Mock Final Examination II — Math 1151

Tae Eun Kim Autumn 2019

Practice problems for comprehensive final exam.

Problem 1. (Derivative techniques)

Answer the following questions.

(a) Compute the derivative; you do not need to simplify.

$$\frac{d}{dx}\left(e^{4x}\sqrt{2+\tan^{-1}(3x^2)}\right)$$

$$= \boxed{4e^{4x}\sqrt{2+\tan^{-1}(3x^2)} + \frac{e^{4x}}{2\sqrt{2+\tan^{-1}(3x^2)}}\cdot\frac{6x}{1+9x^4}}$$

(b) Consider the curve given by

$$\sin(xy) = y(x-3).$$

i. Verify that the point $(3,\pi)$ lies on the curve.

$(\text{LHS}) = \sin(3\cdot\pi) = 0$

$(\text{RHS}) = \pi(3-3) = 0$

$\Rightarrow$ (LHS) = (RHS). So $(3,\pi)$ indeed lies on the curve.

ii. Write an equation of the line tangent to the curve at the point $(3,\pi)$.

By implicit differentiation:

$$\cos(xy)(y+xy') = y'(x-3) + y$$

$$[x\cos(xy) - (x-3)]\,y' = y - y\cos(xy)$$

$$\therefore\ y' = \frac{y(1-\cos(xy))}{3 + x(1+\cos(xy))}$$

$$y'\Big|_{(x,y)=(3,\pi)} = \frac{\pi(\overbrace{1-\cos 3\pi}^{"\,2})}{3 + 3(\underbrace{1+\cos 3\pi}_{0})} = \frac{2\pi}{3}$$

Thus, by the point-slope formula,

$$\boxed{y-\pi = \frac{2\pi}{3}(x-3)}$$

or

$$\boxed{y = \frac{2\pi}{3}x - \pi}$$

Problem 2. (L'Hôpital's rule)

For each limit below

- state the **form** of the limit;
- indicate whether the form is **indeterminate** or not;
- evaluate the limit, if it exists or if it is $+\infty$ or $-\infty$. Otherwise, write "does not exist". If the form is **indeterminate**, show your work. You may use L'Hôpital's rule.

(a) $\lim_{x\to 1^+}\left[\ln(x)\right]^x$ FORM: " 0^1 ", determinate

$= \boxed{0}$

(b) $\lim_{x\to e}\frac{\ln(x)-1}{x-e}$ FORM: " $\frac{0}{0}$ ", indeterminate

$\overset{L'H}{=} \lim_{x\to e} \frac{1}{x} = \boxed{\frac{1}{e}}$

(c) $\lim_{x\to 0^+}\left(\sin(x)\ln(x)\right)$ FORM: " $0\cdot\infty$ ", indeterminate

$= \lim_{x\to 0^+} \frac{\ln(x)}{\csc(x)}$ FORM: " $\frac{\infty}{\infty}$ "

$\overset{L'H}{=} \lim_{x\to 0^+} \frac{1}{-x\csc(x)\cot(x)}$

$= -\lim_{x\to 0^+} \frac{\sin(x)}{x}\cdot\tan(x)$

$= -\underbrace{\left(\lim_{x\to 0^+}\frac{\sin(x)}{x}\right)}_{=1}\underbrace{\left(\lim_{x\to 0^+}\tan(x)\right)}_{=0}$

$= \boxed{0}$

Problem 3. (Various topics)

Answer the following questions.

(a) Solve the initial value problem (IVP),

$$\begin{cases} y' = \sec^2(x) + 10\sin(5x) & \text{(DE)} \\ y(0) = 4 & \text{(IC)} \end{cases}.$$

By antidifferentiation,

$$y(x) = \int \left(\sec^2(x) + 10\sin(5x)\right) dx$$

$$= \tan(x) - \frac{\cancel{10}^{\,2}}{\cancel{5}} \cos(5x) + C$$

Using (IC),

$$y(0) = \cancel{\tan}(0)^{0} - 2\underbrace{\cos(0)}_{=1} + C = 4$$

$$\therefore \; C = 6$$

Therefore,

$$\boxed{y(x) = \tan(x) - 2\cos(5x) + 6}$$

is the solution to the IVP.

(b) Let g be the function given by

$$g(x) = \begin{cases} 3e^{x-2} & \text{if } x \le 2, \\ 6\cos\left(\frac{\pi}{6}x\right) & \text{if } x > 2. \end{cases}$$

State the **definition of continuity**. Use the definition of continuity to determine whether the function g is continuous at $x = 2$. Show your work.

