More Examples of Induction

Recap Induction

- Used to prove sentences of form

$$(\forall n \in \mathbb{N})\ P(n)$$

- based on principle of math. induction

$$(\forall P)\left\{ \underbrace{P(1)}_{\text{Base Case}} \wedge \underbrace{(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]}_{\text{Inductive Step}} \Longrightarrow \underbrace{(\forall n \in \mathbb{N})\ P(n)}_{\text{Conclusion}} \right\}$$

- Template

Declaration : Let $P(n)$ be the sentence ...

Base Case : $P(1)$ is true because ...

Inductive Step : Let $n \in \mathbb{N}$ such that $P(n)$ is true. ...

Conclusion : Therefore, by induction, for each $n \in \mathbb{N}$, $P(n)$ is true.

Contents

$\mathbb{N} = \{1, 2, 3, \dots\}$

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

From $\mathbb{N}$ to $\mathbb{Z}$

- relate to the positive half.

- Induction
- handle this first

Context: Want to prove $(\forall x \in \mathbb{Z})\, P(x)$.

Issue: To use induction, need a "starting point".
But where on $\mathbb{Z}$?

Solution: Cut in half.

$\{\dots, -3, -2, -1\}, \quad \{0, 1, 2, \dots\}$

Example 1

Prove that for each $x \in \mathbb{Z}$, x is even or x is odd.

Proof Let $x \in \mathbb{Z}$. Then $x \leq -1$ or $x \geq 0$.

Let $P(x)$ be the sentence

x is even or x is odd.

PART 1 We wish to show for each integer $x \geq 0$, $P(x)$ is true using induction.

BASE CASE $P(0)$ is true because 0 is even.

Show $P(0)$ is true.

Notation : $\omega = \{0, 1, 2, 3, \dots\}$ whole numbers

INDUCTIVE STEP Let $x \in \omega$ such that $P(x)$ is true.

Show $(\forall x \in \omega)[P(x) \Rightarrow P(x+1)]$

Then x is even or x is odd.

In the case where x is even, $x+1$ is odd.

In the case where x is odd, $x+1$ is even.

Thus in either case, $x+1$ is even or $x+1$ is odd.

Hence $P(x+1)$ is true.

CONCLUSION Therefore, by induction, for each $x \in \omega$, $P(x)$ is true.

PART 2 We wish to show for each integer $x \leq -1$, $P(x)$ is true using PART 1.

positive integer, aka natural number

Let x be a negative integer. Then $-x \in \omega$, so $P(-x)$ is true by PART 1, so $-x$ is even or $-x$ is odd.

Case 1 Suppose $-x$ is even. Then $x = \overbrace{(-x)}^{\text{even}}\overbrace{(-1)}^{\text{odd}}$ is even because an even number times any integer is even.

Case 2 Suppose $-x$ is odd. Then $x = \overbrace{(-x)}^{\text{odd}}\overbrace{(-1)}^{\text{odd}}$ is odd because the product of two odd numbers is odd.

Thus in either case, x is even or x is odd. Therefore for each integer $x \leq -1$, $P(x)$ is true.

Therefore, by PART 1 and PART 2, $(\forall x \in \mathbb{Z})\ P(x)$ is true. $\square$

Notes: Induction over $\mathbb{Z}$.

- Induction may start from a number other than 1.
  - $(\forall x \in \omega)\, P(x)$
  - $S = \{3, 4, 5, \ldots\}$. $(\forall x \in S)\, P(x)$ can be proved by induction as well.
- To prove a universal sentence $(\forall x \in \mathbb{Z}) P(x)$:
  1. Prove by induction that $P(x)$ is true for each nonnegative integer[1] x.
  2. Prove that $P(x)$ is true for each negative integer x.

Pascal's Triangle and the Binomial Theorem

Pascal's Triangle

The following infinite array of numbers is known as *Pascal's tirangle*:

$$
\begin{array}{lccccccccccc}
0^{\text{th}}\text{ row} \longrightarrow & & & & & & 1 & & & & & \\
1^{\text{st}}\text{ row} \longrightarrow & & & & & 1 & & 1 & & & & \\
2^{\text{nd}}\text{ row} \longrightarrow & & & & 1 & & 2 & & 1 & & & \\
3^{\text{rd}}\text{ row} \longrightarrow & & & 1 & & 3 & & 3 & & 1 & & \\
4^{\text{th}}\text{ row} \longrightarrow & & 1 & & 4 & & 6 & & 4 & & 1 & \\
5^{\text{th}}\text{ row} \longrightarrow & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & & & & & & \vdots & & & & &
\end{array}
$$

Notation. For all $n \in \omega$ and all $k \in \{0, \dots, n\}$, let $\binom{n}{k}$ denote the k-th number on the n-th row; this notation is read *n choose k*. This is also called a *binomial coefficient*.

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
 & & & \vdots & & &
\end{array}
$$

Key Features.

1. $\binom{0}{0} = 1.$
2. *Boundary conditions*: For each $n \in \mathbb{N}$, $\binom{n}{0} = \binom{n}{n} = 1.$
3. *Recurrence relation*: For each $n \in \omega$ and all $k \in \{1, \dots, n\}$, $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$

Example

Application of Key Features

Use **Key Features** above to compute $\binom{4}{2}$.

Solution.

$$
\begin{aligned}
\binom{4}{2} &= \binom{3}{2} + \binom{3}{1} && \text{by } 3 \\
&= \underbrace{\binom{2}{2} + \binom{2}{1}} + \underbrace{\binom{2}{1} + \binom{2}{0}} && \text{by } 3 \\
&= 1 + 2\binom{2}{1} + 1 && \text{by } 2 \\
&= 1 + 2\left[\binom{1}{1} + \binom{1}{0}\right] + 1 && \text{by } 3 \\
&= 1 + 2(1 + 1) + 1 \\
&= 6.
\end{aligned}
$$

Naming: *n choose k*

Question: Why is $\binom{n}{k}$ called "n choose k"?

Number of Subsets

List all subsets of the set $\{a, b, c, d\}$ with exactly 2 elements.

Solution.

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$$

There are $6 = \binom{4}{2}$ subsets with two elements. In other words, there are $\binom{4}{2} = 6$ ways to choose 2 elements from a set with 4 elements.

In Section 14, we will prove that for each $n \in \omega$, for each n-element set A, for each $k \in \{0, \ldots, n\}$, the number of k-element subsets of A is $\binom{n}{k}$.

Naming: Binomial Coefficients

Question: Why is $\binom{n}{k}$ called a binomial coefficient?

Expansion of $(a+b)^3$

1. Compute $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$, and $\binom{3}{3}$.
2. Expand the cube of the binomial $a+b$, that is, expand $(a+b)^3$.

Binomial Theorem

The example above suggests:

The Binomial Theorem

For each $n \in \omega$ and all $a, b \in \mathbb{R}$,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$
$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$$

- **Convention**: For each $x \in \mathbb{R}$, $x^0 = 1$.