Lecture 3: Limit Laws (LL)

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The limit laws

- Recall the definition of **continuity**: f is continuous at a if

$$\lim_{x \to a} f(x) = f(a)\,.$$

- In other words, continuity of a function allows us to calculate its limits simply by function evaluation.
- In addition, we learned that many famous functions are continuous on their natural domains.
- Today, using limit laws, we can expand the library of continuous functions even further.

Theorem (Limit laws)

Suppose that $\lim_{x \to a} f(x) = L$ *and* $\lim_{x \to a} g(x) = M$, *i.e., these limits exist.*

- ***Sum/Difference Law***: $\lim_{x \to a} (f(x) \pm g(x)) = L \pm M$.
- ***Product Law***: $\lim_{x \to a} (f(x)g(x)) = LM$.
- ***Quotient Law***: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, *provided that* $M \neq 0$.

Remark

Using these laws, we can show that polynomial and rational functions are also continuous on their natural domains.

Question. Compute the following limit using limit laws:

$$\lim_{x \to 2} (5x^2 + 3x - 2)$$

Question. Compute the following limit using limit laws:

$$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 2}$$

Where is $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$ continuous?

Theorem (Composition limit law)

If $f(x)$ is continuous at $b = \lim_{x\to a} g(x)$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) .$$

Consequently, if g is continuous at $x = a$, and if f is continuous at $g(a)$, then $f \circ g$ is continuous at $x = a$.

Question. Compute the following limit using limit laws:

$$\lim_{x\to 0} \sqrt{\cos(x)}$$

Question. Determine if the following limits can be directly computed using limit laws.

1. $\lim\limits_{x \to 2} \dfrac{x^2 - 3x + 2}{x - 2}$

2. $\lim\limits_{x \to 0} \dfrac{2^x - 1}{3^{x-1}}$

3. $\lim\limits_{x \to 0} x \sin(1/x)$

4. $\lim\limits_{x \to 0} \cot(x^3)$

5. $\lim\limits_{x \to 0} (1 + x)^{1/x}$

The Squeeze Theorem

Theorem (The Squeeze Theorem)

Suppose that

$$g(x) \leq f(x) \leq h(x)$$

for all x *close to* a *but not necessarily equal to* a. *If*

$$\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x)\,,$$

then $\lim_{x \to a} f(x) = L$.

- This theorem is often called the **sandwich theorem**.

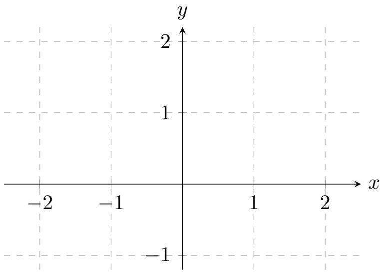
Question. Suppose we have a function $f(x)$ defined for all x in the open interval $(-2, 2)$ and all I know about f is that

$$0 \leq f(x) \leq x^2 \,,$$

in the interval. Can I say anything about $\lim_{x \to 0} f(x)$ with this limited knowledge?

Question. Consider the three functions, g, f, and h, defined on the interval $(-2, 2)$. Given that

$$g(x) = \cos(\pi x), \quad h(x) = x^2 + 1 \quad \text{and } g(x) \leq f(x) \leq h(x),$$



1. Sketch and label the graph of g and h, and a possible graph of f.
2. Use the Squeeze Theorem to evaluate $\lim_{x \to 0} f(x)$.

Question. Compute $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$.

- The answer is 1!
- Please read the textbook for a detailed solution.
- Later in the course, we will learn an alternate method to calculate this limit.