

Hints for S11 E20

Recap

Let A and B be sets.

Let $f: A \rightarrow B$.

- f is a surjection from A to B

iff $(\forall y \in B) (f(x) = y \text{ has } \underline{\text{at least}} \text{ one soln. } x \in A)$

- f is an injection from A to B

iff $(\forall y \in B) (f(x) = y \text{ has } \underline{\text{at most}} \text{ one soln. } x \in A)$

- f is a bijection from A to B

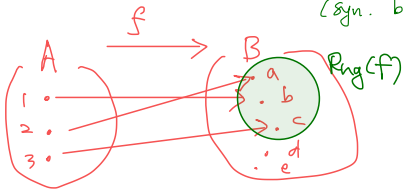
iff $(\forall y \in B) (f(x) = y \text{ has } \underline{\text{a unique}} \text{ soln. } x \in A)$

Q. Fill in the blank.

Let f be a function on A . Suppose f is an injection.
Then f is a bijection from A to $\text{Rng}(f)$.

Reason Recall that any function is a surjection from its domain to its range. In other words, $f: A \rightarrow \text{Rng}(f)$ is a surjection. But f was assumed to be injective. Hence, f is a one-to-one correspondence between A and B .

(syn. bijection)



S11E20(a) Let $g(x) = \frac{x}{1-x}$ for all $x \in [0, 1)$.

① Show $g: [0, 1) \rightarrow [0, \infty)$.

(NTS: for each $x \in [0, 1)$, $g(x) \in [0, \infty)$, i.e., $g(x) \geq 0$)

Pf. Let $x \in [0, 1)$. Then $0 \leq x < 1$, so

$$0 \geq -x > -1,$$

$$\text{so } 1 \geq 1-x > 0,$$

$$\text{so } 1 \leq \frac{1}{1-x}.$$

Since $x \geq 0$ and $\frac{1}{1-x} \geq 1$, we have $g(x) = x \cdot \frac{1}{1-x} \geq 0$. □

② Show g is an injection. (Find $g^{-1}(y)$ along the way.)

(Strategy: Pick $y \in \text{Rng}(g)$ and show $g(x) = y$ has a unique soln.)

PF. Let $y \in \text{Rng}(g)$. Then $y = g(x)$ for some $x \in [0, 1)$.


So we have

$$y = \frac{x}{1-x}$$

$$\Rightarrow (1-x)y = x$$

$$\Rightarrow y - xy = x$$

$$\Rightarrow x(1+y) = y$$


$$\Rightarrow x = \frac{y}{1+y}.$$

Thus, the eqn $y = g(x)$ has a unique soln

$$x = \frac{y}{1+y} = g^{-1}(y),$$

So g is an injection. □

$$\textcircled{1} \quad g: [0, 1) \rightarrow [0, \infty)$$

② Show $\text{Rng}(g) = [0, \infty)$.

(Note: This is equivalent to showing that g is surjective.)

Pf. We have already shown that $\text{Rng}(g) \subseteq [0, \infty)$ in part ①.

So it remains to show that $[0, \infty) \subseteq \text{Rng}(g)$.

$g^{-1}(y)$ from
②

Let $y \in [0, \infty)$. (NTS: there exists $x \in [0, 1)$ s.t. $g(x) = y$.)

Take $x = \frac{y}{1+y} = g^{-1}(y)$. From ②, we know that $g(x) = y$. (check)

So all that is remaining is to confirm that $x \in [0, 1)$.

Since $y \in [0, \infty)$, $y \geq 0$, so $1+y \geq 1$, so $\frac{1}{1+y} \leq 1$.

Now

$$\begin{aligned}x &= \frac{y}{1+y} = \frac{1+y-1}{1+y} \\&= 1 - \frac{1}{1+y}.\end{aligned}$$

Since $0 < \frac{1}{1+y} \leq 1$, $0 > -\frac{1}{1+y} \geq -1$, so $1 > \underbrace{1 - \frac{1}{1+y}}_x \geq 0$.

In other words, $0 \leq x < 1$.

This shows that $[0, \infty) \subseteq \text{Rng}(g)$.



Section 13.

