Lecture 8: Definition of the Derivative (DOTD)

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Autumn 2021

Midterm 1 (Gradescope)

- Monday, 09/13/2021
- $\boxed{\text{Timeline}}$
  - 7:55 PM : download exam
  - 8:00 PM : start working on it
  - 8:40 PM : upload exam
  - 8:55 PM : done

- Re-read all relevant announcements
- Old exams and solns posted
- Tips & review problems in WK3 module

Tae's

- Extended Office hours : M 4:15 ~ 6:15 usual zoom link
- ~~Short recordings over the weekend~~ → Review video avail. on Carmen
- Schedule conflict? Contact me.
  (class or work ; written documents)
- Be sure to contact me using

  $\boxed{\text{kim.3562@osu.edu}}$

Rates of change (instantaneous)

Friday.

The rate of change of

physics
- **position** of an object in time: **velocity**
- **velocity** of an object in time: **acceleration**

business
- **revenue** generated by selling objects: **marginal revenue**
- **cost** to produce objects: **marginal cost**
- **profit** gained by selling objects: **marginal profit**

$$\text{avg. vel.} = \frac{\text{change in position}}{\text{change in time.}}$$

$$\downarrow$$

$$\text{(instant) vel.} = \lim_{(\text{change in time}) \to 0} \left[\text{avg. vel.}\right]$$

From slopes of secant lines ...

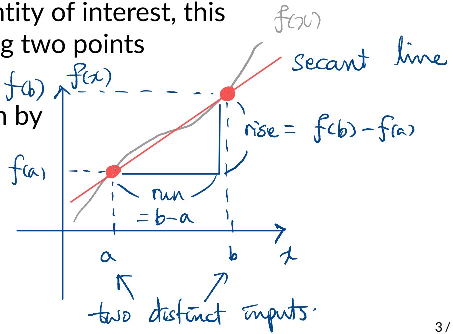
avg. vel. form.

The general formula for average rate of change is given by

$$\frac{\text{change in the function}}{\text{change in the input to the function}}.$$

- In order to produce this rate of change, we need two distinct input values, e.g., two distinct points in time, and their corresponding outputs.
- On the graph of the function f representing the quantity of interest, this rate is exactly the slope of the straight line connecting two points $(a, f(a))$ and $(b, f(b))$.
- Such a line is called a **secant line** whose slope is given by

$$\boxed{m_{\text{sec}} = \frac{f(b) - f(a)}{b - a}.}$$



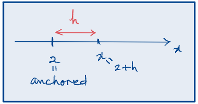
Question. If $f(x) = 2x^2 + 3$, find the slope of the secant line through $(2, f(2))$ and $(x, f(x))$ in terms of x. Do the same when x is expressed as $2 + h$. The answer must be written in terms of h.

b

a

$f(2) = 2\cdot 2^2+3$
$= 8+3 = 11$

$$m_{sec} = \frac{f(x) - f(2)}{x-2}$$
$$= \frac{(2x^2+3) - 11}{x-2}$$
$$= \frac{2x^2-8}{x-2}$$
$$= \frac{2(x^2-4)}{x-2}$$
$$= \frac{2\cancel{(x-2)}(x+2)}{\cancel{x-2}} \quad \text{Since } x \neq 2.$$
$$= \boxed{2(x+2)}$$



$$m_{sec} = \frac{f(2+h) - f(2)}{(2+h)-2} \quad = h$$
$$= \frac{[2(2+h)^2+3] - 11}{h}$$
$$= \frac{2(2+h)^2-8}{h}$$
$$= \frac{2(h^2+4h+4)-8}{h}$$
$$= \frac{2h^{\cancel{2}}+8\cancel{h}}{\cancel{h}}$$

- As a function, its domain is all reals except $h=0$.
- For $h \neq 0$, same as $2h+8$.

since $h \neq 0$

$$= 2h + 8 = \boxed{2(h+4)}$$

$\longrightarrow$ if x is replaced by $2+h$ $\longrightarrow$ $2(2+h+2) = 2(h+4)$

…to slopes of tangent lines

Now, an important question is: how do we get an instantaneous rate of change out of this?

- The slope of so-called **tangent line** represents this rate.
- It is given in terms of limit of slope of secant lines [1]:

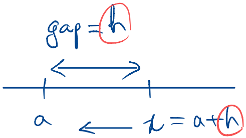
$x = a+h$

$$m_{\tan} = \lim_{x\to a} \underbrace{\frac{f(x)-f(a)}{x-a}}_{\substack{m_{sec} \\ \text{(slope of secant line)}}} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

- $x \; (= a+h) \to a$

is equivalent to

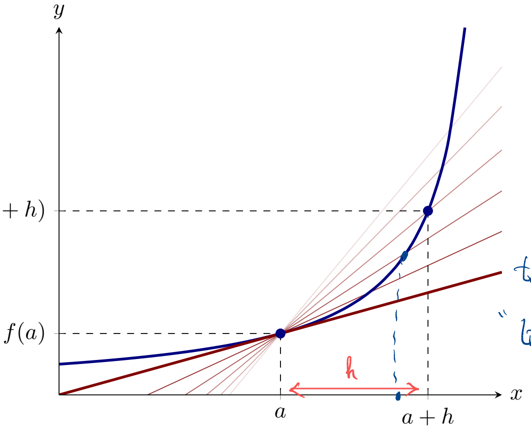
- $h \to 0$



[1] **Note.** We have two equivalent characterizations of this instantaneous rate of change depending on how we solved the previous problem.

