Insight vs. Induction

Sums of Powers Revisited

Verification vs. Discovery

Denote by $S_r(n)$ the sum $1^r + 2^r + \cdots + n^r$, *e.g.*,

$$\begin{aligned} S_1(n) &= 1\ \ + 2\ \ + 3\ \ + \cdots + n\ \ , \\ S_2(n) &= 1^2 + 2^2 + 3^2 + \cdots + n^2, \\ S_3(n) &= 1^3 + 2^3 + 3^3 + \cdots + n^3, \\ &\ \ \vdots \end{aligned}$$

In Section 5, we were given formulas for some of the sums and they were verified by induction. But how would one discover such formulas in the first place?

Derivation of $S_1(n)$ Formula

Observe that

$$
\begin{array}{rrcccccccccc}
 & S_1(n) & = & 1 & + & 2 & + & 3 & + & \cdots & + & n \\
+) & S_1(n) & = & n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \\
\hline
 & 2S_1(n) & = & (n+1) & + & (n+1) & + & (n+1) & + & \cdots & + & (n+1)
\end{array}
$$

Therefore,

Another Derivation of $S_1(n)$ Formula (Telescoping Sums)

Let $T(n) = \sum_{k=1}^{n}[k^2 - (k-1)^2]$.

On the one hand,

On the other hand, since $k^2 - (k-1)^2 =$

Therefore,

Derivation of $S_2(n)$ Formula via Telescoping Sum

S06E01

Let $n \in \mathbb{N}$. Let $T(n) = \sum_{k=1}^{n}[k^3 - (k-1)^3]$. By writing $T(n)$ out in long form, show that it is a telescoping sum and that $T(n) = n^3$. Then, by evaluating $T(n)$ in a different way, deduce without explicit induction that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Geometric Sums Revisited

Sum of Geometric Progression

Let $x \in \mathbb{R}$. Suppose that $x \neq 1$. Let $n \in \mathbb{N}$. In S05E04, you were given the geometric series formula

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x},$$

and were asked to verify it using induction. Let's derive it here.

Derivation of Geometric Sum Formula

Let

$$S = 1 + x + x^2 + \cdots + x^{n-1}.$$

Observe that

$$\begin{array}{rrcccccccccccc} & S & = & 1 & + & x & + & x^2 & + & \cdots & + & x^{n-1} & & \\ -) & xS & = & & & x & + & x^2 & + & \cdots & + & x^{n-1} & + & x^n \\ \hline & (1-x)S & = & 1 & + & 0 & + & 0 & + & \cdots & + & 0 & + & x^n \end{array}$$

Therefore,

Another Derivation Using Summation Notation

In summation notation,

$$S = \sum_{k=0}^{n-1} x^k = 1 + \sum_{k=1}^{n-1} x^k,$$

while

$$xS = \sum_{k=0}^{n-1} x^{k+1} = \sum_{k=1}^{n} x^k = \sum_{k=1}^{n-1} x^k + x^n.$$

Therefore,