Practical - 3

Solution of differential equations using method of variation of parameters.

Question - 1

Solve the following second order ODE using method of variation of parameters.

$$y'' + 9y = \text{Sec}(3t)$$

Solution :

The corresponding homogeneous equation is y'' + 9 y = 0 with general solution yh = c1 Cos3t + c2 Sin3t. Then, a fundamental set of solutions is S = {Cos3t, Sin3t} and W (S) = 3, as we see using Det, and Simplify.

In[4]:=
```
DSolve[y''[t] + 9 y[t] == 0, y[t], t]
```

Out[4]=
```
{{y[t] → C[1] Cos[3 t] + C[2] Sin[3 t]}}
```

In[5]:=
```
yh = c1 Cos[3 t] + c2 Sin[3 t]
```

Out[5]=
```
c1 Cos[3 t] + c2 Sin[3 t]
```

In[12]:=
```
fs = {Cos[3 t], Sin[3 t]};
wm = {fs, D[fs, t]};
wm // MatrixForm
wd = Simplify[Det[wm]]
```

Out[14]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[3t] & \text{Sin}[3t] \\ -3\,\text{Sin}[3t] & 3\,\text{Cos}[3t] \end{pmatrix}$$

Out[15]= 3

Now using the formulae :

$$u'_1 = -\frac{y2(t) * f(t)}{W(S)} \text{ and } u'_2 = -\frac{y1(t) * f(t)}{W(S)}$$

Here y1 (t) = Cos (3 t), y2 (t) = Sin (3 t), f (t) = Sec (3 t) and W (S) = 3.

In[11]:=
```
u1 = Integrate[-Sin[3 t] Sec[3 t] / 3, t]
u2 = Integrate[Cos[3 t] Sec[3 t] / 3, t]
```

Out[11]= $\frac{1}{9}\text{Log}[\text{Cos}[3t]]$

Out[12]= $\frac{t}{3}$

In[19]:=
```
yp = u1 * Cos[3 t] + u2 * Sin[3 t]
```

Out[19]= $\frac{1}{9}\text{Cos}[3t]\,\text{Log}[\text{Cos}[3t]] + \frac{1}{3}t\,\text{Sin}[3t]$

In[20]:=
```
y = yh + yp
```

Out[20]= $c1\,\text{Cos}[3t] + \frac{1}{9}\text{Cos}[3t]\,\text{Log}[\text{Cos}[3t]] + c2\,\text{Sin}[3t] + \frac{1}{3}t\,\text{Sin}[3t]$

Solution by DSolve function :

In[1]:=
```
DSolve[y''[t] + 9 * y[t] == Sec[3 * t], y[t], t]
```

Out[1]= $\left\{\left\{y[t] \to C[1]\,\text{Cos}[3t] + C[2]\,\text{Sin}[3t] + \frac{1}{9}(\text{Cos}[3t]\,\text{Log}[\text{Cos}[3t]] + 3t\,\text{Sin}[3t])\right\}\right\}$

Question – 2
Solve the following second order ODE using method of variation of paramters
y'' + y = Tan (x)

Solution :

The corresponding homogeneous equation is y'' + y = 0 with general solution yh = c1 Cosx + c2 Sinx. Then, a fundamental set of solutions is S = {Cosx, Sinx} and W (S) = 1, as we see using Det, and Simplify.

```
In[3]:= DSolve[y''[x] + y[x] == 0, y[x], x]

Out[3]= {{y[x] → C[1] Cos[x] + C[2] Sin[x]}}

In[4]:= yh = c1 Cos[x] + c2 Sin[x]

Out[4]= c1 Cos[x] + c2 Sin[x]

In[8]:= fs = {Cos[x], Sin[x]};
        wm = {fs, D[fs, x]};
        wm // MatrixForm
        wd = Simplify[Det[wm]]

Out[10]//MatrixForm=
```

$$\begin{pmatrix} \text{Cos}[x] & \text{Sin}[x] \\ -\text{Sin}[x] & \text{Cos}[x] \end{pmatrix}$$

```
Out[11]= 1
```

