Homework 6 (Solution)

Math 3607

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```
clear, close all, format short g
```

Problem 1 (LM 12.5--3)

In what follows, $\mathbf{a}_j \in \mathbb{R}^m$ denotes the jth column of A.

(a) $\mathbf{r}\mathbf{w}^{\mathrm{T}}$ is an $n \times n$ matrix whose ith row is

$$r_i\mathbf{w}^{\mathrm{T}} = [r_iw_1, r_iw_2, \ldots, r_iw_n] \in \mathbb{R}^{1\times n}.$$

(b) $\mathbf{b}^{\mathrm{T}}A$ is a row vector with n element whose jth element is

$$\mathbf{b}^{\mathrm{T}}\mathbf{a}_j = \sum_{i=1}^{m} b_i a_{ij}.$$

(c) C is an $m \times p$ matrix whose (i, j)-entry is given by

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}.$$

(d) D is a $p \times m$ matrix whose (i, j)-entry is given by

$$d_{i,j} = \sum_{k=1}^{n} b_{k,i}a_{j,k}.$$

(e) Let $E = C^{\mathrm{T}}$. Then E is a $p \times m$ matrix whose (i, j)-entry equals the (j, i)-entry of C, that is,

$$e_{i,j} = c_{j,i} = \sum_{k=1}^{n} a_{j,k} b_{k,i}.$$

Note that it equals $d_{i,j}$ for all i and j. Since both D and E are of the same size and all their entries match, $D = E$.

(f) $A^{\mathrm{T}}A$ is an $n \times n$ matrix whose (i, j)-entry is given by

$$\sum_{k=1}^{m} a_{k,i} a_{k,j}.$$

(g) AA^{T} is an $m \times m$ matrix whose (i, j)-entry is given by

$$\sum_{k=1}^{n} a_{i,k} a_{j,k}.$$

(h) $AA^{\mathrm{T}}\mathbf{b}$ is a column vector with m elements whose ith element is given by

$$\sum_{j=1}^{m} \sum_{k=1}^{n} a_{i,k} a_{j,k} b_j.$$

Suggestion for Further Study. Having done this exercise, please read the derivation of the normal equation from LLS formulation using calculus (Appendix to Lecture 17). The very last step of the proof requires your familiarity with this exercise.

Problem 2 (LM 12.6--2)

Below is a version which is more or less a direct translation of the Gram-Schmidt procedure:

```
function [Q,R] = gs(A)
    [m,n] = size(A);
    Q = zeros(m,n);
    R = zeros(n,n);
    for j = 1:n
        if j > 1
            R(1:j-1,j) = Q(:,1:j-1).' * A(:,j);
        end
        v = A(:,j) - Q(:,1:j-1)*R(1:j-1,j);
        R(j,j) = norm(v);
        Q(:,j) = v/R(j,j);
    end
end
```

We can have `A` overwirtten by `Q` to save memory storages. In addition, we can also get by without the auxilliary vector `v` introduced right after the if statement. The code included at the end of this document incorporates all these modifications.

Let's check if the code works correctly.

```
m = 1000; n = 30;
A = rand(m,n);
```

Using `gs.m`:

```
[Q,R] = gs(A);
istriu(R)                    % Is R upper triangular?
```

```
ans = logical

   1
```

```
norm(Q'*Q - eye(n), 'inf')   % Is Q orthogonal?
```

```
ans =
    2.639e-14
```

```
norm(Q*R - A, 'inf')         % Is Q*R = A?
```

```
ans =
    4.996e-16
```

Using `qr`:

```
[Q0,R0] = qr(A,0);
istriu(R0)                   % Is R upper triangular?
```

```
ans = logical

   1
```

```
norm(Q0'*Q0 - eye(n), 'inf') % Is Q orthogonal?
```

```
ans =
   2.9764e-15
```

```
norm(Q0*R0 - A, 'inf')       % Is Q*R = A?
```

```
ans =
   3.7814e-13
```

Note 1.

Several runs with random matrices shows that both code works as expected. However, Q and R produced the codes are not necessarily the same.

