Lecture 24: Extreme and Mean Value Theorems (MVT)

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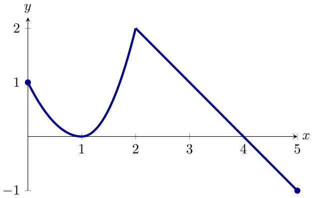
Extreme Values of a Function

Definition

- A function f has a **global maximum** at a if $f(a) \geq f(x)$ for every x in the domain of the function.
- A function f has a **global minimum** at a if $f(a) \leq f(x)$ for every x in the domain of the function.

A **global extremum** is either a global maximum or a global minimum.

Question. Let f be the function given by the graph below.

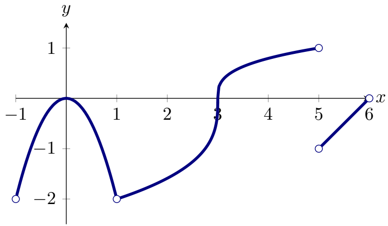


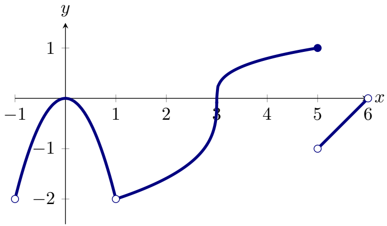
1. Find the x-coordinate of the point where the function f has a global maximum and a global minimum.

2. Find the x-coordinate(s) of the point(s) where the function f has a local minimum and a local maximum.

Caution

A function may not attain a global extremum on its domain. Consider the following graph.





Extreme Value Theorem

So when do we know for sure that a function attains a global extremum?

Theorem (The Extreme Value Theorem)

If f is continuous on the closed interval $[a,b]$, then there are points c and d in $[a,b]$, such that $(c, f(c))$ is a global maximum and $(d, f(d))$ is a global minimum on $[a,b]$.

Remark.

- The theorem does not hold if we work on an open interval (a,b). Can you come up with an example?
- The theorem does not hold if we work on a closed interval $[a,b]$ but f is not continuous. Can you come up with an example?
- Do the previous examples invalidate the EVT?

Remarks

In finding global extrema:

- The EVT guarantees the existence of global extrema when we work with a function f that is continuous on a closed interval.
- The global extrema may occur either at the end points of the interval or in the interior.
- If a global extremum occurs at an interior points, then it is also a local extremum thus a critical point.
- Hence, all **interior critical points** as well as the **end points** are candidates for global extrema.
- In sum, in order to locate global extrema, we evaluate the function at these candidate points and compare them to determine global maxima and global minima.

Question. Let $f(x) = x^2e^{-x}$, for $-2 \leq x \leq 1$. Locate the global extrema of f on the closed interval $[-2, 1]$.

Rolle's Theorem

- We now explore an intricate relation between **average rate** and **instantaneous rate of change**.
- In some context, this can be translated as relation between *average velocity* and *instantaneous velocity* or *slope of secant line* vs. *slope of tangent line*.

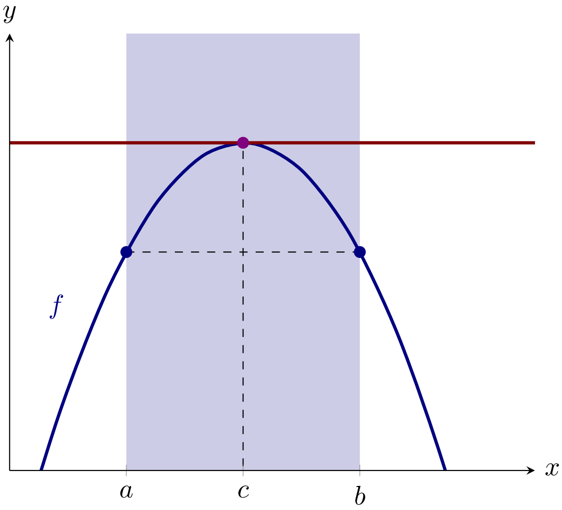
Theorem (Rolle's Theorem)

Suppose that f is differentiable on the interval (a, b), is continuous on the interval $[a, b]$, and $f(a) = f(b)$. Then

$$f'(c) = 0$$

for some $a < c < b$.

Illustration of Rolle’s Theorem



The Mean Value Theorem

A generalization of Rolle's theorem is coined as the mean value theorem.

Theorem (The Mean Value Theorem)

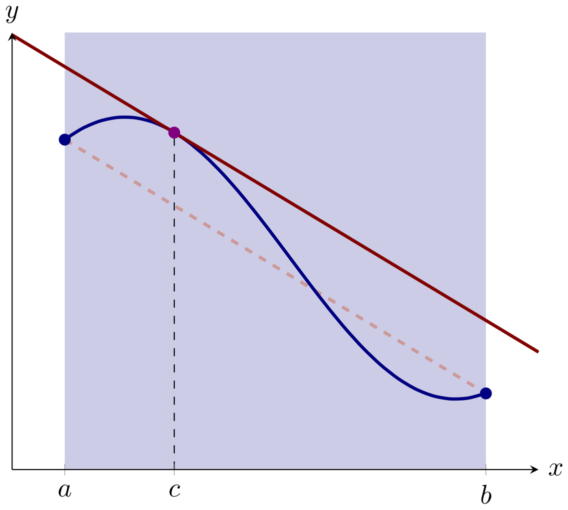
Suppose that f has a derivative on the interval (a, b) and is continuous on the interval $[a, b]$. Then

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some $a < c < b$.

- The mean value theorem says that when a function is continuous on a closed interval and differentiable in its interior, then its average rate of change must be achieved at some point as an instantaneous rate of change.

Illustration of the Mean Value Theorem



There are many interesting questions that can be answered using the mean value theorem.

Example

Suppose you toss a ball into the air and then catch it. Must the ball's vertical velocity have been zero at some point?

Example

Suppose you drive a car from toll booth on a toll road to another toll booth 30 miles away in half of an hour. Was there a moment that you violated the speed limit of 55 mph?

Example

Suppose the derivative of a function is 0 on an open interval I. What can we say about f?

Example

Suppose two different functions have the same derivative. What can you say about the relationship between the two functions?