Tae Eun Kim Autumn 2019

Review problems for Midterm 2 covering materials from AAOL to CFGF.

Problem 1. (Differentiation techniques)

(a) Use implicit differentiation to find $\frac{dy}{dx}$.

$$e^{xy} = e + x - y\,.$$

(b) Find the derivative $\frac{dy}{dx}$. (a is a nonzero constant.)

i. $y = 5\sin(x)\tan^{-1}\left(\dfrac{x}{x+1}\right)$

ii. $y = \dfrac{1}{\sqrt{\tan(x^2)+5}}$

iii. $y = xe^{ax} + \sin^{-1}(ax)$

iv. $y = (\sin(x))^{ax}$

(c) Given that $h(x) = \ln(\sec(x)+\tan(x))$, show that $h'(x) = \sec(x)$

(a) On differentiation, we have

$$e^{xy}(y + xy') = 1 - y'.$$

Rearranging terms,

$$(1 + xe^{xy})\,y' = 1 - ye^{xy}.$$

Finally we solve for y':

$$y' = \frac{1 - ye^{xy}}{1 + xe^{xy}}$$

(b) i. $$\frac{dy}{dx} = 5\cos(x)\tan^{-1}\left(\frac{x}{x+1}\right) + 5\sin(x)\cdot\frac{1}{1+\left(\frac{x}{x+1}\right)^2}\cdot\frac{(\cancel{x}+1)-\cancel{x}}{(x+1)^2}$$

$$= 5\cos(x)\tan^{-1}\left(\frac{x}{x+1}\right) + 5\,\frac{\sin(x)}{(x+1)^2+x^2}$$

ii. $$\frac{dy}{dx} = \frac{d}{dx}\left[(\tan(x^2)+5)^{-1/2}\right]$$

$$= -\frac{1}{2}(\tan(x^2)+5)^{-3/2}\cdot\sec^2(x^2)\cdot 2x$$

iii. $$\frac{dy}{dx} = e^{ax} + x\cdot e^{ax}\cdot a + \frac{a}{\sqrt{1-a^2x^2}}$$

iv. Note that it is a "tower function".
We proceed by logarithmic differentiation:

① Take ln:

$$\ln(y) = \ln\left[(\sin(x))^{ax}\right]$$

$$\ln(y) = ax\ln(\sin(x))$$

③ Take d/dx:

$$\frac{y'}{y} = a\ln(\sin(x)) + ax\,\frac{\cos(x)}{\sin(x)}$$

$\cot(x)$ //

③ Solve for y':

$$y' = y\,(a\ln(\sin(x)) + ax\cot(x))$$

$$= (\sin(x))^{ax}\left(a\ln(\sin(x)) + ax\cot(x)\right)$$

(c) $$h'(x) = \frac{(\sec(x)+\tan(x))'}{\sec(x)+\tan(x)}$$

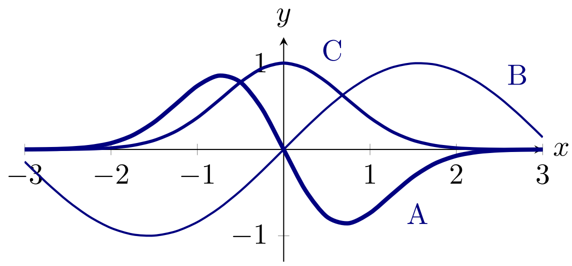
$$= \frac{\sec(x)\tan(x) + \sec^2(x)}{\sec(x)+\tan(x)}$$

$$= \frac{\sec(x)(\sec(x)+\tan(x))}{\sec(x)+\tan(x)}$$

$$= \sec(x)$$

Problem 2. (Graphs of f, f', and f'')

The figure below shows the graphs of f, f', and another function g.



Determine which curve is which.

