Math 3345 (Section 21138, 11:30 class)

$\boxed{\text{Final Exam}}$ (12:00 ~ 13:45, Friday, April 29)

- Take-at-home as midterms / upload to Gradescope
- Cumulative
- Released at 11:55 am ; Closed at 2:00 pm.

04 / 25 / 2022

Lecture 38 Review for Final Exam

Tautologies & Conditional proof

- Lec 3
- Exam 1 #1

$\boxed{\text{Exercise.}}$ $(P \Rightarrow Q) \Rightarrow \{[P \Rightarrow (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)\}$

Rational and Irrational Numbers

- Lec. 9
- Exam 1 #2

Exercise. Let x and y be irrational. Prove that $x+y$ is irrational or $x-y$ is irrational.

Note: Proof by contradiction

z is irrational means ① z is real
② z is not rational.

Rational Roots

- Sec. 4 Exercises 17, 18, 19, 20

$\boxed{\text{Exam 2 \#2}}$ Let $x \in \mathbb{Q}$ and $x > 0$ such that $rx^2 = s$, where $r, s \in \mathbb{N}$. Prove that $x = a/b$ where $a, b \in \mathbb{N}$ and $a \mid s$ and $b \mid r$.

Proof: Since x is a positive rational number, we can pick $a, b \in \mathbb{N}$ such that $x = a/b$ and the fraction a/b is in lowest terms. On substituting $x = a/b$ into $rx^2 = s$, we obtain

$$r\left(a/b\right)^2 = s, \quad \text{so} \quad ra^2/b^2 = s, \quad \text{so} \quad \underbrace{ra^2 = sb^2}_{(*)}.$$

Now writing $(*)$ as $(ra)(a) = s\cdot b\cdot b$, we see that a divides $s\cdot b\cdot b$ because $ra \in \mathbb{N}$. Thus using the given fact, we can pick $a_1, a_2, a_3 \in \mathbb{N}$ such that $a_1 \mid s$, $a_2 \mid b$, $a_3 \mid b$, and $a = a_1 a_2 a_3$. But $a_2 \mid a$ and $a_2 \mid b$, so a_2 is a common factor of a and b. But since a/b is in lowest terms, the only common factor of a and b is 1. Thus $a_2 = 1$. Similarly, $a_3 = 1$. It follows that $a = a_1 a_2 a_3 = a_1 \cdot 1 \cdot 1 = a_1$, and since $a_1 \mid s$, a divides s.

Now writing (*) as $(sb)b = r \cdot a \cdot a$, ...

$\boxed{\text{DIY}}$ Similar argument w/ appropriate adjustments.

Induction and Complete Induction

- Lecs 13, 14, 16, 19
- Exam 1 # 5, Exam 2 # 3
- Also review problems involving the binomial theorem.

divisibility

recursively defined sequences

(Fibonacci, Pell numbers ...)

Induction vs Insight

- Lec 18
- Exam 2 # 4

Sums of powers

$1+2+3+\cdots+n$

$1^2+2^2+3^2+\cdots+n^2$

$1^3+2^3+3^3+\cdots+n^3$

geometric sums

$1+x+x^2+\cdots+x^{n-1}, \quad x\neq 1$

Exercise Let $n\in\mathbb{N}$, let $x\in\mathbb{R}$, and assume $x\neq 1$.

Derive formulas for

$$S=\sum_{k=0}^{n-1} x^k$$ ← done in Lec 18

$$T=\sum_{k=1}^{n} k x^k$$ ← done in office hour

$$U=\sum_{k=1}^{n} k^2 x^k$$

$1-x\neq 0$

justifies the division by $1-x$.

$$
\begin{aligned}
T &= x + 2x^2 + 3x^3 + \cdots + nx^n \\
-)\quad xT &= \phantom{x + {}} x^2 + 2x^3 + \cdots + (n-1)x^n + nx^{n+1} \\
\hline
(1-x)T &= \underbrace{x + x^2 + x^3 + \cdots + x^n} - nx^{n+1} \\
&= xS = x\frac{1-x^n}{1-x}
\end{aligned}
$$

Since $x \neq 1$, $1-x \neq 0$. So by dividing both sides by $1-x$, we obtain

$$
T = \frac{x\dfrac{1-x^2}{1-x} - nx^{n+1}}{1-x} \quad \cdots
$$

Set operations

- Lec. 22 ~ 28
- Exam 2 #5

(a) Prove two sets are equal, say

$$\star = ☺$$

Proof For any object x,

$x \in \star$

iff $x \in \ldots$

iff $x \in \ldots$

$\vdots$

iff $x \in ☺$.

Therefore, $\star = ☺$. $\square$

(b) Deduce ....

Proof By the previous result,

$\vdots$ (set equalities instead of iff's.)

Exam 2 #5

(a) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$. ✓

(b) Deduce that

$$\boxed{(A \cup B) \setminus B = A \setminus B.}$$

Soln

$$(A \cup B) \setminus B = (A \setminus B) \cup (B \setminus B) \quad \text{(using (a) with } C \text{ by } B\text{)}$$

$$= (A \setminus B) \cup \emptyset \quad \text{(by Fact (c))}$$

$$= A \setminus B \quad \text{(by Fact (a))}$$

Functions

- Review Lec 33 (problem solving session)
- S11 E09 (Determining Rng(f))
- S11 E15 (Set-valued functions)

Infinite sets

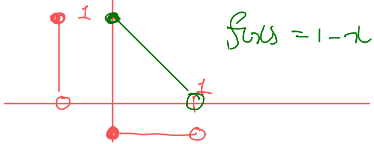
- Equinumerousness
- Cardinality
- Examples of infinite sets
  - A proper subset of an infinite set which is equinumerous to the whole sets.

    (Describe bijections)
- Cantor's Diagonal Lemma

Exercise Describe bijections

- from $\mathbb{Z}$ to $\mathbb{N}$
- from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$
- from $[1, 3]$ to $[-2, 1]$
- from $[0, 1)$ to $(0, 1]$

$$f(x) = \begin{cases} 2x & \text{for } x = 1, 2, 3, \ldots \\ 2(-x)+1 & \text{for } x = 0, -1, -2, \ldots \end{cases}$$



$f(x) = 1 - x$