Quantifiers (II)

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Generalized De Morgan's Laws and Distributive Laws

Generalized De Morgan's Laws

Recall De Morgan's laws:

- $\neg(P_1 \wedge P_2) \equiv \neg P_1 \vee \neg P_2$
- $\neg(Q_1 \wedge Q_2) \equiv \neg Q_1 \vee \neg Q_2$

Theorem 1 (The Generalized De Morgan's Laws)

Let $P(x)$ and $Q(x)$ be statements about x and let A be a subcollection of the universe of discourse. Then:

1. $\neg(\forall x \in A)P(x) \equiv (\exists x \in A)\neg P(x)$.
2. $\neg(\exists x \in A)Q(x) \equiv (\forall x \in A)\neg Q(x)$.

Generalized De Morgan's Laws (cont')

Proof of 1.

$$
\begin{aligned}
\neg(\forall x \in A)P(x) \text{ is true} \quad & \text{iff} \quad (\forall x \in A)P(x) \text{ is false} \\
& \text{iff} \quad P(x) \text{ is false for at least one value of } x \text{ in } A \\
& \text{iff} \quad \neg P(x) \text{ is true for at least one value of } x \text{ in } A \\
& \text{iff} \quad (\exists x \in A)P(x) \text{ is true.} \qquad \square
\end{aligned}
$$

Examples

For each of the following, write down a sentence that is logically equivalent to the given.

1. $\neg(\forall x \in \mathbb{R})(x^2 - 6x + 12 > 0)$

2. $\neg(\forall x)(\exists y)R(x, y)$

3. $\neg(\exists x)(\forall y)S(x, y)$ (See next page for an example.)

Example: Upper Bound

Let S be a subset of $\mathbb{R}$. To say that S is *bounded above* means that there exists $b \in \mathbb{R}$ such that for each $x \in S$, $x \leq b$. That is,

$$S \text{ is bounded above} \Leftrightarrow (\exists b \in \mathbb{R})(\forall x \in S)(x \leq b).$$

Then to say that S is *not* bounded above means that for each $b \in \mathbb{R}$, there exists $x \in S$ such that $x > b$. That is,

$$S \text{ is not bounded above} \Leftrightarrow (\forall b \in \mathbb{R})(\exists x \in S)(x > b).$$

Generalized Distributive Laws

Recall the distributive laws:

- $P \wedge (Q_1 \vee Q_2) \equiv (P \wedge Q_1) \vee (P \wedge Q_2)$
- $P \vee (Q_1 \wedge Q_2) \equiv (P \vee Q_1) \wedge (P \vee Q_2)$

Theorem 2 (The Generalized Distributive Laws)

Let $Q(x)$ be a statement about x, let P be a sentence that is not a statement about x, and let A be a subcollection of the universe of discourse. Then:

1. $P \wedge (\exists x \in A)Q(x) \equiv (\exists x \in A)[P \wedge Q(x)]$.
2. $P \vee (\forall x \in A)Q(x) \equiv (\forall x \in A)[P \vee Q(x)]$.

Note. P is not a statement about x!

Generalized Distributive Laws (cont')

Proof of 2. Suppose $P \vee (\forall x \in A)Q(x)$ is true. Then P is true or $(\forall x \in A)Q(x)$ is true.

Case 1. Suppose P is true. Consider any $x_0 \in A$. Then $P \vee Q(x_0)$ is true, because P is true. Since $x_0 \in A$ was chosen arbitrarily, it follows that $(\forall x \in A)[P \vee Q(x)]$ is true.

Case 2. Suppose $(\forall x \in A)Q(x)$ is true. Consider any $x_0 \in A$. Then $Q(x_0)$ is true, so $P \vee Q(x_0)$ is true. Since $x_0 \in A$ was chosen arbitrarily, $(\forall x \in A)[P \vee Q(x)]$ is true.

Thus in either case, $(\forall x \in A)[P \vee Q(x)]$ is true.

(Continued on the next page.)

Generalized Distributive Laws (cont')

Conversely, suppose $(\forall x \in A)[P \vee Q(x)]$ is true. Now either P is true or P is false.

Case 1. Suppose P is true. Then $P \vee (\forall x \in A)Q(x)$ is true.

Case 2. Suppose P is false. Consider any $x_0 \in A$. Then $P \vee Q(x_0)$ is true, because $(\forall x \in A)[P \vee Q(x)]$. But P is false, so $Q(x_0)$ must be true. Since $x_0 \in A$ was chosen arbitrarily, it follows that $(\forall x \in A)Q(x)$ is true. Hence $P \vee (\forall x \in A)Q(x)$ is true.

Thus in either case, $P \vee (\forall x \in A)Q(x)$ is true. $\square$

Variations to GDL

Note that

- $P \wedge (Q_1 \wedge Q_2) \equiv (P \wedge Q_1) \wedge (P \wedge Q_2)$
- $P \vee (Q_1 \vee Q_2) \equiv (P \vee Q_1) \vee (P \vee Q_2)$

which can be generalized as follows:

Theorem 3

Let $Q(x)$ be a statement about x, let P be a sentence that is not a statement about x, and let A be a subcollection of the universe of discourse. Then:

1. $P \wedge (\forall x \in A)Q(x) \equiv (\forall x \in A)[P \wedge Q(x)]$.
2. $P \vee (\exists x \in A)Q(x) \equiv (\exists x \in A)[P \vee Q(x)]$.

Order of Quantifiers

Overview

Let $P(x, y)$ be a sentences that depends of x and y.

In a statement involving two identical quantifiers, such as in

$$(\forall x)(\forall y)P(x, y) \quad \text{or} \quad (\exists x)(\exists y)P(x, y),$$

the order of the quantifiers does not matter.

However, the order of quantifiers matters in a statement with *mixed* quantifiers such as

$$(\forall x)(\exists y)P(x, y) \quad \text{or} \quad (\exists x)(\forall y)P(x, y).$$

Order Matters in Mixed Quantifiers

Example. Suppose the universe of discourse is the set of all student in the classroom. Let $P(x, y)$ be the sentence "x and y are friends.". Then

- $(\forall x)(\exists y)P(x, y)$ says that

- $(\exists x)(\forall y)P(x, y)$ says that

Example. Determine the truth value of each of the following.

- $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y = 0)$ is a ________ statement.

- $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x + y = 0)$ is a __________ statement.

Examples: Order Matters in Mixed Quantifiers

Example. Moving quantifiers within a statement can make difference as well.

- $(\forall x \in \mathbb{R})[(\forall y \in \mathbb{R})(y > 0) \Rightarrow x > 0]$ is true.

- $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[y > 0 \Rightarrow x > 0]$ is false.