# Lecture 1: Review of Precalculus

Tae Eun Kim, Ph.D.

Autumn 2021

# Overview

1. Understanding Functions (UF)

2. Review of Famous Functions (ROFF)

# Understanding Functions (UF)

# “For each input, exactly on output”

## Definition

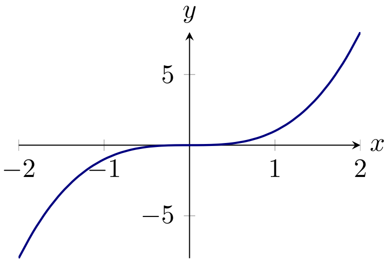
- **function**: a relation between sets where for each input, there is exactly one output
- **domain**: the set of the inputs of a function
- **range**: the set of the outputs of a function

# Representation of functions

- **formula:** $f(x) = x^3$
- **table:**

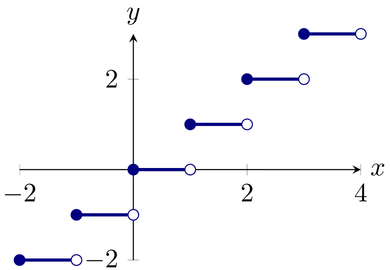
| input | 1 | -2 | 1.5 | $\cdots$ |
|---|---|---|---|---|
| output | 1 | -8 | 3.375 | $\cdots$ |

- **graph:**



## Example: Greatest Integer Function

- maps any real number $x$ to the greatest integer less than or equal to $x$.
- a.k.a. *floor function*
- denoted by $\lfloor x \rfloor$
- many inputs to one output



**Theorem (Vertical line test)**

*The curve $y = f(x)$ represents $y$ as a function of $x$ at $x = a$ if and only if the vertical line $x = a$ intersects the curve $y = f(x)$ at exactly one point. This is called the **vertical line test**.*

## Distinguishing two functions

- Do they have the same domain?
- Do they display the same relation?

**Question.** Determine if the two function are the same.

1. $f(x) = \sqrt{x^2}$ and $g(x) = |x|$

2. $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$ and $g(x) = x - 1$

# Composition of functions

## Composite functions

- can be thought of as putting one function inside another
- **Notation:** $(f \circ g)(x) = f(g(x))$
- **Warning:** The range of inner function must be contained in the domain of outer function.

**Question.** Study the composition $f \circ g$ where

$$\begin{aligned} f(x) &= x^2 && \text{for } -\infty < x < \infty, \\ g(x) &= \sqrt{x} && \text{for } 0 \le x < \infty. \end{aligned}$$

**Question.** Study the composition $f \circ g$.

$$f(x) = \sqrt{x} \qquad \text{for } 0 \leq x < \infty,$$
$$g(x) = x^2 \qquad \text{for } -\infty < x < \infty.$$

# Inverses of functions

**Definition**

Let $f$ be a function with domain $A$ and range $B$:

$$f : A \to B$$

Let $g$ be a function with domain $B$ and range $A$:

$$g : B \to A$$

We say that $f$ and $g$ are **inverses** of each other if $f(g(b)) = b$ for all $b$ in $B$, and also $g(f(a)) = a$ for all $a$ in $A$. Sometimes we write $g = f^{-1}$ in this case.

We could rephrase these conditions as

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

## Warning: notations

Pay attention to where we put the superscript:

$$
\begin{aligned}
f^{-1}(x) &= \text{the inverse function of } f(x). \\
f(x)^{-1} &= \text{the reciprocal of } f(x).
\end{aligned}
$$

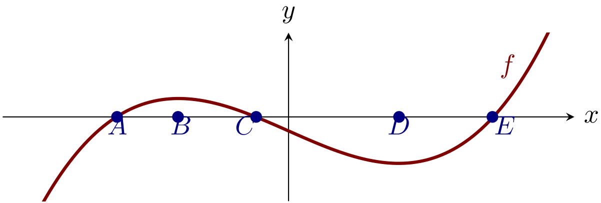
## Definition

A function is called **one-to-one** if each output value corresponds to exactly one input value.

**Theorem (Horizontal line test)**

*A function is one-to-one at $x = a$ if the horizontal line $y = f(a)$ intersects the curve $y = f(x)$ in exactly one point. This is called the* ***horizontal line test****.*

**Question.** Consider the graph of the function $f$ below:




On which of the following intervals is $f$ one-to-one?

