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Selected Solutions to Exercise Problems

Section 2: Propositional Calculus.

S02E15. Use the method of conditional proof to explain in words why the sentence

$$\{(P \vee Q) \wedge [(P \Rightarrow R) \wedge (Q \Rightarrow S)]\} \Rightarrow (R \vee S)$$

is a tautology. Be explicit about discharging assumptions.

Proof.

A1: Suppose $(P \vee Q) \wedge [(P \Rightarrow R) \wedge (Q \Rightarrow S)]$ is true.
(We wish to show that $R \vee S$ is true.)
Then both of $P \vee Q$ and $(P \Rightarrow R) \wedge (Q \Rightarrow S)$ are true.
Since $P \vee Q$ is true, at least one of P and Q is true.

Case 1. Suppose P is true.
Since $(P \Rightarrow R) \wedge (Q \Rightarrow S)$ is true, $P \Rightarrow R$ is true.
Thus $P \Rightarrow R$ is true and P is true.
Hence, by modus ponens, R is true.

Case 2. Suppose Q is true.
Since $(P \Rightarrow R) \wedge (Q \Rightarrow S)$ is true, $Q \Rightarrow S$ is true.
Thus $Q \Rightarrow S$ is true and Q is true.
Hence, by modus ponens, S is true.

Thus in either case, R is true or S is true.
We have shown that $R \vee S$ is true under the assumption A1 that $(P \vee Q) \wedge [(P \Rightarrow R) \wedge (Q \Rightarrow S)]$ is true.

Discharging A1, we see that $\{(P \vee Q) \wedge [(P \Rightarrow R) \wedge (Q \Rightarrow S)]\} \Rightarrow (R \vee S)$ is true under no assumptions, so it is a tautology. ☐

S02E17. Use the method of conditional proof to explain in words why the sentence

$$(P \Rightarrow Q) \Rightarrow \{[P \Rightarrow (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)\}$$

is a tautology. Be explicit about discharging assumptions.

Proof.

A1: Suppose $A_1 : P \Rightarrow Q$ is true. (We wish to show that $C_1 : [P \Rightarrow (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$ is true.)

A2: Suppose $A_2 : P \Rightarrow (Q \Rightarrow R)$ is true. (We wish to show that $C_2 : P \Rightarrow R$ is true.)

A3: Suppose $A_3 : P$ is true. (We wish to show that $C_3 : R$ is true.)
From A1 and A3, we see that Q is true, by modus ponens.
From A2 and A3, we see that $Q \Rightarrow R$ is true, by modus ponens.
From this and the fact that Q is true, we see that R is true, by modus ponens.
We have shown that C_3 is true under A1, A2, and A3.

Discharging A3, we see that $C_2 : A_3 \Rightarrow C_3$ is true under A1 and A2.

Discharging A2, we see that $C_1 : A_2 \Rightarrow C_2$ is true under A1 alone.

Finally, discharging A1, we see that $A_1 \Rightarrow C_1$ is true under no assumptions, so it is a tautology. ☐

Section 3: Quantifiers.

S03E07. Let P be the sentence

$$(\exists x \in \mathbb{R})(x \geqslant 0 \text{ and } \sqrt{x+2} < \sqrt{x} + \sqrt{2}).$$

(a) Use one of the generalized De Morgan's laws to show that $\neg P$ is logically equivalent to

$$(\forall x \in \mathbb{R})(x < 0 \text{ or } \sqrt{x+2} \geqslant \sqrt{x} + \sqrt{2}).$$

Proof.

$$\begin{aligned}
& \neg(\exists x \in \mathbb{R})(x \geqslant 0 \text{ and } \sqrt{x+2} < \sqrt{x} + \sqrt{2}) & \\
\text{iff}\quad & (\forall x \in \mathbb{R})\neg(x \geqslant 0 \text{ and } \sqrt{x+2} < \sqrt{x} + \sqrt{2}) & \text{(by a generalized De Morgan's law)} \\
\text{iff}\quad & (\forall x \in \mathbb{R})(x < 0 \text{ or } \sqrt{x+2} \geqslant \sqrt{x} + \sqrt{2}) & \text{(by a De Morgan's law)}
\end{aligned}$$

□

(b) The sentence $P : (\exists x \in \mathbb{R})(x \geqslant 0 \text{ and } \sqrt{x+2} < \sqrt{x} + \sqrt{2})$ is true because $2 \geqslant 0$ and $\sqrt{2+2} = \sqrt{4} = 2 < \sqrt{2} + \sqrt{2}$. □

S03E10. For each of the following sentences, write out what it means in words, state whether it is true or false, and prove your statement.

