

Binomial Coefficients

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Warm-Up

Rational and Irrational Numbers Revisited

Remark 4.50

Let $d \in \mathbb{N}$, $x, y \in \mathbb{Z}$, and p be a prime number.

- 1 If $p \mid xy$, then $p \mid x$ or $p \mid y$.
- 2 If $d \mid xy$, then there exist $d_1, d_2 \in \mathbb{N}$ such that $d_1 \mid x$, $d_2 \mid y$, and $d = d_1 d_2$.

The proofs of these facts require *complete induction*.

③ Let $d \in \mathbb{N}$, $x_1, x_2, \dots, x_n \in \mathbb{Z}$.

If $d \mid \underbrace{x_1 x_2 \cdots x_n}_{\substack{\text{product} \\ \text{of } n \text{ numbers}}}$, then there exist

$d_1, d_2, \dots, d_n \in \mathbb{N}$ such that $d_1 \mid x_1, d_2 \mid x_2, \dots,$
 $d_n \mid x_n$ and $d = d_1 d_2 \cdots d_n$.

e.g.

$$6 \mid 18$$

$$6 = 2 \cdot 3$$

$$18 = 2 \cdot 9$$

Rational and Irrational Numbers Revisited (cont')

Context: $\sqrt{2}$ is irrational. How about \sqrt{c} ?

SO4E17 (assigned)

$$x^3 = c$$

Example 4.52

- 1 Let x be a rational number such that $x^2 = c$, where c is a whole number. Then x is an integer.
- 2 Let c be a whole number which is not a perfect square. Then \sqrt{c} is irrational.

Proof of ① Since x is a rational number, we can pick two numbers $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that $x = a/b$. By Fact ☺, suppose that the fraction a/b is written in lowest terms. Then

$$(a/b)^2 = c, \text{ so } a^2/b^2 = c, \text{ so } a^2 = b^2 c, \text{ so } a^2 = \underbrace{(bc)}_{\in \mathbb{Z}} b.$$

Since bc is an integer, b divides a^2 . Then by Rmk. 4.50, we can pick $b_1, b_2 \in \mathbb{N}$ such that $b_1 | a$, $b_2 | a$, and $b = b_1 b_2$.

But then $b_1 \mid a$ and $b_1 \mid b$. But since a/b is in lowest terms, it must be the case that $b_1 = 1$.

Similarly, $b_2 = 1$. So $b = b_1 b_2 = 1 \cdot 1 = 1$. It follows that $x = a/b = a/1 = a$. Since a is an integer, so is x . □

Proof of ② Read the proof.

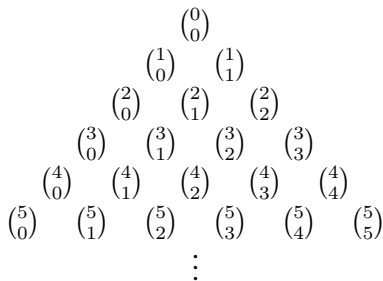
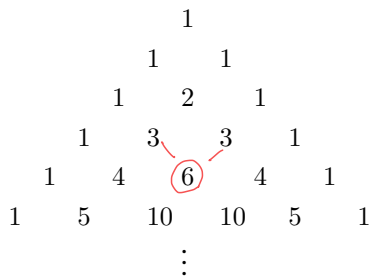
Binomial Coefficients

Pascal's Triangle

$A[1, 3]$
row col.

$A(1, 3)$

$A_{1,3}$



Pascal's Triangle and Binomial Coefficients

Recall. For all $n \in \omega$ and all $k \in \{0, \dots, n\}$, the binomial coefficient $\binom{n}{k}$ denotes the k -th number on the n -th row of Pascal's triangle.

Key features

① $\binom{0}{0} = 1.$

② *Boundary conditions:* For each $n \in \mathbb{N}$,

$$\binom{n}{0} = \binom{n}{n} = 1.$$

③ *Recurrence relation:* For each $n \in \omega$ and all $k \in \{1, \dots, n\}$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

$$\begin{array}{ccc} \binom{n}{k-1} & & \binom{n}{k} \\ & \searrow \quad \swarrow & \\ & \binom{n+1}{k} & \end{array}$$

Why n choose k ?

