Lecture 39: The Idea of Substitution (TIOS)

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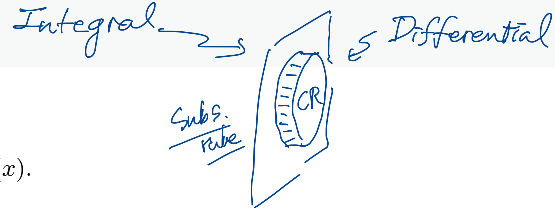
To: Kim, Tae Eun

Thank you for cheering Dr. Kim. I made a piece of meme just for fun. I hope you to not take it too seriously. Please enjoy it.

Courtesy of J Song.

Two Sides of a Coin



Recall that from the chain rule that

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

So by the fundamental theorem of calculus, we have

$$\int_a^b f'(g(x))g'(x)\,dx \overset{\text{FTC2}}{=} \Big[f(g(x))\Big]_a^b = f(g(b)) - f(g(a)).$$

equal

Using the fundamental theorem in reverse direction once again, the last line can be thought of as

$$\Big[f(u)\Big]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f'(u)\,du.$$

$$\int_a^b f'(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du$$

This is the gist of the integration technique known as **substitution rule** or u-**substitution**[1].

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[1] This name is due to a popular and customary choice of substitution variable u. The choice, however, is not an absolute rule written on a stone. Any variable of your choice such as v or ☺ works if used consistently.

Substitution Rule

Theorem (Integral Substitution Formula)

If g is differentiable on the interval $[a, b]$ and f is differentiable on the interval $[g(a), g(b)]$, then

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du.$$

In sum, the substitution rule is the integral counterpart of differential chain rule and the fundamental theorem of calculus serves as a bridge between the two.

Procedures

In integrating a function which we suspect to be the derivative of another obtained by the chain rule:

1. Look for a candidate for the inner function; call it u.
2. Rewrite the given function completely in terms of u leaving no trace of the original variable.
3. Integrate this new function of u. (If necessary, you may need to go back to Step 1 and modify your choice of u.)
4. In dealing with an indefinite integral, make sure to replace u by the equivalent expression of the original variable.
5. Working with a definite integral, you may evaluate the result of Step 3 at the transformed bounds of u or evaluate the antiderivate obtained in Step 4 at the original bounds.

Question. Compute $\int_1^3 x\cos(\boxed{x^2})\,dx$.

"inner"

$$\begin{cases} u = x^2 \\ du = (2x)\,dx \end{cases}$$

deriv. of u.

$x\,dx = \frac{du}{2}$

Limits

x	$u=x^2$
3	9
1	1

$$= \int_1^9 \cos(u)\,\frac{1}{2}\,du$$

(Want: Everything to be written in terms of u.)

$$\overset{\text{FTC2}}{=} \frac{1}{2}\Big[\sin(u)\Big]_1^9 = \boxed{\frac{1}{2}\big(\sin(9)-\sin(1)\big)}$$

- Subs. Rule is useful in handling integrals of product/quotient.
- $\int_a^b \underbrace{f'(g(x))}\,\underbrace{g'(x)}\,dx$.

  deriv. of "inner" appears outside.

Question. Compute $\int \sec^2(x)\,\tan(x)\,dx.$

u du d/dx

- Observation: $\frac{d}{dx}\tan(x) = \sec^2(x)$

- Tip: Set $\begin{cases} u = \tan(x). \\ du = \underline{\sec^2(x)\,dx} \end{cases}$

- indef. integ.: no need to translate limits.

Retrieve orig. var.

$$= \int u\,du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\tan^2(x) + C}$$

Recap Substitution Rule

- an integration technique
  - works for both definite & indefinite integ.
- integral counterpart of Chain Rule
- $$\int_a^b f(\underbrace{\boxed{g(x)}}_{=u})\,\underbrace{g'(x)\,dx}_{=du} = \int_{g(a)}^{g(b)} f(\underline{u})\,d\underline{u}$$
- Practical note: useful when an integrand is in prod./quot. form.

