Surjections, Injections, and Inverses

Restriction and Extension

Restriction

Definition 1

Let f be a function and let $C \subseteq \mathrm{Dom}(f)$. Then *the restriction of f to C* is the function, denoted $f \restriction C$, defined by $(f \restriction C)(x) = f(x)$ for all $x \in C$.

- Note that $\mathrm{Dom}(f \restriction C) = C$.

Examples.

- Let $f(x) = x^{1/3}$ for all $x \in \mathbb{R}$ and let $g(x) = x^{1/3}$ for all $x \in [1, 5)$. Then $g = f \restriction [1, 5)$.
- Let $f(x) = \sqrt{x}$ for all $x \in [0, \infty)$, $g(x) = 1 - x^2$ for all $x \in \mathbb{R}$, and $h(x) = 1 - x$ for all $x \in \mathbb{R}$. Then $g \circ f = h \restriction [0, \infty)$.

Extension

Definition 2

Let f and g be functions. To say that f *is an extension of* g means that $\mathrm{Dom}(f) \supseteq \mathrm{Dom}(g)$ and for each $x \in \mathrm{Dom}(g)$, $f(x) = g(x)$.

- Note f is an extension of g iff $\mathrm{Dom}(f) \supseteq \mathrm{Dom}(g)$ and $f \restriction \mathrm{Dom}(g) = g$.

Surjections, Injections, and Inverses

Surjections

Definition 3

Let A and B be sets. To say that f *is a surjection from* A *to* B means that f is a function from A to B and for each $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Notes.

- A surjection from A to B is also said to be a function from A *onto* B.
- Any function is a surjection from its domain to its range.
- f is a surjection from A to B

  iff f is a function, $\mathrm{Dom}(f) = A$, and $\mathrm{Rng}(f) = B$

  iff for each $y \in B$, the equation $f(x) = y$ has at least one solution x in A.

Surjections (cont')

Example 4

- Let $f(x) = \sin(x)$ for all $x \in \mathbb{R}$. Then f is a surjection from $\mathbb{R}$ to $[-1, 1]$, but f is not a surjection from $\mathbb{R}$ to $\mathbb{R}$.
- Let $g(x) = \arctan(x)$ for all $x \in \mathbb{R}$. Then f is a surjection from $\mathbb{R}$ to $(-\pi/2, \pi/2)$.

Injections

Definition 5

To say that f *is an injection* means that f is a function and for all $x_1, x_2 \in \mathrm{Dom}(f)$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Note.

- To say that f *is an injection from* A *to* B means that f is a function from A to B and f is an injection.
- An injection is also said to be a *one-to-one* function.
- f is an injection from A to B

  iff for each $y \in B$, the equation $f(x) = y$ has *at most* one solution x in A

  iff for all $x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Injections (cont')

Example 6

Let $f(x) = x^2$ for all $x \in \mathbb{R}$ and let $g(x) = \sqrt{x}$ for all $x \in [0, \infty)$. Then:

- f is not an injection from $\mathbb{R}$ to $[0, \infty)$ because
- g is an injection from $[0, \infty)$ to $[0, \infty)$ because

Inverses

Definition 7

Let f be an injection. Then for each $y \in \mathrm{Rng}(f)$, we shall write $f^{-1}(y)$ for the unique $x \in \mathrm{Dom}(f)$ such that $f(x) = y$. This defines a function f^{-1} from $\mathrm{Rng}(f)$ to $\mathrm{Dom}(f)$. The function f^{-1} is called the *inverse of the function* f.

Notes.

- $\mathrm{Dom}(f^{-1}) = \mathrm{Rng}(f)$ and $\mathrm{Rng}(f^{-1}) = \mathrm{Dom}(f)$.

Example 8

For each of the following functions, state whether it is an injection. If it is an injection, determine its inverse function. If it is not an injection, find an interval such that the restriction of the function to that interval is an injection whose inverse function has a standard name, and determine that inverse function.

1. $f(x) = x^2$ for all $x \in \mathbb{R}$.
2. $f(x) = x^3$ for all $x \in \mathbb{R}$.
3. $f(x) = \sin(x)$ for all $x \in \mathbb{R}$.

Composition of Surjections

Theorem 9

Let A, B, and C be sets. Suppose that f is a surjection from A to B and g is a surjection from B to C. Then $g \circ f$ is a surjection from A to C.

Proof. Let $c \in C$. Since g is a surjection from B to C, there exists $b \in B$ such that $g(b) = c$. Since f is a surjection from A to B, there exists $a \in A$ such that $f(a) = b$. It follows that

$$(g \circ f)(a) = g(f(a)) = g(b) = c.$$

We have shown that for any $c \in C$, there exists $a \in A$ such that $(g \circ f)(a) = c$. In other words, $g \circ f$ is a surjection from A to C. □

Composition of Injections

Theorem 10

Let f and g be injections. Then $g \circ f$ is an injection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

See Theorem 11.72.