Lecture 9: Derivatives as Functions (DAF)

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Recap

Slope of secant line

Derivative of a function at a point.

$$\left[\frac{d}{dx} f(x)\right]_{x=a} = \lim_{x \to a} \boxed{\frac{f(x)-f(a)}{x-a}} \quad x \to a \text{ char.}$$

$$= \lim_{h \to 0} \boxed{\frac{f(a+h)-f(a)}{h}} \quad h \to 0 \text{ char.}$$

- Compact notation : $f'(a)$

The derivative as a function

- Recall from last time that the **derivative** of a function f at a point a is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- If we replace a by a variable x, we now have the following function:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

this is now a function of x.

(of $f(x)$ w.r.t. x)

- This function gives us the *instantaneous rate of change* at any variable point x. Calculation of this derived function is called **differentiation**.
- **Notation.**

$$f'(x) = \frac{d}{dx} f(x).$$

"Leibniz notation"

Differentiability implies continuity

Punch line: $\lim_{x\to a}(f(x)-f(a))=0$

$\lim_{x\to a} f(x) - \lim_{x\to a} f(a) = 0$

$\lim_{x\to a} f(x) = f(a)$

Differentiability is a property that is stronger than continuity.

Theorem (Differentiability implies continuity)

If f is a differentiable function at a, then f is continuous at a.

$$\lim_{x\to a}\left(f(x)-f(a)\right) = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\cdot x-a\right] = \underbrace{\left[\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right]}_{= f'(a)}\underbrace{\left[\lim_{x\to a}(x-a)\right]}_{=0} = 0$$

$x \neq a$

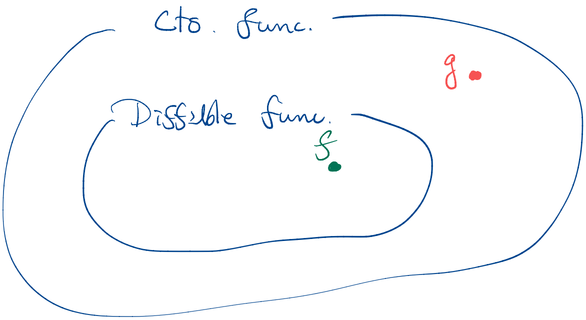
Prod. Law

by assumption

Notes.

- The contrapositive of the theorem is stated as follows:
  If f is not continuous at a, then f is not differentiable at a.
- Consequently, all differentiable functions are continuous, but not all continuous functions are differentiable.

Notes Diff. $\Rightarrow$ Cts.

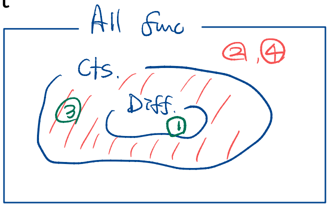


- There are functions (like g) which are cts. at a but not diff'ble at a.

- If f is diff'ble at a, then it is automatically cts at a.

Question. Which of the following functions are continuous but not differentiable on $\mathbb{R}$?

1. $\cancel{x^2}$
2. $\cancel{\lfloor x \rfloor}$
3. $\boxed{|x|}$
4. $\cancel{\dfrac{\sin(x)}{x}}$

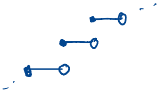


Rule out non-cts. func.

2 $\lfloor x \rfloor$ floor function

(greatest integer smaller than or equal to x)

$\lfloor 2.7 \rfloor = 2$, $\lfloor -3.15 \rfloor = -4$



not cts at all integers

4 $\dfrac{\sin(x)}{x}$ is not continuous at $x = 0$. because it is not in the domain.

Let's inspect ① & ② closely.

① $f(x) = x^2$.

- It is cts. because it's a power func.
  (or a poly.)

- Is it diff'ble? To answer, let's use the definition.

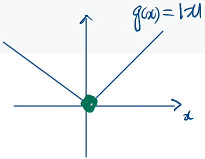
$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \qquad \text{“}\frac{0}{0}\text{” form}$$

$$= \lim_{h\to 0} \frac{(\cancel{x^2} + 2xh + h^2) - \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \boxed{2x = f'(x)}$$

This limit exists, so f(x) is diff'ble and $f'(x) = 2x$

③ $g(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$



$|-5| = -(-5) = 5$

• It is cts everywhere.

Intuition: graph w/ corners $\longrightarrow$ non-diff.

• At $x = 0$.

$$\lim_{h \to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^+} \frac{|h| - |0|}{h} = \lim_{h \to 0^+} \frac{\cancel{h}}{\cancel{h}} = 1.$$

h > 0

$$\neq$$

$$\lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^-} \frac{|h| - |0|}{h} = \lim_{h \to 0^-} \frac{-\cancel{h}}{\cancel{h}} = -1$$

h < 0

• Upshot: $\lim_{h \to 0} \frac{g(h) - g(0)}{h}$ does not exist, i.e., $g(x)$ is not diff'ble at 0.

Question. Consider

$$f(x) = \begin{cases} x^2 & \text{if } x < 3, \\ mx + b & \text{if } x \geq 3. \end{cases}$$

What values of m and b make f differentiable at $x = 3$?

Requirements

① Continuity at $x=3$ $\longrightarrow$ Eqn. 1

② differentiability at $x=3$ $\longrightarrow$ Eqn. 2

① Need

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$$

- $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 = 9$
- $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (mx+b) = 3m+b.$

So, for the limit to exist,

$\boxed{\text{Eqn 1.}}$ $3m + b = 9$

② For differentiability at 3, need

$$\lim_{h\to 0^+} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^-} \frac{f(3+h)-f(3)}{h}$$

- $$\lim_{h\to 0^+} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^+} \frac{\left[\overbrace{m(3+h)}^{\cancel{3m}+mh} + \cancel{b}\right] - \left[\cancel{3m}+\cancel{b}\right]}{h}$$

$$= \lim_{h\to 0^+} \frac{m\cancel{h}}{\cancel{h}} = m$$

- $$\lim_{h\to 0^-} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^-} \frac{(3+h)^2-3^2}{h} = \lim_{h\to 0^-} \frac{(\cancel{3^2}+2\cdot 3\cdot h+h^2)-\cancel{3^2}}{h}$$

$$= \lim_{h\to 0^-} \frac{\cancel{h}(6+h)}{\cancel{h}} = 6$$

So, we need

$\boxed{\text{Eqn } 2}$ $\quad m = 6$

We now have two eqns for two unknowns:

$$\begin{cases} \text{Eqn 1}: & 3m + b = 9 \\ \text{Eqn 2}: & \underline{m \qquad = 6} \quad (\text{done}) \end{cases}$$

Ans.

$m = 6, \quad b = -9$

Plugging in $m=6$ into Eqn 1:

$$3 \cdot 6 + b = 9$$

$$18 + b = 9$$

$$\therefore \underline{b = -9}$$