Ordered Pairs and Cartesian Products

Recap

$a \le b$

- Intervals: $[a,b]$, (a,b), $[a,b)$, $(a,b]$

  $[c,\infty)$, (c,∞), $(-\infty,c)$, $(-\infty,c]$, $(-\infty,\infty)$

  If $a=b$, then

  - $[a,b]=[a,a]=\{a\}$
  - $(a,b)=[a,b)=(a,b]=\varnothing$

  $\Big\}$ "degenerate intervals"

- $\mathcal{A}$ a set of sets
  - $\bigcup \mathcal{A} = \{x : x \in A \text{ for some } A \in \mathcal{A}\}$
  - $\bigcap \mathcal{A} = \{x : x \in A \text{ for all } A \in \mathcal{A}\}$ $\longleftarrow$ $\mathcal{A}$ must be nonempty!

"finer points"

① $\mathcal{A} = \{1, \{2,3\}\}$. $\bigcup \mathcal{A}$ and $\bigcap \mathcal{A}$ are undefined, because $\mathcal{A}$ is not a set of sets.

② $\bigcap \{\ \} = \bigcap \emptyset$ is undefined.

• $\mathcal{P}(A)$ is the set of all subsets of A.

script P   set

This is an example of a set of sets.

<u>Example</u> Let $A = \{1, 2\}$.

- $\mathcal{P}(A) = \{ \emptyset, \{1\}, \{2\}, \{1, 2\} \}$

$$\boxed{\#\text{ of elem. of } \mathcal{P}(A) = 2^{\#\text{ of elem. of } A}}$$

- $\bigcup \mathcal{P}(A) = \emptyset \cup \{1\} \cup \{2\} \cup \{1, 2\} = \{1, 2\} = A$
- $\bigcap \mathcal{P}(A) = \emptyset \cap \{1\} \cap \{2\} \cap \{1, 2\} = \emptyset$

When $S = \emptyset$, $\mathcal{P}(\emptyset) = \{ \emptyset \}$

In general, for a set S,

$$\bigcup \mathcal{P}(S) = S \quad \text{and} \quad \bigcap \mathcal{P}(S) = \emptyset$$

Ordered Pairs

Ordered Pairs

cf) Sets: $\{1,3\} = \{3,1\}$

$\{1,1\} = \{1\}$

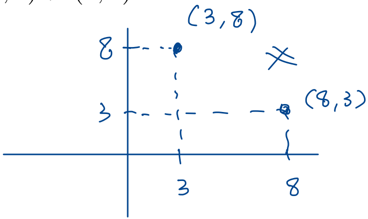
We write (a,b) for the ordered pair whose first entry is a and whose second entry is b. Unlike sets, the order by which the entries are listed does matter.

Theorem 1 (Fundamental Property of Ordered Pairs)

Let a and b be any objects. We have $(a,b) = (a',b')$ if and only if $a = a'$ and $b = b'$.

Example.

- $(1,1) = (1,1)$
- $(3,8) \neq (8,3)$



Note: Proof of $(\Leftarrow)$ is trivial.

Assuming $a = a'$ and $b = b'$, it follows immediately that

$$(a,b) = (a',b')$$

by substituting equals with equals.

Notes on Ordered Pairs

- $\{a, b\} = \{b, a\}$ $\times$
- $\{\{a\}, \{b\}\} = \{\{b\}, \{a\}\}$ $\times$

The proof of the seemingly obvious statement in the theorem relies on a careful definition of the ordered pair (a, b).

Definition 2 (Ordered Pairs as Sets; Kuratowski)

Let a and b be any objects. The *ordered pair* (a, b) is the set $\{\{a\}, \{a, b\}\}$.

