Mock Final Examination I — Math 1151

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Autumn 2019

Practice problems for comprehensive final exam.

Problem 1. (True/false)

(True/False) Circle T if the statement is ALWAYS true; circle F otherwise. No explanation is required.

(a) (T / F) $f(x) = x + 1$ and $g(x) = \dfrac{x^2 - 1}{x - 1}$ are the same functions.

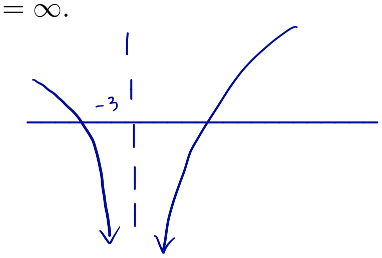
They have different domains.

(b) (T / F) If $\lim_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ exists, then f is continuous at 3.

Because differentiability implies continuity.

(c) (T / F) If f has a vertical asymptote $x = -3$, then $\lim_{x \to -3} f(x) = \infty$.

Counterexample: $\lim_{x \to -3} f(x) = -\infty$

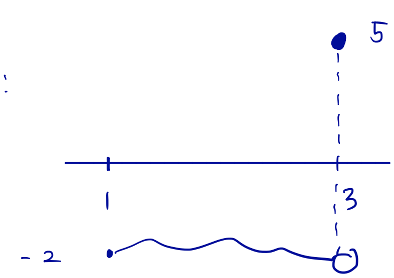


(d) (T / F) A function may possess three distinct horizontal asymptotes.

Because we can have at most two distinct limits at infinity!

(e) (T / F) Let f be continuous on $[1, 3)$. If $f(1) = -2$ and $f(3) = 5$, then the equation $f(x) = 0$ must have a solution between 1 and 3.

Counterexample:



- continuous on $[1, 3)$ ✓
- $f(x)$ has no root in $[1, 3]$.

Problem 2. (Multiple choice)

Select correct answers. A question may have multiple correct answers. No partial credit is given for this problem.

(a) At what point(s) c does the conclusion of the Mean Value Theorem hold for $f(x) = x^3$ on the interval $[-3, 3]$?

A. $-\sqrt{3}$
B. $-1/\sqrt{3}$
C. 0
D. $1/\sqrt{3}$
E. $\sqrt{3}$
F. None of the above

$f'(c) = \frac{f(3) - f(-3)}{3 - (-3)} = \frac{27 + 27}{6} = 9$

$\Rightarrow 3c^2 = 9$

$\Rightarrow c = \pm\sqrt{3}$

(b) The equation of the line that represents the linear approximation to the function $f(x) = \ln(x)$ at $a = 1$ is

A. $y = x - 1$
B. $y = x + 1$
C. $y = -x - 1$
D. $y = -x + 1$
E. None of the above

$L(x) = f(1) + f'(1)(x-1)$

$= 0 + (x-1)$

$\therefore y = x-1$

$f'(x) = \frac{1}{x}.$

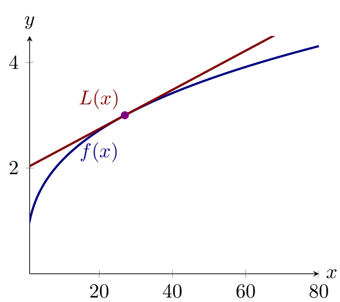
$\cdot f(1) = \ln(1) = 0$

$\cdot f'(1) = 1$

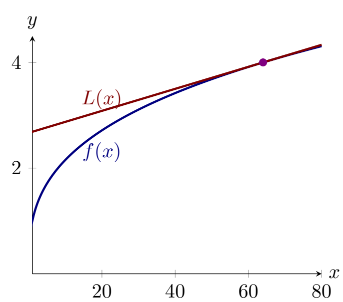
(c) Let $f(x) = \sqrt[3]{x}$ and let $L(x)$ be the linear approximation of $f(x)$ at $a = 64$.

i. Select the figure which includes the correct graph of $L(x)$.

