Lecture 24: Extreme and Mean Value Theorems (MVT)

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Overview

Two existence theorems.

Global min / Max

- Extreme Value Theorem (EVT)
- Mean Value Theorem (MVT)

average value (of rate of change)

2 versions of MVT in Math 1151

- differential — avg. rate of change
- integral — avg. func. value

Extreme Values of a Function

cf) Local extrema

Definition

- A function f has a **global maximum** at a if $f(a) \geq f(x)$ for every x in the domain of the function. G.M.
- A function f has a **global minimum** at a if $f(a) \leq f(x)$ for every x in the domain of the function. G.m.

A **global extremum** is either a global maximum or a global minimum.

Terminology

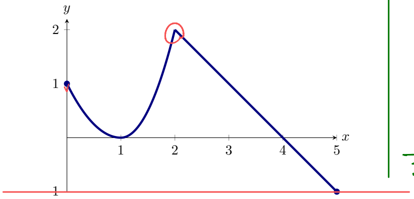
- local extrema = relative extrema
- global extrema = absolute extrema

Question. Let f be the function given by the graph below.

Important

Do not consider endpoints in the discussion of
- local extrema
- crit. pts.

But they can be global extrema!



1. Find the x-coordinate of the point where the function f has a global maximum and a global minimum.

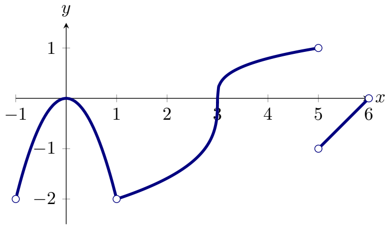
   G.M. at $x =$ 2 ; G.m. at $x =$ 5

2. Find the x-coordinate(s) of the point(s) where the function f has a local minimum and a local maximum.

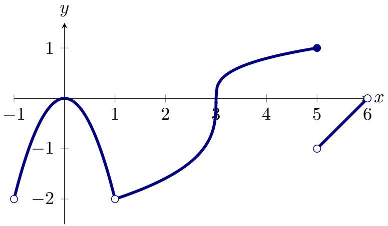
   L.M. at $x =$ 2 ; L.m. at $x =$ 1

Caution

A function may not attain a global extremum on its domain. Consider the following graph.



No G.M.; no G.m.



G.M. at $x=5$; no G.m.

Extreme Value Theorem

So when do we know for sure that a function attains a global extremum?

Theorem (The Extreme Value Theorem)

If f is continuous on the closed interval $[a, b]$, then there are points c and d in $[a, b]$, such that $(c, f(c))$ is a global maximum and $(d, f(d))$ is a global minimum on $[a, b]$.

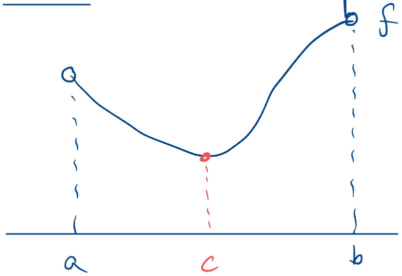
- guarantees existence of global extrema.
- doesn't tell us where they are. ⟶ we need work!

Remark.

- The theorem does not hold if we work on an open interval (a, b). Can you come up with an example?
- The theorem does not hold if we work on a closed interval $[a, b]$ but f is not continuous. Can you come up with an example?
- Do the previous examples invalidate the EVT?

EVT not applicable

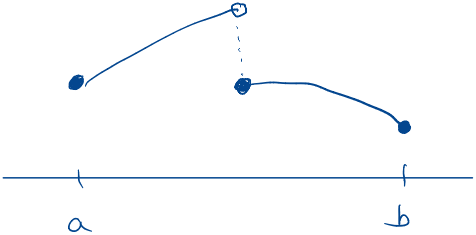
Case 1 f cts on an open interval



G.m. at $x = c$

No G.M.

Case 2 f is not cts on an closed interval



G.m. at $x = b$

No G.M.

Remarks

because EVT only guarantees the existence.

