Lecture 28-29: L'Hôpital's Rule (LHR)

Math 1151

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Autumn 2021

Limits $\longrightarrow$ Differential

$\downarrow$

Integral

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Applications of derivatives

- related rates
- graphing functions
- optimization (max./min.)
- linear approximation
- computation of hard limit problems (L'Hôpital's Rule)

Today $\longrightarrow$ computation of hard limit problems also spelled as "L'Hospital"

This is our final application of derivatives: using derivatives to calculate difficult limits. Enter L'Hôpital's rule.

Theorem (L'Hôpital's Rule)

Let $f(x)$ and $g(x)$ be functions that are differentiable near a. If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \qquad \text{or } \pm\infty,$$

and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, and $g'(x) \neq 0$ for all x near a, then

$$\underline{\lim_{x\to a} \frac{f(x)}{g(x)}} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Indeterminate Form

"$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

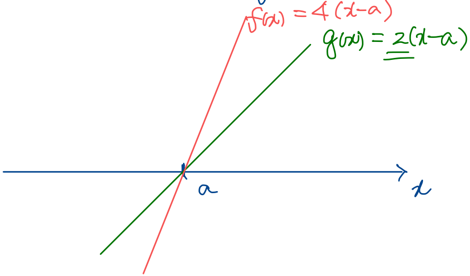
work further

- not Quotient Rule.
- Simply the ratio of individual derivatives.

Geometry of L'H

- Sample scenario a is some number

$$f(x) = 4(x-a), \quad g(x) = 2(x-a) \quad \text{(linear)}$$



Observe

- $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$

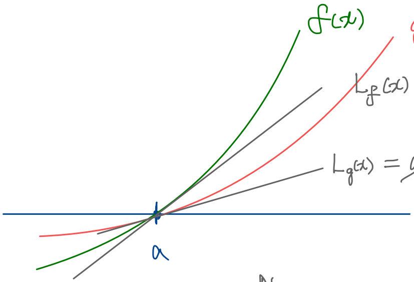
- $\underbrace{\lim_{x\to a} \frac{f(x)}{g(x)}}_{\text{Form: } \text{"}\frac{0}{0}\text{"}} = \lim_{x\to a} \frac{4\cancel{(x-a)}}{2\cancel{(x-a)}}$

$$= \lim_{x\to a} \frac{4}{2} = 2.$$

"ratio of slopes"

- General scenario

Now assume f, g are not linear.



$L_f(x) = \cancel{f(a)}^{\,0} + f'(a)(x-a)$

$L_g(x) = \cancel{g(a)}^{\,0} + g'(a)(x-a)$

Near a

$$\frac{f(x)}{g(x)} \approx \frac{L_f(x)}{L_g(x)} = \frac{f'(a)\cancel{(x-a)}}{g'(a)\cancel{(x-a)}}$$

(ratio of slopes of tang. lines)

- $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$

- $$\underbrace{\lim_{x\to a} \frac{f(x)}{g(x)}}_{\text{Form: } \text{"}\frac{0}{0}\text{"}} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

L'H

Caveats

$$\underbrace{\lim_{x \to a} \frac{f(x)}{g(x)}}_{\text{“}\frac{0}{0}\text{” or “}\frac{\infty}{\infty}\text{”}} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on RHS exists and $g'(x) \neq 0$ near a.

1. RHS is the limit of ratio of derivatives. (Not Q.R.)

2. L'H works even for
   - "$x \to a^{\pm}$" (one-sided limits)
   - "$x \to \pm\infty$" (limits at infinity)

3. May be applied repeatedly until you obtain an answer.
   (simplify before next application of L'H.)

List of Indeterminate Forms

can be handled by L'H eventually.

- $\frac{0}{0}$ — L'H — Lv. 0
- $\frac{\infty}{\infty}$
- $0 \cdot \infty$ — Lv. 1
- $\infty - \infty$ — Lv. 2
- 1^{∞} — Lv. 3 (Final stage boss)
- 0^{0}
- ∞^{0}

Symbols, not numbers.
So use double quotes.

In each of these cases, the value of the limit is **not** immediately obvious.
Hence, a careful analysis is required!

Examples: Basic Indeterminate Forms

Question. Compute $\lim_{x\to 0} \frac{\sin(x)}{x}$.

<u>Form</u> : " $\frac{0}{0}$ "

<u>Ans.</u> 1

(taught in Sqz. Thm. Sec. relied on geometry.)

