7

Interactions Among Continuous Variables

7.1 INTRODUCTION

In this chapter we extend MR analysis to interactions among continuous predictors. By *interactions* we mean an interplay among predictors that produces an effect on the outcome Y that is different from the sum of the effects of the individual predictors. Many theories in the social sciences hypothesize that two or more continuous variables interact; it is safe to say that the testing of interactions is at the very heart of theory testing in the social sciences. Consider as an example how ability (X) and motivation (Z) impact achievement in graduate school (Y). One possibility is that their effects are additive. The combined impact of ability and motivation on achievement equals the sum of their separate effects; there is no interaction between X and Z. We might say that the whole equals the sum of the parts. A second alternative is that ability and motivation may interact synergistically, such that graduate students with both high ability and high motivation achieve much more in graduate school than would be expected from the simple sum of the separate effects of ability and motivation. Graduate students with both high ability and high motivation become "superstars"; we would say that the whole is greater than the sum of the parts. A third alternative is that ability and motivation compensate for one another. For those students who are extremely high in ability, motivation is less important to achievement, whereas for students highest in motivation, sheer native ability has less impact. Here we would say that the whole is less than the sum of the parts; there is some partial trade-off between ability and motivation in the prediction of achievement. The second and third alternatives exemplify interactions between predictors, that is, combined effects of predictors that differ from the sum of their separate effects.

When two predictors in regression analysis interact with one another, the regression of Y on one of those predictors *depends on* or is *conditional on* the value of the other predictor. In the second alternative, a *synergistic interaction* between ability X and motivation Z, the regression coefficient for the regression of achievement Y on ability X increases as motivation Z increases. Under the synergistic model, when motivation is very low, ability has little effect because the student is hardly engaged in the graduate school enterprise. When motivation is higher, then more able students exhibit greater achievement.

Continuous variable interactions such as those portrayed in alternatives two and three can be tested in MR analysis, treating both the original variables and their interaction as continuous predictors. In this chapter we explore how to specify interactions between continuous variables in multiple regression equations, how to test for the statistical signficance of interactions, how to plot them, and how to interpret them through post hoc probing.

We suspect that some readers are familiar with the testing, plotting, post hoc probing, and interpretation of interactions between categorical variables in the analysis of variance (ANOVA) context. Historically, continuous variable interactions have often been analyzed by breaking the continuous variables into categories, so that interactions between them can be examined in ANOVA. For example, an analyst might perform median splits on ability and motivation to create four combinations (hi-hi, hi-lo, lo-hi, and lo-lo) of ability and motivation that could be examined in a 2×2 ANOVA. *This dichotomization strategy is ill-advised, and we strongly recommend against it.* The strategy evolved because methods were fully developed for probing interactions in ANOVA long before they were fully developed in MR. Dichotomization is problematic first because it decreases measured relationships between variables. For example, dichotomization at the median of a single continuous normally distributed predictor X reduces its squared correlation with a normally distributed dependent variable Y to .64 of the original correlation (Cohen, 1983). Dichotomization of a single predictor is equivalent to throwing out over a third of the cases in the data set. Dichotomization of two continuous variables X and Z so that their interaction can be examined in ANOVA lowers the power for detecting a true nonzero interaction between the two continuous predictors. As Maxwell and Delaney (1993) point out, if loss of power were the only impact of dichotomization and researchers found significance nonetheless after dichotomization, the practice might not seem so undesirable from a theoretical standpoint. But Maxwell and Delaney (1993) show much more deleterious effects from a validity standpoint. Carrying out median splits on two continuous predictors X and Z can produce spurious main effects, that is, effects of the individual predictors that are "significant" when the dichotomized data are analyzed, although the effects do not, in fact, exist in the population. Moreover, in one special circumstance in which there is no true interaction between two continuous predictors X and Z, a spurious interaction may be produced between the dichotomized predictors. This can happen if one of the predictors X or Z has a quadratic relationship to Y.

In this chapter we provide prescriptions for specifying, plotting, testing, post hoc probing, and interpretating interactions among continuous variables. In Chapter 8, we introduce the implementation of true categorical predictors (e.g., gender, ethnicity) in MR. In Chapter 9, we extend MR to interactions among categorical variables and between categorical and continuous variables.

7.1.1 Interactions Versus Additive Effects

Regression equations that contain as IVs only predictors taken separately signify that the effects of continuous variables such as X and Z are *additive* in their impact on the criterion, that is,

$$\hat{Y} = B_1X + B_2Z + B_0. \tag{7.1.1}$$

Note that Eq. (7.1.1) is the same equation as Eq. (3.2.1), except that the notation X and Z has been substituted for X_1 and X_2, respectively.

For a specific instance, consider the following numerical example:

$$\hat{Y} = .2X + .6Z + 2.$$

The estimated DV increases .2 points for each 1-point increase in X and another .6 points for each 1-point increase in Z. (Strictly speaking, this is correct only if X and Z are uncorrelated. If they are correlated, these effects hold only when the two IVs are used together to estimate Y.) The effects of X and Z are additive. By *additivity* is meant that the regression of the criterion on one predictor, say predictor X, is constant over all values of the other predictor Z.

Interactions as Joint Effects

In Eq. (7.1.2) we add a predictor XZ to carry an interaction between X and Z:

$$\hat{Y} = B_1X + B_2Z + B_3XZ + B_0. \tag{7.1.2}$$

Literally, the predictor is the product of scores on predictors X and Z, calculated for each case. While the interaction is carried by the XZ product term, the interaction itself is actually that part of XZ that is independent of X and Z, from which X and Z have been partialed (more about this in Section 7.6).

Consider our numerical example, but with the product term added:

$$\hat{Y} = .2X + .6Z + .4XZ + 2.$$

If X and Z are uncorrelated, the criterion Y increases .2 points for each 1-point increase in X and an additional .6 points for each 1-point increase in Z. Moreover, the criterion Y increases an additional .4 points for a 1-point increment in the part of the cross-product XZ that is independent of X and Z. The partialed component of the cross-product represents a unique combined effect of the two variables working together, above and beyond their separate effects; here a *synergistic* effect, as in the example of ability X and motivation Z as predictors of graduate school achievement Y. Thus two variables X and Z are said to interact in their accounting for variance in Y when *over and above* any additive combination of their separate effects, they have a *joint effect.*

We can compare the joint or interactive effect of X and Z with the simple additive effects of X and Z in three-dimensional graphs. For data, we plot 36 cases for which we have scores on predictors X and Z (see Table 7.1.1A). Both X and Z take on the values 0, 2, 4, 6, 8, and 10; the 36 cases were created by forming every possible combination of one X value and one Z value. This method of creating cases makes X and Z uniformly distributed, that is, produces an equal number of scores at each value of X and Z. The method also assures that X and Z are uncorrelated. These special properties facilitate the example but are not at all necessary or typical for the inclusion of interactions in MR equations. The means and standard deviations of X and Z, as well as their correlations with Y, are given in Table 7.1.1B.

Figure 7.1.1(A) illustrates the additive effects (absent any interaction) of X and Z from the equation $\hat{Y} = .2X + .6Z + 2$. Predictors X and Z form the axes on the floor of the graph; all 36 cases (i.e., points representing combinations of values of the predictors) lie on the floor. Predicted $\hat{Y}$s for each case (unique combinations of X and Z) were generated from the regression equation. The *regression plane*, the tilted plane above the floor, represents the location of $\hat{Y}$ for every possible combination of values of X and Z. Note that the regression plane is a flat surface. Regardless of the particular combination of values of X and Z, the $\hat{Y}$ is incremented (geometrically raised off the floor) by a constant value relative to the values of X and Z, that is, by the value $(.2X + .6Z)$.

The regression plane in Fig. 7.1.1(B) illustrates the additive effects of X and Z plus the interaction between X and Z in the equation $\hat{Y} = .2X + .6Z + .4XZ + 2$. The same 36 combinations of X and Z were used again. However, $\hat{Y}$s were generated from the equation containing the interaction. Table 7.1.1B gives the mean and standard deviation of the product term that carries the interaction, and its correlation with the criterion Y. In Fig. 7.1.1(B) the regression plane is now a stretched surface, pulled up in the corner above the height of the

TABLE 7.1.1
Multiple Regression Equations Containing Interactions: Uncentered Versus Centered Predictors

A. Thirty-six cases generated from every possible combination of scores on predictors X and Z.

$$X\ (0, 2, 4, 6, 8, 10)$$
$$Z\ (0, 2, 4, 6, 8, 10)$$

Cases (X, Z combinations)

$$(0,0), (0,2), \ldots, (4,6), \ldots, (6,8), \ldots, (10,10)$$

B. Summary Statistics for X, Z, and XZ (uncentered, in raw score form).

Means and standard deviations			Correlation matrix				
	M	sd		X	Z	XZ	Y
X	5.000	3.464	X	1.00	0.00	.637	.600
Z	5.000	3.464	Z		1.00	.637	.709
XZ	25.000	27.203	XZ			1.00	.995

C. Unstandardized regression equations: prediction of Y from X and Z, and from X, Z, and XZ (uncentered, in raw score form).

1. Uncentered regression equation, no interaction:

$$\hat{Y} = .2X + .6Z + 2$$

2. Uncentered regression equation, with interaction:

$$\hat{Y} = .2X + .6Z + .4XZ + 2$$

D. Simple regression equations for Y on X at values of Z with uncentered predictors and criterion.

$$\text{At } Z_{\text{high}}: \quad \hat{Y} = 3.4X + 6.8$$
$$\text{At } Z_{\text{mean}}: \quad \hat{Y} = 2.2X + 5.0$$
$$\text{At } Z_{\text{low}}: \quad \hat{Y} = 1.0X + 3.2$$

E. Summary statistics for x, z and xz (centered, in deviation form).

Means and standard deviations			Correlation matrix				
	M	sd		x	z	xz	Y
x	0.000	3.464	x	1.00	.000	.000	.600
z	0.000	3.464	z		1.00	.000	.709
xz	0.000	11.832	xz			1.00	.372

F. Unstandardized regression equations: prediction of Y from x and z, and from x, z, and xz (centered, in deviation form).

1. Centered regression equation, no interaction:

$$\hat{Y} = .2x + .6z + 6$$

2. Centered regression equation, with interaction:

$$\hat{Y} = 2.2x + 2.6z + .4xz + 16$$

G. Simple regression equations for Y on x at values of z with centered predictors and criterion.

$$\text{At } z_{\text{high}}: \quad \hat{Y} = 3.4x + 23.8$$
$$\text{At } z_{\text{mean}}: \quad \hat{Y} = 2.2x + 16.0$$
$$\text{At } z_{\text{low}}: \quad \hat{Y} = 1.0x + 8.2$$

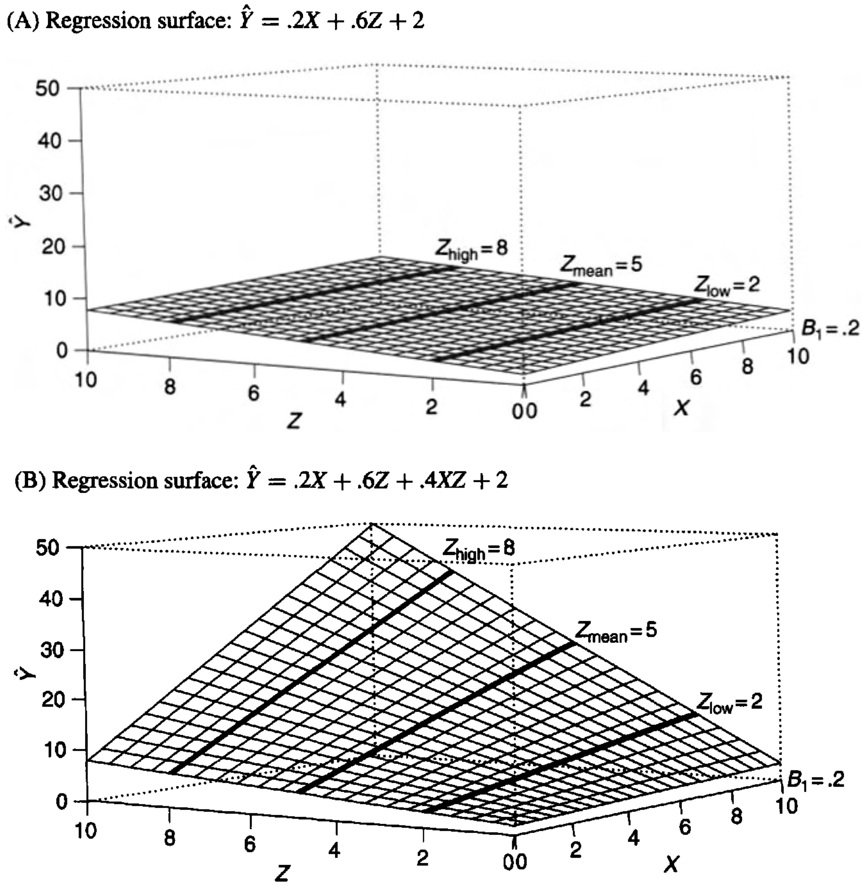


FIGURE 7.1.1 Regression surface predicated in (A) an additive regression equation containing no interaction and (B) a regression equation containing an interaction. Predictors and criterion are in raw score (uncentered) form.

flat regression plane in Fig. 7.1.1(A). The amount by which the stretched surface is lifted above the flat regression plane represents unique variance due to the interaction of X and Z, over and above the individual additive effects of X and Z. What is the source of the upward stretching? The stretching occurs because the increment in Y depends not only on additive values of X and Z but also on their product XZ, and the product XZ increases in a curvilinear fashion as X and Z increase linearly. Note the dramatic rise in the product XZ relative to the sum $X + Z$:

X	0	2	4	6	8	10
Z	0	2	4	6	8	10
$X+Z$	0	4	8	12	16	20
XZ	0	4	16	36	64	100

7.1.2 Conditional First-Order Effects in Equations Containing Interactions

As in polynomial regression explained in Chapter 6, we make the distinction between *first-order effects* and *higher order effects* in regression equations containing interactions. First-order effects refer to the effects of the individual predictors on the criterion. Higher order effects refer to the partialed effects of multiplicative functions of the individual predictors, for example the XZ term with X and Z partialed out in Eq. (7.1.2).

When the effects of individual predictors are purely additive, as in Eq. (7.1.1), the first-order regression coefficient for each predictor is constant over all values of the other predictor (again, this is the definition of *additivity*). The constancy is illustrated in Fig. 7.1.1(A). In Fig. 7.1.1(A), three lines on the regression plane defined by $\hat{Y} = .2X + .6Z + 2$ are darkened: at $Z = 2, Z = 5$ (i.e., the mean of Z, M_Z) and $Z = 8$. These lines show the regression of Y on X at each of these three values of Z: $Z_{\text{low}}, Z_{\text{mean}}$, and Z_{high}, respectively. These three regression lines are parallel, signifying that the regression of Y on X is constant over values of Z. Thus the regression coefficient for the X predictor applies equally across the range of Z. The only characteristic that varies across the three regression lines is the overall height of the regression line (distance from the floor of the graph). The displacement upward of the lines as Z increases signifies that as Z increases, the criterion Y increases as well (a first-order effect). On average, values of Y are higher for higher values of Z.

Figure 7.1.1(B) represents the regression equation $\hat{Y} = .2X + .6Z + .4XZ + 2$. Regression lines for the regression of Y on X are drawn at the same three values of Z as in Fig. 7.1.1(A): $Z = 2, 5, 8$. We see immediately that the regression line for Y on X becomes steeper as Z increases. The regression of Y on X is not constant over all values of Z but depends specifically on the particular value of Z at which the regression of Y on X is taken. Predictors X and Z are no longer additive in their effects on Y; they are interactive. The regression of Y on X is *conditional upon* (i.e., depends upon) the value of Z. In regression equations containing interactions, the *first-order effects* of variables are conditional on (depend upon, or are *moderated by*) the values of the other predictors with which they interact.

We have cast this discussion of *conditional effects* in terms of the regression of Y on X at values of Z. However, the interaction between X and Z is symmetric. We could examine the regression of Y on Z at values of X. The result would be the same: the regression of Y on Z would differ as a function of X; that is, the regression of Y on Z is again conditional upon the value of X.

Now we focus on the angle formed between the regression plane and the floor of Fig. 7.1.1(A). This angle is best seen at the right edge of Fig. 7.1.1(A) (predictor X), where $Z = 0$. In Fig. 7.1.1(A), with no interaction, the slope of the regression of Y on X equals .2 at $Z = 0$. Recall that .2 is the regression coefficient for Y on X in Eq. (7.1.1). This same angle is maintained across the range of Z, which is another way of saying that the regression of Y on X is constant across all values of Z, meeting the definition of additivity.

Examine the right edge of Fig. 7.1.1(B) (predictor X), where $Z = 0$. The regression of Y on X also equals .2 at $Z = 0$ in Fig. 7.1.1(B), and the regression coefficient B_1 for Y on X in our numerical example containing an interaction is .2. However, in Fig. 7.1.1(B), the slope of the regression of Y on X is only .2 at $Z = 0$. As Z increases, the slope of Y on X also increases. Thus the numerical value of the regression coefficient $B_1 = .2$ is only an accurate representation of the regression of Y on X at one point on the regression plane. In general, in a regression equation containing an interaction, the first-order regression coefficient for each predictor involved in the interaction represents the regression of Y on that predictor, *only at the value of zero on all other individual predictors with which the predictor interacts*. The first-order coefficients have different meanings depending on whether the regression equation does or does not include interactions. To reiterate, without an interaction term the B_1 coefficient for X represents the overall effect of X on Y across the full range of Z. However, in Eq. (7.1.2), the B_1 coefficient for X represents the effect of X on the criterion only at $Z = 0$.

7.2 CENTERING PREDICTORS AND THE INTERPRETATION OF REGRESSION COEFFICIENTS IN EQUATIONS CONTAINING INTERACTIONS

The interpretation of the first-order coefficients B_1 and B_2 in the presence of interactions is usually problematic in typical social science data. The B_1 coefficient represents the regression of Y on X at $Z = 0$, and the B_2 coefficient represents the regression of Y on Z at $X = 0$. Only rarely in the social sciences is zero a meaningful point on a scale. For example, suppose, in a developmental psychology study, we predict a level of language development (Y) of children aged 2 to 6 years from mother's language development (D), child's age (A), and the interaction of mother's language development and child's age, carried by the DA term. In the regression equation $\hat{Y} = B_1D + B_2A + B_3DA + B_0$, the regression coefficient B_1 of child's language development on mother's language development D is at child's age $A = 0$, not a useful value in that all children in the study fall between ages 2 and 6. To interpret this B_1 coefficient, we would have to extrapolate from our sample to newborns in whom the process of language development has not yet begun. (Our comments about the dangers of extrapolation in Section 6.2.5 apply here as well.)