```
norm(Q-Q0, 'inf')
```

```
ans =
```

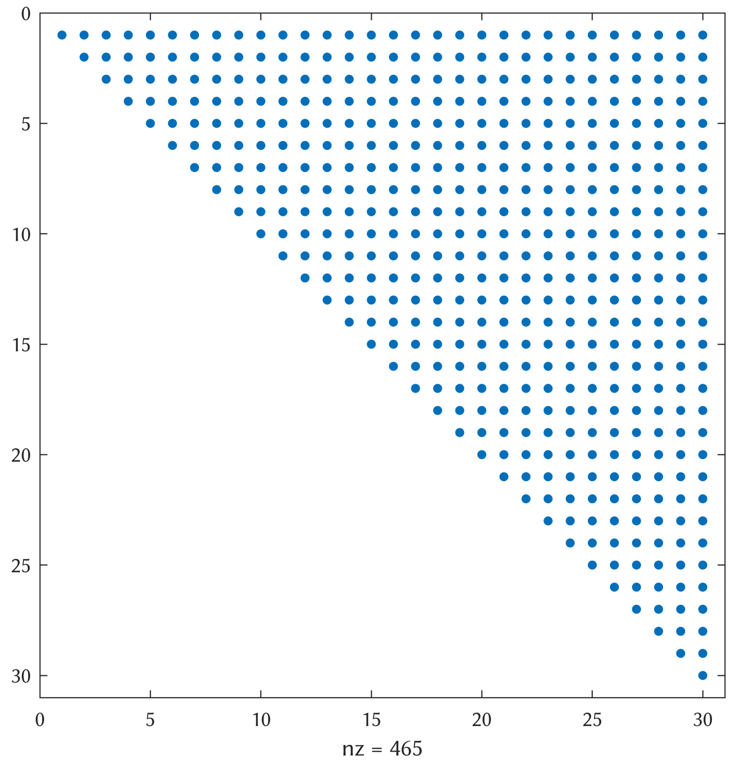
```
        1.247
```

as more is going on for MATLAB's `qr` to ensure numerical stability.

Note 2.

There is a visually convenient way to confirm the triangularity of R:

```
spy(R)
```



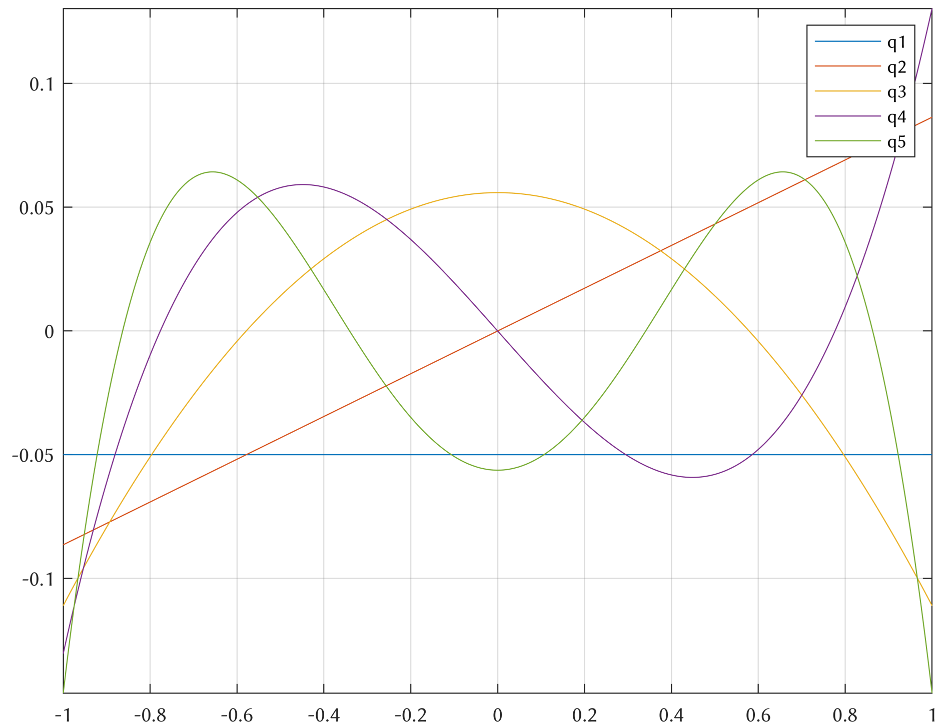
To learn about this function, see