<u>justification</u>

<u>Soln</u> $\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\cos(x)}{1}$

$= \frac{\cos(0)}{1} = \boxed{1}$

drop "lim" symbol since the limit has been evaluated.

Exercise The following important identities were presented a while back (expected to memorize)

- ✓ $$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

- $$\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} = \lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = 0$$

- $$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

• Definition "e"

• $(e^x)' = e^x$

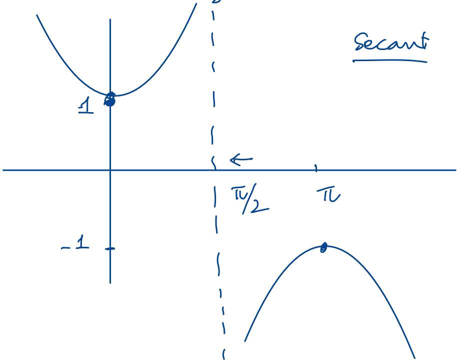
Confirm these using L'H.

Question. Compute $\displaystyle\lim_{x\to\pi/2+} \frac{\sec(x)}{\tan(x)}$.

Form: "$\frac{\infty}{\infty}$"

$$\lim_{x\to\frac{\pi}{2}^+} \sec(x) = -\infty.$$

$$\lim_{x\to\frac{\pi}{2}^+} \tan(x) = -\infty$$



$$\overset{L'H}{=} \lim_{x\to\frac{\pi}{2}^+} \frac{\cancel{\sec(x)}\tan(x)}{\sec^{\cancel{2}}(x)}$$

$$= \lim_{x\to\frac{\pi}{2}^+} \frac{\tan(x)}{\sec(x)} = \cdots$$

Simplify:

$$\frac{\sec(x)}{\tan(x)} = \frac{1}{\cancel{\cos(x)}} \cdot \frac{\cancel{\cos(x)}}{\sin(x)} = \frac{1}{\sin(x)}$$

So $\displaystyle\lim_{x\to\pi/2^+} \frac{1}{\sin(x)} = \boxed{1}$

Lv. 1

The following $\mathbf{0} \cdot \infty$ can be reduced to one of the two previous ones. For instance:

Question. Compute $\lim\limits_{x \to 0^+} x \ln x$.

$$= \lim_{x \to 0^+} \frac{\overset{\to -\infty}{\ln x}}{\underset{\to +\infty}{\frac{1}{x}}}$$

Form: "$\frac{\infty}{\infty}$"

$$\overset{L'H}{=} \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x \to 0^+} (-x) = \boxed{0}$$

Key: Double reciprocation

$$\text{☺} = \frac{1}{\frac{1}{\text{☺}}}$$

recip. once

recip. once more

if $\text{☺} \to 0$, then $\frac{1}{\text{☺}} \to \pm\infty$

Examples: Indeterminate Forms Involving Subtraction

Lv. 2

"$\infty - \infty$"

The name of the game once again is reduction. We will transform differences into either quotients or products then apply L'Hôpital's rule on the basic forms.

Question. Compute $\lim\limits_{x\to 0^+}(\cot(x) - \csc(x))$. ("$\infty - \infty$")

Note: $\lim\limits_{x\to 0^+} \cot(x) = \infty$, $\lim\limits_{x\to 0^+} \csc(x) = \infty$

$$= \lim_{x\to 0^+}\left(\frac{\cos(x)}{\sin(x)} - \frac{1}{\sin(x)}\right)$$

$$= \lim_{x\to 0^+}\frac{\cos(x) - 1}{\sin(x)}$$

Form: "$\frac{0}{0}$" (Lv. 0)

$$\overset{L'H}{=} \lim_{x\to 0^+}\frac{-\sin(x)}{\cos(x)} = -\frac{0}{1} = \boxed{0}$$

Alternate method (w/o using L'H)

$$\lim_{x \to 0^+} \frac{\cos(x) - 1}{\sin(x)} = \lim_{x \to 0^+} \underbrace{\frac{\cos(x) - 1}{x}}_{\downarrow \, 0} \; \underbrace{\frac{x}{\sin(x)}}_{\downarrow \, 1}$$

