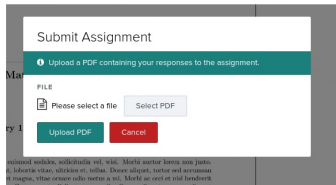
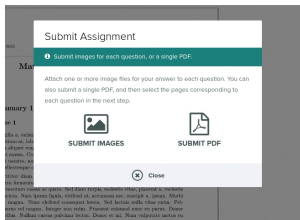


## Proof Techniques

- HW 1 due 11:59 PM.
- Office Hour : 4:30 PM - 6:00 PM (Zoom)
- Quiz 2 on Friday (noon - 11:59 PM)

# Uploading HW to Gradescope



HW01 | Assign Questions and Pages

SUBMITTED AT: JANUARY 15, 9:29 AM

Select questions and pages to indicate where your responses are located. Use **ABC** to deselect all items and **XYZ** to select multiple questions.

QUESTION OUTLINE	
TITLE	POINTS
1 From Wednesday (Section 2: 1, 2, 3, 5, 7)	0.0 pts
2 From Friday (Section 2: 8, 9, 10, 15, 17)	0.0 pts

Select a question or a page.

1 2 3 4

Select pages for Wed. problems.

## HW01 | Assign Questions and Pages

SUBMITTED AT: JANUARY 19, 9:09 AM

Select questions and pages to indicate where your responses are located. Use **esc** to deselect all items and hold **shift** to select multiple questions.

### Question Outline

Select pages to assign to Question 1.

**TITLE**

### POINTS

1 From Wednesday (Section 2: 1, 2, 3, 5, 7)

0.0 pts

P1 x P2 x

**2** From Friday (Section 2: 8, 9, 10, 15, 17)

0.0 pts

[illegible]

Wiederholen, um Wissen zu festigen, ist ein zentraler Aspekt des Lernens. Es gibt verschiedene Methoden, um das Gelernte zu wiederholen und zu vertiefen. Hier sind einige Beispiele:

- Wiederholungsfragen:** Fragen, die das Gelernte abfragen, um das Verständnis zu überprüfen.
- Wiederholungsübungen:** Übungen, die das Gelernte anwenden, um das Verständnis zu vertiefen.
- Wiederholungsprojekte:** Projekte, die das Gelernte anwenden, um das Verständnis zu vertiefen.
- Wiederholungspräsentationen:** Präsentationen, die das Gelernte anwenden, um das Verständnis zu vertiefen.

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[illegible][illegible]

C

Select questions and pages to indicate where your responses are located. Use **esc** to deselect all items and hold **shift** to select multiple questions.

Select pages to assign to Question 2.

TITLE	POINTS
<b>1</b> From Wednesday (Section 2: 1, 2, 3, 5, 7) <b>P1</b> * <b>P2</b> *	0.0 pts
<b>2</b> From Friday (Section 2: 8, 9, 10, 15, 17) <b>P3</b> * <b>P4</b> *	0.0 pts

[illegible][illegible][illegible][illegible]

# Contents

① Logic of Solving Equations

② Proof by Contradiction

③ Proof by Contraposition

# Logic of Solving Equations

# Solving Equations

Logically speaking, to say that  $x = a$  is a solution of the equation  $f(x) = 0$  is to state

$$f(x) = 0 \iff x = a$$

which usually can be seen by a chain of biconditionals.

For example, we see that  $x^2 = 5x - 6$  if and only if  $x = 2$  or  $x = 3$  by:

$$\begin{aligned} x^2 = 5x - 6 &\iff x^2 - 5x + 6 = 0 \\ &\iff (x - 2)(x - 3) = 0 \\ &\iff x - 2 = 0 \text{ or } x - 3 = 0 \\ &\iff x = 2 \text{ or } x = 3. \end{aligned}$$

One needs to be careful to confirm that all steps are true biconditional sentences.

# Examples


## Rational Equation

Solve the equation

$$\frac{x-2}{x^2+2x-8} = \frac{1}{8}.$$

$\parallel$   
 $f(x)$

**Erroneous solution.**


$$\begin{aligned} \Rightarrow x-2 &= (1/8)(x^2+2x-8) \\ \Leftrightarrow 8x-16 &= x^2+2x-8 \\ \Leftrightarrow 0 &= x^2-6x+8 = (x-2)(x-4) \\ \Leftrightarrow x &= 2, 4 \end{aligned}$$

Which step is not a true biconditional sentence?

If  $x=2$ , then

$$x^2+2x-8 = 4+4-8 = 0.$$

(division by zero).

