Lecture 16: Derivatives of Inverse Functions (DOIF)

Tae Eun Kim, Ph.D.

Autumn 2021

The Derivatives of Inverse Trig Functions

Theorem (Derivatives of inverse trigonometric functions)

- $\frac{d}{dx}\arcsin(x) = \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$ *for* $|x| < 1$
- $\frac{d}{dx}\arccos(x) = \frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$ *for* $|x| < 1$
- $\frac{d}{dx}\arctan(x) = \frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}\mathrm{arcsec}(x) = \frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$ *for* $|x| > 1$
- $\frac{d}{dx}\mathrm{arccsc}(x) = \frac{d}{dx}\csc^{-1}(x) = \frac{-1}{|x|\sqrt{x^2-1}}$ *for* $|x| > 1$
- $\frac{d}{dx}\mathrm{arccot}(x) = \frac{d}{dx}\cot^{-1}(x) = \frac{-1}{1+x^2}$

Question. Compute:

1. $\dfrac{d}{dx}\tan^{-1}(\sqrt{x})$

2. $\dfrac{d}{dx}\sec^{-1}(3x)$

Explanation

Remark

In the derivation of the above formulas, we repeatedly used the following form of implicit differentiation

$$\frac{d}{dx} f(y) = f'(y) \cdot y' \,,$$

which requires that the function $y = f^{-1}(x)$ has a derivative. The differentiability of the inverse function is guaranteed by the following theorem.

Inverse Function Theorem

Theorem (The inverse function theorem)

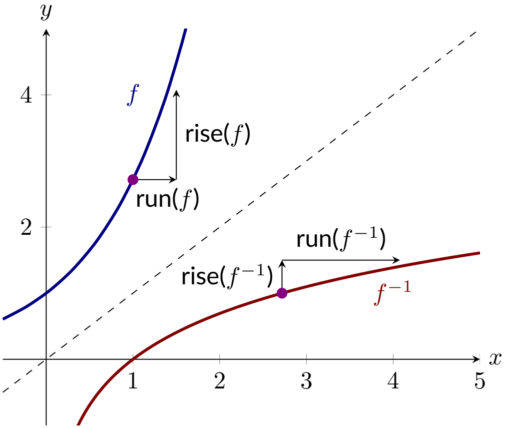
Suppose f is a differentiable function that is one-to-one near a and $f'(a) \neq 0$ and let $b = f(a)$. Then

1. $f^{-1}(x)$ *is* ***defined*** *for x near b,*
2. $f^{-1}(x)$ *is* ***differentiable*** *near b,*
3. *last, but not least:*

$$\left[\frac{d}{dx} f^{-1}(x)\right]_{x=b} = \frac{1}{f'(a)} \qquad \text{where} \qquad b = f(a)\,.$$

Illustration

Besides verifying the last result using implicit differentiation, convince yourselves by considering the following diagram of a function f and its inverse f^{-1}:



Question. Let f be a differentiable function that has an inverse. In the table below we give several values for both f and f':

x	f	f'
2	0	2
3	1	5
4	3	0

Compute

$$\frac{d}{dx} f^{-1}(x) \text{ at } x = 1.$$