Lecture 10: Rules of Differentiation (ROD)

Tae Eun Kim, Ph.D.

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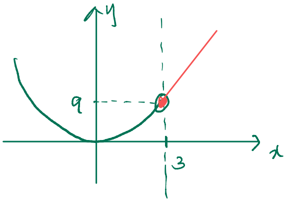
Last question from Wed.

Question. Consider

$$f(x) = \begin{cases} x^2 & \text{if } x < 3, \\ mx + b & \text{if } x \geq 3. \end{cases}$$

What values of m and b make f differentiable at $x = 3$?



Requirements

① Continuity at $x=3$ $\longrightarrow$ Eqn. 1

② differentiability at $x=3$ $\longrightarrow$ Eqn. 2

① Need

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

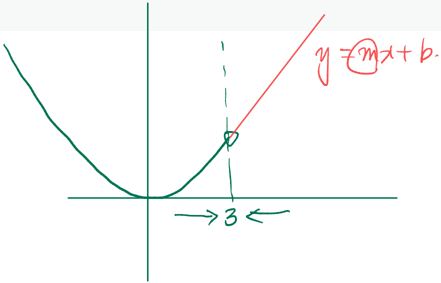
- $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2 = 9$
- $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (mx+b) = 3m+b.$

So, for the limit to exist,

Eqn 1. $3m + b = 9$

② For differentiability at 3, need

$$\lim_{h\to0^+}\frac{f(3+h)-f(3)}{h}=\lim_{h\to0^-}\frac{f(3+h)-f(3)}{h}$$



- $$\lim_{h\to0^+}\frac{f(3+h)-f(3)}{h}=\lim_{h\to0^+}\frac{[\overbrace{m(3+h)}^{\cancel{3m}+mh}+\cancel{b}]-[\cancel{3m}+\cancel{b}]}{h}$$

$$=\lim_{h\to0^+}\frac{m\cancel{h}}{\cancel{h}}=m$$

- $$\lim_{h\to0^-}\frac{f(3+h)-f(3)}{h}=\lim_{h\to0^-}\frac{(3+h)^2-3^2}{h}=\lim_{h\to0^-}\frac{(\cancel{3^2}+2\cdot3\cdot h+h^2)-\cancel{3^2}}{h}$$

$$=\lim_{h\to0^-}\frac{\cancel{h}(6+h)}{\cancel{h}}=6$$

So, we need

$\boxed{\text{Eqn 2}}$ $\quad m=6$

We now have two eqns for two unknowns:

$$\begin{cases} \text{Eqn 1}: & 3m + b = 9 \\ \text{Eqn 2}: & \underline{m \quad = 6} \end{cases}$$

cont.

(done)

diff.

Plugging in $m=6$ into Eqn 1:

$$3 \cdot 6 + b = 9$$

$$18 + b = 9$$

$$\therefore \underline{b = -9}$$

Ans.

$$\boxed{m = 6, \quad b = -9}$$

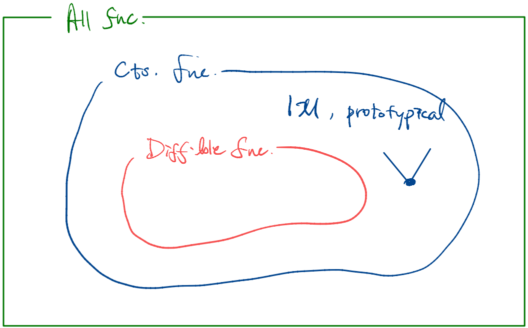
Formula: eqn. of line tangent to the graph of $f(x)$ at $(a, f(a))$

Idea: Point-slope formula w/

- $m = f'(a)$
- point at $(a, f(a))$

$$\boxed{y = f(a) + f'(a)(x-a)}$$

Differentiability $\Rightarrow$ Continuity



Basic rules of differentiation

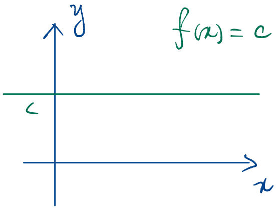
- We have learned the definition of derivative;
- We understand that the derivative of a function can be interpreted as another function;
- Today, we will learn a bunch of differentiation shortcuts which will help us to avoid tedious calculations and focus on more important issues.

Question. Let $f(x) = c$ (constant). Find $f'(x)$, using the def'n.

Soln

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h} = \boxed{0}$$



A horiz. line has slope 0 everywhere.

derivative!

- Limit laws
- Derivative rules.

