Section 3

Quantifiers (I)

Quiz 2 (noon ~ 11:59 pm)

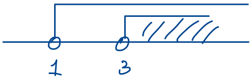
Contents

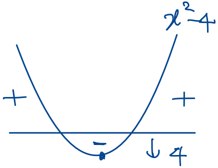
Basics of Quantifiers

Motivation

Let x be a real number. Consider the following sentences.

- $A(x)$: If $x > 3$, then $x > 1$.
- $B(x)$: $x^2 - 4 > 0$.





The truth value of each sentence depends on the value of the variable x.

- $A(x)$ is true for all x.
- $B(x)$ is true for $x < -2$ or $x > 2$.

In general, they can be rephrased using *quantifiers* as:

- For all x, $A(x)$ is true.
- For some x, $B(x)$ is true.

If he is drinking beer, then he is over 21.

Quantifiers

The quantifiers $\forall$ and $\exists$, along with the logical connectives, are main ingredients of modern symbolic logic.

Quantifier	Symbol	Technical Name
"for each"	$\forall$	*universal quantifier*
"for some"	$\exists$	*existential quantifier*

Example. Let x be a person in this class room. Let $P(x)$ stands for "x likes ramen." Then

- $(\forall x)P(x)$: "For each x, x likes ramen." or "Everybody likes ramen."
- $(\exists x)P(x)$: "For some x, x likes ramen." or "Somebody likes ramen."

Notes

Alternate ways to read.

$(\forall x)P(x)$:

For each x, $P(x)$.
For all x, $P(x)$.
For every x, $P(x)$.
For any x, $P(x)$.

$(\exists x)P(x)$:

For some x, $P(x)$.
For at least one x, $P(x)$.
There exists x such that $P(x)$.

Universe of Discourse

$(\forall x) P(x)$,

$(\exists x) P(x)$.

The collection over which the variable x ranges is called *the universe of discourse*. When clear from context, it is omitted in notation; if not, specify the universe of discourse using the following notation.

$$(\forall x \in U) P(x) \quad \text{or} \quad (\exists x \in U) P(x).$$

Frequently used collections.

- $\mathbb{N}$: the set of natural numbers, $\{1, 2, 3, \ldots\}$
- $\mathbb{Z}$: the set of integers, $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $\mathbb{Q}$: the set of rational numbers
- $\mathbb{R}$: the set of real numbers
- $\mathbb{C}$: the set of complex numbers

Free and Bound Variables

- In $P(x)$, x can stand for any particular element of the universe of discourse; it is called a *free variable*.
- In $\boxed{(\forall x)P(x)}$ or $\boxed{(\exists x)P(x)}$, x varies over the universe of discourse, not standing for any particular element; it is called a *bound variable* or a *dummy variable*.

same as $(\forall y)\,P(y)$

for $\cancel{i}$ j = 1 : 10

$\quad 3 * \cancel{i}\ j$;

end

Universal and Existential Quantifiers

#2 on Quiz 2

1, C, 2, B

If $\underbrace{\text{a card has B on one side}}_{P}$, then $\underbrace{\text{it has 2 on the other side}}_{Q}$.

	P	Q	$P \Rightarrow Q$
1st	?	F	
2nd	F	?	T
3rd	?	T	T
4th	T	?	

} Know this w/o needing to flip cards

Universal Quantifier $(\forall)$

Let U be the universe of discourse.

- $(\forall x)P(x)$ is true when $P(x)$ is true for all values of x in U.
- To show $(\forall x)P(x)$ is false, it suffices to show that $P(x)$ is false for at least one value of x in U; such x is said to be a *counterexample that disproves the universal sentence.*

Example. State whether each of the following sentences is true or false. Prove the claims.

1. $(\forall x \in \mathbb{R})(x - 2 = 5)$ T / F

counterexample: $x = 3$

Proof: Claim the given is false. Assume $(\forall x \in \mathbb{R})(x-2=5)$ is true.
Since $x = 3$ is a real number, $3-2=5$.
But $3-2=1 \neq 5$. This is a contradiction.
Hence, $(\forall x \in \mathbb{R})(x-2=5)$ is false.

