Lecture 26-27: Optimization (O & AO)

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Basic Idea and Terminology

An **optimization problem** is a problem where you need to maximize or minimize some quantity under some constraints. This can be accomplished using the tools of differential calculus that we have already developed.

Terminology.

- **constraints:** conditions imposed on variables
- **objective functions:** the quantities desired to be optimized

A Solitary Local Extremum

- The extreme value theorem guarantees the existence of global extrema only on a closed interval.
- On intervals that are not closed, the theorem is not applicable. Yet, when there is only one local extreme value, we can say something about global extrema.

Theorem

Suppose f is continuous on an interval I that contains exactly one local extremum at c.

- *If a local maximum occurs at c,*
  then $f(c)$ is the global maximum of f on I.
- *If a local minimum occurs at c,*
  then $f(c)$ is the global minimum of f on I.

Example (Maximum area rectangles)

Of all rectangles of perimeter 12, which side lengths give the greatest area?

Example (Minimum perimeter rectangles)

Of all rectangles of area 100, which has the smallest perimeter?

Example (Rectangles beneath a semicircle)

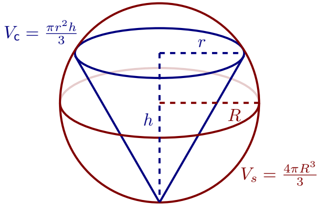
A rectangle is constructed with its base on the diameter of a semicircle with radius 5 and its two other vertices on the semicircle. What are the dimensions of the rectangle with maximum area?

Example (Minimum distance)

Find the point P on the curve $y = x^2$ that is closest to the point $(18, 0)$. What is the least distance between P and $(18, 0)$?

Example

If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by “cone” we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)



Example

Suppose you want to reach a point A that is located across the sand from a nearby road. Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?