Induction

Section 5

Proof by Induction

another proof technique

The method of *proof by induction* is based on the following principle.

Principle of Mathematical Induction

Let $P(n)$ be any statement about n. Suppose we have proved that

$$P(1) \text{ is true} \tag{1}$$

and that

$$\text{for each natural number } n, \text{ if } P(n) \text{ is true, then } P(n+1) \text{ is true.} \tag{2}$$

$(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$

Then we may conclude that for each natural number n, $P(n)$ is true.

$(\forall n \in \mathbb{N})P(n)$

- This is a commonly used technique to prove a universal sentence $(\forall x \in A)P(x)$ when A is $\mathbb{N}$.

Steps in Proof by Induction

Sum of Odd Natural Numbers

For each $n \in \mathbb{N}$, $1 + 3 + \cdots + (2n - 1) = n^2$.

$P(n)$

$(\forall n \in \mathbb{N})$ (sum of first n positive odd numbers $= n^2$)

Proof. Let $P(n)$ be the sentence

$$1 + 3 + \cdots + (2n - 1) = n^2.$$

Declare $P(n)$.

BASE CASE: Observe that $P(1)$ is true because if $n = 1$, then the left-hand side is just 1 and the right-hand side is $1^2 = 1$.

Show $P(1)$ is true. (1)

INDUCTIVE STEP: Let $n \in \mathbb{N}$ such that $P(n)$ is true. Then

$$\begin{aligned} 1 + 3 + \cdots + (2n-1) + [2(n+1) - 1] & \\ = n^2 + [2(n+1) - 1] & \qquad (\star) \\ = n^2 + 2n + 2 - 1 = n^2 + 2n + 1 & \\ = (n+1)^2 & \end{aligned}$$

Thus $P(n+1)$ is true.

Show $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))$. (2)

The first sentence in this paragraph is called *the inductive hypothesis*.

CONCLUSION: Therefore, by induction, for each $n \in \mathbb{N}$, $P(n)$ is true. That is, for each $n \in \mathbb{N}$, $1 + 3 + \cdots + (2n - 1) = n^2$. □

Use induction to conclude.

Example 1

Prove by induction that for each $n \in \mathbb{N}$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Example 2

Prove by induction that for each $n \in \mathbb{N}$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Example 3

Prove by induction that for each $n \in \mathbb{N}$, 3 divides $4^n - 1$.