Fundamental Principle of Counting

Equinumerousness

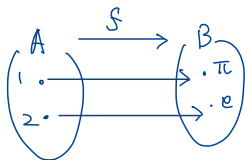
Equinumerousness

Definition 1

Let A and B be sets. To say that A is equinumerous to B (denoted $A \approx B$) means that there exists a bijection from A to B .

Examples.

- The sets $\{1, 2\}$ and $\{\pi, e\}$ are equinumerous because the function f on $\{1, 2\}$ defined by $f(1) = \pi$ and $f(2) = e$ is a bijection from $\{1, 2\}$ to $\{\pi, e\}$.
- The set \mathbb{Z} is equinumerous to the set of all even integers $B = \{2k : k \in \mathbb{Z}\}$ because the function $g : \mathbb{Z} \rightarrow B$ defined by $g(n) = 2n$ is a bijection.



Note B is a proper subset of \mathbb{Z} ,
but still has the same number
of elements as in \mathbb{Z} .

Proposition 1

Equinumerousness is reflexive, symmetric, and transitive. In other words:

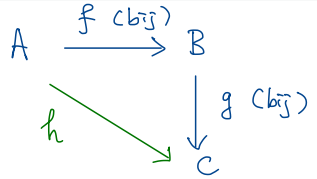
- ① (Reflexivity) For each set A , we have A is equinumerous to A .
- ② (Symmetry) For all sets A and B , if A is equinumerous to B , then B is equinumerous to A .
- ③ (Transitivity) For all sets A , B , and C , if A is equinumerous to B and B is equinumerous to C , then A is equinumerous to C .

Idea of proof

- ① $A \approx A$ because id_A is a bijection from A to itself.

- ② Since $A \approx B$, we can find a bijection f from A to B . Let $g = f^{-1}$. Then g is a bijection from B to A . So $B \approx A$.
(See Thm 7 in lec 32.)

③ Since $A \approx B$ and $B \approx C$,
we can find a bijection f from A to B
and another bijection g from B to C .
Let $h = g \circ f$. It turns out that this h
is a bijection from A to C . Thus $A \approx C$.



Number of Elements and Equinumerousness

$$\omega = \{0, 1, 2, \dots\}$$

Definition 2

Let A be a set and let $n \in \omega$. To say that A has n elements means that A is equinumerous to $\{1, \dots, n\}$.

- A set has 0 elements iff it is empty. (See p.12 of Lec. 30)
- Saying that A is an n -element set is synonymous to saying that A has n elements.

$$A \approx \emptyset$$

Let A be a set with 0 elements.

Then we can find a bijection f from A to \emptyset . NTS: $A = \emptyset$.

Convention

- If $n=0$, then $\{1, \dots, n\} = \emptyset$
- If $n=1$, then $\{1, \dots, n\} = \{1\}$.
- If $n=2$, then $\{1, \dots, n\} = \{1, 2\}$
- \vdots

Def'n Let $n \in \mathbb{N}$. To say that a_1, \dots, a_n are distinct means that the function f defined by

$$f(k) = a_k, \text{ for all } k = 1, \dots, n,$$

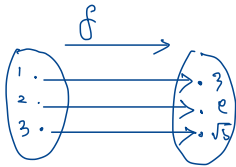
is an injection.

E.g. Distinct

$$a_1 = 3$$

$$a_2 = e$$

$$a_3 = \sqrt{5}$$



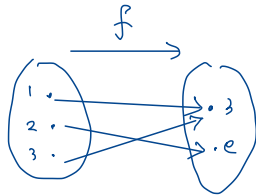
one-to-one

Not distinct

$$a_1 = 3$$

$$a_2 = e$$

$$a_3 = 3$$



not one-to-one

n -element Sets

Some omitted details

A has n elements $\Rightarrow A \approx \{1, \dots, n\}$

Sym.

$\Rightarrow \{1, \dots, n\} \approx A$

Def'n

Def'n

\Rightarrow there is a bijection f from $\{1, \dots, n\}$ to A .

Proposition 2

Let A be a set and let $n \in \omega$. Then the following are equivalent.

