Lecture 39: The Idea of Substitution (TIOS)

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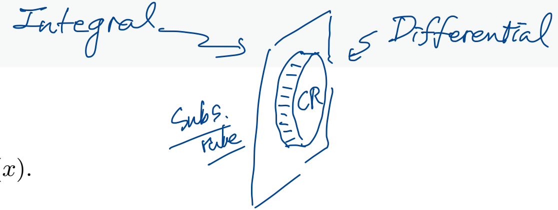
To: Kim, Tae Eun

Thank you for cheering Dr. Kim. I made a piece of meme just for fun. I hope you to not take it too seriously. Please enjoy it.

Courtesy of J Song.

Two Sides of a Coin



Recall that from the chain rule that

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

So by the fundamental theorem of calculus, we have

$$\int_a^b f'(g(x))g'(x)\,dx \overset{\text{FTC2}}{=} \Big[f(g(x))\Big]_a^b = f(g(b)) - f(g(a)).$$

equal

Using the fundamental theorem in reverse direction once again, the last line can be thought of as

$$\Big[f(u)\Big]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f'(u)\,du.$$

$$\int_a^b f'(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du$$

This is the gist of the integration technique known as **substitution rule** or u**-substitution**[1].

[1] This name is due to a popular and customary choice of substitution variable u. The choice, however, is not an absolute rule written on a stone. Any variable of your choice such as v or ☺ works if used consistently.

Substitution Rule

Theorem (Integral Substitution Formula)

If g is differentiable on the interval $[a, b]$ and f is differentiable on the interval $[g(a), g(b)]$, then

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du.$$

In sum, the substitution rule is the integral counterpart of differential chain rule and the fundamental theorem of calculus serves as a bridge between the two.

Procedures

In integrating a function which we suspect to be the derivative of another obtained by the chain rule:

1. Look for a candidate for the inner function; call it u.
2. Rewrite the given function completely in terms of u leaving no trace of the original variable.
3. Integrate this new function of u. (If necessary, you may need to go back to Step 1 and modify your choice of u.)
4. In dealing with an indefinite integral, make sure to replace u by the equivalent expression of the original variable.
5. Working with a definite integral, you may evaluate the result of Step 3 at the transformed bounds of u or evaluate the antiderivate obtained in Step 4 at the original bounds.

Question. Compute $\int_1^3 x\cos(\boxed{x^2})\,dx$.

"u"

"inner"

$$\begin{cases} u = x^2 \\ du = (2x)\,dx \end{cases}$$

deriv. of u.

$x\,dx = \frac{du}{2}$

Limits

x	$u = x^2$
3	9
1	1

$$= \int_1^9 \cos(u)\,\frac{1}{2}\,du$$

(Want: Everything to be written in terms of u.)

$$\overset{\text{FTC2}}{=} \frac{1}{2}\Big[\sin(u)\Big]_1^9 = \boxed{\frac{1}{2}\big(\sin(9) - \sin(1)\big)}$$

- Subs. Rule is useful in handling integrals of product/quotient.
- $\int_a^b \underbrace{f'(g(x))}\,\underbrace{g'(x)}\,dx$.

  deriv. of "inner" appears outside.

Question. Compute $\int \underbrace{\sec^2(x)}\, \overset{u}{\tan(x)}\, dx.$ du

$\frac{d}{dx}$

- Observation : $\frac{d}{dx}\tan(x) = \sec^2(x)$

- Tip : Set $\begin{cases} u = \tan(x). \\ du = \sec^2(x)\,dx \end{cases}$

- indef. integ. : no need to translate limits.

Retrieve orig. var.

$$= \int u\,du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\tan^2(x) + C}$$

Question. Compute $\int x^4(x^5+1)^9\,dx$.

Question. Compute $\int_{\pi/3}^{\pi/2} \sin(x) \sec^2(\cos(x))\, dx$.

Question. Compute $\displaystyle\int_{-2}^{1} t^2 \sin(t^3)\, dt$.

Question. Compute $\displaystyle\int_0^{1/2} \frac{13e^x}{3e^x - 5}\, dx.$