Algebra of Set Operations

- union $(A \cup B)$
- intersection $(A \cap B)$
- relative complement $(A \setminus B)$

Algebra of Set Operations

dist $P \wedge (Q \wedge R)$

$\equiv (P \wedge Q) \wedge (P \wedge R)$

Recall Useful laws from propositional calculus

- Commutativity of $\vee$ and $\wedge$: $P \vee Q \equiv Q \vee P$, $P \wedge Q \equiv Q \wedge P$
- Associativity of $\vee$ and $\wedge$: $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$

  $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
- Distributive laws : $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

  $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
- De Morgan's laws : $\neg (P \vee Q) \equiv \neg P \wedge \neg Q$

  $\neg (P \wedge Q) \equiv \neg P \vee \neg Q$

Not an Element

Recap • $A = B$ means $(\forall x)(x \in A \Leftrightarrow x \in B)$

• $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Proposition 1

Let A and B be sets and let x be any object. Then:

1. $x \notin A \cup B$ iff $x \notin A$ and $x \notin B$.
2. $x \notin A \cap B$ iff $x \notin A$ or $x \notin B$.
3. $x \notin A \backslash B$ iff $x \notin A$ or $x \in B$.

Proof of (1) Recall that $x \in A \cup B$ means $x \in A$ or $x \in B$. So

$$\begin{aligned} x \notin A \cup B \quad & \text{iff} \quad \neg (x \in A \cup B) \\ & \text{iff} \quad \neg (x \in A \text{ or } x \in B) \\ & \text{iff} \quad \neg (x \in A) \text{ and } \neg (x \in B) \qquad \text{(by De Morgan's laws)} \\ & \text{iff} \quad x \notin A \text{ and } x \notin B \end{aligned}$$

$\square$

De Morgan's Laws for Sets

"$S\setminus$" should be thought of an analogue of negation.
$\cup \leftrightarrow \vee$, $\cap \leftrightarrow \wedge$

Theorem 1 (De Morgan's Laws for Sets)

Let S, A, and B be sets. Then:

1. $S\setminus(A \cup B) = (S\setminus A) \cap (S\setminus B)$.
2. $S\setminus(A \cap B) = (S\setminus A) \cup (S\setminus B)$.

Recall: $A\setminus B = \{x : x \in A$ and $x \notin B\}$.

Proof of ① For each element x,

$x \in S\setminus(A\cup B)$ iff $x \in S$ and $x \notin A\cup B$

iff $x \in S$ and ($x \notin A$ and $x \notin B$) (by Prop. 1①)

iff ($x \in S$ and $x \notin A$) and ($x \in S$ and $x \notin B$) (by Dist. Laws)

iff $x \in S\setminus A$ and $x \in S\setminus B$

iff $x \in (S\setminus A) \cap (S\setminus B)$ $\square$

Tip (proof template) To show that two sets are equal, e.g.,

$$X = Y,$$ which means $(\forall x)(x \in X \iff x \in Y)$,

use the following template.

Proof. For each element x,

$x \in X$ iff $\cdots$

iff $\cdots$

$\vdots$

iff $x \in Y$ $\square$

Distributive Laws for Unions and Intersections

Theorem 2 (Distributive Laws for Unions and Intersections)

Let S, A, and B be sets. Then:

1. $S \cap (A \cup B) = (S \cap A) \cup (S \cap B)$.
2. $S \cup (A \cap B) = (S \cup A) \cap (S \cup B)$.

Proof of ① For each (element) x,

$x \in S \cap (A \cup B)$ iff $x \in S$ and $x \in A \cup B$

iff $x \in S$ and ($x \in A$ or $x \in B$)

iff ($x \in S$ and $x \in A$) or ($x \in S$ and $x \in B$) (by Dist. Laws)

iff $x \in S \cap A$ or $x \in S \cap B$

iff $x \in (S \cap A) \cup (S \cap B)$ $\square$

Associative Laws for Unions and Intersections

Proposition 2 (Associative Laws for Unions and Intersections)

Let A, B, and C be sets. Then:

1. $(A \cup B) \cup C = A \cup (B \cup C)$
2. $(A \cap B) \cap C = A \cap (B \cap C)$

Proof of ①: For each element x,

$$
\begin{aligned}
x \in (A \cup B) \cup C \quad & \text{iff} \quad x \in A \cup B \ \text{or} \ x \in C \\
& \text{iff} \quad (x \in A \ \text{or} \ x \in B) \ \text{or} \ x \in C \\
& \text{iff} \quad x \in A \ \text{or} \ (x \in B \ \text{or} \ x \in C) \quad (\text{by Assoc. of prop. calc.}) \\
& \text{iff} \quad x \in A \ \text{or} \ x \in B \cup C \\
& \text{iff} \quad x \in A \cup (B \cup C) \qquad \square
\end{aligned}
$$

Commutative Laws for Unions and Intersections

Proposition 3 (Commutative Laws for Unions and Intersections)

Let A and B be sets. Then:

1. $A \cup B = B \cup A$
2. $A \cap B = B \cap A$