(a) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x + y = x)$ means "There exists a real number y such that for each real number x, $x + y = x$." We claim that this sentence is true.

Proof. It suffices to exhibit a value of y such that the universal sentence $(\forall x \in \mathbb{R})(x + y = x)$ is true. We claim that 0 is such a value of y. To see this, let x_0 be any real number. Then $\mathrm{x}_0 + 0 = \mathrm{x}_0$. Now x_0 is an arbitrary element of $\mathbb{R}$. Hence $(\forall x \in \mathbb{R})(x + y = x)$ is true. This proves the claim. Therefore $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x + y = x)$ is true. □

(b) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y = x)$ means "For each real number x, there exists a real number y such that, $x + y = x$." We claim that this sentence is true.

Proof. Let x_0 be any real number. Then $x_0 + 0 = x_0$. Hence $(\exists y \in \mathbb{R})(x_0 + y = x_0)$ is true, because 0 is such a value of y. Now x_0 is an arbitrary element of $\mathbb{R}$. Therefore $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y = x)$ is true. □

(e) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(xy = 1)$ means "There exists a real number y such that for each real number x, $xy = 1$." We claim that this sentence is false.

Proof. Suppose it is true. Then we can pick $y_0 \in \mathbb{R}$ such that $(\forall x \in \mathbb{R})(xy_0 = 1)$. But then in particular, $0 \cdot y_0 = 1$, so $0 = 1$. But $0 \neq 1$. This is a contradiction. Hence $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(xy = 1)$ must be false. □

(f) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 1)$ means "For each real number x, there exists a real number y such that $xy = 1$." We claim that this sentence is false.

Proof. Suppose it is true. Then in particular, since 0 is a real number, $(\exists y \in \mathbb{R})(0 \cdot y = 1)$ is true, so we can pick $y_0 \in \mathbb{R}$ such that $0 \cdot y_0 = 1$, so $0 = 1$. But $0 \neq 1$. This is a contradiction. Hence $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 1)$ is false. □

S03E11. Let S be a subset of $\mathbb{R}$.

(a) Let S be the set of all real numbers. We claim that this S is not bounded above. By Example 3.15, S *is not bounded above* if and only if $(\forall b \in \mathbb{R})(\exists x \in S)(x > b)$, which we use in the proof below.

Proof. Let $b_0 \in \mathbb{R}$ be arbitrary. Since $b_0 + 1 \in \mathbb{R}$ and $b_0 + 1 > b_0$, it is an example value of x for which $(\exists x \in S)(x > b)$. Since b_0 is arbitrary, it follows that $(\forall b \in \mathbb{R})(\exists x \in S)(x > b)$. Therefore S is not bounded above. $\square$

(b) Let S be the set of all numbers x such that some person on earth has x hairs on his or her head. We claim that this S is bounded above. Recall that S *is bounded above* if and only if $(\exists b \in \mathbb{R})(\forall x \in S)(x \leqslant b)$, which we use in the proof below.

Proof. S is a finite set because there are finitely many people on earth. So S has a maximal element; call it m. Then, for any $x_0 \in S$, $x_0 \leqslant m$. Thus $(\forall x \in S)(x \leqslant m)$ because x_0 is arbitrary. Hence $(\exists b \in \mathbb{R})(\forall x \in S)(x \leqslant b)$ because m is an example value of b. Therefore S is bounded above. $\square$

S03E14. For each of the following sentences, write out what it means in words, state whether it is true or false, and prove your statement.

