Maxima Minima

Lecture 21: Maximums and Minimums (MAM)

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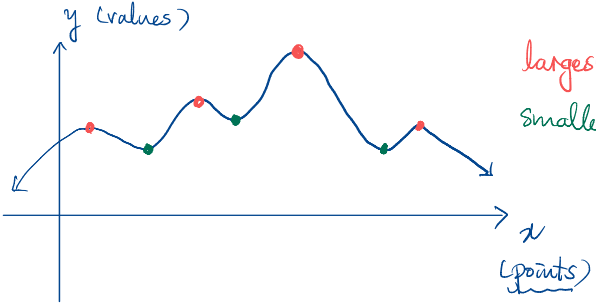
Autumn 2021

Overview What's the largest/smallest?



largest in "small neighborhood"

smallest in "small neighborhood"

1. Common features? → in terms of derivatives.

2. "Candidates" for min/max → Derivative Tests.
   - 1st D.T.
   - 2nd D.T.

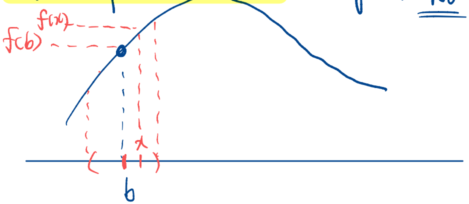
Local Extrema and Critical Points

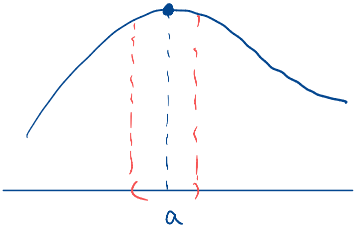
Definition

1. A function f has a **local maximum** at a, if $f(a) \geq f(x)$ for every x in some *open interval* containing a.
2. A function f has a **local minimum** at a, if $f(a) \leq f(x)$ for every x in some *open interval* containing a.

A **local extremum** is either a local maximum or a local minimum.

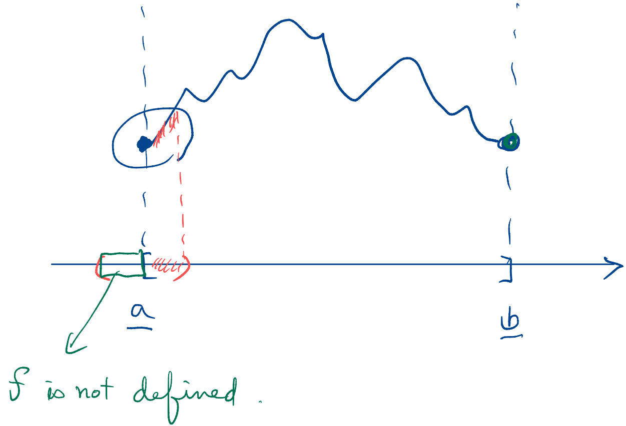
Is $f(b) \geq f(x)$ for every x in some open interval containing b? No





<u>Note</u> Endpoints are never local extremum points.

f is defined on a closed interval $[a, b]$.



Connection to Derivatives

local max

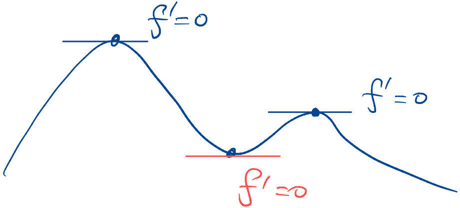
When the function under consideration has a “nice” graph and has a peak or a trough, the tangent line at this local extremum must be horizontal.

local min

Theorem (Fermat’s Theorem)

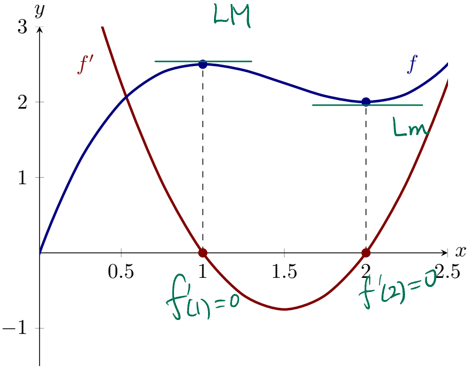
If f has a local extremum at a and f is differentiable at a, then $f'(a) = 0$.

horizontal tangent



Example: horizontal tangent line

Consider the plots of $f(x) = x^3 - 4.5x^2 + 6x$ and $f'(x) = 3x^2 - 9x + 6$.

