Lecture 11: Product Rule and Quotient Rule (PRAQR)

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Freshmen dream

$$f'(x)\, g'(x)$$

Product Rule

$$\frac{d}{dx}\left[f(x)\, g(x) \right] = f'(x)\, g(x) + f(x)\, g'(x)$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)} \right] = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{[g(x)]^2}$$

$$\frac{f'(x)}{g'(x)}$$

Motivation

Question. Let $f(x) = (x^2 + 1)$ and $g(x) = (x^3 - 3x)$. Suppose that you want to compute

$$\frac{d}{dx}\left[f(x)g(x)\right].$$

- We can proceed by expanding $f(x)g(x)$ then differentiating the result using the sum rule and power rule.
- This can get very tedious.
- At times, the strategy may not even be applicable.

Product Rule

Theorem (Product Rule)

If f and g are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)\,.$$

Sketch of derivation

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

Divide and conquer! (See textbook)

Let's revisit the opening example:

Question. Let $f(x) = (x^2 + 1)$ and $g(x) = (x^3 - 3x)$. Using the product rule, compute

$$\frac{d}{dx}[f(x)g(x)]\,.$$

$$= f'(x)\,g(x) + f(x)\,g'(x)$$

$$= 2x(x^3 - 3x) + (x^2+1)(3x^2 - 3)$$

Side calc

$f'(x) = 2x$

$g'(x) = 3x^2 - 3$

Question. Compute

$$\frac{d}{dx}\left(xe^x - e^x\right).$$

$$= \underbrace{\frac{d}{dx}\left(x e^x\right)} - \underbrace{\frac{d}{dx} e^x}_{= e^x}$$

$$= \underbrace{\left(\frac{d}{dx} x\right)}_{=1} e^x + x \underbrace{\left(\frac{d}{dx} e^x\right)}_{= e^x} - e^x$$

$$= \cancel{e^x} + x e^x - \cancel{e^x}$$

$$= \underline{x e^x}$$

Quotient Rule

Theorem (Quotient Rule)

If f and g are differentiable, then

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

- Viewing the quotient as a product $f(x)(1/g(x))$, we can use the product rule to derive the above.
- But in order to do that, we need to know what $\frac{d}{dx}(1/g(x))$ is.

$$\frac{\text{Lo De Hi} - \text{Hi De Lo}}{\text{Lo Lo}}$$

Proof of Q.R. (For those interested)

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \lim_{h\to 0}\frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x+h)} + \frac{f(x)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{g(x+h)}\,\frac{f(x+h)-f(x)}{h}\right] - \lim_{h\to 0}\left[\frac{f(x)}{g(x+h)\,g(x)}\,\frac{g(x+h)-g(x)}{h}\right]$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)\,g'(x)}{[g(x)]^2}$$

$$= \frac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}.$$

✓

Question. Compute:

Hi

$$\frac{d}{dx}\frac{x^2+1}{x^3-3x}.$$

Lo

$$= \frac{(x^3-3x)(2x)-(x^2+1)(3x^2-3)}{(x^3-3x)^2}$$

Side

- DeHi $= 2x$
- DeLo $= 3x^2-3$

Question. Compute:

$$\frac{d}{dx}\frac{625 - x^2}{\sqrt{x}} = h(x)$$

Exercise.

in two ways. First using the quotient rule and then using the product rule.

Using Quotient Rule

DeHi $= -2x$

DeLo $= \frac{d}{dx} x^{1/2}$

$= \frac{1}{2} x^{-1/2}$

$$h'(x) = \boxed{\frac{\sqrt{x}\,(-2x) - (625 - x^2)\left(\frac{1}{2}x^{-1/2}\right)}{\sqrt{x}^{\,2}}}$$

Using Product Rule

Rewrite first! Note $\frac{1}{\sqrt{x}} = x^{-1/2}$

$$h(x) = \underbrace{x^{-1/2}}_{f(x)} \underbrace{(625 - x^2)}_{g(x)}$$

$f'(x) = -\frac{1}{2} x^{-3/2}$

$g'(x) = -2x$

$$\downarrow$$

$$h'(x) = \boxed{-\frac{1}{2} x^{-3/2}(625 - x^2) + x^{-1/2}(-2x)}$$

Do they agree? Let's check on next page.

$$h'(x) = \frac{\sqrt{x}\,(-2x) - (625-x^2)\left(\frac{1}{2}x^{-1/2}\right)}{\sqrt{x}^2} \qquad \text{from Q.R.}$$

$$= \frac{-2x\sqrt{x} - \frac{1}{2\sqrt{x}}(625-x^2)}{x}$$

$$= -2\sqrt{x} - \frac{1}{2x\sqrt{x}}(625-x^2)$$

$$h'(x) = -\frac{1}{2}x^{-3/2}(625-x^2) + x^{-1/2}(-2x) \qquad \text{from P.R.}$$

$$= -2x^{-1/2+1} - \frac{1}{2x^{3/2}}(625-x^2)$$

$$x^{3/2} = x^{1+1/2}$$

$$= x^1 \cdot x^{1/2} = x\sqrt{x}$$

$$= -2\sqrt{x} - \frac{1}{2x\sqrt{x}}(625-x^2)$$

Confirmed!