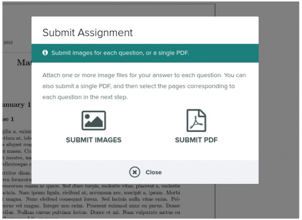
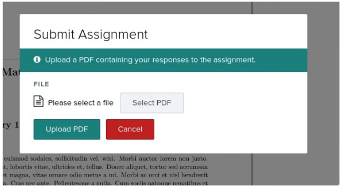
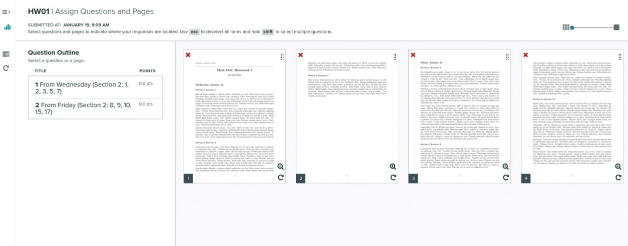
Proof Techniques

- HW 1 due 11:59 PM.
- Office Hour : 4:30 pm – 6:00 pm (Zoom)
- Quiz 2 on Friday (noon – 11:59 pm)

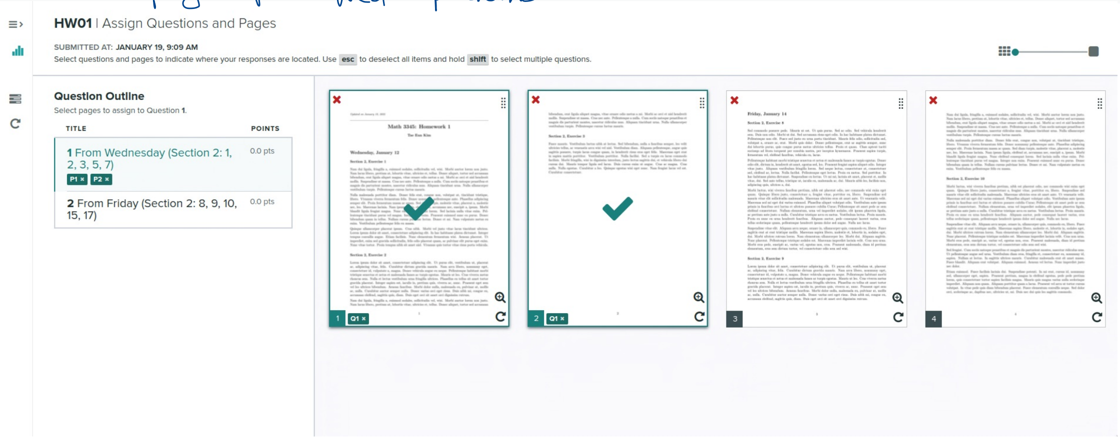
Uploading HW to Gradescope



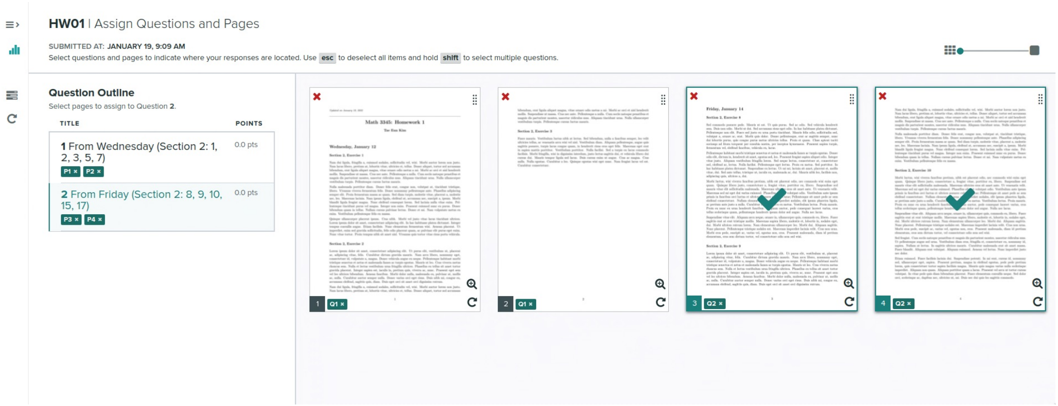




Select pages for Wed. problems.



Select pages for Fri. problems.



Contents

Logic of Solving Equations

Solving Equations

Logically speaking, to say that $x = a$ is a solution of the equation $f(x) = 0$ is to state

$$f(x) = 0 \iff x = a$$

which usually can be seen by a chain of biconditionals.

