# Hints for S11 E20

**Recap** Let $A$ and $B$ be sets.

Let $f: A \to B$.

- $f$ is a surjection from $A$ to $B$

  iff $(\forall y \in B)(\, f(x) = y$ has <u>at least</u> one soln. $x \in A)$

- $f$ is an injection from $A$ to $B$

  iff $(\forall y \in B)(\, f(x) = y$ has <u>at most</u> one soln. $x \in A)$

- $f$ is a bijection from $A$ to $B$

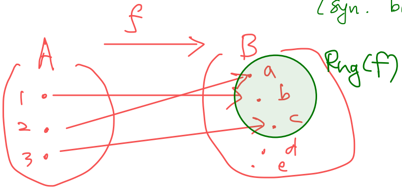
  iff $(\forall y \in B)(\, f(x) = y$ has <u>a unique</u> soln. $x \in A)$

Q. Fill in the blank.

Let $f$ be a function on $A$. Suppose $f$ is an injection.
Then $f$ is <u>a bijection</u> from $A$ to $\mathrm{Rng}(f)$.

<u>Reason</u> Recall that any function is a surjection from its domain to its range. In other words, $f : A \to \mathrm{Rng}(f)$ is a surjection. But $f$ was assumed to be injective.
Hence, $f$ is a <u>one-to-one correspondence</u> between $A$ and $B$.

(syn. bijection)



S11E20 (a) Let $g(x) = \dfrac{x}{1-x}$ for all $x \in [0,1)$.

① Show $g : [0,1) \to [0,\infty)$.

(NTS: for each $x \in [0,1)$, $g(x) \in [0,\infty)$, i.e., $g(x) \geqslant 0$)

Pf. Let $x \in [0,1)$. Then $\underline{0 \leq x} < 1$, so

$$0 \geqslant -x > -1,$$

so $$1 \geqslant 1-x > 0,$$

so $$1 \leq \frac{1}{1-x}.$$

Since $x \geqslant 0$ and $\dfrac{1}{1-x} \geqslant 1$, we have $g(x) = x \cdot \dfrac{1}{1-x} \geqslant 0$. $\square$

② Show $g$ is an injection. (Find $g^{-1}(y)$ along the way.)

(Strategy: Pick $y \in \mathrm{Rng}(g)$ and show $g(x) = y$ has a unique soln.)

Pf. Let $y \in \mathrm{Rng}(g)$. Then $y = g(x)$ for some $x \in [0, 1)$.

So we have

$$y = \frac{x}{1-x}$$

$$\Rightarrow (1-x)y = x$$

$$\Rightarrow y - xy = x$$

$$\Rightarrow x(1+y) = y$$

$$\Rightarrow x = \frac{y}{1+y}.$$

Thus, the eqn $y = g(x)$ has a unique soln

$$x = \frac{y}{1+y} = g^{-1}(y),$$

So $g$ is an injection. $\square$

① $g : [0,1) \to [0,\infty)$

② Show $\mathrm{Rng}(g) = [0,\infty)$.

(Note: This is equivalent to showing that $g$ is surjective.)

Pf. We have already shown that $\mathrm{Rng}(g) \subseteq [0,\infty)$ in part ①.

So it remains to show that $[0,\infty) \subseteq \mathrm{Rng}(g)$.

$g^{-1}(y)$ from ②

Let $y \in [0,\infty)$. (NTS: there exists $x \in [0,1)$ s.t. $g(x) = y$.)

Take $x = \frac{y}{1+y} = g^{-1}(y)$. From ②, we know that $g(x) = y$. (Check)

So all that is remaining is to confirm that $x \in [0,1)$.

Since $y \in [0,\infty)$, $y \geq 0$, so $1+y \geq 1$, so $\frac{1}{1+y} \leq 1$.

Now

$$x = \frac{y}{1+y} = \frac{1+y-1}{1+y}$$

$$= 1 - \frac{1}{1+y}.$$

Since $0 < \frac{1}{1+y} \leq 1$, $0 > -\frac{1}{1+y} \geq -1$, So $1 > \underbrace{1 - \frac{1}{1+y}}_{\overset{\shortparallel}{x}} \geq 0$.

In other words, $0 \leq x < 1$.

This shows that $[0, \infty) \subseteq \operatorname{Rng}(g)$. $\square$

Section 13.

# Fundamental Principle of Counting

# Equinumerousness

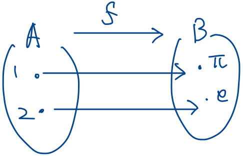
# Equinumerousness

## Definition 1

Let $A$ and $B$ be sets. To say that $A$ *is equinumerous to* $B$ (denoted $A \approx B$) means that there exists a bijection from $A$ to $B$.

**Examples.**

- The sets $\{1, 2\}$ and $\{\pi, e\}$ are equinumerous because the function $f$ on $\{1, 2\}$ defined by $f(1) = \pi$ and $f(2) = e$ is a bijection from $\{1, 2\}$ to $\{\pi, e\}$.
- The set $\mathbb{Z}$ is equinumerous to the set of all even integers $B = \{2k : k \in \mathbb{Z}\}$ because the function $g : \mathbb{Z} \to B$ defined by $g(n) = 2n$ is a bijection.




