Math 3607: Week 14

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Spring 2020

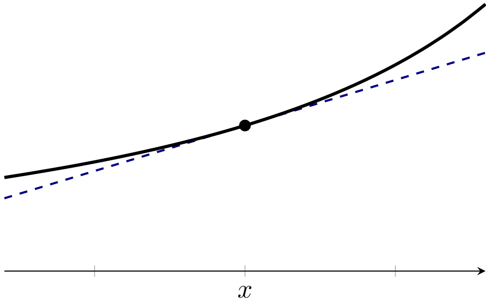
Overview

1. 14.1 Numerical Differentiation

2. 14.2 Numerical Integration

14.1 Numerical Differentiation

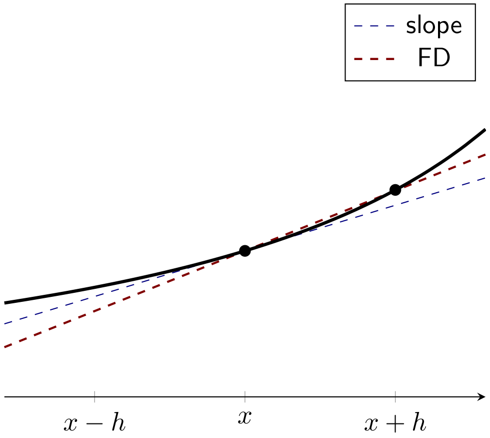
Introduction



Let f be a smooth function. Analytically, the derivative is calculated by

$$D\{f\}(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Introduction

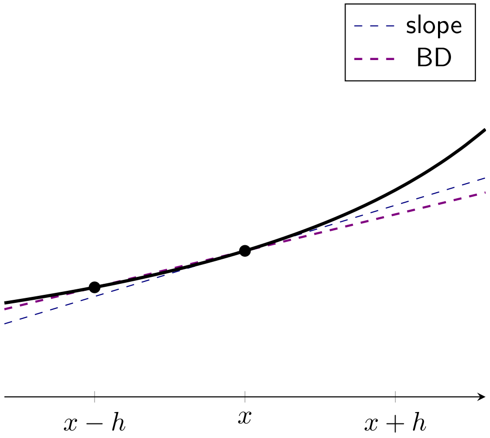


Numerically, $f'(x)$ can be approximately calculated by:

- Forward difference

$$D_h^{[1f]}\{f\}(x) = \frac{f(x+h) - f(x)}{h}$$

Introduction



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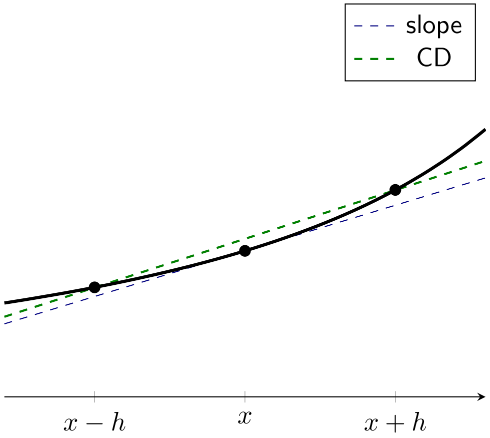
- Backward difference

$$D^{[1b]}\{f\}(x) = \frac{f(x)-f(x-h)}{h}$$

Introduction



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- Forward difference

$$D_h^{[1f]}\{f\}(x) = \frac{f(x+h) - f(x)}{h}$$

- Backward difference

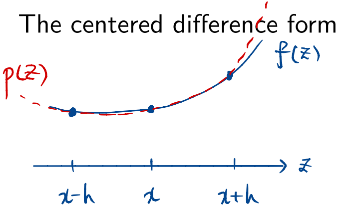
$$D^{[1b]}\{f\}(x) = \frac{f(x) - f(x-h)}{h}$$

- Centered difference

$$D_h^{[2c]}\{f\}(x) = \frac{f(x+h) - f(x-h)}{2h}$$

Interpolation and Centered Difference Formula

The centered difference formula can be understood from the viewpoint of 3-point interpolation.