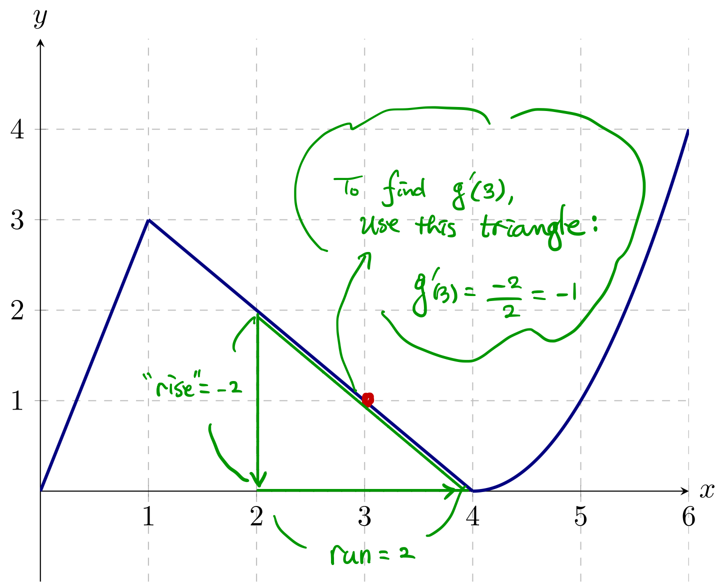
Assume C were the graph of f. Since it is increasing on (-3, 0), the graph of its derivative must be above the x-axis; A is so while B is not. Similarly, while C is decreasing on (0, 3), A is below the x-axis. Hence,

C must be the graph of f;
A must be the graph of f';
B must be the graph of g.

Problem 3. (Derivatives of composed and inverse functions)

A table of values for $f(x)$ and $f'(x)$, along with a graph of a function $g(x)$ is shown below.

x	$f(x)$	$f'(x)$
1	2	4
2	3	5
3	4	1



Compute the following:

(a) $\frac{d}{dx}g(x)$ at $x = 1$

(b) $\frac{d}{dx}f(g(x))$ at $x = 3$

(c) $\frac{d}{dx}g(f(x))$ at $x = 2$

(d) $f^{-1}(3)$

(e) $\frac{d}{dx}f^{-1}(x)$ at $x = 3$

(a) The graph of g has a corner at $x=1$.
So $g'(1)$ does not exist.

(b) $$\left[\frac{d}{dx} f(g(x))\right]_{x=3} = f'(g(3))\underbrace{g'(3)}_{=-1}$$
$$= \underbrace{f'(1)}_{=4}\cdot(-1)$$
$$= -4$$

(c) $$\left[\frac{d}{dx} g(f(x))\right]_{x=2} = g'(f(2))\underbrace{f'(2)}_{=5}$$
$$= \underbrace{g'(3)}_{=-1}\cdot 5$$
$$= -5$$

(d) $f^{-1}(3) = 2$
Since $f(2) = 3$ from the table.

(e) $$\left[\frac{d}{dx} f^{-1}(x)\right]_{x=3} = \frac{1}{f'(\underbrace{f^{-1}(3)}_{=2})}$$
$$= \frac{1}{f'(2)} = \frac{1}{5}$$

Problem 4. (Various properties of a function and its derivatives)

The function f is defined by
$$f(x) = \frac{x}{\sqrt{x^2-9}}.$$

(a) Find the domain of f.

(b) Is f even, odd, or neither?

(c) Calculate $f'(x)$.

(d) Calculate $f''(x)$.

(e) List the intervals on which f is increasing/decreasing/concave up/concave down.

(f) List the x-coordinates of all critical and inflection points.

(a) For $\sqrt{x^2-9}$ to make sense, we need $x^2-9 \geq 0$. But when $x^2-9=0$, the denominator is zero, which must be avoided. So we need
$$0 < x^2-9 = (x-3)(x+3) \implies x<-3 \text{ or } x>3.$$
Thus the domain of f is
$$(-\infty,-3) \cup (3,\infty).$$

(b) Since $f(-x) = \dfrac{-x}{\sqrt{(-x)^2-9}} = -\dfrac{x}{\sqrt{x^2-9}} = -f(x)$, f is odd.

