Set Operations

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<u>Overview</u> Set Theory

- Introduction : collection of objects

  "$\in$" notation

  set - builder notation

- Subsets : $A \subseteq B \iff (\forall x)(x \in A \Rightarrow x \in B)$

  $\emptyset \subseteq A$ for any set A.

Unions, Intersections, and Relative Complements

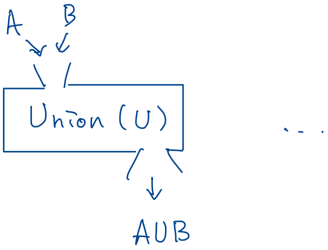
Definition 1 (Set Operations)

Let A and B be sets.

- The *union of A and B* (denoted $A \cup B$) is the set of all things that belong to at least one of the sets A and B; in other words,
$$A \cup B = \{x : x \in A \boxed{\text{or}}\ x \in B\}.$$
- The *intersection of A and B* (denoted $A \cap B$) is the set of all things that belong to both of the sets A and B; in other words,
$$A \cap B = \{x : x \in A \boxed{\text{and}}\ x \in B\}.$$
- The *relative complement of B in A* (denoted $A \backslash B$) is the set of all things that belong to A but not to B; in other words,
$$A \backslash B = \{x : x \in A \text{ and } x \notin B\}.$$

Notes on Set Operations

- Short ways to read $A \cup B$, $A \cap B$, and $A \backslash B$ are "A union B," "A intersect B," and "A less B" respectively.
- $A \cup B$ should not be read "A or B." $A \cap B$ should not be read "A and B." We use the connectives "and" and "or" to connect sentences, not nouns.
- The results of set operations are another sets, so they are nouns. Hence, one must not write something like "$A \cup B$ iff $x \in A$ or $x \in B$." Instead, write "$x \in A \cup B$ iff $x \in A$ or $x \in B$."



Set Inclusion and Set Operations

Example 2

Let A and B be sets. Then:

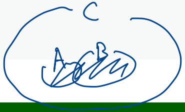
1. $A \subseteq A \cup B$ and $B \subseteq A \cup B$.
2. $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

Proof of 1. We will only show $A \subseteq A \cup B$; $B \subseteq A \cup B$ can be shown similarly.

Let $x \in A$. (We need to show that $x \in A \cup B$.) Then $\underbrace{x \in A}_{T \text{ by assumption}}$ or $x \in B$.

Let x be arbitrary. Assume $x \in A$.

Thus $x \in A \cup B$. Therefore $A \subseteq A \cup B$. $\square$



Example 3

Let A, B, and C be sets. Then:

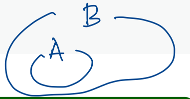
1. If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.
2. If $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

<u>Proof of ①</u> Suppose $A \subseteq C$ and $B \subseteq C$. (NTS: $A \cup B \subseteq C$)

Let x be arb.
Assume $x \in A \cup B$.

Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. In the case where $x \in A$, $x \in C$, because $A \subseteq C$. In the case where $x \in B$, $x \in C$, because $B \subseteq C$. In either case, $x \in C$. (We have shown that for each x, if $x \in A \cup B$, then $x \in C$.) Therefore, $A \cup B \subseteq C$. $\square$

Set Inclusion and Set Operations (cont')



Example 4 (Equivalence to Set Inclusion)

Let A and B be sets. Then:

1. $A \subseteq B$ iff $A \cup B = B$.
2. $A \subseteq B$ iff $A \cap B = A$.
3. $A \subseteq B$ iff $A \setminus B = \varnothing$.

<u>Proof of ①</u> ($\Rightarrow$) Suppose $A \subseteq B$. It is clear that $B \subseteq A \cup B$ from Ex.2①. Now, $A \subseteq B$ by assumption and $B \subseteq B$ by reflexivity. Thus $A \cup B \subseteq B$ by Ex.3①. Hence $A \cup B = B$.

($\Leftarrow$) Conversely, suppose that $A \cup B = B$. By Ex.2①, $A \subseteq A \cup B$, but $A \cup B = B$. Thus, $A \subseteq B$.

$\square$