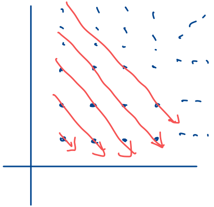
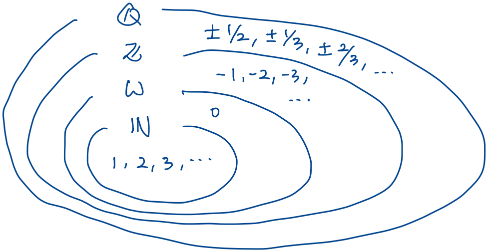
Cantor's Diagonal Lemma

Last time: Infinite sets are not rigid!



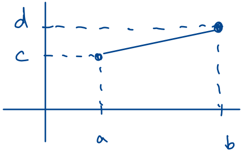
$\boxed{\text{An infinite set can be equinumerous to its } \underline{\text{proper}} \text{ subsets.}}$

E.g.



E.g. $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$

E.g. $[a, b] \approx [c, d]$

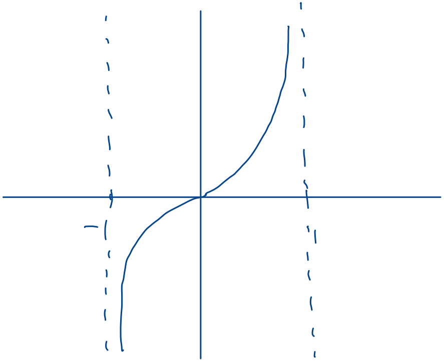


$y = m(x-a)+c$

$(-1,1) \approx (0,1)$

E.g. $(-1,1) \approx \mathbb{R}$

$$\varphi(x) = \frac{x}{1-|x|}$$



Question $\mathbb{N} \approx \mathbb{R}$?

$\hookrightarrow$ Cantor's answer.

Cantor’s Diagonal Lemma

Cantor's Diagonal Lemma

Infinitely many rooms in Hilbert Hotel

Infinitely many names ABAB··· ABBAA···

Cantor's Diagonal Lemma

Let f be any function from $\mathbb{N}$ to $(0,1)$. Then there exists $y \in (0,1)$ such that y does not belong to the range of f.

Below is a key consequence of Cantor's diagonal lemma.

Theorem 1 (Cantor, 1873)

$\mathbb{R}$ is not equinumerous to $\mathbb{N}$.

Proof. Suppose $\mathbb{R} \approx \mathbb{N}$. By symmetry of $\approx$, $\mathbb{N} \approx \mathbb{R}$. We know from Wed that $\mathbb{R} \approx (0,1)$. Hence, by transitivity of $\approx$, $\mathbb{N} \approx (0,1)$. So there is a bijection f from $\mathbb{N}$ to $(0,1)$. This f is a function from $\mathbb{N}$ to $(0,1)$. Thus, by CDL, there is $y \in (0,1)$ such that $y \notin \operatorname{Rng}(f)$. Hence f is not a surj. from $\mathbb{N}$ to $(0,1)$, so f is not a bijection from $\mathbb{N}$ to $(0,1)$. This is a contradiction.

Cantor's Diagonal Lemma: Idea of Proof

To prove Cantor's diagonal lemma, we need to find/construct $y \in (0, 1)$ such that $y \notin \text{Rng}(f) = \{f(n) : n \in \mathbb{N}\}$.

does not end in repeating 9's.

Decimal expansion of $f(n)$

For each $n \in \mathbb{N}$, $f(n) \in (0, 1)$ so it has the standard decimal expansion

$$f(n) = 0.x_{n1}x_{n2}x_{n3}x_{n4}\ldots.$$

e.g. $\frac{1}{2} = 0.5$

$= \cancel{0.4999\ldots}$

That is,

$$\begin{aligned} f(1) &= 0.x_{11}x_{12}x_{13}x_{14}\ldots, \\ f(2) &= 0.x_{21}x_{22}x_{23}x_{24}\ldots, \\ f(3) &= 0.x_{31}x_{32}x_{33}x_{34}\ldots, \\ f(4) &= 0.x_{41}x_{42}x_{43}x_{44}\ldots, \\ &\text{and so on.} \end{aligned}$$

