Subsets

Subsets

Office Hours (unusual schedule)

- M 4:45 ~ 6:15
- T 9:00 ~ 10:30

Subsets

If $A \subseteq B$, A can be the same as B!

Definition 1 (Subsets)

Let A and B be sets.

- To say that A *is a subset of* B (denoted $\boxed{A \subseteq B}$) means that for each x, if $x \in A$, then $x \in B$.
- To say that A *is a proper subset of* B ~~(denoted $A \subseteq B$)~~ means that $A \subseteq B$ and $A \neq B$.

e.g. $A = \{1, 2, 3\}$, $B = \{1\}$: B is a proper subset of A.

Notes.

- The relation $\subseteq$ is called *set inclusion*.
- The notation $B \supseteq A$ means the same as $A \subseteq B$ and is read "B is a superset of A."

Set Inclusion

Def'n $A \subseteq B \iff (\forall x)[x \in A \Rightarrow x \in B]$.

Fri.

$(\forall A, B)[A = B \iff (\forall x)(x \in A \iff x \in B)]$

Proposition 1 (Set Inclusion as Relation)

Set inclusion is reflexive, antisymmetric, and transitive. In other words

1. *For each set A, we have $A \subseteq A$. (Reflexivity.)*
2. *For all sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$. (Antisymmetry.)*
3. *For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. (Transitivity.)*

Proof

(1) Let A be a set. Let x be an element. Then if $x \in A$, then clearly $x \in A$. Thus, $A \subseteq A$.

(2) Let A and B be sets. Suppose that $A \subseteq B$ and $B \subseteq A$. Let x be an element. Suppose $x \in A$. Then $x \in B$, because $A \subseteq B$. Conversely, suppose $x \in B$. Then $x \in A$, because $B \subseteq A$. Thus, for each x, $x \in A$ iff $x \in B$. That is, $A = B$.

$(\forall A, B, C)\left[A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C \right]$

③ Let A, B, and C be sets. Suppose $A \subseteq B$ and $B \subseteq C$,

Let x be an arbitrary element. Suppose that $x \in A$.
Then $x \in B$, because $A \subseteq B$. But then $x \in C$, because
$B \subseteq C$. Thus, for each x, if $x \in A$, then $x \in C$.

Work to show $A \subseteq C$

In other words, $A \subseteq C$.

$\square$

$P \Rightarrow Q$

Proposition 2

For each set A, we have $\varnothing \subseteq A$.

- The proof involves a *vacuously true* statement.
- Conversely, if a set is a subset of any set, then it must be the empty set. In other words,

  (S10E05) Let A be a set such that for each set B, we have $A \subseteq B$. Then $A = \varnothing$.

HW

Proof

Let A be a set. WTS: $\phi \subseteq A$. In other words, we wish to show that

for each x, if $x \in \phi$, then $x \in A$.

Let x be arbitrary. But the antecedent of the conditional sentence, $x \in \phi$, is false. Therefore, the conditional sentence is true. Thus, $\phi \subseteq A$. $\square$

Exercise 1 (Subsets)

Answer the following questions.

1. Is $\{3,5\}$ a subset of $\{2,3,5\}$? Yes.
2. Is $\underbrace{\{2,\{3,5\}\}}_{A=}$ a subset of $\underbrace{\{2,3,5\}}_{B=}$? No, because $\{3,5\} \in A$ but $\{3,5\} \notin B$.
3. Write down all subsets of $\{1,2,3\}$.

Note: $\{3,5\} \notin B$, but $\{3,5\} \subseteq B$.

③

ϕ, → s/set w/ 0 elem.

$\{1\}, \{2\}, \{3\}$, → s/sets w/ 1 elem.

$\{1,2\}, \{2,3\}, \{1,3\}$, → s/sets w/ 2 elem.

$\{1,2,3\}$ → s/subs w/ 3 elem.

Exercise 2 ($\in$ vs. $\subseteq$)

Find two sets A and B such that:

1. $A \in B$ and $A \subseteq B$.
2. $A \in B$ and $A \nsubseteq B$.
3. $A \notin B$ and $A \subseteq B$.
4. $A \notin B$ and $A \nsubseteq B$.