Find the polynomial $p(z)$ satisfying

$$\begin{cases} p(x-h) = f(x-h) = y_1 \\ p(x) = f(x) = y_2 \\ p(x+h) = f(x+h) = y_3 \end{cases}$$

and then use $p'(x)$ to approximate $f'(x)$

Lagrange interp. form.

$$p(z) = y_1 \underbrace{\frac{(z-x)(z-x-h)}{2h^2}}_{\ell_1} - y_2 \underbrace{\frac{(z-x+h)(z-x-h)}{h^2}}_{\ell_2} + y_3 \underbrace{\frac{(z-x+h)(z-x)}{2h^2}}_{\ell_3}$$

$$p'(z) = \frac{y_1}{2h^2}\left[z-x + z-x-h\right] - \frac{y_2}{h^2}\left[z-x+h + z-x-h\right] + \frac{y_3}{2h^2}\left[z-x+h + z-x\right]$$

$$\therefore\ p'(x) = \frac{y_1}{2h^{\cancel{2}}}(-\cancel{h}) + \frac{y_3}{2h^{\cancel{2}}}(\cancel{h}) = \frac{y_3 - y_1}{2h} \checkmark$$

Error Analysis – First-Order Difference Formulas

The formula $D_h^{[1f]}\{f\}$ is said to be a **first-order** method because …

"error"

$$e_h^{[1f]} = D_h^{[1f]}\{f\}(x) - f'(x)$$

$$= \frac{f(x+h) - f(x)}{h} - f'(x)$$

Taylor expand!

$$= \frac{1}{h}\left[\cancel{f(x)} + f'(x)h + \frac{f''(x)}{2}h^2 + \cdots - \cancel{f(x)}\right] - f'(x)$$

$$= \left[\cancel{f'(x)} + \frac{f''(x)}{2}h + \frac{f'''(x)}{6}h^2 + \cdots\right] \cancel{- f'(x)}$$

Therefore,

leading error term

$$e_h^{[1f]} = \frac{f''(x)}{2}h^{\boxed{1}} + O(h^2)$$

terms of order higher than 2

Error Analysis – Second-Order Difference Formulas

The formula $D_h^{[2c]}\{f\}$ is said to be a **second-order** method because ...

First, work out $f(x+h) - f(x-h)$:

$$
\begin{aligned}
f(x+h) &= \cancel{f(x)} + f'(x)h + \cancel{\frac{f''(x)}{2}h^2} + \frac{f'''(x)}{3!}h^3 + \cancel{\frac{f^{(4)}(x)}{4!}h^4} + \cdots \\
-)\quad f(x-h) &= \cancel{f(x)} - f'(x)h + \cancel{\frac{f''(x)}{2}h^2} - \frac{f'''(x)}{3!}h^3 + \cancel{\frac{f^{(4)}(x)}{4!}h^4} - \cdots
\end{aligned}
$$

$$f(x+h) - f(x-h) = 2f'(x)h + 2\frac{f'''(x)}{3!}h^3 + \cdots$$

So,

$$e_h^{[2c]} = D_h^{[2c]}\{f\}(x) - f'(x) = \frac{2f'(x)h + 2\frac{f'''(x)}{3!}h^3 + 2\frac{f^{(5)}(x)}{5!}h^5 + \cdots}{2h} - f'(x)$$

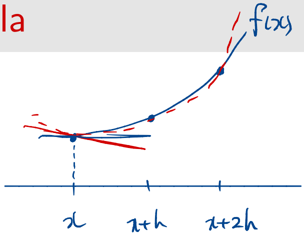
$$= \frac{2f'(x)h + \frac{2f'''(x)}{3!}h^3 + \frac{2f^{(5)}(x)}{5!}h^5 + \cdots}{2h} - f'(x)$$

$$= \left(\cancel{f'(x)} + \frac{f'''(x)\,h^2}{3!} + \frac{f^{(5)}(x)\,h^4}{5!} + \cdots \right) - \cancel{f'(x)}$$

$$= \frac{f'''(x)}{3!} h^2 + \underbrace{O(h^4)}$$

$\longrightarrow$ 4th-order and higher.

leading error term is 2nd-order!



Seeking $D_h^{[2f]}\{f\}(x) = a_1 f(x) + a_2 f(x+h) + a_3 f(x+2h)$.

