Lecture 4: (In)determinate Forms (IF)

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Limits of the form zero over zero – indeterminate form

Definition

A limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is said to be of the form $\dfrac{\mathbf{0}}{\mathbf{0}}$ if

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0\,.$$

- **Warning!** The symbol $\frac{\mathbf{0}}{\mathbf{0}}$ is NOT the number 0 divided by 0.
- A key trick to handle limits in $\frac{\mathbf{0}}{\mathbf{0}}$ form is to cancel out vanishing factors.

Question. Compute the following limits:

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$$

Question. Compute the following limits:

$$\lim_{x\to 1} \frac{\frac{1}{x+1} - \frac{3}{x+5}}{x-1}$$

Question. Compute the following limits:

$$\lim_{x \to -1} \frac{\sqrt{x+5}-2}{x+1}$$

Remark

- Limits of the form $\frac{0}{0}$ can take any value!
- Having this particular form does not give us enough information to determine whether a function has a limit or not;
- Even if the limit exists, the value of the limit is not apparent without further manipulation.
- That is why such a limit is said to be in an **indeterminate form**.

Limits of the form nonzero over zero – determinate form

Definition

A limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

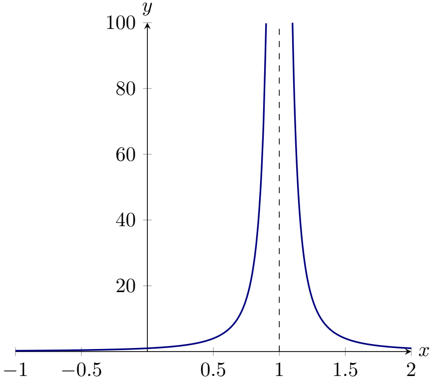
is said to be of the form $\frac{\#}{\mathbf{0}}$ if

$$\lim_{x \to a} f(x) = k \quad \text{and} \quad \lim_{x \to a} g(x) = 0\,,$$

where k is some nonzero constant.

- When a fixed nonzero number is divided by a small number, the quotient is generally large.
- As the denominator get smaller and smaller, the quotient gets larger and larger.

Illustration. The following graph of $f(x) = 1/(x-1)^2$ near $x = 1$ displays the behavior of limits of the form $\frac{\#}{\mathbf{0}}$.



Definition

- If $f(x)$ grows arbitrarily large for all x sufficiently close, but not equal, to a, we write
$$\lim_{x \to a} f(x) = \infty$$
and say that the limit of $f(x)$ as x approaches a **is infinity**.
- If $f(x) < 0$ and $|f(x)|$ grows arbitrarily large for all x sufficiently close, but not equal, to a, we write
$$\lim_{x \to a} f(x) = -\infty$$
and say that the limit of $f(x)$ as x approaches a **is negative infinity**.

Note. We can analogously define one-sided infinite limits, e.g.,

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty\,.$$

Question. Compute

$$\lim_{x\to 0} \frac{e^x}{1-\cos(x)}\,.$$

Question. Compute

$$\lim_{x\to 3} \frac{x^2 - 9x + 14}{x^2 - 5x + 6} \, .$$