Functions

Basics

Function and Its Domain

A function f is a correspondence which to each suitable object x associates an object $f(x)$.

- $f(x)$ is called the value of f at x, or the value that f takes on at x.
- The set of all x such that $f(x)$ is defined is called the *domain of* f, denoted $\mathrm{Dom}(f)$.

Definition 1

Let A and B be sets.

- To say that f *is a function on* A means that f is a function and $\mathrm{Dom}(f) = A$.
- To say that f *is a function from* A *to* B (denoted $f : A \to B$) means that f is a function, $\mathrm{Dom}(f) = A$, and for each x, if $x \in A$, then $f(x) \in B$.

Examples

Example 2

Let $A = \{x : x \text{ is a web page on the WWW}\}$. For each $x \in A$, define:

$$\begin{aligned} \ell(x) &= \text{the number of web pages which link to } x, \\ L(x) &= \text{the set of all web pages which link to } x. \end{aligned}$$

Then both ℓ and L are functions on A, but

$$\ell : A \to \omega \quad \text{while} \quad L : A \to \mathcal{P}(A).$$

Examples (cont')

Example 3

Let $f(x) = x^2$ for all $x \in \mathbb{R}$. Then the following are all true.

- $f : \mathbb{R} \to \mathbb{R}$.
- $f : \mathbb{R} \to [0, \infty)$.
- $f : \mathbb{R} \to B$, where B is any set such that $[0, \infty) \subseteq B$.

Range of a Function

Definition 4

Let f be a function. The *range of* f (denoted $\mathrm{Rng}(f)$) is the set of all values of f; in other words,

$$\begin{aligned}\mathrm{Rng}(f) &= \{f(x) : x \in \mathrm{Dom}(f)\} \\ &= \{y : y = f(x) \text{ for some } x \in \mathrm{Dom}(f)\}.\end{aligned}$$

Remark.

- Let A and B be sets. Then $f : A \to B$ iff f is a function, $\mathrm{Dom}(f) = A$, and $\mathrm{Rng}(f) \subseteq B$.

Equality of Functions

Two functions f and g are equal when they have the same domain and for each x in their domain, $f(x) = g(x)$.

Example 5

Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$. Let $g : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ defined by $g(x) = (x^2 - 1)/(x - 1)$. For any $x \in \mathbb{R} \setminus \{1\}$,

$$g(x) = \frac{(x+1)(x-1)}{x-1} = x + 1 = f(x).$$

Nonetheless, $g \neq f$ because $\mathrm{Dom}(f) \neq \mathrm{Dom}(g)$.

Some Examples of Functions

Constant Functions

Example 6 (Constant Function)

Let A be a set. A function f on A is said to be a *constant function* when there exists y_0 such that for each $x \in A$, $f(x) = y_0$.

Question. For each $x \in \mathbb{R}$, let $f(x) = \pi$. What is $\mathrm{Rng}(f)$?

Indicator Functions

Example 7 (Indicator Function)

Let A be a set and let S be a subset of A. Then *the indicator function of* S, denoted 1_S, is the function on A defined by

$$1_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S, \end{cases}$$

for all $x \in A$.

Question. Let 1_S be as defined above. What are $\mathrm{Dom}(1_S)$ and $\mathrm{Rng}(1_S)$?

Identity Function

Example 8 (Identity Function)

Let A be a set. Then *the identity function on* A, denoted id_A, is the function from A to A defined by $\mathrm{id}_A(x) = x$ for all $x \in A$.

- Note that $\mathrm{id}_A : A \to B$ for any set B such that $A \subseteq B$, in which case id_A is called *the inclusion function from* A *to* B.

The Empty Function

Example 9 (The Empty Function)

The function whose domain is the empty set is called *the empty function*.

- **Existence**? Yes, for instance, take $\mathrm{id}_{\varnothing}$ as an example.
- **Uniqueness**? Yes.

  Proof. Let f and g be functions such that $\mathrm{Dom}(f) = \varnothing = \mathrm{Dom}(g)$. For any x, the sentence

  $$\text{if } x \in \varnothing, \text{ then } f(x) = g(x)$$

  is *vacuously true*. In other words, for each $x \in \varnothing$, $f(x) = g(x)$. Thus $f = g$. □
- Let f be a function. Then f is the empty function iff $\mathrm{Rng}(f) = \varnothing$.

Projections

A function of two variables is a function whose domain is a set of ordered pairs. In general, a function of n variables is a function whose domain is a set of n-tuples.

Example 10 (Projection)

Let A and B be sets and let $\pi_A(x, y) = x$ and $\pi_B(x, y) = y$ for all $(x, y) \in A \times B$. Then $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$. The functions π_A and π_B are called the *projections* from $A \times B$ to A and B respectively.

- For convenience of notation, it is customary to practice a slight abuse of notation such as $\pi_A(x, y)$ instead of $\pi_A((x, y))$ as shown above.

Composition of Functions

Composition of Functions

Definition 11

Let f and g be functions. Then *the composition of* g *with* f is the function, denoted $g \circ f$, that is defined by

$$(g \circ f)(x) = g(f(x))$$

for all $x \in \mathrm{Dom}(f)$ such that $f(x) \in \mathrm{Dom}(g)$.

- Note that $\mathrm{Dom}(g \circ f) = \{x \in \mathrm{Dom}(f) : f(x) \in \mathrm{Dom}(g)\}$.
- The short way to read $g \circ f$ is "g composed with f."
- Composition of functions is associative (see Theorem 11.37) but not commutative.

Examples

Example 12

Let $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ and $g(x) = x - 1$. Then

$$(g \circ f)(x) = x^2 - 1 \quad \text{and} \quad (f \circ g)(x) = (x-1)^2,$$

with $\mathrm{Dom}(g \circ f) = \mathrm{Dom}(f \circ g) = \mathbb{R}$.

Example 13

Let $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = \sqrt{x}$ and let $g : \mathbb{R} \to [-1, 1]$ defined by $g(x) = \sin(x)$. Then

$$(g \circ f)(x) = \sin(\sqrt{x}) \quad \text{with } \operatorname{Dom}(g \circ f) = [0, \infty),$$

and

$$(f \circ g)(x) = \sqrt{\sin(x)} \quad \text{with } \operatorname{Dom}(f \circ g) = \bigcup \left\{ [2n\pi, (2n+1)\pi] : n \in \mathbb{Z} \right\}.$$

Exercise

Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -(x-1)^2$ and let $g : [0,\infty) \to [0,\infty)$ defined by $g(x) = \sqrt{x}$. Find $\mathrm{Dom}(g \circ f)$. Justify your answer.