Lecture 7: An Application of Limits (AAOL)

Tae Eun Kim, Ph.D.

Autumn 2021

MATH 1151 - AUTUMN 2021 THE OHIO STATE UNIVERSITY

Monday	Tuesday	Wednesday	Thursday	Friday
August 23	24 **First day of Classes** Worksheet: UF, ROFF	25 Understanding Functions (UF) Review of Famous Functions (ROFF)	26 Worksheet: ROFF	27 What is a Limit? (WIAL)
30 Limit Laws (LL)	31 Worksheet: WIAL, LL **HW: Precalc Rev**	**September 1** (In)determinate Forms (IF)	2 Worksheet: IF **HW: WIAL, LL**	3 Using Limits to Detect Asymptotes (ULTDA)
6 **Labor Day** **No Classes**	7 Worksheet: ULTDA **WH1 due**	8 Continuity and the Intermediate Value Theorem (CATIVT)	9 Worksheet: CATIVT **HW: IF, ULTDA**	10 An Application of Limits (AAOL)
13 Definition of the Derivative (DOTD) **Midterm 1** **8:00-8:40PM** **UF – CATIVT**	14 Worksheet: AAOL, DOTD **HW: CATIVT**	15 Derivatives as Functions (DAF)	16 Worksheet: DAF **HW: AAOL, DOTD**	17 **Last day to drop w/o a “W”** Rules of Differentiation (ROD)

Coverage of exam

Midterm 1 (Gradescope)

- Monday, 09/13/2021
- Timeline
  - 7:55 PM : download exam
  - 8:00 PM : start working on it
  - 8:40 PM : upload exam
  - 8:55 PM : done

- Re-read all relevant announcements
- Old exams and solns posted
- Tips & review problems in WK3 module

Tae's

- Extended Office hours : M 4:15 ~ 6:15 usual zoom link
- Short recordings over the weekend
- Schedule conflict? Contact me.

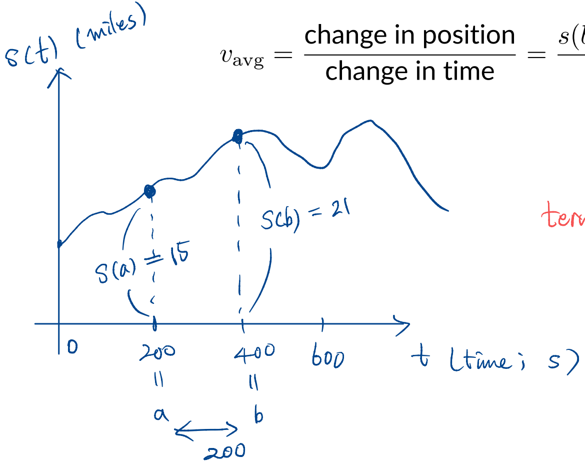
  (class or work ; written documents)
- Be sure to contact me using

  kim.3562@osu.edu

Average velocity

- Let $s(t)$ denote the position of an object moving along a vertical (or a horizontal) line at time t.
- The **average velocity** of the object on the time interval $[a, b]$ is given by

$$v_{\text{avg}} = \frac{\text{change in position}}{\text{change in time}} = \frac{s(b) - s(a)}{b - a}.$$



$s(b) - s(a) = 21 - 15 = 6$

term. pos. init. pos.

$$V_{avg} = \frac{s(b) - s(a)}{b - a} = \frac{6}{200}$$

$$= \frac{3}{100} \text{ (miles/s)}$$

Example

Suppose you are throwing a ball straight upward into the air with velocity 64 ft/sec. Its height (in feet) after t seconds is given by

$$s(t) = 64t - 16t^2 \,.$$

Answer the following questions.

Vertex

Preliminary calculation

- $s(t) = 64t - 16t^2$

$\quad = 16t(4 - t)$

$\Rightarrow$ t-intercepts : $t = 0, t = 4$

- $s(t) = -16t^2 + 64t$

$\quad = -16(\underbrace{t^2 - 4t + 4} - 4)$

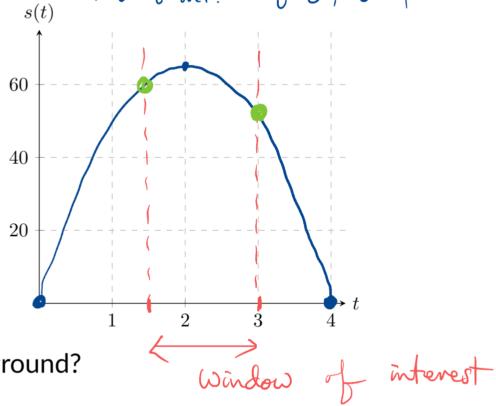
completion of squares $= -16(t - 2)^2 + 64$

$\hookrightarrow$ vertex @ $(2, 64)$

Questions.

1. Sketch the graph of $s(t)$.

- parab. open down
- vertex at (2, 64)
- t-int. : t=0, t=4



2. When will it hit the ground?

After 4 seconds.