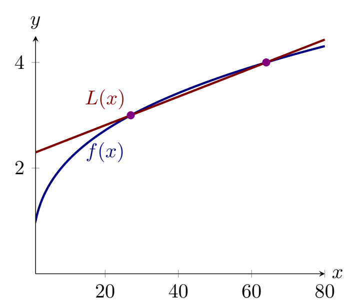
A.



B.



C.



ii. If $L(50)$ is used to approximate $\sqrt[3]{50}$,

A. it gives an overestimate.
B. it gives an underestimate.
C. it gives an exact value of $\sqrt[3]{50}$.
D. it cannot be determined.

Problem 3. (Limit computation)

(a) Evaluate the following limits. You may use L'Hôpital's rule.

i. $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$ $\left(\text{"}\frac{0}{0}\text{"}\right)$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{e^x - 1}{2x} \quad \left(\text{"}\frac{0}{0}\text{"}\right)$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{e^x}{2}$$

$$= \boxed{\frac{1}{2}}$$

ii. $\lim_{x\to\infty} \left(\dfrac{x+3}{x}\right)^x$

$$= \lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^x \quad \left(\text{"}1^\infty\text{"}\right)$$

$$= \lim_{x\to\infty} e^{x\ln(1+\frac{3}{x})}$$

$$= e^{\lim_{x\to\infty} x\ln(1+\frac{3}{x})} \quad \left(\text{"}0\cdot\infty\text{"}\right)$$

$$= e^{\lim_{x\to\infty} \frac{\ln(1+\frac{3}{x})}{\frac{1}{x}}} \quad \left(\text{"}\frac{0}{0}\text{"}\right)$$

Introducing $t = 1/x$,

$$= e^{\lim_{t\to 0^+} \frac{\ln(1+3t)}{t}}$$

$$\overset{L'H}{=} e^{\lim_{t\to 0^+} \frac{3}{1+3t}}$$

$$= \boxed{e^3}$$

iii. $\lim_{x\to-\infty} \dfrac{\sqrt{16x^6 + 8x^3 - 4}}{3x^3 - 7x}$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^6(16 + 8/x^3 - 4/x^6)}}{x^3(3 - 7/x^2)}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^6}\sqrt{16 + 8/x^3 - 4/x^6}}{x^3(3 - 7/x^2)}$$

$\sqrt{x^6} = \sqrt{(x^3)^2}$

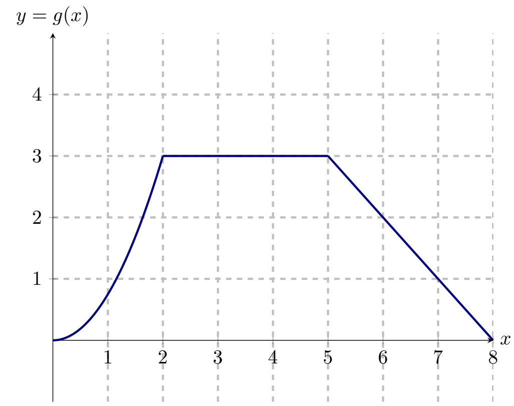
$= |x^3| = -x^3$

Since x eventually is negative as $x\to-\infty$.

$$= \lim_{x\to-\infty} \frac{\cancel{-x^3}\sqrt{16 + \underbrace{8/x^3 - 4/x^6}_{\to 0}}}{\cancel{x^3}(3 - \underbrace{7/x^2}_{\to 0})} = \boxed{-\frac{4}{3}}$$

(b) A table of values for $f(x)$ and $f'(x)$, along with a graph of a function $g(x)$ is shown below.

x	$f(x)$	$f'(x)$
1	2	3
2	4	1
3	6	5



Compute the following or state "DNE". There is no partial credit for this problem.

i. $\dfrac{d}{dx}g(x)$ at $x = 5$

$\boxed{\text{DNE}}$ Since the graph of $g(x)$ has a corner at $x = 5$.

