Section 6.

Insight vs. Induction

Sums of Powers Revisited

Verification vs. Discovery

Denote by $S_r(n)$ the sum $1^r + 2^r + \cdots + n^r$, *e.g.*,

$$S_0(n) = 1^0 + 2^0 + \cdots + n^0$$

$$= \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n$$

$$
\begin{aligned}
S_1(n) &= 1 \;\; + 2 \;\; + 3 \;\; + \cdots + n \;\; , \\
S_2(n) &= 1^2 + 2^2 + 3^2 + \cdots + n^2, \\
S_3(n) &= 1^3 + 2^3 + 3^3 + \cdots + n^3, \\
&\;\;\vdots
\end{aligned}
$$

In Section 5, we were given formulas for some of the sums and they were verified by induction. But how would one discover such formulas in the first place?

Derivation of $S_1(n)$ Formula

$S_1(n) = 1+2+\cdots+n$

Observe that

$$
\begin{array}{rrcccccccccccc}
 & S_1(n) & = & 1 & + & 2 & + & 3 & + & \cdots & + & n \\
+) & S_1(n) & = & n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \\
\hline
 & 2S_1(n) & = & (n+1) & + & (n+1) & + & (n+1) & + & \cdots & + & (n+1)
\end{array}
$$

$$\underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1)}_{n \text{ copies}}$$

$= n(n+1)$

Therefore,

$$S_1(n) = \frac{n(n+1)}{2}.$$

Would it work w/ $S_2(n)$?

$$
\begin{array}{rrcccccccc}
 & S_2(4) & = & 1^2 & + & 2^2 & + & 3^2 & + & 4^2 \\
+) & S_2(4) & = & 4^2 & + & 3^2 & + & 2^2 & + & 1^2 \\
\hline
 & 2S_2(4) & = & 17 & + & 13 & + & 13 & + & 17
\end{array}
$$

$$\underbrace{17 + 13 + 13 + 17}_{\text{non-uniform "column" sums.}}$$

Another Derivation of $S_1(n)$ Formula (Telescoping Sums)

$+1$ $+1$

Let $T(n) = \sum_{k=1}^{n}[k^2 - (k-1)^2]$.

On the one hand,

$$T(n) = [\cancel{1^2} - 0^2] + [\cancel{2^2} - \cancel{1^2}] + [\cancel{3^2} - \cancel{2^2}] + \cdots + [n^2 - \cancel{(n-1)^2}]$$

$$= -0^2 + n^2 = \underline{n^2}$$

On the other hand, since $k^2 - (k-1)^2 = \cancel{k^2} - (\cancel{k^2} - 2k + 1) = 2k - 1$,

$$T(n) = \sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

$$= 2(\underbrace{1+2+\cdots+n}_{n \text{ terms}}) - (\underbrace{1+1+\cdots+1}_{n \text{ copies}}) = \underline{2S_1(n) - n}$$

Therefore,

$$2S_1(n) - n = n^2 \implies S_1(n) = \frac{n^2+n}{2} = \frac{n(n+1)}{2}.$$

Derivation of $S_2(n)$ Formula via Telescoping Sum

S06E01

Let $n \in \mathbb{N}$. Let $T(n) = \sum_{k=1}^{n}[k^3 - (k-1)^3]$. By writing $T(n)$ out in long form, show that it is a telescoping sum and that $T(n) = n^3$. Then, by evaluating $T(n)$ in a different way, deduce without explicit induction that

$$S_2(n) = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

In general, to derive $S_r(n)$, use

$$T(n) = \sum_{k=1}^{n}\left[k^{r+1} - (k-1)^{r+1}\right]$$

Geometric Sums Revisited

Sum of Geometric Progression

Let $x \in \mathbb{R}$. Suppose that $x \neq 1$. Let $n \in \mathbb{N}$. In S05E04, you were given the geometric series formula

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x},$$

and were asked to verify it using induction. Let's derive it here.

Derivation of Geometric Sum Formula

Geometric progression/sequence.

$$1, \; x, \; x^2, \; x^3, \; x^4, \; \ldots$$

$$\times x \quad \times x \quad \times x \quad \cdots$$

"x : common ratio."

Let

$$S = 1 + x + x^2 + \cdots + x^{n-1}.$$

Observe that

$$\begin{array}{rrcccccccccccc}
 & S & = & 1 & + & \cancel{x} & + & \cancel{x^2} & + & \cdots & + & \cancel{x^{n-1}} & & \\
-) & xS & = & & & \cancel{x} & + & \cancel{x^2} & + & \cdots & + & \cancel{x^{n-1}} & + & x^n \\
\hline
 & (1-x)S & = & 1 & + & 0 & + & 0 & + & \cdots & + & 0 & \overset{-}{+} & x^n
\end{array}$$

$$(1-x)S = 1 - x^n$$

Therefore, since $x \neq 1$, $1 - x \neq 0$, and thus

$$S = \frac{1-x^n}{1-x}.$$

Another Derivation Using Summation Notation

In summation notation,

$$S = \sum_{k=0}^{n-1} x^k = 1 + \sum_{k=1}^{n-1} x^k,$$

while

$$xS = \sum_{k=0}^{n-1} x^{k+1} = \sum_{k=1}^{n} x^k = \sum_{k=1}^{n-1} x^k + x^n.$$

Therefore,