

Section 14.

Infinite Sets

Recap Equinumerousness

A and B have the same number of elements

- Def'n let A and B be sets. To say that A is equinumerous to B ($A \approx B$) means that there exists a bijection from A to B .

- \approx is an equivalence relation:

- ① $(\forall A)(A \approx A)$
- ② $(\forall A, B)[A \approx B \Rightarrow B \approx A]$
- ③ $(\forall A, B, C)[A \approx B \wedge B \approx C \Rightarrow A \approx C]$

- The rigidity prop. of finite sets A finite set cannot be equinumerous to any of its proper subsets.

Equinumerousness and Infinite Sets

Infinite Sets Are Not Rigid

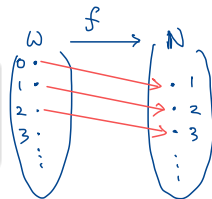
$$\omega = \mathbb{N} \cup \{0\}$$



By the rigidity property, a finite set cannot be equinumerous to any of its proper subsets. But this is not the case with an infinite set.

Example 1 (Natural Numbers and Whole Numbers)

\mathbb{N} is a proper subset of ω , but ω is equinumerous to \mathbb{N} because the function f defined by $f(n) = n + 1$ for all $n \in \omega$ is a bijection from ω to \mathbb{N} .



$$f(n) = n + 1$$

Question. Is f above the only bijection from ω to \mathbb{N} ? If not, construct another one of your own.

Infinite Sets Are Not Rigid (cont')

$$A = B \cup \{1\}$$

Example 2 (Intervals)

Let $A = [1, \infty)$ and $B = (1, \infty)$. B is a proper subset of A , but A is equinumerous to B . Find an example of a bijection f from A to B .

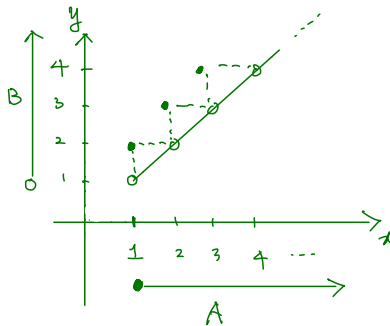
For each $x \in A = [1, \infty)$, define

$$f(x) = \begin{cases} x+1 & \text{if } x \text{ is a natural number} \\ x & \text{otherwise.} \end{cases}$$

This is a bijection from A to B .

Exercise Confirm it.

Illustration



Infinite Sets Are Not Rigid (cont')

$$\begin{pmatrix} \mathbb{N} \\ 1 \cdot \\ 2 \cdot \\ 3 \cdot \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} A \\ \cdot 2 \\ \cdot 4 \\ \cdot 6 \\ \vdots \end{pmatrix}$$

Example 3 (Even and Odd Natural Numbers)

Let $A = \{2, 4, 6, \dots\}$ be the set of even natural numbers and let $B = \{1, 3, 5, \dots\}$ be the set of odd natural numbers. Verify the following by finding suitable bijections.

- ① $A \approx B$. (Note A and B are disjoint.)
- ② $\mathbb{N} \approx A$. (Note A is a proper subset of \mathbb{N} .)
- ③ $\mathbb{N} \approx B$. (Note B is a proper subset of \mathbb{N} .)

① $f: A \rightarrow B$ defined by

$$f(n) = n-1$$

is a bijection.

② $g: \mathbb{N} \rightarrow A$ defined by

$$g(n) = 2n$$

would do.

③ $h: \mathbb{N} \rightarrow B$ defined by

$$h(n) = 2n-1$$

would do.

Note $h(n) = 2n+1$ will not do.

Infinite Sets Are Not Rigid (cont')

Exercise

Show that $\mathbb{Z} \approx \mathbb{N}$.

Hint

$g: \mathbb{Z} \rightarrow \mathbb{N}$ defined by

$$g(n) = \begin{cases} \text{"even"} & \text{if } n = 1, 2, 3, \dots \\ \text{"odd"} & \text{if } n = 0, -1, -2, -3, \dots \end{cases}$$

More Counter-Intuitive Examples

Example 4 (Perfect Squares)

Let $S = \{n^2 : n \in \mathbb{N}\}$ be the set of all perfect squares. Then $S \approx \mathbb{N}$.

