Section 14.

# Infinite Sets

## Recap: Equinumerousness

A and B have the same number of elements.

- **Def'n** Let $A$ and $B$ be sets. To say that $A$ is equinumerous to $B$ $(A \approx B)$ means that there exists a bijection from $A$ to $B$.

- $\approx$ is an equivalence relation:

  ① $(\forall A)(A \approx A)$

  ② $(\forall A, B)[A \approx B \Rightarrow B \approx A]$

  ③ $(\forall A, B, C)[A \approx B \wedge B \approx C \Rightarrow A \approx C]$

- **The rigidity prop. of finite sets** A finite set cannot be equinumerous to any of its proper subsets.

# Equinumerousness and Infinite Sets

# Infinite Sets Are Not Rigid

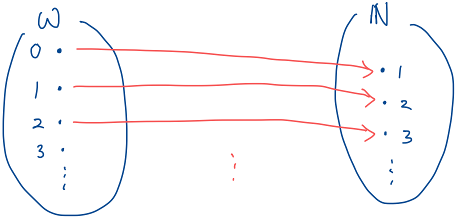
$$\omega = \{0, \underbrace{1, 2, 3, \dots}_{= \mathbb{N}}\} = \mathbb{N} \cup \{0\}$$

By the rigidity property, a finite set cannot be equinumerous to any of its proper subsets. But this is not the case with an infinite set.

## Example 1 (Natural Numbers and Whole Numbers)

$\mathbb{N}$ is a proper subset of $\omega$, but $\omega$ is equinumerous to $\mathbb{N}$ because the function $f$ defined by $f(n) = n + 1$ for all $n \in \omega$ is a bijection from $\omega$ to $\mathbb{N}$.

**Question.** Is $f$ above the only bijection from $\omega$ to $\mathbb{N}$? If not, construction another one of your own.



# Infinite Sets Are Not Rigid (cont')

$A = B \cup \{1\}$

## Example 2 (Intervals)

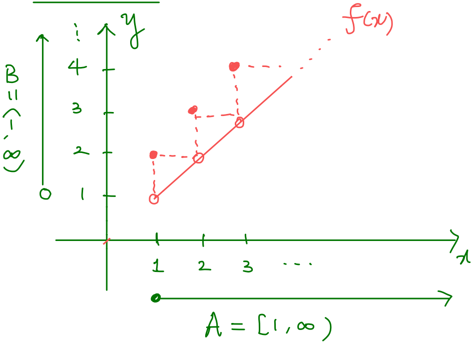
Let $A = [1, \infty)$ and $B = (1, \infty)$. $B$ is a proper subset of $A$, but $A$ is equinumerous to $B$. Find an example of a bijection $f$ from $A$ to $B$.

$f : A \to B$ defined by

$$f(x) = \begin{cases} x + 1 & \text{for } x = 1, 2, 3, \dots \\ x & \text{otherwise} \end{cases}$$

This is a bijection. (Confirm it.)

Illustration



## Kishore's idea

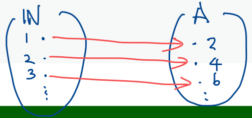
The function $f$ defined by

$$f(x) = x+1 \quad \text{for all } x \in [1, \infty)$$

is a bijection from $[1, \infty)$ to $[2, \infty)$.

Thus $[1, \infty) \approx [2, \infty)$.

# Infinite Sets Are Not Rigid (cont’)




## Example 3 (Even and Odd Natural Numbers)

Let $A = \{2, 4, 6, \ldots\}$ be the set of even natural numbers and let $B = \{1, 3, 5, \ldots\}$ be the set of odd natural numbers. Verify the following by finding suitable bijections.

1. $A \approx B$. (Note A and B are disjoint.)
2. $\mathbb{N} \approx A$. (Note A is a proper subset of $\mathbb{N}$.)
3. $\mathbb{N} \approx B$. (Note B is a proper subset of $\mathbb{N}$.)

① $f : A \to B$ defined by

$$f(n) = n - 1$$

is a bijection.

② $g : \mathbb{N} \to A$ defined by

$$g(n) = 2n$$

will do.

③ $h : \mathbb{N} \to B$ defined by

$$h(n) = 2n - 1$$

will do.

# Infinite Sets Are Not Rigid (cont')

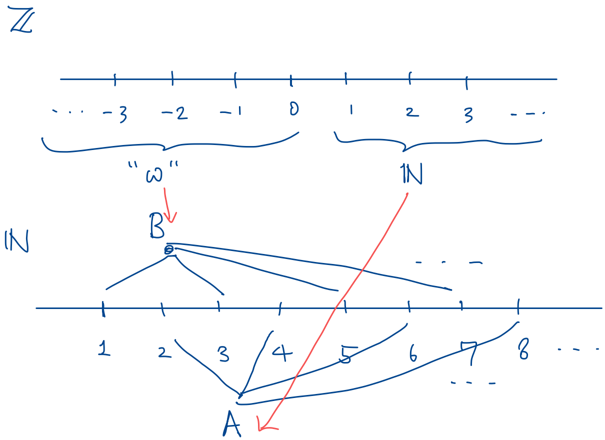
**Exercise**

Show that $\mathbb{Z} \approx \mathbb{N}$.

