Lecture 2: What Is A Limit (WIAL)

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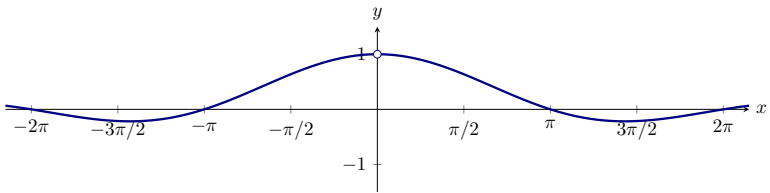
What is a limit?

Basic idea. Consider the function

$$f(x) = \frac{\sin(x)}{x}.$$



Question.

- Is f defined at $x = 0$?
- Where is $f(x)$ approaching as x gets closer to 0?

Definition

Intuitively, we say that

$$\text{the } \textbf{limit} \text{ of } f(x) \text{ as } x \text{ approaches } a \text{ is } L,$$

written

$$\lim_{x \to a} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all x sufficiently close, but not equal to, a.

Definition

Intuitively,

the **limit from the right** of f as x approaches a is L,

written

$$\lim_{x \to a^+} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all $x > a$ sufficiently close, but not equal to, a.
Similarly,

the **limit from the left** of $f(x)$ as x approaches a is L,

written

$$\lim_{x \to a^-} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all $x < a$ sufficiently close, but not equal to, a.

Theorem

A limit

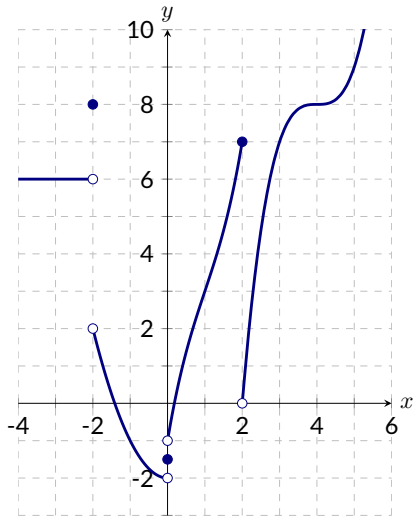
$$\lim_{x \to a} f(x)$$

exists if and only if

- $\lim_{x \to a^-} f(x)$ *exists*
- $\lim_{x \to a^+} f(x)$ *exists*
- $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

In this case, $\lim_{x \to a} f(x)$ *is equal to the common value of the two one sided limits.*

Question. Study limits of the following graph at various points.



Continuity

Definition

A function f is **continuous at a point** a if

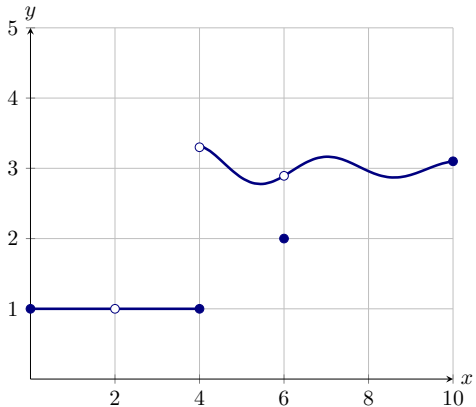
$$\lim_{x \to a} f(x) = f(a).$$

We can unpack the single equation above as:

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

Question. How can a function be discontinuous at a point?

Question. Find the discontinuities.



Definition

- A function f is **left continuous** at a point a if $\lim_{x \to a^-} f(x) = f(a)$.
- A function f is **right continuous** at a point a if $\lim_{x \to a^+} f(x) = f(a)$.

We can talk about continuity on intervals now.

Definition

A function f is

- **continuous on an open interval** (a, b) if $\lim_{x \to c} f(x) = f(c)$ for all c in (a, b);
- **continuous on a closed interval** $[a, b]$ if f is continuous on (a, b), right continuous at a, and left continuous at b.

Continuity of Famous Functions

The following functions are continuous on their natural domains, for k a real number and b a positive real number:

- **Constant function** $f(x) = k$
- **Identity function** $f(x) = x$
- **Power function** $f(x) = x^b$
- **Exponential function** $f(x) = b^x$
- **Logarithmic function** $f(x) = \log_b(x)$
- **Sine and cosine functions** $f(x) = \sin(x)$ and $f(x) = \cos(x)$

Question. (Revisiting the previous graph) What are the *largest intervals* of continuity?