7.2.1 Regression With Centered Predictors

We can make a simple linear transformation of the age predictor that renders zero on the age scale meaningful. Simply, we *center* age, that is, put age in deviation form by subtracting M_A from each observed age (i.e., $a = A - M_A$). If age were symmetrically distributed over the values 2, 3, 4, 5, and 6 years, $M_A = 4$ years, and the centered age variable a would take on the values $-2, -1, 0, 1, 2$. The mean of the centered age variable a necessarily would be zero. When a is used in the regression equation $\hat{Y} = B_1D + B_2a + B_3Da + B_0$, the B_1 coefficient represents the regression of child's language development on mother's language development at the mean age of the children in the sample. This strategy of centering to make the regression coefficients of first-order terms meaningful is identical to the use of centering in polynomial regression (Section 6.2.3.).

The symmetry in interactions applies to centering predictors. If we center mother's language development into variable $d = D - M_D$ and estimate the regression equation $\hat{Y} = B_1d + B_2A + B_3dA + B_0$, then the B_2 coefficient represents the regression of child's language development on child's age at the mean of mother's language development in the sample.

Finally, suppose we wish to assess the interaction between age and mother's language development. We center both predictors and form the product of the centered variables da to carry the interaction and estimate the regression equation $\hat{Y} = B_1d + B_2a + B_3da + B_0$. Both the B_1 and B_2 coefficients represent the first-order relationships at the *centroid* (mean on both predictors) of the sample. The regression equation characterizes the typical case. In sum, if all the predictors in a regression equation containing interactions are centered, then each first-order coefficient has an interpretation that is meaningful in terms of the variables under investigation: the regression of the criterion on the predictor at the sample means of all other variables in the equation.

With centered predictors, each first-order regression coefficient has yet a second meaningful interpretation, as the *average regression* of the criterion on the predictor across the range of the other predictors. In the developmental study, if the d by a interaction were nonzero, then the regression of child's language development on mother's language development would differ at each age. Assume that there were an equal number of children at each age. Imagine computing the regression coefficient B_1 of child's language development on mother's language development separately at each age and then averaging all these B_1 coefficients. The B_1 coefficient for the impact of mother's language development in the overall centered regression equation containing all ages would equal the average of the individual B_1 coefficients at each child's age. If there were an unequal number of children at each age, then the overall B_1 coefficient would equal the weighted average of the individual B_1 coefficients, where the weights were the number of children at each age. In sum, when predictors are centered, then each first-order coefficient in a regression equation containing interactions is the *average regression of the criterion on a predictor* across the range of the other predictors in the equation.

7.2.2 Relationship Between Regression Coefficients in the Uncentered and Centered Equations

As noted in Chapter 2, correlational properties of variables do not change under linear transformation of variables. Linear transformations include adding or subtracting constants, and multiplying and dividing by constants. If we correlate height in inches with weight in pounds, we obtain the same value as if we correlate height in inches with weight in ounces or kilograms. *Centering*, or putting predictors in deviation score form by subtracting the mean of the predictor from each score on the predictor, is a linear transformation. *Thus our first intuition might be that if predictors were centered before they were entered into a regression equation, the resulting regression coefficients would equal those from the uncentered equation. This intuition is correct only for regression equations that contain no interactions.*

As we have seen, centering predictors provides tremendous interpretational advantages in regression equations containing interactions, but centering produces a very puzzling effect. When predictors are centered and entered into regression equations containing interactions, the regression coefficients for the first-order effects B_1 and B_2 are different numerically from those we obtain performing a regression analysis on the same data in raw score or *uncentered* form. We encountered an analogous phenomenon in Chapter 6 in polynomial regression; when we centered the predictor X, the regression coefficient for all but the highest order polynomial term changed (see Section 6.2.3). The explanation of this phenomenon is straightforward and is easily grasped from three-dimensional representations of interactions such as Fig. 7.1.1(B). An understanding of the phenomenon provides insight into the meaning of regression coefficients in regression equations containing interactions.

7.2.3 Centered Equations With No Interaction

We return to the numerical example in Table 7.1.1 and Fig. 7.1.1. The means of both predictors X and Z equal 5.00. Uncentered and centered X and Z would be as follows:

$X_{\text{uncentered}}$	0	2	4	6	8	10
x_{centered}	−5	−3	−1	1	3	5

and

$Z_{\text{uncentered}}$	0	2	4	6	8	10
z_{centered}	−5	−3	−1	1	3	5

Now, assume that we keep the criterion Y in its original uncentered metric, but we use x and z, and re-estimate the regression equation without an interaction. The resulting regression equation is

$$\hat{Y} = .2x + .6z + 6.$$

The regression coefficients for x and z equal those for uncentered X and Z. Only the regression intercept has changed. From Chapter 3, Eq. (3.2.6), the intercept is given as

$B_0 = M_Y - B_1M_X - B_2M_Z$. Centering X and Z changed their means from 5.00 to 0.00, leading to the change in B_0. In fact, there is a simple algebraic relationship between B_0 in the centered versus uncentered equations. For the uncentered regression equation $\hat{Y} = B_1X + B_2Z + B_0$ versus the centered regression equation $\hat{Y} = B_1x + B_2z + B_0$,

$$B_{0,\text{centered}} = B_{0,\text{uncentered}} + B_{1,\text{uncentered}}M_{X,\text{uncentered}} + B_{2,\text{uncentered}}M_{Z,\text{uncentered}} \tag{7.2.1}$$

For our example, this is

$$B_{0,\text{centered}} = 2 + .2(5.00) + .6(5.00) = 6.$$

The centered regression equation is plotted in Fig. 7.2.1(A). The only difference between Fig. 7.1.1(A) and Fig. 7.2.1(A) is that the scales of the X and Z axes in Fig. 7.2.1(A) have been changed from those in Fig. 7.1.1(A) to reflect centering. Note that $x = 0$ and $z = 0$ in Fig. 7.2.1(A) are now in the *middle of the axes*, rather than at one end of the axes, as in Fig. 7.1.1(A). Note also that the criterion Y is left uncentered.

Figure 7.2.1(A) confirms the numerical result that the regression coefficients B_1 and B_2 do not change when we center predictors in regression equations containing no interactions.

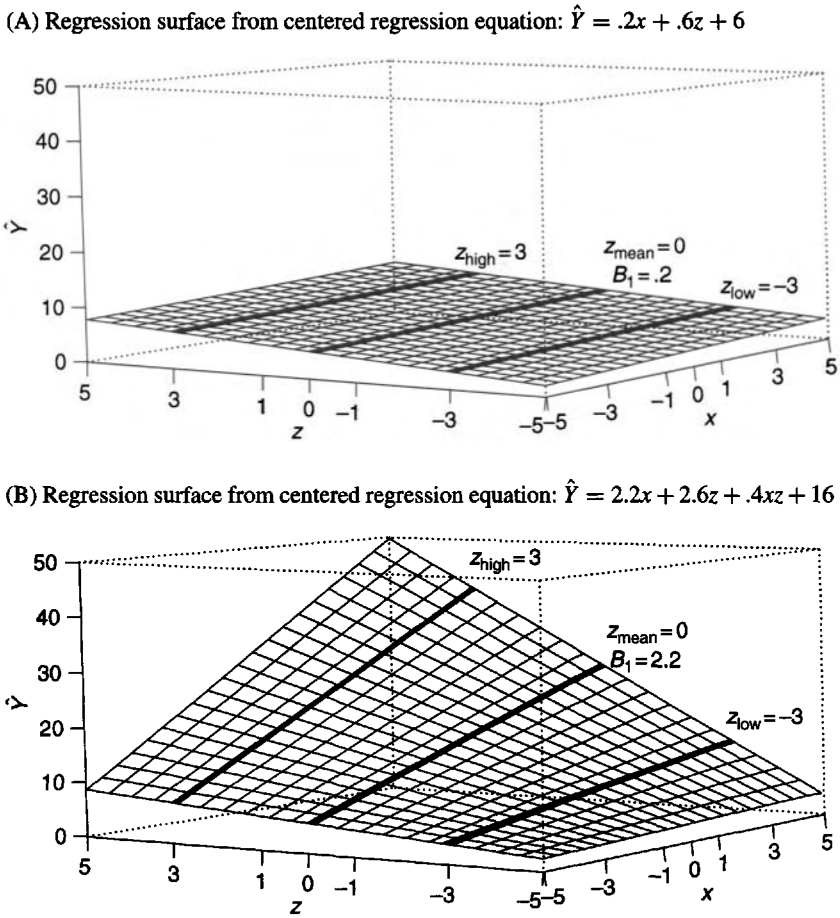


FIGURE 7.2.1 Regression surface predicted in (A) an additive regression equation containing no interaction and (B) a regression equation containing an interaction. Predictors are in centered (deviation) form.

Consider B_1; the slope of Y on X at $Z = 0$ in Fig. 7.1.1(A) is the same as that for Y on x at $z = 0$ in Figure 7.2.1(A), though the location of $Z = 0$ differs across the figures.

A comparison of Figures 7.1.1(A) and 7.2.1(A) also confirms the change in intercept. The intercept is the height of the regression plane from the floor at the point $X = 0, Z = 0$. In Fig. 7.1.1(A) for the uncentered equation, this point is in the lower right-hand corner of the plane; here the plane is only two units from the floor, so $B_0 = 2$. As pointed out earlier, in Fig. 7.2.1(A) for the centered equation, the point $x = 0, z = 0$ is now in the center of the regression plane. (When all predictors have been centered, the value 0, 0 is the *centroid* of the predictor space.) The overall elevation of the plane is farther from the floor at this point, specifically six units from the floor, so $B_0 = 6$. The change in the location of the point $X = 0, Z = 0$ produced by centering also produces the change in the intercept.

7.2.4 Essential Versus Nonessential Multicollinearity

The correlation matrix among the centered predictors, including xz, and of the centered predictors with the criterion is given in Table 7.1.1E. This correlation matrix should be compared with that in Table 7.1.1B for the uncentered predictors. There is one dramatic change when predictors are uncentered versus centered. The correlations of the X and Z terms with XZ ($r = .637$ in each case) are substantial in the uncentered case but fall to zero in the centered case. This drop is another example of *essential* versus *nonessential multicollinearity* (Marquardt, 1980), also encountered in our work with polynomial regression equations (Section 6.2.3).

Algebraically, the covariance (numerator of the correlation coefficient) between X and XZ is in part a function of the arithmetic means of X and Z. If X and Z are each completely symmetrical, as in our numerical example, then the covariance (cov) between X and XZ is as follows (Aiken & West, 1991, p. 180, eq. A.15):

$$\text{cov}(XZ, X) = sd_X^2 M_Z + \text{cov}(X, Z) M_X$$

If X and Z are centered, then M_X and M_Z are both zero, and the covariance between x and xz is zero as well. Thus the correlation between x and xz is also zero. The same holds for the correlation between z and xz. The amount of correlation that is produced between X and XZ or Z and XZ by the nonzero means of X and Z, respectively, is referred to as *nonessential multicollinearity* (Marquardt, 1980). This nonessential multicollinearity is due purely to scaling—when variables are centered, it disappears. The amount of correlation between X and XZ that is due to skew in X cannot be removed by centering. This source of correlation between X and XZ is termed *essential multicollinearity*. The same is true for the correlation between Z and XZ.

7.2.5 Centered Equations With Interactions

Now consider the use of centered predictors in an equation containing an interaction. How do the coefficients of the uncentered equation $\hat{Y} = B_1X + B_2Z + B_3XZ + B_0$ relate to those in the centered equation $\hat{Y} = B_1x + B_2z + B_3xz + B_0$? For the same reason as in the equation without an interaction, the intercept changes here. However, B_1 and B_2 also change, often dramatically, in association with the changes in the correlation matrix of predictors just described.

In the numerical example of Table 7.1.1, the centered equation is

$$\hat{Y} = 2.2x + 2.6z + .4xz + 16.$$

This equation was found by retaining the criterion Y in raw score form, centering X and Z into x and z, respectively, forming the cross-product of centered X and Z (i.e., xz), and predicting

Y from x, z, and xz. Note that the intercept B_0 has changed from $B_0 = 2$ in the uncentered regression equation to 16 in the centered equation. Coefficients B_1 and B_2 have changed from .2 and .6, respectively, to 2.2 and 2.6, respectively. As we will see, these changes do not mean that the relationships of X and Z to the criterion Y have somehow changed with centering.

The centered regression equation containing an interaction is plotted in Fig. 7.2.1(B). As noted earlier, the value $x = 0, z = 0$ has moved from the lower right-hand corner of the regression plane in Fig. 7.1.1(B) to the middle of the regression plane in Fig. 7.2.1(B) due to centering.

A comparison of Fig. 7.1.1(B) with Fig. 7.2.1(B) gives insight into the source of the change in regression coefficients. In the uncentered equation, the B_1 coefficient represented the regression of Y on X at $Z = 0$, at the far right edge of Fig. 7.1.1(B). For higher values of Z (moving left along Fig. 7.1.1(B), the regression of Y on X became increasingly steep. With centered z, in Fig. 7.2.1(B), the value $z = 0$ is no longer at the right edge of the figure; it is halfway up the regression plane. At $z_{\text{mean}} = 0$ the regression of Y on x has risen to 2.2, the value of B_1 in the centered regression equation.

In general, centering predictors moves the value of zero on the predictors along the regression surface. If the regression surface is a flat plane (i.e., the regression equation contains no interaction), then the regression of Y on X is constant at all locations on the plane. Moving the value of zero by linear transformation has no effect on the regression coefficient for the predictor. If the regression surface is not flat (i.e., the regression equation contains an interaction), then the regression of Y on X varies across locations on the plane. The value of the B_1 regression coefficient will always be the slope of Y on X at $Z = 0$ on the plane, but the location of $Z = 0$ on the plane will change with centering.

What about the interpretation of B_1 as the *average* regression slope of Y on x across all values of z in the centered regression equation, $\hat{Y} = B_1x + B_2z + B_3xz + B_0$? A closer examination of Fig. 7.2.1(B) confirms this interpretation. In Fig. 7.2.1, the far right-hand edge now is at $z = -5$; at this point the regression of Y on X is .2. At the far left edge, $z = 5$ and the slope of the regression of Y on X is 4.2. The distribution of Z is uniform, so the average slope across all cases represented in the observed regression plane is $(.2 + 4.2)/2 = 2.2$; this is the value of the B_1 coefficient. Thus B_1 is the average slope of the regression of Y on X across all values of centered predictor Z.

There are straightforward algebraic relationships between the B_0, B_1, and B_2 coefficients in the uncentered versus centered regression equation containing the interactions:

(7.2.2)
$$B_{1,\text{centered}} = B_{1,\text{uncentered}} + B_{3,\text{uncentered}} M_{Z,\text{uncentered}};$$
$$B_{2,\text{centered}} = B_{2,\text{uncentered}} + B_{3,\text{uncentered}} M_{X,\text{uncentered}}.$$

For our numerical example,

$$B_{1,\text{centered}} = .2 + .4(5.00) = 2.20, \quad \text{and} \quad B_{2,\text{centered}} = .6 + .4(5.00) = 2.60.$$

Note that if there is no interaction (i.e., $B_3 = 0$), then the B_1 and B_2 coefficients would remain the same if X and Z were centered versus uncentered. This confirms what we know—*only if there is an interaction does rescaling a variable by a linear transformation change the first order regression coefficients.*

For the relationship of the intercept $B_{0,\text{centered}}$ to $B_{0,\text{uncentered}}$, we have

(7.2.3)
$$B_{0,\text{centered}} = B_{0,\text{uncentered}} + B_{1,\text{uncentered}} M_{X,\text{uncentered}} + B_{2,\text{uncentered}} M_{Z,\text{uncentered}} + B_3 M_{X,\text{uncentered}} M_{Z,\text{uncentered}}.$$

For our numerical example

$$B_{0,\text{centered}} = 2 + .2(5.00) + .6(5.00) + .4(5.00)(5.00) = 16.$$

Equations (7.2.1), (7.2.2), and (7.2.3) pertain only to Eq. (7.1.1). These relationships differ for every form of regression equation containing at least one interaction term; they would be different for more complex equations, for example, Eqs. (7.6.1) and (7.9.2) given below. Aiken and West (1991, Appendix B) provide an extensive mapping of uncentered to centered regression equations.

7.2.6 The Highest Order Interaction in the Centered Versus Uncentered Equation

By inspection the shapes of the regression surfaces in Fig. 7.1.1(B) for uncentered data and Fig. 7.2.1(B) for centered data are identical. Consistent with this, there is no effect of centering predictors on the value of regression coefficient B_3 in Eq. (7.1.2). The B_3 coefficient is for the highest order effect in the equation; that is, there are no three-way or higher order interactions. The interaction, carried by the XZ term, reflects the shape of the regression surface, specifically how this shape differs from the flat regression plane associated with regression equations having only first-order terms. This shape does not change when variables are centered. In general, *centering predictors has no effect on the value of the regression coefficient for the highest order term* in the regression equation. For Eq. (7.1.2) we have

$$B_{3,\text{centered}} = B_{3,\text{uncentered}}. \tag{7.2.4}$$

7.2.7 Do Not Center *Y*

In computing the centered regression equations and in displaying the regression surfaces in Figs. 7.1.1 and 7.2.1, Y has been left in uncentered form. There is no need to center Y because when it is in its original scale, predicted scores will also be in the units of the original scale and will have the same arithmetic mean as the observed criterion scores.

