Optimal h

Question. Suppose that a function $f(x)$ is numerically calculated by the following procedure.

arithmetic-geometric mean.

```
function y = f(x)
    a = 1; b = cos(x);
    for i = 1:5
        c = b;
        b = sqrt(a.*b);
        a = (a + c)/2;
    end
    y = (pi/2)./a;
end
```

AGM$(1, \cos(x))$

Note: This program is related to the "AGM" function.

$$\Longrightarrow \quad f(x) = \frac{\pi}{2\,\mathrm{AGM}(1,\cos(x))}$$

Compute $f'(\pi/4)$ as accurately as possible using a method of numerical differentiation.

optimal step size.

For 2nd-order centered difference,

$$h \approx \boxed{\text{eps}}^{1/3} \approx (10^{-16})^{1/3}$$

Optimal h (step size)

ignore

$$D_h^{[2C]}\{f\}(x) \qquad \overbrace{D_h^{[2C]}\{f\}(x)} = f'(x) + \underbrace{Ch^2} + K\frac{\boxed{eps}}{h} + O(\boxed{eps}) + O(h^3)$$

2^{nd}-order C.D. formula

Computation on computer (round-off errors)

leading truncation error

Minimize the leading error:

$$F(h) = \alpha h^2 + \beta \frac{\boxed{eps}}{h} \xrightarrow{\text{minimize}} F'(h) = 0$$

Balancing

$$h^2 \approx \frac{\boxed{eps}}{h}$$

$$\Rightarrow h^3 \approx \boxed{eps} \Rightarrow h \approx \boxed{eps}^{1/3}$$

Suggested Exercises (related to this problem)

1. Determine optimal h for different num. diff. schemes.

method	optimal h
1^{st} - order	?
2^{nd} - order	$h \approx \boxed{eps}^{1/3}$
3^{rd} - order	?
4^{th} - order	?

shown in lecture.

Can you figure out a pattern?

2. Rewrite multidimensional Newton's method code using num. differentiation.

- Jacobian matrix (consisting of partial derivatives)

  "finite differences" (w/ optimal h)

- computation of partial deriv.

Logarithmic Integral

The **logarithmic integral** is a special mathematical function defined by the equation

$$\mathrm{li}(x) = \int_2^x \frac{dt}{\ln t}.$$

Find $\mathrm{li}(200)$ by means of the composite trapezoid method.

$$\int_2^{200} \frac{dt}{\ln t} = \int_2^{200} \underbrace{g(t)}_{\parallel} \, dt$$

$$g(t) = 1/\ln(t)$$

$$\approx \left[\frac{g(t_1) + g(t_n)}{2} + \sum_{j=2}^{n-1} g(t_j) \right] h, \quad t_1 = 2, \; t_n = 200$$

Side note:

$$\mathrm{Li}(x) = PV \int_0^x \frac{dt}{\ln t}$$

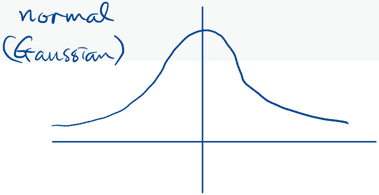
due to singularity at $t=1$.

Side note: Compare to Euler spiral

$$x(t) = \int_0^t \cos(z^2)\, dz$$

$$y(t) = \int_0^t \sin(z^2)\, dz$$

Quadrature Exercise



① Compute

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

by using small and large values for the limits of integration and applying a numerical method. ② Then compute it by making the change of variable

$$x = -\ln t.$$

L'Hopital

$w(0) = 0$

Subs.

$dx = -\dfrac{dt}{t}$

x	$t = e^{-x}$
∞	0
0	1

$$\to \int_1^0 \frac{e^{-(\ln t)^2}}{-t}\,dt = \underbrace{\int_0^1 \overbrace{\frac{e^{-(\ln t)^2}}{t}}^{w(t)}\,dt}_{\text{num. integ.}}$$

Quadrature Exercise

Find the area of the ellipse $y^2 + 4x^2 = 1$.

Airplane Velocity

The radar stations A and B, separated by the distance $a = 500$ m, track a plane C by recording the angles α and β at one-second intervals. Your goal, back at air traffic control, is to determine the speed of the plane.