For example, we see that $x^2 = 5x - 6$ if and only if $x = 2$ or $x = 3$ by:

$$\begin{aligned} x^2 = 5x - 6 &\iff x^2 - 5x + 6 = 0 \\ &\iff (x-2)(x-3) = 0 \\ &\iff x - 2 = 0 \text{ or } x - 3 = 0 \\ &\iff x = 2 \text{ or } \ x = 3. \end{aligned}$$

One needs to be careful to confirm that all steps are true biconditional sentences.

Examples

Rational Equation

Solve the equation

$$\frac{x-2}{x^2+2x-8} = \frac{1}{8}.$$

Erroneous solution.

$$\Longrightarrow x - 2 = (1/8)(x^2 + 2x - 8)$$

$$\Longleftrightarrow 8x - 16 = x^2 + 2x - 8$$

$$\Longleftrightarrow 0 = x^2 - 6x + 8 = \underbrace{(x-2)}_{=0}\underbrace{(x-4)}_{=0}$$

$$\Longleftrightarrow x = 2, 4$$

When $x=2$,

$$x^2+2x-8 = 2^2+2\cdot 2-8$$

$$= 4+4-8 = 0.$$

✓ Which step is not a true biconditional sentence?

intermediate steps omitted

Correct solution.

- $\dfrac{x-2}{x^2+2x-8} = \dfrac{1}{8} \implies x = 2$ or $\underline{x=4}$

- Now if $x=2$, then $x^2+2x-8 = 4+4-8 = 0$,

  so $\dfrac{x-2}{x^2+2x-8}$ is undefined. So $x=2$ is not a solution.

- If $x=4$, then $x^2+2x-8 = 16+8-8 = 16$ and $x-2 = 4-2 = 2$,

  so $\dfrac{x-2}{x^2+2x-8} = \dfrac{2}{16} = \dfrac{1}{8}$. So $x=4$ is a solution.

Examples

Equation Involving Radicals

Solve the equation
$$x = -\sqrt{x+6}$$

An erroneous solution:

$$\Rightarrow x^2 = x + 6 \qquad \text{squaring both sides}$$
$$\Leftrightarrow x^2 - x - 6 = 0$$
$$\Leftrightarrow (x+2)(x-3) = 0$$
$$\Leftrightarrow x = -2, 3$$

$$x^2 = 4$$
$$\Leftrightarrow x = \pm\sqrt{4} = \pm 2$$

Is $x = 3$ a solution of the original equation?

No, because $-\sqrt{3+6} = -\sqrt{9} = -3 \neq 3$.

Correct solution.

If $x = -\sqrt{x+6}$, then $x = -2$ or $x = 3$.

Now if $x = -2$, then $x = -\sqrt{x+6}$.

If $x = 3$, then $x \neq -\sqrt{x+6}$.

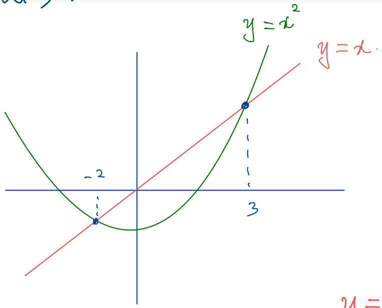
Therefore, $x = -\sqrt{x+6}$ iff $x = -2$. ($x = -2$ is the soln of eqn.)

Practical approach (for insight).

One view

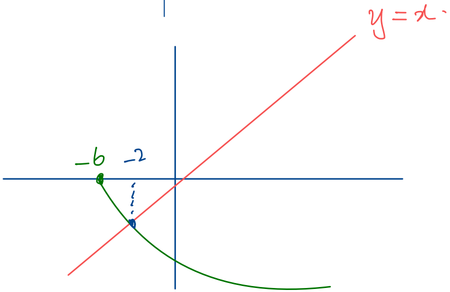
$$x^2 = x + 6$$

$$x^2 - 6 = x$$



Orig.

$$x = -\sqrt{x+6}$$



Proof by Contradiction

Overview: Proof Techniques

The follow is the list of proof techniques discussed in Section 2 of the textbook:

- ☑ Conditional proof   Last Fri.
- ☐ Proof by contradiction
- ☐ Proof by contraposition

- used to prove $P \Rightarrow Q$.
- Template

  A1: Assume P is true.

  Work to show Q is true.

  Discharging A1, $P \Rightarrow Q$ is true under no assumption.

P	Q	$P \Rightarrow Q$
T	T	T
~~T~~	~~F~~	~~F~~
F	T	T
F	F	T

Contradictions

A *contradiction* is a sentence of the form $Q \land \neg Q$, which is false regardless of the truth value of Q.

Why?

Q	$\neg Q$	$Q \land \neg Q$
T	F	F
F	T	F

Uses in proofs.