```
help spy
```

```
 spy Visualize sparsity pattern.
    spy(S) plots the sparsity pattern of the matrix S.

    spy(S,'LineSpec') uses the color and marker from the line
    specification string 'LineSpec' (See PLOT for possibilities).

    spy(S,markersize) uses the specified marker size instead of
    a size which depends upon the figure size and the matrix order.

    spy(S,'LineSpec',markersize) sets both.

    spy(S,markersize,'LineSpec') also works.

    Documentation for spy
```

Problem 3 (Orthogonal Polynomials)

```
m = 400; n = 5;
x = linspace(-1, 1, m)';
V = x.^(0:n-1);
[Q,R] = qr(V,0);    % thin QR factorization

clf
plot(x, Q)
axis tight, grid on
legend('q1', 'q2', 'q3', 'q4', 'q5')
```



Note 1. The `plot` function smartly plots all columns of `Q` as functions of the common `x` when called with `plot(x, Q)`. This is the same as

```
for j = 1:size(Q, 2)
    plot(x, Q(:,j)), hold on
end
```

Where does one learn about such tricks? Please read the very first paragraph of the help document for `plot`:

```
help plot
```

```
 plot   Linear plot.
    plot(X,Y) plots vector Y versus vector X. If X or Y is a matrix,
    then the vector is plotted versus the rows or columns of the matrix,
    whichever line up.  If X is a scalar and Y is a vector, disconnected
    line objects are created and plotted as discrete points vertically at
    X.

    plot(Y) plots the columns of Y versus their index.
```

If Y is complex, **plot**(Y) is equivalent to **plot**(real(Y),imag(Y)). In all other uses of **plot**, the imaginary part is ignored.

Various line types, plot symbols and colors may be obtained with **plot**(X,Y,S) where S is a character string made from one element from any or all the following 3 columns:

b	blue	.	point	-	solid
g	green	o	circle	:	dotted
r	red	x	x-mark	-.	dashdot
c	cyan	+	plus	--	dashed
m	magenta	*	star	(none)	no line
y	yellow	s	square		
k	black	d	diamond		
w	white	v	triangle (down)		
		^	triangle (up)		
		<	triangle (left)		
		>	triangle (right)		
		p	pentagram		
		h	hexagram		

For example, **plot**(X,Y,'c+:') plots a cyan dotted line with a plus at each data point; **plot**(X,Y,'bd') plots blue diamond at each data point but does not draw any line.

plot(X1,Y1,S1,X2,Y2,S2,X3,Y3,S3,...) combines the plots defined by the (X,Y,S) triples, where the X's and Y's are vectors or matrices and the S's are strings.

For example, **plot**(X,Y,'y-',X,Y,'go') plots the data twice, with a solid yellow line interpolating green circles at the data points.

The **plot** command, if no color is specified, makes automatic use of the colors specified by the axes ColorOrder property. By default, **plot** cycles through the colors in the ColorOrder property. For monochrome systems, **plot** cycles over the axes LineStyleOrder property.

Note that RGB colors in the ColorOrder property may differ from similarly-named colors in the (X,Y,S) triples. For example, the second axes ColorOrder property is medium green with RGB [0 .5 0], while **plot**(X,Y,'g') plots a green line with RGB [0 1 0].

If you do not specify a marker type, **plot** uses no marker.
If you do not specify a line style, **plot** uses a solid line.

plot(AX,...) plots into the axes with handle AX.

plot returns a column vector of handles to lineseries objects, one handle per plotted line.

The X,Y pairs, or X,Y,S triples, can be followed by parameter/value pairs to specify additional properties of the lines. For example, **plot**(X,Y,'LineWidth',2,'Color',[.6 0 0]) will create a plot with a dark red line width of 2 points.

Example
```
   x = -pi:pi/10:pi;
   y = tan(sin(x)) - sin(tan(x));
   plot(x,y,'--rs','LineWidth',2,...
                   'MarkerEdgeColor','k',...
                   'MarkerFaceColor','g',...
                   'MarkerSize',10)
```

See also plottools, semilogx, semilogy, loglog, plotyy, plot3, grid,