(c) $(\exists! x \in \mathbb{Z})(x^2 - 4x + 3 < 0)$ means "There exists a unique integer x such that $x^2 - 4x + 3$ is less than 0." We claim that this sentence is true.

Proof. 2 is an integer and $2^2 - 4 \cdot 2 + 3 = -1 < 0$. Now suppose x is another integer such that $x^2 - 4x + 3 < 0$. (We wish to show that $x = 2$.) By completing the square, $x^2 - 4x + 3 = (x-2)^2 - 1 \geqslant -1$. Thus $-1 \leqslant (x-2)^2 - 1 < 0$. Since x is an integer, $(x-2)^2 - 1$ is an integer, thus it must be the case that $(x-2)^2 - 1 = -1$. It follows that $(x-2)^2 = 0$, so $x - 2 = 0$, so $x = 2$. $\square$

(e) $(\exists! x \in \mathbb{R})(x^2 - 4x + 5 = 0)$ means "There exists a unique real number x such that $x^2 - 4x + 5$ is 0." We claim that this sentence is false.

Proof. To disprove $(\exists! x \in \mathbb{R})(x^2 - 4x + 5 = 0)$, we will show that $(\exists x \in \mathbb{R})(x^2 - 4x + 5 = 0)$ is false. Assume $(\exists x \in \mathbb{R})(x^2 - 4x + 5 = 0)$ is true. Then in particular, we can pick a real number x_0 such that $x_0^2 - 4x_0 + 5 = 0$. But $x_0^2 - 4x_0 + 5 = (x_0 - 2)^2 + 1 \geqslant 1$. So $x_0^2 - 4x_0 + 5 \neq 0$. This is a contradiction. So $(\exists x \in \mathbb{R})(x^2 - 4x + 5 = 0)$ is false, and it follows that $(\exists! x \in \mathbb{R})(x^2 - 4x + 5 = 0)$ is false. $\square$

(i) $(\forall x \in \mathbb{R})(\exists! y \in \mathbb{R})(xy = 0)$ means "For each real number x, there exists a unique real number y such that xy is 0." We claim that this sentence is false.

Proof. Suppose $(\forall x \in \mathbb{R})(\exists! y \in \mathbb{R})(xy = 0)$ is true. Then in particular, since 0 is a real number, $(\exists! y \in \mathbb{R})(0 \cdot y = 0)$ is true. But 1 and -1 are two different real values of y for which $0 \cdot y = 0$. So $(\exists! y \in \mathbb{R})(0 \cdot y = 0)$ is false. This is a contradiction. Therefore $(\forall x \in \mathbb{R})(\exists! y \in \mathbb{R})(xy = 0)$ is false. $\square$

(j) $(\forall x \in \mathbb{R})[\text{ if } x \neq 0, \text{ then } (\exists! y \in \mathbb{R})(xy = 0)]$ means "For each real number x, if x is nonzero, then there exists a unique real number y such that xy is 0." We claim that this sentence is true.

Proof. Let $x \in \mathbb{R}$ be arbitrary. Assume that $x \neq 0$. (Here we are proceeding by way of conditional proof. We wish to show that $(\exists! y \in \mathbb{R})(xy = 0)$ is true.) Note that 0 is a real number, $x \cdot 0 = 0$, and if y is another real number such that $xy = 0$, then $y = 0$ because $x \neq 0$. This shows that $(\exists! y \in \mathbb{R})(xy = 0)$ is true. Since x is arbitrary, we conclude that $(\forall x \in \mathbb{R})[\text{ if } x \neq 0, \text{ then } (\exists! y \in \mathbb{R})(xy = 0)]$ is true. $\square$

Section 4: First Examples of Mathematical Proofs.

S04E03. Let x be an integer. Prove that $x(x+1)$ is even.

Proof. Since x is an integer, x is even or x is odd.

Case 1. Suppose x is even. Then we can pick an integer k such that $x = 2k$. Then $x(x+1) = 2k(2k+1) = 2[k(2k+1)]$. Since $k(2k+1)$ is an integer, it follows that $x(x+1)$ is even.

