Midterm 3 — Math 1151

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Review problems for Midterm 3 covering materials from the Mean Value Theorem to Definite Integrals.

Problem 1. (Multiple choice)

Select correct answers. A question may have multiple correct answers. No partial credit is given for this problem.

(a) (5 points) At what point(s) c does the conclusion of the Mean Value Theorem hold for $f(x) = x^3$ on the interval $[-3, 3]$?

- A. $-\sqrt{3}$
- B. $-1/\sqrt{3}$
- C. 0
- D. $1/\sqrt{3}$
- E. $\sqrt{3}$
- F. None of the above

$$f'(c) = \frac{f(3) - f(-3)}{3 - (-3)}$$

$$\Rightarrow 3c^2 = \frac{27 + 27}{6} = \frac{54}{6} = 9$$

$$\Rightarrow c^2 = \frac{9}{3} = 3 \quad \therefore c = \pm\sqrt{3}$$

Side comp.

$f'(x) = 3x^2 \Rightarrow f'(c) = 3c^2$

$f(3) = 3^3 = 27$

$f(-3) = (-3)^3 = -27$

(b) (5 points) The equation of the line that represents the linear approximation to the function $f(x) = \ln(x)$ at $a = 1$ is

- A. $y = x - 1$
- B. $y = x + 1$
- C. $y = -x - 1$
- D. $y = -x + 1$
- E. None of the above

$$L(x) = f(1) + f'(1)(x-1) = 0 + 1\cdot(x-1) = x - 1$$

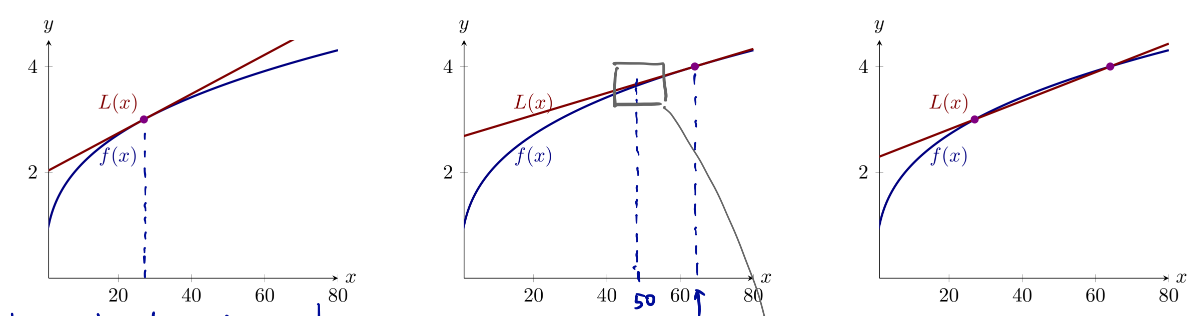
- $f(1) = \ln(1) = 0$
- $f'(x) = \frac{1}{x}$

$\hookrightarrow f'(1) = \frac{1}{1} = 1$

(c) Let $f(x) = \sqrt[3]{x}$ and let $L(x)$ be the linear approximation of $f(x)$ at $a = 64$.

i. (2 points) Select the figure which includes the correct graph of $L(x)$.

~~A.~~ B. ~~C.~~

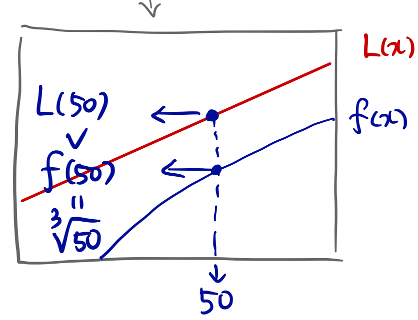


tangent at a wrong place

secant line.

ii. (3 points) If $L(50)$ is used to approximate $\sqrt[3]{50}$,

- A. it gives an overestimate.
- B. it gives an underestimate.
- C. it gives an exact value of $\sqrt[3]{50}$.
- D. it cannot be determined.



Problem 2. (L'Hôpital's rule)

Evaluate the following limits. You may use L'Hôpital's rule.