Definition: We say that g is continuous at a if

$$g(a) = \lim_{x \to a} g(x).$$

To check continuity using the definition above, we note:

① g is defined at 2, i.e., $g(2)$ exists:

$$g(2) = 3e^{2-2} = 3$$

②
$$\begin{cases} \lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} 3e^{x-2} = 3 \\ \lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} 6\cos\left(\frac{\pi}{6}x\right) = 6\cos\left(\frac{\pi}{3}\right) = 3 \end{cases}$$

Hence, $\lim_{x \to 2} g(x) = 3$.

③ From ① and ②,

$$g(2) = 3 = \lim_{x \to 2} g(x).$$

Therefore, g is continuous at 2.

TYPO!

Problem 4. (Integral exercises)

Evaluate the following integrals. Show your work.

(a) $\displaystyle\int_0^{\cancel{\pi}\,\frac{\pi}{3}} \frac{\sin x}{(2\cos x)^2}\,dx$ $\displaystyle = \int_0^{\pi/3} \frac{1}{4}\,\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}\,dx$

$$= \frac{1}{4}\int_0^{\pi/3} \sec x \cdot \tan x \; dx$$

$$= \frac{1}{4}\Big[\sec x\Big]_0^{\pi/3}$$

$$= \frac{1}{4}\Big(\underbrace{\sec \pi/3}_{=2} - \underbrace{\sec 0}_{=1}\Big) = \boxed{\frac{1}{4}}$$

(b) $\displaystyle\int \frac{x-2}{\sqrt{x+3}}\,dx$ $\displaystyle = \int \frac{u^2-5}{\cancel{u}}\,2\cancel{u}\,du = 2\int (u^2-5)\,du$

$$= 2\left(\frac{u^3}{3} - 5u\right) + C$$

$$= \boxed{\frac{2}{3}(x+3)^{3/2} - 10\,(x+3)^{1/2} + C}$$

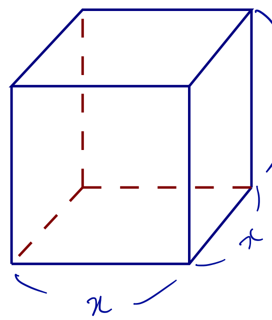
$\left\{\begin{array}{ll} u = \sqrt{x+3} & \longrightarrow \quad x = u^2 - 3 \\ du = \dfrac{1}{2\sqrt{x+3}}\,dx = \dfrac{1}{2u}\,dx & \longrightarrow \quad dx = 2u\,du \end{array}\right.$

(c) $\displaystyle\underbrace{\int_{-\pi}^{\pi}}_{\text{sym. int.}} \Big(\underbrace{x^3\cos x}_{\text{odd}} + \underbrace{5}_{\text{even}} + \underbrace{\frac{2x}{3+x^2}}_{\text{odd}}\Big)\,dx$ $\displaystyle = \cancel{\int_{-\pi}^{\pi} x^3\cos x\,dx}^{\,0} + \int_{-\pi}^{\pi} 5\,dx + \cancel{\int_{-\pi}^{\pi}\frac{2x}{3+x^2}\,dx}^{\,0}$

$$= 2\pi\cdot 5 = \boxed{10\,\pi}$$

Problem 5. (Optimization)

Suppose an airline policy states that all baggage must be box-shaped with a sum of length, width, and height not exceeding 64 in. What are the dimensions and volume of a square-based box with the greatest volume under these conditions?



- constraint : $2x + h = 64 \implies h = 64 - 2x$
- objective function (to be maximized)

$$V = x^2 h$$

$$\hookrightarrow \quad V(x) = x^2(64 - 2x)$$

domain : $0 < x < 32$

Why? Both x and h represent lengths, so they must be positive :

$$x > 0 \quad \text{and} \quad h = 64 - 2x > 0$$

$$\implies x > 0 \quad \text{and} \quad x < 32 \quad \checkmark$$

1. Critical points

$$V'(x) = 128x - 6x^2$$

$$= 2x(64 - 3x) = 0$$

$$\implies \cancel{x = 0} \quad \text{or} \quad x = 64/3$$

not in the domain.

$\therefore$ $x = 64/3$ is the only crit. pt. w/in the domain.

2. Derivative Test. (2^{nd} DT)

$$V''(x) = 128 - 12x$$

$$V''(64/3) = 128 - 12 \cdot \frac{64}{3} < 0$$

$\therefore$ $V(x)$ attains a local maximum at $x = 64/3$

3. Conclusion.

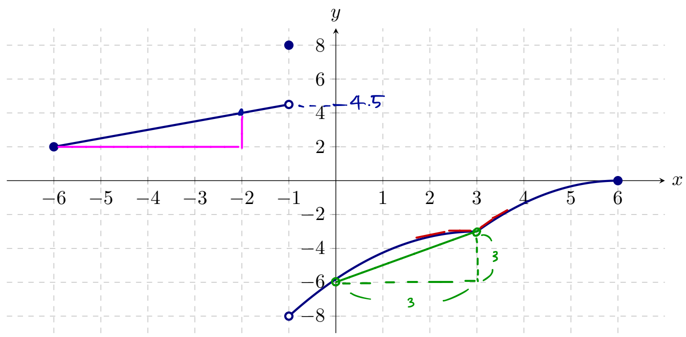
Since $V(x)$ has a unique local maximum at $x = 64/3$ within the domain, it attains the global maximum at

$$x = \frac{64}{3}, \quad h = 64 - \frac{2}{3}64 = \frac{64}{3},$$

i.e. when the baggage is a cube!

