Intervals, Sets of Sets, and Power Set

Intervals

Intervals

An *interval in* $\mathbb{R}$ is a subset of $\mathbb{R}$ that contains all the points between any two of its points.

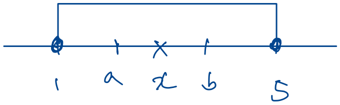
Definition 1 (Interval)

To say that I is an interval of $\mathbb{R}$ means that $I \subseteq \mathbb{R}$ and for each $u, v \in I$, for each $x \in \mathbb{R}$, if $u < x < v$, then $x \in I$.

The set $\{1, 2, 3\}$ is not an interval in $\mathbb{R}$ because

1.5 lies between 1 and 2

but it is not in the set $\{1, 2, 3\}$.



Bounded Intervals

Bounded Intervals

Let $a, b \in \mathbb{R}$ such that $a \leqslant b$. Then the following sets are intervals in $\mathbb{R}$.

$$
\begin{aligned}
[a, b] &= \{x \in \mathbb{R} : a \leqslant x \leqslant b\} && \text{(closed)} \\
(a, b) &= \{x \in \mathbb{R} : a < x < b\} && \text{(open)} \\
[a, b) &= \{x \in \mathbb{R} : a \leqslant x < b\} && \text{(left-closed or right-open)} \\
(a, b] &= \{x \in \mathbb{R} : a < x \leqslant b\} && \text{(left-open or right-closed)}
\end{aligned}
$$

These are called *bounded intervals*. If $a = b$, these bounded intervals yield

$$
\begin{aligned}
[a, b] &= \{a\} \\
(a, b) &= [a, b) = (a, b] = \varnothing
\end{aligned}
$$

These are called *degenerate intervals*.

Unbounded Intervals

Let $c \in \mathbb{R}$. Then the following sets are intervals in $\mathbb{R}$.

$$[c, \infty) = \{x \in \mathbb{R} : c \leqslant x\} \qquad \text{(closed half-line)}$$

$$(c, \infty) = \{x \in \mathbb{R} : c < x\} \qquad \text{(open half-line)}$$

$$(-\infty, c] = \{x \in \mathbb{R} : x \underline{<} c\} \qquad \text{(closed half-line)}$$

$$(-\infty, c) = \{x \in \mathbb{R} : x \underset{<}{\cancel{\leqslant}} c\} \qquad \text{(open half-line)}$$

The whole real line

$$(-\infty, \infty) = \mathbb{R}$$

is also an interval in $\mathbb{R}$. Half-lines and the whole real line are *unbounded intervals*.

Examples

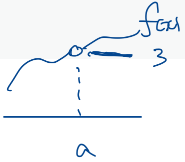
Question. Determine whether each of the following is an interval.

1. $(-1, 1) \cup [0, 4]$ $= (-1, 4]$
2. $[-1, 1] \cap (0, 2)$ $= (0, 1]$
3. $[1, 2] \cup [3, 4]$ not an interval
4. $[2, 5) \cap [5, 6]$ $= \emptyset$ (it is an interval)
5. $(3, 5) \setminus [4, 7]$
6. $(4, 8] \setminus [5, 6)$

Unions and Intersections of Sets of Sets

Unions of Sets of Sets

$\mathcal{A}$ $\mathcal{B}$ $\mathcal{C}$ $\mathcal{D}$ ...



Definition 2

Let $\mathcal{A}$ be a set of sets. Then *the union of $\mathcal{A}$* (denoted $\bigcup \mathcal{A}$) is the set of all things that belong to at least one of the sets in $\mathcal{A}$; in other words,

$$\bigcup \mathcal{A} = \{x : x \in A \text{ for some } A \in \mathcal{A}\}.$$

set

$\lim_{x \to a} f(x) = 3$

$f(a)$ DNE

Example. Let A, B, and C be sets. Then

- $\bigcup\{\ \} = \bigcup \varnothing = \varnothing$
- $\bigcup\{A\} = A$
- $\bigcup\{A, B\} = A \cup B$
- $\bigcup\{A, B, C\} = A \cup B \cup C$

Example. Let $\mathcal{A} = \{[1/n, 5] : n \in \mathbb{N}\}$. Then

$$\bigcup \mathcal{A} = (0, 5].$$

$\mathcal{A} = \{[1, 5], [1/2, 5], [1/3, 5], \cdots\}$

Intersections of Sets of Sets

Definition 3

Let $\mathcal{A}$ be a nonempty set of sets. Then *the intersection of $\mathcal{A}$* (denoted $\bigcap \mathcal{A}$) is the set of all things that belong to all of the sets in $\mathcal{A}$; in other words,

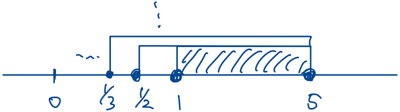
$$\bigcap \mathcal{A} = \{x : x \in A \text{ for each } A \in \mathcal{A}\}.$$

Example. Let A, B, and C be sets. Then

- $\bigcap\{A\} = A$
- $\bigcap\{A, B\} = A \cap B$
- $\bigcap\{A, B, C\} = A \cap B \cap C$

Example. Let $\mathcal{A} = \{[1/n, 5] : n \in \mathbb{N}\}$. Then

$$\overset{\bigcap}{\cancel{\bigcup}} \mathcal{A} = [1, 5].$$



Example 4

For each of the following, find $\bigcup \mathcal{A}$ and $\bigcap \mathcal{A}$. State clearly if either/both of the two is/are undefined.

