

Section 3

Quantifiers (I)

Quiz 2 (noon ~ 11:59 pm)

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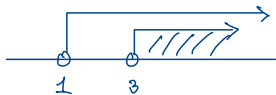
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Basics of Quantifiers

Motivation

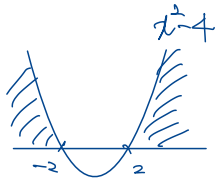
Let x be a real number. Consider the following sentences.

- $A(x)$: If $x > 3$, then $x > 1$.
- $B(x)$: $x^2 - 4 > 0$.



The truth value of each sentence depends on the value of the variable x .

- $A(x)$ is true for all x . (\top)
- $B(x)$ is true for $x < -2$ or $x > 2$. (\top)



In general, they can be rephrased using *quantifiers* as:

- For all x , $A(x)$ is true.
- For some x , $B(x)$ is true.

Quantifiers

The quantifiers \forall and \exists , along with the logical connectives, are main ingredients of modern symbolic logic.

Quantifier	Symbol	Technical Name
"for each"	\forall	<i>universal quantifier</i>
"for some"	\exists	<i>existential quantifier</i>

Example. Let x be a person in this class room. Let $P(x)$ stands for " x likes ramen." Then

- $(\forall x)P(x)$: "For each x , x likes ramen." or "Everybody likes ramen."
- $(\exists x)P(x)$: "For some x , x likes ramen." or "Somebody likes ramen."

Alternate ways to read.

$(\forall x)P(x)$:

For each x , $P(x)$.

For all x , $P(x)$.

For every x , $P(x)$.

For any x , $P(x)$.

$(\exists x)P(x)$:

For some x , $P(x)$.

For at least one x , $P(x)$.

There exists x such that $P(x)$.

Universe of Discourse

The collection over which the variable x ranges is called *the universe of discourse*. When clear from context, it is omitted in notation; if not, specify the universe of discourse using the following notation.

$$(\forall x \in U)P(x) \quad \text{or} \quad (\exists x \in U)P(x).$$

"is a member of"

Frequently used collections.

- \mathbb{N} : the set of natural numbers, $\{1, 2, 3, \dots\}$
- \mathbb{Z} : the set of integers, $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- \mathbb{Q} : the set of rational numbers
- \mathbb{R} : the set of real numbers
- \mathbb{C} : the set of complex numbers

Free and Bound Variables

- In $P(x)$, x can stand for any particular element of the universe of discourse; it is called a *free variable*.
- In $(\forall x)P(x)$ or $(\exists x)P(x)$, x varies over the universe of discourse, not standing for any particular element; it is called a *bound variable* or a *dummy variable*.

• $P(x) : x \text{ likes ramen.} \rightarrow x \text{ is a free variable.}$

• $(\forall x)P(x) : \text{For each } x, x \text{ likes ramen.} \rightarrow x \text{ is a dummy variable.}$

same as $(\forall y)P(y)$

Universal and Existential Quantifiers

#2 on Quiz 2

1, C, 2, B

If a card has B on one side, then it has 2 on the other side.

P

Q

	P	Q	$P \Rightarrow Q$
1 st	?	F	
2 nd	F	?	T
3 rd	?	T	T
4 th	T	?	

} Know this w/o needing to flip cards

Universal Quantifier (\forall)

Let U be the universe of discourse.

- $(\forall x)P(x)$ is true when $P(x)$ is true for all values of x in U .
- To show $(\forall x)P(x)$ is false, it suffices to show that $P(x)$ is false for at least one value of x in U ; such x is said to be a counterexample that disproves the universal sentence.

↓
universal sentence-

Universal Quantifier (cont')

Example. State whether each of the following sentences is true or false. Then prove your claim.

- ① $(\forall x \in \mathbb{R})(x - 2 = 5)$ T / **F** We claim $(\forall x \in \mathbb{R})(x - 2 = 5)$ is false.

Proof Assume that $(\forall x \in \mathbb{R})(x - 2 = 5)$ is true.

Then since 2 is a real number, $2 - 2 = 5$.

But $2 - 2 = 0 \neq 5$. This is a contradiction.

