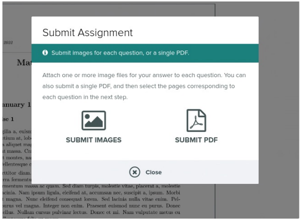
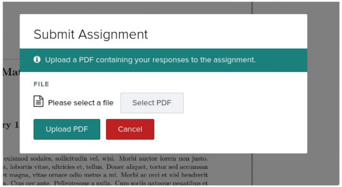
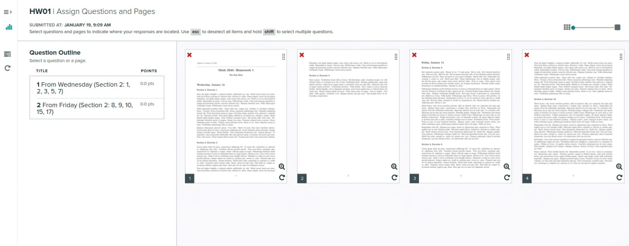
Proof Techniques

- HW 1 due 11:59 PM.
- Office Hour : 4:30 pm – 6:00 pm (Zoom)
- Quiz 2 on Friday (noon – 11:59 pm)

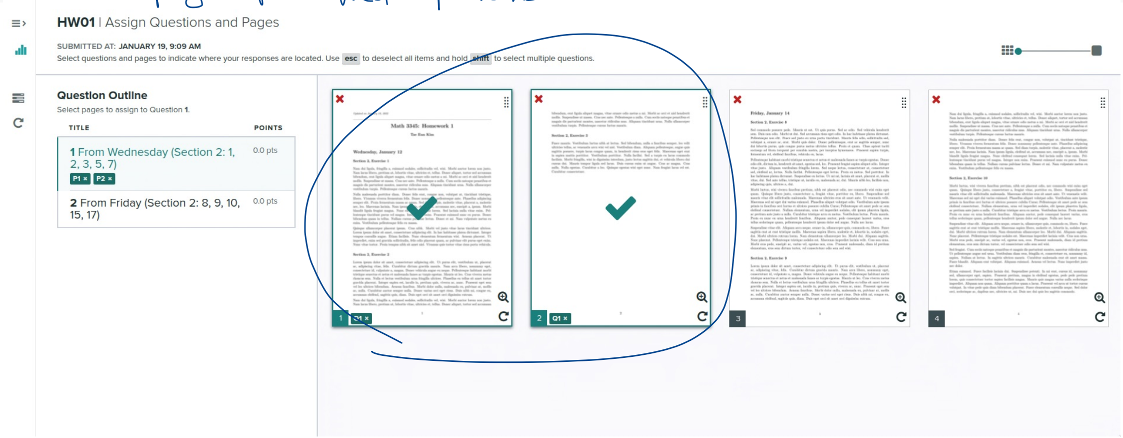
Uploading HW to Gradescope



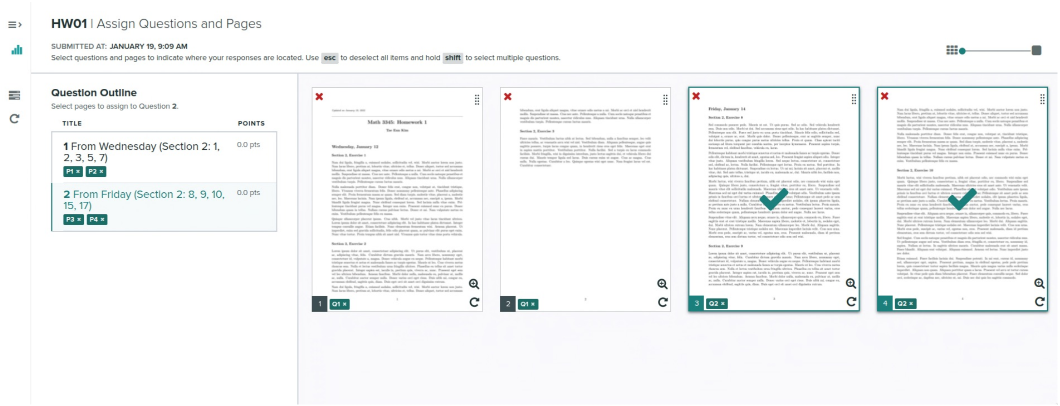




Select pages for Wed. problems.



Select pages for Fri. problems.



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2. Proof by Contradiction

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Logic of Solving Equations

Solving Equations

Logically speaking, to say that $x = a$ is a solution of the equation $f(x) = 0$ is to state

$$f(x) = 0 \iff x = a$$

which usually can be seen by a chain of biconditionals.

For example, we see that $x^2 = 5x - 6$ if and only if $x = 2$ or $x = 3$ by:

$$\begin{aligned} x^2 = 5x - 6 &\iff x^2 - 5x + 6 = 0 \\ &\iff (x-2)(x-3) = 0 \\ &\iff x - 2 = 0 \text{ or } x - 3 = 0 \\ &\iff x = 2 \text{ or } \ x = 3. \end{aligned}$$

One needs to be careful to confirm that all steps are true biconditional sentences.

