

Lec 34: Problem Solving Session (2/2)

Exercise Problems

Optimal h

Question. Suppose that a function $f(x)$ is numerically calculated by the following procedure.

```
function y = f(x)
    a = 1; b = cos(x);
    for i = 1:5
        c = b;
        b = sqrt(a.*b);
        a = (a + c)/2;
    end
    y = (pi/2)./a;
end
```

AGM(1, cos(x))

Note: This program is related to the "AGM" function.

arithmetic-geometric mean.

$$\Rightarrow f(x) = \frac{\pi}{2 \operatorname{AGM}(1, \cos(x))}$$

Compute $f'(\pi/4)$ as accurately as possible using a method of numerical differentiation.

For optimal step size.

✓ 2nd-order centered difference,

$$h \approx \boxed{\text{eps}}^{\frac{1}{3}} \approx (10^{-16})^{\frac{1}{3}}$$

Optimal h (step size)

$$D_h^{[2c]} \{f\}(x)$$



2^{nd} -order C.D.
formula

$$D_h^{[2c]} \{f\}(x) = f'(x) + \underbrace{Ch^2}_{\text{computation}} + K \frac{\boxed{\text{eps}}}{h} + O(\boxed{\text{eps}})$$

↑
on computer
(round-off errors)

$$+ O(h^3)$$

Ignore

↑
leading
truncation
error

Minimize the leading error:

$$F(h) = \alpha h^2 + \beta \frac{\boxed{\text{eps}}}{h} \quad \xrightarrow{\text{minimize}} \quad F'(h) = 0$$

Balancing

$$h^2 \approx \frac{\boxed{\text{eps}}}{h}$$

$$\Rightarrow h^3 \approx \boxed{\text{eps}} \Rightarrow h \approx \boxed{\text{eps}}^{1/3}$$

Suggested Exercises (related to this problem)

1. Determine optimal h for different num. diff. schemes.

method	optimal h
1 st - order	?
2 nd - order	$h \approx \boxed{\text{eps}}^{1/3}$
3 rd - order	?
4 th - order	?



shown in lecture.

Can you figure out a pattern?

2. Rewrite multidimensional Newton's method code using num. differentiation.

- Jacobian matrix
(consisting of partial derivatives)
- "finite differences"
(w/ optimal h)
- computation of partial deriv.

Logarithmic Integral

The logarithmic integral is a special mathematical function defined by the equation

$$\text{li}(x) = \int_2^x \frac{dt}{\ln t}.$$

Find $\text{li}(200)$ by means of the composite trapezoid method.

Side note:

$$\text{li}(x) = \text{PV} \int_0^x \frac{dt}{\ln t}$$

due to singularity at $t=1$.

Side note: Compare to Euler spiral

$$x(t) = \int_0^t \cos(z^2) dz$$

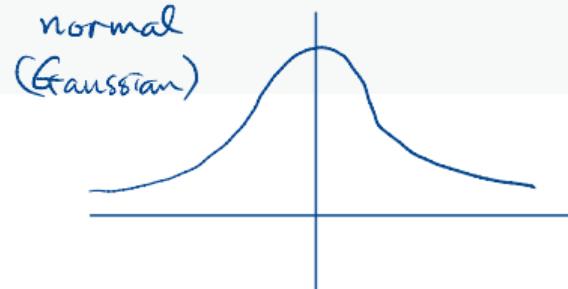
$$y(t) = \int_0^t \sin(z^2) dz$$

$$\begin{aligned} & \rightarrow \int_2^{200} \frac{dt}{\ln t} = \int_2^{200} g(t) dt \\ & \approx \left[\frac{g(t_1) + g(t_n)}{2} + \sum_{j=2}^{n-1} g(t_j) \right] h \end{aligned}$$

Quadrature Exercise

① Compute

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$



by using small and large values for the limits of integration and applying a numerical method. Then compute it by making the change of variable

L'Hopital

$$W(0) = 0$$

Subs.

$$dx = -\frac{dt}{t}$$

x	$t = e^{-x}$
∞	0
0	1

$$x = -\ln t.$$

$$\int_1^\infty \frac{e^{-(\ln t)^2}}{-t} dt = \int_0^1 \frac{e^{-(\ln t)^2}}{t} dt$$

$w(t)$

num. integ.

Quadrature Exercise

Find the area of the ellipse $y^2 + 4x^2 = 1$.

Airplane Velocity

The radar stations A and B , separated by the distance $a = 500$ m, track a plane C by recording the angles α and β at one-second intervals. Your goal, back at air traffic control, is to determine the speed of the plane.

