Lec 25: Piecewise Interpolation – Piecewise Linear

Interpolation Problem

Problem Statement

General Interpolation Problem

Given a set of n data points $\{(x_j, y_j) \mid j \in \mathbb{N}[1, n]\}$ with $x_1 < x_2 < \ldots < x_n$, find a function $p(x)$, called the **interpolant**, such that

$$p(x_j) = y_j, \quad \text{for } j = 1, 2, \ldots, n\,.$$

The ordered pair (x_j, y_j) is called the **data point**.

- x_j is called the **abscissa** or the **node**.
- y_j is called the **ordinate**.

Polynomials

One approach is to find an interpolating *polynomial* of degree (at most) $n-1$,

$$p(x) = c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}.$$

- The unknown coefficients $c_1, \ldots, c_n$ are determined by solving the square linear system $V\mathbf{c} = \mathbf{y}$ where

$$V = \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-2} & x_2^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} & x_n^{n-1} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

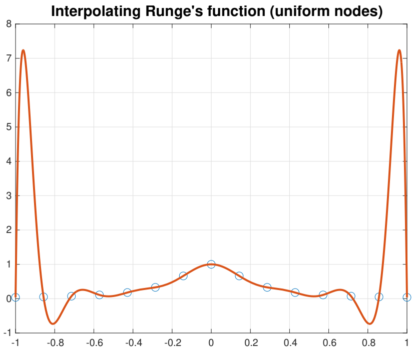
  the matrix V is called the **Vandermonde matrix**; see Lecture 13.

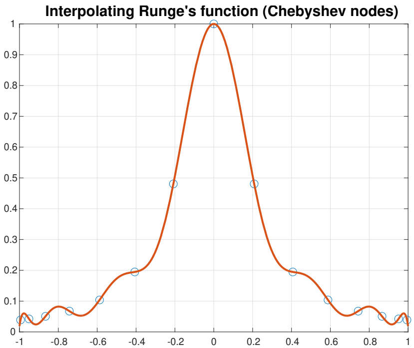
- A polynomial interpolant has severe oscillations as n grows large, unless nodes are special; see illustration in the next slide.

Illustration Runge’s Phenomenon

Polynomial Interpolation of 15 data points collected from the same function

$$f(x) = \frac{1}{1 + 25x^2}.$$



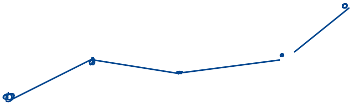


Piecewise Polynomials

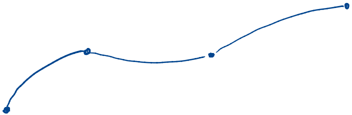
To handle real-life data sets with large n and unrestricted node distribution:

- An alternate approach is to use a low-degree polynomial between each pair of data points; it is called the **piecewise polynomial interpolation**.
- The simplest case is **piecewise linear interpolation** (degree 1) in which the interpolant is linear between each pair of consecutive nodes.
- The most commonly used method is **cubic spline interplation** (degree 3).
- The endpoints of the low-degree polynomials are called **breakpoints** or **knots**.
- The breakpoints and the data points almost always coincide.

nodes



piecewise linear



piecewise cubic

MATLAB Function: `INTERP1`

In MATLAB, piecewise polynomials are constructed using `interp1` function. Suppose the x and y data are stored in vectors `xdp` and `ydp`. To evaluate the piecewise interpolant at `x` (an array):

- By default, it finds a piecewise linear interpolant.

```
y = interp1(xdp, ydp, x);
```

evaluation pts.

- To obtain a smoother interpolant that is piecewise cubic, use `'spline'` option.

```
y = interp1(xdp, ydp, x, 'spline');
```

Demonstration: Piecewise Polynomial Interplation

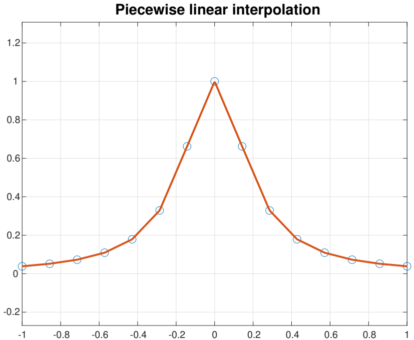
To interpolate data obtained from [1]

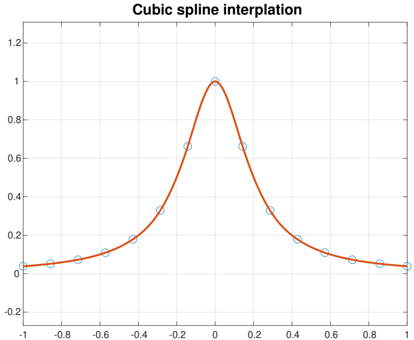
$$f(x) = \frac{1}{1+25x^2}.$$

```
% Generate data and eval pts
n = 15;
xdp = linspace(-1,1,n)';
ydp = 1./(1+25*xdp.^2);
x = linspace(-1,1,400)';

% PL
plot(xdp,ydp,'o'), hold on
plot(x, interp1(xdp,ydp,x))

% Cubic spline
plot(xdp,ydp,'o'), hold on
plot(x, interp1(xdp,ydp,x,'spline'));
```

[1] This function is often called the Runge's function.

Demonstration: Piecewise Polynomial Interplation (cont')





Conditioning

of general interpolation

$a = x_1 < x_2 < x_3 < \cdots < x_n = b$

Set-up for analysis. (both for polynom. and p.polynom.)

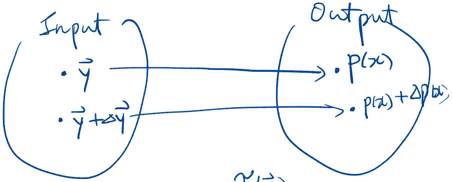
- Let (x_j, y_j) for $j = 1, \ldots, n$ denote the data points. Assume that the nodes x_j are fixed and let $a = x_1$, $b = x_n$.
- View the interpolation problem as a mathematical function $\mathcal{I}$ with
  - Input: a vector $\mathbf{y}$ (ordinates, or y-data ponits)
  - Output: a function $p(x)$ such that $p(x_j) = y_j$ for all j.

  $\mathcal{I} : \vec{y} \longmapsto p(x)$
  (y-data) (interpolant)

  (That is, $\mathcal{I}$ is a *black box* that produces the interpolant from a data vector.)
- For the interpolation methods under consideration (polynomial or piecewise polynomial), $\mathcal{I}$ is *linear*:

$$\mathcal{I}(\alpha\mathbf{y} + \beta\mathbf{z}) = \alpha\mathcal{I}(\mathbf{y}) + \beta\mathcal{I}(\mathbf{z}),$$

for all vectors $\mathbf{y}, \mathbf{z}$ and scalars α, β.



$p = \mathcal{I}(\vec{y})$

Theorem 1 (Conditioning of General Interpolation)

Suppose that $\mathcal{I}$ is a linear interpolation method. Then the absolute condition number of $\mathcal{I}$ satisfies

$$\max_{1\leqslant j\leqslant n}\left\|\mathcal{I}(\mathbf{e}_j)\right\|_\infty \leqslant \kappa(\mathbf{y}) \leqslant \sum_{j=1}^{n}\left\|\mathcal{I}(\mathbf{e}_j)\right\|_\infty ,$$

where all vectors and functions are measured in the infinity norm.

- $\vec{e}_j$ is the j^{th} unit basis vector in $\mathbb{R}^n$,
  i.e., it is the j^{th} column of $n\times n$ identity matrix
- $$\kappa(\vec{y}) = \max_{\Delta\vec{y}} \frac{\| \mathcal{I}(\vec{y}+\Delta\vec{y}) - \mathcal{I}(\vec{y})\|_\infty}{\|\Delta\vec{y}\|_\infty}$$
  → functional ∞-norm
  → vector ∞-norm

Conditioning: Notes

- The functional infinity norm is defined by

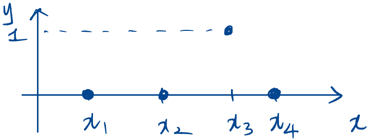
$$\boxed{\|f\|_\infty = \max_{x\in[a,b]} \left|f(x)\right|,}$$

cf. $\|\vec{v}\|_\infty = \max_j |v_j|$

in a manner similar to vector infinity norm.

- The expression $\mathcal{I}(\mathbf{e}_j)$ represents the interpolant $p(x)$ which is *on* at x_j and $\underline{\vec{y} = \vec{e}_3}$ *off* elsewhere, *i.e.*, y-values

$$\boxed{p(x_k) = \delta_{k,j} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}.}$$



Such interpolants are known as **cardinal functions**.

- The theorem says that the (absolute) condition number is larger than the largest of $\left\|\mathcal{I}(\mathbf{e}_j)\right\|_\infty$, but smaller than the sum of these.

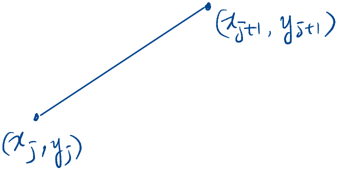
Piecewise Linear Interpolation

Piecewise Linear Interpolation

Assume that $x_1 < x_2 < \cdots < x_n$ are fixed. The function $p(x)$ defined piecewise[2] by

$$\boxed{p(x) = y_j + \frac{y_{j+1} - y_j}{x_{j+1} - x_j}(x - x_j)\,, \quad \text{for } x \in [x_j, x_{j+1}],\, 1 \leqslant j \leqslant n-1}$$

- is linear on each interval $[x_j, x_{j+1}]$; is continuous on $[x_1, x_n]$
- connects any two consecutive data points (x_j, y_j) and (x_{j+1}, y_{j+1}) by a straight line.



$$\left\{ \begin{array}{l} \text{slope} = \dfrac{\text{rise}}{\text{run}} = \dfrac{y_{j+1} - y_j}{x_{j+1} - x_j} \\ \text{point} : (x_j, y_j) \end{array} \right.$$

$\rightarrow$ point-slope formula

[2] Note the formula changes depending on which interval x lies in.

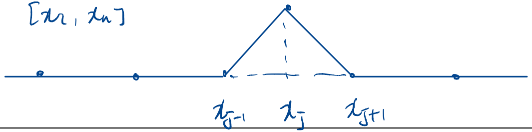
Hat Functions

Denote by $H_j(x)$ the jth *piecewise linear* cardinal function:

$$H_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j], \\ \dfrac{x_{j+1} - x}{x_{j+1} - x_j}, & x \in [x_j, x_{j+1}], \\ 0, & \text{otherwise}, \end{cases} \qquad j = 1, 2, \ldots, n.$$

- The functions $H_1, \ldots, H_n$ are called **hat functions** or **tent functions**.
- Each H_j is globally continuous and is linear inside each interval $[x_j, x_{j+1}]$

$[x_1, x_n]$



Note: The definitions of $H_1(x)$ and $H_n(x)$ require additional nodes x_0 and x_{n+1} for x outside of $[x_1, x_n]$, which is not relevant in the discussion of interpolation.

Hat Functions (cont')