Lecture 40: Working with Substitution (WWS)

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Substitution Procedures

Let's recall that in integrating a function which we suspect to be the derivative of another obtained by the chain rule:

1. Look for a candidate for the inner function; call it u.
2. Rewrite the given function completely in terms of u leaving no trace of the original variable.
3. Integrate this new function of u. (If necessary, you may need to go back to Step 1 and modify your choice of u.)
4. In dealing with an indefinite integral, make sure to replace u by the equivalent expression of the original variable.
5. Working with a definite integral, you may evaluate the result of Step 3 at the transformed bounds of u or evaluate the antiderivate obtained in Step 4 at the original bounds.

For the remainder of the lecture, we will work out practice examples.

$$\int \frac{1}{x\ln(\ln x)}\,dx$$

Example

Compute:

1. $$\int_2^3 \frac{1}{x\ln(x)}\,dx$$

2. $$\int_0^{16} \sqrt{4-\sqrt{x}}\,dx$$

$= du$

$= u$

- What's embedded?
- Does its deriv. appear outside?

$$\frac{1}{x\ln x} = \frac{1}{\ln x}\cdot\frac{1}{x}$$

$\ln x$ put inside $\frac{1}{☺}$

① Set $u = \ln x$

$du = \frac{1}{x}\,dx$

Limits

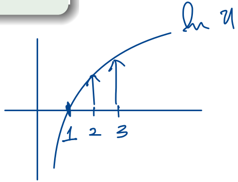
x	$u=\ln x$
3	$\ln 3$
2	$\ln 2$

$$(\text{Integ } ①) = \int_{\ln 2}^{\ln 3} \frac{1}{u}\,du$$

$$\overset{FTC2}{=} \Big[\ln|u|\Big]_{\ln 2}^{\ln 3}$$

$$= \ln|\ln 3| - \ln|\ln 2|$$

$$= \boxed{\ln(\ln 3) - \ln(\ln 2)} = \ln\left(\frac{\ln 3}{\ln 2}\right)$$



Example

Compute:

1. $$\int \frac{\sec(y)\tan(y) + \sec^2(y)}{\sec(y) + \tan(y)}\, dy$$
2. $$\int \tan(x)\, dx$$

Example

Compute:

1. $\displaystyle \int \frac{u}{1-u^2}\,du$
2. $\displaystyle \int \frac{e^{2x}}{1-e^{2x}}\,dx$

Example

Compute:

1. $\displaystyle\int x^3\sqrt{1-x^2}\,dx$

Work

Suppose the force $F(s)$ is applied to an object and moves it from $s = s_0$ to $s = s_1$. Then the work done on the object over the course of motion is given by

$$W = \int_{s_0}^{s_1} F(s)\, ds.$$

The international standard unit of force is a **Joule**, which is defined to be

$$1\,\mathrm{J} = 1\,\mathrm{N} \cdot \mathrm{m}.$$

In words, work measures the accumulated force over a distance. Note that we only accumulate force in the *direction* or *opposite direction* of motion.

Example

If an apple has a mass of $0.1\,\mathrm{kg}$, how much work is required to lift this 1 meter above the ground? Assume that the gravitational acceleration is $-9.8\,\mathrm{m/\,s^2}$.

Kinetic Energy

Now suppose that an object of mass m is moving at velocity $v(t)$. The **kinetic energy** E_k is the amount of *energy* that an object possesses from its motion. It is defined by

$$E_k = \frac{mv^2}{2}.$$

The SI unit of energy is also a **Joule** since $1\,\mathrm{N} = 1\,\mathrm{kg} \cdot \mathrm{m}/\,\mathrm{s}^2$:

$$1\,\mathrm{J} = 1\,\mathrm{kg} \cdot \mathrm{m}^2/\,\mathrm{s}^2.$$

Example

Now the apple is dropped from the height of 1 meter. How much kinetic energy is released when it hits the ground?

Work-Energy Theorem

We observe that the work and the energy calculated above are the same with the same unit. This is not a coincidence and this phenomenon can be explained via the **Work-Energy Theorem**.

Theorem (Work-Energy Theorem)

Suppose that an object of mass m is moving along a straight line. If s_0 and s_1 are the the starting and ending positions, v_0 and v_1 are the the starting and ending velocities, and $F(s)$ is the force acting on the object at any given position, then

$$W = \int_{s_0}^{s_1} F(s)\, ds = \frac{mv_1^2}{2} - \frac{mv_0^2}{2}.$$

In other words, the total work done by the force is equal to the net change in kinetic energy.

Explanation

Let $t_{0,1}$ be the initial and terminal time of the motion respectively, so that we can write $s_{0,1} = s(t_{0,1})$ respectively. Then the work formula can be written as

$$W = \int_{s(t_0)}^{s(t_1)} F(s)\,ds = \int_{t_0}^{t_1} F(s(t))s'(t)\,dt,$$

where the second equality is due to the substitution rule with $u = s(t)$. By Newton's second law of motion $F(s(t)) = ma(t)$, we can write

$$\int_{t_0}^{t_1} F(s(t))s'(t)\,dt = \int_{t_0}^{t_1} ma(t)s'(t)\,dt = m\int_{t_0}^{t_1} a(t)s'(t)\,dt.$$

Remembering that $a(t) = v'(t)$ and $s'(t) = v(t)$, we can write the integrand solely in terms of v and its derivative, i.e.,

$$m\int_{t_0}^{t_1} a(t)s'(t)\,dt = m\int_{t_0}^{t_1} v(t)v'(t)\,dt.$$

Another substitution $u = v(t)$ with new bounds $v_0 = v(t_0)$ and $v_1 = v(t_1)$ yields the desired result.

$$m\int_{t_0}^{t_1} v(t)v'(t)\,dt = m\int_{v_0}^{v_1} u\,du = m\left[\frac{u^2}{2}\right]_{v_0}^{v_1} = \frac{mv_1^2}{2} - \frac{mv_0^2}{2}.$$