

Lec 36: Spectral Theory

Singular Value Decomposition

Singular Value Decomposition: Overview

Singular Value Decomposition

Theorem 1 (SVD)

Let $A \in \mathbb{C}^{m \times n}$. Then A can be written as

$$A = U\Sigma V^*, \quad (\text{SVD})$$

where $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are unitary and $\Sigma \in \mathbb{R}^{m \times n}$ is diagonal. If A is real, then so are U and V .

- The columns of U are called the **left singular vectors** of A ;
- The columns of V are called the **right singular vectors** of A ;
- The diagonal entries of Σ , written as $\sigma_1, \sigma_2, \dots, \sigma_r$, for $r = \min\{m, n\}$, are called the **singular values** of A and they are nonnegative numbers ordered as

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq 0.$$

Singular Value Decomposition (cont')

Thick vs Thin SVD

Suppose that $m > n$ and observe that:

SVD in MATLAB

- Thick SVD: $[U, S, V] = \text{svd}(A);$
- Thin SVD: $[U, S, V] = \text{svd}(A, 0);$

Understanding SVD

Geometric Perspective

Write $A = U\Sigma V^*$ as $AV = U\Sigma$:

$$A\mathbf{v}_k = \sigma_k \mathbf{u}_k, \quad k = 1, \dots, r = \min\{m, n\}.$$

The image of the unit sphere under any $m \times n$ matrix is a hyperellipse.

Algebraic Perspective

Alternately, note that $\mathbf{y} = A\mathbf{z} \in \mathbb{C}^m$ for any $\mathbf{z} \in \mathbb{C}^n$ can be written as

$$(U^*\mathbf{y}) = \Sigma(V^*\mathbf{z}).$$

Any matrix $A \in \mathbb{C}^{m \times n}$ can be viewed as a diagonal transformation from \mathbb{C}^n (source space) to \mathbb{C}^m (target space) with respect to suitably chosen orthonormal bases for both spaces.

SVD vs. EVD

Recall that a diagonalizable $A = VDV^{-1} \in \mathbb{C}^{n \times n}$ satisfies

$$\mathbf{y} = A\mathbf{z} \quad \longrightarrow \quad \left(V^{-1}\mathbf{y}\right) = D\left(V^{-1}\mathbf{z}\right).$$

This allowed us to view any diagonalizable square matrix $A \in \mathbb{C}^{n \times n}$ as a diagonal transformation from \mathbb{C}^n to itself¹ with respect to the basis formed by a set of eigenvector of A .

Differences.

- **Basis:** SVD uses two ONBs (left and right singular vectors); EVD uses one, usually non-orthogonal basis (eigenvectors).
- **Universality:** all matrices have an SVD; not all matrices have an EVD.
- **Utility:** SVD is useful in problems involving the behavior of A or A^+ ; EVD is relevant to problems involving A^k .

¹The source and the target spaces of the transformation coincide.