7.2.8 A Recommendation for Centering

We recommend that continuous predictors be centered before being entered into regression analyses containing interactions. Doing so has no effect on the estimate of the highest order interaction in the regression equation. Doing so yields two straightforward, meaningful interpretations of each first-order regression coefficient of predictors entered into the regression equation: (1) effects of the individual predictors at the mean of the sample, and (2) average effects of each individual predictors across the range of the other variables. Doing so also eliminates nonessential multicollinearity between first-order predictors and predictors that carry their interaction with other predictors.[1]

There is one exception to this recommendation: If a predictor has a meaningful zero point, then one may wish to keep the predictor in uncentered form. Lét us return to the example of language development. Suppose we keep the predictor of child's age (A). Our second predictor

[1]The issue of centering is not confined to continuous variables; it also comes into play in the coding of categorical variables that interact with other categorical variables or with continuous variables in MR analysis, a topic developed in Chapter 9.

is number of siblings (S). Following our previous argument, we center age. However, zero siblings is a meaningful number of siblings; we decide to retain number of siblings S in its uncentered form. We expect age and number of siblings to interact; we form the cross-product aS of centered a with uncentered S and estimate the following regression equation:

$$\hat{Y} = B_1 a + B_2 S + B_3 aS + B_0.$$

The interpretation of the two first-order effects differs. The effect of number of siblings is at $a = 0$; since a is centered, B_2 is the regression of language development on number of siblings at the mean age of children in the sample. The effect of child's age is at $S = 0$, where $S = 0$ stands for zero siblings. Hence B_1 is the regression of language development on age *for children with no siblings.* If this is a meaningful coefficient from the perspective of data summarization or theory testing, then centering is not advised. But even if the variable has a meaningful zero point, it may be centered for interpretational reasons. If number of siblings had been centered, then B_1 would be interpreted as the regression of language development on age at mean number of siblings. Finally, B_3 is not affected by predictor scaling and provides an estimate of the interaction between the predictors regardless of predictor scaling.

Our discussion of centering predictors has been confined to those predictors that are included in the interaction. But it is entirely possible that we include a predictor that is not part of any interaction in a regression equation that contains interactions among other variables. Suppose in the example of language development, we wish to control for mother's education level (E) while studying the interaction between child's age and number of siblings in predicting child's language development. Assume we wish to center number of siblings for interpretational reasons. We estimate the following regression equation:

$$\hat{Y} = B_1 a + B_2 s + B_3 as + B_4 E + B_0.$$

It is not necessary to center E. The B_1, B_2, and B_3 coefficients will not be affected by the scaling of E because E does not interact with any other predictors in the equation. In addition, since E does not interact with the other predictors, the B_4 coefficient will be completely unaffected by changes in scaling of age and number of siblings. In fact, the only effect of centering E is on the intercept B_0. However, we recommend that for simplicity, if one is centering the variables entering the interaction, one should also center the remaining variables in the equation.

To reiterate our position on centering, *we strongly recommend the centering of all predictors* that enter into higher order interactions in MR prior to analysis. The cross-product terms that carry the interactions should be formed from the centered predictors (i.e., center each predictor first and then form the cross-products). Centering all predictors has interpretational advantages and eliminates confusing nonessential multicollinearity.

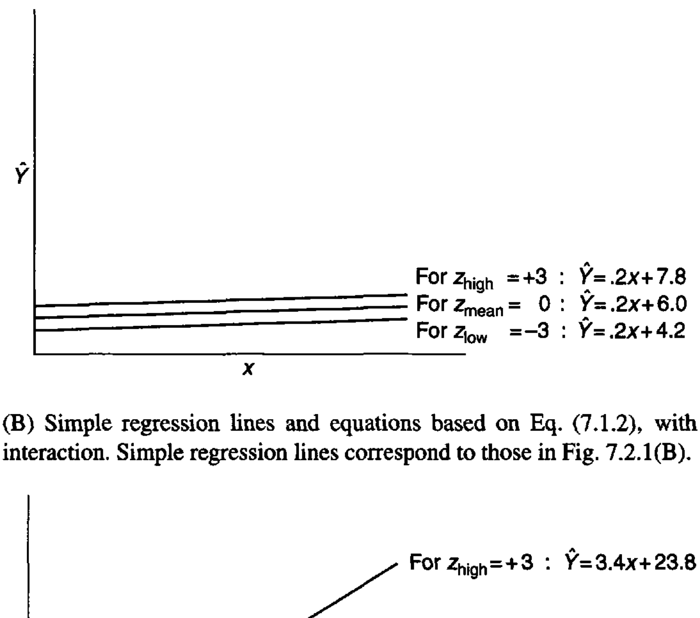
There is only one exception to this recommendation to center. If a predictor has a meaningful zero point, then one may wish to have regression coefficients in the overall regression equation refer to the regression of the criterion on predictors at this zero point. For the remainder of this chapter, we will assume that all predictors in regression equations containing an interaction have been centered, unless otherwise specified.

7.3 SIMPLE REGRESSION EQUATIONS AND SIMPLE SLOPES

If an interaction is found to exist in a regression equation, the issue becomes one of interpretation of the interaction. The approach we take harkens back to the idea of conditional effects in MR with interactions: When X and Z interact, the regression of each predictor depends on the value of the other predictor. To characterize interactions, we examine the regression of the criterion Y on one predictor X at each of several values of the other predictor Z, as when we examine the regression of Y on x at z_{low}, z_{mean}, and z_{high} in Fig. 7.2.1(B). Following Aiken and West (1991), we call the regression line of Y on X at one value of Z a *simple regression line*. Hence, Figs. 7.2.1(A) and 7.2.1(B) each contain three simple regression lines.

In Fig. 7.3.1, we plot the centered simple regression lines of Fig. 7.2.1 in more familiar two-dimension representations. In Fig. 7.3.1(A), the regression lines of Y on x at z_{low}, z_{mean}, and z_{high} are reproduced from Fig. 7.2.1(A). Similarly, the three regression lines of Y on x in Fig. 7.3.1(B) are those from Fig. 7.2.1(B). Each line in Figs. 7.3.1(A) and 7.3.1(B) is the regression of Y on x at one value of the other predictor z, *a simple regression line*. The rule for discerning the presence of an interaction is straightforward. If the lines are parallel, there is no interaction, since the regression of Y on X is constant across all values of Z. If the lines are not parallel, there is an interaction, since the regression of Y on X is changing as a function of Z.

(A) Simple regression lines and equations based on Eq. (7.1.1), no interaction. Simple regression lines correspond to those in Fig. 7.2.1(A).



(B) Simple regression lines and equations based on Eq. (7.1.2), with interaction. Simple regression lines correspond to those in Fig. 7.2.1(B).

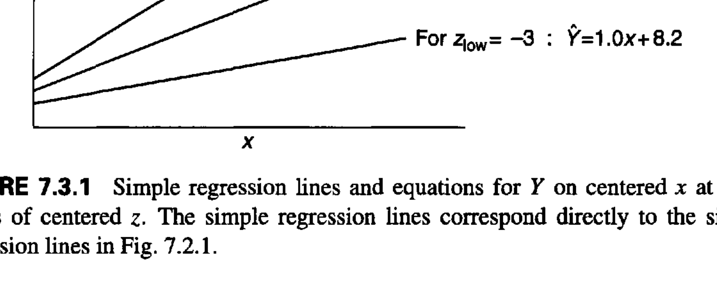


FIGURE 7.3.1 Simple regression lines and equations for Y on centered x at three values of centered z. The simple regression lines correspond directly to the simple regression lines in Fig. 7.2.1.

7.3.1 Plotting Interactions

Plotting interactions is the first step to their interpretation. We recommend plotting the regression of Y on X at three values of Z: the mean of Z plus a low and a high value of Z. Often a convenient set of values to choose are the mean of Z (Z_{mean}), one standard deviation below the mean of Z (Z_{low}), and one standard deviation above the mean of Z (Z_{high}). However, there may be specific meaningful values of Z—for example, clinical cutoffs for diagnostic levels of illness, or the income in dollars defined as the poverty level for a family of four. The symmetry of interactions means that the choice of plotting Y on X at values of Z as compared to Y on Z at values of X will depend on the theoretically more meaningful characterization of the data.

7.3.2 Moderator Variables

Psychological theories often hypothesize that a relationship between two variables will depend on a third variable. The third variable is referred to as a *moderator* (Baron & Kenny, 1986). These third variables may be organismic (e.g., gender, ethnicity, personality traits, abilities) or situational (e.g., controllable versus uncontrollable stressful events). They may be merely observed or manipulated. Of course, they are characterized statistically in terms of interactions. If a theory predicts that a variable M will moderate the relationship of another variable X to the criterion, then it is appropriate to plot regression of Y on X at meaningful values of the *moderator* M.

7.3.3 Simple Regression Equations

We can write a *simple regression equation* for each of the simple regression lines of Figs. 7.3.1(A) and 7.3.1(B). The use of simple regression equations is the key to the interpretation of interactions in MR analysis. *A simple regression equation* is the equation for the regression of the criterion on one predictor at a specific value of the other predictor(s), here Y on x at specific values of z.

For Figs. 7.2.1(A) and 7.3.1(A) with centered x and z, we place brackets in the regression equation with no interaction, $\hat{Y} = .2x + .6z + 6$, to show the regression of Y on x at values of z, in the form of a *simple regression equation*:

$$\hat{Y} = .2x + [.6z + 6].$$

Here the intercept of the simple regression equation $[.6z + 6]$ depends on the value of z; the slope of .2 does not. For each of the three values of z, we generate a simple regression equation. Recall that centered z takes on the values $(-5, -3, -1, 1, 3, 5)$, with $z_{\text{mean}} = 0$. We choose $z_{\text{low}} = -3$, and $z_{\text{high}} = 3$.

$$
\begin{array}{ll}
\text{For } z_{\text{low}} = -3: & \hat{Y} = .2x + [.6(-3) + 6] = .2x + 4.2; \\
\text{For } z_{\text{mean}} = 0: & \hat{Y} = .2x + [.6(0) + 6] = .2x + 6.0; \\
\text{For } z_{\text{high}} = 3: & \hat{Y} = .2x + [.6(3) + 6] = .2x + 7.8.
\end{array}
$$

We note that in all three equations the regression coefficient for x has the constant value .2. The intercept increases from 4.2 to 6.0 to 7.8, as z increases from -3 to 0 to 3.

To plot a simple regression line, we follow standard practice for plotting lines: we substitute into the equation two values of x, and find $\hat{Y}$ corresponding to those two values, giving us two points for plotting. For example, for z_{high}, where $\hat{Y} = .2x + 7.8$, if $x = -3$, then $\hat{Y} = 7.2$; if $x = 3$, $\hat{Y} = 8.4$. To plot the simple regression line for z_{high} in Fig. 7.3.1(A), we used the points $(-3, 7.2)$ and $(3, 8.4)$.

The numerical result corresponds completely with the graphical results in Figs. 7.2.1(A) and 7.3.1(A). The *simple slopes* of the simple regression lines (i.e., the regression coefficients for Y on x in the simple regression equations) are constant at .2. The *simple intercepts*, that is, the regression constants in the simple regression equations (values of Y at $x = 0$ for specific values of z), increase with increasing values of z.

For Figs. 7.2.1(B) and 7.3.1(B), we first rearrange the regression equation containing the interaction, $\hat{Y} = 2.2x + 2.6z + .4xz + 16$, placing the terms involving x at the beginning of the equation:

$$\hat{Y} = 2.2x + .4xz + 2.6z + 16$$

We then factor out x and include some brackets to show the regression of Y on x at z in the form of a simple regression equation:

$$\hat{Y} = [2.2 + .4z]x + [2.6z + 16]$$

The expression $[2.2 + .4z]$ is the simple slope of the regression of Y on x at a particular value of z; $[2.6z + 16]$ is the simple intercept. In an equation with an xz interaction, both the simple slope and simple intercept for the regression of Y on x depend on the value of z.

For each of the three values of z, we generate a simple regression equation:

$$\begin{array}{ll} \text{For } z_{\text{low}} = -3: & \hat{Y} = [2.2 + .4(-3)]x + [2.6(-3) + 16] = 1.0x + 8.2; \\ \text{For } z_{\text{mean}} = 0: & \hat{Y} = [2.2 + .4(0)]x + [2.6(0) + 16] = 2.2x + 16.0; \\ \text{For } z_{\text{high}} = -3: & \hat{Y} = [2.2 + .4(3)]x + [2.6(3) + 16] = 3.4x + 23.8. \end{array}$$

The numerical result is the same as the graphical results in Fig. 7.2.1(B) and 7.3.1(B): The simple slopes of the simple regression lines increase from 1.0 to 2.2 to 3.4 as z increases; the simple intercepts (values of Y at $x = 0$), increase from 8.2 to 16.0 to 23.8 as z increases. To plot a simple regression we follow the same approach as described earlier, that is, to substitute two values of x and solve for $\hat{Y}$. For z_{low}, where $\hat{Y} = 1.0x + 8.2$, if $x = -3$, then $\hat{Y} = 5.2$; if $x = 3$, then $\hat{Y} = 11.2$. To plot the regression line for z_{low} in Fig 7.3.1(B), we used the points $(-3, 5.2)$ and $(3, 11.2)$.

7.3.4 Overall Regression Coefficient and Simple Slope at the Mean

The overall regression coefficient B_1 for the regression of Y on x in the centered regression equation containing the interaction is 2.2 and represents the regression of Y on x at $z = 0$. The simple regression coefficient for Y on centered x at $z_{\text{mean}} = 0$ is also 2.2. This equality of coefficients is expected, since both coefficients represent the regression of Y on x at $z = 0$. In general, the simple regression coefficient for the regression of Y on x at the mean of z will equal the overall regression coefficient of Y on x in the centered regression equation.

We may cast simple regression equations in a general form. First, we have the overall regression equation containing predictors X and Z and their interaction:

$$\hat{Y} = B_1X + B_2Z + B_3XZ + B_0, \tag{7.1.2}$$

where B_3 is the regression coefficient for the interaction. We rearrange Eq. (7.1.2) to show the regression of Y on X at values of Z:

$$\begin{aligned} \hat{Y} &= [B_1X + B_3XZ] + [B_2Z + B_0] \\ \hat{Y} &= [B_1 + B_3Z]X + [B_2Z + B_0], \end{aligned} \tag{7.3.1}$$

the simple regression equation for the regression of Y on X at specific values of Z. The coefficient $[B_1 + B_3Z]$ for X in Eq. (7.3.1) is the simple slope, an expression for the slopes of simple regression lines such as those in Fig. 7.3.1(B). If B_3 is nonzero, meaning that there is an interaction between X and Z, then the value of this simple slope $[B_1 + B_3Z]$ will differ for every value of Z. If B_3 is zero, signifying that there is no interaction between X and Z, then the simple slope will always equal B_1, the coefficient for predictor X, regardless of the value of Z.

We stated earlier that the interaction between X and Z is symmetric. Thus we can also rearrange Eq. (7.1.2) to show the regression of Y on Z at values of X:

$$\begin{aligned}\hat{Y} &= [B_2Z + B_3XZ] + [B_1X + B_0] \\ \hat{Y} &= [B_2 + B_3X]Z + [B_1X + B_0].\end{aligned} \tag{7.3.2}$$

The simple slope $[B_2 + B_3X]$ for the regression of Y on Z shows that if B_3 is nonzero, the regression of Y on Z will differ for each value of X. If B_3 is zero, meaning that there is no interaction, the regression of Y on Z is constant for all values of X. The symmetry is complete. It should also be noted that the expressions for simple slopes depend completely on the regression equation for the entire sample, including both main effects and interactions.

7.3.5 Simple Slopes From Uncentered Versus Centered Equations Are Identical

We learned in Section (7.2.6) that the regression coefficient B_3 for the highest order interaction term XZ in Eq. (7.1.2) remains invariant when predictors are centered; this is so because the shape of the regression surface is unchanged by centering. Simple slopes are regressions of Y on a predictor, say X, at particular points on that surface, defined by the other predictor, here Z. If simple regression equations are computed at analogous values of Z in the centered and uncentered case, then the slopes of these simple regression lines will be identical in the centered versus uncentered case; only the intercepts will differ. This point cannot be overemphasized. The interpretation of the interaction remains identical across the centered versus the uncentered form of a regression equation. This is why we can move between the uncentered and centered forms of an equation without jeopardizing interpretation.

In our example, uncentered Z has the values (0, 2, 4, 6, 8, 10) and the corresponding values of centered z are $(-5, -3, -1, 1, 3, 5)$. Uncentered $Z = 2$, for example, corresponds to centered $z = -3$. We rearrange the uncentered equation $\hat{Y} = .2X + .4XZ + .6Z + 2$ into the simple regression equation $\hat{Y} = (.2 + .4Z)X + (.6Z + 2)$ and substitute $Z = 2$, yielding $\hat{Y} = (.2 + .4(2))X + (.6(2) + 2)$, or $\hat{Y} = 1.0X + 3.2$. The simple regression of Y on x at $z = -3$ in the centered equation $\hat{Y} = 2.2x + 2.6z + .4xz + 16$ is $\hat{Y} = 1.0x + 8.2$. The simple slopes are identical; only the intercept has changed. Simple slopes from the uncentered and centered regression equations are given in Table 7.1.1D and G, respectively.

7.3.6 Linear by Linear Interactions

The interaction between X and Z in Eq. (7.1.2) is a *linear by linear interaction*. This means that the regression of Y on X is linear at every value of Z or, equivalently, that the regression coefficient of Y on X changes at a constant rate as a function of changes in Z. Thus we find the symmetric fanning of simple regression lines illustrated in Fig. 7.3.1(B). All the simple regression equations characterize straight lines; they change slope at a constant rate as Z increases. This linearity is symmetric: The regression of Y on Z is linear at every value of X; the regression of Y on Z changes at a constant rate as a function of changes in X.

Chapter 6 explored the treatment of curvilinear relationships of individual variables through polynomial regression. Interactions of curvilinear functions of one predictor with linear or curvilinear components of another predictor are possible. In Section 7.9 we will take up more complex interactions that include curvilinear relationships.

7.3.7 Interpreting Interactions in Multiple Regression and Analysis of Variance

The reader familiar with ANOVA will note the similarity of the proposed strategy to the well-developed strategy used in ANOVA for the interpretation of interactions. In ANOVA with factors A and B interacting, the approach involves examining the effect of one factor A involved in the interaction at each of the several levels of the other factor B involved in the interaction. The purpose of the analysis is to determine the levels of factor B at which factor A manifests an effect. In ANOVA, the effect of a factor on the outcome, confined to one level of another factor, is termed a *simple main effect* (e.g., Kirk, 1995; Winer, Brown, & Michels, 1991). The format of Fig. 7.3.1 is highly similar to that typically used in ANOVA to illustrate the effects of two factors simultaneously. In ANOVA we have plots of means of one factor at specific levels of another factor. In MR, we have plots of simple regression lines of the criterion on one predictor at specific values of another predictor. In Section 7.4 we present a method for post hoc probing of simple slopes of simple regression lines in MR that parallels post hoc probing of simple main effects in ANOVA.

7.4 POST HOC PROBING OF INTERACTIONS

Plotting interactions provides substantial information about their nature. In addition to inspecting simple slopes to describe the specific nature of interactions, we may also create confidence intervals around simple slopes. Further we may test whether a specific simple slope, computed at one value of the other predictor(s), differs from zero (or from some other value).

