# Homework 10 (Solution)

Math 3607

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Several problems in this homework set involves lengthy by-hand calculations. These solutions are written in a separate document. This mlx file contains solutions for computer exercises.

## Problem 1 (Higher-Order Forward Difference)

See the attached document.

## Problem 2 (LM 14.1--12)

### (a) Extrapolation for 4th-order centerend difference

The fourth-order centered difference formula is given by

$$D_h^{[4c]}\{f\}(x) = \frac{f(x-2h) - 8f(x-h) + 8(x+h) - f(x+2h)}{12h}.$$

See the attached document for derivation. We modify the textbook script `diff1` by implementing this formula as below:

```
%% script m-file: diff1 (modified)
f = @(x) sin(x.^2);
fdrv = @(x) 2*x.*cos(x.^2);
Df = @(x,h) (f(x+h) - f(x))./h;           % 1st-order FD
Dfc = @(x,h) (f(x+h) - f(x-h))./(2*h);    % 2nd-order CD
Df4c = @(x,h) ...                          % 4th-order CD
    (f(x-2*h) - 8*f(x-h) + 8*f(x+h) - f(x+2*h))./(12*h);
x = 1/3;
h0 = 0.1;
nr_h = 35;
h = h0*2.^(-[0:nr_h]');
A(:,1) = h;
A(:,2) = Df(x,h) - fdrv(x);
A(:,3) = Dfc(x,h) - fdrv(x);
```

```
A(:,4) = Df4c(x,h) - fdrv(x);
disp('      h          errors')
```

```
      h          errors
```

