Division Lemma

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The Division Lemma (Euclid)

Let $d \in \mathbb{N}$. Then for each $x \in \mathbb{Z}$, there exist unique numbers $q \in \mathbb{Z}$ and $r \in \{0, \ldots, d-1\}$ such that $x = qd + r$.

Outline of Proof.

1. Prove existence of q and r when $x \in \boldsymbol{\omega}$ by induction.
2. Prove existence of q and r when $x \in \mathbb{Z}$.
3. Prove uniqueness of q and r.

Part 1: Proof of Existence of q and r when $x \in \boldsymbol{\omega}$

Let $P(x)$ be the sentence

> There exist numbers $q \in \mathbb{Z}$ and $r \in \{0, \ldots, d-1\}$ such that $x = qd + r$.

BASE CASE:

INDUCTIVE STEP: Let $x \in \boldsymbol{\omega}$ such that $P(x)$ is true.

CONCLUSION: Therefore, by induction, for each $x \in \boldsymbol{\omega}$, $P(x)$ is true.

Part 2: Proof of Existence of q and r when $x \in \mathbb{Z}$

Consider any $x \in \mathbb{Z}$. Then either $x \geqslant 0$ or $x \leqslant -1$.

Case 1. Suppose $x \geqslant 0$. Then $x \in \boldsymbol{\omega}$, so $P(x)$ is true by Part 1.

Case 2. Suppose $x \leqslant -1$.

Part 3: Proof of Uniqueness of q and r

Consider any $x \in \mathbb{Z}$. Suppose $q_1, q_2 \in \mathbb{Z}$, $r_1, r_2 \in \{0, \ldots, d-1\}$, $x = q_1 d + r_1$, and $x = q_2 d + r_2$. We wish to show that $q_1 = q_2$ and $r_1 = r_2$.