Lecture 40: Working with Substitution (WWS)

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Substitution Procedures

Let's recall that in integrating a function which we suspect to be the derivative of another obtained by the chain rule:

1. Look for a candidate for the inner function; call it u.
2. Rewrite the given function completely in terms of u leaving no trace of the original variable.
3. Integrate this new function of u. (If necessary, you may need to go back to Step 1 and modify your choice of u.)
4. In dealing with an indefinite integral, make sure to replace u by the equivalent expression of the original variable.
5. Working with a definite integral, you may evaluate the result of Step 3 at the transformed bounds of u or evaluate the antiderivate obtained in Step 4 at the original bounds.

For the remainder of the lecture, we will work out practice examples.

$$\int \frac{1}{x\ln(\ln x)}\,dx$$

Example

Compute:

1. $\displaystyle\int_2^3 \frac{1}{x\ln(x)}\,dx$ $= du$, $= u$
2. $\displaystyle\int_0^{16} \sqrt{4-\sqrt{x}}\,dx$

- What's embedded?
- Does its deriv. appear outside?

$$\frac{1}{x\ln x} = \frac{1}{\ln x}\cdot\frac{1}{x}$$

$\ln x$ put inside $\frac{1}{☺}$

(1) Set $u = \ln x$

$du = \frac{1}{x}\,dx$

Limits

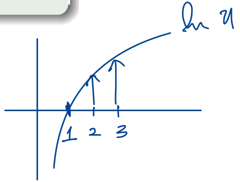
x	$u=\ln x$
3	$\ln 3$
2	$\ln 2$

$$(\text{Integ } 1) = \int_{\ln 2}^{\ln 3} \frac{1}{u}\,du$$

$$\overset{FTC2}{=} \Big[\ln|u|\Big]_{\ln 2}^{\ln 3}$$

$$= \ln|\ln 3| - \ln|\ln 2|$$

$$= \boxed{\ln(\ln 3) - \ln(\ln 2)} = \ln\left(\frac{\ln 3}{\ln 2}\right)$$



Example

Compute:

1. $\displaystyle\int_2^3 \frac{1}{x\ln(x)}\,dx$

2. $\displaystyle\int_0^{16} \sqrt{4-\sqrt{x}}\,dx$

u $\quad -2(4-u)\,du$

Remember In finding "u":

1. Is it embedded?
2. Does its deriv. appear "outside"?

Set $u = 4-\sqrt{x} \longrightarrow \sqrt{x} = 4-u$

$du = \left(-\frac{1}{2\sqrt{x}}\right)dx \longrightarrow dx = -2\sqrt{x}\,du$

$= -2(4-u)\,du$

Limits

x	$u = 4-\sqrt{x}$
16	$4-\sqrt{16} = 0$
0	$4-\sqrt{0} = 4$

$$= \int_4^0 \sqrt{u}\,(-2)(4-u)\,du$$

$$= 2\int_0^4 (4\sqrt{u} - u\sqrt{u})\,du$$

FTC2

$$= 2\left[4\cdot\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^4$$

$$= 2\left[\frac{8}{3}4^{3/2} - \frac{2}{5}4^{5/2}\right]$$

$8 \quad 32$

$$= 2\left[\frac{1}{3} - \frac{1}{5}\right]64$$

$$= \frac{2\cdot 2\cdot 64}{15} = \boxed{\frac{256}{15}}$$

Example

Compute:

1. $$\int \frac{\sec(y)\tan(y)+\sec^2(y)}{\sec(y)+\tan(y)}\,dy$$

du

u

2. $$\int \tan(x)\,dx$$

Set $u = \sec(y) + \tan(y)$

$du = (\sec(y)\tan(y) + \sec^2(y))\,dy$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \boxed{\ln|\sec(y)+\tan(y)| + C}$$

* $\int \frac{1}{u}\,du = \ln|u| + C$

  Don't forget abs. val.

* Final answer in terms of orig. var. (indef. integ.)

* Add C!!!

