Section 11

Functions

Next HW due next Wed.

(Prob. assigned on last M & W and today.)

Basics

Function and Its Domain

A function f is a correspondence which to each suitable object x associates an object $f(x)$.

function

- $f(x)$ is called the value of f at x, or the value that f takes on at x.
- The set of all x such that $f(x)$ is defined is called the *domain of f*, denoted $\mathrm{Dom}(f)$.

Definition 1

Let A and B be sets.

- To say that f *is a function on A* means that f is a function and $\mathrm{Dom}(f) = A$.

This is a sentence!

- To say that *f is a function from A to B* (denoted $f : A \to B$) means that f is a function, $\mathrm{Dom}(f) = A$, and for each x, if $x \in A$, then $f(x) \in B$.

Example 2

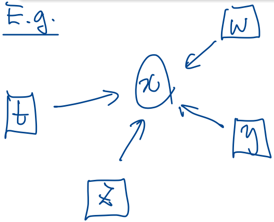
Let $A = \{x : x \text{ is a web page on the WWW}\}$. For each $x \in A$, define:

$$\ell(x) = \text{the number of web pages which link to } x,$$
$$L(x) = \text{the set of all web pages which link to } x.$$

Then both ℓ and L are functions on A, but

$$\ell : A \to \omega \quad \text{while} \quad L : A \to \mathcal{P}(A).$$

E.g.



$\ell(x) = 4 \in \omega$

$L(x) = \{w, y, z, t\} \in \mathcal{P}(A)$

Examples (cont')

- Let f be a function on $\mathbb{R}$ defined by $f(x) = x^2$.
- Let $f(x) = x^2$. (Implicit domain $= \mathbb{R}$)

Example 3

Let $f(x) = x^2$ for all $x \in \mathbb{R}$. Then the following are all true.

- $f : \mathbb{R} \to \mathbb{R}$.
- $f : \mathbb{R} \to [0, \infty)$.
- $f : \mathbb{R} \to B$, where B is any set such that $[0, \infty) \subseteq B$.

$f : A \to B$ means

- f is a function
- $\mathrm{Dom}(f) = A$
- $(\forall x)(x \in A \Rightarrow f(x) \in B) \iff \mathrm{Rng}(f) \subseteq B$

Range of a Function

Definition 4

Let f be a function. The *range of* f (denoted $\mathrm{Rng}(f)$) is the set of all values of f; in other words,

$f(x)$ for $x \in \mathrm{Dom}(f)$

$$\begin{aligned}\mathrm{Rng}(f) &= \{f(x) : x \in \mathrm{Dom}(f)\}\\ &= \{y : y = f(x) \text{ for some } x \in \mathrm{Dom}(f)\}.\end{aligned}$$

Remark.

- Let A and B be sets. Then $f : A \to B$ iff f is a function, $\mathrm{Dom}(f) = A$, and $\mathrm{Rng}(f) \subseteq B$.

(See the prev. page.)

Equality of Functions

$f = g$ means $\mathrm{Dom}(f) = \mathrm{Dom}(g) \wedge (\forall x \in \mathrm{Dom}(f))(f(x) = g(x))$

① ②

Two functions f and g are equal when they have the same domain and for each x in their domain, $f(x) = g(x)$.

Example 5

Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$. Let $g : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ defined by $g(x) = (x^2 - 1)/(x - 1)$. For any $x \in \mathbb{R} \setminus \{1\}$, $x \neq 1$

$$g(x) = \frac{(x+1)(x-1)}{x-1} = x + 1 = f(x).$$

because $x - 1 \neq 0$

Nonetheless, $g \neq f$ because $\mathrm{Dom}(f) \neq \mathrm{Dom}(g)$.

Some Examples of Functions

Constant Functions

$\mathrm{Dom}(f) = A$

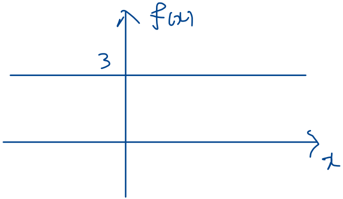
Example 6 (Constant Function)

Let A be a set. A function f on A is said to be a *constant function* when there exists y_0 such that for each $x \in A$, $f(x) = y_0$.

Question. For each $x \in \mathbb{R}$, let $f(x) = \pi$. What is $\mathrm{Rng}(f)$?

$\mathrm{Rng}(f) = \{\pi\}.$

E.g. If $A = \mathbb{R}$ and $y_0 = 3$, then the function f as defined above has



Indicator Functions

Example 7 (Indicator Function)

Let A be a set and let S be a subset of A. Then *the indicator function of S*, denoted 1_S, is the function on A defined by

Dom(1_S)

$$1_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S, \end{cases}$$

for all $x \in A$.

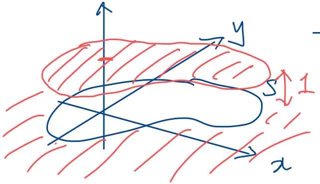
Question. Let 1_S be as defined above. What are $\mathrm{Dom}(1_S)$ and $\mathrm{Rng}(1_S)$?

