



Lecture 25: Linear Approximation (LA)

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Applications of derivatives

- related rates
- local / global extreme values
(optimization)
- graphics
- MVT
- linear approx.

Linear Approximation

- Recall that the derivative contains the slope information of tangent lines to a given curve.
- Using this, we spent a lot of time calculating the equations of lines tangent to curves.

Let's formalize this discussion.

Definition

If f is a function differentiable at $x = a$, then a **linear approximation** for f at $x = a$ is given by

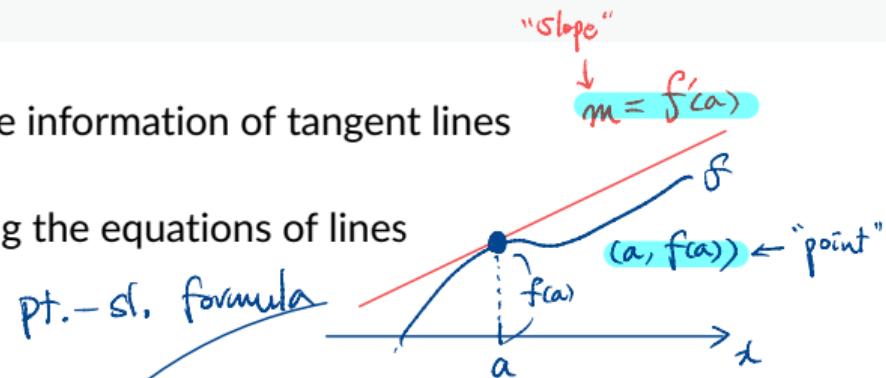
$$L(x) = f(a) + f'(a)(x - a).$$

Note that the graph of L is simply the tangent line to the curve $y = f(x)$ at $x = a$.

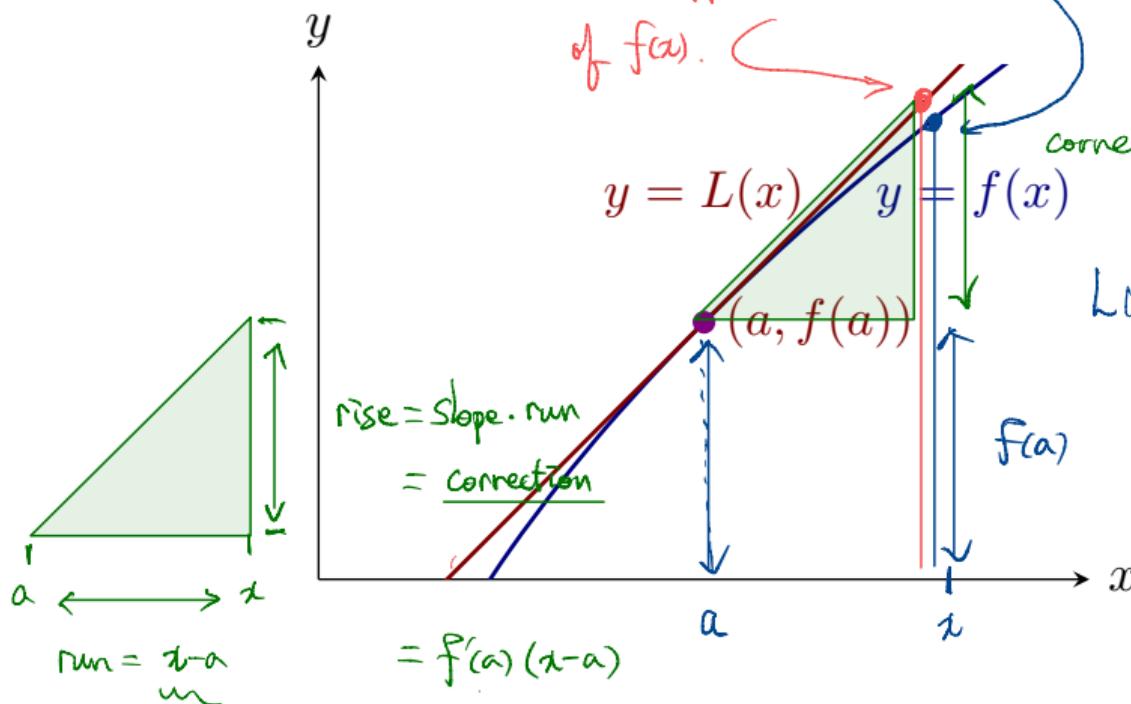
Derivation Generic Pt.-Sl. formula: $y - y_0 = m(x - x_0)$