```
title, xlabel, ylabel, axis, axes, hold, legend, subplot, scatter.

Documentation for plot
Other functions named plot
```

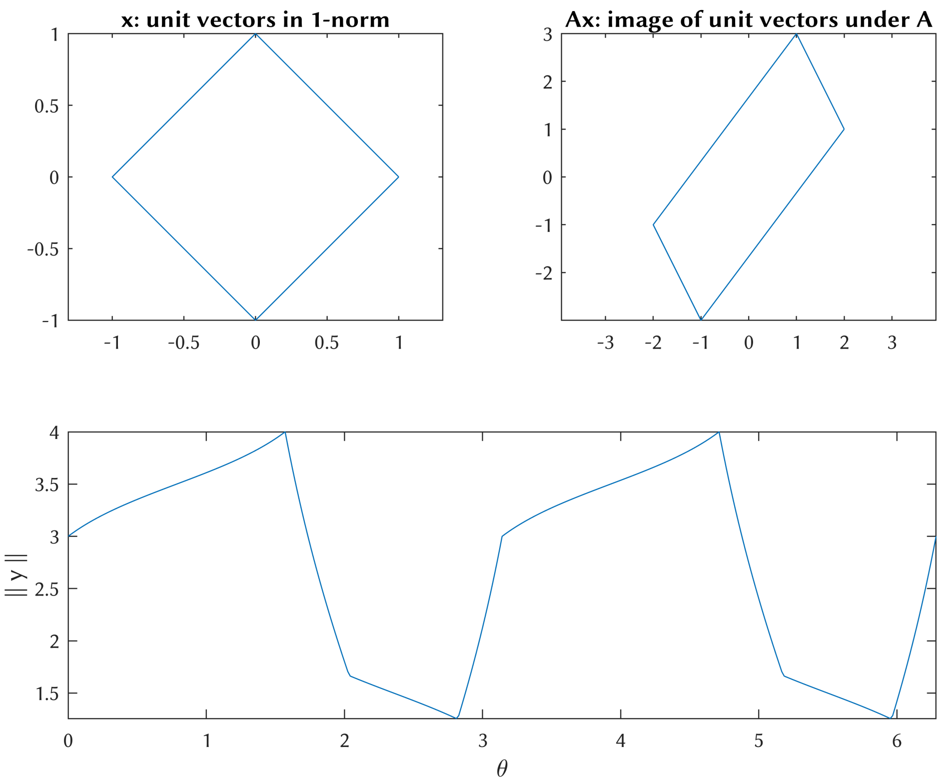
Problem 4 (Visualizing Matrix Norms)

See the function `visMatrixNorms` at the end of this document.