$\mathrm{Dom}(1_S) = A$

$\mathrm{Rng}(1_S) = \{0, 1\}$

$A = \mathbb{R}^2$, S is a region

The graph of 1_S is shaded in



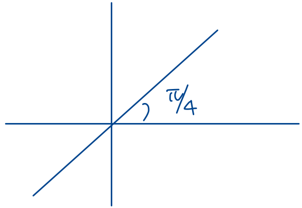
Identity Function

Example 8 (Identity Function)

Let A be a set. Then *the identity function on* A, denoted id_A, is the function from A to A defined by $\mathrm{id}_A(x) = x$ for all $x \in A$.

- Note that $\mathrm{id}_A : A \to B$ for any set B such that $A \subseteq B$, in which case id_A is called *the inclusion function from* A *to* B.

E.g. $A = \mathbb{R}$: $\mathrm{id}_A(x) = x$



The Empty Function

$f : \phi \to A$

Example 9 (The Empty Function)

The function whose domain is the empty set is called *the empty function*.

- **Existence**? Yes, for instance, take $\mathrm{id}_\varnothing$ as an example.
- **Uniqueness**? Yes.

  Proof. Let f and g be functions such that $\mathrm{Dom}(f) = \varnothing = \mathrm{Dom}(g)$. For any x, the sentence

  $$\text{false} \longrightarrow \text{if } \underbrace{x \in \overset{\overset{\mathrm{Dom}(f)}{\shortparallel}}{\varnothing}}, \text{ then } f(x) = g(x)$$

  is *vacuously true*. In other words, for each $x \in \varnothing$, $f(x) = g(x)$. Thus $f = g$. □

- Let f be a function. Then f is the empty function iff $\mathrm{Rng}(f) = \varnothing$.

(Proof of the above follows from $\mathrm{Dom}(f) = \phi \iff \mathrm{Rng}(f) = \phi$.)

Projections

A function of two variables is a function whose domain is a set of ordered pairs. In general, a function of n variables is a function whose domain is a set of n-tuples.

Example 10 (Projection)

Let A and B be sets and let $\pi_A(x, y) = x$ and $\pi_B(x, y) = y$ for all $(x, y) \in A \times B$. Then $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$. The functions π_A and π_B are called the *projections* from $A \times B$ to A and B respectively.

- For convenience of notation, it is customary to practice a slight abuse of notation such as $\pi_A(x, y)$ instead of $\pi_A((x, y))$ as shown above.

Example : Function of functions

Composition of Functions

Composition of Functions

Definition 11

Let f and g be functions. Then *the composition of g with f* is the function, denoted $g \circ f$, that is defined by

$$(g \circ f)(x) = g(f(x))$$

for all $x \in \mathrm{Dom}(f)$ such that $f(x) \in \mathrm{Dom}(g)$.

- Note that $\mathrm{Dom}(g \circ f) = \{x \in \mathrm{Dom}(f) : f(x) \in \mathrm{Dom}(g)\}$.
- The short way to read $g \circ f$ is "g composed with f."
- Composition of functions is associative (see Theorem 11.37) but not commutative.

$(h \circ g) \circ f = h \circ (g \circ f)$ $\qquad$ $g \circ f \neq f \circ g$

Examples

Example 12

Let $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ and $g(x) = x - 1$. Then

$$(g \circ f)(x) = x^2 - 1 \quad \text{and} \quad (f \circ g)(x) = (x-1)^2,$$

with $\mathrm{Dom}(g \circ f) = \mathrm{Dom}(f \circ g) = \mathbb{R}$.

Examples (cont')

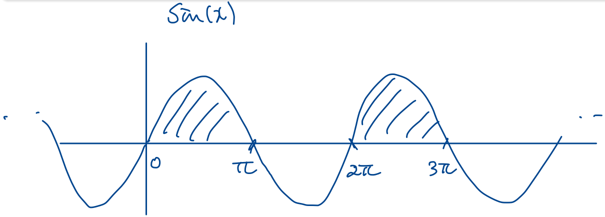
Example 13

Let $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = \sqrt{x}$ and let $g : \mathbb{R} \to [-1, 1]$ defined by $g(x) = \sin(x)$. Then

$$(g \circ f)(x) = \sin(\sqrt{x}) \quad \text{with } \operatorname{Dom}(g \circ f) = [0, \infty),$$

and

$$(f \circ g)(x) = \sqrt{\sin(x)} \quad \text{with } \operatorname{Dom}(f \circ g) = \bigcup \{[2n\pi, (2n+1)\pi] : n \in \mathbb{Z}\}.$$



Exercise

Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -(x-1)^2$ and let $g : [0, \infty) \to [0, \infty)$ defined by $g(x) = \sqrt{x}$. Find $\mathrm{Dom}(g \circ f)$. Justify your answer.