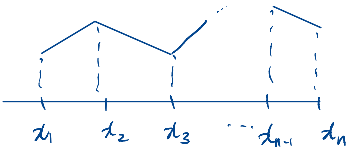
Hat Functions As Basis

Assume $x_1 < x_2 < \cdots < x_n$ fixed.

- Any linear combination of hat functions is continuous and is linear inside each interval $[x_j, x_{j+1}]$.
- Conversely, *any* such function is expressible as a unique linear combination of hat functions, *i.e.*,

$$\boxed{\sum_{j=1}^{n} c_j H_j(x),} \quad \text{for some choice of } c_1, \ldots, c_n.$$

- No smaller set of functions has the same properties.



The hat functions form a **basis** of the set of functions that are continuous and piecewise linear relative to $\mathbf{x}$ (the vector of nodes).

Cardinality Conditions

- By construction, the hat functions are cardinal functions for piecewise linear (PL) interpolation, *i.e.*, they satisfy

$$H_j(x_k) = \delta_{j,k} := \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \quad \text{(cardinality condition)}$$

- Key consequence of this property is that the piecewise linear interpolant $p(x)$ for the data values in $\mathbf{y}$ is trivially expressed by

$$p(x) = \sum_{j=1}^{n} y_j H_j(x).$$

The formula for interpolant is readily available, without needing to solve any eqn.

$$p(x_k) = \sum_{j=1}^{n} y_j H_j(x_k) = y_k \underbrace{H_k(x_k)}_{=1} = y_k.$$

Since all other terms vanish.

Recipe for PL Interpolant

Piecewise Linear Interpolant

The piecewise linear polynomial

$$p(x) = \sum_{j=1}^{n} y_j H_j(x)$$

is the unique such function which passes through all the data points.

Proof: It is easy to check the interpolating property:

$$p(x_k) = \sum_{j=1}^{n} y_j H_j(x_k) = \sum_{j=1}^{n} y_j \delta_{j,k} = y_k \quad \text{for every } k \in \mathbb{N}[1,n].$$

To show uniqueness, suppose $\tilde{p}$ is another such function in the form

$$\tilde{p}(x) = \sum_{j=1}^{n} c_j H_j(x)\,.$$

Then $p(x_k) - \tilde{p}(x_k) = 0$ for all $k \in \mathbb{N}[1,n]$. This implies that $c_k = y_k$ for all k $\square$

Conditioning

(cf. general theorem from Wed.)

Lemma

Let $\mathcal{I}$ is the piecewise linear interpolation operator and $\mathbf{z} \in \mathbb{R}^n$. Then

$$\underbrace{\|\mathcal{I}(\mathbf{z})\|_\infty}_{\text{functional norm}} = \underbrace{\|\mathbf{z}\|_\infty}_{\text{vector norm}}.$$

- It follows from the lemma that the absolute condition number of piecewise linear interpolation in the infinity norm equals one.

abs. cond. number

Def'n

$$\kappa(\vec{y}) = \max_{\Delta\vec{y}} \frac{\|\mathcal{I}(\vec{y}+\Delta\vec{y}) - \mathcal{I}(\vec{y})\|_\infty}{\|\Delta\vec{y}\|_\infty} = \max_{\Delta\vec{y}} \frac{\|\mathcal{I}(\Delta\vec{y})\|_\infty}{\|\Delta\vec{y}\|_\infty} = \max_{\Delta\vec{y}} \frac{\|\Delta\vec{y}\|_\infty}{\|\Delta\vec{y}\|_\infty} = 1$$

func.

vector

linearity of $\mathcal{I}$

lemma

Conditioning (cont')

Proof of lemma. Let

$$p(x) = \mathcal{I}(\mathbf{z}) = \sum_{j=1}^{n} z_j H_j(x).$$

Let k be the index corresponding to the element of $\mathbf{z}$ with the largest absolute value, that is, $z_k = \|\mathbf{z}\|_\infty$. Since $z_k = p(x_k)$, it follows that $|p(x_k)| = \|\mathbf{z}\|_\infty$ and so $\|p\|_\infty \leqslant \|\mathbf{z}\|_\infty$.

To show the other inequality, note that

$$|p(x)| = \left| \sum_{j=1}^{n} z_j H_j(x) \right| \leqslant \sum_{j=1}^{n} |z_j| \, H_j(x) \leqslant \|\mathbf{z}\|_\infty \sum_{j=1}^{n} H_j(x) = \|\mathbf{z}\|_\infty \, ,$$

where the final step uses the fact[3] that $\sum_{j=1}^{n} H_j(x) = 1$. It implies that $\|p\|_\infty \leqslant \|\mathbf{z}\|_\infty$. Therefore, $\|p\|_\infty = \|\mathbf{z}\|_\infty$. □

[3] This property is called the *partition of unity*. Confirm it!

