implicit differentiation
logarithmic "

Lecture 12: Chain Rule (CR)

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Autumn 2021

6 **Labor Day** **No Classes**	7 Worksheet: ULTDA **WH1 due**	8 Continuity and the Intermediate Value Theorem (CATIVT)	9 Worksheet: CATIVT **HW: IF, ULTDA**	10 An Application of Limits (AAOL)
13 Definition of the Derivative (DOTD) **Midterm 1** **8:00-8:40PM** **UF – CATIVT**	14 Worksheet: AAOL, DOTD **HW: CATIVT**	15 Derivatives as Functions (DAF)	16 Worksheet: DAF **HW: AAOL, DOTD**	17 **Last day to drop w/o a "W"** Rules of Differentiation (ROD)
20 Product Rule and Quotient Rule (PRAQR) **WH2 due**	21 Worksheet: ROD, PRAQR **HW: DAF**	22 Chain Rule (CR)	23 Worksheet: PRAQR, CR **HW: ROD, PRAQR**	24 Higher Order Derivatives and Graphs (HODAG)
27 Implicit Differentiation (ID) **Midterm 2** **8:00-8:40PM** **AAOL-CR**	28 Worksheet: HODAG, ID **HW: CR**	29 Logarithmic Differentiation (LD)	30 Worksheet: ID, LD **HW: HODAG, ID**	**October 1** Derivatives of Inverse Functions (DOIF)
4	5	6	7	8

- Syllabus
- Carmen announcement
  (by math department)
- Gradescope formatting
- Make-up exam
  ↳ contact me
  for permission

- Timeline
  - 7:55 PM : download exam
  - 8:00 PM : start working on it
  - 8:40 PM : upload exam
  - 8:55 PM : done

40 min.

- After this, late!
- Do not email your exam to instructor!

Chain Rule

outer inner

$$f(g(x)) = f \circ g\,(x)$$

The **chain rule** spells out the method of differentiating composite functions.

Theorem (Chain Rule)

If f and g are differentiable, then

$$\boxed{\frac{d}{dx} f(g(x)) = \underbrace{f'(g(x))}\underbrace{g'(x)}\,.}$$

(deriv. of outer at inner) $\times$ (deriv. of inner)

Proof

$$\frac{d}{dx} f(g(x)) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

$= \text{DIY}.$

Question. Compute $\frac{d}{dx}\sin(1+2x)$.

outer : $f(x) = \sin(x) \longrightarrow f'(x) = \cos(x)$

inner : $g(x) = 1+2x \longrightarrow g'(x) = 2$

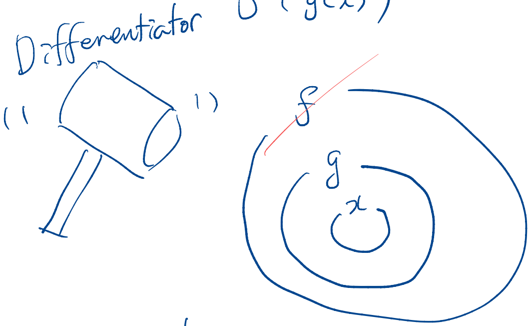
$$\overset{CR}{=} f'(g(x))\, g'(x)$$

$$= \boxed{\cos(1+2x)\cdot 2} \quad \text{or} \quad \boxed{2\cos(1+2x)}$$

Freshmen dream:

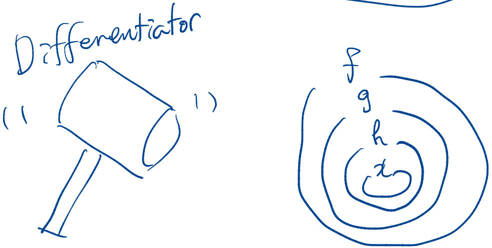
$$\frac{d}{dx} f(g(x)) = f'(g'(x))$$

$f(g(x))$



$$f'(g(x))\,g'(x)$$

Crack one shell at a time, from out to in.



$$f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

Question. Compute $\frac{d}{dx}\sqrt{1+\sqrt{x}}$.

$$\begin{cases} \text{outer}: f(x) = \sqrt{x} \longrightarrow f'(x) = \frac{1}{2\sqrt{x}} \\ \text{inner}: g(x) = 1+\sqrt{x} \longrightarrow g'(x) = \frac{1}{2\sqrt{x}} \end{cases}$$

CR

$$= f'(g(x))\, g'(x)$$

$$= \boxed{\frac{1}{2\sqrt{1+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}}}$$

$g(x)$

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2\sqrt{x}}$$

$$\boxed{\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}}$$

$$x \xmapsto{g} 1+\sqrt{x} \xmapsto{f} \sqrt{1+\sqrt{x}}.$$

$$\frac{d}{dx}\frac{1}{x} = \frac{d}{dx} x^{-1} = (-1)\cdot x^{-2}$$

$$= -\frac{1}{x^2}$$

$$\boxed{\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}}$$

Question. Compute $\frac{d}{dx} e^{\sin(x^2)}$.

$$\begin{cases} \text{outer} : f(x) = e^x \longrightarrow f'(x) = e^x \\ \text{middle} : g(x) = \sin(x) \longrightarrow g'(x) = \cos(x) \\ \text{inner} : h(x) = x^2 \longrightarrow h'(x) = 2x \end{cases}$$

3-fnc version of CR

$$= f'(\underbrace{g(h(x))}_{= \sin(x^2)}) \cdot g'(\underbrace{h(x)}_{= x^2}) \cdot h'(x)$$

$$= \boxed{e^{\sin(x^2)} \cdot \cos(x^2) \cdot 2x}$$

or

$$\boxed{2x \cos(x^2)\, e^{\sin(x^2)}}$$

Exercise

Question. Derive the quotient rule using the power rule, the product rule, and the chain rule.

