Lecture 5: Using Limites to Detect Asymptotes (ULTDA)

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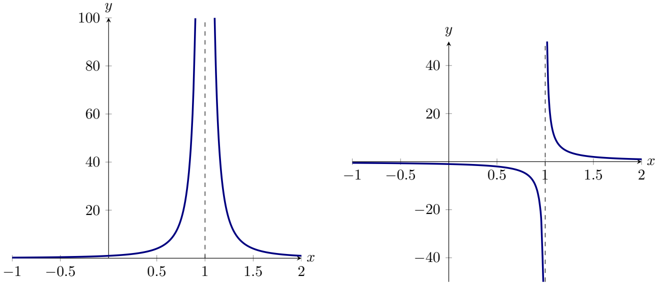
Vertical asymptotes – infinite limits

Consider the graphs of the following two functions near $x = 1$:

$$f(x) = \frac{1}{(x-1)^2} \quad \text{and} \quad g(x) = \frac{1}{x-1} .$$



- In both cases, the graphs get closer and closer to the vertical line $x = 1$, but they never touch it.
- Such a line is called a vertical asymptote.

Definition

If at least one of the following holds:

- $\lim_{x\to a} f(x) = \pm\infty$,
- $\lim_{x\to a^+} f(x) = \pm\infty$,
- $\lim_{x\to a^-} f(x) = \pm\infty$,

then the line $x = a$ is a **vertical asymptote** of f.

Question. Find the vertical asymptotes of

$$f(x) = \frac{x^2 - 9x + 14}{x^2 - 5x + 6}\,.$$

Question. Find the vertical asymptotes of

$$f(x) = \frac{\sqrt{x^2 - 3x + 2}}{x - 2}\,, \quad x > 2\,.$$

Horizontal asymptotes – limits at infinity

Definition

- If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large, we write

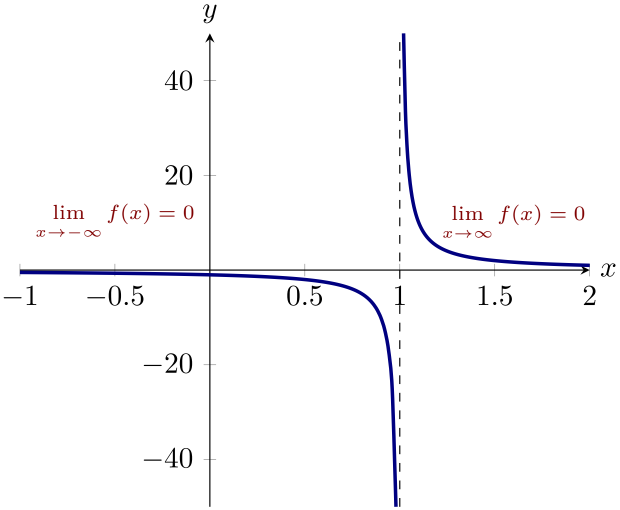
$$\lim_{x \to \infty} f(x) = L$$

  and we say that the **limit at infinity** of $f(x)$ is L.
- If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large and negative, we write

$$\lim_{x \to -\infty} f(x) = L$$

  and we say that the **limit at negative infinity** of $f(x)$ is L.

Illustration. The function $f(x) = 1/(x-1)$ once again provides us with a valuable insight:



The graph suggests that having finite limits at infinity has a lot to do with horizontal asymptotes, thus the following definition:

Definition

If

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L,$$

then the line $y = L$ is a **horizontal asymptote** of $f(x)$.

Question. Find the horizontal asymptotes of

$$f(x) = \frac{6x - 9}{x - 1}.$$

Question. Find the horizontal asymptotes of

$$f(x) = \frac{x^3 + 1}{\sqrt{x^6 + 6}}.$$

Question. Compute

$$\lim_{x\to\infty} \frac{\sin(7x) + 4x}{x}.$$

Hint. Use the squeeze theorem.

For your viewing pleasure: