Lecture 13: Higher Order Derivatives and Graphs (HODAG)

Tae Eun Kim, Ph.D.

Autumn 2021

6 **Labor Day** **No Classes**	7 Worksheet: ULTDA **WH1 due**	8 Continuity and the Intermediate Value Theorem (CATIVT)	9 Worksheet: CATIVT **HW: IF, ULTDA**	10 An Application of Limits (AAOL)
13 Definition of the Derivative (DOTD) **Midterm 1** **8:00-8:40PM** **UF – CATIVT**	14 Worksheet: AAOL, DOTD **HW: CATIVT**	15 Derivatives as Functions (DAF)	16 Worksheet: DAF **HW: AAOL, DOTD**	17 **Last day to drop w/o a "W"** Rules of Differentiation (ROD)
20 Product Rule and Quotient Rule (PRAQR) **WH2 due**	21 Worksheet: ROD, PRAQR **HW: DAF**	22 Chain Rule (CR)	23 Worksheet: PRAQR, CR **HW: ROD, PRAQR**	24 Higher Order Derivatives and Graphs (HODAG)
27 Implicit Differentiation (ID) **Midterm 2** **8:00-8:40PM** **AAOL-CR**	28 Worksheet: HODAG, ID **HW: CR**	29 Logarithmic Differentiation (LD)	30 Worksheet: ID, LD **HW: HODAG, ID**	**October 1** Derivatives of Inverse Functions (DOIF)
4	5	6	7	8

- Syllabus
- Carmen announcement (by math department)
- Gradescope formatting
- Make-up exam
  ↳ contact me for permission
  (Schedule conflict w/ class or work)

- $\boxed{\text{Timeline}}$
  - 7:55 PM : download exam
  - 8:00 PM : start working on it
  - 8:40 PM : upload exam

  40 min.

  - 8:55 PM : done

- After this, late!
- Do not email your exam to instructor!

Tae's

- Extended Office hours : M 4:15 ~ 6:15 usual zoom link
- Review video will be recorded this afternoon.

Higher-Order Derivatives

- The derivative of a function is often called the **first derivative**;
- The derivative of the derivative the **second derivative**;
- The derivative of the second derivative the **third derivative**, and so on.
- Derivatives of derivatives are called **higher-order derivatives** with the following notations:

First derivative: $\dfrac{d}{dx} f(x) = f'(x) = f^{(1)}(x).$

Second derivative: $\dfrac{d^2}{dx^2} f(x) = f''(x) = f^{(2)}(x).$

Third derivative: $\dfrac{d^3}{dx^3} f(x) = f'''(x) = f^{(3)}(x).$

2nd deriv.

$$\frac{d}{dx}\left(\frac{d}{dx} f(x)\right) = \frac{d^2}{dx^2} f(x).$$

Drawback w/ primes:

12th-deriv. of f:

$$f''''''''''''(x) = f^{(12)}(x)$$

Question. Compute the first, second, and third derivatives of:

1. $f(x) = x^2 + 3x - 8$

$f'(x) = 2x + 3, \quad f''(x) = 2, \quad f'''(x) = 0$

2. $g(x) = e^{2x}$

$g'(x) \overset{CR}{=} e^{2x} \cdot 2 = 2e^{2x}, \quad g''(x) \overset{CR}{=} \underbrace{2}_{\text{teenager}} \cdot \underbrace{2e^{2x}}_{\text{Same as before}} = 4e^{2x}, \quad g'''(x) \overset{CR}{=} 4 \cdot 2e^{2x} = 8e^{2x}$

3. $h(x) = \sin(x^2)$

$h'(x) \overset{CR}{=} \cos(x^2) \cdot 2x = 2x \cos(x^2)$

$\frac{d}{dx} \cos(x^2)$

$h''(x) \overset{PR}{=} 2 \cdot \cos(x^2) + 2x\left(-\sin(x^2) \cdot 2x\right)$

$= 2\cos(x^2) - 4x^2 \sin(x^2)$

$h'''(x) = 2\left(-\sin(x^2) \cdot 2x\right) - 4\left[2x \cdot \sin(x^2) + x^2 \cos(x^2) \cdot 2x\right]$