(a) $\lim_{x\to 0^+}(e^x-1)^{\frac{1}{x}}$

(b) $\lim_{x\to 0^+}\tan(x)^{x^2}$

(c) $\lim_{x\to\infty}\frac{\ln(x^{10})}{\sqrt{x}}$

(d) $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$

(e) $\lim_{x\to\frac{\pi}{2}^-}\frac{\cos(x)\sin(2x)}{(x-\frac{\pi}{2})^2}$

(f) $\lim_{x\to\infty}(x-\sqrt{x^2+4x})$

(a) $\lim_{x\to 0^+}(e^x-1)^{\frac{1}{x}} = \lim_{x\to 0^+} e^{\boxed{\frac{1}{x}\ln(e^x-1)}} = 0$

because $\lim_{x\to 0^+}\boxed{\frac{1}{x}\ln(e^x-1)} = -\infty$

Note: The original problem is in "0^∞" form. Limits in this form will end up with either 0 or ∞. Hence, this form is not considered indeterminate.

(b) $\lim_{x\to 0^+}\tan(x)^{x^2}$ ["0^0" form]

$= \lim_{x\to 0^+} e^{x^2\ln\tan(x)}$

$= e^{\boxed{\lim_{x\to 0^+} x^2\ln\tan(x)}} = e^0 = \boxed{1}$ ($= L$ for the boxed limit)

$L = \lim_{x\to 0^+}\frac{\ln\tan(x)}{\frac{1}{x^2}} \overset{L'H}{=} \lim_{x\to 0^+}\frac{\sec^2 x}{\tan x}\cdot\left(-\frac{x^3}{2}\right)$

$= -\lim_{x\to 0^+}\frac{x^3}{2\sin(x)\cos(x)} = -\lim_{x\to 0^+}\frac{x^3}{\sin(2x)}$ (by double-angle formula.)

$= -\lim_{x\to 0^+}\underbrace{\left(\frac{2x}{\sin(2x)}\right)}_{\to 1}\cdot\underbrace{\left(\frac{x^2}{2}\right)}_{\to 0} = 0.$

(c) $\lim_{x\to\infty}\frac{\ln(x^{10})}{\sqrt{x}} = \lim_{x\to\infty}\frac{\ln(\sqrt{x}^{20})}{\sqrt{x}}$

$= 20\lim_{x\to\infty}\frac{\ln(\sqrt{x})}{\sqrt{x}}$

Introduce $t=\sqrt{x}$. Note that $t\to\infty$ as $x\to\infty$.

$= 20\lim_{t\to\infty}\frac{\ln(t)}{t} \overset{L'H}{=} 20\lim_{t\to\infty}\frac{1}{t} = \boxed{0}$

Note: Recall that $\ln^p x \ll x^q$ as $x\to\infty$ for $p,q>0$. By this growth rate, we can obtain the answer immediately.

(d) Applying L'H twice:

$\overset{L'H}{=} \lim_{x\to 0}\frac{e^x-1}{2x} \overset{L'H}{=} \lim_{x\to 0}\frac{e^x}{2} = \boxed{\frac{1}{2}}$

(e) $= \lim_{x\to\frac{\pi}{2}^-}\frac{2\sin(x)\cos^2(x)}{(x-\frac{\pi}{2})^2}$

Since $\lim_{x\to\frac{\pi}{2}^-}\frac{\cos(x)}{x-\frac{\pi}{2}} \overset{L'H}{=} \lim_{x\to\frac{\pi}{2}^-}(-\sin(x)) = -1$,

we can apply the product Law of limits:

$= \left(\lim_{x\to\frac{\pi}{2}^-}2\sin(x)\right)\left(\lim_{x\to\frac{\pi}{2}^-}\frac{\cos^2(x)}{(x-\frac{\pi}{2})^2}\right)$

$= 2\cdot(-1)^2 = \boxed{2}$

(f) Since $x\to\infty$, can rewrite the given as

$\lim_{x\to\infty} x\left(1-\sqrt{1+\frac{4}{x}}\right)$

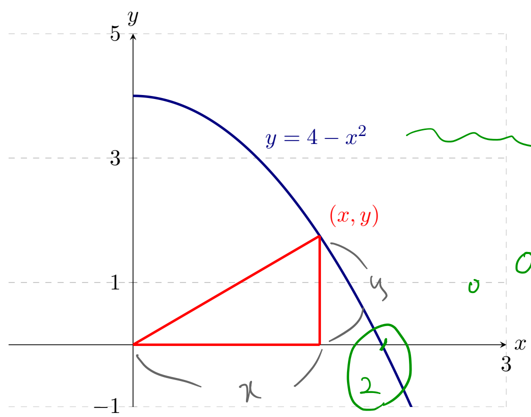
$= \lim_{x\to\infty}\frac{1-\sqrt{1+\frac{4}{x}}}{\frac{1}{x}}$