2. $(\forall x \in \mathbb{R})(x^2 + 6x + 10 > 0)$ T / F

Alex: $\underline{x^2 + 6x + 10 = (x+3)^2 + 1}$

Proof: Let $x_0 \in \mathbb{R}$ be arbitrary. Then

$$x_0^2 + 6x_0 + 10 = \underbrace{x_0^2 + \boxed{6}x_0 + 9}_{\text{complete square}} + 1$$

$$= (x_0+3)^2 + 1 \geq 0 + 1 = 1 > 0.$$

Since x_0 was chosen arbitrarily, it shows that $(\forall x \in \mathbb{R})(x^2 + 6x + 10 > 0)$.

Existential Quantifier $(\exists)$

Let U be the universe of discourse.

- $(\exists x)P(x)$ is true when $P(x)$ is true for at least one value of x in U; such x is said to be an *example that proves the existential sentence*.
- To show $(\exists x)P(x)$ is false, it is necessary to show that $P(x)$ is false for all values of x in U.

Example. State whether each of the following sentences is true or false.

- $(\exists x \in \mathbb{R})(x - 2 = 5)$ T / F

example: $x = 7$

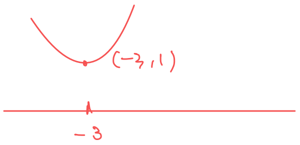
Proof Let $x = 7$, a real number. Then

$$x - 2 = 7 - 2 = 5.$$

Therefore, $(\exists x \in \mathbb{R})(x-2=5)$ is true.

- $(\exists x \in \mathbb{R})(x^2 + 6x + 10 < 0)$ T / F

From prev. work: $x^2 + 6x + 10 = (x+3)^2 + 1$



Proof To show that the given is false, we must show that $x^2 + 6x + 10 < 0$ is false for all $x \in \mathbb{R}$. That is, we must show

$$(\forall x \in \mathbb{R})(x^2 + 6x + 10 \geq 0).$$

[Finish the proof.]

Notes on Quantifiers

Connections to Logical Connectives

Suppose the universe of discourse consists only of two objects $\{a, b\}$. Note that

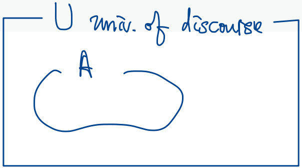
- $(\forall x)P(x)$ is true exactly when $P(a) \wedge P(b)$ is true.
- $(\exists x)P(x)$ is true exactly when $P(a) \vee P(b)$ is true.

In general, when the universe of discourse is a finite set $\{a_1, a_2, \ldots, a_n\}$, then

- $(\forall x)P(x)$ has the same truth value as $P(a_1) \wedge P(a_2) \wedge \cdots \wedge P(a_n)$.
- $(\exists x)P(x)$ has the same truth value as $P(a_1) \vee P(a_2) \vee \cdots \vee P(a_n)$.

Upshot:
- $\forall$ is a generalization of $\wedge$.
- $\exists$ is a generalization of $\vee$

Notation



Suppose A is a subcollection of the universe of discourse. Then

- $(\forall x \in A)P(x)$ is a shorthand notation for $(\forall x)[(x \in A) \Rightarrow P(x)]$.
- $(\exists x \in A)P(x)$ is a shorthand notation for $(\exists x)[(x \in A) \wedge P(x)]$.

When the universe of discourse is $\mathbb{R}$, a subcollection may be characterized by an inequality in which case one may use notations *e.g.*,

- $(\forall x > 0)(2x + 7 = 3)$ T / F
- $(\exists x \geq 7)(x^2 - 4x + 3 > 0)$ T / F

Scope of Quantifiers

The scope of a quantifier is specified using appropriate delimiters.

Example. Let n be an element in $\{2, 3, 5, 7\}$ and let

$$P(n): \; n \text{ is a prime number.}$$
$$Q(n): \; n \text{ is an even number.}$$

dummy / bound variable.

free variable.

Then

- $(\forall n)P(n) \wedge Q(n)$ stands for

  "Every number in {2, 3, 5, 7} is a prime number and n is an even number." (True when n=2)

dummy var.

- $(\forall n)[P(n) \wedge Q(n)]$ stands for

  "Every number in {2, 3, 5, 7} is both a prime number and an even number." (False)

Homework (1/21; due Wed 1/26)

Section 3: # 1(e–k), 3, 4