Lecture 28-29: L'Hôpital's Rule (LHR)

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Basic Ideas

This is our final application of derivatives: using derivatives to calculate difficult limits. Enter L'Hôpital's rule.

Theorem (L'Hôpital's Rule)

Let $f(x)$ and $g(x)$ be functions that are differentiable near a. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \qquad \textit{or } \pm\infty,$$

and $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, and $g'(x) \neq 0$ for all x near a, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

List of Indeterminate Forms

- $\frac{\mathbf{0}}{\mathbf{0}}$
- $\frac{\infty}{\infty}$
- $\mathbf{0} \cdot \infty$
- $\infty - \infty$
- $\mathbf{1}^{\infty}$
- $\mathbf{0}^{\mathbf{0}}$
- $\infty^{\mathbf{0}}$

In each of these cases, the value of the limit is **not** immediately obvious.
Hence, a careful analysis is required!

Examples: Basic Indeterminate Forms

Question. Compute $\lim_{x \to 0} \frac{\sin(x)}{x}$.

Question. Compute $\lim_{x\to\pi/2^+} \dfrac{\sec(x)}{\tan(x)}$.

The following $\mathbf{0} \cdot \infty$ can be reduced to one of the two previous ones. For instance:

Question. Compute $\lim\limits_{x \to 0^+} x \ln x$.

Examples: Indeterminate Forms Involving Subtraction

The name of the game once again is reduction. We will transform differences into either quotients or products then apply L'Hôpital's rule on the basic forms.

Question. Compute $\lim_{x \to 0} (\cot(x) - \csc(x))$.

Question. Compute $\lim_{x \to \infty} \left(\sqrt{x^2 + x} - x \right)$.

Examples: Exponential Indeterminate Form

This pertains to the forms

$$\mathbf{1}^{\boldsymbol{\infty}}, \quad \mathbf{0}^{\mathbf{0}}, \quad \boldsymbol{\infty}^{\mathbf{0}}$$

Suppose we have functions $u(x)$ and $v(x)$ such that

$$\lim_{x \to a} u(x)^{v(x)}$$

falls into one of the forms described above. We use the inverse relation between $\exp$ and $\log$ functions to rewrite the limit as

$$\lim_{x \to a} e^{v(x)\ln(u(x))} \, .$$

Using the fact that the exponential function is continuous, the limit equals to

$$\exp[\lim_{x \to a} v(x)\ln(u(x))] \, .$$

Note that the limit now is in one of the previously presented forms.

Question. First determine the form of the limit, then compute the limit.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}.$$