1. $[A, B]$
2. $[A, C]$
3. $[B, D]$
4. $[C, E]$
5. $[C, D]$

# Review of Famous Functions (ROFF)

These are important functions for Math 1151:

- polynomial functions
- rational functions
- trigonometric functions and their inverses
- exponential and logarithmic functions

# Polynomial functions

## Definition

A **polynomial function** in the variable $x$ is a function which can be written in the form

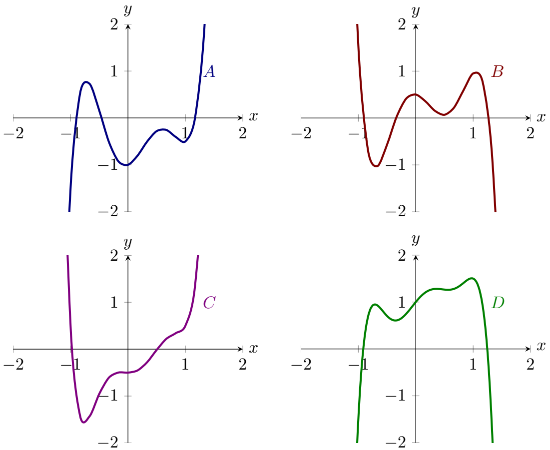
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the $a_i$'s are all constants (called the **coefficients**) and $n$ is a whole number (called the **degree** when $n \neq 0$). The domain of a polynomial function is $(-\infty, \infty)$.

**Question.** Which of the following are polynomial functions?

1. $f(x) = 7$
2. $f(x) = 3x + 1$
3. $f(x) = x^{1/2} - x + 8$
4. $f(x) = x^{-4} - 3x^{-2} + 7 + 12x^3$
5. $f(x) = (x + \pi)(x - \pi) + e^x - e^x$
6. $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$
7. $f(x) = x^7 - 32x^6 - \pi x^3 + 3/7$

## Some possible graphs of polynomials.



# Rational functions

## Definition

A **rational function** in the variable $x$ is a function the form

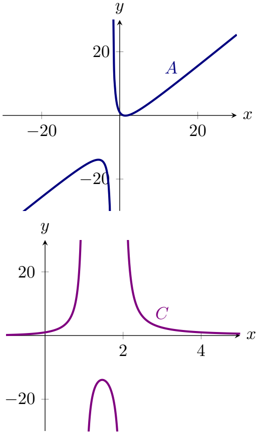
$$f(x) = \frac{p(x)}{q(x)}$$

where $p$ and $q$ are polynomial functions. The domain of a rational function is all real numbers except for where the denominator is equal to zero.

**Question.** Which of the following are rational functions?

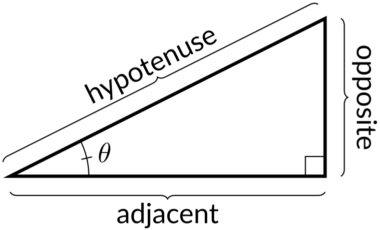
1. $f(x) = 0$
2. $f(x) = \dfrac{3x+1}{x^2 - 4x + 5}$
3. $f(x) = e^x$
4. $f(x) = \dfrac{\sin(x)}{\cos(x)}$
5. $f(x) = -4x^{-3} + 5x^{-1} + 7 - 18x^2$
6. $f(x) = x^{1/2} - x + 8$
7. $f(x) = \dfrac{\sqrt{x}}{x^3 - x}$

# Some possible graphs of rational functions.



# Trigonometric functions

A **trigonometric function** is a function that relates a measure of an angle of a right triangle to a ratio of the triangle's sides.



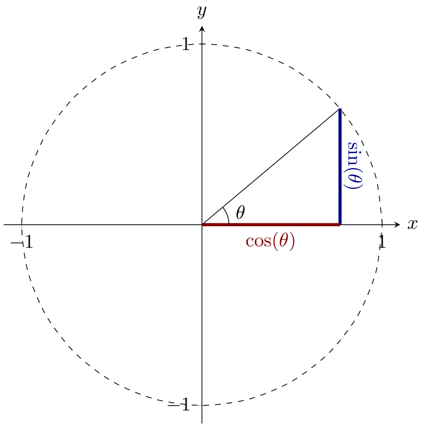
## Definition

The trigonometric functions are:

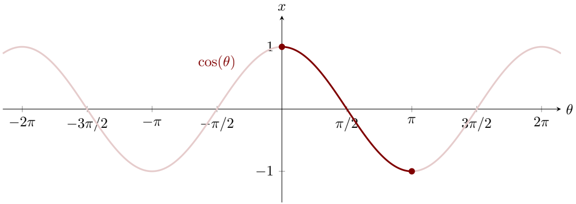
$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} \qquad \sin(\theta) = \frac{\text{opp}}{\text{hyp}} \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

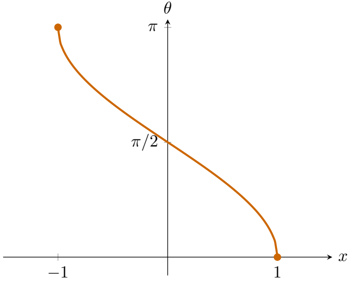
where the domain of sine and cosine is all real numbers, and the other are defined precisely when their denominators are nonzero.