Hence $(\forall x \in \mathbb{R})(x - 2 = 5)$ is false. \square

- ② $(\forall x \in \mathbb{R})(x^2 + 6x + 10 > 0)$ **T** / F

Proof Let $x_0 \in \mathbb{R}$ be arbitrary.

Then

$$x_0^2 + 6x_0 + 10 = x_0^2 + 6x_0 + 9 + 1$$

$$= (x_0 + 3)^2 + 1 \geq 0 + 1 = 1 > 0.$$

that the given
sentence is true.

We showed that $x_0^2 + 6x_0 + 10 > 0$. Since x_0 is arbitrary, we conclude \checkmark

"Completion of the square."

Existential Quantifier (\exists)

Let U be the universe of discourse.

- $(\exists x)P(x)$ is true when $P(x)$ is true for at least one value of x in U ; such x is said to be an example that proves the existential sentence.
- To show $(\exists x)P(x)$ is false, it is necessary to show that $P(x)$ is false for all values of x in U .

Existential Quantifier (cont')

Example. State whether each of the following sentences is true or false.

- $(\exists x \in \mathbb{R})(x - 2 = 5)$ $\textcircled{T} / \text{F}$

Proof Take x to be 7, i.e. $x=7$.

Then

$$x - 2 = 7 - 2 = 5.$$

This proves $(\exists x \in \mathbb{R})(x - 2 = 5)$.

- $(\exists x \in \mathbb{R})(x^2 + 6x + 10 < 0)$ $\text{T} / \textcircled{\text{F}}$

Proof To show that the given is false, we must prove that

$x^2 + 6x + 10 < 0$ is false for all $x \in \mathbb{R}$.

That is, we need to show

$$(\forall x \in \mathbb{R})(x^2 + 6x + 10 \geq 0).$$

[Finish this.]

Notes on Quantifiers

Connections to Logical Connectives

Suppose the universe of discourse consists only of two objects $\{a, b\}$. Note that

- $(\forall x)P(x)$ is true exactly when $P(a) \wedge P(b)$ is true.
- $(\exists x)P(x)$ is true exactly when $P(a) \vee P(b)$ is true.

In general, when the universe of discourse is a finite set $\{a_1, a_2, \dots, a_n\}$, then

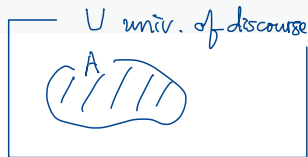
- $(\forall x)P(x)$ has the same truth value as $P(a_1) \wedge P(a_2) \wedge \dots \wedge P(a_n)$.
- $(\exists x)P(x)$ has the same truth value as $P(a_1) \vee P(a_2) \vee \dots \vee P(a_n)$.

Upshot: • \forall is a generalization of \wedge .
• \exists is a generalization of \vee

Notation

Suppose A is a subcollection of the universe of discourse. Then

- $(\forall x \in A)P(x)$ is a shorthand notation for $(\forall x)[(x \in A) \Rightarrow P(x)]$.
- $(\exists x \in A)P(x)$ is a shorthand notation for $(\exists x)[(x \in A) \wedge P(x)]$.



When the universe of discourse is \mathbb{R} , a subcollection may be characterized by an inequality in which case one may use notations e.g.,

- $(\forall x > 0)(2x + 7 = 3)$ T / F
- $(\exists x \geq 7)(x^2 - 4x + 3 > 0)$ T / F
 - Use comp. of sq.
 - Graph.

Scope of Quantifiers

The scope of a quantifier is specified using appropriate delimiters.

Example. Let n be an element in $\{2, 3, 5, 7\}$ and let

$P(n)$: n is a prime number.

$Q(n)$: n is an even number.

Then

bound/dummy
variable

free variable.

- $(\forall n)P(n) \wedge Q(\underline{n})$ stands for

" Every number in $\{2, 3, 5, 7\}$ is a prime number
and n is an even number." (True when $n = 2$.)

- $(\forall n)[P(n) \wedge Q(n)]$ stands for

" Every number in $\{2, 3, 5, 7\}$ is both a prime
number and an even number." (False)

Homework (1/21 ; due Wed 1/26)

Section 3 : # 1 (e-k), 3, 4