Introduction to Fundamental Theorem of Calculus

Lecture 35: Antiderivatives and Area (AAA)

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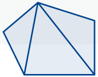
Relating antiderivatives and areas

$\int \square \, dt \qquad \int \square \, dt$ "quadrature"



In investigating connection between antiderivatives and area, we will use our favorite position-velocity-acceleration triple for illustration.

$d/dt \qquad d/dt$

Displacement vs. distance.

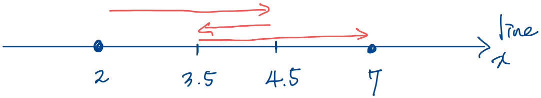
Consider a moving object (in 1-D) from time $t = a$ to $t = b$. The *displacement* measures the difference in position. In other words,

$$\boxed{(\text{displacement}) = (\text{terminal position}) - (\text{initial position}) = s(b) - s(a)\,.}$$

Note:

- When an object moves without changing directions, the (traveled) distance equals the absolute value of displacement.
- However, when it changes directions along the course of movement, they are going to be different.
- In particular, distance is always going to be positive, but displacement may be negative.

@ $t = a$ @ $t = b$



✓ (displacement) = (term. position) − (init. position)

$= 7 - 2 \quad = +5$

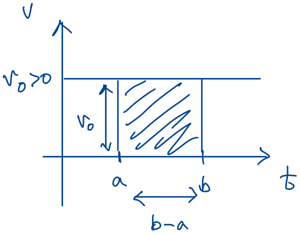
(distance) $= 2.5 + 1 + 3.5 = 7$

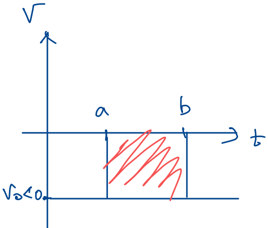
Simple case: uniform velocity

Now consider a simple situation where an object is moving at a constant velocity v_0 for $a \le t \le b$. Then the displacement is simply the velocity multiplied by the time traveled, i.e.,

$$(\text{displacement}) = v_0(b-a)\,. = \int_a^b v_0 \, dt \qquad \text{(constant velocity)}$$

- The graph of velocity against time is a horizontal line.
- The displacement is exactly equal to the (signed) area of rectangle between the velocity curve (i.e., the straight line) and the horizontal time axis on $[a, b]$.





Motion with changing velocity

Then how would we calculate the displacement when the object is moving at a varying velocity?

- Assuming that it moves at a constant velocity over a small interval of time, we can approximate displacement using Riemann sums;
- The quality of approximation improves as we increase the number of approximating rectangles;
- We obtain the exact displacement once we take the limit of general Riemann sum as the number n of rectangles approaches infinity, that is

Riemann

$$\boxed{(\text{displacement}) = \int_a^b v(t)\, dt\,.} \qquad \text{(variable velocity)}$$

as $n \to \infty$, approx. disp. $\longrightarrow$ exact disp.

# of rect.

The connection

Convention: $\begin{cases} s(t): \text{position} \\ v(t): \text{velocity} \\ a(t): \text{acceleration} \end{cases}$

- But recall that the displacement is the difference between the terminal and initial positions, i.e., $\underline{s(b) - s(a)}$. Thus

an antiderivative of $v(t)$

$$(\text{displacement}) = \int_a^b v(t)\, dt = s(b) - s(a)\,. \qquad (\star)$$

$s'(t) = v(t)$

- Noting that $s'(t) = v(t)$, i.e., $s(t)$ is an antiderivative of $v(t)$, we may interpret the equation ($\star$) in a general setting as:

> *The net **area** between the curve $y = f(x)$ and the x-axis on $[a, b]$ is the difference of values of its **antiderivative** at the endpoints.*

- The statement above can be written as

an antiderivative of $f(x)$

$$\int_a^b f(x)\, dx = F(b) - F(a)\,,$$

2^{nd} part of

where F is an antiderivative of f. This is the celebrated Fundamental Theorem of Calculus.

