Section 15.

Infinite Sets

Equinumerousness

A and B have the same number of elements

- **Def'n** Let A and B be sets. To say that <u>A is equinumerous to B</u>   (A ≈ B)

   means that there exists a bijection from A to B.

- ≈ is an equivalence relation:

   ① $(\forall A)(A \approx A)$

   ② $(\forall A, B)[A \approx B \Rightarrow B \approx A]$

   ③ $(\forall A, B, C)[A \approx B \land B \approx C \Rightarrow A \approx C]$

- <u>The rigidity prop. of finite sets</u> A finite set cannot be equinumerous to any of its proper subsets.

Equinumerousness and Infinite Sets
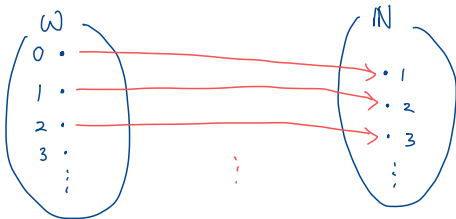
Infinite Sets Are Not Rigid

$$\omega = \{0, \underbrace{1, 2, 3, \cdots}_{= \mathbb{N}}\} = \mathbb{N} \cup \{0\}$$

By the rigidity property, a finite set cannot be equinumerous to any of its proper subsets. But this is not the case with an infinite set.

Example 1 (Natural Numbers and Whole Numbers)

$\mathbb{N}$ is a proper subset of ω, but ω is equinumerous to $\mathbb{N}$ because the function f defined by $f(n) = n + 1$ for all $n \in \omega$ is a bijection from ω to $\mathbb{N}$.

Question. Is f above the only bijection from ω to $\mathbb{N}$? If not, construction another one of your own.
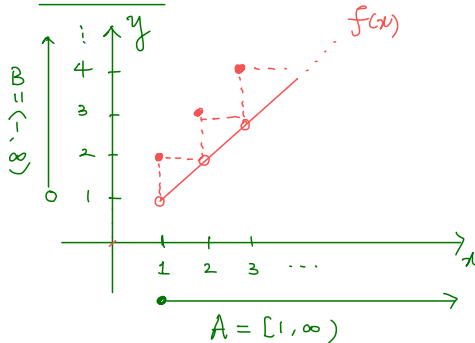
$$A = B \cup \{1\}$$

Example 2 (Intervals)

Let $A = [1, \infty)$ and $B = (1, \infty)$. B is a proper subset of A, but A is equinumerous to B. Find an example of a bijection f from A to B.

$f : A \to B$ defined by

$$f(x) = \begin{cases} x + 1 & \text{for } x = 1, 2, 3, \cdots \\ \\ x & \text{otherwise} \end{cases}$$

This is a bijection. (Confirm it.)

Illustration
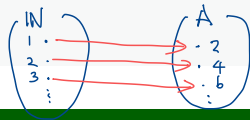
$B = (1, \infty)$

$A = [1, \infty)$

The function f defined by

$$f(x) = x+1 \qquad \text{for all} \quad x \in [1, \infty)$$

is a bijection from $[1, \infty)$ to $[2, \infty)$.

Thus $[1, \infty) \approx [2, \infty)$.

Example 3 (Even and Odd Natural Numbers)

Let $A = \{2, 4, 6, \ldots\}$ be the set of even natural numbers and let
$B = \{1, 3, 5, \ldots\}$ be the set of odd natural numbers. Verify the following by
finding suitable bijections.

1 $A \approx B$. (Note A and B are disjoint.)

2 $\mathbb{N} \approx A$. (Note A is a proper subset of $\mathbb{N}$.)

3 $\mathbb{N} \approx B$. (Note B is a proper subset of $\mathbb{N}$.)

① $f : A \to B$ defined by

$\quad f(n) = n-1$

is a bijection.

② $g : \mathbb{N} \to A$ defined by

$\quad g(n) = 2n$

Will do.

③ $h : \mathbb{N} \to B$ defined by
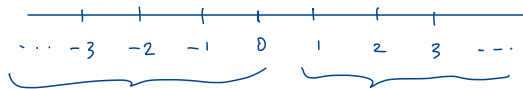
$\quad h(n) = 2n - 1$

will do.

Exercise

Show that $\mathbb{Z} \approx \mathbb{N}$.

Hint $\quad f : \mathbb{Z} \to \mathbb{N} \quad$ defined

by

$$f(n) = \begin{cases} \text{"even"} & \text{if } n = 1, 2, 3, \cdots \\ \\ \\ \text{"odd"} & \text{if } n = 0, -1, -2, \cdots \end{cases}$$
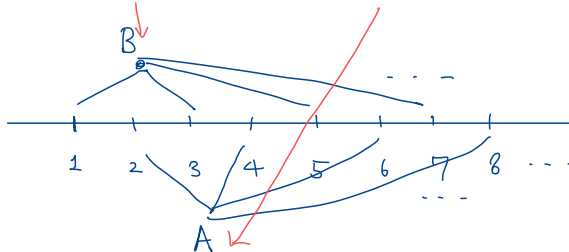
Scheme

$\mathbb{Z}$

$\cdots \ -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad \cdots$

"ω" $\mathbb{N}$

$\mathbb{N}$

B

$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad \cdots$

A

More Counter-Intuitive Examples

Example 4 (Perfect Squares)

Let $S = \{n^2 : n \in \mathbb{N}\}$ be the set of all perfect squares. Then $S \approx \mathbb{N}$.

