Lecture 36: First Fundamental Theorem of Calculus (FFTOC)

Tae Eun Kim, Ph.D.

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Accumulation Function

Accumulation functions

Definition

Given a function f, an **accumulation function** is

$$F(x) = \int_a^x f(t)\, dt$$

- It calculates the signed area of the region between $y = f(t)$ and t-axis over the interval $[a, x]$ where the location of right-endpoint is now a variable.

Example (Rectangles)

Let $F(x) = \displaystyle\int_{-3}^{x} 4\ dt$. What is $F(5)$? What is $F(-5)$? What is $F(x)$?

Example (Trapezoid)

Let $F(x) = \int_0^x (2t+1)\, dt$. Find $F(x)$.

Example (Monotonicity of accumulation function)

Let $F(x) = \int_{-1}^{x} t^3 \, dt$. On the interval $[-1, 1]$,

1. Where is F increasing/decreasing?
2. When does F have local extrema?
3. Answer the same questions with the interval replaced by $(-\infty, \infty)$.

The First Fundamental Theorem of Calculus

Motivation

Let f be a continuous function on the real numbers and consider

$$F(x) = \int_a^x f(t)\,dt\,.$$

We know that

- F is increasing when f is positive;
- F is decreasing when f is negative.

It is also clear that

- F is concave up when f' is positive;
- F is concave down when f' is negative.

There must be a deep connection between F' and f.

The First Fundamental Theorem of Calculus

Theorem (First Fundamental Theorem of Calculus, FTC1)

Suppose that f is continuous on the real numbers and let

$$F(x) = \int_a^x f(t)\,dt\,.$$

Then

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)\,.$$

Interpretation.

- An accumulation function of f is an antiderivative of f.
- The rate at which the accumulated area under a curve grows is precisely described by the curve itself.

The idea of proof. Assume $h > 0$. Note that $F(x+h) - F(x)$ is the net area of the region whose base is $[x, x+h]$ since

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\, dt\,.$$

For sufficiently small h, the region is approximately rectagular and so this region is approximately $f(x)h$, i.e.,

$$F(x+h) - F(x) \approx f(x)h\,.$$

Upon division by h, we obtain

$$\frac{F(x+h) - F(x)}{h} \approx f(x)\,,$$

which, in the limit as $h \to 0$, yields

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)\,,$$

as required. $\square$

Derivatives of composed accumulation functions

The following variation of the FTC1 is noteworthy:

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))g'(x)\,.$$

Question. Find the derivative of

1. $\displaystyle\int_2^{x^2} \ln t \, dt.$

2. $\displaystyle\int_{\cos x}^{5} t^3 \, dt.$

3. $\displaystyle\int_{x^2}^{x} f(t) \, dt.$