Example

Question. Assume an object is moving along a straight line with the velocity $v(t) = 3 - 3t^2$ for $0 \le t \le 2$. Find the displacement of the object over the time interval $[0, 4]$.

(★) on p.5

$$(\text{displacement}) \overset{\text{def'n}}{=} S(2) - S(0) \overset{(\star)}{=} \int_0^2 v(t)\, dt$$

$$\boxed{-2} = (-2 + \cancel{C}) - \cancel{C} = \int_0^2 (3 - 3t^2)\, dt$$

What is $S(t)$? An antiderivative of $v(t)$.

$$S(t) = \int (3 - 3t^2)\, dt = 3t - 3 \cdot \frac{t^3}{3} + C = 3t - t^3 + C$$

indefinite

$$S(0) = 3 \cdot 0 - 0^3 + C = C$$

$$S(2) = 3 \cdot 2 - 2^3 + C = 6 - 8 + C = -2 + C$$

#3 of Review MT5 $\leftarrow$ Wk. 13 module.

(b) $\lim_{x \to 0^+} (\tan(x))^{x^2}$

$\tan(x) \to 0$, $x^2 \to 0$

Form: " 0^0 "

Strategy

① Compute $L = \lim_{x\to 0^+} x^2 \ln \tan(x)$

② Ans. $= e^L$

Justification

$$\to \lim_{x\to 0^+} \left(e^{\ln \tan x}\right)^{x^2}$$

$$= \lim_{x\to 0^+} e^{x^2 \ln \tan x}$$

$$= e^{\boxed{\lim_{x\to 0^+} x^2 \ln \tan x}}$$

$\boxed{\lim_{x\to 0^+} x^2 \ln \tan x} = L$

$$L = \lim_{x\to 0^+} x^2 \ln \tan(x)$$

$x^2 \to 0$, $\ln \tan(x) \to -\infty$

Form: " $0 \cdot \infty$ "

Side $\ln \tan(x)$

$$= \ln \frac{\sin(x)}{\cos(x)}$$

$$= \ln \sin(x) - \ln \cos(x)$$

$$= \lim_{x\to 0^+} x^2 \left(\ln \sin(x) - \ln \cos(x)\right)$$

$$= \lim_{x\to 0^+} \underbrace{x^2 \ln \sin(x)}_{0 \quad -\infty} - \lim_{x\to 0^+} \underbrace{x^2 \ln \cos(x)}_{0 \quad 0 \quad 1}$$

"doub. recip.") Form: "$0 \cdot \infty$"

$= 0$

$$= \lim_{x\to 0^+} \frac{\ln \sin(x)}{\frac{1}{x^2}}$$

Form: "$\frac{\infty}{\infty}$"

L'H

$$= \lim_{x\to 0^+} \frac{\frac{\cos(x)}{\sin(x)}}{-\frac{2}{x^3}} = \lim_{x\to 0^+} \left[-\frac{x^3 \cos(x)}{2\sin(x)} \right],$$

Form: "$\frac{0}{0}$"

either apply L'H once more or .

$$= \lim_{x\to 0^+}\left[-\frac{x}{\sin(x)} \cdot \frac{x^2 \cos(x)}{2} \right] = 0 = L$$

$\frac{x}{\sin(x)} \to 1$ $\qquad$ $\frac{x^2\cos(x)}{2} \to 0$

from $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

Thus

$$\text{Ans.} = e^L = e^0 = \boxed{1}$$

(e) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$ Form: "$\frac{0}{0}$"

$\overset{L'H}{=} \displaystyle\lim_{x\to 0} \frac{e^x - 1}{2x}$ Form: "$\frac{0}{0}$"

$\overset{L'H}{=} \displaystyle\lim_{x\to 0} \frac{e^x}{2} = \boxed{\frac{1}{2}}$