Lecture 31-32: Approximating the Area Under a Curve (ATAUAC)

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Overview of Math 1151

Limits

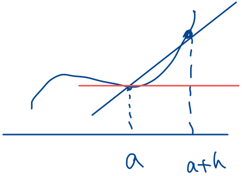
Q. What happens to $f(x)$ as x gets close to a?

$$\lim_{x \to a} f(x) = ?$$

Differentiation

Q. What is the slope of tangent line?

slope of secant.

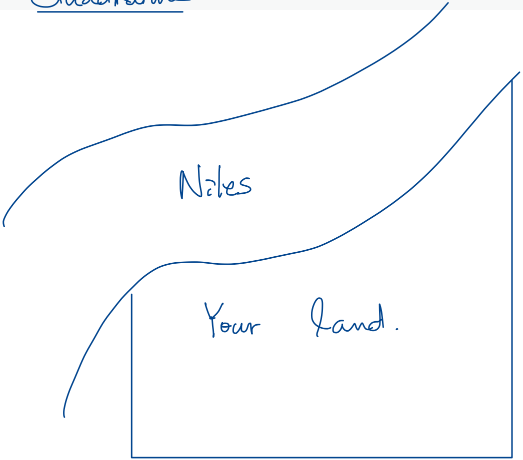


Integration

Q. "Quadrature problem"

i.e. finding area of a 2D-shape.

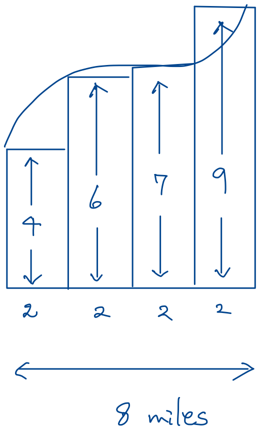
Quadrature



- Flooding → important to know how much land you owned.
- Know: how to measure areas of "regular" shapes.

- Want: find area under curve. (river shore)

<u>Strategy</u> Approximate using rectangles.



Area $\approx$ sum of areas of 4 rectangles

$$= 2 \cdot 4 + 2 \cdot 6 + 2 \cdot 7 + 2 \cdot 9$$

$$= 2(4+6+7+9)$$

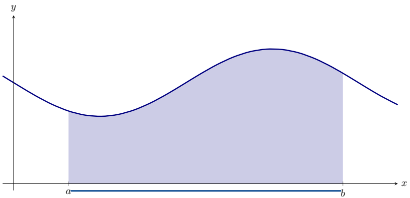
$$= 2 \cdot 26 = \underline{52}$$

Q. Can I improve the quality of the approximation?

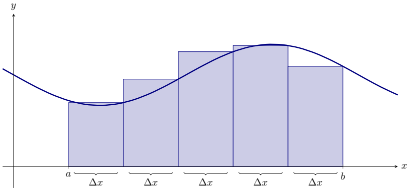
A. More thinner rectangles.

Rectangles and Areas

Our goal here is to compute the area between the curve $y = f(x)$ and the horizontal axis on the interval $[a, b]$.

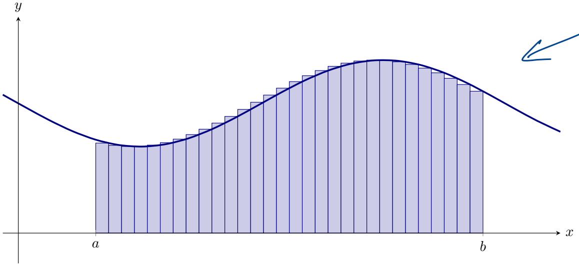


We approximate this area with n rectangles of equal with $\Delta x = (b - a)/n$ as follows:



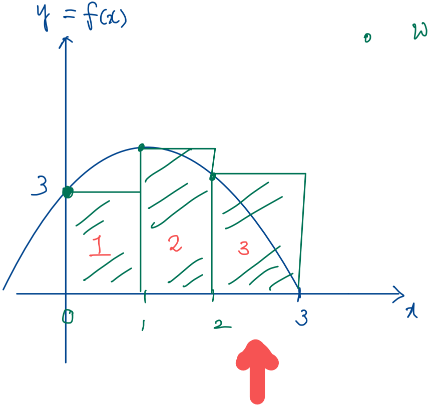
More Rectangles

As we divide the interval $[a, b]$ into finer and finer subintervals and construct more approximating rectangles, we are more closely approximating the exact area we are interested in:



height are determined by function values at right endpoints.

Main Example Approximate the area under the curve $y = f(x) = -x^2 + 2x + 3$ on $[0, 3]$ using 3 rectangles of equal width whose heights are determined by the function values at the left endpoints.



- Width of each s/int $= \dfrac{\text{total width}}{\#\text{ of rect.}} \leftarrow n$

$$= \frac{3-0}{3} = 1 = \Delta x$$

-

i (index)	x_i^* (sample)	$f(x_i^*)$ (height)	Δx (base)
1	0	$f(0) = 3$	1
2	1	$f(1) = 4$	1
3	2	$f(2) = 3$	1

i (index)	x_i^* (sample)	$f(x_i^*)$ (height)	Δx (base)
1	0	$f(0) = 3$	1
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3	2	$f(2) = 3$	1

So the sum of areas of all three rectangles is

$$A \approx (\text{area } 1) + (\text{area } 2) + (\text{area } 3)$$

exact area

$$= f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1$$

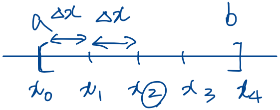
$$= 3\cdot 1 + 4\cdot 1 + 3\cdot 1 = \underline{10}$$

Key Idea

(Generalization of procedure)

We could find the area exactly if we could compute the limit as the width of the rectangles goes to zero and the number of rectangles goes to infinity. In order to make this argument precise, we need to establish some notations to help with calculation.

