Lec 16: Square Linear Systems – Further Analysis

Vector and Matrix Norms

Vector Norms

The "length" of a vector $\mathbf{v}$ can be measured by its **norm**.

Definition 1 (p-Norm of a Vector)

Let $p \in [1, \infty)$. The p-norm of $\mathbf{v} \in \mathbb{R}^m$ is denoted by $\|\mathbf{v}\|_p$ and is defined by

$$\boxed{\|\mathbf{v}\|_p = \left(\sum_{i=1}^{m} |v_i|^p\right)^{1/p}.}$$

when $p=2$, Pythagorean formula.

When $p = \infty$,

$$\|\mathbf{v}\|_\infty = \max_{1 \leqslant i \leqslant m} |v_i| .$$

The most commonly used p values are 1, 2, and ∞:

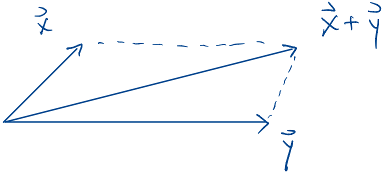
$$\|\mathbf{v}\|_1 = \sum_{i=1}^{m} |v_i| , \quad \|\mathbf{v}\|_2 = \sqrt{\sum_{i=1}^{m} |v_i|^2}.$$

Vector Norms

(generalization of "length")

In general, any function $\|\cdot\| : \mathbb{R}^m \to \mathbb{R}^+ \cup \{0\}$ is called a **vector norm** if it satisfies the following three properties:

1. $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = 0$.
2. $\|\alpha\mathbf{x}\| = |\alpha|\,\|\mathbf{x}\|$ for any constant α and any $\mathbf{x} \in \mathbb{R}^m$.
3. $\|\mathbf{x}+\mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. This is called the *triangle inequality*.



$\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

Unit Vectors

- A vector $\mathbf{u}$ is called a **unit vector** if $\|\mathbf{u}\| = 1$.
- Depending on the norm used, unit vectors will be different.
- For instance: $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$

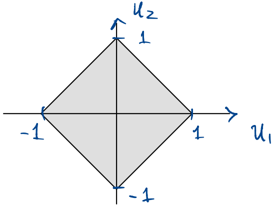


Figure 1: 1-norm

$\|\vec{u}\|_1 = |u_1| + |u_2| = 1$

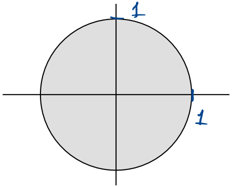


Figure 2: 2-norm

$\|\vec{u}\|_2 = \sqrt{u_1^2 + u_2^2} = 1$

i.e. $u_1^2 + u_2^2 = 1$

(circle eqn.)

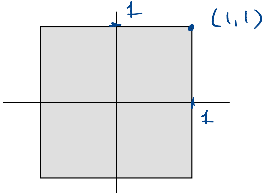


Figure 3: ∞-norm

$\|\vec{u}\|_\infty = \max\{|u_1|, |u_2|\} = 1$

Matrix Norms

The "size" of a matrix $A \in \mathbb{R}^{m \times n}$ can be measured by its **norm** as well. As above, we say that a function $\|\cdot\| : \mathbb{R}^{m \times n} \to \mathbb{R}^+ \cup \{0\}$ is a **matrix norm** if it satisfies the following three properties:

1. $\|A\| = 0$ if and only if $A = 0$.
2. $\|\alpha A\| = |\alpha| \, \|A\|$ for any constant α and any $A \in \mathbb{R}^{m \times n}$.
3. $\|A + B\| \leqslant \|A\| + \|B\|$ for any $A, B \in \mathbb{R}^{m \times n}$. This is called the *triangle inequality*.

Same 3 requirements

Matrix Norms (Cont')

- If, in addition to satisfying the three conditions, it satisfies

$$\|AB\| \leqslant \|A\|\,\|B\| \quad \text{for all } A \in \mathbb{R}^{m\times n} \text{ and all } B \in \mathbb{R}^{n\times p},$$

  it is said to be **consistent**.

- If, in addition to satisfying the three conditions, it satisfies

  vector matrix vector.

$$\|A\mathbf{x}\| \leqslant \|A\|\,\|\mathbf{x}\| \quad \text{for all } A \in \mathbb{R}^{m\times n} \text{ and all } \mathbf{x} \in \mathbb{R}^{n},$$

  then we say that it is **compatible** with a vector norm.

Induced Matrix Norms

Definition 2 (p-Norm of a Matrix)

Let $p \in [1, \infty]$. The p-norm of $A \in \mathbb{R}^{m\times n}$ is given by

in terms of vector p-norm.

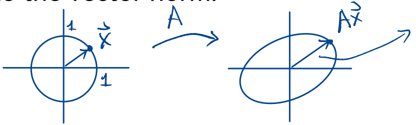
$$\boxed{\|A\|_p = \max_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p} = \max_{\|\mathbf{x}\|_p = 1} \|A\mathbf{x}\|_p \,.}$$

unit vectors.

