Rootfinding

FZERO to Solve Complex Problem

- **FNC** 4.1.5 (Kepler's Law)

4.1.5. The most easily observed properties of the orbit of a celestial body around the sun are the period τ and the elliptical eccentricity ϵ. (A circle has $\epsilon = 0$.) From these it is possible to find at any time t the angle $\theta(t)$ made between the body's position and the major axis of the ellipse. This is done through

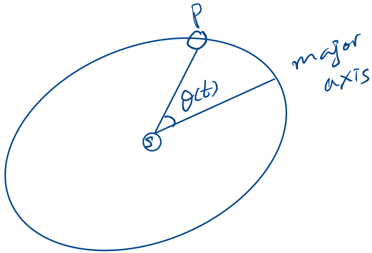
$$\tan\frac{\theta}{2} = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\tan\frac{\psi}{2}, \tag{4.1.2}$$

where the eccentric anomaly ψ satisfies Kepler's equation:

$$\psi - \epsilon\sin\psi - \frac{2\pi t}{\tau} = 0. \tag{4.1.3}$$

"fzero"

Equation (4.1.3) must be solved numerically to find $\psi(t)$, and then (4.1.2) can be solved analytically to find $\theta(t)$.
The asteroid Eros has $\tau = 1.7610$ years and $\epsilon = 0.2230$. Using `fzero` for (4.1.3), make a plot of $\theta(t)$ for 100 values of t between 0 and τ (one full period).



ψ : psi

θ : theta

initial guess

$$\theta(t) = 2\arctan\left[\sqrt{\frac{1+\epsilon}{1-\epsilon}}\tan\frac{\psi(t)}{2}\right]$$

fzero(@(psi) ... , 0)

More With Lambert W-Function

$y = W(x)$ iff $x = y e^y$

Question. Show by hand that solutions of the equation $2^x = 5x$

$$r = -\frac{W\left(-\log(2)/5\right)}{\log 2}.$$

(Here, as usual in this class, $\log(\,\cdot\,) = \ln(\,\cdot\,)$ is the natural logarithmic function.)
Then numerically verify the result using `fzero`[1]

$2^x = 5x \longrightarrow$

$\cancel{x \log 2 = \log 5 + \log x}$

$$e^{x \log 2} = 5x$$

$$\frac{1}{5} = x e^{-x \log 2}$$

$$-\frac{\log 2}{5} = -x \log 2 \, e^{-x \log 2}$$

$$-x \log 2 = W\left(-\frac{\log 2}{5}\right)$$

$$x = -\frac{W\left(-\frac{\log 2}{5}\right)}{\log 2} \quad \checkmark$$

[1] Two real-valued solutions, $r_1 \approx 0.2355$ and $r_2 \approx 4.488$.

FPI: When Convergence Is Faster Than Expected

- **FNC** 4.2.6

$$\boxed{\begin{array}{l}\underline{\text{Fixed point problem}}\text{: Given a function } g,\\ \text{find } x \text{ satisfying}\\ \qquad\qquad x = g(x).\end{array}}$$

Solution strategy (Iteration)

$$\begin{cases} x_0 : \text{ initial guess} \\ x_{n+1} = g(x_n) \quad \text{iteration formula.} \end{cases}$$

$$\boxed{\begin{array}{l}\underline{\text{Key note}}\\ \text{If } |g'(r)| < 1, \text{ the}\\ \text{convergence is } \underline{\text{linear}}.\end{array}}$$

$\Rightarrow x_0, x_1, x_2, \cdots$

If $\lim_{k\to\infty} x_k = r$, then $g(r) = r$, i.e. r is a <u>fixed point</u> of g.

(a) $g(x) = 2x - 3x^2$.

WTS : $r = 1/3$ is a f-p.

i.e., NTS : $1/3 = g(1/3)$

Soln: $g(1/3) = 2\cdot\frac{1}{3} - 3\left(\frac{1}{3}\right)^2 = \left(2 - \frac{3}{3}\right)\frac{1}{3} = \frac{1}{3}$ ✓

(b) $g'(1/3) = ?$

$g'(x) = 2 - 6x \Rightarrow g'(1/3) = 2 - 6\cdot\frac{1}{3} = \boxed{0}$

Since $|g'(1/3)| = 0$, the convergence of FPI near $1/3$ is <u>superlinear</u>!

