Inverses and Bijections

Recall the following definitions. Let A and B be sets.

"f is a surjection from A to B."

means

① $f : A \to B$

② for each $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Note: ② $\Leftrightarrow$ $\mathrm{Rng}(f) = B$

$\Leftrightarrow$ for each $y \in B$, the eqn $f(x) = y$ has at least one solution $x \in A$.

"f is an injection from A to B."

means

① $f : A \to B$

② for any $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Note ② $\Leftrightarrow$ for any $x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ (contrapositive)

$\Leftrightarrow$ for each $y \in B$, the eqn $f(x) = y$ has at most one soln $x \in A$.

Compositions of Surjections and Injections

Composition of Surjections

$$\begin{tikzcd} A \ni a \arrow[r, "f", "\text{surj}"'] \arrow[rr, bend right] & B \ni b \arrow[r, "g", "\text{surj}"'] & C \ni c = (g \circ f)(a) \end{tikzcd}$$

Theorem 1 $g \circ f$ surj.

Let A, B, and C be sets. Suppose that f is a surjection from A to B and g is a surjection from B to C. Then $g \circ f$ is a surjection from A to C.

WTS: for each $c \in C$, there exists $a \in A$ such that $(g \circ f)(a) = c$.

Proof. Let $c \in C$. Since g is a surjection from B to C, there exists $b \in B$ such that $g(b) = c$. Since f is a surjection from A to B, there exists $a \in A$ such that $f(a) = b$. It follows that

$$(g \circ f)(a) = g(\underbrace{f(a)}_{= b}) = g(b) = c.$$

We have shown that for any $c \in C$, there exists $a \in A$ such that $(g \circ f)(a) = c$. In other words, $g \circ f$ is a surjection from A to C. □

Composition of Injections

Theorem 2

Let f and g be injections. Then $g \circ f$ is an injection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

See Theorem 11.72. (Full proof requires knowledge on bijections.)

We can see easily why $g \circ f$ is an injection as shown below.

Partial proof. Let $x_1, x_2 \in \mathrm{Dom}(g \circ f) = \{ x \in \mathrm{Dom}(f) : f(x) \in \mathrm{Dom}(g) \}$.

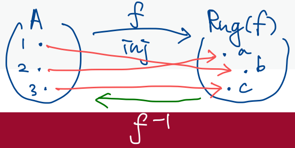
Suppose that $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then $g(f(x_1)) = g(f(x_2))$.

Since g is an injection, $f(x_1) = f(x_2)$.

Since f is an injection, $x_1 = x_2$.

Therefore, $g \circ f$ is an injection. $\square$

Inverses



Definition 3

Let f be an injection. Then for each $y \in \mathrm{Rng}(f)$, we shall write $f^{-1}(y)$ for the unique $x \in \mathrm{Dom}(f)$ such that $f(x) = y$. This defines a function f^{-1} from $\mathrm{Rng}(f)$ to $\mathrm{Dom}(f)$. The function f^{-1} is called the *inverse of the function* f.

Note.

- $\mathrm{Dom}(f^{-1}) = \mathrm{Rng}(f)$ and $\mathrm{Rng}(f^{-1}) = \mathrm{Dom}(f)$.

obvious from def'n

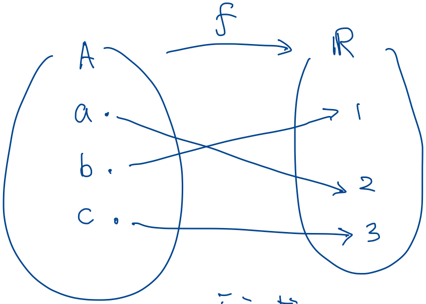
$\mathrm{Rng}(f^{-1}) \subseteq \mathrm{Dom}(f)$ obvious from def'n.

$\mathrm{Dom}(f) \subseteq \mathrm{Rng}(f^{-1})$

Pf. Let $x \in \mathrm{Dom}(f)$. Write $y = f(x)$. Then $f^{-1}(y) = x$, so $x \in \mathrm{Rng}(f^{-1})$. $\square$

Example 4

Let $A = \{a, b, c\}$ where a, b, and c are distinct. Define $f : A \to \mathbb{R}$ by $f(a) = 2, f(b) = 1$, and $f(c) = 3$. Is f an injection? If so, what is the inverse of f?



injection.

$$f^{-1} : \overset{\mathrm{Rng}(f)}{\overset{\parallel}{\{1, 2, 3\}}} \longrightarrow \overset{\mathrm{Dom}(f)}{\overset{\parallel}{A}}$$

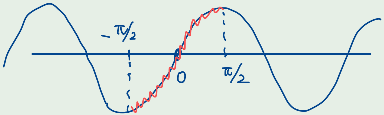
defined by

$$f^{-1}(1) = b$$
$$f^{-1}(2) = a$$
$$f^{-1}(3) = c.$$

Example 5

For each of the following functions, state whether it is an injection. If it is an injection, determine its inverse function. If it is not an injection, find an interval such that the restriction of the function to that interval is an injection whose inverse function has a standard name, and determine that inverse function.

