Surjections, Injections, ~~and Inverses~~

Office hours schedule change
(till the end of semester)

TW 4:45 ~ 6:15 pm

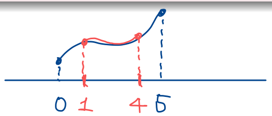
Restriction and Extension

Restriction

Definition 1

Let f be a function and let $C \subseteq \mathrm{Dom}(f)$. Then *the restriction of f to C* is the function, denoted $f\restriction C$, defined by $(f\restriction C)(x) = f(x)$ for all $x \in C$.

- Note that $\mathrm{Dom}(f\restriction C) = C$.



$f : [0,5] \to \mathbb{R}$

$g : [1,4] \to \mathbb{R}$

Examples.

- Let $f(x) = x^{1/3}$ for all $x \in \mathbb{R}$ and let $g(x) = x^{1/3}$ for all $x \in [1,5)$. Then $g = f\restriction[1,5)$.
- Let $f(x) = \sqrt{x}$ for all $x \in [0,\infty)$, $g(x) = 1 - x^2$ for all $x \in \mathbb{R}$, and $h(x) = 1 - x$ for all $x \in \mathbb{R}$. Then $g \circ f = h\restriction[0,\infty)$.

$g = f\restriction[1,4]$

because for all $x \in [1,4]$,

$g(x) = f(x)$.

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = 1 - \sqrt{x}^2 = 1 - x$

for all $x \in [0,\infty)$

Extension

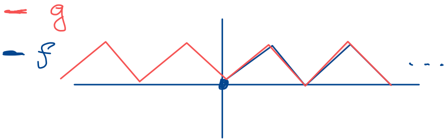
Definition 2

Let f and g be functions. To say that f *is an extension of* g means that $\mathrm{Dom}(f) \supseteq \mathrm{Dom}(g)$ and for each $x \in \mathrm{Dom}(g)$, $f(x) = g(x)$.

- Note f is an extension of g iff $\mathrm{Dom}(f) \supseteq \mathrm{Dom}(g)$ and $f \restriction \mathrm{Dom}(g) = g$.

- Example. (Even or odd periodic extensions)

Let $f : [0, \infty) \to \mathbb{R}$ whose graph is as shown below



Let $g : \mathbb{R} \to \mathbb{R}$ defined by

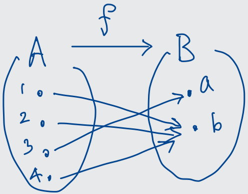
$$g(x) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$$

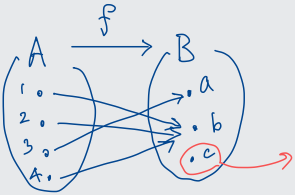
By construction, g is an extension of f.

Note that g is an even function, so g is said to be an even extension of f.

- one-to-one
- horizontal line test

Surjections, Injections, and Inverses





not a function value

f is a surjection from A to B

f is <u>not</u> a surjection from A to B.

Surjections

Definition 3

Let A and B be sets. To say that f *is a surjection from A to B* means that f is a function from A to B and for each $y \in B$, there exists $x \in A$ such that $f(x) = y$.

$(\forall y \in B)(\exists x \in A)(f(x) = y)$

Notes.

- A surjection from A to B is also said to be a function from A *onto* B.
- Any function is a surjection from its domain to its range.
- f is a surjection from A to B

  iff f is a function, $\mathrm{Dom}(f) = A$, and $\mathrm{Rng}(f) = B$

  iff for each $y \in B$, the equation $f(x) = y$ has at least one solution x in A.

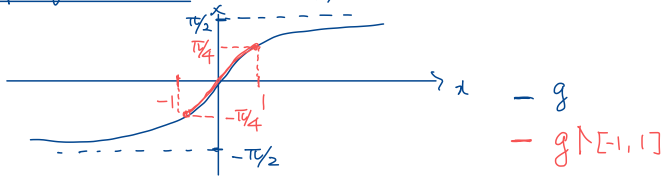
Surjections (cont')

Example 4

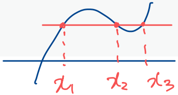
- Let $f(x) = \sin(x)$ for all $x \in \mathbb{R}$. Then f is a surjection from $\mathbb{R}$ to $[-1, 1]$, but f is not a surjection from $\mathbb{R}$ to $\mathbb{R}$.
- Let $g(x) = \arctan(x)$ for all $x \in \mathbb{R}$. Then f is a surjection from $\mathbb{R}$ to $(-\pi/2, \pi/2)$.

Q. $g \restriction [-1,1]$ is a surjection from $[-1, 1]$ to $[-\pi/4, \pi/4]$.

Graph of $\arctan(x)$



Injections



Note: $x_1 \neq x_2$ but $f(x_1) = f(x_2)$

So this is not an injection.

Definition 5

To say that *f is an injection* means that f is a function and for all $x_1, x_2 \in \mathrm{Dom}(f)$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

$(\forall x_1, x_2 \in \mathrm{Dom}(f))[\, f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \,] \equiv (\forall x_1, x_2 \in \mathrm{Dom}(f))[\, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \,]$

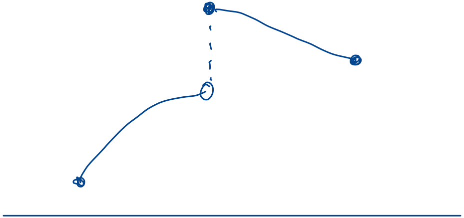
Note.

- To say that *f is an injection from A to B* means that f is a function from A to B and f is an injection.
- An injection is also said to be a *one-to-one* function.
- f is an injection from A to B

  iff for each $y \in B$, the equation $f(x) = y$ has *at most* one solution x in A

  iff for all $x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. (contrapositive)

cf. Similar cond. from injection



Example 6

Let $f(x) = x^2$ for all $x \in \mathbb{R}$ and let $g(x) = \sqrt{x}$ for all $x \in [0, \infty)$. Then:

- f is not an injection from $\mathbb{R}$ to $[0, \infty)$ because, for instance,

  $f(2) = 4 = f(-2)$.

- g is an injection from $[0, \infty)$ to $[0, \infty)$ because

  for any $x_1, x_2 \in [0, \infty)$, if $g(x_1) = g(x_2)$, then $\sqrt{x_1} = \sqrt{x_2}$,

  so $x_1 = \sqrt{x_1}^2 = \sqrt{x_2}^2 = x_2$.