Logical Connectives (II)

Contents

Interplay of Negation, Conjunction, and Disjunction

De Morgan's Laws

The following rules pertain to the negation of conjunctive and disjunctive sentences.

Theorem 1 (De Morgan's Laws)

Let P and Q be sentences. Then

1. $\neg(P \wedge Q)$ *is logically equivalent to* $\neg P \vee \neg Q$.
2. $\neg(P \vee Q)$ *is logically equivalent to* $\neg P \wedge \neg Q$.

Proof of 1. (using a truth table)

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	T	F			
T	F	F	T			
F	T	F	T			
F	F	F	T			

De Morgan's Laws (cont')

Proof of 1. (in words)

Suppose $\neg(P \wedge Q)$ is true. Then …

Conversely, suppose $\neg P \vee \neg Q$ is true. Then …

It follows that $\neg(P \wedge Q)$ is true exactly when $\neg P \vee \neg Q$ is true. Then by elimination, $\neg(P \wedge Q)$ is false exactly when $\neg P \vee \neg Q$ is false. Therefore, $\neg(P \wedge Q)$ is logically equivalent to $\neg P \vee \neg Q$. $\square$

Example

Let x be a real number. The negation of the sentence $1 \leq x < 3$ is logically equivalent to $(x < 1) \vee (x \geq 3)$.

- **In words:**

- **Using logical symbols:**

$$\begin{aligned} \neg(1 \leq x < 3) &\equiv \neg\left[(1 \leq x) \wedge (x < 3)\right] \\ &\equiv \neg(1 \leq x) \vee \neg(x < 3) && \text{by De Morgan's Law} \\ &\equiv (x < 1) \vee (x \geq 3). \end{aligned}$$

Example (cont')

Let x be a real number. The negation of the sentence $1 \leq x < 3$ is logically equivalent to $(x < 1) \vee (x \geq 3)$.

- **Visually:**

The Distributive Laws

The following laws pertain to the conjunction of two disjunctive sentences or the disjunction of two conjunctive sentences.

Theorem 2 (The Distributive Laws)

Let P, Q, *and* R *be sentences. Then:*

1. $P \wedge (Q \vee R)$ *is logically equivalent to* $(P \wedge Q) \vee (P \wedge R)$.
2. $P \vee (Q \wedge R)$ *is logically equivalent to* $(P \vee Q) \wedge (P \vee R)$.

The Distributive Laws (cont')

Proof of 2. (using a truth table)

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T		T	T	
T	T	F	F		T	T	
T	F	T	F		T	T	
T	F	F	F		T	T	
F	T	T	T		T	T	
F	T	F	F		T	F	
F	F	T	F		F	T	
F	F	F	F		F	F	

The column headed by $P \vee (Q \wedge R)$ is identical to the one headed by $(P \vee Q) \wedge (P \vee R)$.

The Distributive Laws (cont')

Proof of 2. (in words)

Suppose that $P \vee (Q \wedge R)$ is true. Then at least one of P and $Q \wedge R$ is true.

- Case 1. Suppose P is true. Then both of $P \vee Q$ and $P \vee R$ are ________, so $(P \vee Q) \wedge (P \vee R)$ is ________.
- Case 2. Suppose $Q \wedge R$ is true. Then both of Q and R are ________, so both of $P \vee Q$ and $P \vee R$ are ________, so $(P \vee Q) \wedge (P \vee R)$ is ________.

Thus in either case, $(P \vee Q) \wedge (P \vee R)$ is true.

Conversely, suppose $(P \vee Q) \wedge (P \vee R)$ is true. Then both of $P \vee Q$ and $P \vee R$ are true.

- Case 1. Suppose P is true. Then the sentence $P \vee (Q \wedge R)$ is ________.
- Case 2. Suppose P is false. Then since $P \vee Q$ is true, Q must be ________. Similarly, since the sentence $P \vee R$ is true, R must be ________. Thus both of Q and R are true, so $Q \wedge R$ is ________, so $P \vee (Q \wedge R)$ is ________.

