Lecture 16: Derivatives of Inverse Functions (DOIF)

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Implicit Differentiation

- Used to differentiate functions defined / expressed implicitly

- Steps

① Take d/dx (Differentiate with respect to x)

Regard y as a function of x. (need to use the CR.)

② Solve for $dy/dx = y'$. → always appears linearly

Recap Derivative of natural log. func.

Let $y = \ln x$. ← function to be differentiated.

"Invert" : $e^y = x$

Imp. Diff : ① Take d/dx : $e^y \cdot y' = 1$

② Solve for y' : $y' = \dfrac{1}{e^y} = \boxed{\dfrac{1}{x}}$

Know the answer now:

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} x = 1$$

The Derivatives of Inverse Trig Functions

$\frac{d}{dx} \sin x = \cos x$

Theorem (Derivatives of inverse trigonometric functions)

- $\dfrac{d}{dx}\arcsin(x) = \dfrac{d}{dx}\sin^{-1}(x) = \dfrac{1}{\sqrt{1-x^2}}$ *for* $|x| < 1$
- $\dfrac{d}{dx}\arccos(x) = \dfrac{d}{dx}\cos^{-1}(x) = \dfrac{-1}{\sqrt{1-x^2}}$ *for* $|x| < 1$
- $\dfrac{d}{dx}\arctan(x) = \dfrac{d}{dx}\tan^{-1}(x) = \dfrac{1}{1+x^2}$
- $\dfrac{d}{dx}\operatorname{arcsec}(x) = \dfrac{d}{dx}\sec^{-1}(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$ *for* $|x| > 1$
- $\dfrac{d}{dx}\operatorname{arccsc}(x) = \dfrac{d}{dx}\csc^{-1}(x) = \dfrac{-1}{|x|\sqrt{x^2-1}}$ *for* $|x| > 1$
- $\dfrac{d}{dx}\operatorname{arccot}(x) = \dfrac{d}{dx}\cot^{-1}(x) = \dfrac{-1}{1+x^2}$

Arcsine

Q. What is $\frac{d}{dx} \sin^{-1}(x)$?

Let $y = \sin^{-1}(x)$.

Apply "sin"

"Invert" : $\sin(y) = x$

Imp. Diff. : $\cos(y) \cdot y' = 1$

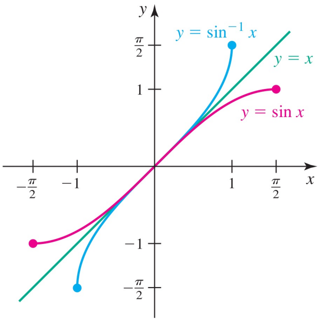
$$y' = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

Need to write cos(y) in terms of x.

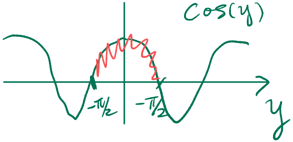
[Pythagorean identity] $\quad \underset{\substack{\uparrow \\ x^2}}{\sin^2(y)} + \underset{\substack{\uparrow \\ ?}}{\cos^2(y)} = 1 \Rightarrow$

$$\cos^2(y) = 1 - x^2$$

$$\cos(y) = \pm\sqrt{1-x^2}$$



Domain of $\sin^{-1} x$: $-1 \le x \le 1$
Range of $\sin^{-1} x$: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



Take $+\sqrt{\cdots}$

Discard $-\sqrt{\cdots}$

Arctangent Q. What is $\frac{d}{dx}\tan^{-1}(x)$?

"Alias": Let $y = \tan^{-1}(x) \longrightarrow y' = ?$

"Invert": Apply "tan" on both sides:

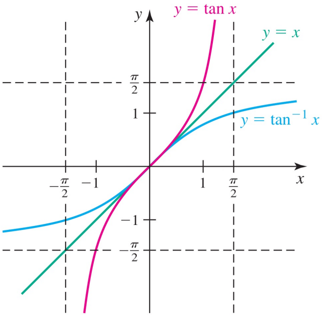
$$\tan(y) = x$$

Imp. Diff.: $\sec^2(y)\, y' = 1.$

$$y' = \frac{1}{\sec^2(y)} = \boxed{\frac{1}{1+x^2}}$$

Using Pythagorean ID

$$\tan^2(y) + 1 = \sec^2(y) \longrightarrow x^2 + 1 = \sec^2(y)$$



Domain of $\tan^{-1} x$: $-\infty < x < \infty$
Range of $\tan^{-1} x$: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Pythagorean Identities

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$\div \sin^2\theta$

$\div \cos^2\theta$

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}$$

$$\boxed{1 + \cot^2\theta = \csc^2\theta}$$

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

Arcsecant

Q. What is $\frac{d}{dx}\sec^{-1}(x)$?

"Alias" : Let $y = \sec^{-1}(x)$.

"Invert" : Apply sec on both sides.

$$\sec(y) = x$$

Imp. Diff. : ① Take d/dx

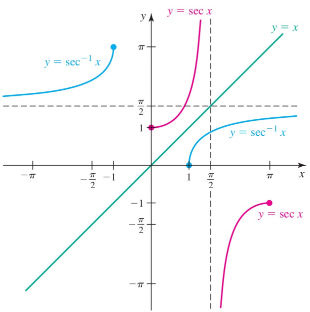
$$\sec(y)\tan(y)\,y' = 1$$

② Solve for y'

$$y' = \frac{1}{\sec(y)\tan(y)} = \left\{\begin{array}{ll} \dfrac{1}{x\sqrt{x^2-1}}, & \text{if } x \geq 1 \\[2ex] -\dfrac{1}{x\sqrt{x^2-1}}, & \text{if } x \leq -1 \end{array}\right\} = \boxed{\dfrac{1}{|x|\sqrt{x^2-1}}}$$

one-to-one

To write tan(y) in terms of x, use Pythagorean IDs.



