Lecture 37: Second Fundamental Theorem of Calculus (SFTOC)

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Recall FTC 1

Suppose f is continuous. Then

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable with

$$\boxed{F'(x) = f(x)}$$

can be rephrased as:

$$F'(x) = \underbrace{\frac{d}{dx}\underbrace{\int_a^x \underbrace{f(t)}_{①}\,dt}_{②}}_{③} = f(x).$$

Integration followed by differentiation does nothing.

Important "Variation" (FTC 1 + CR)

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$$

The Second Fundamental Theorem of Calculus

Here comes the second form of the Fundamental Theorem of Calculus.

Theorem (Second Fundamental Theorem of Calculus, FTC2)

Let f be continuous on $[a,b]$. If F is ***any*** *antiderivative of f, then*

$$\underbrace{\int_a^b f(x)\,dx}_{\text{definite integral}} = F(b) - F(a)\,. \tag{FTC2}$$

$\int_a^b f(x)\,dx$ ✓

To evaluate a def. integ.

1. Find an antiderivative of f
2. Evaluate the antiderivative at limits of integration.
3. Take the difference.

- An alternate interpretation of (FTC2) is to write it as

$$\underbrace{\int_a^b \underbrace{\frac{d}{dx}\underbrace{f(x)}_{①}}_{②}\,dx}_{③} = f(b) - f(a).$$

- The above reads as

The accumulation of a rate is given by the change in the amount.

Physics The integral of a velocity yields displacement.
(S(term. time) − S(init. time))

Notation

- FTC2 is useful in computing a definite integral:
  1. find an antiderivative of the integrand;
  2. evaluate it at the limits of integration;
  3. take the difference.
- In the differencing process, you may find the following notation convenient:

$$\underbrace{\Big[F(x)\Big]_a^b = F(x)\Big|_a^b}_{\text{notation}} = F(b) - F(a)\,.$$

cf) $\left[\frac{d}{dx} f(x)\right]_{x=a} = \left(\text{evaluate } \frac{d}{dx} f(x) \text{ at } x = a\right) = f'(a)$

Skip : Read on your own.

Proof. Let $a \leq c \leq b$ and write

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \\ &= \int_c^b f(x)\,dx - \int_c^a f(x)\,dx.\end{aligned}$$

By the First Fundamental Theorem of Calculus, we have

$$F(b) = \int_c^b f(x)\,dx \qquad \text{and} \qquad F(a) = \int_c^a f(x)\,dx$$

for some antiderivative F of f. So

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

for this antiderivative. However, **any** antiderivative could have be chosen, as antiderivatives of a given function differ only by a constant, and this constant *always* cancels out of the expression when evaluating $F(b) - F(a)$. $\square$

Question. Compute:

Know this already by symmetry.

1. $\displaystyle\int_{-2}^{2} x^3\,dx = 0$

sym. interval

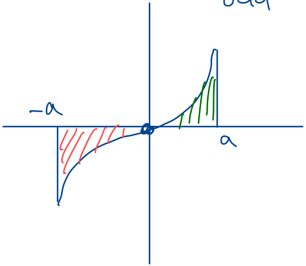
odd

using FTC

$$(\text{Given}) = \left[\frac{x^4}{4}\right]_{-2}^{2}$$

$$= \frac{2^4}{4} - \frac{(-2)^4}{4} = \boxed{0}$$

odd



2. $\displaystyle\int_0^1 \frac{\pi}{3}\sin\frac{\pi}{3}\theta\,d\theta$

$$a\theta + b = \tfrac{\pi}{3}\theta + 0$$

$$= \frac{\pi}{3}\int_0^1 \sin\left(\boxed{\tfrac{\pi}{3}\theta}\right)d\theta$$

$$= \cancel{\frac{\pi}{3}}\cdot\left[-\frac{1}{\cancel{\pi/3}}\cos\left(\frac{\pi}{3}\theta\right)\right]_0^1$$

$$= \left(-\cos\left(\frac{\pi}{3}\cdot 1\right)\right) - \left(-\cos\left(\frac{\pi}{3}\cdot 0\right)\right) = -\cos\left(\frac{\pi}{3}\right) + \cos(0) = -\frac{1}{2} + 1 = \boxed{\frac{1}{2}}$$

Digression

Should we add C or not?

