Lecture 33-34: Definite Integrals (DI)

Tae Eun Kim, Ph.D.

Autumn 2021

Recall :

Indefinite integral of f :

- the infinite family of all antiderivatives of f.
- notation : $\int f(x)\,dx = F(x) + C$

an antiderivative

Definite Integrals

hint generalization: the curve $y=f(x)$ doesn't need to be above x-axis

Definition

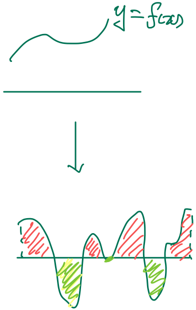
Let f be a function which is continuous on the interval $[a, b]$. We define the **definite integral** of f on $[a, b]$ by

$$\underbrace{\int_a^b f(x)\,dx}_{\text{notation}} = \underbrace{\lim_{n\to\infty} \boxed{\sum_{k=1}^{n} f(x_k^*)\Delta x}}_{\text{definition}}.$$

general Riemann sum

"exact area"

The definite integral is a number that gives the **net area** of the region between the curve $y = f(x)$ and the x-axis on the interval $[a, b]$.



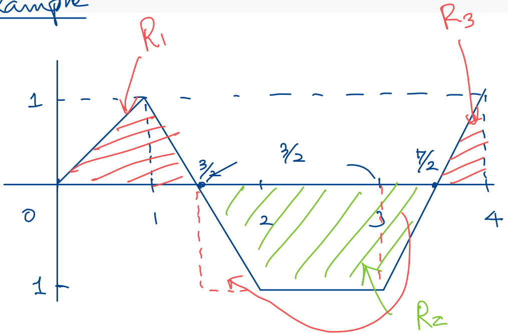
Convention: Signed area

- region above x-axis : positive
- region below x-axis : negative

: region above; (+)

: region below; (−)

Example



R_1, R_2, R_3 : regions

- $\text{Area}(R_1) = \frac{1}{2} \cdot \frac{3}{2} \cdot 1 = \frac{3}{4}$
- $\text{Area}(R_2) = \frac{3}{2} \cdot 1 = \frac{3}{2}$
- $\text{Area}(R_3) = \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4}$

$$\int_0^4 f(x)\,dx = \text{Sum of Signed areas}$$

$$= \frac{3}{4} - \frac{3}{2} + \frac{1}{4} = \boxed{-\frac{1}{2}}$$

	(Geometric) Area	Signed area.
R_1	$3/4$	$+3/4$
R_2	$3/2$	$-3/2$
R_3	$1/4$	$1/4$

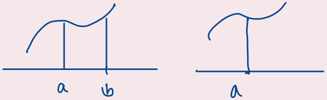
Basic Properties

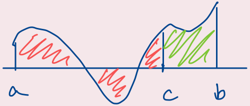
<u>Def'n</u> $\left(\begin{array}{l}\text{definite integral} \\ \text{of } f \text{ on } [a,b]\end{array}\right) \overset{\text{not'n}}{=} \int_a^b f(x)\,dx \overset{\text{def'n}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$

Theorem (Properties of the definite integral)

Let f and g be defined on a closed interval $[a, b]$ that contains the value c, and let k be a constant. The following hold:

1. $\displaystyle\int_a^a f(x)\,dx = 0$
2. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$
3. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
4. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ (scaling)
5. $\displaystyle\int_a^b \{f(x) \pm g(x)\}\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ (adding/subtracting)

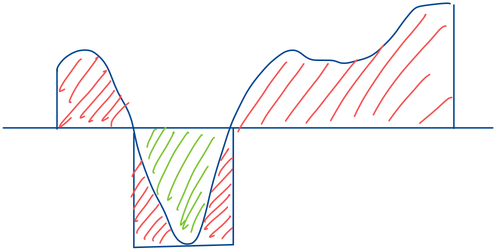




$$x_k^* = \begin{cases} x_{k-1}, & \text{LRS} \\ x_k, & \text{RRS} \\ \dfrac{x_{k-1}+x_k}{2}, & \text{MRS} \end{cases}$$

Exact net area <u>btw</u> $y = f(x)$ and x-axis.

Region b/w $y = f(x)$ & x-axis



misunderstood !

Definite Integrals Using Geometry vs. Definition

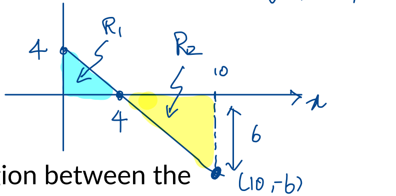
(net area) (limit of Riemann sum)

Question. Compute the integral

$$\int_0^{10} (4-x)\,dx$$

in two ways:

1. by interpreting the integral as the net area of the region between the curve $y = 4 - x$ and the interval $[0, 10]$ on the x-axis;
2. using the definition of the definite integral, i.e. by computing the limit of Riemann sums.



