Cantor’s Diagonal Lemma

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Let f be any function from $\mathbb{N}$ to $(0,1)$. Then there exists $y \in (0,1)$ such that y does not belong to the range of f.

Below is a key consequence of Cantor's diagonal lemma.

Theorem 1 (Cantor, 1873)

$\mathbb{R}$ is not equinumerous to $\mathbb{N}$.

Cantor's Diagonal Lemma: Idea of Proof

To prove Cantor's diagonal lemma, we need to find/construct $y \in (0,1)$ such that $y \notin \mathrm{Rng}(f) = \{f(n) : n \in \mathbb{N}\}$.

Decimal expansion of $f(n)$

For each $n \in \mathbb{N}$, $f(n) \in (0,1)$ so it has the standard decimal expansion

$$f(n) = 0.x_{n1}x_{n2}x_{n3}x_{n4}\ldots.$$

That is,

$$\begin{aligned}
f(1) &= 0.x_{11}x_{12}x_{13}x_{14}\ldots,\\
f(2) &= 0.x_{21}x_{22}x_{23}x_{24}\ldots,\\
f(3) &= 0.x_{31}x_{32}x_{33}x_{34}\ldots,\\
f(4) &= 0.x_{41}x_{42}x_{43}x_{44}\ldots,\\
&\text{and so on.}
\end{aligned}$$

Cantor's Diagonal Lemma: Idea of Proof (cont')

Construction of y

For each $n \in \mathbb{N}$, let

$$y_n = \begin{cases} 5 & \text{if } x_{nn} \neq 5, \\ 4 & \text{if } x_{nn} = 5. \end{cases}$$

Then for each $n \in \mathbb{N}$, $y_n \neq x_{nn}$. Now let y be the number whose standard decimal expansion is

$$y = 0.y_1 y_2 y_3 y_4 \ldots.$$

Observation

- $y \in (0, 1)$; in fact, $0.444\ldots \leqslant y \leqslant 0.555\ldots$.
- $y \notin \operatorname{Rng}(f)$ because for each $n \in \mathbb{N}$, $y \neq f(n)$.

Higher Orders of Infinity

Denumerable, Countable, and Uncountable

Definition 2

Let A be a set.

1. To say that A *is denumerable* means that A is equinumerous to $\mathbb{N}$.
2. To say that A *is countable* means that A is finite or denumerable.
3. To say that A *is uncountable* means that A is not countable.

Example.

- Each of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$, and $\mathbb{Q}$ is denumerable.
- $\mathbb{R}$ is uncountable.

Cardinality

Definition 3

Let A and B be sets.

1. To say that *the cardinality of A is less than or equal to the cardinality of B* (denoted $\overline{\overline{A}} \leqslant \overline{\overline{B}}$) means that A is equinumerous to a subset of B.
2. To say that *the cardinality of A is strictly less than the cardinality of B* (denoted $\overline{\overline{A}} < \overline{\overline{B}}$) means that A is equinumerous to a subset of B but A is not equinumerous to B.
3. To say that *the cardinality of A is equal to the cardinality of B* (denoted $\overline{\overline{A}} = \overline{\overline{B}}$) means that A is equinumerous to B.

Example. $\overline{\overline{\mathbb{N}}} < \overline{\overline{\mathbb{R}}}$.

Cardinality (cont')

Notes.

- Let A and B be sets. Then $\overline{\overline{A}} \leqslant \overline{\overline{B}}$ iff there exists an injection from A to B.

- Let A be any set. Then $\overline{\overline{A}} \leqslant \overline{\overline{\mathcal{P}(A)}}$.

Cantor's Generalized Diagonal Lemma

Cantor's Generalized Diagonal Lemma

Let A be a set and let f be a function on A such that for each $x \in A$, $f(x)$ is a set. Then there exists a subset $C \subseteq A$ such that C does not belong to the range of f.

Below is a key consequence of Cantor's generalized diagonal lemma.

Theorem 4 (Cantor, 1891)

Any set has strictly smaller cardinality than its power set.