$f: \mathbb{N} \rightarrow S$ defined by $f(n) = n^2$ is a bijection.

- Surjection: clear from construction.

- Injection:

Let $n, m \in \mathbb{N}$. If $f(n) = n^2 = m^2 = f(m)$, then $n = m$.

Example 5 (Cartesian Product of \mathbb{N})

$\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$ because the function g defined by

$$g(1, 1) = 1,$$

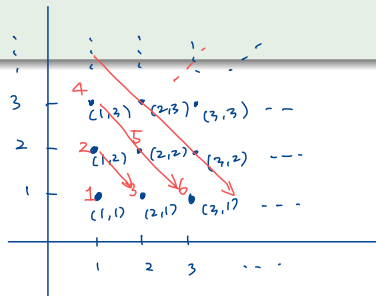
$$g(1, 2) = 2, \quad g(2, 1) = 3,$$

$$g(1, 3) = 4, \quad g(2, 2) = 5, \quad g(3, 1) = 6,$$

$$g(1, 4) = 7, \quad g(2, 3) = 8, \quad g(3, 2) = 9, \quad g(4, 1) = 10,$$

and so on,

is a bijection from $\mathbb{N} \times \mathbb{N}$ to \mathbb{N} .



Example 6 (Positive Rational Numbers)

Let $A = \{x \in \mathbb{Q} : x > 0\}$. Each element $x \in A$ can be expressed uniquely as $x = a/b$ where $a, b \in \mathbb{N}$ and the fraction is in lowest terms. We can list all such fractions in lowest terms as follows:

