Lecture 1: Review of Precalculus

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Overview

1. Understanding Functions (UF)

2. Review of Famous Functions (ROFF)

Understanding Functions (UF)

“For each input, exactly ~~on~~ one output”

Definition

- **function**: a relation between sets where for each input, there is exactly one output → Vertical line test
- **domain**: the set of the inputs of a function
- **range**: the set of the outputs of a function

Tips

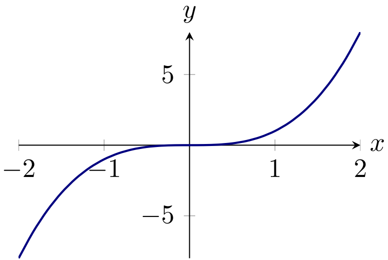
- domain : shadow of graph on horizontal ground
- range : Shadow of graph on vertical wall

Representation of functions

- **formula:** $f(x) = x^3$
- **table:**

input	1	-2	1.5	$\cdots$
output	1	-8	3.375	$\cdots$

- **graph:**



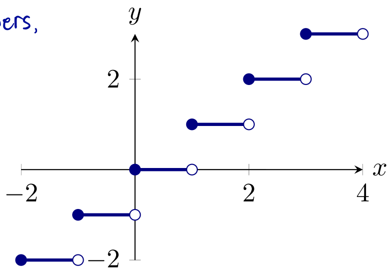
Example: Greatest Integer Function

- maps any real number x to the greatest integer less than or equal to x.
- a.k.a. *floor function*
- denoted by $\lfloor x \rfloor$
- many inputs to one output

domain : all real numbers,

$(-\infty, \infty)$

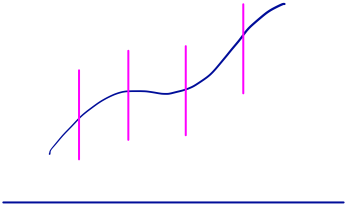
range : all integers

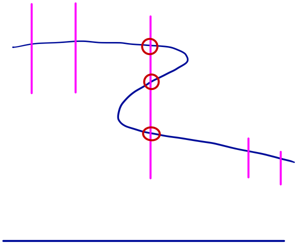


Theorem (Vertical line test)

*The curve $y = f(x)$ represents y as a function of x at $x = a$ if and only if the vertical line $x = a$ intersects the curve $y = f(x)$ at exactly one point. This is called the **vertical line test**.*

Recall: "for each input, exactly one output"





For this input,
three outputs!

Distinguishing two functions

- Do they have the same domain?
- Do they display the same relation?

Question. Determine if the two function are the same.

1. $f(x) = \sqrt{x^2}$ and $g(x) = |x|$

Both are defined on all real numbers, i.e., they share the same domain.
Moreover, since $\sqrt{x^2} = |x|$, they follow the same rule. So the two are the same.

2. $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$ and $g(x) = x - 1$

f is defined for all real numbers except for 2, i.e., $(-\infty, 2) \cup (2, \infty)$
while g is defined for the entire real numbers. So they are different.

Note: For $x \neq 2$, $f(x) = \dfrac{\cancel{(x-2)}(x-1)}{\cancel{x-2}} = x-1 = g(x)$.

Composition of functions

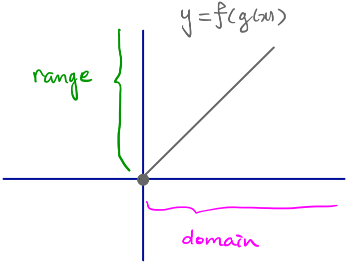
Composite functions

- can be thought of as putting one function inside another
- **Notation:** $(f \circ g)(x) = f(g(x))$
- **Warning:** The range of inner function must be contained in the domain of outer function.

Question. Study the composition $f \circ g$ where

$$f(x) = x^2 \qquad \text{for } -\infty < x < \infty,$$
$$g(x) = \sqrt{x} \qquad \text{for } 0 \le x < \infty.$$

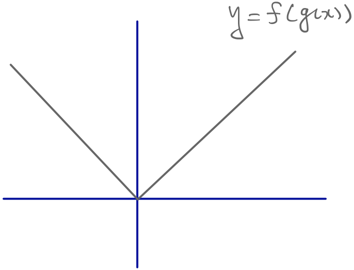
- the range of g is $[0, \infty)$, which is contained in the domain of f. So the domain of $f \circ g$ is also $[0, \infty)$.
- the range of $f \circ g$ is $[0, \infty)$
- $f \circ g\,(x) = f(g(x))$
  $= f(\sqrt{x})$
  $= \sqrt{x}^2 = x$.



