Lecture 26-27: Optimization (O & AO)

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Basic Idea and Terminology

An **optimization problem** is a problem where you need to maximize or minimize some quantity under some constraints. This can be accomplished using the tools of differential calculus that we have already developed.

Terminology.

- **constraints:** conditions imposed on variables
- **objective functions:** the quantities desired to be optimized

A Solitary Local Extremum

- The extreme value theorem guarantees the existence of global extrema only on a closed interval. f cts on a closed interval
- On intervals that are not closed, the theorem is not applicable. Yet, when there is only one local extreme value, we can say something about global extrema.

Theorem

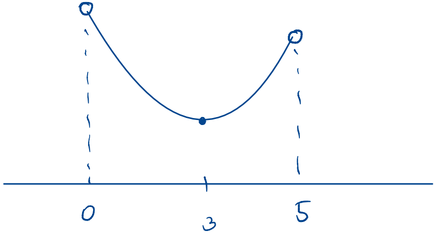
Suppose f is continuous on an interval I that contains exactly one local extremum at c.

generic interval

- *If a local maximum occurs at c, then $f(c)$ is the global maximum of f on I.*
- *If a local minimum occurs at c, then $f(c)$ is the global minimum of f on I.*

Ex 1

$f(x) = (x-3)^2 + 1$ on $(0,5)$ ← open

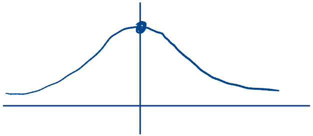


f has a unique local min. at $x=3$.

f attains the global min. value at $x=3$

Ex 2 $f(x) = \dfrac{1}{1+x^2} = \dfrac{d}{dx}\left(\tan^{-1}(x)\right)$ on $(-\infty, \infty)$

(Runge's function)



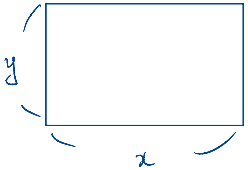
- uniq. loc. max. at $x=0$

$\Rightarrow$ f attains G.M. at $x=0$.

Example (Maximum area rectangles)

Of all rectangles of perimeter 12, which side lengths give the greatest area?

1. Pict., Not'n, and Ident.



$\underset{\text{perimeter}}{P} = \underline{2x + 2y = 12}$ (constraint)

$\underset{\text{area}}{A} = xy$ (obj. func.; to be maximized)

2. Write obj. func. as a single variable func.

Using the constraint $2x+2y=12$,

$\underline{y = 6 - x}$

$A(x) = x(6-x)$

domain: $0 < x < 6$

i.e. $(0, 6)$

3. Do calculus.

Find the global maximum of $A(x) = x(6-x)$ on $(0,6)$

$A(x) = 6x - x^2$

- $A'(x) = 6 - 2x$

  ↳ crit. pts.
  - $A'(x) = 0$: $6 - 2x = 0 \rightarrow \underline{x = 3}$ is the only crit. pts. on $(0,6)$
  - $A'(x)$ undefined : NONE

- 2nd DT

  $A''(x) = -2 < 0$ (regardless of x)

  $A''(3) = -2 < 0 \Rightarrow$  $\Rightarrow$ L.M. at $x=3$ $\xRightarrow{\text{unique}}$ G.M. at $x=3$

  c.p.

4. Conclusion

We attain the maximal area

$$A(3) = \underbrace{3}_{x}\underbrace{(6-3)}_{y} = 3 \cdot 3 = \boxed{9}$$

when

$$\begin{cases} \text{base } (x) = 3 \\ \text{height } (y) = 3 \end{cases}$$

That is, when the rectangle is actually a <u>square</u>.

Example (Minimum perimeter rectangles)

Of all rectangles of area 100, which has the smallest perimeter?

1. Prep.

$$y = \frac{100}{x}$$

Given: $A = xy = 100$ (constraint)

Want: Minimize $P = 2x + 2y$ (objective func.)

2. Write obj. func. as a single variable function.

Determine its domain.

$$P(x) = 2x + 2\cdot\frac{100}{x} = 2x + \frac{200}{x}, \quad \text{for } 0 < x < \infty$$

3. Do calculus.

Find the global minimum of $p(x) = 2x + \frac{200}{x}$ on $(0, \infty)$

- $p'(x) = 2 - \frac{200}{x^2}$ $\qquad (200x^{-2})' = 200(-2)x^{-3} = -\frac{400}{x^3}$

↳ Crit. pts.

