Chain rule plays a key role!

Lecture 14: Implicit Differentiation (ID)

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Autumn 2021

6 **Labor Day** **No Classes**	7 Worksheet: ULTDA **WH1 due**	8 Continuity and the Intermediate Value Theorem (CATIVT)	9 Worksheet: CATIVT **HW: IF, ULTDA**	10 An Application of Limits (AAOL)
13 Definition of the Derivative (DOTD) **Midterm 1** **8:00-8:40PM** **UF – CATIVT**	14 Worksheet: AAOL, DOTD **HW: CATIVT**	15 Derivatives as Functions (DAF)	16 Worksheet: DAF **HW: AAOL, DOTD**	17 **Last day to drop w/o a "W"** Rules of Differentiation (ROD)
20 Product Rule and Quotient Rule (PRAQR) **WH2 due**	21 Worksheet: ROD, PRAQR **HW: DAF**	22 Chain Rule (CR)	23 Worksheet: PRAQR, CR **HW: ROD, PRAQR**	24 Higher Order Derivatives and Graphs (HODAG)
27 Implicit Differentiation (ID) **Midterm 2** **8:00-8:40PM** **AAOL-CR**	28 Worksheet: HODAG, ID **HW: CR**	29 Logarithmic Differentiation (LD)	30 Worksheet: ID, LD **HW: HODAG, ID**	**October 1** Derivatives of Inverse Functions (DOIF)
4	5	6	7	8

- Read important docs and announcements on exam policies, formatting, etc.
- Cover page is available
- Extended office hours: 4:15 ~ 6:15 pm
- Review video

- Timeline
  - 7:55 PM : download exam
  - 8:00 PM : start working on it
  - 8:40 PM : upload exam

  40 min.

  - 8:55 PM : done

- After this, late!
- Do not email your exam to instructor!

Motivation

Depending on whether or not the dependent variable is written explicitly in terms of the independent variable, a function can be classified as an **explicit function** or an **implicit function**. For example:

Any explicit functions can be expressed implicitly!

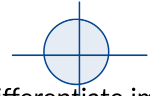
- Explicit functions ($y = f(x)$ form)

$$y = 3x^2 - 2x + 1\,, \qquad y = e^{3x}\,, \qquad y = \frac{x-2}{x^2-3x+2}\,, \ldots$$

$$\underbrace{e^{3x} - y}_{F(x,y)} = 0$$

- Implicit functions ($F(x,y) = 0$ form) $\rightarrow$ exotic (closed) curves.

$$y = \pm\sqrt{4-x^2} \leftarrow \underbrace{x^2 + y^2 = 4}_{\text{circle.}}\,, \qquad x^3 + y^3 = 9xy\,, \qquad x^4 + 3x^2 = x^{2/3} + y^{2/3} + 1\,, \ldots$$



Today's goals are:

- to learn how to differentiate implicit functions
- to derive more differentiation shortcuts using the new technique

Implicit Differentiation

Procedures

In order to differentiate an implicit function:

"input variable!"

1. Differentiate the entire equation with respect to x.
2. Solve for $\boxed{\frac{dy}{dx}}$.

Note.

- In Step 1, keep in mind that y is actually a function of x.
- This inevitably requires an application of the chain rule.

even though the relation cannot be explicitly written.

Question. Consider the curve defined by:

y is a fnc of x.

$$x^2 + y^2 = 1\,. \qquad (\text{unit circle})$$

radius 1.

by CR.

❶ Compute $\dfrac{dy}{dx}$.

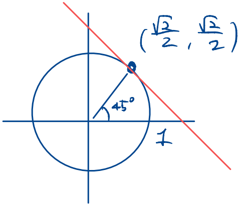
- Hit it w/ d/dx: $\quad 2x + 2y \cdot \dfrac{dy}{dx} = 0$
- Solve for $\dfrac{dy}{dx}$: $\quad \dfrac{dy}{dx} = \dfrac{-\cancel{2}x}{\cancel{2}y}$

$$\therefore \boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

slope of tangents.

❷ Find the slope of the tangent line at $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

x-coord. y-coord.



$$\left[\frac{dy}{dx}\right]_{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})} = -\frac{\sqrt{2}/2}{\sqrt{2}/2} = \boxed{-1}$$

Question. Consider the curve defined by:

$$x^3 + y^3 = 9xy.$$

y is a fnc of x.

$y^3 = [☺(x)]^3$

1. Compute $\dfrac{dy}{dx}$.

- Take d/dx: $3x^2 + \underbrace{3y^2 \cdot \frac{dy}{dx}}_{CR} = \underbrace{9y + 9x \cdot \frac{dy}{dx}}_{PR}$

- Solve for dy/dx:

$$(-9x + 3y^2)\frac{dy}{dx} = -3x^2 + 9y$$

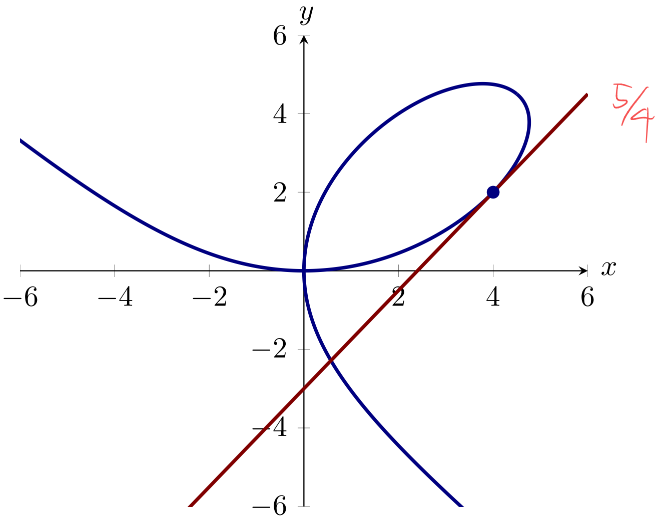
$$\boxed{\frac{dy}{dx} = \frac{-3x^2 + 9y}{-9x + 3y^2}}$$

