Lec 29: Problem Solving Session 2

Exercise with $\underbrace{\text{Series Analysis}}$

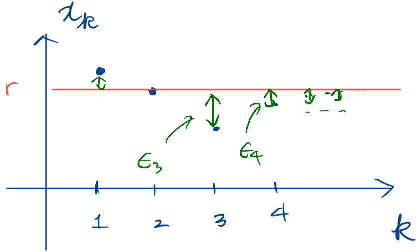
methodology employed
to study convergence of iterations.

- Taylor series

Convergence of Newton's method

Setting Assume Newton iteration $x_1, x_2, \cdots$ generated by

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$



converge to a root r of $f(x)$.

- $\lim_{k \to \infty} x_k = r$ and $f(r) = 0$

Assume $f'(r), f''(r) \neq 0$.

Notation $\epsilon_k = x_k - r$

i.e., $x_k = r + \epsilon_k$ (Note $\lim_{k\to\infty} \epsilon_k = 0$)

ϵ_k can be made arb. small for suff. large k.

Error analysis

- Iter. form. (subs. $x_k = r + \epsilon_k$)

$$\cancel{r} + \epsilon_{k+1} = \cancel{r} + \epsilon_k - \frac{f(r+\epsilon_k)}{f'(r+\epsilon_k)}$$

- Taylor expand at r

since r is a root of $f(x)$.

$$\epsilon_{k+1} = \epsilon_k - \frac{\overset{0}{\overset{\|}{f(r)}} + f'(r)\epsilon_k + \frac{f''(r)}{2}\epsilon_k^2 + O(\epsilon_k^3)}{f'(r) + f''(r)\epsilon_k + O(\epsilon_k^2)}$$

$$= \epsilon_k - \frac{\cancel{f'(r)}\,\epsilon_k\left[1 + \frac{f''(r)}{2f'(r)}\epsilon_k + O(\epsilon_k^2)\right]}{\cancel{f'(r)}\left[1 + \frac{f''(r)}{f'(r)}\epsilon_k + O(\epsilon_k^2)\right]}$$

Side note

$$\frac{1}{1+\boxed{\frac{f''(r)}{f'(r)}\epsilon_k+O(\epsilon_k^2)}} = 1-\frac{f''(r)}{f'(r)}\epsilon_k + O(\epsilon_k^2)$$

think of it as $-\alpha$

Recall Geometric series

$$\frac{1}{1-\alpha} = \sum_{k=0}^{\infty}\alpha^k = 1+\alpha+\alpha^2+\cdots$$

for $|\alpha|<1$

(Cont')

$$= \epsilon_k - \epsilon_k\left[1+\frac{f''(r)}{2f'(r)}\epsilon_k+O(\epsilon_k^2)\right]\left[1-\frac{f''(r)}{f'(r)}\epsilon_k+O(\epsilon_k^2)\right]$$

$$= \epsilon_k - \epsilon_k\left[1+\left(\frac{1}{2}-1\right)\frac{f''(r)}{f'(r)}\epsilon_k+O(\epsilon_k^2)\right]$$

$$\epsilon_{k+1} = \frac{1}{2}\frac{f''(r)}{f'(r)}\epsilon_k^2 + O(\epsilon_k^3).$$ (quad. convergence)

Linear Convergence of Newton's Method

Newton's Method for Multiple Roots

Assume that $f \in C^{m+1}[a,b]$ has a root r of multiplicity m. Then Newton's method is locally convergent to r, and the error ϵ_k at step k satisfies

$$\lim_{k\to\infty} \frac{\epsilon_{k+1}}{\epsilon_k^{1}} = \frac{m-1}{m} < 1 \qquad \text{(linear convergence)}$$

cf. general case

$$\lim_{k\to\infty} \frac{\epsilon_{k+1}}{\epsilon_k^{2}} = \frac{f''(r)}{2f'(r)}$$

- See Problem 4 of HW07 (**FNC** 4.3.7)
- Remedy: Modify the iteration formula

modified Newton $\leftarrow$ $\boxed{x_{k+1} = x_k - \dfrac{m f(x_k)}{f'(x_k)}}$

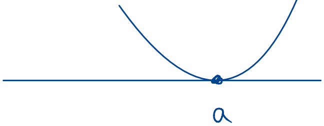
$\Rightarrow$ quadratic convergence

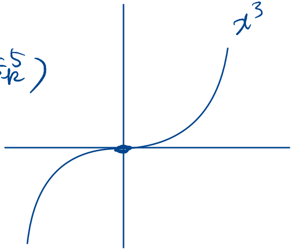
Use $m = 3$ i.e., r is a root w/ multiplicity 3

$$f(r) = 0, \quad f'(r) = 0, \quad f''(r) = 0, \quad f'''(r) \neq 0$$

Example

$f(x) = (x-a)^2$





$\boxed{\text{Key consequence}}$

- $$\underbrace{f(r + \epsilon_k)}_{\epsilon_k \text{ small}} = \cancel{f(r)} + \cancel{f'(r)\epsilon_k} + \cancel{\frac{f''(r)}{2!}}\epsilon_k^2 + \frac{f'''(r)}{3!}\epsilon_k^3 + \frac{f^{(4)}(r)}{4!}\epsilon_k^4 + O(\epsilon_k^5)$$

$$= \frac{f'''(r)}{6}\epsilon_k^3 + \frac{f^{(4)}(r)}{24}\epsilon_k^4 + O(\epsilon_k^5)$$

- $$f'(r + \epsilon_k) = \frac{f'''(r)}{2}\epsilon_k^2 + \frac{f^{(4)}(r)}{6}\epsilon_k^3 + O(\epsilon_k^4)$$

Newton iteration

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\Big\downarrow \; x_k = r + \epsilon_k$$

$$r + \epsilon_{k+1} = r + \epsilon_k - \frac{f(r+\epsilon_k)}{f'(r+\epsilon_k)}$$

Side note

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$f(\underbrace{r+\epsilon}_{\substack{\shortparallel \\ x}}) = f(r) + f'(r)\,\epsilon + \cdots$$

$$r = a$$

$$x - a = (r+\epsilon) - r = \epsilon$$