Section 6.

Insight vs. Induction

Sums of Powers Revisited

Verification vs. Discovery

Denote by $S_r(n)$ the sum $1^r + 2^r + \cdots + n^r$, *e.g.*,

$S_0(n) = 1^0 + 2^0 + \cdots + n^0$

$= \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n$

$$
\begin{aligned}
S_1(n) &= 1 \;\; + 2 \;\; + 3 \;\; + \cdots + n \;\; , \\
S_2(n) &= 1^2 + 2^2 + 3^2 + \cdots + n^2, \\
S_3(n) &= 1^3 + 2^3 + 3^3 + \cdots + n^3, \\
&\;\;\vdots
\end{aligned}
$$

In Section 5, we were given formulas for some of the sums and they were verified by induction. But how would one discover such formulas in the first place?

Derivation of $S_1(n)$ Formula

$S_1(n) = 1^1 + 2^1 + \cdots + n^1 = 1 + 2 + \cdots + n$

Observe that

$$
\begin{array}{rrcccccccccc}
 & S_1(n) & = & 1 & + & 2 & + & 3 & + & \cdots & + & n \\
+) & S_1(n) & = & n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \\
\hline
 & 2S_1(n) & = & (n+1) & + & (n+1) & + & (n+1) & + & \cdots & + & (n+1)
\end{array}
$$

$$= \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1)}_{n \text{ copies}} = n(n+1).$$

Therefore, $S_1(n) = \dfrac{n(n+1)}{2}.$

Would it work with $S_2(n)$?

$$
\begin{array}{rrl}
 & S_2(n) = & 1^2 + 2^2 + \cdots + n^2 \\
+) & S_2(n) = & n^2 + (n-1)^2 + \cdots + 1^2 \\
\hline
 & 2S_2(n) = & (1^2+n^2) + [\underbrace{2^2+(n-1)^2}_{\neq 1^2+n^2}] + \cdots + (n^2+1^2)
\end{array}
$$

The previous trick doesn't work!

Another Derivation of $S_1(n)$ Formula (Telescoping Sums)

Let $T(n) = \sum_{k=1}^{n}[k^2 - (k-1)^2]$.

On the one hand,

$$T(n) = [\cancel{1^2} - 0^2] + [\cancel{2^2} - \cancel{1^2}] + [\cancel{3^2} - \cancel{2^2}] + \cdots + [n^2 - \cancel{(n-1)^2}] = n^2$$

On the other hand, since $k^2 - (k-1)^2 = k^2 - (k^2 - 2k + 1) = 2k - 1$,

$$T(n) = \sum_{k=1}^{n}(2k-1) = 2\underbrace{\sum_{k=1}^{n} k}_{= S_1(n)} - \underbrace{\sum_{k=1}^{n} 1}_{= S_0(n) = n}$$

Therefore,

$$T(n) = 2S_1(n) - n \implies S_1(n) = \frac{n^2+n}{2} = \frac{n(n+1)}{2}.$$

$$\implies n^2 = 2S_1(n) - n$$

$$T(n) = \sum_{k=1}^{n} \left[k^2 - (k-1)^2 \right]$$

$$= \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} (k-1)^2$$

$$= \left(\cancel{1^2} + \cancel{2^2} + \cancel{3^2} + \cdots + \cancel{(n-1)^2} + n^2 \right)$$
$$- \left(0^2 + \cancel{1^2} + \cancel{2^2} + \cancel{3^2} + \cdots + \cancel{(n-1)^2} \right)$$

$$= n^2 - 0^2 = n^2$$

Derivation of $S_2(n)$ Formula via Telescoping Sum

S06E01

Let $n \in \mathbb{N}$. Let $T(n) = \sum_{k=1}^{n}[k^3 - (k-1)^3]$. By writing $T(n)$ out in long form, show that it is a telescoping sum and that $T(n) = n^3$. Then, by evaluating $T(n)$ in a different way, deduce without explicit induction that

$$S_2(n) = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

To handle $S_3(n)$, use $T(n) = \sum_{k=1}^{n}\left[k^4 - (k-1)^4\right]$

$\vdots$

Geometric Sums Revisited

Sum of Geometric Progression

Let $x \in \mathbb{R}$. Suppose that $x \neq 1$. Let $n \in \mathbb{N}$. In S05E04, you were given the geometric series formula

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x},$$

and were asked to verify it using induction. Let's derive it here.

Derivation of Geometric Sum Formula

Let

$$S = 1 + x + x^2 + \cdots + x^{n-1}.$$

(x is the common ratio.)

Observe that

$$\begin{array}{rrcccccccccccc}
 & S & = & 1 & + & x & + & x^2 & + & \cdots & + & x^{n-1} & & \\
-) & xS & = & & & x & + & x^2 & + & \cdots & + & x^{n-1} & + & x^n \\
\hline
 & (1-x)S & = & 1 & + & 0 & + & 0 & + & \cdots & + & 0 & \overset{-}{\cancel{+}} & x^n
\end{array}$$

Therefore, Since $x \neq 1$, $1 - x \neq 0$, so

$$S = \frac{1 - x^n}{1 - x}$$

Another Derivation Using Summation Notation

In summation notation,

$$S = \sum_{k=0}^{n-1} x^k = 1 + \sum_{k=1}^{n-1} x^k,$$

while

$$xS = \sum_{k=0}^{n-1} x^{k+1} = \sum_{k=1}^{n} x^k = \sum_{k=1}^{n-1} x^k + x^n.$$

Therefore,