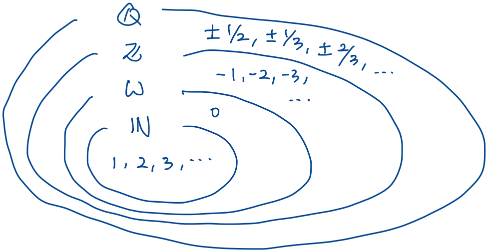
Cantor's Diagonal Lemma

Cantor's Diagonal Lemma

Last time Infinite sets are not rigid!

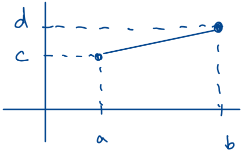
> An infinite set can be equinumerous to its proper subsets.

E.g.



E.g. $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$

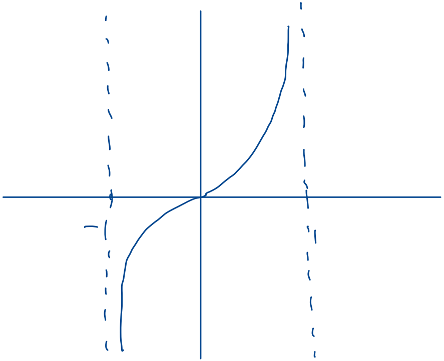
E.g. $[a, b] \approx [c, d]$



$y = m(x-a)+c$

E.g. $(-1, 1) \approx \mathbb{R}$

$$\varphi(x) = \frac{x}{1-|x|}$$



Question $\mathbb{N} \approx \mathbb{R}$?

$\hookrightarrow$ Cantor's answer.

Cantor's Diagonal Lemma

Let $A = \{\langle a_n \rangle : a_n \in \{0,1\}\}$.
Let $f : \mathbb{N} \to A$. Then there exists $y \in A$ s.t. $y \notin Rng(f)$.

Cantor's Diagonal Lemma

Let f be any function from $\mathbb{N}$ to $(0,1)$. Then there exists $y \in (0,1)$ such that y does not belong to the range of f.

Below is a key consequence of Cantor's diagonal lemma.

Theorem 1 (Cantor, 1873)

$\mathbb{R}$ is not equinumerous to $\mathbb{N}$.

S'pose otherwise, i.e., $\mathbb{R} \approx \mathbb{N}$. By symmetry of $\approx$, $\mathbb{N} \approx \mathbb{R}$.
We know from Wed that ~~(0,1)~~ $\mathbb{R} \approx (0,1)$.
~~So~~ Thus ~~$(0,1) \approx \mathbb{N}$~~ $\mathbb{N} \approx (0,1)$ by transitivity of $\approx$.
Hence $\exists$ a bij. f from $\mathbb{N}$ to $(0,1)$. In particular f is a func. from $\mathbb{N}$ to $(0,1)$.
Then by CDL, $\exists\, y \in (0,1)$ s.t. $y \notin Rng(f)$. It follows that f is not a surjection from $\mathbb{N}$ to $(0,1)$, and so f is not a bij. from $\mathbb{N}$ to $(0,1)$. $\times$

Cantor's Diagonal Lemma: Idea of Proof

To prove Cantor's diagonal lemma, we need to find/construct $y \in (0,1)$ such that $y \notin \mathrm{Rng}(f) = \{f(n) : n \in \mathbb{N}\}$.

expansion that does not end ~~with~~ in ~~infinitely many~~ repeating 9's ~~at the end~~

<u>Decimal expansion of $f(n)$</u>

For each $n \in \mathbb{N}$, $f(n) \in (0,1)$ so it has the standard decimal expansion

$$f(n) = 0.x_{n1}x_{n2}x_{n3}x_{n4}\dots.$$

$$\frac{1}{2} = 0.5 = 0.4999\cdots$$

That is,

$$\begin{aligned}
f(1) &= 0.\color{red}{x_{11}}\color{black}x_{12}x_{13}x_{14}\dots,\\
f(2) &= 0.x_{21}\color{red}{x_{22}}\color{black}x_{23}x_{24}\dots,\\
f(3) &= 0.x_{31}x_{32}\color{red}{x_{33}}\color{black}x_{34}\dots,\\
f(4) &= 0.x_{41}x_{42}x_{43}\color{red}{x_{44}}\color{black}\dots,
\end{aligned}$$

and so on.

