

More Examples of Induction

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From \mathbb{N} to \mathbb{Z}

Example 1 (:B_{example}:)

Prove that for each $x \in \mathbb{Z}$, x is even or x is odd.

Notes: Induction over \mathbb{Z} .

- Induction may start from a number other than 1.
- To prove a universal sentence $(\forall x \in \mathbb{Z})P(x)$:
 - 1 Prove by induction that $P(x)$ is true for each nonnegative integer x .
 - 2 Prove that $P(x)$ is true for each negative integer x .

Notation. We use ω , the boldface Greek omega, to denote the set $\{0, 1, 2, \dots\}$ of nonnegative integers.

Pascal's Triangle and the Binomial Theorem

Pascal's Triangle

The following infinite array of numbers is known as *Pascal's triangle*:

0 th row →						1					
1 st row →					1		1				
2 nd row →				1		2		1			
3 rd row →			1		3		3		1		
4 th row →		1		4		6		4		1	
5 th row →	1		5		10		10		5		1
⋮											

Notation. For all $n \in \omega$ and all $k \in \{0, \dots, n\}$, let $\binom{n}{k}$ denote the k -th number on the n -th row; this notation is read n choose k . This is also called a *binomial coefficient*.

Key Features.

- ① $\binom{0}{0} = 1$.
- ② *Boundary conditions:* For each $n \in \mathbb{N}$, $\binom{n}{0} = \binom{n}{n} = 1$.
- ③ *Recurrence relation:* For each $n \in \omega$ and all $k \in \{1, \dots, n\}$, $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Example

Application of Key Features

Use **Key Features** above to compute $\binom{4}{2}$.

Solution.

$$\begin{aligned}\binom{4}{2} &= \binom{3}{2} + \binom{3}{1} \\ &= \binom{2}{2} + \binom{2}{1} + \binom{2}{1} + \binom{2}{0} \\ &= 1 + 2\binom{2}{1} + 1 \\ &= 1 + 2\left[\binom{1}{1} + \binom{1}{0}\right] + 1 \\ &= 1 + 2(1 + 1) + 1 \\ &= 6.\end{aligned}$$

Naming: n choose k

Question: Why is $\binom{n}{k}$ called “ n choose k ”?

Number of Subsets

List all subsets of the set $\{a, b, c, d\}$ with exactly 2 elements.

Solution.

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$$

There are $6 = \binom{4}{2}$ subsets with two elements. In other words, there are

$\binom{4}{2} = 6$ ways to choose 2 elements from a set with 4 elements.

In Section 14, we will prove that for each $n \in \omega$, for each n -element set A , for each $k \in \{0, \dots, n\}$, the number of k -element subsets of A is $\binom{n}{k}$.

Naming: Binomial Coefficients

Question: Why is $\binom{n}{k}$ called a binomial coefficient?

Expansion of $(a + b)^3$

- 1 Compute $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$, and $\binom{3}{3}$.
- 2 Expand the cube of the binomial $a + b$, that is, expand $(a + b)^3$.

Binomial Theorem

The example above suggests:

The Binomial Theorem

For each $n \in \omega$ and all $a, b \in \mathbb{R}$,

$$(a + b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n$$
$$\sum_{k=0}^n \binom{n}{k} a^{n-k} b^k.$$

- **Convention:** For each $x \in \mathbb{R}$, $x^0 = 1$.