Domain of $\sec^{-1} x$: $|x| \geq 1$
Range of $\sec^{-1} x$: $0 \leq y \leq \pi$, $y \neq \frac{\pi}{2}$

Known to be x^2 Want

ID: $\sec^2(y) = 1 + \tan^2(y)$

$$\tan^2(y) = \sec^2(y) - 1$$

$$\tan^2(y) = x^2 - 1$$

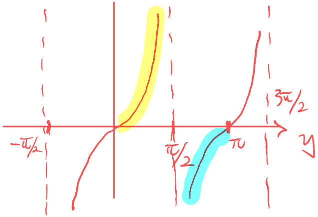
$$\therefore \tan(y) = \pm\sqrt{x^2 - 1}$$

Reality check

- $x^2 - 1 \overset{?}{\geq} 0$

Since $|x| \geq 1$ (domain),

$= |x|^2$ $x^2 \geq 1$. ✓



Sign determination

Case 1 $x \geq 1 \Rightarrow 0 \leq y < \pi/2$

$$0 < \tan(y) = \sqrt{x^2 - 1}$$

Case 2 $x \leq -1 \Rightarrow \pi/2 < y \leq \pi$

$$0 > \tan(y) = -\sqrt{x^2 - 1}$$

Question. Compute:

1. $$\frac{d}{dx}\tan^{-1}(\sqrt{x}) = \underbrace{\frac{1}{1+(\sqrt{x})^2}}_{\text{deriv. of outer @ inner}} \cdot \underbrace{\frac{1}{2\sqrt{x}}}_{\text{deriv. of inner}} = \frac{1}{(1+x)\cdot 2\sqrt{x}}$$

   outer   inner

2. $$\frac{d}{dx}\sec^{-1}(3x) = \underbrace{\frac{1}{|3x|\sqrt{(3x)^2-1}}}_{\text{deriv. of outer @ inner}} \cdot \underbrace{3}_{\text{deriv. of inner}}$$

   outer   inner

$$= \boxed{\frac{3}{|3x|\sqrt{9x^2-1}}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$$

Explanation

Remark

In the derivation of the above formulas, we repeatedly used the following form of implicit differentiation

① $y = f^{-1}(x)$

③ Imp. Diff.

② $f(y) = x$

$$\frac{d}{dx} f(y) = f'(y) \cdot y',$$

which requires that the function $y = f^{-1}(x)$ has a derivative. The differentiability of the inverse function is guaranteed by the following theorem.

$$f'(y)\, y' = 1$$

$$y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Note:

$\frac{d}{dx} f^{-1}$ can be obtained w/o direct differentiation of f^{-1}.

Inverse Function Theorem

Theorem (The inverse function theorem)

Suppose f is a differentiable function that is one-to-one near a and $f'(a) \neq 0$ and let $b = f(a)$. Then

→ f' is legit. → f^{-1} exists ↘ to avoid division by 0.

1. $f^{-1}(x)$ *is* ***defined*** *for x near b,*
2. $f^{-1}(x)$ *is* ***differentiable*** *near b,*
3. *last, but not least:*

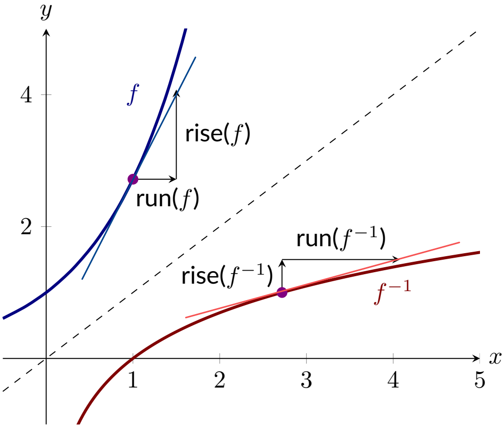
$$\left[\frac{d}{dx} f^{-1}(x)\right]_{x=b} = \frac{1}{f'(a)} \qquad \text{where} \qquad b = f(a)\,.$$

$$b = f(a) \xrightarrow{f^{-1}} f^{-1}(b) = a$$

$$= \frac{1}{f'(f^{-1}(b))}$$

Illustration

Besides verifying the last result using implicit differentiation, convince yourselves by considering the following diagram of a function f and its inverse f^{-1}:



$$\frac{1}{f'(a)} = (f^{-1})'(b)$$

Question. Let f be a differentiable function that has an inverse. In the table below we give several values for both f and f':

x	f	f'
2	0	2
3	1	5
4	3	0

- $f^{-1}(1) = \left(x \text{ value such that } f(x) = 1. \right)$

$= 3$

Compute

$$\frac{d}{dx} f^{-1}(x) \text{ at } x = 1.$$

$$\left[\frac{d}{dx} f^{-1}(x) \right]_{x=1} = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(3)} = \boxed{\frac{1}{5}}$$

this plays the role of b.