(c) $$f'(x) = \frac{\sqrt{x^2-9} - \dfrac{x^2}{\sqrt{x^2-9}}}{x^2-9} \quad \text{(Quot. Rule)}$$

Side calculation:
$$\frac{d}{dx}\sqrt{x^2-9} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-9}} = \frac{x}{\sqrt{x^2-9}}$$

$$= \frac{(\cancel{x^2}-9) - \cancel{x^2}}{(x^2-9)^{3/2}} \quad \left(\text{Multiplying by } \frac{\sqrt{x^2-9}}{\sqrt{x^2-9}}\right)$$

$$= -\frac{9}{(x^2-9)^{3/2}} \quad \text{or} \quad -9(x^2-9)^{-3/2}$$

(d) $$f''(x) = (-9)\left(-\frac{3}{\cancel{2}}\right)(x^2-9)^{-5/2}\cdot \cancel{2}x$$
$$= 27x(x^2-9)^{-5/2} \quad \text{or} \quad \frac{27x}{(x^2-9)^{5/2}}$$

(e) Note that $(x^2-9)^{3/2} > 0$ and $(x^2-9)^{5/2} > 0$ on the domain found in (a). So

- $f'(x) < 0$ and $f''(x) > 0$, i.e. f DEC & CU, on $(3,\infty)$
- $f'(x) < 0$ and $f''(x) < 0$, i.e. f DEC & CD, on $(-\infty,-3)$

(f) Consequently f does not have any critical points nor inflection points.

Problem 5. (Critical points)

Do not use graphs to justify your answer.

(a) Let $f(x) = x^4 - 8x^2 + 10$. Find the critical points of f.

(b) Let $f(x) = x \ln x$.

i. Find the domain of f.

ii. Locate the critical points of f.

(c) Let $f(x) = x\sqrt{6 - x^2}$

i. Find the domain of f.

ii. State the interval(s) of continuity of f.

iii. Find the critical points of f.

(a) $f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$

$= 4x(x-2)(x+2) = 0$ Set $f'(x)$ equal to 0 and solve for x.

$\Rightarrow$ $x = 0$ or $x = \pm 2$ are critical points

(Note: f', as a cubic polynomial, is defined for all real numbers. So there is no critical point arising due to f' not being defined.)

(b) i. $(0, \infty)$ because of $\ln x$.

ii. $f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1 = 0$ Set $f'(x)$ equal to 0 and solve for x.

$\Rightarrow$ $\ln x = -1 \underset{\text{exponentiate}}{\Longrightarrow} x = e^{-1}$ or $x = \frac{1}{e}$

(Note: f' is defined on $(0, \infty)$ as well, so f' is well defined on the entire domain of the original function.)

(c) i. $[-\sqrt{6}, \sqrt{6}]$ since we need $6 - x^2 \geq 0$.

ii. f is continuous on its domain $[-6, 6]$.

iii. $f'(x) = \sqrt{6-x^2} + x \dfrac{-\cancel{2}x}{\cancel{2}\sqrt{6-x^2}} = \sqrt{6-x^2} - \dfrac{x^2}{\sqrt{6-x^2}} = 0$ Set $f'(x)$ equal to 0 and solve for x.

$\Rightarrow \sqrt{6-x^2} = \dfrac{x^2}{\sqrt{6-x^2}} \underset{\text{multiply by } \sqrt{6-x^2}}{\Longrightarrow} 6 - x^2 = x^2$

$\Rightarrow 2x^2 = 6$ So $x = \pm\sqrt{3}$

(Note: In the interior of the domain $(-\sqrt{6}, \sqrt{6})$, f' is well defined.)

Problem 6. (Position, velocity, and acceleration (1))

The position, $s(t)$, of an object moving along a horizontal line is given by $s(t) = 3\sin(\pi t/4)$, $t \geq 0$, where s is measured in feet and t in seconds.

(a) The position of the particle at time $t = 2$ is $\boxed{}$ ft.

(b) Find the average velocity, v_{av}, of the object over the interval $[0, t]$.

(c) Using the expression found in part (b), evaluate $\lim_{t\to 0^+} v_{\text{av}}$. What does this limit represent?

(d) Find the velocity of the particle.