Examples

Rational Equation

Solve the equation
$$\frac{x-2}{x^2+2x-8} = \frac{1}{8}.$$

$\| f(x)$

Erroneous solution.

$\Longrightarrow x - 2 = (1/8)(x^2+2x-8)$

$\Longleftrightarrow 8x - 16 = x^2 + 2x - 8$

$\Longleftrightarrow 0 = x^2 - 6x + 8 = (x-2)(x-4)$

$\Longleftrightarrow x = 2, 4$

Which step is not a true biconditional sentence?

If $x=2$, then

$x^2+2x-8 = 4+4-8 = 0$.

(division by zero).

This work shows:

$\boxed{\text{If } f(x) = \frac{1}{8}, \text{ then } x=2 \text{ or } x=4.}$

Correct solution.

- Previous work showing

  $$f(x) = \frac{1}{8} \Rightarrow x=2 \text{ or } x=4$$

- Now if $x=2$, then $x^2+2x-8 = 4+4-8 = 0$,

  so $f(x)$ is undefined, so $f(x) = \frac{1}{8}$ is not true.

- If $x=4$, then $x-2 = 4-2 = 2$ and $x^2+2x-8 = 16+8-8 = 16$,

  so $f(x) = \frac{x-2}{x^2+2x-8} = \frac{2}{16} = \frac{1}{8}$. That is, if $x=4$, then $f(x) = \frac{1}{8}$.

Therefore, $f(x) = \frac{1}{8}$ if and only if $x=4$.

Think

$$f(x) = \frac{x+1}{x^2 - x - 2} \qquad g(x) = \frac{1}{x-2}$$

$$= \frac{x+1}{(x+1)(x-2)} = \cancel{g(x)}$$

Equation Involving Radicals

Solve the equation

$$x = -\sqrt{x+6} =: g(x)$$

An erroneous solution:

$$\Rightarrow x^2 = x + 6 \quad \text{squaring both sides}$$
$$\Leftrightarrow x^2 - x - 6 = 0$$
$$\Leftrightarrow (x+2)(x-3) = 0$$
$$\Leftrightarrow x = -2, 3$$

If $x = g(x)$,

then $x = -2$ or $x = 3$.

Is $x = 3$ a solution of the original equation?

$$(\text{LHS}) = 3 \neq -\sqrt{3+6} = -\sqrt{9} = -3 = (\text{RHS})$$

Correct solution.

- Prev. work

- Now if $x=-2$, then $g(x) = -\sqrt{x+6} = -\sqrt{-2+6} = -\sqrt{4} = -2$,

  So $x = g(x)$. That is, if $x=-2$, then $x=g(x)$.

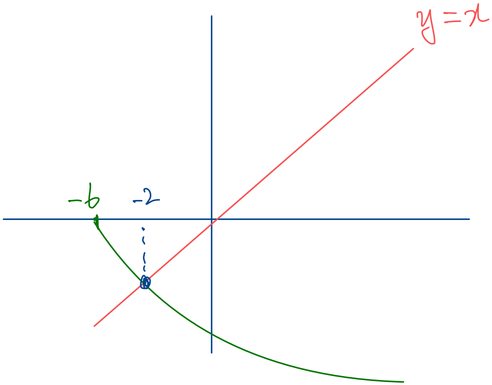
- If $x=3$, then $g(x) = -\sqrt{x+6} = -\sqrt{3+6} = -\sqrt{9} = -3$,

  So $x \neq g(x)$.

Therefore, $x = g(x)$ iff $x = -2$.

Practical concerns (insight)

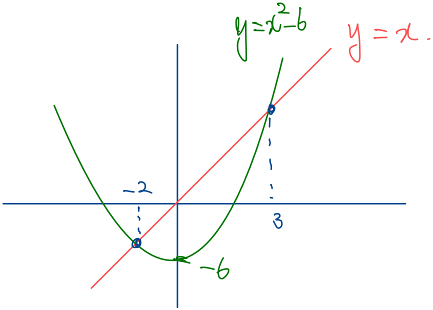
Orig. eqn $\quad x = -\sqrt{x+6}$



Mod. eqn (upon squaring)

$$x^2 = x + 6$$

$$x^2 - 6 = x$$



Proof by Contradiction

Overview: Proof Techniques

The follow is the list of proof techniques discussed in Section 2 of the textbook:

- [x] Conditional proof
- [ ] Proof by contradiction
- [ ] Proof by contraposition

- used to show $P \Rightarrow Q$ is a tautology.
- template.

A1: Assume P is true

Show Q is true.

Discharging A1, $P \Rightarrow Q$ under no assumptions.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Contradictions

A *contradiction* is a sentence of the form $Q \wedge \neg Q$, which is false regardless of the truth value of Q.

