Quantifiers (I)

Contents

Basics of Quantifiers

Motivation

Let x be a real number. Consider the following sentences.

- $A(x)$: If $x > 3$, then $x > 1$.
- $B(x)$: $x^2 - 4 > 0$.

The truth value of each sentence depends on the value of the variable x.

- $A(x)$ is true for all x.
- $B(x)$ is true for $x < -2$ or $x > 2$.

In general, they can be rephrased using *quantifiers* as:

- For all x, $A(x)$ is true.
- For some x, $B(x)$ is true.

Quantifiers

The quantifiers $\forall$ and $\exists$, along with the logical connectives, are main ingredients of modern symbolic logic.

Quantifier	Symbol	Technical Name
"for each"	$\forall$	*universal quantifier*
"for some"	$\exists$	*existential quantifier*

Example. Let x be a person in this class room. Let $P(x)$ stands for "x likes ramen." Then

- $(\forall x)P(x)$: "For each x, x likes ramen." or "Everybody likes ramen."
- $(\exists x)P(x)$: "For some x, x likes ramen." or "Somebody likes ramen."

Notes

Alternate ways to read.

$(\forall x)P(x)$:

For each x, $P(x)$.
For all x, $P(x)$.
For every x, $P(x)$.
For any x, $P(x)$.

$(\exists x)P(x)$:

For some x, $P(x)$.
For at least one x, $P(x)$.
There exists x such that $P(x)$.

Universe of Discourse

The collection over which the variable x ranges is called *the universe of discourse*. When clear from context, it is omitted in notation; if not, specify the universe of discourse using the following notation.

$$(\forall x \in U)P(x) \quad \text{or} \quad (\exists x \in U)P(x).$$

Frequently used collections.

- $\mathbb{N}$: the set of natural numbers, $\{1, 2, 3, \ldots\}$
- $\mathbb{Z}$: the set of integers, $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $\mathbb{Q}$: the set of rational numbers
- $\mathbb{R}$: the set of real numbers
- $\mathbb{C}$: the set of complex numbers

Free and Bound Variables

- In $P(x)$, x can stand for any particular element of the universe of discourse; it is called a *free variable*.
- In $(\forall x)P(x)$ or $(\exists x)P(x)$, x varies over the universe of discourse, not standing for any particular element; it is called a *bound variable* or a *dummy variable*.

Universal and Existential Quantifiers

Universal Quantifier

Let U be the universe of discourse.

- $(\forall x)P(x)$ is true when $P(x)$ is true for all values of x in U.
- To show $(\forall x)P(x)$ is false, it suffices to show that $P(x)$ is false for at least one value of x in U; such x is said to be a *counterexample that disproves the universal sentence*.

Universal Quantifier (cont')

Example. State whether each of the following sentences is true or false.

1. $(\forall x \in \mathbb{R})(x - 2 = 5)$

2. $(\forall x \in \mathbb{R})(x^2 + 6x + 10 > 0)$

Existential Quantifier

Let U be the universe of discourse.

- $(\exists x)P(x)$ is true when $P(x)$ is true for at least one value of x in U; such x is said to be an *example that proves the existential sentence*.
- To show $(\exists x)P(x)$ is false, it is necessary to show that $P(x)$ is false for all values of x in U.

Existential Quantifier (cont')

Example. State whether each of the following sentences is true or false.

- $(\exists x \in \mathbb{R})(x - 2 = 5)$

- $(\exists x \in \mathbb{R})(x^2 + 6x + 10 < 0)$

Notes on Quantifiers

Connections to Logical Connectives

Suppose the universe of discourse consists only of two objects $\{a, b\}$. Note that

- $(\forall x)P(x)$ is true exactly when $P(a) \wedge P(b)$ is true.
- $(\exists x)P(x)$ is true exactly when $P(a) \vee P(b)$ is true.

In general, when the universe of discourse is a finite set $\{a_1, a_2, \ldots, a_n\}$, then

- $(\forall x)P(x)$ has the same truth value as $P(a_1) \wedge P(a_2) \wedge \cdots \wedge P(a_n)$.
- $(\exists x)P(x)$ has the same truth value as $P(a_1) \vee P(a_2) \vee \cdots \vee P(a_n)$.

Notation

Suppose A is a subcollection of the universe of discourse. Then

- $(\forall x \in A)P(x)$ is a shorthand notation for $(\forall x)[(x \in A) \Rightarrow P(x)]$.
- $(\exists x \in A)P(x)$ is a shorthand notation for $(\exists x)[(x \in A) \wedge P(x)]$.

When the universe of discourse is $\mathbb{R}$, a subcollection may be characterized by an inequality in which case one may use notations *e.g.*,

- $(\forall x > 0)(2x + 7 = 3)$
- $(\exists x \geq 7)(x^2 - 4x + 3 > 0)$

Scope of Quantifiers

The scope of a quantifier is specified using appropriate delimiters.

Example. Let n be an element in $\{2, 3, 5, 7\}$ and let

$$P(n): \ n \text{ is a prime number.}$$
$$Q(n): \ n \text{ is an even number.}$$

Then

- $(\forall n)P(n) \wedge Q(n)$ stands for

- $(\forall n)[P(n) \wedge Q(n)]$ stands for