1. $\mathcal{A} = \{\varnothing, \{1,2\}, \{2,3\}\}$.
2. $\mathcal{A} = \varnothing$.
3. $\mathcal{A} = \{1, \{2\}\}$.

① $\bigcup \mathcal{A} = \varnothing \cup \{1,2\} \cup \{2,3\} = \{1,2,3\}$

$\bigcap \mathcal{A} = \varnothing \cap \{1,2\} \cap \{2,3\} = \varnothing$

② $\bigcup \mathcal{A} = \bigcup \{\ \} = \varnothing$

$\bigcap \mathcal{A}$ is undefined.

③ $\bigcup \mathcal{A}$ and $\bigcap \mathcal{A}$ are both undefined because $\mathcal{A}$ is not a set of sets.

Q. What if $\mathcal{A} = \{\emptyset\}$?

A.

$$\bigcup \mathcal{A} = \bigcup \{\emptyset\} = \emptyset$$

$$\bigcap \mathcal{A} = \bigcap \{\emptyset\} = \emptyset$$

Proposition 1

Let $\mathcal{A}$ be a nonempty set of sets and let $A_0 \in \mathcal{A}$. Then

$$\bigcap \mathcal{A} \subseteq A_0 \subseteq \bigcup \mathcal{A}.$$

Proof: Consider any $x \in \bigcap \mathcal{A}$. Then for each $A \in \mathcal{A}$, $x \in A$. In particular, $x \in A_0$ because $A_0 \in \mathcal{A}$. This shows that

$\bigcap \mathcal{A} \subseteq A_0$.

Consider any $x \in A_0$. Since $A_0 \in \mathcal{A}$, $x \in A$ for some $A \in \mathcal{A}$ is true. That is, $x \in \bigcup \mathcal{A}$. This shows that $A_0 \subseteq \bigcup \mathcal{A}$. $\square$

Proposition 2

Let $\mathcal{A}$ be a nonempty set of sets and let x be any object. Then:

1. $x \notin \bigcup \mathcal{A}$ *iff for each* $A \in \mathcal{A}$, $x \notin A$.
2. $x \notin \bigcap \mathcal{A}$ *iff there exists* $A \in \mathcal{A}$ *such that* $x \notin A$.

<u>Proof</u> We have

$$x \notin \bigcup \mathcal{A}$$

iff $\neg(x \in \bigcup \mathcal{A})$

iff $\neg(\exists A \in \mathcal{A})(x \in A)$

iff $(\forall A \in \mathcal{A}) \neg (x \in A)$ (by GDM)

iff $(\forall A \in \mathcal{A})(x \notin A)$

$\square$

De Morgan's Laws Again

Theorem 5 (Generalized De Morgan's Laws for Sets of Sets)

Let S be a set and let $\mathcal{A}$ be a nonempty set of sets. Then:

1. $S \setminus \bigcup \mathcal{A} = \bigcap \{S \setminus A : A \in \mathcal{A}\}$.
2. $S \setminus \bigcap \mathcal{A} = \bigcup \{S \setminus A : A \in \mathcal{A}\}$.

Proof Let x be any object. Then we have

$$x \in S \setminus \bigcup \mathcal{A}$$

iff $x \in S$ and $x \notin \bigcup \mathcal{A}$

iff $x \in S$ and $(\forall A \in \mathcal{A})(x \notin A)$ (by Prop. 2)

iff $(\forall A \in \mathcal{A})(x \in S \text{ and } x \notin A)$ (by dist. law)

iff $(\forall A \in \mathcal{A})(x \in S \setminus A)$

iff $x \in \bigcap \{S \setminus A : A \in \mathcal{A}\}$ $\square$

Distributive Laws Again

Theorem 6 (Generalized Distributive Laws for Sets of Sets)

Let S be a set and let $\mathcal{A}$ be a nonempty set of sets. Then:

1. $S \cap \bigcup \mathcal{A} = \bigcup\{S \cap A : A \in \mathcal{A}\}$.
2. $S \cup \bigcap \mathcal{A} = \bigcap\{S \cup A : A \in \mathcal{A}\}$.

Power Set of a Set

Power Set of a Set

Definition 7

Let A be a set. The *power set of A* (denoted $\mathcal{P}(A)$) is the set of all subsets of A; in other words, $\mathcal{P}(A) = \{S : S \subseteq A\}$.

Example.

$$\mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}.$$

Note.

- If A is a finite set with n elements, then $\mathcal{P}(A)$ has 2^n elements.

Above: $n=2$, $2^n = 2^2 = 4$.

Example: (Recursive) Power Sets of the Empty Set

Let $V_0 = \varnothing$ and for each $n \in \omega$, let $V_{n+1} = \mathcal{P}(V_n)$. That is,

$$
\begin{aligned}
V_0 &= \varnothing \\
V_1 &= \mathcal{P}(\varnothing) = \{\varnothing\} \\
V_2 &= \mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\} \\
V_3 &= \mathcal{P}(\{\varnothing, \{\varnothing\}\}) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\
&\vdots
\end{aligned}
$$

Counting the number of elements:

Set	# of Elem.	Set	# of Elem.
V_0	0	V_4	$2^4 = 16$
V_1	$2^0 = 1$	V_5	$2^{16} = 65536$
V_2	$2^1 = 2$	V_6	$2^{65536} \approx 2 \times \underset{10}{\cancel{2}}^{19,728}$
V_3	$2^2 = 4$	$\cdots$	