(e) Find the acceleration of the particle:

(a) $$s(2) = 3\sin(\pi \cdot 2/4) = 3\underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{=1} = 3$$

(b) $$v_{av} = \frac{s(t) - s(0)}{t - 0} = \frac{3\sin\left(\frac{\pi}{4}t\right) - \cancel{3\sin(0)}^{\,0}}{t}$$

$$= 3\,\frac{\sin\left(\frac{\pi}{4}t\right)}{t}$$

(c) $$\lim_{t\to 0^+} 3\,\frac{\sin\left(\frac{\pi}{4}t\right)}{t} = 3\lim_{t\to 0^+}\frac{\sin\left(\frac{\pi}{4}t\right)}{\frac{\pi}{4}t}\cdot\frac{\pi}{4}$$

$$= \frac{3\pi}{4}\lim_{t\to 0^+}\frac{\sin\left(\frac{\pi}{4}t\right)}{\frac{\pi}{4}t} = \frac{3\pi}{4}\underbrace{\lim_{\theta\to 0^+}\frac{\sin(\theta)}{\theta}}_{=1} = \frac{3\pi}{4}$$

Set $\theta = \frac{\pi}{4}t$

This is the instantaneous velocity of the object at $t=0$.

(d) $$v(t) = s'(t) = \frac{3\pi}{4}\cos\left(\frac{\pi}{4}t\right)$$

(e) $$a(t) = v'(t) = -\frac{3\pi^2}{16}\sin\left(\frac{\pi}{4}t\right)$$

Note: In (c), we calculated the velocity at $t=0$ by taking "limit" of an average velocity. Using the result in part (d), we readily confirm this limit calculation:

$$v(0) = \frac{3\pi}{4}\underbrace{\cos(0)}_{=1} = \frac{3\pi}{4} \quad \checkmark$$

Problem 7. (Position, velocity, and acceleration (2))

The position function for an object at time t, $0 \le t \le 10$, is given by

$$s(t) = -40t^2 + 160t + 480\,. = -40\,(t^2 - 4t - 12)$$
$$= -40\,(t-6)(t+2)$$

Use this information to answer the following questions.

(a) Find the velocity $v(t)$ and the acceleration $a(t)$ at any time t, $0 < t < 10$.

(b) At what time is the object furthest from the origin in the positive direction? What are the velocity and acceleration at that time?

(c) At what (positive) time is this object at the origin? What are the velocity and acceleration at that time?

(d) Find the time intervals when the velocity is decreasing?

(a) $v(t) = s'(t) = -80t + 160 = -80(t-2)$

$a(t) = v'(t) = -80$

(b) From (a), we see that $t=2$ is a critical point. Furthermore, since $s''(2) = a(2) < 0$, by the 2nd Derivative Test, $s(t)$ attains a local maximum value at $t=2$:

$$s(2) = -40\cdot 2^2 + 160\cdot 2 + 480 = 640 > 0$$

Since this is a solitary local extremum, it is the global maximum and so the object is furthest from the origin in the positive direction at $t=2$ with

$$v(2) = 0 \quad \text{and} \quad a(2) = -80$$

(c) Since $s(t)$ is factored as

$$s(t) = -40(t-6)(t+2)$$

the object is at the origin (i.e. $s(t)=0$) at $t=6$; recall we want a positive time. At $t=6$,

$$v(6) = -320 \quad \text{and} \quad a(6) = -80.$$

(d) The velocity is decreasing when its derivative is negative, i.e. when the acceleration is negative. We found that $a(t) = -80 < 0$ at all time. So $(0, 10)$.

Problem 8. (Implicit differentiation)

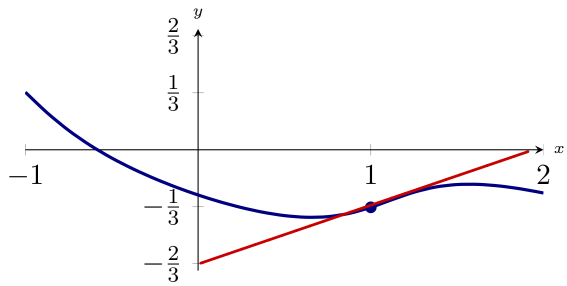
A curve is given by the equation

$$\tan(3y + x) = x^2 - 1\,. \qquad (\star)$$

(a) Use <u>implicit differentiation</u> to find the derivative $\dfrac{dy}{dx}$.

