"Related rates"

Lecture 19: More Than One Rate (MTOR)

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Computational Test

- Monday, October 11

  (8 AM ~ 11:59 PM)

no later than
11:29 PM

- 8 questions & 30 min time limit
- One attempt ; no retake
- Closed-book, closed-note

  Writing on iPad or other tablet device Not Allowed!

- Proctorio — Computer ~~iPad~~

  Chrome ~~Safari~~, ~~Firefox~~, ~~..~~ .. → install Proctorio extension

- Under 18 ? Contact me!

Related rates problems

explicit or implicit

- Suppose two variables x and y are both dependent on time t.
- Moreover, assume that these two are related to each other.
- In this context, the rate of change of y with respect to time is expected to be *related* to that of x; $\frac{dy}{dt} \sim \frac{dx}{dt}$
- when one of the rates is known and the other is to be found, we have a **related rates** problem.

Key idea.

If y is written in terms of x and we are given $\frac{dx}{dt} = x'(t)$, then we can find $\frac{dy}{dt} = y'(t)$ using the chain rule:

$y = y(x)$

$$\frac{dy}{dt} = y'(x(t)) \cdot x'(t)\,.$$

$\frac{dy}{dt} = y'(x) \cdot \frac{dx}{dt}$

y is a func of x.

Problem-solving strategies

General procedure

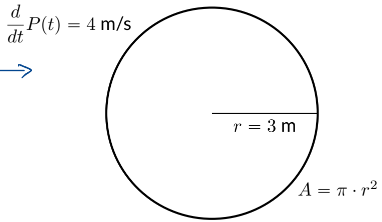
1. **Draw a picture.** If possible, draw a schematic picture with all the relevant information. Introduce notation.
2. **Find equations.** We want equations that relate all relevant functions. explicit or implicit
3. **Differentiate the equations.** Here we will often use implicit differentiation. Chain rule because all variables depend on t.
4. **Evaluate.** Evaluate each quantity at the relevant moment.
5. **Solve.** Solve for the relevant rate at the relevant moment.

Example 1. (Circular geometry)

Imagine an expanding circle. If we know that the perimeter is expanding at a rate of 4 m/s, what rate is the area changing when the radius is 3 meters?

1. Introduce var. & draw pic.

- Known: $\dfrac{dP}{dt} = 4 \ (m/s)$
- Want: $\left[\dfrac{dA}{dt}\right]_{r=3} = \ ?$



2. Find ~~an~~ eqn relating P and A.

- $P = 2\pi r \longrightarrow r = \dfrac{P}{2\pi}$
- $A = \pi r^2 \longrightarrow A = \pi \dfrac{P^2}{4\pi^2}$

$$A = \pi\left(\frac{P}{2\pi}\right)^2 = \pi \frac{P^2}{4\pi^2}$$

$$\therefore A = \frac{1}{4\pi} P^2$$

3. Take d/dt. $\quad A = \frac{1}{4\pi} P^2$

Keep in mind that both A & P depend on t. (use of the C.R. required)

$$\frac{dA}{dt} = \frac{1}{4\pi} 2P \cdot \frac{dP}{dt}$$

$$= \frac{P}{2\pi} \underbrace{\frac{dP}{dt}}_{= 4 \ (m/s)}$$

4. Evaluate / Solve for answer.

Note: For evaluation, we just need to figure out the value of P when $r = 3$.

when $r = 3$, $P = 2\pi \cdot 3 = 6\pi$. Thus,

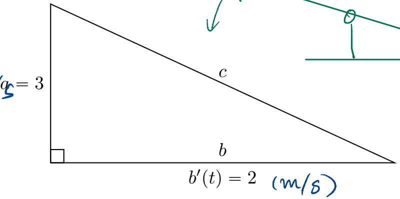
$$\left[\frac{dA}{dt}\right]_{r=3} = \frac{6\pi}{2\pi} \cdot 4 = \boxed{12 \ (m^2/s)}$$

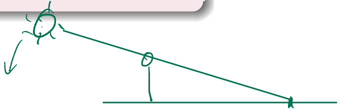
Example 2. (Right triangles)

Imagine an expanding right triangle. If one leg has a fixed length of 3 m, one leg is increasing with a rate of 2 m/s, and the hypotenuse is expanding to accommodate the expanding leg, at what rate is the hypotenuse expanding when both legs are 3 m long?

