Binomial Coefficients

Contents

Warm-Up

Remark 4.50

Let $d \in \mathbb{N}$, $x, y \in \mathbb{Z}$, and p be a prime number.

1. If $p \mid xy$, then $p \mid x$ or $p \mid y$.
2. If $d \mid xy$, then these exist $d_1, d_2 \in \mathbb{N}$ such that $d_1 \mid x$, $d_2 \mid y$, and $d = d_1 d_2$.

The proofs of these facts require *complete induction*.

3. Let $d \in \mathbb{N}$, $x_1, x_2, \cdots, x_n \in \mathbb{Z}$.

If $d \mid \underbrace{x_1 x_2 \cdots x_n}_{\text{product of } n \text{ numbers}}$, then there exist $d_1, d_2, \cdots, d_n \in \mathbb{N}$ such that $d_1 \mid x_1$, $d_2 \mid x_2, \cdots,$ $d_n \mid x_n$ and $d = d_1 d_2 \cdots d_n$.

e.g.

$6 \mid 18$

$6 = 2 \cdot 3$

$18 = 2 \cdot 9$

Rational and Irrational Numbers Revisited (cont')

S04 E1T (assigned)

$x^3 = c$

Context: $\sqrt{2}$ is irrational. How about $\sqrt{c}$?

Example 4.52

1. Let x be a rational number such that $x^2 = c$, where c is a whole number. Then x is an integer.
2. Let c be a whole number which is not a perfect square. Then $\sqrt{c}$ is irrational.

Proof of ① Since x is a rational number, we can pick two numbers $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that $x = a/b$. By Fact ☺, suppose that the fraction a/b is written in lowest terms. Then

$$(a/b)^2 = c, \quad \text{so} \quad a^2/b^2 = c, \quad \text{so} \quad a^2 = b^2 c, \quad \text{so} \quad a^2 = \underbrace{(bc)}_{\in \mathbb{Z}} b.$$

Since bc is an integer, b divides a^2. Then by Rmk. 4.50, we can pick $b_1, b_2 \in \mathbb{N}$ such that $b_1 \mid a$, $b_2 \mid a$, and $b = b_1 b_2$.

But then $b_1 \mid a$ and $b_1 \mid b$. But since a/b is in lowest terms, it must be the case that $b_1 = 1$.

Similarly, $b_2 = 1$. So $b = b_1 b_2 = 1 \cdot 1 = 1$. It follows that $x = a/b = a/1 = a$. Since a is an integer, so is x.

$\square$

<u>Proof of (2)</u> Read the proof.

Binomial Coefficients

Pascal's Triangle

$A[1,3]$ $\quad A(1,3)$ $\quad A_{1,3}$

row col.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
 & & & & & \vdots & & & & &
\end{array}
$$

Pascal's Triangle and Binomial Coefficients

Recall. For all $n \in \omega$ and all $k \in \{0, \ldots, n\}$, the binomial coefficient $\binom{n}{k}$ denotes the k-th number on the n-th row of Pascal's triangle.

Key features

1. $$\binom{0}{0} = 1.$$
2. *Boundary conditions*: For each $n \in \mathbb{N}$,
$$\binom{n}{0} = \binom{n}{n} = 1.$$
3. *Recurrence relation*: For each $n \in \omega$ and all $k \in \{1, \ldots, n\}$,
$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

$\binom{n}{k-1}$ $\binom{n}{k}$

$\binom{n+1}{k}$

Why *n choose k*?

All 2-element subsets of the 4-element set $\{1,2,3,4\}$ are

$$\underbrace{\{1,2\},\{1,3\},\{2,3\}}_{\text{ones without } 4},\ \underbrace{\{1,\underline{4}\},\{2,\underline{4}\},\{3,\underline{4}\}}_{\text{ones with } 4}.$$

Note that

- the number of subsets without 4 is $3 = \binom{3}{2}$
- the number of subsets with 4 is $3 = \binom{3}{1}$

Thus the total number of 2-element subsets of the 4-element set is

$$6 = 3 + 3 = \underbrace{\binom{3}{2} + \binom{3}{1} = \binom{4}{2}}_{\text{recurrence relation.}}$$

Why *n choose k*? (cont')

$0 \le k \le n+1$ or $k \in \{0, 1, \cdots, n+1\}$.

In general, one can count the number of k-element subsets of the $(n+1)$-element set

$$\{1, 2, \ldots, n, n+1\}$$

$$\underbrace{\_\ \_\ \_\ \cdots\ \_}_{\substack{(k-1) \\ \text{avail. spots}}}\ \ \underset{\substack{\uparrow \\ \text{last spot} \\ \text{occupied}}}{\overset{n+1}{\_}}$$

in an analogous fashion:

- the number of subsets without $n+1$ is $\binom{n}{k}$.
- the number of subsets with $n+1$ is $\binom{n}{k-1}$.

Thus the total number of k-element subsets of the $(n+1)$-element set is

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

Why *n choose k*? (cont')

The idea above is key to a proof by induction of the following theorem.

Number of Subsets (cf. S14E03)

For each $n \in \boldsymbol{\omega}$, for each $k \in \{0, \ldots, n\}$, the number of k-element subsets of an n-element set is $\binom{n}{k}$.

Why Binomial Coefficients?

Consider the expansion of $(a+b)^2$:

$$\begin{aligned}
(a+b)^2 &= (a+b)(a+b) \\
&= (a+b)a + (a+b)b \\
&= a^2 + ba \\
&\qquad + ab + b^2 \\
&= a^2 + 2ab + b^2. \quad = \binom{2}{0} a^2 b^0 + \binom{2}{1} a^1 b^1 + \binom{2}{2} a^0 b^2
\end{aligned}$$

Note that the coefficients of a^2, ab, and b^2 are

$$1, 2, 1,$$

respectively, which are precisely the numbers on row 2 of Pascal's triangle:

$$\binom{2}{0}, \binom{2}{1}, \binom{2}{2}.$$

Binomial Expansion

Expansion of $(a+b)^3$

Work out the expansion of $(a+b)^3$ and compare the coefficients with the numbers in row 3 of Pascal's triangle.

$$(a+b)^3 = (a+b)^2(a+b)$$

$$= (a+b)^2 a + (a+b)^2 b$$

$$= (a^2+2ab+b^2)a + (a^2+2ab+b^2)b$$

$$= a^3 + 2a^2b + b^2a$$
$$a^2b + 2ab^2 + b^3$$

$$= a^3 + 3a^2b + 3ab^2 + b^3$$

$$= \binom{3}{0}a^3b^0 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}a^0b^3$$

Binomial Expansion (cont')

Expansion of $(a+b)^4$

Work out the expansion of $(a+b)^3$ and compare the coefficients with the numbers in row 3 of Pascal's triangle.

The Binomial Theorem

The examples above suggest the following general result.

The Binomial Theorem (S05E11)

For each $n \in \omega$ and all $a, b \in \mathbb{R}$,

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

- **Convention**: For each $x \in \mathbb{R}$, $x^0 = 1$.