$n = 4$



- When approximating an area with n rectangles, the **grid points** (# of rectangles)

$$x_0, x_1, x_2, \ldots, x_n$$

  are the x-coordinates that determine the edges of the rectangles. In particular,

$$x_k = a + k\Delta x \,. \quad \text{where} \quad \Delta x = \frac{b-a}{n} \quad (\text{width of rect.})$$

- When approximating an area with rectangles, a **sample point** is the x-coordinate that determines the height of k^{th} rectangle. For $k = 1, \ldots, n$, we denote a sample point as x_k^* and the value $f(x_k^*)$ is the height of the k^{th} rectangle.
- We will use either left-endpoints, right-endpoints, or midpoints as sample points.

Introduction to Sigma Notation

Sigma notation is a way of writing a sum of many terms in a concise form. For example,

Greek capital S

$$\sum_{k=1}^{5} 3k. = 3 + 6 + 9 + 12 + 15$$

- The Σ (sigma) indicates that a sum is being taken.
- The variable k is called the (summation) *index*. → it changes!
- The numbers at the top and bottom of the Σ are called the *upper and lower limits* of the summation.
- The expression following Σ is called the *summand* formula which gives a recipe for the terms to be added up. → cf. integrand $\int f(x)\,dx$
- The notation altogether means that we will take every integer value of k between 1 and 5, plug them each into $3k$, and then add them all up:

$$\sum_{k=1}^{5} 3k = 3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + 3 \cdot 4 + 3 \cdot 5 = 45\,.$$

- The index does not have to be k. Popular choices include i, j, k, m, and n. For instance:

Question. Write out what is meant by the following:

$$\sum_{k=0}^{3} \frac{1}{k+1}. = \frac{1}{0+1} + \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1}$$

Question. Write out what is meant by the following:

$$\sum_{i=1}^{8} (-1)^i. = (-1)^1 + (-1)^2 + \cdots + (-1)^8$$

$$= (-1) + 1 + (-1) + 1$$

$$+ (-1) + 1 + (-1) + 1 = 0$$

Now let's work backward.

Question. Write the following sum in sigma notation.

$$2 + 4 + 6 + 8 + \ldots + 22 + 24$$

$+2 \quad +2 \quad \cdots$

changes by 1

$$= 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + \cdots + 2\cdot 11 + 2\cdot 12$$

$$= \sum_{k=1}^{12} 2k$$

Question. Write the following sum in sigma notation.

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{1}{64} - \frac{1}{128} = \frac{1}{2^0} - \frac{1}{2^1} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots + \frac{1}{2^6} - \frac{1}{2^7}$$

$$= \frac{(-1)^0}{2^0} + \frac{(-1)^1}{2^1} + \cdots + \frac{(-1)^7}{2^7} = \sum_{k=0}^{7} \frac{(-1)^k}{2^k} \checkmark$$

$$= \sum_{k=0}^{7} \left(-\frac{1}{2}\right)^k \checkmark$$

Question. Write the following sum in sigma notation.

$$\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{3}} + \frac{3}{\sqrt{4}} - \frac{4}{\sqrt{5}} + \ldots + \frac{51}{\sqrt{52}} - \frac{52}{\sqrt{53}}$$

Calculating with Sigma Notation

The following formulas written in terms of sigma notation will be useful.

Formula 1. $\sum_{k=1}^{n} C = nC.$

$$\underbrace{\underset{k=1}{C} + \underset{k=2}{C} + \cdots + \underset{k=n}{C}}_{n \text{ copies}} = nC$$

Formula 2. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$

Formula 3. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$

Formula 3^+. $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

Formula 4. $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$

Commutativity & associativity

Formula 5. $\sum_{k=1}^{n} c \cdot a_k = c \sum_{k=1}^{n} a_k.$

$$(a_1+b_1)+(a_2+b_2) = a_1 + (b_1+a_2) + b_2 = a_1 + (a_2+b_1) + b_2 = (a_1+a_2)+(b_1+b_2)$$

Question. Find the value of the sum $\displaystyle\sum_{k=1}^{10}\left(2k^2+5\right)$.

Question. Find the value of the sum $\displaystyle\sum_{k=1}^{200}\left(-6k^2+3\right)$.

Riemann Sums and Approximating Area

We now have a general formula for approximation of area with n rectangles:

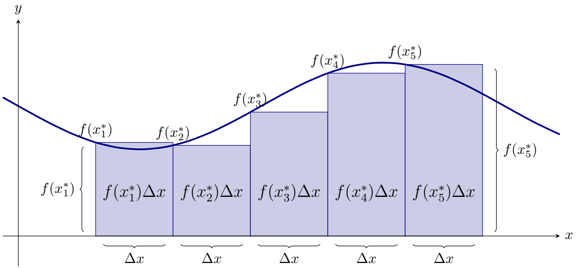
$$\text{Area} \approx \sum_{k=1}^{n} (\text{height of } k\text{th rectangle}) \times (\text{width of } k\text{th rectangle})$$

$$= f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x\,.$$

This approximating sum is called a **Riemann sum** of f on the interval $[a, b]$.

Schematic

The following is a schematic of a left Riemann sum where sample points are collected from left-endpoints of each subinterval:



The associated Riemann sum is

$$\sum_{k=1}^{5} f(x_k^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x + f(x_5^*)\Delta x.$$

Approximate the area under

$$y = f(x) = -x^2 + 2x + 3 \quad \text{on } [0, 3]$$

using a right Riemann sum.

Question

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1-\left(\frac{i}{n}\right)^2}\;\frac{1}{n} = ?$$