R_1, R_2 denote the shaded regions.

Soln ① (net area) = Signed Area (R_1) + Signed Area (R_2)

$= +\text{Area}(R_1) - \text{Area}(R_2)$

$= \frac{1}{2}\cdot 4\cdot 4 - \frac{1}{2}6\cdot 6 = 8 - 18 = \boxed{-10}$

② $\int_{0 = a}^{10 = b} (4-x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x$

RRS $(x_k^* = x_k)$

$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{10k}{n}\right)\frac{10}{n}$

Keep "n" as is ...

b/c you will be sending it to ∞.

$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(4 - \frac{10k}{n}\right)\frac{10}{n}$

- k running index
- from viewpoint of Σ, n is const.

$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{40}{n} - \frac{100k}{n^2}\right)$

"Dictionary"

Given: $f(x) = 4-x$, $a = 0$, $b = 10$

- $\Delta x = \frac{b-a}{n} = \frac{10-0}{n} \Rightarrow \Delta x = \frac{10}{n}$
- $x_k = a + k\Delta x$
  $= 0 + k\cdot\frac{10}{n} \Rightarrow x_k = \frac{10k}{n}$
- Rmk. For computation of limit of RR, the choice of sample points doesn't affect the result.

Doesn't matter whether you use LRS, RRS, or MRS.

(Cont')

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{40}{n} - \frac{100\,k}{n^2} \right)$$

$$\frac{100}{n^2} \sum_{k=1}^{n} k$$

$$= \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{40}{n} - \sum_{k=1}^{n} \frac{100\,k}{n^2} \right]$$

Const

Recall

- $\displaystyle\sum_{k=1}^{n} C = nC$
- $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$$= \lim_{n\to\infty} \left[\cancel{n}\,\frac{40}{\cancel{n}} - \frac{\cancel{100}^{\,50}}{n^2} \cdot \frac{n(n+1)}{\cancel{2}} \right]$$

$$= 40 - 50 \underbrace{\lim_{n\to\infty} \frac{n(n+1)}{n^2}}_{=1}$$

See next page.

$$\text{"}\frac{\infty}{\infty}\text{"} \text{ Form}$$

$$= 40 - 50 = \boxed{-10}$$

Detailed explanation

$$\lim_{n\to\infty} \frac{n(n+1)}{n^2} = \lim_{n\to\infty} \frac{n^2/n^2 + n/n^2}{n^2/n^2} \quad \text{“}\frac{\infty}{\infty}\text{”}$$

Divide by highest power of DENOM.

September

$$= \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{1} \quad \frac{1}{n} \to 0$$

$$= 1$$

November

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{2n+1}{2n} \quad \text{“}\frac{\infty}{\infty}\text{”}$$

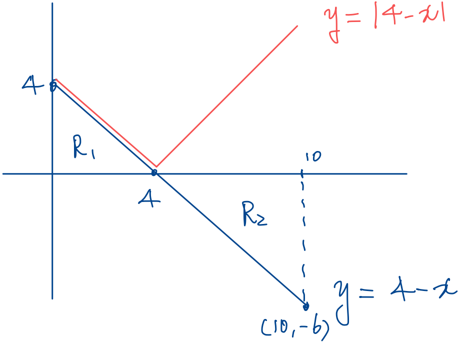
$$\overset{L'H}{=} \lim_{n\to\infty} \frac{2}{2} = 1$$

Question. Compute the integral

$$\int_0^{10} |4 - x|\, dx\,. = \text{Area}(R_1) + \text{Area}(R_2)$$

$$= 8 + 18 = \boxed{26}$$

Hint Using geometry is simpler.



Observation

$\int_a^b |f(x)|\, dx$ yield the exact geometric area, not the signed, net area.

Note: Net Areas vs. Geometric Areas

We know that the net area of the region between a curve $y = f(x)$ and the x-axis on $[a, b]$ is given by

$$\int_a^b f(x)\,dx.$$

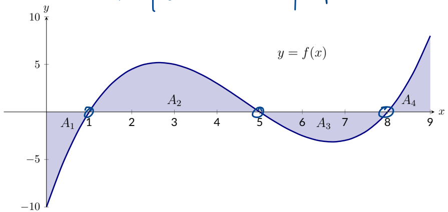
On the other hand, if we want to know the *geometric area*, meaning the "actual" area, we compute

$$\int_a^b |f(x)|\,dx.$$

Question. The graph of a function f is given in the figure.