Example

Compute:

1. $$\int \frac{\sec(y)\tan(y) + \sec^2(y)}{\sec(y) + \tan(y)}\, dy$$

2. $$\int \tan(x)\, dx$$

$\cancel{u = \sin(x)}$

$\cancel{du = \cos(x)\, dx.}$

$$= \int \frac{\underbrace{\sin(x)\,dx}_{= -du}}{\boxed{\cos(x)}_{=u}}$$

Set $u = \cos(x)$

$du = -\sin(x)\, dx.$

$$= -\int \frac{du}{u} = -\ln|u| + C$$

$$= \boxed{-\ln|\cos(x)| + C}$$

may be written as

$\ln|\sec(x)| + C.$

Example

Compute:

1. $\displaystyle\int \frac{u}{1-u^2}\,du$ $u\,du = -\frac{1}{2}\,d\text{☺}$, $1-u^2 = \text{☺}$

2. $\displaystyle\int \frac{e^{2x}}{1-e^{2x}}\,dx$

Unconventional yet creative:

Set $\begin{cases} \text{☺} = 1-u^2 \\ d\text{☺} = (-2u)\,du \end{cases}$

$$= \int -\frac{1}{2}\,\frac{d\text{☺}}{\text{☺}}$$

$$= -\frac{1}{2}\int \frac{1}{\text{☺}}\,d\text{☺}$$

$$= -\frac{1}{2}\ln|\text{☺}| + C$$

$$= \boxed{-\frac{1}{2}\ln|1-u^2| + C}$$

Example

Compute:

1. $\displaystyle\int \frac{u}{1-u^2}\,du$

2. $\displaystyle\int \frac{e^{2x}}{1-e^{2x}}\,dx = \int \frac{(e^x)^2}{1-(e^x)^2}\,dx$

$e^x = u$, $e^x \cdot e^x$, $= du$

Recall: $e^{ab} = (e^a)^b$

Set $u = e^x$

$du = e^x\,dx$

$$= \int \frac{e^x \cdot \boxed{e^x\,dx}}{1-(e^x)^2}$$

$e^x = u$, $e^x\,dx = du$, $e^x = u$

~~Split!~~

$$= \int \frac{u}{1-u^2}\,du \quad (\text{same as } ①)$$

$$= -\frac{1}{2}\ln|1-u^2| + C$$

$$= \boxed{-\frac{1}{2}\ln|1-e^{2x}| + C}$$

Example

Compute:

1. $\int x^3\sqrt{1-x^2}\,dx$

$$\begin{cases} u = 1-x^2 \longrightarrow x^2 = 1-u \\ du = -2x\,dx \longrightarrow x\,dx = -\frac{1}{2}\,du. \end{cases}$$

split !

$$= \int \sqrt{\underbrace{\boxed{1-x^2}}_{=u}}\;\underbrace{\overbrace{\boxed{x^2}}^{=1-u}\cdot x}_{=x^3}\,\overbrace{\boxed{x\,dx}}^{=-\frac{1}{2}du}$$

$$= -\frac{1}{2}\int \sqrt{u}\,(1-u)\,du$$

$$= -\frac{1}{2}\int \left(u^{1/2} - u^{3/2}\right) du$$

$$= \ldots \quad (\text{DIY})$$

Work

Suppose the force $F(s)$ is applied to an object and moves it from $s = s_0$ to $s = s_1$. Then the work done on the object over the course of motion is given by

$$W = \int_{s_0}^{s_1} F(s)\, ds.$$

The international standard unit of force is a **Joule**, which is defined to be

$$1\,\mathrm{J} = 1\,\mathrm{N} \cdot \mathrm{m}.$$

In words, work measures the accumulated force over a distance. Note that we only accumulate force in the *direction* or *opposite direction* of motion.

Example

If an apple has a mass of $0.1\,\mathrm{kg}$, how much work is required to lift this 1 meter above the ground? Assume that the gravitational acceleration is $-9.8\,\mathrm{m/s^2}$.

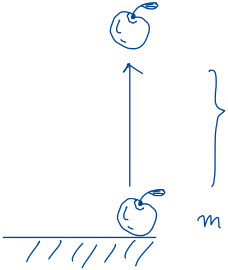
constant

$$W = \int_0^1 \boxed{mg}\, ds$$

$$= mg \int_0^1 ds$$

$$= mg\,[s]_0^1$$

$$= mg(1-0) = (0.1\,\mathrm{kg})(9.8\,\mathrm{m/s^2})(1\,\mathrm{m}) = \boxed{0.98\,\underbrace{\mathrm{kg\,m^2/s^2}}\ \text{or}\ 0.98\,\mathrm{J}}$$



distance : 1m

$m = 0.1\,\mathrm{kg}$

$g = 9.8\,\mathrm{m/s^2}$

↑ positive.

