Algebra of Set Operations

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Recall Useful laws from propositional calculus

- Commutativity of $\vee$ and $\wedge$: $P \vee Q \equiv Q \vee P$, $P \wedge Q \equiv Q \wedge P$
- Associativity of $\vee$ and $\wedge$: $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$

  $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
- Distributive laws : $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

  $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
- De Morgan's laws : $\neg (P \vee Q) \equiv \neg P \wedge \neg Q$

  $\neg (P \wedge Q) \equiv \neg P \vee \neg Q$

Not an Element

Proposition 1

Let A and B be sets and let x be any object. Then:

1. $x \notin A \cup B$ iff $x \notin A$ and $x \notin B$.
2. $x \notin A \cap B$ iff $x \notin A$ or $x \notin B$.
3. $x \notin A \backslash B$ iff $x \notin A$ or $x \in B$.

Proof of 1: Recall that $x \in A \cup B$ means that $x \in A$ or $x \in B$. Then

$$x \notin A \cup B \quad \text{iff} \quad \neg(x \in A \text{ or } x \in B)$$

$$\text{iff} \quad x \notin A \text{ and } x \notin B \qquad \text{(by De Morgan's laws)}$$

De Morgan's Laws for Sets

Tip: Think of "$S\setminus$" as an analogue of negation.

Theorem 1 (De Morgan's Laws for Sets)

Let S, A, and B be sets. Then:

1. $S\setminus(A \cup B) = (S\setminus A) \cap (S\setminus B)$.
2. $S\setminus(A \cap B) = (S\setminus A) \cup (S\setminus B)$.

Note: Recall that $A = B$ iff $(\forall x)(x \in A \Leftrightarrow x \in B)$.

Proof of ① For each element x,

$x \in S\setminus(A \cup B)$ iff $x \in S$ and $x \notin A \cup B$

iff $x \in S$ and $(x \notin A$ and $x \notin B)$ (by Prop. 1①)

iff $(x \in S$ and $x \notin A)$ and $(x \in S$ and $x \notin B)$ (by dist. law)

iff $x \in S\setminus A$ and $x \in S\setminus B$

iff $x \in (S\setminus A) \cap (S\setminus B)$

This shows that $S\setminus(A \cup B)$ and $(S\setminus A) \cap (S\setminus B)$ have the same elements

$\square$

Distributive Laws for Unions and Intersections

Theorem 2 (Distributive Laws for Unions and Intersections)

Let S, A, and B be sets. Then:

1. $S \cap (A \cup B) = (S \cap A) \cup (S \cap B)$.
2. $S \cup (A \cap B) = (S \cup A) \cap (S \cup B)$.

<u>Proof of ①</u> For each x,

$x \in S \cap (A \cup B)$ iff $x \in S$ and $x \in A \cup B$

iff $x \in S$ and ($x \in A$ or $x \in B$)

iff ($x \in S$ and $x \in A$) or ($x \in S$ and $x \in B$) (Dist. laws for prop. calculus)

iff $x \in S \cap A$ or $x \in S \cap B$

iff $x \in (S \cap A) \cup (S \cap B)$ $\square$

<u>One way</u> to prove that two sets are the same is:

$\langle X = Y \rangle$

<u>Proof</u> For each x,

$x \in X$ iff $\ldots$

iff $\ldots$

$\vdots$

iff $x \in Y$.

Associative Laws for Unions and Intersections

Proposition 2 (Associative Laws for Unions and Intersections)

Let A, B, and C be sets. Then:

1. $(A \cup B) \cup C = A \cup (B \cup C)$
2. $(A \cap B) \cap C = A \cap (B \cap C)$

Proof of ① For each (object) x,

$$
\begin{aligned}
x \in (A \cup B) \cup C \quad &\text{iff} \quad x \in A \cup B \text{ or } x \in C \\
&\text{iff} \quad (x \in A \text{ or } x \in B) \text{ or } x \in C \\
&\text{iff} \quad x \in A \text{ or } (x \in B \text{ or } x \in C) \quad \text{(by assoc. of prop. calc.)} \\
&\text{iff} \quad x \in A \text{ or } x \in B \cup C \\
&\text{iff} \quad x \in A \cup (B \cup C) \qquad \square
\end{aligned}
$$

Commutative Laws for Unions and Intersections

Proposition 3 (Commutative Laws for Unions and Intersections)

Let A and B be sets. Then:

1. $A \cup B = B \cup A$
2. $A \cap B = B \cap A$