Problem 6. (Understanding functions from graphs)

The graph of the function f on its domain $[-6, 6]$ is shown in the figure below.



Note: Grids are not drawn in scale.

Use the graph of f to answer the questions below.

(a) Determine the **range** of f. Write your answer in interval notation.

$\boxed{(-8, 0] \cup [2, 4.5) \cup \{8\}}$

(b) Let f^{-1} be the inverse of f. Determine the **range** of f^{-1}. Write your answer in interval notation.

= domain of f.

$\boxed{[-6, 6]}$

(c) Find the following values or say "does not exist".

i. $f^{-1}(0) = 6$

iii. $f'(3)$ DNE

v. $f^{-1}(4) = -2$

ii. $f^{-1}(8) = -1$

iv. $f'(-2) = \dfrac{4-2}{-2+6} = \dfrac{1}{2}$

vi. $\dfrac{df^{-1}}{dx}(4) = \dfrac{1}{f'(f^{-1}(4))} = \dfrac{1}{f'(-2)} = 2$

(d) In the figure above, sketch the graph of f^{-1}.

(e) Find the x-coordinates of all **critical points** of f on the interval $(-6, 6)$ or say "no critical points".

$\boxed{x = -1, \quad x = 3}$

(f) Find the x-coordinates of all **local maxima** of f on the interval $(-6, 6)$ or say "no local maxima".

$\boxed{x = -1}$

(g) Order the following four numbers from smallest to largest:

$$f'(2),\ f'(2.6),\ f'(3.2),\ f'(3.3)$$

$$\boxed{f'(2.6) < f'(2) < f'(3.3) < f'(3.2)}$$

(h) Find the limit if it exists. Otherwise, write "does not exist".

i. $\lim_{x\to 0} f(x) = \boxed{-6}$

ii. $\lim_{x\to 6^-} f(x) = \boxed{0}$

iii. $\lim_{x\to -1^+} f(x) = \boxed{-8}$

iv. $\lim_{x\to -1} f(x)$ $\boxed{\text{DNE}}$

(i) Find the **average rate of change** of the function f on the interval $[0, 3]$. Show work.

$$\left(\begin{array}{l}\text{average}\\\text{rate of change}\end{array}\right) = \frac{f(3) - f(0)}{3 - 0} = \frac{-3 - (-6)}{3 - 0} = \boxed{1}$$

· diff'ble on open

· cont. on closed

(j) Circle the interval on which the function f satisfies the conditions of the mean value theorem.

i. $[-6, 4]$
ii. $[-1, 4]$
iii. $[0, 4]$
iv. $[1, 3]$
v. $[2, 4]$
vi. No previous answer is correct.

Problem 7. (Understanding functions from tables)

Let f be a function that is **differentiable** on the interval $(0, 6)$. Particular values of f and f' are given in the table below. We also know that the function f', the derivative of f, is continuous on the interval $[1, 5]$. Use the table below to answer the following parts. Show your work.

x	1	2	3	4	5
$f(x)$	-2	1	-1	2	3
$f'(x)$	-4	3	4	-2	-1

(a) Find the limit below or say "does not exist".

$$\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = f'(3) = \boxed{4}$$

(b) Find $L(x)$, the linear approximation to f at $a = 3$.

$$L(x) = f(3) + f'(3)(x-3)$$

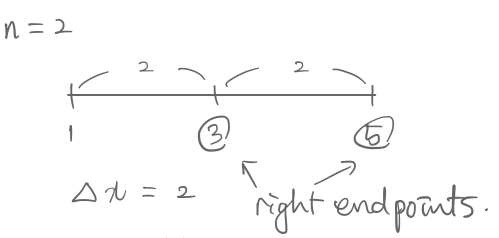
$$= \boxed{-1 + 4(x-3)}$$

(c) Use the linearization from part (b) to estimate $f(2.7)$.

$$f(2.7) \approx L(2.7) = -1 + 4(2.7-3)$$

$$= -1 + 4(-0.3) = \boxed{-2.2}$$

(d) Compute the **right** Riemann sum of f', the derivative of f, on $[1, 5]$ for $n = 2$. Show your work.