ii. $\dfrac{d}{dx}g(f(x))$ at $x = 2$

$$g'(f(2))\, f'(2) = g'(4) = \boxed{0}$$

iii. $f^{-1}(6)$ $= \boxed{3}$

iv. $\dfrac{d}{dx}f^{-1}(x)$ at $x = 6$

$$\frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(3)} = \boxed{\frac{1}{5}}$$

v. $\dfrac{d}{dx}\left[f(x)\,e^{g(x)}\right]$ at $x = 3$

$$f'(3)\, e^{g(3)} + f(3)\, g'(3)\, e^{g(3)}$$

$$= 5e^{3} + 6 \cdot 0 \cdot e^{3} = \boxed{5e^{3}}$$

Problem 4. (Integral exercises)

Compute the following integrals.

(a) $\dfrac{d}{dx}\underbrace{\displaystyle\int_0^{\pi/2} \sin^7 t\, dt}_{\text{constant}} = \boxed{0}$

(b) $\displaystyle\int_0^{\pi/2} \frac{d}{dx}(\sin^7 x)\, dx \overset{\text{FTC 2}}{=} \Big[\sin^7 x\Big]_0^{\pi/2}$

$= \sin^7(\pi/2) - \sin^7(0) = \boxed{1}$

(c) $\dfrac{d}{dx}\displaystyle\int_0^{\sin(x)} \ln(t^2+1)\, dt \overset{\text{FTC 1 + CR}}{=} \boxed{\ln\left(\sin^2(x)+1\right)\cdot\cos(x)}$

(d) $\underbrace{\int_{-1}^{1}}_{\text{symmetric interval}} \underbrace{\frac{\theta^5 + \sin\theta}{\sqrt{1+\cos^2\theta}}}_{\text{odd}} \, d\theta = \boxed{0}$

(e) $\int (4x-6)\sqrt{x^2-3x}\, dx = \int 2\sqrt{u}\, du$

$$\begin{cases} u = x^2 - 3x \\ du = (2x-3)\,dx \end{cases}$$

$= 2 \cdot \frac{2}{3} u^{3/2} + C$

$= \boxed{\frac{4}{3}(x^2-3x)^{3/2} + C}$

$\frac{\tan\theta}{\sec\theta} = \frac{\sin\theta}{\cancel{\cos\theta}} \cdot \frac{\cos\theta}{1}$

(f) $\int_0^{\pi/4} \frac{1+\tan\theta}{\sec\theta}\, d\theta = \int_0^{\pi/4} \frac{1}{\sec\theta}\, d\theta + \int_0^{\pi/4} \frac{\tan\theta}{\sec\theta}\, d\theta$

$= \int_0^{\pi/4} \cos\theta\, d\theta + \int_0^{\pi/4} \sin\theta\, d\theta$

$= \left[\sin\theta\right]_0^{\pi/4} + \left[-\cos\theta\right]_0^{\pi/4}$

$= \left(\frac{\sqrt{2}}{2} - 0\right) + \left(1 - \frac{\sqrt{2}}{2}\right) = \boxed{1}$

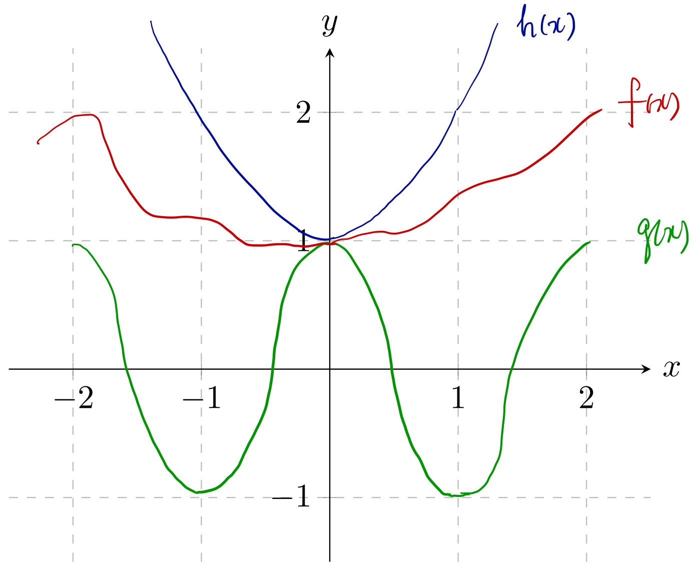
Problem 5. (Squeeze theorem and L'Hôpital)

Consider the three functions, g, f, and h, defined on the interval $(-2, 2)$. Given that

$$g(x) = \cos(\pi x), \quad h(x) = x^2 + 1 \quad \text{and } g(x) \le f(x) \le h(x),$$

answer the following questions.

