

More Examples of Induction

Recap Induction

- used to prove sentences of form

$$(\forall n \in \mathbb{N}) P(n)$$

- based on principle of math. induction

$$(\forall P) \left\{ \underbrace{P(1)}_{\text{Base Case}} \wedge \underbrace{(\forall n \in \mathbb{N}) [P(n) \Rightarrow P(n+1)]}_{\text{Inductive Step}} \Rightarrow \underbrace{(\forall n \in \mathbb{N}) P(n)}_{\text{Conclusion}} \right\}$$

- Template

Declaration : Let $P(n)$ be the sentence ...

Base Case : $P(1)$ is true because ...

Inductive step : Let $n \in \mathbb{N}$ such that $P(n)$ is true. ...

Conclusion : Therefore, by induction, for each $n \in \mathbb{N}$, $P(n)$ is true.

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$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \}$$

From \mathbb{N} to \mathbb{Z}

- relate to the positive half.

- induction
- handle this first

Context Want to prove $(\forall x \in \mathbb{Z}) P(x)$.

Issue To use induction, need a "starting point".
But where on \mathbb{Z} ?

Solution Cut in half.

$$\{ \dots, -3, -2, -1 \} , \{ 0, 1, 2, \dots \}$$

From \mathbb{N} to \mathbb{Z}

Example 1

Prove that for each $x \in \mathbb{Z}$, x is even or x is odd.

Proof Let $x \in \mathbb{Z}$. Then $x \leq -1$ or $x \geq 0$.

Let $P(x)$ be the sentence

x is even or x is odd.

PART 1 We wish to show for each integer $x \geq 0$, $P(x)$ is true
using induction.

BASE CASE $P(0)$ is true because 0 is even.

Show $P(0)$ is true.

Notation : $\omega = \{0, 1, 2, 3, \dots\}$ whole numbers

INDUCTIVE STEP Let $x \in \omega$ such that $P(x)$ is true,

show $(\forall x \in \omega)[P(x) \Rightarrow P(x+1)]$

Then x is even or x is odd.

In the case where x is even, $x+1$ is odd.

In the case where x is odd, $x+1$ is even.

Thus in either case, $x+1$ is even or $x+1$ is odd.

Hence $P(x+1)$ is true.

CONCLUSION Therefore, by induction, for each $x \in \omega$,
 $P(x)$ is true.

PART 2 We wish to show for each integer $x \leq -1$, $P(x)$ is true using PART 1.

positive integer,
aka natural number

Let x be a negative integer. Then $-x \in \mathbb{N}$, so $P(-x)$ is true by PART 1, so $-x$ is even or $-x$ is odd.

Case 1 Suppose $-x$ is even. Then $x = \overbrace{(-x)}^{\text{even}} \overbrace{(-1)}^{\text{odd}}$ is even because an even number times any integer is even.

Case 2 Suppose $-x$ is odd. Then $x = \overbrace{(-x)}^{\text{odd}} \overbrace{(-1)}^{\text{odd}}$ is odd because the product of two odd numbers is odd.

Thus in either case, x is even or x is odd. Therefore for each integer $x \leq -1$, $P(x)$ is true.

Therefore, by PART 1 and PART 2, $(\forall x \in \mathbb{Z}) P(x)$ is true.



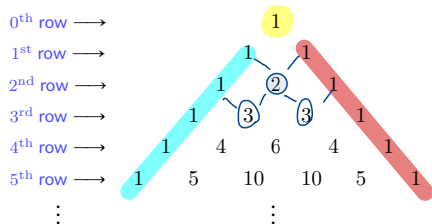
Notes: Induction over \mathbb{Z} .

- Induction may start from a number other than 1.
 - $(\forall x \in \omega) P(x)$
 - $S = \{3, 4, 5, \dots\}$. $(\forall x \in S) P(x)$ can be proved by induction as well.
- To prove a universal sentence $(\forall x \in \mathbb{Z})P(x)$:
 - 1 Prove by induction that $P(x)$ is true for each nonnegative integer¹ x .
 - 2 Prove that $P(x)$ is true for each negative integer x .

Pascal's Triangle and the Binomial Theorem

Pascal's Triangle

The following infinite array of numbers is known as *Pascal's triangle*:



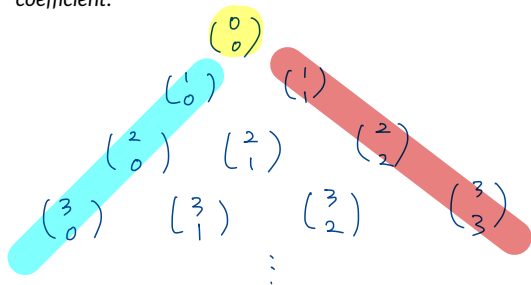
Key Features.

① $\binom{0}{0} = 1.$

② *Boundary conditions:* For each $n \in \mathbb{N}$, $\binom{n}{0} = \binom{n}{n} = 1.$

③ *Recurrence relation:* For each $n \in \omega$ and all $k \in \{1, \dots, n\}$, $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$

Notation. For all $n \in \omega$ and all $k \in \{0, \dots, n\}$, let $\binom{n}{k}$ denote the k -th number on the n -th row; this notation is read n choose k . This is also called a *binomial coefficient*.

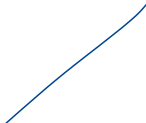


Example

Application of Key Features

Use **Key Features** above to compute $\binom{4}{2}$.

Solution.

$$\begin{aligned}\binom{4}{2} &= \binom{3}{2} + \binom{3}{1} && \text{by } \textcircled{3} \\ &= \binom{2}{2} + \binom{2}{1} + \binom{2}{1} + \binom{2}{0} && \text{by } \textcircled{3} \\ &= 1 + 2\binom{2}{1} + 1 && \text{by } \textcircled{2} \\ &= 1 + 2\left[\binom{1}{1} + \binom{1}{0}\right] + 1 && \text{by } \textcircled{3} \\ &= 1 + 2(1 + 1) + 1 \\ &= 6.\end{aligned}$$


Naming: n choose k

Question: Why is $\binom{n}{k}$ called “ n choose k ”?

Number of Subsets

List all subsets of the set $\{a, b, c, d\}$ with exactly 2 elements.

Solution.

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$$

There are $6 = \binom{4}{2}$ subsets with two elements. In other words, there are

$\binom{4}{2} = 6$ ways to choose 2 elements from a set with 4 elements.

In Section 14, we will prove that for each $n \in \omega$, for each n -element set A , for each $k \in \{0, \dots, n\}$, the number of k -element subsets of A is $\binom{n}{k}$.

Naming: Binomial Coefficients

Question: Why is $\binom{n}{k}$ called a binomial coefficient?

Expansion of $(a + b)^3$

- 1 Compute $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$, and $\binom{3}{3}$.
- 2 Expand the cube of the binomial $a + b$, that is, expand $(a + b)^3$.

Binomial Theorem

The example above suggests:

The Binomial Theorem

For each $n \in \omega$ and all $a, b \in \mathbb{R}$,

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$
$$\sum_{k=0}^n \binom{n}{k} a^{n-k} b^k.$$

- **Convention:** For each $x \in \mathbb{R}$, $x^0 = 1$.