Section 15.

Infinite Sets

Recap Equinumerousness

A and B have the same number of elements.

- **Def'n** Let A and B be sets. To say that A is equinumerous to B $(A \approx B)$ means that there exists a bijection from A to B.

- $\approx$ is an equivalence relation:

  ① $(\forall A)(A \approx A)$

  ② $(\forall A, B)[A \approx B \Rightarrow B \approx A]$

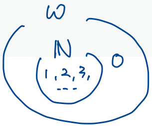
  ③ $(\forall A, B, C)[A \approx B \wedge B \approx C \Rightarrow A \approx C]$

- The rigidity prop. of finite sets: A finite set cannot be equinumerous to any of its proper subsets.

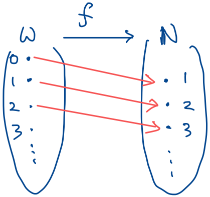
Equinumerousness and Infinite Sets

Infinite Sets Are Not Rigid

$\omega = \mathbb{N} \cup \{0\}$



By the rigidity property, a finite set cannot be equinumerous to any of its proper subsets. But this is not the case with an infinite set.



Example 1 (Natural Numbers and Whole Numbers)

$\mathbb{N}$ is a proper subset of ω, but ω is equinumerous to $\mathbb{N}$ because the function f defined by $f(n) = n + 1$ for all $n \in \omega$ is a bijection from ω to $\mathbb{N}$.

$f(n) = n+1$

Question. Is f above the only bijection from ω to $\mathbb{N}$? If not, construction another one of your own.

Infinite Sets Are Not Rigid (cont')

$A = B \cup \{1\}$

Example 2 (Intervals)

Let $A = [1, \infty)$ and $B = (1, \infty)$. B is a proper subset of A, but A is equinumerous to B. Find an example of a bijection f from A to B.

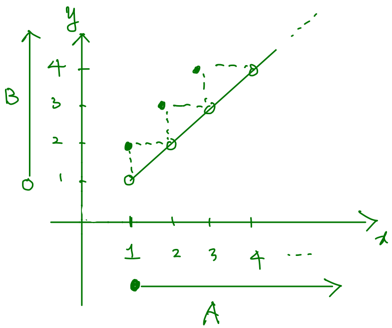
For each $x \in A = [1, \infty)$, define

$$f(x) = \begin{cases} x + 1 & \text{if } x \text{ is a natural number} \\ x & \text{otherwise.} \end{cases}$$

This is a bijection from A to B.

Exercise Confirm it.

Illustration



Infinite Sets Are Not Rigid (cont')

Example 3 (Even and Odd Natural Numbers)

Let $A = \{2, 4, 6, \ldots\}$ be the set of even natural numbers and let $B = \{1, 3, 5, \ldots\}$ be the set of odd natural numbers. Verify the following by finding suitable bijections.

1. $A \approx B$. (Note A and B are disjoint.)
2. $\mathbb{N} \approx A$. (Note A is a proper subset of $\mathbb{N}$.)
3. $\mathbb{N} \approx B$. (Note B is a proper subset of $\mathbb{N}$.)

① $f : A \to B$ defined by

$$f(n) = n - 1$$

is a bijection.

② $g : \mathbb{N} \to A$ defined by

$$g(n) = 2n$$

would do.

③ $h : \mathbb{N} \to B$ defined by

$$h(n) = 2n - 1$$

would do.

Note $h(n) = 2n + 1$ will not do.

Infinite Sets Are Not Rigid (cont')

Exercise

Show that $\mathbb{Z} \approx \mathbb{N}$.

Hint

$g : \mathbb{Z} \to \mathbb{N}$ defined by

$$g(n) = \begin{cases} \text{"even"} & \text{if } n = 1, 2, 3, \dots \\ \text{"odd"} & \text{if } n = 0, -1, -2, -3, \dots \end{cases}$$

More Counter-Intuitive Examples

Example 4 (Perfect Squares)

Let $S = \{n^2 : n \in \mathbb{N}\}$ be the set of all perfect squares. Then $S \approx \mathbb{N}$.

