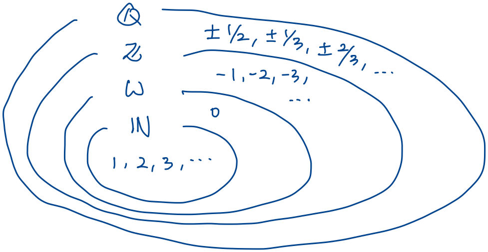
Cantor's Diagonal Lemma

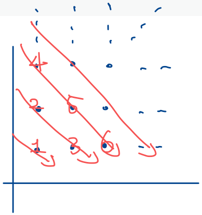
Last time: Infinite sets are not rigid!

> An infinite set can be equinumerous to its proper subsets.

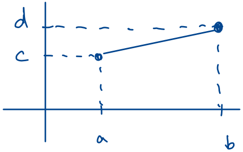
E.g.



E.g. $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$



E.g. $[a, b] \approx [c, d]$



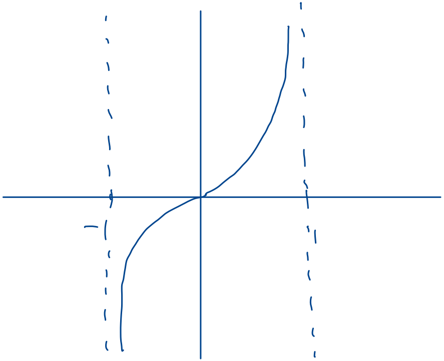
$y = m(x-a)+c$

Since $(-1,1) \approx (0,1)$,
by symmetry & transitivity of $\approx$, we know

E.g. $(-1,1) \approx \mathbb{R}$

$\mathbb{R} \approx (0,1)$.

$$\varphi(x) = \frac{x}{1-|x|}$$



Question $\mathbb{N} \approx \mathbb{R}$?

↳ Cantor's answer.

Hilbert Hotel

people whose names consist of infinitely many characters (A or B)

Cantor’s Diagonal Lemma

Cantor's Diagonal Lemma

R1 : $\boxed{A}$ B A B . . .
R2 : B $\boxed{A}$ A B . . .
R3 : A B $\boxed{B}$ A . . .

B B A . . .

Cantor's Diagonal Lemma

Let f be any function from $\mathbb{N}$ to $(0, 1)$. Then there exists $y \in (0, 1)$ such that y does not belong to the range of f.

Below is a key consequence of Cantor's diagonal lemma.

Theorem 1 (Cantor, 1873)

$\mathbb{R}$ is not equinumerous to $\mathbb{N}$.

Proof: Suppose $\mathbb{R} \approx \mathbb{N}$. By symmetry of $\approx$, $\mathbb{N} \approx \mathbb{R}$. We know that $\mathbb{R} \approx (0,1)$. Thus, by transitivity of $\approx$, $\mathbb{N} \approx (0,1)$. So there is a bijection f from $\mathbb{N}$ to $(0,1)$. This f is a function from $\mathbb{N}$ to $(0,1)$. By CDL, there is $y \in (0,1)$ such that $y \notin \mathrm{Rng}(f)$. Thus f is not a surjection from $\mathbb{N}$ to $(0,1)$, so f is not a bijection from $\mathbb{N}$ to $(0,1)$. This is a contradiction. $\square$

Cantor's Diagonal Lemma: Idea of Proof

To prove Cantor's diagonal lemma, we need to find/construct $y \in (0,1)$ such that $y \notin \mathrm{Rng}(f) = \{f(n) : n \in \mathbb{N}\}$.

does not end in repeating 9's.

e.g. $1/2 = 0.5 = 0.4999\cdots$

Decimal expansion of $f(n)$

For each $n \in \mathbb{N}$, $f(n) \in (0,1)$ so it has the standard decimal expansion

$$f(n) = 0.x_{n1}x_{n2}x_{n3}x_{n4}\ldots.$$

That is,

$$\begin{aligned} f(1) &= 0.{\color{red}x_{11}}x_{12}x_{13}x_{14}\ldots, \\ f(2) &= 0.x_{21}{\color{red}x_{22}}x_{23}x_{24}\ldots, \\ f(3) &= 0.x_{31}x_{32}{\color{red}x_{33}}x_{34}\ldots, \\ f(4) &= 0.x_{41}x_{42}x_{43}{\color{red}x_{44}}\ldots, \\ &\text{and so on.} \end{aligned}$$

