Section 10.

Introduction to Set Theory

Basics of Set Theory

Sets

A *set* is a collection of objects, considered as an object in its own right.

Notation.

- $\boxed{x \in A}$: x is one of the objects in the set A.
  "x is an element of A", "x belong to A", "x is a member of A", or "x is in A."
- $\boxed{x \notin A:}$ x is not in the set A.

LaTex $ x \in A $

$ x \notin A $

Ways to Denote Sets

We denote a set by

- listing its elements between braces, *e.g.*,

$$\{2, 3, 5, 7, 11\}$$

- using the *set-builder notation*, *e.g.*,

$$\{\boxed{x} : \boxed{x \text{ is prime and } x \leqslant 11}\} = \{2, 3, 5, 7, 11\} \cancel{13}, \cancel{17}, \ldots$$

"form" "such that" "membership criteria"

More on Set-Builder Notation

In set-builder notation, we describe a set in terms of *membership criteria*.

- $\{x : P(x)\}$: "the set of all x such that $P(x)$"

$$\{x : x \text{ is a natural number and } x \text{ is even}\} = \{2, 4, 6, 8, \dots\}$$

- $\{x \in A : P(x)\} = \{x : x \in A \text{ and } P(x)\}$: "the set of all x in A such that $P(x)$"

$$\{x \in \mathbb{N} : x \text{ is even}\} = \{2, 4, 6, 8, \dots\}$$

- $\{f(x) : P(x)\} = \{y : y = f(x) \text{ for some } x \text{ such that } P(x)\}$: "the set of all $f(x)$ such that $P(x)$"

$$\{2x : x \in \mathbb{N}\} = \{2, 4, 6, 8, \dots\}$$

Notes

$(\forall A)(\forall B)$ same as $(\forall A, B)$

1. Sets having the same elements are equal, *i.e.*,

   hidden $\to$ (For all A, B,) If $\underbrace{\text{for each } x,\ x \in A \text{ iff } x \in B}_{\text{"having the same elements"}}$, then $A = B$.

   Consequently,

   - The order in which the elements of a set are listed is unimportant. $\{1, 3\} = \{3, 1\}$
   - Repetitions in the description of a set do not count. $\{1, 2, 3, 3, 5\} = \{1, 2, 3, 5\}$.

2. Equal sets have the same elements, *i.e.*,

   For all sets A and B, if $A = B$, then for each x, $x \in A$ iff $x \in B$.

3. Equal objects are elements of the same sets, *i.e.*,

   For all x and y, if $x = y$, then for each set A, $x \in A$ iff $y \in A$.

① $(\forall A, B)[(\forall x)(x \in A \Leftrightarrow x \in B) \Rightarrow A = B]$

② $(\forall A, B)[A = B \Rightarrow (\forall x)(x \in A \Leftrightarrow x \in B)]$

①② $(\forall A, B)[(\forall x)(x \in A \Leftrightarrow x \in B) \Leftrightarrow A = B]$

③ $(\forall x, y)[x = y \Rightarrow (\forall A)(x \in A \Leftrightarrow y \in A)]$

Example

S10E01

Which of the sets A, B, C, D, and E below are the same?

$$A = \{3\}, \quad B = \{2,4\}, \quad C = \{x : x \text{ is prime, } x \text{ is odd, and } x < 5\}, = \{3\}$$

$$D = \{x - 1 : \underbrace{x \text{ is prime, } x \text{ is odd, and } x \leqslant}_{3,\,5} 5\}, \quad E = \{x^2 + 2 : x \in \{-1, 1\}\}.$$

$= \{2,4\}$ $\qquad = \{3\}$

How many different sets are named here?

$A = C = E = \{3\}$

$B = D = \{2,4\}$

The Number of Elements

Question

How many elements does $\{a, b\}$ have?

If $a = b$, then $\{a, b\} = \{a, a\} = \{a\}$, so there is one element.

If $a \neq b$, then $\{a, b\}$ has two (distinct) elements.

The Number of Elements (cont')

S10E02

How many elements does $\{a, b, c\}$ have?

Read the example/remark right above this exercise.

Sets as Elements of other Sets

Since sets are objects as well, they can be elements of other sets.

Example. Study the elements of each of the following sets.

- $\{1, 2, \{3, 4\}\}$ 3 elements, one of which is a set. Q. $3 \in \{1, 2, \{3, 4\}\}$? A. No. $3 \notin \{1, 2, \{3, 4\}\}$
- $\{\{1, 2, 3, \ldots\}\}$ One element which is an infinite set.
- $\{\{1\}, \{2\}, \{3\}, \ldots\}$ infinite set each of whose elements is a singleton.
- $\{\{1, 2, 3, \ldots\}, \{2, 4, 6, \ldots\}, \{3, 6, 9, \ldots\}, \ldots\}$ infinite set each of whose elements is an infinite set.

The Empty Set

The *empty set* is the set that has no elements, usually denoted by $\varnothing$.

- $\{x : P(x)\} = \varnothing$ if there are no values of x for which $P(x)$ is true. For example,

$$\{x : x \text{ is even and } x \text{ is odd}\} = \varnothing.$$

$\{x : x$ is prime, x is even, and $x \geq 3\} = \phi$

- The empty set is unique.

Proof. Suppose $\varnothing'$ is another set with no elements. Then for each x, $x \in \varnothing$ and $x \in \varnothing'$ are both false, so $x \in \varnothing \Leftrightarrow x \in \varnothing'$. Hence $\varnothing = \varnothing'$. □

- Tip: To prove that a set A is the empty set, show that for each x, $x \notin A$.

Homework Coaching

Rational Roots (Cubic Polynomials)

S04E17 (Collected)

1. Let x be a rational number such that $x^3 = c$, where c is an integer. Prove that x is an integer.
2. Let c be an integer which is not a perfect cube. Prove that $\sqrt[3]{c}$ is irrational.

S04E18 (Collected)

Let $x \in \mathbb{R}$ such that $x^3 = rx^2 + sx + t$, where $r, s, t \in \mathbb{Z}$.

1. Prove that if x is rational, then x is an integer.
2. Prove that if x is not an integer, then x is irrational.

Rational Roots (General Polynomials)

S04E19 (To be turned in)

Let x be a real number such that

$$x^n + c_n x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where $n \in \mathbb{N}$ and $c_0, c_1, \ldots, c_{n-1} \in \mathbb{Z}$.

1. Prove that if x is rational, then x is an integer.
2. Prove that if x is not an integer, then x is irrational.

Binomial Theorem and Applications

S05E11 (The Binomial Theorem; collected)

Let $a, b \in \mathbb{R}$. Then for each $n \in \boldsymbol{\omega}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Convention. $x^0 = 1$ for each $x \in \mathbb{R}$.

S05E12 (Collected)

Let $n \in \boldsymbol{\omega}$. Show that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

$a = b = 1$

(Do not use induction.)

Application of Binomial Theorem

S05E13 (To be turned in)

Let $n \in \mathbb{N}$. Show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

(Do not use induction.)