↓ negative

Kinetic Energy

Now suppose that an object of mass m is moving at velocity $v(t)$. The **kinetic energy** E_k is the amount of *energy* that an object possesses from its motion. It is defined by

$$E_k = \frac{mv^2}{2}.$$

The SI unit of energy is also a **Joule** since $1\,\mathrm{N} = 1\,\mathrm{kg}\cdot\mathrm{m}/\,\mathrm{s}^2$:

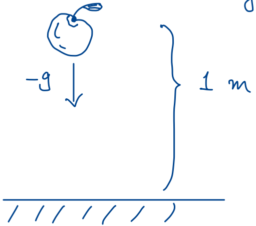
$$1\,\mathrm{J} = 1\,\mathrm{kg}\cdot\mathrm{m}^2/\,\mathrm{s}^2.$$

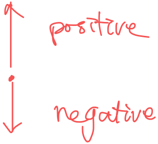
Example

Now the apple is dropped from the height of 1 meter. How much kinetic energy is released when it hits the ground?

$m = 0.1$ kg , $g = 9.8$ m/s^2

Find the velocity of apple when it hits the ground.





- $V(t) = \underbrace{V(0)}_{\substack{\| \\ \text{initial velocity} \\ \| \\ 0}} + \int_0^t \underbrace{V'(s)}_{\substack{\| \\ \text{acceleration} = -g \text{ (falling down)}}} \, ds$

$$= 0 + \int_0^t (-g)\, ds = \left[-g s\right]_0^t = -gt.$$

- To find the time when it hits the ground, calculate $S(t)$ function.

$$S(t) = \underbrace{S(0)}_{\substack{\text{initial position} \\ = 1}} + \int_0^t \underbrace{S'(\tau)}_{= v(\tau) = -g\tau} \, d\tau$$

$$= 1 - \int_0^t g\tau \, d\tau$$

$$= 1 - \left[g \frac{\tau^2}{2} \right]_0^t$$

$$= 1 - \frac{1}{2} g t^2$$

and set it equal to 0.

$$S(t) = 1 - \frac{1}{2} g t^2 = 0$$

$$\Rightarrow \quad t = \pm \sqrt{\frac{2}{g}}$$

Discarding the negative root,

$$t = \sqrt{\frac{2}{g}}$$

So the velocity when the apple hits the ground is

$$v\left(\sqrt{\frac{2}{g}}\right) = -g\sqrt{\frac{2}{g}} = -\sqrt{2g}$$

Therefore, the kinetic energy released at the point is

$$E_K = \frac{1}{2} m v\left(\frac{\sqrt{2}}{g}\right)^2 = \frac{1}{2} m \cdot 2g = mg = \boxed{0.98\ kg\ m^2/s^2 \text{ or } 0.98\ J}$$

units : kg $\quad (m/s)^2 = kg \cdot m^2/s^2$

Work-Energy Theorem

We observe that the work and the energy calculated above are the same with the same unit. This is not a coincidence and this phenomenon can be explained via the **Work-Energy Theorem**.

Theorem (Work-Energy Theorem)

Suppose that an object of mass m is moving along a straight line. If s_0 and s_1 are the the starting and ending positions, v_0 and v_1 are the the starting and ending velocities, and $F(s)$ is the force acting on the object at any given position, then

$$W = \int_{s_0}^{s_1} F(s)\, ds = \frac{mv_1^2}{2} - \frac{mv_0^2}{2}.$$

In other words, the total work done by the force is equal to the net change in kinetic energy.

Explanation

Let $t_{0,1}$ be the initial and terminal time of the motion respectively, so that we can write $s_{0,1} = s(t_{0,1})$ respectively. Then the work formula can be written as

$$W = \int_{s(t_0)}^{s(t_1)} F(s)\,ds = \int_{t_0}^{t_1} F(s(t))s'(t)\,dt,$$

where the second equality is due to the substitution rule with $u = s(t)$. By Newton's second law of motion $F(s(t)) = ma(t)$, we can write

$$\int_{t_0}^{t_1} F(s(t))s'(t)\,dt = \int_{t_0}^{t_1} ma(t)s'(t)\,dt = m\int_{t_0}^{t_1} a(t)s'(t)\,dt.$$

Remembering that $a(t) = v'(t)$ and $s'(t) = v(t)$, we can write the integrand solely in terms of v and its derivative, i.e.,

$$m\int_{t_0}^{t_1} a(t)s'(t)\,dt = m\int_{t_0}^{t_1} v(t)v'(t)\,dt.$$

Another substitution $u = v(t)$ with new bounds $v_0 = v(t_0)$ and $v_1 = v(t_1)$ yields the desired result.

$$m\int_{t_0}^{t_1} v(t)v'(t)\,dt = m\int_{v_0}^{v_1} u\,du = m\left[\frac{u^2}{2}\right]_{v_0}^{v_1} = \frac{mv_1^2}{2} - \frac{mv_0^2}{2}.$$