Lec 22: Rootfinding Problem – One Dimension

Fixed Point Iteration

Fixed Point

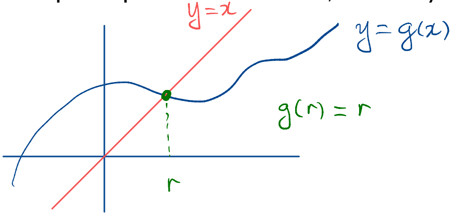
Definition 1 (Fixed Point)

The real number r is a **fixed point** of the function g if $g(r) = r$.

- The rootfinding problem $f(x) = 0$ can always be written as a fixed point problem $g(x) = x$ by, *e.g.*, setting[1]

$$g(x) = x - f(x).$$

- The fixed point problem is true at, and only at, a root of f.



If r is a fixed point of g, then

$$g(r) = r - f(r) = r$$

$$\Rightarrow \cancel{r} - f(r) = \cancel{r}$$

$$\Rightarrow +f(r) = 0$$

[1] This is not the only way to transform the rootfinding problem. More on this later.

Fixed Point Iteration

A fixed point problem $g(x) = x$ naturally provides an iteration scheme:

$$\begin{cases} x_0 = \text{initial guess} \\ x_{k+1} = g(x_k), \quad k = 0, 1, 2, \ldots. \end{cases} \qquad \text{(fixed point iteration)}$$

- The sequence $\{x_k\}$ may or may not converge as $k \to \infty$.
- If g is continuous and $\{x_k\}$ converges to a number r, then r is a fixed point of g.

$$g(r) = g\left(\lim_{k\to\infty} x_k\right) = \lim_{k\to\infty} g(x_k) = \lim_{k\to\infty} x_{k+1} = r.$$

$\lim_{k\to\infty} x_k = r$

cont. of g

iter. form.

Fixed Point Iteration Algorithm

```
function x = fpi(g, x0, n)
% FPI x = fpi(g, x0, n)
% Computes approximate solution of g(x)=x
% Input:
%    g     function handle
%    x0    initial guess
%    n     number of iteration steps
    x = x0;
    for k = 1:n
        x = g(x);  ←  iteration step
    end            (replacement/updating)
end
```

Examples

initial iterate x_0.

- To find a fixed point of $g(x) = 0.3\cos(2x)$ near 0.5 using `fpi`:

```
g = @(x) 0.3*cos(2*x);
xc = fpi(g,0.5,20)
```

```
xc = 0.260266319627758
```

fnc x_0 # of iteration

Not All Fixed Point Problems Are The Same

The rootfinding problem $f(x) = x^3 + x - 1 = 0$ can be transformed to various fixed point problems:

- $g_1(x) = x - f(x) = 1 - x^3$
- $g_2(x) = \sqrt[3]{1-x}$
- $g_3(x) = \dfrac{1+2x^3}{1+3x^2}$

$x^3 + x - 1 = 0$ (RF)

$x^3 = 1 - x$

$x = \sqrt[3]{1-x} = g_2(x)$

Note that all $g_j(x) = x$ are equivalent to $f(x) = 0$. However, not all these find a fixed point of g, that is, a root of f on the computer.

Exercise. Run `fpi` with g_j and $x_0 = 0.5$. Which fixed point iterations converge?

$\underbrace{x = g_3(x)}_{\text{FP}} = \dfrac{1+2x^3}{1+3x^2} \Rightarrow x(1+3x^2) = 1 + 2x^3$

$3x^3 + x = 2x^3 + 1$

$x^3 + x - 1 = 0$ (RF) ✓

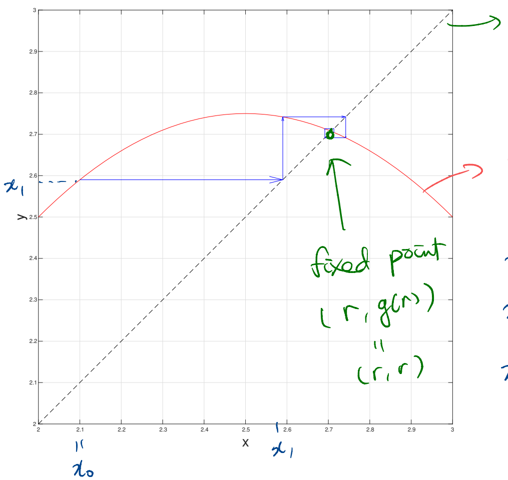
Geometry of Fixed Point Iteration

FP : $g(x) = x - f(x)$
$= -x^2 + 5x - 3.5$

The following script[2] finds a root of $f(x) = x^2 - 4x + 3.5$ via FPI.