$$\frac{4}{10} + \underbrace{\frac{9}{10^2} + \frac{9}{10^3} + \cdots}_{} = \frac{5}{10} = \frac{1}{2}$$

$$= \frac{9}{10^2}\cdot\left(1 + \frac{1}{10} + \cdots\right)$$

$$= \frac{9}{10^2}\,\frac{1}{1-\frac{1}{10}} = \frac{\cancel{9}}{10^{\cancel{2}}}\,\frac{1}{\frac{\cancel{9}}{\cancel{10}}} = \frac{1}{10}$$

Cantor's Diagonal Lemma: Idea of Proof (cont')

Not the only

Construction of y

For each $n \in \mathbb{N}$, let

$$y_n = \begin{cases} 5 & \text{if } x_{nn} \neq 5, \\ \cancel{4} & \text{if } x_{nn} = 5. \end{cases}$$

$$y_n = \begin{cases} 1 & \text{if } x_{nn} \neq 1 \\ 2 & \text{if } x_{nn} = 1 \end{cases}$$

1, 2, 3, 4, ~~5~~, 6, 7, 8, ~~9~~

Then for each $n \in \mathbb{N}$, $y_n \neq x_{nn}$. Now let y be the number whose standard decimal expansion is

$$y = 0.y_1 y_2 y_3 y_4 \ldots.$$

Observation

- $y \in (0, 1)$; in fact, $0.444\ldots \leqslant y \leqslant 0.555\ldots$.
- $y \notin \mathrm{Rng}(f)$ because for each $n \in \mathbb{N}$, $y \neq f(n)$.

because they differ in the n^{th} dec. place.

Higher Orders of Infinity

Denumerable, Countable, and Uncountable

Definition 2

Let A be a set.

1. To say that A *is denumerable* means that A is equinumerous to $\mathbb{N}$.
2. To say that A *is countable* means that A is finite or denumerable.
3. To say that A *is uncountable* means that A is not countable.

infinite & not denumerable.

$\approx \mathbb{N}$.

Example.

- Each of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$, and $\mathbb{Q}$ is denumerable.
- $\mathbb{R}$ is uncountable.

Cardinality

Definition 3

Let A and B be sets.

1. To say that *the cardinality of A is less than or equal to the cardinality of B* (denoted $\overline{\overline{A}} \leqslant \overline{\overline{B}}$) means that A is equinumerous to a subset of B.

2. To say that *the cardinality of A is strictly less than the cardinality of B* (denoted $\overline{\overline{A}} < \overline{\overline{B}}$) means that A is equinumerous to a subset of B but A is not equinumerous to B.

3. To say that *the cardinality of A is equal to the cardinality of B* (denoted $\overline{\overline{A}} = \overline{\overline{B}}$) means that A is equinumerous to B.

Example. $\overline{\overline{\mathbb{N}}} < \overline{\overline{\mathbb{R}}}$.

Notes.

- Let A and B be sets. Then $\overline{\overline{A}} \leqslant \overline{\overline{B}}$ iff there exists an injection from A to B.

($\Rightarrow$) S'pose $\bar{\bar{A}} \le \bar{\bar{B}}$. Then $\exists\, C \subseteq B$ s.t. $A \approx C$, so there is a bij. f from A to C.
In particular, f is a func from A to C and f is an injection.
Since f is a func from A to C and $C \subseteq B$, f is a func from A to B.
Thus f is an inj. from A to B.

- Let A be any set. Then $\overline{\overline{A}} \leqslant \overline{\overline{\mathcal{P}(A)}}$.

