Euclid’s Lemma

Division by a Prime

Recap

Principle of Complete Mathematical Induction (PCMI)

Let $P(n)$ be any statement about n. Suppose we have proved that

$$P(1) \text{ is true} \tag{1}$$

and that

$$\text{for each } n \in \mathbb{N}, \text{ if } P(1), \ldots, P(n) \text{ are all true, then } P(n+1) \text{ is true.} \tag{2}$$

Then we may conclude that for each natural number n, $P(n)$ is true.

Proof by Complete Induction (Template)

To prove $(\forall x \in \mathbb{N})P(n)$ using complete induction:

- Declaration: *Let* $P(n)$ *be the sentence* ...

- BASE CASE: $P(1)$ *is true because* ...

- INDUCTIVE STEP: *Let* $n \in \mathbb{N}$ *such that* $P(1), \ldots, P(n)$ *are all true.*

- Conclusion: *Therefore, by complete induction, for each* $n \in \mathbb{N}$, $P(n)$ *is true.*

Example: Division by a Prime

Theorem 1 (Euclid's Lemma)

Let p be a prime number. Then for all integers x and y, if p divides xy, then p divides x or p divides y.

- We have been using this result without proof for a while; see Remark 4.50.
- It can now be proved using complete induction.

Proof of Euclid's Lemma