# The unit circle and trig functions



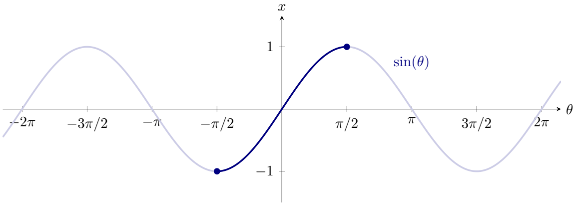
# Cosine and its inverse.

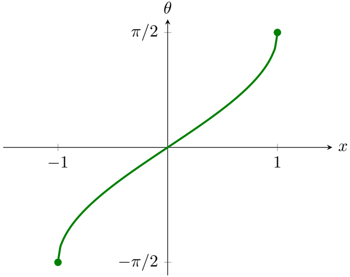



Here we see a plot of $\arccos(x)$, the inverse function of $\cos(\theta)$ when the domain is restricted to the interval $[0, \pi]$.

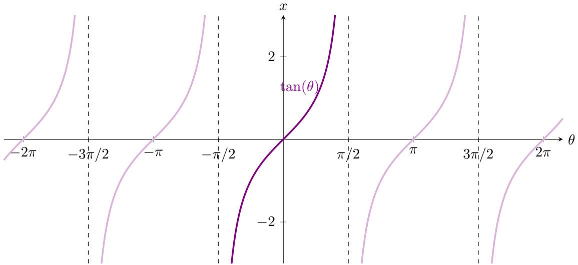
## Sine and its inverse.

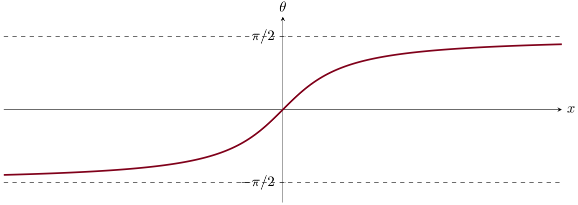



Here we see a plot of $\arcsin(x)$, the inverse function of $\sin(\theta)$ when the domain is restricted to the interval $[-\pi/2, \pi/2]$.

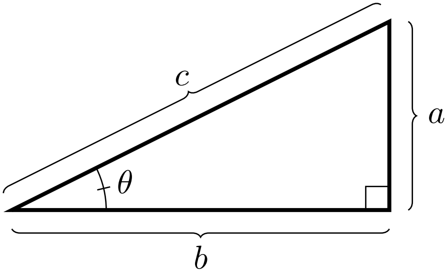
# Tangent and its inverse.




Here we see a plot of $\arctan(x)$, the inverse function of $\tan(\theta)$ when the domain is restricted to the interval $(-\pi/2, \pi/2)$.

## Pythagorean theorem and identities.




Pythagorean theorem:

- $a^2 + b^2 = c^2$

Pythagorean identities:

- $\cos^2\theta + \sin^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $\cot^2\theta + 1 = \csc^2\theta$

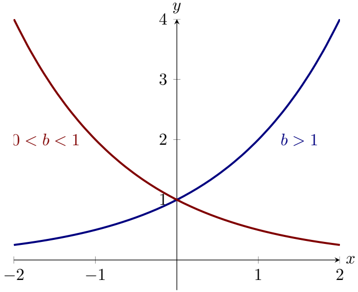
# Exponential and logarithmic functions

## Definition

An **exponential function** is a function of the form

$$f(x) = b^x$$

where $b \neq 1$ is a positive real number. The domain of an exponential function is $(-\infty, \infty)$. (Special: $f(x) = e^x$.)

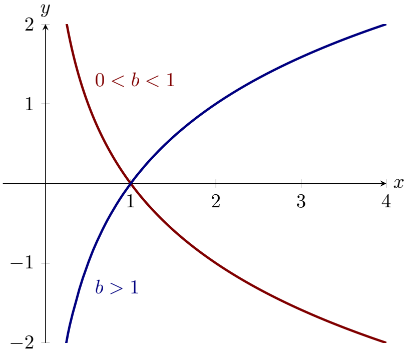


## Definition

A **logarithmic function** is a function defined as follows

$$\log_b(x) = y \qquad \text{means that} \qquad b^y = x$$

where $b \neq 1$ is a positive real number. The domain of a logarithmic function is $(0, \infty)$. (Special: $f(x) = \ln(x)$.)



## Properties of exponents

Let $b$ be a positive real number with $b \neq 1$.

- $b^m \cdot b^n = b^{m+n}$
- $b^{-1} = \dfrac{1}{b}$
- $(b^m)^n = b^{mn}$

## Properties of logarithms

Let $b$ be a positive real number with $b \neq 1$.

- $\log_b(m \cdot n) = \log_b(m) + \log_b(n)$
- $\log_b(m^n) = n \cdot \log_b(m)$
- $\log_b(1/m) = \log_b(m^{-1}) = -\log_b(m)$
- $\log_a(m) = \dfrac{\log_b(m)}{\log_b(a)}$