Tautologies and Conditional Proofs

Announcements

- . Quiz 1: today (20-min; noon 11:59 PM)
- · HW1: next Wed (probs assigned this week)
- · No class next Mon (MLK)
- 0 OH next week: TW 4:30 PM 6:00 PM (Zoom)

Contents

1 Some Remarks on Logical Connectives

2 Conditional Proofs

Some Remarks on Logical Connectives

The order of priority of the logical connectives (from highest to lowest):

$$\neg$$
, \land , \lor , \Rightarrow , \Leftrightarrow

Examples.

$$\begin{array}{ccc} \neg P \wedge \neg Q & \text{means} & (\neg P) \wedge (\neg Q) \\ P \wedge Q \vee R & \text{means} & (P \wedge Q) \vee R \\ P \wedge Q \Rightarrow P \vee Q & \text{means} & (P \wedge Q) \Rightarrow (P \vee Q) \\ P \Rightarrow (Q \Rightarrow R) \Leftrightarrow P \wedge Q \Rightarrow R & \text{means} & [P \Rightarrow (Q \Rightarrow R)] \Leftrightarrow [(P \wedge Q) \Rightarrow R] \end{array}$$

- Economical writing by omitting some parentheses
- Judicious inclusion of some dispensable parenthesis may enhance readability

The logical connective \Rightarrow is not associative, that is,

$$P \Rightarrow (Q \Rightarrow R)$$
 and $(P \Rightarrow Q) \Rightarrow R$

are logically *in*equivalent . Furthermore, neither of the two is logically equivalent to

$$(P\Rightarrow Q)\wedge (Q\Rightarrow R)$$

For further discussions, see Exercise 11 and Remarks 2.18 and 2.19.

However, people commonly use
$$P \Rightarrow Q \Rightarrow R$$

to mean

$$(P \Rightarrow Q) \land (Q \Rightarrow R)$$

Associativity of Biconditionals

The logical connective \Leftrightarrow is associative, that is,

$$P \Leftrightarrow (Q \Leftrightarrow R)$$
 and $(P \Leftrightarrow Q) \Leftrightarrow R$

are logically equivalent. However, neither of the two is logically equivalent to

$$(P \Leftrightarrow Q) \land (Q \Leftrightarrow R)$$

For further discussions, see Exercise 12 and Remark 2.20.

However, people commonly use
$$P \Leftrightarrow Q \Leftrightarrow R$$
to mean
$$(P \Leftrightarrow Q) \land (Q \Leftrightarrow R)$$

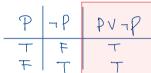
Tautologies

In logic, a tautology is a sentence which is true under any possible truth values of its propositional variables.

Examples.

- \bullet $P \vee \neg P$
- $[\neg (P \land Q)] \Leftrightarrow [\neg P \lor \neg Q]$ $(P \land Q) \Rightarrow P$
- $P \Rightarrow (P \lor Q)$
- If $R \equiv S$, then the sentence $R \Leftrightarrow S$ is a tautology. (Why?)

Exercise. Construct a tautology using three sentences P, Q, and R.



Conditional Proofs

Conditional Proofs

"always" true, i.e., is a tautology

Conditional Proofs

To show that $A \Rightarrow B$ is true, it suffices to consider the case where A is true and to show that in this case, B must also be true. This approach is known as the method of conditional proof.

Template of Conditional Proof. To show $A \Rightarrow B$ is a tautology:

A1: Suppose that A is true.

Work to show that B is true under A1.

Discharging A1, we see that $A \Rightarrow B$ is true under no assumptions. Therefore $A \Rightarrow B$ is a tautology.

Euppose A is folse.
Regardless of the
Fouth value of B,
A>B is true,

Suppose A is true.
Only when B is true,
A => B is true.

Conditional Proofs (cont')

Example

Use the method of conditional proof to explain in words why

$$P \Rightarrow (P \lor Q)$$

is a tautology. (Do not use cases. Be explicit about discharging assumptions.)

Solution.

A1: Suppose P is true.

Then $P \vee Q$ is true.

We have shown that $P \lor Q$ is true under the assumption A1 that P is true.

Discharging A1, we see that $P \Rightarrow (P \lor Q)$ is true under no assumptions. Therefore $P \Rightarrow (P \lor Q)$ is a tautology, because we have shown that it is true under no assumptions on the truth values of P and Q.

 $(P \land Q) \Rightarrow P$ is a tantology. $(P \land Q) \Rightarrow Q$ as well.

Módus Pónens

The following is useful in proofs.

Modus Ponens

If $P\Rightarrow Q$ is true and P is also true, then Q must be true. This rule of inference is called *modus ponendo ponens* or, more commonly, *modus ponens*.

Exercise 1

Use the method of conditional proof to explain in words why the sentence

$$[(P\Rightarrow Q)\wedge[(Q\Rightarrow R)]\Rightarrow(P\Rightarrow R)$$

is a tautology. Be explicit about discharging assumptions.

nested cond. proof (z lev.)

We wish to show that P=>R is true. AZ: Suppose P is true. We wish to show that R is true. From A1, it follows that P > Q is true. From this and AZ, we see that Q is true, by modus ponens. | From AI again, it follows that Q=>R is true. From this and the fact that Q is true, we see that R Ts true, by modus We have shown that P=> R is true under A1 and A2. Discharging A2, we see that P => R is true under A1 alone. Discharging A1, we see that $A \Rightarrow C$ is true under no assumptions. So It is a tautelogy.

IAI: Suppose (P⇒Q) ∧ (Q⇒R) is true.

Solution

Exercise 2

Use the method of conditional proof to explain in words why the sentence

$$[P\Rightarrow (Q\Rightarrow R)]\Rightarrow [(P\Rightarrow Q)\Rightarrow (P\Rightarrow R)]$$

is a tautology. Be explicit about discharging assumptions.

nested

Homework (1/14; due Wed 1/19)

Section 2: #8,9,10,15,17