Week 1: Preliminaries

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Weekly Overview

1. Understanding Functions (UF)
2. Review of Famous Functions (ROFF)
3. What Is a Limit? (WIAL)

Understanding Functions (UF)

“For each input, exactly on output”

Definition

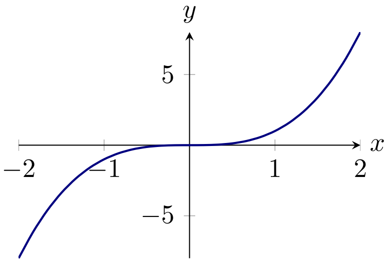
- **function**: a relation between sets where for each input, there is exactly one output
- **domain**: the set of the inputs of a function
- **range**: the set of the outputs of a function

Representation of functions

- **formula**: $f(x) = x^3$
- **table**:

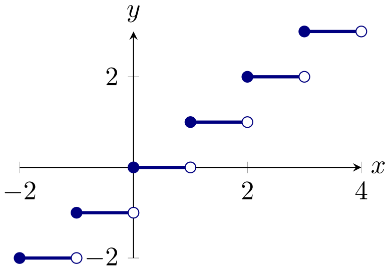
input	1	-2	1.5	$\cdots$
output	1	-8	3.375	$\cdots$

- **graph**:



Example: Greatest Integer Function

- maps any real number x to the greatest integer less than or equal to x.
- a.k.a. *floor function*
- denoted by $\lfloor x \rfloor$
- many inputs to one output



Theorem (Vertical line test)

*The curve $y = f(x)$ represents y as a function of x at $x = a$ if and only if the vertical line $x = a$ intersects the curve $y = f(x)$ at exactly one point. This is called the **vertical line test**.*

Distinguishing two functions

- Do they have the same domain?
- Do they display the same relation?

Question. Determine if the two function are the same.

1. $f(x) = \sqrt{x^2}$ and $g(x) = |x|$

2. $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$ and $g(x) = x - 1$

Composition of functions

Composite functions

- can be thought of as putting one function inside another
- **Notation:** $(f \circ g)(x) = f(g(x))$
- **Warning:** The range of inner function must be contained in the domain of outer function.

Question. Study the composition $f \circ g$ where

$$
\begin{aligned}
f(x) &= x^2 && \text{for } -\infty < x < \infty, \\
g(x) &= \sqrt{x} && \text{for } 0 \le x < \infty.
\end{aligned}
$$

Question. Study the composition $f \circ g$.

$$f(x) = \sqrt{x} \qquad \text{for } 0 \leq x < \infty,$$
$$g(x) = x^2 \qquad \text{for } -\infty < x < \infty.$$

Inverses of functions

Definition

Let f be a function with domain A and range B:

$$f : A \to B$$

Let g be a function with domain B and range A:

$$g : B \to A$$

We say that f and g are **inverses** of each other if $f(g(b)) = b$ for all b in B, and also $g(f(a)) = a$ for all a in A. Sometimes we write $g = f^{-1}$ in this case.

We could rephrase these conditions as

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

Warning: notations

Pay attention to where we put the superscript:

$$f^{-1}(x) = \text{the inverse function of } f(x).$$
$$f(x)^{-1} = \text{the reciprocal of } f(x).$$

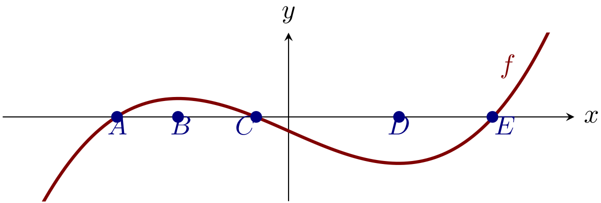
Definition

A function is called **one-to-one** if each output value corresponds to exactly one input value.

Theorem (Horizontal line test)

*A function is one-to-one at $x = a$ if the horizontal line $y = f(a)$ intersects the curve $y = f(x)$ in exactly one point. This is called the **horizontal line test**.*

Question. Consider the graph of the function f below:



On which of the following intervals is f one-to-one?