```
A = [2 1;
     1 3];
```

$p = 1$:

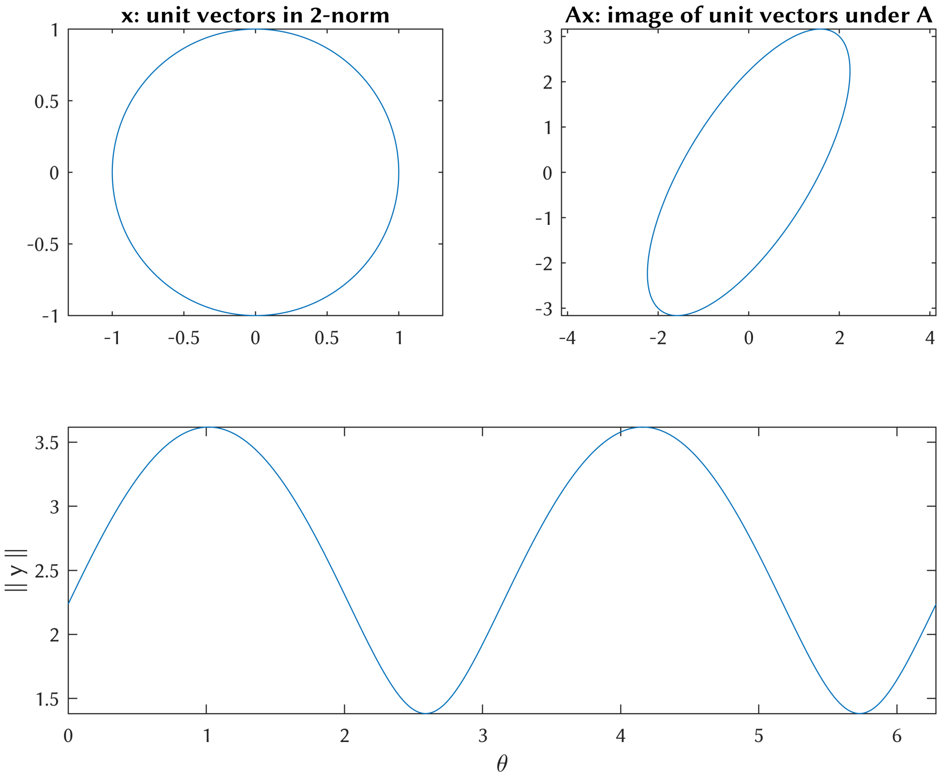
```
visMatrixNorms(A, 1);
```




```
p = 1
approx. norm: 4.0000000000000000
actual  norm: 4.0000000000000000
```

$p = 2$:

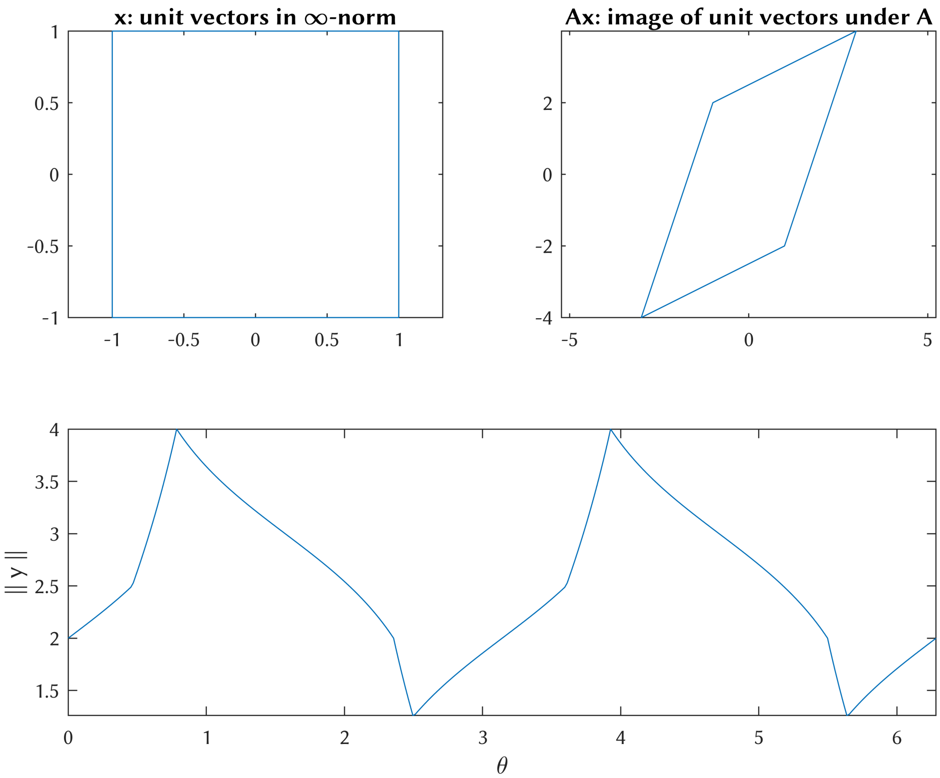
```
visMatrixNorms(A, 2);
```