Case 2. Suppose x is odd. Then we can pick an integer k such that $x = 2k$. Then $x(x+1) = (2k+1)((2k+1)+1) = (2k+1)(2k+2) = 2[(2k+1)(k+1)]$. Since $(2k+1)(k+1)$ is an integer, it follows that $x(x+1)$ is even.

Thus in either case, $x(x+1)$ is even. □

S04E04.

(a) The sentence "For each real number x, if x is an even number, then x is not an odd number." is true.

Proof. Let $x \in \mathbb{R}$ be arbitrary. Suppose that x is even. We wish to show that x is not odd. Suppose x is odd. Then x is both even and odd. But, by (a) of Remark 4.12, x is not both even and odd. Thus we have reached a contradiction. Thus it must be that x is not odd. Since x is arbitrary, it follows that for each $x \in \mathbb{R}$, if x is even, then x is not odd. □

(b) The sentence "For each real number x, if x is not an odd number, then x is an even number." is false.

Proof. Suppose it is true. Then in particular, $1/2$ is a real number and $1/2$ is not odd, so $1/2$ is even. Then we can find $k \in \mathbb{Z}$ such that $1/2 = 2k$, so $1 = 2(2k)$. Thus 1 is even. But $1 = 2 \cdot 0 + 1$, so 1 is odd. So 1 is both even and odd. But since 1 is an integer, by (a) of Remark 4.12, 1 is not both even and odd. This is a contradiction. Therefore, the sentence must be false. □

In the proof above, we used the number $1/2$ is not odd without proving it. Though obvious, let's prove it here.

Claim 1. *The number* $1/2$ *is not an odd number.*

Proof. Suppose $1/2$ is odd. Then we can pick $k \in \mathbb{Z}$ such that $1/2 = 2k+1$, so $1 = 2(2k+1)$. Thus 1 is even. But $1 = 2 \cdot 0 + 1$, so 1 is odd. Since 1 is an integer, by part (c) of Remark 4.12, 1 is not even. This is a contradiction. So $1/2$ is not even. □

S04E08. Let u, v, and w be rational numbers.

(a) $-v$ is a rational number.

Proof. Since v is a rational number, we can pick $a, b \in \mathbb{Z}$ such that $b \neq 0$ and $v = a/b$. Then $-v = -(a/b) = (-a)/b$. Since $-a$ is an integer and b is an integer that is not zero, $-v$ is a rational number. □

(b) $u - v$ is a rational number.

Proof. (Using definition) Since u and v are rational numbers, we can pick $a, b, c, d \in \mathbb{Z}$ such that $b, d \neq 0$ and $u = a/b$ and $v = c/d$. Then

$$u - v = \frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}.$$

Since $ad - bc$ is an integer and bd is an integer that is not zero as a product of two nonzero integers, $u - v$ is a rational number. ☐

Proof. (Using other results) By part (a), since v is a rational number, $-v$ is a rational number. By Example 4.21, since u and $-v$ are both rational numbers, $u + (-v) = u - v$ is a rational number. ☐

(d) If $w \neq 0$, then $1/w$ is a rational number.

Proof. Let $w \neq 0$ be a rational number. Then we can pick $a, b \in \mathbb{Z}$ such that $a, b \neq 0$ and $w = a/b$. (Note that $a \neq 0$ because $w \neq 0$.) Then $1/w = 1/(a/b) = b/a$. Since $a, b \in \mathbb{Z}$ and $a \neq 0$, it follows that $1/w$ is a rational number. ☐

S04E10. Let x be a rational number and let y be an irrational number.

(a) $-y$ is irrational.

Proof. Since y is irrational, y is real and y is not rational. Since y is real, $-y$ is also real. It remains to show that $-y$ is not rational. Suppoe that $-y$ is rational. Then by Exercise 8(a), $-(-y) = y$ is rational. So y is not rational and y is rational. This is a contradiction. Thus $-y$ is not rational. Hence $-y$ is irrational. ☐

(b) $x - y$ is irrational.