Illustration

$f(x) =$



$= x$

tangent line

"line tangent to $y = f(x)$
at $x = a$"

the curve and the tangent line
share : ① function value ② slope.

Definition of derivative

Definition

The **derivative** of f at a is

$$\begin{aligned}\left[\frac{d}{dx}f(x)\right]_{x=a} &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \qquad (h\to 0 \text{ characterization}) \\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \qquad (x\to a \text{ characterization}).\end{aligned}$$

If this limit exists, then we say that f is **differentiable** at a. If this limit does not exist for a given value of a, then f is **non-differentiable** at a.

Notation

The following are equivalent notations for derivatives:

$$\left[\frac{d}{dx}f(x)\right]_{x=a} = f'(a).$$

Example

If $f(x) = x^2 - 2x$, find the derivative of f at 2 using the $h \to 0$ characterization.

$$\left[\frac{d}{dx} f(x)\right]_{x=2} = f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

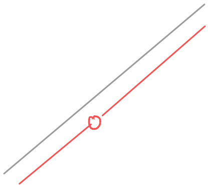
a

$$A^2 - 2A = A(A-2)$$

$$= \lim_{h \to 0} \frac{\left[(2+h)^2 - 2(2+h)\right] - \cancel{\left[2^2 - 2\cdot 2\right]}}{h} \rightarrow \text{“}\frac{0}{0}\text{”}$$

$$= \lim_{h \to 0} \frac{(2+h)(\cancel{2}+h-\cancel{2})}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2+h)}{\cancel{h}} = \lim_{h \to 0} (2+h) = \boxed{2}$$



Example

Find the derivative of $f(x) = x^2 + x + 1$ at $x = -1$ using the $x \to a$ characterization.

$$\left[\frac{d}{dx} f(x)\right]_{x=-1} = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)}$$

or

$$f'(-1) = \lim_{x \to -1} \frac{(x^2+x+1) - \underbrace{((-1)^2+(-1)+1)}_{\text{"1"}}}{x+1}$$

$$= \lim_{x \to -1} \frac{x^2 + x}{x+1} \qquad \left[\text{"}\frac{0}{0}\text{" form}\right]$$

$$= \lim_{x \to -1} \frac{x\cancel{(x+1)}}{\cancel{x+1}} = \lim_{x \to -1} x = \boxed{-1}$$

Example

Find an equation for the line tangent to the curve $y = f(x) = 1/(3-x)$ at the point $(2, 1)$.

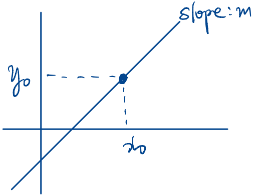
Point-slope formula

- slope : m
- point: (x_0, y_0)

$$\left.\begin{array}{l}\text{slope} : m \\ \text{point} : (x_0, y_0)\end{array}\right\} \longrightarrow y - y_0 = m(x - x_0)$$

or

$$y = m(x - x_0) + y_0$$



- We already have $(x_0, y_0) = (2, 1)$.
- Need the slope at $x = 2$.

$$f(2) = \frac{1}{3-2} = 1$$

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2} \frac{\frac{1}{3-x} - 1}{x - 2}$$

$$= \lim_{x \to 2} \frac{\frac{1 - (3-x)}{3-x}}{x-2}$$

$$= \lim_{x \to 2} \frac{\cancel{x-2}}{(3-x)\cancel{(x-2)}} = 1 = m$$

- Using Pt.-Sl. formula:

$$y = (x-2) + 1 \longrightarrow \boxed{y = x - 1}$$

Example

The position of an object moving along a straight line is given by $s(t) = \sqrt{t+3}$. Find its (instantaneous) velocity at time $t = 6$.

$$v(6) = \left[\frac{d}{dt} s(t)\right]_{t=6} \quad \text{or} \quad s'(6)$$

$$= \lim_{h \to 0} \frac{s(6+h) - s(6)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(6+h)+3} - \sqrt{6+3}}{h}$$

$\sqrt{9} = 3$

$$= \lim_{h \to 0} \frac{\sqrt{h+9} - 3}{h}$$

["$\frac{0}{0}$" form]

Conjugate

Conjugate

$$= \lim_{h \to 0} \frac{\sqrt{h+9} - 3}{h} \cdot \frac{\sqrt{h+9} + 3}{\sqrt{h+9} + 3}$$

$$= \lim_{h \to 0} \frac{(h+9) - 9}{h\left(\sqrt{h+9} + 3\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{h+9} + 3\right)}$$

$$= \frac{1}{\sqrt{9} + 3} = \boxed{\frac{1}{6}}$$