$$\left[F(x) + C\right]_a^b = (F(b) + \cancel{C}) - (F(a) + \cancel{C})$$

$$= F(b) - F(a) = \left[F(x)\right]_a^b$$

Tips Working w/ $\left[F(x)\right]_a^b$

1. $$\left[F(x)+G(x)\right]_a^b = \left[F(x)\right]_a^b + \left[G(x)\right]_a^b$$

e.g. $$\left[x+\frac{1}{2}\sin(2\pi x)\right]_0^1 = \left(1+\frac{1}{2}\sin(2\pi)\right) - \left(0+\frac{1}{2}\sin(0)\right)$$

$$= \left[x\right]_0^1 + \left[\frac{1}{2}\sin(2\pi x)\right]_0^1$$

$$= (1-0) + \left(\frac{1}{2}\sin(2\pi) - \frac{1}{2}\sin(0)\right)$$

2. $$\left[c\,F(x)\right]_a^b = c\left[F(x)\right]_a^b$$

e.g. $$\left[\frac{1}{2}\sin(2\pi x)\right]_0^1 = \frac{1}{2}\left[\sin(2\pi x)\right]_0^1 = \frac{1}{2}\left(\sin(2\pi) - \sin(0)\right)$$

2'. $\left[-F(x)\right]_a^b = F(a) - F(b)$

Change the order of differencing.

Why?

$$= \left(-F(b)\right) - \left(-F(a)\right)$$

$$= -F(b) + F(a)$$

$$= F(a) - F(b)$$

Question. Compute:

1. $\displaystyle\int_0^5 e^t\,dt \quad = \left[e^t \right]_0^5 = e^5 - e^0 = \boxed{e^5 - 1}$

2. $\displaystyle\int_1^2 \left(x^9 + \frac{1}{x}\right) dx$

<u>Exercise.</u>

<u>Recall</u> $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

FTC2 $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is an antiderivative of f.

i.e., $F'(x) = f(x)$

$\star$ $\int_a^b F'(x)\,dx = F(b) - F(a)$

Net Change and Future Value

$Q(t)$ represents some quantity.

- $\int_a^b \underbrace{Q'(t)}_{\text{rate of change of } Q}\,dt = \underbrace{Q(b) - Q(a)}_{\text{net change in } Q}$

- $\int_a^t Q'(s)\,ds = Q(t) - Q(a) \implies Q(t) = Q(a) + \int_a^t Q'(s)\,ds$ (future value)

some point in future time.

Displacement and net change

Let's recall that

- The derivative of a position function s is a velocity function v.
- The derivative of a velocity function v is an acceleration function a.

In other words,

- A velocity function v is an antiderivative of an acceleration function a.
- A position function s is an antiderivative of a velocity function v.

In particular, by FTC2,

$$\int_a^b v(t)\,dt = s(b) - s(a)\,,$$

which measures a **change in position**, or **displacement** as already introduced on ~~Monday~~.

Wednesday.

Net change and future value

- In general, FTC2 states that the definite integral of a rate of change of a certain quantity Q is the **net change** in its amount between two limits of integration:

$$\int_a^b Q'(s)\,ds = Q(b) - Q(a)\,. \qquad \text{(Net change)}$$

- If we replace $a = 0$ and $b = t$, we have a formula for **future value**:

$$Q(t) = Q(0) + \int_0^t Q'(s)\,ds\,. \qquad \text{(Future value)}$$

Soln of IVP.

Question. A book publisher estimates that the marginal cost of a particular title (in dollars/book) is given by

$$C'(x) = 12 - 0.0002x\,,$$

where $0 \le x \le 50,000$ is the number of books printed. What is the cost of producing the 12,001st through 15,000th book?

$C(x)$: cost of producing x items

$C'(x)$: marginal cost of producing 1 more item after x items have been produced

$$C(15K) - C(12K) = \int_{12K}^{15K} C'(x)\,dx$$

net change

$$= \int_{12K}^{15K} \left(12 - \frac{2}{10K}x\right) dx$$

FTC2

$$= \left[12x - \frac{\cancel{2}}{10K}\frac{x^2}{\cancel{2}}\right]_{12K}^{15K}$$

$$= 12(15K - 12K) - \frac{1}{10K}\left((15K)^2 - (12K)^2\right) = \boxed{27.9K}$$

Confirm

Summary of three different integrals

1. An **indefinite integral**, a.k.a. an antiderivative computes a family of functions:

$$\boxed{\int f(x)\,dx = F(x) + C}$$

   where $F'(x) = f(x)$.

2. An **accumulation function** computes an accumulated area:

$$\boxed{F(x) = \int_a^x f(t)\,dt}$$

   FTC1 says that $F'(x) = f(x)$.

3. A **definite integral** computes a signed area:

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$