Intervals, Sets of Sets, and Power Set

Intervals

Intervals

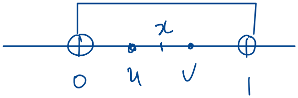
An *interval in* $\mathbb{R}$ is a subset of $\mathbb{R}$ that contains all the points between any two of its points.

Definition 1 (Interval)

To say that I is an interval of $\mathbb{R}$ means that $I \subseteq \mathbb{R}$ and for each $u, v \in I$, for each $x \in \mathbb{R}$, if $u < x < v$, then $x \in I$.

$(\forall u, v \in I)(\forall x \in \mathbb{R})[\, u < x < v \Rightarrow x \in I \,]$

The set $\{1, 2, 3\}$ is not an interval in $\mathbb{R}$ because



1.5 lies between 1 and 2,

yet $1.5 \notin \{1, 2, 3\}$.

Bounded Intervals

Bounded Intervals

Let $a, b \in \mathbb{R}$ such that $a \leqslant b$. Then the following sets are intervals in $\mathbb{R}$.

$$
\begin{aligned}
[a, b] &= \{x \in \mathbb{R} : a \leqslant x \leqslant b\} && \text{(closed)} \\
(a, b) &= \{x \in \mathbb{R} : a < x < b\} && \text{(open)} \\
[a, b) &= \{x \in \mathbb{R} : a \leqslant x < b\} && \text{(left-closed or right-open)} \\
(a, b] &= \{x \in \mathbb{R} : a < x \leqslant b\} && \text{(left-open or right-closed)}
\end{aligned}
$$

These are called *bounded intervals*. If $a = b$, these bounded intervals yield

$$
\begin{aligned}
[a, b] &= \{a\} \\
(a, b) &= [a, b) = (a, b] = \varnothing
\end{aligned}
$$

These are called *degenerate intervals*.

Unbounded Intervals

Let $c \in \mathbb{R}$. Then the following sets are intervals in $\mathbb{R}$.

$$\begin{aligned}
[c, \infty) &= \{x \in \mathbb{R} : c \leqslant x\} && \text{(closed half-line)} \\
(c, \infty) &= \{x \in \mathbb{R} : c < x\} && \text{(open half-line)} \\
(-\infty, c] &= \{x \in \mathbb{R} : x \leqslant c\} && \text{(closed half-line)} \\
(-\infty, c) &= \{x \in \mathbb{R} : x < c\} && \text{(open half-line)}
\end{aligned}$$

The whole real line

$$(-\infty, \infty) = \mathbb{R}$$

is also an interval in $\mathbb{R}$. Half-lines and the whole real line are *unbounded intervals*.

Examples

Question. Determine whether each of the following is an interval.

1. $(-1, 1) \cup [0, 4]$

2. $[-1, 1] \cap (0, 2)$

3. $[1, 2] \cup [3, 4]$

4. $[2, 5) \cap [5, 6]$

5. $(3, 5) \setminus [4, 7]$

6. $(4, 8] \setminus [5, 6)$

Unions and Intersections of Sets of Sets

Unions of Sets of Sets

Cursive caps: $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \dots$ for set of sets.

Definition 2

Let $\mathcal{A}$ be a set of sets. Then *the union of* $\mathcal{A}$ (denoted $\bigcup \mathcal{A}$) is the set of all things that belong to at least one of the sets in $\mathcal{A}$; in other words,

$$\bigcup \mathcal{A} = \{x : \underline{x \in A \text{ for some } A \in \mathcal{A}}\}.$$

$(\exists A \in \mathcal{A})(x \in A)$

Example. Let A, B, and C be sets. Then

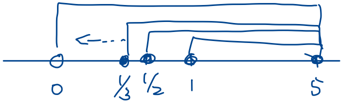
- $\bigcup\{\,\} = \bigcup\varnothing = \varnothing$
- $\bigcup\{A\} = A$
- $\bigcup\{A, B\} = A \cup B$
- $\bigcup\{A, B, C\} = A \cup B \cup C$

Example. Let $\mathcal{A} = \{[1/n, 5] : n \in \mathbb{N}\}$. Then

$$\bigcup \mathcal{A} = (0, 5].$$

$\mathcal{A}$ is a set of infinitely many intervals.

$= \{[1,5], [1/2,5], [1/3,5], [1/4,5], \dots\}$



Intersections of Sets of Sets

Definition 3

Let $\mathcal{A}$ be a nonempty set of sets. Then *the intersection of* $\mathcal{A}$ (denoted $\bigcap \mathcal{A}$) is the set of all things that belong to all of the sets in $\mathcal{A}$; in other words,

$$\bigcap \mathcal{A} = \{x : \underline{x \in A \text{ for each } A \in \mathcal{A}}\}.$$

$(\forall A \in \mathcal{A})(x \in A)$

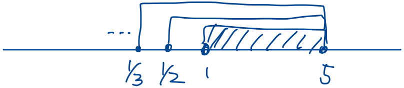
Example. Let A, B, and C be sets. Then

- $\bigcap\{A\} = A$
- $\bigcap\{A, B\} = A \cap B$
- $\bigcap\{A, B, C\} = A \cap B \cap C$

Example. Let $\mathcal{A} = \{[1/n, 5] : n \in \mathbb{N}\}$. Then

$$\cancel{\bigcup}\mathcal{A} = [1, 5].$$

$\bigcap$



Example 4

For each of the following, find $\bigcup \mathcal{A}$ and $\bigcap \mathcal{A}$. State clearly if either/both of the two is/are undefined.

