Lecture 33-34: Definite Integrals (DI)

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Definite Integrals

Definition

Let f be a function which is continuous on the interval $[a, b]$. We define the **definite integral** of f on $[a, b]$ by

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x\,.$$

The definite integral is a number that gives the **net area** of the region between the curve $y = f(x)$ and the x-axis on the interval $[a, b]$.

Basic Properties

Theorem (Properties of the definite integral)

Let f and g be defined on a closed interval $[a, b]$ that contains the value c, and let k be a constant. The following hold:

1. $\displaystyle\int_a^a f(x)\,dx = 0$
2. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$
3. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
4. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
5. $\displaystyle\int_a^b \{f(x) \pm g(x)\}\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Definite Integrals Using Geometry vs. Definition

Question. Compute the integral

$$\int_0^{10} (4 - x)\, dx$$

in two ways:

1. by interpreting the integral as the net area of the region between the curve $y = 4 - x$ and the interval $[0, 10]$ on the x-axis;
2. using the definition of the definite integral, i.e. by computing the limit of Riemann sums.

Question. Compute the integral

$$\int_0^{10} |4 - x| \, dx \, .$$

Note: Net Areas vs. Geometric Areas

We know that the net area of the region between a curve $y = f(x)$ and the x-axis on $[a, b]$ is given by

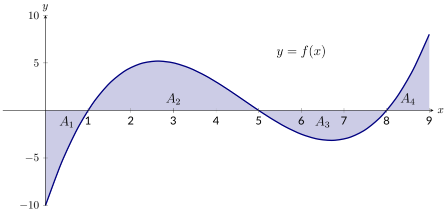
$$\int_a^b f(x)\,dx.$$

On the other hand, if we want to know the *geometric area*, meaning the "actual" area, we compute

$$\int_a^b |f(x)|\,dx.$$

Question. The graph of a function f is given in the figure.

1. Express the geometric area of the region between the curve $y = f(x)$ and the $x-$axis on the interval $[0, 9]$ as a definite integral.
2. Express the geometric area of the region between the curve $y = f(x)$ and the $x-$axis on the interval $[0, 9]$ in terms of definite integrals of f.
3. Express the geometric area of the region between the curve $y = f(x)$ and the $x-$axis on the interval $[0, 9]$ in terms of areas A_1, A_2, A_3 and A_4.



Question. Compute the limit:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{1 - \left(-1 + \frac{2k}{n}\right)^2} \right) \left(\frac{2}{n} \right)$$

Question. Express the following limit of Riemann sum as a definite integral:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k\pi}{n} + \cos \frac{k\pi}{n} \right) \frac{\pi}{n}\,.$$

Definite Integrals of Symmetric Functions

Recall that a function f is

- an **odd** function if $f(-x) = -f(x)$;
- an **even** function if $f(-x) = f(x)$.

Theorem

Let f be a symmetric function on a symmetric interval $[-a, a]$ where $a > 0$. Then

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\displaystyle\int_{0}^{a} f(x)\,dx & \text{if } f \text{ is even} \\ 0 & \text{if } f \text{ is odd.} \end{cases}$$

Question.

1. Find the following definite integral:

$$\int_{-4}^{4} \frac{x^2 \sin^3(x)}{\sqrt{x^4+1}}\, dx\,.$$

2. Suppose that f is an even function. Given that $\int_0^6 f(x)\, dx = 13$, find $\int_{-6}^{6} (5f(x) + 14)\, dx$.