Induction

Section 5

Proof by Induction

another proof technique

The method of *proof by induction* is based on the following principle.

Principle of Mathematical Induction

Let $P(n)$ be any statement about n. Suppose we have proved that

$$P(1) \text{ is true} \tag{1}$$

and that

$$\text{for each natural number } n \text{, if } P(n) \text{ is true, then } P(n+1) \text{ is true.} \tag{2}$$

$(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$

Then we may conclude that for each natural number n, $P(n)$ is true.

$(\forall n \in \mathbb{N})P(n)$

- This is a commonly used technique to prove a universal sentence $(\forall x \in A)P(x)$ when A is $\mathbb{N}$.

Steps in Proof by Induction

$P(n)$: $1 + \cdots + (2n-1) = n^2$

$P(n+1)$: $1 + \cdots + (2n-1) + (2(n+1)-1)$

$= (n+1)^2$

Sum of Odd Natural Numbers

For each $n \in \mathbb{N}$, $\underbrace{1 + 3 + \cdots + (2n-1) = n^2}_{P(n)}$.

$(\forall n \in \mathbb{N})$ (sum of first n positive odd numbers $= n^2$)

Proof. Let $P(n)$ be the sentence

$$1 + 3 + \cdots + (2n-1) = n^2.$$

Declare $P(n)$.

BASE CASE: Observe that $P(1)$ is true because if $n = 1$, then the left-hand side is just 1 and the right-hand side is $1^2 = 1$.

Show $P(1)$ is true. (1)

INDUCTIVE STEP: Let $n \in \mathbb{N}$ such that $P(n)$ is true. Then

$$\begin{aligned} 1 + 3 + \cdots + (2n-1) + [2(n+1) - 1] \\ = n^2 + [2(n+1) - 1] \\ = n^2 + 2n + 2 - 1 = n^2 + 2n + 1 \\ = (n+1)^2 \end{aligned} \qquad (\star)$$

Thus $P(n+1)$ is true.

Show $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))$. (2)

The first sentence in this paragraph is called *the inductive hypothesis*.

CONCLUSION: Therefore, by induction, for each $n \in \mathbb{N}$, $P(n)$ is true. That is, for each $n \in \mathbb{N}$, $1 + 3 + \cdots + (2n-1) = n^2$. □

Use induction to conclude.

__Tip__ Do scratch work by writing out $P(n+1)$ is true!

Example 1

Prove by induction that for each $n \in \mathbb{N}$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof Let $P(n)$ be the sentence

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

BASE CASE $P(1)$ is true because $1+2+\cdots+n$ is simply 1 and $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{1\cdot 2}{2} = 1$.

Show $P(1)$ is true

INDUCTIVE STEP Let $n \in \mathbb{N}$ such that $P(n)$ is true. ✱

Show $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$

$$\underbrace{1 + 2 + \cdots + n}_{= \frac{n(n+1)}{2}} + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

by ✱

$P(n+1)$:

$$1+2+\cdots+n+(n+1) = \frac{(n+1)[(n+1)+1]}{2} = \frac{(n+1)(n+2)}{2}$$

this may be something that you encounter.

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$= \frac{(n+2)(n+1)}{2} = \frac{(n+1)[(n+1)+1]}{2}.$$

Thus $P(n+1)$ is true.

<u>CONCLUSION</u> Therefore, by induction, for each $n \in \mathbb{N}$, $P(n)$ is true. That is, for each $n \in \mathbb{N}$, $1+2+\cdots+n = n(n+1)/2$. $\square$

Example 2

Prove by induction that for each $n \in \mathbb{N}$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof Let $P(n)$ be the sentence

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

BASE CASE $P(1)$ is true because $1^2 + 2^2 + \cdots + n^2$ is

Show $P(1)$ is true

Simply $1^2 = 1$ and $\frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2+1)}{6} = 1$.

INDUCTIVE STEP Let $n \in \mathbb{N}$ such that $P(n)$ is true.

Show $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

Scratch:

$P(n+1)$:

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2$$

$$= \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6} \checkmark$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6}$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)\left[n(2n+1) + 6(n+1)\right]}{6}$$

$$= \frac{(n+1)(2n^2+7n+6)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6} = \frac{(n+1)\left[(n+1)+1\right]\left[2(n+1)+1\right]}{6}$$

Thus $P(n+1)$ is true.

<u>CONCLUSION</u> Therefore, by induction, for each $n \in \mathbb{N}$, $P(n)$ is true. That is, for each $n \in \mathbb{N}$, $1^2+2^2+\cdots+n^2 = n(n+1)(2n+1)/6$.

Example 3

Prove by induction that for each $n \in \mathbb{N}$, 3 divides $4^n - 1$.

Hint In the inductive step...

WTS: 3 divides $4^{n+1} - 1$

(Under inductive hypothesis)

$$4^{n+1} - 1 = 4 \cdot 4^n - 1$$

$$= (3+1) \cdot 4^n - 1$$

$$= \underbrace{3 \cdot 4^n} + \underbrace{4^n - 1}$$