Prod. Law

$$= \left(\lim_{x \to 0^+} \frac{\cos(x) - 1}{x} \right) \left(\lim_{x \to 0^+} \frac{x}{\sin(x)} \right) \longrightarrow \text{known limits}$$

$$= 0 \cdot 1 = 0$$

Question. Compute $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$. Form: "$\infty-\infty$" (Lv. 2)

$$= \lim_{x\to\infty}\left(\sqrt{x^2(1+\tfrac{1}{x})} - x\right)$$

$$= \sqrt{x^2}\sqrt{1+1/x}$$

Very important esp. when examining limit as $x\to-\infty$

$$= |x|\sqrt{1+1/x}$$

$$= x\sqrt{1+1/x}.$$ Since $x\to\infty$, $x>0$ eventually

$$= \lim_{x\to\infty} x\left(\sqrt{1+1/x}-1\right),$$ Form: "$0\cdot\infty$" (Lv. 1)

$x\to\infty$

$\sqrt{1+1/x}-1 \to 0$

$$\overset{\text{D.R. }x}{=} \lim_{x\to\infty}\frac{\sqrt{1+1/x}-1}{1/x},$$ Form: "$\frac{0}{0}$" (Lv. 0)

Wait ! Before applying L'H, let's introduce $t = \frac{1}{x}$.

$$\lim_{x \to \infty} \frac{\sqrt{1 + 1/x} - 1}{1/x}$$

Rewrite

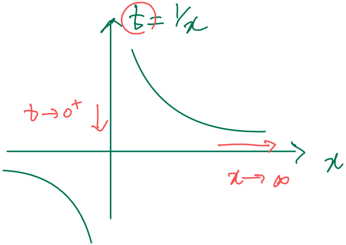
$$= \lim_{t \to 0^+} \frac{\sqrt{1+t} - 1}{t}$$

L'H

$$= \lim_{t \to 0^+} \frac{\frac{1}{2\sqrt{1+t}}}{1} = \boxed{\frac{1}{2}}$$

Observe that as $x \to \infty$,

$$t \to 0^+$$



Examples: Exponential Indeterminate Form

Lv. 3 final boss

This pertains to the forms

$$\mathbf{1}^{\infty}, \quad \mathbf{0}^{\mathbf{0}}, \quad \infty^{\mathbf{0}}$$

Suppose we have functions $u(x)$ and $v(x)$ such that

$$\lim_{x\to a} u(x)^{v(x)}$$

" tower "

$u(x) = e^{\ln u(x)}$

falls into one of the forms described above. We use the inverse relation between $\exp$ and $\log$ functions to rewrite the limit as

$$\lim_{x\to a} e^{v(x)\ln(u(x))}.$$

" Composite limit law "

Using the fact that the exponential function is continuous, the limit equals to

$$e^{\lim_{x\to a} v(x)\ln u(x)} = \exp\left[\lim_{x\to a} v(x)\ln(u(x))\right].$$

Note that the limit now is in one of the previously presented forms.

Procedure

Assume $\lim_{x \to a} u(x)^{v(x)}$ has the indeterminate form "1^∞", "0^0", or "∞^0".

1. Evaluate $L = \lim_{x \to a} v(x) \ln u(x)$. (Lv. 1 ; "$0 \cdot \infty$")

D.R.

This limit can be put in Lv. 0 form (Apply L'H.)

2. Then

$$\lim_{x \to a} u(x)^{v(x)} = e^L .$$

Question. First determine the form of the limit, then compute the limit.

$$\lim_{x\to\infty}\left(\underbrace{1+\frac{1}{x}}_{1}\right)^{\overbrace{x}^{\infty}}. \qquad \text{Form: } "1^{\infty}" \quad (\text{Lv. } 3)$$

Identification: $u(x) = 1 + 1/x$, $v(x) = x$.

1. $\underline{L} = \lim_{x\to\infty} v(x) \ln u(x)$

$= \lim_{x\to\infty} \underbrace{x}_{\infty} \underbrace{\ln(1+1/x)}_{\ln(1)=0}$, Form: "$0 \cdot \infty$" (Lv. 1)

$= \lim_{x\to\infty} \dfrac{\ln(1+1/x)}{1/x}$, Form: "$\frac{0}{0}$" (Lv. 0)

$t = 1/x$

$t \to 0^+$

$= \lim_{t\to 0^+} \dfrac{\ln(1+t)}{t}$

$\overset{L'H}{=} \lim_{t\to 0^+} \dfrac{1/(1+t)}{1} = \underline{1}$ $\therefore \underline{L=1}$

2. thus,

(answer) $= e^{L} = \boxed{e^{1}}$

Exercise Evaluate

$$\lim_{x \to 0^+} x^x$$

(tower function)