Topics in Number Theory

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Preface

Contents

Chapter 1

Arithmetical Functions

Definition 1.1. A function is said to be arithmetical, if $f : \mathbb{N} \to \mathbb{C}$.

- f is said to be additive if $f(mn) = f(m) + f(n) \forall m, n$ such that $(m, n) = 1$.
- f is said to be multiplicative if $f(mn) = f(m)f(n) \forall m, n$ such that $(m, n) = 1$.
- f is said to be completely additive or multiplicative if additive or multiplicative property holds for all $m, n \in \mathbb{N}$

Examples. Some arithmetical functions:

1. $\omega : \mathbb{N} \to \mathbb{C}, \omega(n) =$ #distincty prime factors of n.
   • Additive.
2. $\Omega : \mathbb{N} \to \mathbb{C}, \Omega(n) =$ #prime factors of n with multiplicity.
   • Completely additive.
3. $\log : \mathbb{N} \to \mathbb{C}$
   • Completely additive.
4. $\mu : \mathbb{N} \to \mathbb{C}$ - Mobius function

$$\mu(n) = \begin{cases} (-1)^{\omega(n)} & n \text{ is squarefree} \\ 0 & \text{otherwise} \end{cases}$$

   • Multiplicative.

5. $\Lambda : \mathbb{N} \to \mathbb{C}$ - von Mongoldt function

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^{\alpha} \text{ for some prime p} \\ 0 & \text{otherwise} \end{cases}$$

• Neither multiplicative nor additive.

6. $\lambda(n) : \mathbb{N} \to \mathbb{C}$ - Lioville's function

$$\lambda(n) = \begin{cases} 1 & n = 1 \\ (-1)^{\alpha_1+\alpha_2+\cdots+\alpha_k} & \alpha_i \text{ such that } n = \prod_{i\le k} p_i^{\alpha_i} \end{cases}$$

• Completely multiplicative.

7. $\varphi : \mathbb{N} \to \mathbb{C}$ - Euler's totient function

$$\begin{aligned} \varphi(n) &= |\{1 \le k \le n : (n,k) = 1\}| \\ &= n\prod_{p|n}\left(1 - \frac{1}{p}\right), \text{ and } \varphi(1) = 1. \end{aligned}$$

•$\varphi(mn) = \varphi(m)\varphi(n)\frac{(m,n)}{\varphi(m,n)}$

8. $f : \mathbb{N} \to \mathbb{C}$, $f(n) = n^{-s}$, $s \in \mathbb{C}$.
   • Completely multiplicative.

Theorem 1.1. *If* $n \ge 1$, $\sum_{d|n} \mu(d) = I(n) := \left[\frac{1}{n}\right]$.

Chapter 2

Asymptopes, Orders, and Inequalities