Review for Exam 1

Fundamentals

- Logical connectives
- Tautologies
- Proof techniques
- Quantifiers
- De Morgan's Laws and Distributive Laws

Definitions

Write down the definitions of the following sentences exactly as provided in the textbook. Write down preambles whenever needed, such as "Let $a, b, m \in \mathbb{Z}$.".

- x is even.
- x is odd.
- x is rational.
- x is irrational.
- d divides x.
- x is a prime number.
- a is congruent to b modulo m.

Tautologies

Example 1

Use the method of conditional proof to explain in words why the sentence

$$[P \Rightarrow (Q \Rightarrow R)] \Rightarrow [(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)]$$

is a tautology. Be explicit about discharging assumptions.

Dichotomies and the Universe of Discourse

Let $x \in \mathbb{Z}$.

- If x is odd, then x is not even.
- If x is not even, then x is odd.

Let $x \in \mathbb{R}$.

- If x is odd, then x is not even.
- If x is not even, then x is odd.

Let $x \in \mathbb{R}$.

- If x is rational, then x is not irrational.
- If x is not irrational, then x is rational.

Let $x \in \mathbb{C}$.

- If x is rational, then x is not irrational.
- If x is not irrational, then x is rational.

Example 2 (Cf. S04E12.)

It is known that π is an irrational number. From this, prove that $\pi + 2e$ is irrational or $\pi - 3e$ is irrational.

Classical Showcases of Proof by Contradiction

- $\sqrt{2}$ is irrational.
- There are infinitely many prime numbers.

When Prime Divides Product

Euclid's Lemma (Remark 4.50)

Let p be a prime number and let $x, y \in \mathbb{Z}$. If $p \mid xy$, then $p \mid x$ or $p \mid y$.

In general, we have:

> Let $d \in \mathbb{N}$ and let $x, y \in \mathbb{Z}$. If $d \mid xy$, then there exist $d_1, d_2 \in \mathbb{N}$ such that $d_1 \mid x$, $d_2 \mid y$, and $d = d_1 d_2$.

The converse of Remark 4.50 is also true.

Example 3 (S04E26(b))

Let m, a_1, b_1, a_2, and b_2 be integers. Suppose that

$$a_1 \equiv b_1 \mod m \quad \text{and} \quad a_2 \equiv b_2 \mod m.$$

Prove that $a_1 a_2 \equiv b_1 b_2 \mod m$.

Induction

- Principle of mathematical induction.
- Declaration, base case, induction step, and conclusion.
- Inductive hypothesis.
- Proving $(\forall x \in \mathbb{Z}) P(x)$.

Induction Examples

For each $n \in \mathbb{N}$,

- $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$
- $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
- $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

For each $n \in \mathbb{N}$,

- 3 divides $4^n - 1$.

For each $x \in \mathbb{Z}$,

- x is even or x is odd.