See. 7 Complete induction

Euclid's Lemma

Next bectave: Section 10.

Division by a Prime

Recap

Principle of Complete Mathematical Induction (PCMI)

Let P(n) be any statement about n. Suppose we have proved that

$$P(1)$$
 is true (1)

and that

for each
$$n \in \mathbb{N}$$
, if $P(1), \dots, P(n)$ are all true, then $P(n+1)$ is true. (2)

Then we may conclude that for each natural number n, P(n) is true.

Proof by Complete Induction (Template)

To prove $(\forall x \in \mathbb{N})P(n)$ using complete induction:

• Declaration: Let P(n) be the sentence ...

• BASE CASE: P(1) is true because ...

• INDUCTIVE STEP: Let $n \in \mathbb{N}$ such that $P(1), \dots, P(n)$ are all true.

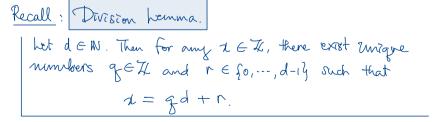
• Conclusion: Therefore, by complete induction, for each $n \in \mathbb{N}$, P(n) is true.

Example: Division by a Prime

Theorem 1 (Euclid's Lemma)

Let p be a prime number. Then for all integers x and y, if p divides xy, then p divides x or p divides y.

- We have been using this result without proof for a while; see Remark 4.50.
- It can now be proved using complete induction.



Proof of Euclid's Lemma TT = f all prime numbers fThe theorem in symbols: $(\forall y \in \mathbb{Z})(\forall x \in \mathbb{Z})[p|xy \Rightarrow p|x \vee p|y]$ Proof of Euclid's Lemma TT = f all prime numbers fThe theorem in symbols: $(\forall y \in \mathbb{Z})(\forall x \in \mathbb{Z})[p|xy \Rightarrow p|x \vee p|y]$

Proof Let $y \in \mathbb{Z}$. Let P(x) be the sentence

If play, then pla or ply. We wish to show that for each XEZ, p(x) is true.

We wish to show that for each x Ex, p(x) is true.
Since play iff p(-x)y, and plx iff p(-x),

We see that p(x) and p(-x) share the same fruth value. So it suffices to show that for each $x \in W$, p(x) is true.

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We shall do so using complete induction. Goal: $(\forall x \in \omega) f(x)$ BASE CASE P(0) is true because P(0)

IND. STEP Let $2 \in W$ such that $P(0), \dots, P(2)$ are all true. WTS P(2+1) is true. In other words, we WTS that

if p (941) y, then p(641) or ply.

Suppose p (12+1)y. (We need to show that p (12+11) or ply.)

Now either $p \le x + 1$ or p > x + 1.

Case 1 Case 2.

Case 1 Suppose $p \le d+1$. Dividing d+1 by p, we get d+1 = qp+r where $q \in IN$ and $r \in \{0, ..., p-1\}$. Note that $0 \le r \le p-1 \le d$

Now ry = (a+1) y - gpy and 80 p | ry.

divible divible by p.

by p

by assumption

So, by the hypothesis, plr or ply.

$$P(x): P|xy \Rightarrow P|x \vee P|y$$
 $P(-x): P|(-x)y \Rightarrow P|(-x) \vee P|y$
 $P(0): P|0.y \Rightarrow P|0 \text{ or } P|y$
 $P(0): P|0.y \Rightarrow P|0 \text{ or } P|y$
 $P(0): P|0.y \Rightarrow P|0 \text{ or } P|y$