

Ordered Pairs and Cartesian Products

Ordered Pairs

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We write (a, b) for the ordered pair whose first entry is a and whose second entry is b . Unlike sets, the order by which the entries are listed does matter.

Theorem 1 (Fundamental Property of Ordered Pairs)

Let a and b be any objects. We have $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.

Example.

- $(1, 1) = (1, 1)$
- $(3, 8) \neq (8, 3)$

Notes on Ordered Pairs

The proof of the seemingly obvious statement in the theorem relies on a careful definition of the ordered pair (a, b) .

Definition 2 (Ordered Pairs as Sets; Kuratowski)

Let a and b be any objects. The *ordered pair* (a, b) is the set $\{\{a\}, \{a, b\}\}$.

Notes on Ordered Pairs (cont')

- The ordered triple (a, b, c) can be defined as $((a, b), c)$. The analogue of the fundamental property of ordered pairs holds for ordered triples. Namely,

$$(a, b, c) = (a', b', c') \text{ iff } a = a', b = b', \text{ and } c = c'.$$

- The ordered quadruple (a, b, c, d) can be defined as $((a, b, c), d)$. We have

$$(a, b, c, d) = (a', b', c', d') \text{ iff } a = a', b = b', c = c', \text{ and } d = d'.$$

- Continuing in the same fashion, the ordered n -tuple (a_1, a_2, \dots, a_n) can be defined for each natural number $n \geq 2$. The fundamental property of ordered n -tuple states

$$(a_1, a_2, \dots, a_n) = (a'_1, a'_2, \dots, a'_n) \text{ iff for each } j \in \{1, 2, \dots, n\}, a_j = a'_j.$$

Cartesian Products

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Definition 3 (Cartesian Products)

Let A and B be sets. Then the *Cartesian product of A and B* (denoted $A \times B$) is the set of all ordered pairs (x, y) such that $x \in A$ and $y \in B$; in other words,

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}.$$

Example.

- For a set A , the Cartesian product of A with itself, $A \times A$, is also denoted A^2 .
- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, the Cartesian product of \mathbb{R} with itself. This is the coordinate plane in analytical geometry.
- $[1, 4] \times [2, 3]$ is a rectangle in \mathbb{R}^2 . Specifically, it is the set

$$\{(x, y) : 1 \leq x \leq 4 \text{ and } 2 \leq y \leq 3\}.$$

Example 4

Let $A = \{1, 3\}$ and $B = \{2, 4, 6\}$. Find $A \times B$.

Cartesian Products of More Than Two Sets

The Cartesian product of three sets A , B , and C (denoted $A \times B \times C$) is defined in a similar way, namely,

$$A \times B \times C = \{(x, y, z) : x \in A, y \in B, \text{ and } z \in C\}.$$

In general:

Definition 5 (Cartesian Product of n Sets)

Let $n \in \mathbb{N}$ such that $n \geq 2$ and let A_1, \dots, A_n be sets. Then the *Cartesian product of A_1, A_2, \dots, A_n* (denoted $A_1 \times A_2 \times \dots \times A_n$) is the set of all ordered n -tuples (x_1, x_2, \dots, x_n) such that $x_1 \in A_1, x_2 \in A_2, \dots, x_n \in A_n$; in other words,

$$A_1 \times A_2 \times \dots \times A_n = \{(x_1, x_2, \dots, x_n) : \text{for each } j \in \{1, 2, \dots, n\}, x_j \in A_j\}.$$

Cartesian Products of More Than Two Sets (cont')

Example 6

Let $A = \{0, 1\}$. Find $A^3 = A \times A \times A$.