Proof. Since x is rational and y is irrational, both x and y are real, so $x - y$ is real. It remains to show that $x - y$ is not rational. Suppose that $x - y$ is rational. Then $x - (x - y) = y$ is rational, because the difference of two rational numbers is rational; see Exercise 8(b). But y is not rational because y is irrational. This is a contradiction. Thus $x - y$ is not rational. Therefore $x - y$ is irrational. ☐

(d) If $x \neq 0$, then xy is irrational.

Proof. Assume that $x \neq 0$. (We wish to show that xy is irrational.) Since x is rational and y is irrational, both x and y are real, so xy is real. It remains to show that xy is not rational. Suppose xy is rational. Then by Exercise 8(e), $(xy)/x = y$ is rational. (Note that Exercise 8(b) is applicable since both x and xy are rational and $x \neq 0$.) But y is not rational because y is irrational. This is a contradiction. Thus xy is not rational. Therefore xy is irrational. ☐

S04E12. For each $x \in \mathbb{R}$, $\pi + x$ is irrational or $\pi - x$ is irrational.

Proof. Let $x \in \mathbb{R}$. Assume, by way of contradiction, that $\pi + x$ is rational and $\pi - x$ is rational. Since the sum of two rational numbers is a rational number, $(\pi + x) + (\pi - x) = 2\pi$ is a rational number. Since the quotient of rational numbers (with nonzero denominator) is a rational number, $(2\pi)/2 = \pi$ is a rational number. But π is an irrational number. This is a contradiction. Hence $\pi + x$ is irrational or $\pi - x$ is irrational. ☐

S04E14. Let $a, b, c \in \mathbb{Z}$.

(b) If a divides b and b divides a, then $b = a$ or $b = -a$.

Proof. Since a divides b, we can pick $k \in \mathbb{Z}$ such that $b = ka$. Since b divides a, we can pick $\ell \in \mathbb{Z}$ such that $a = \ell b$. On substitution, $b = k(\ell b) = (k\ell)b$, so $b - (k\ell)b = b(1 - k\ell) = 0$, so $b = 0$ or $k\ell = 1$.

Case 1. Suppose $b = 0$. Then $a = \ell b = \ell \cdot 0 = 0$, so $b = a$.

Case 2. Suppose $k\ell = 1$. Then $k = \ell = 1$ or $k = \ell = -1$, because $k, \ell \in \mathbb{Z}$. In particular, $k = 1$ or $k = -1$. Since $b = ka$, it follows that $b = a$ or $b = -a$.

Thus in either case, $b = a$ or $b = -a$. $\square$

(c) If a divides b and b divides c, then a divides c.

Proof. Since a divides b and b divides c, we can pick $k, \ell \in \mathbb{Z}$ such that $b = ka$ and $c = \ell b$. But then $c = \ell b = \ell(ka) = (\ell k)a$. Since $\ell k \in \mathbb{Z}$, it follows that a divides c. $\square$

S04E16. Let $n \in \mathbb{N}$. Prove that there exists a prime number q such that $n < q \leqslant 1 + n!$.

Proof. Let $x = 1 + n!$. We claim that none of $2, 3, \ldots, n$ divides x. By way of contradiction, assume that one of $2, 3, \ldots, n$ divides x; call it k. But then k divides $x - 1 = n!$ because $n!$ is the product of $1, 2, \ldots, n$, one of which is k. Thus k divides x and k divides $x - 1$, so k divides $x - (x - 1) = 1$. This is a contradiction because $k \geqslant 2$ because k is one of $2, 3, \ldots, n$ and $k \leqslant 1$ because k divides 1. Hence none of $2, 3, \ldots, n$ divides x. Now $x \in \mathbb{N}$ and $x \neq 1$, so there must exists a prime number q such that q divides x. But since none of $2, 3, \ldots n$ divides x, q is not one of them, and so $q > n$. Since q divides x and q is a prime number, it must be the case that $q \leqslant x$. Therefore $n < q \leqslant x = 1 + n!$. $\square$