Cantor's Diagonal Lemma: Idea of Proof (cont')

$f(1) = 0.\boxed{5}2314\ldots$

$f(2) = 0.1\boxed{5}314\ldots$

$\vdots \qquad 0.27\boxed{5}\ldots$

$\boxed{5}$

Construction of y

For each $n \in \mathbb{N}$, let

$$y_n = \begin{cases} 5 & \text{if } x_{nn} \neq 5, \\ 4 & \text{if } x_{nn} = 5. \end{cases}$$

$0, 1, 2, 3, 4, \cancel{5}, 6, 7, 8, \cancel{9}$

Then for each $n \in \mathbb{N}$, $y_n \neq x_{nn}$. Now let y be the number whose standard decimal expansion is

$$y = 0.y_1 y_2 y_3 y_4 \ldots.$$

Observation

- $y \in (0, 1)$; in fact, $0.444\ldots \leqslant y \leqslant 0.555\ldots$.
- $y \notin \mathrm{Rng}(f)$ because for each $n \in \mathbb{N}$, $y \neq f(n)$.

$(\forall n \in \mathbb{N})(y_n \neq x_{nn})$

they differ in the n^{th} decimal place

Example

$$\left.\begin{aligned} f(1) &= 0.\boxed{4}732651\cdots \\ f(2) &= 0.3\boxed{4}96781\cdots \\ f(3) &= 0.72\boxed{6}1238\cdots \\ f(4) &= 0.555\boxed{5}555\cdots \\ &\;\vdots \end{aligned}\right\} \Rightarrow \left.\begin{aligned} y_1 &= 5 \\ y_2 &= 5 \\ y_3 &= 5 \\ y_4 &= 4 \\ &\;\vdots \end{aligned}\right\} \Rightarrow y = 0.5554\cdots$$

Higher Orders of Infinity

Denumerable, Countable, and Uncountable

Definition 2

Let A be a set.

1. To say that A *is denumerable* means that A is equinumerous to $\mathbb{N}$.
2. To say that A *is countable* means that A is finite or denumerable.
3. To say that A *is uncountable* means that A is not countable.

meaning infinite and not denumerable.

$\not\approx \mathbb{N}$.

Example.

- Each of $\mathbb{N}$, ω, $\mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$, and $\mathbb{Q}$ is denumerable.
- $\mathbb{R}$ is uncountable.

Cardinality

Definition 3

Let A and B be sets.

1. To say that *the cardinality of A is less than or equal to the cardinality of B* (denoted $\overline{\overline{A}} \leqslant \overline{\overline{B}}$) means that A is equinumerous to a subset of B.
2. To say that *the cardinality of A is strictly less than the cardinality of B* (denoted $\overline{\overline{A}} < \overline{\overline{B}}$) means that A is equinumerous to a subset of B but A is not equinumerous to B.
3. To say that *the cardinality of A is equal to the cardinality of B* (denoted $\overline{\overline{A}} = \overline{\overline{B}}$) means that A is equinumerous to B.

$\overline{\overline{\mathbb{N}}} \leqslant \overline{\overline{\mathbb{Q}}}$

$\overline{\overline{\mathbb{N}}} \leqslant \overline{\overline{\mathbb{Q}}}$

$\overline{\overline{A}} \leqslant \overline{\overline{\mathbb{N}}}$

even natural numbers

Example. $\overline{\overline{\mathbb{N}}} < \overline{\overline{\mathbb{R}}}$.

Cardinality (cont')

Notes.

- Let A and B be sets. Then $\overline{\overline{A}} \leqslant \overline{\overline{B}}$ iff there exists an injection from A to B.

- Let A be any set. Then $\overline{\overline{A}} \leqslant \overline{\overline{\mathcal{P}(A)}}$.

Cantor's Generalized Diagonal Lemma

Cantor's Generalized Diagonal Lemma

Let A be a set and let f be a function on A such that for each $x \in A$, $f(x)$ is a set. Then there exists a subset $C \subseteq A$ such that C does not belong to the range of f.

Below is a key consequence of Cantor's generalized diagonal lemma.

Theorem 4 (Cantor, 1891)

Any set has strictly smaller cardinality than its power set.