Lecture 22-23: Graphing Functions (COGF & CFGF)

Tae Eun Kim, Ph.D.

Autumn 2021

Calculus and Graphs

Let's put together all the tools we've learned so far with graphical implications:

limits

- **Infinite limits** indicate the presence of vertical asymptotes. $\lim_{x \to a} f(x) = \pm\infty$
- **Limits at infinity** describe "far-field" behavior of the function, e.g., horizontal asymptotes. $\lim_{x \to \pm\infty} f(x) = L$
- **The sign of the first derivative** tells us about the monotonicity, i.e., whether the graph is increasing or decreasing.
- **The sign of the second derivative** conveys the concavity information, i.e., whether it is concave up or down.

On Monotonicity and Concavity

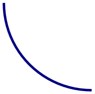
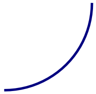
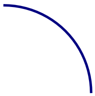
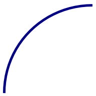
INC
DEC

CU
CD

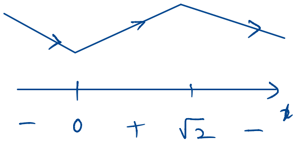
Combining two possible monotonicity and two possible concavity modes, we came up with the following four signature of curves:

- $f' > 0$ and $f'' > 0$: increasing and concave up
- $f' > 0$ and $f'' < 0$: increasing and concave down
- $f' < 0$ and $f'' > 0$: decreasing and concave up
- $f' < 0$ and $f'' < 0$: decreasing and concave down

Recall the following table from couple weeks ago.

	$f'(x) < 0$	$f'(x) > 0$
$f''(x) > 0$	 The function f is decreasing, while the rate itself is increasing. In this case the curve $y = f(x)$ is **concave up**.	 The function f is increasing, while the rate itself is increasing. In this case the curve $y = f(x)$ is **concave up**.
$f''(x) < 0$	 The function f is decreasing, while the rate itself is decreasing. In this case the curve $y = f(x)$ is **concave down**.	 The function f is increasing, while the rate itself is decreasing. In this case the curve $y = f(x)$ is **concave down**.

f, f',



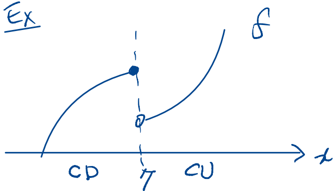
On Critical and Inflection Points

I have several important remarks on **critical points** and **inflection points**:

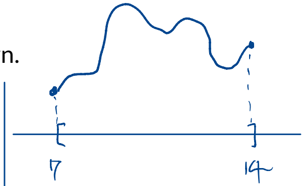
Critical points are interior points. (follows from def'n : "open interval containing a")

- There are two types of critical points – one at which $f' = 0$ (**the nice ones**) and the other at which f' is not defined (**the exotic ones**). Do not neglect the second kind.
- Being a critical point is merely a requirement to be a local extremum. It is not guaranteed that a critical point must be a local minimum or a local maximum. $\longrightarrow$ Use DT's to classify crit. pts.
- An **inflection point** is a point at which
  - f is continuous AND
  - f changes concavity from concave down to up or up to down.

Ex



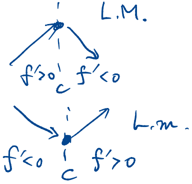
Even though the graph of f switches from CD to CU at $x=7$, $x=7$ is not an I.P. because f is not cts. at that point.



On Derivative Tests

Lastly, on the derivative tests:

- These are used to classify critical points into local maxima or local minima. Once again, understand that a critical point may be neither one of them.
- The key idea of these derivative tests is as follows:
  Suppose c is a critical points of f.
  - If a graph shifts from an increasing to a decreasing phase about c, then it is a local maximum.
  - If a graph shifts from a decreasing to an increasing phase about c, then it is a local minimum.



- In the 1st Derivative Test, we look out for the change in sign of f' **about** c.
- In the 2nd Derivative Test, we look out for the sign of f'' **at** c.

Sign table/chart.

$f(a) = 0 \rightarrow x = a$ is an x-intercept.
$f(0) = b \rightarrow y = b$ is the y-intercept.

Example

Sketch the graph of a function f which has the following properties:

- $f(0) = 0$ goes through $(0,0)$
- $\lim_{x \to 10^+} f(x) = +\infty$
- $\lim_{x \to 10^-} f(x) = -\infty$

V.A. at $x = 10$

- $f'(x) < 0$ on $(-\infty, 0) \cup (6, 10) \cup (10, 14)$ DEC
- $f'(x) > 0$ on $(0, 6) \cup (14, \infty)$ INC

mono.

- $f''(x) < 0$ on $(4, 10)$ CD
- $f''(x) > 0$ on $(-\infty, 4) \cup (10, \infty)$ CU

concav.