Adopted version : $y - f(a) = f'(a)(x - a)$

$$(x_0 = a, y_0 = f(a), m = f'(a)) \quad \underline{y = f(a) + f'(a)(x - a)}$$



Illustration



$f(x)$, the exact value (somehow difficult to attain directly)

Advanced notes

→ "Taylor polynomial"

higher-order
derivatives.

Higher-order approximations are possible

provided that f is "nice". For instance,

We can approximate $f(x)$ using a quadratic function

$$Q(x) := \underline{f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}$$
$$L(x)$$

Prop

$$f(a) = Q(a)$$

$$f'(a) = Q'(a)$$

$$f''(a) = Q''(a)$$

Examples

One major advantage of linear approximation is in computation of various mathematical functions.

Question. Use a linear approximation of $f(x) = \sqrt[3]{x}$ at $a = 64$ to approximate $\sqrt[3]{50}$. What would happen if we chose $a = 27$ instead?

To form $L(x)$, need:

$$\cdot f(a) = f(64) = \sqrt[3]{64} = 4$$

$$\cdot f'(x) = \frac{d}{dx} x^{1/3} = \frac{1}{3} x^{-2/3}$$

$$\hookrightarrow f'(a) = f'(64) = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{64}^2}$$

$$= \frac{1}{3} \cdot \frac{1}{4^2} = \frac{1}{48}$$

$$\begin{aligned} L(x) &= f(64) + f'(64)(x - 64) \\ &= 4 + \frac{1}{48}(x - 64) \\ \text{So, } \sqrt[3]{50} &= f(50) \approx L(50) \quad \text{approximately equal} \\ &\quad // \\ &\quad 3.684... \quad = 4 + \frac{1}{48}(50 - 64) \\ &\quad = 4 - \frac{14}{48} = \boxed{3.7...} \end{aligned}$$

Examples

Question. Use a linear approximation of $f(x) = \sin(x)$ at $a = 0$ to approximate $\sin(0.3)$.

$$\underline{\sin(0.3)} = f(0.3) \underset{\approx}{\textcircled{z}} L(0.3) = \frac{0.3}{\underline{}}$$
$$\underline{\sin(0.3) \approx 0.3}$$

To form $L(x)$, need $\partial a=0$.

- $f(a) = f(0) = \sin(0) = 0$
- $f'(x) = \frac{d}{dx} \sin(x) = \cos(x)$
- $f'(a) = f'(0) = \cos(0) = 1$

$$\left. \begin{array}{l} L(x) = 0 + 1(x-0) \\ L(x) = x \end{array} \right\}$$

Note Physicists use this all the time. ("small angle approx.")

$\sin(\theta) \approx \theta$. for small $\theta \approx 0$

Alternate form/way of doing linear approx. (notation)

