Lecture 37: Second Fundamental Theorem of Calculus (SFTOC)

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The Second Fundamental Theorem of Calculus

Here comes the second form of the Fundamental Theorem of

Theorem (Second Fundamental Theorem of Calculus, FTC2)

*Let f be continuous on $[a, b]$. If F is **any** antiderivative of f, then*

$$\int_a^b f(x)\,dx = F(b) - F(a)\,. \tag{FTC2}$$

- An alternate interpretation of (FTC2) is to write it as

$$\int_a^b \frac{d}{dx} f(x)\,dx = f(b) - f(a).$$

- The above reads as

The accumulation of a rate is given by the change in the amount.

Notation

- FTC2 is useful in computing a definite integral:
  1. find an antiderivative of the integrand;
  2. evaluate it at the limits of integration;
  3. take the difference.
- In the differencing process, you may find the following notation convenient:

$$\Big[F(x)\Big]_a^b = F(x)\Big|_a^b = F(b) - F(a)\,.$$

Proof. Let $a \leq c \leq b$ and write

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \\ &= \int_c^b f(x)\,dx - \int_c^a f(x)\,dx.\end{aligned}$$

By the First Fundamental Theorem of Calculus, we have

$$F(b) = \int_c^b f(x)\,dx \qquad \text{and} \qquad F(a) = \int_c^a f(x)\,dx$$

for some antiderivative F of f. So

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

for this antiderivative. However, **any** antiderivative could have be chosen, as antiderivatives of a given function differ only by a constant, and this constant *always* cancels out of the expression when evaluating $F(b) - F(a)$. □

Question. Compute:

1. $\displaystyle\int_{-2}^{2} x^3\,dx$

2. $\displaystyle\int_{0}^{1} \frac{\pi}{3} \sin \frac{\pi}{3}\theta\,d\theta$

Question. Compute:

1. $$\int_0^5 e^t \, dt$$

2. $$\int_1^2 \left(x^9 + \frac{1}{x} \right) dx$$

Net Change and Future Value

Displacement and net change

Let's recall that

- The derivative of a position function s is a velocity function v.
- The derivative of a velocity function v is an acceleration function a.

In other words,

- A velocity function v is an antiderivative of an acceleration function a.
- A position function s is an antiderivative of a velocity function v.

In particular, by FTC2,

$$\int_a^b v(t)\, dt = s(b) - s(a)\,,$$

which measures a **change in position**, or **displacement** as already introduced on Monday.

Net change and future value

- In general, FTC2 states that the definite integral of a rate of change of a certain quantity Q is the **net change** in its amount between two limits of integration:
$$\int_a^b Q'(s)\,ds = Q(b) - Q(a)\,. \qquad \text{(Net change)}$$
- If we replace $a = 0$ and $b = t$, we have a formula for **future value**:
$$Q(t) = Q(0) + \int_0^t Q'(s)\,ds\,. \qquad \text{(Future value)}$$

Question. A book publisher estimates that the marginal cost of a particular title (in dollars/book) is given by

$$C'(x) = 12 - 0.0002x\,,$$

where $0 \leq x \leq 50,000$ is the number of books printed. What is the cost of producing the 12,001st through 15,000th book?

Summary of three different integrals

1. An **indefinite integral**, a.k.a. an antiderivative computes a family of functions:

$$\boxed{\int f(x)\,dx = F(x) + C}$$

   where $F'(x) = f(x)$.

2. An **accumulation function** computes an accumulated area:

$$\boxed{F(x) = \int_a^x f(t)\,dt}$$

   FTC1 says that $F'(x) = f(x)$.

3. A **definite integral** computes a signed area:

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$