

Lecture 30: Antiderivatives (A)

Tae Eun Kim, Ph.D.

Autumn 2021

Basic Antiderivatives

• [one of infinitely many possible answers]

Antidifferentiation is a process where we undo differentiation. Precisely:

Definition

A function F is called an antiderivative of f on an interval if

$$F'(x) = f(x)$$

for all x in the interval.

[All antiderivatives differ by const.]

Example $f(x) = 2x - 7$.

Q. What is an antiderivative of $f(x)$?

i.e., Find a function $F(x)$ whose derivative equals $f(x)$.

A. $F(x) = x^2 - 7x$

check: $F'(x) = 2x - 7 = f(x)$. ✓

Q. Can you find any other?

A.

Student A: $F(x) = x^2 - 7x + \pi$

Student D: $F(x) = x^2 - 7x + 2$

Since the derivative of a constant is zero, we can add it to any antiderivative of f and it will still be an antiderivative.

Theorem (The family of antiderivatives)

If F is an antiderivative of f , then the function f has a whole **family of antiderivatives**. Each antiderivative of f is the sum of F and some constant C . The family of all antiderivatives of f is denoted by

$$\left(\begin{array}{c} \text{family of all} \\ \text{antiderivatives} \end{array} \right) = \int f(x) dx.$$

This is called the **indefinite integral** of f .

family w/ ∞ -many members.

"representative"

infinitude.

It follows that

$$\int f(x) dx = F(x) + C,$$

 Do not forget +C.

where F is any antiderivative of f and C is an arbitrary constant.

Ex $\left(\begin{array}{c} \text{The indefinite integral} \\ \text{of } f(x) = 2x - 7 \end{array} \right) = \int (2x - 7) dx = \underbrace{x^2 - 7x + C}_{\text{an antiderivative}}$

Basic Indefinite Integrals

- $\int k dx = kx + C$
- $\int \frac{1}{x} dx = \ln|x| + C$
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\ln(a)} + C$
- $\int \cos(x) dx = \sin(x) + C$
- $\int \sin(x) dx = -\cos(x) + C$
- $\int \sec^2(x) dx = \tan(x) + C$
- $\int \csc^2(x) dx = -\cot(x) + C$
- $\int \sec(x) \tan(x) dx = \sec(x) + C$
- $\int \csc(x) \cot(x) dx = -\csc(x) + C$
- $\int \frac{1}{x^2+1} dx = \arctan x + C$
- $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

Confirm: Differentiate RHS and

see if it equals the "integrand" on LHS.

"integrand": function to be antidiifferentiated.

$\int \boxed{f(x)} dx$

$$\frac{d}{dx} \frac{x^{n+1}}{n+1} = \frac{\cancel{n+1} x^n}{\cancel{n+1}} = x^n$$

Basic Antiderivative Rules

We have the following rules that mirror basic derivative rules.

Theorem

If F is an antiderivative of f and G is an antiderivative of g , then $F + G$ is an antiderivative of $f + g$. Moreover, for any constant k , kF is an antiderivative of kf . We can write equivalently, using indefinite integrals,

$$\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx, \quad (\text{sum rule})$$

$$\int kf(x) dx = k \int f(x) dx. \quad (\text{constant multiple rule})$$

Q. How about "differences"?

$$\begin{aligned} \text{A. } \int (f(x) - g(x)) dx &= \int (f(x) + (-1)g(x)) dx \\ &= \int f(x) dx + \int (-1)g(x) dx \quad (\text{sum rule}) \\ &= \int f(x) dx + (-1) \int g(x) dx \quad (\text{const. mult.}) \\ &= \int f(x) dx - \int g(x) dx. \end{aligned}$$

Question. Compute:

$$\int (x^4 + 5x^2 - \cos(x)) dx$$

Sum rule

$$= \int x^4 dx + \int 5x^2 dx + \int (-\cos x) dx$$

const. mult.

$$= \int x^4 dx + 5 \int x^2 dx - \int \cos(x) dx$$

↓ power rule ↓

b/c $\frac{d}{dx} \sin(x) = \cos(x)$.

$$= \frac{x^5}{5} + 5 \frac{x^3}{3} - \sin(x) + C$$

"representatives"

"package" all as-many of them.

Question. A student claims that $\int 2x \cos(x) dx = x^2 \sin(x) + C$. Determine whether the student is correct or incorrect.

