Lab: EVD and SVD

Fibonacci Sequence and EVD

Consider the familiar Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, \ldots,$$

which can be defined recursively as

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad \text{for } n = 1, 2, 3, \ldots.$$

Find the general formula for the kth Fibonacci number F_k.

Define the sequence in terms of matrices and vectors as follows. For $k \geq 1$, define

$$\mathbf{u}_k = \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}$$

and observe that

$$\mathbf{u}_{k+1} = A\mathbf{u}_k, \quad \text{where} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

To find the kth term in the Fibonacci sequence, use the fact that

$$\mathbf{u}_{k+1} = A\mathbf{u}_k = A^2\mathbf{u}_{k-1} = \cdots = A^k\mathbf{u}_1. \tag{1}$$

To calculate A^k, we will see if we can diagonalize A. A routine calculation shows that the eigenvalues of A are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

and the corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix},$$

respectively (You should check this yourself!). Since the eigenvectors are linearly independent, they form an eigenbasis, and A is indeed diagonalizable, that is, it can be written as $A = VDV^{-1}$, where

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \text{and} \quad V^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}.$$

The kth Fibonacci number F_k is the second entry of $\mathbf{u}_{k+1}$ which, by (1) and the EVD of A, is given by

$$\mathbf{u}_{k+1} = A^k \mathbf{u}_1 = VD^kV^{-1}\mathbf{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Multiplying it all out, we find that

$$F_k = \frac{\lambda_1^k - \lambda_2^k}{\sqrt{5}}.$$

Remark 1. This formula is known as Binet's formula. Note that $\lambda_1 = (1+\sqrt{5})/2$ is the golden ratio ϕ.

Exercise 2. The Pell numbers $0, 1, 2, 5, 12, 29, 70, 169, 208, 985, \ldots$ are defined by recursively by

$$P_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2P_{n-1} + P_{n-2} & \text{otherwise} \end{cases}$$

Find the general formula for the kth Pell number.

Spectra and Pseudospectra

(Adapted from **FNC** 7.2.7.) The eigenvalues of *Toeplitz* matrices, which have a constant value on each diagonal, have beautiful connections to complex analysis. Define a 64×64 Toeplitz matrix using

```
z = zeros(1,60);
A = toeplitz( [0,0,-4,-2i,z], [0,2i,-1,2,z] );
```

(a) Plot the eigenvalues of `A` as red dots in the complex plane. (Set `'MarkerSize'` to be 3.)

(b) Let E and F be 64×64 random matrices generated by `randn`. On top of the plot from part (a), plot the eigenvalues of the matrix $A + \varepsilon E + i\varepsilon F$ as blue dots, where $\varepsilon = 10^{-3}$. (Set `'MarkerSize'` to be 1.)

(c) Repeat part (b) 49 more times (generating a single plot).

(d) Compute $\kappa(V)$ for an eigenvector matrix V and relate your picture to the conclusion of the Bauer-Fike theorem.

Theorem 1. *Let $A \in \mathbb{C}^{n \times n}$ be diagonalizable, $A = VDV^{-1}$, with eigenvalues $\lambda_1, \ldots, \lambda_n$. If μ is an eigenvalue of $A + \delta A$ for a complex matrix δA, then*

$$\min_{1 \le j \le n} \left|\mu - \lambda_j\right| \le \kappa_2(V) \left\|\delta A\right\|_2 .$$

Singular Values of Image Matrices

MATLAB ships with some sample images for trying out ideas. You can get one of these by using

```
load mandrill
imshow(X,map)
```

Make a log-linear plot of the singular values of X. (The shape of this graph is surprisingly similar across a wide range of images.)