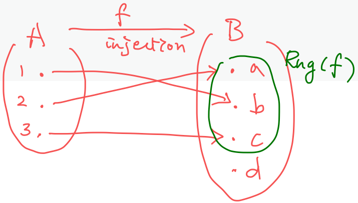
Hints for S11 E20

Recap Let A and B be sets. Let $f : A \to B$.

- f is a surjection from A to B
  iff $(\forall y \in B)(f(x) = y$ has <u>at least one</u> soln. $x \in A)$
- f is an injection from A to B
  iff $(\forall y \in B)(f(x) = y$ has <u>at most one</u> soln. $x \in A)$
- f is a bijection from A to B
  iff $(\forall y \in B)(f(x) = y$ has <u>a unique</u> soln. $x \in A)$



Question. Let f be a function on A.

Suppose f is an injection.

Then f is <u>a bijection</u> from A to $Rng(f)$.

(a) an injection

(b) a surjection

(c) a bijection

All of the options make the statement true.

Reason f, by assumption, is injective.

Since any function is a surjection from its domain to its range,

$f : A \to Rng(f)$ is a surjection.

Thus, $f : A \to Rng(f)$ is a bijection.

S11E20 (a) Let $g(x) = \dfrac{x}{1-x}$ for all $x \in [0,1)$.

① Show $g : [0,1) \to [0,\infty)$.

(NTS : for any $x \in [0,1)$, $g(x) \in [0,\infty)$, i.e., $g(x) \geqslant 0$)

Pf. Let $x \in [0,1)$. Since $x \in [0,1)$, $0 \leq x < 1$, so

$$0 \geqslant -x > -1,$$

so $$1 \geqslant 1-x > 0,$$

so $$1 \leq \frac{1}{1-x}.$$

Thus $g(x) = x \cdot \dfrac{1}{1-x} \geqslant 0 \cdot 1 = 0$, i.e., $g(x) \in [0,\infty)$. $\square$

② Show g is an injection. (Along the way, find $g^{-1}(y)$.)

Use the following formulation of an injection:

- g is an injection iff $(\forall y \in B)(\ g(x) = y$ has at most one soln $x \in A)$
- g is a bijection from $[0,1)$ to $\mathrm{Rng}(g)$
  iff $(\forall y \in \mathrm{Rng}(g))(g(x) = y$ has a unique soln $x \in A)$

Pf Let $y \in \mathrm{Rng}(g)$. So $y = g(x)$ for some $x \in [0,1)$. Then we have

$$y = \frac{x}{1-x}$$

$$\Rightarrow (1-x)y = x$$

$$\Rightarrow y - xy = x$$

$$\Rightarrow y = x + xy$$

$$\Rightarrow x(1+y) = y$$

$$\Rightarrow x = \frac{y}{1+y} = g^{-1}(y).$$

We found a unique soln $x \in [0,1)$ to $y = g(x)$.

So g is an injection. $\square$

③ Show $Rng(g) = [0, \infty)$.

Note By ①, ②, and ③, we conclude that g is a bijection from $[0,1)$ to $[0,\infty)$.

Pf It was already shown in ① that $Rng(g) \subseteq [0,\infty)$. Thus it remains to show $[0,\infty) \subseteq Rng(g)$.

Let $y \in [0,\infty)$. (WTS: $y \in Rng(g)$ which means $y = g(x)$ for some $x \in [0,1)$.)

We propose that $x = \frac{y}{1+y}$ (found in ②) would do.

First, check $g(x) = y$.

$$g(x) = g\left(\frac{y}{1+y}\right) = \frac{y/_{1+y}}{1 - y/_{1+y}} = \frac{\frac{y}{\cancel{1+y}}}{\frac{1+\cancel{y}-\cancel{y}}{\cancel{1+y}}} = \frac{y}{1} = y.$$

Lastly, we check $x = \frac{y}{1+y} \in [0,1)$:

Since $y \in [0, \infty)$, $y \geq 0$, so $1+y \geq 1$, so $\underline{0 < \frac{1}{1+y} \leq 1}$.