All 2-element subsets of the 4-element set $\{1, 2, 3, 4\}$ are

$$\underbrace{\{1, 2\}, \{1, 3\}, \{2, 3\}}_{\text{ones without 4}}, \underbrace{\{1, 4\}, \{2, 4\}, \{3, 4\}}_{\text{ones with 4}}.$$

Note that

- the number of subsets without 4 is $3 = \binom{3}{2}$
- the number of subsets with 4 is $3 = \binom{3}{1}$

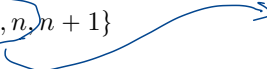
Thus the total number of 2-element subsets of the 4-element set is

$$6 = 3 + 3 = \underbrace{\binom{3}{2} + \binom{3}{1}}_{\text{recurrence relation}} = \binom{4}{2}$$

Why n choose k ? (cont')

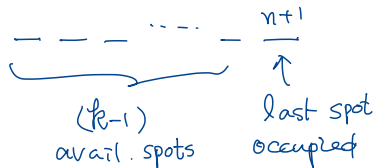
$$0 \leq k \leq n+1 \quad \text{or} \quad k \in \{0, 1, \dots, n+1\}.$$

In general, one can count the number of k -element subsets of the $(n+1)$ -element set

$$\{1, 2, \dots, n, n+1\}$$


in an analogous fashion:

- the number of subsets without $n+1$ is $\binom{n}{k}$.
- the number of subsets with $n+1$ is $\binom{n}{k-1}$.



Thus the total number of k -element subsets of the $(n+1)$ -element set is

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

Why n choose k ? (cont')

The idea above is key to a proof by induction of the following theorem.

Number of Subsets (cf. S14E03)

For each $n \in \omega$, for each $k \in \{0, \dots, n\}$, the number of k -element subsets of an n -element set is $\binom{n}{k}$.

Why Binomial Coefficients?

Consider the expansion of $(a + b)^2$:

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b) \\&= \underbrace{(a + b)a} + \underbrace{(a + b)b} \\&= a^2 + ba + ab + b^2 \\&= \underline{a^2} + 2\underline{ab} + \underline{b^2}. \quad = \binom{2}{0} a^2 b^0 + \binom{2}{1} a^1 b^1 + \binom{2}{2} a^0 b^2\end{aligned}$$

Note that the coefficients of a^2 , ab , and b^2 are

$$1, 2, 1,$$

respectively, which are precisely the numbers on row 2 of Pascal's triangle:

$$\binom{2}{0}, \binom{2}{1}, \binom{2}{2}.$$

Binomial Expansion

Expansion of $(a + b)^3$

Work out the expansion of $(a + b)^3$ and compare the coefficients with the numbers in row 3 of Pascal's triangle.

$$\begin{aligned}(a+b)^3 &= (a+b)^2(a+b) \\&= (a+b)^2a + (a+b)^2b \\&= (a^2 + 2ab + b^2)a + (a^2 + 2ab + b^2)b \\&= a^3 + 2a^2b + b^2a \\&\quad a^2b + 2ab^2 + b^3 \\&= a^3 + 3a^2b + 3ab^2 + b^3 \\&= \binom{3}{0} a^3 b^0 + \binom{3}{1} a^2 b^1 + \binom{3}{2} a^1 b^2 + \binom{3}{3} a^0 b^3\end{aligned}$$

Binomial Expansion (cont')

Expansion of $(a + b)^4$

Work out the expansion of $(a + b)^3$ and compare the coefficients with the numbers in row 3 of Pascal's triangle.

The Binomial Theorem

The examples above suggest the following general result.

The Binomial Theorem (S05E11)

For each $n \in \omega$ and all $a, b \in \mathbb{R}$,

$$\begin{aligned}(a + b)^n &= \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n \\ &= \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k.\end{aligned}$$

- **Convention:** For each $x \in \mathbb{R}$, $x^0 = 1$.