Reduction of Dimensions

Accompanying livescript from lec. 36.

Recap $\boxed{\text{SVD}}$

$A \in \mathbb{C}^{m \times n}$, $m \geq n$ (tall rectangle)

hermitian = conjugate transpose

SVD: $A = U \Sigma V^{*}$

$$\left\{ \begin{array}{llll} \cdot\ U \in \mathbb{C}^{m \times m} & \text{unitary} & \longrightarrow & \text{columns: left singular vectors} \\ \cdot\ \Sigma \in \mathbb{R}^{m \times n} & \text{diagonal} & \longrightarrow & \text{diagonal entries: singular values} \\ \cdot\ V \in \mathbb{C}^{n \times n} & \text{unitary} & \longrightarrow & \text{columns: right singular vectors} \end{array} \right\} \text{ of } A$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \\ & & & \sigma_n \\ \hline & 0 & \end{bmatrix}, \qquad \sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_n \geq 0$$

primary singular value (of A)

Note (MATLAB)

- [U, S, V] = svd(A); (thick)
- [U, S, V] = svd(A, 0); (thin)

  zero
- hermitian: V'

Low-Rank Approximations

$\boxed{A} = \boxed{\hat{U}}\ \boxed{\hat{\Sigma}}\ \boxed{V}^*$

$m\times n \quad m\times n \quad n\times n \quad n\times n$

Let $A \in \mathbb{C}^{m\times n}$ with $m \geqslant n$. Its thin SVD $\underline{A = \widehat{U}\widehat{\Sigma}V^*}$ can be written as

$$A = \boxed{\underbrace{\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}}_{\text{first } n \text{ left-sing. vectors}} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}} \left.\begin{bmatrix} \mathbf{v}_1^* \\ \vdots \\ \mathbf{v}_n^* \end{bmatrix}\right\} \text{right-sing. vectors}^*$$

$$= \underbrace{\begin{bmatrix} \sigma_1\mathbf{u}_1 & \cdots & \sigma_n\mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^* \\ \vdots \\ \mathbf{v}_n^* \end{bmatrix}}_{\text{“inner product”}} = \sum_{j=1}^{r = \min\{m,n\} = n} \sigma_j \boxed{\mathbf{u}_j\mathbf{v}_j^*},$$

$\vec{u}_j \quad \vec{v}_j^{\,*}$

$m\times 1 \quad 1\times n$

“outer product”

where r is the rank of A.

- Each outer product $\mathbf{u}_j\mathbf{v}_j^*$ is a <u>rank-1 matrix</u>.
- Since $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r > 0$, important contributions to A come from terms with small j.

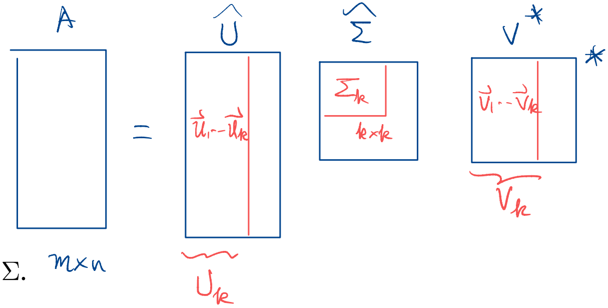
Low-Rank Approximations (cont')

For $1 \leqslant k \leqslant r$, define

$$A_k = \sum_{j=1}^{k} \sigma_j \mathbf{u}_j \mathbf{v}_j^* = U_k \Sigma_k V_k^*,$$

(rank-k approx. of A)

where



- U_k is the first k columns of U;
- V_k is the first k columns of V;
- Σ_k is the upper-left $k \times k$ submatrix of Σ.

This is a rank-k approximation of A.

$$A \approx A_k = U_k \Sigma_k V_k^*$$

Best Rank-k Approximation

Theorem 6 (Eckart-Young)

Let $A \in \mathbb{C}^{m\times n}$. Suppose A has rank r and let $A = U\Sigma V^*$ be an SVD. Then

- $\|A - A_k\|_2 = \sigma_{k+1}$, *for* $k = 1, \ldots, r-1$.
- *For any matrix B with* $\operatorname{rank}(B) \leqslant k$, $\|A - B\|_2 \geqslant \sigma_{k+1} = \|A - A_k\|$

first omitted sing. value