→ This work shows:

$$\text{If } f(x) = \frac{1}{8}, \text{ then } x=2 \text{ or } x=4.$$



## Examples (cont')

### Correct solution.

- Previous work showing

$$f(x) = \frac{1}{8} \Rightarrow x=2 \text{ or } x=4$$

- Now if  $x=2$ , then  $x^2+2x-8 = 4+4-8=0$ ,

so  $f(x)$  is undefined, so  $f(x) = \frac{1}{8}$  is not true.

- If  $x=4$ , then  $x-2 = 4-2=2$  and  $x^2+2x-8 = 16+8-8=16$ ,

so  $f(x) = \frac{x-2}{x^2+2x-8} = \frac{2}{16} = \frac{1}{8}$ . That is, if  $x=4$ , then  $f(x) = \frac{1}{8}$ .

Therefore,  $f(x) = \frac{1}{8}$  if and only if  $x=4$ .

Think

$$f(x) = \frac{x+1}{x^2-x-2}$$

$$= \frac{x+1}{(x+1)(x-2)} = \cancel{g(x)}$$

$$g(x) = \frac{1}{x-2}$$

# Examples

## Equation Involving Radicals

Solve the equation

$$x = -\sqrt{x+6} \quad \text{where } x = g(x)$$

An erroneous solution:

$$\begin{aligned} \Rightarrow x^2 &= x + 6 && \text{squaring both sides} \\ \Leftrightarrow x^2 - x - 6 &= 0 \\ \Leftrightarrow (x+2)(x-3) &= 0 \\ \Leftrightarrow x &= -2, 3 \end{aligned}$$

Is  $x = 3$  a solution of the original equation?

$$(LHS) = 3 \neq -\sqrt{3+6} = -\sqrt{9} = -3 = (RHS)$$

If  $x = g(x)$ ,  
then  $x = -2$  or  
 $x = 3$ .

## Examples (cont')

Correct solution.

' Prev. work

• Now if  $x = -2$ , then  $g(x) = -\sqrt{x+6} = -\sqrt{-2+6} = -\sqrt{4} = -2$ ,

so  $x = g(x)$ . That is, if  $x = -2$ , then  $x = g(x)$ .

• If  $x = 3$ , then  $g(x) = -\sqrt{x+6} = -\sqrt{3+6} = -\sqrt{9} = -3$ ,

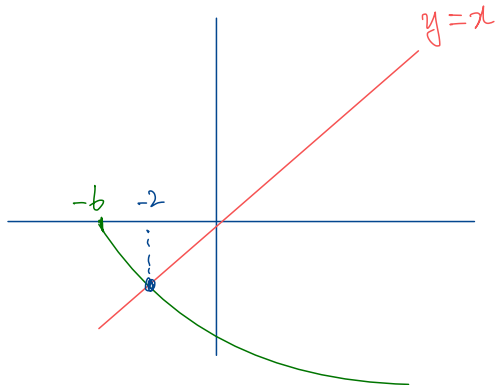
so  $x \neq g(x)$ .

Therefore,  $x = g(x)$  iff  $x = -2$ .

## Practical concerns (insight)

Orig. eqn

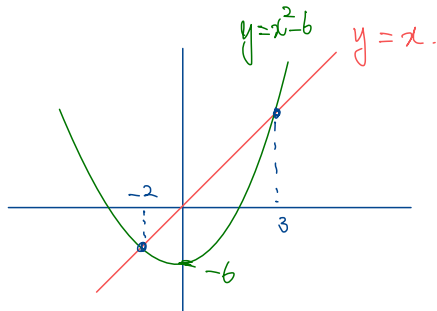
$$x = -\sqrt{x+6}$$



Mod. eqn (upon squaring)

$$x^2 = x + 6$$

$$x^2 - 6 = x$$



# Proof by Contradiction

# Overview: Proof Techniques

The follow is the list of proof techniques discussed in Section 2 of the textbook:

- ☒ Conditional proof
- ☐ Proof by contradiction
- ☐ Proof by contraposition

• used to show  $P \Rightarrow Q$  is a tautology.

• template.

|     A1: Assume  $P$  is true  
|     |     Show  $Q$  is true.  
|     Discharging A1,  $P \Rightarrow Q$  under no assumptions.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

# Contradictions

A *contradiction* is a sentence of the form  $Q \wedge \neg Q$ , which is false regardless of the truth value of  $Q$ .

$Q$	$\neg Q$	$Q \wedge \neg Q$
T	F	F
F	T	F



# Proof by Contradiction

## Proof by Contradiction

To prove a sentence  $P$ , assume  $\neg P$  and deduce a contradiction. This approach is known as the method of *proof by contradiction*.

**Template.** To prove  $P$ :

- Begin with “Assume  $\neg P$  is true.”
- Deduce a contradiction.
- Conclude that  $P$  is true.

a contradiction

pf. by contradiction

Why does it work?