(b) Check algebraically that the point $(1, -1/3)$ lies on the curve.

Part of the curve that contains the point $(1, -1/3)$ is shown in the figure below.



(c) Find an explicit expression for y represented by the graph above.

(d) Using the explicit expression in Part (c), find the derivative $\dfrac{dy}{dx}$.

(e) Evaluate $\dfrac{dy}{dx}$ at $x = 1$.

(f) In the figure above, draw the tangent line to the curve at point $(1, -1/3)$.

(g) Write an equation of this tangent line.

(a) Taking d/dx of both sides of the given equation:

$$\sec^2(3y+x)\cdot(3y'+1) = 2x$$

Rearranging and solving for y':

$$3\sec^2(3y+x)\,y' = 2x - \sec^2(3y+x)$$

$$\therefore\ y' = \frac{2x - \sec^2(3y+x)}{3\sec^2(3y+x)}$$

(b) Plugging in $x=1$ and $y=-1/3$:

(Left-hand side) $= \tan(3\cdot(-1/3)+1)$
$= \tan(0) = 0$

(Right-hand side) $= 1^2 - 1 = 0$

Since (LHS) = (RHS), the point is indeed on the curve

(d) $3y + x = \tan^{-1}(x^2-1)$

$\Rightarrow\ y = \frac{1}{3}\left(-x + \tan^{-1}(x^2-1)\right)$

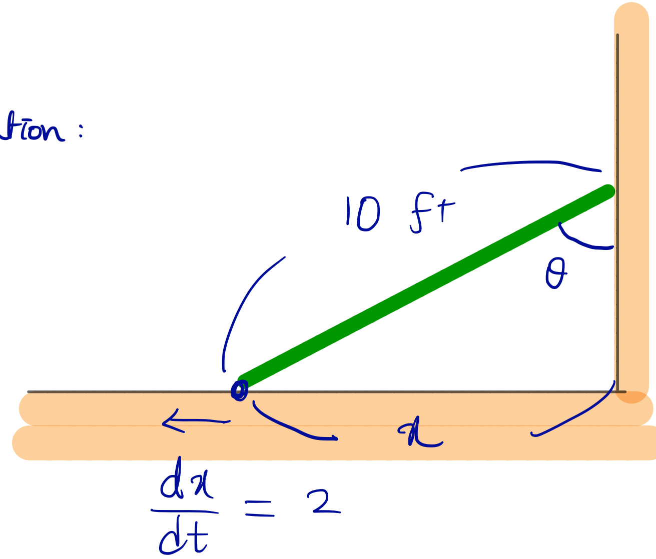
(e) $$\left[\frac{dy}{dx}\right]_{x=1} = \left[\frac{1}{3}\left(-1 + \frac{2x}{1+(x^2-1)^2}\right)\right]_{x=1}$$

$$= \frac{1}{3}(-1+2) = \frac{1}{3}$$

(g) slope : $\left[dy/dx\right]_{x=1} = \frac{1}{3}$
point : $(1, -\frac{1}{3})$

Thus, $y + \frac{1}{3} = \frac{1}{3}(x-1)$

A 10-foot-long ladder rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 ft/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\pi/3$ radians?



- Equation relevant to the configuration:

  (☺) $\sin(\theta) = \dfrac{x}{10}$

- Know: $\dfrac{dx}{dt} = 2$
- Want: $\left[\dfrac{d\theta}{dt}\right]_{\theta = \frac{\pi}{3}}$

Taking $\frac{d}{dt}$ of (☺):

$$\cos(\theta)\,\frac{d\theta}{dt} = \frac{1}{10}\,\frac{dx}{dt}$$

$$\frac{d}{dt}\left(\frac{x}{10}\right) = \frac{d}{dt}\left(\frac{1}{10}\cdot x\right) = \frac{1}{10}\,\frac{dx}{dt}$$

