Lecture 3: Limit Laws (LL)

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Announcement

- Homework quizzes: Carmen, not Ximera
  - timed: 10 minutes
  - only one attempt.

The limit laws

- Recall the definition of **continuity**: f is continuous at a if

$$\lim_{x \to a} f(x) = f(a)\,.$$

- In other words, continuity of a function allows us to calculate its limits simply by function evaluation.
- In addition, we learned that many famous functions are continuous on their natural domains.
- Today, using limit laws, we can expand the library of continuous functions even further.

Theorem (Limit laws)

Suppose that $\lim_{x \to a} f(x) = L$ *and* $\lim_{x \to a} g(x) = M$*, i.e., these limits exist.*

- ***Sum/Difference Law***: $\lim_{x \to a} (f(x) \pm g(x)) = L \pm M$. $= \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
- ***Product Law***: $\lim_{x \to a} (f(x)g(x)) = LM$.
- ***Quotient Law***: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, *provided that* $M \neq 0$.

Remark

Using these laws, we can show that polynomial and rational functions are also continuous on their natural domains.

Question. Compute the following limit using limit laws: (Polynomial)

$$\lim_{x\to 2}(5x^2+3x-2)$$

$$= \lim_{x\to 2} 5x^2 + \lim_{x\to 2} 3x - \lim_{x\to 2} 2 \qquad \text{[Sum / Diff.]}$$

$$= \underbrace{\left(\lim_{x\to 2} 5\right)}_{= 5}\underbrace{\left(\lim_{x\to 2} x^2\right)}_{= 2^2 = 4} + \underbrace{\left(\lim_{x\to 2} 3\right)}_{= 3}\underbrace{\left(\lim_{x\to 2} x\right)}_{= 2} - \underbrace{\lim_{x\to 2} 2}_{= 2} \qquad \text{[Prod.]}$$

: cont. of const. fnc

: cont. of power fnc.

(identity included)

$$= 5\cdot 4 + 3\cdot 2 - 2 = \boxed{24}$$

Question. Compute the following limit using limit laws: (rational function)

$$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 2}$$

Where is $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$ continuous?

Soln

$$= \frac{\lim_{x\to 1} (x^2 - 3x + 2)}{\lim_{x \to 1} (x-2)} \qquad [\text{Quot.}]$$

$$= \frac{\lim_{x\to 1} x^2 - \left(\lim_{x\to 1} 3\right)\left(\lim_{x\to 1} x\right) + \lim_{x \to 1} 2}{\lim_{x\to 1} x - \lim_{x\to 1} 2} \qquad [\text{Sum/Diff./Prod}]$$

$$= \frac{1^2 - 3\cdot 1 + 2}{1 - 2} = \boxed{0} \qquad [\text{continuity}]$$

Composition of fnc. : $f \circ g(x) = f(g(x))$

Theorem (Composition limit law)

If $f(x)$ is continuous at $b = \lim_{x\to a} g(x)$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) .$$

Consequently, if g is continuous at $x = a$, and if f is continuous at $g(a)$, then $f \circ g$ is continuous at $x = a$.

$$\lim_{x\to a} f(g(x))$$
$$= f(\lim_{x\to a} g(x))$$
$$= f(g(a))$$

Question. Compute the following limit using limit laws:

$$\lim_{x\to 0} \sqrt{\cos(x)}$$

$$\left.\begin{aligned} f(x) &= \sqrt{x} \\ g(x) &= \cos(x) \end{aligned}\right\} \Rightarrow f(g(x)) = \sqrt{\cos x}$$

Note $\lim_{x\to 0} g(x) = \lim_{x\to 0} \cos(x) = \cos(0) = 1$. Since $\sqrt{x}$ is cts at 1, $\lim_{x\to 0} \sqrt{\cos x} = \sqrt{\lim_{x\to 0} \cos x} = \sqrt{1}$

Question. Determine if the following limits can be directly computed using limit laws.

1. $\lim_{x \to 2} \dfrac{x^2 - 3x + 2}{x - 2}$ No, since $\lim_{x \to 2} (x-2) = 0$.

