Lecture 2: What Is A Limit (WIAL)

Tae Eun Kim, Ph.D.

Autumn 2021

Where are we?

MATH 1151 - AUTUMN 2021 THE OHIO STATE UNIVERSITY

Monday	Tuesday	Wednesday	Thursday	Friday
August 23	~~24 **First day of Classes** Worksheet: UF, ROFF~~	~~25 Understanding Functions (UF) Review of Famous Functions (ROFF)~~	~~26 Worksheet: ROFF~~	27 What is a Limit? (WIAL)
30 Limit Laws (LL)	31 Worksheet: WIAL, LL **HW: Precalc Rev**	**September 1** (In)determinate Forms (IF)	2 Worksheet: IF **HW: WIAL, LL**	3 Using Limits to Detect Asymptotes (ULTDA)

Saturday 28

Syllabus Quiz (by 11:59 PM)

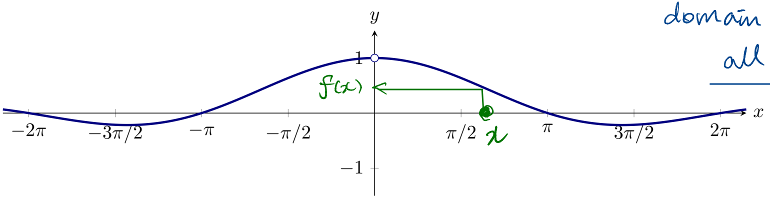
Any questions before we begin?

What is a limit?

Basic idea. Consider the function

$$f(x) = \frac{\sin(x)}{x}.$$

$x = 0$ makes DENOM vanish.
So we need to avoid this.
i.e.
domain of f is
all real #'s except for 0.



Question.

- Is f defined at $x = 0$? No, because DEN $= 0$ at $x = 0$.
- Where is $f(x)$ approaching as x gets closer to 0? 1.

↳ Note that we can answer this question even if f is not defined at $x = 0$.

Definition

Intuitively, we say that

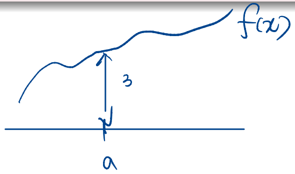
the **limit** of $f(x)$ as x approaches a is L,

written

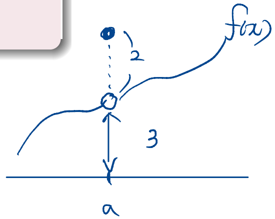
$$\lim_{x\to a} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all x sufficiently close, but not equal to, a.

because of this,
we can still talk about
$\lim_{x\to a} f(x)$
even if $f(x)$ is not defined at a.



$$\lim_{x\to a} f(x) = 3$$



$$f(a) = 5$$

$$\lim_{x\to a} f(x) = 3$$

One-sided limits

Definition

Intuitively,

the **limit from the right** of f as x approaches a is L,

written

$$\lim_{x\to a^+} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all $x > a$ sufficiently close, but not equal to, a.

Similarly,

the **limit from the left** of $f(x)$ as x approaches a is L,

written

$$\lim_{x\to a^-} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all $x < a$ sufficiently close, but not equal to, a.

Notation

$+$: from R

$-$: from L

Connection b/w limit & one-sided limits.

Theorem

A limit

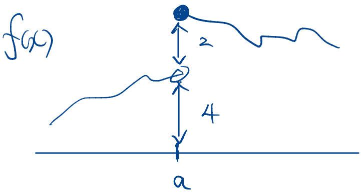
$$\lim_{x\to a} f(x)$$

exists if and only if

- $\lim_{x\to a^-} f(x)$ *exists*
- $\lim_{x\to a^+} f(x)$ *exists*
- $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

In this case, $\lim_{x\to a} f(x)$ *is equal to the common value of the two one sided limits.*

Does not exist



- $\lim_{x\to a^-} f(x)$ exists and equals 4. ✓
- $\lim_{x\to a^+} f(x)$ exists and equals 6. ✓
- However, these two one-sided limits are different.

∴ The limit DNE.

Digression

Previous theorem is of the form:

$(P \Leftrightarrow Q)$

Equivalence P if and only if Q

statements statements.