1. $f(x) = x^2$ for all $x \in \mathbb{R}$.
2. $f(x) = x^3$ for all $x \in \mathbb{R}$.
3. $f(x) = \sin(x)$ for all $x \in \mathbb{R}$.



(3) f is not an injection because

$$f(x) = f(x+2\pi)$$

for any $x \in \mathbb{R}$.

However, $f \restriction [-\pi/2, \pi/2]$ is an injection.

The inverse of $f \restriction [-\pi/2, \pi/2]$ is the arc sine function, that is,

$$\left(f \restriction [-\pi/2, \pi/2]\right)^{-1} = \arcsin$$

Bijections

Bijections

Definition 6

Let A and B be sets. To say that *f is a bijection from A to B* means that f is both a surjection from A to B and an injection.

Note.

- Another name for a bijection from A to B is *a one-to-one correspondence between A and B*.

Bijections and Inverses

$A \underset{g}{\overset{f}{\rightleftarrows}} B$

To show (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c),
need to show (a) $\Leftrightarrow$ (b) and (b) $\Leftrightarrow$ (c)
or
(a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c)
$\vdots$

Theorem 7

Let A and B be sets, let f be a function on A, and let g be a function on B. Then the following are equivalent.

Swap $f \leftrightarrow g$ & $A \leftrightarrow B$

- (a) *f is a bijection from A to B and $g = f^{-1}$.*
- (b) *For all x and all y, we have $x \in A$ and $f(x) = y$ iff $y \in B$ and $x = g(y)$.*
- (c) *g is a bijection from B to A and $f = g^{-1}$.*

Proof. We shall prove that (c) implies (b) and that (b) implies (c). (Why would it be sufficient?)

(c) $\Rightarrow$ (b): Let g be a bijection from B to A and $f = g^{-1}$. Then in particular, $g : B \to A$ and $f : A \to B$. Consider any x and any y. Suppose $x \in A$ and $f(x) = y$. Then $y \in B$ because $f : A \to B$. Furthermore $g^{-1}(x) = y$ because $f = g^{-1}$. Hence $x = g(y)$. This proves the forward implication in (b).

Bijections and Inverses (cont')

continued

Conversely, suppose $y \in B$ and $x = g(y)$. Then $x \in A$ because $g : B \to A$. Furthermore $g^{-1}(x) = y$. But $f = g^{-1}$. Hence $f(x) = y$. This proves the reverse implication in (b).

(b) $\Rightarrow$ (c): Suppose that for all x and all y we have $x \in A$ and $f(x) = y$ iff $y \in B$ and $x = g(y)$. We wish to show that g is a bijection from B to A and $f = g^{-1}$. If $y \in B$, then letting $x = g(y)$, we get $x \in A$ and $f(x) = y$. Hence $g : B \to A$ and for each $y \in B$, we have $y = f(g(y))$. If $x \in A$, then letting $y = f(x)$, we get $y \in B$ and $g(y) = x$. Hence g is a surjection from B to A. If $y_1, y_2 \in B$ and $g(y_1) = g(y_2)$, then $y_1 = f(g(y_1)) = f(g(y_2)) = y_2$. Hence g is an injection. Since g is both an injection and a surjection from B to A, g is a bijection from B to A. Hence g^{-1} is defined and is a function from A to B. For each $x \in A$, letting $y = f(x)$, we have $g(y) = x$, so $y = g^{-1}(x)$, so $f(x) = g^{-1}(x)$. Hence $f = g^{-1}$. $\square$

Consequences

- Let A and B be sets and let f be a bijection from A to B. Then f^{-1} is a bijection from B to A.
- Let f be an injection. Then f^{-1} is an injection too and $\left(f^{-1}\right)^{-1} = f$.

Let $g = f^{-1}$. Follows from ⓐ $\Leftrightarrow$ ⓒ.

Let $g = f^{-1}$. Follows from ⓐ $\Leftrightarrow$ ⓒ.

$g = f^{-1}$ is a bijection. $\Rightarrow$ f^{-1} is an injection.

$g^{-1} = \underline{(f^{-1})^{-1} = f}$

Example 8

Let $f(x) = 1 - x$ for all $x \in [0, 1)$. Show that:

1. $f : [0, 1) \to (0, 1]$.
2. f is an injection.
3. $f^{-1}(y) = 1 - y$ for all $y \in \mathrm{Rng}(f)$.
4. $\mathrm{Rng}(f) = (0, 1]$.

- From these parts, we conclude that f is a bijection from $[0, 1)$ to $(0, 1]$.