$f : \mathbb{N} \to S$ defined by $f(n) = n^2$ is a bijection.

- Surjection : clear from construction.

- Injection :

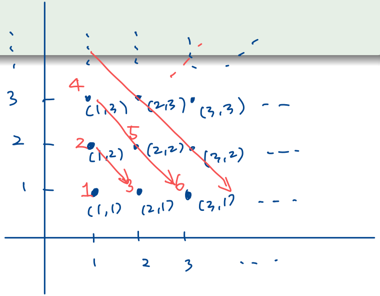
  Let $n, m \in \mathbb{N}$. If $f(n) = n^2 = m^2 = f(m)$, then $n = m$.

Example 5 (Cartesian Product of $\mathbb{N}$)

$\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$ because the function g defined by

$$
\begin{aligned}
&g(1,1) = 1,\\
&g(1,2) = 2, \quad g(2,1) = 3,\\
&g(1,3) = 4, \quad g(2,2) = 5, \quad g(3,1) = 6,\\
&g(1,4) = 7, \quad g(2,3) = 8, \quad g(3,2) = 9, \quad g(4,1) = 10,\\
&\text{and so on,}
\end{aligned}
$$

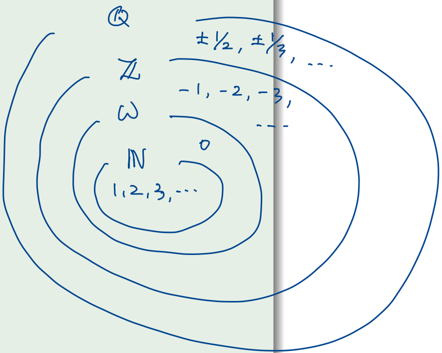
is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.



Example 6 (Positive Rational Numbers)

Let $A = \{x \in \mathbb{Q} : x > 0\}$. Each element $x \in A$ can be expressed uniquely as $x = a/b$ where $a, b \in \mathbb{N}$ and the fraction is in lowest terms. We can list all such fractions in lowest terms as follows:

$$
\begin{array}{llllll}
\frac{1}{1}, & & & & & \\
\frac{1}{2}, & \frac{2}{1}, & & & & \\
\frac{1}{3}, & \frac{3}{1}, & & & & \\
\frac{1}{4}, & \frac{2}{3}, & \frac{3}{2}, & \frac{4}{1}, & & \\
\frac{1}{5}, & \frac{5}{1}, & & & & \\
\frac{1}{6}, & \frac{2}{5}, & \frac{3}{4}, & \frac{4}{3}, & \frac{5}{2}, & \frac{6}{1}, \\
\text{and so on,} & & & & &
\end{array}
$$



Define $f(n)$ to be the n-th term in this list. Then f is a bijection from $\mathbb{N}$ to A.

Exercise

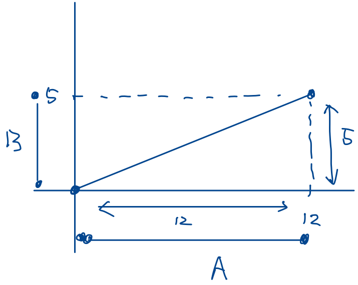
Show that $\mathbb{Q} \approx \mathbb{N}$.

Example 7

Let $A = [0, 12]$ and let $B = [0, 5]$. Then $A \approx B$ because ~~of~~ if we let $f(x) = 5x/12$ for all $x \in A$, then f is a bijection from A to B.

Note. In general, for any $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$,

$$[a, b] \approx [c, d], \quad (a, b) \approx (c, d), \quad (a, b] \approx (c, d], \quad \text{and} \quad [a, b) \approx [c, d).$$



$$y = mx$$
$$= \frac{5}{12} x$$

Example 8

Let $\varphi(x) = x/(1 - |x|)$ for all $x \in (-1, 1)$. In S11E23, we showed that φ is a bijection from $(-1, 1)$ to $\mathbb{R}$. Hence $(-1, 1) \approx \mathbb{R}$.

Note. By this example and the note from the previous example, we deduce that $(0, 1) \approx \mathbb{R}$. This fact will be useful later.