Lecture 14: Implicit Differentiation (ID)

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Motivation

Depending on whether or not the dependent variable is written explicitly in terms of the independent variable, a function can be classified as an **explicit function** or an **implicit function**. For example:

- Explicit functions ($y = f(x)$ form)

$$y = 3x^2 - 2x + 1\,, \qquad y = e^{3x}\,, \qquad y = \frac{x-2}{x^2 - 3x + 2}\,, \ldots$$

- Implicit functions ($F(x, y) = 0$ form)

$$x^2 + y^2 = 4\,, \qquad x^3 + y^3 = 9xy\,, \qquad x^4 + 3x^2 = x^{2/3} + y^{2/3} + 1\,, \ldots$$

Today's goals are:

- to learn how to differentiate implicit functions
- to derive more differentiation shortcuts using the new technique

Implicit Differentiation

Procedures

In order to differentiate an implicit function:

1. Differentiate the entire equation with respect to x.
2. Solve for $\frac{dy}{dx}$.

Note.

- In Step 1, keep in mind that y is actually a function of x.
- This inevitably requires an application of the chain rule.

Question. Consider the curve defined by:

$$x^2 + y^2 = 1\,.$$

1. Compute $\dfrac{dy}{dx}$.

2. Find the slope of the tangent line at $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

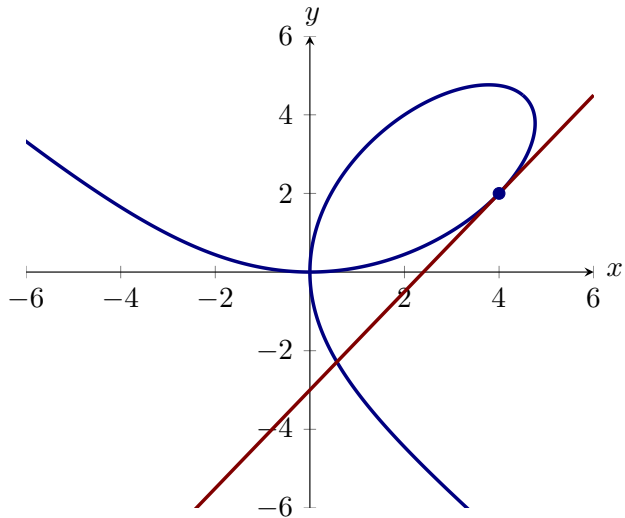
Question. Consider the curve defined by:

$$x^3 + y^3 = 9xy\,.$$

1. Compute $\dfrac{dy}{dx}$.

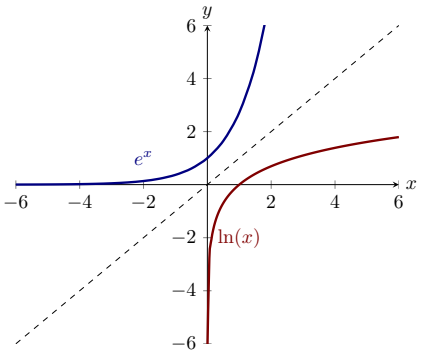
2. Find the slope of the tangent line at $(4, 2)$.

Graph for the previous problem.



Derivatives of Logarithmic Functions

- Recall that e^x and $\ln(x)$ are inverses and thus their graphs are reflections of each other about $y = x$.
- In particular, we can write $y = \ln(x)$ as $x = e^y$.
- And we know that $\dfrac{d}{dx}e^x = e^x$.



Question: What is $\frac{d}{dx}\ln(x)$?

Theorem (The derivative of logarithm)

Let $b > 0$ and $b \neq 1$. Then

$$\frac{d}{dx} \log_b(x) = \frac{1}{x \ln(b)}.$$

In particular, when $b = e$, the formula reduces to

$$\frac{d}{dx} \ln(x) = \frac{1}{x}.$$

Explanation.

Question. Compute:

1. $\dfrac{d}{dx}\left(-\ln(\cos(x))\right)$

2. $\dfrac{d}{dx}\log_7(x)$

The Derivative of an Exponential Function

Theorem (The derivative of an exponential function)

Let a be a positive real number. Then

$$\frac{d}{dx}a^x = a^x \cdot \ln(a)\,.$$

Explanation.

Question. Compute

$$\frac{d}{dx}7^x.$$