Lecture 39: The Idea of Substitution (TIOS)

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Two Sides of a Coin

Recall that from the chain rule that

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

So by the fundamental theorem of calculus, we have

$$\int_a^b f'(g(x))g'(x)\,dx = \Big[f(g(x))\Big]_a^b = f(g(b)) - f(g(a)).$$

Using the fundamental theorem in reverse direction once again, the last line can be thought of as

$$\Big[f(u)\Big]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f'(u)\,du.$$

This is the gist of the integration technique known as **substitution rule** or u**-substitution**[1].

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[1] This name is due to a popular and customary choice of substitution variable u. The choice, however, is not an absolute rule written on a stone. Any variable of your choice such as v or ☺ works if used consistently.

Substitution Rule

Theorem (Integral Substitution Formula)

If g is differentiable on the interval $[a, b]$ and f is differentiable on the interval $[g(a), g(b)]$, then

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du.$$

In sum, the substitution rule is the integral counterpart of differential chain rule and the fundamental theorem of calculus serves as a bridge between the two.

Procedures

In integrating a function which we suspect to be the derivative of another obtained by the chain rule:

1. Look for a candidate for the inner function; call it u.
2. Rewrite the given function completely in terms of u leaving no trace of the original variable.
3. Integrate this new function of u. (If necessary, you may need to go back to Step 1 and modify your choice of u.)
4. In dealing with an indefinite integral, make sure to replace u by the equivalent expression of the original variable.
5. Working with a definite integral, you may evaluate the result of Step 3 at the transformed bounds of u or evaluate the antiderivate obtained in Step 4 at the original bounds.

Question. Compute $\int_1^3 x\cos(x^2)\,dx$.

Question. Compute $\int \sec^2(x)\tan(x)\,dx$.

Question. Compute $\int x^4(x^5+1)^9\,dx$.

Question. Compute $\displaystyle\int_{\pi/3}^{\pi/2} \sin(x)\sec^2(\cos(x))\,dx$.

Question. Compute $\int_{-2}^{1} t^2 \sin(t^3)\, dt$.

Question. Compute $\displaystyle\int_0^{1/2} \frac{13e^x}{3e^x - 5}\,dx$.