

To see why Say  $r$  is a root w/ multiplicity  $m$ . of  $f(x)$ :

$$f(r) = f'(r) = \dots = f^{(m-1)}(r) = 0, \quad f^{(m)}(r) \neq 0 \quad | \quad (\text{from the def'n.})$$

Consequently:

$$\bullet \quad f(r + \epsilon_k) = \cancel{f(r) + f'(r)\epsilon_k + \frac{f''(r)}{2!}\epsilon_k^2} + \dots + \cancel{\frac{f^{(m-1)}(r)}{(m-1)!}\epsilon_k^{m-1}} + \frac{f^{(m)}(r)}{m!}\epsilon_k^m + O(\epsilon_k^{m+1})$$

$$= \frac{f^{(m)}(r)}{m!}\epsilon_k^m + \frac{f^{(m+1)}(r)}{(m+1)!}\epsilon_k^{m+1} + O(\epsilon_k^{m+2})$$

$$\bullet \quad f'(r + \epsilon_k) = \cancel{f'(r) + f''(r)\epsilon_k} + \dots + \cancel{\frac{f^{(m-1)}(r)}{(m-2)!}\epsilon_k^{m-2}} + \frac{f^{(m)}(r)}{(m-1)!}\epsilon_k^{m-1} + O(\epsilon_k^m)$$

$$= \frac{f^{(m)}(r)}{(m-1)!}\epsilon_k^{m-1} + \frac{f^{(m+1)}(r)}{m!}\epsilon_k^m + O(\epsilon_k^{m+1})$$

So the Newton's iteration formula

(in terms of  $\epsilon_k$ 's) is now  
written as:

$$\epsilon_{k+1} = \epsilon_k - \frac{m f(r + \epsilon_k)}{f'(r + \epsilon_k)}$$

$$\therefore \frac{\epsilon_{k+1}}{\epsilon_k} \rightarrow \frac{m-1}{m} \epsilon(0,1)$$

(linear convergence)

$$= \epsilon_k - \frac{m \left[ \frac{f^{(m)}(r)}{m!} \epsilon_k^m + \frac{f^{(m+1)}(r)}{(m+1)!} \epsilon_k^{m+1} + O(\epsilon_k^{m+2}) \right]}{\frac{f^{(m)}(r)}{(m-1)!} \epsilon_k^{m-1} + \frac{f^{(m+1)}(r)}{m!} \epsilon_k^m + O(\epsilon_k^{m+1})}$$

confirm  
(fill in the  
details)

$$= \epsilon_k - \left( \frac{1}{m} \epsilon_k \right) + O(\epsilon_k^2) = \frac{m-1}{m} \epsilon_k + O(\epsilon_k^2)$$



$$\epsilon_{k+1} = * \epsilon_k^2 + O(\epsilon_k^3)$$

$$\therefore \frac{\epsilon_{k+1}}{\epsilon_k^2} \rightarrow *$$

as  $k \rightarrow \infty$

(quad. conv.)

# Calculating $n$ th Roots

**Question.** Let  $n$  be a positive integer. Use Newton's method to produce a quadratically convergent method for calculating the  $n$ th root of a positive number  $a$ . Prove quadratic convergence.

Given  $n \in \mathbb{N}$ ,  $a > 0$ , calculate  $\sqrt[n]{a}$  using Newton.

Note that  $\sqrt[n]{a}$  is a root of

$$x^n = a$$

So, it is a root of

$$f(x) = x^n - a$$

$$x_{k+1} = \frac{n-1}{n} x_k + \frac{a}{n x_k^{n-1}}$$

e.g. ( $n=2$ )

$$x_{k+1} = \frac{1}{2} x_k + \frac{a}{2 x_k}$$

Newton

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^n - a}{n x_k^{n-1}} = \frac{n-1}{n} x_k + \frac{a}{n x_k^{n-1}}$$

## Predicting Next Error

$$x(x^2 - 4) = x(x-2)(x+2)$$

Question. Let  $f(x) = x^3 - 4x$ .  $\hat{=}$

- (a) • The function  $f(x)$  has a root at  $r = 2$ . If the error  $\epsilon_k = x_k - r$  after four steps of Newton's method is  $\underline{\epsilon_4} = 10^{-6}$ , estimate  $\epsilon_5$ .
- (b) • Do the same to the root  $r = 0$ . (Exercise)

---

Recall: Newton's method

$$\epsilon_{k+1} = \frac{f'(r)}{2f''(r)} \epsilon_k^2 + O(\epsilon_k^3)$$

(a)  $r = 2$ ,

$$\begin{cases} f'(x) = 3x^2 - 4 \\ f''(x) = 6x \end{cases} \Rightarrow \begin{cases} f'(2) = 8 \\ f''(2) = 12 \end{cases}$$

So at  $r=2$

$$\epsilon_{k+1} = \frac{8}{24} \epsilon_k^2 + O(\epsilon_k^3)$$

$$k=4, \epsilon_k = 10^{-6}$$

$$\epsilon_5 = \frac{1}{3} 10^{-12} + O(10^{-18}) \approx 3.33 \times 10^{-11}$$

## Secant Method

Assume that iterates  $x_1, x_2, \dots$  generated by the secant method converges to a root  $r$  and  $f'(r) \neq 0$ . Let  $\epsilon_k = x_k - r$ .

**Exercise.**<sup>1</sup> Show that

- ① The error  $\epsilon_k$  satisfies the approximate equation

$$|\epsilon_{k+1}| \approx \left| \frac{f''(r)}{2f'(r)} \right| |\epsilon_k| |\epsilon_{k-1}|.$$

- ② If in addition  $\lim_{k \rightarrow \infty} |\epsilon_{k+1}| / |\epsilon_k|^\alpha$  exists and is nonzero for some  $\alpha > 0$ , then

$$|\epsilon_{k+1}| \approx \left| \frac{f''(r)}{2f'(r)} \right|^{\alpha-1} |\epsilon_k|^\alpha, \quad \text{where } \alpha = \frac{1 + \sqrt{5}}{2}.$$

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<sup>1</sup>This exercise is from Lecture 22.

Hints Error analysis for Secant Method.

- $x_k = r + \epsilon_k$ ,  $(\epsilon_k \rightarrow 0 \text{ as } k \rightarrow \infty)$
- Taylor expansion
- big-O notation for simplification

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \approx f'(x_k)$$

Recall: iter. form. for secant method

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})f(x_k)}{f(x_k) - f(x_{k-1})}$$

## Lec 29: Problem Solving Session 3

- MATLAB
  - interp1 (PL, cubic splines)
  - spline
- 2-D Splines

## Exercise with Piecewise Interpolation

- piecewise linear (simplest: "connect the dots")
  - \* hat functions
- piecewise cubic (most common; smooth outcome)
  - \* Hermite cubic interpolation
  - \* cubic spline.
    - . not-a-knot B.C. (MATLAB's default)
    - . clamped B.C. ("spline")

B-splines

# Derivation of Hermite Cubic Interpolation

## Cubic Spline by Hand

(Ignore)

- FNC 5.3.1

## Cubic Spline with Periodic Boundary Conditions

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- FNC 5.3.7
- Also see the appendix to Lecture 26 slides for various other boundary conditions.

## Error Analysis

( 4<sup>th</sup>-order accuracy)

log-log graph

- FNC 5.3.4