$p'(x) = 0$: $2 - \frac{200}{x^2} = 0 \Rightarrow x^2 = 100 \Rightarrow x = \pm 10$.

$p'(x)$ is defined everywhere on $(0, \infty)$ → no "exotic" crit. pts.

Discarding $x = -10$ which is out of the domain, we are left with a unique crit. pt. $\underline{x = 10}$

- $\underline{2^{nd}\ \text{D.T.}}$

$p''(x) = \frac{400}{x^3} \Rightarrow p''(10) = \frac{400}{1000} = \frac{2}{5} > 0 \Rightarrow$  $\Rightarrow$ $\underline{\text{Uniq.}}$ L.m. at $x=10$

$\underline{\text{G.m. at } x=10}$

4. Conclusion

The minimal perimeter is attained when $x = 10$.

- $P(10) = 2 \cdot 10 + \frac{200}{10} = 20 + 20 = 40$

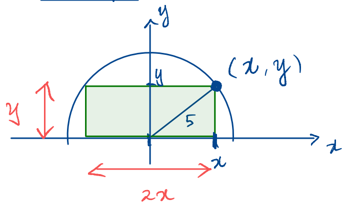
- Configuration:
  - base $(x) = 10$
  - height $(y) = 10$

that is, the rectangle is actually a square.

Example (Rectangles beneath a semicircle)

A rectangle is constructed with its base on the diameter of a semicircle with radius 5 and its two other vertices on the semicircle. What are the dimensions of the rectangle with maximum area?

1. Prep



point in 1st quad

- Known: (x, y) is on the semicircle centered at $(0,0)$ with radius 5.

$$x^2 + y^2 = 5^2 \qquad \text{(constraint)}$$

- Want: Maximize $A = 2xy$ (obj. func.)

$$y^2 = 25 - x^2 \Rightarrow y = + \sqrt{25 - x^2}$$

2. Rewrite

$A(x) = 2x\sqrt{25 - x^2}$

domain: $0 < x < 5$

3. Do calculus

Find the global maximum of

$A(x) = 2x\sqrt{25-x^2}$ on $(0,5)$.

- $A'(x) = 2\sqrt{25-x^2} + 2x \cdot \dfrac{-\cancel{2}x}{\cancel{2}\sqrt{25-x^2}}$

$= 2\sqrt{25-x^2} - \dfrac{2x^2}{\sqrt{25-x^2}}$

where $25-x^2 \leq 0$

Crit. pts.

$A'(x)$ is not defined at $\underbrace{x \geq 5,\ x \leq -5}$, i.e.,

$A'(x)$ is well defined on $(0,5)$

$A'(x) = 0$: $2\sqrt{25-x^2} - \dfrac{2x^2}{\sqrt{25-x^2}} = 0$

$A(x)$ has a uniq. crit. pt. $x = \frac{5}{\sqrt{2}}$

inside $(0, 5)$.

- **1^{st} D.T.**

$$A'(x) = x\,\frac{(25-x^2) - x^2}{\sqrt{25-x^2}}$$

$$= x\,\frac{25-2x^2}{\sqrt{25-x^2}}$$

$0 <$ > 0 on $(0,5)$

x	0		$5/\sqrt{2}$		5
A'		$+$	0	$-$	
A		$\nearrow$		$\searrow$	
			L.M.		

Side calc

$$\cancel{x}\sqrt{25-x^2} - \frac{\cancel{x}x^2}{\sqrt{25-x^2}} = 0$$

$$\sqrt{25-x^2} = \frac{x^2}{\sqrt{25-x^2}}$$

$$25 - x^2 = x^2$$

$$2x^2 = 25$$

$$x = \pm\frac{5}{\sqrt{2}}$$

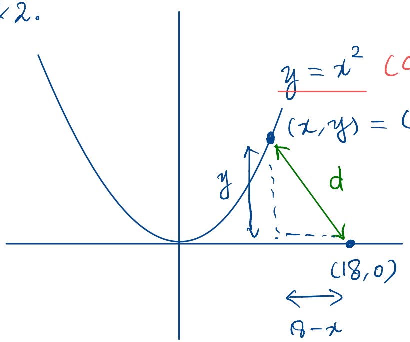
$A(x)$ has a uniq. L.M. at $x = 5/\sqrt{2}$, thus G.M. at $x = 5/\sqrt{2}$.

4. Conclusion

Example (Minimum distance)

Find the point P on the curve $y = x^2$ that is closest to the point $(18, 0)$. What is the least distance between P and $(18, 0)$?

