Lecture 25: Linear Approximation (LA)

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Applications of derivatives

- related rates
- local / global extreme values
  (optimization)
- graphics
- MVT
- linear approx.

Linear Approximation

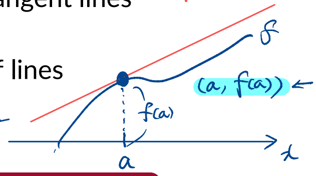
"slope"

$m = f'(a)$

- Recall that the derivative contains the slope information of tangent lines to a given curve.
- Using this, we spent a lot of time calculating the equations of lines tangent to curves.

Let's formalize this discussion.

pt.-sl. formula



"point"

Definition

If f is a function differentiable at $x = a$, then a **linear approximation** for f at $x = a$ is given by

$$\boxed{L(x) = f(a) + f'(a)(x - a)\,.}$$

Note that the graph of L is simply the tangent line to the curve $y = f(x)$ at $x = a$.

Derivation   Generic Pt.-Sl. formula: $y - y_0 = m(x - x_0)$

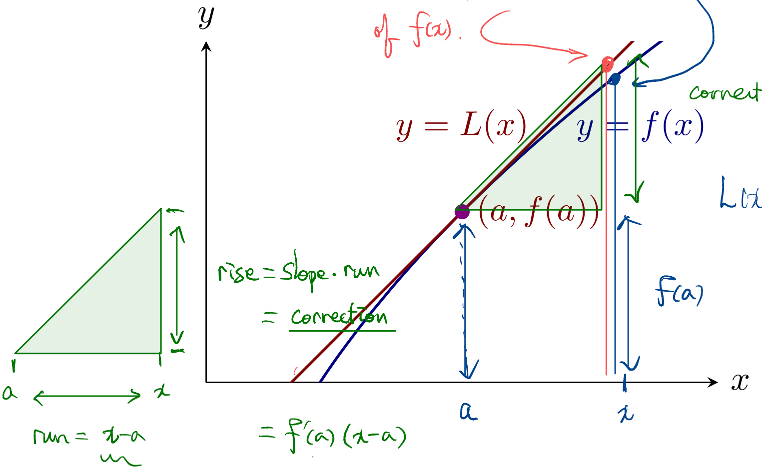
Adapted version : $y - f(a) = f'(a)(x - a)$

$(x_0 = a, y_0 = f(a), m = f'(a))$ $y = f(a) + f'(a)(x - a)$

Illustration

$f(x)$, the exact value (somehow difficult to attain directly)

$L(x)$, approx. of $f(a)$.



$$L(x) = \underbrace{f(a)}_{\text{initial (coarse) approximation}} + \underbrace{f'(a)(x-a)}_{\text{correction (improvement)}}$$

Advanced notes "Taylor polynomial"

higher-order derivatives.

Higher-order approximations are possible provided that f is "nice". For instance,

we can approximate $f(x)$ using a quadratic function

$$Q(x) := \underbrace{f(a) + f'(a)(x-a)}_{L(x)} + \frac{f''(a)}{2}(x-a)^2$$

Prop

$$\begin{cases} f(a) = Q(a) \\ f'(a) = Q'(a) \\ f''(a) = Q''(a) \end{cases}$$

Examples

One major advantage of linear approximation is in computation of various mathematical functions.

Question. Use a linear approximation of $f(x) = \sqrt[3]{x}$ at $a = 64$ to approximate $\sqrt[3]{50}$. What would happen if we chose $a = 27$ instead?

To form $L(x)$, need:

- $f(a) = f(64) = \sqrt[3]{64} = 4$
- $f'(x) = \frac{d}{dx} x^{1/3} = \frac{1}{3} x^{-2/3}$

$\hookrightarrow f'(a) = f'(64) = \frac{1}{3} \frac{1}{\sqrt[3]{64}^2}$

$= \frac{1}{3} \cdot \frac{1}{4^2} = \frac{1}{48}$

$L(x) = f(64) + f'(64)(x-64)$

$= 4 + \frac{1}{48}(x-64)$

So,

approximately equal

$\sqrt[3]{50} = f(50) \approx L(50)$

$= 4 + \frac{1}{48}(50-64)$

$= 4 - \frac{14}{48} = \boxed{3.7\ldots}$

$\sqrt[3]{50}$ // $3.684\ldots$

Examples

Question. Use a linear approximation of $f(x) = \sin(x)$ at $a = 0$ to approximate $\sin(0.3)$.

$$\sin(0.3) = f(0.3) \approx L(0.3) = \boxed{0.3}$$

$$\sin(0.3) \approx 0.3$$

To form $L(x)$, need @ $a = 0$.