1. Express the geometric area of the region between the curve $y = f(x)$ and the $x-$axis on the interval $[0, 9]$ as a definite integral.
2. Express the geometric area of the region between the curve $y = f(x)$ and the $x-$axis on the interval $[0, 9]$ in terms of definite integrals of f.
3. Express the geometric area of the region between the curve $y = f(x)$ and the $x-$axis on the interval $[0, 9]$ in terms of areas A_1, A_2, A_3 and A_4.

they are already positive



① $\int_0^9 |f(x)|\,dx$

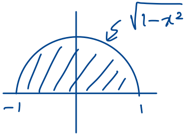
② $-\int_0^1 f(x)\,dx + \int_1^5 f(x)\,dx - \int_5^8 f(x)\,dx + \int_8^9 f(x)\,dx$

③ $A_1 + A_2 + A_3 + A_4$.

From Riemann Sums to Definite Integrals

Question. Compute the limit:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\underbrace{\sqrt{1-\left(-1+\frac{2k}{n}\right)^2}}_{f(x_k^*)}\right)\underbrace{\left(\frac{2}{n}\right)}_{\Delta x}$$



$$= \int_{-1}^{1} \sqrt{1-x^2}\, dx$$

$$= \left(\text{Area of semicircle of radius } 1\right)$$

$$= \frac{1}{2}\pi \cdot 1^2 = \boxed{\frac{\pi}{2}}$$

This limit can be viewed as the limit of a Riemann sum of $f(x) = \sqrt{1-x^2}$ on $[-1, 1]$.

- $\Delta x = \dfrac{b-a}{n} = \dfrac{2}{n} \Rightarrow b-a=2$
- $x_k = a + k\Delta x = -1 + k\cdot\frac{2}{n} \Rightarrow \underline{a=-1}$

$\Rightarrow \underline{b=1}$

Question. Express the following limit of Riemann sum as a definite integral:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\underbrace{\left(\overset{x_k}{\frac{k\pi}{n}}+\cos\frac{k\pi}{n}\right)}_{f(x_k)}\underbrace{\frac{\pi}{n}}_{\Delta x}.$$

$$f(x) = x + \cos(x)$$

$$\begin{cases} \cdot\ \Delta x = \dfrac{b-a}{n} = \dfrac{\pi}{n} \implies b-a=\pi \\ \cdot\ x_k = a + k\Delta x = \dfrac{k\pi}{n} \\ \qquad \implies a = 0 \end{cases}$$

$\longrightarrow a=0,\ b=\pi.$

$$= \int_0^{\pi} \left(x + \cos(x)\right) dx$$

Definite Integrals of Symmetric Functions

Recall that a function f is

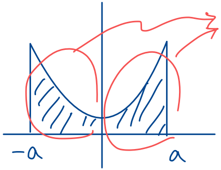
- an **odd** function if $f(-x) = -f(x)$;
- an **even** function if $f(-x) = f(x)$.

Theorem

Let f be a symmetric function on a symmetric interval $[-a, a]$ where $a > 0$. Then

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\displaystyle\int_0^a f(x)\,dx & \text{if } f \text{ is even} \\ 0 & \text{if } f \text{ is odd.} \end{cases}$$

even

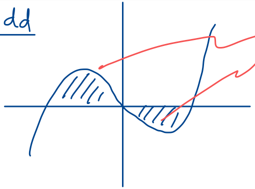


2 congruent copies

⇓

Compute half and double!

odd



equal & opposite signs.

⇓

cancel to 0.

Question.

1. Find the following definite integral: $f(x)$

$$\int_{-4}^{4} \frac{x^2 \sin^3(x)}{\sqrt{x^4+1}}\,dx. = \boxed{0}$$

symmetric interval

odd

Why odd?

$$f(-x) = \frac{(-x)^2[\sin(-x)]^3}{\sqrt{(-x)^4+1}}$$

$$= \frac{x^2[-\sin(x)]^3}{\sqrt{x^4+1}}$$

$$= -\frac{x^2[\sin(x)]^3}{\sqrt{x^4+1}} = -f(x).$$

2. Suppose that f is an even function. Given that $\int_0^6 f(x)\,dx = 13$, find

$$\int_{-6}^{6} (5f(x)+14)\,dx.$$

$$= 5\int_{-6}^{6} f(x)\,dx + \int_{-6}^{6} 14\,dx$$

even ↓ ↓ geometry

$$= 5\cdot 2\int_0^6 f(x)\,dx + 12\cdot 14$$

$$= 10\cdot 13 + 168$$

$$= 130 + 168$$

$$= \boxed{298}$$