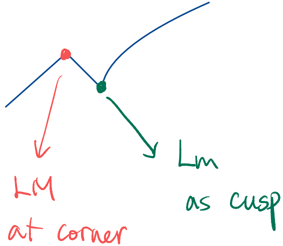


Connection to Derivatives (cont'd)

Question. Is it still possible for a function to have a peak or a trough without having a horizontal tangent line there? If so, draw a graph.

The only way out is to not have any tangent line at all.

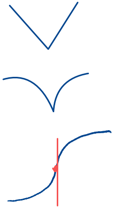
Example



examples

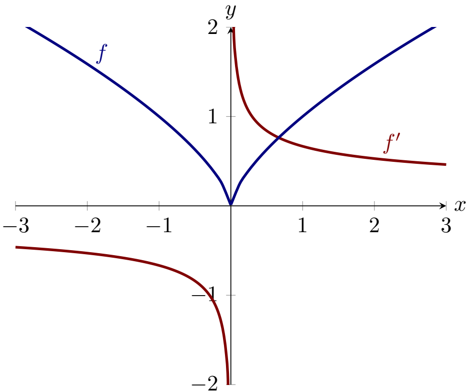
nondifferentiable

- discontinuity
- Corner
- cusp
- vert. tang.



Example: undefined derivative

Consider the plots of $f(x) = x^{2/3}$ and $f'(x) = \dfrac{2}{3x^{1/3}}$:



Critical Points

The following definition captures the two scenarios previously presented:

Definition (Critical points)

Assume that a function f is defined on an open interval I that contains a point a. The function f has a **critical point** at a if

$$f'(a) = 0 \qquad \text{or} \qquad f'(a) \text{ does not exist.}$$

nice ones

exotic ones

smooth "bumps"

sharp corners, cusps, mix-and-match

Question. Find all critical points of $f(x) = e^{\frac{1}{3}x^3 - 4x + 5}$.

- $f'(x) = 0$

or

- ~~$f'(x)$ DNE.~~

First, compute $f'(x)$.

$$f'(x) = e^{\frac{1}{3}x^3 - 4x + 5}\left(x^2 - 4\right)$$

$$= \underbrace{e^{\frac{1}{3}x^3 - 4x + 5}}_{\neq 0}(x-2)(x+2)$$

Note that $f'(x)$ is defined everywhere.

Setting $f'(x) = 0$, we find that

$$f'(x) = 0 \Rightarrow (x-2)(x+2) = 0$$

because $\exp \neq 0$.

Therefore, we have 2 c.p.

$$\boxed{\begin{aligned} x &= -2, \\ x &= 2 \end{aligned}}$$

Question. Find all critical points of $g(x) = |x-5|$.

Abs. val. func.

$\longrightarrow$ Case by case.

Case 1 $x-5>0$ i.e. $x>5$

$g(x) = x-5$

$g'(x) = 1$

No c.p. on $(5,\infty)$

Case 2 $x-5<0$ i.e. $x<5$

$g(x) = -(x-5)$

$g'(x) = -1$

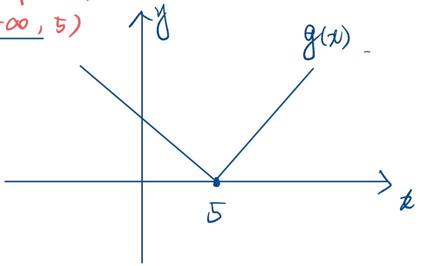
No c.p. on $(-\infty, 5)$

Case 3 $x-5=0$ i.e., $x=5$

$g(x)$ is not diff'ble at $x=5$.