- $f(a) = f(0) = \sin(0) = 0$
- $f'(x) = \frac{d}{dx}\sin(x) = \cos(x)$

  $f'(a) = f'(0) = \cos(0) = 1$

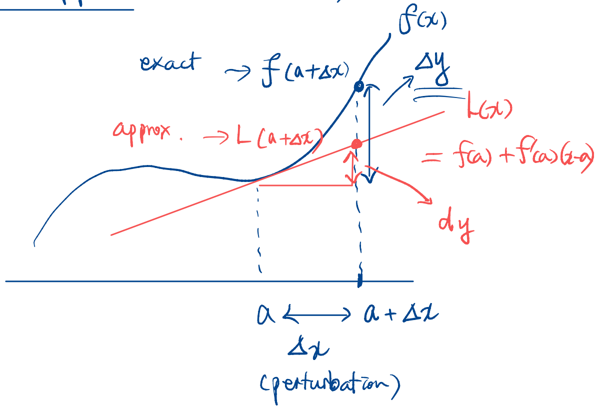
$$L(x) = 0 + 1\,(x - 0)$$

$$L(x) = x$$

Note Physicists use this all the time. ("small angle approx.")

$\sin(\theta) \approx \theta$. for small $\theta \approx 0$.

Alternate form/way of doing linear approx. (notation)



$$
\begin{aligned}
f(a+\Delta x) &\approx L(a+\Delta x) \\
&= f(a) + f'(a)\left(\cancel{a}+\Delta x - \cancel{a}\right) \\
&= f(a) + f'(a)\,\Delta x
\end{aligned}
$$

$$f(a+\Delta x) \approx f(a) + f'(a)\,\Delta x$$

$$\underbrace{f(a+\Delta x) - f(a)}_{\text{exact difference } = \Delta y} \approx \underbrace{\overset{\text{slope}}{f'(a)}\ \overset{\text{run}}{\Delta x}}_{\text{rise } = dy}$$

In short

$$\Delta y \approx dy$$

differential

Differentials

Note

- $\Delta x = dx$
- $\Delta y \approx dy$

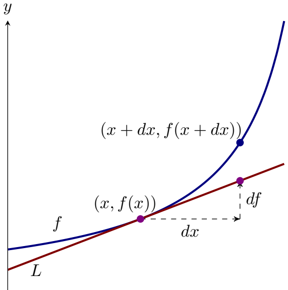
Definition

Let f be a differentiable function. We define df, a differential of f, at a point x by

$$\overset{dy\,=}{df} = f'(x)\ \overset{=\,\Delta x}{dx}$$

where dx is an independent variable that is called a **differential of** x.

Geometrically, differentials can be interpreted via the diagram below.



$$\boxed{\begin{array}{ccc}\Delta f & \approx & df \\ \shortparallel & & \shortparallel \\ f(x+dx) - f(x) & & f'(x)\ dx\end{array}}$$

Remark

- It is important to remember that the notations dx and $dy = df$ represent variables.
- Observe from the picture that

$$f(x + dx) \approx f(x) + df\ .$$

- Noting that the right-hand side is identical to $L(x)$ given above, we once again confirm the validity of the linear approximation formula.

Question. Use differentials to approximate $\sqrt[3]{50}$ and $\sin(0.3)$.

$\nwarrow$ Same as before, written differently.