1. $[A, B]$
2. $[A, C]$
3. $[B, D]$
4. $[C, E]$
5. $[C, D]$

Review of Famous Functions (ROFF)

These are important functions for Math 1151:

- polynomial functions
- rational functions
- trigonometric functions and their inverses
- exponential and logarithmic functions

Polynomial functions

Definition

A **polynomial function** in the variable x is a function which can be written in the form

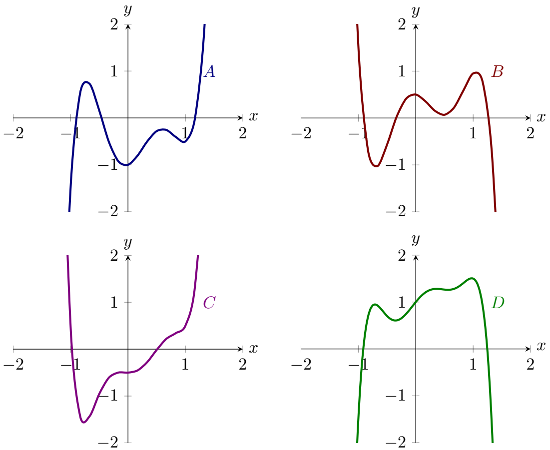
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are all constants (called the **coefficients**) and n is a whole number (called the **degree** when $n \neq 0$). The domain of a polynomial function is $(-\infty, \infty)$.

Question. Which of the following are polynomial functions?

1. $f(x) = 7$
2. $f(x) = 3x + 1$
3. $f(x) = x^{1/2} - x + 8$
4. $f(x) = x^{-4} - 3x^{-2} + 7 + 12x^3$
5. $f(x) = (x + \pi)(x - \pi) + e^x - e^x$
6. $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$
7. $f(x) = x^7 - 32x^6 - \pi x^3 + 3/7$

Some possible graphs of polynomials.



Rational functions

Definition

A **rational function** in the variable x is a function the form

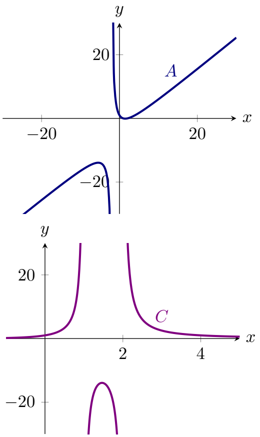
$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions. The domain of a rational function is all real numbers except for where the denominator is equal to zero.

Question. Which of the following are rational functions?

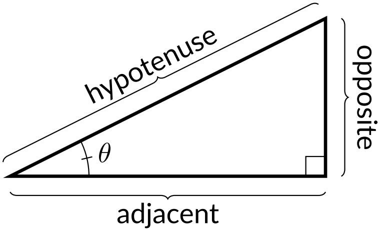
1. $f(x) = 0$
2. $f(x) = \dfrac{3x+1}{x^2 - 4x + 5}$
3. $f(x) = e^x$
4. $f(x) = \dfrac{\sin(x)}{\cos(x)}$
5. $f(x) = -4x^{-3} + 5x^{-1} + 7 - 18x^2$
6. $f(x) = x^{1/2} - x + 8$
7. $f(x) = \dfrac{\sqrt{x}}{x^3 - x}$

Some possible graphs of rational functions.



Trigonometric functions

A **trigonometric function** is a function that relates a measure of an angle of a right triangle to a ratio of the triangle's sides.



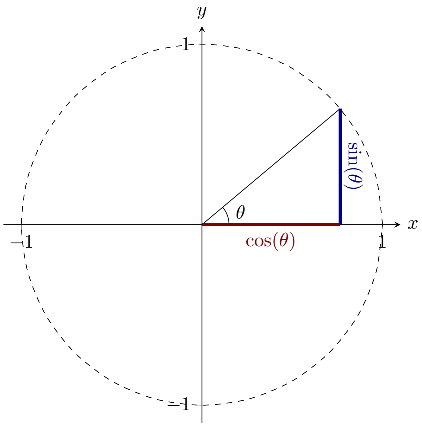
Definition

The trigonometric functions are:

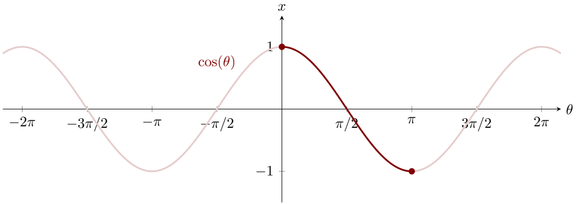
$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} \qquad \sin(\theta) = \frac{\text{opp}}{\text{hyp}} \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

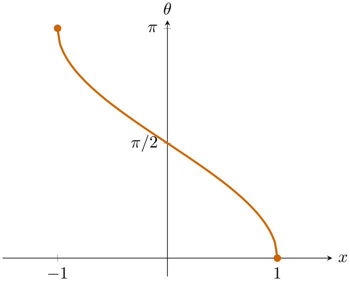
where the domain of sine and cosine is all real numbers, and the other are defined precisely when their denominators are nonzero.