Proof by Contradiction

Proof by Contradiction

To prove a sentence P, assume $\neg P$ and deduce a contradiction. This approach is known as the method of *proof by contradiction*.

Template. To prove P:

- Begin with "Assume $\neg P$ is true."
- Deduce a contradiction.
- Conclude that P is true.

Why does it work?

a contradiction

P	$\neg P$	$\wedge$	$\neg P \Rightarrow \wedge$
T	F	F	T
F	T	F	F

Proof by Contradiction (cont')

Example

Let n be an integer. Using the method of proof by contradiction, prove that

If n^2 is an odd number, then n is an odd number.

A1: Assume n^2 is an odd number.

We wish to show n is an odd number.

Assume (towards a contradiction) n is not an odd number.

Since n is an integer, n must be an even number.

So n^2 is an even number, that is, n^2 is not an odd number.

This leads to a contradiction, so we must reject the assumption that n is not an odd number. So n is an odd number.

Discharging A1, we see that if n^2 is an odd number, then n is an odd number under no assumption. $\square$

Proof of a Negative Sentence

The usual way to prove a negative sentence $\neg P$ to prove by contradiction, that is, assume P and deduce a contradiction.

Why does it work?

Proof by contradiction on $\neg P$.

Section 2, Exercise 23

Use the method of conditional proof to explain in words why

$$\underbrace{[(P \Rightarrow Q) \wedge \neg Q]}_{A_1} \Rightarrow \underbrace{\neg P}_{C_1}$$

is a tautology.

Suggestion: Conditional proof.

A1: Assume A_1 is true.

NTS C_1 is true.

⋮

} proof of a negative sentence.

Proof of a Negative Sentence (cont')

Proof by Contraposition

Overview: Proof Techniques

The follow is the list of proof techniques discussed in Section 2 of the textbook:

- ☑ Conditional proof
- ☑ Proof by contradiction
- ☐ Proof by contraposition

To prove $P \Rightarrow Q$,
it suffices to assume P is true
and show Q is true.

To prove P,
assume $\neg P$ is true
and deduce a contradiction $(Q \wedge \neg Q)$.

Contrapositive

Given $P \Rightarrow Q$, the related conditional sentence $\neg Q \Rightarrow \neg P$ is called *the contrapositive of* $P \Rightarrow Q$. Note that $P \Rightarrow Q$ is logically equivalent to $\neg Q \Rightarrow \neg P$. (Confirm this using a truth table.)

Example. Given the conditional sentence

A: If $\underbrace{\text{today is Sunday}}_{P}$, then $\underbrace{\text{I do not have to go to work today}}_{Q}$.

- Converse of A: $Q \Rightarrow P$

  If I do not have to go to work today, then today is Sunday.

- Contrapositive of A: $\neg Q \Rightarrow \neg P$

  If I have to go to work today, then today is not Sunday.

No work

Sunday

Proof by Contraposition

Proof by Contraposition

To prove $P \Rightarrow Q$, it suffices to prove $\neg Q \Rightarrow \neg P$.

Confirm using a truth table that

$$(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P).$$

Proof by Contraposition (cont')

Example (revisited)

Let n be an integer. Using the method of proof by contraposition, prove that

If n^2 is an odd number, then n is an odd number.

Solution. The given sentence is logically equivalent to the sentence

$$\text{If } \underbrace{n \text{ is not an odd number}}_{A}, \text{ then } \underbrace{n^2 \text{ is not an odd number}}_{C}. \qquad (\star)$$

Contrapositive of the given.

which we will prove.

A1: Assume that n is not an odd number.
(We wish to show that n^2 is not an odd number.)
Since n is an integer but n is not an odd number, n is an even number.
Hence n^2 is an even number, so n^2 is not an odd number.
We have shown this under A1.

Discharging A1, we conclude that the conditional sentence $(\star)$ is true under no assumptions. This completes the proof by contraposition that the original conditional sentence is true. □

Proof by Contradiction vs Proof by Contraposition

Let's examine the two proof techniques in proving $P \Rightarrow Q$.

Proof by contradiction.

Assume P. (NTS Q is true)
Assume $\neg Q$.
Show $\neg P$.
Contradiction, $\underline{P \wedge \neg P}$!
So Q must be true.
Therefore, $P \Rightarrow Q$.

Proof by contraposition. (show $\neg Q \Rightarrow \neg P$)

Assume $\neg Q$.
Show $\neg P$. (if this can be done w/o P.)
So $\neg Q \Rightarrow \neg P$.
Therefore, $P \Rightarrow Q$, by contraposition.

Homework (1/19 ; due Wed 1/26)

Section 2 : # 19, 20, 24