$f(x) = \sqrt[3]{x}$

$L(x)$

$\parallel$

$$\sqrt[3]{50} = f(50) \approx f(\underset{a}{64}) + df = f(64) - 14 f'(64) = \boxed{}.$$

$x-a$

$$\text{where} \quad df = f'(64)\, dx$$

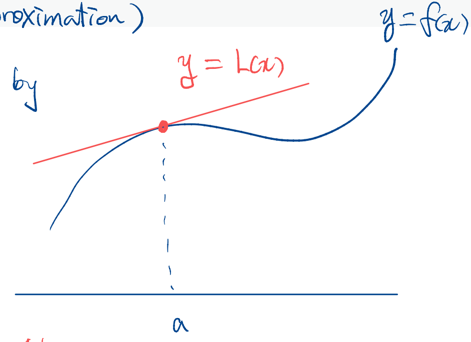
$$= f'(64)\,(50-64)$$

$$= -14\, f'(64)$$

Recap Linear approximation (tangent line approximation)

of a function $f(x)$ at a is given by

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$



- For x near a,

$$f(x) \approx L(x)$$

approx. change / exact change $= \Delta x$

Recap Differentials (df, dx) "d for difference"

$$\boxed{df = f'(x)\,dx}$$

"approx" change in f — change in x

$$df \approx f(x+dx) - f(x)$$

exact change in f

Error Approximation

- We now consider how *linear approximation* or *differentials* can be used to estimate errors in various computations.
- The quantity that we are interested in is the difference or *error* $E(x)$ between the quantity $f(x)$ and the quantity $f(x+dx)$ with a slightly perturbed input $x+dx$, i.e., for small dx.
- Using differentials, we can estimate this error by

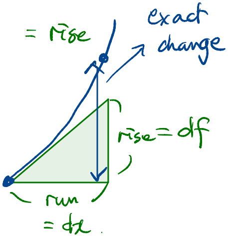
slope × run = rise

$$E(x) = \underline{f(x+dx) - f(x)} \approx f(x) + \underbrace{df}_{= f'(x)dx} - f(x) = df = \underline{f'(x)\,dx}.$$

exact error (or change) in f as input changes from x to $x + dx$.

Linear approx.

perturbation.



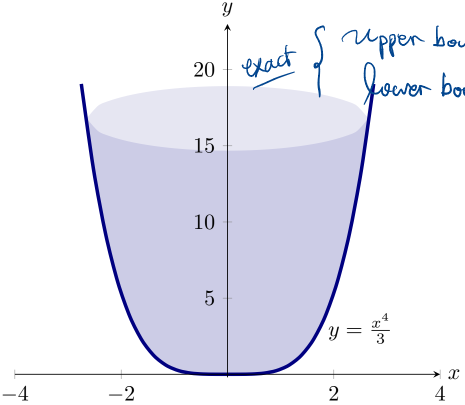
Example

The cross-section of a 250 ml glass can be modeled by the function $r(x) = x^4/3$:

We want to approximate:

At 16.8 cm from the base of the glass, there is a mark indicating when the glass is filled to 250 ml. If the glass is filled within ± 2 millimeters of the mark, what are the bounds on the volume? The volume in milliliters, as a function of the height of water in centimeters, y, is given by

$$V(y) = \frac{2\pi y^{3/2}}{\sqrt{3}}.$$



exact Upper bound : $V(16.8 + 0.2)$

lower bound : $V(16.8 - 0.2)$

$$V'(y) = \frac{2\pi}{\sqrt{3}} \cdot \frac{3}{2} y^{1/2} = \underline{\pi\sqrt{3}\, y^{1/2}}$$

Approximate Let $dy = 0.2$, $y = 16.8$

- upper : $V(16.8 + dy) \approx V(16.8) + dV$

$$= V(16.8) + V'(y)\,dy$$

$$= V(16.8) + V'(16.8)(0.2) = \boxed{254.46}$$

- lower : $V(16.8 \underbrace{-\,dy}_{= +(-dy)}) \approx V(16.8) - V'(y)\,dy.$

$$= V(16.8) - V'(16.8)(0.2) = \boxed{245.54}$$

Affluent benefactor

Double your balance each day.

Day	Balance
1	\$1. $= 2^0$
2	\$2. $= 2^1$
3	\$4 $= 2^2$
4	\$8 $= 2^3$
⋮	⋮
31	$\boxed{\$2^{30}}$

Approximate

2^{30}

w/o calculator.