Now using the formulae :

$$u'_1 = -\frac{y2\,(t) * f\,(t)}{W\,(S)} \text{ and } u'_2 = -\frac{y1\,(t) * f\,(t)}{W\,(S)}$$

Here y1 (x) = Cos (x), y2 (x) = Sin (x), f (x) = Tan (x) and W (S) = 1.

```
In[3]:= u1 = Integrate[-Sin[x] Tan[x] / 1, x]
        u2 = Integrate[Cos[x] Tan[x] / 1, x]
```

Out[3]= $\text{Log}\left[\text{Cos}\left[\frac{x}{2}\right] - \text{Sin}\left[\frac{x}{2}\right]\right] - \text{Log}\left[\text{Cos}\left[\frac{x}{2}\right] + \text{Sin}\left[\frac{x}{2}\right]\right] + \text{Sin}[x]$

```
Out[4]= -Cos[x]

In[6]:= yp = u1 * Cos[x] + u2 * Sin[x]
```

Out[6]= $-\text{Cos}[x]\,\text{Sin}[x] + \text{Cos}[x]\left(\text{Log}\left[\text{Cos}\left[\frac{x}{2}\right] - \text{Sin}\left[\frac{x}{2}\right]\right] - \text{Log}\left[\text{Cos}\left[\frac{x}{2}\right] + \text{Sin}\left[\frac{x}{2}\right]\right] + \text{Sin}[x]\right)$

```
In[5]:= Cos[x] Log[Cos[x]] + x Sin[x]

Out[5]= Cos[x] Log[Cos[x]] + x Sin[x]

In[19]:= y = yh + yp
```

Out[19]= $\text{c1}\,\text{Cos}[x] + \frac{\text{Cos}[x]^3}{2} + \text{c2}\,\text{Sin}[x] + \text{Sin}[x]\left(\frac{x}{2} + \frac{1}{4}\text{Sin}[2\,x]\right)$

Solution by DSolve function :

```
In[1]:= DSolve[y''[x] + y[x] == Tan[x], y[x], x]
```

Out[1]= $\{\{y[x] \to C[1]\,\text{Cos}[x] + \text{Cos}[x]\,\text{Log}\left[\text{Cos}\left[\frac{x}{2}\right] - \text{Sin}\left[\frac{x}{2}\right]\right] - \text{Cos}[x]\,\text{Log}\left[\text{Cos}\left[\frac{x}{2}\right] + \text{Sin}\left[\frac{x}{2}\right]\right] + C[2]\,\text{Sin}[x]\}\}$

```
In[2]:= ClearAll

Out[2]= ClearAll
```

Question - 3

Solve the following second order ODE using the method of variation of parameters.

$y'' - 4y = xe^x$

Solution :

The corresponding homogeneous equation is $y'' - 4y = 0$ with general solution $yh = c1\,e^{2x} + c2\,e^{-2x}$. Then, a fundamental set of solutions is $S = \{e^{2x},\ e^{-2x}\}$ and $W(S) = -4$, as we see using Det, and Simplify.

In[3]:=
```
DSolve[y''[x] - 4 y[x] == 0, y[x], x]
```

Out[3]= $\{\{y[x] \to e^{2x} C[1] + e^{-2x} C[2]\}\}$

In[4]:=
```
yh = c1 Exp[2 x] + c2 Exp[-2 x]
```

Out[4]= $c2\, e^{-2x} + c1\, e^{2x}$

In[5]:=
```
fs = {Exp[2 x], Exp[-2 x]};
wm = {fs, D[fs, x]};
wm // MatrixForm
wd = Simplify[Det[wm]]
```

Out[7]//MatrixForm=

$$\begin{pmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{pmatrix}$$

Out[8]= -4

Now using the formulae :

$$u'_1 = -\frac{y2(t) * f(t)}{W(S)} \text{ and } u'_2 = -\frac{y1(t) * f(t)}{W(S)}$$

Here $y1(x) = e^{2x}$, $y2(x) = e^{-2x}$, $f(x) = xe^x$ and $W(S) = -4$.

In[9]:=
```
u1 = Integrate[-Exp[-2 x] * x * Exp[x] / (-4), x]
u2 = Integrate[Exp[2 x] * x * Exp[x] / (-4), x]
```

Out[9]= $\frac{1}{4} e^{-x} (-1 - x)$

Out[10]= $-\frac{1}{4} e^{3x} \left(-\frac{1}{9} + \frac{x}{3}\right)$

In[11]:=
```
yp = u1 Exp[2 x] + u2 Exp[-2 x]
```

Out[11]= $\frac{1}{4} e^{x} (-1 - x) - \frac{1}{4} e^{x} \left(-\frac{1}{9} + \frac{x}{3}\right)$

In[12]:=
```
y = yh + yp
```

Out[12]= $c2\, e^{-2x} + c1\, e^{2x} + \frac{1}{4} e^{x} (-1 - x) - \frac{1}{4} e^{x} \left(-\frac{1}{9} + \frac{x}{3}\right)$

In[13]:=
```
Simplify[y]
```

Out[13]= $\frac{1}{9} e^{-2x} \left(9\, c2 + e^{3x} (-2 + 9\, c1\, e^{x} - 3x)\right)$

Solution by DSolve function :

In[1]:=
```
DSolve[y''[x] - 4 * y[x] == x * Exp[x], y[x], x]
```

Out[1]= $\left\{\left\{y[x] \to -\frac{1}{9} e^{x} (2 + 3 x) + e^{2 x} C[1] + e^{-2 x} C[2]\right\}\right\}$

Question - 4

Solve the following second order ODE using the method of variation of parameters.

$$y'' + 3 y' + 2 y = 4 e^{x}$$

Solution :

The corresponding homogeneous equation is $y'' + 3 y' + 2 y = 0$ with general solution $yh = c1\, e^{2 x} + c2\, e^{-x}$. Then, a fundamental set of solutions is $S = \{e^{-2 x}, e^{-x}\}$ and $W(S) = e^{-3 x}$, as we see using Det, and Simplify.

In[2]:=
```
DSolve[y''[x] + 3 * y'[x] + 2 * y[x] == 0, y[x], x]
```

Out[2]= $\left\{\left\{y[x] \to e^{-2 x} C[1] + e^{-x} C[2]\right\}\right\}$

In[17]:=
```
yh = c1 Exp[-2 x] + c2 Exp[-x]
```

Out[17]= $c1\, e^{-2 x} + c2\, e^{-x}$

In[18]:=
```
fs = {Exp[-2 x], Exp[-x]};
wm = {fs, D[fs, x]};
wm // MatrixForm
wd = Simplify[Det[wm]]
```

Out[20]//MatrixForm=

$$\begin{pmatrix} e^{-2 x} & e^{-x} \\ -2 e^{-2 x} & -e^{-x} \end{pmatrix}$$

Out[21]= $e^{-3 x}$

Now using the formulae :

$$u'_1 = -\frac{y2(t) * f(t)}{W(S)} \text{ and } u'_2 = -\frac{y1(t) * f(t)}{W(S)}$$

Here $y1(x) = e^{-x}$, $y2(x) = e^{-2 x}$, $f(x) = 4 e^{x}$ and $W(S) = -e^{-3 x}$.