$= \lim_{t\to 0^+}\frac{1-\sqrt{1+4t}}{t}$ $\quad\left(t=\frac{1}{x}\right)$

$\overset{L'H}{=} \lim_{t\to 0^+} -\frac{4}{2\sqrt{1+4t}} = \boxed{-2}$

Problem 3. (Optimization: right triangle)

The figure shows a right triangle in the first quadrant. One side of the triangle is on the x-axis; its hypotenuse runs from the origin to a point on the parabola $y = 4 - x^2$. Find the coordinates x and y that maximize the area of the triangle.



x-intercept:
$0 = 4 - x^2$
$\therefore x = \pm 2$

1. Set-up:

- Known : $y = 4 - x^2$
- Want : maximize $A = \frac{1}{2} x y$

$$A(x) = \frac{1}{2} x (4 - x^2) \quad ; \quad \text{domain} : [0, 2]$$

2. Calculus:

- Crit. pt. : $A'(x) = \frac{1}{2}\left((4 - x^2) - 2x^2 \right)$

$$= \frac{1}{2}(4 - 3x^2) = 0$$

$\therefore x = \pm \frac{2}{\sqrt{3}}$. Only $\underline{x = \frac{2}{\sqrt{3}}}$ is w/in the domain.

- Candidates : $x = 0, 2$ (end points) and $x = \frac{2}{\sqrt{3}}$ (crit. pt.)

x	$A(x)$	Note
0	0	
$2/\sqrt{3}$	$8/3\sqrt{3}$	max
2	0	

$$A(2/\sqrt{3}) = \frac{1}{\cancel{2}} \frac{\cancel{2}}{\sqrt{3}} \left(4 - \frac{4}{3}\right)$$

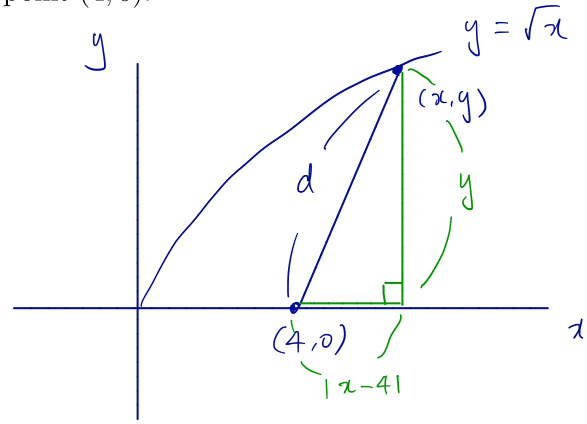
$$= \frac{4 \cdot 2}{\sqrt{3} \cdot 3} = \frac{8}{3\sqrt{3}}$$

3. Conclusion: The maximal area is attained when

$$\boxed{x = 2/\sqrt{3} \quad \text{and} \quad y = 4 - 4/3 = 8/3}$$

Problem 4. (Optimization: minimum distance)

Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(4, 0)$.



1. Set-up:

- Known: $y = \sqrt{x}$
- Want: minimize $d = \sqrt{(x-4)^2 + y^2}$.

* Will minimize $d^2 = (x-4)^2 + (\sqrt{x})^2$ instead; call it $f(x)$.

$$f(x) = (x-4)^2 + x \quad ; \quad \text{domain: } [0, \infty)$$

2. Calculus:

- Crit. pt.: $f'(x) = 2(x-4) + 1 = 2x - 7 = 0$

  $\therefore\ x = 7/2$ is the only crit. pt. in $(0, \infty)$.

- D.T.: $f''(x) = 2 > 0$. So $x = 7/2$ yields a local minimum.

  Since it is the only local minimum, it is the global minimum.