1. & 2.



- Known: $y = x^2$
- Want: Minimize $d = \sqrt{(18-x)^2 + y^2}$

$$d(x) = \sqrt{(18-x)^2 + x^4}, \quad (-\infty, \infty)$$

Note: $d(x)$ is minimized if and only if $[d(x)]^2$ is minimized.

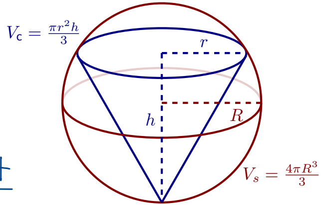
3. Do calculus.

Do it yourself

Example

If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)

1. Picture, notation provided →



- objective function (maximize): $V_c = \frac{\pi r^2 h}{3}$

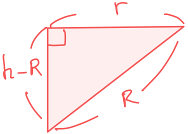
  (because we want to fit the largest possible cone inside)

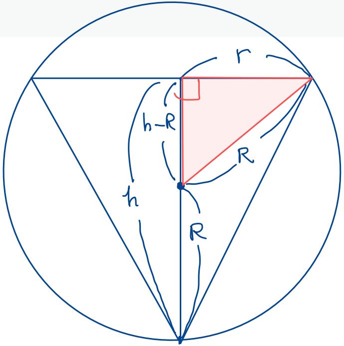
- constraint: the cone is inside the sphere

  → this has a restricting effect on r & h.

From the Consider the cross-section,

note the right triangle shaded in red:





By Pythagorean theorem, $R^2 = r^2 + (h-R)^2$

$$\Longrightarrow \underline{r^2 = R^2 - (h-R)^2}$$

2. Write the objective function as a single variable function

$$V_c(h) = \frac{1}{3}\pi\underbrace{\left(R^2 - (h-R)^2\right)}_{= r^2} h \quad , \quad \text{domain}: (0, 2R)$$

$$0 < h < 2R.$$

3. Do calculus.

Maximize $V_c(h) = \frac{1}{3}\pi\left(R^2 - (h-R)^2\right)h$ on $(0, 2R)$.

- Simplify $V_c(h)$:

$$V_c(h) = \frac{1}{3}\pi\left(\cancel{R^2} - (h^2 - 2hR + \cancel{R^2})\right)h$$
$$= \frac{1}{3}\pi\left(2hR - h^2\right)h$$
$$= \frac{1}{3}\pi\left(2Rh^2 - h^3\right)$$

only crit. point inside $(0,2R)$.

- $V_c'(h) = \frac{1}{3}\pi\left(4Rh - 3h^2\right)$
$= \frac{1}{3}\pi h(4R - 3h) = 0 \implies h = 0,\ h = \frac{4R}{3}$

- $V_c''(h) = \frac{1}{3}\pi(4R - 6h)$, $V_c''\left(\frac{4R}{3}\right) = \frac{1}{3}\pi\left(4R - \cancel{6}^{\,2} \cdot \frac{4R}{\cancel{3}}\right) = \frac{1}{3}\pi(-4R) < 0$

So, by 2^{nd} DT, $V_c(h)$ has a LM at $h = \frac{4R}{3}$

- Since $V_c(h)$ has a unique LM at $h = \frac{4R}{3}$, it has the GM there.

4. Conclusion.

The volume of the largest possible cone inside the sphere is

$$V_c\left(\frac{4R}{3}\right) = \frac{1}{3}\pi\left(2R - \frac{4R}{3}\right)\left(\frac{4R}{3}\right)^2$$

$$= \frac{1}{3}\pi \; \frac{2R}{3} \cdot \frac{16R^2}{9}$$

$$V(h) = \frac{1}{3}\pi\left(2hR - h^2\right)h$$

$$= \frac{1}{3}\pi(2R-h)h^2$$

So the ratio of the volume of cone to that of the sphere is

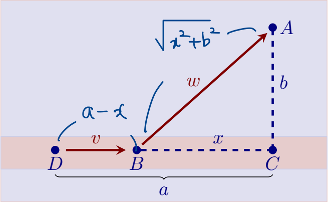
$$\frac{\text{Vol.(cone)}}{\text{Vol (sphere)}} = \frac{\frac{1}{\cancel{3}}\cancel{\pi} \cdot \frac{\overset{8}{\cancel{32}}\cancel{R^3}}{27}}{\frac{\cancel{4}}{\cancel{3}}\cancel{\pi}\;\cancel{R^3}} = \boxed{\frac{8}{27}}$$

$w < v$

Example

Suppose you want to reach a point A that is located across the sand from a nearby road. Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?

