Updated on November 8, 2021

Math 3607: Exam 3

Due: 11:59PM, Wednesday, November 10, 2021

Please read the statements below and sign your name.

Disclaimers and Instructions

- You are **not** allowed to use MATLAB commands and functions **other than** the ones discussed in lectures, accompanying live scripts, textbooks, and homework/practice problem solutions.
- You may be requested to explain your code to me, in which case a proper and satisfactory explanation must be provided to receive any credits on relevant parts.
- You are **not** allowed to search online forums or even MathWorks website for this exam.
- You are **not** allowed to collaborate with classmates, unlike for homework.
- If any code is found to be plagiarized from the internet or another person, you will receive a zero on the *entire* exam and will be reported to the COAM.
- Do not carry out computations using *Symbolic Math Toolbox*. Any work done using `sym`, `syms`, `vpa`, and such will receive NO credit.
- **Notation.** Problems marked with are to be done by hand; those marked with are to be solved using a computer.
- Answers to analytical questions (ones marked with) without supporting work or justification will not receive any credit.

Academic Integrity Statements

- All of the work shown on this exam is my own.
- I will not consult with any resources (MathWorks website, online searches, etc.) other than the textbooks, lecture notes, and supplementary resources provided on the course Carmen pages.
- I will not discuss any part of this exam with anyone, online or offline.
- I understand that academic misconduct during an exam at The Ohio State University is very serious and can result in my failing this class or worse.
- I understand that any suspicious activity on my part will be automatically reported to the OSU Committee on Academic Misconduct (COAM) for their review.

Signature ____________________________

1 EVD of Householder Matrices [12 points]

Choose the correct answer for each of the blanks below. You do not need to justify your answers for this problem; there is no partial credits.

Let $\mathbf{u} \in \mathbb{R}^m$ be a unit vector and define a Householder matrix $H \in \mathbb{R}^{m\times m}$ by $H = I - 2\mathbf{u}\mathbf{u}^{\mathrm{T}}$. Observe that

$$H\mathbf{u} = \left(I - 2\mathbf{u}\mathbf{u}^{\mathrm{T}}\right)\mathbf{u} = \mathbf{u} - 2\mathbf{u} = -\mathbf{u}.$$

This implies that

$$\lambda = \boxed{\text{(a)}}$$

is an eigenvalue of H and $\mathbf{u}$ is a corresponding eigenvector. Now let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{m-1}\}$ be an orthonormal basis of the orthogonal complement $\mathbf{u}^{\perp}$, which is an $(m-1)$-dimensional subspace of $\mathbb{R}^m$ consisting of all vectors perpendicular to $\mathbf{u}$; in particular, $\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_i = \delta_{k,j}$ (Kronecker delta) and

$$\mathbf{u}^{\mathrm{T}}\mathbf{v}_j = \boxed{\text{(b)}}.$$

Consequently, for any $1 \leq j \leq m-1$,

$$H\mathbf{v}_j = \left(I - 2\mathbf{u}\mathbf{u}^{\mathrm{T}}\right)\mathbf{v}_j = \boxed{\text{(c)}}.$$

This implies that $\mathbf{v}_j$ is $\boxed{\text{(d)}}$ of H corresponding to the eigenvalue $\lambda_j = \boxed{\text{(e)}}$. Since all eigenvectors are orthogonal and thus linearly $\boxed{\text{(f)}}$, H has an EVD $H = VDV^{-1}$, where

$$V = \left[\begin{array}{c|c|c|c|c} \mathbf{u} & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_{m-1} \end{array}\right] \quad \text{and} \quad D = \begin{bmatrix} -1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}.$$

(a) ☐ $+1$ ☐ -1

(b) ☐ 0 ☐ 1

(c) ☐ $+\mathbf{v}_j$ ☐ $-\mathbf{v}_j$

(d) ☐ an eigenvector ☐ a singular vector

(e) ☐ $+1$ ☐ -1

(f) ☐ dependent ☐ independent

2 Demystifying Recursion using EVD [18 points]

Consider the sequence (a_n) defined recursively by

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2a_{n-1} + a_{n-2} & \text{if } n \geq 2 \end{cases}$$

Find the general formula for the nth term a_n. Show all your work.

3 Newton's Method Faster Than Quadratic? [20 points]

Suppose that f has a simple root at r at which f'' vanishes, that is,

$$f(r) = 0, \quad f'(r) \neq 0, \quad f''(r) = 0.$$

Assuming that f is (at least) three times continuously differentiable near r, determine the rate of convergence of Newton's method. That is, express the relationship between ϵ_{k+1} and ϵ_k in the form

$$\epsilon_{k+1} = C\epsilon_k^p + O(\epsilon_k^{p+1}),$$

where C and p are to be determined. Recall that $\epsilon_k = x_k - r$.

4 Air Resistance, Rootfinding, and Lambert W [25 points]

The function

$$h(t) = -30t + 780(1 - e^{-t/3})$$

models the height of a rocket in the air at time t, subject to air resistance; see the figure below. Let $t_{\max}$ be the time at which the rocket reaches its maximum height and let t_{ground} be the time at which the rocket hits the ground.

