Lecture 20: Applied Related Rates (ARR)

Tae Eun Kim, Ph.D.

Autumn 2021

- Office Hours 4 ~ 6 PM, Tue., Oct 12

  (no OH on Mon. and Wed. this week.)

- Midterm 3 Monday after Autumn break.

  Review video to be posted before break.

Applied Related Rates Problems

Continuation of MTOR.

↳ with respect to time.

General procedures

1. **Introduce variables and identify the given and unknown rates.** Assign a variable to each quantity that changes in time.
2. **Draw a picture.** If possible, draw a schematic picture with all the relevant information.
3. **Find equations.** Write equations that relate all relevant variables.
4. **Differentiate with respect to time t.** Here we will often use implicit differentiation and obtain an equation that relates the given rate and the unknown rate. (d/dt)
5. **Evaluate.** Evaluate each quantity at the relevant moment.
6. **Solve.** Solve for the unknown rate at that moment.

Example 1. (Cylindrical geometry)

A hand-tossed pizza crust starts off as a ball of dough with a volume of 400π cm^3. First, the cook stretches the dough to the shape of a cylinder of radius 12 cm. Next the cook tosses the dough.

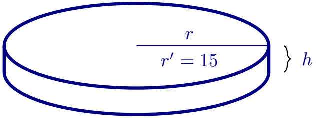
Volume

If during tossing, the dough maintains the shape of a cylinder and the radius is increasing at a rate of 15 cm/min, how fast is its thickness changing when the radius is 20 cm?

h

1. Picture, notation, known & unknown

$V = 400\pi$ cm^3



$V = \underbrace{\pi \cdot r^2}_{\text{(Area of base)}} \cdot \underbrace{h}_{\text{(height)}}$ cm^3

- Known: $\dfrac{dr}{dt} = 15$ (cm/min)
- Want: $\underbrace{\left[\dfrac{dh}{dt}\right]_{r=20}}_{\text{value of } \frac{dh}{dt} \text{ when } r=20}$

2. Eqn: $V = \pi r^2 h. \Rightarrow 400\cancel{\pi} = \cancel{\pi} r^2 h \Rightarrow \boxed{400 = r^2 h}$

3. Take d/dt : $400 = \underbrace{r^2}\underbrace{h}$ product rule

Keep in mind that both r & h depend on t.

$$0 = 2r \cdot \underbrace{\frac{dr}{dt}}_{=15} \cdot h + r^2 \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = -\frac{2rh}{r^2}\frac{dr}{dt} = -\frac{2h}{r}\frac{dr}{dt}$$

Note To find $\left[\frac{dh}{dt}\right]_{r=20}$, need

- $h = ?$
- $r = 20$
- $\frac{dr}{dt} = 15$ (given)

4. Evaluate and Solve: When $r = 20$, the corresponding h value is found by using the volume eqn.

$$V = \pi r^2 h$$

$$400\cancel{\pi} = \cancel{\pi} \cdot \underbrace{20^2}_{=400} \cdot h \Rightarrow \underline{h = 1}$$

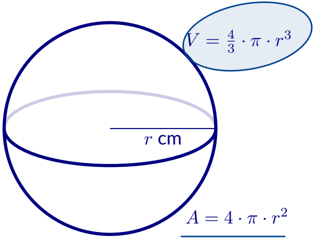
$$\left[\frac{dh}{dt}\right]_{r=20} = -\frac{2 \cdot 1}{20} \cdot 15$$

$$= \boxed{-\frac{3}{2}\ (\text{cm}/\text{min})}$$

Example 2. (Spherical geometry)

Consider a melting snowball. We will assume that the rate at which the snowball is melting is proportional to its surface area. Show that the radius of the snowball is changing at a constant rate.

1. Picture, notation, known & unknown



(surface area)

- known: $\frac{dV}{dt} = k \cdot 4\pi r^2$ constant of proportionality
- Want: $\frac{dr}{dt}$ is const.

