Lecture 6: Continuity and the Intermediate Value Theorem (CATIVT)

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Recall ...

Recall that f is said to be continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Below is our up-to-date library of functions that are continuous on their natural domains:

- Constant functions
- Power functions
- Polynomial functions
- Rational functions
- Exponential functions
- Logarithmic functions
- Trigonometric functions
- Inverse trigonometric functions

Continuity of piecewise functions

Today, we will consider continuity of functions obtained by patching two or more functions. In doing so, pay attention to

- any possible discontinuities of individual *pieces*
- (dis)continuity at junctions where two pieces are joined

Question. Consider the function defined piecewise as

$$f(x) = \begin{cases} \dfrac{x}{x-1} & \text{if } x < 0, \\ e^{-x} + c & \text{if } x \geq 0. \end{cases}$$

Find c so that f is continuous at $x = 0$.

Question. Consider the following piecewise defined function

$$f(x) = \begin{cases} x+4 & \text{if } x < 1, \\ ax^2 + bx + 2 & \text{if } 1 \leq x < 3, \\ 6x + a - b & \text{if } x \geq 3. \end{cases}$$

Find a and b so that f is continuous at both $x = 1$ and $x = 3$.

For those interested ...

You can create interactive plots of such functions in **Mathematica**. For example:

```
Manipulate[Plot[
    Piecewise[{ {x + 4, x < 1},
                {a*x^2 + b*x + 2, 1 <= x < 3},
                {6*x + a - b, x >= 3} }],
    {x, 0, 4}], {a, 0, 2}, {b, 1, 3}]
```

The Intermediate Value Theorem

Theorem (Intermediate Value Theorem)

If

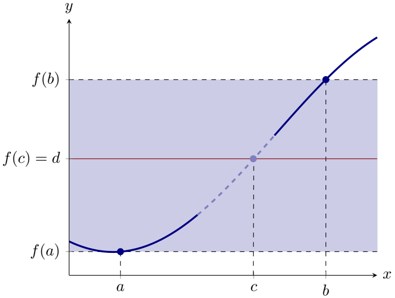
- f *is a continuous function for all* x *in* $[a, b]$ *and*
- d *is between* $f(a)$ *and* $f(b)$,

then there is a number c *in* $[a, b]$ *such that*

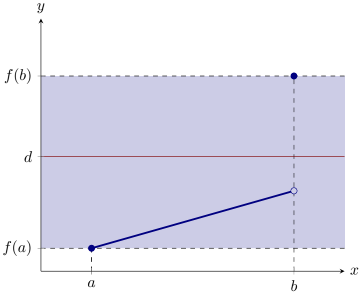
$$f(c) = d\,.$$

Illustration

Case 1: f is continuous on $[a, b]$



Case 2: f is continuous on $[a, b)$



Question. Demonstrate, using the IVT, that the function

$$f(x) = x^3 + 3x^2 + x - 2$$

has a root [1] between 0 and 1.

[1] It can be shown that $f(x) = (x+2)(x^2+x-1)$ and so we know precisely that f has three roots

$$-2\,, \frac{-1+\sqrt{5}}{2}\,, \text{ and } \frac{-1-\sqrt{5}}{2}\,.$$

Question. Explain why the functions

$$f(x) = x^2 \ln(x)$$
$$g(x) = 2x \cos(\ln(x))$$

intersect on the interval $[1, e]$.