Introduction to Set Theory

Basics of Set Theory

Sets

A *set* is a collection of objects, considered as an object in its own right.

Notation.

- $x \in A$: x is one of the objects in the set A.
  "x is an element of A", "x belong to A", "x is a member of A", or "x is in A."
- $x \notin A$: x is not in the set A.

Ways to Denote Sets

We denote a set by

- listing its elements between braces, *e.g.*,

$$\{2, 3, 5, 7, 11\}$$

- using the *set-builder notation*, *e.g.*,

$$\{x : x \text{ is prime and } x \leqslant 11\}$$

More on Set-Builder Notation

In set-builder notation, we describe a set in terms of *membership criteria*.

- $\{x : P(x)\}$: "the set of all x such that $P(x)$"

$$\{x : x \text{ is a natural number and } x \text{ is even}\}$$

- $\{x \in A : P(x)\} = \{x : x \in A \text{ and } P(x)\}$: "the set of all x in A such that $P(x)$"

$$\{x \in \mathbb{N} : x \text{ is even}\}$$

- $\{f(x) : P(x)\} = \{y : y = f(x) \text{ for some } x \text{ such that } P(x)\}$: "the set of all $f(x)$ such that $P(x)$"

$$\{2x : x \in \mathbb{N}\}$$

Notes

1. Sets having the same elements are equal, *i.e.*,

   $$\text{If for each } x,\ x \in A \text{ iff } x \in B, \text{ then } A = B.$$

   Consequently,

   - The order in which the elements of a set are listed is unimportant.
   - Repetitions in the description of a set do not count.

2. Equal sets have the same elements, *i.e.*,

   $$\text{For all sets } A \text{ and } B, \text{ if } A = B, \text{ then for each } x,\ x \in A \text{ iff } x \in B.$$

3. Equal objects are elements of the same sets, *i.e.*,

   $$\text{For all } x \text{ and } y, \text{ if } x = y, \text{ then for each set } A,\ x \in A \text{ iff } y \in A.$$

Example

S10E01

Which of the sets A, B, C, D, and E below are the same?

$$A = \{3\}, \quad B = \{2, 4\}, \quad C = \{x : x \text{ is prime, } x \text{ is odd, and } x < 5\},$$
$$D = \{x - 1 : x \text{ is prime, } x \text{ is odd, and } x \leqslant 5\}, \quad E = \left\{x^2 + 2 : x \in \{-1, 1\}\right\}.$$

How many different sets are named here?

The Number of Elements

Question

How many elements does $\{a, b\}$ have?

The Number of Elements (cont')

S10E02

How many elements does $\{a, b, c\}$ have?

Sets as Elements of other Sets

Since sets are objects as well, they can be elements of other sets.

Example. Study the elements of each of the following sets.

- $\{1, 2, \{3, 4\}\}$
- $\{\{1, 2, 3, \ldots\}\}$
- $\{\{1\}, \{2\}, \{3\}, \ldots\}$
- $\{\{1, 2, 3, \ldots\}, \{2, 4, 6, \ldots\}, \{3, 6, 9, \ldots\}, \ldots\}$

The Empty Set

The *empty set* is the set that has no elements, usually denoted by $\varnothing$.

- $\{x : P(x)\} = \varnothing$ if there are no values of x for which $P(x)$ is true. For example,
$$\{x : x \text{ is even and } x \text{ is odd}\} = \varnothing.$$
- The empty set is unique.

  Proof. Suppose $\varnothing'$ is another set with no elements. Then for each x, $x \in \varnothing$ and $x \in \varnothing'$ are both false, so $x \in \varnothing \Leftrightarrow x \in \varnothing'$. Hence $\varnothing = \varnothing'$. $\square$
- Tip: To prove that a set A is the empty set, show that for each x, $x \notin A$.

Homework Coaching

Rational Roots (Cubic Polynomials)

S04E17 (Collected)

1. Let x be a rational number such that $x^3 = c$, where c is an integer. Prove that x is an integer.
2. Let c be an integer which is not a perfect cube. Prove that $\sqrt[3]{c}$ is irrational.

S04E18 (Collected)

Let $x \in \mathbb{R}$ such that $x^3 = rx^2 + sx + t$, where $r, s, t \in \mathbb{Z}$.

1. Prove that if x is rational, then x is an integer.
2. Prove that if x is not an integer, then x is irrational.

Rational Roots (General Polynomials)

S04E19 (To be turned in)

Let x be a real number such that

$$x^n + c_n x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where $n \in \mathbb{N}$ and $c_0, c_1, \ldots, c_{n-1} \in \mathbb{Z}$.

1. Prove that if x is rational, then x is an integer.
2. Prove that if x is not an integer, then x is irrational.

Binomial Theorem and Applications

S05E11 (The Binomial Theorem; collected)

Let $a, b \in \mathbb{R}$. Then for each $n \in \boldsymbol{\omega}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Convention. $x^0 = 1$ for each $x \in \mathbb{R}$.

S05E12 (Collected)

Let $n \in \boldsymbol{\omega}$. Show that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

(Do not use induction.)

S05E13 (To be turned in)

Let $n \in \mathbb{N}$. Show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

(Do not use induction.)