Strategy: Find the polynomial $p(z)$ satisfying

$$p(x) = \underbrace{f(x)}_{y_1}, \quad p(x+h) = \underbrace{f(x+h)}_{y_2}, \quad p(x+2h) = \underbrace{f(x+2h)}_{y_3}.$$

(Lagrange) $$p(z) = y_1 \frac{(z-x-h)(z-x-2h)}{+2h^2} + y_2 \frac{(z-x)(z-x-2h)}{-h^2} + y_3 \frac{(z-x)(z-x-h)}{2h^2}$$

$$p'(z) = \frac{y_1}{2h^2}(2z-2x-3h) + \frac{y_2}{-h^2}(2z-2x-2h) + \frac{y_3}{2h^2}(2z-2x-h)$$

$$\therefore p'(x) = \frac{-3y_1}{2h} + \frac{2y_2}{h} - \frac{y_3}{2h} = \boxed{\frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} = D_h^{[2f]}\{f\}(x)}$$

Determining Optimal h

$*$ floating point evaluation of $f(x)$:

$$\underbrace{\left| \frac{\hat{f}(x) - f(x)}{f(x)} \right|}_{\text{relative error}} \le \frac{1}{2} \boxed{\text{eps}}$$

machine epsilon.

$*$

$$|\hat{f}(x)| = | f(x) + \hat{f}(x) - f(x)|$$

triangle ineq.

$$\le |f(x)| + |\hat{f}(x) - f(x)|$$

$$\le |f(x)| + \frac{1}{2}\boxed{\text{eps}}\, |f(x)|$$

$$\Rightarrow \hat{f}(x) = f(x)(1+\epsilon), \quad |\epsilon| \le \frac{1}{2}\boxed{\text{eps}}$$

Determine h minimizing

$$\left| \underbrace{\hat{D}_h^{[2c]}\{f\}(x)}_{} - f'(x) \right|.$$

numerical eval. of $D_h^{[2c]}\{f\}(x)$

Assume: only round-off errors occur in eval. $f(x \pm h)$

Determining Optimal h

Write:

$$\begin{cases} \hat{f}(x+h) = f(x+h)(1+\epsilon_+) \\ \hat{f}(x-h) = f(x-h)(1+\epsilon_-) \end{cases}$$

where $|\epsilon_\pm| \le \frac{1}{2}\boxed{\text{eps}}$

Then

$$\hat{D}_h^{[2c]}\{f\}(x) - f'(x)$$

$$= \frac{\hat{f}(x+h) - \hat{f}(x-h)}{2h} - f'(x)$$

Taylor @ x

$$= \frac{f(x+h)(1+\epsilon_+) - f(x-h)(1+\epsilon_-)}{2h} - f'(x)$$

$$= \underbrace{\left(\frac{f(x+h) - f(x-h)}{2h} - f'(x)\right)}_{\text{already worked out}} + \frac{f(x+h)\epsilon_+ - f(x-h)\epsilon_-}{2h}$$

$$= \frac{f'''(x)}{3!}h^2 + O(h^4) + \frac{f(x)(\epsilon_+ - \epsilon_-)}{2h} + O(\boxed{\text{eps}})$$

$$\therefore \left|\hat{D}_h^{[2c]}\{f\}(x) - f'(x)\right| \lesssim \frac{|f'''(x)|}{6}h^2 + \frac{|f(x)|}{2h}\boxed{\text{eps}}$$

ignoring higher-order terms

Determining Optimal h

Let $g(h) = \underbrace{\frac{|f'''(x)|}{6}}_{=\,\alpha} h^2 + \frac{|f(x)|}{2h}\boxed{\text{eps}}$, $\quad \frac{|f(x)|}{2} = \beta$

$\boxed{\text{eps}} \approx 10^{-16}$

$$= \alpha h^2 + \beta \frac{\boxed{\text{eps}}}{h}$$

Want to minimize $g(h)$:

$$g'(h) = 2\alpha h - \frac{\beta \boxed{\text{eps}}}{h^2} = 0$$

$$\Rightarrow \quad h^3 = \frac{\beta}{2\alpha}\boxed{\text{eps}} \quad \therefore h = \left(\frac{\beta}{2\alpha}\boxed{\text{eps}}\right)^{1/3} \approx 10^{-5},\ 10^{-6} \quad \checkmark$$

