Tautologies and Conditional Proofs

Contents

Some Remarks on Logical Connectives

Parentheses

The order of priority of the logical connectives (from highest to lowest):

$$\neg,\ \wedge,\ \vee,\ \Rightarrow,\ \Leftrightarrow$$

Examples.

$$\begin{aligned}
\neg P \wedge \neg Q \quad & \text{means} \quad (\neg P) \wedge (\neg Q) \\
P \wedge Q \vee R \quad & \text{means} \quad (P \wedge Q) \vee R \\
P \wedge Q \Rightarrow P \vee Q \quad & \text{means} \quad (P \wedge Q) \Rightarrow (P \vee Q) \\
P \Rightarrow (Q \Rightarrow R) \Leftrightarrow P \wedge Q \Rightarrow R \quad & \text{means} \quad [P \Rightarrow (Q \Rightarrow R)] \Leftrightarrow [(P \wedge Q) \Rightarrow R]
\end{aligned}$$

- Economical writing by omitting some parentheses
- Judicious inclusion of some dispensable parenthesis may enhance readability

Associativity of Conditionals

The logical connective $\Rightarrow$ is not associative, that is,

$$P \Rightarrow (Q \Rightarrow R) \quad \text{and} \quad (P \Rightarrow Q) \Rightarrow R$$

are logically *in*equivalent . Furthermore, neither of the two is logically equivalent to

$$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$$

For further discussions, see Exercise 11 and Remarks 2.18 and 2.19.

Associativity of Biconditionals

The logical connective $\Leftrightarrow$ is associative, that is,

$$P \Leftrightarrow (Q \Leftrightarrow R) \quad \text{and} \quad (P \Leftrightarrow Q) \Leftrightarrow R$$

are logically equivalent. However, neither of the two is logically equivalent to

$$(P \Leftrightarrow Q) \wedge (Q \Leftrightarrow R)$$

For further discussions, see Exercise 12 and Remark 2.20.

Tautologies

In logic, a *tautology* is a sentence which is true under any possible truth values of its propositional variables.

Examples.

- $P \vee \neg P$
- $[\neg(P \wedge Q)] \Leftrightarrow [\neg P \vee \neg Q]$
- $(P \wedge Q) \Rightarrow P$
- $P \Rightarrow (P \vee Q)$
- If $R \equiv S$, then the sentence $R \Leftrightarrow S$ is a tautology. (Why?)

Exercise. Construct a tautology using three sentences P, Q, and R.

Conditional Proofs

Conditional Proofs

To show that $A \Rightarrow B$ is true, it suffices to consider the case where A is true and to show that in this case, B must also be true. This approach is known as the method of *conditional proof*.

Template of Conditional Proof. To show $A \Rightarrow B$ is a tautology:

A1: Suppose that A is true.

Discharging A1, we see that $A \Rightarrow B$ is true under no assumptions. Therefore $A \Rightarrow B$ is a tautology.

Conditional Proofs (cont')

Example

Use the method of conditional proof to explain in words why

$$P \Rightarrow (P \vee Q)$$

is a tautology. (Do not use cases. Be explicit about discharging assumptions.)

Solution.

A1: Suppose P is true.
Then $P \vee Q$ is true.
We have shown that $P \vee Q$ is true under the assumption A1 that P is true.

Discharging A1, we see that $P \Rightarrow (P \vee Q)$ is true under no assumptions.
Therefore $P \Rightarrow (P \vee Q)$ is a tautology, because we have shown that it is true under no assumptions on the truth values of P and Q. □

Modus Ponens

The following is useful in proofs.

Modus Ponens

If $P \Rightarrow Q$ is true and P is also true, then Q must be true. This rule of inference is called *modus ponendo ponens* or, more commonly, *modus ponens*.

Exercise 1

Use the method of conditional proof to explain in words why the sentence

$$[(P \Rightarrow Q) \wedge [(Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$$

is a tautology. Be explicit about discharging assumptions.

Exercise 2

Use the method of conditional proof to explain in words why the sentence

$$[P \Rightarrow (Q \Rightarrow R)] \Rightarrow [(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)]$$

is a tautology. Be explicit about discharging assumptions.