- **FNC** 4.2.7

$$(\ast) \quad x_{k+1} = x_k - \frac{f(x_k)}{c}, \qquad f(r) = 0, \quad f'(r) \text{ exists.}$$

Define $g(x) = x - \dfrac{f(x)}{c}$.

By $f(r) = 0$, note that

$$g(r) = r - \frac{f(r)}{c} = r,$$

i.e., r is a fixed point of $g(x)$

and $(\ast)$ generates fixed point iterates.

Some considerations on c

- $c \neq 0$
- If $f'(r) > 0$,

  $c > \frac{1}{2} f'(r)$
- If $f'(r) < 0$

  $c < \frac{1}{2} f'(r)$

<u>Recall:</u> FPI converges to a f-p r

$$\boxed{\text{if} \quad |g'(r)| < 1}$$

In our case,

$$g(x) = x - \frac{f(x)}{c}$$

$$\Rightarrow g'(x) = 1 - \frac{f'(x)}{c}$$

So for convergence, we need

$$|g'(r)| = \left| 1 - \frac{f'(r)}{c} \right| < 1$$

$$-1 < 1 - \frac{f'(r)}{c} < 1$$

$$-2 < -\frac{f'(r)}{c} < 0$$

$$0 < \frac{f'(r)}{c} < 2$$

$$\frac{1}{2} < \frac{c}{f'(r)} < \infty$$

$$\Rightarrow \begin{cases} \frac{1}{2} f'(r) < c & \text{if } f'(r) > 0 \\ \frac{1}{2} f'(r) > c & \text{if } f'(r) < 0 \end{cases}$$

Stopping Criteria

- **FNC** 4.3.8

Find a sequence $\{x_k\}$ such that

- $\lim_{k\to\infty} x_k$ does not exist (i.e., $\{x_k\}$ diverges)
- $\lim_{k\to\infty} (x_{k+1} - x_k) = 0$

Hint: Calc 2.

Exercise with $\underbrace{\text{Series Analysis}}$

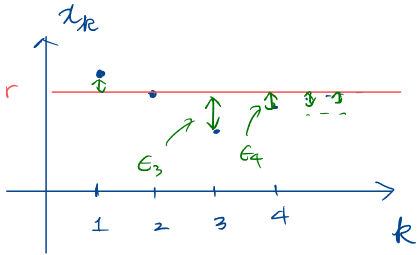
methodology employed
to study convergence of iterations.

- Taylor series

Convergence of Newton's method

Setting Assume Newton iteration $x_1, x_2, \cdots$ generated by

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$



converge to a root r of $f(x)$.

- $\lim_{k\to\infty} x_k = r$ and $f(r) = 0$

Assume $f'(r), f''(r) \neq 0$.

Notation $\epsilon_k = x_k - r$

i.e., $x_k = r + \epsilon_k$ (Note $\lim_{k\to\infty} \epsilon_k = 0$)

ϵ_k can be made arb. small for suff. large k.

Error analysis

- Iter. form. (subs. $x_k = r + \epsilon_k$)

$$\cancel{r} + \epsilon_{k+1} = \cancel{r} + \epsilon_k - \frac{f(r+\epsilon_k)}{f'(r+\epsilon_k)}$$

- Taylor expand at r

since r is a root of $f(x)$.

$$\epsilon_{k+1} = \epsilon_k - \frac{\overset{0}{\overset{\shortparallel}{f(r)}} + f'(r)\epsilon_k + \frac{f''(r)}{2}\epsilon_k^2 + O(\epsilon_k^3)}{f'(r) + f''(r)\epsilon_k + O(\epsilon_k^2)}$$

$$= \epsilon_k - \frac{\cancel{f'(r)}\epsilon_k\left[1 + \frac{f''(r)}{2f'(r)}\epsilon_k + O(\epsilon_k^2)\right]}{\cancel{f'(r)}\left[1 + \frac{f''(r)}{f'(r)}\epsilon_k + O(\epsilon_k^2)\right]}$$