Richardson Extrapolation

Set-up

$$V_h \approx V$$

$\uparrow$ discretized numerical approx. $\qquad$ $\uparrow$ analytical value

i.e. $V_h = V + \underbrace{c_1 h^{p_1} + c_2 h^{p_2} + \cdots}_{\text{error}}$

$\Big\{ \quad 0 < p_1 < p_2 < p_3 \cdots$

Richardson extrapolation is a procedure to improve accuracy of V_h

Idea: Form a suitable linear comb. of V_h and V_{2h} (or $V_{h/2}$) that eliminates the leading error term.

Example: Take $V = f'(x)$, $V_h = D_h^{[1f]}\{f\}(x)$

Recall:

$$D_h^{[1f]}\{f\}(x) - f'(x) = \frac{f''(x)}{2} h + \frac{f'''(x)}{6} h^2 + \cdots$$

① : $V_h = V + c_1 h + c_2 h^2 + \cdots$

② : $V_{2h} = V + 2c_1 h + 4c_2 h^2 + \cdots$

Goal: Get rid of 1st-order terms!

Richardson Extrapolation

$$① : V_h = V + c_1 h + c_2 h^2 + \cdots$$

$$② : V_{2h} = V + 2c_1 h + 4c_2 h^2 + \cdots$$

$$2\times① - ② : \underbrace{2V_h - V_{2h}}_{} = V + 0 - 2c_2 h^2 + \cdots$$

1st-order term gone!

New algorithm approximating V whose leading error term is now of 2nd-order!

Recall:

$$V_h = D_h^{[1f]}\{f\}(x) = \frac{f(x+h) - f(x)}{h}$$

$$V_{2h} = D_{2h}^{[1f]}\{f\}(x) = \frac{f(x+2h) - f(x)}{2h}$$

$$c_2 = \frac{f'''(x)}{6}, \quad c_3 = \frac{f^{(4)}(x)}{4!}, \cdots$$

$$\therefore \quad D_h^{[2f]}\{f\}(x) = 2D_h^{[1f]} - D_{2h}^{[1f]} = 2\,\frac{f(x+h) - f(x)}{h} - \frac{f(x+2h) - f(x)}{2h}$$

Same as earlier!

$$= \frac{4f(x+h) - 4f(x) - f(x+2h) + f(x)}{2h} = \boxed{\frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}}$$

14.2 Numerical Integration

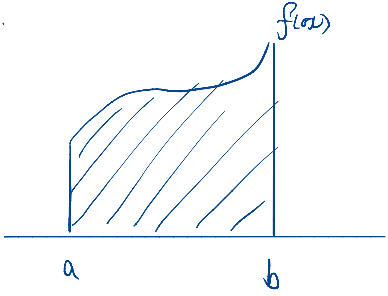
Introduction

Consider the definite integral

antiderivative of f.

$$I\{f\} = \int_a^b f(x)\, dx = F(b) - F(a)$$

(FTC)



which is approximated numerically by

$$I\{f\} \approx \sum_{i=1}^{n} \omega_i f(x_i).$$

weighted sum.

" quadrature "

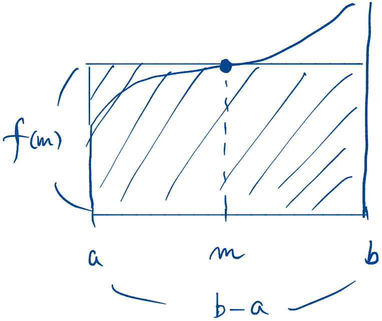
- ω_i's are called the *weights*;
- x_i's are called the *nodes* for the particular numerical method used.

Newton-Cotes Formulas

In the *Newton-Cotes methods*, the nodes are equally spaced in $[a, b]$.

- **Midpoint Method**: one node

$$\int_a^b f(x)\,dx \approx f(m)(b-a), \quad m = \frac{1}{2}(a+b).$$



Idea: Replace f(x) by a constant function and use its integral instead!