$$(RRS) = (f'(3) + f'(5))\, 2$$

$$= (4-1)\cdot 2 = \boxed{6}$$

(e) Express the **limit of right Riemann sums** on $[1, 5]$ as a definite integral, then evaluate the definite integral.

$$\int_1^5 f'(x)\,dx \overset{\text{FTC2}}{=} \Big[f(x)\Big]_1^5 = f(5) - f(1)$$

$$= 3-(-2) = \boxed{5}$$

(f) Compute the derivative below. Explain.

$$\frac{d}{dx}\underbrace{\int_1^4 f(t)\,dt}_{\text{constant}} = \boxed{0}$$

(g) Compute the value of the derivative

$$\left[\frac{d}{dx}\int_1^x \sqrt{f(t)+7}\,dt\right]_{x=4} \overset{\text{FTC1}}{=} \left[\sqrt{f(x)+7}\right]_{x=4}$$

$$= \sqrt{\underbrace{f(4)}_{=2}+7} = \boxed{3}$$

(h) Compute the value of the derivative

$$\left[\frac{d}{dx}\int_1^{\sqrt{x}} f'(t)\,dt\right]_{x=4} \overset{\text{FTC1+CR}}{=} \left[\frac{f'(\sqrt{x})}{2\sqrt{x}}\right]_{x=4}$$

$$= \frac{f'(2)}{2\cdot 2} = \boxed{\frac{3}{4}}$$

(i) Compute the value of the derivative

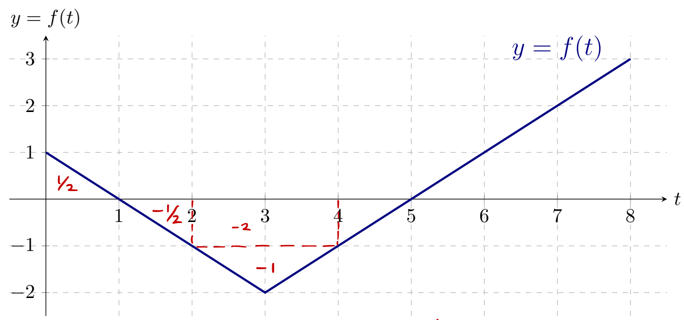
$$\left[\frac{d}{dx}\left(\frac{xf(x)}{7}\right)\right]_{x=2} = \frac{1}{7}\left[f(x) + xf'(x)\right]_{x=2}$$

$$= \frac{1}{7}\left(f(2) + 2f'(2)\right)$$

$$= \frac{1}{7}(1 + 2\cdot 3) = \boxed{1}$$

Problem 8. (Accumulation function and the fundamental theorem of calculus)

The function f is continuous on $[0,8]$. The graph of f is shown below.



Numbers in red represent signed areas.

Let $A(x) = \int_0^x f(t)\,dt$ for $0 \le x \le 8$.

(a) Find the value.

i. $A(0) = \int_0^0 f(t)\,dt = \boxed{0}$

ii. $A(4) = \int_0^4 f(t)\,dt = \frac{1}{2} + \left(-\frac{1}{2}\right) + (-3) = \boxed{-3}$

iii. $A'(4) = f(4) = \boxed{-1}$

(b) Complete the following sentence.

The function A attains its minimum value on $[0,8]$ at $x =$ 5.

Since $A' = f$ changes signs from (−) to (+) at $x = 5$.

1st DT

(c) Complete the following sentence.

The function A is both **decreasing** and **concave up** on the interval $(3, 5)$.

Since $A' < 0$ and A' is increasing on that interval

(d) Sketch the graph of A in the figure below.

Important markers

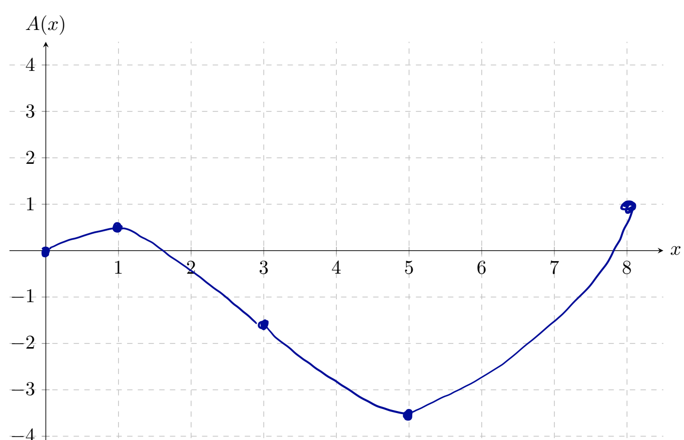
$A(0) = 0$

$A(1) = \frac{1}{2}$

$A(3) = -\frac{3}{2}$

$A(5) = -\frac{7}{2}$

$A(8) = 1$



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