Lecture 8: Definition of the Derivative (DOTD)

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Autumn 2021

Rates of change

The rate of change of

- **position** of an object in time: **velocity**
- **velocity** of an object in time: **acceleration**
- **revenue** generated by selling objects: **marginal revenue**
- **cost** to produce objects: **marginal cost**
- **profit** gained by selling objects: **marginal profit**

From slopes of secant lines ...

The general formula for average rate of change is given by

$$\frac{\text{change in the function}}{\text{change in the input to the function}}\,.$$

- In order to produce this rate of change, we need two distinct input values, e.g., two distinct points in time, and their corresponding outputs.
- On the graph of the function f representing the quantity of interest, this rate is exactly the slope of the straight line connecting two points $(a, f(a))$ and $(b, f(b))$.
- Such a line is called a **secant line** whose slope is given by

$$m_{\text{sec}} = \frac{f(b) - f(a)}{b - a}\,.$$

Question. If $f(x) = 2x^2 + 3$, find the slope of the secant line through $(2, f(2))$ and $(x, f(x))$ in terms of x. Do the same when x is expressed as $2 + h$. The answer must be written in terms of h.

…to slopes of tangent lines

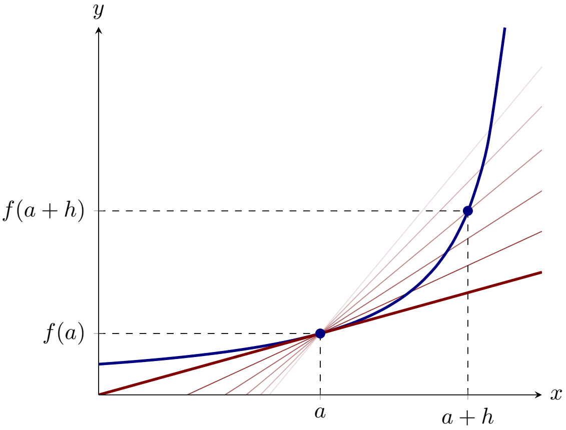
Now, an important question is: how do we get an instantaneous rate of change out of this?

- The slope of so-called **tangent line** represents this rate.
- It is given in terms of limit of slope of secant lines [1]:

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

[1] **Note.** We have two equivalent characterizations of this instantaneous rate of change depending on how we solved the previous problem.

Illustration



Definition of derivative

Definition

The **derivative** of f at a is

$$\left[\frac{d}{dx}f(x)\right]_{x=a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \qquad (h \to 0 \text{ characterization})$$
$$= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \qquad (x \to a \text{ characterization})\,.$$

If this limit exists, then we say that f is **differentiable** at a. If this limit does not exist for a given value of a, then f is **non-differentiable** at a.

Notation

The following are equivalent notations for derivatives:

$$\left[\frac{d}{dx}f(x)\right]_{x=a} = f'(a)\,.$$

Example

If $f(x) = x^2 - 2x$, find the derivative of f at 2 using the $h \to 0$ characterization.

Example

Find the derivative of $f(x) = x^2 + x + 1$ at $x = -1$ using the $x \to a$ characterization.

Example

Find an equation for the line tangent to the curve $y = f(x) = 1/(3 - x)$ at the point $(2, 1)$.

Example

The position of an object moving along a straight line is given by $s(t) = \sqrt{t+3}$. Find its (instantaneous) velocity at time $t = 6$.