Lecture 30: Antiderivatives (A)

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Basic Antiderivatives

• [one of infinitely many possible answers

Antidifferentiation is a process where we undo differentiation. Precisely:

Definition

A function F is called an **antiderivative** of f on an interval if

$$F'(x) = f(x)$$

for all x in the interval.

[All antiderivatives differ by const.

Example $f(x) = 2x - 7$.

Q. What is an antiderivative of $f(x)$?

i.e., Find a function $F(x)$ whose derivative equals $f(x)$.

A. $F(x) = x^2 - 7x$

check: $F'(x) = 2x - 7 = f(x)$. ✓

Q. Can you find any other?

A.

Student A: $F(x) = x^2 - 7x + \pi$

Student D: $F(x) = x^2 - 7x + 2$

Since the derivative of a constant is zero, we can add it to any antiderivative of f and it will still be an antiderivative.

family w/ ∞-many members.

Theorem (The family of antiderivatives)

*If F is an antiderivative of f, then the function f has a whole **family of antiderivatives**. Each antiderivative of f is the sum of F and some constant C. The family of all antiderivatives of f is denoted by*

"representative" infinitude.

$$\begin{pmatrix}\text{family of all} \\ \text{antiderivatives}\end{pmatrix} = \int f(x)\, dx.$$

*This is called the **indefinite integral** of f.*

It follows that

$$\int f(x)\, dx = F(x) + C,$$

Do not forget $+C$.

where F is any antiderivative of f and C is an arbitrary constant.

Ex $\begin{pmatrix}\text{The indefinite integral} \\ \text{of } f(x) = 2x-7\end{pmatrix} = \int (2x-7)\, dx = \underbrace{x^2-7x}_{\text{an antiderivative}} + C$

Basic Indefinite Integrals

- $\int k\,dx = kx + C$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
- $\int e^x\,dx = e^x + C$
- $\int a^x\,dx = \frac{a^x}{\ln(a)} + C$
- $\int \cos(x)\,dx = \sin(x) + C$
- $\int \sin(x)\,dx = -\cos(x) + C$
- $\int \sec^2(x)\,dx = \tan(x) + C$
- $\int \csc^2(x)\,dx = -\cot(x) + C$
- $\int \sec(x)\tan(x)\,dx = \sec(x) + C$
- $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$
- $\int \frac{1}{x^2+1}\,dx = \arctan x + C$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$

Confirm: Differentiate RHS and see if it equals the "integrand" on LHS.

"integrand": function to be antidifferentiated.

$$\int \boxed{f(x)}\,dx$$

$$\frac{d}{dx}\frac{x^{n+1}}{n+1} = \frac{\cancel{(n+1)}\,x^n}{\cancel{n+1}} = x^n$$

Basic Antiderivative Rules

We have the following rules that mirror basic derivative rules.

Theorem

If F is an antiderivative of f and G is an antiderivative of g, then $F+G$ is an antiderivative of $f+g$. Moreover, for any constant k, kF is an antiderivative of kf. We can write equivalently, using indefinite integrals,

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx\,, \qquad \text{(sum rule)}$$

$$\int kf(x)\,dx = k\int f(x)\,dx\,. \qquad \text{(constant multiple rule)}$$

Q. How about "differences"?

A. $\int (f(x) - g(x))\,dx = \int (f(x) + (-1)g(x))\,dx$

$= \int f(x)\,dx + \int (-1)g(x)\,dx$ (sum rule)

$= \int f(x)\,dx + (-1)\int g(x)\,dx$ (const. mult.)

$= \int f(x)\,dx - \int g(x)\,dx.$

Question. Compute:

$$\int \left(x^4 + 5x^2 - \cos(x)\right) dx$$

Sum rule

$$= \int x^4 \, dx + \int 5x^2 \, dx + \int (-\cos x) \, dx$$

const. mult.

$$= \int x^4 \, dx + 5 \int x^2 \, dx - \int \cos(x) \, dx$$

power rule

b/c $\frac{d}{dx} \sin(x) = \cos(x)$.

$$= \boxed{\frac{x^5}{5} + 5\,\frac{x^3}{3} - \sin(x) + C}$$

"representatives"

"package" all ∞-many of them.