Convergence: Error Analysis

Set-up for analysis.

- Generate a set of data points using a "nice" function f on an interval containing all nodes, *i.e.*, $y_j = f(x_j)$. (The *niceness* of a function is described in precise terms below.)
- Then perform PL interpolation of the data to obtain the interpolant p.
- **Question.** How close is p to f?

Notation (Space of Differentiable Functions)

Let $C^n[a, b]$ denote the set of all functions that are n-times continuously differentiable on $[a, b]$. That is, if $f \in C^n[a, b]$, then $f^{(n)}$ exists and is continuous on $[a, b]$, where derivatives at the end points are taken to be one-sided derivatives.

Convergence: Error Analysis (cont')

$$\|f\|_\infty = \max_{[a,b]} |f(x)|$$

Theorem 2 (Error Theorem for PL Interpolation)

Suppose that $f \in C^2[a,b]$. *Let* p_n *be the piecewise linear interpolant of* $(x_j, f(x_j))$ *for* $j = 1, \ldots, n$, *where*

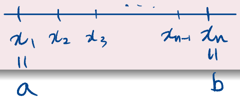
$$x_j = a + \overset{(j-1)h}{\cancel{jh}} \quad \text{and} \quad h = \frac{b-a}{n-1}.$$

step size

(uniform nodes)

Then

$$\|f - p_n\|_\infty \leqslant \|f''\|_\infty h^2.$$

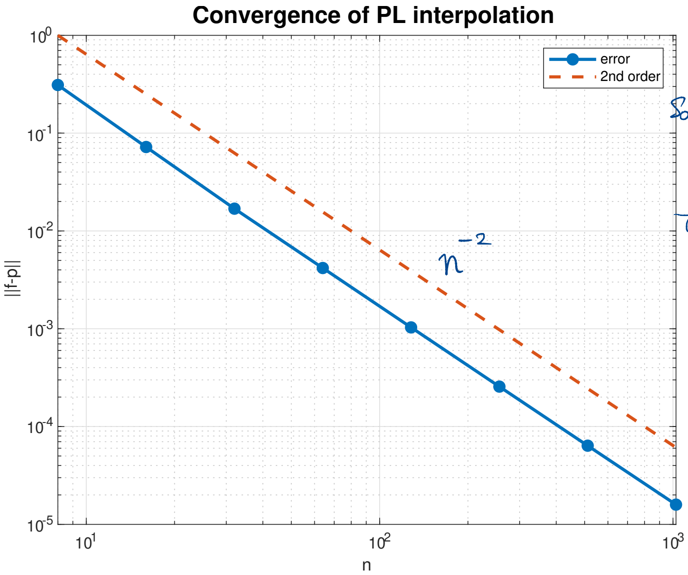


- The theorem pertains to the interpolation on equispaced nodes.
- The significance of the theorem is that the error in the interpolant is $O(h^2)$ as $h \to 0$. (We say that PL interpolation is *second-order accurate*.)
- **Practical implication:** If n is doubled, the PL interpolant becomes about four times more accurate. A log-log graph (`loglog`) of error against n is a straight line.

h will be halved

with slope -2

Convergence: Error Analysis (cont')



$$h = \frac{b-a}{n-1}$$

So for large n,

$$h \approx \frac{b-a}{n}.$$

Thus

$$\|f-p\|_\infty \le M h^2$$

$$\lesssim \frac{M'}{n^2} \approx M' n^{-2}$$

$$\log \|f-p\|_\infty \lesssim \boxed{(-2)} \log(n) + \text{const.}$$

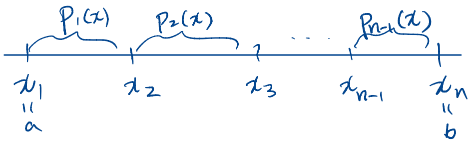
Lec 26: Piecewise Interpolation – Cubic Splines

- piecewise linear (1) : easy, not smooth
- piecewise quadratic (2) : more work, not common
- piecewise cubic (3) : even more work, smooth.
    - Hermite cubic
    - cubic spline (✱)

Hermite Cubic Interpolation

Notation (Set-up)

"piecewise"



i^{th} local cubic polynomial: $\quad 1 \le i \le n-1$

$$P_i(x) = c_{i,1} + c_{i,2}(x - x_i) + c_{i,3}(x - x_i)^2 + c_{i,4}(x - x_i)^3, \quad [x_i, x_{i+1})$$

Note The defn of $P_{n-1}(x)$ must hold on $[x_{n-1}, x_n]$

Interpolant

$$\mathbb{P}(x) = \begin{cases} P_1(x), & [x_1, x_2) \\ P_2(x), & [x_2, x_3) \\ \vdots & \\ P_{n-1}(x), & [x_{n-1}, x_n] \end{cases}$$

Hermite Cubic Interpolation

We now seek a piecewise cubic polynomial p where $p_i(x)$ on $[x_i, x_{i+1}]$ is written in shifted power form as

breakpoints (nodes, knots) ordinates slopes

$$p_i(x) = c_{i,1} + c_{i,2}(x - x_i) + c_{i,3}(x - x_i)^2 + c_{i,4}(x - x_i)^3 \,.$$

x_i, y_i, σ_i

usual interp.