```
p = 2
approx. norm: 3.6179964204609893
actual  norm: 3.6180339887498953
```

$p = \infty$:

```
visMatrixNorms(A, Inf);
```




```
p = Inf
approx. norm: 3.9999999999999996
actual  norm: 4.0000000000000000
```

Explore.

Modify the script/function even further so that it can handle any $p \in [1, \infty]$. MATLAB's `norm` function can calculate vector p-norm for any value p, but it can only calculate $\|A\|_p$ for $p = 1, 2, \infty$ and $\|A\|_F$ (Frobenius norm). So for p other than $1, 2, \infty$, your program must only output the approximate $\|A\|_p$.

Functions Used

Gram-Schmidt

```
function [Q,R] = gs(A)
    n = size(A,2);
    Q = A;
    R = zeros(n);
    for j = 1:n
        if j > 1
            R(1:j-1,j) = Q(:,1:j-1)' * Q(:,j);
            Q(:,j) = Q(:,j) - Q(:,1:j-1)*R(1:j-1,j);
        end
        R(j,j) = norm(Q(:,j));
        Q(:,j) = Q(:,j)/R(j,j);
    end
end
```

Visualization of Matrix Norms

```
function normA = visMatrixNorms(A, p)
% VISMATRIXNORM visMatrixNorms(A, p)
% Generate plots of unit vectors (in p-norm) and their images under A.
% Also approximates the matrix p-norm of A according to the definition of
% induced norms.
%
% Input:
%   A       2-by-2 matrix
%   p       1, 2, or Inf
% Output:
%  normA    matrix p-norm of A (approximation)
    if size(A,1)~=2 || size(A,2)~=2
        error('A must be a 2-by-2 matrix.')
    elseif p~=1 && p~=2 && p~=Inf
        error('p must be either 1, 2, or Inf.')
    end


    theta = linspace(0, 2*pi, 361);
    U = [cos(theta); sin(theta)];    % unit circle

    % Calculate p-norm of vectors on the unit circle
    if p == 1
        norm_U = sum( abs(U), 1 );
    elseif p == 2
        norm_U = sqrt( sum(abs(U).^2, 1) );
    else
        norm_U = max( abs(U), [], 1 );
    end

    % Normalize to unit vectors in respective norm
    X = U ./ norm_U;

    % Calculate y's
    Y = A*X;

    % Calculate p-norm of y's
    if p == 1
        norm_Y = sum( abs(Y), 1 );
    elseif p == 2
        norm_Y = sqrt( sum(abs(Y).^2, 1) );
    else
        norm_Y = max( abs(Y), [], 1 );
    end
    normA = max(norm_Y);

    % Plotting routine
    clf
    subplot(2,2,1)
```

```
    plot(X(1,:), X(2,:)), axis equal

    if p ~= Inf
        str = sprintf('x: unit vectors in %g-norm', p);
    else
        str = sprintf('x: unit vectors in \\infty-norm', 'Interpreter', 'latex');
    end
    title(str)

    subplot(2,2,2)
    plot(Y(1,:), Y(2,:)), axis equal
    title('Ax: image of unit vectors under A')

    subplot(2,1,2)
    plot(theta, norm_Y), axis tight
    xlabel('\theta')
    ylabel('|| y ||')

    fprintf(' p = %g\n', p)
    fprintf(' approx. norm: %18.16f\n', normA)
    fprintf(' actual  norm: %18.16f\n', norm(A, p))
end
```