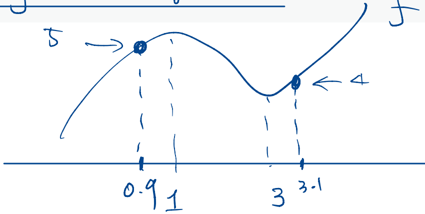
Combined sign chart

V.A.

x: 0, 4, 6, 10, 14

$f'(x)$	$-$	$+$	$+$	$-$	$-$	$+$
$f''(x)$	$+$	$+$	$-$	$-$	$+$	$+$
f	↘	↗	↗	↘	↘	↗

Answering Chris' question



$f(0.9) > f(3.1)$

$f(a) \geq f(b)$ if $a > b$.

- $f' > 0$ if $x < 1$ or $x > 3$ $\Rightarrow (-\infty, 1) \cup (3, \infty)$

  or $\downarrow$ union

- f is increasing on $(-\infty, 1)$, $(3, \infty)$

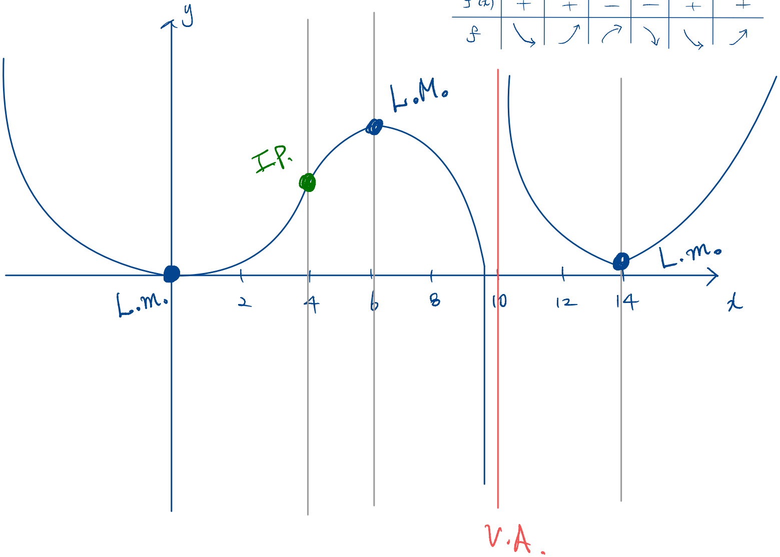
  not on $\cancel{(-\infty, 1) \cup (3, \infty)}$

- $f(0) = 0$
- $\lim_{x \to 10^+} f(x) = +\infty$
- $\lim_{x \to 10^-} f(x) = -\infty$
- $f'(x) < 0$ on $(-\infty, 0) \cup (6, 10) \cup (10, 14)$
- $f'(x) > 0$ on $(0, 6) \cup (14, \infty)$
- $f''(x) < 0$ on $(4, 10)$
- $f''(x) > 0$ on $(-\infty, 4) \cup (10, \infty)$

Combined sign chart

V.A.

x		0		4		6		10		14	
$f'(x)$	−		+		+		−		−		+
$f''(x)$	+		+		−		−		+		+
f	↘		↗		↗		↘		↘		↗



Example

The graph of f' (the derivative of f) is shown below. Assume f is continuous for all real numbers.

Use commas

1. On which of the following intervals is f increasing?
   $(-\infty, 0)$, $(2, 3)$

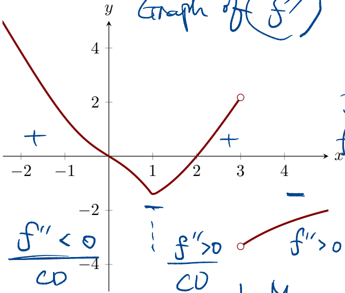
2. Which ~~of the following~~ are critical points of f?
   $\underbrace{x=0,\ x=2}_{\text{Nice}}$, $\underbrace{x=3}_{\text{Exotic}}$

3. Where do the local maxima occur?
   $x=0$, $x=3$

4. Where does a point of inflection occur?
   $x=1$ (CD to CU)

5. On which of the following intervals is f concave down?
   $(-\infty, 1)$

Graph of f'



$+$ $-$ $+$ $-$

$\frac{f''<0}{CD}$ $\frac{f''>0}{CU}$ $f''>0$

LM lm LM

x: 0, 2, 3

f': $+$ $-$ $+$ $-$

f: $\nearrow$ $\searrow$ $\nearrow$ $\searrow$

$\begin{cases} f'(a) = 0 \\ f'(a) \text{ DNE} \end{cases}$

L.M. when f' transitions from $(+)$ to $(-)$.

Easy

Example

Let $f(x) = \dfrac{1}{1+x^2}$. Find the following for f:

1. f' and f''

2. Critical points

3. Local extrema

4. Inflection points

① $$f'(x) = \boxed{-\frac{2x}{(1+x^2)^2}}$$

$$f''(x) = -2\,\frac{(1+x^2)^{\cancel{2}\,1} - 4x^2\cancel{(1+x^2)}}{(1+x^2)^{\cancel{4}\,3}}$$

$$= -2\,\frac{-3x^2+1}{(1+x^2)^3}$$

$$= \boxed{2\,\frac{3x^2-1}{(1+x^2)^3}}$$

② $$f'(x) = -\frac{2x}{(1+x^2)^2}$$

Note that (DEN) $\geq 1 > 0$.

$$0 = f'(x) \Rightarrow 2x = 0 \Rightarrow \boxed{x=0}$$

③ Use 2nd DT:

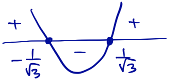
$$f''(0) = 2\,\frac{3\cdot 0^2 - 1}{(1+0^2)^3} = -2 < 0$$

$\boxed{\text{L.M. @ } x=0}$

④ $$f''(x) = 2\,\frac{3x^2-1}{(1+x^2)^3}$$

$(1+x^2)^3 > 0$

Sign of f'' coincides w/ that of $3x^2-1$



$\downarrow$

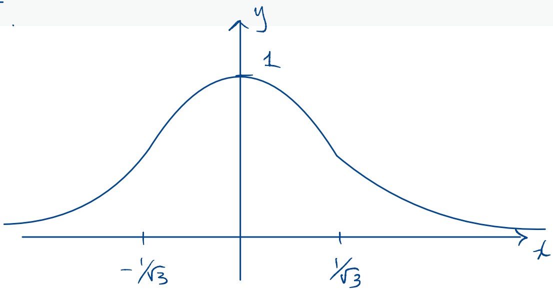
f : CU $\to$ CD $\to$ CU

$\therefore \boxed{x = \pm\frac{1}{\sqrt{3}}}$

Sketch the graph of $f(x) = \frac{1}{1+x^2}$.

- No x-intercept because $f(x) > 0$.
- $f(0) = \frac{1}{1+0} = 1$. So y-intercept at $y=1$.
- $\lim_{x \to \pm\infty} f(x) = 0$. So HA at $y=0$
- $f(-x) = f(x)$. So f is even.
- Combined sign chart based on previous work

x		$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	
f'	+	+	−	−	
f''	+	−	−	+	
f	↗	↗	↘	↘	



Easy, tedious

Example

Sketch the plot of $\underbrace{2x^3 - 3x^2 - 12x}_{f(x)}$.

x-intercepts

Note that

$$f(x) = x(2x^2 - 3x - 12) = 0$$

when

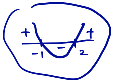
$$x = 0 \quad \text{or} \quad 2x^2 - 3x - 12 = 0$$

i.e.

$$x = 0 \quad \text{or} \quad x = \frac{3 \pm \sqrt{9 + 96}}{4} = \frac{3 \pm \sqrt{105}}{4}$$

Study f'

$$f'(x) = 6x^2 - 6x - 12 \quad \circ \quad {\scriptstyle 0}$$

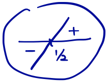


$$= 6(x^2 - x - 2) = 6(x-2)(x+1)$$

$\Rightarrow$ Crit. pts.: $x = -1$, $x = 2$

Study f''

$$f''(x) = 12x - 6 \quad \circ \quad {\scriptstyle 0}$$



$\Rightarrow \quad f''(x) = 0$ at $x = \frac{1}{2}$.

Combined sign chart

x	-1	$\frac{1}{2}$	2	
f'	$+$	$-$	$-$	$+$
f''	$-$	$-$	$+$	$+$
f	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

Combined sign chart

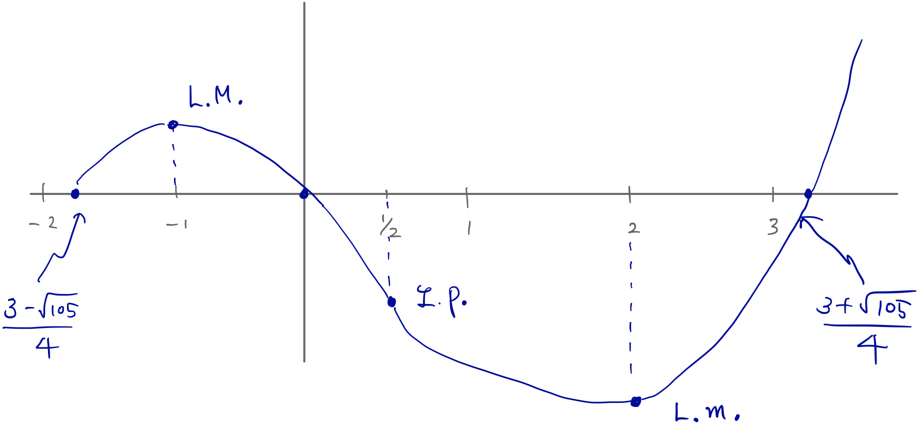
x	-1	$\frac{1}{2}$	2	
f'	$+$	$-$	$-$	$+$
f''	$-$	$-$	$+$	$+$
f	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

x - intercepts :

$x=0$

$x=\dfrac{3+\sqrt{105}}{4} = 3.311\ldots$

$x=\dfrac{3-\sqrt{105}}{4} = -1.811\ldots$



Recitation

Example

Sketch the plot of

$$f(x) = \begin{cases} xe^x + 2 & \text{if } x < 0 \\ x^4 - x^2 + 3 & \text{if } x \geq 0. \end{cases}$$

Horiz. Asymp.

Hint: $\lim_{x \to -\infty} x e^x = 0$.

- $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (xe^x + 2) = 2$

  using the hint given above.

- $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} (x^4 - x^2 + 3) = \infty$

$\therefore$ f has one HA at $y = 2$.

Continuity at $x = 0$

- $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (xe^x + 2) = 2$
- $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^4 - x^2 + 3) = 3$

So $\lim_{x \to 0} f(x)$ does not exist and so $f(x)$ is not cts. at $x = 0$.

Study f'

$$f'(x) = \begin{cases} e^x(x+1), & x<0 \\ 4x^3-2x, & x>0 \end{cases}$$

- Note that f' is not defined at $x=0$.

  So $x=0$ is a crit. pt.

- To solve $f'(x)=0$:

  Case 1: $x<0$

  $e^x(x+1) = 0 \Rightarrow x=-1$

  Case 2: $x>0$

  $4x^3-2x = 2x(2x^2-1) = 0$

  $\Rightarrow \underbrace{x=0,\ x=-\frac{1}{\sqrt{2}}}_{\text{discard since } x \text{ needs to be } >0}, \ x=\frac{1}{\sqrt{2}}$

So we have three crit. pts.:

$x=-1$, $x=0$, $x=\frac{1}{\sqrt{2}}$

Sign chart for f'

x		-1		0		$\frac{1}{\sqrt{2}}$	
f'	$-$		$+$		$-$		$+$
f	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

Study f''

$$f''(x) = \begin{cases} e^{x}(x+2), & x < 0 \\ 12x^2 - 2, & x > 0 \end{cases}$$

Note that $f''(x) = 0$ when

$x = -2$, $x = \frac{1}{\sqrt{6}}$.

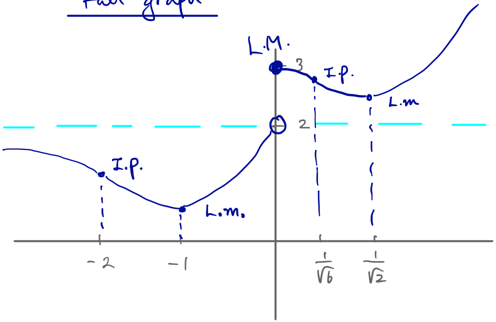
Sign chart for f''

x		-2		0		$\frac{1}{\sqrt{6}}$	
f''	$-$		$+$		$-$		$+$
f	CD		CU		CD		CU

Combined sign chart

x		-2		-1		0		$\frac{1}{\sqrt{6}}$		$\frac{1}{\sqrt{2}}$	
f'	$-$		$-$		$+$		$-$		$-$		$+$
f''	$-$		$+$		$+$		$-$		$+$		$+$
f	$\searrow$		$\searrow$		$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

Full graph



Summary

The following is the list of all the tools at our finger tips to sketch the graph of $y = f(x)$

- Compute f' and f''.
- Find the y-intercept, this is the point $(0, f(0))$. Place this point on your graph.
- Find any vertical asymptotes, these are points $x = a$ where $f(x)$ goes to infinity as x goes to a (from the right, left, or both).
- If possible, find the x-intercepts, the points where $f(x) = 0$. Place these points on your graph.
- Analyze end behavior: as $x \to \pm\infty$, what happens to the graph of f? Does it have horizontal asymptotes, increase or decrease without bound, or have some other kind of behavior?
- Find the critical points (the points where $f'(x) = 0$ or $f'(x)$ is undefined).
- Use either the first or second derivative test to identify local extrema and/or find the intervals where your function is increasing/decreasing.
- Find the candidates for inflection points, the points where $f''(x) = 0$ or $f''(x)$ is undefined.
- Identify inflection points and concavity.
- Determine an interval that shows all relevant behavior.