Rational and Irrational Numbers

Contents

1 Rational Numbers

2 Irrational Numbers

Rational Numbers

Definition

$(\exists m \in \mathbb{Z})(\exists n \in \mathbb{Z})(n \neq 0 \wedge x = m/n)$

Definition 1 (Rational Numbers)

To say that *x is a rational number* means that there exist integers m and n such that $n \neq 0$ and $x = m/n$.

- Any integer is a rational number.

$$7 = 7/1 = 49/7 = -49/-7 = \cdots$$

- $1/2, \ 3/4, \ -5/17, \ \ldots$

$$\boxed{1/2} = 2/4 = 10/20 = \cdots$$

in lowest terms (num. & denom. has no common factor other than 1)

Examples

Example 2 (Sum of Rational Numbers)

Let u and v be rational numbers. Then $u+v$ is a rational number.

Proof Since u is rational, we can pick integers a and b such that

$$b \neq 0 \quad \text{and} \quad u = a/b.$$

Since v is rational, we can pick integers c and d such that

$$d \neq 0 \quad \text{and} \quad v = c/d.$$

Then

$$u + v = \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}.$$

Since $a, b, c,$ and d are integers, $ad+bc$ is an integer and bd is an integer.

Furthermore, since $b \neq 0$ and $d \neq 0$, $bd \neq 0$. Hence, $u+v$ is a rational number. $\square$

Let u and v be rational numbers.

- $u + v$ is rational. ⟵ just shown
- $u - v$ is rational.
- uv is rational.
- If $v \neq 0$, then u/v is rational.

HW

Special Forms of Rational Numbers

A given rational number x can be expressed in many different ways. For example,

$$\frac{7}{3} = \frac{-7}{-3} = \frac{14}{6} = \frac{-14}{-6} = \cdots = \frac{350}{150} = \cdots.$$

Fact ☺

The fact that $\boxed{\text{each rational number can be written in lowest terms}}$ such as $7/3$ can be proved later once we learn complete induction. For now, we can prove the following:

Proof strategy:

Consider cases.

Rational Number as An Integer Divided by A Natural Number

Let x be a rational number. Then there exists an integer a and a natural number b such that $x = a/b$.

Example

$x = 2/3$: $a = 2$ integers
$b = 3$ natural number

$x = -2/3 = 2/(-3) = (-2)/3 = \cdots$

$a = -2$ integer
$b = 3$ natural number.

Irrational Numbers

Definition

Definition 3 (Irrational Numbers)

To say that $\boxed{x \textit{ is an irrational number}}$ means that $\underline{x \text{ is a real number}}$ and $\underline{x \text{ is not a rational number.}}$

Note

To show x is irrational:

- x is real
- x is not rational

Remember that each irrational number is a real number!

$$\text{“}x\text{ is an irrational number.”} \not\equiv \text{“}x\text{ is not a rational number.”}$$

Consider the following question.

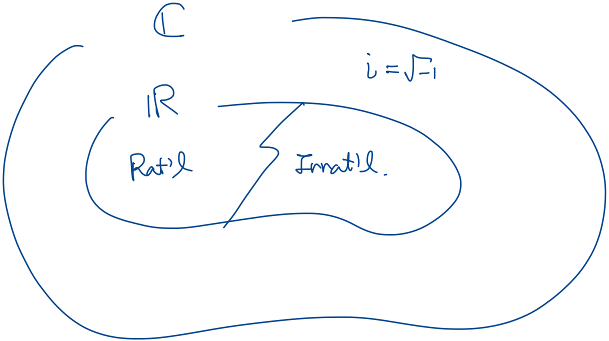
Question. Determine whether each of the following is true or false. Explain your answers.

1. For each $x \in \mathbb{C}$, if x is an irrational number, then x is not a rational number. (T)

2. For each $x \in \mathbb{C}$, if x is not a rational number, then x is an irrational number. (F)

Counterexample: x is not rational and x is not irrational

$x = \sqrt{-1} = i$.


$\mathbb{C}$
$i = \sqrt{-1}$
$\mathbb{R}$
Rat'l
Irrat'l.

Examples

Example 4 (Sum of Rational and Irrational Numbers)

Let x be a rational number and let y be an irrational number. Then $x + y$ is an irrational number.

Proof: Since x and y are real, $x+y$ is a real number.

So it remains to show that $x+y$ is not a rat'l number.

Suppose $x+y$ is a rational number.

Then

$$(x+y) - x$$

is a rat'l number as a difference of two rat'l numbers.

But

$$(x+y) - x = y.$$

Hence y is a rational number. But since y is irrational, y is not rational.

This is a contradiction. So the assumption that $x+y$ is rational is wrong.

Thus $x+y$ is not rational. Therefore, $x+y$ is irrational. $\square$

Let x be a rational number and let y be an irrational number.

- $x + y$ is irrational. $\leftarrow$ Shown

HW
- $x - y$ is irrational.
- $y - x$ is irrational.
- If $x \neq 0$, then xy is irrational.
- If $x \neq 0$, then x/y is irrational.
- If $x \neq 0$, then y/x is irrational.

$y \neq 0$ not needed because y is irrational and 0 is rational.

Examples (cont')

Question. Let x and y be real numbers. Determine whether each of the following is true or false. Explain.

1. If xy is rational, then x and y are rational.

2. If $x + y$ is rational, then x and y are rational.

Irrationality of $\sqrt{2}$

Theorem 5

1. *Let x be a rational number. Then $x^2 \neq 2$.*
2. *$\sqrt{2}$ is irrational.*

Proof Since x is rational, we can pick integers a and b such that

$$b \neq 0 \quad \text{and} \quad x = a/b.$$

Using $\boxed{\text{Fact } ☺}$, we assume $x = a/b$ is in lowest terms.

(WTS $x^2 \neq 2$) Suppose $x^2 = 2$.

Then

$$\left(\frac{a}{b}\right)^2 = 2, \quad \text{So} \quad \frac{a^2}{b^2} = 2, \quad \text{So} \quad a^2 = 2b^2.$$

But a^2 is an even number. Thus a must be an even number.

$a^2 = 2b^2$

So we can find an integer k such that $a = 2k$.

Then

$$(2k)^2 = 2b^2, \quad \text{so} \quad 4k^2 = 2b^2, \quad \text{so} \quad b^2 = 2k^2.$$

a

Thus b^2 is an even number. Hence b must be even.

This is a contradiction to $x = a/b$ is in lowest terms because a and b have 2 as a common factor.

Hence $x^2 \neq 2$. $\square$