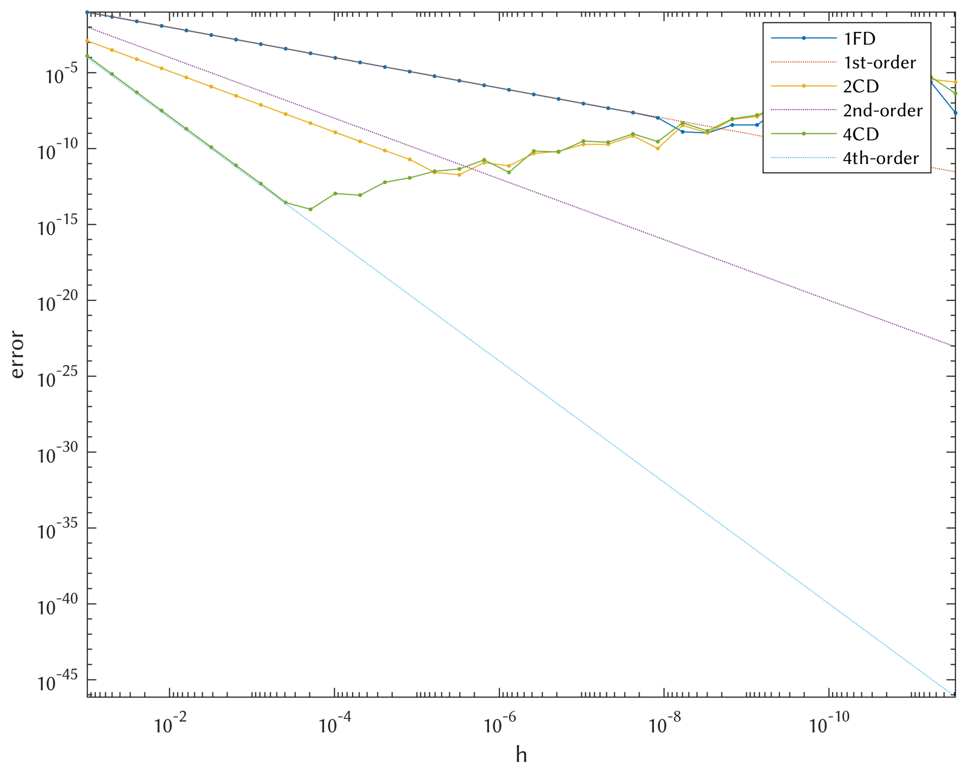
```
disp(A)
```

```
                     0.1        0.0953800307503522       -0.0012624336679421      0.000129128211100116
                    0.05        0.0481156495713236      -0.00030952874788992      8.10622546054685e-06
                   0.025        0.0241485170712513      -7.70018004009332e-05     5.07182095432768e-07
                  0.0125        0.0120951469886871      -1.9226669620287e-05      3.17073064470819e-08
                 0.00625       0.00605258336365655      -4.80518102807803e-06     1.98183547350794e-09
                0.003125       0.00302751828247738      -1.2012023592467e-06      1.23864696277565e-10
               0.0015625          0.00151406259914      -3.00294775157361e-07     7.75568498312396e-12
              0.00078125       0.00075710676755103      -7.50733292198547e-08     4.85944617878431e-13
             0.000390625      0.000378572201298888      -1.87683507624214e-08    -2.76445533131664e-14
            0.0001953125      0.000189290798871533      -4.6920906049408e-09     -9.88098491916389e-15
            9.765625e-05      9.46465732593049e-05      -1.17295007040497e-09     1.08468789505878e-13
           4.8828125e-05      4.73235799486327e-05      -2.9315605498681e-10       8.4821039081362e-14
          2.44140625e-05      2.36618628118856e-05      -7.37404581840906e-11    -6.02073946254222e-13
         1.220703125e-05      1.18309488211787e-05      -1.91707760777149e-11     -1.1705081348623e-12
         6.103515625e-06      5.91548045714152e-06      -2.68618460808057e-12     3.18756132600129e-12
        3.0517578125e-06      2.95774229608359e-06       1.86128890078407e-12      4.5138337512185e-12
       1.52587890625e-06      1.47885843626572e-06      -1.17811316258098e-11    -1.78445036524977e-11
       7.62939453125e-07      7.39430148777309e-07      -7.23365811694521e-12    -2.68629563038303e-12
      3.814697265625e-07      3.69756932294685e-07       4.73360239894305e-11     6.85574930159305e-11
     1.9073486328125e-07      1.84911229106355e-07       6.55259180248891e-11     5.94625459982012e-11
     9.5367431640625e-08       9.2215529101658e-08      -1.89132598471531e-10    -3.10398706737658e-10
    4.76837158203125e-08       4.57949195231677e-08      -1.89132598471531e-10    -2.61892285635668e-10
    2.38418579101563e-08      2.33849700714828e-08        6.8398231523048e-10     9.26514087673525e-10
    1.19209289550781e-08      1.05792846705199e-08       1.01905706095806e-10    -2.86145440675512e-10
    5.96046447753906e-09      1.26605892436515e-09      -3.39055394871224e-09     -4.9427583137529e-09
    2.98023223876953e-09     -1.06224751217354e-09      -1.06224751217354e-09    -1.45029865894486e-09
    1.49011611938477e-09      3.59436536090385e-09       8.25097823398124e-09     9.02708030547927e-09
    7.45058059692383e-10      3.59436536090385e-09       1.29075911070586e-08     1.60119996150954e-08
    3.72529029846191e-10     -5.22849891160249e-08      -5.22849891160249e-08    -6.78070321002977e-08
    1.86264514923096e-10     -5.22849891160249e-08      -5.22849891160249e-08    -5.22849892270472e-08
    9.31322574615479e-11        1.7123242879169e-07      2.45738234760928e-07     3.57496943603763e-07
    4.65661287307739e-11      2.22208168532134e-08        1.7123242879169e-07     2.45738234649906e-07
     2.3283064365387e-11     -5.73825630900693e-07      -5.73825630900693e-07    -6.73166705600359e-07
    1.16415321826935e-11      -1.1698720786546e-06       -1.1698720786546e-06    -1.26921315335426e-06
    5.82076609134674e-12      2.40640660786884e-06       3.59849950337665e-06     5.38663884652735e-06
    2.91038304567337e-12      2.22208168532134e-08       2.40640660786884e-06     4.19585115207788e-07
```