7.4.1 Standard Error of Simple Slopes

In Chapter 3 (Section 3.6.1), we introduced the *standard error of a partial regression coefficient*, SE_{B_i}, a measure of the expected instability of a partial regression coefficient from one random sample to another. The square of the standard error is the *variance of the regression coefficient*.

We also may measure the *standard error of a simple slope*, that is, of the simple regression coefficient for the regression of Y on X at a particular value of Z. For example, if Z were a 7-point attitude scale ranging from 1 to 7, $M_Z = 4$, and we centered Z into z, ranging from -3 to $+3$, we might examine the simple slope of Y on x at values of z across the centered attitude scale, say at the values $[-3 \quad -1 \quad 1 \quad 3]$. The numerical value of the standard error of the simple slope of Y on x is different at each value of z.

In Eq. (7.3.1) for the regression of Y on X at values of z, the simple slope is $[B_1 + B_3 Z]$. The standard error of the simple slope depends upon the variances of both B_1 and B_3. It also varies as a function of the covariance between the estimates of B_1 and B_3. This is a new concept—that regression coefficients from the same equation may be more or less related to one another. Some intuition can be gained if one imagines carrying out the same regression analysis on repeated random samples and making note of the values of B_1 and B_3 in each sample. Having carried out the analysis many times, we could measure the covariance between the B_1 and B_3 coefficients across the many samples; this is the covariance we seek.

For Eq. (7.1.2), the variances of B_1, B_2, and B_3 and their covariances are organized into a matrix called the *covariance matrix of the regression coefficients*; it appears as follows:

$$\mathbf{S}_B = \begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline B_1 & SE^2_{B_{11}} & \mathrm{cov}_{B_{12}} & \mathrm{cov}_{B_{13}} \\ B_2 & \mathrm{cov}_{B_{21}} & SE^2_{B_{22}} & \mathrm{cov}_{B_{23}} \\ B_3 & \mathrm{cov}_{B_{31}} & \mathrm{cov}_{B_{32}} & SE^2_{B_{33}} \end{array} \tag{7.4.1}$$

where $SE^2_{B_{ii}}$ is the variance of regression coefficient B_i and $\mathrm{cov}_{B_{ij}}$ is the covariance between regression coefficients B_i and B_j. This matrix is provided by standard programs for multiple regression, including SAS, SPSS, and SYSTAT.

The standard error of the simple slope for the regression of Y on X at a particular value of Z is given as follows:

$$SE_{B \text{ at } Z} = [SE^2_{B_{11}} + 2Z\mathrm{cov}_{B_{13}} + Z^2 SE^2_{B_{33}}]^{1/2}. \tag{7.4.2}$$

Specific values are taken from the covariance matrix in Eq. (7.4.1) and from the predictor Z itself. This equation applies whether centered or uncentered variables are used. However, the values in the $\mathbf{S}_B$ matrix will differ depending on predictor scaling, just as do the values in the correlation matrix of the predictors themselves.

Each standard error of a simple slope only applies to a particular regression coefficient in a particular regression equation. For the regression of Y on Z at values of X, the standard error is

$$SE_{B \text{ at } X} = [SE^2_{B_{22}} + 2X\mathrm{cov}_{B_{23}} + X^2 SE^2_{B_{33}}]^{1/2}. \tag{7.4.3}$$

7.4.2 Equation Dependence of Simple Slopes and Their Standard Errors

As was stated earlier, the expressions for both the simple slopes depend on the particular regression equation for the full sample. This is also the case for the standard errors of simple slopes. The simple slopes determined by Eqs. (7.3.1) and (7.3.2), and their respective standard errors in Eqs. (7.4.2) and (7.4.3), apply *only* to equations with two-variable linear interactions such as Eq. (7.1.2). *These expressions are not appropriate for more complex equations with higher order terms or interactions involving quadratic terms*, such as Eqs. (7.6.1) and (7.9.2). Aiken and West (1991, pp. 60 and 64) provide expressions for both the simple slopes and the standard errors of simple slopes for a variety of regression equations.

7.4.3 Tests of Significance of Simple Slopes

Tests of significance of individual predictors in a multiple regression equation are given in Chapter 3 (Section 3.6.4). These tests generalize directly to tests of significance of simple slopes. Suppose we wish to test the hypothesis that the simple slope of Y on X is zero at some particular value of Z. The t test for this hypothesis is

$$t_{B \text{ at } Z} = (B_1 + B_3 Z)/SE_{B \text{ at } Z} \quad \text{with} \quad (n-k-1)\,df, \tag{7.4.4}$$

where k is the number of predictors. For the significance of difference from zero of the regression of Y on Z at values of X, the appropriate t test is

$$t_{B \text{ at } X} = (B_2 + B_3 X)/SE_{B \text{ at } X} \quad \text{with} \quad (n-k-1)\,df. \tag{7.4.5}$$

7.4.4 Confidence Intervals Around Simple Slopes

The structure of the confidence interval for a simple slope follows that described in Section 2.8.2 for the confidence interval on the predictor in a one-predictor equation and in Section 3.6.1 for the confidence interval on a regression coefficient in a multiple prediction equation. For a two-tailed confidence interval for the regression of Y on X at a specific level of confidence $(1 - \alpha)$ the *margin of error (me)* is given as follows:

$$me = t_{1-\alpha/2} SE_{B \text{ at } Z}, \tag{7.4.6}$$

where $t_{1-\alpha/2}$ refers to a two-tailed critical value of t for specified α, with $(n - k - 1)\,df$. The critical value of t is the same value as for the t test for the significance of each regression coefficient in the overall regression analysis and for the significance of the simple slope.

The confidence interval is given as

$$CI = [(B_1 + B_3 Z) - me \leq \beta^*_{Y \text{ on } X \text{ at } Z} \leq (B_1 + B_3 Z) + me], \tag{7.4.7}$$

where $\beta^*_{Y \text{ on } X \text{ at } Z}$ is the value of the simple slope in the population. For the regression of Y on Z at values of X, the margin of error and confidence interval are as follows:

$$me = t_{1-\alpha/2} SE_{B \text{ at } X}, \tag{7.4.8}$$

where $t_{1-\alpha/2}$ is as in Eq. (7.4.6).

The confidence interval on the simple slope is

$$CI = [(B_2 + B_3 X) - me \leq \beta^*_{Y \text{ on } Z \text{ at } X} \leq (B_2 + B_3 X) + me], \tag{7.4.9}$$

where $\beta^*_{Y \text{ on } Z \text{ at } X}$ is the value of the simple slope in the population.

The interpretation of the *CI* for a simple slope follows that for a regression coefficient. For example, for level of confidence 95%, we can be 95% confident that the true simple slope $\beta^*_{Y \text{ on } X \text{ at } Z}$ falls within the interval we have calculated from our observed data. An alternative frequentist interpretation is that if we were to draw a large number of random samples from the same population, carry out the regression analysis, and compute the confidence interval of Y on X at one specific value of Z, 95% of those intervals would be expected to contain the value $\beta^*_{Y \text{ on } X \text{ at } Z}$. Of course, the *CI* on the simple slope provides all the information provided by the null hypothesis significance tests given in Eq. (7.4.4) when the α selected in determining the *me* is equivalent. If the confidence interval on the simple slope includes zero, we do not reject the null hypothesis that the simple slope differs from zero.

Some caution in our thinking is required here. Consider once again the developmental example of the prediction of child's language development (Y) from child's age (A) as a function of number of siblings (S). In each of a large number of samples of children from the same population, we might construct the 95% confidence interval for the regression of Y on A for $S = 1$ sibling. Our frequentist interpretation would be that across a large number of samples, 95% of the confidence intervals would include the true population value of the slope for the regression of child's language development on child's age for children with only one sibling. Suppose, however, we computed the regression of Y on A for the mean number of siblings in each sample; that is, we would not pick a specific number of siblings, but would rather use the mean number of siblings in a particular sample as the value of S for examining the regression of Y on age (A) at S. The mean number of siblings varies across samples. Thus the *CI* on the simple slope would be for a different value of S in each sample. We could not strictly use the frequentist interpretation of the *CI* calculated at the mean of any particular sample. Put another way, in comparing the simple regression of Y on X at a value of Z across different samples, the value of Z must be held constant (fixed) across the samples.

7.4.5 A Numerical Example

In Table 7.4.1 and Fig. 7.4.1 we present an example in which physical endurance (Y) of $n = 245$ adults is predicted from their age (X) and the number of years of vigorous physical exercise (Z) in which they have engaged. In the sample, the mean age is 49.18 ($sd = 10.11$, range 20 to 82), and the mean number of years of vigorous physical exercise is 10.67 ($sd = 4.78$, range 0 to 26 years). Physical endurance is measured as the number of minutes of sustained jogging on a treadmill. The mean number of minutes of sustained performance is 26.53 ($sd = 10.82$, range 0 to 55 minutes, a sample with noteworthy stamina).

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Centered and Uncentered Scale in Plots

In Fig. 7.4.1 we have adopted a convention of plotting on the x axis both the original raw score scale of the predictor and the centered scale in which data are being analyzed. This strategy is useful for conceptually retaining the meaning of the original scale units of the predictor during analysis and interpretation. Since the criterion is not centered for analysis, it is shown only in raw score form in the graph. We plot the range of the variable on the x axis from one standard deviation below the mean (age = 39.07 years) to one standard deviation above the mean (age = 59.29 years). This range is smaller than the full range of the X variable of 20 to 82 years.

In the overall centered regression of endurance (Y) on centered age (x) and centered years of exercise (z), $\hat{Y} = -.262x + .973z + .047xz + 25.888$. Endurance, not surprisingly, declines with age. Since the predictors are centered, the amount of decline with age signified by the regression coefficient ($B_1 = -.26$) is a loss in endurance of .26 minutes on the treadmill test for a one-year increase in age for people at the mean level of years of exercise in the sample (uncentered $M_Z = 10.67$ years). Endurance, in contrast, increases with exercise ($B_2 = .97$), with the amount of increase of .97 minutes on the endurance test for each year of vigorous exercise, applicable to people at the mean age of the sample (uncentered $M_X = 49.18$ years).

The XZ interaction signifies that the decline in endurance with age depends on a history of exercise, as illustrated in Fig. 7.4.1(A); the regression of endurance (Y) on age (x) is plotted at three values of exercise (z). In fact, the decline in endurance with age is buffered by a history of exercise; that is, the more vigorous is exercise across the life span, the less dramatic the decline in endurance with age. In general, if one variable weakens the impact of another variable on the criterion, that variable is said to *buffer* the effect of the other variable. One can intuit the numerical workings of the interaction by considering some cross-product values of centered x and z. If a person is above the mean age and above the mean exercise, then the cross-product is positive and increases predicted endurance; that is, the person's predicted endurance is higher than would be expected from his/her age alone. If the person is above the mean age but below the mean exercise, the cross-product is negative and decreases predicted endurance below that expected for people of that age at the average exercise level in the sample.

A comment is in order about the magnitude of the interaction. As shown in Table 7.4.1B, R^2 with only x and z but without the interaction as predictors is .17; inclusion of the interaction increases R^2 to .21. Thus the interaction accounts for 4% of the variance in the criterion, over and above the main effects, $F_{\text{gain}}(1, 241) = 12.08, p < .01$. This may seem to be a small amount, but it is of the order of magnitude typically found in behavioral research (Chaplin, 1991; Champoux & Peters, 1987; Jaccard & Wan, 1995). While this is "only" 4% of the variance accounted for, the buffering effect is strong indeed, as shown in Fig. 7.4.1(A). With a short history of exercise, there is a decline of .49 minutes in treadmill performance per year of age; yet with a long history of exercise, there is essentially no decline in treadmill performance (a bit of wishful thinking on the part of the creator of this example).

TABLE 7.4.1
Regression Analysis of Physical Endurance (Y) as a Function of Age (X) and Years of Vigorous Exercise (Z), $n = 245$

A. Summary Statistics for centered x and z and cross-product xz.

	Means and standard deviations			Correlation matrix			
	M	sd		x	z	xz	Y
x	0.00	10.11	x	1.00	.28	.01	−.13
z	0.00	4.78	z		1.00	−.12	.34
xz	13.59	46.01	xz			1.00	.15
Y	26.53	10.82	Y				1.00

B. Centered regression equations:

1. Prediction of Y from centered x and z:

$$\hat{Y} = -.257x^{**} + .916z^{**} + 26.530$$
$$R^2 = .17$$

2. Prediction of Y from centered x and z, and xz:

$$\hat{Y} = -.262x^{**} + .973z^{**} + .047xz^{**} + 25.888$$
$$R^2 = .21$$

C. Covariance matrix of the regression coefficients in the centered regression equation containing interactions (Part B2):

	B_1	B_2	B_3
B_1	.00410	−.00248	−.00001806
B_2	−.00248	.01864	.0002207
B_3	−.00001806	.0002207	.0001848

D. Analysis of simple regression equations for regression of uncentered endurance (Y) on centered age (x) at three values of centered years of exercise (z):

Value of z	Simple regression equation	Standard error of simple slope	t test	95% CI
At $z_{low} = -4.78$	$\hat{Y} = -.487x + 21.24$	.092	−5.29**	[−.67, −.31]
At $z_{mean} = 0.00$	$\hat{Y} = -.262x + 25.89$	.064	−4.09**	[−.39, −.14]
At $z_{high} = +4.78$	$\hat{Y} = -.036x + 30.53$	.090	− .40**	[−.21, .14]

E. Analysis of simple regression equations for regression of uncentered endurance (Y) on centered years of exercise (z) at three values of centered age (x):

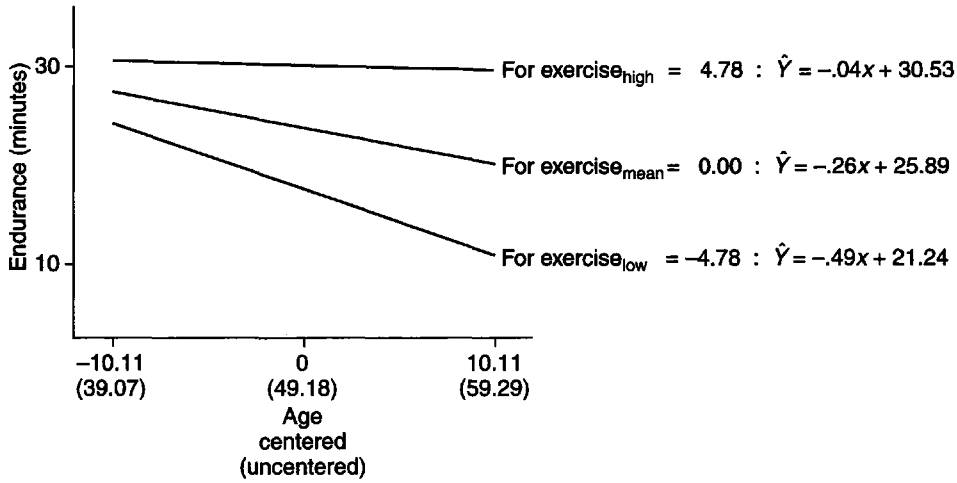
Value of x	Simple regression equation	Standard error of simple slope	t test	95% CI
At $x_{low} = -10.11$	$\hat{Y} = .495z + 28.53$	.182	2.72**	[.14, .85]
At $x_{mean} = 0.00$	$\hat{Y} = .973z + 25.89$	.137	7.12**	[.70, 1.24]
At $x_{high} = +10.11$	$\hat{Y} = 1.450z + 23.24$	.205	7.08**	[1.05, 1.85]

F. Regression equation with uncentered data ($M_x = 49.18; M_z = 10.67$)

$$\hat{Y} = -.766X^{**} - 1.35\,Z^{*} + .047\,XZ^{**} + 53.18$$
$$R^2 = .21$$

$^{**}p < .01$; $^{*}p < .05$.

(A) Regression of endurance (Y) on age (x) at three levels of exercise (z). Simple regression equations are for centered data.



(B) Regression of endurance (Y) on exercise (z) at three levels of age (x). Simple regression equations are for centered data.

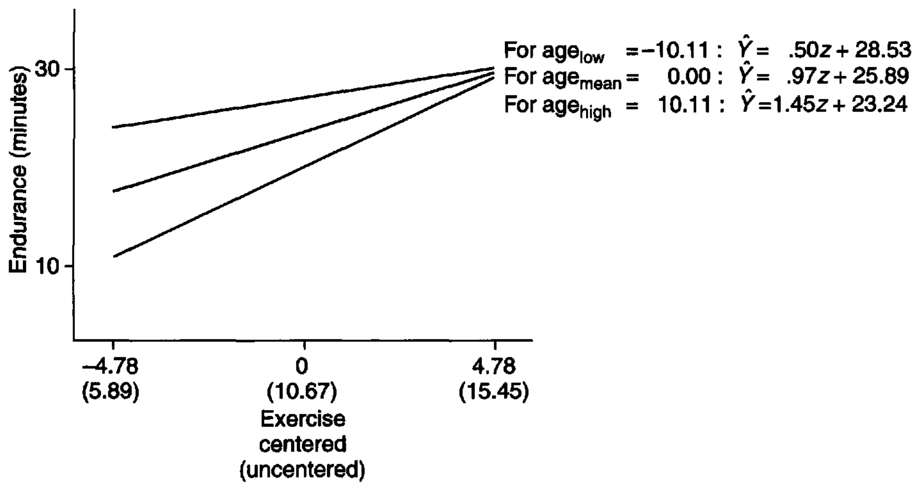


FIGURE 7.4.1 Simple slope analysis for example in Table 7.4.1, based on centered regression equation.

Tests of Significance of Simple Slopes

The analysis of simple slopes of endurance (Y) on centered age (x) at various years of centered exercise (z), given in Table 7.4.1D corroborates our inspection of Fig. 7.4.1(A). First, we rearrange the overall equation to show the regression of Y on x at values of z:

$$\hat{Y} = (-.262 + .047z)x + (.973z + 25.888).$$

Then we compute three simple regression equations, using the mean of z ($z_{mean} = 0$ for centered z), and the values one standard deviation above and below the mean of centered z ($sd = 4.78$).

