# Lecture 4: (In)determinate Forms (IF)

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# Where are we?

MATH 1151 - AUTUMN 2021 THE OHIO STATE UNIVERSITY

| Monday | Tuesday | Wednesday | Thursday | Friday |
|---|---|---|---|---|
| **August 23** | 24<br>**First day of Classes**<br><br>Worksheet: UF, ROFF | 25<br>Understanding Functions (UF)<br><br>Review of Famous Functions (ROFF) | 26<br>Worksheet: ROFF | 27<br>What is a Limit? (WIAL) |
| 30<br>Limit Laws (LL) | 31<br>Worksheet: WIAL, LL<br><br>**HW: Precalc Rev** | **September 1**<br>(In)determinate Forms (IF) | 2<br>Worksheet: IF<br><br>**HW: WIAL, LL** | 3<br>Using Limits to Detect Asymptotes (ULTDA) |
| **6**<br>**Labor Day**<br>**No Classes** | 7<br>Worksheet: ULTDA<br><br>**WH1 due** | 8<br>Continuity and the Intermediate Value Theorem (CATIVT) | 9<br>Worksheet: CATIVT<br><br>**HW: IF, ULTDA** | 10<br>An Application of Limits (AAOL) |

- One attempt for HW quizzes
- Gradescope set up
  ↓
  for WH & Exams

# Review

- Limit laws : compute limits
  - involving arithmetic ops $(+, -, \times, \div)$
  - involving composition of func.
- Continuity of famous function. (building blocks)

$\Rightarrow$ continuity of polynomials & rational func.

- Squeeze Theorem. (Sandwich theorem)

## Overview

Recall: $\boxed{\text{Quotient law:}}$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad \text{provided that } \lim_{x\to a} g(x) \neq 0.$$

Q. What if $\lim_{x\to a} g(x) = 0$ ?

A. Possible scenarios are ...

Case 1: $\lim_{x\to a} f(x) = 0$

Case 2: $\lim_{x\to a} f(x) \neq 0$

Spoiler

cannot predict (indeterminate)

can predict (definite)

# Limits of the form zero over zero – indeterminate form

**Definition**

A limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

is said to be of the $\boxed{\text{form } \frac{\mathbf{0}}{\mathbf{0}}}$ if

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0\,.$$

- **Warning!** The symbol $\frac{\mathbf{0}}{\mathbf{0}}$ is NOT the number 0 divided by 0.
- A key trick to handle limits in $\frac{\mathbf{0}}{\mathbf{0}}$ form is to cancel out vanishing factors.

" $\frac{0}{0}$ "

**Question.** Compute the following limits:

Form : "$\frac{0}{0}$"

$$\lim_{x\to 2}\frac{x^2-3x+2}{x-2}$$

Initial assessment

- $\lim_{x\to 2}(\text{NUM.}) = \lim_{x\to 2}(x^2-3x+2)$

$= 2^2-3\cdot 2+2$ (by cont. of poly.)

$= 4-6+2 = 0$

- $\lim_{x\to 2}(\text{DEN.}) = \lim_{x\to 2}(x-2)$

$= 2-2$ (by cont. of poly.)

$= 0$

Solution : Do algebra and simplify.

$= \lim_{x\to 2}\frac{\cancel{(x-2)}(x-1)}{\cancel{x-2}}$

$= \lim_{x\to 2}(x-1)$

$= 2-1$

$= \boxed{1}$

**Question.** Compute the following limits:

$$\lim_{x\to 1} \frac{\frac{1}{x+1} - \frac{3}{x+5}}{x-1}$$

Initial assessment

- $\lim_{x\to 1}(\text{NUM}) = \frac{1}{1+1} - \frac{3}{1+5}$ (cont. of rat'l)

  $= \frac{1}{2} - \frac{3}{6} = 0$

- $\lim_{x\to 1}(\text{DEN}) = 1 - 1$ (cont. of poly.)

  $= 0$

Form: "$\frac{0}{0}$" (indeterminate)

Solution: Do algebra to simplify and cancel.

$$(3x+3) = 3(x+1)$$

$$= \lim_{x\to 1} \frac{\frac{x+5 - 3(x+1)}{(x+1)(x+5)}}{x-1}$$

$$= \lim_{x\to 1} \frac{\frac{-2x+2}{(x+1)(x+5)}}{x-1}$$

$$= \lim_{x\to 1} \frac{-2\cancel{(x-1)}}{(x+1)(x+5)\cancel{(x-1)}}$$

$$= \frac{-2}{(1+1)(1+5)} = \boxed{-\frac{1}{6}}$$

## Sidenote: Stacked fractions

- num $$\frac{\;\frac{a}{b}\;}{\;\frac{c}{d}\;} = \frac{ad}{bc}$$ den.

