Topics in Number Theory

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January Semester 2026

Contents

Chapter 1

Elementary Methods

1.1 Arithmetic Functions

Definition 1.1. A function is said to be arithmetic, if $f : \mathbb{N} \to \mathbb{C}$.

- f is said to be additive if $f(mn) = f(m)+f(n)$ $\forall m, n$ such that $(m, n) = 1$.
- f is said to be multiplicative if $f(mn) = f(m)f(n)$ $\forall m, n$ such that $(m, n) = 1$.
- f is said to be completely additive or multiplicative if additive or multiplicative property holds for all $m, n \in \mathbb{N}$

Examples. Some arithmetic functions:

1. $\omega : \mathbb{N} \to \mathbb{C}$, $\omega(n) =$ #distincty prime factors of n.
   Additive.
2. $\Omega : \mathbb{N} \to \mathbb{C}$, $\Omega(n) =$ #prime factors of n with multiplicity.
   Completely additive.
3. $\log : \mathbb{N} \to \mathbb{C}$
   Completely additive.
4. $\mu : \mathbb{N} \to \mathbb{C}$ - Mobius function

   $$\mu(n) = \begin{cases} (-1)^{\omega(n)} & n \text{ is squarefree} \\ 0 & \text{otherwise} \end{cases}$$

   Multiplicative.

5. $\Lambda : \mathbb{N} \to \mathbb{C}$ - von Mongoldt function

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^{\alpha} \text{ for some prime } p \\ 0 & \text{otherwise} \end{cases}$$

Neither multiplicative nor additive.

6. $\lambda(n) : \mathbb{N} \to \mathbb{C}$ - Lioville's function

$$\lambda(n) = \begin{cases} 1 & n = 1 \\ (-1)^{\alpha_1 + \alpha_2 + \cdots + \alpha_k} & \alpha_i \text{ such that } n = \prod_{i \leqslant k} p_i^{\alpha_i} \end{cases}$$

Completely multiplicative.

7. $\varphi : \mathbb{N} \to \mathbb{C}$ - Euler's totient function

$$\begin{aligned} \varphi(n) &= |\{1 \leqslant k \leqslant n : (n,k) = 1\}| \\ &= n \prod_{p|n} \left(1 - \frac{1}{p}\right), \text{ and } \varphi(1) = 1. \end{aligned}$$

$\varphi(mn) = \varphi(m)\varphi(n)\frac{(m,n)}{\varphi(m,n)}$

8. $f : \mathbb{N} \to \mathbb{C}$, $f(n) = n^{-s}$, $s \in \mathbb{C}$.
Completely multiplicative.

Theorem 1.1. *If* $n \geqslant 1$, $\sum_{d|n} \mu(d) = I(n) := \left[\frac{1}{n}\right]$.

Proof. When $n = 1$, the summation is equal to $1 = I(1)$. Let $n = \prod_{i=1}^{m} p_i^{\alpha_i}$ for some $m \in \mathbb{N}$, and let $N = \prod_{i=1}^{m} p_i$. Now,

$$\begin{aligned} \sum_{d|n} \mu(d) = \sum_{d|N} \mu(d) &= 1 + \sum_{1 \leqslant i \leqslant m} \mu(p_i) + \sum_{1 \leqslant i,j \leqslant m} \mu(p_i p_j) + \cdots + \mu(N) \\ &= 1 - \binom{m}{1} + \binom{m}{2} - \cdots + (-1)^m \\ &= (1 + (-1))^m = 0 = I(n) \end{aligned}$$

■

Definition 1.2 (Number Field)**.** A finite field extension of Q is known as a number field.

Definition 1.3. An integer α is said to be algebraic if $f(\alpha) = 0$ for some monic irreducible $f \in \mathbb{Z}[x]$.

Definition 1.4. Given a number field K, the set

$$\theta_k := \{\alpha \in K : \alpha \text{ is an algebraic integer}\}$$

is known as its ring of integers.

Theorem 1.2. *Given $\zeta(s) > 0 \forall s > 1$, there are inifinetely many primes.*

Proof. By Euler product formula,

$$\begin{aligned} \log \zeta(s) &= \sum_p \log\left(1 - \frac{1}{p^s}\right) = \sum_p \sum_{n=1}^{\infty} \frac{1}{np^{ns}} \\ &= \sum_p \frac{1}{p^s} + \sum_p \sum_{n \geqslant 2} \frac{1}{np^{ns}} \end{aligned}$$

If $s > 1$,

$$\begin{aligned} \sum_p \sum_{n \geqslant 2} \frac{1}{np^{ns}} &\leqslant \sum_p \sum_{n \geqslant 2} \frac{1}{p^n} \\ &= \sum_p \frac{1}{p(p-1)} \leqslant \sum_p \frac{2}{p^2} < \infty \end{aligned}$$

$$\begin{aligned} &\implies \lim_{s \to 1^+} \log \zeta(s) = +\infty \\ &\implies \sum_p \frac{1}{p} = +\infty \end{aligned}$$

Hence, there are infintely many primes. ■

1.1.1 Infinite Products

1.2 Ordering of Arithmetical Functions

1.2.1 Summation Tools

1.2.2 Abel Summation

1.2.3 Euler-Maclaurin Summation

1.3 Prime Numbers

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Chapter 2

Analytic Theory

Chapter 3

Advanced Topics

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References

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