Lagrange Polynomial Form

For $k \in \mathbb{N}$, denote by Π_k the set of all polynomials of degree $\leqslant k$, that is,

$$\Pi_k = \{a_1 + a_2 x + a_3 x^2 + \cdots + a_{k+1} x^k \mid a_i \in \mathbb{R}\}\,.$$

Note that Π_k is a $(k+1)$-dimensional vector space. It is well-known that the powers $\{1, x, x^2, \cdots, x^k\}$ form a basis for this space. Polynomial interpolation with respect with this power basis gives rise to a *Vandermonde* linear system. In this note, we will show that Lagrange polynomials, to be defined, form another basis for Π_k and that the Lagrange-basis solves an interpolation problem trivially.

Definition 1. For the given set of n data points $\{(x_j, y_j) \mid j \in \mathbb{N}[1, n]\}$ with distinct x_j's, the set of *Lagrange polynomials* is denoted by $\{\ell_j \mid j \in \mathbb{N}[1, n]\}$ where

$$\ell_j(x) = \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k} = \frac{x - x_1}{x_j - x_1} \cdot \frac{x - x_2}{x_j - x_2} \cdots \frac{x - x_{j-1}}{x_j - x_{j-1}} \cdot \frac{x - x_{j+1}}{x_j - x_{j+1}} \cdots \frac{x - x_n}{x_j - x_n}\,.$$

Note 2. Each ℓ_j is of degree $n - 1$ and

$$\ell_j(x_k) = \delta_{j,k} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}\,.$$

Here $\delta_{j,k}$ is the *Kronecker's delta.*

Theorem 1. *The set of Lagrange polynomials $\{\ell_j(x) \mid j \in \mathbb{N}[1, n]\}$ is a basis for Π_{n-1}.*

Proof. Consider the linear combination

$$c_1 \ell_1(x) + c_2 \ell_2(x) + \cdots + c_n \ell_n(x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

Since $\ell_j(x_k) = \delta_{j,k}$, upon substituting $x = x_k$, we have

$$c_1 \cdot 0 + \cdots + c_{k-1} \cdot 0 + c_k \cdot 1 + \cdots + c_n \cdot 0 = 0\,,$$

so $c_k = 0$ for every $k \in \mathbb{N}[1, n]$. This implies that these polynomials are linearly independent. In addition, all these n polynomials are in Π_{n-1}, which is an n-dimensional space, so

$$\Pi_{n-1} = \mathrm{Span}\{\ell_1(x), \ell_2(x), \cdots, \ell_n(x)\}\,,$$

that is, they form a basis for the space. $\square$

Lemma 2. *The polynomial*

$$p_{n-1}(x) = \sum_{j=1}^{n} y_j \ell_j(x)$$

is the unique polynomial of degree $\leqslant n$ which passes through all the data points.

Proof. It is easy to check the interpolating property:

$$p_{n-1}(x_k) = \sum_{j=1}^{n} y_j \ell_j(x_k) = \sum_{j=1}^{n} y_j \delta_{j,k} = y_k \quad \text{for every } k \in \mathbb{N}[1,n].$$

To show uniqueness, suppose $\widetilde{p}_{n-1}(x)$ is another such polynomial in the form

$$\widetilde{p}_{n-1}(x) = \sum_{j=1}^{n} c_j \ell_j(x)\,.$$

Then $p_{n-1}(x_k) - \widetilde{p}_{n-1}(x_k) = 0$ for all $k \in \mathbb{N}[1,n]$. This implies that $c_k = y_k$ for all k, completing the proof. □

Example 3. Interpolate the data set $\{(x_j, y_j) \mid j \in \mathbb{N}[1,3]\} = \{(1,4),(3,2),(5,1)\}$ using Lagrange polynomials.

Solution. First, find Lagrange polynomials associated with the given data set:

$$\begin{aligned}
\ell_1(x) &= \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-3)(x-5)}{(1-3)(1-5)} = \frac{1}{8}(x-3)(x-5)\,, \\
\ell_2(x) &= \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x-1)(x-5)}{(3-1)(3-5)} = -\frac{1}{4}(x-1)(x-5)\,, \\
\ell_3(x) &= \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x-1)(x-3)}{(5-1)(5-3)} = \frac{1}{8}(x-1)(x-3)\,.
\end{aligned}$$

Then the interpolating polynomial is

$$\begin{aligned}
p_2(x) &= 4\ell_1(x) + 2\ell_2(x) + \ell_3(x) \\
&= \underbrace{\frac{1}{2}(x-3)(x-5) - \frac{1}{2}(x-1)(x-5) + \frac{1}{8}(x-1)(x-3)}_{\text{Lagrange form}} = \underbrace{\frac{1}{8}x^2 - \frac{3}{2}x + \frac{43}{8}}_{\text{power form}}\,.
\end{aligned}$$