Ordered Pairs and Cartesian Products

Recap

- Intervals : $[a,b]$, (a,b), $[a,b)$, $(a,b]$

  $[c,\infty)$, (c,∞), $(-\infty,c]$, $(-\infty,c)$, $(-\infty,\infty)$

  when $a=b$, $[a,b]=[a,a]=\{a\}$ and

  $(a,b)=[a,b)=(a,b]=\phi$.

$\mathcal{A}$ set of sets.

- $\bigcup \mathcal{A}$
- $\bigcap \mathcal{A}$ ← need $\mathcal{A}$ to be nonempty

e.x. · If $\mathcal{A}=\{1,\{2,3\}\}$, then $\bigcup\mathcal{A}$ and $\bigcap\mathcal{A}$ are undefined

· $\bigcap\{\ \}=\bigcap\phi$ is not defined.

· Powerset : $\mathcal{P}(A)$ ← set is the set of all subsets of A.

E.g.

$$\mathcal{P}(\underbrace{\{1,2\}}_{A}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$$

- $\bigcup \mathcal{P}(\underbrace{\{1,2\}}_{\overset{\shortparallel}{A}}) = \emptyset \cup \{1\} \cup \{2\} \cup \{1,2\} = \{1,2\} = A$

- $\bigcap \mathcal{P}(\{1,2\}) = \emptyset \cap \{1\} \cap \{2\} \cap \{1,2\} = \emptyset$

In general, for a set A,

$$\bigcup \mathcal{P}(A) = A \quad , \quad \bigcap \mathcal{P}(A) = \emptyset.$$

Ordered Pairs

Ordered Pairs

Sets: $\{1,3\} = \{3,1\}$

$\{1,1\} = \{1\}$

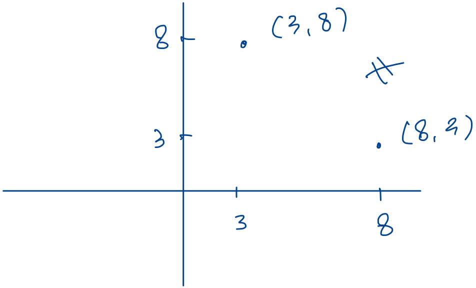
We write (a, b) for the ordered pair whose first entry is a and whose second entry is b. Unlike sets, the order by which the entries are listed does matter.

Theorem 1 (Fundamental Property of Ordered Pairs)

Let a and b be any objects. We have $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.

Example.

- $(1, 1) = (1, 1)$
- $(3, 8) \neq (8, 3)$



Notes on Ordered Pairs

$\{a, b\} = \{b, a\}$

The proof of the seemingly obvious statement in the theorem relies on a careful definition of the ordered pair (a, b).

Definition 2 (Ordered Pairs as Sets; Kuratowski)

Let a and b be any objects. The *ordered pair* (a, b) is the set $\{\{a\}, \{a, b\}\}$.

Notes on Ordered Pairs (cont')

- The ordered triple (a, b, c) can be defined as $((a, b), c)$. The analogue of the fundamental property of ordered pairs holds for ordered triples. Namely,

$$(a, b, c) = (a', b', c') \text{ iff } a = a', b = b', \text{ and } c = c'.$$

- The ordered quadruple (a, b, c, d) can be defined as $((a, b, c), d)$. We have

$$(a, b, c, d) = (a', b', c', d') \text{ iff } a = a', b = b', c = c', \text{ and } d = d'.$$

- Continuing in the same fashion, the ordered n-tuple $(a_1, a_2, \ldots, a_n)$ can be defined for each natural number $n \geqslant 2$. The fundamental property of ordered n-tuple states

$$(a_1, a_2, \ldots, a_n) = (a_1', a_2', \ldots, a_n') \text{ iff for each } j \in \{1, 2, \ldots, n\}, a_j = a_j'.$$

Sketch of proof (of Fund. Prop. of ordered pairs)

$$\begin{array}{c}(a,b)=(a',b')\\ \Updownarrow \\ a=a',\ b=b'\end{array}$$

- ($\Leftarrow$) Backward implication is clear because equals can be replaced by equals.

- ($\Rightarrow$) Assume $(a,b)=(a',b')$.

  That is, $\underbrace{\{\{a\},\{a,b\}\}}_{=S} = \underbrace{\{\{a'\},\{a',b'\}\}}_{=S'}$. (Kuratowski's def'n)

  - $\{a'\}\in S'$ and $S=S'$ $\Longrightarrow$ $\{a'\}=\{a\}$ or $\{a'\}=\{a,b\}$ $\xRightarrow{\text{[Work]}}$ $\boxed{a=a'}$

  - $\{a',b'\}\in S'$ and $S=S'$ $\Longrightarrow$ $\{a',b'\}=\{a\}$ or $\{a',b'\}=\{a,b\}$ $\xRightarrow{\text{[Work]}}$ $\boxed{b=b'}$

Cartesian Products

Cartesian Products

read as "A times B"

Definition 3 (Cartesian Products)

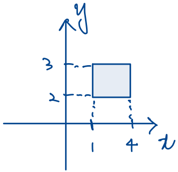
Let A and B be sets. Then the *Cartesian product of A and B* (denoted $A \times B$) is the set of all ordered pairs (x, y) such that $x \in A$ and $y \in B$; in other words,

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$$

Example.

- For a set A, the Cartesian product of A with itself, $A \times A$, is also denoted A^2.
- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, the Cartesian product of $\mathbb{R}$ with itself. This is the coordinate plane in analytical geometry. $\mathbb{R}^2 = \{(x,y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$.
- $[1, 4] \times [2, 3]$ is a rectangle in $\mathbb{R}^2$. Specifically, it is the set

$$\{(x, y) : 1 \leqslant x \leqslant 4 \text{ and } 2 \leqslant y \leqslant 3\}.$$



Example 4

Let $A = \{1, 3\}$ and $B = \{2, 4, 6\}$. Find $A \times B$.

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$$

$$= \{(1,2), (1,4), (1,6), (3,2), (3,4), (3,6)\}$$

$A \backslash B$	2	4	6
1	(1,2)	(1,4)	(1,6)
3	(3,2)	(3,4)	(3,6)

Cartesian Products of More Than Two Sets

The Cartesian product of three sets A, B, and C (denoted $A \times B \times C$) is defined in a similar way, namely,

$$A \times B \times C = \{(x, y, z) : x \in A, y \in B, \text{ and } z \in C\}.$$

In general:

Definition 5 (Cartesian Product of n Sets)

Let $n \in \mathbb{N}$ such that $n \geqslant 2$ and let $A_1, \ldots, A_n$ be sets. Then the *Cartesian product of* $A_1, A_2, \ldots, A_n$ (denoted $A_1 \times A_2 \times \cdots A_n$) is the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ such that $x_1 \in A_1$, $x_2 \in A_2$, …, $x_n \in A_n$; in other words,

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) : \text{for each } j \in \{1, 2, \ldots, n\}, x_j \in A_j\}.$$

Cartesian Products of More Than Two Sets (cont')

Example 6

Let $A = \{0, 1\}$. Find $A^3 = A \times A \times A$.