Q	$\neg Q$	$Q \wedge \neg Q$
T	F	F
F	T	F

Proof by Contradiction

Proof by Contradiction

To prove a sentence P, assume $\neg P$ and deduce a contradiction. This approach is known as the method of *proof by contradiction*.

Template. To prove P:

- Begin with "Assume $\neg P$ is true."
- Deduce a contradiction.
- Conclude that P is true.

a contradiction

pf. by contradiction

Why does it work?

P	$\neg P$	Λ	$\neg P \Rightarrow \Lambda$
T	F	F	T
F	T	F	F

Example

Let n be an integer. Using the method of proof by contradiction, prove that

$$\underbrace{\text{If } n^2 \text{ is an odd number}}_{A_1}\text{, then } \underbrace{n \text{ is an odd number}}_{C_1}.$$

Soln

A1 : Assume n^2 is an odd number.

We wish to show that n is an odd number.

Assume (towards a contradiction) n is not an odd number. Since n is an integer, n must be an even number, so n^2 is an even number, so n^2 is not an odd number. This leads to a contradiction. We reject the assumption that n is not an odd number. Therefore, n is an odd number under A1.

proof by contrad.

Discharging A1, we see that $A_1 \Rightarrow C_1$ under no assumptions.

Proof of a Negative Sentence

The usual way to prove a negative sentence $\neg P$ to prove by contradiction, that is, assume P and deduce a contradiction.

Why does it work?

Proof by Contradiction on $\neg P$.

Proof of a Negative Sentence (cont')

Section 2, Exercise 23

Use the method of conditional proof to explain in words why

$$\underbrace{[(P \Rightarrow Q) \wedge \neg Q]}_{A_1} \Rightarrow \underbrace{\neg P}_{C_1}$$

is a tautology.

Suggestion: Conditional proof.

A1: Assume A_1 is true.

NTS C_1 is true.

⋮

} proof of a negative sentence.

Proof of a Negative Sentence (cont')

Proof by Contraposition

Overview: Proof Techniques

The follow is the list of proof techniques discussed in Section 2 of the textbook:

- ☑ Conditional proof
- ☑ Proof by contradiction
- ☐ Proof by contraposition

To prove $P \Rightarrow Q$,

assume P is true and

strive to show Q is true.

To prove P,

assume $\neg P$ is true and

deduce a contradiction $(Q \wedge \neg Q)$.

Contrapositive

Given $P \Rightarrow Q$, the related conditional sentence $\neg Q \Rightarrow \neg P$ is called *the contrapositive of* $P \Rightarrow Q$. Note that $P \Rightarrow Q$ is logically equivalent to $\neg Q \Rightarrow \neg P$. (Confirm this using a truth table.) ← Exercise.

Example. Given the conditional sentence

A: If $\overbrace{\text{today is Sunday}}^{P}$, then $\overbrace{\text{I do not have to go to work today}}^{Q}$.

- Converse of A: $Q \Rightarrow P$

  If I don't have to go to work today, then today is Sunday.

- Contrapositive of A: $\neg Q \Rightarrow \neg P$

  If I have to go to work today, then today is not Sunday.

Proof by Contraposition

Proof by Contraposition

To prove $P \Rightarrow Q$, it suffices to prove $\neg Q \Rightarrow \neg P$.

relies on the fact that

$$(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$$

Proof by Contraposition (cont')

Example (revisited)

Let n be an integer. Using the method of proof by contraposition, prove that

If n^2 is an odd number, then n is an odd number.

Solution. The given sentence is logically equivalent to the sentence

contrapositive of given

$$\text{If } \underbrace{n \text{ is not an odd number}}_{A}\text{, then } \underbrace{n^2 \text{ is not an odd number}}_{C}. \qquad (\star)$$

which we will prove.

A1: Assume that n is not an odd number.

(We wish to show that n^2 is not an odd number.) "short-term goal"

Since n is an integer but n is not an odd number, n is an even number.

Hence n^2 is an even number, so n^2 is not an odd number.

We have shown this under A1.

Discharging A1, we conclude that the conditional sentence $(\star)$ is true under no assumptions. This completes the proof by contraposition that the original conditional sentence is true. □

Proof by Contradiction vs Proof by Contraposition

Let's examine the two proof techniques in proving $P \Rightarrow Q$.

Proof by contradiction.

Assume P. (NTS Q is true.)
Assume $\neg Q$.
Show $\neg P$.
Contradiction, $P \wedge \neg P$!
So Q must be true.
Therefore, $P \Rightarrow Q$.

Proof by Contrad.

Proof by contraposition.

Assume $\neg Q$.
Show $\neg P$. (if this can be done w/o P.)
So $\neg Q \Rightarrow \neg P$.
Therefore, $P \Rightarrow Q$, by contraposition.

<u>Homework</u> (1/19 ; due Wed 1/26)

Section 2 : # 19, 20, 24