- Proof by induction used to prove $(\forall n \in \mathbb{N})\, P(n)$

More Examples of Induction

- Principle of Math. Induction.

$$(\forall P)\left[\left\{ \underbrace{P(1)}_{\text{Base case}} \wedge \underbrace{(\forall n \in \mathbb{N})\left[P(n) \Rightarrow P(n+1)\right]}_{\text{Inductive step}} \right\} \Longrightarrow (\forall n \in \mathbb{N})\, P(n)\right]$$

- Template

Declaration : Let $P(n)$ be the sentence ...

Base Case : $P(1)$ is true because ...

Inductive step : Let $n \in \mathbb{N}$ such that $P(n)$ is true

Inductive hypothesis

Conclusion : Therefore, by induction, for each $n \in \mathbb{N}$, $P(n)$ is true.

Contents

From $\mathbb{N}$ to $\mathbb{Z}$

$\mathbb{N} = \{1, 2, 3, \ldots\}$

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Example 1

Prove that for each $x \in \mathbb{Z}$, $\underbrace{x \text{ is even or } x \text{ is odd}}_{P(x)}$.

$W = \{0, 1, 2, \ldots\}$ whole numbers.

Plan

Let $x \in \mathbb{Z}$. Then $x \geq 0$ or $x \leq -1$.

PART 1 Show $(\forall x \in W)\ P(x)$ is true using induction.

PART 2 Show $(\forall x \leq -1)\ P(x)$ is true using PART 1.

Proof. Let $x \in \mathbb{Z}$. Then $x \geq 0$ or $x \leq -1$.

PART 1 We wish to show $(\forall x \in \omega)(x$ is even or x is odd$)$

using induction.

Let $P(x)$ be the sentence

x is even or x is odd.

BASE CASE $P(0)$ is true, because 0 is even.

Show $P(0)$ is true.

INDUCTIVE STEP Let $x \in \omega$ such that $P(x)$ is true.

Show $(\forall x \in \omega)[P(x) \Rightarrow P(x+1)]$.

So x is even or x is odd. In the case where x is even, $x+1$ is odd. In the case

x is odd, $x+1$ is even. In either case, $x+1$ is even or $x+1$ is odd. Thus $P(x+1)$ is true

CONCLUSION Therefore, by induction, for each $x \in \omega$, $P(x)$ is true.

PART 2 We wish to show that for each $x \leq -1$, $P(x)$ is true.

Let x be a negative integer. Then $-x \in \omega$. Thus, by PART 1, $P(-x)$ is true, that is, $-x$ is even or $-x$ is odd.

Case 1 Suppose $-x$ is even. Then $x = (-x)(-1)$.

So x is even because an even number times any integer is even.

Case 2 Suppose $-x$ is odd. Then $x = (-x)(-1)$.

So x is odd because the product of two odd numbers is odd.

Thus in either case, x is even or x is odd.

Hence for each integer $x \le -1$, $P(x)$ is true.

Finally, by PART 1 and PART 2, we conclude that for each $x \in \omega$, $P(x)$ is true. $\square$

Notes: Induction over $\mathbb{Z}$.

- Induction may start from a number other than 1.

e.g. $\omega = \{0, 1, 2, \cdots\}$

$S = \{3, 4, 5, \cdots\}$ $\quad (\forall x \in S)\, P(x)$: $\quad P(3) \wedge (\forall x \in S)[P(x) \Rightarrow P(x+1)]$

- To prove a universal sentence $(\forall x \in \mathbb{Z})P(x)$:

1. Prove by induction that $P(x)$ is true for each nonnegative integer[1] x.
2. Prove that $P(x)$ is true for each negative integer x.