In finding global extrema:

- The EVT guarantees the existence of global extrema when we work with a function f that is continuous on a closed interval.
- The global extrema may occur either at the end points of the interval or in the interior.
- If a global extremum occurs at an interior points, then it is also a local extremum thus a critical point.
- Hence, all **interior critical points** as well as the **end points** are candidates for global extrema. (same as for local ext.) (new for global extrema)
- In sum, in order to locate global extrema, we evaluate the function at these candidate points and compare them to determine global maxima and global minima.

General Strategy

1. Find f' and find crit. pts. inside the interior of given interval.
2. Evaluate f at interior crit. pts & endpoints
3. Look for the largest / smallest. (No DT needed)

Exercise

Question. Let $f(x) = x^2 e^{-x}$, for $-2 \le x \le 1$. Locate the global extrema of f on the closed interval $[-2, 1]$.

Note that f is continuous everywhere.
In particular, it is cts. on $[-2, 1]$.
So, by EVT, f has both GM and Gm in the interval.

Strategy

1. Find all interior crit. pts.
   (select ones btw -2 and 1)
2. Evaluate f at:
   - end points: $x=-2$, $x=1$
   - found crit. pts.
3. Compare & conclude.

Soln $f(x) = x^2 e^{-x}$

1. $f'(x) = e^{-x}(2x - x^2)$, $\rightarrow$ confirm
   which is defined everywhere.

   $0 = f'(x) \Rightarrow 2x - x^2 = 0$

   $\Rightarrow \underbrace{x = 0}_{\text{Keep}}$ or $\cancel{x = 2}$. discard

2. Evaluate:

   $f(-2) = 4e^2$

   $f(0) = 0$

   $f(1) = e^{-1}$

3. Compare

   $\rightarrow$ G.M. at $x=-2$

   $\rightarrow$ G.m. at $x=0$

Rolle's Theorem

- We now explore an intricate relation between **average rate** and **instantaneous rate of change**.

  mean

- In some context, this can be translated as relation between *average velocity* and *instantaneous velocity* or *slope of secant line* vs. *slope of tangent line*.

Theorem (Rolle's Theorem)

Suppose that ① f *is differentiable on the interval* (a, b), *is* ② *continuous on the interval* $[a, b]$, *and* ③ $f(a) = f(b)$. *Then*

$$\underbrace{f'(c) = 0}_{\text{horizontal tangent.}}$$

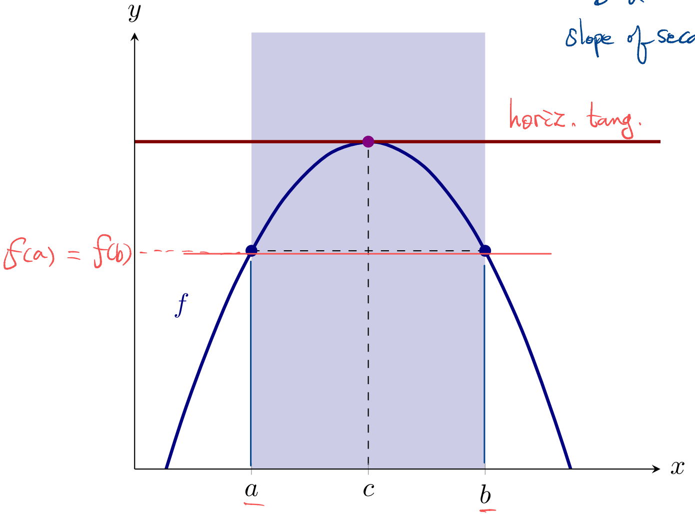
for some $a < c < b$.

Illustration of Rolle's Theorem

$$0 = \frac{f(b)-f(a)}{b-a} = f'(c) \text{ at some point } c \text{ in } (a,b)$$

slope of secant

Rolle's thm.



The Mean Value Theorem

A generalization of Rolle's theorem is coined as the mean value theorem.

