horizontal, vertical, ~~slant~~

Lecture 5: Using Limit~~e~~s to Detect Asymptotes (ULTDA)

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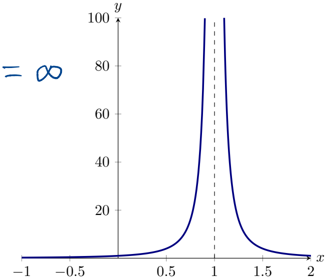
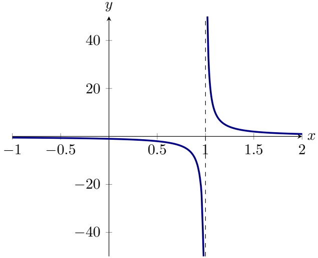
Vertical asymptotes – infinite limits ← definite form "$\frac{\#}{0}$"

Consider the graphs of the following two functions near $x = 1$:

$$f(x) = \frac{1}{(x-1)^2} \quad \text{and} \quad g(x) = \frac{1}{x-1}\,.$$

$\lim_{x \to 1} f(x) = \infty$





$\lim_{x \to 1^+} g(x) = \infty$

$\lim_{x \to 1^-} g(x) = -\infty$

- In both cases, the graphs get closer and closer to the vertical line $x = 1$, but they never touch it.
- Such a line is called a vertical asymptote.

Definition

If at least one of the following holds:

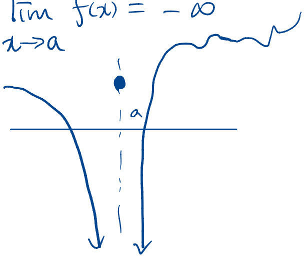
- $\lim_{x\to a} f(x) = \pm\infty$,
- $\lim_{x\to a^+} f(x) = \pm\infty$,
- $\lim_{x\to a^-} f(x) = \pm\infty$,

then the line $x = a$ is a **vertical asymptote** of f.

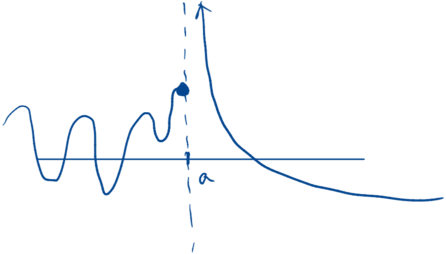
also the case that $\lim_{x\to a^+} f(x) = -\infty$

Examples

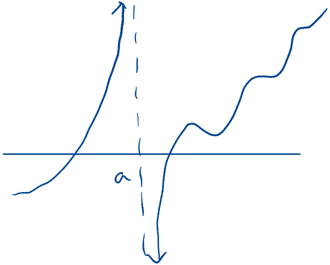
- $\lim_{x\to a} f(x) = -\infty$



- $\lim_{x\to a^+} f(x) = +\infty$



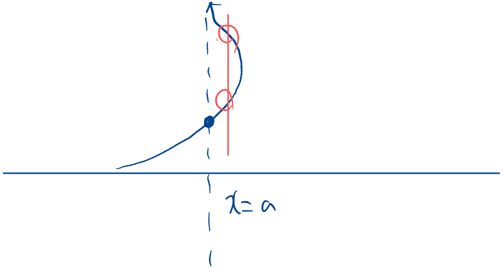
$\lim_{x\to a^-} f(x) = +\infty$



Q. Can a graph of a func cross a vertical asymptote?

Note: A function may be defined at $x=a$ (V.A.) as long as the conditions of def'n is satisfied.

A. No. Vertical line test.



Question. Find the vertical asymptotes of

$$f(x) = \frac{x^2 - 9x + 14}{x^2 - 5x + 6}.$$

Potential locations of V.A. are the zeros (or roots) of the denominator.

Soln Note that

$$x^2 - 5x + 6 = (x-2)(x-3).$$

So $x=2$ and $x=3$ are candidates for V.A.