First Derivatives and Monotonicity

(Sign of f') $\longrightarrow$ monotonicity

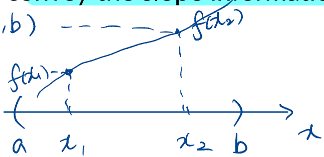
(Sign of f'') $\longrightarrow$ concavity

Definition (Monotonicity)

- We say that a function f is **increasing** on an interval I if $f(x_1) < f(x_2)$ for all pairs of numbers x_1, x_2 in I such that $x_1 < x_2$.
- We say that a function f is **decreasing** on an interval I if $f(x_1) > f(x_2)$ for all pairs of numbers x_1, x_2 in I such that $x_1 < x_2$.

- We say that a function is **monotonic** on an interval if it is either increasing or decreasing there.
- The notion of monotonicity is closely related to the derivatives as they convey the slope information of tangent lines to the curve.

$I = (a, b)$



Assume f is differentiable on I.

Theorem

A function f is ***increasing*** *on any interval I where $f'(x) > 0$, for all x in I. A function f is* ***decreasing*** *on any interval I where $f'(x) < 0$, for all x in I.*

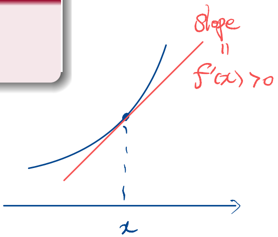
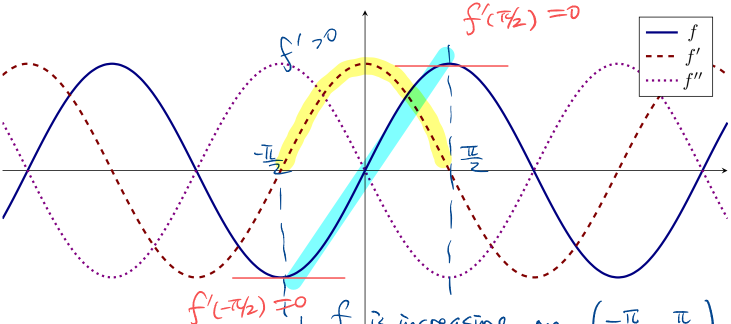
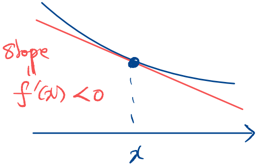


Illustration. Here we have graphs of $f(x) = \sin(x)$, $f'(x) = \cos(x)$, and $f''(x) = -\sin(x)$:



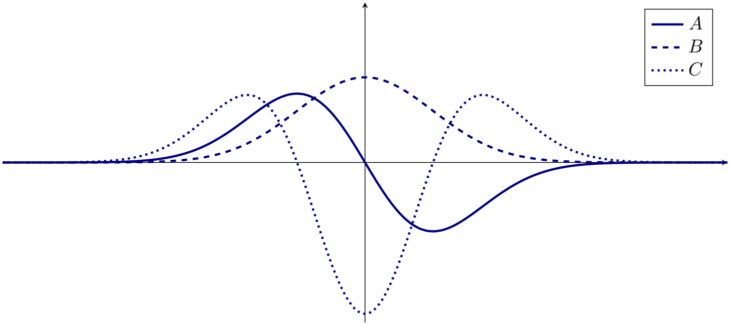


f is increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

f' is positive (i.e. above x-axis) there.

Exercise

Question. Below we have unlabeled graphs of f, f', and f''. Identify each curve above as a graph of f, f', or f''.



Second Derivatives and Concavity

Definition (Concavity)

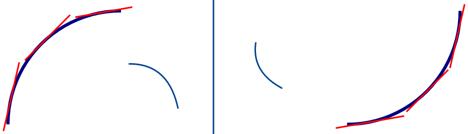
Let f be a function differentiable on an open interval I.

