Set Operations

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Overview Set theory

- Introduction
  - A set is a collection of objects.
  - $x \in A$, $x \notin A$
  - $A = B$ means $(\forall x)(x \in A \Leftrightarrow x \in B)$
  - Empty set ($\emptyset$)

- Subsets
  - $A \subseteq B$ means

    $(\forall x)(x \in A \Rightarrow x \in B)$
  - For each set A,

    $\emptyset \subseteq A$.

Unions, Intersections, and Relative Complements

Definition 1 (Set Operations)

Let A and B be sets.

- The *union of A and B* (denoted $A \cup B$) is the set of all things that belong to at least one of the sets A and B; in other words,

$$A \cup B = \{x : x \in A \boxed{\text{or}}\ x \in B\}.$$

- The *intersection of A and B* (denoted $A \cap B$) is the set of all things that belong to both of the sets A and B; in other words,

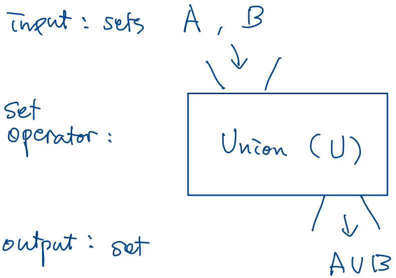
$$A \cap B = \{x : x \in A \boxed{\text{and}}\ x \in B\}.$$

- The *relative complement of B in A* (denoted $A \backslash B$) is the set of all things that belong to A but not to B; in other words,

$$A \backslash B = \{x : x \in A \text{ and } x \notin B\}.$$

Notes on Set Operations

- Short ways to read $A \cup B$, $A \cap B$, and $A \backslash B$ are “A union B,” “A intersect B,” and “A less B” respectively.
- $A \cup B$ should not be read “A or B.” $A \cap B$ should not be read “A and B.” We use the connectives “and” and “or” to connect sentences, not nouns.
- The results of set operations are another sets, so they are nouns. Hence, one must not write something like “$\cancel{A \cup B \text{ iff } x \in A \text{ or } x \in B}$.” Instead, write “$x \in A \cup B$ iff $x \in A$ or $x \in B$.”



Set Inclusion and Set Operations

Example 2

Let A and B be sets. Then:

1. $A \subseteq A \cup B$ and $B \subseteq A \cup B$.
2. $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

Recall $A \subseteq B$ means $(\forall x)\,(x \in A \Rightarrow x \in B)$.

Proof of ① We will show $A \subseteq A \cup B$; $B \subseteq A \cup B$ is shown similarly.

Let $x \in A$. (WTS: $x \in A \cup B$) Then $x \in A$ or $x \in B$.

Let x be arbitrary. Assume $x \in A$.

T by assumption

Thus $x \in A \cup B$. This shows that $A \subseteq A \cup B$. $\square$

$$(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))$$

Let $n \in \mathbb{N}$. Assume $P(n)$ is true.

Let $n \in \mathbb{N}$ such that $P(n)$ is true.

Example 3

Let A, B, and C be sets. Then:

1. If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.
2. If $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

Proof of ① Suppose $A \subseteq C$ and $B \subseteq C$. (WTS: $A \cup B \subseteq C$)

Let $x \in A \cup B$. Then $x \in A$ or $x \in B$.

In the case where $x \in A$, we know $x \in C$, because $A \subseteq C$.

In the case where $x \in B$, we know $x \in C$, because $B \subseteq C$.

Thus in either case, $x \in C$. Hence $A \cup B \subseteq C$. $\square$

Set Inclusion and Set Operations (cont')

Example 4 (Equivalence to Set Inclusion)

Let A and B be sets. Then:

1. $A \subseteq B$ iff $A \cup B = B$.
2. $A \subseteq B$ iff $A \cap B = A$.
3. $A \subseteq B$ iff $A \setminus B = \varnothing$.

✗ $\underline{A \cup B} \subseteq \underline{B}$

✗ $B \subseteq A \cup B$

Proof of ①

($\Rightarrow$) Suppose $A \subseteq B$. (WTS: $A \cup B = B$). Note that $B \subseteq A \cup B$ by Ex 2①. Now $A \subseteq B$ by assumption and $B \subseteq B$ by reflexivity, so $A \cup B \subseteq B$ by Ex. 3①. Thus $A \cup B = B$.

($\Leftarrow$) Conversely, suppose $A \cup B = B$. (WTS: $A \subseteq B$). Note that $A \subseteq A \cup B$ by Ex. 2①. But $A \cup B = B$ by assumption. Thus $A \subseteq B$. $\square$