Lecture 38: Applications of Integration (AOI)

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Fundamental Theorem of Calculus

"the bridge btw. two departments of calculus."

① $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

notation for differencing

② $$\int_a^b \underbrace{\frac{d}{dx}f(x)}_{f'(x)}\,dx = f(b) - f(a) = \Big[f(x)\Big]_a^b = f(x)\Big|_a^b$$

Net Change and Future Value

Net change and future value

Q : a func. representing quant. of interest.

Q' : the rate of change of the quant.

Recall the following formulas for net change and future value:

- **Net change:** $\displaystyle\int_{\cancel{a}\ 0}^{\cancel{b}\ t} Q'(s)\,ds = Q(\cancel{b}^{\,t}) - Q(\cancel{a}^{\,0})$
- **Future value:** $\displaystyle Q(t) = Q(0) + \int_0^t Q'(s)\,ds$

Equipped with this, we will now refine our understanding of the relationship between position and velocity.

- a re-write of FTC2.
- In particular, if you set $a=0$, $b=t$ in Net Change formula and solve for $Q(t)$, you obtain Future Value formula.

Velocity and displacement

$Q(t) = S(t)$

Let $v(t)$ be the **velocity** of an object at time t. This represents the "rate of change in position" at time t. Let $s(t)$ be the **position** of an object at time t. This gives location with respect to the origin.

$\rightsquigarrow V(t) = S'(t)$

- If we can assume that $s(a) = 0$, then by the future value formula

$$s(t) = \int_a^t v(x)\, dx.$$

$$S(t) = \cancel{S(a)}^{\,0} + \int_a^t \underbrace{S'(\tau)}_{= v(\tau)}\, d\tau$$

- $s(b) - s(a)$ is the **displacement**, the distance between the starting and finishing locations.

[net change] $\quad S(b) - S(a) = \int_a^b V(t)\, dt = (\text{displacement})$

Speed and distance

right (+) / left (−)

Velocity and displacement are values containing not only information about "magnitude" but also of "direction" that is relative to some fixed point. On the other hand, there are values without "direction" information. For instance:

- $|v(t)|$ is the **speed**.
- $\displaystyle\int_a^b |v(t)|\, dt$ is the **distance** traveled.

} always non-negative

- $v(t) = -5$: moving $\underbrace{\text{to the left}}_{\text{direction}}$ $\underbrace{\text{at 5 m/s}}_{\text{magnitude}}$. $\xrightarrow{\text{integrate}}$ displacement
  (velocity)
- $|v(t)| = 5$: moving $\underbrace{\text{at 5 m/s}}_{\text{magnitude.}}$ $\xrightarrow{\text{integrate}}$ distance
  (speed)

Summary

	direction & magnitute (+, 0, −)	magnitude (+, 0)
physics	$\int_a^b v(t)\,dt$ = displacement	$\int_a^b \lvert v(t)\rvert\,dt$ = distance
geometry	$\int_a^b f(x)\,dx$ = net area sum of signed areas	$\int_a^b \lvert f(x)\rvert\,dx$ = geometric area

The following integral is going to be useful.

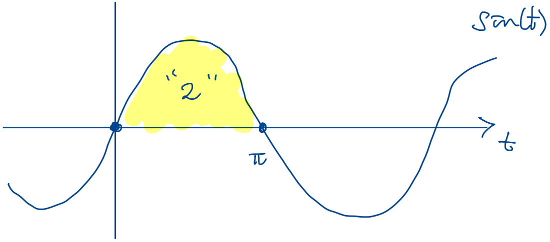
$$\int_0^{\pi} \sin(t)\, dt$$

$$\overset{\text{FTC 2}}{=} \Big[-\cos(t) \Big]_0^{\pi}$$

$$= \Big[\cos(t) \Big]_{\pi}^{0} = \cos(0) - \cos(\pi)$$

$$= 1 - (-1) = \boxed{2}$$

See notes from Lec 30.



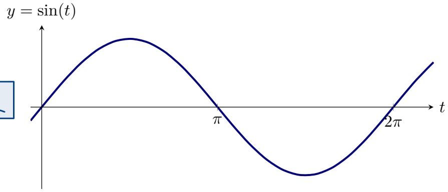
Example

Consider a particle whose velocity at time t is given by $v(t) = \sin(t)$.