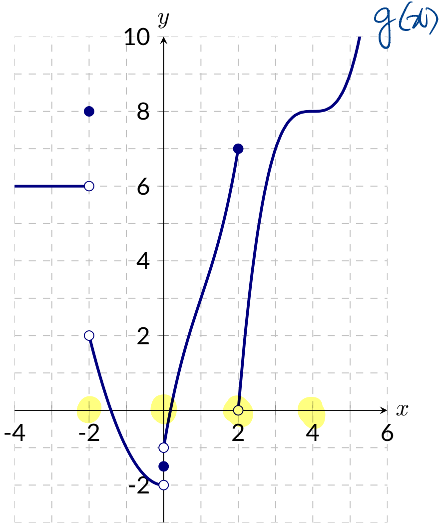
Combo
$$\begin{cases} P \text{ if } Q & (P \Leftarrow Q) \\ P \text{ only if } Q & (P \Rightarrow Q) \end{cases}$$

Question. Study limits of the following graph of $g(x)$ at various points. at $x = -2, 0, 2, 4$

(a) $\lim_{x \to -2} g(x)$

- $\lim_{x \to -2^-} g(x) = 6$ ✓
- $\lim_{x \to -2^+} g(x) = 2$ ✓
- Are they equal? $6 \neq 2$ ✗

No (DNE)



(d) $\lim_{x \to 4} g(x) = \boxed{8}$

- $\lim_{x \to 4^-} g(x) = 8$
- $\lim_{x \to 4^+} g(x) = 8$
- Are they equal? Yes

Argue similarly for $x = 0$, $x = 2$.

Continuity

"for f to be cts. at a, it has to be defined at a."

Definition

A function f is **continuous at a point** a if

$$\lim_{x \to a} f(x) = f(a).$$

We can unpack the single equation above as:

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

Question. How can a function be discontinuous at a point?

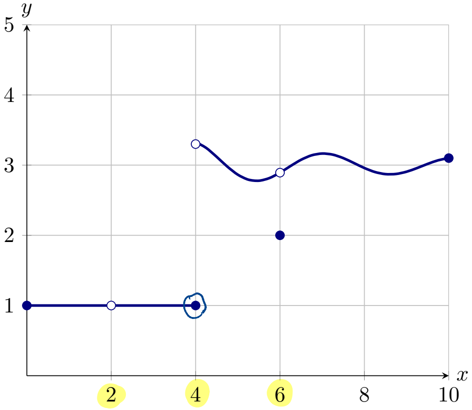
if one or more of these conditions are violated, then f is discontinuous at a.

Question. Find the discontinuities.

① $f(a)$ defined

② $\lim_{x\to a} f(x)$ exists

③ $f(a) = \lim_{x\to a} f(x)$



Weekend:

Try to classify / identify different types of discont.

① is not met.

① : $f(4)=1$

② : $\lim_{x\to 4^-} f(x) = 1 \neq \lim_{x\to 4^+} f(x) = 3.4$. So $\lim_{x\to 4} f(x)$ DNE.

① : $f(6)=2$

② : $\lim_{x\to 6} f(x) = 2.8$

$\neq$ ③ is not met.

one-sided continuity

Definition

- A function f is **left continuous** at a point a if $\lim_{x\to a^-} f(x) = f(a)$.
- A function f is **right continuous** at a point a if $\lim_{x\to a^+} f(x) = f(a)$.

We can talk about continuity on intervals now.

Definition

A function f is

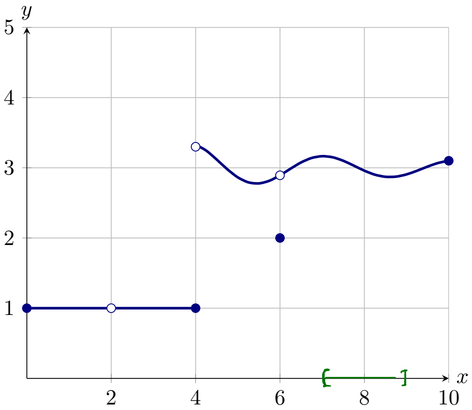
- **continuous on an open interval** (a, b) if $\underbrace{\lim_{x\to c} f(x) = f(c)}_{f \text{ is cts. at } c}$ for all c in (a, b);
- **continuous on a closed interval** $[a, b]$ if f is continuous on (a, b), right continuous at a, and left continuous at b.

Continuity of Famous Functions

The following functions are continuous on their natural domains, for k a real number and b a positive real number:

- **Constant function** $f(x) = k$
- **Identity function** $f(x) = x$
- **Power function** $f(x) = x^b$ $\rightarrow$ $x^2, x^3, \cdots, x^{1/2} = \sqrt{x}, x^{1/3} = \sqrt[3]{x}, \cdots$
- **Exponential function** $f(x) = b^x$
- **Logarithmic function** $f(x) = \log_b(x)$
- **Sine and cosine functions** $f(x) = \sin(x)$ and $f(x) = \cos(x)$

Question. (Revisiting the previous graph) What are the *largest intervals* of continuity?



- Consider $[7,9]$.

  On this interval, f is cts.

  However, we can find a larger interval containing $[7,9]$ on which f is still continuous.