We then solve for $\frac{d\theta}{dt}$:

$$\frac{d\theta}{dt} = \frac{1}{10\cos(\theta)}\,\overbrace{\frac{dx}{dt}}^{=2}$$

Finally, evaluate at the desired moment.

$$\left[\frac{d\theta}{dt}\right]_{\theta=\frac{\pi}{3}} = \frac{1}{10\underbrace{\cos\left(\frac{\pi}{3}\right)}_{=\frac{1}{2}}}\cdot 2 = \boxed{\frac{2}{5}}$$

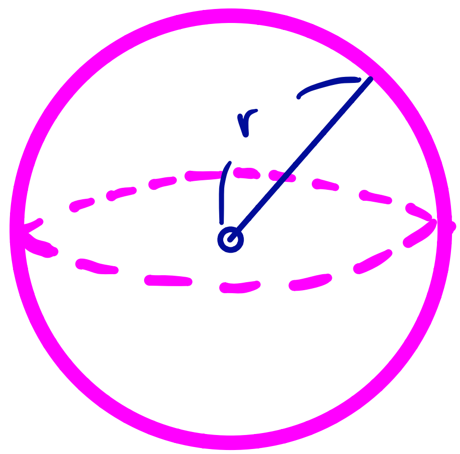
Problem 10. (Related rates: spherical geometry)

We are inflating a spherical balloon at a rate of 3 cm^3/sec. At what rate is the radius increasing when the radius is 6 cm? At what rate is the surface area increasing at that moment?

- Relevant equations:

$$V = \frac{4}{3}\pi r^3 \quad \text{(Volume)}$$

$$S = 4\pi r^2 \quad \text{(Surface area)}$$



- Know: $\frac{dV}{dt} = 3$
- Want: $\left[\frac{dr}{dt}\right]_{r=6}$ and $\left[\frac{dS}{dt}\right]_{r=6}$.

Begin by taking d/dt of (Volume) equation:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Solving for $\frac{dr}{dt}$:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

At last, evaluate at $r=6$ using given values:

$$\left[\frac{dr}{dt}\right]_{r=6} = \frac{1}{4\pi \cdot 6^2}\cdot 3 = \frac{1}{48\pi}$$

To answer the second part, take $\frac{d}{dt}$ of (surface area) equation:

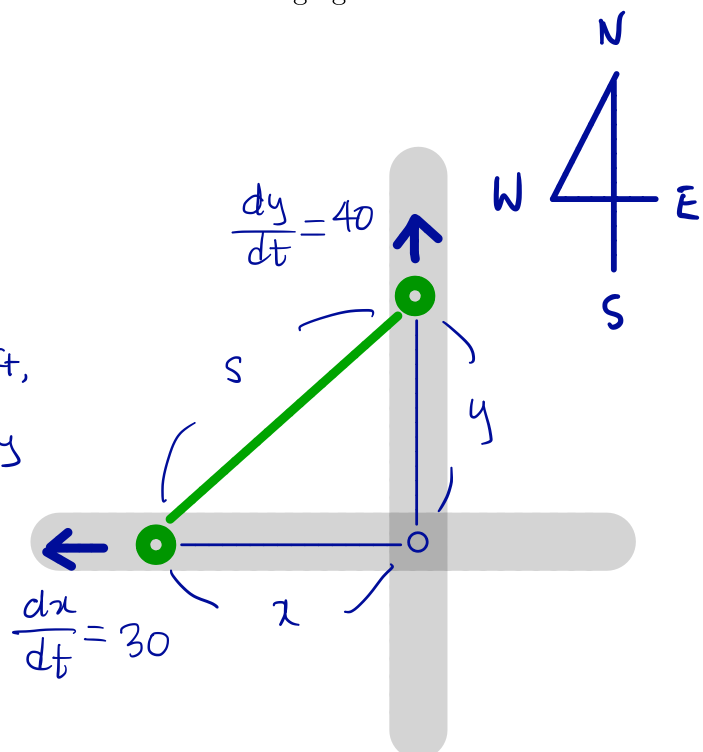
$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

We can evaluate this right away:

$$\left[\frac{dS}{dt}\right]_{r=6} = 8\pi\cdot 6\left[\frac{dr}{dt}\right]_{r=6} = 48\pi\cdot\frac{1}{48\pi} = 1$$

Problem 11. (Related rates: right triangles)

Two cars leave an intersection. One heads west at 30 mi/h; the other leaves half an hour later, heading north at 40 mi/h. How fast is the distance between them changing one hour after the first car left the intersection?