Recap

$$0.4999\cdots = 0.4\overline{9} = 0.4\dot{9} \stackrel{?}{=} \tfrac{1}{2}$$

$$\| $$

$$0.4 + 0.09 + 0.009 + \cdots$$

$$\| $$

$$0.4 + \underbrace{\frac{9}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \cdots}_{\text{geom. series}}$$

$$\| $$

$$0.4 + \frac{9}{10^2}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right) = 0.4 + \boxed{\overbrace{\frac{9}{10^2} \cdot \frac{1}{\underbrace{1 - \frac{1}{10}}_{= \frac{9}{10}}}}^{= \frac{1}{10}}} = \underline{0.5}$$

Construction of y

For each $n \in \mathbb{N}$, let

$$y_n = \begin{cases} 5 & \text{if } x_{nn} \neq 5, \\ 4 & \text{if } x_{nn} = 5. \end{cases}$$

$0, 1, 2, 3, 4, \cancel{5}, 6, 7, 8, \cancel{9}$

Since we are using standard dec. exp.

Then for each $n \in \mathbb{N}$, $y_n \neq x_{nn}$. Now let y be the number whose standard decimal expansion is

$$y = 0.y_1y_2y_3y_4\ldots.$$

Observation

- $y \in (0, 1)$; in fact, $0.444\ldots \leqslant y \leqslant 0.555\ldots$.
- $y \notin \mathrm{Rng}(f)$ because for each $n \in \mathbb{N}$, $y \neq f(n)$.

$(y_n \neq x_{nn})$

because they differ in the n^{th} decimal place

$$\left.\begin{array}{l} f(1) = 0.\boxed{7}84\cdots \\ f(2) = 0.3\boxed{2}7\cdots \\ f(3) = 0.55\boxed{5}\cdots \\ \quad\vdots \end{array}\right\} \longrightarrow \left.\begin{array}{l} y_1 = 5 \\ y_2 = 5 \\ y_3 = 4 \\ \quad\vdots \end{array}\right\} \longrightarrow y = 0.554\cdots$$

Higher Orders of Infinity

Denumerable, Countable, and Uncountable

Definition 2

Let A be a set.

1. To say that A *is denumerable* means that A is equinumerous to $\mathbb{N}$.
2. To say that A *is countable* means that A is finite or denumerable.
3. To say that A *is uncountable* means that A is not countable.

meaning A is infinite and A is not denumerable

$A \not\approx \mathbb{N}$,

Example.

- Each of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$, and $\mathbb{Q}$ is denumerable.
- $\mathbb{R}$ is uncountable.

Cardinality

$\overline{\overline{\mathbb{N}}} \leqslant \overline{\overline{\omega}}$

Definition 3

Let A and B be sets.

1. To say that *the cardinality of A is less than or equal to the cardinality of B* (denoted $\overline{\overline{A}} \leqslant \overline{\overline{B}}$) means that A is equinumerous to a subset of B.
2. To say that *the cardinality of A is strictly less than the cardinality of B* (denoted $\overline{\overline{A}} < \overline{\overline{B}}$) means that A is equinumerous to a subset of B but A is not equinumerous to B.
3. To say that *the cardinality of A is equal to the cardinality of B* (denoted $\overline{\overline{A}} = \overline{\overline{B}}$) means that A is equinumerous to B.

Example. $\overline{\overline{\mathbb{N}}} < \overline{\overline{\mathbb{R}}}$.

Cardinality (cont')

Notes.

- Let A and B be sets. Then $\overline{\overline{A}} \leqslant \overline{\overline{B}}$ iff there exists an injection from A to B.

- Let A be any set. Then $\overline{\overline{A}} \leqslant \overline{\overline{\mathcal{P}(A)}}$.

Cantor's Generalized Diagonal Lemma

Cantor's Generalized Diagonal Lemma

Let A be a set and let f be a function on A such that for each $x \in A$, $f(x)$ is a set. Then there exists a subset $C \subseteq A$ such that C does not belong to the range of f.

Below is a key consequence of Cantor's generalized diagonal lemma.

Theorem 4 (Cantor, 1891)

Any set has strictly smaller cardinality than its power set.