Question. A student claims that $\int 2x\cos(x)\,dx = x^2\sin(x) + C$. Determine whether the student is correct or incorrect.

Verification using $\boxed{\int f(x)\,dx = F(x) + c \quad \text{implies} \quad F'(x) = f(x)}$

$$\underbrace{\frac{d}{dx}\left[x^2 \sin(x) + C\right]}_{\text{deriv. of RHS}} = 2x\sin(x) + x^2\cos(x) + 0 \neq \underbrace{2x\cos(x)}_{\text{integrand of LHS.}}$$

Hence, the student's answer is incorrect.

Side

$$\frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$$

Question. Compute:

$$\int \frac{\sqrt{x}+1+x}{x}\,dx$$

Rewriting / Simplifying the integrand:

$$= \int \frac{\sqrt{x}}{x}\,dx + \int \frac{1}{x}\,dx + \int \frac{x}{x}\,dx$$

$$= \int x^{-\frac{1}{2}}\,dx + \int \frac{1}{x}\,dx + \int 1\,dx$$

$-\frac{1}{2}+1 = \frac{1}{2}$

$$= \boxed{\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \ln|x| + x + C}$$

Revisit in couple week.

Question. Compute:

$$\int 3x^2 \sin\left(x^3 - 6\right) dx$$

Revisit later.

Question. Compute:

$$\int \frac{2x^2}{7x^3 + 3}\, dx$$

Guessing Antiderivatives ("linear factors")

a, b constants.

$$\frac{d}{dx} F(ax+b) = a F'(ax+b) \Rightarrow \int F'(ax+b)\, dx = \frac{1}{a} F(ax+b) + C$$

Example

- $\int (7x+5)^5 \, dx = \frac{1}{7} \frac{(7x+5)^6}{6} + C$

- $\int \sin(\pi x) \, dx = -\frac{1}{\pi} \cos(\pi x) + C$

plays the role of "a"

Check: $\frac{d}{dx}\left(-\frac{1}{\pi} \cos(\pi x)\right) = -\frac{1}{\cancel{\pi}}(-\sin(\pi x)) \cdot \cancel{\pi} = \sin(\pi x)$ ✓

Differential Equations

- A *differential equation* is simply an equation with a derivative in it. Here is an example:
$$af'(x) + bf(x) = g(x).$$
- Differential equations show you relationships between rates of functions.
- The theory of differential equation is a very important branch of mathematics with vast real-life applications.

What Does It Mean To Solve A Differential Equation?

When a mathematician solves a differential equation, they are finding *functions* satisfying the equation. For example, consider the following differential equation:

$$f'(x) = f(x)\,.$$

- It turns out that the complete solution to this differential equation is Ce^x, i.e., all the solutions of this differential equation have this form.
- Showing that any function $y = Ce^x$ is a solution of this differential equation is easy,
- but showing that **all** of the solutions have this form is beyond the scope of this course.

General Solution and Initial Value Problems

- In the previous example, a function Ce^x is called a **general solution** of the differential equation.
- Since there are infinitely many solutions to a differential equation, we can impose additional condition, called an **initial condition**, e.g. $f(0) = 1$.

The problem now is to find a function f that satisfies both the differential equation (DE) and the initial condition (IC).

$$f'(x) = f(x) \tag{DE}$$

$$f(0) = 1 \tag{IC}$$

This is called an **initial value problem** (IVP).

Example: IVP and A Falling Object

Here is a classical example of IVP arising in simple physics.

Question. A ball is tossed into the air with an initial velocity of 15 m/s. What is the velocity of the ball after 1 second? How about after 2 seconds?

Question. A ball is tossed into the air with an initial velocity of 15 m/s from a height of 2 meters. When does the ball hit the ground?