Theorem (The Mean Value Theorem)

Suppose that (1) *f has a derivative on the interval (a, b) and is* (2) *continuous on the interval $[a, b]$. Then* i.e. is diff'ble on (a,b)

$$\underbrace{f'(c)}_{\text{inst. r.o.c.}} = \underbrace{\frac{f(b) - f(a)}{b - a}}_{\text{avg. r.o.c.}}$$

for some $a < c < b$.

slope of t-line   slope of s-line

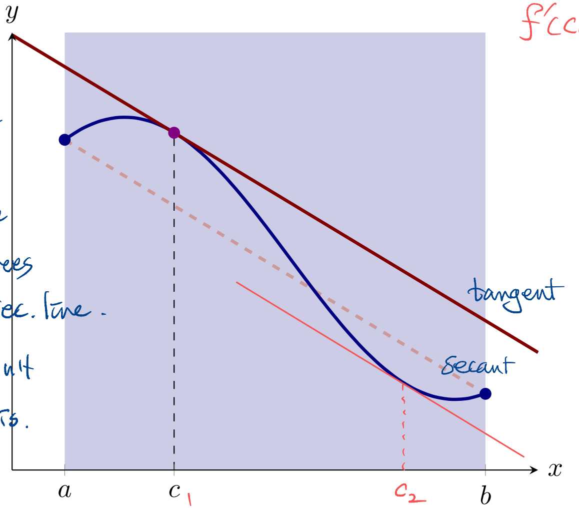
- The mean value theorem says that when a function is continuous on a closed interval and differentiable in its interior, then its average rate of change must be achieved at some point as an instantaneous rate of change.

Illustration of the Mean Value Theorem

Note

MVT guarantees the existence of point c in (a, b) at which the slope of tang. line agrees w/ that of the sec. line.

However, it doesn't tell us where c is.



$$f'(c_1) = f'(c_2) = \frac{f(b) - f(a)}{b - a}$$

There are many interesting questions that can be answered using the mean value theorem.

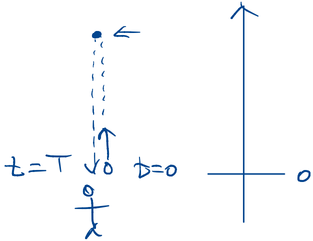
Example

Suppose you toss a ball into the air and then catch it. Must the ball's vertical velocity have been zero at some point?

Let $s(t)$ be the position of the ball and let 0 and T be the times when the ball leaves and is caught, respectively. Assuming the smooth motion of ball,

- $s(t)$ is cts on $[0,T]$
- $s(t)$ is diff'ble on $(0,T)$.

$$S(0) = S(T) = 0$$



So by MVT, there must be a point in time, say t_0, in $(0,T)$ such that

$$s'(t_0) = \frac{S(T) - S(0)}{T - 0}$$

$$\boxed{v(t_0) = 0}$$

Example

Suppose you drive a car from toll booth on a toll road to another toll booth 30 miles away in half of an hour. Was there a moment that you violated the speed limit of 55 mph?

Let $S(t)$ be the position of the car at time t (in hours).

Assuming "smooth" motion:

- S is diff'ble on $(0, \frac{1}{2})$
- S is cts. on $[0, \frac{1}{2}]$

Therefore, by MVT, there must exist some time c such that

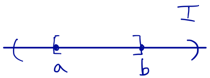
$$S'(c) = \frac{S(\frac{1}{2}) - S(0)}{\frac{1}{2} - 0} = \frac{30}{\frac{1}{2}} = 60 \ (mi/h).$$

i.e. the car was traveling at the instantaneous velocity of 60 mph, violating the speed limit.

Example

Suppose the derivative of a function is 0 on an open interval I. What can we say about f?

Let a and b be two distinct points in I that are chosen arbitrarily.



Then it is clear that

- f is cts. on $[a, b]$ (why?)
- f is diff'ble on (a, b).

So by MVT,

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some c in (a, b). But, by assumption, it implies that $f(a) = f(b)$.

Since a and b were chosen arbitrarily, it must be the case that f is constant.

Example

Suppose two different functions have the same derivative. What can you say about the relationship between the two functions?

I will leave this as an exercise.

Hint. Use the result of previous example.