Lec. 33 : Exercises on Num. Diff. & Integration.

1. Lagrange polynomials

$\boxed{\text{Defn}}$ Given (x_i, y_i) for $i = 1, \cdots, n$, define the Lagrange polynomials by

$$l_k(x) = \prod_{\substack{i=1 \\ i \neq k}}^{n} (x - x_i) \Big/ \prod_{\substack{i=1 \\ i \neq k}}^{n} (x_k - x_i)$$

for $k = 1, \cdots, n$.

Observations

- The construction only depends on x-data, $x_1, x_2, \cdots, x_n$.
- Each of $l_k(x)$ is a degree $(n-1)$ polynomial.
- They are cardinal functions, i.e.,

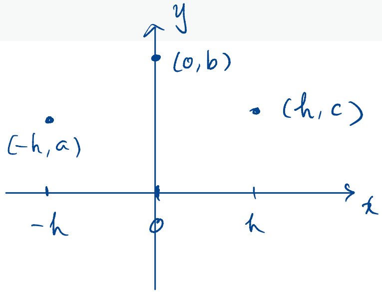
$$l_k(x_j) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

Examples (#2, HW8)

Say $n=3$. With x_1, x_2, x_3

x	$-h$	0	h
y	a	b	c

(x-data)



With the given data, we define

$$l_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-0)(x-h)}{(-h-0)(-h-h)} = \frac{x(x-h)}{2h^2}$$

$$l_2(x) = \frac{(x+h)(x-h)}{h\ (-h)} = -\frac{(x+h)(x-h)}{h^2}$$

$$l_3(x) = \frac{(x+h)\,x}{2h\cdot h} = \frac{x(x+h)}{2h^2}$$

Key theorem on Lag. poly.

The polynomial interpolant $p(x)$ of the data (x_i, y_i) for $i=1,\cdots,n$ is given by

$$p(x) = \sum_{k=1}^{n} y_k \, l_k(x)$$

Recall: In general, we work out the linear system involving the Vandermonde matrix to find the interpolant.

"Power form"

- We've seen a similar result in the context of piecewise linear interpolation w/ hat functions.

2. Summary of Key Formulas

$$\underset{\text{diff.}}{D}\,_{\underset{\text{discretization}}{h}}^{\overset{\text{method}}{[1f]}}\underset{\text{fnc to be diff.}}{\{f\}}\underset{\text{eval. pt}}{(x)}$$

1st-order accurate

$$\underbrace{D_h^{[1f]}\{f\}(x) = \frac{f(x+h)-f(x)}{h}}_{\text{approx. form}} = \underbrace{f'(x)}_{\text{exact}} + \underbrace{\overbrace{\frac{f''(x)}{2}h}^{\text{(leading)}} + O(h^2)}_{\text{error}}$$

$$D_h^{[1b]}\{f\}(x) = \frac{f(x)-f(x-h)}{h} = f'(x) - \frac{f''(x)}{2}h + O(h^2)$$

$$D_h^{[2c]}\{f\}(x) = \frac{f(x+h)-f(x-h)}{2h} = f'(x) + \frac{f'''(x)}{6}h^2 + O(h^4)$$

2nd-order accurate

Summary continued

$$I = \int_a^b f(x)\,dx$$

3rd-order

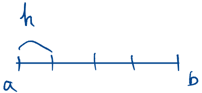
- $I^{[m]}\{f\} = f(m)(b-a) = I - \frac{1}{24} f''(m)(b-a)^3 + O((b-a)^5)$
- $I^{[t]}\{f\} = [f(a)+f(b)]\frac{b-a}{2} = I + \frac{1}{12} f''(m)(b-a)^3 + O((b-a)^5)$

- $I^{[S]}\{f\} = [f(a) + 4f(m) + f(b)]\frac{b-a}{6} = I + \frac{1}{2880} f^{(4)}(m)(b-a)^5 + O((b-a)^7)$

5th-order.

* <u>Composite methods</u> for multiple subintervals in $[a,b]$.

(lose an order of accuracy)



3. Second Derivatives

2^{nd}-order C. D. formula.

$$f''(x) = (f')'(x) \approx D_h^{[2C]}\{D_h^{[2C]}\{f\}\}(x)$$

$$= \frac{D_h^{[2C]}\{f\}(x+h) - D_h^{[2C]}\{f\}(x-h)}{2h}$$

$$= \frac{1}{2h}\left[\frac{f(x+2h)-f(x)}{2h} - \frac{f(x)-f(x-2h)}{2h}\right] \quad \times$$

Not this, but the one w/ $h \to h/2$ will do.

Proper one

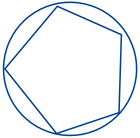
$$f''(x) \approx D^{[2C]}_{h/2}\left\{ D^{[2C]}_{h/2}\{f\} \right\}(x)$$

$$= \boxed{\frac{f(x+h) - 2f(x) + f(x+h)}{h^2}}$$

4. Richardson Extrapolation

technique used to improve the accuracy of an approx. scheme.

Sequences approximating π.



$$a_n = n \sin(\pi/n) \qquad \lim_{n\to\infty} a_n = \pi$$

$$\Big\downarrow \; h = 1/n$$

$$V_h = \frac{1}{h}\sin(\pi h) = \frac{1}{h}\left(\pi h - \frac{(\pi h)^3}{3!} + \frac{(\pi h)^5}{5!} - \cdots\right)$$

(interested in V_h as $h \to 0$)

$$\underbrace{V_h}_{\substack{\text{approx.} \\ \text{form}}} = \underbrace{\pi}_{\text{exact}} \underbrace{- \frac{\pi^3}{6} h^2 + \cdots}_{\substack{\text{error} \\ \shortparallel \\ e_{p_n}}}$$