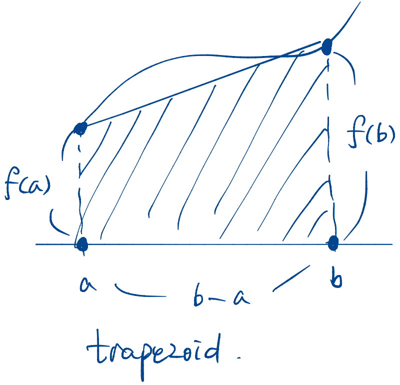
Newton-Cotes Formulas

In the *Newton-Cotes methods*, the nodes are equally spaced in $[a, b]$.

- **Trapezoidal Method**: two nodes

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\,(f(a) + f(b))\,(b-a).$$

Idea: Replace f(x) by a linear function and use its integral instead.

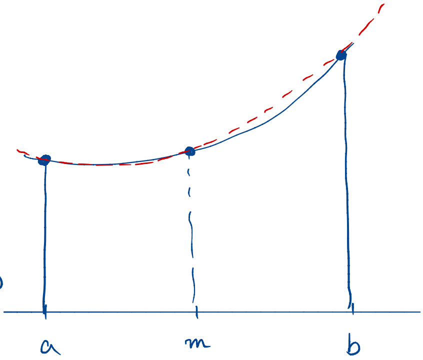


Newton-Cotes Formulas

In the *Newton-Cotes methods*, the nodes are equally spaced in $[a, b]$.

- **Simpson's Method**: three nodes

$$\int_a^b f(x)\,dx \approx \frac{1}{6}\left(f(a) + 4f(m) + f(b)\right)(b-a).$$



Idea: Replace $f(x)$ by a quadratic function (polynomial) and use its integral instead.

passing through $(a, f(a))$, $(m, f(m))$, $(b, f(b))$

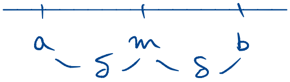
"polynomial interpolation!"

Derivation of Simpson's Rule

Find the quad. $p(x)$ interpolating $f(x)$ at a, m, b. i.e.

$$p(a) = f(a), \quad p(m) = f(m), \quad p(b) = f(b).$$

then use $\int_a^b p(x)\,dx$ to approx. $I\{f\}$.



$$\delta = \frac{b-a}{2}$$

<u>Lagrange</u>: $p(x) = f(a)\,l_1(x) + f(m)\,l_2(x) + f(b)\,l_3(x)$

where

$$l_1(x) = \frac{(x-m)(x-b)}{2\delta^2}$$

$$l_2(x) = \frac{(x-a)(x-b)}{-\delta^2}$$

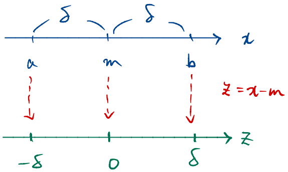
$$l_3(x) = \frac{(x-a)(x-m)}{2\delta^2}$$

Need: $\int_a^b l_j(x)\,dx$

Why? $\int_a^b p(x)\,dx = f(a)\int_a^b l_1(x)\,dx + f(m)\int_a^b l_2(x)\,dx + f(b)\int_a^b l_3(x)\,dx$

$$l_1(x) = \frac{(x-m)(x-b)}{2\delta^2}, \qquad l_2(x) = \frac{(x-a)(x-b)}{-\delta^2}, \qquad l_3(x) = \frac{(x-a)(x-m)}{2\delta^2}$$

① $$\int_a^b l_1(x)\,dx = \frac{1}{2\delta^2}\int_a^b (x-m)(x-b)\,dx$$



$$z = x-m$$

$$x-b = x-m-\delta = z-\delta$$

$$= \frac{1}{2\delta^2}\int_{-\delta}^{\delta} \underbrace{z(z-\delta)}\,dz$$

$$\to \underset{\text{even}}{z^2} - \underset{\text{odd}}{\delta z}$$

$$= \frac{1}{2\delta^2}\cdot 2\int_0^{\delta} z^2\,dz$$

$$= \frac{1}{2\delta^2}\cdot 2\cdot\frac{\delta^3}{3} = \frac{\delta}{3} = \frac{b-a}{6}$$

