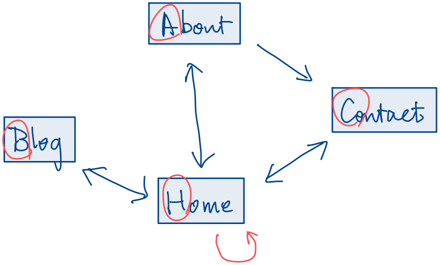
Section 4 First Examples of Mathematical Proofs

Even and Odd Numbers

- Quiz 3 today. Read questions carefully!
- HW1 scores to be released soon.
  Review your graded work.

Example: Order in mixed quantifiers

Consider the following webpages:



$P(x,y)$: x has a link to y.

$P(x,y)$

$x \backslash y$	A	B	C	H
A	F	F	T	T
B	F	F	F	T
C	F	F	F	T
H	T	T	T	F

- $(\forall x)(\exists y)\, P(x,y)$: T
- $(\exists x)(\forall y)\, P(x,y)$: F

free vs bound variables

	x	y
$P(x,y)$	free	free
$(\forall y)\, P(x,y)$	free	bound
$(\exists x)(\forall y)\, P(x,y)$	bound	bound

To determine the truth value of

$(\exists x)(\forall y)\, P(x,y)$,

consider the truth values of $(\forall y)\, P(x,y)$ which depend on x. See the table here →

There exist no entry for which $(\forall y)\, P(x,y)$ is true.

x	$(\forall y)\, P(x,y)$
A	F
B	F
C	F
H	F

Sec 3, Ex ? (not assigned)

Example: Negation of multiply quantified sentences. (Calculus)

Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ and let $a \in \mathbb{R}$.

To say that $\boxed{f \text{ is continuous at } a}$ means that

for each $\varepsilon > 0$, there exists a $\delta > 0$ such that
for each $x \in \mathbb{R}$, if $|x-a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

In symbols:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})\Big[\underbrace{|x-a| < \delta}_{A} \underbrace{\Rightarrow}_{\Rightarrow} \underbrace{|f(x) - f(a)| < \varepsilon}_{B}\Big]$$

To say that $\boxed{f \text{ is not continuous at } a}$ means, in symbols,

$$\boxed{(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})\Big[|x-a| < \delta \wedge |f(x) - f(a)| \geq \varepsilon\Big]}$$

Even Numbers and Odd Numbers

Definitions

Definition 1 (Even and Odd Numbers)

- To say that x *is an even number* means that there exists an integer k such that $x = 2k$.

  $(\exists k \in \mathbb{Z})(x = 2k)$

- To say that x *is an odd number* means that there exists an integer k such that $x = 2k + 1$.

- $6 = 2 \cdot 3$ even
- $0 = 2 \cdot 0$ even
- $7 = 2 \cdot 3 + 1$ odd
- $-7 = 2(-4) + 1$ odd [incorrect : $-7 = 2(-3) - 1$]

Examples

Example 2 (Sum of Odd Numbers)

If x is odd and y is odd, then $x + y$ is even.

Proof. Suppose x is odd and y is odd. (WTS $x+y$ is even.)

Since x is odd, we can find an integer k such that $x = 2k+1$.

Likewise, since y is odd, we can find an integer l such that $y = 2l+1$.

Then
$$x+y = (2k+1) + (2l+1) = 2k + 2l + 2 = 2(k+l+1).$$

Since $k+l+1$ is an integer, $x+y$ is even. Hence, if x is odd and y is odd, then $x+y$ is even. $\square$

x	y	$x+y$	
odd	odd	even	← just showed
odd	even	odd	
even	odd	odd	HW
even	even	even	

How about $x+y+z$?

Example 3 (Product with Even Numbers)

Let x and y be integers. If x is even or y is even, then xy is even.

<u>Proof</u> Suppose x is even or y is even. (WTS xy is even)

<u>Case 1</u> x is even. Then we can find an integer k such that

$$x = 2k.$$

Then

$$xy = (2k)y = 2(ky).$$

Since ky is an integer, xy is even.

<u>Case 2</u> y is even. Then we can find an integer k such that

$$y = 2k.$$

Then

$$xy = x(2k) = (2k)x = 2(kx).$$

Since kx is an integer, xy is even.

In either case, we showed that xy is even. Hence, we conclude that if x is even or y is even, then xy is even. $\square$

x	y	xy	
odd	odd	odd	HW
odd	even	even	
even	odd	even	just showed
even	even	even	

Fundamental Properties

Even/Odd Dichotomy (I)

Let x be an integer. Then:

1. x is even or x is odd.
2. If x is not even, then x is odd.
3. If x is not odd, then x is even.

Note. If P stands for "x is even" and Q stands for "x is odd", then three statements above are $P \vee Q$, $\neg P \Rightarrow Q$, and $\neg Q \Rightarrow P$, respectively. Note that the three sentences are logically equivalent.

P	Q	$P \vee Q$	$\neg P \Rightarrow Q$	$\neg Q \Rightarrow P$
T	T			
T	F			
F	T			
F	F			

Fundamental Properties (cont')

Even/Odd Dichotomy (II)

Let x be an integer. Then:

1. x is not both even and odd.
2. If x is even, then x is not odd.
3. If x is odd, then x is not even.

Note. If P stands for "x is even" and Q stands for "x is odd", then three statements above are $\neg(P \wedge Q)$, $P \Rightarrow \neg Q$, and $Q \Rightarrow \neg P$, respectively. Note that the three sentences are logically equivalent.

P	Q	$\neg(P \wedge Q)$	$P \Rightarrow \neg Q$	$Q \Rightarrow \neg P$
T	T			
T	F			
F	T			
F	F			

Examples

Example 4 (When Sum of Two Integers Is Odd)

Let x and y be integers. If $x + y$ is odd, then x is even or y is even.

Homework (1/28; due Wed 2/2)

Section 4: # 1, 2, 3, 4, 5, 6