3. Conclusion:

The minimal distance is attained when $x = 7/2$:

$$d = \sqrt{\left(\frac{7}{2} - 4\right)^2 + \frac{7}{2}} = \sqrt{\frac{1}{4} + \frac{7}{2}} = \boxed{\frac{\sqrt{15}}{2}}$$

Compute the following:

(a) $\displaystyle\int_0^4 \frac{x-3}{\sqrt{x}}\,dx$

(b) $\displaystyle\int_{-\pi/4}^{\pi/4} x^4 \tan^9 x\,dx$

(c) Assume that f is odd and suppose that $\displaystyle\int_1^3 f(x)\,dx = 4$. Evaluate $\displaystyle\int_{-1}^{-3} f(x)\,dx$.

(a) $$\int \frac{x-3}{\sqrt{x}}\,dx = \int \overset{= x^{1/2}}{\sqrt{x}}\,dx - 3\int \overset{x^{-1/2}}{\frac{1}{\sqrt{x}}}\,dx$$
$$= \boxed{\frac{2}{3}x^{3/2} - 6x^{1/2} + C}$$

(b) Let $f(x) = x^4 \tan^9(x)$ and note that it is an odd function because
$$f(-x) = (-x)^4 \tan^9(-x) = -x^4 \tan^9(x) = -f(x).$$
Therefore, the integral equals $\boxed{0}$.

(c) Consider the integral $\int_{-3}^{3} f(x)\,dx$. On the one hand, we know it is zero by symmetry. On the other hand, we can split it into three integrals as shown below:

$$0 = \int_{-3}^{-1} f(x)\,dx + \underbrace{\int_{-1}^{1} f(x)\,dx}_{=0} + \underbrace{\int_{1}^{3} f(x)\,dx}_{=4}$$

So $\int_{-3}^{-1} f(x)\,dx = -4$ and hence

$$\int_{-1}^{-3} f(x)\,dx = -\int_{-3}^{-1} f(x)\,dx = \boxed{4}$$

Problem 6. (1-D motion of a particle, I)

The acceleration function (in $\mathrm{m/s^2}$) and the velocity, and the position at $t = 0$ are given for a particle moving along a line.

$$a(t) = 2t - \sin(t), \quad 0 \le t \le 8$$
$$v(0) = 3, \quad s(0) = 4.$$

Find:

(a) the velocity function $v(t)$

(b) the distance traveled during $[0, 8]$

(c) the position function $s(t)$

(d) the displacement over $[0, 8]$

(a) $V(t) = \int a(t)\, dt = \int (2t - \sin(t))\, dt$

$= t^2 + \cos(t) + C.$

Since $V(0) = 0 + \cos(0) + C = 3$, $C = 2.$

thus, $\boxed{V(t) = t^2 + \cos(t) + 2}$

(b) $s(t) = \int v(t)\, dt = \int (t^2 + \cos(t) + 2)\, dt$

$= \frac{t^3}{3} + \sin(t) + 2t + C$

Since $s(0) = 0 + \sin(0) + 0 + C = 4$, $C = 4.$

Thus, $\boxed{s(t) = \frac{t^3}{3} + 2t + \sin(t) + 4}$

(C) & (d) Note that $V(t) > 0$ on the entire domain, which implies that the direction of motion never changes. So the distance equals displacement:

$s(8) - s(0) = \boxed{\frac{8^3}{3} + 2 \cdot 8 + \sin(8)}$

Problem 7. (Initial value problems)

Graph several functions that satisfy the differential equation $f'(x) = 3x^2 - 1$. Then find and graph the particular solution that satisfies the initial condition $f(2) = 1$.

- $f'(x) = 3x^2 - 1$

$$\Rightarrow f(x) = \int (3x^2 - 1)\, dx = \boxed{x^3 - x + C}$$

- Using the given condition $f(2) = 1$:

$$2^3 - 2 + C = 1$$
$$6 + C = 1$$
$$\therefore C = -5$$

So $f(x) = \boxed{x^3 - x - 5}$ is the unique solution to the initial value problem.

- One way to graph several functions in the form

$$f(x) = x^3 - x + C$$

is to pick one that is easy to plot and vertically shift it by several different values. For instance, taking $C = 0$, we have

$$f(x) = x^3 - x = x(x-1)(x+1)$$

which is easy to plot: