Lecture 28-29: L'Hôpital's Rule (LHR)

Math 1151

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Applications of derivatives

Limits $\longrightarrow$ Differential

$\downarrow$

Integral

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- related rates
- graphing functions
- optimization (max. / min.)
- linear approximation

Today $\longrightarrow$

- computation of hard limit problems (L'Hôpital's Rule) also spelled as "L'Hospital"

This is our final application of derivatives: using derivatives to calculate difficult limits. Enter L'Hôpital's rule.

Theorem (L'Hôpital's Rule)

Let $f(x)$ and $g(x)$ be functions that are differentiable near a. If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \qquad \text{or } \pm\infty,$$

and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, and $g'(x) \neq 0$ for all x near a, then

$$\underline{\lim_{x\to a} \frac{f(x)}{g(x)}} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Indeterminate Form

work further

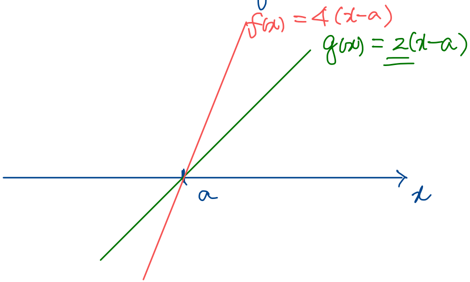
"$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

- not Quotient Rule.
- simply the ratio of individual derivatives.

Geometry of L'H

- Sample scenario a is some number

$$f(x) = 4(x-a), \quad g(x) = 2(x-a) \quad \text{(linear)}$$



Observe

- $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$

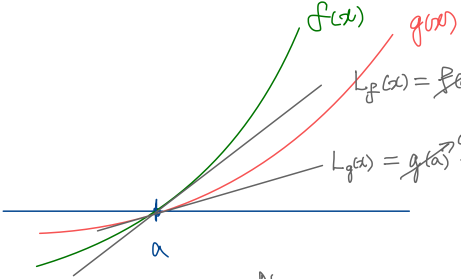
- $$\underbrace{\lim_{x\to a} \frac{f(x)}{g(x)}}_{\text{Form: } \text{"}\frac{0}{0}\text{"}} = \lim_{x\to a} \frac{4\cancel{(x-a)}}{2\cancel{(x-a)}} = \lim_{x\to a} \frac{4}{2} = 2.$$

"ratio of slopes"

- <u>General scenario</u>

Now assume f, g are not linear.

- $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$



$L_f(x) = \cancelto{0}{f(a)} + f'(a)(x-a)$

$L_g(x) = \cancelto{0}{g(a)} + g'(a)(x-a)$

- $\underbrace{\lim_{x\to a} \frac{f(x)}{g(x)}}_{\text{Form: } \text{“}\frac{0}{0}\text{”}} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

L'H

<u>Near a</u>

$$\frac{f(x)}{g(x)} \approx \frac{L_f(x)}{L_g(x)} = \frac{f'(a)\cancel{(x-a)}}{g'(a)\cancel{(x-a)}}$$

(ratio of slopes of tang. lines)

Caveats

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ provided the limit on RHS exists and $g'(x) \neq 0$ near a.

$$\underbrace{\lim_{x \to a} \frac{f(x)}{g(x)}}_{\text{"}\frac{0}{0}\text{" or "}\frac{\infty}{\infty}\text{"}}$$

1. RHS is the limit of ratio of derivatives. (Not Q.R.)

2. L'H works even for
   - "$x \to a^{\pm}$" (one-sided limits)
   - "$x \to \pm\infty$" (limits at infinity)

3. May be applied repeatedly until you obtain an answer.
   (simplify before next application of L'H.)

List of Indeterminate Forms

can be handled by L'H eventually.

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$

L'H

Lv. 0

- $0 \cdot \infty$

Lv. 1

- $\infty - \infty$

Lv. 2

- 1^{∞}
- 0^{0}
- ∞^{0}

Lv. 3 (Final stage boss)

Symbols, not numbers,
So use double quotes.

In each of these cases, the value of the limit is **not** immediately obvious.
Hence, a careful analysis is required!

Examples: Basic Indeterminate Forms

Question. Compute $\lim_{x \to 0} \frac{\sin(x)}{x}$.

Form: "$\frac{0}{0}$"

Ans. 1

(taught in Sqz. Thm. sec. relied on geometry.)

Soln

justification

$$\overset{\text{L'H}}{=} \lim_{x \to 0} \frac{\cos(x)}{1}$$

$$= \frac{\cos(0)}{1} = \boxed{1}$$

drop "lim" symbol since the limit has been evaluated.

Exercise The following important identities were presented a while back (expected to memorize)

- $\displaystyle\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$

- $\displaystyle\lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta} = \lim_{\theta\to 0} \frac{\cos(\theta)-1}{\theta} = 0$

- $\displaystyle\lim_{x\to 0} \frac{e^x-1}{x} = 1$

- Definition "e"
- $(e^x)' = e^x$

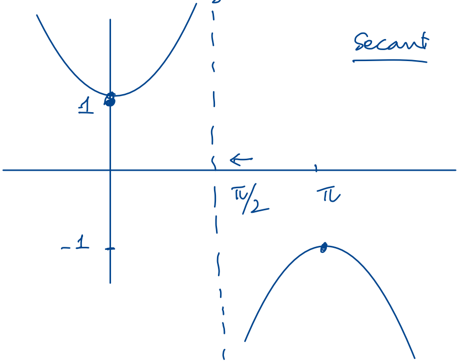
Confirm these using L'H.

Question. Compute $\displaystyle\lim_{x\to\pi/2^+} \frac{\sec(x)}{\tan(x)}$.

Form: "$\dfrac{\infty}{\infty}$"

$$\lim_{x\to\frac{\pi}{2}^+} \sec(x) = -\infty.$$

$$\lim_{x\to\frac{\pi}{2}^+} \tan(x) = -\infty$$



$$\overset{L'H}{=} \lim_{x\to\frac{\pi}{2}^+} \frac{\cancel{\sec(x)}\tan(x)}{\sec^{\cancel{2}}(x)}$$

$$= \lim_{x\to\frac{\pi}{2}^+} \frac{\tan(x)}{\sec(x)} = \dots$$

Simplify:

$$\frac{\sec(x)}{\tan(x)} = \frac{1}{\cancel{\cos(x)}} \cdot \frac{\cancel{\cos}(x)}{\sin(x)} = \frac{1}{\sin(x)}$$

So $\displaystyle\lim_{x\to\pi/2^+} \frac{1}{\sin(x)} = \boxed{1}$

Lv. 1

The following $\mathbf{0} \cdot \infty$ can be reduced to one of the two previous ones. For instance:

Question. Compute $\lim_{x \to 0^+} x \ln x$.

Examples: Indeterminate Forms Involving Subtraction

Lv. 2

"$\infty - \infty$"

The name of the game once again is reduction. We will transform differences into either quotients or products then apply L'Hôpital's rule on the basic forms.

Question. Compute $\lim\limits_{x \to 0} (\cot(x) - \csc(x))$.

Question. Compute $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$.

Examples: Exponential Indeterminate Form

Lv. 3 final boss

This pertains to the forms

$$\mathbf{1}^{\infty}, \quad \mathbf{0}^{\mathbf{0}}, \quad \infty^{\mathbf{0}}$$

Suppose we have functions $u(x)$ and $v(x)$ such that

$$\lim_{x \to a} u(x)^{v(x)}$$

falls into one of the forms described above. We use the inverse relation between $\exp$ and $\log$ functions to rewrite the limit as

$$\lim_{x \to a} e^{v(x)\ln(u(x))} .$$

Using the fact that the exponential function is continuous, the limit equals to

$$\exp[\lim_{x \to a} v(x)\ln(u(x))] .$$

Note that the limit now is in one of the previously presented forms.

Question. First determine the form of the limit, then compute the limit.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}.$$