Lecture 15: Logarithmic Differentiation (LD)

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Introduction

Let's recall:

Properties of logarithms

Let $b > 0$ and $b \neq 1$; let $x, y > 0$.

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b(x/y) = \log_b(x) - \log_b(y)$
- $\log_b(x^y) = y \log_b(x)$

$$\ln(x^2) = 2 \ln x$$

$$\ln(x^{1/3}) = \ln(\sqrt[3]{x}) = \frac{1}{3} \ln x .$$

$$\ln(x^x) = x \ln x$$

Logarithmic differentiation

Useful for differentiating functions w/ prod./quotient/radical structures.

A key point of the logarithmic differentiation is the following application of the chain rule:

$$\boxed{\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}\,.}$$

outer inner

$\frac{1}{f(x)} \cdot f'(x)$

Illustration of method

To differentiate $y = f(x)$, i.e., to find $\frac{dy}{dx} = y'$:

1. Take the logarithm of $y = f(x)$: $\ln y = \ln f(x)$ → Simplify the RHS using prop. of log.
2. Differentiate implicitly: $y'/y = f'(x)/f(x)$
3. Solve for y'.

Imp. Diff.

$$y' = y\,\frac{f'(x)}{f(x)}$$

Question. Compute

$$\frac{d}{dx}\frac{x^9e^{4x}}{\sqrt{x-4}}.$$

$$\sqrt{x}\,' = \frac{1}{2\sqrt{x}}.$$

$$\ln \frac{A^9 B^{4x}}{C^{1/2}} = 9\ln A + 4x\ln B - \tfrac{1}{2}\ln C$$

Let $y = \dfrac{x^9 e^{4x}}{\sqrt{x-4}}$

① Take ln: $\ln y = \ln \dfrac{x^9 e^{4x}}{\sqrt{x-4}}$

$$= 9\ln x + 4x\overset{1}{\ln(e)} - \frac{1}{2}\ln(x-4)$$

② Take d/dx: $\dfrac{y'}{y} = \dfrac{9}{x} + 4 - \dfrac{1}{2}\cdot\dfrac{1}{x-4}$

③ Solve for $\frac{dy}{dx} = y'$: $y' = \boxed{\left(\frac{9}{x} + 4 - \frac{1}{2}\cdot\frac{1}{x-4}\right)\left(\frac{x^9 e^{4x}}{\sqrt{x-4}}\right)}$

y

Question. For $x > 0$, compute

$$\frac{d}{dx} x^x .$$

(fnc)
(fnc)

This function is an example of a *tower function*. [1]

Let $y = x^x$.

① Take ln: $\ln y = \ln x^x$

$$\ln y = \underbrace{x}\,\underbrace{\ln x}$$

$\downarrow$ PR

② Take d/dx: $\frac{y'}{y} = \ln x + \frac{x}{x} = \ln x + 1$

③ Solve for y':

$$\boxed{y' = (\ln x + 1)\underbrace{(x^x)}_{y}}$$

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[1] **Note.** Make sure you are able to distinguish the following functions:

$$a^x , \qquad x^a , \qquad x^x .$$

Question. Compute the derivative

$$\frac{d}{dx}\ln(|x|).$$

Part (a)

Use the result to compute the derivative

$$\frac{d}{dx}\ln(|f(x)|).$$

part (b)

(a) Need to consider two cases.

Case 1: $x > 0 \longrightarrow |x| = x$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

Case 2: $x < 0 \longrightarrow |x| = -x$

$$\frac{d}{dx}\ln(-x) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$$

In all cases,

$$\boxed{\frac{d}{dx}\ln(|x|) = \frac{1}{x}}$$

(b) $\dfrac{d}{dx} \ln(|f(x)|)$

$$\begin{cases} \text{outer : } g(x) = \ln(|x|) \longrightarrow g'(x) = \dfrac{1}{x} \\ \text{inner : } f(x) \longrightarrow f'(x) \end{cases}$$

$\longrightarrow g(f(x)) = \ln(|f(x)|)$

$$\frac{d}{dx} g(f(x)) = g'(f(x)) \cdot f'(x)$$

$$= \frac{1}{f(x)} \cdot f'(x)$$

$$= \boxed{\frac{f'(x)}{f(x)}}$$

Exercise

Question. Using logarithmic differentiation, compute the derivative.

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right).$$

Let $y = \dfrac{\sin x}{x}$

① Take ln: $\ln y = \ln \dfrac{\sin x}{x}$

$= \ln \sin x - \ln x$

② Take d/dx: $\dfrac{y'}{y} = \dfrac{\cos x}{\sin x} - \dfrac{1}{x}$

③ Solve for y': $\boxed{y' = \underbrace{\left(\frac{\cos x}{\sin x} - \frac{1}{x}\right)}_{\text{previous RHS}}\underbrace{\left(\frac{\sin x}{x}\right)}_{y}}$

The Power Rule Revisited

Theorem (The power rule)

For any real number n and a positive real number x,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Let $y = x^n$

① Take ln:

$$\ln y = \ln x^n$$
$$= n \ln x$$

② Take d/dx:

$$\frac{y'}{y} = \frac{n}{x}$$

③ Solve for y':

y

$$y' = \frac{n}{x}(x^n)$$
$$= n\frac{x^n}{x}$$
$$= \boxed{n x^{n-1}}$$