RF

```
f = @(x) x.^2 - 4*x + 3.5;
g = @(x) x - f(x);
fplot(g, [2 3], 'r');
hold on
plot([2 3], [2 3], 'k--')  -> "y=x"
x = 2.1;   -> x_0
y = g(x);
for k = 1:5
    arrow([x y], [y y], 'b');
    x = y; y = g(x);
    arrow([x x], [x y], 'b');
end
```



$x_0 = 2.1$

$x_1 = g(x_0)$

$x_2 = g(x_1)$

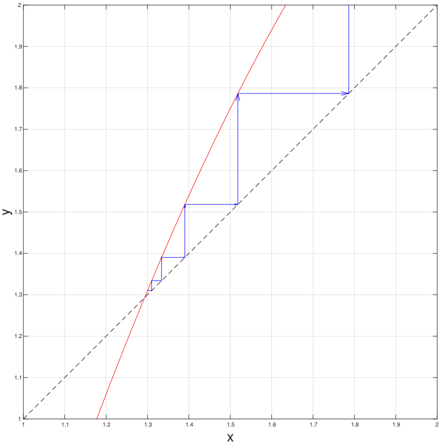
Note the line segments spiral in towards the fixed point.

[2]Modified from FNC.

Geometry of Fixed Point Iteration (cont')

However, with a different starting point, the process does not converge.

```
clf
fplot(g, [1 2], 'r');
hold on
plot([1 2], [1 2], 'k--'),
ylim([1 2])
x = 1.3; y = g(x);
for k = 1:5
    arrow([x y], [y y], 'b');
    x = y; y = g(x);
    arrow([x x], [x y], 'b');
end
```



Custom function: `arrow = @(p1, p2, varargin) quiver(p1(1), p1(2), p2(1)-p1(1), p2(2)-p1(2), 0, varargin{:})`

Series Analysis

fixed pt (soln)

Let $\epsilon_k = x_k - r$ be the sequence of errors. (Assume convergence, i.e. $\epsilon_k \to 0$ for the following analysis.)

$x_k = r + \epsilon_k$

- The iteration formula $x_{k+1} = g(x_k)$ can be written as

$$\begin{aligned} \epsilon_{k+1} + r &= g(\epsilon_k + r) \\ &= g(r) + g'(r)\epsilon_k + \frac{1}{2}g''(r)\epsilon_k^2 + \cdots, \qquad \text{(Taylor series)} \end{aligned}$$

  implying

$$\epsilon_{k+1} = g'(r)\epsilon_k + O(\epsilon_k^2)$$

  assuming sufficient regularity of g.

- Neglecting the second-order term, we have $\epsilon_{k+1} \approx g'(r)\epsilon_k$, which is satisfied if $\epsilon_k \approx C\left[g'(r)\right]^k$ for sufficiently large k.

  (geometric sequence)

- Therefore, the iteration converges if $|g'(r)| < 1$ and diverges if $|g'(r)| > 1$.

Quick check: $\epsilon_{k+1} \approx C[g'(r)]^{k+1} \approx C\, g'(r)[g'(r)]^k \approx C\,\epsilon_k$

Note: Rate of Convergence

Definition 2 (Linear Convergnece)

Suppose $\lim_{k\to\infty} x_k = r$ and let $\epsilon_k = x_k - r$, the error at step k of an iteration method. If

$$\lim_{k\to\infty} \frac{|\epsilon_{k+1}|}{|\epsilon_k|} = \sigma < 1,$$

the method is said to obey **linear convergence** with rate σ.

Note. In general, say

$$\lim_{k\to\infty} \frac{|\epsilon_{k+1}|}{|\epsilon_k|^p} = \sigma$$

for some $p \geqslant 1$ and $\sigma > 0$.

- If $p = 1$ and
  - $\sigma = 1$, the convergence is *sublinear*;
  - $0 < \sigma < 1$, the convergence is *linear*; $\longleftarrow$
  - $\sigma = 0$, the convergence is *superlinear*.
- If $p = 2$, the convergence is *quadratic*;
- If $p = 3$, the convergence is *cubic*, $\ldots$

Convergence of Fixed Point Iteration

Theorem 3 (Convergence of FPI)

Assume that g is continuously differentiable, that $g(r) = r$, and that $\sigma = |g'(r)| < 1$. Then the fixed point iterates x_k generated by

$$x_{k+1} = g(x_k), \quad k = 1, 2, \dots,$$

converge linearly with rate σ to the fixed point r for x_0 sufficiently close to r.

In the previous example with $g(x) = x - f(x) = -x^2 + 5x - 3.5$:

- For the first fixed point, near 2.71, we get $g'(r) \approx -0.42$ (convergence);
- For the second fixed point, near 1.29, we get $g'(r) \approx 2.42$ (divergence).

Note. An iterative method is called **locally convergent** to r if the method converges to r for initial guess sufficiently close to r.

