Surjections and Injections

Restriction and Extension

Restriction

Definition 1

Let f be a function and let $C \subseteq \mathrm{Dom}(f)$. Then *the restriction of f to C* is the function, denoted $f \upharpoonright C$, defined by $(f \upharpoonright C)(x) = f(x)$ for all $x \in C$.

- Note that $\mathrm{Dom}(f \upharpoonright C) = C$.

Examples.

- Let $f(x) = x^{1/3}$ for all $x \in \mathbb{R}$ and let $g(x) = x^{1/3}$ for all $x \in [1, 5)$. Then $g = f \upharpoonright [1, 5)$.
- Let $f(x) = \sqrt{x}$ for all $x \in [0, \infty)$, $g(x) = 1 - x^2$ for all $x \in \mathbb{R}$, and $h(x) = 1 - x$ for all $x \in \mathbb{R}$. Then $g \circ f = h \upharpoonright [0, \infty)$.

Extension

Definition 2

Let f and g be functions. To say that f *is an extension of* g means that $\mathrm{Dom}(f) \supseteq \mathrm{Dom}(g)$ and for each $x \in \mathrm{Dom}(g)$, $f(x) = g(x)$.

- Note f is an extension of g iff $\mathrm{Dom}(f) \supseteq \mathrm{Dom}(g)$ and $f \restriction \mathrm{Dom}(g) = g$.

Surjections and Injections

Surjections

Definition 3

Let A and B be sets. To say that f *is a surjection from* A *to* B means that f is a function from A to B and for each $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Notes.

- A surjection from A to B is also said to be a function from A *onto* B.
- Any function is a surjection from its domain to its range.
- f is a surjection from A to B

  iff f is a function, $\mathrm{Dom}(f) = A$, and $\mathrm{Rng}(f) = B$

  iff for each $y \in B$, the equation $f(x) = y$ has at least one solution x in A.

Surjections (cont')

Example 4

- Let $f(x) = \sin(x)$ for all $x \in \mathbb{R}$. Then f is a surjection from $\mathbb{R}$ to $[-1, 1]$, but f is not a surjection from $\mathbb{R}$ to $\mathbb{R}$.
- Let $g(x) = \arctan(x)$ for all $x \in \mathbb{R}$. Then f is a surjection from $\mathbb{R}$ to $(-\pi/2, \pi/2)$.

Injections

Definition 5

To say that f *is an injection* means that f is a function and for all $x_1, x_2 \in \mathrm{Dom}(f)$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Note.

- To say that f *is an injection from* A *to* B means that f is a function from A to B and f is an injection.
- An injection is also said to be a *one-to-one* function.
- f is an injection from A to B

  iff for all $x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$

  iff for each $y \in B$, the equation $f(x) = y$ has *at most* one solution x in A.

Example 6

Let $f(x) = x^2$ for all $x \in \mathbb{R}$ and let $g(x) = \sqrt{x}$ for all $x \in [0, \infty)$. Then:

- f is not an injection from $\mathbb{R}$ to $[0, \infty)$ because
- g is an injection from $[0, \infty)$ to $[0, \infty)$ because

Composition of Surjections

Theorem 7

Let A, B, and C be sets. Suppose that f is a surjection from A to B and g is a surjection from B to C. Then $g \circ f$ is a surjection from A to C.

Proof. Let $c \in C$. Since g is a surjection from B to C, there exists $b \in B$ such that $g(b) = c$. Since f is a surjection from A to B, there exists $a \in A$ such that $f(a) = b$. It follows that

$$(g \circ f)(a) = g(f(a)) = g(b) = c.$$

We have shown that for any $c \in C$, there exists $a \in A$ such that $(g \circ f)(a) = c$. In other words, $g \circ f$ is a surjection from A to C. □

Composition of Injections

Theorem 8

Let f and g be injections. Then $g \circ f$ is an injection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

See Theorem 11.72.