$P$	$\neg P$	$\wedge$	$\neg P \Rightarrow \wedge$
T	F	F	T
F	T	F	F

# Proof by Contradiction (cont')

## Example

Let  $n$  be an integer. Using the method of proof by contradiction, prove that

If  $n^2$  is an odd number, then  $n$  is an odd number.

$A_1$

$C_1$

Soln

$A_1$ : Assume  $n^2$  is an odd number.

We wish to show that  $n$  is an odd number.

Assume (towards a contradiction)  $n$  is not an odd number. Since  $n$  is an integer,  $n$  must be an even number, so  $n^2$  is an even number, so  $n^2$  is not an odd number.

This leads to a contradiction. We reject the assumption that  $n$  is not an odd number. Therefore,  $n$  is an odd number under  $A_1$ .

Discharging  $A_1$ , we see that  $A_1 \Rightarrow C_1$  under no assumptions.

← proof by contrad.

# Proof of a Negative Sentence

The usual way to prove a negative sentence  $\neg P$  is to prove by contradiction, that is, assume  $P$  and deduce a contradiction.

**Why does it work?**

Proof by Contradiction on  $\neg P$ .

# Proof of a Negative Sentence (cont')

## Section 2, Exercise 23

Use the method of conditional proof to explain in words why

$$\underbrace{[(P \Rightarrow Q) \wedge \neg Q]}_{A_1} \Rightarrow \underbrace{\neg P}_{C_1}$$

is a tautology.

Suggestion: Conditional proof.

A1: Assume  $A_1$  is true.

NTS  $C_1$  is true.

$\left. \begin{array}{l} \vdots \\ \vdots \end{array} \right\} \text{proof of a negative sentence.}$

## Proof of a Negative Sentence (cont')

# Proof by Contraposition

# Overview: Proof Techniques

The follow is the list of proof techniques discussed in Section 2 of the textbook:

- ☒ Conditional proof
- ☒ Proof by contradiction
- ☐ Proof by contraposition

→ To prove  $P \Rightarrow Q$ ,  
| assume  $P$  is true and  
| strive to show  $Q$  is true.

↘ To prove  $P$ ,  
| assume  $\neg P$  is true and  
| deduce a contradiction ( $Q \wedge \neg Q$ ).

# Contrapositive

Given  $P \Rightarrow Q$ , the related conditional sentence  $\neg Q \Rightarrow \neg P$  is called the contrapositive of  $P \Rightarrow Q$ . Note that  $P \Rightarrow Q$  is logically equivalent to  $\neg Q \Rightarrow \neg P$ . (Confirm this using a truth table.)

← Exercise.

**Example.** Given the conditional sentence

A: If  $\underbrace{\text{today is Sunday}}_P$ , then  $\underbrace{\text{I do not have to go to work today}}_Q$ .

- Converse of A:  $Q \Rightarrow P$

If I don't have to go to work today, then today is Sunday.

- Contrapositive of A:  $\neg Q \Rightarrow \neg P$

If I have to go to work today, then today is not Sunday.



# Proof by Contraposition

## Proof by Contraposition

To prove  $P \Rightarrow Q$ , it suffices to prove  $\neg Q \Rightarrow \neg P$ .

relies on the fact that

$$(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$$

# Proof by Contraposition (cont')

## Example (revisited)

Let  $n$  be an integer. Using the method of proof by contraposition, prove that

If  $n^2$  is an odd number, then  $n$  is an odd number.

*Solution.* The given sentence is logically equivalent to the sentence

If  $n$  is not an odd number, then  $n^2$  is not an odd number.

contrapositive of given

(\*)

which we will prove. <sub>A</sub>

A1: Assume that  $n$  is not an odd number.

(We wish to show that  $n^2$  is not an odd number.)

"short-term goal"

Since  $n$  is an integer but  $n$  is not an odd number,  $n$  is an even number.

Hence  $n^2$  is an even number, so  $n^2$  is not an odd number.

We have shown this under A1.

Discharging A1, we conclude that the conditional sentence (\*) is true under no assumptions. This completes the proof by contraposition that the original conditional sentence is true.  $\square$

# Proof by Contradiction vs Proof by Contraposition

Let's examine the two proof techniques in proving  $P \Rightarrow Q$ .

## Proof by contradiction.

Assume  $P$ . (NTS  $Q$  is true.)

Assume  $\neg Q$ .

Show  $\neg P$ .

Contradiction,  $P \wedge \neg P$ !

So  $Q$  must be true.

Therefore,  $P \Rightarrow Q$ .

## Proof by contraposition.

Assume  $\neg Q$ .

Show  $\neg P$ . (if this can be done w/o  $P$ .)

So  $\neg Q \Rightarrow \neg P$ .

Therefore,  $P \Rightarrow Q$ , by contraposition.

Homework (1/19 ; due Wed 1/26)

Section 2 : # 19, 20, 24