Question. Study the composition $f \circ g$.

$$f(x) = \sqrt{x} \qquad \text{for } 0 \le x < \infty,$$
$$g(x) = x^2 \qquad \text{for } -\infty < x < \infty.$$

- The range of g is $[0, \infty)$, which coincides with the domain of f. So the domain of $f \circ g$ is the entire domain of g, that is, $(-\infty, \infty)$
- The range of $f \circ g$ is $[0, \infty)$
- $f \circ g(x) = f(g(x))$
  $= f(x^2)$
  $= \sqrt{x^2} = |x|$



Inverses of functions

Definition

Let f be a function with domain A and range B:

$$f : A \to B$$

Let g be a function with domain B and range A:

$$g : B \to A$$

We say that f and g are **inverses** of each other if $f(g(b)) = b$ for all b in B, and also $g(f(a)) = a$ for all a in A. Sometimes we write $g = f^{-1}$ in this case.

We could rephrase these conditions as

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

Warning: notations

Pay attention to where we put the superscript:

$$f^{-1}(x) = \text{the inverse function of } f(x).$$
$$f(x)^{-1} = \text{the reciprocal of } f(x).$$

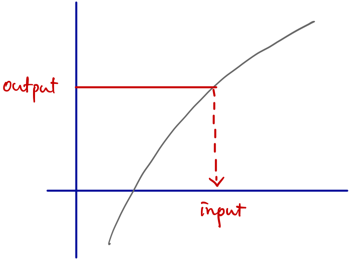
$$\frac{1}{f(x)}$$

$$\boxed{f^{-1}(x) \neq \frac{1}{f(x)} \quad !!!}$$

Definition

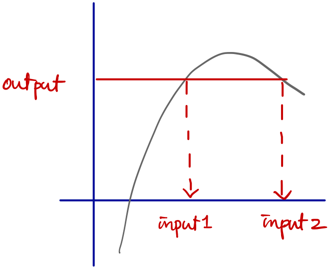
A function is called **one-to-one** if each output value corresponds to exactly one input value.

One-to-one



For each output, exactly one input!

NOT one-to-one



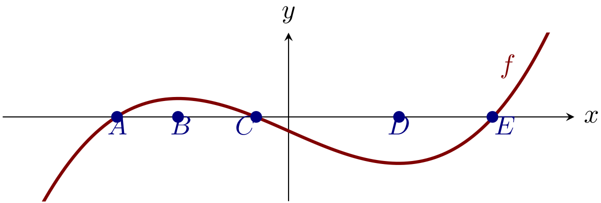
Two distinct inputs giving rise to the same output.

Theorem (Horizontal line test)

*A function is one-to-one at $x = a$ if the horizontal line $y = f(a)$ intersects the curve $y = f(x)$ in exactly one point. This is called the **horizontal line test**.*

See illustrations on the previous slide.

Question. Consider the graph of the function f below:

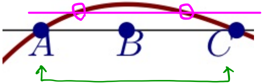


On which of the following intervals is f one-to-one?

1. $[A, B]$: Yes
2. $[A, C]$: No
3. $[B, D]$: Yes
4. $[C, E]$: No
5. $[C, D]$: Yes

Why not?

Horiz. Line test fails!



[A, C]

Ans. : 1, 3, 5

Review of Famous Functions (ROFF)

These are important functions for Math 1151:

- polynomial functions
- rational functions
- trigonometric functions and their inverses
- exponential and logarithmic functions

Calculus 1

↳ branch of maths which studies the nature of functions

Polynomial functions

$a_j x^{j} \rightarrow$ non-negative integers.

Definition

A **polynomial function** in the variable x is a function which can be written in the form

x^1 $\qquad x^0 = 1$

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are all constants (called the **coefficients**) and n is a whole number (called the **degree** when $n \neq 0$). The domain of a polynomial function is $(-\infty, \infty)$.

Yes

- $f(x) = 5$ (constant)
- $f(x) = 2x - 1$ (linear)
- $f(x) = x^2 + 2x - 5$ (quadratic)

No

not an integer

- $f(x) = \underbrace{x^{1/2}}_{\text{square root}} + 3x$
- $f(x) = x^{-2} + x^{-1} - \pi^7$

$\frac{1}{x}$

$\frac{1}{x^2}$

negative integers

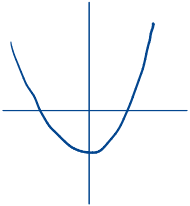
Question. Which of the following are polynomial functions?

