Practice Problems on EVD and SVD — Math 3607

Tae Eun Kim Autumn 2020

Below are some practice problems covering eigenvalue and singular value decompositions.

Problem 1. (Checking Understanding – EVD and SVD)

(**True/False**) Circle T if the statement is ALWAYS true; circle F otherwise.

(a) (T / F) Given a square matrix $A \in \mathbb{R}^{n \times n}$, we can always find an orthogonal matrix $V \in \mathbb{R}^{n \times n}$ and a diagonal matrix $D \in \mathbb{R}^{n \times n}$ such that $AV = VD$.

> **False.** For $AV = VD$ with a diagonal D, the columns of V must be eigenvectors of A; see p. 4 of Lecture 31. In general, a matrix A need not have an orthonormal set of eigenvectors.

(b) (T / F) If $A \in \mathbb{R}^{5 \times 5}$ has 5 distinct eigenvalues, then A has an EVD.

> **True.** By Theorem 1 of Lecture 31, A is diagonalizable, *i.e.,* A has an EVD.

(c) (T / F) If $A \in \mathbb{R}^{5 \times 5}$ has 3 distinct eigenvalues, then A does not have an EVD.

> **False.** As long as A has 5 linearly independent eigenvectors, it has an EVD.

(d) (T / F) A square matrix $A \in \mathbb{R}^{m \times m}$ with $\det(A) = 0$ does not have an SVD.

> **False.** Any matrix has an SVD; in particular, any square matrix has an SVD regardless of invertibility.

(e) (T / F) A rank deficient matrix $A \in \mathbb{R}^{m \times n}$ has an SVD.

> **True.** Any matrix has an SVD; in particular, any rectangular matrix has an SVD regardless of its rank.

(f) (T / F) Let $A \in \mathbb{R}^{m \times n}$. Then $B = AA^{\mathrm{T}} \in \mathbb{R}^{m \times m}$ is a diagonalizable matrix.

> **True.** B is a symmetric matrix and so it has an EVD; see p. 9 of Lecture 33.

Let $A \in \mathbb{R}^{n\times n}$ has an EVD $A = VDV^{-1}$ and suppose that all its eigenvalues are either positive or negative ones. Show that $A^2 = I$.

Note. To gain a geometric intuition about this problem, think about the eigenvalue decomposition of a Householder reflector $H = I - 2\mathbf{u}\mathbf{u}^{\mathrm{T}}$.

Solution: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of A, so that the matrix D, up to re-ordering, can be written as

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Since $A = VDV^{-1}$,

$$A^2 = VD\left(V^{-1}V\right)DV^{-1} = VD^2V^{-1}.$$

Note that

$$D^2 = \begin{bmatrix} \lambda_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix} = I,$$

because all eigenvalues are assumed to be either $+1$ or -1. It follows that $A^2 = VIV^{-1} = VV^{-1} = I$, as desired.

Calculate the singular values of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

by solving a 2×2 eigenvalue problem.

Solution: Recall that the nonzero singular values of A are the square roots of the nonzero eigenvalues of $A^{\mathrm{T}}A$; see Theorem 3 of Lecture 33. First, compute $A^{\mathrm{T}}A$:

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then find its eigenvalues:

$$\left\{ \begin{aligned} \det\left(\lambda I - A^{\mathrm{T}}A\right) &= \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} \\ &= (\lambda - 2)^2 - 1 \\ &= \lambda^2 - 4\lambda + 3. \end{aligned} \right\} \implies \lambda_1 = 3,\ \lambda_2 = 1.$$

(Note that all eigenvalues are nonnegative; I order them in descending order so that the singular values are ordered properly.)
We conclude that the two singular values of A are

$$\begin{aligned} \sigma_1 &= \sqrt{\lambda_1} = \sqrt{3}, \\ \sigma_2 &= \sqrt{\lambda_2} = 1. \end{aligned}$$

Problem 4. (SVD and the 2-Norm)

Let $A \in \mathbb{R}^{n\times n}$. Show that

(a) A and A^{T} have the same singular values.

(b) $\|A\|_2 = \|A^{\mathrm{T}}\|_2$.

Solution:

(a) Suppose that $A = U\Sigma V^{\mathrm{T}}$ is an SVD of A. Then

$$A^{\mathrm{T}} = \left(U\Sigma V^{\mathrm{T}}\right)^{\mathrm{T}} = V\Sigma^{\mathrm{T}}U^{\mathrm{T}} = V\Sigma U^{\mathrm{T}}.$$

Note that $\Sigma^{\mathrm{T}} = \Sigma$ since it is an $(n \times n)$ diagonal matrix. Since U and V are orthogonal matrices, the last factorization is an SVD of A^{T}. In particular, the singular values of A^{T} are the diagonal entries of Σ which are also the singular values of A.

(b) From the previous part, we know that both matrices share the same set of singular values. Since the 2-norm of a matrix is its largest singular value, it follows that $\|A\|_2 = \|A^{\mathrm{T}}\|_2$.

Problem 5. (Rayleigh Quotient)

Let

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix}.$$

and define a function $R_A : \mathbb{R}^2 \to \mathbb{R}$ by

$$R_A(\mathbf{x}) = \frac{\mathbf{x}^{\mathrm{T}} A\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}.$$

(a) Write out $R_A(\mathbf{x})$ explicitly as a function of x_1 and x_2.

(b) Find $R_A(\mathbf{x})$ for $x_1 = 1$, $x_2 = 2$.

(c) Confirm that $\mathbf{x} = (1, 2)^{\mathrm{T}}$ is an eigenvector of A, whose corresponding eigenvalue is equal to the value computed in part (b).

Solution:

(a) Let $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$. Then

$$\begin{aligned} \mathbf{x}^{\mathrm{T}} A\mathbf{x} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^{\mathrm{T}}\mathbf{x} = x_1^2 + x_2^2. \\ &= 3x_1^2 - x_1 x_2, \end{aligned}$$

Thus,

$$R_A(\mathbf{x}) = \frac{3x_1^2 - 4x_1 x_2}{x_1^2 + x_2^2}.$$

(b) Let $\mathbf{x} = (1, 2)^{\mathrm{T}}$. Then by the expression found above,

$$R_A(\mathbf{x}) = \frac{3(1)^2 - 4(1)(2)}{1^2 + 2^2} = -1.$$

(c) It follows immediately from the following calculation

$$A\mathbf{x} = \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\mathbf{x},$$

that $\lambda = -1$ is an eigenvalue of A and $\mathbf{x} = (1, 2)^{\mathrm{T}}$ is a corresponding eigenvector of A.

Note. The map R_A constructed above is known as the *Rayleigh quotient*. As confirmed in part (c), this map is known to send an eigenvector of A to its associated eigenvalue. Below are some more exercise problems related to this map.

1. Find the gradient vector $\nabla R_A(\mathbf{x})$.
2. Show that the gradient vector is zero when $x_1 = 1$, $x_2 = 2$.