Confirm the accuracy on the log-log graph below.

```
clf
% 1FD
loglog(A(:,1), abs(A(:,2)), '.-'), hold on
loglog(A(:,1), A(:,1), ':')
% 2CD
loglog(A(:,1), abs(A(:,3)), '.-')
loglog(A(:,1), A(:,1).^2, ':')
% 4CD
loglog(A(:,1), abs(A(:,4)), '.-')
loglog(A(:,1), A(:,1).^4, ':')
% Prettifying
xlabel('h'),ylabel('error')
axis tight
legend('1FD', '1st-order', '2CD', '2nd-order', '4CD', '4th-order', 'Location', 'best')
```

```
set(gca, 'xdir', 'Reverse')
```




**Question.** Can you confirm from the previous graph the optimal $h$ for each method?

## (b) Second Derivatives

```
clear A
```

The second-order centered difference method for $f''(x)$ is given by

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

By Richardson extrapolation, we obtained the fourth-order centered difference formula for $f''(x)$

$$f''(x) \approx \frac{-f(x-2h) + 16f(x-h) - 6f(x) + 16f(x+h) - f(x+2h)}{12h^2}.$$

```
%% script m-file: diff1 (modified)
f = @(x) sin(x.^2);
f2drv = @(x) 2*cos(x.^2) - 4*x.^2.*sin(x.^2);
D2fc = @(x,h) (f(x+h) - 2*f(x) + f(x-h))./(h.^2);    % 2nd-order CD
D2f4c = @(x,h) ...                                    % 4th-order CD
    (-f(x-2*h) + 16*f(x-h) - 30*f(x) + 16*f(x+h) - f(x+2*h))./(12*h.^2);
x = 1/3;
h0 = 0.1;
```

```
nr_h = 20;
h = h0*2.^(-[0:nr_h]');
A(:,1) = h;
A(:,2) = D2fc(x,h) - f2drv(x);
A(:,3) = D2f4c(x,h) - f2drv(x);
disp('      h          errors')
```

```
      h          errors
```

