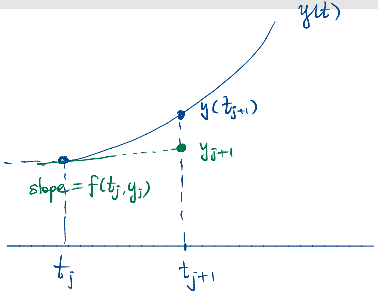
More Accurate Methods – Runge-Kutta Method



- Seek algorithm w/ improved accuracy.
- Idea: come up w/ better "slope" approx.

Prev $\quad y_{j+1} = y_j + \underbrace{f(t_j, y_j)}_{\text{slope}} \cdot \underbrace{h}_{\text{run}}$

want $\quad y_{j+1} = y_j + \phi(t_j, y_j) \cdot h$

- Model: Runge-Kutta method.

k steps
$$\begin{cases} s_1 = f(t_j, y_j) \\ s_2 = f(t_j + c_1 h,\ y_j + a_{11} s_1 h) \\ \vdots \\ s_k = f(t_j + c_{k-1} h,\ y_j + (a_{k-1,1} s_1 + a_{k-1,2} s_2 + \cdots + a_{k-1,k-1} s_{k-1}) h) \end{cases}$$

$$y_{j+1} = y_j + (b_1 s_1 + b_2 s_2 + \cdots + b_k s_k) h$$

~~find~~ determine $c_1, c_2, \ldots, c_{k-1}$,
a_{11},
a_{21}, a_{22}
$\vdots \quad \vdots$
$a_{k-1,1}, a_{k-1,2}, \ldots, a_{k-1,k-1}$
$b_1, b_2, \ldots, b_k$

to reduce errors.

Series-based derivation   2 steps for 2^{nd}-order accuracy

$$s_1 = f(t_j, y_j)$$

$$s_2 = f(t_j + c_1 h,\ y_j + a_{11} s_1 h)$$

$$y_{j+1} = y_j + (b_1 s_1 + b_2 s_2) h$$

First, T-expand s_2 at (t_j, y_j)

$$s_2 = f(t_j, y_j) + f_t(t_j, y_j) c_1 h + f_y(t_j, y_j) a_{11} s_1 h + O(h^2)$$

Then

$$y_{j+1} = y_j + \big(b_1 f(t_j, y_j) + b_2 f(t_j, y_j)\big) h + \big(b_2 c_1 f_t(t_j, y_j) + b_2 a_{11} f_y(t_j, y_j) f(t_j, y_j)\big) h^2 + O(h^3)$$

On the one hand: (algorithm)

$$y_{j+1} = y_j + \left(b_1 f(t_j, y_j) + b_2 f(t_j, y_j)\right) h + \left(b_2 c_1 f_t(t_j, y_j) + b_2 a_{11} f_y(t_j, y_j) f(t_j, y_j)\right) h^2 + O(h^3)$$

On the other hand: (exact)

$$\underbrace{y(t_{j+1})}_{t_{j+1} = t_j + h} = \underbrace{y(t_j)}_{= y_j} + \underbrace{y'(t_j)}_{= f(t_j, y_j)} h + \underbrace{\frac{y''(t_j)}{2}}_{= \frac{1}{2} \frac{d}{dt} f(t,y) \big|_{(t_j, y_j)} = \frac{1}{2}\left(f_t(t_j, y_j) + f_y(t_j, y_j) \cdot f(t_j, y_j) \right)} h^2 + O(h^3)$$

$\frac{dy}{dt}$

Now match coeffs:

$$\begin{cases} b_1 + b_2 = 1 \\ b_2 c_1 = \frac{1}{2} = b_2 a_{11} \end{cases}$$

<u>many answers!</u>

4 unknows, 3 eqns.

Runge-Kutta Method – 2nd-Order Methods

$$\begin{cases} b_1 + b_2 = 1 \\ b_2 c_1 = \frac{1}{2} = b_2 a_{11} \end{cases}$$

Some popular choices are:

$\boxed{1}$ $\begin{cases} a_{11} = c_1 = 1 \\ b_1 = b_2 = \frac{1}{2} \end{cases}$

Euler-Trap. method

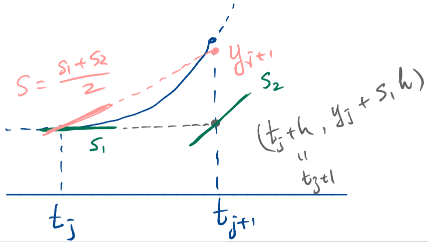
$\boxed{2}$ $\begin{cases} a_{11} = c_1 = \frac{1}{2} \\ b_1 = 0, \quad b_2 = 1 \end{cases}$

Euler-Midpt. method

$\boxed{3}$ $\begin{cases} a_{11} = c_1 = \frac{2}{3} \\ b_1 = \frac{1}{4}, \quad b_2 = \frac{3}{4} \end{cases}$

Heun's method

$$\begin{cases} s_1 = f(t_j, y_j) \\ s_2 = f(t_j + h, y_j + s_1 h) \\ y_{j+1} = y_j + \frac{1}{2}(s_1 + s_2) h \end{cases}$$



Runge-Kutta Method – 4th-Order Method, RK4

Euler : 1^{st}-order : one step

E-T : 2^{nd}-order : 2 steps

4^{th}-order : 4 steps

- Will not derive it.
- 4 step method:

$$s_1 = f(t_j, y_j)$$

$$s_2 = f(t_j + \tfrac{1}{2}h,\ y_j + \tfrac{1}{2}s_1 h)$$

$$s_3 = f(t_j + \tfrac{1}{2}h,\ y_j + 0\cdot s_1 h + \tfrac{1}{2}s_2 h)$$

$$s_4 = f(t_j + 1\cdot h,\ y_j + 0\cdot s_1 h + 0\cdot s_2 h + 1\cdot s_3 h)$$

$$y_{j+1} = y_j + \left(\tfrac{1}{6}s_1 + \tfrac{1}{3}s_2 + \tfrac{1}{3}s_3 + \tfrac{1}{6}s_4\right)h$$

"ode45"

"The" 4^{th}-order Runge-Kutta method (RK4)

Error in video fixed