Review for Exam 1

Fundamentals

- Logical connectives $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$
- Tautologies
- Proof techniques
- Quantifiers $\forall$, $\exists$, mixed quantifiers
- De Morgan's Laws and Distributive Laws

$\neg(P \Rightarrow Q) \equiv$?

"Vacuously true"

- Drinks & ages.
- Cards

- cond'l proofs
- proof by contradiction
- proof by contraposition
- proof by induction

(See summary at the end of Sec. 4.)

Definitions (from Sec. 4)

Write down the definitions of the following sentences exactly as provided in the textbook. Write down preambles whenever needed, such as "Let $a, b, m \in \mathbb{Z}$.".

- x is even.
- x is odd.
- x is rational.
- x is irrational.
- d divides x.
- x is a prime number.
- a is congruent to b modulo m.

Example 1

Use the method of conditional proof to explain in words why the sentence

$$\underbrace{[P \Rightarrow (Q \Rightarrow R)]}_{A_1} \Rightarrow \underbrace{[\underbrace{(P \Rightarrow Q)}_{A_2} \Rightarrow \underbrace{(\overbrace{P}^{A_3} \Rightarrow \overbrace{R}^{C_3})}_{C_2}]}_{C_1}$$

is a tautology. Be explicit about discharging assumptions.

Proof

A1: Suppose $A_1: P \Rightarrow (Q \Rightarrow R)$ is true. (WTS: C_1 is true.)

A2: Suppose $A_2: P \Rightarrow Q$ is true. (WTS: C_2 is true.)

A3: Suppose $A_3: P$ is true. (WTS: $C_3: R$ is true.)

From A2 and A3, we see that Q is true, by modus ponens.

From A1 and A3, we see that $Q \Rightarrow R$ is true, by modus ponens.

From this and the fact that Q is true, R is true, by modus ponens.

We have shown that R is true under A1, A2, and A3.

Discharging A3, C_2 is true under A1 and A2.

Discharging A2, C_1 is true under A1 alone.

Discharging A1, $A_1 \Rightarrow C_1$ is true under no assumptions. Thus it is a tautology

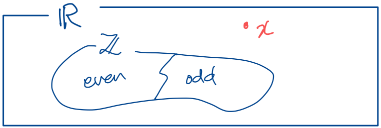
Dichotomies and the Universe of Discourse

Let $x \in \mathbb{Z}$.

- If x is odd, then x is not even. T
- If x is not even, then x is odd. T

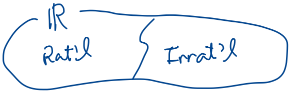
Let $x \in \mathbb{R}$.

- If x is odd, then x is not even. T
- If x is not even, then x is odd. F



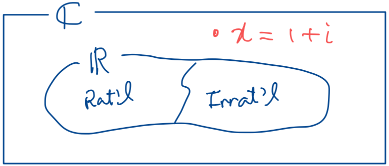
Let $x \in \mathbb{R}$.

- If x is rational, then x is not irrational. T
- If x is not irrational, then x is rational. T



Let $x \in \mathbb{C}$.

- If x is rational, then x is not irrational. T
- If x is not irrational, then x is rational. F



Scratch

$\neg(P \vee Q)$

$\equiv \neg P \wedge \neg Q$

Example 2 (Cf. S04E12.)

It is known that π is an irrational number. From this, prove that $\underbrace{\pi + 2e \text{ is irrational}}_{P}$ or $\underbrace{\pi - 3e \text{ is irrational}}_{Q}$.

Proof (by contradiction)

Assume $\pi + 2e$ is not irrational and $\pi - 3e$ is not irrational.

Since both $\pi + 2e$ and $\pi - 3e$ are real numbers,

$\pi + 2e$ is rational and $\pi - 3e$ is rational.

Note that $3(\pi + 2e)$ and $2(\pi - 3e)$ are rational because the product of two rat'l numbers is rational. Then

$$3(\pi + 2e) + 2(\pi - 3e) = 3\pi + 6e + 2\pi - 6e = 5\pi$$

is rational because the sum of two rat'l numbers is rational.

Then

$$\frac{5\pi}{5} = \pi$$

is rational because a rat'l number divided by a non-zero rat'l number is rational. This is a contradiction because π is rational and π is not rational, because π is real and π is irrational.

Therefore, $\pi + 2e$ is irrational or $\pi - 3e$ is irrational. $\square$

Classical Showcases of Proof by Contradiction

Quiz 4: $\sqrt{3}$ is irrational.

- $\sqrt{2}$ is irrational.
- There are infinitely many prime numbers. (S04E16)

When Prime Divides Product

Euclid's Lemma (Remark 4.50)

Let p be a prime number and let $x, y \in \mathbb{Z}$. If $p \mid xy$, then $p \mid x$ or $p \mid y$.

In general, we have:

> Let $d \in \mathbb{N}$ and let $x, y \in \mathbb{Z}$. If $d \mid xy$, then there exist $d_1, d_2 \in \mathbb{N}$ such that $d_1 \mid x$, $d_2 \mid y$, and $d = d_1 d_2$.

The converse of Remark 4.50 is also true.

Congruence

Example 3 (S04E26(b))

Let m, a_1, b_1, a_2, and b_2 be integers. Suppose that

$$a_1 \equiv b_1 \mod m \quad \text{and} \quad a_2 \equiv b_2 \mod m.$$

Prove that $a_1 a_2 \equiv b_1 b_2 \mod m$.

Soln will be posted.

Induction

- Principle of mathematical induction.
- Declaration, base case, inducti~~on~~ step, and conclusion.

  ive

- Inductive hypothesis.

  "by induction"

- Proving $(\forall x \in \mathbb{Z})P(x)$.

Induction Examples

For each $n \in \mathbb{N}$,

- $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$
- $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
- $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

For each $n \in \mathbb{N}$,

- 3 divides $4^n - 1$.

For each $x \in \mathbb{Z}$,

- x is even or x is odd.

Prove by induction that for each $n \in \mathbb{N}$,

5 divides $8^n - 3^n$.

Prove by induction that for each $n \in \mathbb{N}$,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$