Binomial Coefficients

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Warm-Up

Remark 4.50

Let $d \in \mathbb{N}$, $x, y \in \mathbb{Z}$, and p be a prime number.

1. If $p \mid xy$, then $p \mid x$ or $p \mid y$.
2. If $d \mid xy$, then these exist $d_1, d_2 \in \mathbb{N}$ such that $d_1 \mid x$, $d_2 \mid y$, and $d = d_1 d_2$.

The proofs of these facts require *complete induction*.

e.g. $6 \mid (2 \cdot 9)$

$6 = 2 \cdot 3$

$2 \mid 2$, $3 \mid 9$

③ Let $x_1, \cdots, x_n \in \mathbb{Z}$.

If $d \mid x_1 \cdots x_n$, then there exist $d_1, d_2, \cdots, d_n \in \mathbb{N}$ such that

$d_1 \mid x_1$, $d_2 \mid x_2$, $\cdots$, $d_n \mid x_n$, and

$d = d_1 d_2 \cdots d_n$.

S04.E17 (assigned)

$x^3 = c$

Context: We proved $\sqrt{2}$ is irrational. How about $\sqrt{c}$ for general c?

Example 4.52

1. Let x be a rational number such that $x^2 = c$, where c is a whole number. Then x is an integer.
2. Let c be a whole number which is not a perfect square. Then $\sqrt{c}$ is irrational.

" Fact ☺ "

Proof of ①. Since x is rational, we can pick two numbers $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that $x = a/b$ and the fraction is in lowest terms. Then

$$(a/b)^2 = c, \quad \text{so} \quad a^2/b^2 = c, \quad \text{so} \quad a^2 = cb^2, \quad \text{so} \quad a^2 = (cb)\,b.$$

Then $b \mid a^2$, because cb is an integer. By Rmk. 4.50, we can pick $b_1, b_2 \in \mathbb{N}$ such that $b_1 \mid a$, $b_2 \mid a$, and $b = b_1 b_2$. So $b_1 \mid a$ and $b_1 \mid b$. But then $b_1 = 1$ since a/b is in lowest terms.

Similarly, $b_2 = 1$. Then it follows that $b = b_1 b_2 = 1 \cdot 1 = 1$.
Then $x = a/b = a/1 = a$. So x is an integer, because a is an integer. $\square$

Binomial Coefficients

Pascal's Triangle

$A[1,3]$, $A(1,3)$, $A_{1,3}$

row index column index

$\binom{n}{k}$ row col.

$n \in \omega = \{0, 1, 2, \cdots\}$

$k \in \{0, 1, \cdots, n\}$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
 & & & & & \vdots & & & & &
\end{array}
$$

$$\binom{4}{3} = \binom{3}{3} + \binom{3}{2}$$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Pascal's Triangle and Binomial Coefficients

Recall. For all $n \in \omega$ and all $k \in \{0, \ldots, n\}$, the binomial coefficient $\binom{n}{k}$ denotes the k-th number on the n-th row of Pascal's triangle.

Key features

1. $\dbinom{0}{0} = 1.$
2. *Boundary conditions*: For each $n \in \mathbb{N}$,
$$\binom{n}{0} = \binom{n}{n} = 1.$$
3. *Recurrence relation*: For each $n \in \omega$ and all $k \in \{1, \ldots, n\}$,
$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Why *n choose k*?

the only avail. slot

$$\underline{\ \ }\ \underline{4}$$

All 2-element subsets of the 4-element set $\{1,2,3,4\}$ are

$$\underbrace{\{1,2\},\{1,3\},\{2,3\}}_{\text{ones without } 4},\ \underbrace{\{1,4\},\{2,4\},\{3,4\}}_{\text{ones with } 4}.$$

Note that

- the number of subsets without 4 is $3 = \binom{3}{2}$
- the number of subsets with 4 is $3 = \binom{3}{1}$

Thus the total number of 2-element subsets of the 4-element set is

$$6 = 3 + 3 = \binom{3}{2} + \binom{3}{1} = \binom{4}{2}$$

Why *n choose k*? (cont')

In general, one can count the number of k-element subsets of the $(n+1)$-element set

$$\{\underline{1, 2, \ldots, n}, n+1\}$$

$$\overbrace{\underbrace{\_\_\ \_\_\ \_\_\ \cdots\ \_\_}_{(k-1)\text{ avail.}}\ \underline{n+1}}^{k\text{ slots}}$$

in an analogous fashion:

- the number of subsets without $n+1$ is $\binom{n}{k}$
- the number of subsets with $n+1$ is $\binom{n}{k-1}$

Thus the total number of k-element subsets of the $(n+1)$-element set is

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

rec. rel'n

Why *n choose k*? (cont')

The idea above is key to a proof by induction of the following theorem.

Number of Subsets (cf. S14E03)

For each $n \in \boldsymbol{\omega}$, for each $k \in \{0, \ldots, n\}$, the number of k-element subsets of an n-element set is $\binom{n}{k}$.

Why Binomial Coefficients?

Consider the expansion of $(a+b)^2$:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= \underline{(a+b)a} + \underline{(a+b)b}\\ &= a^2 + ba\\ &\qquad + ab + b^2\\ &= a^2 + 2ab + b^2. \quad = \binom{2}{0} a^2 b^0 + \binom{2}{1} a^1 b^1 + \binom{2}{2} a^0 b^2\end{aligned}$$

Note that the coefficients of a^2, ab, and b^2 are

$$1, 2, 1,$$

respectively, which are precisely the numbers on row 2 of Pascal's triangle:

$$\binom{2}{0}, \binom{2}{1}, \binom{2}{2}.$$

Binomial Expansion

Expansion of $(a+b)^3$

Work out the expansion of $(a+b)^3$ and compare the coefficients with the numbers in row 3 of Pascal's triangle.

$$
\begin{aligned}
(a+b)^3 &= (a+b)^2(a+b) \\
&= (a+b)^2 a + (a+b)^2 b \\
&= (a^2+2ab+b^2)\,a + (a^2+2ab+b^2)\,b \\
&= a^3 + 2a^2b + b^2a \\
&\quad + a^2b + 2ab^2 + b^3 \\
&= a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3 \\
&= \binom{3}{0} a^3 b^0 + \binom{3}{1} a^2 b^1 + \binom{3}{2} a^1 b^2 + \binom{3}{3} a^0 b^3
\end{aligned}
$$

Binomial Expansion (cont')

Expansion of $(a+b)^4$

Work out the expansion of $(a+b)^3$ and compare the coefficients with the numbers in row 3 of Pascal's triangle.

The Binomial Theorem

The examples above suggest the following general result.

The Binomial Theorem (S05E11)

For each $n \in \omega$ and all $a, b \in \mathbb{R}$,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

- **Convention**: For each $x \in \mathbb{R}$, $x^0 = 1$.