Review for Exam 2

- TW 4:45 ~ 6:15 pm (Zoom)
- W no class / extra in-person OH
  in classroom.

Key Topics to Review

- Exercises leading up to the rational roots theorem (S4)
- Binomial coefficients and binomial theorem (S4)
- Complete induction (S7)
- Insight vs. induction (S6)
- Algebra with set operations (S10)

Rational Roots Theorem (S04: 17, 18, 19, 20)

<u>S04E20</u> Let $x \in \mathbb{Q}$ such that

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where $n \in \mathbb{N}$ and $c_0, c_1, \cdots, c_n \in \mathbb{Z}$.

Prove that x can be written in the form $x = a/b$ where $a \in \mathbb{Z}$ that divides c_0 and $b \in \mathbb{N}$ that divides c_n.

<u>Rmk 4.50</u>

Let $d \in \mathbb{N}$ and $x_1, \cdots, x_n \in \mathbb{Z}$.

If d divides $x_1 x_2 \cdots x_n$, then there exist $d_1, d_2, \cdots, d_n \in \mathbb{N}$ such that

for each $j \in \{1, 2, \cdots, n\}$,

d_j divides x_j

and

$$d = d_1 d_2 \cdots d_n.$$

Rational Root Theorem

Proof. Since x is rational, we can pick $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that $x = a/b$ and the fraction a/b is in lowest terms. On substitution, we have

$$c_n \left(a/b\right)^n + c_{n-1}\left(a/b\right)^{n-1} + \cdots + c_1\left(a/b\right) + c_0 = 0$$

$$\frac{c_n a^n}{b^n} + \frac{c_{n-1} a^{n-1}}{b^{n-1}} + \cdots + \frac{c_1 a}{b} + \frac{c_0}{1} = 0$$

$$\frac{c_n a^n + c_{n-1} a^{n-1} b + \cdots + c_1 a b^{n-1} + c_0 b^n}{b^n} = 0$$

Rational Root Theorem

$$\underbrace{\underbrace{c_n}_{1 \text{ integer}} a^n = c_n \cdot \underbrace{a \cdot a \cdots a}_{n \text{ integers}}}_{n+1 \text{ integers}}$$

By multiplying both sides by b^n, we obtain

$$c_n a^n + c_{n-1} a^{n-1} b + \cdots + c_1 a b^{n-1} + c_0 b^n = 0. \qquad (*)$$

Now we write $(*)$ as

$$c_n a^n = -\left(c_{n-1} a^{n-1} b + \cdots + c_1 a b^{n-1} + c_0 b^n\right)$$

$$= -\left(c_{n-1} a^{n-1} + \cdots + c_1 a b^{n-2} + c_0 b^{n-1}\right) b,$$

so b divides $c_n a^n$. product of $n+1$ integers. By the part of Rmk. 4.50 written above,

we can pick $b_1, b_2, \cdots, b_{n+1} \in \mathbb{N}$ such that

$$b_1 \mid c_n,\ b_2 \mid a,\ b_3 \mid a,\ \cdots,\ b_{n+1} \mid a \quad \text{and} \quad b = b_1 b_2 \cdots b_{n+1}.$$

Rational Root Theorem

Complete the argument to show that $b_1 = b$ and so $b \mid c_n$

Now, we can also rewrite (★) as

$$c_0 b^n = -\left(c_n a^n + c_{n-1} a^{n-1} b + \cdots + c_1 a b^{n-1}\right)$$

$$= -\left(c_n a^{n-1} + c_{n-1} a^{n-2} b + \cdots + c_1 b^{n-1}\right) a;$$

So a divides $c_0 b^n$. ...

$\square$

Binomial Coefficients and Binomial Theorem

<u>Pascal's triangle</u>

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\vdots$$

$\binom{n}{k}$: n^{th} row (starting from 0^{th})

k^{th} entry (starting from 0^{th})

$0 \leq k \leq n$.

- $\binom{0}{0} = 1$
- $\binom{n}{0} = \binom{n}{n} = 1$, $\quad n \in \omega$
- $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

Useful in proof by induction.

Binomial Coefficients and Binomial Theorem

$a, b \in \mathbb{R}, \quad n \in \omega.$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$*$ $\quad x^0 = 1$ for any $x \in \mathbb{R}$.

Recursively Defined Sequences and Complete Induction

Let $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$.

Prove using complete induction that for any $n \in \omega$,

$$F_n = \frac{1}{\sqrt{5}}\Bigg(\Big(\underbrace{\frac{1+\sqrt{5}}{2}}_{=\varphi}\Big)^n - \Big(\underbrace{\frac{1-\sqrt{5}}{2}}_{=\hat{\varphi}}\Big)^n \Bigg).$$

(Hint: $\frac{1 \pm \sqrt{5}}{2}$ are roots of $x^2 - x - 1 = 0$.)

Note:

Since $\varphi = \frac{1+\sqrt{5}}{2}$ and $\hat{\varphi} = \frac{1-\sqrt{5}}{2}$ are roots of $x^2 - x - 1$,

- $\varphi^2 - \varphi - 1 = 0 \Rightarrow \varphi^2 = \varphi + 1$
- $\hat{\varphi}^2 - \hat{\varphi} - 1 = 0 \Rightarrow \hat{\varphi}^2 = \hat{\varphi} + 1$.

These will be useful.

Proof. Let $P(n)$ be the sentence

$$F_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \hat{\varphi}^n\right).$$

WTS that for each $n \in \omega$, $P(n)$ is true.

Recursively Defined Sequences and Complete Induction

BASE CASES

- $P(0)$ is true ...
- $P(1)$ is true ...

INDUCTIVE STEP Let $n \in \omega$ such that $n \geq 1$ and $P(0), \dots, P(n)$ are all true. To show $P(n+1)$ is true, we examine

$$F_{n+1} = F_n + F_{n-1}$$

$$= \frac{1}{\sqrt{5}}\left(\varphi^n - \hat{\varphi}^n\right) + \frac{1}{\sqrt{5}}\left(\varphi^{n-1} - \hat{\varphi}^{n-1}\right) \quad \text{(by ind. hyp.)}$$

$$= \frac{1}{\sqrt{5}}\left[\left(\varphi^n + \varphi^{n-1}\right) - \left(\hat{\varphi}^n + \hat{\varphi}^{n-1}\right)\right]$$

Recursively Defined Sequences and Complete Induction

$$= \frac{1}{\sqrt{5}}\left[\varphi^{n-1} \underbrace{(\varphi+1)}_{=\varphi^2} - \hat{\varphi}^{n-1} \underbrace{(\hat{\varphi}+1)}_{=\hat{\varphi}^2} \right]$$

$$= \frac{1}{\sqrt{5}}\left(\varphi^{n+1} - \hat{\varphi}^{n+1} \right)$$

where we used the fact that φ and $\hat{\varphi}$ are roots of $x^2 - x - 1 = 0$ in the second from the last step.

<u>CONCLUSION</u> Therefore, by complete induction, for each $n \in \omega$, $P(n)$ is true. $\square$