These simple regression equations are given in Table 7.4.1D. For example, for $z_{\text{high}} = 4.78$, within rounding error we have

$$\hat{Y}_{\text{at } z_{\text{high}}} = (-.262 + .047(4.78))x + (.973(4.78) + 25.888)$$

$$\hat{Y}_{\text{at } z_{\text{high}}} = (-.262 + .224)x + (4.646 + 25.888)$$

$$\hat{Y}_{\text{at } z_{\text{high}}} = -.036x + 30.53$$

To test for the significance of difference of each simple slope from zero, we compute the standard error of the simple slope of Y on x at a particular value of z (Eq. 7.4.2). The covariance matrix of the predictors is given in Table 7.4.1C. We need the values

$$SE^2_{B_{11}} = .00410, \text{ cov}_{B_{13}} = -.00001806, \text{ and } SE^2_{B_{33}} = .0001848. \text{ Then at } z_{\text{high}} = 4.78$$

$$SE_{B \text{ at } z_{\text{high}}} = [SE^2_{B_{11}} + 2Z\text{cov}_{B_{13}} + Z^2 SE^2_{B_{33}}]^{1/2}$$

$$SE_{B \text{ at } z_{\text{high}}=4.78} = [.00410 + 2(4.78)(-.00001806) + (4.78)^2 .0001848]^{1/2}$$

$$= (.00410 - .000172 + .004221) = .008149^{1/2} = .090.$$

Finally, the t test is

$$t_{B \text{ at } Z} = (B_1 + B_3 Z)/SE_{B \text{ at } Z} = (-.262 + .047(4.78))/.090$$

$$= -.036/.090 = -.40.$$

With $(n - k - 1) = 245 - 3 - 1 = 241$ df, at $\alpha = .05$, two tailed, $t_{\text{critical}} = 1.97$; there is no evidence of decline in endurance with age when there is a long history of exercise.

Confidence Intervals Around Simple Slopes

We may estimate a confidence interval on the simple slope at z_{high} for the decline in endurance (minutes on the treadmill) with age at one standard deviation above the mean on exercise. Using Eq. (7.4.6) for the margin of error (me) for $\alpha = .05$, we find

$$me = t_{1-\alpha/2} SE_{B \text{ at } Z_{\text{high}}=4.78} = 1.97\ (.090) = .177.$$

From (7.4.7) the 95% confidence interval is given as

$$CI = [(B_1 + B_3 Z) - me \le \beta^*_{Y \text{ on } x \text{ at } z_{\text{high}}} \le (B_1 + B_3 Z) + me]$$

$$CI = [(-.036) - .177 \le \beta^*_{Y \text{ on } x \text{ at } z_{\text{high}}} \le (-.036) + .177]$$

$$CI = [-.213 \le \beta^*_{Y \text{ on } x \text{ at } z_{\text{high}}} \le .141]$$

where $\beta^*_{Y \text{ on } X \text{ at } Z}$ is a population value of a simple slope.

The confidence interval includes zero, indicating a lack of statistical evidence for decline in endurance with increasing age for people with a substantial history of exercise (one *sd* above the mean), consistent with the outcome of the statistical test.

We need not confine ourselves to values of z like z_{high} or z_{low}. We might wish to estimate a confidence interval on the decline with age in endurance for people who have exercised a particular numbers of years. Here, as examples, we choose decline for people who have exercised not at all, in comparison to those who have exercised a full decade (10 years). The choice of 0 years and 10 years is arbitrary; researchers might pick other values on the basis of theory or because the values represent practical points of interest. Note that 0 years and 10 years are on the uncentered scale (in the original units of number of years of exercise). To center years of exercise, we subtracted $M_Z = 10.67$ from each score on raw years. Now we convert 0 and 10 raw years to the centered scale by subtracting 10.67 from each number of raw years:

Raw years of exercise (Z)	0	10
Centered years of exercise (z)	-10.67	$-.67$

We form confidence intervals at each number of centered years. First we compute the simple slope for the regression of Y on x at $z = -10.67$, corresponding to 0 years of exercise:

$$(B_1 + B_3 z) = (-.262 + .047(-10.67)) = -.763.$$

Then we compute the standard errors at each number of centered years, using

$$SE_{B \text{ at } z=i} = [SE^2_{B_{11}} + 2z\,\text{cov}_{B_{13}} + z^2 SE^2_{B_{33}}]^{1/2} \tag{7.4.2}$$

For zero years of exercise (equal to -10.67 years on the centered scale),

$$\begin{aligned} SE_{B \text{ at } Z=-10.67} &= [.00410 + 2(-10.67)(-.00001806) + (-10.67)^2 .0001848]^{1/2} \\ &= (.00410 + .0003854 + .0210392)^{1/2} \\ &= .0255246^{1/2} = .160. \end{aligned}$$

Then we compute the margin of error for $\alpha = .05$, using

$$me = t_{1-\alpha/2} SE_{B \text{ at } z=-10.67} = (1.97)(.160) = .315.$$

Finally, we compute the confidence interval using

$$\begin{aligned} CI &= [(B_1 + B_3 z) - me \le \beta^*_{Y \text{ on } X \text{ at } Z} \le (B_1 + B_3 z) + me] \\ CI &= [-.763 - .315 \le \beta^*_{Y \text{ on } X \text{ at } Z} \le -.763 + .315] \\ CI &= [-1.078 \le \beta^*_{Y \text{ on } X \text{ at } Z} \le -.448] \end{aligned}$$

This confidence interval indicates that we can be 95% certain that there is somewhere between a half minute ($-.448$) and full minute (-1.078) decline on the endurance test for each year of increasing age for individuals who have no history of exercise.

What is the decline for individuals who have a 10-year history of exercise? Ten raw-score years translates into $-.67$ years on the centered scale (since the mean years of exercise is 10.67).

The simple slope for the regression of Y on x at $z = -.67$ is

$$(B_1 + B_3 z) = (-.262 + .047(-.67)) = -.293.$$

The standard error of this simple slope is given as

$$\begin{aligned} SE_{B \text{ at } Z=-.67} &= [.00410 + 2(-.67)(-.00001806) + (-.67)^2 .0001848]^{1/2} \\ &= (.00410 + .0000242 + .0000829)^{1/2} = .004207^{1/2} = .065. \end{aligned}$$

Then we compute the margin of error, $\alpha = .05$, using

$$me = t_{1-\alpha/2} SE_{B \text{ at } Z=-.67} = (1.97)(.065) = .128.$$

Finally, we compute the 95% confidence interval using

$$CI = [(B_1 + B_3z) - me \leq \beta^*_{Y \text{ on } X \text{ at } Z=-.67} \leq (B_1 + B_3z) + me]$$
$$CI = [-.293 - .128 \leq \beta^*_{Y \text{ on } X \text{ at } Z=-.67} \leq -.293 + .128]$$
$$CI = [-.421 \leq \beta^*_{Y \text{ on } X \text{ at } Z=-.67} \leq -.165].$$

This confidence interval indicates that we can be 95% certain that for individuals who have exercised for 10 years, the true decline is from about a sixth (i.e., $-.165$) of a minute to at most 4/10 (i.e., $-.421$) of a minute for each year of age. If we compare the two confidence intervals, that for zero versus 10 years of exercise, we see that the intervals do not overlap. Thus we may also conclude that there is materially less decline in endurance among people with a 10-year history of exercise than among those with no exercise.

No Tests of Significance of Difference Between Simple Slopes

We might be tempted to say that the simple slope for the decline of endurance with increasing age at 10 years of exercise ($-.293$) is "significantly less" than the decline of endurance with increasing age at 0 years of exercise ($-.763$). We cannot say this, however. There exists no test of significance of difference between simple slopes computed at single values (points) along a continuum (e.g., along the age continuum). The issue then comes down to a matter of meaningfulness. Here we would ask if the savings of almost a half minute in a measure of endurance with each year of exercise is meaningful (judging how long people live, a half a minute a year translates into a lot of endurance over the life span). We said there was a material difference in the two measures of decline in endurance since the *CI*s did not overlap. However, the *CI*s might overlap and the difference in simple slopes be material from a substantive perspective.

Regression of Y on z at Values of x

We also display the *XZ* interaction in Fig. 7.4.1(B), but now showing the regression of endurance (Y) on exercise (z) at three values of age (x). That is, instead of Y on x at values of z, we have Y on z at values of x. This display tells us about the impact of exercise on endurance as a function of age. We would expect exercise to have a more profound effect on endurance as age increases. The appearance of the interaction is quite different, but tells the same story in a different way. Figure 7.4.1(B) shows us that as age increases, the positive impact of exercise on endurance becomes more pronounced (i.e., a steeper positive slope, even though, overall, younger individuals have greater endurance than older individuals. The corresponding simple slope analysis is carried out in Table 7.4.1E. In fact, there is a significant gain in endurance with increased length of exercise history at the mean age ($M_X = 49.18$ years, which is equivalent to centered $M_x = 0.00$ years), at one *sd* below the mean age ($X_{\text{low}} = 39.07$ years, equivalent to centered $x_{\text{low}} = -10.11$ years) and at one *sd* above the mean age ($X_{\text{high}} = 59.29$ years, $x_{\text{high}} = 10.11$ years).

Simple Slope Analysis by Computer

The numerical example develops the analysis of simple slopes for Y on X at values of Z by hand computation for Eq. (7.1.2). The complete analysis of simple slopes can easily be carried out by computer using standard regression analysis software. Aiken and West (1991) explain the computer method and provide computer code for computing simple slopes, standard errors, and tests of significance of simple slopes.

7.4.6 The Uncentered Regression Equation Revisited

We strongly recommend working with centered data. We visit the uncentered regression equation briefly to show just how difficult it may be to interpret uncentered regression coefficients. Uncentered equations are primarily useful when there are true zeros on the predictor scales.

As shown in Table 7.4.1F, the regression equation based on uncentered data is

$$\hat{Y} = -.766X - 1.351Z + .047XZ + 53.18.$$

This equation is different from the centered equation in that now the regression of endurance on exercise (Z) is significantly *negative*: the more exercise, the less endurance. How can this be? A consideration of the regression of endurance on exercise at values of age (Y on uncentered Z at values of uncentered X) provides clarification. The simple regressions are computed from the overall uncentered regression equation by:

$$\hat{Y} = (-1.351 + .047X)Z + (-.766X + 53.18).$$

We choose meaningful values of age on the uncentered scale: $X_{\text{mean}} = 49.18$ years, $X_{\text{low}} = 39.07$ (one standard deviation below the mean age), and $X_{\text{high}} = 59.29$. (Again, these values correspond to centered ages of -10.11, 0, and 10.11, respectively in the simple slope analysis of Table 7.4.1E. For ages 39.07 to 59.29, the regression of endurance on exercise is positive, as we expect. At $X_{\text{low}} = 39.08$ years, $(B_2 + B_3Z) = [-1.351 + (.047)(39.08)] = .96$; at $X_{\text{high}} = 59.29$, $(B_2 + B_3Z) = 1.44$. As we already know, the slopes of the simple regression lines from the uncentered equation and those from the corresponding simple regression lines from the centered regression equation are the same. The interpretation of the interaction is unchanged by centering.

The significantly negative B_2 coefficient (-1.351) from the uncentered equation represents the regression of endurance (Y) on exercise (Z), for individuals at age zero ($X = 0$). At $X = 0$, the simple slope $(B_2 + B_3Z) = [-1.351 + (.047)(0)] = -1.351$. We know that this simple regression line for age zero with its negative slope is nonsensical, because it represents the number of years of exercise completed by people of age zero years (i.e., newborns). We may compute simple regression lines for regions of the regression plane that exist mathematically, since mathematically the regression plane extends to infinity in all directions. However, the simple regression equations only make sense in terms of the meaningful range of the data. It is reasonable, for example, that someone 29.01 years of age might have a 12-year history of strenuous exercise, if he played high school and college football or she ran in high school and college track. However, in this example, years of exercise is limited by age. Moreover, the age range studied is adults who have had an opportunity to exercise over a period of years.

In computing simple regression lines we must consider the *meaningful range of each variable* in the regression equation and limit our choice of simple regression lines to this meaningful range. This is why we caution about the uncentered regression equation once again—zero is often not a meaningful point on scales in the behavioral sciences. We do not mean to say that the use of uncentered variables produces incorrect results; rather, uncentered regression equations often produce interpretation difficulties for behavioral science data, difficulties that are eliminated by centering.

7.4.7 First-Order Coefficients in Equations Without and With Interactions

Suppose we have a data set that contains an interaction, as in the example of endurance, age, and exercise in Table 7.4.1. We compare the centered regression equation without the interaction versus with the interaction in Table 7.4.1B. We note that the B_1 coefficient for x

is $-.257$ versus $-.262$ in the equation without versus with the interaction, respectively. The B_2 coefficient is .916 versus .973, respectively. Why do these coefficients change when the interaction term is added? These coefficients are partial regression coefficients, and x and z are both slightly correlated with the cross-product term xz, as shown in Table 7.4.1A. These very slight correlations reflect essential multicollinearity (Section 7.2.4) due to very slight nonsymmetry of X and Z. If X and Z were perfectly symmetric, then $r_{x,xz} = 0$ and $r_{z,xz} = 0$. In this latter case, the addition of the cross-product xz term would have no effect on the B_1 and B_2 coefficients.

The result of adding the interaction term in the uncentered equation is dramatically different. The uncentered equation containng the interaction is given in Table 7.4.1F. The uncentered equation without the interaction is $\hat{Y} = -.257X + .916Z + 29.395$. The large changes in B_1 and B_2 when the XZ term is added are due to the fact that the B_1 and B_2 coefficients in the uncentered equation without versus with the interaction represent different things. In the equation without the interaction they are overall effects; in the equation with the interaction, the B_1 and B_2 coefficients are conditional, at the value of zero on the other predictor. In the centered equation, the B_1 and B_2 coefficients are again conditional at zero on the other predictor. However, they also represent the average effect of a predictor across the range of the predictor, much more closely aligned with the meaning of the B_1 and B_2 coefficients in the overall centered equation without an interaction.

The reader is cautioned that this discussion pertains to predicting the same dependent variable from only first-order effects and then from first-order effects plus interactions, as in the age, exercise, and endurance example. The example in Section 7.2 is not structured in this manner, but rather is a special pedagogical case—the dependent variables are different for the equation without versus with interactions, so the principles articulated here do not apply.

7.4.8 Interpretation and the Range of Data

A principle is illustrated in our cautions about interpreting coefficients in the uncentered data—regression analyses should be interpreted only within the range of the observed data. This is so whether or not equations contain interactions, and whether or not variables are centered. In graphical characterizations of the nature of the interaction between age and years of exercise on endurance, we confined the range of the x and y axes in Figs. 7.4.1(A) and 7.4.1(B) to well within the range of the observed data. We certainly would not extrapolate findings beyond the youngest and beyond the oldest participant ages (20 years, 82 years). Beyond limiting our interpretations to the confines of the range of the observed data, we encounter the issue of sparseness (very few data points) near the extremes of the observed data, just as we did in polynomial regression (Section 6.2.5). The limitation that sparseness places on interpretation of regression results is further discussed in Section 7.7.1.

7.5 STANDARDIZED ESTIMATES FOR EQUATIONS CONTAINING INTERACTIONS

To create a standardized solution for regression equations containing interactions, we must take special steps. First, we must standardize X and Z into z_x and z_z. Then we must form the cross-product term $z_x z_z$ to carry the interaction. The appropriate standardized solution has as the cross-product term the *cross-product of the z-scores for the individual predictors entering the interaction.* What happens if we simply use the "standardized" solution that accompanies the centered solution in usual regression analysis output? This "standardized" solution is improper in the interaction term. The XZ term that purportedly carries the interaction in the "standardized" solution reported in standard statistical packages is formed from the XZ term standardizing subjects' scores on the XZ product *after* the product is formed. It is the z-score of the product XZ, rather than the correct product of the z-scores $z_X z_Z$. The "standardized" solution that accompanies regression analyses containing interactions should be ignored. Instead, X and Z should be standardized first, then the cross-product of the z-scores should be computed, and these predictors should be entered into a regression analysis. The ***"raw" coefficients from the analysis based on z-scores*** are the proper standardized solution (Friedrich, 1982; Jaccard, Turrisi, & Wan, 1990). The improper and proper standardized solutions are given in Table 7.5.1 for the endurance example. The improper solution is given in Table 7.5.1A, the proper solution in Table 7.5.1B. There are two differences between the two solutions. First, the value of the

TABLE 7.5.1
Standardized Solution for the Regression of Endurance on Age and Years of Strenuous Exercise ($n = 245$)

A. Improper standardized solution taken from computer printout. The solution is the "standardized" solution that accompanies the centered regression analysis reported in Table 7.4.1.

$$\hat{Y} = -.244X + .429Z + .201XZ$$

Coefficient	SE	t test
$B_1 = -.244$	.060	−4.085
$B_2 = .429$	.060	7.124
$B_3 = .201$	.058	3.476
$B_0 = 0.00$		

B. Proper standardized solution. The solution is computed by forming z-scores from centered predictors and forming the cross-product of the z-scores.

1. Summary statistics for centered x and z and cross-product xz.

	Means and standard deviations			Correlation matrix			
	M	sd		x	z	xz	y
x	0.00	1.000	x	1.00	.28	.01	−.13
z	0.00	1.000	z		1.00	−.12	.34
xz	.28	.953	xz			1.00	.15
y	0.00	1.000	y				1.00

2. Proper standardized regression equation containing interaction:

$$\hat{Y} = -.244X + .429Z + .211XZ - .059.$$

Coefficient	SE	t test
$B_1 = -.244$	.060	−4.085
$B_2 = .429$	.060	7.124
$B_3 = .211$	.061	3.476
$B_0 = -.059$		

coefficient for the interaction B_3 changes slightly. The change, however slight in the present example, is important, because it affects the values of the simple slopes. In other circumstances, the difference may be more pronounced. Second, there is a nonzero intercept in the proper solution, since the $z_X z_Z$ term will have a nonzero mean to the extent that X and Z are correlated.

