Section 3

Quantifiers (I)

Quiz 2 (noon ~ 11:59 pm)

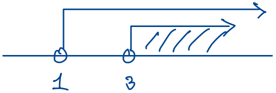
Contents

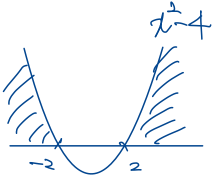
Basics of Quantifiers

Motivation

Let x be a real number. Consider the following sentences.

- $A(x)$: If $x > 3$, then $x > 1$.
- $B(x)$: $x^2 - 4 > 0$.





The truth value of each sentence depends on the value of the variable x.

- $A(x)$ is true for all x. (T)
- $B(x)$ is true for $x < -2$ or $x > 2$. (T)

In general, they can be rephrased using *quantifiers* as:

- For all x, $A(x)$ is true.
- For some x, $B(x)$ is true.

Quantifiers

The quantifiers $\forall$ and $\exists$, along with the logical connectives, are main ingredients of modern symbolic logic.

Quantifier	Symbol	Technical Name
"for each"	$\forall$	*universal quantifier*
"for some"	$\exists$	*existential quantifier*

Example. Let x be a person in this class room. Let $P(x)$ stands for "x likes ramen." Then

- $(\forall x)P(x)$: "For each x, x likes ramen." or "Everybody likes ramen."
- $(\exists x)P(x)$: "For some x, x likes ramen." or "Somebody likes ramen."

Notes

Alternate ways to read.

$(\forall x)P(x)$:

For each x, $P(x)$.
For all x, $P(x)$.
For every x, $P(x)$.
For any x, $P(x)$.

$(\exists x)P(x)$:

For some x, $P(x)$.
For at least one x, $P(x)$.
There exists x such that $P(x)$.

Universe of Discourse

The collection over which the variable x ranges is called *the universe of discourse*. When clear from context, it is omitted in notation; if not, specify the universe of discourse using the following notation.

$$\boxed{(\forall x \in U) P(x) \quad \text{or} \quad (\exists x \in U) P(x).}$$

"is a member of"

Frequently used collections.

- $\mathbb{N}$: the set of natural numbers, $\{1, 2, 3, \ldots\}$
- $\mathbb{Z}$: the set of integers, $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $\mathbb{Q}$: the set of rational numbers
- $\mathbb{R}$: the set of real numbers
- $\mathbb{C}$: the set of complex numbers

Free and Bound Variables

- In $P(x)$, x can stand for any particular element of the universe of discourse; it is called a *free variable*.
- In $(\forall x)P(x)$ or $(\exists x)P(x)$, x varies over the universe of discourse, not standing for any particular element; it is called a *bound variable* or a *dummy variable*.

- $P(x)$: x likes ramen. $\longrightarrow$ x is a free variable.

- $(\forall x)P(x)$: For each x, x likes ramen. $\longrightarrow$ x is a dummy variable.

Same as $(\forall y)P(y)$

Universal and Existential Quantifiers

Universal Quantifier ($\forall$)

Let U be the universe of discourse.

- $(\forall x)P(x)$ is true when $P(x)$ is true for all values of x in U.
- To show $(\forall x)P(x)$ is false, it suffices to show that $P(x)$ is false for at least one value of x in U; such x is said to be *a counterexample that disproves the universal sentence.*

universal sentence.

Universal Quantifier (cont')

Example. State whether each of the following sentences is true or false. Then prove your claim.

1. $(\forall x \in \mathbb{R})(x - 2 = 5)$ T / F We claim $(\forall x \in \mathbb{R})(x-2=5)$ is false.

<u>Proof</u> Assume that $(\forall x \in \mathbb{R})(x-2=5)$ is true.

Then since x is a real number, $x-2=5$.

But $x-2=0 \neq 5$. This is a contradiction.

Hence $(\forall x \in \mathbb{R})(x-2=5)$ is false. $\square$

2. $(\forall x \in \mathbb{R})(x^2 + 6x + 10 > 0)$ T / F

<u>Proof</u>

"Completion of the square"

Existential Quantifier

Let U be the universe of discourse.

- $(\exists x)P(x)$ is true when $P(x)$ is true for at least one value of x in U; such x is said to be an *example that proves the existential sentence*.
- To show $(\exists x)P(x)$ is false, it is necessary to show that $P(x)$ is false for all values of x in U.

Existential Quantifier (cont')

Example. State whether each of the following sentences is true or false.

- $(\exists x \in \mathbb{R})(x - 2 = 5)$ T / F

Proof

- $(\exists x \in \mathbb{R})(x^2 + 6x + 10 < 0)$ T / F

Proof

Notes on Quantifiers

Connections to Logical Connectives

Suppose the universe of discourse consists only of two objects $\{a, b\}$. Note that

- $(\forall x)P(x)$ is true exactly when $P(a) \wedge P(b)$ is true.
- $(\exists x)P(x)$ is true exactly when $P(a) \vee P(b)$ is true.

In general, when the universe of discourse is a finite set $\{a_1, a_2, \ldots, a_n\}$, then

- $(\forall x)P(x)$ has the same truth value as $P(a_1) \wedge P(a_2) \wedge \cdots \wedge P(a_n)$.
- $(\exists x)P(x)$ has the same truth value as $P(a_1) \vee P(a_2) \vee \cdots \vee P(a_n)$.

Upshot:
- $\forall$ is a generalization of $\wedge$.
- $\exists$ is a generalization of $\vee$

Notation

Suppose A is a subcollection of the universe of discourse. Then

- $(\forall x \in A)P(x)$ is a shorthand notation for $(\forall x)[(x \in A) \Rightarrow P(x)]$.
- $(\exists x \in A)P(x)$ is a shorthand notation for $(\exists x)[(x \in A) \wedge P(x)]$.

When the universe of discourse is $\mathbb{R}$, a subcollection may be characterized by an inequality in which case one may use notations *e.g.*,

- $(\forall x > 0)(2x + 7 = 3)$ T / F
- $(\exists x \geq 7)(x^2 - 4x + 3 > 0)$ T / F

Scope of Quantifiers

The scope of a quantifier is specified using appropriate delimiters.

Example. Let n be an element in $\{2, 3, 5, 7\}$ and let

$$P(n): \; n \text{ is a prime number.}$$
$$Q(n): \; n \text{ is an even number.}$$

Then

- $(\forall n)P(n) \wedge Q(n)$ stands for

- $(\forall n)[P(n) \wedge Q(n)]$ stands for

Homework (1/21 ; due Wed 1/26)

Section 3 : # 1 (e–k), 3, 4