Pascal's Triangle and the Binomial Theorem

Pascal's Triangle

The following infinite array of numbers is known as *Pascal's tirangle*:

$$
\begin{array}{lccccccccccc}
0^{\text{th}} \text{ row} \longrightarrow & & & & & & 1 & & & & & \\
1^{\text{st}} \text{ row} \longrightarrow & & & & & 1 & & 1 & & & & \\
2^{\text{nd}} \text{ row} \longrightarrow & & & & 1 & & 2 & & 1 & & & \\
3^{\text{rd}} \text{ row} \longrightarrow & & & 1 & & 3 & & 3 & & 1 & & \\
4^{\text{th}} \text{ row} \longrightarrow & & 1 & & 4 & & 6 & & 4 & & 1 & \\
5^{\text{th}} \text{ row} \longrightarrow & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & & & & & & \vdots & & & & &
\end{array}
$$

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$$

$n = 4, \quad k = 2$

Notation. For all $n \in \omega$ and all $k \in \{0, \ldots, n\}$, let $\binom{n}{k}$ denote the k-th number on the n-th row; this notation is read *n choose k*. This is also called a *binomial coefficient*.

$$
\begin{array}{ccccc}
 & & \binom{0}{0} & & \\
 & \binom{1}{0} & & \binom{1}{1} & \\
\binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} \\
 & & \vdots & &
\end{array}
$$

Key Features.

1. $\binom{0}{0} = 1.$
2. *Boundary conditions*: For each $n \in \mathbb{N}$, $\binom{n}{0} = \binom{n}{n} = 1.$
3. *Recurrence relation*: For each $n \in \omega$ and all $k \in \{1, \ldots, n\}$, $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$

Example

Application of Key Features

Use **Key Features** above to compute $\binom{4}{2}$.

Solution.

$$
\begin{aligned}
\binom{4}{2} &= \binom{3}{2} + \binom{3}{1} && \text{by } 3 \\
&= \underbrace{\binom{2}{2} + \binom{2}{1}} + \underbrace{\binom{2}{1} + \binom{2}{0}} && \text{by } 3 \\
&= 1 + 2\binom{2}{1} + 1 && \text{by } 2 \\
&= 1 + 2\left[\binom{1}{1} + \binom{1}{0}\right] + 1 && \text{by } 3 \\
&= 1 + 2(1+1) + 1 \\
&= 6.
\end{aligned}
$$

$$\binom{5}{4} = \binom{4}{4} + \binom{4}{3}$$

$$= 1 + \binom{3}{3} + \binom{3}{2}$$

$$\vdots$$

Naming: *n choose k*

Question: Why is $\binom{n}{k}$ called "n choose k"?

Number of Subsets

List all subsets of the set $\{a, b, c, d\}$ with exactly 2 elements.

Solution.

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$$

There are $6 = \binom{4}{2}$ subsets with two elements. In other words, there are $\binom{4}{2} = 6$ ways to choose 2 elements from a set with 4 elements.

In Section 14, we will prove that for each $n \in \omega$, for each n-element set A, for each $k \in \{0, \ldots, n\}$, the number of k-element subsets of A is $\binom{n}{k}$.

Naming: Binomial Coefficients

$0^{th} \to 1$
$1^{st} \to 1 \quad 1$
$2^{nd} \to 1 \quad 2 \quad 1$
$3^{rd} \to 1 \quad 3 \quad 3 \quad 1$

Question: Why is $\binom{n}{k}$ called a binomial coefficient?

Expansion of $(a+b)^3$

1. Compute $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$, and $\binom{3}{3}$.
2. Expand the cube of the binomial $a+b$, that is, expand $(a+b)^3$.

① $\binom{3}{0} = 1$, $\binom{3}{1} = 3$, $\binom{3}{2} = 3$, $\binom{3}{3} = 1$

② $$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$= \binom{3}{0} a^3 b^0 + \binom{3}{1} a^2 b^1 + \binom{3}{2} a^1 b^2 + \binom{3}{3} a^0 b^3$$

Binomial Theorem

The example above suggests:

The Binomial Theorem

For each $n \in \omega$ and all $a, b \in \mathbb{R}$,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$
$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$$

- **Convention**: For each $x \in \mathbb{R}$, $x^0 = 1$.