Now we note that

$$x = \frac{y}{1+y} = \frac{1+y-1}{1+y} = 1 - \frac{1}{1+y}.$$

Then it follows that

$$0 > -\frac{1}{1+y} \geq -1 \implies 1 > \underbrace{1 - \frac{1}{1+y}}_{\substack{\shortparallel \\ x}} \geq 0$$

$$\implies x \in [0, 1).$$

$\square$

Section 13

Fundamental Principle of Counting

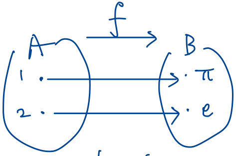
Equinumerousness

Definition 1

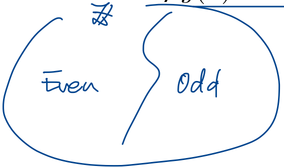
Let A and B be sets. To say that *A is equinumerous to B* (denoted $A \approx B$) means that there exists a bijection from A to B.

Examples. A B

- The sets $\{1, 2\}$ and $\{\pi, e\}$ are equinumerous because the function f on $\{1, 2\}$ defined by $f(1) = \pi$ and $f(2) = e$ is a bijection from $\{1, 2\}$ to $\{\pi, e\}$.
- The set $\mathbb{Z}$ is equinumerous to the set of all even integers $B = \{2k : k \in \mathbb{Z}\}$ because the function $g : \mathbb{Z} \to B$ defined by $g(n) = 2n$ is a bijection.



bijection,



Yet $\mathbb{Z} \approx B$!

B is a proper subset of $\mathbb{Z}$.

Equinumerousness (cont')

is an equivalence relation.

Proposition 1

Equinumerousness is reflexive, symmetric, and transitive. In other words:

1. *(Reflexivity) For each set A, we have A is equinumerous to A.*
2. *(Symmetry) For all sets A and B, if A is equinumerous to B, then B is equinumerous to A.*
3. *(Transitivity) For all sets A, B, and C, if A is equinumerous to B and B is equinumerous to C, then A is equinumerous to C.*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "f^{-1}=g"] \end{tikzcd}$$

(Thm 7, Lec 32)

Idea of proof

① $A \approx A$ because id_A is a bijection from A to itself.

② Since $A \approx B$, there exists a bijection f from A to B, Let $g = f^{-1}$. Then g is a bijection from B to A. Thus $B \approx A$.

③ Since $A \approx B$ and $B \approx C$, there exist

- a bijection f from A to B
- a bijection g from B to C

Let $h = g \circ f$. Then h is a bijection from A to C. Thus $A \approx C$.

this needs a proof. (HW)

Diagram

$$\begin{tikzcd} A \arrow[r, "f\ (\text{bij})"] \arrow[dr, "h = g \circ f"'] & B \arrow[d, "g\ (\text{bij})"] \\ & C \end{tikzcd}$$

Number of Elements and Equinumerousness

$\omega = \{0, 1, 2, \dots\}$ "whole numbers" or "non-negative integers"

Definition 2

Let A be a set and let $n \in \omega$. To say that *A has n elements* means that A is equinumerous to $\{1, \dots, n\}$.

Convention

- If $n=0$, then $\{1, \dots, n\} = \emptyset$.
- If $n=1$, then $\{1, \dots, n\} = \{1\}$.
- If $n=2$, then $\{1, \dots, n\} = \{1, 2\}$.

$\vdots$

- A set has 0 elements iff it is empty. (See p. 12 of Lec. 30)
- Saying that *A is an n-element set* is synonymous to saying that A has n elements.

Let A be a set.

A has 0 elements $\iff A \approx \emptyset$

$\iff$ there exists a bijection f from A to $\emptyset$.

$\overset{?}{\iff} A = \emptyset$

Def'n Let $n \in \omega$. To say that $a_1, \dots, a_n$ are distinct means that the function f defined by

$$f(k) = a_k, \quad \text{for all } k = 1, \dots, n,$$

is an injection.

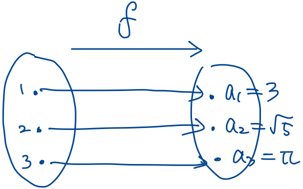
Example

$a_1 = 3$

$a_2 = \sqrt{5}$

$a_3 = \pi$

distinct



an injection

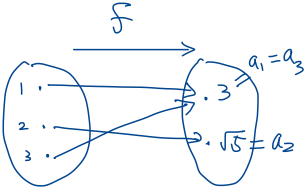
Non example

$a_1 = 3$

$a_2 = \sqrt{5}$

$a_3 = 3$

not distinct



not an injection

Some omitted details btw the first two sentences:

A has n elem. $\overset{}{\underset{\text{Def'n}}{\Longrightarrow}}$ $A \approx \{1,\dots,n\}$ $\underset{\text{Sym}}{\Longrightarrow}$ $\{1,\dots,n\} \approx A$ $\underset{\text{Def'n}}{\Longrightarrow}$ there exists a bijection f from $\{1,\dots,n\}$ to A.