For details, see prev. lect.

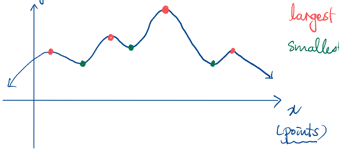
So $x=5$ is a c.p.



Recap

* Critical points are interior points.

Overview What's the largest/smallest?



largest in "small neighborhood"

smallest in "small neighborhood"

1. Common features? → in terms of derivatives.
2. "Candidates" for min/max → Derivative Tests.
   - 1st D.T.
   - 2nd D.T.

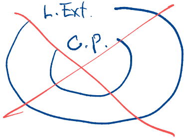
Critical points

Assume f is defined on an open interval I containing a point "a". We says that f has a critical point at a if

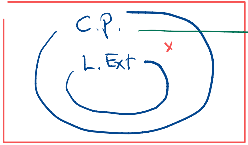
- $f'(a) = 0$ or
- $f'(a)$ DNE

Endpoints cannot be crit. pts.!

Question Are all critical points local extreme points?



VS

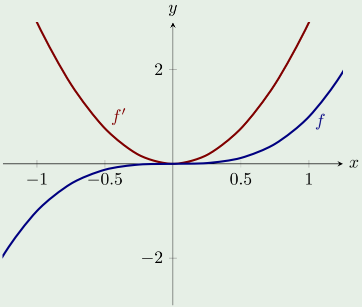


Candidates for loc. extreme points.

Necessary but not sufficient

Scenario 1: $f'(a) = 0$ with no local extremum

Consider the plots of $f(x) = x^3$ and $f'(x) = 3x^2$.

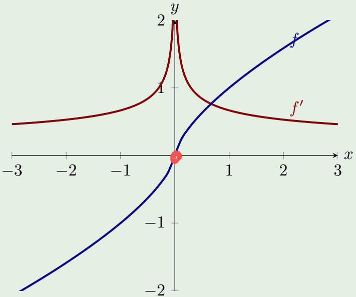


$x = 0$ is a critical point since $f'(0) = 0$, yet f has neither a local minimum or a local maximum at 0.

Necessary but not sufficient (cont'd)

$f'(a)$ DNE with no local extremum

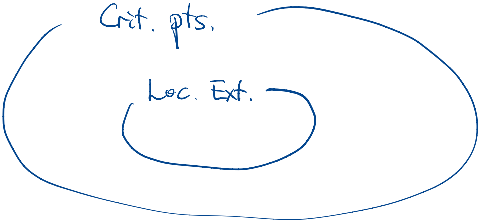
Consider the plots of $f(x) = \sqrt[3]{x}$ and $f'(x) = \frac{1}{3}x^{-2/3}$.



$x = 0$ is a critical point of f since $\lim_{x \to 0} f'(x) = \infty$ (vertical tangent), yet $f(0)$ is neither a local minimum nor a local maximum.

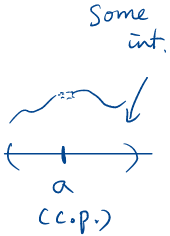
Upshot Critical points are candidates for local extreme points.

max
min.



The First Derivative Test

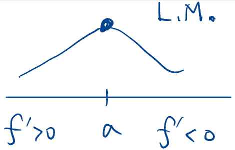
So the real question is how we classify a critical point. The following answer provides a method of finding relative extrema.

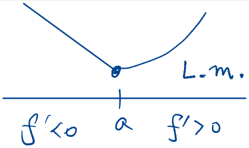


Theorem (First Derivative Test)

Suppose that f is continuous on an interval that contains a critical point a and assume f is differentiable on an interval containing a, except possibly at a.

- *If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then f has a* ***local maximum*** *at a.*
- *If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then f has a* ***local minimum*** *at a.*
- *If $f'(x)$ has the same sign to the left and right of a, then f has no local extreme value at a.*





Question. Find all local maximum and minimum points for the function $f(x) = x^3 - x$.

x-values or input values.

$$f'(x) = 3x^2 - 1 = 3\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right)$$

- f' is defined everywhere because it is a polynomial.