② $\int_a^b l_2(x)\,dx = -\frac{1}{\delta^2}\int_a^b (x-a)(x-b)\,dx$

$$= -\frac{1}{\delta^2}\int_{-\delta}^{\delta} \underbrace{(z-\delta)(z+\delta)}_{\substack{\shortparallel \\ z^2-\delta^2}}\,dx$$

$$= -\frac{2}{\delta^2}\int_0^{\delta} (z^2-\delta^2)\,dx = \cdots = \frac{4\delta}{3} = \frac{2}{3}(b-a)$$

③ $\int_a^b l_3(x)\,dx = \cdots = \frac{b-a}{6}$

Therefore,

$$\int_a^b f(x)\,dx \approx \int_a^b p(x)\,dx$$

$$= f(a)\underbrace{\int_a^b l_1(x)\,dx}_{=\frac{b-a}{6}} + f(m)\underbrace{\int_a^b l_2(x)\,dx}_{=\frac{2}{3}(b-a)} + f(b)\underbrace{\int_a^b l_3(x)\,dx}_{=\frac{b-a}{6}}$$

$$= \boxed{\frac{b-a}{6}\left(f(a) + 4f(m) + f(b) \right)}$$

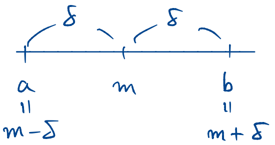
Simpson's method!

"Trapezoidal method"

$$I^{[t]}\{f\} = \frac{b-a}{2}\left(f(a)+f(b)\right)$$

$$I\{f\} = \int_a^b f(x)\,dx$$

Strategy: Taylor expansion at $m = \frac{a+b}{2}$



$$\begin{cases} f(a) = f(m-\delta) = f(m) - \cancel{f'(m)\delta} + \dfrac{f''(m)}{2}\delta^2 - \cancel{\cdots} \\ f(b) = f(m+\delta) = f(m) + \cancel{f'(m)\delta} + \dfrac{f''(m)}{2}\delta^2 + \cancel{\cdots} \end{cases}$$

$$f(a)+f(b) = 2\left[f(m) + \frac{f''(m)}{2}\delta^2 + \frac{f^{(4)}(m)}{4!}\delta^4 + \cdots \right]$$

$$\therefore\ I^{[t]} = \underbrace{\frac{b-a}{2}}_{=\delta}\left(f(a)+f(b)\right)$$

$$= 2\delta\left[f(m) + \frac{f''(m)}{2}\delta^2 + \frac{f^{(4)}(m)}{4!}\delta^4 + \cdots \right]$$

Error Analysis

- $$I = \int_a^b f(x)\,dx$$

$$= \int_a^b f(m + \underline{x-m})\,dx$$

$$= \int_a^b \sum_{k=0}^{\infty} \frac{f^{(k)}(m)}{k!}(x-m)^k\,dx$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(m)}{k!} \int_a^b (x-m)^k\,dx$$

$\underline{z = x-m}$

$$= \int_{-\delta}^{\delta} z^k\,dz$$

$$= \begin{cases} 0 & , k \text{ odd} \\ \dfrac{2\delta^{k+1}}{k+1} & , k \text{ even} \end{cases}$$

$\underline{\underline{k=2j}}$

$$= \sum_{j=0}^{\infty} \frac{f^{(2j)}(m)}{(2j)!} \cdot \frac{2\delta^{2j+1}}{2j+1} = 2\delta\left[f(m) + \frac{f''(m)}{3!}\delta^2 + \frac{f^{(4)}(m)}{5!}\delta^4 + \cdots \right]$$

Finally, $e^{[t]} = I^{[t]} - I$

$$= 2\delta \left[\cancel{f(m)} + \frac{f''(m)}{2}\delta^2 + \frac{f^{(4)}(m)}{4!}\delta^4 + \cdots \right]$$

$$- 2\delta \left[\cancel{f(m)} + \frac{f''(m)}{3!}\delta^2 + \frac{f^{(4)}(m)}{5!}\delta^4 + \cdots \right]$$

$$= 2\delta \left[\underbrace{\left(\frac{1}{2} - \frac{1}{3!}\right)}_{= \frac{1}{3}} f''(m)\delta^2 + \underbrace{\left(\frac{1}{4!} - \frac{1}{5!}\right)}_{= \frac{4}{5!} = \frac{1}{3!\,5} = \frac{1}{30}} f^{(4)}(m)\delta^4 + \cdots \right]$$