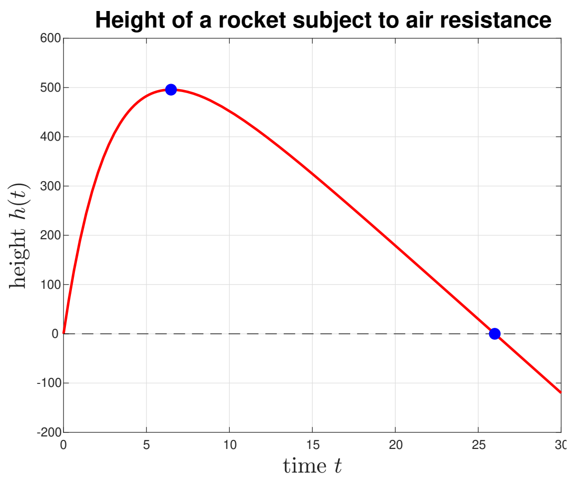

Figure 1: Example output.

(a) Find $t_{\max}$ analytically. That is, find an exact expression for $t_{\max}$. Justify all your steps.

(b) Find $t_{\max}$ numerically, using `fzero`. Then compare the result against the analytical answer obtained in part (a).

(c) (Optional/Bonus) Find t_{ground} analytically. That is, find an exact expression for t_{ground}. Justify all your steps.

(d) Find t_{ground} numerically, using `fzero`. Then compare the result against the analytical answer obtained in part (c), only if you did it.

(e) Produce a well-labeled plot of $h(x)$ as shown in the figure. Clearly mark the maximum height and the time the rocket hits the ground.

Hints for (a) and (c). To find $t_{\max}$ which maximizes the height $h(t)$, use just what you know from Calculus. To find t_{ground} analytically, you need to solve an equation of the form

$$At + e^{Bt} = 1 \tag{1}$$

for t. To do this, introduce a new variable u by the transformation

$$t = -\frac{1}{B}u + \frac{1}{A}. \tag{2}$$

Substitute this into (1) and solve for u, using Lambert W function. Then use the transformation (2) to solve for t.

5 Embedding Watermark inside Image [25 points]

Let A and W be m-by-n matrices representing two grayscale images of the same pixel dimensions as shown below. The image W is *embedded* inside the image A using an SVD-based scheme described below.

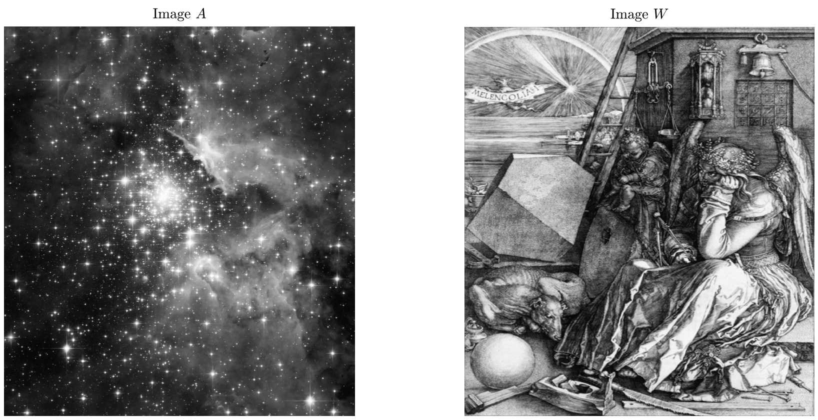

Figure 2: Two original images. The image represented by A on the left and the image represented by W on the right.

Let $A = U\Sigma V^{\mathrm{T}}$ be an SVD of A. For some small scaling factor $\alpha > 0$, construct a new matrix $\Sigma + \alpha W$ and consider its SVD

$$\Sigma + \alpha W = U_w \Sigma_w V_w^{\mathrm{T}}. \tag{3}$$

Note that both Σ and Σ_w are diagonal matrices and, for a suitably chosen α, $\Sigma \approx \Sigma_w$. Thus, A_w obtained by

$$A_w = U \Sigma_w V^{\mathrm{T}} \tag{4}$$

is approximately equal to A and so represents an image which looks nearly identical to that of A, while containing information about the image W as well.

Your mission, should you choose to accept it, is to establish an algorithm to retrieve (an approximation of) W embedded in A_w given U_w, V_w, Σ, and α.

Begin by loading the data with

```
>> load watermark  % download and save 'watermark.mat' first
```

This loads `Aw`, `Uw`, `Vw`, `S`, and `alpha`, which correspond to A_w, U_w, V_w, Σ, and α as described above, respectively. Note that `Aw` is a matrix of double-precision floating point numbers whose elements range from 0 to 255.

(a) Write a MATLAB code to find a matrix W_0, an *approximation* of W, out of A_w.

(b) Display the images represented by A_w and W_0 side by side using `subplot`. Do they look similar to their respective original images in Figure 2?

Hint for (a). Begin by computing an SVD of A_w. This will not give you the same Σ_w as in (4), but will give you something quite close. (This is why the best you can do is to find an estimate of W, which we call W_0.) Then substitute this approximate Σ_w into (3) and solve for W.