Lec. 33. Problem Solving Session

04/08/2022

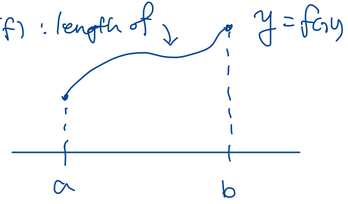
S11E09 Let $a, b \in \mathbb{R}$ such that $a < b$.

$$L : C^1[a,b] \to \mathbb{R}$$

$$L(f) = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \left(\begin{array}{l} \text{the length of} \\ \text{the curve } y = f(x) \\ \text{over } [a,b] \end{array} \right)$$

continuously differentiable on $[a,b]$

$L(f)$: length of



① For any $f \in C^1[a,b]$, $L(f) \geq b-a$.

$$L(f) = \int_a^b \sqrt{1+[f'(x)]^2}\, dx.$$

Pf. Let $f \in C^1[a,b]$. Then for any $x \in [a,b]$, $f'(x)$ exists and $f'(x) \in \mathbb{R}$, so $[f'(x)]^2 \geq 0$, so $1+[f'(x)]^2 \geq 1$, thus $\sqrt{1+[f'(x)]^2} \geq \sqrt{1} = 1$. It follows that

$$L(f) = \int_a^b \underbrace{\sqrt{1+[f'(x)]^2}}_{\geq 1 \text{ for any } x \in [a,b]}\, dx \geq \int_a^b 1\, dx = b-a.$$

Therefore, $L(f) \geq b-a$. $\square$

Note: $\operatorname{Rng}(L) \subseteq [b-a, \infty)$.

② Let $m \in [0, \infty)$ and let $h(x) = m(x-a)$ for all $x \in [a,b]$.

Find $L(h)$.

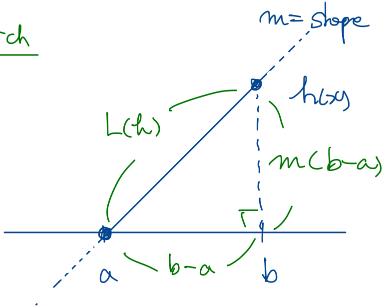
Pf (Calculus approach) Note that $h'(x) = m$.

$$\underline{L(h)} = \int_a^b \sqrt{1 + m^2}\, dx$$

$$= \sqrt{1+m^2} \underbrace{\int_a^b 1\, dx}_{= \, b-a}$$

$$= \boxed{\sqrt{1+m^2}\,(b-a)}$$

Sketch



Using geometry,

$$L(h) = \sqrt{(b-a)^2 + m^2(b-a)^2}$$

$$= \sqrt{1+m^2}\,(b-a)$$

Scratch work

Want $f \in C^1[a,b]$ of the form $f(x) = m(x-a)$, $m \in [0,\infty)$.

Know from ②:

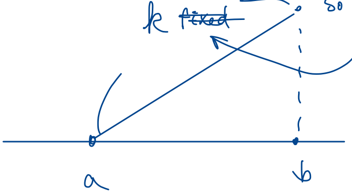
$$L(f) = \sqrt{1+m^2}\,(b-a) = k$$

$$1 + m^2 = \left(\frac{k}{b-a}\right)^2$$

$$m = \sqrt{\left(\frac{k}{b-a}\right)^2 - 1}$$

find slope m so that

k ~~fixed~~



③ $\mathrm{Rng}(L) = [b-a, \infty)$.

<u>Pf</u> We shall show $\mathrm{Rng}(L) \subseteq [b-a, \infty)$ and $[b-a,\infty) \subseteq \mathrm{Rng}(L)$.
The first set inclusion was established in ①, so it remains to show $[b-a, \infty) \subseteq \mathrm{Rng}(L)$.

Let $k \in [b-a, \infty)$, i.e., $k \geq b-a$. (WTS: $k \in \mathrm{Rng}(L)$, which means we can find $f \in C^1[a,b]$ s.t. $L(f) = k$.)
We propose that $f(x) = \sqrt{\left(\frac{k}{b-a}\right)^2 - 1}\,(x-a)$ will do.

// $f'(x)$

$$L(f) = \int_a^b \sqrt{\cancel{1} + \left(\frac{k}{b-a}\right)^2 \cancel{-1}}\,dx = \int_a^b \frac{k}{b-a}\,dx = \frac{k}{b-a}\int_a^b 1\,dx = k.$$

$\square$

S11E12&14 (Tips on how to prove / disprove surjection / injection.)

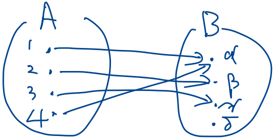
Let A and B be sets and let $f : A \to B$.

To prove that f is a surjection:

$$(\forall y \in B)(\exists x \in A)(y = f(x))$$

To disprove that f is a surjection:

$$(\exists y \in B)(\forall x \in A)(y \neq f(x))$$



To prove that f is an injection:

$$(\forall x_1, x_2 \in A)[f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$$

or $(\forall x_1, x_2 \in A)[x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)]$

To disprove that f is an injection:

$$(\exists x_1, x_2 \in A)[f(x_1) = f(x_2) \wedge x_1 \neq x_2]$$

or $(\exists x_1, x_2 \in A)[x_1 \neq x_2 \wedge f(x_1) = f(x_2)]$

§11 E15 Let S and T be sets. Define

$$f : \mathcal{P}(S) \times \mathcal{P}(T) \longrightarrow \mathcal{P}(S \cup T)$$

by $f(A, B) = A \cup B$ for all $A \subseteq S$ and all $B \subseteq T$.

(a) Show that f is a surjection.

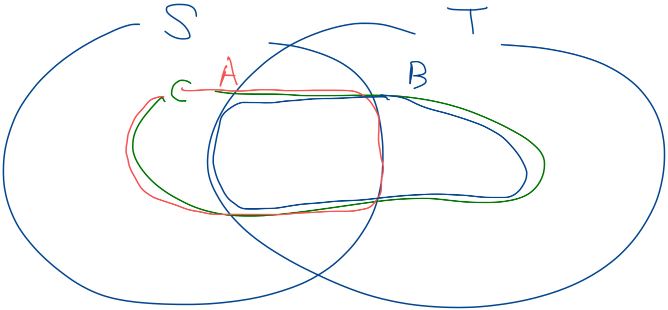
(WTS: for any $C \in \mathcal{P}(S \cup T)$, there exists $(A, B) \in \mathcal{P}(S) \times \mathcal{P}(T)$ such that $\underbrace{f(A,B)}_{A \cup B} = C$)

Pf Let $C \in \mathcal{P}(S \cup T)$, i.e., $C \subseteq S \cup T$.

Need to show that there exist $A \subseteq S$ and $B \subseteq T$ such that $A \cup B = C$.

$\vdots$

Intuition



Let $A = S \cap C$ and $B = T \cap C$.

Then show:

1. $A \subseteq S$
3. $B \subseteq T$
2. $A \cup B = C$