1. Introduce var. & draw pic.

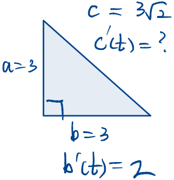
- Known: $\frac{db}{dt} = 2$ m/s , $\frac{da}{dt} = 0$ m/s
- Want: $\left[\frac{dc}{dt}\right]_{a=b=3}$





2. Find eqn relating

$$c^2 = a^2 + b^2$$



3. Take d/dt

$$c^2 = a^2 + b^2 \qquad \text{(eqn)}$$

$d/dt \downarrow$

$$\cancel{2}c\frac{dc}{dt} = \cancel{2}a\frac{da}{dt} + \cancel{2}b\frac{db}{dt}$$

$$\frac{dc}{dt} = \frac{a}{c}\underbrace{\frac{da}{dt}}_{=0} + \frac{b}{c}\underbrace{\frac{db}{dt}}_{=2}$$

4. Evaluate when $a = b = 3$.

At this moment, we know that $c = \sqrt{a^2+b^2} = \sqrt{3^2+3^2} = 3\sqrt{2}$.

length

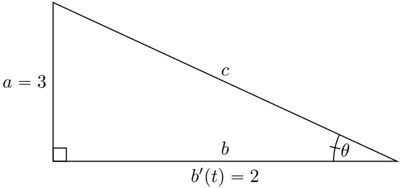
Thus, $\left[\frac{dc}{dt}\right]_{a=b=3} = \frac{3}{3\sqrt{2}}2 = \frac{2}{\sqrt{2}} = \boxed{\sqrt{2}\ (m/s)}$

time

Example 3. (Angular rates)

Imagine an expanding right triangle. If one leg has a fixed length of 3 m, one leg is increasing with a rate of 2 m/s, and the hypotenuse is expanding to accommodate the expanding leg, at what rate is the angle opposite the fixed leg changing when both legs are 3 m long?

1. Introduce var & Draw pic. →



(Identify known & unknown)

- known: $\frac{da}{dt} = 0$, $\frac{db}{dt} = 2$
- Want: $\left[\frac{d\theta}{dt}\right]_{a=b=3}$

2. Find eqn.

$$\tan(\theta) = \frac{a}{b} = \frac{3}{b}$$

$$\frac{d}{dx}\frac{1}{u(x)} = -\frac{u'(x)}{u(x)}$$

3. Take d/dt

(eqn) $\quad \tan(\theta) = \dfrac{3}{b}$

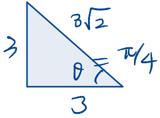
$d/dt \downarrow$

$$\sec^2(\theta)\frac{d\theta}{dt} = -\frac{3}{b^2}\underbrace{\frac{db}{dt}}_{=2} \Rightarrow \frac{d\theta}{dt} = -\frac{3\cos^2(\theta)}{b^2}\frac{db}{dt}$$

$$\frac{d\theta}{dt} = -\frac{3}{b^2\sec^2(\theta)}\frac{db}{dt}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

4. Evaluate when $a = b = 3$.



At this moment, $\cos(\theta) = \dfrac{3}{3\sqrt{2}} = \dfrac{1}{\sqrt{2}}$. Thus,

$$\left[\frac{d\theta}{dt}\right]_{a=b=3} = \frac{-3\cdot\frac{1}{2}}{3^2}\cdot 2 = \boxed{-\frac{1}{3}\ (\text{rad}/s)}$$

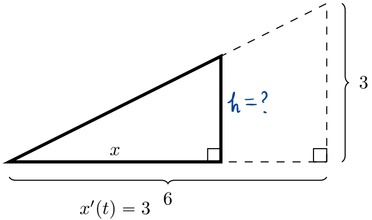
Example 4. (Similar triangles)

Imagine two right triangles that share an angle. If x is growing from the vertex with a rate of 3 m/s, what rate is the area of the smaller triangle changing when $x = 5$ m?

1. Notation & picture

A for area

- Known: $\frac{dx}{dt} = 3$ m/s
- Want: $\left[\frac{dA}{dt}\right]_{x=5} = ?$



Similar triangles:

$$\frac{h}{x} = \frac{3}{6} \Rightarrow h = \frac{1}{2}x$$

2. Eqn.

$$A = \frac{1}{2} \cdot x \cdot \frac{x}{2} = \frac{x^2}{4}$$

3. Take d/dt :

(eqn) $$A = \frac{x^2}{4}$$

d/dt ↓

$$\frac{dA}{dt} = \frac{\cancel{2}x}{\cancel{4}_2} \underbrace{\frac{dx}{dt}}_{=3}$$

4. Evaluate when $x = 5$

$$\left[\frac{dA}{dt}\right]_{x=5} = \frac{5}{2} \cdot 3 = \boxed{\frac{15}{2} \; (m^2/s)}$$