In[22]:=
```
u1 = Integrate[-Exp[-x] * 4 * Exp[x] / -Exp[-3 * x], x]
u2 = Integrate[Exp[-2 * x] * 4 * Exp[x] / -Exp[-3 * x], x]
```

Out[22]= $\frac{4 e^{3 x}}{3}$

Out[23]= $-2 e^{2 x}$

In[24]:=
```
yp = u1 Exp[-2 x] + u2 Exp[-x]
```

Out[24]= $-\frac{2 e^{x}}{3}$

In[28]:=
```
y = yh + yp
```

Out[28]= $c1\, e^{-2 x} + c2\, e^{-x} - \frac{2 e^{x}}{3}$

In[27]:=

Solution by DSolve function :

In[1]:= `DSolve[y''[x] + 3 y'[x] + 2 y[x] == 4 * Exp[x], y[x], x]`

Out[1]= $\left\{\left\{y[x] \to \frac{2 e^{x}}{3} + e^{-2x} C[1] + e^{-x} C[2]\right\}\right\}$

In[2]:= `ClearAll`

Out[2]= ClearAll

Question – 5

Solve the following second order ODE using the method of variation of parameters.

$y'' + 4y = \cos(x)$

Solution :

The corresponding homogeneous equation is $y'' + y = 0$ with general solution yh = c1 Cosx + c2 Sinx. Then, a fundamental set of solutions is S = {Cosx, Sinx} and W (S) = 1, as we see using Det, and Simplify.

In[3]:= `DSolve[y''[x] + 4 * y[x] == 0, y[x], x]`

Out[3]= {{y[x] → C[1] Cos[2 x] + C[2] Sin[2 x]}}

In[9]:= `yh = c1 Cos[2 x] + c2 Sin[2 x]`

Out[9]= c1 Cos[2 x] + c2 Sin[2 x]

In[10]:=
```
fs = {Cos[2 x], Sin[2 x]};
wm = {fs, D[fs, x]};
wm // MatrixForm
wd = Simplify[Det[wm]]
```

Out[12]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[2x] & \mathrm{Sin}[2x] \\ -2\,\mathrm{Sin}[2x] & 2\,\mathrm{Cos}[2x] \end{pmatrix}$$

Out[13]= 2

Now using the formulae :

$$u'_1 = -\frac{y2(t) * f(t)}{W(S)} \text{ and } u'_2 = -\frac{y1(t) * f(t)}{W(S)}$$

Here y1 (x) = Cos (2 x), y2 (x) = Sin (2 x), f (x) = Cos (x) and W (S) = 2.

In[14]:=
```
u1 = Integrate[-Sin[2 x] Cos[x] / 2, x]
u2 = Integrate[Cos[2 x] Cos[x] / 2, x]
```

Out[14]= $\frac{\mathrm{Cos}[x]^3}{3}$

Out[15]= $\frac{1}{2}\left(\frac{\mathrm{Sin}[x]}{2} + \frac{1}{6}\mathrm{Sin}[3x]\right)$

In[16]:= `yp = u1 * Cos[2 x] + u2 * Sin[2 x]`

Out[16]= $\frac{1}{3}\mathrm{Cos}[x]^3\,\mathrm{Cos}[2x] + \frac{1}{2}\mathrm{Sin}[2x]\left(\frac{\mathrm{Sin}[x]}{2} + \frac{1}{6}\mathrm{Sin}[3x]\right)$

In[17]:= `y = yh + yp`

Out[17]= $c1\,\mathrm{Cos}[2x] + \frac{1}{3}\mathrm{Cos}[x]^3\,\mathrm{Cos}[2x] + c2\,\mathrm{Sin}[2x] + \frac{1}{2}\mathrm{Sin}[2x]\left(\frac{\mathrm{Sin}[x]}{2} + \frac{1}{6}\mathrm{Sin}[3x]\right)$

Solution by DSolve function :

In[1]:=
```
DSolve[y''[x] + 4 * y[x] == Cos[x], y[x], x]
```

Out[1]=

$$\left\{\left\{y[x] \to C[1]\,\text{Cos}[2\,x] + C[2]\,\text{Sin}[2\,x] + \frac{1}{12}\left(4\,\text{Cos}[x]^3\,\text{Cos}[2\,x] + 3\,\text{Sin}[x]\,\text{Sin}[2\,x] + \text{Sin}[2\,x]\,\text{Sin}[3\,x]\right)\right\}\right\}$$