

Binomial Coefficients

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Warm-Up

Rational and Irrational Numbers Revisited

Remark 4.50

Let $d \in \mathbb{N}$, $x, y \in \mathbb{Z}$, and p be a prime number.

- ① If $p \mid xy$, then $p \mid x$ or $p \mid y$.
- ② If $d \mid xy$, then there exist $d_1, d_2 \in \mathbb{N}$ such that $d_1 \mid x$, $d_2 \mid y$, and $d = d_1 d_2$.

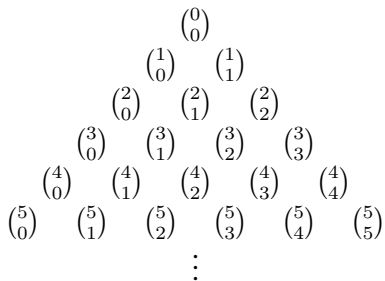
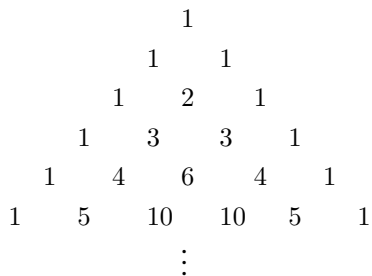
The proofs of these facts require *complete induction*.

Example 4.52

- 1 Let x be a rational number such that $x^2 = c$, where c is a whole number. Then x is an integer.
- 2 Let c be a whole number which is not a perfect square. Then \sqrt{c} is irrational.

Binomial Coefficients

Pascal's Triangle



Pascal's Triangle and Binomial Coefficients

Recall. For all $n \in \omega$ and all $k \in \{0, \dots, n\}$, the binomial coefficient $\binom{n}{k}$ denotes the k -th number on the n -th row of Pascal's triangle.

Key features

① $\binom{0}{0} = 1.$

② *Boundary conditions:* For each $n \in \mathbb{N}$,

$$\binom{n}{0} = \binom{n}{n} = 1.$$

③ *Recurrence relation:* For each $n \in \omega$ and all $k \in \{1, \dots, n\}$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Why n choose k ?

All 2-element subsets of the 4-element set $\{1, 2, 3, 4\}$ are

$$\underbrace{\{1, 2\}, \{1, 3\}, \{2, 3\}}_{\text{ones without 4}}, \underbrace{\{1, 4\}, \{2, 4\}, \{3, 4\}}_{\text{ones with 4}}.$$

Note that

- the number of subsets without 4 is
- the number of subsets with 4 is

Thus the total number of 2-element subsets of the 4-element set is

Why n choose k ? (cont')

In general, one can count the number of k -element subsets of the $(n + 1)$ -element set

$$\{1, 2, \dots, n, n + 1\}$$

in an analogous fashion:

- the number of subsets without $n + 1$ is
- the number of subsets with $n + 1$ is

Thus the total number of k -element subsets of the $(n + 1)$ -element set is

Why n choose k ? (cont')

The idea above is key to a proof by induction of the following theorem.

Number of Subsets (cf. S14E03)

For each $n \in \omega$, for each $k \in \{0, \dots, n\}$, the number of k -element subsets of an n -element set is $\binom{n}{k}$.

Why Binomial Coefficients?

Consider the expansion of $(a + b)^2$:

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b) \\ &= (a + b)a + (a + b)b \\ &= a^2 + ba \\ &\quad + ab + b^2 \\ &= a^2 + 2ab + b^2.\end{aligned}$$

Note that the coefficients of a^2 , ab , and b^2 are

$$1, 2, 1,$$

respectively, which are precisely the numbers on row 2 of Pascal's triangle:

$$\binom{2}{0}, \binom{2}{1}, \binom{2}{2}.$$

Binomial Expansion

Expansion of $(a + b)^3$

Work out the expansion of $(a + b)^3$ and compare the coefficients with the numbers in row 3 of Pascal's triangle.

Binomial Expansion (cont')

Expansion of $(a + b)^4$

Work out the expansion of $(a + b)^3$ and compare the coefficients with the numbers in row 3 of Pascal's triangle.

The Binomial Theorem

The examples above suggest the following general result.

The Binomial Theorem (S05E11)

For each $n \in \omega$ and all $a, b \in \mathbb{R}$,

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$
$$\sum_{k=0}^n \binom{n}{k}a^{n-k}b^k.$$

- **Convention:** For each $x \in \mathbb{R}$, $x^0 = 1$.