Lecture 25: Linear Approximation (LA)

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Linear Approximation

- Recall that the derivative contains the slope information of tangent lines to a given curve.
- Using this, we spent a lot of time calculating the equations of lines tangent to curves.

Let's formalize this discussion.

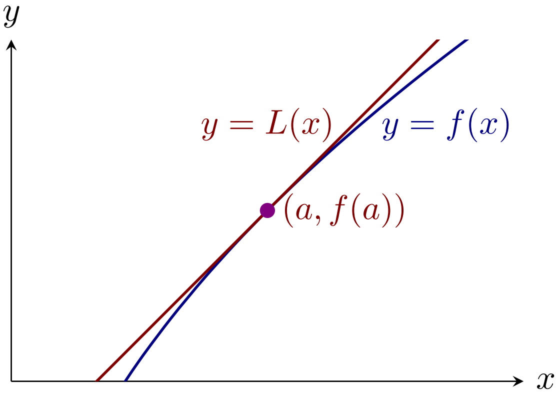
Definition

If f is a function differentiable at $x = a$, then a **linear approximation** for f at $x = a$ is given by

$$L(x) = f(a) + f'(a)(x - a)\,.$$

Note that the graph of L is simply the tangent line to the curve $y = f(x)$ at $x = a$.

Illustration



Examples

One major advantage of linear approximation is in computation of various mathematical functions.

Question. Use a linear approximation of $f(x) = \sqrt[3]{x}$ at $a = 64$ to approximate $\sqrt[3]{50}$. What would happen if we chose $a = 27$ instead?

Examples

Question. Use a linear approximation of $f(x) = \sin(x)$ at $a = 0$ to approximate $\sin(0.3)$.

[REVIEW] Over or Under?

At times, we need to determine whether a linear approximation is an overestimate or an underestimate without any access to a graph. The following may be useful.

- If $f''(a) > 0$, then $L(x) \leq f(x)$ for x *near* a, i.e., $L(x)$ is an underestimate of $f(x)$.
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The phrase "x near a" is vague. The following statements are less ambiguous.

- If $f'' > 0$ on an interval I containing a, then $L(x) \leq f(x)$ on I, that is, $L(x)$ is an *underestimate* of $f(x)$.
- If $f'' < 0$ on an interval I containing a, then $L(x) \geq f(x)$ on I, that is, $L(x)$ is an *overestimate* of $f(x)$.

Differentials

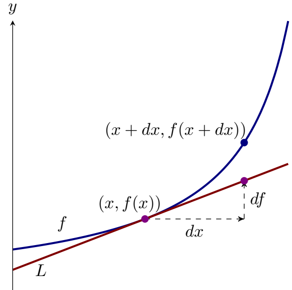
Definition

Let f be a differentiable function. We define df, a differential of f, at a point x by

$$df = f'(x)\ dx$$

where dx is an independent variable that is called a **differential of** x.

Geometrically, differentials can be interpreted via the diagram below.



Remark

- It is important to remember that the notations dx and $dy = df$ represent variables.
- Observe from the picture that

$$f(x + dx) \approx f(x) + df \; .$$

- Noting that the right-hand side is identical to $L(x)$ given above, we once again confirm the validity of the linear approximation formula.

Question. Use differentials to approximate $\sqrt[3]{50}$ and $\sin(0.3)$.

Error Approximation

- We now consider how *linear approximation* or *differentials* can be used to estimate errors in various computations.
- The quantity that we are interested in is the difference or *error* $E(x)$ between the quantity $f(x)$ and the quantity $f(x+dx)$ with a slightly perturbed input $x+dx$, i.e., for small dx.
- Using differentials, we can estimate this error by

$$E(x) = f(x+dx) - f(x) \approx f(x) + df - f(x) = df = f'(x)\,dx\,.$$

Example

The cross-section of a 250 ml glass can be modeled by the function $r(x) = x^4/3$:

At 16.8 cm from the base of the glass, there is a mark indicating when the glass is filled to 250 ml. If the glass is filled within ± 2 millimeters of the mark, what are the bounds on the volume? The volume in milliliters, as a function of the height of water in centimeters, y, is given by

$$V(y) = \frac{2\pi y^{3/2}}{\sqrt{3}}.$$