$\delta = \frac{b-a}{2}$

$$= \frac{2}{3} f''(m)\left(\frac{b-a}{2}\right)^3 + \underbrace{\frac{1}{15} f^{(4)}(m)\left(\frac{b-a}{2}\right)^5} + O\left((b-a)^7\right)$$

$$= \frac{1}{12} f''(m)(b-a)^3 + O\left((b-a)^5\right) \qquad \frac{1}{15 \cdot 32} f^{(4)}(m)(b-a)^5$$

leading error.

$\therefore$ 3rd-order accurate.

Error Analysis

Note :

$$\begin{cases} \bullet \quad e^{[t]} = I^{[t]} - I = \frac{1}{12} f''(m)(b-a)^3 + \frac{1}{480} f^{(4)}(m)(b-a)^5 + O((b-a)^7) \\ \bullet \quad e^{[m]} = I^{[m]} - I = -\frac{1}{24} f''(m)(b-a)^3 - \frac{1}{1920} f^{(4)}(m)(b-a)^5 + O((b-a)^7) \end{cases}$$

both are 3rd-order accurate!

$$\bullet \quad e^{[s]} = I^{[s]} - I = \frac{1}{2880} f^{(4)}(m)(b-a)^5 + O((b-a)^7)$$

5th-order accurate!

Know:
$$I^{[m]} = I - \frac{1}{24} f''(m)(b-a)^3 + \frac{1}{1920} f^{(4)}(m)(b-a)^5 + \cdots$$

$$I^{[t]} = I + \frac{1}{12} f''(m)(b-a)^3 + \frac{1}{480} f^{(4)}(m)(b-a)^5 + \cdots$$

Want: Construct a higher-order method based on the two.

Strategy: Extrapolate! Form a suitable linear comb. of $I^{[m]}$ and $I^{[t]}$ to remove their leading errors.

In our case:
$$\frac{2I^{[m]}}{3} + \frac{I^{[t]}}{3} = \cancel{3}\, I + \underbrace{\frac{1}{3}\left(\frac{1}{960} + \frac{1}{480}\right)}_{=\frac{1}{2880}} f^{(4)}(m)(b-a)^5 + O\left((b-a)^7\right)$$

Simpson Revisted – Derivation Via Extrapolation

Upshot:

$$\underbrace{\frac{2}{3} I^{[m]} + \frac{1}{3} I^{[t]}}_{\text{new algorithm approx. "}I\text{"}} = I + \underbrace{\frac{1}{2880} f^{(4)}(m)\,(b-a)^5 + O\left((b-a)^7\right)}_{\text{with } 5^{th}\text{-order error.}}$$

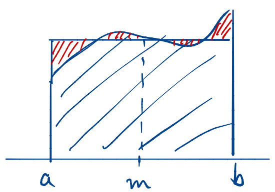
$$\frac{2}{3} f(m)(b-a) + \frac{1}{3}\left(f(a)+f(b)\right)\cdot\frac{b-a}{2}$$

$$= \frac{b-a}{6}\left(4f(m) + f(a) + f(b)\right)$$

✓ Simpson's method!

Composite Methods

Midpoint Rule

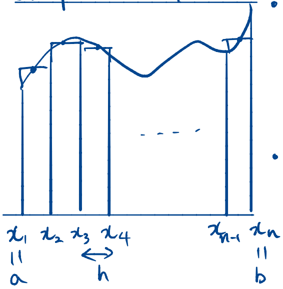


$$I^{[m]} = I + \underbrace{O((b-a)^3)}_{\text{3rd-order accurate}}$$

$$I^{[m]} - I \longrightarrow 0 \quad \text{as} \quad b-a \to 0$$

divide into multiple subintervals $\longrightarrow$

Composite Midpoint Rule



- n grid points:

$$x_j = a + j\,\frac{b-a}{n-1}$$

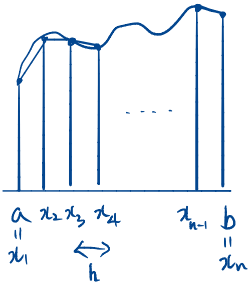
where $1 \le j \le n$.

