# Lecture 35: Antiderivatives and Area (AAA)

Tae Eun Kim, Ph.D.

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# Relating antiderivatives and areas

In investigating connection between antiderivatives and area, we will use our favorite position-velocity-acceleration triple for illustration.

**Displacement vs. distance.**

Consider a moving object (in 1-D) from time $t = a$ to $t = b$. The *displacement* measures the difference in position. In other words,

$$(\text{displacement}) = (\text{terminal position}) - (\text{initial position}) = s(b) - s(a)\,.$$

**Note:**

- When an object moves without changing directions, the (traveled) distance equals the absolute value of displacement.
- However, when it changes directions along the course of movement, they are going to be different.
- In particular, distance is always going to be positive, but displacement may be negative.

# Simple case: uniform velocity

Now consider a simple situation where an object is moving at a constant velocity $v_0$ for $a \leq t \leq b$. Then the displacement is simply the velocity multiplied by the time traveled, i.e.,

$$(\text{displacement}) = v_0(b-a)\,. \qquad \text{(constant velocity)}$$

- The graph of velocity against time is a horizontal line.
- The displacement is exactly equal to the (signed) area of rectangle between the velocity curve (i.e., the straight line) and the horizontal time axis on $[a, b]$.

# Motion with changing velocity

Then how would we calculate the displacement when the object is moving at a varying velocity?

- Assuming that it moves at a constant velocity over a small interval of time, we can approximate displacement using Riemann sums;
- The quality of approximation improves as we increase the number of approximating rectangles;
- We obtain the exact displacement once we take the limit of general Riemann sum as the number $n$ of rectangles approaches infinity, that is

$$(\text{displacement}) = \int_a^b v(t)\, dt\,. \qquad \text{(variable velocity)}$$

# The connection

- But recall that the displacement is the difference between the terminal and initial positions, i.e., $s(b) - s(a)$. Thus

$$\int_a^b v(t)\, dt = s(b) - s(a)\,. \qquad (\star)$$

- Noting that $s'(t) = v(t)$, i.e., $s(t)$ is an antiderivative of $v(t)$, we may interpret the equation ($\star$) in a general setting as:

  *The net **area** between the curve $y = f(x)$ and the $x$-axis on $[a, b]$ is the difference of values of its **antiderivative** at the endpoints.*

- The statement above can be written as

$$\int_a^b f(x)\, dx = F(b) - F(a)\,,$$

  where $F$ is an antiderivative of $f$. This is the celebrated Fundamental Theorem of Calculus.

# Example

**Question.** Assume an object is moving along a straight line with the velocity $v(t) = 3 - 3t^2$ for $0 \leq t \leq 2$. Find the displacement of the object over the time interval $[0, 4]$.