Lecture 26-27: Optimization (O & AO)

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Autumn 2021

Basic Idea and Terminology

An **optimization problem** is a problem where you need to maximize or minimize some quantity under some constraints. This can be accomplished using the tools of differential calculus that we have already developed.

Terminology.

- **constraints:** conditions imposed on variables
- **objective functions:** the quantities desired to be optimized

A Solitary Local Extremum

- The extreme value theorem guarantees the existence of global extrema only on a closed interval. f cts on a closed interval
- On intervals that are not closed, the theorem is not applicable. Yet, when there is only one local extreme value, we can say something about global extrema.

Theorem

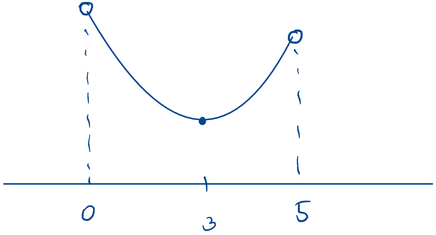
Suppose f is continuous on an interval I that contains exactly one local extremum at c.

generic interval

- *If a local maximum occurs at c, then $f(c)$ is the global maximum of f on I.*
- *If a local minimum occurs at c, then $f(c)$ is the global minimum of f on I.*

Ex 1

$f(x) = (x-3)^2 + 1$ on $(0, 5)$ ← open

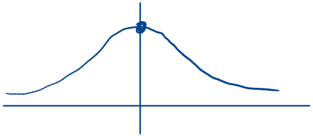


f has a unique local min. at $x=3$.

f attains the global min. value at $x=3$

Ex 2 $f(x) = \dfrac{1}{1+x^2} = \dfrac{d}{dx}\left(\tan^{-1}(x)\right)$ on $(-\infty, \infty)$

(Runge's function)



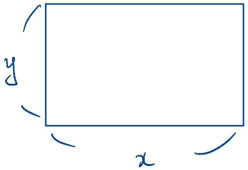
- uniq. loc. max. at $x=0$

$\Rightarrow$ f attains G.M. at $x=0$.

Example (Maximum area rectangles)

Of all rectangles of perimeter 12, which side lengths give the greatest area?

1. Pict., Not'n, and Ident.



$$\left\{\begin{array}{l} \underset{\text{perimeter}}{P} = \underline{2x+2y=12} \\ \\ \underset{\text{area}}{A} = xy \end{array}\right.$$

(constraint)

(obj. func. ; to be maximized)

2. Write obj. func. as a single variable func.

Using the constraint $2x+2y=12$,

$$\underline{y = 6-x}$$

$A(x) = x(6-x)$

domain: $0<x<6$

i.e. $(0,6)$

3. Do calculus.

Find the global maximum of $A(x) = x(6-x)$ on $(0,6)$

- $A'(x) = 6 - 2x$ $\qquad$ $A(x) = 6x - x^2$

  $\hookrightarrow$ crit. pts. $\begin{cases} A'(x)=0 : \quad 6-2x=0 \rightarrow \underline{x=3} \text{ is the only crit. pts. on } (0,6) \\ A'(x) \text{ undefined} : \text{ NONE} \end{cases}$

- $\underline{2^{nd} \text{ DT}}$

  $A''(x) = -2 < 0$ (regardless of x)

  $A''(\underset{\text{c.p.}}{3}) = -2 < 0 \Rightarrow$  $\Rightarrow$ L.M. at $x=3$ $\xRightarrow{\text{unique}}$ $\underline{\text{G.M. at } x=3}$

4. Conclusion

We attain the maximal area

$$A(3) = \underbrace{3}_{x}\underbrace{(6-3)}_{y} = 3 \cdot 3 = \boxed{9}$$

when

$$\begin{cases} \text{base } (x) = 3 \\ \text{height } (y) = 3 \end{cases}$$

That is, when the rectangle is actually a <u>square</u>.

Example (Minimum perimeter rectangles)

Of all rectangles of area 100, which has the smallest perimeter?

1. Prep.

Given: $A = xy = 100$ (constraint) $\quad y = \frac{100}{x}$

Want: Minimize $P = 2x + 2y$ (objective func.)

2. Write obj. func. as a single variable function.

Determine its domain.

$$P(x) = 2x + 2 \cdot \frac{100}{x} = 2x + \frac{200}{x}, \quad \text{for } 0 < x < \infty$$

3. Do calculus.

Find the global minimum of $p(x) = 2x + \frac{200}{x}$ on $(0,\infty)$

- $p'(x) = 2 - \frac{200}{x^2}$ $\qquad (200x^{-2})' = 200(-2)x^{-3} = -\frac{400}{x^3}$

  ↳ Crit. pts.
  - $p'(x) = 0$: $2 - \frac{200}{x^2} = 0 \Rightarrow x^2 = 100 \Rightarrow x = \pm 10$.
  - $p'(x)$ is defined everywhere on $(0,\infty)$ → no "exotic" crit. pts.