- $n-1$ s/int. of length $\boxed{h = \dfrac{b-a}{n-1}}$

$$I \approx \sum_{j=1}^{n-1} \underbrace{f(x_{j+\frac{1}{2}})}_{\text{height}} \cdot \underbrace{h}_{\text{width}} = I_h^{[m]}$$

$$\underset{j^{\text{th}} \text{ midpoint}}{x_{j+\frac{1}{2}}} = \frac{x_j + x_{j+1}}{2}$$

Composite Trapezoidal Rule



$h = \frac{b-a}{n-1}$

j^{th} slint.

L.E.P. R.E.P.

$$I = \sum_{j=1}^{n-1} \int_{x_j}^{x_{j+1}} f(x)\,dx \approx \sum_{j=1}^{n-1} \frac{h}{2}\left(f(x_j) + f(x_{j+1})\right)$$

$1 \to n-1$ $\qquad$ $2 \to n$

$$= \frac{h}{2}\left(\sum_{j=1}^{n-1} f(x_j) + \sum_{j=1}^{n-1} f(x_{j+1})\right)$$

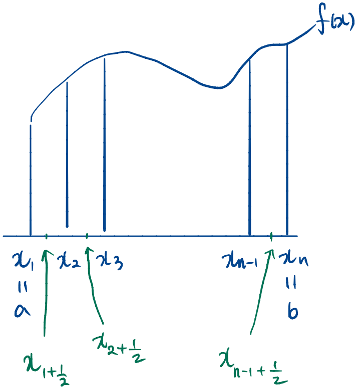
$$= \frac{h}{2}\left(f(x_1) + 2\sum_{j=2}^{n-1} f(x_j) + f(x_n)\right)$$

$$= h\left(\frac{f(x_1)+f(x_n)}{2} + \sum_{j=2}^{n-1} f(x_j)\right)$$

∗ Note: if f is periodic w/ period $b-a$, then $f(x_1) = f(x_n)$ ($x_1 = a$, $x_n = b$)

$$I_h^{[t]} = h \sum_{j=1}^{n-1} f(x_j)$$

Composite Simpson's Rule



$$I^{[S]}_{h} = \sum_{j=1}^{n-1} \frac{h}{6} \left(f(x_j) + 4 f(x_{j+\frac{1}{2}}) + f(x_{j+1}) \right)$$

Composite method with $h = \frac{b-a}{n-1}$

Errors for Composite Methods

Comp. Trapezoidal Method

"$I^{[t]} = I + O((b-a)^3)$"

$$I = \sum_{j=1}^{n-1} \underbrace{\int_{x_j}^{x_{j+1}} f(x)\,dx}_{I_j}$$

$$= \sum_{j=1}^{n-1} \left(I_j^{[t]} + O(h^3) \right)$$

$$= \underbrace{\sum_{j=1}^{n-1} I_j^{[t]}}\; + \sum_{j=1}^{n-1} O(h^3)$$

$$= \underset{\text{(Comp. T.M)}}{I_h^{[t]}} + \underbrace{(n-1)\,O(h^3)}_{\text{error}}$$

Notice that $n-1 = \dfrac{b-a}{h}$. Thus

$$(\text{error}) = \frac{b-a}{\cancel{h}}\, O(h^{\cancel{3}})$$

$$= \underset{\text{const.}}{(b-a)}\, O(h^2) = O(h^2)$$

$\therefore$ $\begin{cases} \text{trap. method : } 3^{rd}\text{-order accurate} & \text{(local error)} \\ \text{Comp. trap. method : } 2^{nd}\text{-order accurate} & \text{(global error)} \end{cases}$

Errors for Composite Methods

Precisely:

$$e_h^{[m]} = -\frac{f''(\xi_m)}{24}(b-a)\,h^2$$

$$e_h^{[t]} = \frac{f''(\xi_t)}{12}(b-a)\,h^2$$

2^{nd}-order accurate

$$e_h^{[s]} = \frac{f^{(4)}(\xi_s)}{2880}(b-a)\,h^4$$

4^{th}-order accurate