Thus in either case, $P \vee (Q \wedge R)$ is true.

From the previous two paragraphs, it follows that $P \vee (Q \wedge R)$ is true exactly when $(P \vee Q) \wedge (P \vee R)$ is true. Hence $P \vee (Q \wedge R)$ is logically equivalent to $(P \vee Q) \wedge (P \vee R)$. $\square$

Conditional and Biconditional Sentences

Conditional Sentences

A sentence of the form $P \Rightarrow Q$ is called a *conditional sentence*.

Conditional Sentences

Given P and Q:

- When P and Q are both true, $P \Rightarrow Q$ is considered to be true.
- When P is true and Q is false, $P \Rightarrow Q$ is considered to be false.
- Whenever P is false, $P \Rightarrow Q$ is considered to be true.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Terminology. In a conditional sentence $P \Rightarrow Q$, P is called the *antecedent* and Q is call the *consequent*.

Conditional Sentences (cont')

The sentence $P \Rightarrow Q$ stands for "If P, then Q" which is synonymous to

P implies Q.

P is sufficient for Q.

Q is necessary for P.

Q if P.

Careful!

- Do NOT write "If $P \Rightarrow Q$."
- Do NOT use "$\Rightarrow$" for "therefore" or for "so".

Example

Let x be any real number. Consider the sentence

$$\text{“If } \underbrace{x < 1}_{P}\text{, then } \underbrace{x < 3}_{Q}\text{.”}$$

which is always true.

	P	Q	$P \Rightarrow Q$
$x < 1$	T	T	T
$1 \leq x < 3$	F	T	T
$x \geq 3$	F	F	T

Negation of a Conditional Sentence

Theorem 3 (Negation of a Conditional Sentence)

Let P and Q be sentences. Then $\neg(P \Rightarrow Q)$ is logically equivalent to $P \wedge \neg Q$.

Proof. (in words)

Converse of a Conditional Sentence

Given a conditional sentence $P \Rightarrow Q$, the sentence $Q \Rightarrow P$ is called *the converse of* $P \Rightarrow Q$. Note that $Q \Rightarrow P$ is not logically equivalent to $P \Rightarrow Q$.

Examples.

- Let x be a real number.

$$P \Rightarrow Q: \qquad x > 3 \implies x^2 > 9 \qquad \text{(always true)}$$
$$Q \Rightarrow P: \qquad x^2 > 9 \implies x > 3 \qquad \text{(not always true)}$$

- Consider an infinite series $\sum_n a_n$.

$$P \Rightarrow Q: \qquad \sum_{n=1}^{\infty} a_n < \infty \implies \lim_{n\to\infty} a_n = 0 \qquad \text{(always true)}$$
$$Q \Rightarrow P: \qquad \lim_{n\to\infty} a_n = 0 \implies \sum_{n=1}^{\infty} a_n < \infty \qquad \text{(not always true)}$$

Biconditional Sentences

A sentence of the form $P \Leftrightarrow Q$ is called a *biconditional sentence*.

Biconditional Sentences

Given P and Q, the sentence $P \Leftrightarrow Q$ is considered to be true just when both of P and Q have the same truth value.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Notation. $P \Leftrightarrow Q$ stands for "P if and only if Q" or "P iff Q".

Example

Let x be a real number. Then $x^2 = 5x - 6$ if and only if $x = 2$ or $x = 3$, which can be seen by a chain of biconditionals:

Conditional and Biconditional

Theorem 4

Let P and Q be sentences. Then $P \Leftrightarrow Q$ is logically equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

Proof. (Using a truth table)

P	Q	$P \Leftrightarrow Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T		
T	F	F	F		
F	T	F	T		
F	F	T	T		

As a consequence of this theorem, $P \Leftrightarrow Q$ is synonymous to saying

"P is necessary and sufficient for Q".