($\Leftarrow$) S'pose $\exists$ an injection f from A to B. Since any function is a surj. from its dom. to its range, f is a surj. from A to $Rng(f)$.
It follows that f is a bije. from A to $Rng(f)$. We have shown that $\exists$ a bij. from A to a subset of B. Thus $\bar{\bar{A}} \le \bar{\bar{B}}$.

For each $x \in A$, define $h(x) = \{x\}$. Then $h : A \to \mathcal{P}(A)$. Note that for any $x_1, x_2 \in A$, if $h(x_1) = h(x_2)$, then $\{x_1\} = \{x_2\}$, so $x_1 \in \{x_2\}$, so $x_1 = x_2$. Thus h is an inj.

Cantor's Generalized Diagonal Lemma

Cantor's Generalized Diagonal Lemma

Let A be a set and let f be a function on A such that for each $x \in A$, $f(x)$ is a set. Then there exists a subset $C \subseteq A$ such that C does not belong to the range of f.

Below is a key consequence of Cantor's generalized diagonal lemma.

Theorem 4 (Cantor, 1891)

Any set has strictly smaller cardinality than its power set.

Let A be a set. Know $\bar{\bar{A}} \leq \overline{\overline{\mathcal{P}(A)}}$. So just need to show $\bar{\bar{A}} \neq \overline{\overline{\mathcal{P}(A)}}$.

Suppose otherwise. $\bar{\bar{A}} = \overline{\overline{\mathcal{P}(A)}}$. $\exists f : A \to \mathcal{P}(A)$ a bijection. It is a functi[on] $\mathcal{P}(A)$ //.

and $f(x) \in \mathcal{P}(A)$ $\forall x \in A$, i.e. $f(x)$ is a set. Thus by CGDL, $\exists C \subseteq A$ s.t. $C \notin Rng(f)$

But C is a subset of A, so $C \in \mathcal{P}(A)$. Contr.

Generalized Diag. Lem. A set, f on A, $f(x)$ is a set $\forall x \in A$.

$\exists\, C \subseteq A$ s.t. $C \notin \mathrm{Rng}(f)$.

Let $C = \{x \in A : x \notin f(x)\}$.

① $C \subseteq A$ ✓

② NTS: $C \notin \mathrm{Rng}(f)$ $\qquad x_0 \in C$ or $x_0 \notin C$

↓ S'pose $C \in \mathrm{Rng}(f)$. Then $C = f(x_0)$ for some $x_0 \in A$.

__Case 1__ S'pose $x_0 \in C$. Then by def'n of C, $x_0 \notin f(x_0)$. But $f(x_0) = C$.
Hence $x_0 \notin C$. A contradiction.

__Case 2__ S'pose $x_0 \notin C$. Then $x_0 \in C$ because $x_0 \in A$ and $x_0 \notin f(x_0)$.
Since $C = f(x_0)$ ~~that is~~ $x_0 \notin f(x_0)$. But f

__Ex__ $A = \{1, 2, 3\}$, $f : A \to \wp(A)$

$f(1) = \{1, 2\}$
$f(2) = \{2, 3\}$
$f(3) = \{3\}$

$C = \{x \in A : \cancel{f(x)}\, x \notin f(x)\} = \cancel{\{\ \}}\ \emptyset$

$\mathrm{Rng}(f) = \{\{1,2\}, \{2,3\}, \{3\}\}$

$\cancel{\emptyset}\ C \notin \mathrm{Rng}(f)$ ✓

$f(1) = \{2\}$
$f(2) = \{1, 3\}$
$f(3) = \{1, 2\}$

$C = \{x \in A : x \notin f(x)\} = \{1, 2, 3\}$

$\mathrm{Rng}(f) = \{\{2\}, \{1,2\}, \{1,3\}\}$

$\cancel{\{1,2\}}$, $C \notin \mathrm{Rng}(f)$

Confirmed!