3. Compute the average velocity of the ball on the time interval $[1.5, 3]$.

$$v_{avg,[1.5,3]} = \frac{\text{change in position}}{\text{change in time}}$$

$$= \frac{s(3) - s(1.5)}{3 - 1.5}$$

$$= \frac{48 - 60}{3/2}$$

$$= -12 \cdot \frac{2}{3} = \boxed{-8 \ (ft/s)}$$

Confirm:

$s(3) = 48$

$s(1.5) = 60$

4 Compute the average velocity of the ball on the time interval $[t, 3]$ for $0 < t < 3$.

$$v_{\text{avg},[t,3]} = \frac{s(3) - s(t)}{3 - t}$$

$s(3) = 48$

$$= \frac{48 - (-16t^2 + 64t)}{3 - t}$$

$$= \frac{16t^2 - 64t + 48}{3 - t}$$

$$= \frac{16(t^2 - 4t + 3)}{-(t-3)}$$

$$= \frac{16(\cancel{t-3})(t-1)}{-(\cancel{t-3})} = \boxed{-16(t-1)}$$

5 Finally, do the same with $[3, t]$ for $3 < t < 4$.

Fill in details yourselve.

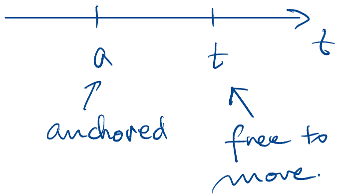
$$v_{\text{avg},[3,t]} = \boxed{-16(t-1)}$$

Instantaneous velocity

- An average velocity over a shorter time interval yields a better approximation of the **instantaneous velocity** at a moment contained in the time interval.
- This statement can be made precise using limits:

$$\underbrace{v(a)}_{\text{inst. vel.}} = \lim_{t \to a} \underbrace{\frac{s(t) - s(a)}{t - a}}_{\text{avg. vel.}}$$

where $v(a)$ is the (instantaneous) velocity at time a.



❻ Using the results of parts 4 and 5, compute the velocity of the ball at $t = 3$.

$$\underbrace{\frac{s(t) - s(3)}{t - 3}}_{V_{avg,[3,t]}} = \underbrace{\frac{s(3) - s(t)}{3 - t}}_{V_{avg,[t,3]}} = -16(t-1)$$

(inst.) ✓

$$\underbrace{V(3)}_{\text{velocity at } t=3} = \lim_{t \to 3} \frac{s(t) - s(3)}{t - 3}$$

continuity of polynom.

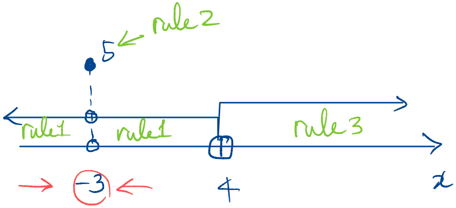
$$= \lim_{t \to 3} \left[-16(t-1)\right] = -16(3-1) = \boxed{-32 \text{ (ft/s)}}$$

Problem 4 *Let*

Found in Wk 3 module.

Piecewise function

$$f(x) = \begin{cases} \dfrac{x^2 - x - 12}{x+3} & \text{if } x < 4,\ x \neq -3 \\ 5 & \text{if } x = -3 \\ \dfrac{x}{x-4} & \text{if } x > 4. \end{cases}$$



(a) $\lim\limits_{x \to -3} f(x) = ?$

Soln Prelim. algebra before lim. comp.

$$\frac{x^2 - x - 12}{x+3} = \frac{(\cancel{x+3})(x-4)}{\cancel{x+3}} = x - 4$$

for $x \neq -3$

For all x in our interest,

$$\frac{x^2 - x - 12}{x+3} = x - 4$$

Therefore,

$$\lim_{x \to -3} \frac{x^2 - x - 12}{x+3} = \lim_{x \to -3} (x-4)$$

$$= \boxed{-7}$$

Post-Lecture Discussion

Limits at infinity cf. Infinite limits

$$\lim_{x \to \pm\infty} \frac{p(x)}{q(x)}$$

Case $p(x)$, $q(x)$ polynom.

Divide through by the highest power of $q(x)$.

Case $p(x)$, $q(x)$ involve sqrts.

Caveat: $\sqrt{☺^2} = |☺|$

Case $p(x)$, $q(x)$ involve e^x

Divide through by the most dominant term in $q(x)$.

Tame all terms $\to \pm\infty$

Useful Facts $\quad p > 0$

- $\displaystyle\lim_{x \to \pm\infty} \frac{1}{x^p} = 0$

- $\displaystyle\lim_{x \to \infty} e^{-px} = 0$

- $\displaystyle\lim_{x \to -\infty} e^{px} = 0$

$$e^{2x} = (e^x)^2$$

CAT IVT

continuity of piecewise function.

e.g.

$$f(x) = \begin{cases} \text{rule } 1, & \text{if } x \text{ in Int. } 1 \\ \text{rule } 2, & \text{if } x \text{ in Int. } 2 \\ \vdots & \vdots \end{cases}$$

- continuity at junctions

IVT

- rootfinding
- finding intersections

- digest the theorem
- examples in lecture..