BeyondAI: Introduction to AI and Research programme

Problems to be submitted via Overleaf

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Problem 1: A remaining piece of cardboard has the form of a standard parabola. The length from the vertex to the edge is $24\,cm$. We wish to cut out a rectangle from this piece with the greatest possible area. What are the width and the height of this rectangle, and what is its area? Make sure to state all your assumptions carefully. Justify them.

Solution:
Length from the vertex to the edge = 24 cm
Vertex of the parabola at the origin is $(0, 0)$
Assuming the parabola is opening downward

Equation of the parabola :

$$y = -kx^2$$

where k is a positive constant.
When $x = 12$, $y = 0$:

$$0 = -k(12^2) \implies k = \frac{1}{144}$$

Thus, the equation of the parabola is:

$$y = -\frac{1}{144}x^2 + 24$$

Width of the rectangle = $2x$
Height of the rectangle = y
Area A of the rectangle :

$$A = \text{width} \times \text{height} = 2x \cdot y = 2x\left(-\frac{1}{144}x^2 + 24\right)$$

$$A = 2x\left(-\frac{1}{144}x^2 + 24\right) = -\frac{1}{72}x^3 + 48x$$

To find the maximum area, we will take the derivative of A with respect to x and set it to zero:

$$A' = -\frac{1}{24}x^2 + 48$$

Setting A' to zero:

$$-\frac{1}{24}x^2 + 48 = 0 \implies \frac{1}{24}x^2 = 48 \implies x^2 = 1152 \implies x = 24$$

Next, calculate the second derivative A'':

$$A'' = -\frac{1}{12}x$$

A'' at $x = 24$:

$$A''(24) = -\frac{1}{12}(24) = -2 < 0$$

Since $A'' < 0$, the area function has a maximum at $x = 24$

Width of the rectangle is:

$$\text{Width} = 2x = 2(24) = 48$$

Height of the rectangle is:

$$y = -\frac{1}{144}(24)^2 + 24 = -\frac{1}{144}(576) + 24 = -4 + 24 = 20$$

Area of the rectangle is:

$$A = 2x \cdot y = (48) \cdot 20 = 960$$

Answer:
Width: $48\,\text{cm}$
Height: $20\,\text{cm}$
Area: $960\,\text{cm}^2$

1. Find all the points on the surface that have the coordinates $x = 1$ and $y = 1$.
2. Choose one of the points and call it P. Find the tangent plane at P.

Problem 2: Consider the surface given by the equation

$$\frac{x^2}{4} + \frac{y^2}{2} + z^2 + xy^2z = 1$$

1. **Find all the points on the surface that have the coordinates $x = 1$ and $y = 1$.**
2. **Choose one of the points and call it P. Find the tangent plane at P.**

Solution:
1. We know, $x = 1$ and $y = 1$,

$$\frac{1^2}{4} + \frac{1^2}{2} + z^2 + (1)(1^2)z = 1$$

Gives,

$$\frac{1}{4} + \frac{1}{2} + z^2 + z = 1$$

Gives,

$$\frac{3}{4} + z^2 + z = 1 \implies z^2 + z = \frac{1}{4}$$

Gives,

$$z^2 + z - \frac{1}{4} = 0$$

Finding the roots using the formula,

$$z = \frac{-1 \pm \sqrt{1+1}}{2} = \frac{-1 \pm \sqrt{2}}{2}$$

Points on the surface are:

$$P_1 = \left(1, 1, \frac{-1+\sqrt{2}}{2}\right), \quad P_2 = \left(1, 1, \frac{-1-\sqrt{2}}{2}\right)$$

2. Let $F(x, y, z) = \frac{x^2}{4} + \frac{y^2}{2} + z^2 + xy^2z - 1$

After calculating,

$$F_x = \frac{x}{2} + y^2 z, \quad F_y = y + 2yz, \quad F_z = 2z + xy^2$$

P_1:
$F_x(1, 1, z_1) = \frac{1}{2} + \left(\frac{-1+\sqrt{2}}{2}\right) = \frac{1+\sqrt{2}}{2}$
$F_y(1, 1, z_1) = 1 + 2\left(\frac{-1+\sqrt{2}}{2}\right) = \sqrt{2}$
$F_z(1, 1, z_1) = 2\left(\frac{-1+\sqrt{2}}{2}\right) + 1 = \sqrt{2}$

$$\nabla F = \left(\frac{1+\sqrt{2}}{2}, \sqrt{2}, sqrt2\right)$$

Tangent plane equation at point $P_1(1, 1, z_1)$:

$$\frac{1+\sqrt{2}}{2}(x-1) + \sqrt{2}(y-1) + \sqrt{2}\left(z - \frac{-1+\sqrt{2}}{2}\right) = 0$$