Updated on October 11, 2021

Math 3607: Exam 2

Due: 12:00PM (noon), Monday, October 18, 2021

Please read the statements below and sign your name.

Disclaimers and Instructions

- You are **not** allowed to use MATLAB commands and functions **other than** the ones discussed in lectures, accompanying live scripts, textbooks, and homework/practice problem solutions.
- You may be requested to explain your code to me, in which case a proper and satisfactory explanation must be provided to receive any credits on relevant parts.
- You are **not** allowed to search online forums or even MathWorks website for this exam.
- You are **not** allowed to collaborate with classmates, unlike for homework.
- If any code is found to be plagiarized from the internet or another person, you will receive a zero on the *entire* exam and will be reported to the COAM.
- Do not carry out computations using *Symbolic Math Toolbox*. Any work done using `sym`, `syms`, `vpa`, and such will receive NO credit.
- Answers to analytical questions (ones marked with) without supporting work or justification will not receive any credit.

Academic Integrity Statements

- All of the work shown on this exam is my own.
- I will not consult with any resources (MathWorks website, online searches, etc.) other than the textbooks, lecture notes, and supplementary resources provided on the course Carmen pages.
- I will not discuss any part of this exam with anyone, online or offline.
- I understand that academic misconduct during an exam at The Ohio State University is very serious and can result in my failing this class or worse.
- I understand that any suspicious activity on my part will be automatically reported to the OSU Committee on Academic Misconduct (COAM) for their review.

Signature ____________________________

Notation. Problems marked with ✎ are to be done by hand; those marked with 💻 are to be solved using a computer.

1 Catastrophic Cancellation [25 points]

Let

$$f(x) = \begin{cases} \dfrac{1-\cos x}{x^2} & \text{if } x \neq 0, \\ \dfrac{1}{2} & \text{if } x = 0. \end{cases}$$

We are interested in a stable numerical evaluation of $f(x)$ for small x.

(a) ✎ Find the condition number $\kappa_f(x)$; simplify it as much as you can. Then compute

$$\lim_{x \to 0} \kappa_f(x).$$

Based on the limit computation, is the evaluation of $f(x)$ for small x well-conditioned or ill-conditioned?

(b) ✎ For small x, the "obvious" evaluation algorithm

```
f1 = @(x) (1-cos(x))./(x.^2);
```

suffers from catastrophic cancellation. Explain why.

(c) ✎ Using the first three nonzero terms of the Taylor series expansion of $f(x)$, establish an alternate algorithm `f2` to compute $f(x)$ stably for small x. Fully justify the derivation of your algorithm.

(d) 💻 Evaluate $f(x)$ for $x = 10^{-k}$ for $k \in \mathbb{N}[1, 10]$ using the two algorithms `f1` and `f2`. Tabulate the results neatly.

Notes. The table should have three columns with the first being `x`, the second being `f1`, and the third being `f2`. Use either `format long g` or an appropriate `fprintf` statement to display full accuracy. Do it as efficiently as you can, avoiding the use of a loop whenever possible.

(e) ✎ Comment on the results obtained in the previous part.

2 PLU Factorization [20 points]

Let

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 10 & -7 & 10 \\ -6 & 4 & -5 \end{bmatrix}.$$

(a) Find the PLU factorization of A. Justify all your steps.

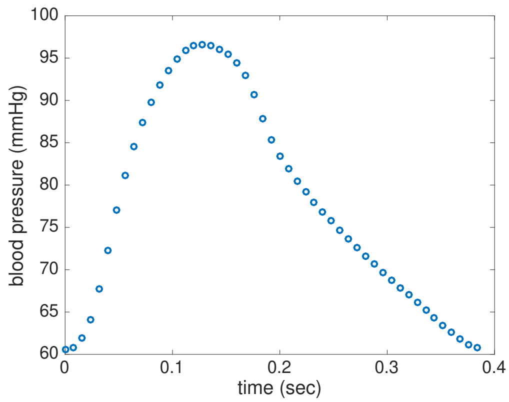
Hint. You may (and should) use, without proof, the properties of Gaussian transformation matrices presented in Problem 7 of Module 2 practice problem set (click here). Be sure to give a clear reference to the properties used in your solution.

(b) Compute $\det(P)$, where P is the permutation matrix found in part (a).

(c) Compute $\det(A)$ using parts (a) and (b) alone. The result obtained by a direct calculation will not be accepted. Be sure to justify all your steps clearly.

3 Least Squares for Periodic Data [25 points]

The graph below represents arterial blood pressure collected at 8 ms (milliseconds) intervals over one heart beat from an infant patient:



Denote the data points by (t_i, y_i) for $i = 1, \ldots, m$. The data can be fit using a low-degree polynomial of the form

$$f(t) = c_1 + c_2 t + \cdots + c_n t^{n-1}, \quad n < m. \tag{1}$$

In the most general terms, the fitting function takes the form

$$f(t) = c_1 f_1(t) + \cdots + c_n f_n(t), \tag{2}$$

where $f_1, \ldots, f_n$ are known functions while $c_1, \ldots, c_n$ are to be determined to optimize the fit to the data. This optimization can be formulated as an $m \times n$ LLS problem of minimizing the 2-norm of the residual $\|\mathbf{y} - A\mathbf{c}\|_2$, where $A_{i,j} = f_j(t_i)$.

(a) Download the data file `pressuredata.mat` and load into MATLAB using

```
load pressuredata
```

This creates two vectors `t` and `y` containing time and blood pressure data, respectively. Use them to regenerate the plot above.