Side:

$$x^2 - 9x + 14 = (x-2)(x-7)$$

@ $x=2$ $\quad \lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{\cancel{(x-2)}(x-7)}{\cancel{(x-2)}(x-3)} = \frac{-5}{-1} = 5$

"$\frac{0}{0}$"

It is not a V.A.

@ $x=3$

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{(x-2)(x-7)}{(x-2)(x-3)}$$

$$\begin{cases} \lim_{x\to 3}(NUM) = -4 \\ \lim_{x\to 3}(DEN) = 0 \end{cases}$$

This limit is in "$\frac{\#}{0}$" form, hence we will end up with an infinite limit.

Therefore, $x=3$ is a V.A.

"$\frac{\#}{0}$" form.

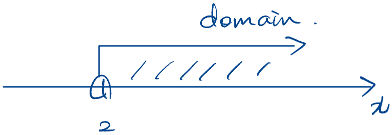
$\lim_{x\to 3^+} f(x) = -\infty$, $\lim_{x\to 3^-} f(x) = +\infty$, $\lim_{x\to 3} f(x)$ DNE.

NUM: $(-)$

DEN: $(+)$, small.

Question. Find the vertical asymptotes of

$$f(x)=\frac{\sqrt{x^2-3x+2}}{x-2}, \quad \underbrace{x>2}_{\text{domain}}.$$



Soln Candidates for V.A.:

$\boxed{\text{Only } x=2.}$

@ $x=2$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{\sqrt{(x-2)(x-1)}}{x-2} = \lim_{x\to 2^+} \frac{\cancel{\sqrt{x-2}}\,\sqrt{x-1}}{\left(\sqrt{x-2}\right)^{\cancel{2}}}$$

$$= \lim_{x\to 2^+} \frac{\sqrt{x-1}}{\sqrt{x-2}}$$

$$\left\{\begin{array}{l} \bullet\ (\text{NUM}) \to 1 > 0 \\ \bullet\ (\text{DEN}) \to 0, \text{ from positive} \end{array}\right\} \text{ "}\frac{\#}{0}\text{" form}$$

$$= +\infty$$

Hence, $x=2$ is a V.A.

Horizontal asymptotes – limits at infinity

$$\lim_{x\to a} f(x) = \pm\infty$$

$$\lim_{x\to\pm\infty} f(x) = L$$

Definition

- If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large, we write

$$\boxed{\lim_{x\to\infty} f(x) = L}$$

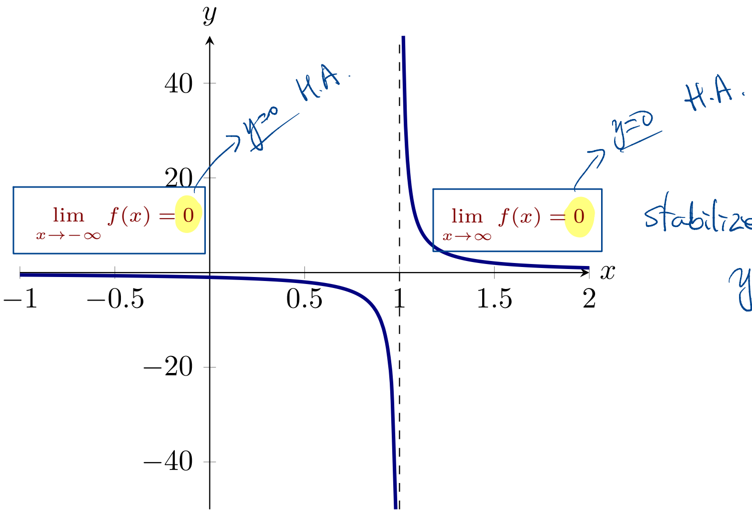
  and we say that the **limit at infinity** of $f(x)$ is L.

- If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large and negative, we write

$$\boxed{\lim_{x\to-\infty} f(x) = L}$$

  and we say that the **limit at negative infinity** of $f(x)$ is L.