- 1 A has n elements.
- 2 There exist distinct objects a_1, \dots, a_n such that $A = \{a_1, \dots, a_n\}$.

$(\textcircled{1} \Rightarrow \textcircled{2})$

Proof. Suppose A has n elements. Then there is a bijection f from $\{1, \dots, n\}$ to A . Let $a_k = f(k)$ for $k = 1, \dots, n$. Since f is a surjection from $\{1, \dots, n\}$ to A , we have $A = \{a_1, \dots, a_n\}$. Since f is an injection, the objects a_1, \dots, a_n are distinct.

$\text{Rng}(f)$.

$(\textcircled{2} \Rightarrow \textcircled{1})$

Conversely, suppose there exist distinct objects a_1, \dots, a_n such that $A = \{a_1, \dots, a_n\}$. Then $A = \text{Rng}(f)$ where f is a function on $\{1, \dots, n\}$ defined by $f(k) = a_k$. Since a_1, \dots, a_n are distinct, f is injective, so f is a bijection from $\{1, \dots, n\}$ to A , so A has n elements. □

Finite and Infinite Sets

Definition 3

Let A be a set.

- To say that A is *finite* means that there exists $n \in \omega$ such that A has n elements.
- To say that A is *infinite* means that A is not finite.

Preliminary Lemmas

Comparing the Sizes of Sets

Lemma A

Let A and B be sets. Suppose that A is equinumerous to B .

- 1 If $s \notin A$ and $t \notin B$, then $A \cup \{s\}$ is equinumerous to $B \cup \{t\}$.
- 2 If $s \in A$ and $t \in B$, then $A \setminus \{s\}$ is equinumerous to $B \setminus \{t\}$.

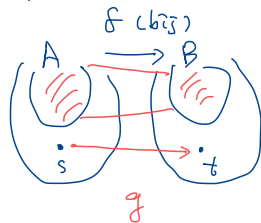
Proof of ① Since $A \approx B$, there exists a bijection f from A to B .

Define g by

$$g(x) = \begin{cases} f(x) & \text{if } x \in A \\ t & \text{if } x = s \end{cases}$$

S11E22(c) \rightarrow Then g is a bijection from $A \cup \{s\}$ to $B \cup \{t\}$.

Thus $A \cup \{s\} \approx B \cup \{t\}$.



Comparing the Sizes of Sets (cont')

Lemma B

For all $m, n \in \omega$, if $\{1, \dots, m\}$ is equinumerous to $\{1, \dots, n\}$, then $m = n$.

Note For each $n \in \omega$,

$$\text{for each } m \in \omega, \quad \underbrace{\{1, \dots, m\} \approx \{1, \dots, n\} \Rightarrow m = n}_{= P(m)}.$$

↓
Prove this using induction.

Key Results

Uniqueness of the Number of Elements

Theorem 4

The number of elements in a finite set is uniquely determined.

- According to the theorem, for each finite set A , there is a unique $n \in \omega$ such that A has n elements; we write $\overline{\overline{A}}$ for the unique n . The notation $\overline{\overline{A}}$ is read the number of elements in A or the cardinality of A .

Subsets of a Finite Set is Finite

Theorem 5

A subset of a finite set is finite and has at most as many elements as the whole set.

A precise rephrase of the theorem is: For each $n \in \omega$,

for each set B , if B has n elements, then for each $A \subseteq B$, A is finite and $\overline{A} \leq n$.

Thus one can prove this using induction.

Rigidity Property of Finite Sets

Theorem 6

A finite set cannot be equinumerous to a proper subset of itself.

Proof. Let B be a finite set and let A be a proper subset of B . We wish to show that B is not equinumerous to A . Suppose B is equinumerous to A . Since B is finite, B has n elements for some $n \in \omega$. Since B is equinumerous to A , A also has n elements. Since A is a proper subset of B , $B \setminus A$ is not empty. Let $b \in B \setminus A$ and let $C = A \cup \{b\}$. Then $C \subseteq B$ and C has $n + 1$ elements. Hence by Theorem 13.30, $n + 1 \leq n$. But $n + 1 > n$. Thus we have reached a contradiction. Therefore B must not be equinumerous to A .

Theorem 5 on prev. page