$f : \mathbb{N} \to S$ defined by

$\qquad f(n) = n^2$

is a bijection.

$1, 4, 9, 16, 25, \cdots, 100, \cdots, 10000$

$3/10 = 30\%$

$10/100 = 10\%$

$100/10000 = 1\%$

$\vdots$

Example 5 (Cartesian Product of $\mathbb{N}$)

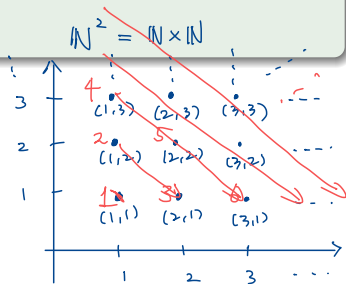$\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$ because the function g defined by

$$g(1,1) = 1,$$
$$g(1,2) = 2, \quad g(2,1) = 3,$$
$$g(1,3) = 4, \quad g(2,2) = 5, \quad g(3,1) = 6,$$
$$g(1,4) = 7, \quad g(2,3) = 8, \quad g(3,2) = 9, \quad g(4,1) = 10,$$
and so on,

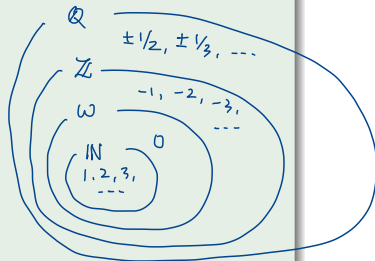is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

Example 6 (Positive Rational Numbers)

Let $A = \{x \in \mathbb{Q} : x > 0\}$. Each element $x \in A$ can be expressed uniquely as $x = a/b$ where $a, b \in \mathbb{N}$ and the fraction is in lowest terms. We can list all such fractions in lowest terms as follows:

$$\frac{1}{1},$$

$$\frac{1}{2}, \quad \frac{2}{1},$$

$$\frac{1}{3}, \quad \frac{3}{1},$$

$$\frac{1}{4}, \quad \frac{2}{3}, \quad \frac{3}{2}, \quad \frac{4}{1},$$

$$\frac{1}{5}, \quad \frac{5}{1},$$

$$\frac{1}{6}, \quad \frac{2}{5}, \quad \frac{3}{4}, \quad \frac{4}{3}, \quad \frac{5}{2}, \quad \frac{6}{1},$$

and so on,

Define $f(n)$ to be the n-th term in this list. Then f is a bijection from $\mathbb{N}$ to A.

$\mathbb{Q}$ — $\pm 1/2, \pm 1/3, \cdots$
$\mathbb{Z}$ — $-1, -2, -3, \cdots$
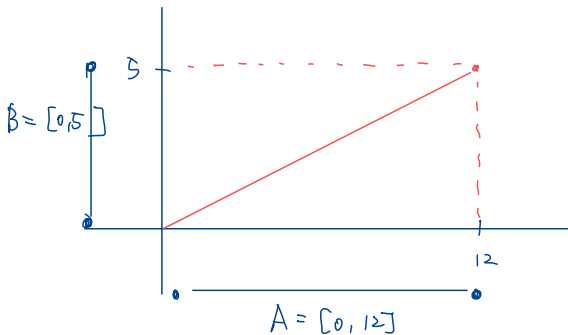ω — 0
$\mathbb{N}$ — $1, 2, 3, \cdots$

Exercise

Show that $\mathbb{Q} \approx \mathbb{N}$.

Example 7

Let $A = [0, 12]$ and let $B = [0, 5]$. Then $A \approx B$ because of we let $f(x) = 5x/12$ for all $x \in A$, then f is a bijection from A to B. *if*

Note. In general, for any $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$,

$$[a, b] \approx [c, d], \quad (a, b) \approx (c, d), \quad (a, b] \approx (c, d], \quad \text{and} \quad [a, b) \approx [c, d).$$

$$y = mx$$

$$= \frac{5}{12} x$$

$B = [0, 5]$

$A = [0, 12]$

Example 8

Let $\varphi(x) = x/(1 - |x|)$ for all $x \in (-1, 1)$. In S11E23, we showed that φ is a bijection from $(-1, 1)$ to $\mathbb{R}$. Hence $(-1, 1) \approx \mathbb{R}$.

Note. By this example and the note from the previous example, we deduce that $(0, 1) \approx \mathbb{R}$. This fact will be useful later.