1. $\mathcal{A} = \{\varnothing, \{1,2\}, \{2,3\}\}$.
2. $\mathcal{A} = \varnothing$.
3. $\mathcal{A} = \{1, \{2\}\}$.

Note

- $(\forall A)(A \cup \phi = A)$
- $(\forall A)(A \cap \phi = \phi)$

① $\bigcup \mathcal{A} = \phi \cup \{1,2\} \cup \{2,3\} = \{1,2,3\}$

$\bigcap \mathcal{A} = \phi \cap \{1,2\} \cap \{2,3\} = \phi$

② $\bigcup \mathcal{A} = \phi$, $\bigcap \mathcal{A}$ is undefined.

③ $\bigcup \mathcal{A}$ and $\bigcap \mathcal{A}$ are undefined because $\mathcal{A}$ is not a set of sets.

Q. What if $\mathcal{A} = \{\emptyset\}$?

A.

$$\bigcup\{\emptyset\} = \emptyset = \bigcap\{\emptyset\} \neq \bigcap \emptyset$$

which is undefined.

Set Inclusion

Proposition 1

Let $\mathcal{A}$ be a nonempty set of sets and let $A_0 \in \mathcal{A}$. Then

(1) (2)

$$\bigcap \mathcal{A} \subseteq A_0 \subseteq \bigcup \mathcal{A}.$$

Proof

(1) Consider any $x \in \bigcap \mathcal{A}$. Then for each $A \in \mathcal{A}$, $x \in A$. In particular, $x \in A_0$ because $A_0 \in \mathcal{A}$. This shows $\bigcap \mathcal{A} \subseteq A_0$.

(2) Now consider $x \in A_0$. Since $A_0 \in \mathcal{A}$, it is true that $x \in A$ for some $A \in \mathcal{A}$. (Simply take $A = A_0$). In other words, $x \in \bigcup \mathcal{A}$. This shows $A_0 \subseteq \bigcup \mathcal{A}$. □

Not an Element

Proposition 2

Let $\mathcal{A}$ be a nonempty set of sets and let x be any object. Then:

1. $x \notin \bigcup \mathcal{A}$ *iff for each* $A \in \mathcal{A}$, $x \notin A$.
2. $x \notin \bigcap \mathcal{A}$ *iff there exists* $A \in \mathcal{A}$ *such that* $x \notin A$.

Proof We have

$$x \notin \bigcup \mathcal{A}$$

iff $\neg (x \in \bigcup \mathcal{A})$

iff $\neg (\exists A \in \mathcal{A})(x \in A)$

iff $(\forall A \in \mathcal{A}) \neg (x \in A)$ (by GDM)

iff $(\forall A \in \mathcal{A})(x \notin A)$

$\square$

De Morgan's Laws Again

Theorem 5 (Generalized De Morgan's Laws for Sets of Sets)

Let S be a set and let $\mathcal{A}$ be a nonempty set of sets. Then:

1. $S \setminus \bigcup \mathcal{A} = \bigcap \{S \setminus A : A \in \mathcal{A}\}$.
2. $S \setminus \bigcap \mathcal{A} = \bigcup \{S \setminus A : A \in \mathcal{A}\}$.

Proof. Let x be any object. Then we have

$$x \in S \setminus \bigcup \mathcal{A}$$

iff $x \in S$ and $x \notin \bigcup \mathcal{A}$

iff $x \in S$ and $(\forall A \in \mathcal{A})(x \notin A)$ (by Prop. 2)

iff $(\forall A \in \mathcal{A})(x \in S$ and $x \notin A)$ (by dist. law)

iff $(\forall A \in \mathcal{A})(x \in S \setminus A)$

$\square$

Distributive Laws Again

Theorem 6 (Generalized Distributive Laws for Sets of Sets)

Let S be a set and let $\mathcal{A}$ be a nonempty set of sets. Then:

1. $S \cap \bigcup \mathcal{A} = \bigcup\{S \cap A : A \in \mathcal{A}\}$.
2. $S \cup \bigcap \mathcal{A} = \bigcap\{S \cup A : A \in \mathcal{A}\}$.

Power Set of a Set

Power Set of a Set

Definition 7

Let A be a set. The *power set of* A (denoted $\mathcal{P}(A)$) is the set of all subsets of A; in other words, $\mathcal{P}(A) = \{S : S \subseteq A\}$.

Example.

$n=2, \quad 2^n = 2^2 = 4$

$$\mathcal{P}(\{1,2\}) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}.$$

Note.

- If A is a finite set with n elements, then $\mathcal{P}(A)$ has 2^n elements.

Example: (Recursive) Power Sets of the Empty Set

Let $V_0 = \varnothing$ and for each $n \in \boldsymbol{\omega}$, let $V_{n+1} = \mathcal{P}(V_n)$. That is,

$$
\begin{aligned}
V_0 &= \varnothing \\
V_1 &= \mathcal{P}(\varnothing) = \{\varnothing\} \\
V_2 &= \mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\} \\
V_3 &= \mathcal{P}(\{\varnothing, \{\varnothing\}\}) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\
&\vdots
\end{aligned}
$$

Counting the number of elements:

Set	# of Elem.	Set	# of Elem.
V_0	0	V_4	$2^4 = 16$
V_1	$2^0 = 1$	V_5	$2^{16} = 65536$
V_2	$2^1 = 2$	V_6	$2^{65536} \approx 2 \times \underset{10}{\cancel{2}}^{19,728}$
V_3	$2^2 = 4$	$\cdots$	