Lecture 22-23: Graphing Functions (COGF & CFGF)

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Calculus and Graphs

Let's put together all the tools we've learned so far with graphical implications:

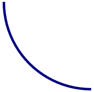
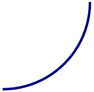
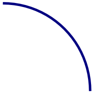
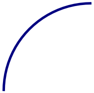
- **Infinite limits** indicate the presence of vertical asymptotes.
- **Limits at infinity** describe "far-field" behavior of the function, e.g., horizontal asymptotes.
- **The sign of the first derivative** tells us about the monotonicity, i.e., whether the graph is increasing or decreasing.
- **The sign of the second derivative** conveys the concavity information, i.e., whether it is concave up or down.

On Monotonicity and Concavity

Combining two possible monotonicity and two possible concavity modes, we came up with the following four signature of curves:

- $f' > 0$ and $f'' > 0$: increasing and concave up
- $f' > 0$ and $f'' < 0$: increasing and concave down
- $f' < 0$ and $f'' > 0$: decreasing and concave up
- $f' < 0$ and $f'' < 0$: decreasing and concave down

Recall the following table from couple weeks ago.

	$f'(x) < 0$	$f'(x) > 0$
$f''(x) > 0$	 The function f is decreasing, while the rate itself is increasing. In this case the curve $y = f(x)$ is **concave up**.	 The function f is increasing, while the rate itself is increasing. In this case the curve $y = f(x)$ is **concave up**.
$f''(x) < 0$	 The function f is decreasing, while the rate itself is decreasing. In this case the curve $y = f(x)$ is **concave down**.	 The function f is increasing, while the rate itself is decreasing. In this case the curve $y = f(x)$ is **concave down**.

On Critical and Inflection Points

I have several important remarks on **critical points** and **inflection points**:

- **Critical points** are interior points.
- There are two types of critical points – one at which $f' = 0$ (**the nice ones**) and the other at which f' is not defined (**the exotic ones**). Do not neglect the second kind.
- Being a critical point is merely a requirement to be a local extremum. It is not guaranteed that a critical point must be a local minimum or a local maximum.
- An **inflection point** is a point at which
  - f is continuous AND
  - f changes concavity from concave down to up or up to down.

On Derivative Tests

Lastly, on the derivative tests:

- These are used to classify critical points into local maxima or local minima. Once again, understand that a critical point may be neither one of them.
- The key idea of these derivative tests is as follows:
  Suppose c is a critical points of f.
  - If a graph shifts from an increasing to a decreasing phase about c, then it is a local maximum.
  - If a graph shifts from a decreasing to an increasing phase about c, then it is a local minimum.
- In the 1st Derivative Test, we look out for the change in sign of f' **about** c.
- In the 2nd Derivative Test, we look out for the sign of f'' **at** c.

Example

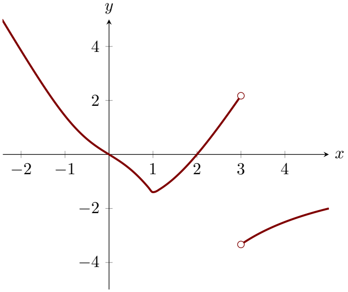
Sketch the graph of a function f which has the following properties:

- $f(0) = 0$
- $\lim_{x \to 10^+} f(x) = +\infty$
- $\lim_{x \to 10^-} f(x) = -\infty$
- $f'(x) < 0$ on $(-\infty, 0) \cup (6, 10) \cup (10, 14)$
- $f'(x) > 0$ on $(0, 6) \cup (14, \infty)$
- $f''(x) < 0$ on $(4, 10)$
- $f''(x) > 0$ on $(-\infty, 4) \cup (10, \infty)$

Example

The graph of f' (the derivative of f) is shown below. Assume f is continuous for all real numbers.

1. On which of the following intervals is f increasing?
2. Which of the following are critical points of f?
3. Where do the local maxima occur?
4. Where does a point of inflection occur?
5. On which of the following intervals is f concave down?



Example

Let $f(x) = \dfrac{1}{1+x^2}$. Find the following for f:

1. f' and f''

2. Critical points

3. Local extrema

4. Inflection points

Example

Sketch the plot of $2x^3 - 3x^2 - 12x$.

Example

Sketch the plot of

$$f(x) = \begin{cases} xe^x + 2 & \text{if } x < 0 \\ x^4 - x^2 + 3 & \text{if } x \geq 0. \end{cases}$$

Summary

The following is the list of all the tools at our finger tips to sketch the graph of $y = f(x)$

- Compute f' and f''.
- Find the y-intercept, this is the point $(0, f(0))$. Place this point on your graph.
- Find any vertical asymptotes, these are points $x = a$ where $f(x)$ goes to infinity as x goes to a (from the right, left, or both).
- If possible, find the x-intercepts, the points where $f(x) = 0$. Place these points on your graph.
- Analyze end behavior: as $x \to \pm\infty$, what happens to the graph of f? Does it have horizontal asymptotes, increase or decrease without bound, or have some other kind of behavior?
- Find the critical points (the points where $f'(x) = 0$ or $f'(x)$ is undefined).
- Use either the first or second derivative test to identify local extrema and/or find the intervals where your function is increasing/decreasing.
- Find the candidates for inflection points, the points where $f''(x) = 0$ or $f''(x)$ is undefined.
- Identify inflection points and concavity.
- Determine an interval that shows all relevant behavior.