c, d]^{-1} \end{aligned}$$

Let h be a function such that $h(x) = \frac{f(x)}{f(d)} \forall x$. The composite function $h \circ f$ maps d to 1, keeping th eimages of a and b unchanged. Hence, it is the defining function of the cross-ratio $[a, b, d, c]$. That is,

$$\begin{aligned} [a, b, d, c] &= h \circ f(c) \\ \implies [a, b, d, c] &= \frac{f(c)}{f(d)} \\ &= [a, b, c, d]^{-1} \end{aligned}$$

■

Remark. If $[a, b, c, d] = k$, the 24 cross-ratios obtained by permuting the four points take one of these six values:

$$k \;,\; \frac{1}{k} \;,\; 1-k \;,\; 1-\frac{1}{k} \;,\; \frac{1}{1-k} \;,\; \frac{k}{k-1}$$

▲▼▲

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