Lecture 6: Continuity and the Intermediate Value Theorem (CATIVT)

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Recall ...

Recall that f is said to be continuous at $x = a$ if

$$\boxed{\lim_{x \to a} f(x) = f(a).}$$

Unpack into 3 conditions
- $f(a)$ defined
- $\lim_{x \to a} f(x)$ exists
- two values to be the same

Continuity makes limit computation trivial.

Below is our up-to-date library of functions that are continuous on their natural domains:

- Constant functions
- Power functions
- Polynomial functions
- Rational functions
- Exponential functions
- Logarithmic functions
- Trigonometric functions
- Inverse trigonometric functions

limit law

Overview

Continuity

- more delicate examples
- piecewise functions

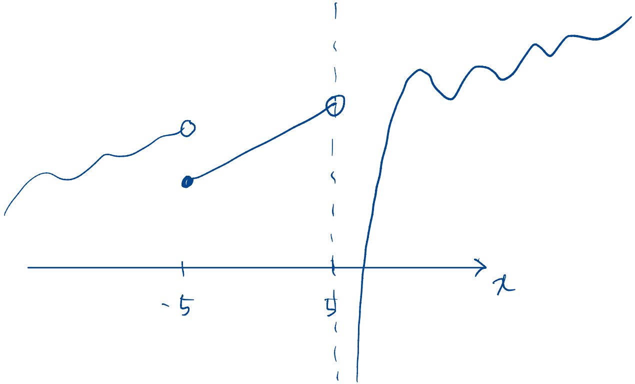
IVT

- key consequences of continuity
- applications

Continuity of piecewise functions

Today, we will consider continuity of functions obtained by patching two or more functions. In doing so, pay attention to

- any possible discontinuities of individual *pieces*
- (dis)continuity at junctions where two pieces are joined



Question. Consider the function defined piecewise as

"parameter"

$$f(x) = \begin{cases} \dfrac{x}{x-1} & \text{if } x < 0, \\ e^{-x} + c & \text{if } x \geq 0. \end{cases}$$

Find c so that f is continuous at $x = 0$.

Notes

- $f(x)$ is continuous on $(-\infty, 0)$ because $x/x-1$ is a rational function.
- $f(x)$ is continuous on $(0, \infty)$ because $e^{-x} + c$ is the sum of two continuous function.

exp.

const.

- Depending on the value of c, $f(x)$ may be cts. at $x=0$.

Soln For f to be ~~to be~~ cts. at $x = 0$, $= a$

$$\lim_{x \to 0} f(x) = f(0).$$

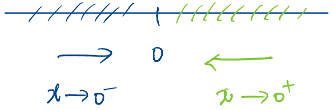
Now, enforce 3 conditions:

① Is $f(0)$ defined?

Yes. $f(0) = e^{-0} + c$

$= \underline{1 + c}$.

② Does $\lim_{x \to 0} f(x)$ exist?



To answer, inspect one-sided limits,

- $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x}{x-1} = \underline{0}$
- $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(e^{-x} + c \right) = e^{-0} + c = \underline{1 + c}$

For the limit to exist, these two one-sided limits must coincide.

$$0 = 1 + c \quad \Rightarrow \quad \boxed{c = -1}$$

③ $\lim_{x \to 0} f(x) \overset{?}{=} f(0)$

$f(0) = 1 + c$

$= 1 + (-1)$

$= 0$

$\lim_{x \to 0} f(x)$ exists when $c = -1$,

in which case

$\lim_{x \to 0} f(x) = 0$

Confirmed !

Question. Consider the following piecewise defined function

$$f(x) = \begin{cases} x+4 & \text{if } x < 1, \\ ax^2 + bx + 2 & \text{if } 1 \leq x < 3, \\ 6x + a - b & \text{if } x \geq 3. \end{cases}$$

Junctions: $x = 1$

$x = 3$

Find a and b so that f is continuous at both $x = 1$ and $x = 3$.

Soln

@ $x=1$ All we need to require is the existence of limit.

$$\begin{cases} \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+4) = 5 \\ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (ax^2+bx+2) = a+b+2. \end{cases}$$

Need: $\boxed{a+b+2=5}$

@ $x=3$ Again, need $\lim_{x \to 3} f(x)$ to exist.

$$\begin{cases} \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax^2+bx+2) = 9a+3b+2 \\ \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x+a-b) = a-b+18 \end{cases}$$

Need: $\boxed{a-b+18 = 9a+3b+2}$

Let's solve the sys. of eqns.