  (No c.p. of "exotic" kind)

- $f'(x) = 0$

$$3x^2 - 1 = 0$$

$$3x^2 = 1$$

$$x^2 = \frac{1}{3} \implies x = \pm\frac{1}{\sqrt{3}} \quad \text{(two c.p.)}$$

To implement the 1st D.T., let's build a sign-chart for f'.

x		$-\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$	
$x + \frac{1}{\sqrt{3}}$	$-$	0	$+$	$+$	$+$
$x - \frac{1}{\sqrt{3}}$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$
f	$\nearrow$	$\rightarrow$	$\searrow$	$\rightarrow$	$\nearrow$

$\boxed{\text{L.M. at } x = -\frac{1}{\sqrt{3}}, \quad \text{L.m. at } x = \frac{1}{\sqrt{3}}}$

Concavity and Inflection Points

Recall that second derivatives carry concavity information:

Theorem (Test for Concavity)

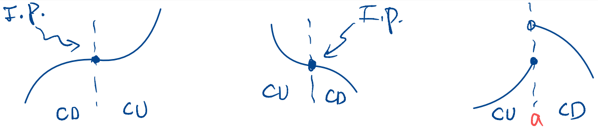
Suppose that $f''(x)$ exists on an interval.

1. *If $f''(x) > 0$ on an interval, then f is concave up on that interval.*
2. *If $f''(x) < 0$ on an interval, then f is concave down on that interval.*

And there are points at which the concavity changes from up to down or down to up. → concavity transition.

Definition (Inflection point)

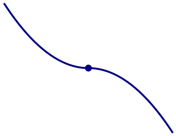
If f <u>is continuous at $x = a$</u> and its concavity changes either from up to down or down to up at $x = a$, then f has an **inflection point** at a.



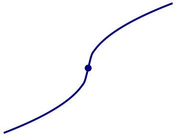
a is not an I.P. because f is not cts. at a.

Illustration: inflection points

Examples



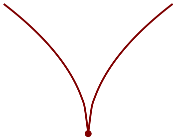
This is an inflection point. The concavity changes from concave up to concave down.



This is an inflection point. The concavity changes from concave up to concave down.

Non-examples

This is **not** an inflection point. The curve is concave down on either side of the point.



This is **not** an inflection point. The curve is concave down on either side of the point.

Warning. Even if f'' vanishes at a, the point $(a, f(a))$ may **not** be an inflection point.

The Second Derivative Test

"a" is a C.P. of "nice" kind.

Theorem (Second Derivative Test)

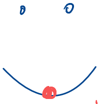
Suppose that $f''(x)$ is continuous on an open interval and that $f'(a) = 0$ for some value of a in that interval.

- *If $f''(a) < 0$, then f has a local maximum at a.*
- *If $f''(a) > 0$, then f has a local minimum at a.*
- *If $f''(a) = 0$, then the test is inconclusive. In this case, f may or may not have a local extremum at $x = a$.*

Note: 2nd DT is not applicable to crit. pts. of "exotic" kind, i.e. the ones with undefined derivatives.

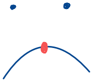
Smiley & Frowne

$f''(a) > 0$



L.m.

$f''(a) < 0$



L.M.

Question. Consider the function

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Using the second derivative test, find the intervals on which f is increasing and decreasing and identify the local extrema of f.

"nice" ✓

3 c.p.

- $f'(x) = x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x+2)(x-1) = 0 \Rightarrow x = 0, -2, 1$
- $f''(x) = 3x^2 + 2x - 2$

Eval. f'' at c.p.

$f''(-2) = 3 \cdot 4 + 2 \cdot (-2) - 2 = 6 > 0$ (L.m.)

$f''(0) = -2 < 0$ (L.M.)

$f''(1) = 3 + 2 - 2 = 3 > 0$ (L.m.)