Proving Uniqueness

Contents

Uniqueness

Introduction

At times, one wishes to show that there exists exactly one value of x in the universe of discourse for which $P(x)$ is true.

"There exists a unique x such that $P(x)$."

denoted as

$$(\exists! x) P(x).$$

This uniqueness statement can be rephrased as

$$\underbrace{(\exists x) P(x)}_{\text{at least one}} \wedge \underbrace{(\forall x_1)(\forall x_2)[P(x_1) \wedge P(x_2) \Rightarrow x_1 = x_2]}_{\text{at most one}},$$

or

$$(\exists x_1)\,[\underbrace{P(x_1)}_{\text{existence}} \wedge \underbrace{(\forall x_2)\,(P(x_2) \Rightarrow x_1 = x_2)}_{\text{uniqueness}}]\,.$$

Examples

- $(\exists! x \in \mathbb{R})(7x - 1 = 0)$ is true because

  $1/7$ is a real number such that $7(1/7) - 1 = 0$ and $\leftarrow$ existence

  if $x \in \mathbb{R}$ such that $7x - 1 = 0$, then $x = 1/7$. $\leftarrow$ Uniqueness.

- $(\exists! x \in \mathbb{Z})(7x - 1 = 0)$ is false because there is no integer x for which $7x - 1 = 0$.

  (due to non-existence)

Examples (cont')

- $(\exists! x \in \mathbb{R})(x^2 - 8x + 16 = 0)$ is true because

$$\left.\begin{aligned} & x^2 - 8x + 16 = 0 \\ \iff\ & (x-4)^2 = 0 \\ \iff\ & x - 4 = 0 \\ \iff\ & x = 4. \end{aligned}\right\} \text{There exist a unique solution.}$$

- $(\exists! x \in \mathbb{R})(x^2 - 8x + 12 = 0)$ is false because

(due to non-uniqueness)

$$\left.\begin{aligned} & x^2 - 8x + 12 = 0 \\ \iff\ & (x-4)^2 - 4 = 0 \\ \iff\ & (x-4)^2 = 4 \\ \iff\ & (x - 4 = -2) \vee (x - 4 = 2) \\ \iff\ & (x = 2) \vee (x = 6). \end{aligned}\right\} \text{There are two solutions.}$$

Examples: Mixed with Universal Quantifier

- $(\forall a > 0)(\exists! x > 0)(x^2 = a)$ is true.

Proof. Let $a_0 > 0$ be arbitrary. Then $(\exists! x > 0)(x^2 = a_0)$ is true because

$\sqrt{a_0}$ is a positive real number such that $(\sqrt{a_0})^2 = a_0$ and $\leftarrow$ existence

if x is a positive real number such that $x^2 = a_0$, then $x = \sqrt{a_0}$, discarding the negative square root. $\leftarrow$ uniqueness

Since a_0 is an arbitrary positive real number, $(\forall a > 0)(\exists! x > 0)(x^2 = a)$ is true. □

Q. How about $(\forall a > 0)(\exists! x \in \mathbb{R})(x^2 = a)$ or

$(\forall a \in \mathbb{R})(\exists! x > 0)(x^2 = a)$?

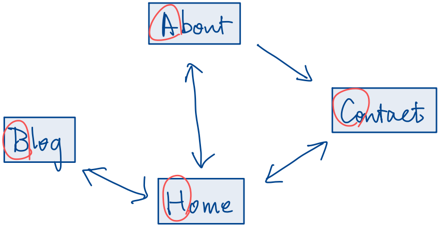
Homework (1/26; due Wed 2/2)

Section 3: # 11, 12, 14

Exercises with Quantifiers

Example: Order in mixed quantifiers

Consider the following webpages:



$P(x,y)$

$x \backslash y$	A	B	C	H
A	F	F	T	T
B	F	F	F	T
C	F	F	F	T
H	T	T	T	F

$P(x,y)$: x has a link to y.

- $(\forall x)(\exists y)\, P(x,y)$
- $(\exists x)(\forall y)\, P(x,y)$

Example: Negation of multiply quantified sentences.

Let f be a function from $\mathbb{R}$ to $\mathbb{R}$

and let $a \in \mathbb{R}$.

To say that $\boxed{f \text{ is continuous at } a}$

means that