Updated on November 30, 2021

Math 3607: Homework 11

(No due date)

This assignment will not be collected. Use this problem set to prepare for the final exam. Full solutions will be released sooner than usual (on Friday, December 3) so that you have enough time to study. However I highly recommend you give them a try on your own before looking at solutions.

1. (Derivation of the third-order forward difference formula) Find the third-order forward difference approximation to $f'(x)$, which can be written as

$$D_h^{[3f]}\{f\}(x) \approx c_1 f(x) + c_2 f(x+h) + c_3 f(x+2h) + c_4 f(x+3h).$$

   You may use any one of the approaches presented in lecture[1] or follow the directions found in **LM** 14.1–5.

2. (Another derivation exercise; **LM** 14.1–12)

   (a) Use the second-order centered difference formula for the first derivative and Richardson extrapolation to obtain a fourth-order centered difference formula.

   (b) Verify that the formula obtained in part (a) is fourth-order accurate by modifying the script `diff1` on p. 1767 of **LM**.

   (c) Repeat the previous parts for the second derivative.

   Hint. The second-order centered difference formula for $f''(x)$, with the leading error term, is given by

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + \frac{1}{12} f''''(x) h^2 + O(h^4).$$

   See Lecture 35 or **LM** p. 1766-7. You may use it without derivation for part (c).

3. (Approximating π again; **LM** 14.1–17) Archimedes' algorithm for approximating π uses the perimeter of the inscribed polygon with n sides, $p_n = n\sin(\pi/n)$, and the circumscribed polygon, $P_n = n\tan(\pi/n)$; see Lecture 35 or **LM** Section 11.4.1.1. Let $h = 1/n$.

   (a) Use the Taylor series expansions for $\sin(\pi h)/h$ and $\tan(\pi h)/h$ to show that

$$\begin{aligned} p_n &= \pi + a_1 h^2 + a_2 h^4 + \cdots \\ P_n &= \pi + b_1 h^2 + b_2 h^4 + \cdots, \end{aligned}$$

   where you are to calculate the four coefficients a_1, a_2, b_1, b_2 explicitly.

[1] Three approaches were presented – interpolation-based, series-based, and Richardson extrapolation.

(b) A "better" approximation to π is obtained by averaging the two:

$$\mathfrak{B}_n \equiv \frac{1}{2}(p_n + P_n) = \pi + c_1 h^2 + c_2 h^4 + \cdots ,$$

Calculate these two coefficients c_1, c_2 explicitly.

(c) Using Richardson extrapolation, find an "even better" approximation, $\mathfrak{R}_n$, to π which is fourth-order accurate, that is, it must satisfy

$$\mathfrak{R}_n = \pi + d_1 h^4 + \cdots ,$$

where you are also to calculate the coefficient d_1 explicitly. (The answer for this part is not unique.)

(d) Archimedes approximated π by letting $n = 96$. Calculate $p_n, P_n, \mathfrak{B}_n$, and $\mathfrak{R}_n$ for $n = 48, 96, 192$. Also print out the error in each.

Further exploration. Watch this video by Veratasium [2] which explains a new approximation algorithm suggested by Sir Isaac Newton a couple millennia later, which is based on quadrature (numerical integration, if you like) and the binomial theorem he invented. Write a MATLAB program which implements Newton's idea presented in the video and see how quickly it converges, that is, how many terms of the series is needed to approximate π to full precision on MATLAB?

4. (Variation of Euler spiral; **LM** 14.2–3(b)) Plot the curve

$$x(w) = \int_0^w \cos\left(\frac{1}{4}z^3 - 5.2z\right)\, dz \quad \text{and} \quad y(w) = \int_0^w \sin\left(\frac{1}{4}z^3 - 5.2z\right)\, dz$$

for $w \in [-S, +S]$; use S of your own choice. Use the symmetry to complete the curve.

Hint. In the first video on Week 15 supplementary resources page, I showed how to plot the Euler spiral (**LM** 14.2–3(a)).

5. (Smoothness and accuracy of quadrature methods; **LM** 14.2–6) If $f(x)$ is a "smooth" function, the errors in the composite trapezoidal and midpoint methods are $O(h^2)$, and the error in the composite Simpson's method is $O(h^4)$. But what if the function is "not smooth enough"?

(a) Show numerically that the errors in the composite trapezoidal method, midpoint method, and Simpson's method are all $O(h^{3/2})$ when calculating

$$I_0 = \int_0^1 \sqrt{x}\, dx.$$

Generate the table with headers

```
h            err-trap            err-mid            err-simp
```

and show that the errors decrease by a factor of approximately $2^{3/2} \approx 2.8$ when h is halved.

[2] If the link above does not work, use https://youtu.be/gMlf1ELvRzc.

(b) Repeat the previous part for

$$I_1 = \int_0^1 x^{3/2}\, dx \quad \text{and} \quad I_2 = \int_0^1 x^{5/2}\, dx.$$

6. (Mechanical vibration using Euler-midpoint method) Suppose that the motion of a certain spring-mass system satisfies the differential equation

$$u'' + u' + \frac{1}{5}u^3 = 3\cos\omega t$$

and the initial conditions

$$u(0) = 2, u'(0) = 0.$$

Write a MATLAB program to plot the trajectory $u(t)$ for $0 \leqslant t \leqslant 100$ using the *Euler-midpoint method*.

Note. The **Euler-midpoint method** is an example of second-order Runge-Kutta methods. It was introduced in Lecture 37 along with Euler-trapezoidal and Heun's methods. The Euler-midpoint method for the IVP $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$, $\mathbf{y}(t_0) = \mathbf{y}_0$ can be written as

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}\mathbf{f}(t_n, \mathbf{y}_n)\right).$$

Confirm for yourself that this agrees with what was shown in lecture.

7. (Lorenz model, butterfly effect, and MATLAB `ode45`) The Lorenz equations are the nonlinear autonomous three-dimensional system

$$\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= x(\rho - z) - y, \\
\dot{z} &= xy - \beta z
\end{aligned}$$

where the dot notation indicates the time-derivative $\frac{d}{dt}$. Using

$$\sigma = 10, \quad \rho = 28, \quad \beta = 8/3,$$

plot the three-dimensional trajectory of the particle initially located at $(x, y, z) = (-8, 8, 27)$ for $0 \leqslant t \leqslant 10$ using `ode45`.