- Relevant equation:

$$s^2 = x^2 + y^2$$

- Know: $\frac{dx}{dt} = 30$, $\frac{dy}{dt} = 40$

- Also know: $t = 1$ hour after first car left,

$x = 30$, $y = 20$ (since this only traveled for ½ hour.)

$s = 10\sqrt{13}$

- Want: $\left[\frac{ds}{dt}\right]_{t=1}$

1 hour after first car left

Differentiating the equation with respect to t:

$$\cancel{2}s\frac{ds}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$$

Isolating $\frac{ds}{dt}$:

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

Evaluating:

$$\left[\frac{ds}{dt}\right]_{t=1} = \frac{1}{10\sqrt{13}}\left(30\cdot 30 + 20\cdot 40\right)$$

$$= \frac{170}{\sqrt{13}}$$

Problem 12. (Rates of change and related rates: miscellaneous)

A water tank is to be drained for cleaning. There are V liters (ℓ) of water left in the tank t minutes after the draining began, where $V = 42(60-t)^2$.

(a) Find the average rate at which water drains during the first 10 minutes.

(b) Find the (instantaneous) rate at which the volume of water is changing 10 minutes after the draining began.

(c) What are the units of

$$\frac{d}{dt}\left(\frac{\frac{d}{dt}V(t)}{V(t)}\right)?$$

Find this expression.

(d) Find the rate of the rate at which the volume of the water is changing 10 minutes after draining began.

(e) Is the rate at which the volume of the water is changing increasing or decreasing (during the draining)?

(f) Assume that the tank has the shape of a rectangular box 7 meters long, 6 meters wide, and 5 meters high. What is the rate of change of the water depth when the water depth is 3 meters? (*Hint*: $1\ell = 0.001\text{m}^3$.)

(a) Avg. rate $= \dfrac{V(10)-V(0)}{10-0} = \dfrac{42}{10}\left(50^2-60^2\right) = \dfrac{42}{10}\cdot(-10)(110) = -4620 \ (\ell/\text{min})$

(b) $V'(t) = -84(60-t) \Rightarrow V'(10) = -84\cdot 50 = -4200 \ (\ell/\text{min})$

(c) Since $\dfrac{V'(t)}{V(t)} = \dfrac{-2}{60-t}$, $\dfrac{d}{dt}\left(\dfrac{V'(t)}{V(t)}\right) = (-2)\cdot\dfrac{(-1)}{(60-t)^2}\cdot(-1) = -\dfrac{2}{(60-t)^2} \ \left(1/\text{min}^2\right)$

(d) $V''(t) = 84 \Rightarrow V''(10) = 84 \ (\ell/\text{min}^2)$

(e) Since $V'' > 0$, it is increasing.

(f) In terms of water depth h, $V = 7\cdot 6\cdot h = 42h$, so $\dfrac{dV}{dt} = 42\dfrac{dh}{dt}$.

Thus $\dfrac{dh}{dt} = \dfrac{1}{42}\dfrac{dV}{dt}$ in general. Now, when $h=3$, $V = 126 \text{ m}^3 = 126000\,\ell$.

Using the given formula for $V(t)$:

$$V(t) = 42(60-t)^2 = 126000 \Rightarrow (60-t)^2 = 3000 \Rightarrow t = 60-\sqrt{3000}$$

Thus,

$$\left.\frac{dh}{dt}\right|_{h=3} = \left.\frac{dh}{dt}\right|_{t=60-\sqrt{3000}} = \frac{1}{42}\cdot V'\left(60-\sqrt{3000}\right) = -2\sqrt{3000} \ \ \ell/\text{m}^2\cdot\text{min}$$

$$= -\frac{\sqrt{3000}}{500} \ \text{m/min}.$$