Section 10.

Introduction to Set Theory

Basics of Set Theory

Sets

LaTeX

A *set* is a collection of objects, considered as an object in its own right.

Notation.

- $\boxed{x \in A}$: x is one of the objects in the set A.
  "x is an element of A", "x belongs to A", "x is a member of A", or "x is in A."
- $\boxed{x \notin A}$: x is not in the set A.

$ x \in A $

Ways to Denote Sets

We denote a set by

- listing its elements between braces, *e.g.*,

$$\{2, 3, 5, 7, 11\}$$

- using the *set-builder notation*, *e.g.*,

$$\{\boxed{x} : \boxed{x \text{ is prime and } x \leqslant 11}\} = \{2, 3, 5, 7, 11\}$$

"form" of elements

"membership criteria"

More on Set-Builder Notation

In set-builder notation, we describe a set in terms of *membership criteria*.

- $\{x : P(x)\}$: "the set of all x such that $P(x)$"

$$\{x : x \text{ is a natural number and } x \text{ is even}\} = \{2, 4, 6, 8, \dots\}$$

- $\{x \in A : P(x)\} = \{x : x \in A \text{ and } P(x)\}$: "the set of all x in A such that $P(x)$"

$$\{x \in \mathbb{N} : x \text{ is even}\} = \{2, 4, 6, 8, \dots\}$$

- $\{f(x) : P(x)\} = \{y : y = f(x) \text{ for some } x \text{ such that } P(x)\}$: "the set of all $f(x)$ such that $P(x)$"

$$\{2x : x \in \mathbb{N}\} = \{2, 4, 6, 8, \dots\}$$

Notes

$(\forall A, B)\left[(\forall x)(x \in A \Leftrightarrow x \in B) \Leftrightarrow A = B\right]$

1. Sets having the same elements are equal, *i.e.*,

   $$\text{If } \underbrace{\text{for each } x,\ x \in A \text{ iff } x \in B}_{\text{“having the same elements”}},\ \text{then } A = B.$$

   Consequently,

   - The order in which the elements of a set are listed is unimportant. $\{1, 3\} = \{3, 1\}$
   - Repetitions in the description of a set do not count. $\{1, 2, 2, 3, 4\} = \{1, 2, 3, 4\}$

2. Equal sets have the same elements, *i.e.*,

   $$\text{For all sets } A \text{ and } B,\ \text{if } A = B,\ \text{then } \underbrace{\text{for each } x,\ x \in A \text{ iff } x \in B}.$$

3. Equal objects are elements of the same sets, *i.e.*,

   $$\text{For all } x \text{ and } y,\ \text{if } x = y,\ \text{then for each set } A,\ x \in A \text{ iff } y \in A.$$

   $$(\forall x, y)\left[x = y \Rightarrow (\forall A)(x \in A \Leftrightarrow y \in A)\right]$$

Example

S10E01

Which of the sets A, B, C, D, and E below are the same?

$$A = \{3\}, \quad B = \{2,4\}, \quad C = \{x : x \text{ is prime, } x \text{ is odd, and } x < 5\}, = \{3\}$$

$$D = \{x - 1 : x \text{ is prime, } x \text{ is odd, and } x \leqslant 5\}, \quad E = \{x^2 + 2 : x \in \{-1, 1\}\}. = \{(-1)^2+2, 1^2+2\}$$

$= \{2, 4\}$ $\quad 3, 5$

$= \{3, 3\}$

$= \{3\}$

How many different sets are named here? Two !

- $A = C = E$
- $B = D$

The Number of Elements

Singleton : 1-element set

Question

How many elements does $\{a, b\}$ have?

- If $a=b$, then $\{a, b\} = \{a, a\} = \{a\}$, so there is one element.
- If $a \neq b$, then $\{a, b\}$ has two elements.

S10E02

How many elements does $\{a, b, c\}$ have?

Read the example right above this exercise carefully.

Sets as Elements of other Sets

Since sets are objects as well, they can be elements of other sets.

Example. Study the elements of each of the following sets.

- $\{1, 2, \{3, 4\}\}$. $3, 4 \notin \{1, 2, \{3, 4\}\}$, $\{3, 4\} \in \{1, 2, \{3, 4\}\}$.
- $\{\{1, 2, 3, \ldots\}\}$
- $\{\{1\}, \{2\}, \{3\}, \ldots\}$ an infinite set containing singletons.

  $= \{ S : S = \{k\} \text{ for some } k \in \mathbb{N} \}$
- $\{\{1, 2, 3, \ldots\}, \{2, 4, 6, \ldots\}, \{3, 6, 9, \ldots\}, \ldots\}$

Exercise: Express using set-builder not'n.

The Empty Set

$\neq \phi$ Greek phi.

The *empty set* is the set that has no elements, usually denoted by $\varnothing$.

- $\{x : P(x)\} = \varnothing$ if there are no values of x for which $P(x)$ is true. For example,

$$\{x : x \text{ is even and } x \text{ is odd}\} = \varnothing.$$

$\{x : x$ is prime, x is even, and $x > 2\} = \phi$

- The empty set is unique.

  Proof. Suppose $\varnothing'$ is another set with no elements. Then for each x, $x \in \varnothing$ and $x \in \varnothing'$ are both false, so $x \in \varnothing \Leftrightarrow x \in \varnothing'$. Hence $\varnothing = \varnothing'$. $\square$

- Tip: To prove that a set A is the empty set, show that for each x, $x \notin A$.

$A = \phi.$

Homework Coaching

Rational Roots (Cubic Polynomials)

S04E17 (Collected)

1. Let x be a rational number such that $x^3 = c$, where c is an integer. Prove that x is an integer.
2. Let c be an integer which is not a perfect cube. Prove that $\sqrt[3]{c}$ is irrational.

S04E18 (Collected)

Let $x \in \mathbb{R}$ such that $x^3 = rx^2 + sx + t$, where $r, s, t \in \mathbb{Z}$.

1. Prove that if x is rational, then x is an integer.
2. Prove that if x is not an integer, then x is irrational.

Rational Roots (General Polynomials)

S04E19 (To be turned in)

Let x be a real number such that

$$x^n + c_n x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where $n \in \mathbb{N}$ and $c_0, c_1, \ldots, c_{n-1} \in \mathbb{Z}$.

1. Prove that if x is rational, then x is an integer.
2. Prove that if x is not an integer, then x is irrational.

S05E11 (The Binomial Theorem; collected)

Let $a, b \in \mathbb{R}$. Then for each $n \in \boldsymbol{\omega}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Convention. $x^0 = 1$ for each $x \in \mathbb{R}$.

S05E12 (Collected)

Let $n \in \boldsymbol{\omega}$. Show that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

(Do not use induction.)

Application of Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

S05E13 (To be turned in)

Let $n \in \mathbb{N}$. Show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

(Do not use induction.)