

Lecture 2: What Is A Limit (WIAL)

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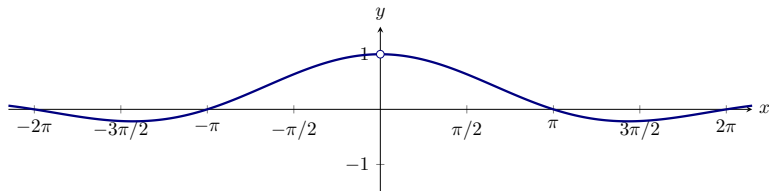
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What Is a Limit? (WIAL)

What is a limit?

Basic idea. Consider the function

$$f(x) = \frac{\sin(x)}{x}.$$



Question.

- Is f defined at $x = 0$?
- Where is $f(x)$ approaching as x gets closer to 0 ?

Definition

Intuitively, we say that

the **limit** of $f(x)$ as x approaches a is L ,

written

$$\lim_{x \rightarrow a} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all x sufficiently close, but not equal to, a .

Definition

Intuitively,

the **limit from the right** of f as x approaches a is L ,

written

$$\lim_{x \rightarrow a^+} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all $x > a$ sufficiently close, but not equal to, a .

Similarly,

the **limit from the left** of $f(x)$ as x approaches a is L ,

written

$$\lim_{x \rightarrow a^-} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all $x < a$ sufficiently close, but not equal to, a .

Theorem

A limit

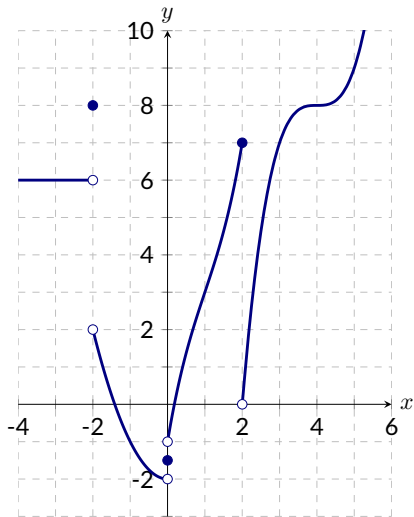
$$\lim_{x \rightarrow a} f(x)$$

exists if and only if

- $\lim_{x \rightarrow a^-} f(x)$ exists
- $\lim_{x \rightarrow a^+} f(x)$ exists
- $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$

In this case, $\lim_{x \rightarrow a} f(x)$ is equal to the common value of the two one sided limits.

Question. Study limits of the following graph at various points.



Continuity

Definition

A function f is **continuous at a point** a if

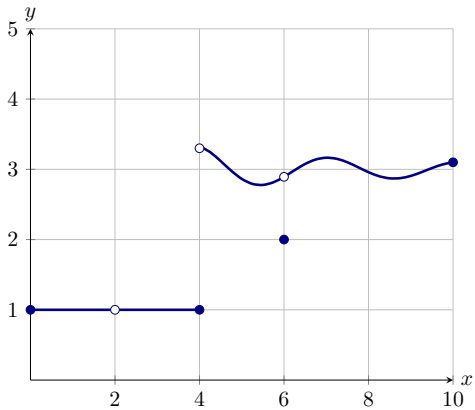
$$\lim_{x \rightarrow a} f(x) = f(a).$$

We can unpack the single equation above as:

- ① $f(a)$ is defined.
- ② $\lim_{x \rightarrow a} f(x)$ exists.
- ③ $\lim_{x \rightarrow a} f(x) = f(a)$.

Question. How can a function be discontinuous at a point?

Question. Find the discontinuities.



Definition

- A function f is **left continuous** at a point a if $\lim_{x \rightarrow a^-} f(x) = f(a)$.
- A function f is **right continuous** at a point a if $\lim_{x \rightarrow a^+} f(x) = f(a)$.

We can talk about continuity on intervals now.

Definition

A function f is

- **continuous on an open interval** (a, b) if $\lim_{x \rightarrow c} f(x) = f(c)$ for all c in (a, b) ;
- **continuous on a closed interval** $[a, b]$ if f is continuous on (a, b) , right continuous at a , and left continuous at b .

Continuity of Famous Functions

The following functions are continuous on their natural domains, for k a real number and b a positive real number:

- **Constant function** $f(x) = k$
- **Identity function** $f(x) = x$
- **Power function** $f(x) = x^b$
- **Exponential function** $f(x) = b^x$
- **Logarithmic function** $f(x) = \log_b(x)$
- **Sine and cosine functions** $f(x) = \sin(x)$ and $f(x) = \cos(x)$

Question. (Revisiting the previous graph) What are the *largest intervals* of continuity?

