

Quantifiers (II)

Contents

① Generalized De Morgan's Laws and Distributive Laws

② Order of Quantifiers

Generalized De Morgan's Laws and Distributive Laws

Generalized De Morgan's Laws

Recall De Morgan's laws:

- $\neg(P_1 \wedge P_2) \equiv \neg P_1 \vee \neg P_2$
- $\neg(Q_1 \vee Q_2) \equiv \neg Q_1 \wedge \neg Q_2$

Theorem 1 (The Generalized De Morgan's Laws)

Let $P(x)$ and $Q(x)$ be statements about x and let A be a subcollection of the universe of discourse. Then:

- 1 $\neg(\forall x \in A)P(x) \equiv (\exists x \in A)\neg P(x).$
- 2 $\neg(\exists x \in A)Q(x) \equiv (\forall x \in A)\neg Q(x).$

Generalized De Morgan's Laws (cont')

Proof of 1.

$\neg(\forall x \in A)P(x)$ is true iff $(\forall x \in A)P(x)$ is false
iff $P(x)$ is false for at least one value of x in A
iff $\neg P(x)$ is true for at least one value of x in A
iff $(\exists x \in A)P(x)$ is true. □

Examples

For each of the following, write down a sentence that is logically equivalent to the given.

$$\begin{aligned} \textcircled{1} \neg(\forall x \in \mathbb{R})(x^2 - 6x + 12 > 0) &\equiv (\exists x \in \mathbb{R}) \neg(x^2 - 6x + 12 > 0) \\ &\equiv (\exists x \in \mathbb{R}) (x^2 - 6x + 12 \leq 0) \end{aligned}$$



$$\begin{aligned} \textcircled{2} \neg(\forall x)(\exists y)R(x, y) &\equiv (\exists x) \neg(\exists y)R(x, y) \\ &\equiv (\exists x) (\forall y) \neg R(x, y) \end{aligned}$$

$$\textcircled{3} \neg(\exists x)(\forall y)S(x, y) \text{ (See next page for an example.)}$$

$$\begin{aligned} &\equiv (\forall x) \neg(\forall y)S(x, y) \\ &\equiv (\forall x) (\exists y) \neg S(x, y) \end{aligned}$$

Example: Upper Bound

Let S be a subset of \mathbb{R} . To say that S is *bounded above* means that there exists $b \in \mathbb{R}$ such that for each $x \in S$, $x \leq b$. That is,

$$S \text{ is bounded above} \Leftrightarrow (\exists b \in \mathbb{R})(\forall x \in S)(x \leq b).$$

Then to say that S is *not* bounded above means that for each $b \in \mathbb{R}$, there exists $x \in S$ such that $x > b$. That is,

$$S \text{ is not bounded above} \Leftrightarrow (\forall b \in \mathbb{R})(\exists x \in S)(x > b).$$



$$S = \{0, 1, 2\}. \quad (\exists b \in \mathbb{R})(\forall x \in S)(x \leq b)$$

Generalized Distributive Laws

Recall the distributive laws:

- $P \wedge (Q_1 \vee Q_2) \equiv (P \wedge Q_1) \vee (P \wedge Q_2)$
- $P \vee (Q_1 \wedge Q_2) \equiv (P \vee Q_1) \wedge (P \vee Q_2)$

Key idea

$$\cdot \quad \forall \quad : \quad \wedge$$

$$\cdot \quad \exists \quad : \quad \vee$$

Theorem 2 (The Generalized Distributive Laws)

Let $Q(x)$ be a statement about x , let P be a sentence that is not a statement about x , and let A be a subcollection of the universe of discourse. Then:

- 1 $P \wedge (\exists x \in A)Q(x) \equiv (\exists x \in A)[P \wedge Q(x)].$
- 2 $P \vee (\forall x \in A)Q(x) \equiv (\forall x \in A)[P \vee Q(x)].$

Note. P is not a statement about x !

Generalized Distributive Laws (cont')

Read the proof.

