# Prime Numbers

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**Definition 1 (Divisibility)**

Let $d$ and $x$ be integers. To say that $d$ *divides* $x$ means that there exists an integer $k$ such that $x = kd$.

- Every integer divides $0$.
- 0 is the only integer that 0 divides. (If $x$ is an integer and $0$ divides $x$, then $x = k \cdot 0$ for some integer $k$, and $k \cdot 0 = 0$, so $x = 0$.)
- Let $x$ be an integer. Then $x$ is even iff $2$ divides $x$.

# Remarks

- Alternate expression for “$d$ divides $x$”: “$x$ is divisible by $d$”
- “$d$ divides $x$” is a sentence while “$d$ divided into $x$” ($x/d$, for $d \neq 0$) is a number.
- **Notation.** $d \mid x$ for “$d$ divides $x$.” and $d \nmid x$ for “$d$ does not divide $x$.”
- Let $m$ and $n$ be integers, with $n \neq 0$. To say that the fraction $m/n$ is in lowest terms means that for each natural number $d$, if $d$ divides $m$ and $d$ divides $n$, then $d = 1$.

# Examples

## Example 2 (Divisibility with Natural Numbers)

Let $d, x \in \mathbb{N}$. Suppose $d$ divides $x$. Then $d \leq x$.

*Proof*. Since $d$ divides $x$, we can pick an integer $k$ such that $x = kd$. Since $k$ is an integer, either $k \geq 1$ or $k \leq 1$. But it is not the case that $k \leq 0$, because if $k \leq 0$, then $x = kd \leq 0$, which contradicts the fact that $x \geq 1$. Hence $k \geq 1$. Therefore $kd \geq d$. In other words, $x \geq d$. $\square$

# Examples (cont')

## Example 3

Let $a, b, c \in \mathbb{Z}$. If $a$ divides $b$ and $a$ divides $c$, then $a$ divides $b + c$ and $a$ divides $b - c$.

# Examples (cont')

### Example 4

Let $a, b, c \in \mathbb{Z}$. If $a$ divides $b$ and $b$ divides $a$, then $b = a$ or $b = -a$.

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# Definitions

## Definition 5 (Prime Numbers)

To say that $x$ *is a prime number* means that $x \in \mathbb{N}$ and $x \neq 1$ and for each $a \in \mathbb{N}$, for each $b \in \mathbb{N}$, if $x = ab$, then $a = 1$ or $b = 1$.

**Exercise.** Write the sentence "$x \in \mathbb{N}$ and $x \neq 1$ and for each $a \in \mathbb{N}$, for each $b \in \mathbb{N}$, if $x = ab$, then $a = 1$ or $b = 1$." using symbols.

# Prime Numbers as Building Blocks

**Fact (Prime Factorization)**

Each natural number, except $1$, is prime or is a product of two or more primes.

- Proof of this fact requires complete induction.
- From this fact, it follows that for each $n \in \mathbb{N}$, if $n \neq 1$, then there exists a prime number $p$ such that $p$ divides $n$.

# How Many Primes?

**Theorem 6 (Euclid, circa 300 B.C.)**

*There are infinitely many prime numbers.*