Lec. 33. Problem Solving Session

04/08/2022

S11E09 Let $a, b \in \mathbb{R}$ such that $a < b$.

$L : C^1[a,b] \to \mathbb{R}$

$$L(f) = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \left(\begin{array}{l}\text{the length of} \\ \text{the curve } y = f(x) \\ \text{over } [a,b]\end{array}\right)$$

continuously differentiable on $[a,b]$

① Show $L(f) \geq b-a$ for all $f \in C^1[a,b]$.

Pf. Let $f \in C^1[a,b]$. Then for any $x \in [a,b]$, $f'(x) \in \mathbb{R}$

and so $[f'(x)]^2 \geq 0$, so $1 + [f'(x)]^2 \geq 1$, so $\sqrt{1+[f'(x)]^2} \geq 1$.

It follows that

$$L(f) = \int_a^b \underbrace{\sqrt{1+[f'(x)]^2}}_{\geq 1}\, dx \geq \int_a^b 1 \, dx = b-a.$$

$\square$

$\mathrm{Rng}(L) = [b-a, \infty)$? To determine, need more work.

② Let $m \in [0, \infty)$ and let $f(x) = m(x-a)$ for all $x \in [a,b]$. Compute $L(f)$.

<u>Soln</u> Since $f'(x) = m$,

$$L(f) = \int_a^b \sqrt{1+m^2}\, dx = \sqrt{1+m^2} \int_a^b 1\, dx = \boxed{\sqrt{1+m^2}\,(b-a)}.$$

③ $\text{Rng}(L) = [b-a, \infty)$.

<u>pf</u>. Will show $\text{Rng}(L) \subseteq [b-a, \infty)$ and $[b-a, \infty) \subseteq \text{Rng}(L)$.

The first set inclusion was shown in ①, so it remains to show $[b-a, \infty) \subseteq \text{Rng}(L)$. Let $y \in [b-a, \infty)$, i.e., $y \geq b-a$.

NTS: $y \in \text{Rng}(L)$, which means we need to find $f \in C^1[a,b]$ s.t. $L(f) = y$.

We propose that $f(x) = \sqrt{\left(\frac{y}{b-a}\right)^2 - 1}\,(x-a)$ will do. First, $f \in C^1[a,b]$ because it is linear. Furthermore, we note that

$$L(f) = \int_a^b \sqrt{\cancel{1} + \left(\frac{y}{b-a}\right)^2 \cancel{-1}}\, dx$$

$$= \int_a^b \frac{y}{b-a}\, dx = \frac{y}{b-a}\int_a^b 1\, dx$$

$$= \frac{y}{b-a}(b-a) = y.$$

$\square$

<u>Side work</u> $m \in [0, \infty)$

$$y = \sqrt{1+m^2}\,(b-a)$$

Solve for m.

$$1 + m^2 = \left(\frac{y}{b-a}\right)^2$$

$$m = \sqrt{\left(\frac{y}{b-a}\right)^2 - 1}$$

S11E12&14 (Tips on how to prove / disprove surjection / injection.)

Let A and B be sets and let $f: A \to B$.

① To prove that f is a surjection from A to B:

$$(\forall y \in B)(\exists x \in A)(f(x) = y).$$

So to disprove that f is a surjection from A to B:

$$(\exists y \in B)(\forall x \in A)(f(x) \neq y)$$

$$\neg(P \Rightarrow Q) \equiv P \land \neg Q$$

② To prove that f is an injection:

$$(\forall x_1, x_2 \in A)[f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$$

or $(\forall x_1, x_2 \in A)[x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)]$

To disprove that f is an injection:

$$(\exists x_1, x_2 \in A)[f(x_1) = f(x_2) \land x_1 \neq x_2]$$

or $(\exists x_1, x_2 \in A)[x_1 \neq x_2 \land f(x_1) = f(x_2)]$

S11 E15 Let S and T be sets. Define

$$f : \mathcal{P}(S) \times \mathcal{P}(T) \longrightarrow \mathcal{P}(S \cup T)$$

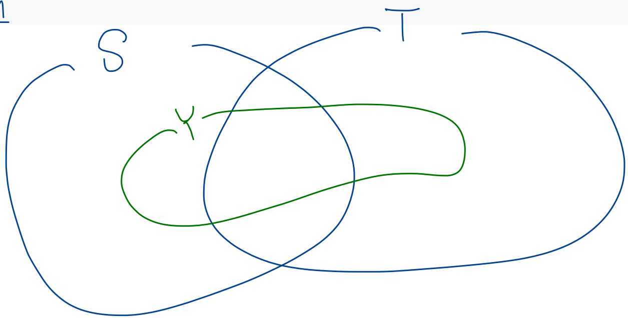
by $f(A, B) = A \cup B$ for all $A \subseteq S$ and all $B \subseteq T$.

(a) Show f is a surjection.

Pf WTS: for any $Y \in \mathcal{P}(S \cup T)$, there exists $(A, B) \in \mathcal{P}(S) \times \mathcal{P}(T)$ such that $f(A, B) = A \cup B = Y$.

Let $Y \in \mathcal{P}(S \cup T)$, i.e., $Y \subseteq S \cup T$.

Suggestion



Consider:

$$A = S \cap Y \quad \text{and} \quad B = T \cap Y$$

Show:

$$A \subseteq S \quad \text{and} \quad B \subseteq T \quad \text{and} \quad Y = A \cup B.$$