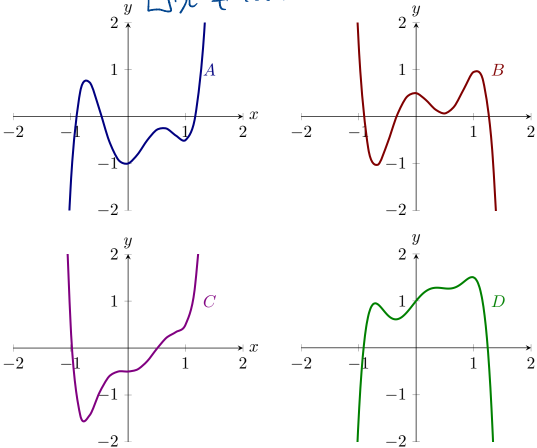
1. $f(x) = 7$ Yes (constant function)
2. $f(x) = 3x + 1$ Yes (linear function)
3. $f(x) = x^{1/2} - x + 8$ No, because of $x^{1/2}$. (fractional power)
4. $f(x) = x^{-4} - 3x^{-2} + 7 + 12x^3$ No, because of x^{-4} and x^{-2}. (neg. powers)
5. $f(x) = (x+\pi)(x-\pi) + \cancel{e^x} - \cancel{e^x}$ $= x^2 - \pi^2$, so Yes.
6. $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$ No, not in the form (☺).
7. $f(x) = x^7 - 32x^6 - \pi x^3 + 3/7$ Yes.

Some possible graphs of polynomials.

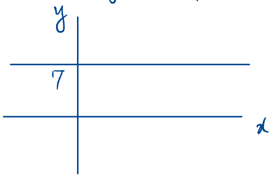
$\square x^5 + \cdots$

quadratic

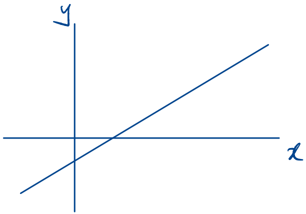




Const. $f(x) = 7$



linear $f(x) = 2x - 1$



Definition

A **rational function** in the variable x is a function of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions. The domain of a rational function is all real numbers except for where the denominator is equal to zero.

frac'l power

✓ $x^{-2} = \frac{1}{x^2}$ (rational function)

Yes $f(x) = \frac{x - \frac{1}{2}}{2x^2 - 3x + 4}$ linear quad.

No $f(x) = \frac{(x+1)^{1/3} - e}{(x+1)^3 + e}$

$f(x) = \frac{\sin(x)}{\ln(x)}$

Note: All polynomials are rational functions since $p(x) = \frac{p(x)}{1}$.

Question. Which of the following are rational functions?

1. $f(x) = 0$ Yes. See note on the previous page.

2. $f(x) = \dfrac{3x+1}{x^2-4x+5}$ Yes.

3. $f(x) = e^x$ No.

4. $f(x) = \dfrac{\sin(x)}{\cos(x)}$ No. Though it assume a fractional structure, numer. and denom. are not polyom.

5. $f(x) = -4x^{-3} + 5x^{-1} + 7 - 18x^2$ Yes, upon combining terms using common denominator.

6. $f(x) = x^{1/2} - x + 8$ No, because of $x^{1/2}$.

7. $f(x) = \dfrac{\sqrt{x}}{x^3 - x}$ No, because of $\sqrt{x}$.

Q. Why is 5 a rational function?

5 $f(x) = -4x^{-3} + 5x^{-1} + 7 - 18x^2$

$$= -\frac{4}{x^3} + \frac{5}{x} + 7 - 18x^2$$

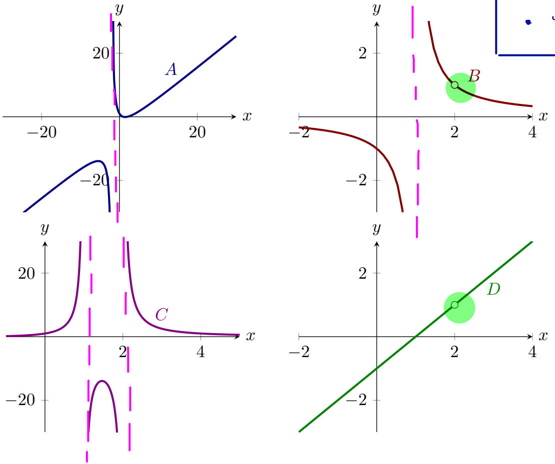
$$= \frac{-4 + 5x^2 + 7x^3 - 18x^5}{x^3}$$

polynom.

polynom.

Yes, it is a rational function!

Some possible graphs of rational functions.

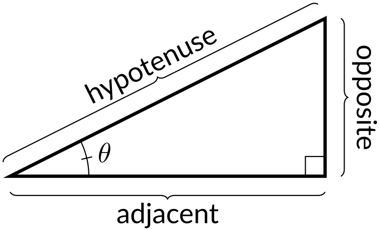


Notable features

- asymptotes
- "holes" on graphs

Trigonometric functions

A **trigonometric function** is a function that relates a measure of an angle of a right triangle to a ratio of the triangle's sides.



In America, "SOH CAH TOA" is a famous mnemonic for trig. defn.

Definition

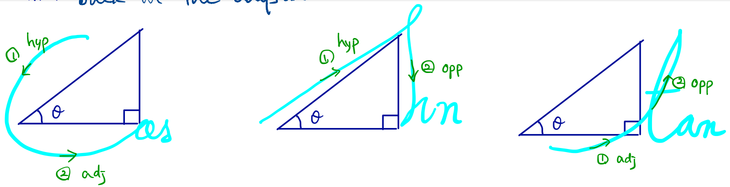
The trigonometric functions are:

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} \qquad \sin(\theta) = \frac{\text{opp}}{\text{hyp}} \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

where the domain of sine and cosine is all real numbers, and the other are defined precisely when their denominators are nonzero.

How I learned back in the days...

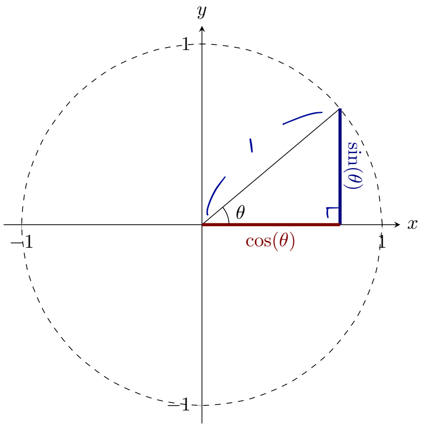


Cultural Note

In many asian languages, denom. comes before numer.,

e.g. "오분의 삼" or "五分之三", which reads "out of 5, 3".

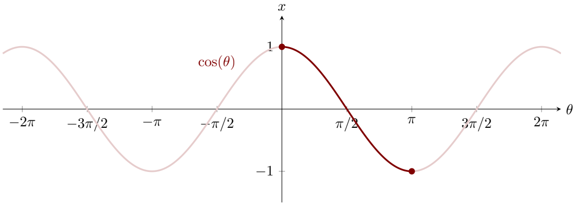
The unit circle and trig functions

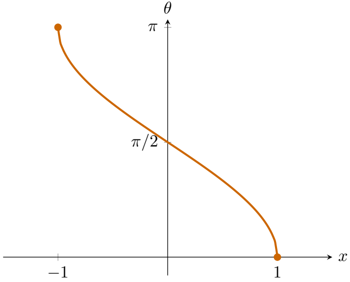


Applying Pythagean theorem, we obtain

$$1^2 = \cos^2\theta + \sin^2\theta$$

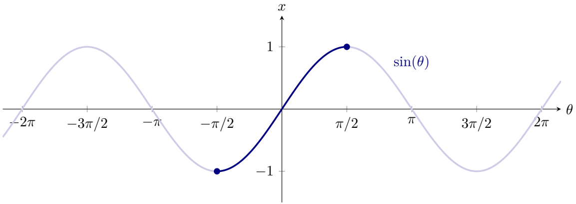
Cosine and its inverse.

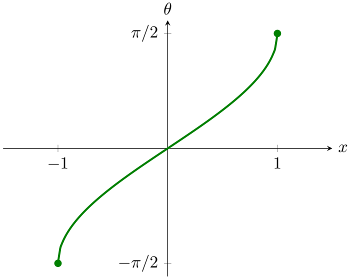




Here we see a plot of $\arccos(x)$, the inverse function of $\cos(\theta)$ when the domain is restricted to the interval $[0, \pi]$.