1. Picture, notation provided



- objective function : minimize travel time.

$$T = T_{road} + T_{sand}$$

total travel time

time on road

time on sand

Recall : $(\text{time}) = \dfrac{(\text{distance})}{(\text{speed})}$

$$T(x) = \frac{a-x}{v} + \frac{\sqrt{x^2+b^2}}{w}, \quad [0, a]$$

No constraint needed.

2. Do calculus.

「Minimize $T(x) = \dfrac{a-x}{v} + \dfrac{\sqrt{x^2+b^2}}{w}$ on $[0, a]$ where $w < v$.」

- $$T'(x) = -\frac{1}{v} + \frac{x}{w\sqrt{x^2+b^2}} = 0$$

$+\frac{1}{v}$ on both

$$\frac{x}{w\sqrt{x^2+b^2}} = \frac{1}{v}$$

cross-multiply

$$v x = w\sqrt{x^2+b^2}$$

square both

$$v^2x^2 = w^2(x^2+b^2)$$

$$(v^2-w^2)x^2 = w^2b^2$$

$$x = \pm\frac{wb}{\sqrt{v^2-w^2}}$$

(since $w, b, v^2-w^2 > 0$)

Discarding the negative root, we have $x = \dfrac{wb}{\sqrt{v^2-w^2}}$.

However, we do not know whether this critical point is in $[0, a]$ or not.

So we need to consider different possibilities.

Before jumping into case-by-case analysis,

let's compute $T''(x)$, which will be useful later.

(from above) $T'(x) = -\frac{1}{v} + \frac{x}{w\sqrt{x^2+b^2}}$

$$T''(x) = \frac{1}{w} \cdot \frac{\sqrt{x^2+b^2} - x\frac{x}{\sqrt{x^2+b^2}}}{x^2+b^2} \cdot \frac{\sqrt{x^2+b^2}}{\sqrt{x^2+b^2}}$$

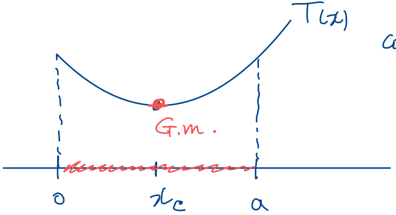
$$= \frac{1}{w} \frac{(x^2+b^2) - x^2}{(x^2+b^2)^{3/2}}$$

$$= \frac{1}{w} \frac{b^2}{(x^2+b^2)^{3/2}} > 0 \quad \text{for any } x>0$$

In other words, the graph of $T(x)$ is concave up for any positive x.

Here comes case studies. Let $x_c = \dfrac{wb}{\sqrt{v^2-w^2}}$, the positive critical point.

<u>Case 1</u> x_c is in $(0, a)$



concave up everywhere

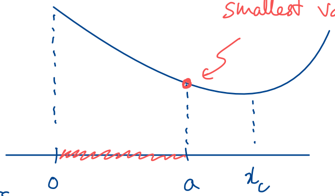
In this case, by the 2nd DT,

$$T''(x_c) > 0$$ (see prev. page)

So T has a unique local min. at $x = x_c$, hence the global min. at $x = x_c$

<u>Case 2</u> x_c is outside of $(0, a)$, i.e.

$$x_c \geq a$$



In this case, x_c is out of bound, and so there is no interior critical point. So the only candidates for global. min. are endpoints

$x=0$ and $x=a$. You may evaluate and compare $T(0)$ and $T(a)$.

A smarter way is to note that

$$\begin{cases} \bullet\ T''(x) > 0 \text{ for all } x \\ \bullet\ T'(x_c) = 0 \end{cases}$$

So $T'(x) < 0$ on $(0, x_c)$

In particular, $T(x)$ is decreasing on $(0, x_c)$. Therefore

$$T(0) > T(x_c)$$

This can be seen immediately if you sketch the graph.

In conclusion, T attains the global minimum at $x=a$.

3. Conclusion.

One can minimize the travel time T by setting

$$x = \begin{cases} x_c & \text{if } x_c = \dfrac{wb}{\sqrt{v^2 - w^2}} < a \\ a & \text{if } x_c \geqslant a \end{cases}$$

<u>Note</u>: The above can be succinctly written as

$$x = \min\left\{ \frac{wb}{\sqrt{v^2 - w^2}},\ a \right\}$$