Applications of LU Factorization

Contents

Applications of PLU Factorization

- Solving system of eqn.
- computing matrix inverse
- computing determinants

Solving a Square System

Multiplying $A\mathbf{x} = \mathbf{b}$ on the left by P we obtain

$$\underbrace{PA}_{=LU}\mathbf{x} = \underbrace{P\mathbf{b}}_{=:\boldsymbol{\beta}} \quad \longrightarrow \quad LU\mathbf{x} = \boldsymbol{\beta}\,,$$

which can be solved in two steps:

- Define $U\mathbf{x} = \mathbf{y}$ and solve for $\mathbf{y}$ in the equation

$$L\mathbf{y} = \boldsymbol{\beta}\,. \qquad \text{(forward elimination)}$$

- Having calculated $\mathbf{y}$, solve for $\mathbf{x}$ in the equation

$$U\mathbf{x} = \mathbf{y}\,. \qquad \text{(backward substitution)}$$

- $A\vec{x} = \vec{b}$
- $PA\vec{x} = \underbrace{P\vec{b}}_{=\vec{\beta}}$
- $L(\underbrace{U\vec{x}}_{\vec{y}}) = \vec{\beta}$
- Solve $L\vec{y} = \vec{\beta}$ using forw. elim.
- Solve $U\vec{x} = \vec{y}$ using back. subs.

Solving a Square System (cont')

- Using the instructional codes (`backsub`, `forelim`, `myplu`):

```
[L,U,P] = myplu(A);
x = backsub( U, forelim(L, P*b) );
```

y b̃

- Using MATLAB's built-in functions:

```
[L,U,P] = lu(A);
x = U \ (L \ (P*b));
```

  - The backslash is designed so that triangular systems are solved with the appropriate substitution.

- The most compact way:

```
x = A \ b;
```

  - The backslash does partial pivoting and triangular substitutions silently and automatically.

Computing Inverses

Observe that

$$(PA)^{-1} = (LU)^{-1} \longrightarrow A^{-1}P^{-1} = U^{-1}L^{-1} \longrightarrow LUA^{-1} = P$$

So solve $LU\mathbf{a}_i = \mathbf{p}_i$ with forward and backward substitution for each column $\mathbf{p}_i$ of P. Then

$$A^{-1} = \left[\begin{array}{c|c|c|c} & & & \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ & & & \end{array}\right].$$

Computing Determinants

Observe that

$$\det(A) = \det(P^{-1}LU) = \det(P^{-1})\det(L)\det(U) = \frac{\det(L)\det(U)}{\det(P)}.$$

Useful facts.

- The determinant of a triangular matrix is the product of its diagonal entries. (What are diagonal entries of L?)
- P is a row permutation of the identity matrix (which has determinant 1), and each row swap negates the determinant. So if s is the number of row swaps, then $\det(P) = (-1)^s$.

It follows that

$$\det(A) = (-1)^s \prod_{i=1}^{n} u_{ii}.$$