Lecture 36: First Fundamental Theorem of Calculus (FFTOC)

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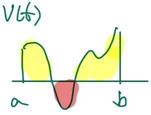
From last time: $S(t)$ position, $V(t) = S'(t)$ velocity

Geometry

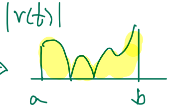
(displacement) $= \int_a^b v(t)\,dt = \underbrace{S(b)}_{\text{terminal position}} - \underbrace{S(a)}_{\text{initial position.}}$ $\longrightarrow$



net area

(can be negative)

(distance) $= \int_a^b |v(t)|\,dt$ $\longrightarrow$



geometric area

(always non-negative)

Note

FTC (part 2)

antiderivative

$$\boxed{\int_a^b S'(t)\,dt = S(b) - S(a)}$$

Accumulation Function

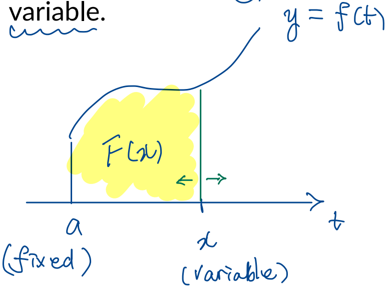
Accumulation functions

Definition

Given a function f, an **accumulation function** is

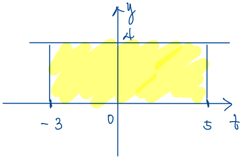
$$F(x) = \int_a^x f(t)\, dt$$

- It calculates the accumulated signed area of the region between $y = f(t)$ and t-axis over the interval $[a, x]$ where the location of right-endpoint is now a variable.



Example (Rectangles)

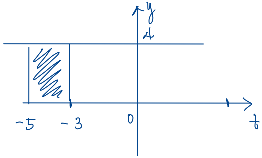
Let $F(x) = \displaystyle\int_{-3}^{x} 4\,dt$. What is $F(5)$? What is $F(-5)$? What is $F(x)$?



(a) $F(5) = \text{Area}\left(\underbrace{\boxed{\quad\quad}}_{8} \}4 \right)$

$= (5-(-3)) \cdot 4$

$= 8 \cdot 4 = \boxed{32}$



(b) $F(-5) = \displaystyle\int_{-3}^{-5} 4\,dt = -\int_{-5}^{-3} 4\,dt$

$= -\text{Area}\left(\underbrace{\boxed{\,}}_{2} \}4 \right) = -2 \cdot 4$

$= \boxed{-8}$

$-(-3-(-5)) \cdot 4$

$= (-5-(-3)) \cdot 4$

$-(a-b) = b-a$

(c) $F(x) = (x-(-3)) \cdot 4$

$= \boxed{4(x+3)}$

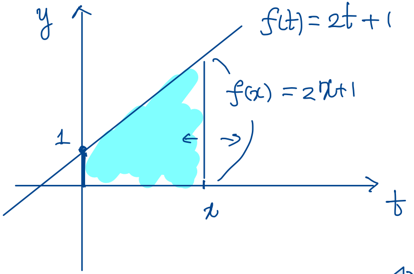
Alternately

Case 1 : $x \geq -3$

Case 2 : $x < -3$

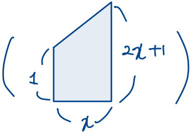
Example (Trapezoid)

Let $F(x) = \int_0^x (2t+1)\, dt$. Find $F(x)$. for $x \geq 0$



Do this for $x < 0$.
See if it yields the same answer.

$F(x) = $ Area



$= \frac{1}{2}(2x+1+1)\,x$

$= \frac{1}{2}(2x+2)\,x = (x+1)\,x = \boxed{x^2+x}$

Skip

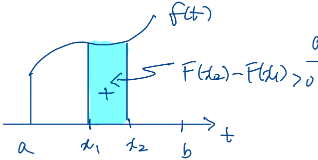
Example (Monotonicity of accumulation function)

Let $F(x) = \displaystyle\int_{-1}^{x} t^3 \, dt$. On the interval $[-1, 1]$,

1. Where is F increasing/decreasing?
2. When does F have local extrema?
3. Answer the same questions with the interval replaced by $(-\infty, \infty)$.