- The definition of this particular matrix norm is **induced** from the vector p-norm.
- By construction, matrix p-norm is a compatible norm. $\|A\vec{x}\| \leq \|A\| \|\vec{x}\|$.
- Induced norms describe how the matrix stretches unit vectors with respect to the vector norm.

Ex. $\mathbb{R}^2$

$p = 2$



"largest stretch" = matrix norm

Induced Matrix Norms

The commonly used p-norms (for $p = 1, 2, \infty$) can also be calculated by

"sum" along jth column.

$$\|A\|_1 = \max_{1 \leqslant j \leqslant n} \sum_{i=1}^{m} |a_{ij}|, \qquad \text{(max. col sum)}$$

singular value

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^{\mathrm{T}}A)} = \sigma_{\max}(A),$$

max. eigenvalue

$$\|A\|_\infty = \max_{1 \leqslant i \leqslant m} \sum_{j=1}^{n} |a_{ij}|. \qquad \text{(max. row sum)}$$

In words,

- The 1-norm of A is the maximum of the 1-norms of all column vectors.
- The 2-norm of A is the square root of the largest eigenvalue of $A^{\mathrm{T}}A$.
- The ∞-norm of A is the maximum of the 1-norms of all row vectors.

Non-Induced Matrix Norm – Frobenius Norm

Definition 3 (Frobenius Norm of a Matrix)

The Frobenius norm of $A \in \mathbb{R}^{m\times n}$ is given by

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

- This is not induced from a vector p-norm.
- However, both p-norm and the Frobenius norm are consistent and compatible.
- For compatibility of the Frobenius norm, the vector norm must be the 2-norm, that is, $\boxed{\|A\mathbf{x}\|_2 \leqslant \|A\|_F \|\mathbf{x}\|_2.}$

$\|AB\|_F \leq \|A\|_F \|B\|_F$

Norms in MATLAB

$$\vec{v}^T\vec{v} = [v_1\ v_2 \cdots v_n]\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\vec{v}\|_2^2$$

$$\boxed{\vec{v}^T\vec{v} = \|\vec{v}\|_2^2}$$

- Vector p-norms can be easily computed:

```
norm(v, 1)      % = sum(abs(v))
norm(v, 2)      % = sqrt(v'*v)  if v is a column
norm(v, 'inf')  % = max(abs(v))
```

$$\|\vec{v}\|_1 = \sum_{i=1}^{n} |v_i|$$

- The same function `norm` is used to calculate matrix p-norms:

```
norm(A, 1)      % = max(sum(abs(A), 1))
norm(A, 2)      % = max(sqrt(eig(A'*A)))
norm(A, Inf)    % = max(sum(abs(A), 2))
```

- To calculate the Frobenius norm:

```
norm(A, 'fro')  % = sqrt(A(:)'*A(:))
                % = norm(A(:), 2)
```

Conditioning

Conditioning of Solving Linear Systems: Overview

- Analyze how robust (or sensitive) the solutions of $\boxed{A\mathbf{x} = \mathbf{b}}$ are to perturbations of A and $\mathbf{b}$.

Given: A, $\vec{b}$ (inputs)

Want: $\vec{x}$ such that $A\vec{x} = \vec{b}$ (output)

- For simplicity, consider separately the cases where
  1. $\mathbf{b}$ changes to $\mathbf{b} + \boxed{\delta\mathbf{b}}$, while A remains unchanged, that is

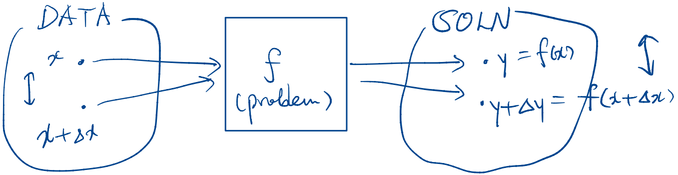
     • small change in $\vec{b}$

     • not $\delta \times \vec{b}$.

$$A\mathbf{x} = \mathbf{b} \quad\longrightarrow\quad A\underbrace{(\mathbf{x} + \delta\mathbf{x})}_{\text{pert. output}} = \underbrace{\mathbf{b} + \delta\mathbf{b}}_{\text{pert. input}}.$$

  2. A changes to $A + \boxed{\delta A}$, while $\mathbf{b}$ remains unchanged, that is

$$A\mathbf{x} = \mathbf{b} \quad\longrightarrow\quad \underbrace{(A + \delta A)}_{\text{pert. input}}\underbrace{(\mathbf{x} + \delta\mathbf{x})}_{\text{pert. output}} = \mathbf{b}.$$



Sensitivity to Perturbation of RHS

Case 1. $A\mathbf{x} = \mathbf{b} \ \rightarrow \ A(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}$

