Lec 28: Problem Solving Session

Rootfinding

FZERO to Solve Complex Problem

- **FNC** 4.1.5 (Kepler's Law)

4.1.5. The most easily observed properties of the orbit of a celestial body around the sun are the period τ and the elliptical eccentricity ϵ. (A circle has $\epsilon = 0$.) From these it is possible to find at any time t the angle $\theta(t)$ made between the body's position and the major axis of the ellipse. This is done through

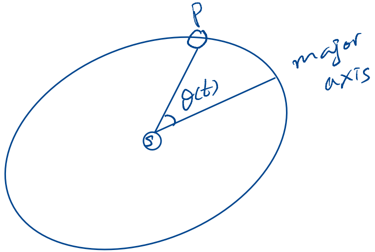
$$\tan\frac{\theta}{2} = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\tan\frac{\psi}{2}, \qquad (4.1.2)$$

where the eccentric anomaly ψ satisfies Kepler's equation:

$$\psi - \epsilon\sin\psi - \frac{2\pi t}{\tau} = 0. \qquad (4.1.3)$$

"fzero"

Equation (4.1.3) must be solved numerically to find $\psi(t)$, and then (4.1.2) can be solved analytically to find $\theta(t)$.
The asteroid Eros has $\tau = 1.7610$ years and $\epsilon = 0.2230$. Using **fzero** for (4.1.3), make a plot of $\theta(t)$ for 100 values of t between 0 and τ (one full period).



ψ : psi

θ : theta

initial guess

$$\theta(t) = 2\arctan\left[\sqrt{\frac{1+\epsilon}{1-\epsilon}}\tan\frac{\psi(t)}{2}\right]$$

fzero(@(psi) ... , 0)

Lambert W-Function

- **FNC** 4.1.6 $\longrightarrow$ Review Prob 2 of HW 7.

Same idea and technique

More With Lambert W-Function

$y = W(x)$ iff $x = ye^y$

Question. Show by hand that solutions of the equation $2^x = 5x$

$$r = -\frac{W\left(-\log(2)/5\right)}{\log 2}.$$

(Here, as usual in this class, $\log(\,\cdot\,) = \ln(\,\cdot\,)$ is the natural logarithmic function.)
Then numerically verify the result using `fzero`[1]

$2^x = 5x \longrightarrow$

$\downarrow$

$\cancel{x \log 2 = \log 5 + \log x}$

$$e^{x \log 2} = 5x$$

$$\frac{1}{5} = x e^{-x \log 2}$$

$$-\frac{\log 2}{5} = -x \log 2 \, e^{-x \log 2}$$

$$-x \log 2 = W\left(-\frac{\log 2}{5}\right)$$

$$x = -\frac{W\left(-\frac{\log 2}{5}\right)}{\log 2} \quad \checkmark$$

[1] Two real-valued solutions, $r_1 \approx 0.2355$ and $r_2 \approx 4.488$.

FPI: When Convergence Is Faster Than Expected

- **FNC** 4.2.6

Fixed point problem: Given a function g, find x satisfying

$$x = g(x).$$

Solution strategy (Iteration)

$$\begin{cases} x_0 : \text{initial guess} \\ x_{n+1} = g(x_n) \quad \text{iteration formula.} \end{cases}$$

Key note

If $|g'(r)| < 1$, the convergence is linear.

$\Rightarrow x_0, x_1, x_2, \cdots$

If $\lim_{k \to \infty} x_k = r$, then $g(r) = r$, i.e. r is a fixed point of g.

(a) $g(x) = 2x - 3x^2$.

WTS : $r = 1/3$ is a f-p.

i.e., NTS : $1/3 = g(1/3)$

Soln: $g(1/3) = 2\cdot\frac{1}{3} - 3\left(\frac{1}{3}\right)^2 = \left(2 - \frac{3}{3}\right)\frac{1}{3} = \frac{1}{3}$ ✓

(b) $g'(1/3) = \,?$

$g'(x) = 2 - 6x \Rightarrow g'(1/3) = 2 - 6\cdot\frac{1}{3} = \boxed{0}$

Since $|g'(1/3)| = 0$, the convergence of FPI near $1/3$ is <u>Superlinear</u>!

FPI: Conditions for Convergence

- **FNC** 4.2.7

$$(\ast) \quad x_{k+1} = x_k - \frac{f(x_k)}{c}, \qquad f(r)=0, \; f'(r) \text{ exists}.$$

Define $g(x) = x - \frac{f(x)}{c}$.

By $f(r) = 0$, note that

$$g(r) = r - \frac{f(r)}{c} = r,$$

i.e., r is a fixed point of $g(x)$

and $(\ast)$ generates fixed point iterates.

Some considerations on c

- $c \neq 0$
- If $f'(r) > 0$,

  $c > \frac{1}{2} f'(r)$
- If $f'(r) < 0$

  $c < \frac{1}{2} f'(r)$

<u>Recall:</u> FPI converges to a f-p r

$$\boxed{\text{if} \quad |g'(r)| < 1}$$

In our case,

$$g(x) = x - \frac{f(x)}{c}$$

$$\Rightarrow g'(x) = 1 - \frac{f'(x)}{c}$$

So for convergence, we need

$$|g'(r)| = \left| 1 - \frac{f'(r)}{c} \right| < 1$$

$$-1 < 1 - \frac{f'(r)}{c} < 1$$

$$-2 < -\frac{f'(r)}{c} < 0$$

$$0 < \frac{f'(r)}{c} < 2$$

$$\frac{1}{2} < \frac{c}{f'(r)} < \infty$$

$$\Rightarrow \begin{cases} \frac{1}{2} f'(r) < c & \text{if } f'(r) > 0 \\ \frac{1}{2} f'(r) > c & \text{if } f'(r) < 0 \end{cases}$$

Stopping Criteria

- **FNC** 4.3.8

Find a sequence $\{x_k\}$ such that

- $\lim_{k\to\infty} x_k$ does not exist (i.e., $\{x_k\}$ diverges)
- $\lim_{k\to\infty} (x_{k+1} - x_k) = 0$

Hint: Calc 2.

Linear Convergence of Newton's Method

Newton's Method for Multiple Roots

Assume that $f \in C^{m+1}[a, b]$ has a root r of multiplicity m. Then Newton's method is locally convergent to r, and the error ϵ_k at step k satisfies

$$\lim_{k \to \infty} \frac{\epsilon_{k+1}}{\epsilon_k} = \frac{m-1}{m} \qquad \text{(linear convergence)}$$

- See Problem 4 of HW07 (**FNC** 4.3.7)
- Remedy: Modify the iteration formula

$$x_{k+1} = x_k - \frac{m f(x_k)}{f'(x_k)}$$