Section 7.

Complete Induction

Sec. 5: Induction

Sec. 6: Insight vs. Induction

Complete Induction

Complete Induction

Below is a refinement of the principle of mathematical induction (PMI).

cf) PMI from Lec. 12.

Principle of Complete Mathematical Induction (PCMI)

Let $P(n)$ be any statement about n. Suppose we have proved that

$$P(1) \text{ is true} \tag{1}$$

and that

$$\text{for each } n \in \mathbb{N}, \text{ if } \underbrace{P(1), \ldots, P(n) \text{ are all true}}_{\text{the only difference from PMI}}, \text{ then } P(n+1) \text{ is true.} \tag{2}$$

Then we may conclude that for each natural number n, $P(n)$ is true.

$$(\forall P)\left\{\underbrace{P(1)}_{(1)} \wedge \underbrace{(\forall n \in \mathbb{N})[P(1) \wedge P(2) \wedge \cdots \wedge P(n) \Rightarrow P(n+1)]}_{(2)} \Rightarrow \underbrace{(\forall n \in \mathbb{N})\, P(n)}_{\text{Conclusion}}\right\}$$

Notes on Complete Induction

- Just like the PMI, the “starting number” can be any integer n_0, not necessarily 1.
- Unlike what the name may suggest, PCMI is logically equivalent to PMI. That is, if we accept PCMI as true, then we can prove PMI, and conversely, if we accept PMI as true, then we can prove PCMI.

Deducing PCMI from PMI

PCMI: $(\forall P)\{\boxed{(1)} \wedge \boxed{(2)} \Longrightarrow (\forall n \in \mathbb{N})P(n)\}$

Claim. PMI implies PCMI.

Proof. Consider any sentence $P(n)$. Suppose we have proved that $P(1)$ is true and that for each $n \in \mathbb{N}$, if $P(1), \ldots, P(n)$ are all true, then $P(n+1)$ is true. We wish to show that for each $n \in \mathbb{N}$, $P(n)$ is true. To this end, we introduce another sentence

$$Q(n)\text{: For each } k \in \{1, \ldots, n\}, P(k) \text{ is true.} \equiv P(1) \wedge P(2) \wedge \cdots \wedge P(n)$$

Comment

Note that we will be done once we show that for each $n \in \mathbb{N}$, $Q(n)$ is true because

$$\begin{aligned}(\forall n \in \mathbb{N})Q(n) &\Leftrightarrow (\forall n \in \mathbb{N})(\forall k \in \{1, \ldots, n\})P(k)\\ &\Rightarrow (\forall n \in \mathbb{N})P(n).\end{aligned}$$

Deducing PCMI from PMI (cont')

because we accepted PMI as true.

Here we shall show that for each $n \in \mathbb{N}$, $Q(n)$ is true by induction.

BASE CASE: $Q(1)$ is true because $P(1)$ is true. (by yellow highlight)

When $n=1$, $\{1, \cdots, n\} = \{1\}$.

INDUCTIVE STEP: Let $n \in \mathbb{N}$ such that $Q(n)$ is true. Since $Q(n)$ is true, $P(1), \ldots, P(n)$ are all true. Hence $P(n+1)$ is true. Thus $P(1), \ldots, P(n), P(n+1)$ are all true. That is, $Q(n+1)$ is true too.

(by blue highlight)

CONCLUSION: Therefore, by induction, for each $n \in \mathbb{N}$, $Q(n)$ is true. In other words, for each $n \in \mathbb{N}$, $P(1), \ldots, P(n)$ are all true. In particular, for each $n \in \mathbb{N}$, $P(n)$ is true. $\square$

Deducing PMI from PCMI

PMI: $(\forall P)\left\{ \underbrace{\boxed{P(1)}}_{\text{Base}} \wedge \underbrace{\boxed{(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]}}_{\text{Inductive}} \Rightarrow \underbrace{\boxed{(\forall n \in \mathbb{N})P(n)}}_{\text{Conclusion}} \right\}$

Exercise. Show that PCMI implies PMI. (HW)

accepted as true.

Proof. Consider any sentence $P(n)$ about n. Suppose that $P(1)$ is true and that for each $n \in \mathbb{N}$, if $P(n)$ is true, then $P(n+1)$ is true. We wish to show that for each $n \in \mathbb{N}$, $P(n)$ is true.

[Your work to show $(\forall n \in \mathbb{N})\, P(n)$ is true using PCMI.]

Homework Coaching

S04E17 (collected)

1. Let x be a rational number such that $x^3 = c$, where c is an integer. Prove that x is an integer.
2. Let c be an integer which is not a perfect cube. Prove that $\sqrt[3]{c}$ is irrational.

Sketch of key steps in proof of ①

- $x = a/b$, $a \in \mathbb{Z}$, $b \in \mathbb{N}$, in lowest terms
- $x^3 = c \implies a^3 = b^3 c = (b^2 c)\, b \implies b \mid a^3$
- Rmk 4.50: $b = b_1 b_2 b_3$ s.t. $b_1 \mid a$, $b_2 \mid a$, $b_3 \mid a$

  $\implies b = 1$ (using a/b in lowest terms)

  $\implies x = a/b = a/1 = a \in \mathbb{Z}$.

② Let $c \in \mathbb{Z}$ which is not a perfect cube. Then $\sqrt[3]{c}$ is irrational.

Proof Since $c \in \mathbb{Z}$, $\sqrt[3]{c}$ is a real number. It remains to show $\sqrt[3]{c}$ is not rational. Suppose $\sqrt[3]{c}$ is rational. Let $x = \sqrt[3]{c}$. Then $x^3 = c$. But by part ①, since x is rational and c is an integer, x is an integer. Since c is not a perfect cube, there is no integer whose cube is c. In particular, $x^3 \neq c$. This is a contradiction. Therefore, $\sqrt[3]{c}$ is not rational. $\square$

S04E18

(to be assigned)

Let $x \in \mathbb{R}$ such that $x^3 = rx^2 + sx + t$, where $r, s, t \in \mathbb{Z}$.

1. Prove that if x is rational, then x is an integer.
2. Prove that if x is not an integer, then x is irrational.

S04E17

$c \in \mathbb{Z}$.

1. $x \in \mathbb{Q}$ s.t. $x^3 = c \Rightarrow x \in \mathbb{Z}$.
2. If $c \in \mathbb{Z}$ is not a perfect cube, then $\sqrt[3]{c}$ is irrational.

S05E11 (The Binomial Theorem)

Let $a, b \in \mathbb{R}$. Then for each $n \in \omega$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Convention. $x^0 = 1$ for each $x \in \mathbb{R}$.

Application of bin. thm.

- $$(\underset{a}{\underline{1}} + \underset{b}{\underline{x}})^n = \sum_{k=0}^{n} \binom{n}{k} \underbrace{1^{n-k}}_{= 1} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$= \binom{n}{0} x^0 + \binom{n}{1} x^1 + \cdots + \binom{n}{n} x^n. \ \ldots$$