↳ ex) $$\frac{\;\frac{a}{b}\;}{c} = \frac{\;\frac{a}{b}\;}{\;\frac{c}{1}\;} = \frac{a \cdot 1}{bc} = \frac{a}{bc}$$

**Question.** Compute the following limits:

$$\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1}$$

<u>observe</u>

$$\left.\begin{aligned} &\lim_{x\to -1} (\text{NUMER.}) = \sqrt{4}-2=0 \\ &\lim_{x\to -1} (\text{DENOM.}) = -1+1 = 0 \end{aligned}\right\} \Rightarrow \text{“}\frac{0}{0}\text{” FORM.}$$

<u>Solution</u>

$$= \lim_{x\to -1} \frac{(\sqrt{x+5}-2)(\sqrt{x+5}+2)}{(x+1)(\sqrt{x+5}+2)}$$

$$= \lim_{x\to -1} \frac{x+5-4}{(x+1)(\sqrt{x+5}+2)}$$

$$= \lim_{x\to -1} \frac{1}{\sqrt{x+5}+2}$$

$$= \frac{1}{\sqrt{4}+2}$$

$$= \boxed{\frac{1}{4}}$$

## Remark

- Limits of the form $\frac{0}{0}$ can take any value!
- Having this particular form does not give us enough information to determine whether a function has a limit or not;
- Even if the limit exists, the value of the limit is not apparent without further manipulation.
- That is why such a limit is said to be in an **indeterminate form**.

# Limits of the form nonzero over zero – determinate form

**Definition**

A limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

is said to be of the $\boxed{\text{form } \frac{\#}{\mathbf{0}}}$ if

$$\lim_{x \to a} f(x) = k \quad \text{and} \quad \lim_{x \to a} g(x) = 0\,,$$

where $k$ is some nonzero constant.

$\frac{1}{0.1} = 10$

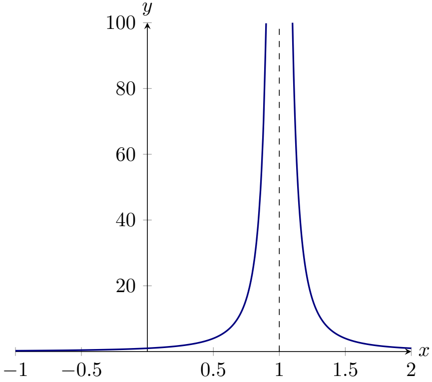
$\frac{1}{0.01} = 100$

$\frac{1}{0.001} = 1000$

$\vdots$

- When a fixed nonzero number is divided by a small number, the quotient is generally large.
- As the denominator get smaller and smaller, the quotient gets larger and larger.

**Illustration.** The following graph of $f(x) = 1/(x-1)^2$ near $x = 1$ displays the behavior of limits of the form $\frac{\#}{\mathbf{0}}$.




As $x \to 1$,

(DEN) $\to 0$ while

(NUM) $= 1$.

## Definition

- If $f(x)$ grows arbitrarily large for all $x$ sufficiently close, but not equal, to $a$, we write

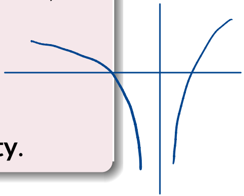
$$\lim_{x \to a} f(x) = \infty$$

and say that the limit of $f(x)$ as $x$ approaches $a$ **is infinity**.

- If $f(x) < 0$ and $|f(x)|$ grows arbitrarily large for all $x$ sufficiently close, but not equal, to $a$, we write

$$\lim_{x \to a} f(x) = -\infty$$

and say that the limit of $f(x)$ as $x$ approaches $a$ **is negative infinity**.



**Note.** We can analogously define one-sided infinite limits, e.g.,

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty \,.$$

**Question.** Compute

$$\lim_{x\to 0} \underbrace{\frac{e^x}{1-\cos(x)}}_{=f(x)}.$$

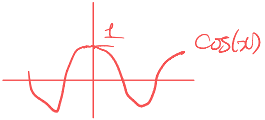
Form : " $\frac{\#}{0}$ " $\rightarrow$ nonzero

## Initial assessment

- $\lim\limits_{x\to 0}(\text{NUM}) = \lim\limits_{x\to 0} e^x \overset{\text{(by cont. of exp.)}}{=} e^0 = 1 \neq 0$

- $\lim\limits_{x\to 0}(\text{DEN}) = \lim\limits_{x\to 0}(1-\cos(x)) = 1-\cos(0) = 0$

(by diff. law. by cont. of cosine




Note: $1-\cos(x)$ approaches 0 from above as $x \to 0$.

## Solution :

As $x \to 0$, the numerator approaches a positive number while the denominator approaches 0 from positive side. So the limit is infinity.

$$\boxed{\lim_{x\to 0} f(x) = \infty}$$

**Question.** Compute

$$\lim_{x\to 3} \underbrace{\frac{x^2-9x+14}{x^2-5x+6}}_{f(x)}.$$

Assess

- $\lim\limits_{x\to 3}(x^2-9x+14) = 9-9\cdot 3+14 = -4 < 0$
- $\lim\limits_{x\to 3}(x^2-5x+6) = 9-5\cdot 3+6 = 0$

$\Rightarrow$ " $\frac{\#}{0}$ " FORM

Note $x^2-5x+6 = (x-3)(x-2) \longrightarrow \begin{cases} 0^+ & \text{as } x\to 3^+ \\ 0^- & \text{as } x\to 3^- \end{cases}$

Thus, $\boxed{\lim\limits_{x\to 3^+} f(x) = -\infty}$, $\boxed{\lim\limits_{x\to 3^-} f(x) = \infty}$. Limit DNE.