Review for Exam 2

- TW 4:45 ~ 6:15 PM (Zoom)
- W during class time (in-person in classroom)

Key Topics to Review

- Exercises leading up to the rational root**s** theorem (S4)
- Binomial coefficients and binomial theorem
- Complete induction (recursively defined sequences)
- Insight vs. induction Summation formulas
- Algebra with set operations

Rational Roots Theorem

(S04: 17, 18, 19, 20)

S04E20 Let $x \in \mathbb{Q}$ such that

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where $n \in \mathbb{N}$ and $c_0, c_1, \cdots, c_n \in \mathbb{Z}$.

Prove that x can be written in the form

$x = a/b$ where $a \in \mathbb{Z}$ that divides c_0 and $b \in \mathbb{N}$ that divides c_n.

Rmk 4.50

Let $d \in \mathbb{N}$ and $x_1, \cdots, x_n \in \mathbb{Z}$.

If d divides $x_1 x_2 \cdots x_n$, then there exist $d_1, d_2, \cdots, d_n \in \mathbb{N}$ such that

for each $j \in \{1, 2, \cdots, n\}$, d_j divides x_j

and

$$d = d_1 d_2 \cdots d_n.$$

Rational Roots Theorem

<u>Proof</u> Since x is rational, we can pick $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that $x = a/b$ and the fraction a/b is in lowest terms.

Now on substituting $x = a/b$ into $c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$, we obtain

$$c_n (a/b)^n + c_{n-1} (a/b)^{n-1} + \cdots + c_1 (a/b) + c_0 = 0$$

$$\frac{c_n a^n}{b^n} + \frac{c_{n-1} a^{n-1}}{b^{n-1}} + \cdots + \frac{c_1 a}{b} + c_0 = 0$$

We multiply both sides by b^n to obtain

Rational Root Theorem

$$c_n a^n + c_{n-1} a^{n-1} b + \cdots + c_1 a b^{n-1} + c_0 b^n = 0. \qquad (\ast)$$

Isolating $c_n a^n$, the eqn $(\ast)$ can be written as

$$c_n a^n = -\left(c_{n-1} a^{n-1} b + \cdots + c_1 a b^{n-1} + c_0 b^n \right)$$

$$= -(c_{n-1} a^{n-1} + \cdots + c_1 a b^{n-2} + c_0 b^{n-1}) b,$$

so b divides $c_n a^n$. By the part of Rmk. 4.50 written above,

product of $n+1$ integers.

we can pick $b_1, b_2, \cdots, b_{n+1} \in \mathbb{N}$ such that

$$b_1 \mid c_n, \; b_2 \mid a, \; b_3 \mid a, \; \cdots, \; b_{n+1} \mid a \quad \text{and} \quad b = \underbrace{b_1 b_2 \cdots b_{n+1}}.$$

Rational Root Theorem

Complete the argument by showing that $b_2 = b_3 = \cdots = b_{n+1} = 1$ and so $b_1 = b$.

Now isolating $c_0 b^n$, we can rewrite $(\star)$ as

$$c_0 b^n = -\left(c_n a^n + c_{n-1} a^{n-1} b + \cdots + c_1 a b^{n-1}\right)$$

$$= -\left(c_n a^{n-1} + c_{n-1} a^{n-1} b + \cdots + c_1 b^{n-1}\right) a,$$

so a divides $c_0 b^n$.

Carry out similar arguments as above to show $a \mid c_0$.

Binomial Coefficients and Binomial Theorem

Pascal's triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\vdots$$

Useful in induction proofs

$\binom{n}{k}$ is the number on row n and column k of Pascal's triangle.

(Row and column indices begin at 0.)

① $\binom{0}{0} = 1$

② $\binom{n}{0} = \binom{n}{n} = 1$, for each $n \in \omega$

③ $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

Binomial Coefficients and Binomial Theorem

Let $a, b \in \mathbb{R}$ and $n \in \omega$. Then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Convention: For each $x \in \mathbb{R}$, $x^0 = 1$.

Recursively Defined Sequences and Complete Induction

Let $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 1$. (Fibonacci sequence)

Prove using complete induction that for any $n \in \omega$,

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\underbrace{\frac{1+\sqrt{5}}{2}}_{=\varphi}\right)^n - \left(\underbrace{\frac{1-\sqrt{5}}{2}}_{=\hat{\varphi}}\right)^n \right).$$

(Hint: $\frac{1 \pm \sqrt{5}}{2}$ are roots of $x^2 - x - 1 = 0$.)

Note:

Since $\varphi = \frac{1+\sqrt{5}}{2}$ and $\hat{\varphi} = \frac{1-\sqrt{5}}{2}$ are roots of x^2-x-1,

- $\varphi^2 - \varphi - 1 = 0 \Rightarrow \varphi^2 = \varphi + 1$
- $\hat{\varphi}^2 - \hat{\varphi} - 1 = 0 \Rightarrow \hat{\varphi}^2 = \hat{\varphi} + 1$.

These will be useful.

Proof. Let $P(n)$ be the sentence

$$F_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \hat{\varphi}^n\right).$$

WTS for each $n \in \omega$, $P(n)$ is true.

Recursively Defined Sequences and Complete Induction

BASE CASE

- $P(0)$ is true because

$$F_0 = \frac{1}{\sqrt{5}}(\varphi^0 - \hat{\varphi}^0) = \frac{1}{\sqrt{5}}(1-1) = 0.$$

- $P(1)$ is true because

...

INDUCTIVE STEP Let $n \in \omega$ such that $n \geq 1$ and $P(0), \cdots, P(n)$ are all true. To show $P(n+1)$ is true, we examine

$$F_{n+1} = F_n + F_{n-1}$$

$$= \frac{1}{\sqrt{5}}(\varphi^n - \hat{\varphi}^n) + \frac{1}{\sqrt{5}}(\varphi^{n-1} - \hat{\varphi}^{n-1}) \quad \text{(by ind. hyp.)}$$

$$= \frac{1}{\sqrt{5}}\left[(\varphi^n + \varphi^{n-1}) - (\hat{\varphi}^n + \hat{\varphi}^{n-1})\right]$$

Recursively Defined Sequences and Complete Induction

$$= \frac{1}{\sqrt{5}} \left[\varphi^{n-1} \underbrace{(\varphi + 1)}_{= \varphi^2} - \hat{\varphi}^{n-1} \underbrace{(\hat{\varphi} + 1)}_{= \hat{\varphi}^2} \right]$$

$$= \frac{1}{\sqrt{5}} \left(\varphi^{n+1} - \hat{\varphi}^{n+1} \right)$$

where we used the fact that φ and $\hat{\varphi}$ are roots of $x^2 - x - 1 = 0$ in the second from the last step.

<u>CONCLUSION</u> Therefore, by complete induction, for each $n \in \omega$, $P(n)$ is true.

$\square$