Lec. 33 : Exercises on Num. Diff. & Integration.

1. Lagrange polynomials

$\boxed{\text{Defn}}$ Given (x_i, y_i) for $i = 1, \cdots, n$, define the Lagrange polynomials by

$$l_k(x) = \prod_{\substack{i=1 \\ i \neq k}}^{n} (x - x_i) \Big/ \prod_{\substack{i=1 \\ i \neq k}}^{n} (x_k - x_i)$$

for $k = 1, \cdots, n$.

Observations

- The construction only depends on x-data, $x_1, x_2, \cdots, x_n$.
- Each of $l_k(x)$ is a degree $(n-1)$ polynomial.
- They are cardinal functions, i.e.,

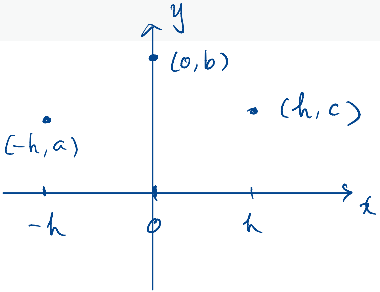
$$l_k(x_j) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

Examples (#2, HW8)

Say $n=3$. With x_1, x_2, x_3

x	$-h$	0	h
y	a	b	c

(x-data)



With the given data, we define

$$\ell_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-0)(x-h)}{(-h-0)(-h-h)} = \frac{x(x-h)}{2h^2}$$

$$\ell_2(x) = \frac{(x+h)(x-h)}{h\ (-h)} = -\frac{(x+h)(x-h)}{h^2}$$

$$\ell_3(x) = \frac{(x+h)\,x}{2h\cdot h} = \frac{x(x+h)}{2h^2}$$

Key theorem on Lag. poly.

The polynomial interpolant $p(x)$ of the data (x_i, y_i) for $i=1,\cdots,n$ is given by

$$p(x) = \sum_{k=1}^{n} y_k \, l_k(x)$$

Recall: In general, we work out the linear system involving the Vandermonde matrix to find the interpolant.

"Power form"

- We've seen a similar result in the context of piecewise linear interpolation w/ hat functions.

2. Summary of Key Formulas

method

$$D_h^{[1f]}\{f\}(x)$$

diff.   discretization   fnc to be diff.   eval. pt

1st-order accurate

approx. form   exact   (leading) error

- $$D_h^{[1f]}\{f\}(x) = \frac{f(x+h) - f(x)}{h} = f'(x) + \frac{f''(x)}{2}h + O(h^2)$$
- $$D_h^{[1b]}\{f\}(x) = \frac{f(x) - f(x-h)}{h} = f'(x) - \frac{f''(x)}{2}h + O(h^2)$$

- $$D_h^{[2c]}\{f\}(x) = \frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{f'''(x)}{6}h^2 + O(h^4)$$

2nd-order accurate

Summary continued

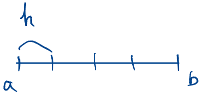
$$I = \int_a^b f(x)\,dx$$

3rd-order

- $I^{[M]}\{f\} = f(m)\,(b-a) = I - \frac{1}{24} f''(m)(b-a)^3 + O((b-a)^5)$
- $I^{[T]}\{f\} = [f(a) + f(b)]\,\frac{b-a}{2} = I + \frac{1}{12} f''(m)(b-a)^3 + O((b-a)^5)$
- $I^{[S]}\{f\} = [f(a) + 4f(m) + f(b)]\,\frac{b-a}{6} = I + \frac{1}{2880} f^{(4)}(m)(b-a)^5 + O((b-a)^7)$

5th-order.

* Composite methods for multiple subintervals in $[a, b]$.



(lose an order of accuracy)

3. Second Derivatives

2^{nd}-order C.D. formula.

$$f''(x) = (f')'(x) \approx D_h^{[2C]}\{D_h^{[2C]}\{f\}\}(x)$$

$$= \frac{D_h^{[2C]}\{f\}(x+h) - D_h^{[2C]}\{f\}(x-h)}{2h}$$

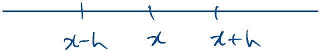
$$= \frac{1}{2h}\left[\frac{f(x+2h)-f(x)}{2h} - \frac{f(x)-f(x-2h)}{2h}\right] \quad \times$$

Not this, but the one w/ $h \to h/2$ will do.

Proper one

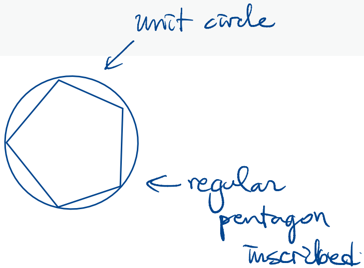
$$f''(x) \approx D_{h/2}^{[2c]}\left\{ D_{h/2}^{[2c]}\{f\} \right\}(x)$$

$$= \boxed{\frac{f(x+h) - 2f(x) + f(x+h)}{h^2}}$$



4. Richardsen Extrapolation

technique used to improve the accuracy of an approx. scheme.