① : $a + b = 3$

$8a + 4b = 16$

② : $\hookrightarrow 2a + b = 4$

Ans : $\boxed{a = 1, \quad b = 2}$

(Do the algebra yourself.)

For those interested . . .

You can create interactive plots of such functions in **Mathematica**. For example:

```
Manipulate[Plot[
    Piecewise[{ {x + 4, x < 1},
                {a*x^2 + b*x + 2, 1 <= x < 3},
                {6*x + a - b, x >= 3} }],
    {x, 0, 4}], {a, 0, 2}, {b, 1, 3}]
```

The Intermediate Value Theorem

closed interval

Theorem (Intermediate Value Theorem)

If

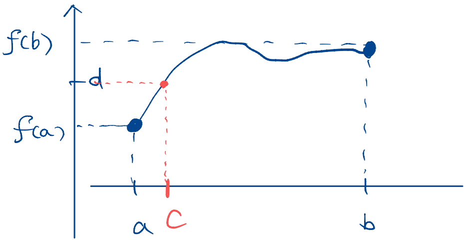
- f *is a continuous function for all* x *in* $[a, b]$ *and*
- d *is between* $f(a)$ *and* $f(b)$,

Conditions

then there is a number c *in* $[a, b]$ *such that*

$$f(c) = d\,.$$

conclusion.

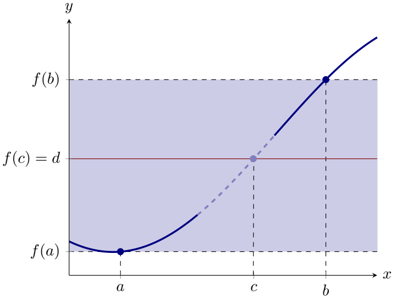


Illustration

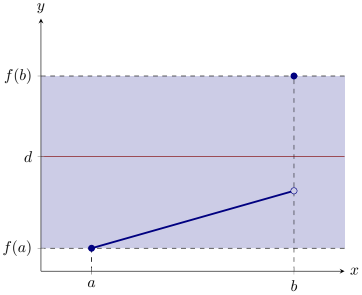
IVT applies

IVT not applicable.

Case 1: f is continuous on $[a, b]$



Case 2: f is continuous on $[a, b)$

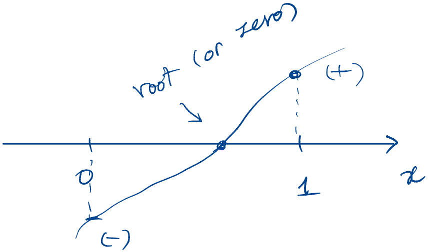


(Root Finding)

Question. Demonstrate, using the IVT, that the function

$$f(x) = x^3 + 3x^2 + x - 2$$

has a root [1] between 0 and 1.



Soln. Since $f(x)$, as a polynomial, is cts everywhere, it is cts on $[0, 1]$.

. observe:

$$f(0) = -2$$

$$f(1) = 3$$

So if we set $d = 0$ in IVT, the theorem guarantees the existence of c in $[0, 1]$ such that $f(c) = 0$. This c is a root btw 0 and 1.

[1] It can be shown that $f(x) = (x+2)(x^2+x-1)$ and so we know precisely that f has three roots

$$-2\,,\ \frac{-1+\sqrt{5}}{2}\,,\ \text{and}\ \frac{-1-\sqrt{5}}{2}\,.$$

Question. Explain why the functions

$$f(x) = x^2 \ln(x)$$
$$g(x) = 2x\cos(\ln(x))$$

intersect on the interval $[1, e]$.

Soln Let $h(x) = f(x) - g(x)$.

Since each of f and g are continuous on $[1, e]$ (confirm this!), h is continuous on $[1, e]$.

Furthermore,

- $h(1) = f(1) - g(1)$
  $= 0 - 2 = -2 < 0$

- $h(e) = f(e) - g(e)$
  $= e^2 - 2e\cos(1)$
  $= e\underbrace{(e - 2\cos(1))}_{>0} > 0$

So, by IVT, there must exist a number c in $[1, e]$ such that

$$h(c) = f(c) - g(c) = 0.$$

This implies that

$$f(c) = g(c),$$

which means that the graphs of f and g intersect at some point c in $[1, e]$