Theorem (Constant Rule)

Given a constant c,

$$\frac{d}{dx}c = 0.$$

This can be confirmed easily using the definition. However, it is equally important to understand this with good intuition:

- The constant function plots a horizontal line, so the slope of the tangent line at any point is 0.
- If $s(t)$ represents the position of an object with respect to time and $s(t)$ is constant, then the object is not moving, so its velocity is zero. Hence $\frac{d}{dt}s(t) = 0$.
- If $v(t)$ represents the velocity of an object with respect to time and $v(t)$ is constant, then the object's acceleration is zero. Hence $\frac{d}{dt}v(t) = 0$.

Question Let $f(x) = x^n$, n a positive integer. Find $f'(x)$, using the defin.

Recall: $A^n - B^n = (A-B)(A^{n-1} + A^{n-2}B + A^{n-3}B^2 + \cdots + AB^{n-2} + B^{n-1})$

e.g. $A^2 - B^2 = (A-B)(A+B)$

$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$

Soln

Derivation or proof

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \rightarrow \text{messy algebra}$$

$(x+h)^n - x^n$

$$= \lim_{z\to x} \frac{f(z) - f(x)}{z - x} \rightarrow \text{more promising}$$

$$= \lim_{z\to x} \frac{z^n - x^n}{z - x} = \lim_{z\to x} \frac{\cancel{(z-x)}(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})}{\cancel{z-x}}$$

$$= x^{n-1} + x^{n-2}\cdot x + \cdots + x\cdot x^{n-2} + x^{n-1} = \underbrace{x^{n-1} + x^{n-1} + \cdots + x^{n-1}}_{n \text{ terms}} = \underline{n x^{n-1}}$$

Q. How many integers are there btw 0 and $n-1$, inclusive?

A. ① $n-1$ ② n

We simply state the result here. For derivation, please read the textbook.

Theorem (Power Rule)

For any real number n,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Remark. Note that the power rule holds for **any real number** n. This allows us to differentiate such functions as

- $f(x) = x^{13}$
- $f(x) = 1/x^4$
- $f(x) = \sqrt[5]{x}$

$\boxed{f'(x) = 13x^{12}}$

$$\boxed{\begin{aligned} f(x) &= \frac{1}{x^4} = x^{-4} \\ \Rightarrow f'(x) &= -4x^{-4-1} \\ &= -4x^{-5} \end{aligned}}$$

$$\boxed{\begin{aligned} f(x) &= \sqrt[5]{x} = x^{1/5} \\ \Rightarrow f'(x) &= \frac{1}{5}x^{1/5-1} \\ &= \frac{1}{5}x^{-4/5} \end{aligned}}$$

Theorem (Sum Rule)

If $f(x)$ and $g(x)$ are differentiable and c is a constant, then

1. $\dfrac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$,
2. $\dfrac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x)$,
3. $\dfrac{d}{dx}(c\cdot f(x)) = c\cdot f'(x)$. $\longrightarrow$ Later, compare to Prod. Rule.

Question. Compute:

$$\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}} - 2\sqrt{x} + \frac{1}{x^7}\right).$$

$$= \frac{d}{dx}\frac{3}{\sqrt[3]{x}} - \frac{d}{dx}2\sqrt{x} + \frac{d}{dx}\frac{1}{x^7}$$

$$= 3\frac{d}{dx}x^{-1/3} - 2\frac{d}{dx}x^{1/2} + \frac{d}{dx}x^{-7}$$

$$= 3\left(-\frac{1}{3}\right)x^{-4/3} - 2\left(\frac{1}{2}\right)x^{-1/2} + (-7)x^{-8}$$

$$= \underline{-x^{-4/3} - x^{-1/2} - 7x^{-8}}$$

The derivative of the natural exponential function

e^x

- The slope of the line tangent to the curve $y = a^x$, $a > 1$, is positive.
- It is easy to observe that this slope increases as a increases.
- Moreover, we can speculate that for a suitably chosen a, the slope at $x = 0$ can be precisely 1.
- Such a number does exist and is one of the most important constants in mathematics.

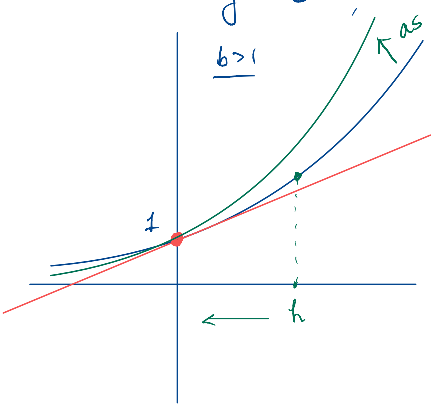
Definition

The number denoted by e, called **Euler's number**, is defined to be the number satisfying the following relation

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1\,.$$

Its approximate value is $e = 2.718281828459045\ldots$.

