Subsets

Subsets

Office Hours (unusual schedule)

- M 4:45 ~ 6:15
- T 9:00 ~ 10:30

Subsets

Definition 1 (Subsets)

Let A and B be sets.

- To say that $\boxed{\textit{A is a subset of } B}$ (denoted $A \subseteq B$) means that for each x, if $x \in A$, then $x \in B$.
- To say that $\boxed{\textit{A is a proper subset of } B}$ (~~denoted $A \subseteq B$~~) means that $A \subseteq B$ and $A \neq B$.

E.g. $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$: $A \subseteq B$ and A is also a proper subset of B.

Notes.

- The relation $\subseteq$ is called *set inclusion*.
- The notation $B \supseteq A$ means the same as $A \subseteq B$ and is read "B is a superset of A."

Set Inclusion

- Def'n $A \subseteq B \iff (\forall x)[x \in A \Rightarrow x \in B]$
- $(\forall A, B)\left[(\forall x)(x \in A \Leftrightarrow x \in B) \Leftrightarrow A = B\right]$

Proposition 1 (Set Inclusion as Relation)

Set inclusion is reflexive, antisymmetric, and transitive. In other words

1. *For each set A, we have $A \subseteq A$. (Reflexivity.)*
2. *For all sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$. (Antisymmetry.)*
3. *For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. (Transitivity.)*

Proof ① Let A be a set. Let x be an element. Suppose $x \in A$.
Then obviously $x \in A$. Hence, we have shown that
for each x, if $x \in A$, then $x \in A$. Thus $A \subseteq A$.

② Let A and B be sets. Suppose $A \subseteq B$ and $B \subseteq A$.
(WTS that $A = B$.) Let x be an element. Suppose $x \in A$.
Then $x \in B$, because $A \subseteq B$. Conversely, suppose $x \in B$.

Then $x \in A$, because $B \subseteq A$. So we have shown that

$$\text{for each } x,\ x \in A \text{ iff } x \in B.$$

In other words, we have shown that $A = B$.

③ NTS: $(\forall A, B, C)[\, A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C \,]$

Let A, B, and C be sets. Suppose $A \subseteq B$ and $B \subseteq C$.

(WTS: $A \subseteq C$.) Let x be arbitrary. Suppose $x \in A$.

Then $x \in B$, because $A \subseteq B$. But then $x \in C$, because $B \subseteq C$.

We have shown that

$$\text{for each } x,\ \text{if } x \in A, \text{ then } x \in C.$$

In other words, we have shown that $A \subseteq C$. □

Empty Set

Proposition 2

For each set A, we have $\varnothing \subseteq A$.

- The proof involves a *vacuously true* statement.
- Conversely, if a set is a subset of any set, then it must be the empty set. In other words,

  (S10E05) Let A be a set such that for each set B, we have $A \subseteq B$. Then $A = \varnothing$.

Proof. Let A be a set. We wish to show that

for each x, if $x \in \varnothing$, then $x \in A$.

Let x be arbitrary. Note that the antecedent $x \in \varnothing$ of the conditional sentence is false. Hence the conditional sentence is (vacuously) true. Therefore, $\varnothing \subseteq A$. □

Exercise 1 (Subsets)

Answer the following questions.

1. Is $\{3,5\}$ a subset of $\{2,3,5\}$? Yes.
2. Is $\underbrace{\{2,\{3,5\}\}}_{A=}$ a subset of $\underbrace{\{2,3,5\}}_{B=}$? No, because $\{3,5\} \in A$ but $\{3,5\} \notin B$.
3. Write down all subsets of $\{1,2,3\}$.

<u>Note</u> $\{3,5\} \subseteq B$.

$\{1, 2, 3\}$

$\boxed{\cdot} \quad \boxed{\cdot} \quad \boxed{\cdot}$

③

$\emptyset$

proper s/sets

$\{1\}, \{2\}, \{3\},$

$\{1,2\}, \{1,3\}, \{2,3\}.$

$\{1,2,3\}$

Exercise 2 ($\in$ vs. $\subseteq$)

Find two sets A and B such that:

1. $A \in B$ and $A \subseteq B$.
2. $A \in B$ and $A \nsubseteq B$.
3. $A \notin B$ and $A \subseteq B$.
4. $A \notin B$ and $A \nsubseteq B$.