Proof Techniques

Contents

Logic of Solving Equations

Solving Equations

Logically speaking, to say that $x = a$ is a solution of the equation $f(x) = 0$ is to state

$$f(x) = 0 \iff x = a$$

which usually can be seen by a chain of biconditionals.

For example, we see that $x^2 = 5x - 6$ if and only if $x = 2$ or $x = 3$ by:

$$\begin{aligned} x^2 = 5x - 6 &\iff x^2 - 5x + 6 = 0 \\ &\iff (x-2)(x-3) = 0 \\ &\iff x - 2 = 0 \text{ or } x - 3 = 0 \\ &\iff x = 2 \text{ or } \ x = 3. \end{aligned}$$

One needs to be careful to confirm that all steps are true biconditional sentences.

Examples

Rational Equation

Solve the equation
$$\frac{x-2}{x^2+2x-8} = \frac{1}{8}.$$

Erroneous solution.

$$x - 2 = (1/8)(x^2 + 2x - 8)$$
$$8x - 16 = x^2 + 2x - 8$$
$$0 = x^2 - 6x + 8 = (x-2)(x-4)$$
$$x = 2, 4$$

Which step is not a true biconditional sentence?

Correct solution.

Examples

Equation Involving Radicals

Solve the equation

$$x = -\sqrt{x+6}$$

An erroneous solution:

$$\begin{aligned} &x^2 = x + 6 \\ &x^2 - x - 6 = 0 \\ &(x+2)(x-3) = 0 \\ &x = -2, 3 \end{aligned}$$

Is $x = 3$ a solution of the original equation?

Examples (cont')

Correct solution.

Proof by Contradiction

Overview: Proof Techniques

The follow is the list of proof techniques discussed in Section 2 of the textbook:

- Conditional proof
- Proof by contradiction
- Proof by contraposition

Contradictions

A *contradiction* is a sentence of the form $Q \wedge \neg Q$, which is false regardless of the truth value of Q.

Proof by Contradiction

To prove a sentence P, assume $\neg P$ and deduce a contradiction. This approach is known as the method of *proof by contradiction*.

Template. To prove P:

- Begin with "Assume $\neg P$ is true."
- Deduce a contradiction.
- Conclude that P is true.

Why does it work?

Proof by Contradiction (cont')

Example

Let n be an integer. Using the method of proof by contradiction, prove that

$$\text{If } n^2 \text{ is an odd number, then } n \text{ is an odd number.}$$

Proof of a Negative Sentence

The usual way to prove a negative sentence $\neg P$ to prove by contradiction, that is, assume P and deduce a contradiction.

Why does it work?

Proof of a Negative Sentence (cont')

Section 2, Exercise 23

Use the method of conditional proof to explain in words why

$$[(P \Rightarrow Q) \wedge \neg Q] \Rightarrow \neg P$$

is a tautology.

Proof of a Negative Sentence (cont')

Proof by Contraposition

Overview: Proof Techniques

The follow is the list of proof techniques discussed in Section 2 of the textbook:

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Contrapositive

Given $P \Rightarrow Q$, the related conditional sentence $\neg Q \Rightarrow \neg P$ is called *the contrapositive of* $P \Rightarrow Q$. Note that $P \Rightarrow Q$ is logically equivalent to $\neg Q \Rightarrow \neg P$. (Confirm this using a truth table.)

Example. Given the conditional sentence

A: If today is Sunday, then I do not have to go to work today.

- Converse of A:

- Contrapositive of A:

Proof by Contraposition

Proof by Contraposition

To prove $P \Rightarrow Q$, it suffices to prove $\neg Q \Rightarrow \neg P$.

Proof by Contraposition (cont')

Example (revisited)

Let n be an integer. Using the method of proof by contraposition, prove that

If n^2 is an odd number, then n is an odd number.

Solution. The given sentence is logically equivalent to the sentence

If n is not an odd number, then n^2 is not an odd number. $(\star)$

which we will prove.

A1: Assume that n is not an odd number.
(We wish to show that n^2 is not an odd number.)
Since n is an integer bu n is not an odd number, n is an even number.
Hence n^2 is an even number, so n^2 is not an odd number.
We have shown this under A1.

Discharging A1, we conclude that the conditional sentence $(\star)$ is true under no assumptions. This completes the proof by contraposition that the original conditional sentence is true. $\square$

Proof by Contradiction vs Proof by Contraposition

Let's examine the two proof techniques in proving $P \Rightarrow Q$.

Proof by contradiction.

Assume P.
Assume $\neg Q$.
Show $\neg P$.
Contradiction, $P \wedge \neg P$!
So Q must be true.
Therefore, $P \Rightarrow Q$.

Proof by contraposition.

Assume $\neg Q$.
Show $\neg P$. (if this can be done w/o P.)
So $\neg Q \Rightarrow \neg P$.
Therefore, $P \Rightarrow Q$, by contraposition.