- We say that the graph of f is **concave up** on I if f' is **increasing** on I.
- We say that the graph of f is **concave down** on I if f' is **decreasing** on I.

tangent line getting steeper

tangent line getting flatter

Illustration.



The function f is increasing, while the rate itself is decreasing. In this case the curve $y = f(x)$ is **concave down**.

The function g is increasing, while the rate itself is increasing. In this case the curve $y = g(x)$ is **concave up**.

We know from the previous section that

f' is increasing/decreasing when its derivative f'' is positive/negative.

In other words, the second derivatives contain concavity information as summarized in the following theorem.

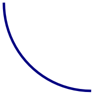
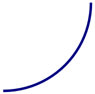
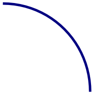
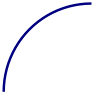
Theorem (Test for concavity)

Let I be an open interval.

1. *If $f''(x) > 0$ for all x in I, then the graph of f is concave up on I.*
2. *If $f''(x) < 0$ for all x in I, then the graph of f is concave down on I.*

$$
\begin{tikzcd}
f' \text{ INC} \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & f'' > 0 \arrow[dl, Leftrightarrow] \\
\text{C.U.} &
\end{tikzcd}
$$

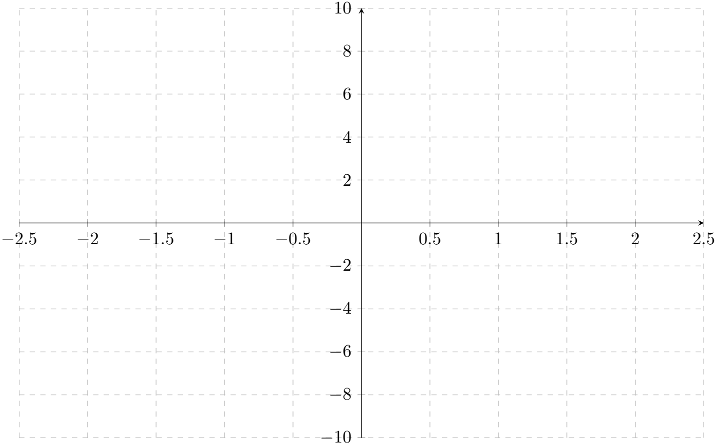
Summary: Derivatives and Graphs

Concavity \ Monotonicity	DEC $f'(x) < 0$	INC $f'(x) > 0$
CU $f''(x) > 0$	 The function f is decreasing, while the rate itself is increasing. In this case the curve $y = f(x)$ is **concave up**.	 The function f is increasing, while the rate itself is increasing. In this case the curve $y = f(x)$ is **concave up**.
CD $f''(x) < 0$	 The function f is decreasing, while the rate itself is decreasing. In this case the curve $y = f(x)$ is **concave down**.	 The function f is increasing, while the rate itself is decreasing. In this case the curve $y = f(x)$ is **concave down**.

Monday

Question. Find the intervals on which f is increasing/decreasing and concave up/down and plot its graph.

$$f(x) = x^3 - x^2 - 4x + 4\,.$$



Example from Physics

monotonicity
Concavity } graphical features.

Motion with constant acceleration

We know from physics that the motion of an object with constant acceleration a is described by the following formulas:

$$\text{position:} \qquad x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \,,$$

$$\text{velocity:} \qquad v(t) = v_0 + at \,,$$

$v(t) = x'(t)$

$\frac{1}{\cancel{2}} a \cdot \cancel{2}t$

$a(t) = v'(t)$

where x_0 and v_0 are the initial position and velocity respectively.

$= a$

In general,

- the derivative of the position function $x(t)$ is the velocity function $v(t)$, i.e., $v(t) = x'(t)$;
- the derivative of the velocity function $v(t)$ is the acceleration function $a(t)$, i.e. $a(t) = v'(t) = x''(t)$.

Question. The position of a moving particle is given by

$$s(t) = 36t^2 - 7t^3 .$$

Find a formula for its acceleration.

$$v(t) = s'(t) = 72t - 21t^2$$

$$a(t) = v'(t) = 72 - 42t .$$