Note: $B$ is a proper subset of $\mathbb{Z}$, but still has the same number of elements as in $\mathbb{Z}$.

# Equinumerousness (cont')

## Proposition 1

*Equinumerousness is reflexive, symmetric, and transitive. In other words:*

1. *(Reflexivity) For each set $A$, we have $A$ is equinumerous to $A$.*
2. *(Symmetry) For all sets $A$ and $B$, if $A$ is equinumerous to $B$, then $B$ is equinumerous to $A$.*
3. *(Transitivity) For all sets $A, B$, and $C$, if $A$ is equinumerous to $B$ and $B$ is equinumerous to $C$, then $A$ is equinumerous to $C$.*

# Number of Elements and Equinumerousness

## Definition 2

Let $A$ be a set and let $n \in \boldsymbol{\omega}$. To say that $A$ *has* $n$ *elements* means that $A$ is equinumerous to $\{1, \ldots, n\}$.

- A set has 0 elements iff it is empty.
- Saying that $A$ *is an n-element set* is synonymous to saying that $A$ has $n$ elements.

## Proposition 2

*Let $A$ be a set and let $n \in \omega$. Then the following are equivalent.*

1. *$A$ has $n$ elements.*
2. *There exist distinct objects $a_1, \ldots, a_n$ such that $A = \{a_1, \ldots, a_n\}$.*

*Proof.* Suppose $A$ has $n$ elements. Then there is a bijection $f$ from $\{1, \ldots, n\}$ to $A$. Let $a_k = f(k)$ for $k = 1, \ldots, n$. Since $f$ is a surjection from $\{1, \ldots, n\}$ to $A$, we have $A = \{a_1, \ldots, a_n\}$. Since $f$ is an injection, the objects $a_1, \ldots, a_n$ are distinct.

Conversely, suppose there exist distinct objects $a_1, \ldots, a_n$ such that $A = \{a_1, \ldots, a_n\}$. Then $A = \mathrm{Rng}(f)$ where $f$ is a function on $\{1, \ldots, n\}$ defined by $f(k) = a_k$. Since $a_1, \ldots, a_n$ are distinct, $f$ is injective, so $f$ is a bijection from $\{1, \ldots, n\}$ to $A$, so $A$ has $n$ elements. □

# Finite and Infinite Sets

**Definition 3**

Let $A$ be a set.

- To say that $A$ *is finite* means that there exists $n \in \omega$ such that $A$ has $n$ elements.
- To say that $A$ *is infinite* means that $A$ is not finite.

# Preliminary Lemmas

**Lemma A**

Let $A$ and $B$ be sets. Suppose that $A$ is equinumerous to $B$.

1. If $s \notin A$ and $t \notin B$, then $A \cup \{s\}$ is equinumerous to $B \cup \{t\}$.
2. If $s \in A$ and $t \in B$, then $A \setminus \{s\}$ is equinumerous to $B \setminus \{t\}$.

# Comparing the Sizes of Sets (cont')

**Lemma B**

For all $m, n \in \omega$, if $\{1, \ldots, m\}$ is equinumerous to $\{1, \ldots, n\}$, then $m = n$.

# Key Results

# Uniqueness of the Number of Elements

**Theorem 4**

*The number of elements in a finite set is uniquely determined.*

- According to the theorem, for each finite set $A$, there is a unique $n \in \boldsymbol{\omega}$ such that $A$ has $n$ elements; we write $\overline{\overline{A}}$ for the unique $n$. The notation $\overline{\overline{A}}$ is read *the number of elements in* $A$ or *the cardinality of* $A$.

# Subsets of a Finite Set is Finite

**Theorem 5**

*A subset of a finite set is finite and has at most as many elements as the whole set.*

A precise rephrase of the theorem is: For each $n \in \boldsymbol{\omega}$,

for each set $B$, if $B$ has $n$ elements, then for each $A \subseteq B$, $A$ is finite and $\overline{\overline{A}} \leqslant n$.

Thus one can prove this using induction.

# Rigidity Property of Finite Sets

**Theorem 6**

*A finite set cannot be equinumerous to a proper subset of itself.*

*Proof*. Let $B$ be a finite set and let $A$ be a proper subset of $B$ . We wish to show that $B$ is not equinumerous to $A$. Suppose $B$ is equinumerous to $A$. Since $B$ is finite, $B$ has $n$ elements for some $n \in \omega$. Since $B$ is equinumerous to $A$, $A$ also has $n$ elements. Since $A$ is a proper subset of $A$, $B \setminus A$ is not empty. Let $b \in B \setminus A$ and let $C = A \cup \{b\}$. Then $C \subseteq B$ and $C$ has $n + 1$ elements. Hence by Theorem 13.30, $n + 1 \leqslant n$. But $n + 1 > n$. Thus we have reached a contradiction. Therefore $B$ must not be equinumerous to $A$.