7.6 INTERACTIONS AS PARTIALED EFFECTS: BUILDING REGRESSION EQUATIONS WITH INTERACTIONS

In Eq. (7.1.2) the regression coefficient for the interaction, B_3, is a partial regression coefficient. It represents the effect of the interaction *if and only if* the two predictors comprising the interaction are included in the regression equation (Cohen, 1978). If only the XZ term were included in the regression equation and the X and Z terms were omitted, then the effect attributed to XZ would include any first order effects of X and Z that were correlated with the XZ term as well. Recall that in our numerical example X and Z each had an effect on the criterion, independent of their interaction. If X and Z had been omitted from regression Eq. (7.1.2), then any first-order effects of X and Z that were correlated with XZ would have been incorrectly attributed to the interaction. Only when X and Z have been linearly partialed from XZ does it, in general, become the interaction predictor we seek; thus,

$$X \text{ by } Z = XZ \cdot X, Z.$$

Interactions in MR analysis may be far more complex than the simple two-way interaction portrayed here. The next order of generalization we make is to more than two predictors. Whatever their nature, the predictors X, Z, and W may form a three-way interaction in their relationship to Y; that is, they may operate jointly in accounting for Y variance *beyond* what is accounted for by X, Z, W, XZ, XW, and ZW. This could mean, for example, that the nature of an interaction between X and Z differs as a function of the value of W. Put another way, the three-way interaction would signal that the Y on X regression varies with differing ZW *joint* values, or is conditional on the specific Z, W combination, being greater for some than others. The symmetry makes possible the valid interchange of X, Z, and W. The X by Z by W interaction is carried by the XZW product, which requires refining by partialing of constituent variables and two-way products of variables; that is,

$$X \text{ by } Z \text{ by } W = XZW \cdot X, Z, W, XZ, XW, ZW.$$

The proper regression equation for assessing the three-way interaction is

$$\hat{Y} = B_1X + B_2Z + B_3W + B_4XZ + B_5XW + B_6ZW + B_7XZW + B_0. \tag{7.6.1}$$

All lower order terms must be included in the regression equation for the B_7 coefficient to represent the effect of the three-way interaction on Y. (Consistent with the discussion of centering predictors in Section 7.2, if we were to center predictors X, Z, and W in Eq. 7.6.1, then only the value of the B_7 coefficient would remain constant across the centered versus uncentered equation, since now B_7 is the invariant highest order term.)

Higher order interactions follow the same pattern, both in interpretation and representation: a d-way interaction is represented by the d-way product from which the constituent main effect variables, the two-way, three-way, etc. up to $(d - 1)$-way products have been partialed, most readily accomplished by including all these lower order terms in the same multiple regression equation with the highest order term.

The fact that the mathematics can rigorously support the analysis of interactions of high order, however, does not mean that they should necessarily be constructed and used. Interactions greater than three-way certainly may exist. The many variables that are controlled to create uniform laboratory environments in the biological, physical, and social sciences are all potential sources of higher order interactions. Nonetheless, our current designs (and theories, to some extent) make it unlikely that we will detect and understand these effects. Recall that in Chapter 6, a similar argument was made about polynomial terms above the cubic (X^3). The reader should recognize that a quadratic polynomial term (X^2) is of the same order as a two-way cross-product term, XZ; both are of order 2. The cubic (X^3) term and the XZW interaction terms are of order 3. Data quality may well not support the treatment of interactions among more than three variables.

7.7 PATTERNS OF FIRST-ORDER AND INTERACTIVE EFFECTS

Thus far, we have encountered two different patterns of first-order and interaction effects in the two numerical examples. In our first numerical example the increases in predictor Z strengthened the relationship of X to Y, as illustrated in Fig. 7.3.1(B). In the second example, illustrated in Fig. 7.4.1(A), the nature of the interaction was quite different, in that a history of exercise weakened the deleterious effect of increased age on endurance.

In fact, a variety of interaction patterns are possible, and are reflected in the possible combinations of values of regression coefficients B_1, B_2 and B_3 in Eq. (7.1.2). We may have any combination whatever of zero, positive, and negative regression coefficients of first-order effects (B_1 and B_2), coupled with positive and negative interactive effects (B_3). The appearance of the interaction will depend on the signs of all three coefficients. Moreover, the precise nature of the interactions will be determined by the relative magnitudes of these coefficients.

7.7.1 Three Theoretically Meaningful Patterns of First-Order and Interaction Effects

We characterize three theoretically meaningful and interesting interaction patterns between two predictors; each pattern depends on the values of B_1, B_2, and B_3 in Eq. (7.1.2). First are *synergistic or enhancing interactions* in which both predictors affect the criterion Y in the same direction, and together they produce a stronger than additive effect on the outcome. As already mentioned, the interaction in the first numerical example (Fig. 7.3.1B) is synergistic; all three regression coefficients in the centered equation are positive. When both the first-order and interactive effects are of the same sign, the interaction is synergistic or enhancing. If all three signs are negative, we have the same synergistic effect. Suppose life satisfaction (Y) is negatively related to job stress (X) and to level of marital problems (Z). Their interaction is negative, so that having both high job stress and high marital problems leads to even less life satisfaction than the sum of X and Z would predict.

A theoretically prominent pattern of first-order and interactive effects is the *buffering interaction*, already defined in Section 7.4.5. Here the two predictors have regression coefficients of opposite sign. In addition, one predictor weakens the effect of the other predictor; that is, as the impact of one predictor increases in value, the impact of the other predictor is diminished. Buffering interactions are discussed in both mental and physical health research in which one predictor may represent a *risk factor* for mental or physical illness while the other predictor represents a *protective factor* that mitigates the threat of the risk factor (e.g., Cleary & Kessler, 1982; Cohen & Wills, 1985; Krause, 1995) . In the second numerical example in this chapter, increasing age (X) is the risk factor for diminished endurance (Y) and vigorous exercise is the protective factor (Z); the negative impact of age on endurance is lessened by a history of vigorous exercise. In this example $B_1 < 0, B_2 > 0$, and $B_3 > 0$.

A third pattern of interaction is an *interference or antagonistic interaction* in which both predictors work on the criterion in the same direction, and the interaction is of opposite sign (Neter, Kutner, Nachtsheim, & Wasserman, 1996). Recall the example mentioned at the outset of Section 7.1, that perhaps ability and motivation have compensatory effects on graduate school achievement. Surely both ability and motivation are each positively related to achievement ($B_1 > 0$ and $B_2 > 0$). Yet the importance of exceptional ability may be lessened by exceptional motivation, and vice versa, a partially "either-or" pattern of influence of the two predictors on the criterion. If so, their interaction is negative ($B_3 < 0$), that is, of the opposite sign of the two first-order effects.

It is clear from these examples that it is not simply the sign of the B_3 regression coefficient for the interaction that determines whether an interaction is enhancing or buffering or antagonistic. Rather the pattern of signs and magnitudes of the coefficients for all three terms in Eq. (7.1.2) determine the form of the interaction.

All the interactions we have considered here are linear by linear (see Section 7.3.4); the simple slopes are all linear in form. Such patterns of interactions may be observed in more complex regression equations, for example, as components of three-way interactions, described in Section 7.8. The patterns of interactions are not confined to linear by linear interactions. These patterns generalize as well to more complex equations with curvilinear relationships as well, described in Section 7.9.

7.7.2 Ordinal Versus Disordinal Interactions

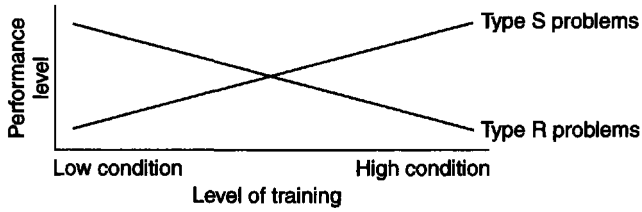
The interactions in both numerical examples [illustrated in Figs. 7.3.1(B) and 7.4.1(A)] are both *ordinal* interactions. Ordinal interactions are those interactions in which the rank order of the outcomes of one predictor is maintained across all levels of the other predictor *within the observed range of the second predictor.* These interactions are typical of interactions obtained in observational studies. In psychological research in which existing variables are measured (there is no manipulation), we most often observe *ordinal interactions.*

Figure 7.7.1 illustrates a *disordinal interaction* between level of problem solving training (X) and type of training (Z) on problem solving performance (Y). Here the rank order of factor Z in relation to the criterion Y changes as a function of the value of factor X (i.e., whether cases with high or low scores on variable Z have higher criterion scores varies with changing X).

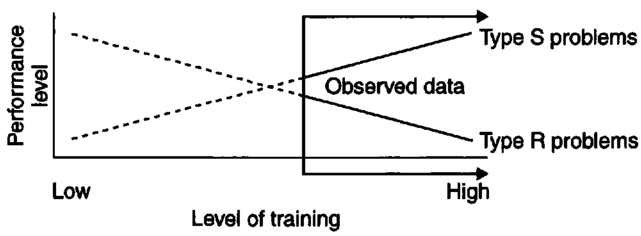
What produces the noncrossing (ordinal) versus crossing (disordinal) appearance of Fig. 7.4.1(A) versus Fig. 7.7.1(A) is the strength of the first-order effects also portrayed in the figures. In Fig. 7.4.1(A), the large first-order effect of exercise on endurance forces the three regression lines apart; people with a long history of exercise have much greater endurance regardless of age than those with moderate exercise histories; the same is true for moderate versus low histories of exercise. Figure 7.4.1(A) actually portrays both the first-order effects of exercise and age and their interaction. If we subtracted out the effects of exercise and age, leaving a pure interaction plot, then the simple regression lines would cross. All the figures in this chapter include both first-order and interactive effects; they are not pure graphs of interactions only and so are better termed plots of simple regression lines than plots of interactions per se.

The more specific term *crossover interaction* is sometimes applied to interactions with effects in opposite directions; hence, Fig. 7.7.1(A) can also be termed a crossover interaction (see Section 9.1 for further discussion). Crossover interactions are often predicted in experimental settings. In a study of teaching methods (lecture versus seminar), we might predict that teaching method interacts with subject matter. For example, our prediction might be that lecture leads to better learning of statistical methods, whereas a seminar format leads to better

(A) Performance in an experiment on Type R versus Type S problems as a function of level of training (Low, High) received in the experiment.



(B) Performance in an observational study as a function of training experienced by individuals prior to participation in the observational study.



(C) Weak interaction with strong first-order effect of problem type.

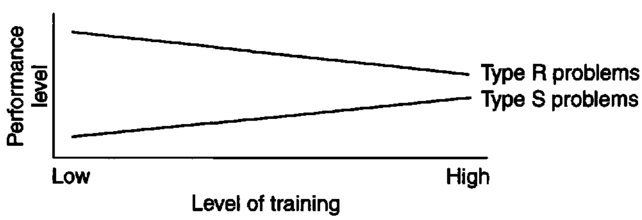


FIGURE 7.7.1 Forms of interactions as a function of sampled range of predictor variables and strength of first-order versus interaction effects.

learning of substantive material, a crossover prediction. We do not mean to imply that crossover interactions never obtain in observational research in which there is no manipulation—they are simply more rare.

The phrase *within the range of the observed data* adds a level of complexity in MR analysis applied to observational data rather than to data gathered in experimental settings. In experiments, the experimental conditions implemented define the range of the predictors (the conditions of experimental variables are translated into predictors in MR). In Fig. 7.7.1(A), suppose there is a crossover interaction between amount of training and performance on two types of problems, Types R and S. Extensive training on Type S facilitates performance but extensive training on Type R leads to boredom and performance deficits. The range of the x axis, amount of training, is fixed by the specific conditions implemented in the experiment. Now consider an observational study in which people are interviewed to assess their amount of previous training on a specific type of task (a continuous variable) and then their performance is assessed on tasks R and S. The study is observational because there is no experimental manipulation of training on the tasks. Instead of a fixed range of training having been manipulated in the experiment, the range of training is determined by the previous experience of the particular subjects sampled. Suppose all subjects have had moderate to extensive levels of training. The simple regression equations might appear as in the right-hand portion of Fig. 7.7.1(B), the portion of "observed data." Within the range of training represented on variable X in the sample, the simple regression lines do not cross. In fact, these simple regression lines would cross at lower levels of extent of training, not observed in the sample. This crossover is represented by the dashed portion of the regression lines in Fig. 7.7.1(B)—if we assume linear relationships across all levels of training. In sum, whether we observe ordinal or disordinal interactions in MR analysis may depend upon the range of values on particular predictors across the subjects of a sample. We say *may* here, because the simple regression lines may cross at some numerical point outside the meaningful range of the variable.

In the experimental setting, the researcher controls the range of manipulated variables such as training level by the nature of the particular conditions implemented. In fact, wise experimenters pilot test various forms of their manipulations until they find ones strong enough to produce effects they seek (this is not to say experimenters "cheat," but rather they structure their experiments to optimize the possibility of observing the predicted relationships.) Hence, an experimenter would wisely create dramatically different training levels in the two training conditions to optimize the possibility of observing the crossover (here, perhaps, by training people not at all or by training them to distraction). In an observational setting, the researcher may also control this range by systematic sampling of cases (Pitts & West, 2001) if there is prior knowledge of subjects' scores on variables of interest—here, level of previous experience. In our training example, if experience scores on a large pool of subjects had been available, then the researcher might have systematically sampled a very wide range of experience, thereby leading to the observation of the disordinal (crossover) interaction.

Crossing Point of Simple Regression Lines

The value of a predictor at which simple regression lines cross can be determined algebraically for any specific regression equation and predictor within that equation. For Eq. (7.1.2), the value of X at which the simple regressions of Y on X cross is

$$X_{\text{cross}} = -B_2/B_3 \tag{7.7.1}$$

for the simple regressions of Y on X at values of Z.

For Eq. (7.1.2), the value of Z at which the simple regressions of Y on Z cross is

$$Z_{\text{cross}} = -B_1/B_3 \tag{7.7.2}$$

for the simple regressions of Y on Z at values of X. Equation (7.7.1) and (7.7.2) are instructive in three ways.

1. First, the denominator in both cases is the regression coefficient B_3 for the interaction. If this interaction is zero, then the simple regression lines will not cross—the simple regression lines are parallel.

2. Second, the numerators of these expressions tell us that the crossing point also depends on the magnitude of the first-order effects relative to the interaction. In Eq. (7.3.1) for the regression of Y on X at values of Z, suppose B_2, the regression coefficient for Z, is very large, relative to the interaction (in ANOVA terms, a large main effect coupled with a small interaction). If so, then if B_3 is positive, the regression lines will cross somewhere near minus infinity for positive B_2 or plus infinity for negative B_2. This is illustrated in Fig. 7.7.1(C) versus Fig. 7.7.1(A). In Fig. 7.7.1(A), there is no first-order effect of type of task, $B_2 = 0$ (on average performance on Type R and Type S problems is equal). In Fig. 7.7.1(C), however, there is a large first-order effect of type of task coupled with a smaller interaction than in Fig. 7.7.1(A).

That Type R problems are so much better solved than are Type S problems in Fig. 7.7.1(C) means that regardless of the boredom level induced by training, performance on the Type R problems will not deteriorate to that on Type S problems.

3. Third, the crossing point of the lines depends on the pattern of signs (positive versus negative) of the first-order and interactive effects. First, consider the centered regression equation corresponding to the illustration of simple slopes in Fig. 7.3.1(B); that is, $\hat{Y} = 2.2x + 2.6z + .4xz + 16$. From Fig. 7.3.1(B) it appears as though the simple regression lines cross at a low numerical value outside the range of centered variable x. In fact, for this regression equation $x_{\text{cross}} = -B_2/B_3 = -2.6/4 = -6.5$, well outside the range of the variables illustrated in Fig. 7.3.1(B). Now consider the centered regression equation for the prediction of endurance as a function of age (x) and years of vigorous exercise (z), $\hat{Y} = -.26x + .97z + .05xz + 25.89$. Simple slopes for the regression of endurance on age as a function of exercise (Y on centered x at centered z) are given in Fig. 7.4.1(A). For this regression equation, $X_{\text{cross}} = -B_2/B_3 = -.97/.05 = -19.40$, or 19.40 years below the mean age of 49.18 years. This suggests that for people of age $49.18 - 19.40 = 30.4$ years of age, endurance does not vary as a function of level of vigorous exercise—all the simple slope lines converge. Note that this crossing point is almost two standard deviations below the mean age ($sd_{\text{age}} = 10.11$). Data would be expected to be very sparse at two sds below the M_{age}; in fact, there are only 7 cases of the 245 who are younger than 30. We would not wish to make generalizations about endurance and age in individuals under 30 with so few data points; there is no guarantee that the relationships are even linear at younger ages. This is an important point to note—in MR we may extend the simple regression lines graphically as far as we wish (as opposed to in an experiment where the experimental conditions set the limits); the issue becomes one of whether there are data points at the extremes. Now consider Fig. 7.4.1(B), which shows the regression of endurance on exercise as a function of age (Y on centered z as a function of centered x). For this representation of the data, $Z_{\text{cross}} = -B_1/B_3 = -(-.26)/.05 = 5.20$, or 5.20 years above the mean level of exercise. With a mean years of exercise of 10.67 this corresponds to $10.67 + 5.20 = 15.87$ years of exercise; the data thus suggest that for individuals who have exercised vigorously for 16 years, endurance is independent of age. Only 23 individuals of the 245 have exercised more than 16 years. Again we do not wish to make inferences beyond this point with so few cases, since the form of the regression equation might be quite different for very long-term exercisers.

It is mathematically true that so long as the interaction term is nonzero (even minuscule!) there is a point at which the simple regression lines will cross. In the case of both numerical examples, if we assume that the x axis as illustrated in Fig. 7.3.1 or the x and z axes in Fig. 7.4.1 represent the meaningful ranges of the variables, then there is no crossover within these meaningful ranges. Putting this all together, whether an interaction is ordinal or disordinal depends on the strength of the interaction relative to the strength of the first-order effects coupled with the presence of cases in the range of predictors where the cross occurs, if B_3 is nonzero. When main effects are very strong relative to interactions, crossing points are at extreme values, even beyond the meaningful range (or actual limits) of the scale on which they are measured, that is, the scale of predictor X in Eq. (7.7.1) or the scale of Z in Eq. (7.7.2). Whether we observe the crossing point of simple slopes depends on where it is relative to the distribution of scores on the predictor in the population being sampled and whether the sample contains any cases at the crossing point. The crossing point may have no real meaning—for example, if we were to find a crossing point at 120 years of age (for a sample of humans, not Galapagos tortoises). In contrast, a crossing point expected to occur at about 70 years of human age might be illuminated by systematically sampling cases in a range around 70 years of age, if the crossing point is of theoretical interest.

Crossing Points Are Equation Specific

There is an algebraic expression for the crossing with regard to each predictor in a regression equation containing higher order terms. This expression differs by equation; see Aiken and West (1991) for crossing-point expressions for more complex equations.

7.8 THREE-PREDICTOR INTERACTIONS IN MULTIPLE REGRESSION

Linear by linear interactions in multiple regression generalize beyond two-variable interactions. We provide a brief treatment of the three-way interaction Eq. (7.6.1):

$$\hat{Y} = B_1X + B_2Z + B_3W + B_4XZ + B_5XW + B_6ZW + B_7XZW + B_0. \tag{7.6.1}$$

First, we reiterate that all lower order terms must be included in the regression equation containing the XZW variable. The X by Z by W interaction is carried by the XZW term but only represents the interaction when all lower order terms have been partialed.