1					
1					
2	2				
2	1				
3	3				
3	1				
4	5	3	4		
4	2	3	1		
5	6	7	8		
5	1	5	4		
6	7	8	9		
6	2	3	4	5	6
7	8	9	10	11	12
7	1	7	6	5	4
8	9	10	11	12	13
8	3	4	5	6	7
9	10	11	12	13	14
9	1	9	8	7	6
10	11	12	13	14	15
10	2	5	4	3	2
11	12	13	14	15	16
11	1	11	10	9	8
12	13	14	15	16	17
12	3	4	5	6	7
13	14	15	16	17	18
13	1	13	12	11	10
14	15	16	17	18	19
14	2	7	6	5	4
15	16	17	18	19	20
15	1	15	14	13	12
16	17	18	19	20	21
16	3	5	4	3	2
17	18	19	20	21	22
17	1	17	16	15	14
18	19	20	21	22	23
18	2	9	8	7	6
19	20	21	22	23	24
19	1	19	18	17	16
20	21	22	23	24	25
20	3	5	4	3	2
21	22	23	24	25	26
21	1	21	20	19	18
22	23	24	25	26	27
22	2	11	10	9	8
23	24	25	26	27	28
23	1	23	22	21	20
24	25	26	27	28	29
24	3	8	7	6	5
25	26	27	28	29	30
25	1	25	24	23	22
26	27	28	29	30	31
26	2	13	12	11	10
27	28	29	30	31	32
27	1	27	26	25	24
28	29	30	31	32	33
28	3	9	8	7	6
29	30	31	32	33	34
29	1	29	28	27	26
30	31	32	33	34	35
30	2	15	14	13	12
31	32	33	34	35	36
31	1	31	30	29	28
32	33	34	35	36	37
32	3	10	9	8	7
33	34	35	36	37	38
33	1	33	32	31	30
34	35	36	37	38	39
34	2	17	16	15	14
35	36	37	38	39	40
35	1	35	34	33	32
36	37	38	39	40	41
36	3	12	11	10	9
37	38	39	40	41	42
37	1	37	36	35	34
38	39	40	41	42	43
38	2	19	18	17	16
39	40	41	42	43	44
39	1	39	38	37	36
40	41	42	43	44	45
40	3	14	13	12	11
41	42	43	44	45	46
41	1	41	40	39	38
42	43	44	45	46	47
42	2	21	20	19	18
43	44	45	46	47	48
43	1	43	42	41	40
44	45	46	47	48	49
44	3	16	15	14	13
45	46	47	48	49	50
45	1	45	44	43	42
46	47	48	49	50	51
46	2	23	22	21	20
47	48	49	50	51	52
47	1	47	46	45	44
48	49	50	51	52	53
48	3	18	17	16	15
49	50	51	52	53	54
49	1	49	48	47	46
50	51	52	53	54	55
50	2	25	24	23	22
51	52	53	54	55	56
51	1	51	50	49	48
52	53	54	55	56	57
52	3	20	19	18	17
53	54	55	56	57	58
53	1	53	52	51	50
54	55	56	57	58	59
54	2	27	26	25	24
55	56	57	58	59	60
55	1	55	54	53	52
56	57	58	59	60	61
56	3	24	23	22	21
57	58	59	60	61	62
57	1	57	56	55	54
58	59	60	61	62	63
58	2	29	28	27	26
59	60	61	62	63	64
59	1	59	58	57	56
60	61	62	63	64	65
60	3	26	25	24	23
61	62	63	64	65	66
61	1	61	60	59	58
62	63	64	65	66	67
62	2	31	30	29	28
63	64	65	66	67	68
63	1	63	62	61	60
64	65	66	67	68	69
64	3	30	29	28	27
65	66	67	68	69	70
65	1	65	64	63	62
66	67	68	69	70	71
66	2	33	32	31	30
67	68	69	70	71	72
67	1	67	66	65	64
68	69	70	71	72	73
68	3	36	35	34	33
69	70	71	72	73	74
69	1	69	68	67	66
70	71	72	73	74	75
70	2	35	34	33	32
71	72	73	74	75	76
71	1	71	70	69	68
72	73	74	75	76	77
72	3	40	39	38	37
73	74	75	76	77	78
73	1	73	72	71	70
74	75	76	77	78	79
74	2	37	36	35	34
75	76	77	78	79	80
75	1	75	74	73	72
76	77	78	79	80	81
76	3	44	43	42	41
77	78	79	80	81	82
77	1	77	76	75	74
78	79	80	81	82	83
78	2	39	38	37	36
79	80	81	82	83	84
79	1	79	78	77	76
80	81	82	83	84	85
80	3	48	47	46	45
81	82	83	84	85	86
81	1	81	80	79	78
82	83	84	85	86	87
82	2	41	40	39	38
83	84	85	86	87	88
83	1	83	82	81	80
84	85	86	87	88	89
84	3	56	55	54	53
85	86	87	88	89	90
85	1	85	84	83	82
86	87	88	89	90	91
86	2	43	42	41	40
87	88	89	90	91	92
87	1	87	86	85	84
88	89	90	91	92	93
88	3	64	63	62	61
89	90	91	92	93	94
89	1	89	88	87	86
90	91	92	93	94	95
90	2	45	44	43	42
91	92	93	94	95	96
91	1	91	90	89	88
92	93	94	95	96	97
92	3	60	59	58	57
93	94	95	96	97	98
93	1	93	92	91	90
94	95	96	97	98	99
94	2	47			

Exercise

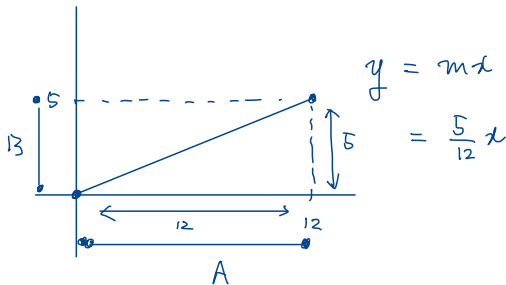
Show that $\mathbb{Q} \approx \mathbb{N}$.

Example 7

Let $A = [0, 12]$ and let $B = [0, 5]$. Then $A \approx B$ because ~~of~~^{if} we let $f(x) = 5x/12$ for all $x \in A$, then f is a bijection from A to B .

Note. In general, for any $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$,

$$[a, b] \approx [c, d], \quad (a, b) \approx (c, d), \quad (a, b] \approx (c, d], \quad \text{and} \quad [a, b) \approx [c, d).$$



Example 8

Let $\varphi(x) = x/(1 - |x|)$ for all $x \in (-1, 1)$. In S11E23, we showed that φ is a bijection from $(-1, 1)$ to \mathbb{R} . Hence $(-1, 1) \approx \mathbb{R}$.

Note. By this example and the note from the previous example, we deduce that $(0, 1) \approx \mathbb{R}$. This fact will be useful later.