- Bound $\|\delta\mathbf{x}\|$ in terms of $\|\delta\mathbf{b}\|$:

$$\left.\begin{aligned} \cancel{A\mathbf{x}} + A\delta\mathbf{x} &= \cancel{\mathbf{b}} + \delta\mathbf{b} \\ A\delta\mathbf{x} &= \delta\mathbf{b} \\ \delta\mathbf{x} &= A^{-1}\delta\mathbf{b} \end{aligned}\right\} \implies \quad \|\delta\mathbf{x}\| \leqslant \|A^{-1}\|\|\delta\mathbf{b}\|.$$

(Compatibility) ✓

provided that A^{-1} exists

- Sensitivity in terms of relative errors:

$$\frac{|\text{rel. err. output}|}{|\text{rel. err. input}|} = \frac{\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|}}{\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}} = \frac{\|\delta\mathbf{x}\|\,\|\mathbf{b}\|}{\|\delta\mathbf{b}\|\,\|\mathbf{x}\|} \leqslant \frac{\|A^{-1}\|\,\cancel{\|\delta\mathbf{b}\|} \cdot \|A\|\,\cancel{\|\mathbf{x}\|}}{\cancel{\|\delta\mathbf{b}\|}\,\cancel{\|\mathbf{x}\|}} = \boxed{\|A^{-1}\|\|A\|.}$$

$$\|\vec{b}\| = \|A\vec{x}\| \leq \|A\|\,\|\vec{x}\|$$

Sensitivity to Perturbation of Matrix

Case 2. $A\mathbf{x} = \mathbf{b} \;\rightarrow\; (A + \delta A)(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b}$

- Bound $\|\delta\mathbf{x}\|$ now in terms of $\|\delta A\|$:

"Approximate" in the limit as $\|\delta A\| \to 0$.

$$
\begin{aligned}
\cancel{A\mathbf{x}} + A\delta\mathbf{x} + (\delta A)\mathbf{x} + (\delta A)\delta\mathbf{x} &= \cancel{\mathbf{b}} \\
A\delta\mathbf{x} &= -(\delta A)\mathbf{x} - (\delta A)\delta\mathbf{x} \\
\delta\mathbf{x} &= -A^{-1}(\delta A)\mathbf{x} - A^{-1}(\delta A)\delta\mathbf{x}
\end{aligned}
\quad\Longrightarrow\quad
\|\delta\mathbf{x}\| \lesssim \|A^{-1}\|\|\delta A\|\,\|\mathbf{x}\|\,.
$$

(first-order truncation)

- Sensitivity in terms of relative errors:

$$
\frac{\;\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|}\;}{\;\frac{\|\delta A\|}{\|A\|}\;} = \frac{\|\delta\mathbf{x}\|\,\|A\|}{\|\delta A\|\,\|\mathbf{x}\|} \lesssim \frac{\|A^{-1}\|\cancel{\|\delta A\|}\,\cancel{\|\mathbf{x}\|}\cdot\|A\|}{\cancel{\|\delta A\|}\,\cancel{\|\mathbf{x}\|}} = \boxed{\|A^{-1}\|\|A\|.}
$$

Matrix Condition Number

- Motivated by the previous estimations, we define the **matrix condition number** by

$$\kappa(A) = \|A^{-1}\|\|A\|,$$

"Compatible norms"

where the norms can be any p-norm or the Frobenius norm.

- A subscript on κ such as 1, 2, ∞, or F(robenius) is used if clarification is needed.

e.g. $\kappa_2(A) = \|A^{-1}\|_2 \|A\|_2$

Matrix Condition Number (Cont')

- We can write

$$\underbrace{\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leqslant \kappa(A)\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}}_{\text{Case 1}}, \quad \underbrace{\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leqslant \kappa(A)\frac{\|\delta A\|}{\|A\|}}_{\text{Case 2}},$$

  where the second inequality is true only in the limit of infinitesimal perturbations δA. $\|\delta A\| \to 0$

- The matrix condition number $\kappa(A)$ is equal to the condition number of solving a linear system of equation $A\mathbf{x} = \mathbf{b}$.
- The exponent of $\kappa(A)$ in scientific notation determines the approximate number of digits of accuracy that will be lost in calculation of $\mathbf{x}$.

  If $\kappa(A) = 6.5 \times 10^7$, 7 accurate digits will be lost

- Since $1 = \|I\| = \|A^{-1}A\| \leqslant \|A^{-1}\|\|A\| = \kappa(A)$, a condition number of 1 is the best we can hope for.
- If $\kappa(A) > \boxed{\text{eps}}^{-1}$, then for computational purposes the matrix is singular.

  loss 16 accurate digits.

Condition Numbers in MATLAB

- Use `cond` to calculate various condition numbers:

$\underbrace{\|A^{-1}\| \|A\|}$

```
cond(A)            % the 2-norm; or  cond(A, 2)
cond(A, 1)         % the 1-norm
cond(A, Inf)       % the infinity-norm
cond(A, 'fro')     % the Frobenius norm
```

costly.

- A condition number estimator (in 1-norm)

```
condest(A)         % faster than cond
```

- The fastest method to estimate the condition number is to use `linsolve` function as below:

```
[x, inv_condest] = linsolve(A, b);
fast_condest = 1/inv_condest;
```