(b) Fit the data to a straight line, $f(t) = c_1 + c_2 t$. Solve for the coefficients using backslash. Superimpose the graph of the fitting line on your graph from the previous step, and compute the 2-norm of the residual $\|\mathbf{y} - A\mathbf{c}\|_2$, where $A_{i,j} = t_i^{j-1}$ is a *Vandermonde*-type matrix.

(c) Repeat part (b) for a quadratic and a cubic polynomial. The residual norm will get smaller in each case, but there is still a room for improvement.

(d) Exploiting the fact that the data come from a periodic phenomenon (heart beats), adapt (2) to a periodic fitting function

$$f(t) = c_1 + c_2 \cos\frac{2\pi t}{\tau} + c_3 \sin\frac{2\pi t}{\tau} + c_4 \cos\frac{4\pi t}{\tau} + c_5 \sin\frac{4\pi t}{\tau}, \quad \text{where } \tau = t_m - t_1. \tag{3}$$

As in the previous parts, solve for the coefficients using backslash, superimpose the graph of $f(t)$ to the plot of data points, and compute the residual norm. Comment on your observation.

4 Visualization of Matrix Norms in 3-D [30 points]

Recall that

$$\|A\|_p = \max_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_p\,, \quad p \in [1, \infty].$$

In this problem, we generate three-dimensional visualization of this definition. This is a direct extension of the most recent homework problem.

(a) Complete the following program[1] which, given $p \in [1, \infty]$ and $A \in \mathbb{R}^{3\times 3}$, approximates $\|A\|_p$ and plots the unit sphere in the p-norm and its image under A. Avoid using loops as much as possible.

```
function norm_A = visMatrixNorms3D(A, p)
    %% Basic checks
    if size(A,1)~=3 || size(A,2)~=3
        error('A must be a 3-by-3 matrix.')
    elseif p < 1
        error('p must be >= 1.')
    end

    %% Step 1: Initialization
    nr_th = 41; nr_ph = 31;
    th = linspace(0, 2*pi, nr_th);
    ph = linspace(0, pi, nr_ph);
    [T, P] = meshgrid(th, ph);
    x1 = cos(T).*sin(P);
    x2 = sin(T).*sin(P);
    x3 = cos(P);
    X = [x1(:), x2(:), x3(:)]';

    %% Step 2: [FILL IN] Normalize columns of X into unit vectors

    %% Step 3: [FILL IN] Form Y = A*X; calculate norms of columns of Y

    %% Step 4: [FILL IN] Calculate p-norm of A (approximate)

    %% Step 5: [FILL IN] Generate surface plots

end
```

The following steps are carried out by the program.

Step 1. Create 3-vectors

$$\mathbf{x}_k = \begin{bmatrix} \cos\theta_i \sin\phi_j \\ \sin\theta_i \sin\phi_j \\ \cos\phi_j \end{bmatrix}, \quad \text{for } 1 \le i \le 41,\ 1 \le j \le 31 \tag{4}$$

using 41 evenly distributed θ_i in $[0, 2\pi]$ and 31 evenly distributed ϕ_j in $[0, \pi]$. Note the use of `meshgrid`, which is useful for surface plots later; see Lecture 8.

[1] The function must be written at the very end of your Live Script.

Step 2. Normalize $\mathbf{x}_k$ into a unit vector in p-norm by $\mathbf{x}_k \leftarrow \mathbf{x}_k / \|\mathbf{x}_k\|_p$ (that is, replacing $\mathbf{x}_k$ with $\mathbf{x}_k / \|\mathbf{x}_k\|_p$).

Step 3. For each k, let $\mathbf{y}_k = A\mathbf{x}_k$. Calculate and store $\|\mathbf{y}_k\|_p$.

Step 4. Approximate $\|A\|_p$ based on the norms $\|\mathbf{y}_k\|_p$ calculated in the previous step.

Step 5. Generate surface plots of the unit sphere in the p-norm and its image under A. Use `surf` function; see Lecture 8. Use `subplot` to put two graphs side by side.

(b) Run the program with $p = 1, \frac{3}{2}, 2, 4$, all with the same matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \cos(\pi/12) & -\sin(\pi/12) \\ 0 & \sin(\pi/12) & \cos(\pi/12) \end{bmatrix}, \tag{5}$$

by executing the following code block.

```
A = [2 0 0;
     0 cos(pi/12) -sin(pi/12);
     0 sin(pi/12)  cos(pi/12)];
visMatrixNorms3D(A, 1);          % Depending on how you write the code,
visMatrixNorms3D(A, 3/2);        % you may need to use 'clf' or 'hold off'
visMatrixNorms3D(A, 2);          % in between function calls.
visMatrixNorms3D(A, 4);
```

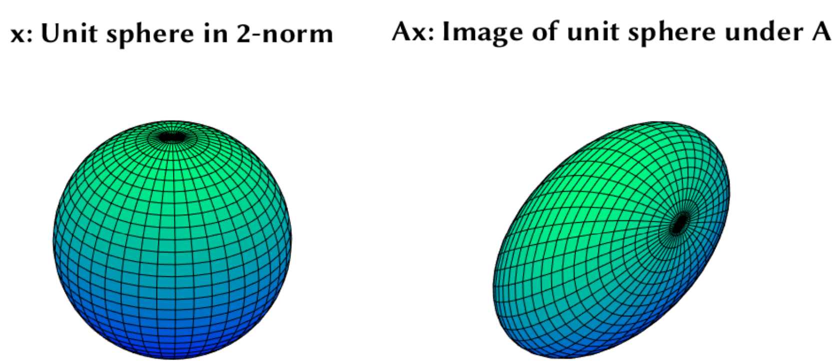


Figure 1: Example output.