Proof of 2. Suppose $P \vee (\forall x \in A)Q(x)$ is true. Then P is true or $(\forall x \in A)Q(x)$ is true.

Case 1. Suppose P is true. Consider any $x_0 \in A$. Then $P \vee Q(x_0)$ is true, because P is true. Since $x_0 \in A$ was chosen arbitrarily, it follows that $(\forall x \in A)[P \vee Q(x)]$ is true.

Case 2. Suppose $(\forall x \in A)Q(x)$ is true. Consider any $x_0 \in A$. Then $Q(x_0)$ is true, so $P \vee Q(x_0)$ is true. Since $x_0 \in A$ was chosen arbitrarily, $(\forall x \in A)[P \vee Q(x)]$ is true.

Thus in either case, $(\forall x \in A)[P \vee Q(x)]$ is true.

(Continued on the next page.)

To show
Idea \vee $A \equiv B$

Suppose A is true.
Show B is true
 $[A \Rightarrow B]$

Conversely, Suppose B is true.
Show A is true
 $[B \Rightarrow A]$

Generalized Distributive Laws (cont')

Conversely, suppose $(\forall x \in A)[P \vee Q(x)]$ is true. Now either P is true or P is false.

Case 1. Suppose P is true. Then $P \vee (\forall x \in A)Q(x)$ is true.

Case 2. Suppose P is false. Consider any $x_0 \in A$. Then $P \vee Q(x_0)$ is true, because $(\forall x \in A)[P \vee Q(x)]$. But P is false, so $Q(x_0)$ must be true. Since $x_0 \in A$ was chosen arbitrarily, it follows that $(\forall x \in A)Q(x)$ is true. Hence $P \vee (\forall x \in A)Q(x)$ is true.

Thus in either case, $P \vee (\forall x \in A)Q(x)$ is true. □

Variations to GDL

Note that

- $P \wedge (Q_1 \wedge Q_2) \equiv (P \wedge Q_1) \wedge (P \wedge Q_2)$
- $P \vee (Q_1 \vee Q_2) \equiv (P \vee Q_1) \vee (P \vee Q_2)$

which can be generalized as follows:

Theorem 3

Let $Q(x)$ be a statement about x , let P be a sentence that is not a statement about x , and let A be a subcollection of the universe of discourse. Then:

- 1 $P \wedge (\forall x \in A)Q(x) \equiv (\forall x \in A)[P \wedge Q(x)].$
- 2 $P \vee (\exists x \in A)Q(x) \equiv (\exists x \in A)[P \vee Q(x)].$

Order of Quantifiers

Overview

Let $P(x, y)$ be a sentences that depends of x and y .

In a statement involving two identical quantifiers, such as in

$$(\forall x)(\forall y)P(x, y) \quad \text{or} \quad (\exists x)(\exists y)P(x, y),$$

the order of the quantifiers does not matter.

However, the order of quantifiers matters in a statement with mixed quantifiers such as

$$(\forall x)(\exists y)P(x, y) \quad \text{or} \quad (\exists x)(\forall y)P(x, y).$$

Order Matters in Mixed Quantifiers

Example. Suppose the universe of discourse is the set of all student in the classroom. Let $P(x, y)$ be the sentence “ x and y are friends.”. Then

- $(\forall x)(\exists y)P(x, y)$ says that
- $(\exists x)(\forall y)P(x, y)$ says that

Example. Determine the truth value of each of the following.

- $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y = 0)$ is a _____ statement.
- $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x + y = 0)$ is a _____ statement.

Examples: Order Matters in Mixed Quantifiers

Example. Moving quantifiers within a statement can make difference as well.

- $(\forall x \in \mathbb{R})[(\forall y \in \mathbb{R})(y > 0) \Rightarrow x > 0]$ is true.
- $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[y > 0 \Rightarrow x > 0]$ is false.

Homework (1/24; due Wed 2/2)

Section 3: # 6, 7, 9, 10