In our example of endurance predicted from age (X) and exercise history (Z), suppose we add a third predictor W, a continuously measured index of healthy life style that includes having adequate sleep, maintaining appropriate weight, not smoking, and the like. We predict endurance from age (X), exercise history (Z), and healthy life style (W), using Eq. (7.6.1). We find that the three-variable interaction is significant and wish to interpret it.

Given three factors, we may break down the interaction into more interpretable form by considering the interaction of two of the factors at different values of the third factor. Suppose we consider the XZ interaction of age with strenuous exercise at two values of W, that is, for those people who have maintained a healthy life style versus not. We choose W_{low} and W_{high} to represent people whose life style practices are one standard deviation below and above the mean life style score. (In a more complete analysis of the XZW interaction, we also would have plotted the XZ interaction at the mean of W, i.e., at W_{mean}). Further, we could have chosen any combination of variables (e.g., the interaction of exercise Z with lifestyle W for different ages).

To plot the three-way interaction as the XZ interaction at values of W, we choose which variable of X and Z will form the x axis. Suppose we follow Fig. 7.4.1(B), which shows the regression of endurance on exercise (Z) at values of age (X). We will make two such graphs, for W_{low} and W_{high}, respectively. This amounts to characterizing the XZW interaction as a series of simple regression equations of Y on Z at values of X and W. We arrange Eq. (7.6.1) to show the regression of Y on Z at values of X and W, yielding the following simple slope expression:

$$\hat{Y} = (B_2 + B_4X + B_6W + B_7XW)Z + (B_1X + B_3W + B_5XW + B_0). \tag{7.8.1}$$

Figure 7.8.1 provides a hypothetical outcome of the three-way interaction. The pattern of regression of Y (endurance) on Z (exercise) as a function of age (X) differs depending on the extent of a healthy life style (W), signaling the presence of a three-way XZW interaction. For individuals who have maintained a healthy life style (at W_{high}), endurance increases with length of exercise history. The amount of increase depends upon age: exercise has an increasingly salutary effect with increasing age. In statistical terms, there is an XZ interaction at W_{high}. We note that the interaction is ordinal in the range of the data (the three simple regression lines do not cross). If individuals older than those represented in Fig. 7.8.1 had been sampled, these simple regression lines would have been observed to cross, yielding a disordinal interaction within the range of the observed data. For individuals who have not maintained a healthy life style (at W_{low}), exercise does not have an increasingly salutary effect

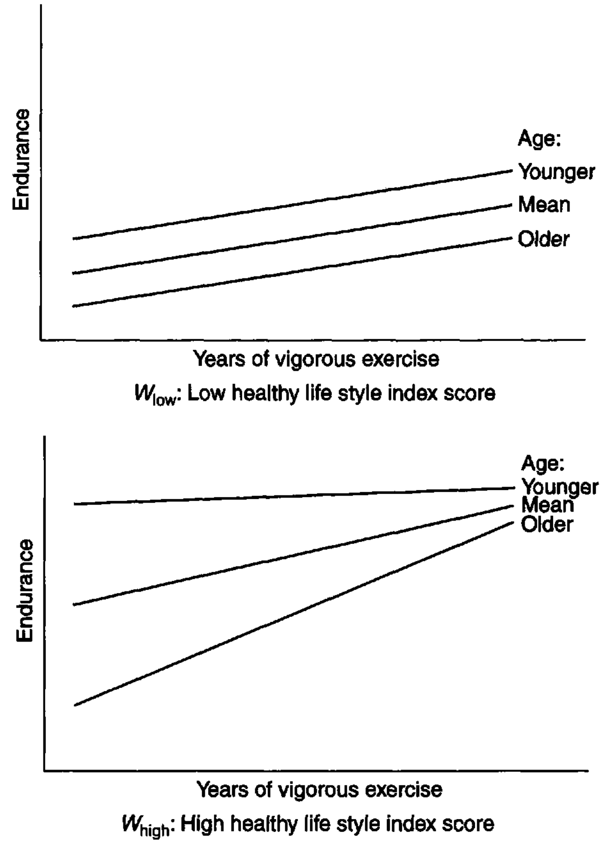


FIGURE 7.8.1 Display of simple slopes for a three-way continuous variable interaction. Regression of endurance (Y) on years of vigorous exercise (Z) is shown at three values of age (X) and at low versus high values of a healthy life style index.

with increasing age. Although endurance increases with increasing exercise, older individuals do not show a stronger benefit from exercise than do younger individuals. Apparently, the special benefit of exercise for older individuals is offset by lack of an otherwise healthy life style. In statistical terms, there is not a significant XZ interaction at W_{low}. We note that the regression lines at W_{low} are parallel; they never cross. ANOVA users will recognize the interactions XZ at W_{high} and XZ at W_{low} to be *simple interactions* (Kirk, 1995; Winer, Brown, & Michaels, 1991), interactions between two variables confined to one level (or value) of a third variable.

As an aid to interpretation of the three-way interaction, one may test each of the six simple slopes in Fig. 7.8.1 for significance. Aiken and West (1991, Chapter 4) provide a full development of post hoc probing of the three-way continuous variable interaction including standard errors and t tests for the simple slope in Eq. (7.8.1), a numerical example of probing a three-way interaction, and computer code for probing the three-way interaction with standard MR software. Beyond simple slopes, one may test for the significance of the interaction between X and Z at varying values of W, that is, the *simple interactions* illustrated in Figure 7.8.1 (Aiken & West, 2000).

7.9 CURVILINEAR BY LINEAR INTERACTIONS

All the interactions we have considered are linear by linear in form. However, curvilinear variables may interact with linear variables. Figure 7.9.1 provides an example of such an interaction. In this hypothetical example, we are predicting intentions to quit smoking from smokers' fear of the negative effects of smoking on health (X). According to theorizing on the impact of fear communication on behavior mentioned in Chapter 6 (Janis, 1967), fear should be curvilinearly related to intention to act. Intention should increase with increasing fear up to a point. Then as the fear becomes more intense, individuals should wish to avoid the whole issue or become so focused on managing the fear itself that intention to quit smoking is lowered. If we considered only the regression of intention on fear, we would have the polynomial regression equation given in Chapter 6:

(6.2.3) $$\hat{Y} = B_{1.2}X + B_{2.1}X^2 + B_0,$$

where the combination of predictors X and X^2 represents the total effect of variable X on Y.

A second predictor is now considered, the individual's self-efficacy for quitting smoking (Z), that is, the individual's belief that he or she can succeed at quitting smoking. Suppose that intentions to quit smoking rose in a constant fashion as self-efficacy increased, but that there was no interaction between fear (X) and self-efficacy (Z). The appropriate regression equation would be as follows:

(7.9.1) $$\hat{Y} = B_1X + B_2X^2 + B_3Z + B_0.$$

Two things would be true about the relationship of X to Y illustrated at different values of Z. First, the shape of the simple regressions of Y on X would be constant across values of Z; put another way, any curvilinearity of the relationship of X to Y represented by the $B_1X + B_2X^2$ terms would be constant over all values of Z. Second, the simple slope regression curves of Y on

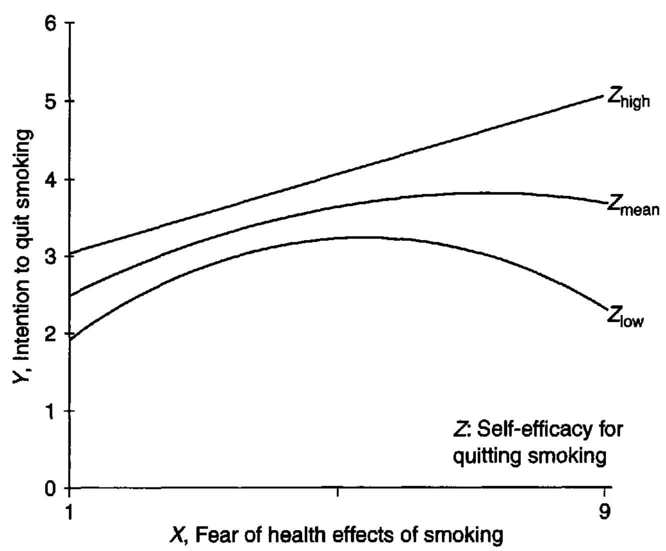


FIGURE 7.9.1 Illustration of a curvilinear by linear interaction. The degree of curvilinearity of the relationship of fear (X) to intention (Y) depends linearly upon the level of self-efficacy for quitting smoking (Z).

X would be parallel for all values of Z, as before. However, it is quite evident from Fig. 7.9.1 that neither of these conditions holds. First, the simple slope equations are not parallel, suggesting the presence of an XZ interaction. Second, the three simple regressions are not of the same shape; variable Z moderates the curvilinearity of the relationship of X to Y. When self-efficacy for quitting is low (at Z_{low}), the relationship of X to Y is strongly curvilinear. As self-efficacy for quitting increases, the relationship of X to Y becomes increasingly linear; high self-efficacy appears to overcome the decrease in intentions at high fear. We have a *curvilinear by linear interaction*; that is, the curvilinear relationship of X to Y changes as variable Z increases linearly. The appropriate regression equation to model this relationship is as follows:

$$\hat{Y} = B_1X + B_2X^2 + B_3Z + B_4XZ + B_5X^2Z + B_0. \tag{7.9.2}$$

Here for the first time we encounter two interaction components of a single pair of variables, XZ and X^2Z. These interaction terms are formed by crossing each of the terms in the "set" of variables that represent the impact of X on Y (i.e., X and X^2) with the predictor Z. Together XZ and X^2Z represent the interaction of X and Z in predicting Y, given that lower order predictors X, X^2, and Z are included in the equation. There are now two components of the interaction, the familiar XZ linear by linear component plus the curvilinear by linear component represented by X^2Z. In Eq. (7.9.2) the highest order term is X^2Z; for this term to represent the curvilinear by linear interaction, all lower order terms must be included. To see these terms clearly we can rewrite X^2Z as XXZ. The lower order terms are all the terms that can be constructed by taking one or two of the three letters of XXZ, that is, X, Z, $X^2 = XX$, and XZ. The equation is estimated by taking centered x and centered z and forming the higher order terms from them. The new term x^2z is formed by squaring centered x and multiplying the square by centered z. This term represents the curvilinear by linear interaction between x and z if and only if all the other terms in Eq. (7.9.2) are included in the model; these are all the lower order terms that can be created from x and z.

In centered Eq. (7.9.2) the role of each term can be separately characterized, given the presence of all other terms in the equation. From Fig. 7.9.1 on average, intention to quit smoking increases on average as fear increases; hence B_1 is positive. The overall relationship of X to Y is concave downward; hence B_2 is negative. Intention to quit increases as self-efficacy (Z) increases; therefore B_3 is positive. As indicated, the XZ term represents the linear by linear component of the XZ interaction. If we ignored the curvilinear relationship of X to Y and just found the best fitting straight line relating X to Y at each value of Z, these simple regression lines would not be parallel; we have the synergistic interaction that as both fear and self-efficacy increase, intentions increase by more than just their sum; thus the B_4 term is positive. Finally, the B_5 term is positive. It is juxtaposed against the negative B_2 term, which carries the curvilinearity of the X–Y relationship. The positive B_5 term in a sense "cancels out" the downward curvilinearity in the B_2 term as Z increases; that is, when Z is high, the curvilinearity in the prediction of Y by X disappears.

Rearranging Eq. (7.9.2) to show the regression of Y on X at values of Z provides further insight.

$$\begin{aligned} \hat{Y} &= B_1X + B_4XZ + B_2X^2 + B_5X^2Z + B_3Z + B_0 \\ \hat{Y} &= (B_1 + B_4Z)X + (B_2 + B_5Z)X^2 + (B_3Z + B_0). \end{aligned} \tag{7.9.3}$$

Equation (7.9.3) is in the form of a second-order polynomial of the regression of Y on X. However, it is more complex than Eq. (6.2.3). In Eq. (7.9.3) we see that the overall linear regression of Y on X, given in the coefficient $(B_1 + B_4Z)$ depends on the value of Z, as do both the extent of curvilinearity of the regression of Y on X^2, given in the coefficient $(B_2 + B_5Z)$, and the intercept $(B_3Z + B_0)$. The coefficients $(B_1 + B_4Z)$, $(B_2 + B_5Z)$, and $(B_3Z + B_0)$ give the global form of the regression of Y on X. These are not simple slope coefficients.

CH07EX02 Table 7.9.1 characterizes the data and summarizes the outcome of the regression analysis. In Table 7.9.1B, Eq. (7.9.1) containing no interaction is estimated. Interestingly, the B_2 coefficient for X^2 is not different from zero. Fitting the data with a regression equation that does not capture the appropriate interaction leads to failure to detect the curvilinearity that exists for some portion (but not all) of the regression surface. Including the two interaction terms in the regression equation, as given in Table 7.9.1C, leads to an increment in R^2 of .03. The full interaction carried by the combination of the XZ and X^2Z terms accounts for 3% of variance in the criterion over and above the X, X^2 and Z terms, $F_{\text{gain}}(2, 244) = 8.30$, $p < .001$.

Putting the full regression equation of Table 7.9.1C into the form of Eq. (7.9.3) quantifies the relationships of x to y at values of Z observed in Fig. 7.9.1. The rearranged equation is given in Table 7.9.1D. Now we see that the term $(B_2 + B_5Z) = (-.052 + .065z)$ becomes less and less negative as the numerical value of z increases, consonant with the decreasing downward curvilinearity in the relationship of X to Y as Z increases. With $sd_z = .76$, substituting the values of $z_{\text{low}} = -.76$, $z_{\text{mean}} = 0.0$, and $z_{\text{high}} = .76$ yields the three simple regression equations given in Table 7.9.1D. At z_{low}, there is a strong curvilinear relationship between X and Y. This curvilinear relationship diminishes as z increases. In contrast, the linear component of the impact of x on y is close to zero at z_{low} and increases as z increases. These simple equations show

TABLE 7.9.1
Regression Analysis of Intention to Quit Smoking (Y) as a Function of Centered Fear of Health Effects of Smoking (x) and Centered Self-Efficacy for Quitting Smoking (z) ($n = 250$)

A. Summary statistics for centered x, x^2, centered z, xz, and x^2z

Means and standard deviations			Correlations						
	M	sd		x	x^2	z	xz	x^2z	Y
x	0.00	1.22	x	1.00	.19	.32	.10	.58	.49
x^2	1.47	2.04	x^2		1.00	.07	.51	.23	.08
z	0.00	.76	z			1.00	−.01	.62	.65
xz	.29	.91	xz				1.00	.07	.14
x^2z	.12	1.95	x^2z					1.00	.58
Y	3.63	.86	Y						1.00

B. Centered regression equation with no interaction ($R^2_{Y.123} = .51$)

$$\hat{Y} = .224x^{**} - .008x^2 + .620z^{**} + 3.642.$$

C. Centered regression equation with curvilinear by linear interaction ($R^2_{Y.12345} = .54$)

$$\hat{Y} = .178x^{**} - .052x^{2*} + .551z^{**} + .164xz^{**} + .065x^2z^{*} + 3.651.$$

D. Centered regression equation with curvilinear by linear interaction showing regression of Y on X.

$$\hat{Y} = (.178 + .164z)x + (-.052 + .065z)x^2 + (.551z + 3.561)$$

For $Z_{\text{low}} = -.76$: $\hat{Y} = .053x - .101x^{2**} + 3.230$

For $Z_{\text{mean}} = 0.00$: $\hat{Y} = .178x^{**} - .052x^{2*} + 3.651$

For $z_{\text{high}} = .76$: $\hat{Y} = .304x^{**} - .003x^2 + 4.072$

$^{**}p < .01$; $^{*}p < .05$.

$R^2_{Y.12345}$ is the squared multiple correlation from all 5 predictors.

quantitatively what we observe in Fig. 7.9.1. Estimating curvilinear by linear interactions has been useful theoretically; for an example, see Krause (1995). Aiken and West (1991) provide a full treatment of the probing of curvilinear by linear interactions, along with computer code for use in standard regression software.

7.10 INTERACTIONS AMONG SETS OF VARIABLES

In the curvilinear by linear interaction, Eq. (7.9.2), the interaction was carried by a set of predictors that included XZ and X^2Z. This is but one example of a broad characterization of interactions. All the statements previously made about interactions among single variables x, z, and w also hold for *sets* of variables $\boldsymbol{X}$, $\boldsymbol{Z}$, and $\boldsymbol{W}$ (here we denote a set of variables in bold italics). Specifically, if $\boldsymbol{X}$ is a set of k IVs $(x_i; i = 1, 2, \ldots, k)$, and $\boldsymbol{Z}$ a different set of j IVs $(z_j; j = 1, 2, \ldots, j)$, then we can form an $\boldsymbol{XZ}$ product set of kj variables by multiplying each x_i by each z_j. The $\boldsymbol{X}$ by $\boldsymbol{Z}$ interaction is found in the same way as

$$\boldsymbol{X} \text{ by } \boldsymbol{Z} = \boldsymbol{XZ} \cdot \boldsymbol{X}, \boldsymbol{Z}.$$

As in Eq. (7.9.2) the interaction is now carried by a set of cross-product terms. Generalizing further, suppose we had two predictors X_1 and X_2 in set $\boldsymbol{X}$, and two predictors Z_1 and Z_2 in set $\boldsymbol{Z}$. Then our regression equation is as follows:

$$\hat{Y} = B_1X_1 + B_2X_2 + B_3Z_1 + B_4Z_2 + B_5X_1Z_1 + B_6X_1Z_2 + B_7X_2Z_1 + B_8X_2Z_2 + B_0. \tag{7.10.1}$$

The interaction between sets $\boldsymbol{X}$ and $\boldsymbol{Z}$ is carried by the set of terms $[B_5X_1Z_1 + B_6X_1Z_2 + B_7X_2Z_1 + B_8X_2Z_2]$. To assess the contribution of the $\boldsymbol{X}$ by $\boldsymbol{Z}$ interaction to the overall regression over and above the first-order effects of the four individual predictors, we must use a hierarchical approach. The contribution of the interaction to variance accounted for in the criterion is the difference between $R^2_{Y.12345678}$ and $R^2_{Y.1234}$.

