Section 6.

Insight vs. Induction

Sums of Powers Revisited

Verification vs. Discovery

Denote by
$$S_r(n)$$
 the sum $1^r+2^r+\cdots+n^r$, e.g.,
$$S_0(n)= \binom{0}{2} + \binom{0}{2} + \binom{0}{2} + \cdots + \binom{0}{2} + \binom{0}{2} + \cdots + \binom{0}{2} + \binom{0}{2}$$

In Section 5, we were given formulas for some of the sums and they were verified by induction. But how would one discover such formulas in the first place?

Derivation of $S_1(n)$ Formula

$$S_1(n) = 1 + 2 + \cdots + n = 1 + 2 + \cdots + n$$

Observe that

$$S_1(n) = 1 + 2 + 3 + \cdots + n$$
+) $S_1(n) = n + (n-1) + (n-2) + \cdots + 1$

$$2S_1(n) = (n+1) + (n+1) + (n+1) + \cdots + (n+1)$$
 \mathcal{U} copies
$$= \mathcal{U}(n+1).$$

Therefore,
$$S_1(n) = \frac{N(n+1)}{2}$$

Would it work with Szem?

$$S_{2}(n) = 1^{2} + 2^{2} + \cdots + n^{2}$$

$$+) S_{2}(n) = n^{2} + (n-1)^{2} + \cdots + 1^{2}$$

$$2 S_{2}(n) = (1+n^{2}) + [2^{2} + (n-1)^{2}] + \cdots + (n^{2} + 1^{2})$$

$$+ 1^{2} + n^{2}$$

The prievous track doesn't work!

Another Derivation of $S_1(n)$ Formula (Telescoping Sums)

Let
$$T(n) = \sum_{k=1}^{n} [k^2 - (k-1)^2]$$
.

On the one hand,

$$T(n) = [x^2 - o^2] + [x^2 - x^2] + [x^2 - x^2] + \dots + [n^2 - (n-1)^2] = n^2$$

On the other hand, since
$$k^2-(k-1)^2=k^2-(k^2-2k+1)=2k-1$$
,

$$T(n) = \sum_{k=1}^{n} (2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

$$S_{1}(n) = \sum_{k=1}^{n} 1$$

$$T(n) = 2S_1(n) - n$$
 $\Rightarrow S_1(n) = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$.

$$S_1(n) = \frac{n^2+1}{2}$$

$$T(m) = \sum_{k=1}^{n} \left[k^2 - (k-1)^2 \right]$$

$$= \sum_{k=1}^{\infty} k^{2} - \sum_{k=1}^{\infty} (k-1)$$

$$= \sum_{k=1}^{m} k^{2} - \sum_{k=1}^{m} (k-1)^{2}$$

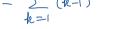
$$= \sum_{k=1}^{\infty} k^{2} - \sum_{k=1}^{\infty} (k-1)^{2}$$

 $= N^2 - D^2 = N^2$

$$= \frac{1}{k-1} + \frac{1}{k-1}$$

$$= \left(\frac{1}{k-1} + \frac{2}{k-1} + \frac{3}{k-1} + \frac{3}{k-1}$$





Derivation of $S_2(n)$ Formula via Telescoping Sum

S06E01

Let $n \in \mathbb{N}$. Let $T(n) = \sum_{k=1}^n [k^3 - (k-1)^3]$. By writing T(n) out in long form, show that it is a telescoping sum and that $T(n) = n^3$. Then, by evaluating T(n) in a different way, deduce without explicit induction that

$$S_2(n) = 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

To handle
$$S_{2}(n)$$
, use $T(n) = \sum_{k=1}^{n} [k^{4} - (k-1)^{4}]$

Geometric Sums Revisited

Sum of Geometric Progression

Let $x \in \mathbb{R}$. Suppose that $x \neq 1$. Let $n \in \mathbb{N}$. In S05E04, you were given the geometric series formula

$$1 + x + x^{2} + \dots + x^{n-1} = \frac{1 - x^{n}}{1 - x},$$

and were asked to verify it using induction. Let's derive it here.

Derivation of Geometric Sum Formula

Let

$$S=1+x+x^2+\cdots+x^{n-1}.$$
 (It is the common ratio.)

Observe that

$$S = 1 + x + x^{2} + \cdots + x^{n-1}$$

$$-) \quad xS = x + x^{2} + \cdots + x^{n-1} + x^{n}$$

$$(1-x)S = 1 + 0 + 0 + \cdots + 0$$

Therefore, Since $\chi \neq 1$, $1-\chi \neq 0$, 80

$$S = \frac{1 - \chi^n}{1 - \chi}$$

Another Derivation Using Summation Notation

In summation notation,

$$S = \sum_{k=0}^{n-1} x^k = 1 + \sum_{k=1}^{n-1} x^k,$$

while

$$xS = \sum_{k=0}^{n-1} x^{k+1} = \sum_{k=1}^{n} x^k = \sum_{k=1}^{n-1} x^k + x^n.$$

Therefore,