

# Lecture 40: Working with Substitution (WWS)

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# Substitution Procedures

Let's recall that in integrating a function which we suspect to be the derivative of another obtained by the chain rule:

- 1 Look for a candidate for the inner function; call it  $u$ .
- 2 Rewrite the given function completely in terms of  $u$  leaving no trace of the original variable.
- 3 Integrate this new function of  $u$ . (If necessary, you may need to go back to Step 1 and modify your choice of  $u$ .)
- 4 In dealing with an indefinite integral, make sure to replace  $u$  by the equivalent expression of the original variable.
- 5 Working with a definite integral, you may evaluate the result of Step 3 at the transformed bounds of  $u$  or evaluate the antiderivate obtained in Step 4 at the original bounds.

For the remainder of the lecture, we will work out practice examples.

## Example

Compute:

①  $\int_2^3 \frac{1}{x \ln(x)} dx$

②  $\int_0^{16} \sqrt{4 - \sqrt{x}} dx$

## Example

Compute:

$$① \int \frac{\sec(y) \tan(y) + \sec^2(y)}{\sec(y) + \tan(y)} dy$$

$$② \int \tan(x) dx$$

## Example

Compute:

$$\textcircled{1} \int \frac{u}{1-u^2} du$$

$$\textcircled{2} \int \frac{e^{2x}}{1-e^{2x}} dx$$

## Example

Compute:

$$\textcircled{1} \int x^3 \sqrt{1-x^2} \, dx$$

# Work

Suppose the force  $F(s)$  is applied to an object and moves it from  $s = s_0$  to  $s = s_1$ . Then the work done on the object over the course of motion is given by

$$W = \int_{s_0}^{s_1} F(s) ds.$$

The international standard unit of force is a **Joule**, which is defined to be

$$1 \text{ J} = 1 \text{ N} \cdot \text{m}.$$

In words, work measures the accumulated force over a distance. Note that we only accumulate force in the *direction* or *opposite direction* of motion.

## Example

If an apple has a mass of 0.1 kg, how much work is required to lift this 1 meter above the ground? Assume that the gravitational acceleration is  $-9.8 \text{ m/s}^2$ .



# Kinetic Energy

Now suppose that an object of mass  $m$  is moving at velocity  $v(t)$ . The **kinetic energy**  $E_k$  is the amount of *energy* that an object possesses from its motion. It is defined by

$$E_k = \frac{mv^2}{2}.$$

The SI unit of energy is also a **Joule** since  $1 \text{ N} = 1 \text{ kg} \cdot \text{m} / \text{s}^2$ :

$$1 \text{ J} = 1 \text{ kg} \cdot \text{m}^2 / \text{s}^2.$$

## Example

Now the apple is dropped from the height of 1 meter. How much kinetic energy is released when it hits the ground?

# Work-Energy Theorem

We observe that the work and the energy calculated above are the same with the same unit. This is not a coincidence and this phenomenon can be explained via the **Work-Energy Theorem**.

## Theorem (Work-Energy Theorem)

*Suppose that an object of mass  $m$  is moving along a straight line. If  $s_0$  and  $s_1$  are the the starting and ending positions,  $v_0$  and  $v_1$  are the the starting and ending velocities, and  $F(s)$  is the force acting on the object at any given position, then*

$$W = \int_{s_0}^{s_1} F(s) ds = \frac{mv_1^2}{2} - \frac{mv_0^2}{2}.$$

*In other words, the total work done by the force is equal to the net change in kinetic energy.*

# Explanation

Let  $t_{0,1}$  be the initial and terminal time of the motion respectively, so that we can write  $s_{0,1} = s(t_{0,1})$  respectively. Then the work formula can be written as

$$W = \int_{s(t_0)}^{s(t_1)} F(s) ds = \int_{t_0}^{t_1} F(s(t))s'(t) dt,$$

where the second equality is due to the substitution rule with  $u = s(t)$ . By Newton's second law of motion  $F(s(t)) = ma(t)$ , we can write

$$\int_{t_0}^{t_1} F(s(t))s'(t) dt = \int_{t_0}^{t_1} ma(t)s'(t) dt = m \int_{t_0}^{t_1} a(t)s'(t) dt.$$

Remembering that  $a(t) = v'(t)$  and  $s'(t) = v(t)$ , we can write the integrand solely in terms of  $v$  and its derivative, i.e.,

$$m \int_{t_0}^{t_1} a(t)s'(t) dt = m \int_{t_0}^{t_1} v(t)v'(t) dt.$$

Another substitution  $u = v(t)$  with new bounds  $v_0 = v(t_0)$  and  $v_1 = v(t_1)$  yields the desired result.

$$m \int_{t_0}^{t_1} v(t)v'(t) dt = m \int_{v_0}^{v_1} u du = m \left[ \frac{u^2}{2} \right]_{v_0}^{v_1} = \frac{mv_1^2}{2} - \frac{mv_0^2}{2}.$$