2. Eqn.

$V = \frac{4}{3}\pi r^3$ may be useful.

3. Take d/dt of $V = \frac{4}{3}\pi r^3$.

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$$

$$= \underbrace{4\pi r^2}_{A} \frac{dr}{dt}.$$

4. Punchline:

Solving for $\frac{dr}{dt}$, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt} = \frac{1}{\cancel{4\pi r^2}} \cdot k \cdot \cancel{4\pi r^2}$$

Therefore, $\frac{dr}{dt} = k$ constant, i.e., radius changes at a constant rate.

Example 3. (Right triangles)

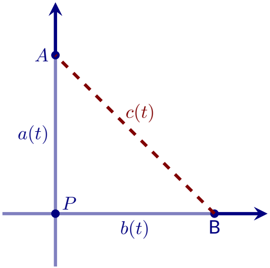
A road running north to south crosses a road going east to west at the point P. Cyclist A is riding north along the first road, and cyclist B is riding east along the second road. At a particular time, cyclist A is 3 kilometers to the north of P and traveling at 20 km/hr, while cyclist B is 4 kilometers to the east of P and traveling at 15 km/hr. How fast is the distance between the two cyclists changing at that time?

1. Picture & notation →

- Known: $\frac{da}{dt} = 20$ kph, $\frac{db}{dt} = 15$ kph
- Want: $\left[\frac{dc}{dt}\right]$ $a=3,\ b=4$.

2. Equation.

$$c^2 = a^2 + b^2$$



3. Differentiate

$$c\,\frac{dc}{dt} = a\,\frac{da}{dt} + b\,\frac{db}{dt}$$

$$\therefore\ \frac{dc}{dt} = \frac{a}{c}\,\frac{da}{dt} + \frac{b}{c}\,\frac{db}{dt}$$

4. Evaluate & solve.

When $a=3$, $b=4$, $\quad c=\sqrt{a^2+b^2} = 5$.

Consequently,

$$\left[\frac{dc}{dt}\right]_{a=3,\,b=4} = \frac{3}{5}\cdot 20 + \frac{4}{5}\cdot 15$$

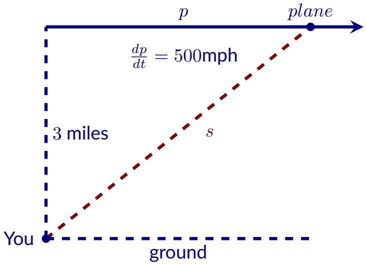
$$= \boxed{24 \text{ km/h}}$$

Example 4. (Right triangles)

A plane is flying at an altitude of 3 miles directly away from you at 500 mph. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

1. Picture and notation $\longrightarrow$

   - Known: $\frac{dp}{dt} = 500$ mph
   - Want: $\left[\frac{ds}{dt}\right]_{p=4}$



2. Equation

$$s^2 = 9 + p^2$$

3. Differentiate

$$s \frac{ds}{dt} = p \frac{dp}{dt} \implies \frac{ds}{dt} = \frac{p}{s} \frac{dp}{dt}$$

4. Evaluate and solve

Note that when $p = 4$, $s = \sqrt{9 + p^2} = 5$.

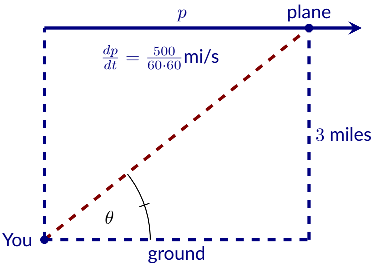
So

$$\left[\frac{ds}{dt}\right]_{p=4} = \frac{4}{5} \cdot 500 = \boxed{400 \text{ (mph)}}$$

Example 5. (Angular rates)

A plane is flying at an altitude of 3 miles directly away from you at 500 mph . Let θ be the **angle of elevation** of the plane, i.e., the angle between the ground and your line of sight to the plane. How fast (in radians per second) is the angle θ decreasing at the moment when the plane is flying over a point on the ground 4 miles from you?