Illustration. The function $f(x) = 1/(x-1)$ once again provides us with a valuable insight:



stabilized fnc values

$y = 0$.

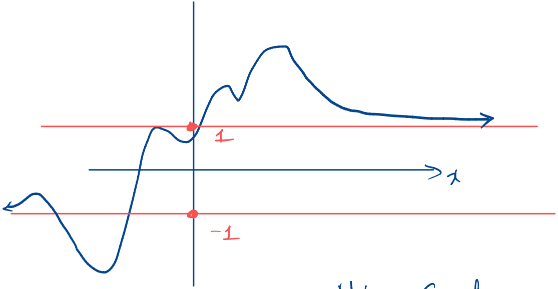
The graph suggests that having finite limits at infinity has a lot to do with horizontal asymptotes, thus the following definition:

Definition

If

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L,$$

then the line $y = L$ is a **horizontal asymptote** of $f(x)$.



$\cdot \lim_{x\to\infty} f(x) = 1$

$\cdot \lim_{x\to-\infty} f(x) = -1$

Two H.A.: $y=1$, $y=-1$

Note: Graph may cross H.A. cf) V.A.

Question. Find the horizontal asymptotes of

$$f(x) = \frac{6x - 9}{x - 1}.$$

Soln Just need to calculate $\lim_{x \to \pm\infty} f(x)$.

- $\lim_{x \to +\infty} \frac{6x - 9}{x - 1}$

Trick: Divide by highest power of the top. (In this case, x)

$$= \lim_{x \to +\infty} \frac{6 - \frac{9}{x}}{1 - \frac{1}{x}}$$

$\frac{9}{x} \to 0$, $\frac{1}{x} \to 0$

$$= \boxed{6}$$

$\therefore$ $y = 6$ is a H.A.

- Likewise, can show that $\lim_{x \to -\infty} f(x) = \boxed{6}$

Building Blocks

- $\lim_{x \to \pm\infty} \frac{1}{x^p} = 0$ where $p > 0$

Question. Find the horizontal asymptotes of

$$f(x) = \frac{x^3+1}{\sqrt{x^6+6}}.$$

Soln Before computing limits at infinity, let's first rewrite $f(x)$: $f(x) = \dfrac{x^3+1}{\sqrt{x^6}\sqrt{1+6/x^6}}$

Sidenote

$$\sqrt{x^6} = |x^3| = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases}$$

- $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3+1}{x^3\sqrt{1+6/x^6}} = \lim_{x\to\infty} \frac{1+\overbrace{1/x^3}^{\to 0}}{\sqrt{1+\underbrace{6/x^6}_{\to 0}}} = \boxed{1}$

- $\displaystyle\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^3+1}{-x^3\sqrt{1+6/x^3}} = \lim_{x\to\infty} \frac{1+\overbrace{1/x^3}^{\to 0}}{-\sqrt{1+\underbrace{6/x^1}_{\to 0}}} = \boxed{-1}$

Therefore, $f(x)$ has two HAs : $\boxed{y = \pm 1}$

Question. Compute

$$\lim_{x\to\infty} \frac{\sin(7x) + 4x}{x}.$$

Hint. Use the squeeze theorem.

Soln $= \lim_{x\to\infty} \left(\frac{\sin(7x)}{x} + \frac{4x}{x} \right)$

$= \underbrace{\lim_{x\to\infty} \frac{\sin(7x)}{x}}_{=0} + \underbrace{\lim_{x\to\infty} 4}_{=4} = 0 + 4 = \boxed{4}$

$-1 \leq \sin(7x) \leq 1$

① $-\frac{1}{x} \leq \frac{\sin(7x)}{x} \leq \frac{1}{x}$, $x > 0$

② $\lim_{x\to\infty} \left(\pm\frac{1}{x}\right) = 0$

$\left.\right\}$ S.T. $\Longrightarrow \lim_{x\to\infty} \frac{\sin(7x)}{x} = 0.$

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