Review for Final Examination — Math 1151

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Review problems for final exam covering materials from **Antiderivatives and Area** (AAA) till the end. For previous topics, please study the three midterm exams as well as their corresponding review problems.

Problem 1. (Integration exercises, I)

Compute the following:

(a) $\displaystyle\int \frac{\tan^3(\theta)+1}{\cos^2(\theta)}\,d\theta$

$$= \int \frac{\tan^3\theta}{\cos^2\theta}\,d\theta + \int \frac{1}{\cos^2\theta}\,d\theta$$

$$= \int \frac{\sin^3\theta}{\cos^5\theta}\,d\theta + \int \sec^2\theta\,d\theta$$

$$= \boxed{\frac{1}{4}\sec^4\theta - \frac{1}{2}\sec^2\theta + \tan\theta + C}$$

$$\int \frac{\sin^2\theta}{\cos^5\theta}\sin\theta\,d\theta = \int \frac{1-\cos^2\theta}{\cos^5\theta}\sin\theta\,d\theta$$

$$\begin{cases} u = \cos\theta \\ du = -\sin\theta\,d\theta \end{cases}$$

$$= -\int \frac{1-u^2}{u^5}\,du = \frac{1}{4}u^{-4} - \frac{1}{2}u^{-2} + C = \frac{1}{4}\sec^4\theta - \frac{1}{2}\sec^2\theta + C$$

(b) $\displaystyle\int \overbrace{(4x-6)}^{2(2x-3)}\sqrt{x^2-3x}\,dx$

$$= \int 2\sqrt{u}\,du$$

$$\begin{cases} u = x^2 - 3x \\ du = (2x-3)\,dx \end{cases}$$

$$= 2\cdot\frac{2}{3}u^{3/2} + C$$

$$= \boxed{\frac{4}{3}(x^2-3x)^{3/2} + C}$$

(c) $\displaystyle\int_0^{\pi/2} \frac{d}{dx}(\sin^7 x)\,dx$

$$\overset{\text{FTC 2}}{=} \Big[\sin^7 x\Big]_0^{\pi/2}$$

$$= \sin^7(\pi/2) - \sin^7(0) = \boxed{1}$$

(d) $\displaystyle\frac{d}{dx}\underbrace{\int_0^{\pi/2}\sin^7 t\,dt}_{\text{constant}} = \boxed{0}$

(e) $\displaystyle\int_0^{\pi/4} \frac{1+\tan\theta}{\sec\theta}\,d\theta$

$$= \int_0^{\pi/4}\frac{1}{\sec\theta}\,d\theta + \int_0^{\pi/4}\frac{\tan\theta}{\sec\theta}\,d\theta$$

$$\frac{\tan\theta}{\sec\theta} = \frac{\sin\theta}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos\theta}}{1}$$

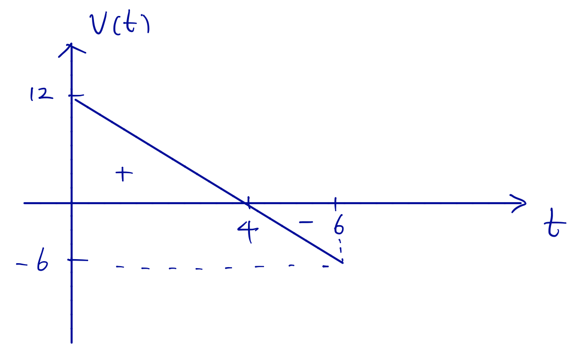
$$= \int_0^{\pi/4}\cos\theta\,d\theta + \int_0^{\pi/4}\sin\theta\,d\theta$$

$$= \Big[\sin\theta\Big]_0^{\pi/4} + \Big[-\cos\theta\Big]_0^{\pi/4}$$

$$= \left(\frac{\sqrt{2}}{2} - 0\right) + \left(1 - \frac{\sqrt{2}}{2}\right) = \boxed{1}$$

Answer the following questions.

