More Examples of Induction

Contents

From $\mathbb{N}$ to $\mathbb{Z}$

From $\mathbb{N}$ to $\mathbb{Z}$

Example 1

Prove that for each $x \in \mathbb{Z}$, x is even or x is odd.

Notes: Induction over $\mathbb{Z}$.

- Induction may start from a number other than 1.

- To prove a universal sentence $(\forall x \in \mathbb{Z})P(x)$:
  1. Prove by induction that $P(x)$ is true for each nonnegative integer[1] x.
  2. Prove that $P(x)$ is true for each negative integer x.

Pascal's Triangle and the Binomial Theorem

Pascal's Triangle

The following infinite array of numbers is known as *Pascal's tirangle*:

$$
\begin{array}{rccccccccccc}
0^{\text{th}} \text{ row} \longrightarrow & & & & & & 1 & & & & & \\
1^{\text{st}} \text{ row} \longrightarrow & & & & & 1 & & 1 & & & & \\
2^{\text{nd}} \text{ row} \longrightarrow & & & & 1 & & 2 & & 1 & & & \\
3^{\text{rd}} \text{ row} \longrightarrow & & & 1 & & 3 & & 3 & & 1 & & \\
4^{\text{th}} \text{ row} \longrightarrow & & 1 & & 4 & & 6 & & 4 & & 1 & \\
5^{\text{th}} \text{ row} \longrightarrow & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & & & & & & \vdots & & & & &
\end{array}
$$

Notation. For all $n \in \omega$ and all $k \in \{0, \ldots, n\}$, let $\binom{n}{k}$ denote the k-th number on the n-th row; this notation is read *n choose k*. This is also called a *binomial coefficient*.

Key Features.

1. $\binom{0}{0} = 1.$
2. *Boundary conditions*: For each $n \in \mathbb{N}$, $\binom{n}{0} = \binom{n}{n} = 1.$
3. *Recurrence relation*: For each $n \in \omega$ and all $k \in \{1, \ldots, n\}$, $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Example

Application of Key Features

Use **Key Features** above to compute $\binom{4}{2}$.

Solution.

$$\begin{aligned}
\binom{4}{2} &= \binom{3}{2} + \binom{3}{1} \\
&= \binom{2}{2} + \binom{2}{1} + \binom{2}{1} + \binom{2}{0} \\
&= 1 + 2\binom{2}{1} + 1 \\
&= 1 + 2\left[\binom{1}{1} + \binom{1}{0}\right] + 1 \\
&= 1 + 2(1+1) + 1 \\
&= 6.
\end{aligned}$$

Naming: *n choose k*

Question: Why is $\binom{n}{k}$ called "n choose k"?

Number of Subsets

List all subsets of the set $\{a, b, c, d\}$ with exactly 2 elements.

Solution.

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$$

There are $6 = \binom{4}{2}$ subsets with two elements. In other words, there are $\binom{4}{2} = 6$ ways to choose 2 elements from a set with 4 elements.

In Section 14, we will prove that for each $n \in \omega$, for each n-element set A, for each $k \in \{0, \ldots, n\}$, the number of k-element subsets of A is $\binom{n}{k}$.

Naming: Binomial Coefficients

Question: Why is $\binom{n}{k}$ called a binomial coefficient?

Expansion of $(a+b)^3$

1. Compute $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$, and $\binom{3}{3}$.
2. Expand the cube of the binomial $a+b$, that is, expand $(a+b)^3$.

Binomial Theorem

The example above suggests:

The Binomial Theorem

For each $n \in \boldsymbol{\omega}$ and all $a, b \in \mathbb{R}$,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$
$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$$

- **Convention**: For each $x \in \mathbb{R}$, $x^0 = 1$.