The unit circle and trig functions



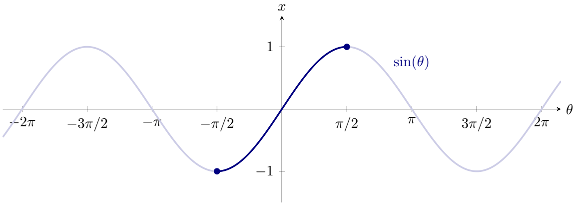
Cosine and its inverse.

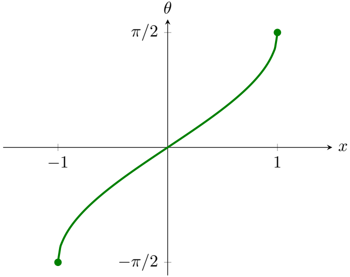




Here we see a plot of $\arccos(x)$, the inverse function of $\cos(\theta)$ when the domain is restricted to the interval $[0, \pi]$.

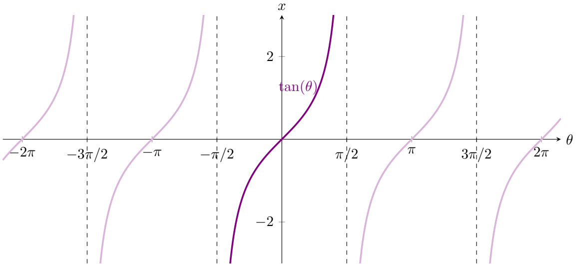
Sine and its inverse.

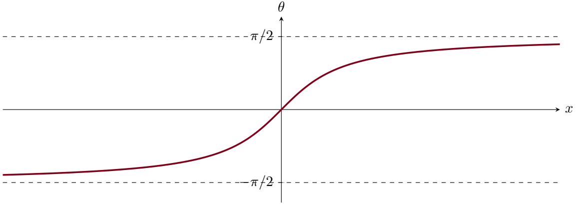




Here we see a plot of $\arcsin(x)$, the inverse function of $\sin(\theta)$ when the domain is restricted to the interval $[-\pi/2, \pi/2]$.

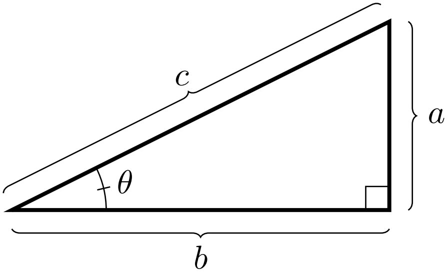
Tangent and its inverse.





Here we see a plot of $\arctan(x)$, the inverse function of $\tan(\theta)$ when the domain is restricted to the interval $(-\pi/2, \pi/2)$.

Pythagorean theorem and identities.



Pythagorean theorem:

- $a^2 + b^2 = c^2$

Pythagorean identities:

- $\cos^2\theta + \sin^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $\cot^2\theta + 1 = \csc^2\theta$

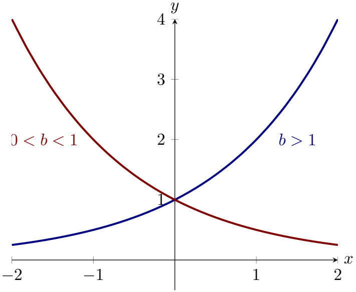
Exponential and logarithmic functions

Definition

An **exponential function** is a function of the form

$$f(x) = b^x$$

where $b \neq 1$ is a positive real number. The domain of an exponential function is $(-\infty, \infty)$. (Special: $f(x) = e^x$.)

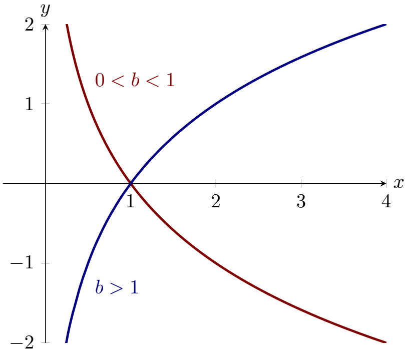


Definition

A **logarithmic function** is a function defined as follows

$$\log_b(x) = y \qquad \text{means that} \qquad b^y = x$$

where $b \neq 1$ is a positive real number. The domain of a logarithmic function is $(0, \infty)$. (Special: $f(x) = \ln(x)$.)



Properties of exponents

Let b be a positive real number with $b \neq 1$.

- $b^m \cdot b^n = b^{m+n}$
- $b^{-1} = \dfrac{1}{b}$
- $(b^m)^n = b^{mn}$

Properties of logarithms

Let b be a positive real number with $b \neq 1$.

