Binomial Coefficients

Contents

Warm-Up

Remark 4.50

Let $d \in \mathbb{N}$, $x, y \in \mathbb{Z}$, and p be a prime number.

1. If $p \mid xy$, then $p \mid x$ or $p \mid y$.
2. If $d \mid xy$, then these exist $d_1, d_2 \in \mathbb{N}$ such that $d_1 \mid x$, $d_2 \mid y$, and $d = d_1 d_2$.

The proofs of these facts require *complete induction*.

Example 4.52

1. Let x be a rational number such that $x^2 = c$, where c is a whole number. Then x is an integer.
2. Let c be a whole number which is not a perfect square. Then $\sqrt{c}$ is irrational.

Binomial Coefficients

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
 & & & & & \vdots & & & & &
\end{array}
$$

Pascal's Triangle and Binomial Coefficients

Recall. For all $n \in \omega$ and all $k \in \{0, \ldots, n\}$, the binomial coefficient $\binom{n}{k}$ denotes the k-th number on the n-th row of Pascal's triangle.

Key features

1. $\binom{0}{0} = 1.$
2. *Boundary conditions*: For each $n \in \mathbb{N}$,
$$\binom{n}{0} = \binom{n}{n} = 1.$$
3. *Recurrence relation*: For each $n \in \omega$ and all $k \in \{1, \ldots, n\}$,
$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Why *n choose k*?

All 2-element subsets of the 4-element set $\{1, 2, 3, 4\}$ are

$$\underbrace{\{1,2\},\{1,3\},\{2,3\}}_{\text{ones without } 4},\ \underbrace{\{1,4\},\{2,4\},\{3,4\}}_{\text{ones with } 4}.$$

Note that

- the number of subsets without 4 is
- the number of subsets with 4 is

Thus the total number of 2-element subsets of the 4-element set is

Why *n choose k*? (cont')

In general, one can count the number of k-element subsets of the $(n+1)$-element set

$$\{1, 2, \ldots, n, n+1\}$$

in an analogous fashion:

- the number of subsets without $n+1$ is
- the number of subsets with $n+1$ is

Thus the total number of k-element subsets of the $(n+1)$-element set is

Why *n choose k*? (cont')

The idea above is key to a proof by induction of the following theorem.

Number of Subsets (cf. S14E03)

For each $n \in \boldsymbol{\omega}$, for each $k \in \{0, \ldots, n\}$, the number of k-element subsets of an n-element set is $\binom{n}{k}$.

Why Binomial Coefficients?

Consider the expansion of $(a+b)^2$:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= (a+b)a + (a+b)b\\ &= a^2 + ba\\ &\qquad + ab + b^2\\ &= a^2 + 2ab + b^2.\end{aligned}$$

Note that the coefficients of a^2, ab, and b^2 are

$$1, 2, 1,$$

respectively, which are precisely the numbers on row 2 of Pascal's triangle:

$$\binom{2}{0}, \binom{2}{1}, \binom{2}{2}.$$

Binomial Expansion

Expansion of $(a+b)^3$

Work out the expansion of $(a+b)^3$ and compare the coefficients with the numbers in row 3 of Pascal's triangle.

Binomial Expansion (cont')

Expansion of $(a+b)^4$

Work out the expansion of $(a+b)^3$ and compare the coefficients with the numbers in row 3 of Pascal's triangle.

The Binomial Theorem

The examples above suggest the following general result.

The Binomial Theorem (S05E11)

For each $n \in \omega$ and all $a, b \in \mathbb{R}$,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$
$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

- **Convention**: For each $x \in \mathbb{R}$, $x^0 = 1$.