Derivation of Hermite Cubic Interpolant

Given (x_i, y_i, σ_i), for $i = 1, 2, \cdots, n$, find the piecewise cubic polynomial

$$\mathbb{P}(x) = \begin{cases} p_1(x), & [x_1, x_2) \\ p_2(x), & [x_2, x_3) \\ \vdots & \vdots \\ p_{n-1}(x), & [x_{n-1}, x_n] \end{cases}$$

where $p_i(x) = c_{i,1} + c_{i,2}(x - x_i) + c_{i,3}(x - x_i)^2 + c_{i,4}(x - x_i)^3$

satisfying

$$\mathbb{P}(x_i) = y_i \quad \text{and} \quad \mathbb{P}'(x_i) = \sigma_i \quad \text{for all} \quad i = 1, 2, \cdots, n.$$

Strategy: For each $i = 1, 2, \cdots, n-1$, determine

$$p_i(x) = c_{i,1} + c_{i,2}(x - x_i) + c_{i,3}(x - x_i)^2 + c_{i,4}(x - x_i)^3$$

such that

$$\begin{cases} ① : p_i(x_i) = y_i \\ ② : p_i'(x_i) = \sigma_i \end{cases} \qquad \begin{cases} ③ : p_i(x_{i+1}) = y_{i+1} \\ ④ : p_i'(x_{i+1}) = \sigma_{i+1} \end{cases}$$

4 eqns
4 unknowns

Solution: Easy to see from ① and ② that

$$\boxed{c_{i,1} = y_i, \qquad c_{i,2} = \sigma_i.}$$

Eqns ③ and ④ are written out as

$$\begin{cases} ③' : y_i + \sigma_i \Delta x_i + c_{i,3}(\Delta x_i)^2 + c_{i,4}(\Delta x_i)^3 = y_{i+1} \\ ④' : \sigma_i + 2c_{i,3}\Delta x_i + 3c_{i,4}(\Delta x_i)^2 = \sigma_{i+1} \end{cases}$$

where $\Delta x_i = x_{i+1} - x_i$. (Analogously, we use $\Delta y_i = y_{i+1} - y_i$ and define $y[x_i, x_{i+1}] = \Delta y_i / \Delta x_i$ in what follows.)

Rewrite:

③$'$: $$c_{i,3} + \Delta x_i \, c_{i,4} = \frac{y_{i+1} - y_i - \sigma_i \Delta x_i}{(\Delta x_i)^2}$$

$$= \frac{\Delta y_i / \Delta x_i - \sigma_i}{\Delta x_i}$$

$$= \frac{y[x_i, x_{i+1}] - \sigma_i}{\Delta x_i}$$

④$'$: $$c_{i,3} + \frac{3}{2}\Delta x_i \, c_{i,4} = \frac{\sigma_{i+1} - \sigma_i}{2\Delta x_i}$$

First, solve for $c_{i,4}$ by eliminating $c_{i,3}$:

④$'$ − ③$'$: $$\frac{1}{2}\Delta x_i \, c_{i,4} = \frac{(\sigma_{i+1} - \sigma_i) - 2(y[x_i, x_{i+1}] - \sigma_i)}{2\Delta x_i}$$

$$= \frac{\sigma_i + \sigma_{i+1} - 2y[x_i, x_{i+1}]}{2\Delta x_i}$$

$$\therefore \boxed{c_{i,4} = \frac{\sigma_i + \sigma_{i+1} - 2y[x_i, x_{i+1}]}{(\Delta x_i)^2}}$$

Plugging this back into 4':

$$c_{i,3} = \frac{\sigma_{i+1} - \sigma_i}{2\Delta x_i} - \frac{3}{2}\cancel{\Delta x_i}\,\frac{\sigma_i + \sigma_{i+1} - 2y[x_i, x_{i+1}]}{(\Delta x_i)^{\cancel{2}}}$$

$$= \frac{(\sigma_{i+1} - \sigma_i) - 3(\sigma_i + \sigma_{i+1} - 2y[x_i, x_{i+1}])}{2\Delta x_i}$$

$$= \frac{6y[x_i, x_{i+1}] - 4\sigma_i - 2\sigma_{i+1}}{2\Delta x_i}$$

$$\therefore \boxed{c_{i,3} = \frac{3y[x_i, x_{i+1}] - 2\sigma_i - \sigma_{i+1}}{\Delta x_i}}$$