

## Math 3607: Homework 6

Due: 11:59PM, Monday, March 1, 2021

**TOTAL: 20 points**

1. Do **LM** 12.5–3. (Understanding matrix multiplication)
2. Do **LM** 12.6–2. (Gram-Schmidt)
3. (Adapted from **FNC** 3.3.3.) Let  $x_1, x_2, \dots, x_m$  be  $m$  equally spaced points in  $[-1, 1]$ . Let  $V$  be the Vandermonde-type matrix appearing on p.10 of Lecture 17 slides for  $m = 400$  and  $n = 5$ . Find the thin QR factorization of  $V = \hat{Q}\hat{R}$ , and, on a single graph, plot every column of  $\hat{Q}$  as a function of the vector  $\mathbf{x} = (x_1, x_2, \dots, x_m)^T$ . (Use MATLAB to solve this problem.)
4. (Visualizing matrix norms; adapted from **LM** 9.4–26.) For  $p \in [1, \infty]$ , recall the definition of the matrix  $p$ -norm,

$$\|A\|_p = \max_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_p.$$

To understand this definition, we will work in two-dimensional space so that we can easily plot the results. For this problem, use

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}. \quad (1)$$

As an illustration, we study the case  $p = 2$  following the steps below.

- Create unit vectors  $\mathbf{x}_j$  in 2-norm,

$$\mathbf{x}_j = \begin{bmatrix} \cos \theta_j \\ \sin \theta_j \end{bmatrix}, \quad 1 \leq j \leq 361 \quad (2)$$

using 361 evenly distributed  $\theta_j$  in  $[0, 2\pi]$ . Make sure  $\mathbf{x}_1 = \mathbf{x}_{361} = (1, 0)^T$ , just as in the spiral polygon problem. Plot these points, which lie on the unit circle. Make sure the plot looks like a circle.

- For each  $j$ , let  $\mathbf{y}_j = A\mathbf{x}_j$ . Plot all points  $\mathbf{y}_j$ . In addition, store  $\|\mathbf{y}_j\|_2$  for all  $j$  in a vector.
- Plot  $\|\mathbf{y}_j\|_2$  as a function of  $\theta_j$ .
- Find the maximum value of  $\|\mathbf{y}_j\|_2$  over all  $j$ . This estimates  $\|A\|_2$ . Compare this against the actual value computed by `norm(A, 2)`.

These steps are carried out by the following script.

```
A = [2 1; 1 3];
theta = linspace(0, 2*pi, 361);
X = [cos(theta); sin(theta)];
Y = A*X;
norm_Y = sqrt(sum(Y.^2, 1));
```

```

clf
subplot(2,2,1)
plot(X(1,:), X(2,:)), axis equal
title('x: unit vectors in 2-norm')
subplot(2,2,2)
plot(Y(1,:), Y(2,:)), axis equal
title('Ax: image of unit vectors under A')
subplot(2,1,2)
plot(theta, norm_Y), axis tight
xlabel('\theta')
ylabel('||y||')
fprintf(' p = 2\n')
fprintf(' approx. norm: %18.16f\n', max(norm_Y))
fprintf(' actual norm: %18.16f\n', norm(A, 2))

```

which generates Figure 1 and the following outputs in the Command Window:

```

p = 2
approx. norm: 3.6179964204609893
actual norm: 3.6180339887498953

```

Modify the script to carry out the same tasks for  $p = 1, \infty$ . Pay particular attention to the lines where  $X$  is defined and  $\text{norm\_Y}$  is calculated.

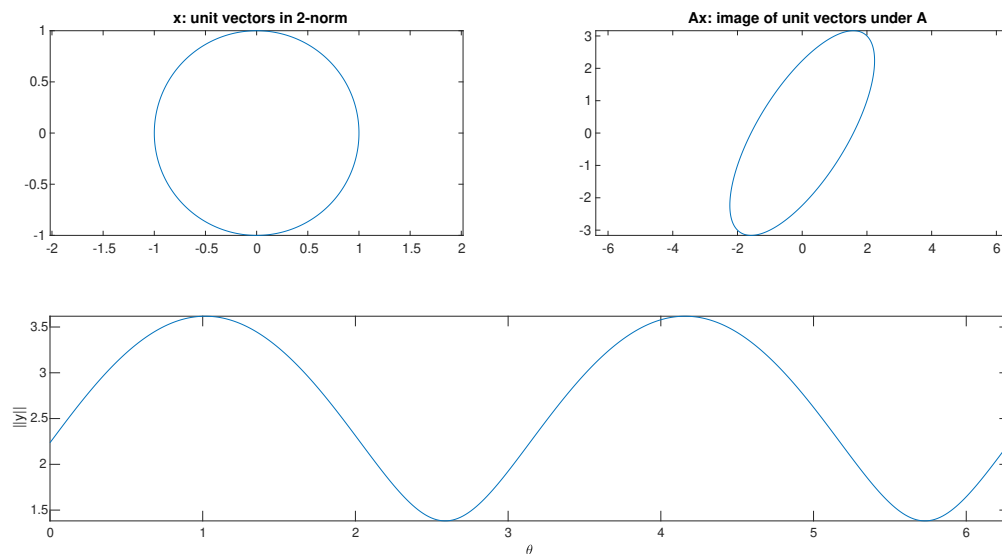


Figure 1: Plots illustrating the definition of matrix norm.

## 5. Graphics exercise of the week: Julia and Mandelbrot Sets

This is an *optional* problem for those interested in further developing programming skills and creating cool graphics. Read **LM** 6.8.3 on Julia and Mandelbrot sets. Then generate fractals by working out **LM** 6.8–69.