Sequences approximating π.

$$a_n = n \sin(\pi/n) \qquad \lim_{n\to\infty} a_n = \pi$$

$$\downarrow \; h = 1/n$$

$$V_h = \frac{1}{h}\sin(\pi h) = \frac{1}{h}\left(\pi h - \frac{(\pi h)^3}{3!} + \frac{(\pi h)^5}{5!} - \cdots\right)$$

(interested in V_h as $h \to 0$)

$$\underbrace{V_h}_{\text{approx. form}} = \underbrace{\pi}_{\text{exact}} \underbrace{- \frac{\pi^3}{6} h^2 + \cdots}_{\substack{\text{error} \\ \| \\ e_{Fn}}}$$

Lec 33. Recitation on Numerical Diff. and Integration

1. Lagrange polynomials (#2, HW 8)

$\boxed{\text{Def'n}}$ Given (x_i, y_i) for $i = 1, \cdots, n$, we define the Lagrange polynomials by

$$l_k(x) = \prod_{\substack{i=1 \\ i \neq k}}^{n} (x - x_i) \Big/ \prod_{\substack{i=1 \\ i \neq k}}^{n} (x_k - x_i)$$

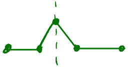
for $k = 1, \cdots, n$.

Observations

- $l_k(x)$ is a degree $(n-1)$ polynomial.
- Lagrange polynomials are cardinal functions w.r.t. the given data:

$$l_k(x_j) = \delta_{kj} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

cf. Hat functions for PL interpolation.



Further props of Lag. polynom.

- they form a basis of the space of polynomials of degree at most $n-1$.
- The interpolant of the data (x_i, y_i) is simply given by

$$\boxed{p(x) = \sum_{i=1}^{n} y_i \, l_i(x)}$$

(no system to be solved)

examples of bases

- $\{1, x, x^2, \cdots, x^{n-1}\}$
- $\{1, x-c, (x-c)^2, \cdots, (x-c)^{n-1}\}$
- $\{l_1(x), l_2(x), \cdots, l_n(x)\}$

Vandermonde matrix

Identity matrix

2. Quick Summary of Key formulas (Differential)

Notation

method

eval. pt.

leading error

$$\underbrace{D_h^{[1f]}\{f\}(x)}_{\text{notation}} = \underbrace{\frac{f(x+h)-f(x)}{h}}_{\text{approx. formula}} = \underbrace{f'(x)}_{\text{exact}} + \underbrace{\boxed{\frac{f''(x)}{2}h} + O(h^2)}_{\text{error}}$$

"derivative"

step size

(discretization)

fnc. to be differentiated

- $$D_h^{[1f]}\{f\}(x) = \frac{f(x+h) - f(x)}{h} = f'(x) + \frac{f''(x)}{2}h + O(h^2)$$

- $$D_h^{[1b]}\{f\}(x) = \frac{f(x) - f(x-h)}{h} = f'(x) - \frac{f''(x)}{2}h + O(h^2)$$

- $$D_h^{[2c]}\{f\}(x) = \frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{f'''(x)}{6}h^2 + O(h^4)$$

leading error

(Integral) $\quad I = \int_a^b f(x)\,dx$

Composite methods omitted

midpoint $\quad I^{[m]}\{f\} = f(m)(b-a) = I - \frac{1}{24} f''(m)(b-a)^3 + O((b-a)^5)$

trapezoid $\quad I^{[t]}\{f\} = \left[f(a) + f(b)\right]\frac{b-a}{2} = I + \frac{1}{12} f''(m)(b-a)^3 + O((b-a)^5)$

Simpson $\quad I^{[S]}\{f\} = \left[f(a) + 4f(m) + f(b)\right]\frac{b-a}{6} = I + \frac{1}{2880} f^{(4)}(m)(b-a)^5 + O((b-a)^7)$

Note: 2 derivations

- interpolation based (Lagrange)
- Richardson extrapolation

3. 2^{nd} Derivative Approximation

$$f''(x) = (f')'(x)$$

$$\cancel{\approx D_h^{[2C]}\left\{D_h^{[2C]}\{f\}\right\}(x)}$$

$x-2h,\ x,\ x+2h$

are used. step size: $2h$.

$$\approx D_{h/2}^{[2C]}\left\{D_{h/2}^{[2C]}\{f\}\right\}(x) = \frac{D_{h/2}^{[2C]}\{f\}(x+h/2) - D_{h/2}^{[2C]}\{f\}(x-h/2)}{h}$$

$$= \frac{1}{h}\left[\frac{f(x+h) - f(x)}{h} - \frac{f(x) - f(x-h)}{h}\right]$$

$$= \boxed{\frac{f(x+h) - 2f(x) + f(x-h)}{h^2}}$$

4. Richardson Extrapolation

- technique used to improve accuracy of an approx. method.
- Key idea: form a suitable linear combination of approx. w/ h and $2h$ (or $h/2$)
- Scatch of technique: V exact, V_h approx

Starting: $V_h = V + (\text{error})$

$= V + c_1 h^{p_1} + c_2 h^{p_2} + \cdots$

- $0 < p_1 < p_2 < p_3 \cdots$
- V_h is p_1^{th}-order accurate.

* **Linear combo.** to cancel the current leading error.

- $V_h = V + c_1 h^{p_1} + c_2 h^{p_2} + \cdots$

- $V_{2h} = V + c_1 2^{p_1} h^{p_1} + c_2 2^{p_2} h^{p_2} + \cdots$

Note that

$$2^{p_1} V_h - V_{2h} = (2^{p_1} - 1) V + O(h^{p_2})$$

new recipe

$$\frac{2^{p_1} V_h - V_{2h}}{2^{p_1} - 1} = V + O(h^{p_2})$$