1. What is the displacement of the particle from $t = 0$ to $t = \pi$?
2. What is the displacement of the particle from $t = 0$ to $t = 2\pi$?
3. What is the distance traveled by the particle from $t = 0$ to $t = \pi$?
4. What is the distance traveled by the particle from $t = 0$ to $t = 2\pi$?

① $S(\pi) - S(0) = \int_0^{\pi} v(t)\,dt$

$= \int_0^{\pi} \sin(t)\,dt = \boxed{2}$



② $S(2\pi) - S(0) = \int_0^{2\pi} v(t)\,dt$

$= \underbrace{\int_0^{\pi} \sin(t)\,dt}_{=\,2} + \underbrace{\int_{\pi}^{2\pi} \sin(t)\,dt}_{=\,-2} = \boxed{0}$

by symmetry. below t-axis.

③ $\displaystyle\int_0^{\pi} |v(t)|\, dt = \int_0^{\pi} |\sin(t)|\, dt$

"speed"

$\displaystyle = \int_0^{\pi} \sin(t)\, dt$ Since $\sin(t) \geq 0$ on $[0, \pi]$,

$|\sin(t)| = \sin(t)$

$= \boxed{2}$

④ $\displaystyle\int_0^{2\pi} |v(t)|\, dt = \int_0^{2\pi} |\sin(t)|\, dt$

$$= \underbrace{\int_0^{\pi} \sin(t)\, dt}_{\substack{\shortparallel \\ 2}} + \underbrace{\int_{\pi}^{2\pi} (-\sin(t))\, dt}_{\substack{\shortparallel \\ 2}}$$ by symmetry

$= \boxed{4}$

Exercise

Example

An experiment is conducted in which a culture of bacteria is grown in a controlled lab environment. The initial population was estimated at 100 cells. The growth rate of the population $P(t)$ is estimated to be $P'(t) = 4/(1+t^2)$ cells per day.

1. By how much has the population grown during the first day of the experiment?
2. Find the population at any time $t \geq 0$.

Useful

$$\int \frac{1}{1+t^2}\,dt = \tan^{-1}(t) + C$$

Average Value

Alternate interpretation of definite integrals

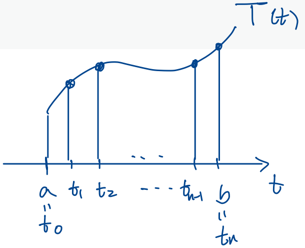
Our framework of choice in interpreting definite integrals was "signed area" between a curve and the horizontal axis, which also provides a very good visualization of integration process. An alternate way to understand integrals is to relate them to *average values*.

- Recall that the average of n discrete data $\{f_1, f_2, f_3, \ldots, f_n\}$ is given by

$$\frac{f_1 + f_2 + \cdots + f_n}{n} = \frac{1}{n} \sum_{k=1}^{n} f_k \,. \qquad \text{(average; discrete)}$$

- Now suppose you want to find the average value of a certain quantity that changes "continuously" over some interval, e.g., the temperature of water in my electric kettle from 6 a.m. to noon.
- In general terms, we want to find the average value of the function $f(t)$ over $[a, b]$.

Approximate average value of a function



A natural way to approximate the average value $f(t)$ over $[a, b]$ is:

1. partition the domain into n equal subintervals,

$$a = t_0 < t_1 < \cdots < t_n = b\,,$$

2. collect data $f(t_1), f(t_2), \cdots, f(t_n)$ at the end of each subinterval,
3. take the average of this n data:

$$\frac{1}{b-a} \cdot \frac{b-a}{n} = \Delta x$$

$$\begin{pmatrix}\text{approx.}\\ \text{avg.}\end{pmatrix} = \frac{1}{n}\sum_{k=1}^{n} f(x_k)\,. \xrightarrow{n\to\infty} \begin{pmatrix}\text{exact}\\ \text{avg.}\end{pmatrix} = \lim_{n\to\infty} \frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

If n is small, we get a coarse estimate of the true average; sampling more frequently, i.e, increasing n, we get a better estimate. When n approaches ∞, we will have the true average value. But before we send this to limit as $n \to \infty$, we need a small touch-up on the previous expression.

