Lecture 13: Higher Order Derivatives and Graphs (HODAG)

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Higher-Order Derivatives

- The derivative of a function is often called the **first derivative**;
- The derivative of the derivative the **second derivative**;
- The derivative of the second derivative the **third derivative**, and so on.
- Derivatives of derivatives are called **higher-order derivatives** with the following notations:

First derivative: $\dfrac{d}{dx}f(x) = f'(x) = f^{(1)}(x).$

Second derivative: $\dfrac{d^2}{dx^2}f(x) = f''(x) = f^{(2)}(x).$

Third derivative: $\dfrac{d^3}{dx^3}f(x) = f'''(x) = f^{(3)}(x).$

Question. Compute the first, second, and third derivatives of:

1. $f(x) = x^2 + 3x - 8$

2. $g(x) = e^{2x}$

3. $h(x) = \sin(x^2)$

First Derivatives and Monotonicity

Definition (Monotonicity)

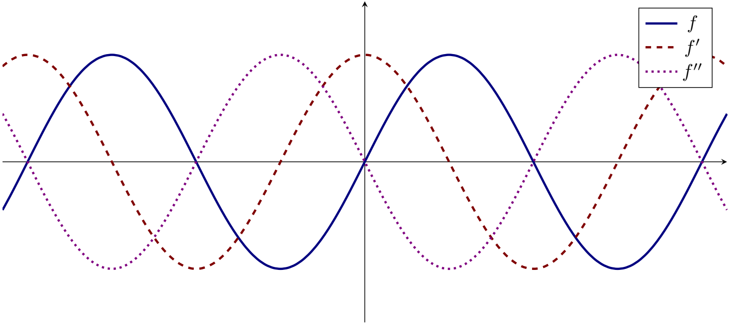
- We say that a function f is **increasing** on an interval I if $f(x_1) < f(x_2)$ for all pairs of numbers x_1, x_2 in I such that $x_1 < x_2$.
- We say that a function f is **decreasing** on an interval I if $f(x_1) > f(x_2)$ for all pairs of numbers x_1, x_2 in I such that $x_1 < x_2$.

- We say that a function is **monotonic** on an interval if it is either increasing or decreasing there.
- The notion of monotonicity is closely related to the derivatives as they convey the slope information of tangent lines to the curve.

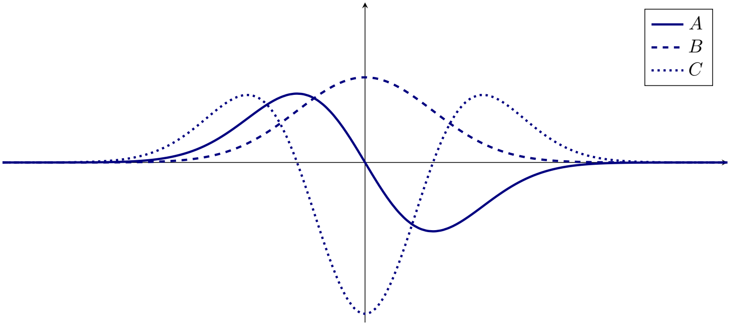
Theorem

*A function f is **increasing** on any interval I where $f'(x) > 0$, for all x in I. A function f is **decreasing** on any interval I where $f'(x) < 0$, for all x in I.*

Illustration. Here we have graphs of $f(x) = \sin(x)$, $f'(x) = \cos(x)$, and $f''(x) = -\sin(x)$:



Question. Below we have unlabeled graphs of f, f', and f''. Identify each curve above as a graph of f, f', or f''.



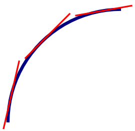
Second Derivatives and Concavity

Definition (Concavity)

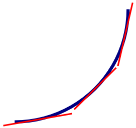
Let f be a function differentiable on an open interval I.

- We say that the graph of f is **concave up** on I if f' is **increasing** on I.
- We say that the graph of f is **concave down** on I if f' is **decreasing** on I.

Illustration.



The function f is increasing, while the rate itself is decreasing. In this case the curve $y = f(x)$ is **concave down**.



The function g is increasing, while the rate itself is increasing. In this case the curve $y = g(x)$ is **concave up**.

We know from the previous section that

> f' is increasing/decreasing when its derivative f'' is positive/negative.

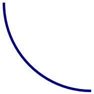
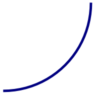
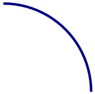
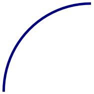
In other words, the second derivatives contain concavity information as summarized in the following theorem.

Theorem (Test for concavity)

Let I be an open interval.

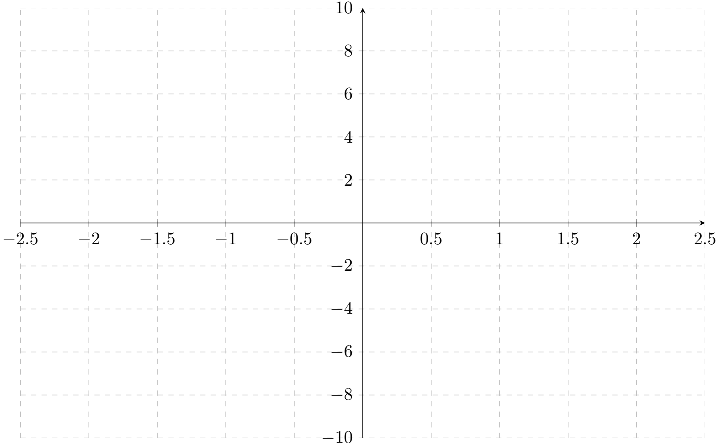
1. *If $f''(x) > 0$ for all x in I, then the graph of f is concave up on I.*
2. *If $f''(x) < 0$ for all x in I, then the graph of f is concave down on I.*

Summary: Derivatives and Graphs

	$f'(x) < 0$	$f'(x) > 0$
$f''(x) > 0$	 The function f is decreasing, while the rate itself is increasing. In this case the curve $y = f(x)$ is **concave up**.	 The function f is increasing, while the rate itself is increasing. In this case the curve $y = f(x)$ is **concave up**.
$f''(x) < 0$	 The function f is decreasing, while the rate itself is decreasing. In this case the curve $y = f(x)$ is **concave down**.	 The function f is increasing, while the rate itself is decreasing. In this case the curve $y = f(x)$ is **concave down**.

Question. Find the intervals on which f is increasing/decreasing and concave up/down and plot its graph.

$$f(x) = x^3 - x^2 - 4x + 4\,.$$



Example from Physics

Motion with constant acceleration

We know from physics that the motion of an object with constant acceleration a is described by the following formulas:

$$\text{position:} \qquad x(t) = x_0 + v_0 t + \frac{1}{2}at^2\,,$$

$$\text{velocity:} \qquad v(t) = v_0 + at\,,$$

where x_0 and v_0 are the initial position and velocity respectively.

In general,

- the derivative of the position function $x(t)$ is the velocity function $v(t)$, i.e., $v(t) = x'(t)$;
- the derivative of the velocity function $v(t)$ is the acceleration function $a(t)$, i.e. $a(t) = v'(t) = x''(t)$.

Question. The position of a moving particle is given by

$$s(t) = 36t^2 - 7t^3 \,.$$

Find a formula for its acceleration.