Hint $f : \mathbb{Z} \to \mathbb{N}$ defined by

$$f(n) = \begin{cases} \text{"even"} & \text{if } n = 1, 2, 3, \dots \\ \text{"odd"} & \text{if } n = 0, -1, -2, \dots \end{cases}$$

Scheme



# More Counter-Intuitive Examples

## Example 4 (Perfect Squares)

Let $S = \{n^2 : n \in \mathbb{N}\}$ be the set of all perfect squares. Then $S \approx \mathbb{N}$.

$f : \mathbb{N} \to S$ defined by

$$f(n) = n^2$$

is a bijection.

$1, 4, 9, 16, 25, \cdots, 100, \cdots, 10000$

$3/10 = 30\%$

$10/100 = 10\%$

$100/10000 = 1\%$

$\vdots$

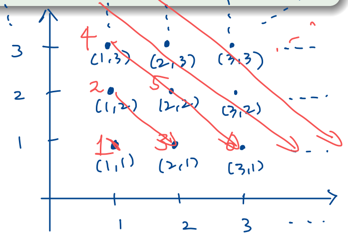
## Example 5 (Cartesian Product of $\mathbb{N}$)

$\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$ because the function $g$ defined by

$$
\begin{aligned}
&g(1,1) = 1, \\
&g(1,2) = 2, \quad g(2,1) = 3, \\
&g(1,3) = 4, \quad g(2,2) = 5, \quad g(3,1) = 6, \\
&g(1,4) = 7, \quad g(2,3) = 8, \quad g(3,2) = 9, \quad g(4,1) = 10, \\
&\text{and so on,}
\end{aligned}
$$

is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

$\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$



## Example 6 (Positive Rational Numbers)

Let $A = \{x \in \mathbb{Q} : x > 0\}$. Each element $x \in A$ can be expressed uniquely as $x = a/b$ where $a, b \in \mathbb{N}$ and the fraction is in lowest terms. We can list all such fractions in lowest terms as follows:

$\frac{1}{1},$ 1

$\frac{1}{2},$ 2 $\quad \frac{2}{1},$ 3

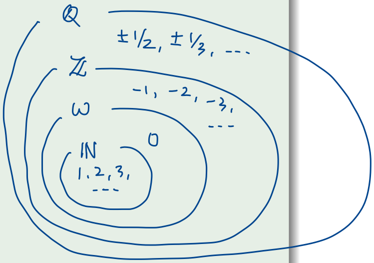
$\frac{1}{3},$ 4 $\quad \frac{3}{1},$ 5

$\frac{1}{4},$ 6 $\quad \frac{2}{3},$ 7 $\quad \frac{3}{2},$ 8 $\quad \frac{4}{1},$ 9

$\frac{1}{5},$ 10 $\quad \frac{5}{1},$ 11

$\frac{1}{6},$ 12 $\quad \frac{2}{5},$ 13 $\quad \frac{3}{4},$ 14 $\quad \frac{4}{3},$ 15 $\quad \frac{5}{2},$ 16 $\quad \frac{6}{1},$ 17

and so on,




Define $f(n)$ to be the $n$-th term in this list. Then $f$ is a bijection from $\mathbb{N}$ to $A$.

## Exercise

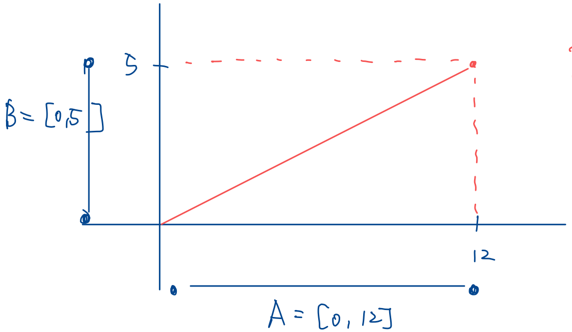
Show that $\mathbb{Q} \approx \mathbb{N}$.

## Example 7

Let $A = [0, 12]$ and let $B = [0, 5]$. Then $A \approx B$ because ~~of~~ if we let $f(x) = 5x/12$ for all $x \in A$, then $f$ is a bijection from $A$ to $B$.

**Note.** In general, for any $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$,

$$[a, b] \approx [c, d], \quad (a, b) \approx (c, d), \quad (a, b] \approx (c, d], \quad \text{and} \quad [a, b) \approx [c, d).$$




$$y = mx$$

$$= \frac{5}{12}x$$

## Example 8

Let $\varphi(x) = x/(1-|x|)$ for all $x \in (-1,1)$. In S11E23, we showed that $\varphi$ is a bijection from $(-1,1)$ to $\mathbb{R}$. Hence $(-1,1) \approx \mathbb{R}$.

**Note**. By this example and the note from the previous example, we deduce that $(0,1) \approx \mathbb{R}$. This fact will be useful later.