(a) Sketch and label the graph of g and h, and a possible graph of f.



(b) Use the Squeeze Theorem to evaluate $\lim_{x\to 0} f(x)$.

- $g(x) \le f(x) \le h(x)$ (Given)
- $\lim_{x\to 0} g(x) = \cos(0) = 1$
  $\lim_{x\to 0} h(x) = 0^2 + 1 = 1$ } equal!

} $\Rightarrow$ By Sqz. Thm.,

$$\boxed{\lim_{x\to 0} f(x) = 1}$$

(c) Evaluate

$$\lim_{x\to 0} \frac{g(x) - 1}{h(x) - 1}$$

(Write "does not exist" only if the limit does not exist and is neither $+\infty$ nor $-\infty$.)

$$= \lim_{x\to 0} \frac{\cos(\pi x) - 1}{x^2} \quad \left(\text{"}\frac{0}{0}\text{"}\right)$$

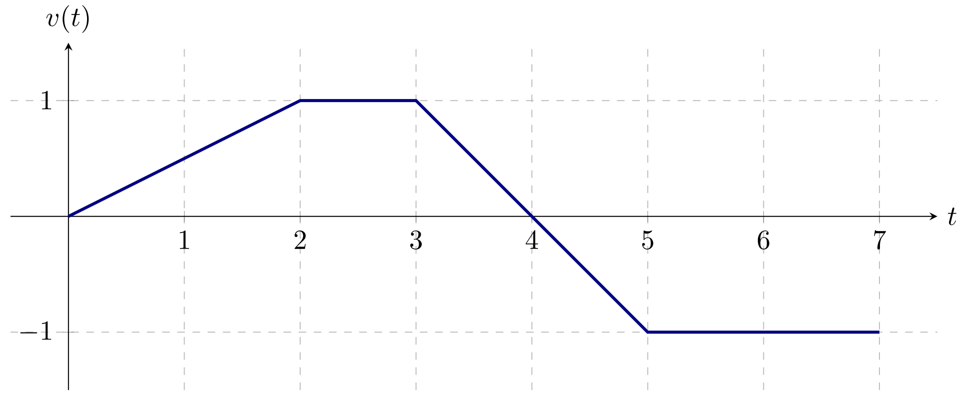
$$\overset{L'H}{=} \lim_{x\to 0} \frac{-\pi \sin(\pi x)}{2x} \quad \left(\text{"}\frac{0}{0}\text{"}\right)$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-\pi^2 \cos(\pi x)}{2}$$

$$= \boxed{-\frac{\pi^2}{2}}$$

Problem 6. (1-D motion)

Consider the motion of a particle moving on a straight line whose velocity v is described in the graph below:



Assume that $s(0) = 0$.

(a) Determine the displacement between $t = 0$ and $t = 7$.

$$(\text{displacement}) = \int_0^7 v(t)\, dt = \boxed{0}$$

(b) Determine the distance traveled between $t = 0$ and $t = 7$.

$$(\text{distance}) = \int_0^7 |v(t)|\, dt = \boxed{5}$$

(c) Determine the position function, $s(t)$, for $5 \le t \le 7$.

$$s(t) = s(5) + \int_5^t \underbrace{v(s)}_{=-1}\, ds = 2 + [-s]_5^t = \boxed{-t+7}$$