(a) Consider an object moving along a straight line with the velocity $v(t) = 12 - 3t$ on $[0, 6]$. Express the distance traveled over the given interval as a sum (or difference) of two definite integrals.



$$(\text{distance}) = \int_0^6 |v(t)|\, dt$$

$$= \int_0^4 v(t)\, dt - \int_4^6 v(t)\, dt$$

$$= \boxed{\int_0^4 (12 - 3t)\, dt - \int_4^6 (12 - 3t)\, dt}$$

Not required, but whatever... Using geometry,

$$= \frac{1}{2} \cdot 4 \cdot 12 + \frac{1}{2} \cdot 2 \cdot 6 = 30$$

(b) An oil refinery produces oil at a variable rate of $A'(t) = 1000 - 10t$ with $0 \le t \le 40$, where t is measured in days and A is measured in barrels. How many barrels are produced in the first 10 days?

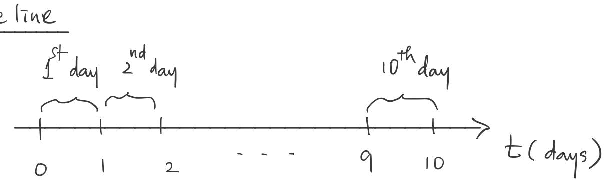
The requested quantity, in mathematical notation, is

$$A(10) - A(0) = \int_0^{10} A'(t)\, dt \quad [\text{Net change}]$$

(the amount of oil at the end of 10^{th} day) (the amount of oil at the beginning of the 1^{st} day)

$$= \int_0^{10} (1000 - 10t)\, dt$$

time line



$$= \Big[1000t - 5t^2\Big]_0^{10}$$

$$= 1000 \cdot 10 - 5 \cdot 100 = \boxed{9500 \text{ (barrels)}}$$

$$(100 - 5) \cdot 100$$

Suppose that $\int_1^3 f(x)\,dx = 4$.

(a) Evaluate the following integrals.

i. $\displaystyle\int_1^9 \frac{3f(\sqrt{x})}{\sqrt{x}}\,dx$

$$= \int_1^3 3\,f(u)\cdot 2\,du$$

$$\begin{cases} u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\,dx \rightarrow \frac{1}{\sqrt{x}}\,dx = 2du \end{cases}$$

x	u
9	3
1	1

$$= 6\underbrace{\int_1^3 f(u)\,du}_{=4}$$

$$= \boxed{24}$$

ii. $\displaystyle\int_0^{\sqrt{2}} 3x f(x^2+1)\,dx$

$$= \int_1^3 3\,f(u)\cdot\frac{1}{2}\,du$$

$$\begin{cases} u = x^2+1 \\ du = 2x\,dx \rightarrow x\,dx = \frac{1}{2}\,du \end{cases}$$

x	u
$\sqrt{2}$	$\sqrt{2}^2+1 = 3$
0	$0^2+1 = 1$

$$= \frac{3}{2}\underbrace{\int_1^3 f(u)\,du}_{=4}$$

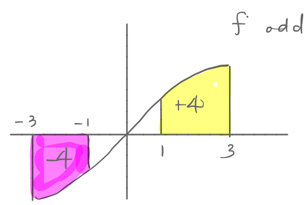
$$= \boxed{6}$$

(b) Assume additionally that f is odd. Evaluate $\int_{-1}^{-3} f(x)\,dx$.

$$\int_{-1}^{-3} f(x)\,dx = -\int_{-3}^{-1} f(x)\,dx$$

$$= -((-4))$$

$$= \boxed{4}$$



OR:

$$\cancel{\int_{-3}^{3} f(x)\,dx}^{\,0} = \underbrace{\int_{-3}^{-1} f(x)\,dx}_{\text{must be } -4} + \cancel{\int_{-1}^{1} f(x)\,dx}^{\,0} + \underbrace{\int_1^3 f(x)\,dx}_{=4}$$

(c) Find f_{avg}, the average value of f, on the interval $[1,3]$.