Proposition 2

Let A be a set and let $n \in \omega$. Then the following are equivalent.

1. *A has n elements.*

$\Updownarrow$

2. *There exist distinct objects $a_1, \dots, a_n$ such that $A = \{a_1, \dots, a_n\}$.*

① $\Rightarrow$ ②

Proof. Suppose A has n elements. Then there is a bijection f from $\{1, \dots, n\}$ to A. Let $a_k = f(k)$ for $k = 1, \dots, n$. Since f is a surjection from $\{1, \dots, n\}$ to A, we have $A = \{a_1, \dots, a_n\}$. Since f is an injection, the objects $a_1, \dots, a_n$ are distinct.

"Rng(f)"

② $\Rightarrow$ ①

Conversely, suppose there exist distinct objects $a_1, \dots, a_n$ such that $A = \{a_1, \dots, a_n\}$. Then $A = \operatorname{Rng}(f)$ where f is a function on $\{1, \dots, n\}$ defined by $f(k) = a_k$. Since $a_1, \dots, a_n$ are distinct, f is injective, so f is a bijection from $\{1, \dots, n\}$ to A, so A has n elements. □

Finite and Infinite Sets

Definition 3

Let A be a set.

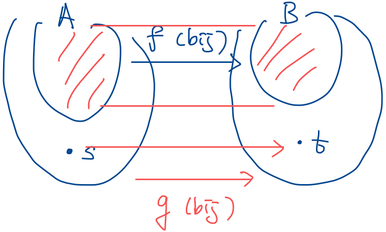
- To say that A *is finite* means that there exists $n \in \omega$ such that A has n elements.
- To say that A *is infinite* means that A is not finite.

Preliminary Lemmas

Lemma A

Let A and B be sets. Suppose that A is equinumerous to B. ($A \approx B$)

1. If $s \notin A$ and $t \notin B$, then $A \cup \{s\}$ is equinumerous to $B \cup \{t\}$.
2. If $s \in A$ and $t \in B$, then $A \setminus \{s\}$ is equinumerous to $B \setminus \{t\}$.



Proof of ① Since $A \approx B$, there exists a bijection f from A to B.

Define g by

$$g(x) = \begin{cases} f(x) & \text{if } x \in A \\ t & \text{if } x = s. \end{cases}$$

SIIE2D(c)

Then g is a bijection from $A \cup \{s\}$ to $B \cup \{t\}$, thus $A \cup \{s\} \approx B \cup \{t\}$. $\square$

Lemma B

For all $m, n \in \omega$, if $\{1, \ldots, m\}$ is equinumerous to $\{1, \ldots, n\}$, then $m = n$.

This obvious result is proved <u>using induction</u>.

Key Results

Uniqueness of the Number of Elements

Theorem 4

The number of elements in a finite set is uniquely determined.

- According to the theorem, for each finite set A, there is a unique $n \in \omega$ such that A has n elements; we write $\overline{\overline{A}}$ for the unique n. The notation $\overline{\overline{A}}$ is read *the number of elements in A* or *the cardinality of A*.

Subsets of a Finite Set is Finite

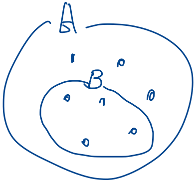
Theorem 5

A subset of a finite set is finite and has at most as many elements as the whole set.

A precise rephrase of the theorem is: For each $n \in \boldsymbol{\omega}$,

for each set B, if B has n elements, then for each $A \subseteq B$, A is finite and $\overline{\overline{A}} \leqslant n$.

Thus one can prove this using induction.



Rigidity Property of Finite Sets

Not true for infinite sets.
e.g.

Theorem 6

A finite set cannot be equinumerous to a proper subset of itself.

$B \subsetneq \mathbb{Z}$
but
$B \approx \mathbb{Z}$.

Proof. Let B be a finite set and let A be a proper subset of B . We wish to show that B is not equinumerous to A. Suppose B is equinumerous to A. Since B is finite, B has n elements for some $n \in \omega$. Since B is equinumerous to A, A also has n elements. Since A is a proper subset of A, $B \setminus A$ is not empty. Let $b \in B \setminus A$ and let $C = A \cup \{b\}$. Then $C \subseteq B$ and C has $n + 1$ elements. Hence by Theorem 13.30, $n + 1 \leqslant n$. But $n + 1 > n$. Thus we have reached a contradiction. Therefore B must not be equinumerous to A.

Theorem 5 on prev. page