2. Find the slope of the tangent line at $(4, 2)$.

$$\left[\frac{dy}{dx}\right]_{(x,y)=(4,2)} = \frac{-3 \cdot 4^2 + 9 \cdot 2}{-9 \cdot 4 + 3 \cdot 2^2}$$

$$= \frac{\cancel{-}\cancel{3}(16-6)}{\cancel{-}\cancel{3}(12-4)} = \boxed{\frac{5}{4}}$$

Graph for the previous problem.



Recall: Implicit differentiation

- Useful in differentiating functions defined "implicitly", e.g.,

$$x^2 + y^2 = 1$$

- Steps: To find $dy/dx = y'$

  ① Take d/dx. ← Regard y as a function of x (Use chain rule)

  ② Solve for $dy/dx = y'$.

Ex

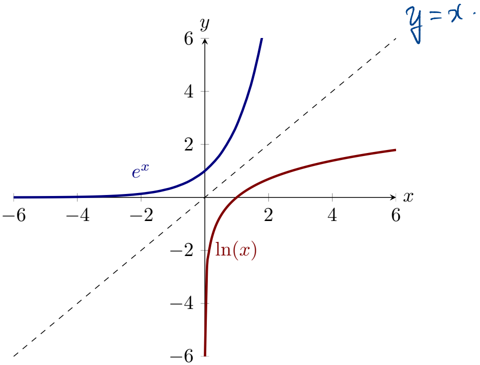
- $\frac{d}{dx} y^2 = 2y \cdot \frac{dy}{dx} = 2y \cdot y'$

- $\frac{d}{dx} x^3 y = 3x^2 y + x^3 \frac{dy}{dx} = 3x^2 y + x^3 y'$

- $\frac{d}{dx} e^{2y+1} = e^{2y+1} \cdot (2y' + 0)$
  $= 2e^{2y+1} y'$

Derivatives of Logarithmic Functions

- Recall that e^x and $\ln(x)$ are inverses and thus their graphs are reflections of each other about $y = x$.
- In particular, we can write $\underline{y = \ln(x)}$ as $\underline{x = e^y}$.
- And we know that $\boxed{\frac{d}{dx}e^x = e^x}$.



$e^y = \cancel{e}^{\cancel{\ln}(x)} = x$

Question: What is $\frac{d}{dx}\ln(x)$?

Let $y = \ln(x)$. Want $\frac{dy}{dx} = y'$.

Rewrite: $x = e^y$

Imp. Diff.

① Take d/dx : $1 = e^y \cdot y'$

② Solve for $dy/dx = y'$: $\boxed{y' = \frac{1}{e^y} = \frac{1}{x}}$

Theorem (The derivative of logarithm)

Let $b > 0$ and $b \neq 1$. Then

Why? $\rightarrow$
$$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}.$$
(General)

In particular, when $b = e$, the formula reduces to

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$
(special)

When $b = e$

$\frac{d}{dx}\log_e(x) = \frac{1}{x\ln(e)}$, $\ln(e) = 1$

$\frac{d}{dx}\ln(x) = \frac{1}{x}$.

Explanation. Let $y = \log_b(x)$. Want $\frac{dy}{dx} = y'$.

Rewrite: $x = b^y = e^{(\ln b)y}$ change of bases.

Imp. Diff.: ① Take d/dx: $1 = e^{(\ln b)y} \cdot \ln b \, \frac{dy}{dx}$, $\frac{dy}{dx} = y'$

② Solve for $\frac{dy}{dx} = y'$: $y' = \frac{1}{e^{(\ln b)y}\ln b} = \boxed{\frac{1}{x\ln b}}$

Question. Compute:

deriv. of outer @ inner

deriv. of inner

1. $\dfrac{d}{dx}(-\ln(\cos(x))) = -\dfrac{1}{\cos(x)} \cdot (-\sin(x))$

outer

inner

$= \dfrac{\sin(x)}{\cos(x)} = \boxed{\tan(x)}$

2. $\dfrac{d}{dx}\log_7(x) = \boxed{\dfrac{1}{x \cdot \ln 7}}$

base

The Derivative of an Exponential Function

Theorem (The derivative of an exponential function)

Let a be a positive real number. Then

$$\frac{d}{dx}a^x = a^x \cdot \ln(a)\,. \quad \text{(General)}$$

$a = e$

$$\frac{d}{dx}e^x = e^x \cdot \underbrace{\ln(e)}_{=1}$$

(Special)

$$= e^x$$

Explanation.

change of bases

Note:

$$a^x = \left(e^{\ln a}\right)^x$$

$$= e^{(\ln a)x}$$

So

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{(\ln a)x} = e^{(\ln a)x} \cdot \ln a = \boxed{a^x \ln a}$$

Question. Compute

$$\frac{d}{dx}7^x. \quad = \quad 7^x \cdot \ln 7$$