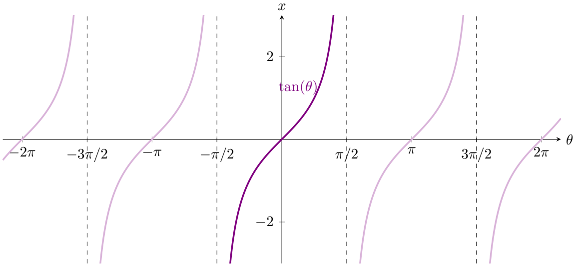
Sine and its inverse.

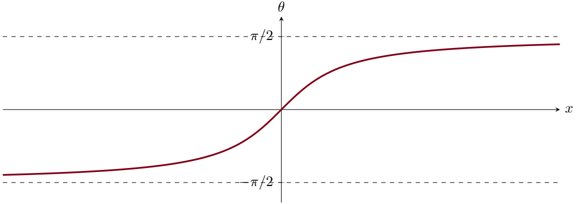




Here we see a plot of $\arcsin(x)$, the inverse function of $\sin(\theta)$ when the domain is restricted to the interval $[-\pi/2, \pi/2]$.

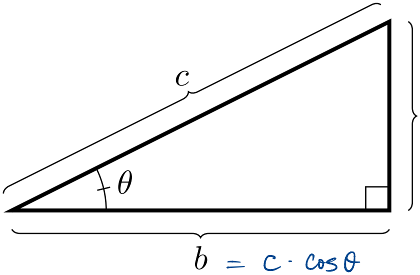
Tangent and its inverse.





Here we see a plot of $\arctan(x)$, the inverse function of $\tan(\theta)$ when the domain is restricted to the interval $(-\pi/2, \pi/2)$.

Pythagorean theorem and identities.



$a = c \cdot \sin\theta$

$b = c \cdot \cos\theta$

Pythagorean theorem:

- $a^2 + b^2 = c^2$

Using $a = c\sin\theta$, $b = c\cos\theta$,

$$a^2 + b^2 = c^2\sin^2\theta + c^2\cos^2\theta$$

$$= c^2(\underbrace{\sin^2\theta + \cos^2\theta}_{=1}) = c^2$$

Pythagorean identities:

- $\cos^2\theta + \sin^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $\cot^2\theta + 1 = \csc^2\theta$

$\div \cos^2\theta$

$\div \sin^2\theta$

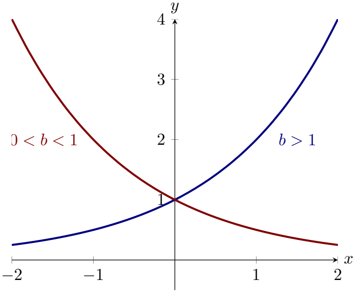
Exponential and logarithmic functions

Definition

An **exponential function** is a function of the form

$$f(x) = b^x$$

where $b \neq 1$ is a positive real number. The domain of an exponential function is $(-\infty, \infty)$. (Special: $f(x) = e^x$.)

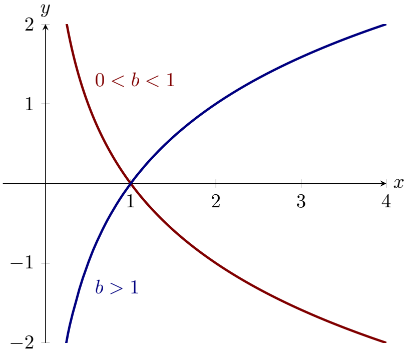


Definition

A **logarithmic function** is a function defined as follows

$$\log_b(x) = y \qquad \text{means that} \qquad b^y = x$$

where $b \neq 1$ is a positive real number. The domain of a logarithmic function is $(0, \infty)$. (Special: $f(x) = \ln(x)$.)



Properties of exponents

Let b be a positive real number with $b \neq 1$.

- $b^m \cdot b^n = b^{m+n}$
- $b^{-1} = \dfrac{1}{b}$
- $(b^m)^n = b^{mn}$

Properties of logarithms

Let b be a positive real number with $b \neq 1$.

- $\log_b(m \cdot n) = \log_b(m) + \log_b(n)$
- $\log_b(m^n) = n \cdot \log_b(m)$
- $\log_b(1/m) = \log_b(m^{-1}) = -\log_b(m)$
- $\log_a(m) = \dfrac{\log_b(m)}{\log_b(a)}$