$$f_{avg} = \frac{1}{3-1}\underbrace{\int_1^3 f(x)\,dx}_{=4}$$

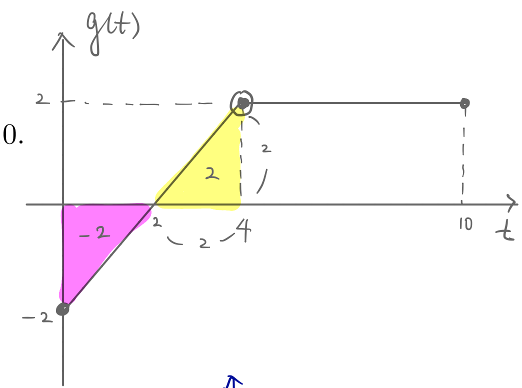
$$= \frac{1}{2}\cdot 4 = \boxed{2}$$

Let g be defined on $[0, 10]$ by

$$g(x) = \begin{cases} x - 2 & 0 \leq x < 4 \\ 2 & 4 \leq x \leq 10 \end{cases}.$$

Define A by

$$A(x) = \int_0^x g(t)\,dt, \quad \text{for } 0 \leq x \leq 10.$$



OR use the graph!

Evaluate:

(a) $A(4)$ $= \int_0^4 g(t)\,dt$

$$= \int_0^4 (t-2)\,dt$$

$$= \left[\frac{t^2}{2} - 2t \right]_0^4$$

$$= \left(\frac{4^2}{2} - 2\cdot 4 \right) - \left(\frac{0^2}{2} - 2\cdot 0 \right) = \boxed{0}$$

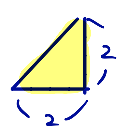
(b) $A'(4)$

First, we note by FTC1 that

$$A'(x) = \frac{d}{dx} \int_0^x g(t)\,dt = g(x).$$

Thus,

$$A'(4) = g(4) = \boxed{2}$$

(c) $\int_0^4 |g(t)|\,dt$ $= 2 \cdot \text{Area}\Big($  $\Big)$

"Geometric area"

$$= 2 \cdot \frac{1}{2} \cdot 2 \cdot 2 = \boxed{4}$$

Problem 5. (Initial value problems)

Answer the following questions.

(a) Graph several functions that satisfy the differential equation $\underbrace{f'(x) = 3x^2 - 1}_{\text{(DE)}}$. Then find and graph the particular solution that satisfies the initial condition $\underbrace{f(2) = 1}_{\text{(IC)}}$. (This was one of Midterm 3 review problems.)

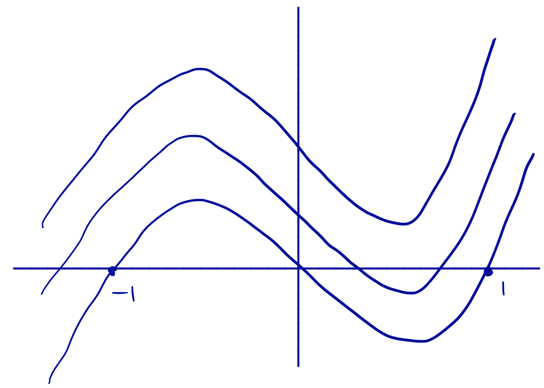
- $f(x) = \int (3x^2 - 1)\, dx$

  $= x^3 - x + C$

- $f(2) = 2^3 - 2 + C = 1$

  $\Longrightarrow 6 + C = 1$

  $\therefore C = -5$

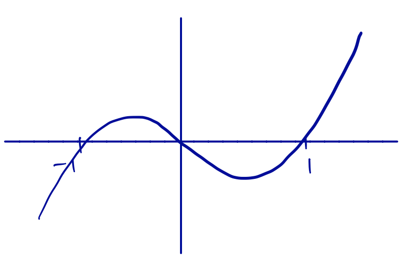


(b) Find and graph the function $A(x) = \displaystyle\int_0^x (3t^2 - 1)\, dt$. Does the function A satisfy the differential equation in the previous part? Explain. Compute $A(2)$. Does the function A satisfy the initial condition given above?