Interpretively, an exactly analogous joint or conditional meaning obtains for interactions of sets $\boldsymbol{X}$ and $\boldsymbol{Z}$: the regression coefficients that relate Y to the X_i of the $\boldsymbol{X}$ set are not all constant, but vary with the changes in the Z_i values of the $\boldsymbol{Z}$ set (and, too, when $\boldsymbol{X}$ and $\boldsymbol{Z}$ are interchanged in this statement). Stated in less abstract terms, this means that the nature and degree of relationship between Y and $\boldsymbol{X}$ varies, depending on $\boldsymbol{Z}$. Note again that if only the first-order effects of set $\boldsymbol{X}$ and $\boldsymbol{Z}$ are included, whatever is found to be true about the relationship of Y with $\boldsymbol{X}$ alone is true across the full range of the $\boldsymbol{Z}$ set. However, when an $\boldsymbol{X}$ by $\boldsymbol{Z}$ interaction is present, the relationship of the $\boldsymbol{X}$ set to the criterion changes with (is conditional on) changes in the Z_j values of $\boldsymbol{Z}$. (Again, symmetry permits interchanging $\boldsymbol{X}$ and X_i with $\boldsymbol{Z}$ and Z_j.)

The importance of this analytic strategy lies in the fact that some of the most interesting findings and research problems in behavioral and social science lie in conditional relationships. For example, the relationship between performance on a learning task (Y) and anxiety ($\boldsymbol{X}$) may vary as a function of psychiatric diagnosis ($\boldsymbol{Z}$). As another example, the relationship between income (Y) and education ($\boldsymbol{X}$) may vary as a function of race ($\boldsymbol{Z}$). As yet another example, in aggregate data where the units of analysis are urban neighborhoods, the relationship between incidence of prematurity at birth (Y) and the female age distribution ($\boldsymbol{X}$) may depend on the distribution of female marital status ($\boldsymbol{Z}$). The reader can easily supply other examples of possible interactions. The reason we represent these research factors as sets is that it may take more than one variable to represent each or, in the language of our system, each research factor may have been represented in more than one aspect of interest to us (as discussed in Chapter 5). As we saw in Chapter 6, and again in Section 7.9, if a research factor has a curvilinear relationship to the criterion, then more than one predictor is required to represent that relationship. (If anxiety bore an inverted U-shaped relationship to performance, we would require two predictors X_i and X_i^2 to represent the linear and quadratic aspects of anxiety in the prediction equation.) Nonnormally distributed variables (say, of age distributions in census tracts) may be represented in terms of their first three moments (X_1 = mean, X_2 = *sd*, X_3 = skew), and any categorical variable of more than two levels requires for complete representation at least two terms (see Chapter 8 for categorical predictors).

A most important feature of the aforementioned procedure is the interpretability of each of the single product terms X_iZ_j. As noted, the multiplication of the k X predictors of the X set by the j Z predictors of the Z set results in a product set that contains kj IVs (for example, the four IVs in Eq. 7.10.1). Each X_i is a specifiable and interpretable aspect of X and each Z_j is a specifiable and interpretable aspect of Z. Thus when partialed, each of these kj IVs, X_i by Z_j represents an interpretable aspect of X by aspect of Z interaction, a *distinct* conditional or joint relationship, and like any other IV, its B, sr, pr, and their common t test are meaningful statements of regression, correlation, and significance status.

There are issues in working with interactions of sets of predictors that bear consideration. First, we note in Eq. (7.10.1) that multiple cross-product terms necessarily include the same variable (e.g., the X_1Z_1 and X_1Z_2 terms both include X_1). If the predictors are not centered, then cross-product terms that share a common predictor will be highly correlated, rendering separate interpretation of the individual components of the interaction difficult. Centering will eliminate much of this correlation. Hence our recommendations for centering apply here. Second, the issue of how Type I error is allocated in testing the interaction must be considered. The omnibus test of the complete interaction is the hierarchical test of gain in prediction from the inclusion of all the cross-product terms that comprise the interaction. In the hierarchical regression, we would assign a nominal Type I error rate, say $\alpha = .05$, to the overall multiple degree of freedom omnibus test of the interaction. However, in Eq. (7.10.1), for example, four terms comprise the single set X by set Z interaction. If these terms are tested in the usual manner in the MR context, then a nominal Type I error rate, say $\alpha = .05$, will be assigned to each of the four components of the interaction. The overall collective error rate for the test of the set X by set Z will exceed the nominal Type I error rate. The issue here is closely related to the issue of multiple contrasts in the ANOVA contrast (see Kirk, 1995, pp. 119–123 for an excellent discussion), where thinking about assignment of Type I error to multiple contrasts versus an omnibus test is well developed. If there is only a global hypothesis that set X and set Z interact, then it is appropriate to assign a nominal Type I error rate, say $\alpha = .05$, to the overall multiple degree of freedom omnibus test of the interaction, and to control the collective error on the set of tests of the individual interaction components that comprise the overall test. If, on the other hand, there are a priori hypotheses about individual components of the overall interaction, then following practice in ANOVA, one may assign a nominal Type I error rate to the individual contrast.

As discussed in Chapter 5, the concept of set is not constrained to represent aspects of a single research factor such as age, psychiatric diagnostic group, or marital status distribution. Sets may be formed that represent a functional class of research factors, for example, a set of variables collectively representing demographic status, or a set made up of the subscales of a personality questionnaire or intelligence scale, or, as a quite different example, one made up of potential common causes, that is, variables that one wishes to statistically control while studying the effects of others. However defined, the global X by Z interactions and their constituent X_iZ_j single-interaction IVs are analyzed and interpreted as described previously.

7.11 ISSUES IN THE DETECTION OF INTERACTIONS: RELIABILITY, PREDICTOR DISTRIBUTIONS, MODEL SPECIFICATION

7.11.1 Variable Reliability and Power to Detect Interactions

The statistical power to detect interaction effects is a serious concern. We pointed out that interactions typically observed in psychological and other social science research often account for only a few percentage points of variance over and above first-order effects (i.e., squared semipartial or part correlations of .01 to .05 or so). J. Cohen (1988) defined squared partial correlations of .02, .13, and .26 of a term in MR with the criterion as representing small, moderate, and large effect sizes, respectively. Large effect size interactions are rarely found in observational studies in social science, business, and education; small to moderate effect size interactions predominate.

If predictors are measured without error (i.e., are perfectly reliable), then the sample size required to detect interactions in Eq. (7.1.2) are 26 for large effect size, 55 for moderate effect size, and 392 for small effect size interaction (J. Cohen, 1988). Even though fixed effects regression analysis assumes error-free predictors (see Chapter 4), in reality predictors are typically less than perfectly reliable. In fact, we are typically pleased if the reliabilities of our predictors reach .80.

The reliability of the XZ cross-product term in Eq. (7.1.2) is a function of the reliabilities of the individual variables. With population reliabilities ρ_{xx} and ρ_{zz} for X and Z, respectively, the reliability $\rho_{xz,xz}$ of the cross-product term of two *centered* predictors x and z with uncorrelated *true scores* (see Section 2.10.2) is the product of the reliabilities of the individual predictors (Bohrnstedt & Marwell, 1978; Busemeyer & Jones, 1983):

$$\rho_{xz,xz} = \rho_{xx}\rho_{zz} \tag{7.11.1}$$

For example, if two uncorrelated predictors X and Z each have an acceptable reliability of .80, the estimate of reliability of their cross-product term according to Eq. (7.11.1) is quite a bit lower at (.8)(.8) = .64. The effect of unreliability of a variable is to reduce or *attenuate* its correlation with other variables (as discussed in Section 2.10.2). If a predictor is uncorrelated with other predictors in an equation, then the effect of unreliability of the predictor is to attenuate its relationship to the criterion, so its regression coefficient is underestimated relative to the true value of the regression coefficient in the population. With centered x and z in Eq. (7.1.2), we expect minimal correlation between the x and xz, and between the z and xz terms; the nonessential multicollinearity has been eliminated by centering (see Section 7.2.4). Thus, when individual predictors are less than perfectly reliable, the interaction term is even more unreliable, and we expect the power to detect the interaction term to be reduced, relative to the power to detect the first-order effects, even if they have equal effect sizes in the population. When predictors X and Z have reliability 1.0 and the true effect size of the interaction is moderate, 55 cases are required for power .80 to detect the interaction. When each predictor (X, Z) has reliability .88, the required sample size for power .80 to detect an interaction ranges from 100 to 150 or more, depending on the amount of variance accounted for by the main effects of X and Z. For a small effect size interaction, the required sample size for .80 power to detect an interaction may exceed 1000 cases when the reliabilities of the individual predictors are each .80! (See Aiken and West, 1991, Chapter 8, for an extensive treatment of reliability, effect sizes, and power to detect interactions between continuous variables.)

7.11.2 Sampling Designs to Enhance Power to Detect Interactions—Optimal Design

Reviews of observational studies indicate that interactional effects may be of only a small magnitude. In contrast, interactions with substantially larger effect sizes are often obtained in experiments. Moreover, the types of interactions in experiments are often disordinal interactions (Section 7.7.1). Such interactions are much easier to detect statistically than are the ordinal interactions typically found in observational studies. In experiments carried out in laboratory settings, the experimental conditions are likely to be implemented in a highly structured fashion, such that all cases in a condition receive a nearly identical treatment manipulation. This same control cannot be exercised in observational studies, where scores on IVs are merely gathered as they exist in the sample (though efforts can be made to have highly reliable measures).

To further compare experiments with observational studies, we need to make a translation from the levels of a treatment factor consisting of a treatment and a control condition to a predictor in MR. In fact, a binary variable (consisting of two values, here corresponding to treatment and control) can be entered as a predictor in MR. A code variable is created that equals $+1$ for all cases who are in the treatment condition and -1 for all cases who are in the control condition, referred to as an *effect code*. The conditions of an experiment are thus translated into a predictor in MR. Chapter 8 is devoted to the treatment of categorical variables in MR and the creation of such code variables.

A second and critical source of difference between experiments and observational studies with regard to detection of interactions is the distribution of the predictor variables. McClelland and Judd (1993) provide an exceptionally clear presentation of the impact of predictor distribution on power to detect interactions; highlights are summarized here. McClelland and Judd draw on *optimal design*, a branch of research design and statistics that characterizes designs that maximize statistical power to detect effects and provide the shortest confidence intervals (i.e., smallest standard errors) of parameter estimates. In experiments, predicting linear effects, the treatment conditions are implemented at the ends of a continuum, as characterized in Section 7.7.2. Again, the IVs corresponding to the coded variables are typically effects codes in which a two-level treatment factor is coded $(+1, -1)$. Thus the scores of all the subjects on the treatment predictor are at one or the other extreme of the continuum from -1 to $+1$; there are no scores in the middle. If there are two 2-level factors (a 2×2 design), the resulting treatment conditions represent *four corners* of a two-dimensional surface—that is, hi-hi $(+1, +1)$; hi-lo $(+1, -1)$; lo-hi $(-1, +1)$, and lo-lo $(-1, -1)$. All the cases fall at one of the four corners based on their scores on the treatment predictors corresponding to the two factors.

The distribution of two predictors X and Z in an observational study is quite another matter. If X and Z are bivariate normally distributed, then cases in the four corners are extremely rare. Instead, most cases pile up in the middle of the joint distribution of X and Z. Given the same population regression equation and the same reliability of predictors, if predictors X and Z are bivariate normally distributed, then about 20 times as many cases are required to achieve the same efficiency to detect the XZ interaction as in the four-corners design!

The reader is warned that dichotomizing continuous predictors is not a way to increase the efficiency of observational studies to detect interactions (Cohen, 1983). Dichotomizing normally distributed predictors merely introduces measurement error because all the cases coded as being at a single value of the artificial dichotomy actually have substantially different true scores (they represent half the range of the predictor's distribution).

A possible strategy for increasing power to detect interactions in observational studies is to oversample extreme cases, if one has prior information about the value of the cases on the predictors (Pitts & West, 2001). Merely randomly sampling more cases won't offer as much improvement, since the most typical cases are in the middle of the distribution. If one oversamples extreme cases, then the efficiency of the data set (the statistical power) for detecting interactions will be improved. A downside of oversampling extreme cases is that the standardized effect size for the interaction in the sample will exceed that in the population, so that sample weights will be needed to be used to generate estimates of the population effects.

This discussion merely opens the door to the area of optimal design applied in multiple regression. As regression models change, optimal designs change. Pitts and West (2001) present an extensive discussion of sampling of cases to optimize the power of tests of interactions in MR, as do McClelland and Judd (1993). This consideration is important in the design of both laboratory and observational studies that seek to examine high-order effects.

7.11.3 Difficulty in Distinguishing Interactions Versus Curvilinear Effects

In Chapter 4 we discussed specification errors, that is, errors of estimating a regression equation in the sample that incorrectly represents the true regression model in the population. One form of specification error is of particular concern when we are studying interactions in MR. This error occurs when we specify an interactive model when the true model in the population is quadratic in form. That is, we estimate the model of Eq. (7.1.2):

$$\hat{Y} = B_1X + B_2Z + B_3XZ + B_0 \quad (7.1.2)$$

when the correct model is as follows (Lubinski & Humphreys, 1990; MacCallum & Mar, 1995):

$$\hat{Y} = B_1X + B_2Z + B_3X^2 + B_0 \quad (7.11.2)$$

or

$$\hat{Y} = B_1X + B_2Z + B_3X^2 + B_4Z^2 + B_0 \quad (7.11.3)$$

or even as follows (Ganzach, 1997):

$$\hat{Y} = B_1X + B_2Z + B_3X^2 + B_4Z^2 + B_5XZ + B_0. \quad (7.11.4)$$

When the true regression model is curvilinear in nature, with no interaction existing in the population, if Eq. (7.1.2) is mistakenly estimated in the sample, a significant interaction can potentially be detected. This possibility arises from the correlation between predictors X and Z. In the common situation in which IVs are correlated, the X^2 and Z^2 terms will be correlated with XZ. Centering X and Z does not eliminate this essential multicollinearity. Table 7.9.1A provides a numerical illustration. The correlation between x and z is .32; between x^2 and xz, .51. Only if x and z are completely uncorrelated will the correlations of x^2 and z^2 with xz be zero. Compounding these inherent correlations is unreliability of X and Z. This unreliability results in unreliability of the terms constructed from X and Z, that is, X^2, Z^2, and XZ (Busemeyer & Jones, 1983). In fact, the reliability of the X^2 and the Z^2 terms will be lower than that of the XZ term, except in the instance in which X and Z are completely uncorrelated (MacCallum & Mar, 1995). This is so because the reliability of the XZ term increases as the correlation between X and Z increases.

Lubinski and Humphreys (1990) suggested that one might choose between X^2, Z^2, and XZ as appropriate terms by assessing which of these three predictors contributed most to prediction over and above the X and Z terms. MacCallum and Mar (1995) in an extensive simulation study showed that this procedure is biased in favor of choosing XZ over the squared terms and attributed this outcome primarily to the lower reliability of the squared terms than the cross-product term.

There is currently debate about whether one should examine regression Eq. (7.11.2) when the analyst's central interest is in the XZ term. To guard against spurious interactions (i.e., interactions detected in a sample that do not exist in the population, particularly in the face of true quadratic effects), Ganzach (1997) has argued in favor of using Eq. (7.11.4); he provides empirical examples of the utility of doing so. Both Aiken and West (1991) and MacCallum and Mar (1995) argue that terms included should be substantively justified and caution that the inclusion of multiple higher order terms in a single regression equation will introduce multicollinearity and instability of the regression equation.

7.12 SUMMARY

Interactions among continuous predictors in MR are examined. Such interactions are interpreted and illustrated as conditional relationships between Y and two or more variables or variable sets; for example, an X by Z interaction is interpreted as meaning that the regression (relationship) of Y to X is conditional on (depends on, varies with, is not uniform over) the status of Z. An interaction between two variable sets X and Z is represented by multiplication of their respective IVs and then linearly partialing out the X and Z sets from the product set. The contribution of the X by Z interaction is the increment to R^2 due to the XZ products over and above the X set and Z set.

The regression equation $\hat{Y} = B_1X + B_2Z + B_3XZ + B_0$ is first explored in depth. The geometric representation of the regression surface defined by this regression equation is provided. The first-order coefficients (B_1, B_2) for predictors X and Z, respectively, in an equation containing an XZ interaction represent the regression of Y on each predictor at the value of zero on the other predictor; thus B_1 represents the regression of Y on X at $Z = 0$. Zero is typically not meaningful on psychological scales. Centering predictors (i.e., putting them in deviation form, $(x = X - M_X), (z = Z - M_Z)$ so that $M_X = M_Z = 0$) renders the interpretation of the first-order coefficients of predictors entering an interaction meaningful: the regression of Y on each predictor at the arithmetic mean of the other predictor (Section 7.2).

Post hoc probing of interactions in MR involves examining simple regression equations. Simple regression equations are expressions of the regression of Y on one predictor at specific values of another predictor, as in the expression $\hat{Y} = [B_1 + B_3Z]X + [B_2Z + B_0]$ for the regression of Y on X at values of Z. Plotting these regression equations at several values across the range of Z provides insight into the nature of the interaction. The simple slope is the value of the regression coefficient for the regression of Y on one predictor at a particular value of the other predictor, here $[B_1 + B_3Z]$. The simple slope of Y on X at a specific value of Z may be tested for significance, and the confidence interval around the simple slope may be estimated (Sections 7.3 and 7.4).

Standardized solutions for equations containing interactions pose special complexities (Section 7.5). The structuring of regression equations containing interactions requires that all lower order terms be included for interaction effects to be accurately measured (Section 7.6).

Interactions in MR may take on a variety of forms. In interactions, predictors may work synergistically, one may buffer the effect of the other, or they may work in an interference pattern (i.e., in an "either-or" fashion). They may be ordinal (preserving rank order) or disordinal (changing rank order) within the meaningful range of the data (Section 7.7).

More complex equations containing interactions may involve three IVs (Section 7.8). They may also involve the interaction between a predictor that bears a curvilinear relationship to the criterion and one that bears a linear relationship to the criterion (Section 7.9). Further, interactions may occur between sets of predictors (Section 7.10).

Issues in the assessment of interactions in MR include the low statistical power for their detection, particularly in the face of predictor unreliability. The power to detect interactions also varies as a function of the distribution of predictors, particularly the extent to which there are scores at the extremes of the predictor distributions. It is difficult to distinguish between models containing an XZ interaction versus quadratic relationships X^2 or Z^2 except in experimental settings where X and Z are uncorrelated (Section 7.11).