$\hookrightarrow$ related to Monday's lecture.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d}{dx}\left[f(x)\cdot(g(x))^{-1}\right]$$

$$\overset{\text{P.R.}}{=} f'(x)\,[g(x)]^{-1} + f(x)\,\frac{d}{dx}\left[(g(x))^{-1}\right]$$

$$\overset{\text{C.R.}}{=} \frac{f'(x)}{g(x)} + f(x)\left[\frac{-1}{(g(x))^2}\right]g'(x)$$

$$\overset{\text{Algebra}}{=} \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

✓

Derivatives of trigonometric functions

- At the moment, we only know that $\boxed{\frac{d}{dx}\sin(x) = \cos(x).}$
- This one fact along with other derivative "shortcuts" will give us the derivative formulas for all other standard trigonometric functions.

Theorem (Derivatives of Trigonometric Functions)

- $\dfrac{d}{dx}\sin(x) = \cos(x).$
- $\dfrac{d}{dx}\sec(x) = \sec(x)\tan(x).$
- $\dfrac{d}{dx}\tan(x) = \sec^2(x).$
- $\dfrac{d}{dx}\cos(x) = -\sin(x).$
- $\dfrac{d}{dx}\csc(x) = -\csc(x)\cot(x).$
- $\dfrac{d}{dx}\cot(x) = -\csc^2(x).$

$$\left(\frac{\sin(x)}{\cos(x)}\right)' \overset{QR}{=} \ldots$$

Deriv. of cosine

Pythag: $\cos^2\theta + \sin^2\theta = 1$

$$\frac{d}{d\theta}\left[\cos^2\theta + \sin^2\theta\right] = \frac{d}{d\theta}1$$

$$\Rightarrow 2\cos\theta \frac{d}{d\theta}\cos\theta + 2\sin\theta \underbrace{\frac{d}{d\theta}\sin\theta}_{=\cos\theta} = 0$$

$$\Rightarrow 2\cos\theta\left(\frac{d}{d\theta}\cos\theta + \sin\theta\right) = 0$$

$$\therefore \frac{d}{d\theta}\cos\theta = -\sin\theta \quad \checkmark$$

This procedure is formalized as so-called the <u>implicit differentiation</u>. We will talk about this in more details next week.

Deriv. of tangent

$$\frac{d}{d\theta} \tan\theta = \frac{d}{d\theta}\left[\frac{\sin\theta}{\cos\theta}\right]$$

$$= \frac{\cos\theta \cdot \cos\theta - \sin\theta \cdot (-\cos\theta)}{\cos^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta} = \sec^2\theta \quad \checkmark$$

Question. Compute $\left[\frac{d}{dx}\underbrace{\cos\left(\frac{x^3}{2}\right)}\right]_{x=\sqrt[3]{\pi}}$.

Find $f'(x)$ first, then evaluate it at $x=\sqrt[3]{\pi}$

$\to f(x) = g(h(x))$ where $\begin{cases} g(x) = \cos(x) \longrightarrow g'(x) = -\sin(x) \\ h(x) = x^3/2 \longrightarrow h'(x) = \frac{3}{2}x^2 \end{cases}$

- $\frac{d}{dx} f(x) \overset{CR}{=} g'(h(x)) \cdot h'(x)$

$$f'(x) = -\sin\left(\frac{x^3}{2}\right)\cdot\frac{3}{2}x^2$$

- $f'(\sqrt[3]{\pi}) = -\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}\cdot\frac{3}{2}\sqrt[3]{\pi}^{\,2}$

$$= \boxed{-\frac{3}{2}\sqrt[3]{\pi}^{\,2}} \quad \text{or} \quad \boxed{-\frac{3}{2}\pi^{2/3}}$$

pull constant out

Question. Compute $\frac{d}{dx}\left(\frac{5x\tan(x)}{x^2-3}\right)$.

$$= \boxed{5 \cdot \frac{(x^2-3)\left[\tan(x) + x\sec^2(x)\right] - x\tan(x)\cdot 2x}{(x^2-3)^2}}$$

Question. Compute $\left[\frac{d}{dx}\underbrace{(\csc(x)\cot(x))}_{= f(x)}\right]_{x=\pi/3}$.

Find $f'(x)$ first.
Then evaluate it at $x = \pi/3$.

- $f'(x) \overset{PR}{=} -\csc(x)\cot^2(x) - \csc^3(x)$

- $f'(\pi/3) = -\csc(\pi/3)\cot^2(\pi/3) - \csc^3(\pi/3)$

$$= -\frac{2}{\sqrt{3}}\cdot\frac{1}{3} - \frac{8}{3\sqrt{3}}$$

$$= \boxed{-\frac{10}{3\sqrt{3}}}$$