# Practice Problems on EVD and SVD — Math 3607

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Below are some practice problems covering eigenvalue and singular value decompositions.

**Problem 1.** (Checking Understanding – EVD and SVD)

(**True/False**) Circle T if the statement is ALWAYS true; circle F otherwise.

(a) ( T / F ) Given a square matrix $A \in \mathbb{R}^{n\times n}$, we can always find an orthogonal matrix $V \in \mathbb{R}^{n\times n}$ and a diagonal matrix $D \in \mathbb{R}^{n\times n}$ such that $AV = VD$.

(b) ( T / F ) If $A \in \mathbb{R}^{5\times 5}$ has 5 distinct eigenvalues, then $A$ has an EVD.

(c) ( T / F ) If $A \in \mathbb{R}^{5\times 5}$ has 3 distinct eigenvalues, then $A$ does not have an EVD.

(d) ( T / F ) A square matrix $A \in \mathbb{R}^{m\times m}$ with $\det(A) = 0$ does not have an SVD.

(e) ( T / F ) A rank deficient matrix $A \in \mathbb{R}^{m\times n}$ has an SVD.

(f) ( T / F ) Let $A \in \mathbb{R}^{m\times n}$. Then $B = AA^{\mathrm{T}} \in \mathbb{R}^{m\times m}$ is a diagonalizable matrix.

**Problem 2.** (EVD and Powers of a Matrix)

Let $A \in \mathbb{R}^{n \times n}$ has an EVD $A = VDV^{-1}$ and suppose that all its eigenvalues are either positive or negative ones. Show that $A^2 = I$.

**Note.** To gain a geometric intuition about this problem, think about the eigenvalue decomposition of a Householder reflector $H = I - 2\mathbf{u}\mathbf{u}^{\mathrm{T}}$.

Calculate the singular values of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

by solving a $2 \times 2$ eigenvalue problem.

**Problem 4.** (SVD and the 2-Norm)

Let $A \in \mathbb{R}^{n\times n}$. Show that

(a) $A$ and $A^{\mathrm{T}}$ have the same singular values.

(b) $\|A\|_2 = \|A^{\mathrm{T}}\|_2$.

**Problem 5.** (Rayleigh Quotient)

Let

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix}.$$

and define a function $R_A : \mathbb{R}^2 \to \mathbb{R}$ by

$$R_A(\mathbf{x}) = \frac{\mathbf{x}^{\mathrm{T}} A \mathbf{x}}{\mathbf{x}^{\mathrm{T}} \mathbf{x}}.$$

(a) Write out $R_A(\mathbf{x})$ explicitly as a function of $x_1$ and $x_2$.

(b) Find $R_A(\mathbf{x})$ for $x_1 = 1$, $x_2 = 2$.

(c) Confirm that $\mathbf{x} = (1, 2)^{\mathrm{T}}$ is an eigenvector of $A$, whose corresponding eigenvalue is equal to the value computed in part (b).

**Note.** The map $R_A$ constructed above is known as the *Rayleigh quotient*. As confirmed in part (c), this map is known to send an eigenvector of $A$ to its associated eigenvalue. Below are some more exercise problems related to this map.

1. Find the gradient vector $\nabla R_A(\mathbf{x})$.
2. Show that the gradient vector is zero when $x_1 = 1$, $x_2 = 2$.