Hermite.

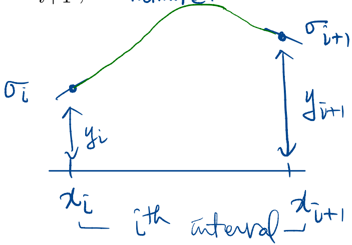
- If the slopes at endpoints are additionally given, *i.e.*,

$$p_i(x_i) = y_i\,, \quad p_i'(x_i) = \sigma_i\,, \quad p_i(x_{i+1}) = y_{i+1}\,, \quad p_i'(x_{i+1}) = \sigma_{i+1}\,,$$

  then we can solve for the four unknown coefficients $c_{i,j}$:

- $\Delta x_i = x_{i+1} - x_i$
- $y[x_i, x_{i+1}] = \dfrac{y_{i+1} - y_i}{x_{i+1} - x_i}$

$$\boxed{\begin{aligned} c_{i,1} &= y_i\,, & c_{i,3} &= \frac{3y[x_i,x_{i+1}] - 2\sigma_i - \sigma_{i+1}}{\Delta x_i}\,, \\ c_{i,2} &= \sigma_i\,, & c_{i,4} &= \frac{\sigma_i + \sigma_{i+1} - 2y[x_i,x_{i+1}]}{(\Delta x_i)^2}\,. \end{aligned}}$$



- This is called **Hermite cubic interpolation**.

Convergence: Error Analysis

Theorem 1 (Error Theorem for Hermite Cubic Interpolation)

Let $f \in C^4[a,b]$ and let $p(x)$ be the Hermite cubic interpolant of

$$\left(x_i, f(x_i), f'(x_i)\right), \quad \text{for } i = 1, \ldots, n,$$

where

$$x_j = a + \cancel{jh}^{\,(j-1)h} \quad \text{and} \quad h = \frac{b-a}{n-1}. \qquad \text{(uniform nodes)}$$

Then

$$\|f - p\|_\infty \leqslant \frac{1}{384}\|f^{(4)}\|_\infty h^4. \qquad \text{(4th-order accurate)}$$

cf) PL : $\leq M h^2$

Cubic Splines

Cubic Splines

Only given (x_i, y_i) requiring $\mathbb{P} \in C^2[a,b]$

Idea: Put together cubic polynomials to make the result as smooth as possible.

- At interior breakpoints: for $j = 2, 3, \cdots, n-1$
  - $\mathbb{P}$ matching values: $p_{j-1}(x_j) = p_j(x_j)$ [$(n-2)$ eqns]
  - $\mathbb{P}'$ matching first derivatives: $p'_{j-1}(x_j) = p'_j(x_j)$ [$(n-2)$ eqns]
  - $\mathbb{P}''$ matching second derivative: $p''_{j-1}(x_j) = p''_j(x_j)$ [$(n-2)$ eqns]
- So, together with the n interpolating conditions, we have total of $(4n-6)$ equations.
- To match up with the number of unknowns $(4n-4)$, we need to impose two more conditions on the boundary:
  1. slopes at each end (clamped cubic spline)
  2. second derivatives at the endpoints (natural cubic spline)
  3. periodic boundary condition
  4. *not-a-knot* boundary condition: $p_1(x) \equiv p_2(x)$ and $p_{n-2}(x) \equiv p_{n-1}(x)$.

Convergence: Error Analysis

Theorem 2 (Error Theorem for Clamped Cubic Splines)

Let $f \in C^4[a,b]$ and let $p(x)$ be the cubic spline interpolant of

$$\left(x_i, f(x_i)\right), \quad \text{for } i = 1, \ldots, n,$$

with the exact boundary conditions

$$\sigma_1 = f'(x_1) \quad \text{and} \quad \sigma_n = f'(x_n),$$

in which

$$x_j = a + jh \quad \text{and} \quad h = \frac{b-a}{n-1}.$$

Then

$$\|f - p\|_\infty \leqslant \frac{5}{384}\|f^{(4)}\|_\infty h^4.$$

5 × that of Hermite cub.

4th - order accurate

Remarks

- Hermite cubic interpolation is about five times as accurate as cubic spline interpolation, yet both have *fourth-order accuracy*.
- Unlike the former, the latter does not require first derivatives.