Proving Uniqueness

Contents

Uniqueness

Introduction

At times, one wishes to show that there exists exactly one value of x in the universe of discourse for which $P(x)$ is true.

"There exists a unique x such that $P(x)$."

one and only

denoted as

$$(\exists! x) P(x).$$

This uniqueness statement can be rephrased as

$$\underbrace{(\exists x) P(x)}_{\text{at least one}} \wedge \underbrace{(\forall x_1)(\forall x_2)[P(x_1) \wedge P(x_2) \Rightarrow x_1 = x_2],}_{\text{at most one}}$$

or

$$(\exists x_1)\left[\underbrace{P(x_1)}_{\substack{\text{existence} \\ \text{"at least one"}}} \wedge \underbrace{(\forall x_2)\left(P(x_2) \Rightarrow x_1 = x_2\right)}_{\substack{\text{unique} \\ \text{"at most one"}}}\right].$$

Examples

an example proving $(\exists x)(7x-1=0)$

- $(\exists! x \in \mathbb{R})(7x - 1 = 0)$ is true because

  $1/7$ is a real number such that $7(1/7) - 1 = 0$ and ← existence

  if $x \in \mathbb{R}$ such that $7x - 1 = 0$, then $x = 1/7$. ← uniqueness.

integers

- $(\exists! x \in \mathbb{Z})(7x - 1 = 0)$ is false because there is no integer x for which $7x - 1 = 0$.

  (due to non-existence)

Examples (cont')

- $(\exists! x \in \mathbb{R})(x^2 - 8x + 16 = 0)$ is true because

$$\left.\begin{aligned} & x^2 - 8x + 16 = 0 \\ \iff \ & (x-4)^2 = 0 \\ \iff \ & x - 4 = 0 \\ \iff \ & x = 4. \end{aligned}\right\} \text{ the eqn has one and only soln.}$$

(due to non-uniqueness)

- $(\exists! x \in \mathbb{R})(x^2 - 8x + 12 = 0)$ is false because

$$\begin{aligned} & x^2 - 8x + 12 = 0 \\ \iff \ & (x-4)^2 - 4 = 0 && \text{completing the square} \\ \iff \ & (x-4)^2 = 4 \\ \iff \ & (x - 4 = -2) \vee (x - 4 = 2) \\ \iff \ & (x = 2) \vee (x = 6). \end{aligned}$$

there are two different values of x satisfying $x^2 - 8x + 12 = 0$.

Examples: Mixed with Universal Quantifier

- $(\forall a > 0)(\exists! x > 0)(x^2 = a)$ is true.

  Proof. Let $a_0 > 0$ be arbitrary. Then $(\exists! x > 0)(x^2 = a_0)$ is true because

  $\sqrt{a_0}$ is a positive real number such that $(\sqrt{a_0})^2 = a_0$ and } existence

  if x is a positive real number such that $x^2 = a_0$, then $x = \sqrt{a_0}$, discarding the negative square root. } uniqueness

  Since a_0 is an arbitrary positive real number, $(\forall a > 0)(\exists! x > 0)(x^2 = a)$ is true. □

Q. How about $(\forall a > 0)(\exists! x \in \mathbb{R})(x^2 = a)$ or

$(\forall a \in \mathbb{R})(\exists! x > 0)(x^2 = a)$?

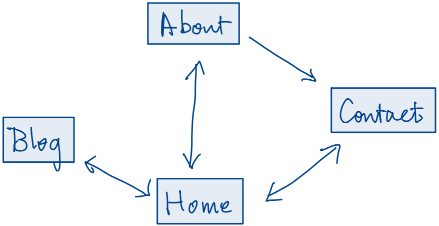
Homework (1/26; due Wed 2/2)

Section 3: # 11, 12, 14

Exercises with Quantifiers

Example: Order in mixed quantifiers

Consider the following webpages:



$P(x, y)$: x has a link to y.

T / F

$(\forall x)(\exists y)\, P(x,y)$

$(\exists x)(\forall y)\, P(x,y)$

Suggestion: Tabulate $P(x,y)$
truth value of.

<u>Example</u>: Negation of multiply quantified sentences.

Let f be a function from $\mathbb{R}$ to $\mathbb{R}$

and let $a \in \mathbb{R}$.

To say that $\boxed{f \text{ is continuous at } a}$

means that