Hints for Homework 6

1. (Understanding matrix multiplication)

This problem is to be done purely by hand and most of your answers will involve some sort of summation. Below are some similar questions and answers.

Example 1. What is the j^{th} column of $\mathbf{r}\mathbf{w}^{\text{T}}$, where $\mathbf{r} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^n$? What is the size of the resulting matrix?

Answer. The expression $\mathbf{r}\mathbf{w}^{\mathrm{T}}$, as an outer product, produces an $m \times n$ matrix

$$\mathbf{rw}^{\mathrm{T}} = egin{bmatrix} r_1 \ r_2 \ dots \ r_m \end{bmatrix} egin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \ = egin{bmatrix} r_1 w_1 & r_1 w_2 & \cdots & r_1 w_j & \cdots & r_1 w_n \ r_2 w_1 & r_2 w_2 & \cdots & r_2 w_j & \cdots & r_2 w_n \ dots & dots & dots & dots & dots & dots \ r_m w_1 & r_m w_2 & \cdots & r_m w_j & \cdots & r_m w_n \end{bmatrix},$$

whose j^{th} column is

$$\begin{bmatrix} r_1 w_j \\ r_2 w_j \\ \vdots \\ r_m w_j \end{bmatrix}.$$

Notation. For a matrix M, denote by $[M]_{ij} = m_{ij}$ the (i, j)-element of M.

Observation. Since rows of M becomes columns of M^{T} , and vice versa, the (i, j)-element of M^{T} is the same as the (j, i)-element of M, i.e.,

$$[M^{\mathrm{T}}]_{ij} = [M]_{ji} = m_{ji}.$$

This is going to be useful in the following example.

Example 2. What is the j^{th} element of the column vector $A^{\text{T}}\mathbf{x}$, where $A \in \mathbb{R}^{m \times n}$ and $\mathbf{x} \in \mathbb{R}^m$? What is the size of the resulting vector?

Answer. Since $A \in \mathbb{R}^{m \times n}$, $A^{\mathrm{T}} \in \mathbb{R}^{n \times m}$ and so $A^{\mathrm{T}}\mathbf{x} \in \mathbb{R}^n$, a column vector with n elements. The j^{th} element of $A^{\mathrm{T}}\mathbf{x}$ is

$$\sum_{i=1}^{m} [A^{\mathrm{T}}]_{ji} x_i = \sum_{i=1}^{m} [A]_{ij} x_i = \sum_{i=1}^{m} a_{ij} x_i.$$

2. (Gram-Schmidt in MATLAB)

- The symbol \mathcal{O} in the third line of the problem stands for the zero matrix, with suitable dimensions according to the context.
- For a summary of the Gram-Schmidt procedure, see "Appendix: Gram-Schmidt Orthogonalization" at the end of Lecture 17 (10/06/21, W). For your convenience, the main algorithm (mathematical) is included below:

$$\mathbf{q}_{1} = \frac{\mathbf{a}_{1}}{r_{11}},$$

$$\mathbf{q}_{2} = \frac{\mathbf{a}_{2} - r_{12}\mathbf{q}_{1}}{r_{22}},$$

$$\mathbf{q}_{3} = \frac{\mathbf{a}_{3} - r_{13}\mathbf{q}_{1} - r_{23}\mathbf{q}_{2}}{r_{33}},$$

$$\vdots$$

$$\mathbf{q}_{n} = \frac{\mathbf{a}_{n} - \sum_{i=1}^{n-1} r_{in}\mathbf{q}_{i}}{r_{nn}},$$
where $r_{ij} = \begin{cases} \mathbf{q}_{i}^{T}\mathbf{a}_{j}, & \text{if } i \neq j \\ \pm \left\|\mathbf{a}_{j} - \sum_{k=1}^{j-1} r_{kj}\mathbf{q}_{k}\right\|_{2}, & \text{if } i = j. \end{cases}$

Here, $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are the columns of A; $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ the columns of Q; r_{ij} the (i, j)-element of R.

• Your goal is to translate the mathematical procedure inside the box to a MATLAB code. Begin the function with

function
$$[Q,R] = GramSchmidt(A)$$

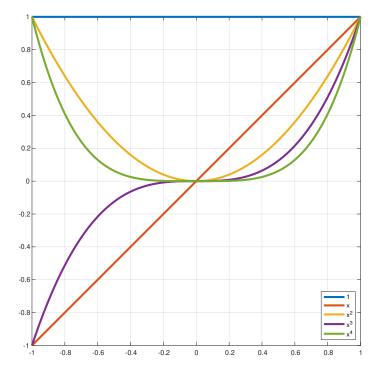
so that it calculates both Q and R. Keep in mind that the Gram-Schmidt procedure yields a **thin QR factorization** of A.

- 3. (Periodic fit; **FNC** 3.1.3) See video demo for hints.
- 4. (Adapted from **FNC** 3.3.3.)

Just to clarify what it means to plot a column of a matrix, consider the Vandermonde matrix

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{bmatrix},$$

where $-1 = x_1 < x_2 < \cdots < x_m = 1$ are equally spaced. So each column of V is a set of evenly distributed samples of a monomial function x^j , $j = 0, \ldots, n-1$. That is, the first column is a discrete representation of the constant function $f_1(x) = 1$, the second column $f_2(x) = x$, and the general j^{th} column $f_j(x) = x^{j-1}$. Thus if every column of V is plotted against $\mathbf{x} = (x_1, x_2, \cdots, x_m)^{\text{T}}$, one obtains something like



Now, thin QR factorization of V will find the orthonormal set of vectors whose span coincides with the span of columns of V and put them in \widehat{Q} (with \widehat{R} being the recovery formula). When the columns of \widehat{Q} are plotted against \mathbf{x} , you will see a visualization of so-called orthogonal polynomials.

5. (Visualizing matrix norms; adapted from LM 9.4–26.)

The key modification must take place in the lines where X is defined and norm_Y is calculated. The rest can be simply re-used in your function with minimal modification.