Rational and Irrational Numbers

Contents

Rational Numbers

$(\exists m \in \mathbb{Z})(\exists n \in \mathbb{Z})(n \neq 0 \wedge x = m/n)$

Definition 1 (Rational Numbers)

To say that x *is a rational number* means that there exist integers m and n such that $n \neq 0$ and $x = m/n$.

- Any integer is a rational number.

  $9 = 9/1$

- $1/2,\ 3/4,\ -5/17,\ \ldots$

* Written in lowest terms (num. & den. has no common factor other than 1)

$\boxed{1/2} = 2/4 = 5/10 = 50/100 = -1/-2 = \cdots$

Example 2 (Sum of Rational Numbers)

Let u and v be rational numbers. Then $u+v$ is a rational number.

Proof Since u is rational, we can find integers a and b such that

$b \neq 0$ and $u = a/b$.

Since v is rational, we can find integers c and d such that

$d \neq 0$ and $v = c/d$.

Then

$$u+v = \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}.$$

Since $ad+bc$ and bd are integers, and since $bd \neq 0$,

$u+v$ is a rational number. $\square$

Let u and v be rational numbers.

- $u + v$ is rational. $\longleftarrow$ just showed.
- $u - v$ is rational.
- uv is rational.
- If $v \neq 0$, then u/v is rational.

HW.

To show $u - v$ is rational:

- Know: $u + v$ is rational
- If you can show: $-v$ is rational.

$$\longrightarrow \quad u - v = u + (-v)$$

Special Forms of Rational Numbers

A given rational number x can be expressed in many different ways. For example,

$$\boxed{\frac{7}{3}} = \frac{-7}{-3} = \frac{14}{6} = \frac{-14}{-6} = \cdots = \frac{350}{150} = \cdots.$$

$\boxed{\text{Fact}}$

will be used later.

The fact that each rational number can be written in lowest terms such as $7/3$ can be proved later once we learn complete induction. For now, we can prove the following:

Rational Number as An Integer Divided by A Natural Number

Let x be a rational number. Then there exists an integer a and a natural number b such that $x = a/b$.

e.g.

$\frac{7}{3}$ a : integer, b : natural number

$-\frac{7}{3} = \frac{-7}{3} = \frac{7}{-3}$ a : integer. b : natural number

Irrational Numbers

Definition

Definition 3 (Irrational Numbers)

To say that $\boxed{x \text{ is an irrational number}}$ means that x is a real number and x is not a rational number.

Note

Remember that each irrational number is a real number!

$$\text{“}x \text{ is an irrational number.”} \neq \text{“}x \text{ is not a rational number.”}$$

Consider the following question.

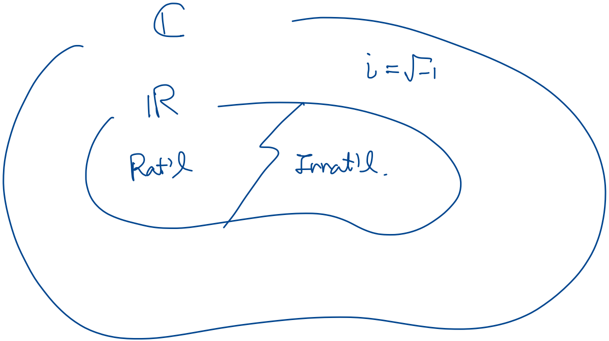
Question. Determine whether each of the following is true or false. Explain your answers.

1. For each $x \in \mathbb{C}$, if x is an irrational number, then x is not a rational number. (T)

2. For each $x \in \mathbb{C}$, if x is not a rational number, then x is an irrational number. (F)

Note: This is the converse of the above.

Counterexample: $x = \sqrt{-1} = i$ not real.
(x is not rational & x is not irrational.)


$\mathbb{C}$
$i = \sqrt{-1}$
$\mathbb{R}$
Rat'l
Irrat'l.

Remember

- ✓ real
- not rational

Example 4 (Sum of Rational and Irrational Numbers)

Let x be a rational number and let y be an irrational number. Then $x + y$ is an irrational number.

Proof: Since x and y are real, $x+y$ is a real number.

So it remains to show $x+y$ is not rational.

Suppose $x+y$ is rational.

Then

$$(x+y) - x$$

is rational as a difference of two rat'l numbers.

But

$$(x+y) - x = y.$$

So y is a rational number. But y was assumed to be irrational, so y is not rational. This is a contradiction.

Hence the assumption that $x+y$ is rational is wrong.

Therefore $x+y$ is irrational. $\square$

Let x be a rational number and let y be an irrational number.

- $x+y$ is irrational. ← Shown
- $x-y$ is irrational.
- $y-x$ is irrational.
- If $x \neq 0$, then xy is irrational.
- If $x \neq 0$, then x/y is irrational.
- If $x \neq 0$, then y/x is irrational.

HW.

Examples (cont') DIY.

Question. Let x and y be real numbers. Determine whether each of the following is true or false. Explain.

1. If xy is rational, then x and y are rational.

2. If $x + y$ is rational, then x and y are rational.

Irrationality of $\sqrt{2}$

Theorem 5

1. *Let x be a rational number. Then $x^2 \neq 2$.*
2. *$\sqrt{2}$ is irrational.*

Proof of ①

Since x is rational, we can pick integers a and b such that $b \neq 0$ and $x = a/b$.

Using Fact (✱), assume that $x = a/b$ is in lowest terms.

WTS $x^2 \neq 2$.

towards a contradiction

Suppose $x^2 = 2$.

Then $x^2 = (a/b)^2 = 2$,

so $a^2/b^2 = 2$, so $a^2 = 2b^2$.

$a^2 = 2b^2$

(Cont'd) But then a^2 is an even number. Hence a must be an even number

So we can find an integer k such that $a = 2k$.

Then

$$a^2 = (2k)^2 = 4k^2$$

but then

$$4k^2 = 2b^2.$$

So

$$b^2 = 2k^2.$$

Thus b^2 is an even number, and so b is an even number.

This is a contradiction to the fact that $x = a/b$ is in lowest terms.

Therefore $x^2 \neq 2$.