Ultimately : Singular Value Decomposition (SVD)

Lec 35: Spectral Theory

Eigenvalue Decomposition

Eigenvalue Decomposition (EVD)

Eigenvalue Problem

$n \times n$ matric w/ complex entries

Give a square matrix $A \in \mathbb{C}^{n\times n}$,

find $\lambda \in \mathbb{C}$ and $\vec{v} \in \mathbb{C}^n$ such that

nonzero

$$A\vec{v} = \lambda \vec{v}$$

Geometry

if $\lambda \in \mathbb{R}$, stretch or compress

$$\underbrace{A\vec{v}}_{\text{"image of } \vec{v} \text{ under } A\text{"}} = \underbrace{\lambda \vec{v}}_{\text{"scalar multiple of } \vec{v}\text{"}}$$

- λ : eigenvalue of A
- $\vec{v}$: eigenvector of A

} eigenpair.

- How do we find them?

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

This eqn has a nontrivial soln $\vec{v}$ iff $A - \lambda I$ is singular. In other words,

$$\boxed{\underbrace{\det(A - \lambda I)}_{\text{poly. in } \lambda \text{ (deg-}n)} = 0}$$

(characteristic eqn)

Terminology

- $A \in \mathbb{C}^{n\times n}$ has <u>n eigenvalues</u> counting multiplicity:

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

- The set of all eigenvalues of A is called the <u>spectrum</u> of A.

$$\operatorname{Spec}(A) = \{\lambda_1, \ldots, \lambda_n\}$$

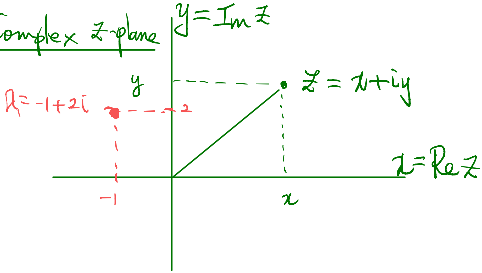
- $\max\limits_{1\le j\le n} |\lambda_j| =$ spectral radius

Digression: Geometry of comp. numbers

$$z = x + iy \in \mathbb{C}$$

$(x, y \in \mathbb{R})$ $\qquad i = \sqrt{-1}$

<u>Complex z-plane</u>



Eigenvalue Decomposition (EVD)

- $A \in \mathbb{C}^{n \times n}$
- $\lambda_1, \lambda_2, \cdots, \lambda_n$: eigenvalues
- $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$: eigenvectors

$$A\vec{v}_j = \lambda_j \vec{v}_j \quad \text{for } j = 1, \cdots, n$$

$$\Rightarrow \underbrace{\begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \end{bmatrix}}_{n \times n \text{ matrix}} = \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \end{bmatrix}$$

$$\Rightarrow A \overbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}}^{V} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \overbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}}^{D}$$

$$\Rightarrow AV = VD \quad \text{(general)}$$

- V : columns are eigenvectors
- D : diagonal matrix w/ eigenvalue on the diagonal.

In a special case where eigenvectors of A are <u>linearly independent</u>, then

rank $(V) = n$

(full rank)

$$V = \left[\vec{v}_1 \,\middle|\, \vec{v}_2 \,\middle|\, \cdots \,\middle|\, \vec{v}_n \right]$$

linearly independent

is nonsingular (or invertible).

<u>Consequence</u>: $AV = VD$ (true always for <u>square</u> A)

$$\Rightarrow \boxed{A = V D V^{-1}} \quad (\text{EVD})$$

Eigenvalue Decomposition (cont')

eigenvectors of A are lin. indep.

If A has an EVD, we say that A is **diagonalizable**; otherwise **nondiagonalizable**.

Theorem 1 (Diagonalizability)

If $A \in \mathbb{C}^{n \times n}$ has n distinct eigenvalues, then A is diagonalizable.

Notes.

- Let $A, B \in \mathbb{C}^{n \times n}$. We say that B is **similar** to A if there exists a nonsingular matrix X such that

$$B = XAX^{-1}.$$

- So *diagonalizability* is *similarity to a diagonal matrix*.
- Similar matrices share the same eigenvalues.

Exercise: Confirm this.

(Review your 2568 notes/text.)

Calculating EVD in MATLAB

- `E = eig(A)`
  produces a column vector `E` containing the eigenvalues of `A`.
- `[V, D] = eig(A)`
  produces V and D in an EVD of A, $A = VDV^{-1}$.