Proving Uniqueness

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Uniqueness

Introduction

At times, one wishes to show that there exists exactly one value of x in the universe of discourse for which $P(x)$ is true.

"There exists a unique x such that $P(x)$."

denoted as

$$(\exists! x)P(x).$$

This uniqueness statement can be rephrased as

$$(\exists x)P(x) \wedge (\forall x_1)(\forall x_2)[P(x_1) \wedge P(x_2) \Rightarrow x_1 = x_2],$$

or

$$(\exists x_1)\left[P(x_1) \wedge (\forall x_2)\left(P(x_2) \Rightarrow x_1 = x_2\right)\right].$$

Examples

- $(\exists! x \in \mathbb{R})(7x - 1 = 0)$ is true because

  $1/7$ is a real number such that $7(1/7) - 1 = 0$ and

  if $x \in \mathbb{R}$ such that $7x - 1 = 0$, then $x = 1/7$.

- $(\exists! x \in \mathbb{Z})(7x - 1 = 0)$ is false because there is no integer x for which $7x - 1 = 0$.

Examples (cont')

- $(\exists! x \in \mathbb{R})(x^2 - 8x + 16 = 0)$ is true because

$$\begin{aligned} & x^2 - 8x + 16 = 0 \\ \iff & (x-4)^2 = 0 \\ \iff & x - 4 = 0 \\ \iff & x = 4. \end{aligned}$$

- $(\exists! x \in \mathbb{R})(x^2 - 8x + 12 = 0)$ is false because

$$\begin{aligned} & x^2 - 8x + 12 = 0 \\ \iff & (x-4)^2 - 4 = 0 \\ \iff & (x-4)^2 = 4 \\ \iff & (x - 4 = -2) \vee (x - 4 = 2) \\ \iff & (x = 2) \vee (x = 6). \end{aligned}$$

Examples: Mixed with Universal Quantifier

- $(\forall a > 0)(\exists! x > 0)(x^2 = a)$ is true.

  Proof. Let $a_0 > 0$ be arbitrary. Then $(\exists! x > 0)(x^2 = a_0)$ is true because

  > $\sqrt{a_0}$ is a positive real number such that $(\sqrt{a_0})^2 = a_0$ and
  >
  > if x is a positive real number such that $x^2 = a_0$, then $x = \sqrt{a_0}$, discarding the negative square root.

  Since a_0 is an arbitrary positive real number, $(\forall a > 0)(\exists! x > 0)(x^2 = a)$ is true. $\square$