2. $\lim_{x \to 0} \dfrac{2^x - 1}{3^{x-1}} = \dfrac{2^0 - 1}{3^{0-1}} = 0$

3. $\lim_{x \to 0} x \sin(1/x)$ No, $\lim_{x \to 0} \frac{1}{x}$ cannot be determined by a limit law.

4. $\lim_{x \to 0} \cot(x^3)$ No, since cot is not continuous at 0.

5. $\lim_{x \to 0} (1 + x)^{1/x}$ None of the limit laws is relevant.

The Squeeze Theorem

Theorem (The Squeeze Theorem)

Suppose that

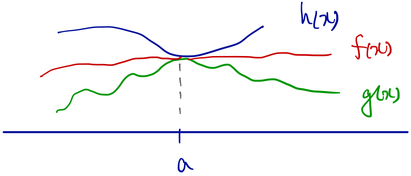
$$g(x) \le f(x) \le h(x)$$

for all x close to a but not necessarily equal to a. If

$$\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)\,,$$

then $\lim_{x\to a} f(x) = L$.

- This theorem is often called the **sandwich theorem.**



Question. Suppose we have a function $f(x)$ defined for all x in the open interval $(-2, 2)$ and all I know about f is that

$$\underbrace{0}_{= g(x)} \leq f(x) \leq \underbrace{x^2}_{= h(x)},$$

in the interval. Can I say anything about $\lim_{x\to 0} f(x)$ with this limited knowledge?

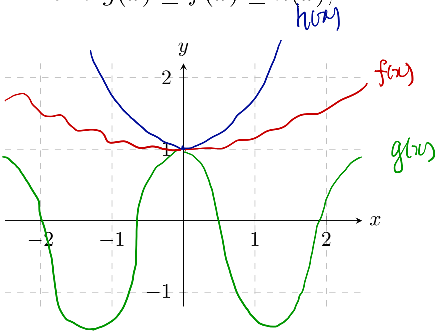
Soln

Since $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} h(x) = \lim_{x\to 0} x^2 = 0$,

by Squeeze Theorem, $\lim_{x\to 0} f(x) = 0$.

Question. Consider the three functions, g, f, and h, defined on the interval $(-2, 2)$. Given that

$$g(x) = \cos(\pi x), \quad h(x) = x^2 + 1 \quad \text{and } g(x) \le f(x) \le h(x),$$



1. Sketch and label the graph of g and h, and a possible graph of f.
2. Use the Squeeze Theorem to evaluate $\lim_{x \to 0} f(x)$.

Since

$\lim_{x \to 0} g(x) = \cos(\pi \cdot 0) = 1$

and

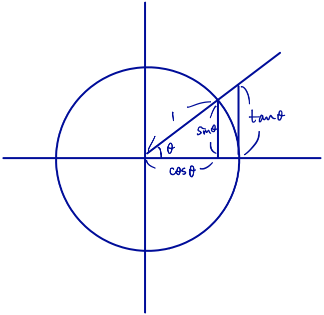
$\lim_{x \to 0} h(x) = 0^2 + 1 = 1$

by Squeeze theorem,

$\lim_{x \to 0} f(x) = 1$.

Question. Compute $\lim_{\theta \to 0} \dfrac{\sin(\theta)}{\theta}$.

- The answer is 1!
- Please read the textbook for a detailed solution.
- Later in the course, we will learn an alternate method to calculate this limit.



Sketch of ideas: For $\theta > 0$

$$\cancel{\frac{1}{2}}\sin\theta\cos\theta \le \cancel{\frac{1}{2}}\theta \le \cancel{\frac{1}{2}}\tan\theta$$

recip.

$$\frac{\cos\theta}{\sin\theta} \le \frac{1}{\theta} \le \frac{1}{\sin\theta\cos\theta}$$

$\times \sin\theta$

$$\cos\theta \le \frac{\sin\theta}{\theta} \le \frac{1}{\cos\theta}$$

$$\lim_{\theta\to 0^+}\cos\theta = \lim_{\theta\to 0^+}\frac{1}{\cos\theta} = 1 \xrightarrow[\text{Sqz. Thm.}]{} \boxed{\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1}$$