Verification

using

$$\int f(x) dx = F(x) + C \quad \text{implies} \quad F'(x) = f(x)$$

$$\frac{d}{dx} \left[\underbrace{x^2 \sin(x) + C}_{\text{deriv. of RHS}} \right] = 2x \sin(x) + x^2 \cos(x) + 0 \neq \underbrace{2x \cos(x)}_{\text{integrand of LHS.}}$$

Hence, the student's answer is incorrect.

Guessing Antiderivatives

Question. Compute:

$$\int \frac{\sqrt{x} + 1 + x}{x} dx$$

Side

$$\frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$$

Rewriting / Simplifying the integrand:

$$= \int \frac{\sqrt{x}}{x} dx + \int \frac{1}{x} dx + \int \frac{x}{x} dx$$

$$= \int x^{-\frac{1}{2}} dx + \int \frac{1}{x} dx + \int 1 dx$$

$$-\frac{1}{2} + 1 = \frac{1}{2}$$

$$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \ln|x| + x + C$$

Revisit in couple week.

Question. Compute:

$$\int 3x^2 \sin(x^3 - 6) dx$$

Revisit later.

Question. Compute:

$$\int \frac{2x^2}{7x^3 + 3} dx$$

Guessing Antiderivatives ("linear factors") a, b constants.

$$\left| \frac{d}{dx} F(ax+b) = a F'(ax+b) \Rightarrow \int F'(ax+b) dx = \frac{1}{a} F(ax+b) + C \right.$$

Example

$$\bullet \int (7x+5)^5 dx = \frac{1}{7} \frac{(7x+5)^6}{6} + C$$

$$\bullet \int \sin(\pi x) dx = -\frac{1}{\pi} \cos(\pi x) + C$$

plays the role
of "a"

$$\text{Check: } \frac{d}{dx} \left(-\frac{1}{\pi} \cos(\pi x) \right) = -\frac{1}{\cancel{\pi}} (-\sin(\pi x)) \cdot \cancel{\pi} = \sin(\pi x) \quad \checkmark$$

Differential Equations

$3x + 2 = 7$. What is x ?

↖ unknown number

$f(x)$: unknown function.

- A differential equation is simply an equation with a derivative in it. Here is an example:

$$\underline{a}f'(x) + \underline{b}f(x) = g(x).$$

↖ given

- Differential equations show you relationships between rates of functions.
- The theory of differential equation is a very important branch of mathematics with vast real-life applications.

What Does It Mean To Solve A Differential Equation?

When a mathematician solves a differential equation, they are finding *functions* satisfying the equation. For example, consider the following differential equation:

$$f'(x) = f(x).$$

Find a function $f(x)$ whose derivative equals itself.

- It turns out that the complete solution to this differential equation is Ce^x , i.e., all the solutions of this differential equation have this form.
- Showing that any function $y = Ce^x$ is a solution of this differential equation is easy, $(LHS) = Ce^x \stackrel{?}{=} (RHS) = Ce^x$
- but showing that **all** of the solutions have this form is beyond the scope of this course.

General Solution and Initial Value Problems

- In the previous example, a function Ce^x is called a **general solution** of the differential equation. ← arbitrary const.
- Since there are infinitely many solutions to a differential equation, we can impose additional condition, called an **initial condition**, e.g. $f(0) = 1$.

The problem now is to find a function f that satisfies both the differential equation (DE) and the initial condition (IC).

$$\left[\begin{array}{ll} f'(x) = f(x) & \text{(DE)} \\ f(0) = 1 & \text{(IC)} \end{array} \right.$$

This is called an **initial value problem (IVP)**.

Example: IVP and A Falling Object

Here is a classical example of IVP arising in simple physics.

Question. A ball is tossed into the air with an initial velocity of 15 m/s. What is the velocity of the ball after 1 second? How about after 2 seconds?

$V(t)$: velocity
at time t .

$$\begin{cases} \text{(DE)} & V'(t) = -g \quad (\text{acceleration}) \\ \text{(IC)} & V(0) = 15 \end{cases}$$

Antidiff.

$$\Rightarrow \text{(G.S.) } V(t) = \int (-g) \, dt = -gt + C$$

$$\Rightarrow \text{Using IC: } V(0) = -g \cdot 0 + C = 15$$

↑ up: +

↓ down: -

Question. A ball is tossed into the air with an initial velocity of 15 m/s from a height of 2 meters. When does the ball hit the ground?

(Exercise)

$$\therefore C = 15 \quad \text{and so} \quad \underline{V(t) = -gt + 15}$$