Binomial Coefficients

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Warm-Up

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Rational and Irrational Numbers Revisited

Remark 4.50

Let $d \in \mathbb{N}$, $x, y \in \mathbb{Z}$, and p be a prime number.

- 1 If $p \mid xy$, then $p \mid x$ or $p \mid y$.
- **2** If $d \mid xy$, then these exist $d_1, d_2 \in \mathbb{N}$ such that $d_1 \mid x, d_2 \mid y$, and $d = d_1 d_2$.

The proofs of these facts require complete induction.

3 Let
$$\chi_1, \dots, \chi_n \in \mathbb{Z}$$
.

If
$$d \mid x_1 \cdots x_n$$
, then there exist $d_1, d_2, \cdots, d_n \in \mathbb{N}$
such that $d_1 \mid x_1, d_2 \mid x_2, \cdots, d_n \mid x_n$, and

$$6 = 2.3$$
 $2[2, 3]9$

Rational and Irrational Numbers Revisited (cont')

304 E17 (assigned)

Fact (i)"

Context: We proved \$\int z\$ is invational. How about \$\int C\$ for general \$c?

Example 4.52

- \rightarrow 1 Let x be a rational number such that $x^2 = c$, where c is a whole number. Then x is an integer.
 - **2** Let c be a whole number which is not a perfect square. Then \sqrt{c} is irrational.

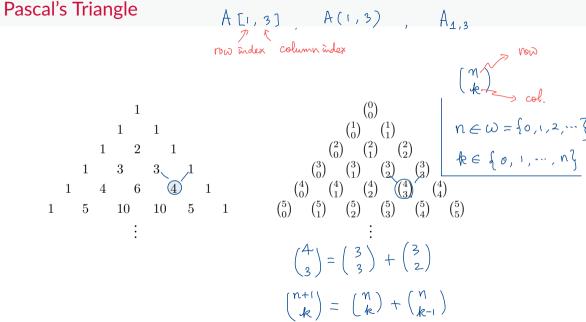
Proof of a. Since it is rational, we can pick two numbers $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that a = a/b and the fraction is in lowest terms. Then $(a/b)^2 = c$, so $a^2/b^2 = c$, so $a^2 = cb^2$, so $a^2 = (cb)b$. Then $b \mid a^2$, because cb is an integer. By Rink, 4.50, we can pick $b_1, b_2 \in \mathbb{N}$ such that $b_1 \mid a$, $b_2 \mid a$, and $b = b_1 b_2$. So $b_1 \mid a$ and $b_1 \mid b$. But then $b_1 = 1$ since a/b is in lowest terms.

Similarly, $b_2=1$. Then if follows that $b=b_1b_2=1\cdot 1=1$.

Then x = a/b = a/1 = a. So it is an integer, because

a is on integer.

Binomial Coefficients



Pascal's Triangle and Binomial Coefficients

Recall. For all $n \in \omega$ and all $k \in \{0, ..., n\}$, the binomial coefficient $\binom{n}{k}$ denotes the k-th number on the n-th row of Pascal's triangle.

Key features

- $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 1.$
- **2** Boundary conditions: For each $n \in \mathbb{N}$,

$$\binom{n}{0} = \binom{n}{n} = 1.$$

3 Recurrence relation: For each $n \in \omega$ and all $k \in \{1, ..., n\}$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Why n choose k?

the only awail. slot

All 2-element subsets of the 4-element set $\{1, 2, 3, 4\}$ are

$$\underbrace{\{1,2\},\{1,3\},\{2,3\}}_{\text{ones without }4},\;\underbrace{\{1,\underline{4}\},\{2,\underline{4}\},\{3,\underline{4}\}}_{\text{ones with }4}.$$

Note that

- the number of subsets without 4 is $3 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
- the number of subsets with 4 is $3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Thus the total number of 2-element subsets of the 4-element set is

$$6 = 3 + 3 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Why n choose k? (cont')

In general, one can count the number of k-element subsets of the (n+1)-element set

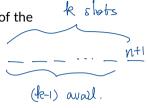
$$\{1,2,\ldots,n,n+1\}$$

in an analogous fashion:

- the number of subsets without n+1 is $\binom{\mathcal{N}}{k}$

Thus the total number of k-element subsets of the (n+1)-element set is

$${\binom{n}{k}} + {\binom{n}{k-1}} = {\binom{n+1}{k}}$$
rec. rel'n



Why *n* choose *k*? (cont')

The idea above is key to a proof by induction of the following theorem.

Number of Subsets (cf. S14E03)

For each $n \in \omega$, for each $k \in \{0, ..., n\}$, the number of k-element subsets of an n-element set is $\binom{n}{k}$.

Why Binomial Coefficients?

Consider the expansion of $(a + b)^2$:

$$(a+b)^{2} = (a+b)(a+b)$$

$$= (a+b)a + (a+b)b$$

$$= a^{2} + ba$$

$$+ ab + b^{2}$$

$$= a^{2} + 2ab + b^{2}. = {2 \choose 0} a^{2}b^{0} + {2 \choose 1} a^{1}b^{1} + {2 \choose 2}a^{0}b^{2}$$

Note that the coefficients of a^2 , ab, and b^2 are

respectively, which are precisely the numbers on row 2 of Pascal's triangle:

$$\binom{2}{0}$$
, $\binom{2}{1}$, $\binom{2}{2}$.

Binomial Expansion

Expansion of $(a + b)^3$

Work out the expansion of $(a+b)^3$ and compare the coefficients with the numbers in row 3 of Pascal's triangle.

$$(a+b)^{3} = (a+b)^{2}(a+b)$$

$$= (a+b)^{2}(a+b)^{2}(a+b)^{2}(b+b)^{2}(b+b)^{2}(a+b)^{2}(b+b)^{2}(a+b)^$$

Binomial Expansion (cont')

Expansion of $(a + b)^4$

Work out the expansion of $(a+b)^3$ and compare the coefficients with the numbers in row 3 of Pascal's triangle.

The Binomial Theorem

The examples above suggest the following general result.

The Binomial Theorem (S05E11)

For each $n \in \omega$ and all $a, b \in \mathbb{R}$,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$
$$= \sum_{k=0}^n \binom{n}{k}a^{n-k}b^k.$$

• Convention: For each $x \in \mathbb{R}$, $x^0 = 1$.