1. Picture & notation $\longrightarrow$



- Known: $\frac{dp}{dt} = \frac{5}{36}$ mi/s
- Want: $\left[\frac{d\theta}{dt}\right]_{p=4}$

2. Equation

$$\tan(\theta) = \frac{3}{p}$$

3. Differentiate

$$\sec^2(\theta)\,\frac{d\theta}{dt} = -\frac{3}{p^2}\,\frac{dp}{dt}$$

$$\therefore\ \frac{d\theta}{dt} = -\frac{3}{p^2}\cos^2(\theta)\,\frac{dp}{dt}$$

4. Evaluate & Solve.

Note that when $p=4$, the hypotenuse has length 5.

So $\cos(\theta) = \frac{4}{5}$. Thus

$$\left[\frac{d\theta}{dt}\right]_{p=4} = -\frac{3}{4^2}\cdot\frac{4^2}{5^2}\cdot\frac{5}{36} = \boxed{-\frac{1}{60}\ (\text{rad}/s)}$$

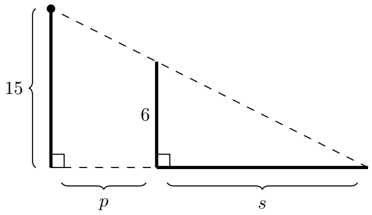
Example 6. (Similar triangles)

It is night. Someone who is 6 feet tall is walking away from a street light at a rate of 3 feet per second. The street light is 15 feet tall. The person casts a shadow on the ground in front of them. How fast is the length of the shadow growing when the person is 7 feet from the street light?

$\left[\frac{ds}{dt}\right]_{p=7}$

1. Picture, not'n, known & unknown.

- Known: $\frac{dp}{dt} = 3$ (ft/s)
- Want: $\left[\frac{ds}{dt}\right]_{p=7}$



$\frac{dp}{dt} = 3$ (ft/s)

2. Eqn. (Similarity)

$\frac{6}{s} = \frac{15}{p+s} \Rightarrow 6(p+s) = 15s \Rightarrow \underline{s = \frac{2}{3}p}$

3. Take d/dt of $s = \frac{2}{3} p$.

$$\frac{ds}{dt} = \frac{2}{3} \frac{dp}{dt}$$

Note: $\frac{ds}{dt}$ does not depend on p.

4. Evaluate & solve:

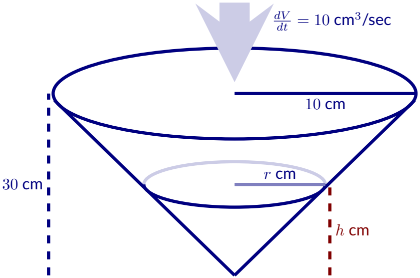
$$\left[\frac{ds}{dt}\right]_{p=7} = \frac{2}{3} \cdot 3 = \boxed{2\ (ft/s)}$$

Example 7. (Similar triangles)

Water is poured into a conical container at the rate of 10 cm^3/s. The cone points directly down, and it has a height of 30 cm and a base radius of 10 cm. How fast is the water level rising when the water is 4 cm deep?

1. Picture and notation $\longrightarrow$

- Known : $\dfrac{dV}{dt} = 10 \text{ cm}^3/s$
- Want : $\left[\dfrac{dh}{dt}\right]_{h=4}$



2. Equation

By similarity of triangles indicated, $r = h/3$

$$V = \frac{1}{3}\pi r^2 h \implies V = \frac{\pi}{27} h^3$$

3. Differentiate

$$\frac{dV}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{9}{\pi h^2} \frac{dV}{dt}$$

4. Evaluate and solve

when $h = 4$,

$$\left[\frac{dh}{dt}\right]_{h=4} = \frac{9}{16\pi} \cdot 10 = \boxed{\frac{45}{8\pi} \text{ cm/s}}$$