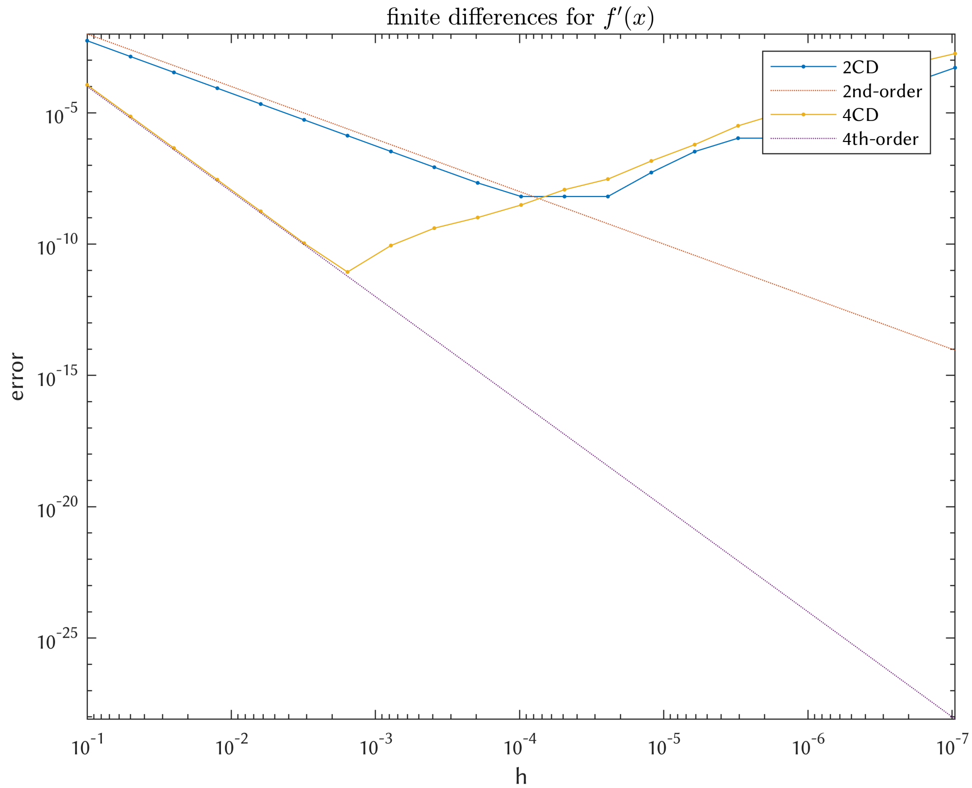
```
disp(A)
```

```
                       0.1     -0.00553656048525486     0.000114478122608119
                      0.05     -0.00137871608259621      7.2320516149027e-06
                     0.025     -0.00034433911895837     4.53202243289041e-07
                    0.0125    -8.60635219601669e-05     2.83435579451208e-08
                   0.00625    -2.15145520576776e-05     1.77094716669046e-09
                  0.003125    -5.37855569637813e-06     1.07981179553462e-10
                 0.0015625    -1.34463973955334e-06    -8.66640093022397e-12
                0.00078125    -3.36197698569407e-07    -8.82471873353552e-11
               0.000390625    -8.42449288107616e-08     -4.0467540429745e-10
              0.0001953125    -2.13988449182523e-08    -1.02616359853869e-09
              9.765625e-05    -6.48313180917626e-09    -3.08768477452759e-09
             4.8828125e-05    -6.48313180917626e-09    -1.18188339115477e-08
            2.44140625e-05    -6.48313180917626e-09    -2.97661961745632e-08
           1.220703125e-05    -5.30492605399502e-08    -1.46181518001498e-07
           6.103515625e-06    -3.32446032924594e-07    -6.11842805309237e-07
          3.0517578125e-06    -1.07750409261698e-06    -3.18850192826403e-06
         1.52587890625e-06    -1.07750409261698e-06    -1.00182008089256e-05
         7.62939453125e-07     4.88296038492209e-06    -2.09457190176732e-05
        3.814697265625e-07    -4.28007554353904e-05     -9.8431757225681e-05
       1.9073486328125e-07    -0.000138168187076015      -0.0007421619207999
       9.5367431640625e-08    -0.000519637913638515     -0.00179120366884677
```

Plot the errors on the log-log graph to confirm the order of accuracy:

```
clf
loglog(A(:,1), abs(A(:,2)), '.-'),hold on
loglog(A(:,1), A(:,1).^2, ':')

loglog(A(:,1), abs(A(:,3)), '.-')
loglog(A(:,1), A(:,1).^4, ':')
xlabel('h'),ylabel('error')
title('finite differences for $f''(x)$', 'Interpreter', 'latex')
axis tight
legend('2CD', '2nd-order', '4CD', '4th-order', 'Location', 'best')
set(gca, 'xdir', 'Reverse')
```




## Problem 3 (LM 14.1--17, Sequences Converging to $\pi$ )

Begin by defining anonymous functions

```
p = @(n) n.*sin(pi./n);       % 2nd-order; underestimate
P = @(n) n.*tan(pi./n);       % 2nd-order; overestimate
B = @(n) (p(n) + P(n))/2;     % 2nd-order; better than p(n) and P(n)
R = @(n) (2*p(n) + P(n))/3;   % 4th-order; obtained from extrapolation
```

The question asks that we calculate the sequences for $n = 48$ and $n = 96$; I will do some more. Below is a quick and dirty way to calculate them.

```
n = 48*2.^(0:9)';
calc = [p(n), P(n), B(n), R(n)];
disp([n, calc])
```

```
          48        3.13935020304687        3.14608621513143        3.14271820908915
          96        3.14103195089051        3.14271459964537        3.14187327526794
         192        3.14145247228546        3.14187304997982        3.14166276113264
         384        3.14155760791186        3.14166274705685        3.14161017748435
         768        3.14158389214832        3.14161017660469         3.1415970343765
        1536        3.14159046322805        3.14159703432153        3.14159374877479
        3072        3.14159210599927        3.14159374877135        3.14159292738531
        6144        3.14159251669216         3.1415929273851        3.14159272203863
       12288        3.14159261936538        3.14159272203861          3.141592670702
       24576        3.14159264503369          3.141592670702        3.14159265786784
```

