Induction

Section 5

Proof by Induction

yet another proof technique

The method of *proof by induction* is based on the following principle.

Principle of Mathematical Induction

Let $P(n)$ be any statement about n. Suppose we have proved that

$$P(1) \text{ is true} \tag{1}$$

and that

$$\text{for each natural number } n, \text{ if } P(n) \text{ is true, then } P(n+1) \text{ is true.} \tag{2}$$

Then we may conclude that for each natural number n, $P(n)$ is true.

- This is a commonly used technique to prove a universal sentence $(\forall x \in A)P(x)$ when A is $\mathbb{N}$.

whose universe of discourse is $\mathbb{N}$.

i.e. $(\forall n \in \mathbb{N})\, P(n)$

Steps in Proof by Induction

$2\cdot\boxed{1}^{st}-1$ $\quad 2\cdot\boxed{2}^{nd}-1$ $\quad n^{th}$ odd.

$(\forall n \in \mathbb{N})\ P(n)$

Sum of Odd Natural Numbers

For each $n \in \mathbb{N}$, $\underbrace{1 + 3 + \cdots + (2n-1)}_{\text{sum of first } n \text{ positive odd integers}} = n^2$.

Proof. Let $P(n)$ be the sentence $$1 + 3 + \cdots + (2n-1) = n^2.$$	Declare $P(n)$.
BASE CASE: Observe that $P(1)$ is true because if $n = 1$, then the left-hand side is just 1 and the right-hand side is $1^2 = 1$.	Show $P(1)$ is true.
INDUCTIVE STEP: Let $n \in \mathbb{N}$ such that $P(n)$ is true. Then $$\begin{aligned}(\text{LHS}) = 1 + 3 + \cdots + (2n-1) + [2(n+1)-1] &\\ = n^2 + [2(n+1)-1] &\\ = n^2 + 2n + 2 - 1 = n^2 + 2n + 1 &\\ = (n+1)^2 = (\text{RHS}) &\end{aligned}$$ Thus $P(n+1)$ is true.	Show $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))$. The first sentence in this paragraph is called *the inductive hypothesis*. $(\star)$
CONCLUSION: Therefore, by induction, for each $n \in \mathbb{N}$, $P(n)$ is true. That is, for each $n \in \mathbb{N}$, $1 + 3 + \cdots + (2n-1) = n^2$. $\square$	Use induction to conclude.

Practical tip

Scratch work: Write out $P(n+1)$

In prev. example:

$$P(n): \quad 1+3+\cdots+(2n-1) = n^2$$

$$P(n+1): \quad 1+3+\cdots+(2n-1)+[2(n+1)-1] = (n+1)^2$$

Example 1

Prove by induction that for each $n \in \mathbb{N}$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof Let $P(n)$ be the sentence

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

BASE CASE $P(1)$ is true because the LHS of $P(1)$ is 1, and the RHS is $\frac{1(1+1)}{2} = \frac{1 \cdot 2}{2} = 1$.

($P(1)$ is true)

INDUCTIVE STEP Let $n \in \mathbb{N}$ such that $P(n)$ is true.

$(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$

$$\underbrace{1 + 2 + \cdots + n}_{\frac{n(n+1)}{2}} + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$= \left(\frac{n}{2} + 1\right)(n+1)$$

$$= \left(\frac{n+2}{2}\right)(n+1) = \frac{(n+1)[(n+1)+1]}{2}.$$

Scratch

$P(n+1)$

$$1 + 2 + \cdots + n + (n+1)$$

$$= \frac{(n+1)[(n+1)+1]}{2}$$

This may show up as

$$\frac{(n+1)(n+2)}{2}$$

Thus $P(n+1)$ is true.

CONCLUSION Therefore, by induction, for each $n \in \mathbb{N}$, $P(n)$ is true. In other words, for each $n \in \mathbb{N}$,

$$1 + 2 + \cdots + n = n(n+1)/2. \qquad \square$$

Example 2

Prove by induction that for each $n \in \mathbb{N}$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof Let $P(n)$ be the sentence

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

BASE CASE ($P(1)$ is true) $P(1)$ is true because the LHS of $P(1)$ is $1^2 = 1$, and the RHS is $\frac{1(1+1)(2\cdot 1+1)}{6} = \frac{1\cdot 2\cdot 3}{6} = 1$.

INDUCTIVE STEP $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$ Let $n \in \mathbb{N}$ such that $P(n)$ is true.

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= (n+1)\left(\frac{n(2n+1)}{6} + \frac{6(n+1)}{6}\right)$$

$$= \frac{(n+1)\left[n(2n+1) + 6(n+1)\right]}{6}$$

Scratch $P(n+1)$

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2$$

$$= \frac{(n+1)\left[(n+1)+1\right]\left[2(n+1)+1\right]}{6}$$

$$\parallel$$

$$\frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{(n+1)(2n^2+n+6n+6)}{6}$$

$$= \frac{(n+1)(2n^2+7n+6)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{(n+1)\left[(n+1)+1\right]\left[2(n+1)+1\right]}{6}.$$

Thus $P(n+1)$ is true.

CONCLUSION Therefore, by induction, for each $n \in \mathbb{N}$, $P(n)$ is true. In other words, for each $n \in \mathbb{N}$,

$$1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6 .$$

$\square$

Example 3

Prove by induction that for each $n \in \mathbb{N}$, 3 divides $4^n - 1$.

Calculation for inductive step

$P(n)$: 3 divides $4^n - 1$

- Assumption : $n \in \mathbb{N}$ s.t. $P(n)$ is true, i.e. 3 divides $4^n - 1$.
- WTS : $P(n+1)$ is true, i.e., 3 divides $4^{n+1} - 1$
- Key observation :

$$
\begin{aligned}
4^{n+1} - 1 &= 4 \cdot 4^n - 1 \\
&= (3+1) \cdot 4^n - 1 \\
&= 3 \cdot 4^n + 4^n - 1
\end{aligned}
$$