Fundamental Principle of Counting

Equinumerousness

Equinumerousness

Definition 1

Let A and B be sets. To say that A *is equinumerous to* B (denoted $A \approx B$) means that there exists a bijection from A to B.

Examples.

- The sets $\{1, 2\}$ and $\{\pi, e\}$ are equinumerous because the function f on $\{1, 2\}$ defined by $f(1) = \pi$ and $f(2) = e$ is a bijection from $\{1, 2\}$ to $\{\pi, e\}$.
- The set $\mathbb{Z}$ is equinumerous to the set of all even integers $B = \{2k : k \in \mathbb{Z}\}$ because the function $g : \mathbb{Z} \to B$ defined by $g(n) = 2n$ is a bijection.

Equinumerousness (cont')

Proposition 1

Equinumerousness is reflexive, symmetric, and transitive. In other words:

1. *(Reflexivity) For each set A, we have A is equinumerous to A.*
2. *(Symmetry) For all sets A and B, if A is equinumerous to B, then B is equinumerous to A.*
3. *(Transitivity) For all sets A, B, and C, if A is equinumerous to B and B is equinumerous to C, then A is equinumerous to C.*

Number of Elements and Equinumerousness

Definition 2

Let A be a set and let $n \in \boldsymbol{\omega}$. To say that A *has* n *elements* means that A is equinumerous to $\{1, \ldots, n\}$.

- A set has 0 elements iff it is empty.
- Saying that A *is an n-element set* is synonymous to saying that A has n elements.

n-element Sets

Proposition 2

Let A be a set and let $n \in \omega$. Then the following are equivalent.

1. *A has n elements.*
2. *There exist distinct objects $a_1, \ldots, a_n$ such that $A = \{a_1, \ldots, a_n\}$.*

Proof. Suppose A has n elements. Then there is a bijection f from $\{1, \ldots, n\}$ to A. Let $a_k = f(k)$ for $k = 1, \ldots, n$. Since f is a surjection from $\{1, \ldots, n\}$ to A, we have $A = \{a_1, \ldots, a_n\}$. Since f is an injection, the objects $a_1, \ldots, a_n$ are distinct.

Conversely, suppose there exist distinct objects $a_1, \ldots, a_n$ such that $A = \{a_1, \ldots, a_n\}$. Then $A = \mathrm{Rng}(f)$ where f is a function on $\{1, \ldots, n\}$ defined by $f(k) = a_k$. Since $a_1, \ldots, a_n$ are distinct, f is injective, so f is a bijection from $\{1, \ldots, n\}$ to A, so A has n elements. $\square$

Finite and Infinite Sets

Definition 3

Let A be a set.

- To say that A *is finite* means that there exists $n \in \omega$ such that A has n elements.
- To say that A *is infinite* means that A is not finite.

Preliminary Lemmas

Comparing the Sizes of Sets

Lemma A

Let A and B be sets. Suppose that A is equinumerous to B.

1. If $s \notin A$ and $t \notin B$, then $A \cup \{s\}$ is equinumerous to $B \cup \{t\}$.
2. If $s \in A$ and $t \in B$, then $A \setminus \{s\}$ is equinumerous to $B \setminus \{t\}$.

Comparing the Sizes of Sets (cont')

Lemma B

For all $m, n \in \boldsymbol{\omega}$, if $\{1, \ldots, m\}$ is equinumerous to $\{1, \ldots, n\}$, then $m = n$.

Key Results

Uniqueness of the Number of Elements

Theorem 4

The number of elements in a finite set is uniquely determined.

- According to the theorem, for each finite set A, there is a unique $n \in \omega$ such that A has n elements; we write $\overline{\overline{A}}$ for the unique n. The notation $\overline{\overline{A}}$ is read *the number of elements in* A or *the cardinality of* A.

Subsets of a Finite Set is Finite

Theorem 5

A subset of a finite set is finite and has at most as many elements as the whole set.

A precise rephrase of the theorem is: For each $n \in \omega$,

for each set B, if B has n elements, then for each $A \subseteq B$, A is finite and $\overline{\overline{A}} \leqslant n$.

Thus one can prove this using induction.

Rigidity Property of Finite Sets

Theorem 6

A finite set cannot be equinumerous to a proper subset of itself.

Proof. Let B be a finite set and let A be a proper subset of B . We wish to show that B is not equinumerous to A. Suppose B is equinumerous to A. Since B is finite, B has n elements for some $n \in \omega$. Since B is equinumerous to A, A also has n elements. Since A is a proper subset of A, $B \setminus A$ is not empty. Let $b \in B \setminus A$ and let $C = A \cup \{b\}$. Then $C \subseteq B$ and C has $n + 1$ elements. Hence by Theorem 13.30, $n + 1 \leqslant n$. But $n + 1 > n$. Thus we have reached a contradiction. Therefore B must not be equinumerous to A.