Tutorial: Row and Column Operations

Notation and Terminology

Notation: Unit Basis Vectors

Throughout this tutorial, suppose $n \in \mathbb{N}$ is fixed. Let I be the $n \times n$ identity matrix and denote by $\mathbf{e}_j$ its jth column, *i.e.*,

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \left[\begin{array}{c|c|c|c} & & & \\ \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ & & & \end{array}\right].$$

That is,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Notation: Concatenation

Let $A \in \mathbb{R}^{n\times n}$. We can view it as a concatenation of its rows or columns as visualized below.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right] = \left[\begin{array}{c} \boldsymbol{\alpha}_1^{\mathrm{T}} \\ \hline \boldsymbol{\alpha}_2^{\mathrm{T}} \\ \hline \vdots \\ \hline \boldsymbol{\alpha}_n^{\mathrm{T}} \end{array}\right].$$

Basic Row and Column Operations

Row or Column Extraction

A row or a column of A can be extracted using columns of I.

Operation	Mathematics	MATLAB
extract the ith row of A	$\mathbf{e}_i^{\mathrm{T}} A$	`A(i,:)`
extract the jth column of A	$A\mathbf{e}_j$	`A(:,j)`
extract the (i,j) entry of A	$\mathbf{e}_i^{\mathrm{T}} A\mathbf{e}_j$	`A(i,j)`

Elementary Permutation Matrices

Definition 1 (Elementary Permutation Matrix)

For $i, j \in \mathbb{N}[1, n]$ distinct, denote by $P(i, j)$ the $n \times n$ matrix obtained by interchanging the ith and jth rows of the $n \times n$ identity matrix. Such matrices are called *elementary permutation matrices*.

Example. ($n = 4$)

$$P(1,2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P(1,3) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \cdots$$

Notable Properties.

- $P(i, j) = P(j, i)$
- $P(i, j)^2 = I$

Row or Column Interchange

Elementary permutation matrices are useful in interchanging rows or columns.

Operation	Mathematics	MATLAB
$\boldsymbol{\alpha}_i^{\mathrm{T}} \leftrightarrow \boldsymbol{\alpha}_j^{\mathrm{T}}$	$P(i,j)A$	`A([i,j],:)=A([j,i],:)`
$\mathbf{a}_i \leftrightarrow \mathbf{a}_j$	$AP(i,j)$	`A(:,[i,j])=A(:,[j,i])`

Permutation Matrices

Definition 2 (Permutation Matrix)

A *permutation matrix* $P \in \mathbb{R}^{n \times n}$ is a square matrix obtained from the same-sized identity matrix by re-ordering of rows.

Notable Properties.

- $P^{\mathrm{T}} = P^{-1}$
- A product of *elementary permutation matrices* is a permutation matrix.

Row and Column Operations. For any $A \in \mathbb{R}^{n \times n}$,

- PA permutes the rows of A.
- AP permutes the columns of A.

Row or Column Rearrangement

Question

Let $A \in \mathbb{R}^{6\times 6}$, and suppose that it is stored in MATLAB. Rearrange rows of A by moving 1st to 2nd, 2nd to 3rd, 3rd to 5th, 4th to 6th, 5th to 4th, and 6th to 1st, that is,

$$\left[\begin{array}{c} \boldsymbol{\alpha}_1^{\mathrm{T}} \\ \hline \boldsymbol{\alpha}_2^{\mathrm{T}} \\ \hline \boldsymbol{\alpha}_3^{\mathrm{T}} \\ \hline \boldsymbol{\alpha}_4^{\mathrm{T}} \\ \hline \boldsymbol{\alpha}_5^{\mathrm{T}} \\ \hline \boldsymbol{\alpha}_6^{\mathrm{T}} \end{array}\right] \longrightarrow \left[\begin{array}{c} \boldsymbol{\alpha}_6^{\mathrm{T}} \\ \hline \boldsymbol{\alpha}_1^{\mathrm{T}} \\ \hline \boldsymbol{\alpha}_2^{\mathrm{T}} \\ \hline \boldsymbol{\alpha}_5^{\mathrm{T}} \\ \hline \boldsymbol{\alpha}_3^{\mathrm{T}} \\ \hline \boldsymbol{\alpha}_4^{\mathrm{T}} \end{array}\right]$$

Row or Column Rearrangement

Solution.

- Mathematically: PA where

$$P = \left[\begin{array}{c} \mathbf{e}_6^{\mathrm{T}} \\ \hline \mathbf{e}_1^{\mathrm{T}} \\ \hline \mathbf{e}_2^{\mathrm{T}} \\ \hline \mathbf{e}_5^{\mathrm{T}} \\ \hline \mathbf{e}_3^{\mathrm{T}} \\ \hline \mathbf{e}_4^{\mathrm{T}} \end{array}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

- MATLAB:

```
A = A([6 1 2 5 3 4], :)
% short for  A([1 2 3 4 5 6], :) = A([6 1 2 5 3 4], :)
```

Gaussian Transformation Matrices

Elementary Row Operation and GTM

Let $1 \leqslant j < i \leqslant n$.

- The row operation $R_i \to R_i + cR_j$ on $A \in \mathbb{R}^{n\times n}$, for some $c \in \mathbb{R}$, can be emulated by a matrix multiplication[1]

$$(I + c\,\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}})A.$$

- In the context of Gaussian elimination, the operation of introducing zeros below the jth diagonal entry can be done via

$$\underbrace{\Big(I + \sum_{i=j+1}^{n} c_{i,j}\,\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}\Big)}_{=G_j} A, \quad 1 \leqslant j < n.$$

The matrix G_j is called a *Gaussian transformation matrix* (GTM).

[1] Many linear algebra texts refer to the matrix in parentheses as an *elementary matrix*.

Elementary Row Operation and GTM (cont')

- To emulate $(I + c\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}})A$ in MATLAB:

```
A(i,:) = A(i,:) + c*A(j,:);
```

- To emulate

$$G_jA = (I + \sum_{i=j+1}^{n} c_{i,j}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}})A$$

  in MATLAB:

```
for i = j+1:n
    c = ....
    A(i,:) = A(i,:) + c*A(j,:);
end
```

  This can be done without using a loop.

Analytical Properties of GTM

- GTMs are *unit* lower triangular matrices.
- The product of GTMs is another unit lower triangular matrix.
- The inverse of a GTM is also a unit lower triangular matrix.