Side note

$$\frac{1}{1+\boxed{\frac{f''(r)}{f'(r)}\epsilon_k+O(\epsilon_k^2)}} = 1-\frac{f''(r)}{f'(r)}\epsilon_k + O(\epsilon_k^2)$$

think of it as $-\alpha$

Recall Geometric series

$$\frac{1}{1-\alpha} = \sum_{k=0}^{\infty}\alpha^k = 1+\alpha+\alpha^2+\cdots$$

for $|\alpha|<1$

(Cont')

$$= \epsilon_k - \epsilon_k\left[1+\frac{f''(r)}{2f'(r)}\epsilon_k+O(\epsilon_k^2)\right]\left[1-\frac{f''(r)}{f'(r)}\epsilon_k+O(\epsilon_k^2)\right]$$

$$= \epsilon_k - \epsilon_k\left[1+\left(\frac{1}{2}-1\right)\frac{f''(r)}{f'(r)}\epsilon_k+O(\epsilon_k^2)\right]$$

$$\epsilon_{k+1} = \frac{1}{2}\frac{f''(r)}{f'(r)}\epsilon_k^2 + O(\epsilon_k^3).$$ (quad. convergence)

Linear Convergence of Newton's Method

← generally, quad. convergent.

Newton's Method for Multiple Roots

Assume that $f \in C^{m+1}[a,b]$ has a root r of multiplicity m. Then Newton's method is locally convergent to r, and the error ϵ_k at step k satisfies

$$\lim_{k\to\infty} \frac{\epsilon_{k+1}}{\epsilon_k} = \frac{m-1}{m} \qquad \text{(linear convergence)}$$

- See Problem 4 of HW07 (**FNC** 4.3.7)
- Remedy: Modify the iteration formula

$$\boxed{x_{k+1} = x_k - \frac{m f(x_k)}{f'(x_k)}}$$

To see why Say r is a root w/ multiplicity m. of $f(x)$:

$$f(r) = f'(r) = \cdots = f^{(m-1)}(r) = 0, \quad f^{(m)}(r) \neq 0$$

(from the def'n.)

Consequently:

- $$f(r+\epsilon_k) = \cancel{f(r)} + \cancel{f'(r)\epsilon_k} + \cancel{\frac{f''(r)}{2!}\epsilon_k^2} + \cdots + \cancel{\frac{f^{(m-1)}(r)}{(m-1)!}\epsilon_k^{m-1}} + \frac{f^{(m)}(r)}{m!}\epsilon_k^m + O(\epsilon_k^{m+1})$$

$$= \frac{f^{(m)}(r)}{m!}\epsilon_k^m + \frac{f^{(m+1)}(r)}{(m+1)!}\epsilon_k^{m+1} + O(\epsilon_k^{m+2})$$

- $$f'(r+\epsilon_k) = \cancel{f'(r)} + \cancel{f''(r)\epsilon_k} + \cdots + \cancel{\frac{f^{(m-1)}(r)}{(m-2)!}\epsilon_k^{m-2}} + \frac{f^{(m)}(r)}{(m-1)!}\epsilon_k^{m-1} + O(\epsilon_k^m)$$

$$= \frac{f^{(m)}(r)}{(m-1)!}\epsilon_k^{m-1} + \frac{f^{(m+1)}(r)}{m!}\epsilon_k^m + O(\epsilon_k^{m+1})$$

So the Newton's iteration formula (in terms of ϵ_k's) is now written as:

$$\epsilon_{k+1} = \epsilon_k - \frac{m f(r+\epsilon_k)}{f'(r+\epsilon_k)}$$

$$= \epsilon_k - \frac{m\left[\frac{f^{(m)}(r)}{m!}\epsilon_k^m + \frac{f^{(m+1)}(r)}{(m+1)!}\epsilon_k^{m+1} + O(\epsilon_k^{m+2})\right]}{\frac{f^{(m)}(r)}{(m-1)!}\epsilon_k^{m-1} + \frac{f^{(m+1)}(r)}{m!}\epsilon_k^m + O(\epsilon_k^{m+1})}$$

