Lecture 26-27: Optimization (O & AO)

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Basic Idea and Terminology

An **optimization problem** is a problem where you need to maximize or minimize some quantity under some constraints. This can be accomplished using the tools of differential calculus that we have already developed.

Terminology.

- **constraints:** conditions imposed on variables
- **objective functions:** the quantities desired to be optimized

A Solitary Local Extremum

- The extreme value theorem guarantees the existence of global extrema only on a closed interval. f cts on a closed interval
- On intervals that are not closed, the theorem is not applicable. Yet, when there is only one local extreme value, we can say something about global extrema.

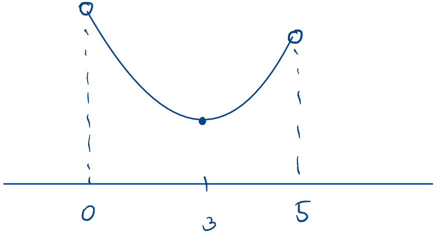
Theorem

Suppose f is continuous on an interval I that contains exactly one local extremum at c. generic interval

- *If a local maximum occurs at c, then $f(c)$ is the global maximum of f on I.*
- *If a local minimum occurs at c, then $f(c)$ is the global minimum of f on I.*

Ex 1

$f(x) = (x-3)^2 + 1$ on $(0, 5)$ ← open



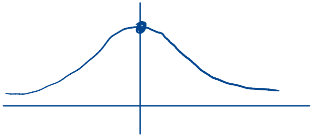
f has a unique local min. at $x = 3$.

f attains the global min. value at $\underline{x=3}$

Ex 2

$f(x) = \dfrac{1}{1+x^2} = \dfrac{d}{dx}\left(\tan^{-1}(x)\right)$ on $(-\infty, \infty)$

(Runge's function)



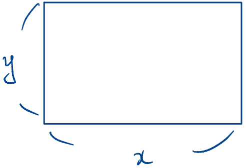
- Uniq. loc. max. at $x = 0$

$\Rightarrow f$ attains G. M. at $x = 0$.

Example (Maximum area rectangles)

Of all rectangles of perimeter 12, which side lengths give the greatest area?

1. Pict., Not'n, and Ident.



$$\begin{cases} \underset{\text{perimeter}}{P} = \underline{2x+2y = 12} & \text{(} \underline{\text{constraint}} \text{)} \\ \underset{\text{area}}{A} = xy & \text{(obj. func.; to be maximized)} \end{cases}$$

2. Write obj. func. as a single variable func.

Using the constraint $2x+2y=12$,

$\underline{y = 6-x}$ $\longrightarrow$ $A(x) = x(6-x)$

domain: $0<x<6$

i.e. $(0,6)$

3. Do calculus.

Example (Minimum perimeter rectangles)

Of all rectangles of area 100, which has the smallest perimeter?

Example (Rectangles beneath a semicircle)

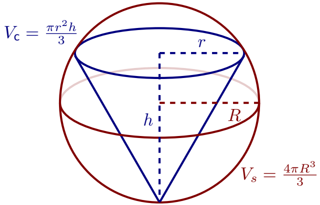
A rectangle is constructed with its base on the diameter of a semicircle with radius 5 and its two other vertices on the semicircle. What are the dimensions of the rectangle with maximum area?

Example (Minimum distance)

Find the point P on the curve $y = x^2$ that is closest to the point $(18, 0)$. What is the least distance between P and $(18, 0)$?

Example

If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by “cone” we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)



Example

Suppose you want to reach a point A that is located across the sand from a nearby road. Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?