Lecture 30: Antiderivatives (A)

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Basic Antiderivatives

Antidifferentiation is a process where we undo differentiation. Precisely:

Definition

A function F is called an **antiderivative** of f on an interval if

$$F'(x) = f(x)$$

for all x in the interval.

Since the derivative of a constant is zero, we can add it to any antiderivative of f and it will still be an antiderivative.

Theorem (The family of antiderivatives)

*If F is an antiderivative of f, then the function f has a whole **family of antiderivatives**. Each antiderivative of f is the sum of F and some constant C. The family of all antiderivatives of f is denoted by*

$$\int f(x)\, dx\,.$$

*This is called the **indefinite integral** of f.*

It follows that

$$\int f(x)\, dx = F(x) + C,$$

where F is any antiderivative of f and C is an arbitrary constant.

Basic Indefinite Integrals

- $\int k\, dx = kx + C$
- $\int \frac{1}{x}\, dx = \ln|x| + C$
- $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
- $\int e^x\, dx = e^x + C$
- $\int a^x\, dx = \frac{a^x}{\ln(a)} + C$
- $\int \cos(x)\, dx = \sin(x) + C$
- $\int \sin(x)\, dx = -\cos(x) + C$
- $\int \sec^2(x)\, dx = \tan(x) + C$
- $\int \csc^2(x)\, dx = -\cot(x) + C$
- $\int \sec(x)\tan(x)\, dx = \sec(x) + C$
- $\int \csc(x)\cot(x)\, dx = -\csc(x) + C$
- $\int \frac{1}{x^2+1}\, dx = \arctan x + C$
- $\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + C$

Basic Antiderivative Rules

We have the following rules that mirror basic derivative rules.

Theorem

If F is an antiderivative of f and G is an antiderivative of g, then $F + G$ is an antiderivative of $f + g$. Moreover, for any constant k, kF is an antiderivative of kf. We can write equivalently, using indefinite integrals,

$$\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx\,, \qquad \text{(sum rule)}$$

$$\int kf(x)\, dx = k \int f(x)\, dx\,. \qquad \text{(constant multiple rule)}$$

Question. Compute:

$$\int \left(x^4 + 5x^2 - \cos(x)\right) dx$$

Question. A student claims that $\int 2x\cos(x)\,dx = x^2\sin(x) + C$. Determine whether the student is correct or incorrect.

Guessing Antiderivatives

Question. Compute:

$$\int \frac{\sqrt{x} + 1 + x}{x} \, dx$$

Question. Compute:

$$\int 3x^2 \sin\left(x^3 - 6\right) dx$$

Question. Compute:

$$\int \frac{2x^2}{7x^3+3}\,dx$$

Differential Equations

- A *differential equation* is simply an equation with a derivative in it. Here is an example:
$$af'(x) + bf(x) = g(x).$$
- Differential equations show you relationships between rates of functions.
- The theory of differential equation is a very important branch of mathematics with vast real-life applications.

What Does It Mean To Solve A Differential Equation?

When a mathematician solves a differential equation, they are finding *functions* satisfying the equation. For example, consider the following differential equation:

$$f'(x) = f(x)\,.$$

- It turns out that the complete solution to this differential equation is Ce^x, i.e., all the solutions of this differential equation have this form.
- Showing that any function $y = Ce^x$ is a solution of this differential equation is easy,
- but showing that **all** of the solutions have this form is beyond the scope of this course.

General Solution and Initial Value Problems

- In the previous example, a function Ce^x is called a **general solution** of the differential equation.
- Since there are infinitely many solutions to a differential equation, we can impose additional condition, called an **initial condition**, e.g. $f(0) = 1$.

The problem now is to find a function f that satisfies both the differential equation (DE) and the initial condition (IC).

$$f'(x) = f(x) \tag{DE}$$

$$f(0) = 1 \tag{IC}$$

This is called an **initial value problem** (IVP).

Example: IVP and A Falling Object

Here is a classical example of IVP arising in simple physics.

Question. A ball is tossed into the air with an initial velocity of 15 m/s. What is the velocity of the ball after 1 second? How about after 2 seconds?

Question. A ball is tossed into the air with an initial velocity of 15 m/s from a height of 2 meters. When does the ball hit the ground?