Lecture 40: Working with Substitution (WWS)

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Substitution Procedures

Let's recall that in integrating a function which we suspect to be the derivative of another obtained by the chain rule:

- **1** Look for a candidate for the inner function; call it u.
- **2** Rewrite the given function completely in terms of u leaving no trace of the original variable.
- **3** Integrate this new function of u. (If necessary, you may need to go back to Step 1 and modify your choice of u.)
- 4 In dealing with an indefinite integral, make sure to replace u by the equivalent expression of the original variable.
- **6** Working with a definite integral, you may evaluate the result of Step 3 at the transformed bounds of u or evaluate the antiderivate obtained in Step 4 at the original bounds.

For the remainder of the lecture, we will work out practice examples.

$$\mathbf{1} \int_{2}^{3} \frac{1}{x \ln(x)} dx$$

$$\mathbf{1} \int_{2}^{3} \frac{1}{x \ln(x)} dx$$

$$\mathbf{2} \int_{0}^{16} \sqrt{4 - \sqrt{x}} dx$$

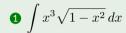
$$\mathbf{1} \int \frac{\sec(y)\tan(y) + \sec^2(y)}{\sec(y) + \tan(y)} dy$$

$$\mathbf{2} \int \tan(x) dx$$

$$\mathbf{1} \int \frac{u}{1-u^2} \, du$$

$$\int \frac{u}{1 - u^2} du$$

$$\int \frac{e^{2x}}{1 - e^{2x}} dx$$



Work

Suppose the force F(s) is applied to an object and moves it from $s=s_0$ to $s=s_1$. Then the work done on the object over the course of motion is given by

$$W = \int_{s_0}^{s_1} F(s) \, ds.$$

The international standard unit of force is a **Joule**, which is defined to be

$$1 J = 1 N \cdot m$$
.

In words, work measures the accumulated force over a distance. Note that we only accumulate force in the *direction* or *opposite direction* of motion.

If an apple has a mass of $0.1\,\mathrm{kg}$, how much work is required to lift this 1 meter above the ground? Assume that the gravitational acceleration is $-9.8\,\mathrm{m/\,s^2}$.

Kinetic Energy

Now suppose that an object of mass m is moving at velocity v(t). The **kinetic** energy E_k is the amount of *energy* that an object possesses from its motion. It is defined by

$$E_k = \frac{mv^2}{2}.$$

The SI unit of energy is also a **Joule** since $1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2$:

$$1 J = 1 kg \cdot m^2 / s^2.$$

Now the apple is dropped from the height of 1 meter. How much kinetic energy is released when it hits the ground?

Work-Energy Theorem

We observe that the work and the energy calculated above are the same with the same unit. This is not a coincidence and this phenomenon can be explained via the **Work-Energy Theorem**.

Theorem (Work-Energy Theorem)

Suppose that an object of mass m is moving along a straight line. If s_0 and s_1 are the the starting and ending positions, v_0 and v_1 are the the starting and ending velocities, and F(s) is the force acting on the object at any given position, then

$$W = \int_{s_0}^{s_1} F(s) \, ds = \frac{mv_1^2}{2} - \frac{mv_0^2}{2}.$$

In other words, the total work done by the force is equal to the net change in kinetic energy.

Explanation

Let $t_{0,1}$ be the initial and terminal time of the motion respectively, so that we can write $s_{0,1}=s(t_{0,1})$ respectively. Then the work formula can be written as

$$W = \int_{s(t_0)}^{s(t_1)} F(s) \, ds = \int_{t_0}^{t_1} F(s(t))s'(t) \, dt,$$

where the second equality is due to the substitution rule with u=s(t). By Newton's second law of motion F(s(t))=ma(t), we can write

$$\int_{t_0}^{t_1} F(s(t))s'(t) dt = \int_{t_0}^{t_1} ma(t)s'(t) dt = m \int_{t_0}^{t_1} a(t)s'(t) dt.$$

Remembering that a(t)=v'(t) and s'(t)=v(t), we can write the integrand solely in terms of v and its derivative, i.e.,

$$m \int_{t_0}^{t_1} a(t)s'(t) dt = m \int_{t_0}^{t_1} v(t)v'(t) dt.$$

Another substitution u=v(t) with new bounds $v_0=v(t_0)$ and $v_1=v(t_1)$ yields the desired result.

$$m \int_{t_0}^{t_1} v(t)v'(t) dt = m \int_{v_0}^{v_1} u du = m \left[\frac{u^2}{2} \right]_{v_0}^{v_1} = \frac{mv_1^2}{2} - \frac{mv_0^2}{2}.$$