★
- Look for an inner func. $\longrightarrow u$
- Look out for u' to appear "outside"

Question. Compute $\int \underbrace{x^4 (\boxed{x^5 + 1})^9 \, dx}_{\frac{1}{5} du}$.

Set $\begin{cases} u = x^5 + 1 \\ du = (5x^4)\, dx \implies x^4\, dx = \frac{1}{5} du \end{cases}$

$$= \int u^9 \, \frac{1}{5} \, du$$

In terms of u alone? Yes!

$$= \frac{1}{5} \, \frac{u^{10}}{10} + C$$

$$= \boxed{\frac{1}{50} (x^5+1)^{10} + C}$$

Ans. must be written in terms of orig. var.!

Question. Compute $\displaystyle\int_{\pi/3}^{\pi/2} \underbrace{\sin(x)\, \sec^2(\overbrace{\boxed{\cos(x)}}^{u})\, dx}_{= -du}$.

- Which func. is embedded?
- Does its deriv. appear outside?

Set $u = \cos(x)$

$du = -\sin(x)\, dx. \Rightarrow \sin(x)\, dx = -du$

Limits

x	$u = \cos(x)$
$\pi/2$	$\cos(\pi/2) = 0$
$\pi/3$	$\cos(\pi/3) = 1/2$

$$= -\int_{1/2}^{0} \sec^2(u)\, du$$

$$= \int_{0}^{1/2} \sec^2(u)\, du$$

$$\overset{\text{FTC2}}{=} \Big[\tan(u)\Big]_0^{1/2} = \tan(1/2) - \tan(0) = \boxed{\tan(1/2)}$$

$$\tan(0) = \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = 0$$

Alternate approach

To work out a definite integral using Subs. rule:

① Find an antiderivative. (Subs. rule)

② Use FTC2 w/ orig. limits of integ.

Ex ① $\int \underbrace{\sin(x)}\, \sec^2(\underbrace{\cos(x)}_{=u})\, \underbrace{dx}$ $= -\int \sec^2(u)\, du$

$\sin(x)\,dx = -du$

$= -\tan(u) + C$

$= -\tan(\cos(x)) + C.$

② $\int_{\pi/3}^{\pi/2} \sin(x) \sec^2(\cos(x))\, dx$

$\overset{FTC2}{=} \Big[-\tan(\cos(x)) \Big]_{\pi/3}^{\pi/2}$

$= \tan(\cos(\pi/3)) - \tan(\cos(\pi/2))$

$= \boxed{\tan(1/2)} - \cancel{\tan(0)}^{\,0}$

Exercise

Question. Compute $\displaystyle\int_{-2}^{1} t^2 \sin(t^3)\, dt$.

Question. Compute $\displaystyle\int_0^{1/2} \frac{13e^x}{\boxed{3e^x - 5}}\,dx.$

$\underset{u}{\boxed{3e^x-5}}$, $e^x\,dx = \frac{1}{3}du$

- Which is embedded?
- Does its derivative appear outside?

Set $u = 3e^x - 5$

$du = (3e^x)\,dx \Rightarrow e^x\,dx = \frac{1}{3}\,du$

Limits

x	$u = 3e^x - 5$
$1/2$	$3e^{1/2} - 5 = ☺$
0	$3e^0 - 5 = 3 - 5 = -2$

negative number

$$= \int_{-2}^{☺} \frac{13}{u}\,\frac{1}{3}\,du$$

$$= \frac{13}{3}\int_{-2}^{☺} \frac{du}{u}$$

$$\overset{\text{FTC2}}{=} \frac{13}{3}\Big[\ln|u|\Big]_{-2}^{☺}$$

$$= \frac{13}{3}\Big(\ln|☺| - \ln|-2|\Big)$$

$$= \boxed{\frac{13}{3}\Big(\ln(5 - 3e^{1/2}) - \ln(2)\Big)}$$