Lecture 33-34: Definite Integrals (DI)

Tae Eun Kim, Ph.D.

Autumn 2021

Recall :

Indefinite integral of f :

- the infinite family of all antiderivatives of f.
- notation : $\int f(x)\,dx = F(x) + C$

an antiderivative

Definite Integrals

hint generalization: the curve $y = f(x)$ doesn't need to be above x-axis

Definition

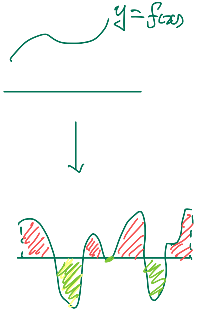
Let f be a function which is continuous on the interval $[a, b]$. We define the **definite integral** of f on $[a, b]$ by

general Riemann sum

$$\underbrace{\int_a^b f(x)\, dx}_{\text{notation}} = \lim_{n\to\infty} \underbrace{\boxed{\sum_{k=1}^{n} f(x_k^*)\Delta x}}_{\text{definition}}.$$

"exact area"

The definite integral is a number that gives the **net area** of the region between the curve $y = f(x)$ and the x-axis on the interval $[a, b]$.



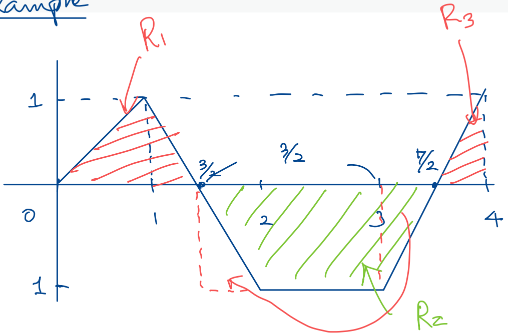
Convention: Signed area

- region above x-axis : positive
- region below x-axis : negative

: region above; (+)

: region below; (−)

Example



R_1, R_2, R_3 : regions

- $\text{Area}(R_1) = \frac{1}{2} \cdot \frac{3}{2} \cdot 1 = \frac{3}{4}$
- $\text{Area}(R_2) = \frac{3}{2} \cdot 1 = \frac{3}{2}$
- $\text{Area}(R_3) = \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4}$

$$\int_0^4 f(x)\,dx = \text{Sum of Signed areas}$$

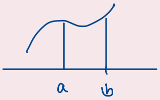
$$= \frac{3}{4} - \frac{3}{2} + \frac{1}{4} = \boxed{-\frac{1}{2}}$$

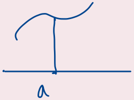
	(Geometric) Area	Signed area.
R_1	$3/4$	$+3/4$
R_2	$3/2$	$-3/2$
R_3	$1/4$	$1/4$

Basic Properties

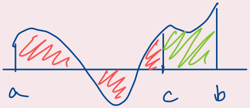
Theorem (Properties of the definite integral)

Let f and g be defined on a closed interval $[a, b]$ that contains the value c, and let k be a constant. The following hold:





1. $\displaystyle\int_a^a f(x)\,dx = 0$
2. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$
3. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
4. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
5. $\displaystyle\int_a^b \{f(x) \pm g(x)\}\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$



Definite Integrals Using Geometry vs. Definition

Question. Compute the integral

$$\int_0^{10} (4 - x)\, dx$$

in two ways:

1. by interpreting the integral as the net area of the region between the curve $y = 4 - x$ and the interval $[0, 10]$ on the x-axis;
2. using the definition of the definite integral, i.e. by computing the limit of Riemann sums.

Question. Compute the integral

$$\int_0^{10} |4 - x| \, dx \, .$$

Note: Net Areas vs. Geometric Areas

We know that the net area of the region between a curve $y = f(x)$ and the x-axis on $[a, b]$ is given by

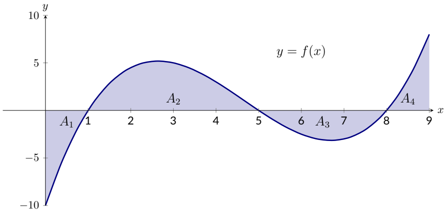
$$\int_a^b f(x)\,dx.$$

On the other hand, if we want to know the *geometric area*, meaning the "actual" area, we compute

$$\int_a^b |f(x)|\,dx.$$

Question. The graph of a function f is given in the figure.

1. Express the geometric area of the region between the curve $y = f(x)$ and the $x-$axis on the interval $[0, 9]$ as a definite integral.
2. Express the geometric area of the region between the curve $y = f(x)$ and the $x-$axis on the interval $[0, 9]$ in terms of definite integrals of f.
3. Express the geometric area of the region between the curve $y = f(x)$ and the $x-$axis on the interval $[0, 9]$ in terms of areas A_1, A_2, A_3 and A_4.



From Riemann Sums to Definite Integrals

Question. Compute the limit:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt{1-\left(-1+\frac{2k}{n}\right)^2}\right)\left(\frac{2}{n}\right)$$

Question. Express the following limit of Riemann sum as a definite integral:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k\pi}{n} + \cos \frac{k\pi}{n} \right) \frac{\pi}{n} \, .$$

Definite Integrals of Symmetric Functions

Recall that a function f is

- an **odd** function if $f(-x) = -f(x)$;
- an **even** function if $f(-x) = f(x)$.

Theorem

Let f be a symmetric function on a symmetric interval $[-a, a]$ where $a > 0$. Then

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\displaystyle\int_0^a f(x)\,dx & \text{if } f \text{ is even} \\ 0 & \text{if } f \text{ is odd.} \end{cases}$$

Question.

1. Find the following definite integral:

$$\int_{-4}^{4} \frac{x^2 \sin^3(x)}{\sqrt{x^4+1}}\, dx\,.$$

2. Suppose that f is an even function. Given that $\displaystyle\int_0^6 f(x)\, dx = 13$, find $\displaystyle\int_{-6}^{6} (5f(x)+14)\, dx$.