The errors:

```
disp([n, calc-pi])
```

```
   48     -0.00224245054292638     0.00449356154164171     0.00112555549935767
   96     -0.000560702699283766    0.00112194605557514     0.000280621678145465
  192     -0.000140181304331577    0.000280396390030191    7.0107542849307e-05
  384     -3.50456779356634e-05    7.00934670549991e-05    1.75238945594458e-05
  768     -8.76144147543556e-06    1.75230148959926e-05    4.38078671027853e-06
 1536     -2.19036174353704e-06    4.38073173247844e-06    1.09518499424865e-06
 3072     -5.47590521815522e-07    1.09518155877453e-06    2.73795518701547e-07
 6144     -1.36897636338063e-07    2.7379530376237e-07     6.84488337121536e-08
12288     -3.42244095286048e-08    6.84488203894773e-08    1.71122054304362e-08
24576     -8.55610249317351e-09    1.7112204986347e-08     4.27805124658676e-09
```

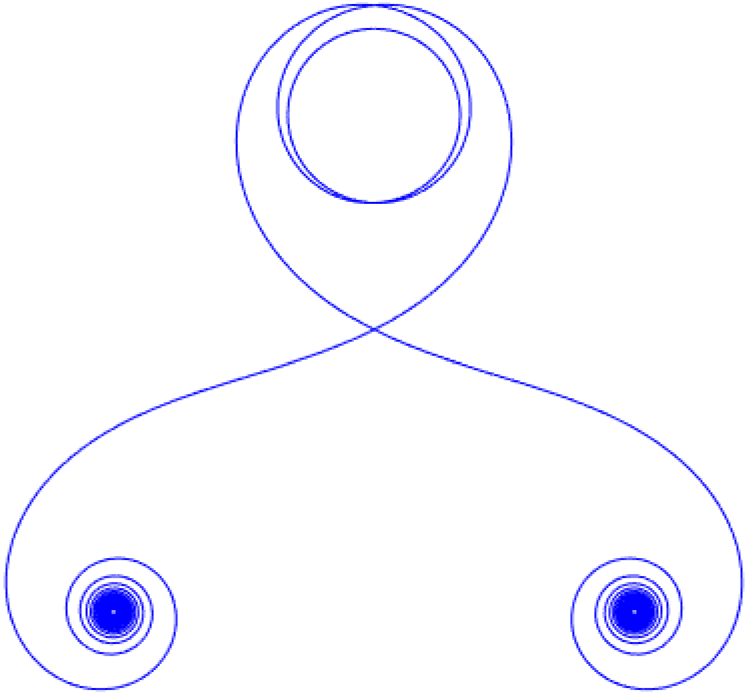
If you want to have a finer control over how the numbers are formatted, use `fprintf` as demonstrated many times in other homework solutions or in class.

## Problem 4 (LM 14.2--3(b), Spiral)

Simply modify the integrands in the Euler spiral code provided for Lecture 31 and 32.

```
T = 15;
h = 0.001;
t = (0:h:T);
tmid = ( t(1:end-1)+t(2:end) )/2;
fx = @(z) cos(0.25*z.^3-5.2*z);
fy = @(z) sin(0.25*z.^3-5.2*z);
Ix = h/6 * ( fx(t(1:end-1)) + 4*fx(tmid) + fx(t(2:end)) );
Iy = h/6 * ( fy(t(1:end-1)) + 4*fy(tmid) + fy(t(2:end)) );
x = cumsum(Ix);
y = cumsum(Iy);

clf
plot(x, y, 'b', -x, y, 'b')
axis equal, axis off
grid on
```



## Problem 5 (LM 14.2--6, Smoothness and Accuracy)

To be updated.

## Problem 6 (LM 14.2--11(a), Extrapolation for Composite Methods)

See the attached document.