$y = b^x$



exponential growth.

$$\text{slope?} = \text{slope of tangent} = \lim_{h \to 0} \frac{b^h - b^0}{h - 0} = \lim_{h \to 0} \frac{b^h - 1}{h}$$

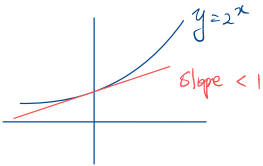
$b^0 = 1$

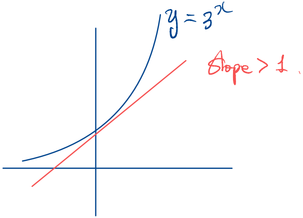
regardless of b.

As b increases, the graph $y = b^x$ displays a steeper tangent line at $x = 0$.

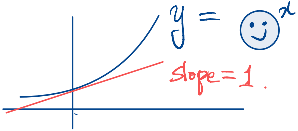
In other words,

$$\lim_{h \to 0} \frac{b^h - 1}{h} \quad \text{increases as } b \text{ increases.}$$





There must be a number btw 2 & 3 such that



This special number is Euler's number, $e \approx 2.7\ldots$

With this definition, we can derive the following important result.

Theorem (Derivative of the Natural Exponential Function)

The derivative of the natural exponential function is the natural exponential function itself. In other words,

$$\boxed{\frac{d}{dx}e^x = e^x\,.}$$

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x e^h - e^x}{h}$$

$$= \lim_{h\to 0}\left[e^x\,\frac{e^h-1}{h}\right]$$

doesn't depend on h. (const.)

$$= e^x \underbrace{\lim_{h\to 0}\frac{e^h-1}{h}}_{=1 \text{ by definition of "}e\text{"}}$$

$$= e^x \quad //$$

Question. Find the slope of the tangent line to the graph of the function $f(x) = e^x$ at $x = 5$.

Soln

$f'(5)$ ...

1. Find the deriv. func. $f'(x)$
2. Plug in $x=5$ into $f'(x)$ $f'(5)$

- $f'(x) = e^x$
- $f'(5) = \boxed{e^5}$ ← slope sought after.

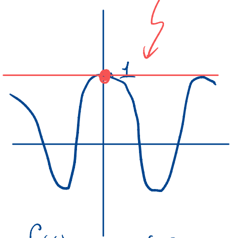
Q. Find the eqn. of the tang. line at $x=5$. ← "a"

A. $y = f(5) + f'(5)(x-5)$

$$\boxed{y = e^5 + e^5(x-5)}$$

The derivative of sine

slope = 0



- In order to derive the derivative of sine function, we need the following results:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \qquad \text{and} \qquad \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0\,.$$

- We derived the first one using the squeeze theorem; the second one follows from the first one. Let's derive it here.
- In addition, let's recall the addition formula for sine:

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)\,.$$

h by θ

- $f(x) = \cos(x)$
- slope at $x=0$

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$f(0) = \cos(0) = 1$

$$= \boxed{\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0}$$

picture.

Now, we are ready to prove the following result:

Theorem (Derivative of Sine)

For any angle θ measured in radians, the derivative of $\sin(\theta)$ with respect to θ is $\cos(\theta)$. In other words,

$$\frac{d}{d\theta}\sin(\theta) = \cos(\theta)\,.$$

$$\frac{d}{d\theta}\sin(\theta) = \lim_{h\to 0}\frac{\sin(\theta+h)-\sin(\theta)}{h}$$

$$\lim_{x\to 1}(7x^4)$$

$$= \lim_{h\to 0}\frac{\sin(\theta)\cos(h)+\sin(h)\cos(\theta)-\sin(\theta)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(\theta)\cos(h)-\sin(\theta)}{h} + \lim_{h\to 0}\frac{\sin(h)\cos(\theta)}{h}$$

$$= \sin(\theta)\underbrace{\lim_{h\to 0}\frac{\cos(h)-1}{h}}_{=0} + \cos(\theta)\underbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}_{=1} = \underline{\cos(\theta)}$$

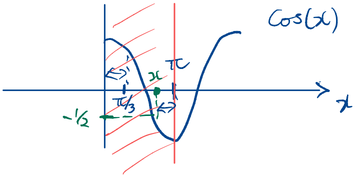
Question. What is the value of x in the interval $[0, \pi]$ where the tangent to the graph of $f(x) = \sin(x)$ has slope $-1/2$?

Soln

(Slope of lines tangent to $f(x) = \sin(x)$)

$$= f'(x) = \cos(x)$$

Rephased Question: Find x in $[0, \pi]$ such that $\cos(x) = -1/2$.



$$\begin{cases} \cos(\pi/3) = 1/2 \\ \cos(2\pi/3) = -1/2. \end{cases}$$

$$\boxed{x = 2\pi/3}$$

For those interested, the following diagram gives a visual interpretation of sine-differentiation.