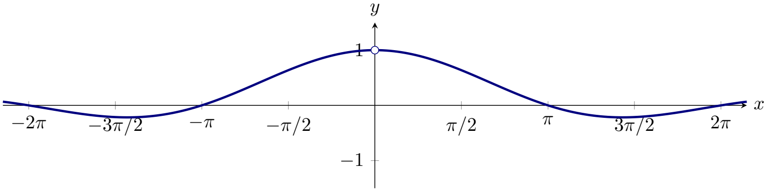
- $\log_b(m \cdot n) = \log_b(m) + \log_b(n)$
- $\log_b(m^n) = n \cdot \log_b(m)$
- $\log_b(1/m) = \log_b(m^{-1}) = -\log_b(m)$
- $\log_a(m) = \dfrac{\log_b(m)}{\log_b(a)}$

What Is a Limit? (WIAL)

What is a limit?

Basic idea. Consider the function

$$f(x) = \frac{\sin(x)}{x}.$$



Question.

- Is f defined at $x = 0$?
- Where is $f(x)$ approaching as x gets closer to 0?

Definition

Intuitively, we say that

the **limit** of $f(x)$ as x approaches a is L,

written

$$\lim_{x \to a} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all x sufficiently close, but not equal to, a.

Definition

Intuitively,

the **limit from the right** of f as x approaches a is L,

written

$$\lim_{x \to a^+} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all $x > a$ sufficiently close, but not equal to, a.
Similarly,

the **limit from the left** of $f(x)$ as x approaches a is L,

written

$$\lim_{x \to a^-} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all $x < a$ sufficiently close, but not equal to, a.

Theorem

A limit

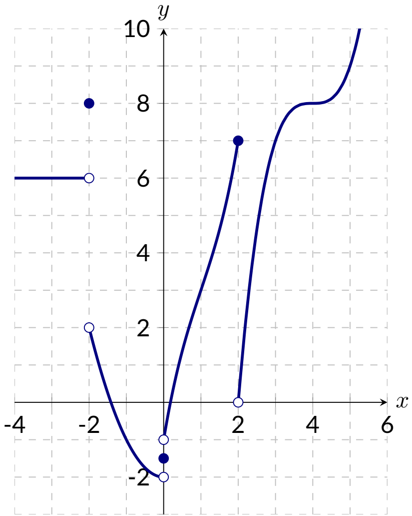
$$\lim_{x \to a} f(x)$$

exists if and only if

- $\lim_{x \to a^-} f(x)$ *exists*
- $\lim_{x \to a^+} f(x)$ *exists*
- $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

In this case, $\lim_{x \to a} f(x)$ *is equal to the common value of the two one sided limits.*

Question. Study limits of the following graph at various points.



Continuity

Definition

A function f is **continuous at a point** a if

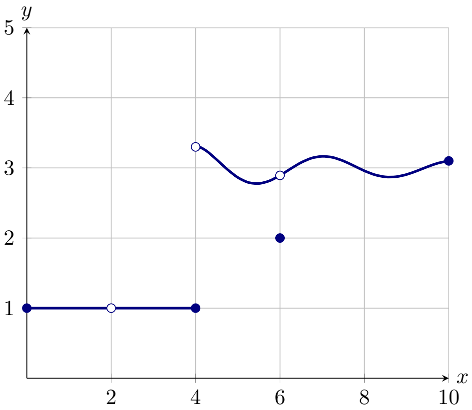
$$\lim_{x \to a} f(x) = f(a).$$

We can unpack the single equation above as:

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

Question. How can a function be discontinuous at a point?

Question. Find the discontinuities.



Definition

- A function f is **left continuous** at a point a if $\lim_{x \to a^-} f(x) = f(a)$.
- A function f is **right continuous** at a point a if $\lim_{x \to a^+} f(x) = f(a)$.

We can talk about continuity on intervals now.

Definition

A function f is

- **continuous on an open interval** (a, b) if $\lim_{x \to c} f(x) = f(c)$ for all c in (a, b);
- **continuous on a closed interval** $[a, b]$ if f is continuous on (a, b), right continuous at a, and left continuous at b.

Continuity of Famous Functions

The following functions are continuous on their natural domains, for k a real number and b a positive real number:

- **Constant function** $f(x) = k$
- **Identity function** $f(x) = x$
- **Power function** $f(x) = x^b$
- **Exponential function** $f(x) = b^x$
- **Logarithmic function** $f(x) = \log_b(x)$
- **Sine and cosine functions** $f(x) = \sin(x)$ and $f(x) = \cos(x)$

Question. (Revisiting the previous graph) What are the *largest intervals* of continuity?