Contraction Maps

Lipschitz Condition

A function g is said to satisfy a **Lipschitz condition** with constant L on the interval $S \subset \mathbb{R}$ if

$$\left|g(s) - g(t)\right| \leqslant L\,|s - t| \quad \text{for all } s, t \in S.$$

- A function satisfying the Lipschitz condition is continuous on S.
- If $L < 1$, g is called a **contraction map**.

When Does FPI Succeed?

Contraction Mapping Theorem

Suppose that g satisfies Lipschitz condition on S with $L < 1$, *i.e.*, g is a contraction map on S. Then S contains exactly one fixed point r of g. If $x_1, x_2, \ldots$ are generated by the fixed point iteration $x_{k+1} = g(x_k)$, and $x_1, x_2, \ldots$ all lie in S, then

$$|x_k - r| \leqslant L^{k-1} |x_1 - r|, \quad k > 1.$$

Newton's Method

FPI, when convergent, is linearly convergent.

Newton's Method

To find the root of f:

Newton's Method (Algorithm)

- Begin at the point $(x_0, f(x_0))$ on the curve and draw the tangent line at the point using the slope $f'(x_0)$:

$$y = f(x_0) + f'(x_0)(x - x_0).$$

- Find the x-intercept of the line and call it x_1: Set $y=0$ and solve for x.

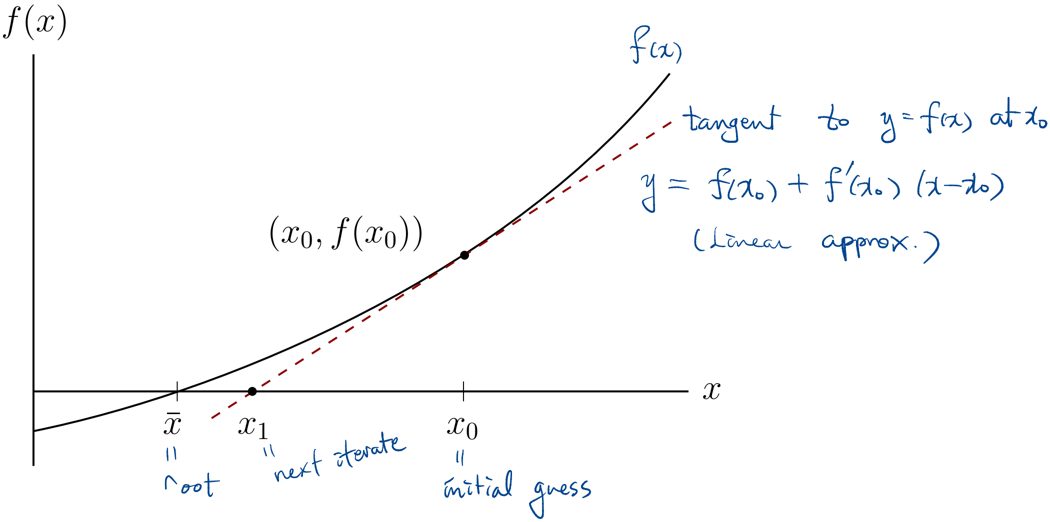
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

- Continue this procedure to find $x_2, x_3, \ldots$ until the sequence converges to the root.

General iterative formula: (Newton iteration)

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad \text{for } k = 0, 1, 2, \ldots} \qquad (\star)$$

Newton's Method: Illustration



Series Analysis

$\lim_{k\to\infty} x_k$

Let $\epsilon_k = x_k - r$, $k = 1, 2, \ldots$, where r is the limit of the sequence and $f(r) = 0$.

Substituting $x_k = r + \epsilon_k$ into the iterative formula (⋆):

$x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$

$$\epsilon_{k+1} = \epsilon_k - \frac{f(r+\epsilon_k)}{f'(r+\epsilon_k)}.$$

Taylor-expand f about $x = r$ and simplify (assuming $f'(r) \neq 0$):

$f(r) = 0$

$$\begin{aligned}
\epsilon_{k+1} &= \epsilon_k - \frac{\cancel{f(r)}^{\,0} + \epsilon_k f'(r) + \frac{1}{2}\epsilon_k^2 f''(r) + O(\epsilon_k^3)}{f'(r) + \epsilon_k f''(r) + O(\epsilon_k^2)} \\
&= \epsilon_k - \epsilon_k \left[1 + \frac{1}{2}\frac{f''(r)}{f'(r)}\epsilon_k + O(\epsilon_k^2)\right]\left[1 + \frac{f''(r)}{f'(r)}\epsilon_k + O(\epsilon_k^2)\right]^{-1} \\
&= \frac{1}{2}\frac{f''(r)}{f'(r)}\epsilon_k^2 + O(\epsilon_k^3).
\end{aligned}$$

Geom. Series

Use: $\dfrac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \cdots$ for $|\alpha| < 1$

Exercise. Check algebra.

Cf $\epsilon_{k+1} \approx C\epsilon_k$