confirm (fill in the details)

$$\frac{m}{m}\epsilon_k = \epsilon_k$$

$$= \epsilon_k - \frac{1}{m}\epsilon_k + O(\epsilon_k^2) = \frac{m-1}{m}\epsilon_k + O(\epsilon_k^2)$$

$$\therefore \frac{\epsilon_{k+1}}{\epsilon_k^1} \to \frac{m-1}{m} \in (0,1)$$

(linear convergence)

$$\epsilon_{k+1} = * \epsilon_k^2 + O(\epsilon_k^3)$$

$$\therefore \frac{\epsilon_{k+1}}{\epsilon_k^2} \to *$$

as $k \to \infty$

(quad. conv.)

Calculating nth Roots

Question. Let n be a positive integer. Use Newton's method to produce a quadratically convergent method for calculating the nth root of a positive number a. Prove quadratic convergence.

Given $n \in \mathbb{N}$, $a > 0$, calculate $\sqrt[n]{a}$ using Newton.

Note that $\sqrt[n]{a}$ is a root of

$$x^n = a$$

So, it is a root of

$$f(x) = x^n - a$$

$$\boxed{x_{k+1} = \frac{n-1}{n} x_k + \frac{a}{n x_k^{n-1}}}$$

e.g. ($n=2$)

$$x_{k+1} = \frac{1}{2} x_k + \frac{a}{2 x_k}$$

Newton

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^n - a}{n x_k^{n-1}} = \frac{n-1}{n} x_k + \frac{a}{n x_k^{n-1}}$$

Predicting Next Error

$x(x^2-4) = x(x-2)(x+2)$

Question. Let $f(x) = x^3 - 4x$.

(a) • The function $f(x)$ has a root at $r = 2$. If the error $\epsilon_k = x_k - r$ after four steps of Newton's method is $\epsilon_4 = 10^{-6}$, estimate ϵ_5.

(b) • Do the same to the root $r = 0$. (Exercise)

Recall: Newton's method

$$\boxed{\epsilon_{k+1} = \frac{f'(r)}{2f''(r)} \epsilon_k^2 + O(\epsilon_k^3)}$$

(a) $r = 2$,

$$\begin{cases} f'(x) = 3x^2 - 4 \\ f''(x) = 6x \end{cases} \Rightarrow \begin{cases} f'(2) = 8 \\ f''(2) = 12 \end{cases}$$

So at $r=2$

$$\epsilon_{k+1} = \frac{8}{24}\epsilon_k^2 + O(\epsilon_k^3)$$

$k=4$, $\epsilon_k = 10^{-6}$

$$\epsilon_5 = \frac{1}{3} 10^{-12} + O(10^{-18}) \approx \boxed{3.33 \times 10^{-11}}$$

Secant Method

Assume that iterates $x_1, x_2, \ldots$ generated by the secant method converges to a root r and $f'(r) \neq 0$. Let $\epsilon_k = x_k - r$.

Exercise.[1] Show that

1. The error ϵ_k satisfies the approximate equation

$$\left|\epsilon_{k+1}\right| \approx \left|\frac{f''(r)}{2f'(r)}\right| \left|\epsilon_k\right| \left|\epsilon_{k-1}\right| .$$

2. If in addition $\lim_{k\to\infty} \left|\epsilon_{k+1}\right| / \left|\epsilon_k\right|^{\alpha}$ exists and is nonzero for some $\alpha > 0$, then

$$\left|\epsilon_{k+1}\right| \approx \left|\frac{f''(r)}{2f'(r)}\right|^{\alpha-1} \left|\epsilon_k\right|^{\alpha}, \quad \text{where } \alpha = \frac{1+\sqrt{5}}{2}.$$

[1]This exercise is from Lecture 22.

Hints Error analysis for Secant Method.

- $x_k = r + \epsilon_k$. $(\epsilon_k \to 0$ as $k \to \infty)$
- Taylor expansion
- big-O notation for simplication

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \approx f'(x_k)$$

Recall: iter. form. for secant method

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1}) f(x_k)}{f(x_k) - f(x_{k-1})}$$