Average value of a function

Definition

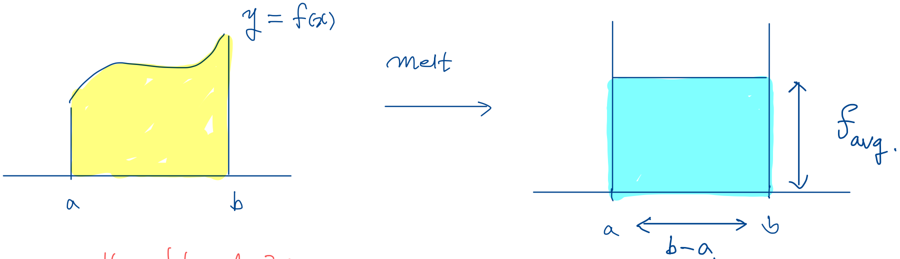
Let f be continuous on $[a, b]$. The **average value** of f on $[a, b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx\,. \quad \checkmark$$

In words,

$$\begin{pmatrix}\text{avg. of} \\ f(x) \text{ on } [a,b]\end{pmatrix} = \frac{\left(\text{def. integral of } f(x) \text{ on } [a,b]\right)}{\left(\text{length of } [a,b]\right)} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Intuition (Visual illustration)



- thin plate of ice
- $x=a$, $x=b$: rigid walls

- area preserved
- top curve leveled.

Area must be preserved!

$$\int_a^b f(x)\,dx = f_{avg}(b-a) \implies \boxed{f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Example

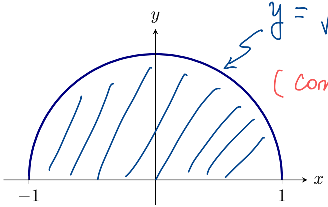
Find the average height of points on the upper-half unit circle.

$$(\text{Avg. height}) = (\text{avg. } y \text{ value})$$

$$= \frac{1}{1-(-1)} \int_{-1}^{1} \sqrt{1-x^2}\,dx$$

$$= \frac{1}{2}\,\text{Area}\left(\quad \right) \qquad 1$$

$$= \frac{1}{2} \cdot \frac{1}{2}\pi \cdot 1^2 = \boxed{\frac{\pi}{4}}$$



$y = \sqrt{1-x^2}$

(comes from $x^2 + y^2 = 1$.)

$$\begin{cases} f(x) = \sqrt{1-x^2} \\ a = -1 \\ b = 1 \end{cases}$$

cf Differential version of MVT.

Mean Value Theorem for Integrals

Having seen how the average value (that is, the mean value) of a function is calculated using a definite integral, we are now ready for the following version of mean value theorem.

iMVT

Theorem (The Mean Value Theorem for integrals)

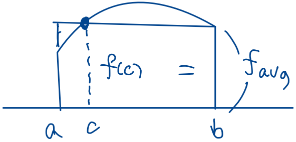
Let f be continuous on $[a, b]$. There exists a value c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

$$\underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{f_{avg}} = f(c)$$

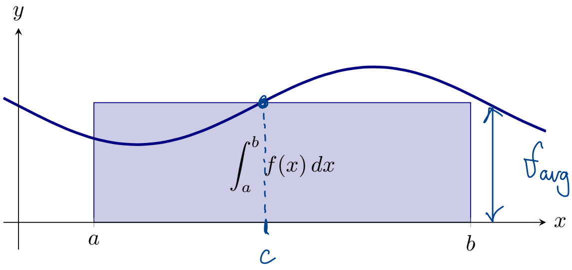
Note. This is an *existence* theorem just as its differential counterpart. It states that

> ***The average value of a continuous function falls within the range of the function.***



Illustration

The following is a visual description of the integral MVT:



$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

the point guaranteed to exist by i MVT.

Example

Consider $\int_0^{\pi} \sin x \, dx$. Find a value c guaranteed by the Mean Value Theorem.

(integral)

$f(x) = \sin(x)$

$a = 0, \quad b = \pi$

iMVT there exists c in $[0, \pi]$ such that

$$\underbrace{\frac{1}{\pi}\int_0^{\pi} \sin x \, dx}_{= 2} = \sin(c)$$

find c.

Find c in $[0, \pi]$ such that

$$\sin(c) = \frac{2}{\pi}$$

$$\Rightarrow \quad c = \sin^{-1}\left(\frac{2}{\pi}\right), \ \pi - \sin^{-1}\left(\frac{2}{\pi}\right)$$