Math 3345 (Section 22786, 10:20 class)

$\boxed{\text{Final Exam}}$ (10:00 ~ 11:45, Thursday, April 28)

- Take-at-home as midterms / upload to Gradescope
- Cumulative
- Released at 9:55 am ; Closed at 12:00 pm.

04 / 25 / 2022

Lecture 38 Review for Final Exam

Tautologies & Conditional proof

- Lec 3
- Exam 1 #1

Exercise. $(P \Rightarrow Q) \Rightarrow \{[P \Rightarrow (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)\}$

Rational and Irrational Numbers

- Lec. 9
- Exam 1 #2

Exercise. Let x and y be irrational. Prove that

$x+y$ is irrational or $x-y$ is irrational.

Note Proof by contradiction.

To show z is irrational, NTS
1. z is real
2. z is not rational.

Rational Roots

- Sec. 4 Exercises 17, 18, 19, 20

Exam 2 #2 Let $x \in \mathbb{Q}$ and $x > 0$ such that $rx^2 = s$, where $r, s \in \mathbb{N}$. Prove that $x = a/b$ where $a, b \in \mathbb{N}$ and $a \mid s$ and $b \mid r$.

Proof: Since x is a positive rational number, we can pick $a, b \in \mathbb{N}$ such that $x = a/b$ and the fraction a/b is in lowest terms. On substituting $x = a/b$ into $rx^2 = s$, we obtain

$$r\left(a/b\right)^2 = s$$

$$\Rightarrow \frac{r a^2}{b^2} = s$$

$$\Rightarrow \boxed{r a^2 = s b^2}. \qquad (*)$$

a_2 is a common factor of a & b.

Now we can write $(*)$ as $(ra)a = s \cdot b \cdot b$, so a divides $s \cdot b \cdot b$ because $ra \in \mathbb{N}$. Thus using the given fact, we can find $a_1, a_2, a_3 \in \mathbb{N}$ such that $a_1 | s$, $a_2 | b$, and $a_3 | b$, and $a = a_1 a_2 a_3$. Since $a_2 | b$ and $a_2 | a$, but a/b is in lowest terms, it must be the case that $a_2 = 1$. Similarly, $a_3 = 1$.

Thus $a = a_1 \cdot a_2 \cdot a_3 = a_1 \cdot 1 \cdot 1 = a_1$, and since

$a_1 \mid s$, <u>a divides s</u>.

[DIY] Show that b divides r by repeating a similar argument w/ <u>appropriate adjustments.</u>

Induction and Complete Induction

- Lees 13, 14, 16, 19
- Exam 1 # 5, Exam 2 # 3
- Also review problems involving the binomial theorem.

divisibility

recursively defined sequence
(Fibonacci, Pell sequences)

Induction vs Insight

- Lec 18 : Sums of powers / geometric sums
- Exam 2 # 4

$$\begin{cases} 1+2+3+\cdots+n \\ 1^2+2^2+3^2+\cdots+n^2 \\ 1^3+2^3+3^3+\cdots+n^3 \end{cases}$$

$$1+x+x^2+\cdots+x^{n-1}$$

Exercise Let $n \in \mathbb{N}$, let $x \in \mathbb{R}$, and assume $x \neq 1$.

Derive formulas for

$$S = \sum_{k=0}^{n-1} x^k \leftarrow$$

$$T = \sum_{k=1}^{n} k x^k \leftarrow$$

$$U = \sum_{k=1}^{n} k^2 x^k$$

$$
\begin{aligned}
T &= x + 2x^2 + 3x^3 + \cdots + nx^n \\
-)\quad xT &= \phantom{x + {}} x^2 + 2x^3 + \cdots + (n-1)x^n + nx^{n+1} \\
\hline
(1-x)T &= \underbrace{x + x^2 + x^3 + \cdots + x^n}_{= xS = x\frac{1-x^n}{1-x}} - nx^{n+1}
\end{aligned}
$$

Set operations

- Lec. 22 ~ 28
- Exam 2 #5

(a) Let x be any element.

$x \in$ ★

iff $x \in \ldots$

iff $x \in \ldots$

$\vdots$

iff $x \in$ ☺. Therefore ★ = ☺ $\square$

(b) Deduce ...

Do not replicate the previous proof but rather use the established result.

Functions

- Review Lec 33 (problem solving session)
- S11 E09 (Determining Rng(f))
- S11 E15 (Set-valued functions)

Infinite sets

- Equinumerousness

- Examples of infinite sets

  - A proper subset of an infinite set which is equinumerous to the whole set.

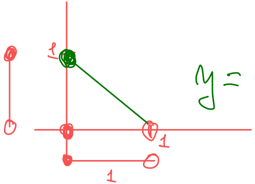
  (Describe bijections)

- Cantor's Diagonal Lemma

Exercise Describe bijections

- from $\mathbb{Z}$ to $\mathbb{N}$
- from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$
- from $[1, 3]$ to $[-2, 1]$
- from $[0, 1)$ to $(0, 1]$

$$f(x) = \begin{cases} 2x & \text{if } x = 1, 2, 3, \cdots \\ 2(-x) + 1 & \text{if } x = 0, -1, -2, \cdots \end{cases}$$



$y = 1 - x$