Section 7.

Complete Induction

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Below is a refinement of the principle of mathematical induction (PMI).

Principle of Complete Mathematical Induction (PCMI)

Let $P(n)$ be any statement about n. Suppose we have proved that

$$P(1) \text{ is true} \tag{1}$$

and that

$$\text{for each } n \in \mathbb{N}, \text{ if } P(1), \ldots, P(n) \text{ are all true, then } P(n+1) \text{ is true.} \tag{2}$$

$\leftarrow$ different from PMI

Then we may conclude that for each natural number n, $P(n)$ is true.

if $P(n)$ is true

$$(\forall P)\left\{\underbrace{P(1)}_{(1)} \wedge \underbrace{(\forall n \in \mathbb{N})\left[P(1) \wedge \cdots \wedge P(n) \Rightarrow P(n+1)\right]}_{(2)} \Rightarrow \underbrace{(\forall n \in \mathbb{N})\, P(n)}_{\text{conclusion}}\right\}$$

Notes on Complete Induction

- Just like the PMI, the "starting number" can be any integer n_0, not necessarily 1.
- Unlike what the name may suggest, PCMI is logically equivalent to PMI. That is, if we accept PCMI as true, then we can prove PMI, and conversely, if we accept PMI as true, then we can prove PCMI.

Deducing PCMI from PMI

Claim. PMI implies PCMI.

Proof. Consider any sentence $P(n)$. Suppose we have proved that $P(1)$ is true and that for each $n \in \mathbb{N}$, if $P(1), \ldots, P(n)$ are all true, then $P(n+1)$ is true. We wish to show that for each $n \in \mathbb{N}$, $P(n)$ is true. To this end, we introduce another sentence

$Q(n)$: For each $k \in \{1, \ldots, n\}$, $P(k)$ is true. $\equiv P(1) \wedge P(2) \wedge \cdots \wedge P(n)$

Note that we will be done once we show that for each $n \in \mathbb{N}$, $Q(n)$ is true because

$$
\begin{aligned}
(\forall n \in \mathbb{N}) Q(n) &\Leftrightarrow (\forall n \in \mathbb{N})(\forall k \in \{1, \ldots, n\}) P(k) \\
&\Rightarrow (\forall n \in \mathbb{N}) P(n).
\end{aligned}
$$

Deducing PCMI from PMI (cont')

Here we shall show that for each $n \in \mathbb{N}$, $Q(n)$ is true by induction.

BASE CASE: $Q(1)$ is true because $P(1)$ is true.

See the highlighted assumption above.

INDUCTIVE STEP: Let $n \in \mathbb{N}$ such that $Q(n)$ is true. Since $Q(n)$ is true, $P(1), \ldots, P(n)$ are all true. Hence $P(n+1)$ is true. Thus $P(1), \ldots, P(n), P(n+1)$ are all true. That is, $Q(n+1)$ is true too.

CONCLUSION: Therefore, by induction, for each $n \in \mathbb{N}$, $Q(n)$ is true. In other words, for each $n \in \mathbb{N}$, $P(1), \ldots, P(n)$ are all true. In particular, for each $n \in \mathbb{N}$, $P(n)$ is true. $\square$

Deducing PMI from PCMI

Exercise. Show that PCMI implies PMI. (HW.)

Suggestion Begin by writing out the "assumptions" of PMI, e.g.

Proof Consider any sentence $P(n)$ about n. Suppose that you have proved - - - - .

We wish to show for each $n \in \mathbb{N}$, $P(n)$ is true.

PMI.

Work to show $(\forall n \in \mathbb{N})\, P(n)$ using PCMI

Homework Coaching

1. Let x be a rational number such that $x^3 = c$, where c is an integer. Prove that x is an integer.

2. Let c be an integer which is not a perfect cube. Prove that $\sqrt[3]{c}$ is irrational.

Key steps in proof: $x = a/b$, $a \in \mathbb{Z}$, $b \in \mathbb{N}$, in lowest terms.

$$x^3 = c \Rightarrow (a/b)^3 = c \Rightarrow a^3 = (b^2 c)\, b.$$

- So $b \mid a^3$.
- Rmk. 4.50: $b = b_1 b_2 b_3$ s.t. $b_1 \mid a$, $b_2 \mid a$, $b_3 \mid a$.

$$\Rightarrow b = 1 \Rightarrow x = a/1 = a.$$

<u>Proof of ②</u> Since c is an integer, $\sqrt[3]{c}$ is a real number.

It remains to show that $\sqrt[3]{c}$ is not rational. Suppose that $\sqrt[3]{c}$ is rational. Let $x = \sqrt[3]{c}$. Since x is rational and $x^3 = c$, by Part ①, x is an integer. Since c is not a perfect cube, there is no integer whose cube is c. In particular, $x^3 \neq c$. We have reached a contradiction. Therefore, $\sqrt[3]{c}$ is not rational. $\square$

S04E18

Let $x \in \mathbb{R}$ such that $x^3 = rx^2 + sx + t$, where $r, s, t \in \mathbb{Z}$.

1. Prove that if x is rational, then x is an integer.
2. Prove that if x is not an integer, then x is irrational.

cf) S04E17

$x^3 = c$, $c \in \mathbb{Z}$

① $x \in \mathbb{Q} \Rightarrow x \in \mathbb{Z}$

② If c is not a perfect cube, then x is irrational.

S05E11 (The Binomial Theorem)

Recurrence relation

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Let $a, b \in \mathbb{R}$. Then for each $n \in \omega$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

$$= \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \cdots + \binom{n}{n} a^0 b^n$$

Convention. $x^0 = 1$ for each $x \in \mathbb{R}$.

Key steps (inductive step)

$$(a+b)^{n+1} = (a+b)^n (a+b) = (a+b)^n a + (a+b)^n b$$

$$= \left[\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\right] a + \left[\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\right] b$$

$$= \left[\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k}\right] a + \left[\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k}\right] b$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^{k} + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}$$

$$= \underbrace{\binom{n}{0} a^{n+1} b^{0}}_{k=0 \text{ from } 1^{st} \text{ sum}} + \underbrace{\sum_{k=1}^{n} \binom{n}{k} a^{n-k+1} b^{k}}_{\text{rest of } 1^{st} \text{ sum}}$$

$$+ \underbrace{\sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1}}_{\text{first } n \text{ terms of } 2^{nd} \text{ sum}} + \underbrace{\binom{n}{n} a^{0} b^{n+1}}_{\text{last term of } 2^{nd} \text{ sum.}}$$

$k' = k+1$

change of indices

$$\sum_{k'=1}^{n} \binom{n}{k'-1} a^{n-(k'-1)} b^{k'}$$

$$= \sum_{k=1}^{n} \binom{n}{k-1} a^{n-k+1} b^{k}$$

= Work out sum using