$$f(a + \Delta x) \approx L(a + \Delta x)$$

$$= f(a) + f'(a) (\underline{a + \Delta x} - a)$$

$$= f(a) + f'(a) \Delta x$$

$$f(a + \Delta x) \approx f(a) + f'(a) \Delta x$$

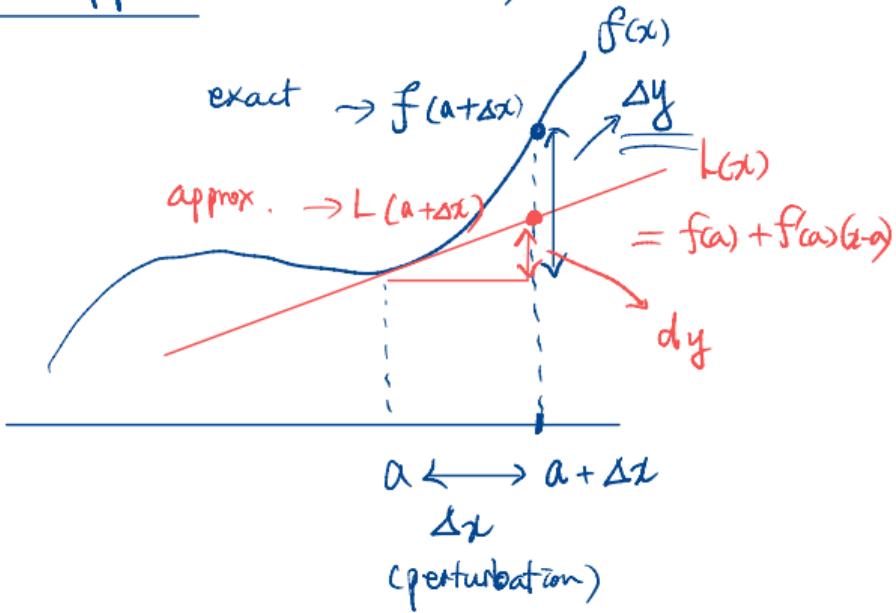
$$\underbrace{f(a + \Delta x) - f(a)}_{\text{exact difference}} \approx \underbrace{\frac{f'(a)}{\Delta x}}_{\text{slope}} \Delta x$$

slope

$$= \Delta y$$

run

$$= \frac{\text{rise}}{\Delta x}$$



In short

$$\frac{\Delta y}{\text{differential}} \approx \frac{dy}{\text{differential}}$$

Differentials

Note

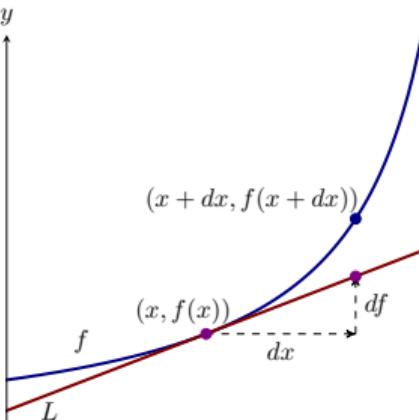
Definition

Let f be a differentiable function. We define df , a differential of f , at a point x by

$$df = f'(x) dx$$

where dx is an independent variable that is called a **differential of x** .

Geometrically, differentials can be interpreted via the diagram below.



$$\Delta f \underset{\text{if}}{\approx} df \underset{\text{if}}{\approx} f(x+dx) - f(x) \underset{\text{if}}{\approx} f'(x) dx$$

Remark

- It is important to remember that the notations dx and $dy = df$ represent variables.
- Observe from the picture that

$$f(x + dx) \approx f(x) + df .$$

- Noting that the right-hand side is identical to $L(x)$ given above, we once again confirm the validity of the linear approximation formula.

Question. Use differentials to approximate $\sqrt[3]{50}$ and $\sin(0.3)$.

Same as before,

written differently.

$$f(x) = \sqrt[3]{x}$$

$$L(x)$$

$$\sqrt[3]{50} = f(50) \approx f(64) + df = f(64) - 14 f'(64) = \boxed{}$$

a $x-a$

where $df = f'(64) dx =$

$$= f'(64) (50 - 64)$$

$$= -14 f'(64)$$

Error Approximation

- We now consider how *linear approximation* or *differentials* can be used to estimate errors in various computations.
- The quantity that we are interested in is the difference or *error* $E(x)$ between the quantity $f(x)$ and the quantity $f(x + dx)$ with a slightly perturbed input $x + dx$, i.e., for small dx .
- Using differentials, we can estimate this error by

$$E(x) = f(x + dx) - f(x) \approx f(x) + df - f(x) = df = f'(x) dx .$$

Example

The cross-section of a 250 ml glass can be modeled by the function $r(x) = x^4/3$:

At 16.8 cm from the base of the glass, there is a mark indicating when the glass is filled to 250 ml. If the glass is filled within ± 2 millimeters of the mark, what are the bounds on the volume? The volume in milliliters, as a function of the height of water in centimeters, y , is given by

$$V(y) = \frac{2\pi y^{3/2}}{\sqrt{3}}.$$