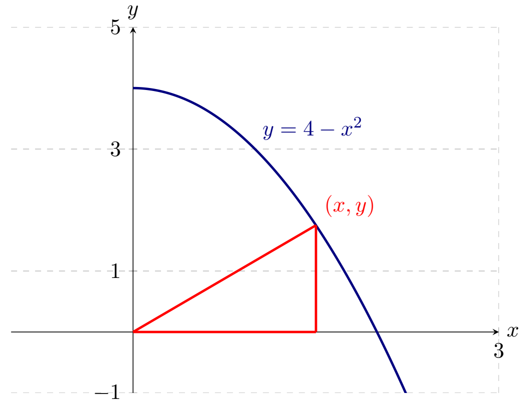
(d) Determine the acceleration, $a(t)$, for $5 < t < 7$.

Since $v(t) = -1$ for $5 < t < 7$,

$$a(t) = v'(t) = \boxed{0}.$$

Problem 7. (Optimization)

The figure shows a right triangle in the first quadrant. One side of the triangle is on the x-axis; its hypotenuse runs from the origin to a point on the parabola $y = 4 - x^2$. Find the coordinates that maximize the area of the triangle.



In your solution:

- State explicitly the domain of objective function.
- Be sure to justify that your answer indeed yields the maximal area.

Set-up

- constraint : $y = 4 - x^2$
- objective function : $A = \frac{1}{2} x y$

$$\Rightarrow A(x) = \frac{1}{2} x (4 - x^2)$$

Domain: $(0, 2)$

Calculus

1. Finding critical points

$$A'(x) = 2 - \frac{3}{2} x^2 = 0$$

$$x = \pm \sqrt{\frac{4}{3}} = \pm \frac{2}{\sqrt{3}}$$

There is only one interior critical point $x = \frac{2}{\sqrt{3}}$.

2. Derivative test

$$A''(x) = -3x$$

$$A''\left(\frac{2}{\sqrt{3}}\right) = -2\sqrt{3} < 0$$

$\therefore$ $A(x)$ attains a local maximum at $x = \frac{2}{\sqrt{3}}$

Conclusion

Since $A(x)$ has a unique local maximum at $x = \frac{2}{\sqrt{3}}$, it attains the global maximum at

$$\boxed{\begin{aligned} x &= \frac{2}{\sqrt{3}} \\ y &= 4 - \frac{4}{3} = \frac{8}{3} \end{aligned}}$$

Problem 8.

(More integrals)

Suppose that $\int_{-1}^{2} f(x)\,dx = 4$. Assume that f is **odd**.

(a) Evaluate $\int_{1}^{2} f(x)\,dx$.

Note that

$$\int_{-1}^{2} f(x)\,dx = \underbrace{\cancel{\int_{-1}^{1} f(x)\,dx}}_{0 \text{ by symmetry}} + \int_{1}^{2} f(x)\,dx$$

Thus,

$$\int_{1}^{2} f(x)\,dx = \boxed{4}$$

(b) Which average value of f is larger, the one over $[-1, 2]$ or the one over $[1, 2]$? Explain.

$$\begin{pmatrix}\text{average of } f \\ \text{over } [-1,2]\end{pmatrix} = \bar{f}_{[-1,2]} = \frac{1}{2-(-1)} \int_{-1}^{2} f(x)\,dx = \frac{4}{3}$$

$$\begin{pmatrix}\text{average of } f \\ \text{over } [1,2]\end{pmatrix} = \bar{f}_{[1,2]} = \frac{1}{2-1} \int_{1}^{2} f(x)\,dx = 4$$

So, $\boxed{\bar{f}_{[1,2]} > \bar{f}_{[-1,2]}}$

(c) Evaluate $\int_{0}^{2\ln 2} e^x f(e^x - 2)\,dx$.

$$\begin{cases} u = e^x - 2 \\ du = e^x\,dx \end{cases}$$

x	u
$2\ln 2$	2
0	-1

$$e^{2\ln 2} - 2 = \left(e^{\ln 2}\right)^2 - 2 = 2^2 - 2 = 2$$

$$= \int_{-1}^{2} f(u)\,du$$

$$= \boxed{4}$$