Discarding $x=-10$ which is out of the domain, we are left with a unique crit. pt. $x=10$

- 2nd D.T.

$p''(x) = \frac{400}{x^3} \Rightarrow p''(10) = \frac{400}{1000} = \frac{2}{5} > 0 \Rightarrow$  $\Rightarrow$ Uniq. L.m. at $x=10$

G.m. at $x=10$

4. Conclusion

The minimal perimeter is attained when $x = 10$.

- $P(10) = 2 \cdot 10 + \frac{200}{10} = 20 + 20 = 40$

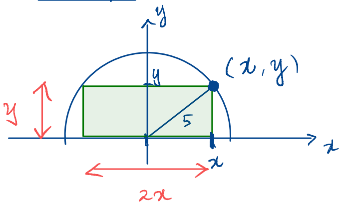
- Configuration:
  - base $(x) = 10$
  - height $(y) = 10$

that is, the rectangle is actually a square.

Example (Rectangles beneath a semicircle)

A rectangle is constructed with its base on the diameter of a semicircle with radius 5 and its two other vertices on the semicircle. What are the dimensions of the rectangle with maximum area?

1. Prep



point in 1st quad

- Known: (x, y) is on the semicircle centered at $(0,0)$ with radius 5.

$$x^2 + y^2 = 5^2 \qquad \text{(constraint)}$$

- Want: Maximize $A = 2xy$ (obj. func.)

$$y^2 = 25 - x^2 \implies y = \pm\sqrt{25 - x^2}$$

2. Rewrite

$A(x) = 2x\sqrt{25 - x^2}$

domain: $0 < x < 5$

3. Do calculus

Find the global maximum of

$A(x) = 2x\sqrt{25-x^2}$ on $(0,5)$.

- $A'(x) = 2\sqrt{25-x^2} + 2x \cdot \dfrac{-\cancel{2}x}{\cancel{2}\sqrt{25-x^2}}$

$= 2\sqrt{25-x^2} - \dfrac{2x^2}{\sqrt{25-x^2}}$

where $25-x^2 \le 0$

Crit. pts.

$A'(x)$ is not defined at $\underbrace{x \geq 5,\ x \leq -5}$, i.e.,

$A'(x)$ is well defined on $(0,5)$

$A'(x) = 0$: $2\sqrt{25-x^2} - \dfrac{2x^2}{\sqrt{25-x^2}} = 0$

$A(x)$ has a uniq. crit. pt. $x = \frac{5}{\sqrt{2}}$

inside $(0,5)$.

Side calc

$\cancel{x}\sqrt{25-x^2} - \frac{\cancel{x}x^2}{\sqrt{25-x^2}} = 0$

$\sqrt{25-x^2} = \frac{x^2}{\sqrt{25-x^2}}$

$25 - x^2 = x^2$

$2x^2 = 25$

$x = \pm\frac{5}{\sqrt{2}}$

- 1st D.T.

$$A'(x) = 2\,\frac{(25-x^2) - x^2}{\sqrt{25-x^2}}$$

$$= 2\,\frac{25-2x^2}{\sqrt{25-x^2}}$$

$2 > 0$, $\sqrt{25-x^2} > 0$ on $(0,5)$

x	0		$5/\sqrt{2}$		5
A'		+	0	−	
A		↗		↘	
			L.M.		

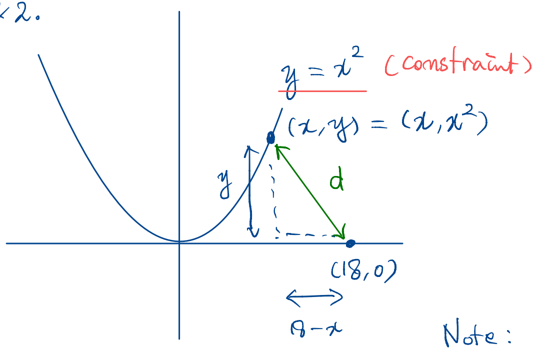
$A(x)$ has a uniq. L.M. at $x = 5/\sqrt{2}$, thus G.M. at $x = 5/\sqrt{2}$.

4. Conclusion

Example (Minimum distance)

Find the point P on the curve $y = x^2$ that is closest to the point $(18, 0)$. What is the least distance between P and $(18, 0)$?

1. & 2.



- Known: $y = x^2$
- Want: Minimize $d = \sqrt{(18-x)^2 + y^2}$

$$d(x) = \sqrt{(18-x)^2 + x^4}, \quad (-\infty, \infty)$$

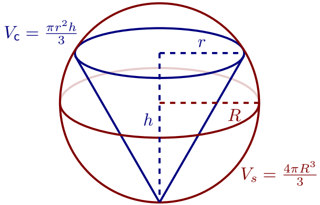
Note: $d(x)$ is minimized if and only if $[d(x)]^2$ is minimized.

3. Do calculus.

Do it yourself

Example

If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by “cone” we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)



Example

Suppose you want to reach a point A that is located across the sand from a nearby road. Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?