Lec 28: Problem Solving Session

Rootfinding

FZERO to Solve Complex Problem

- **FNC** 4.1.5 (Kepler's Law)

4.1.5. The most easily observed properties of the orbit of a celestial body around the sun are the period τ and the elliptical eccentricity ϵ. (A circle has $\epsilon = 0$.) From these it is possible to find at any time t the angle $\theta(t)$ made between the body's position and the major axis of the ellipse. This is done through

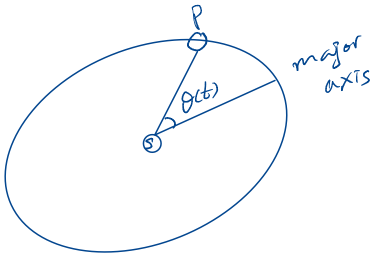
$$\tan\frac{\theta}{2} = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\,\tan\frac{\psi}{2}, \tag{4.1.2}$$

where the eccentric anomaly ψ satisfies Kepler's equation:

$$\psi - \epsilon\sin\psi - \frac{2\pi t}{\tau} = 0. \tag{4.1.3}$$

"fzero"

Equation (4.1.3) must be solved numerically to find $\psi(t)$, and then (4.1.2) can be solved analytically to find $\theta(t)$.
The asteroid Eros has $\tau = 1.7610$ years and $\epsilon = 0.2230$. Using `fzero` for (4.1.3), make a plot of $\theta(t)$ for 100 values of t between 0 and τ (one full period).



ψ : psi

θ : theta

initial guess

$$\theta(t) = 2\arctan\left[\sqrt{\frac{1+\epsilon}{1-\epsilon}}\,\tan\frac{\psi(t)}{2}\right]$$

fzero(@(psi) ... , 0)

Lambert W-Function

- **FNC** 4.1.6 → Review Prob 2 of HW 7.

Same idea and technique

More With Lambert W-Function

$y = W(x)$ iff $x = ye^y$

Question. Show by hand that solutions of the equation $2^x = 5x$

$$r = -\frac{W\left(-\log(2)/5\right)}{\log 2}.$$

(Here, as usual in this class, $\log(\,\cdot\,) = \ln(\,\cdot\,)$ is the natural logarithmic function.)
Then numerically verify the result using `fzero`[1]

$$2^x = 5x \longrightarrow e^{x\log 2} = 5x$$

$$\cancel{x \log x = \log 5 + \log x}$$

$$\frac{1}{5} = x\, e^{-x\log 2}$$

$$-\frac{\log 2}{5} = -x\log 2\; e^{-x\log 2}$$

$$-x\log 2 = W\left(-\frac{\log 2}{5}\right)$$

$$x = -\frac{W\left(-\frac{\log 2}{5}\right)}{\log 2} \quad \checkmark$$

[1] Two real-valued solutions, $r_1 \approx 0.2355$ and $r_2 \approx 4.488$.

FPI: When Convergence Is Faster Than Expected

- **FNC** 4.2.6

<u>Fixed point problem</u>: Given a function g, find x satisfying

$$x = g(x).$$

<u>Solution strategy</u> (Iteration)

$$\begin{cases} x_0 : \text{ initial guess} \\ x_{n+1} = g(x_n) \quad \text{iteration formula.} \end{cases}$$

<u>Key note</u>

If $|g'(r)| < 1$, the convergence is <u>linear</u>.

$\Rightarrow x_0, x_1, x_2, \cdots$

If $\lim_{k\to\infty} x_k = r$, then $g(r) = r$, i.e. r is a <u>fixed point</u> of g.

(a) $g(x) = 2x - 3x^2$.

WTS : $r = 1/3$ is a f-p.

i.e., NTS : $1/3 = g(1/3)$

Soln: $g(1/3) = 2\cdot\frac{1}{3} - 3\left(\frac{1}{3}\right)^2 = \left(2 - \frac{3}{3}\right)\frac{1}{3} = \frac{1}{3}$ ✓

(b) $g'(1/3) = ?$

$g'(x) = 2 - 6x \Rightarrow g'(1/3) = 2 - 6\cdot\frac{1}{3} = \boxed{0}$

Since $|g'(1/3)| = 0$, the convergence of FPI near $1/3$ is <u>superlinear</u>!