- $A(x) = \int_0^x (3t^2 - 1)\, dt$

  $= \left[t^3 - t \right]_0^x$

  $= \boxed{x^3 - x} = \underbrace{x(x-1)(x+1)}_{\text{useful for graphing}}$



- By FTC1,

  $A'(x) = \frac{d}{dx} \int_0^x (3t^2 - 1)\, dx = 3x^2 - 1$.

  So $A(x)$ satisfies the diff. eqn.

- However,

  $A(2) = 2^3 - 2 = 6 \neq 1$.

  So it does not satisfy the init. cond.

Determine the following definite integrals.

(a) $\displaystyle\int_0^4 \frac{x-3}{\sqrt{x}}\,dx$

$$= \int_0^4 \left(\frac{x}{\sqrt{x}} - \frac{3}{\sqrt{x}}\right) dx$$

$$= \int_0^4 \left(x^{1/2} - 3x^{-1/2}\right) dx$$

$$= \left[\frac{2}{3}x^{3/2} - 3\cdot 2x^{1/2}\right]_0^4$$

$$= \left(\frac{2}{3}\cdot 4^{3/2} - 6\cdot 4^{1/2}\right) - \left(\frac{2}{3}\cdot 0^{3/2} - 6\cdot 0^{1/2}\right)$$

$$= \frac{2}{3}\cdot 8 - 6\cdot 2 = \boxed{-\frac{20}{3}}$$

- $4^{3/2} = (4^{1/2})^3 = \sqrt{4}^3 = 2^3 = 8$
- $4^{1/2} = \sqrt{4} = 2$

(b) $\displaystyle\int_0^1 \frac{e^x}{e^x+e^{-x}}\,dx$

$$= \int_0^1 \frac{e^{2x}}{1+e^{2x}}\,dx$$

multiply by $\frac{e^x}{e^x}$

$$\begin{cases} u = 1+e^{2x} \\ du = 2e^{2x}\,dx \end{cases}$$

x	u
1	$1+e^2$
0	2

$$= \int_2^{1+e^2} \frac{1}{2u}\,du$$

$$= \frac{1}{2}\Big[\ln|u|\Big]_2^{1+e^2} = \boxed{\frac{1}{2}\left[\ln(1+e^2) - \ln(2)\right]}$$

(c) $\displaystyle\int_{\underbrace{-\pi/4}_{\text{symmetric interval}}}^{\pi/4} \underbrace{x^4\tan^9 x}_{\text{odd}}\,dx = \boxed{0}$

Why is $f(x) = x^4\tan^9 x$ odd?

$$f(-x) = (-x)^4\left[\tan(-x)\right]^9$$

$$= x^4\left[-\tan(x)\right]^9$$

$$= -x^4\tan^9(x) = -f(x).$$

Let f be given by

$$f(x) = \begin{cases} \frac{x}{2} & 0 \le x < 2 \\ 1 & 2 \le x < 3 \\ -x+4 & 3 \le x < 5 \\ -1 & 5 \le x \le 7 \end{cases}.$$

(a) Find the average value of the function f on the interval $[0, 7]$.

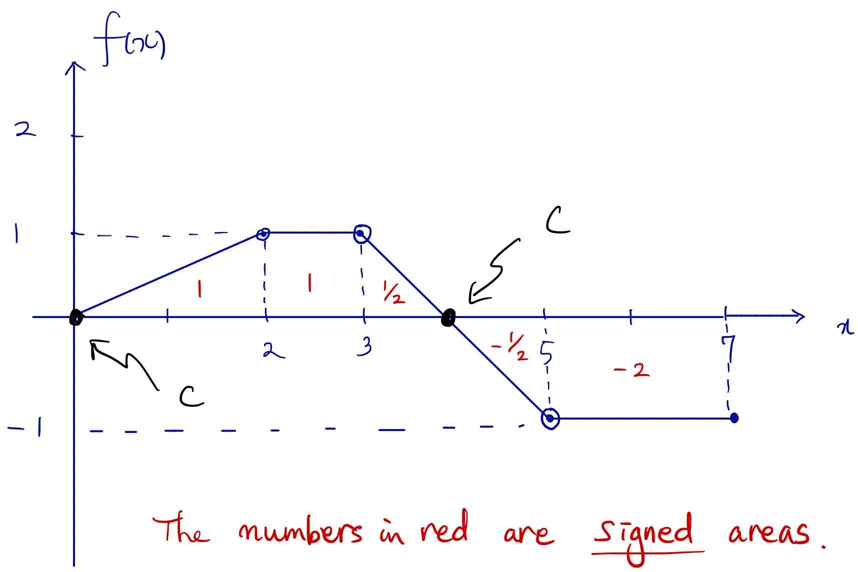
$$f_{avg} = \frac{1}{7-0}\int_0^7 f(x)\,dx$$