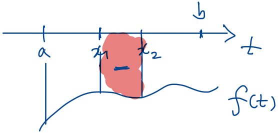
Note Consider the following scenario

Case 1 $f(t) > 0$ on $[a, b]$



$a < x_1 < x_2 < b$

Case 2 $f(t) < 0$ on $[a, b]$



Note: $F(x_2) - F(x_1) > 0$

$\Rightarrow F(x_1) < F(x_2)$

$\Rightarrow F(x)$ is INC on (a, b)

Note: $F(x_2) - F(x_1) < 0$

$\Rightarrow F(x_1) > F(x_2)$

$\Rightarrow F(x)$ is DEC on (a, b)

$$\left\{ \begin{array}{l} \bullet\ f > 0 \Rightarrow F \text{ is INC} \\ \bullet\ f < 0 \Rightarrow F \text{ is DEC} \end{array} \right\} \xrightarrow{\text{reminescent of}} \boxed{\begin{array}{l} F' > 0 \Rightarrow F \text{ INC} \\ F' < 0 \Rightarrow F \text{ DEC} \end{array}}$$

The First Fundamental Theorem of Calculus

Motivation

Let f be a continuous function on the real numbers and consider

$$F(x) = \int_a^x f(t)\,dt\,.$$

We know that

- F is increasing when f is positive;
- F is decreasing when f is negative.

It is also clear that

- F is concave up when f' is positive;
- F is concave down when f' is negative.

There must be a deep connection between F' and f.

The First Fundamental Theorem of Calculus

Theorem (First Fundamental Theorem of Calculus, FTC1)

Suppose that f is continuous on the real numbers and let

$$F(x) = \int_a^x f(t)\, dt\,.$$ (accumulation function)

Then

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\, dt = f(x)\,.$$

$$\underbrace{\cancel{\frac{d}{dx}}}_{\text{diff.}} \underbrace{\underbrace{\cancel{\int_a^x}}_{} \underbrace{f(t)}_{\text{input}} \underbrace{\cancel{dt}}_{}}_{\text{acc. (integ.)}} = \underbrace{f(x)}_{\text{same as input}}$$

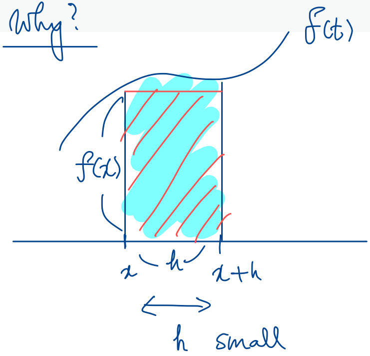
⇒ integ. followed by diff. does nothing.

Interpretation.

- An accumulation function of f (F) is an antiderivative of f.
- The rate at which the accumulated area under a curve grows is precisely described by the curve itself.

$F'(x) = f(x)$

Why?



$$\frac{F(x+h) - F(x)}{h} \approx f(x)$$

$$\downarrow \text{ as } h \to 0$$

$$F'(x)$$

$F(x+h) - F(x) = \text{Area}\Big($  $\Big)$

$\approx \text{Area}\Big($  $\Big) = f(x)\,h$

The idea of proof. Assume $h > 0$. Note that $F(x+h) - F(x)$ is the net area of the region whose base is $[x, x+h]$ since

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\, dt\,.$$

For sufficiently small h, the region is approximately rectagular and so this region is approximately $f(x)h$, i.e.,

$$F(x+h) - F(x) \approx f(x)h\,.$$

Upon division by h, we obtain

$$\frac{F(x+h) - F(x)}{h} \approx f(x)\,,$$

which, in the limit as $h \to 0$, yields

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x)\,,$$

as required. $\square$

Derivatives of composed accumulation functions

The following variation of the FTC1 is noteworthy:

$$\frac{d}{dx}\underline{\int_a^{g(x)} f(t)\,dt} = f(g(x))g'(x)\,. \longleftarrow \text{FTC 1 + CR}$$

$$F(x) = \int_a^{x} f(t)\,dt$$

$$F(g(x)) = \int_a^{g(x)} f(t)\,dt$$

$$\frac{d}{dx} F(g(x)) \overset{CR}{=} F'(g(x)) \cdot g'(x)$$

$$\overset{FTC1}{=} f(g(x))\, g'(x).$$

Question. Find the derivative of

1. $\int_2^{x^2} \ln t \, dt.$ $\xleftarrow{} g(x)$, $f(t)$ $\longrightarrow$ $\frac{d}{dx}\int_2^{x^2} \ln t \, dt = \ln(\underbrace{x^2}_{g(x)}) \cdot \underbrace{2x}_{g'(x)}$

2. $\int_{\cos x}^{5} t^3 \, dt.$ (Exercise)

3. $\int_{x^2}^{x} f(t) \, dt.$ $\rightarrow$ $\frac{d}{dx}\int_{x^2}^{c} f(t)\,dt + \frac{d}{dx}\int_c^x f(t)\,dt$

$$= -\frac{d}{dx}\int_c^{x^2} f(t)\,dt + \frac{d}{dx}\int_c^x f(t)\,dt$$

$$= \boxed{-f(x^2)\cdot 2x + f(x)}$$