Notes on Ordered Pairs (cont')

- The ordered triple (a, b, c) can be defined as $((a, b), c)$. The analogue of the fundamental property of ordered pairs holds for ordered triples. Namely,

$$(a, b, c) = (a', b', c') \text{ iff } a = a', b = b', \text{ and } c = c'.$$

- The ordered quadruple (a, b, c, d) can be defined as $((a, b, c), d)$. We have

$$(a, b, c, d) = (a', b', c', d') \text{ iff } a = a', b = b', c = c', \text{ and } d = d'.$$

- Continuing in the same fashion, the ordered n-tuple $(a_1, a_2, \ldots, a_n)$ can be defined for each natural number $n \geqslant 2$. The fundamental property of ordered n-tuple states

$$(a_1, a_2, \ldots, a_n) = (a'_1, a'_2, \ldots, a'_n) \text{ iff for each } j \in \{1, 2, \ldots, n\}, a_j = a'_j.$$

Sketch of Proof (Fund. Prop. of Ordered Pairs)

$(a,b)=(a',b')$

$\iff$ ~~$a=a'$~~ and $b=b'$

- $(\Leftarrow)$ Backward implication is trivials.
- $(\Rightarrow)$ Assume $(a,b)=(a',b')$.

That is, $\underbrace{\{\{a\},\{a,b\}\}}_{=S} = \underbrace{\{\{a'\},\{a',b'\}\}}_{=S'}$ (Kuratowski's defn.)

$*$ $\{a'\}\in S'$ and $S'=S \Rightarrow \{a'\}=\{a\}$ or $\{a'\}=\{a,b\}$.

$\xrightarrow{\text{(WORK)}} \boxed{a=a'}$

$*$ $\{a',b'\}\in S'$ and $S'=S \Rightarrow \{a',b'\}=\{a\}$ or $\{a',b'\}=\{a,b\}$

$\xrightarrow{\text{(WORK)}} \boxed{b=b'}$

Cartesian Products

Cartesian Products

Definition 3 (Cartesian Products)

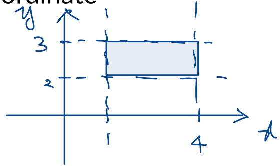
Let A and B be sets. Then the *Cartesian product of A and B* (denoted $A \times B$) is the set of all ordered pairs (x, y) such that $x \in A$ and $y \in B$; in other words,

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}.$$

Example.

- For a set A, the Cartesian product of A with itself, $A \times A$, is also denoted A^2.
- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, the Cartesian product of $\mathbb{R}$ with itself. This is the coordinate plane in analytical geometry.
- $[1, 4] \times [2, 3]$ is a rectangle in $\mathbb{R}^2$. Specifically, it is the set

$$\{(x, y) : 1 \leqslant x \leqslant 4 \text{ and } 2 \leqslant y \leqslant 3\}.$$



Example 4

Let $A = \{1, 3\}$ and $B = \{2, 4, 6\}$. Find $A \times B$.

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$$

$$= \{(1, 2), (1, 4), (1, 6), (3, 2), (3, 4), (3, 6)\}$$

Cartesian Products of More Than Two Sets

The Cartesian product of three sets A, B, and C (denoted $A \times B \times C$) is defined in a similar way, namely,

$$A \times B \times C = \{(x, y, z) : x \in A, y \in B, \text{and } z \in C\}.$$

In general:

Definition 5 (Cartesian Product of n Sets)

Let $n \in \mathbb{N}$ such that $n \geqslant 2$ and let $A_1, \ldots, A_n$ be sets. Then the *Cartesian product of* $A_1, A_2, \ldots, A_n$ (denoted $A_1 \times A_2 \times \cdots A_n$) is the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ such that $x_1 \in A_1$, $x_2 \in A_2$, …, $x_n \in A_n$; in other words,

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) : \text{for each } j \in \{1, 2, \ldots, n\}, x_j \in A_j\}.$$

Cartesian Products of More Than Two Sets (cont')

Example 6

Let $A = \{0, 1\}$. Find $A^3 = A \times A \times A$.