$$= \frac{1}{7}\left\{ \underbrace{\int_0^2 \frac{x}{2}\,dx}_{=1} + \underbrace{\int_2^3 1\,dx}_{=1} + \underbrace{\int_3^5 (-x+4)\,dx}_{=0} + \underbrace{\int_5^7 (-1)\,dx}_{=-2} \right\}$$

See figure below

$$= 0$$

(b) Sketch the graph of $f(x)$ and mark the point (or points) c in $[0, 7]$ where the function attains this average value. Draw (best you can) a rectangle whose net area is equal to $\int_0^7 f(t)\,dt$.



There is no rectangle since $\int_0^7 f(t)\,dt = 0$!

The numbers in red are signed areas.

(c) Compute $\int_0^7 |f(t)|\,dt$. Find the average value of $|f|$.

- $\int_0^7 |f(t)|\,dt = \underbrace{\int_0^4 f(t)\,dt}_{=5/2} - \underbrace{\int_4^7 f(t)\,dt}_{=-5/2} = \boxed{5}$

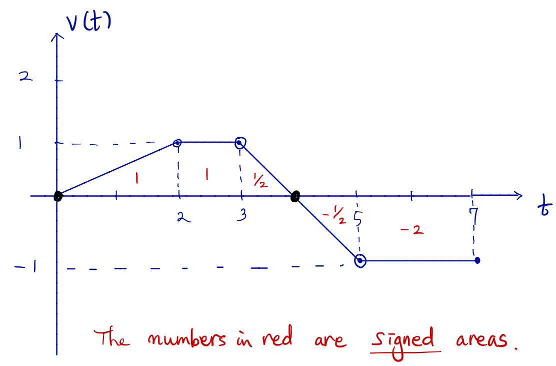
- $(\text{avg. of } |f|) = \frac{1}{7-0}\int_0^7 |f(t)|\,dt = \boxed{\frac{5}{7}}$

Let v be given by

$$v(t) = \begin{cases} \frac{t}{2} & 0 \le t < 2 \\ 1 & 2 \le t < 3 \\ -t+4 & 3 \le t < 5 \\ -1 & 5 \le t \le 7 \end{cases}.$$

Note: The same function as in Problem 7.

Assume that $s(0) = 0$.



The numbers in red are signed areas.

(a) Determine the displacement between $t = 0$ and $t = 7$.

$$(\text{displacement}) = \int_0^7 v(t)\, dt = \boxed{0}$$

(b) Determine the distance traveled between $t = 0$ and $t = 7$.

$$(\text{distance}) = \int_0^7 |v(t)|\, dt = \boxed{5}$$

(c) Determine the position at $t = 3$.

$$s(3) = \underbrace{s(0)}_{=0} + \int_0^3 v(t)\, dt = \boxed{2}$$

(d) Determine the position at $t = 5$.

$$s(5) = \underbrace{s(3)}_{=2} + \underbrace{\int_3^5 v(t)\, dt}_{=0} = \boxed{2}$$

(e) Determine the position function, $s(t)$, for $5 \le t \le 7$.

$$s(t) = s(5) + \int_5^t \underbrace{v(s)}_{=-1}\, ds = 2 + \left[-s\right]_5^t = \boxed{-t+7}$$

(f) Determine the acceleration, $a(t)$, for $5 < t < 7$.

Since $v(t) = -1$ for $5 < t < 7$,

$$a(t) = v'(t) = \boxed{0}.$$