

Conditioning of Square Linear Systems

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Vector and Matrix Norms

Vector Norms

The “length” of a vector \mathbf{v} can be measured by its **norm**.

Definition 1 (p -Norm of a Vector)

Let $p \in [1, \infty)$. The p -norm of $\mathbf{v} \in \mathbb{R}^m$ is denoted by $\|\mathbf{v}\|_p$ and is defined by

$$\|\mathbf{v}\|_p = \left(\sum_{i=1}^m |v_i|^p \right)^{1/p}.$$

When $p = \infty$,

$$\|\mathbf{v}\|_\infty = \max_{1 \leq i \leq m} |v_i|.$$

The most commonly used p values are 1, 2, and ∞ :

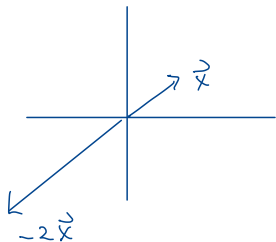
$$\|\mathbf{v}\|_1 = \sum_{i=1}^m |v_i|, \quad \|\mathbf{v}\|_2 = \sqrt{\sum_{i=1}^m |v_i|^2}.$$

Pythagorean.

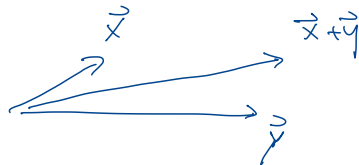
Vector Norms

In general, any function $\|\cdot\| : \mathbb{R}^m \rightarrow \mathbb{R}^+ \cup \{0\}$ is called a **vector norm** if it satisfies the following three properties:

- 1 $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = 0$.
- 2 $\|\alpha\mathbf{x}\| = |\alpha| \|\mathbf{x}\|$ for any constant α and any $\mathbf{x} \in \mathbb{R}^m$.
- 3 $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. This is called the *triangle inequality*.



$$2\|\vec{x}\| = \|-2\vec{x}\|$$



Examples Let $\vec{v} = \begin{bmatrix} 6 \\ -3 \\ 93 \\ 5 \end{bmatrix}$.

$$\begin{aligned} (90+3)^2 &= 8100 + 540 + 9 \\ &= 8649 \end{aligned}$$

$$\begin{aligned} \textcircled{1} \quad \|\vec{v}\|_1 &= |6| + |-3| + |93| + |5| \\ &= 6 + 3 + 93 + 5 = 107 \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad \|\vec{v}\|_2 &= \left(|6|^2 + |-3|^2 + |93|^2 + |5|^2 \right)^{1/2} \\ &= \sqrt{36 + 9 + 93^2 + 25} \end{aligned}$$

$$\textcircled{3} \quad \|\vec{v}\|_\infty = \max \{ |6|, |-3|, |93|, |5| \} = 93$$

Unit Vectors

$$1 \leq p \leq \infty$$

- A vector \mathbf{u} is called a **unit vector** if $\|\mathbf{u}\| = 1$.
- Depending on the norm used, unit vectors will be different.
- For instance: \mathbb{R}^2

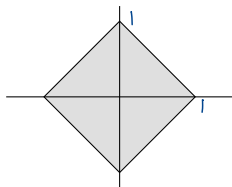


Figure 1: 1-norm

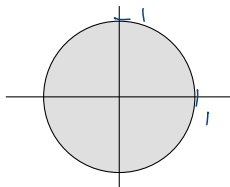


Figure 2: 2-norm

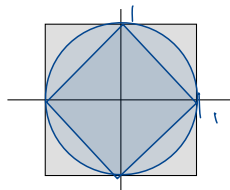


Figure 3: ∞ -norm

Why are they shaped like that? (Work in 2-D)

$p=2$ $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a unit vector in 2-norm, that is,

$$1 = \|\vec{u}\|_2 = \sqrt{u_1^2 + u_2^2} \quad \Rightarrow \quad u_1^2 + u_2^2 = 1 \quad (\text{unit circle})$$

$p=1$

$$1 = \|\vec{u}\|_1 = |u_1| + |u_2|$$



$$\pm u_1 \pm u_2 = 1$$

$$\left\{ \begin{array}{l} u_1 + u_2 = 1 \\ u_1 - u_2 = 1 \\ -u_1 + u_2 = 1 \\ -u_1 - u_2 = 1 \end{array} \right.$$

Matrix Norms

The “size” of a matrix $A \in \mathbb{R}^{m \times n}$ can be measured by its **norm** as well. As above, we say that a function $\|\cdot\| : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^+ \cup \{0\}$ is a **matrix norm** if it satisfies the following three properties:

- 1 $\|A\| = 0$ if and only if $A = 0$.
- 2 $\|\alpha A\| = |\alpha| \|A\|$ for any constant α and any $A \in \mathbb{R}^{m \times n}$.
- 3 $\|A + B\| \leq \|A\| + \|B\|$ for any $A, B \in \mathbb{R}^{m \times n}$. This is called the *triangle inequality*.

Matrix Norms (cont')

- If, in addition to satisfying the three conditions, it satisfies

$$\|AB\| \leq \|A\| \|B\| \quad \text{for all } A \in \mathbb{R}^{m \times n} \text{ and all } B \in \mathbb{R}^{n \times p},$$

it is said to be **consistent**.

- If, in addition to satisfying the three conditions, it satisfies

$$\underbrace{\|A\mathbf{x}\|}_{\text{vec.}} \leq \underbrace{\|A\|}_{\text{mat.}} \underbrace{\|\mathbf{x}\|}_{\text{vec.}} \quad \text{for all } A \in \mathbb{R}^{m \times n} \text{ and all } \mathbf{x} \in \mathbb{R}^n,$$

then we say that it is **compatible** with a vector norm.

Induced Matrix Norms

$$\|A\vec{x}\|_p \leq \|A\|_p \|\vec{x}\|_p.$$

Definition 2 (p -Norm of a Matrix)

Let $p \in [1, \infty]$. The p -norm of $A \in \mathbb{R}^{m \times n}$ is given by

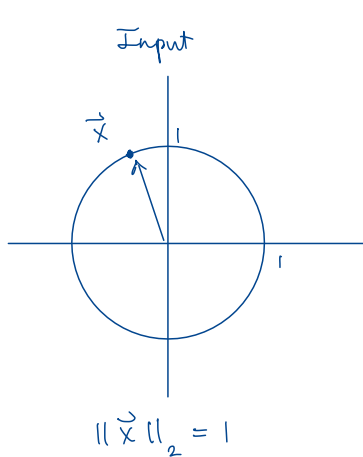
$$\|A\|_p = \max_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p} = \max_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_p.$$

max. over all
nonzero vectors

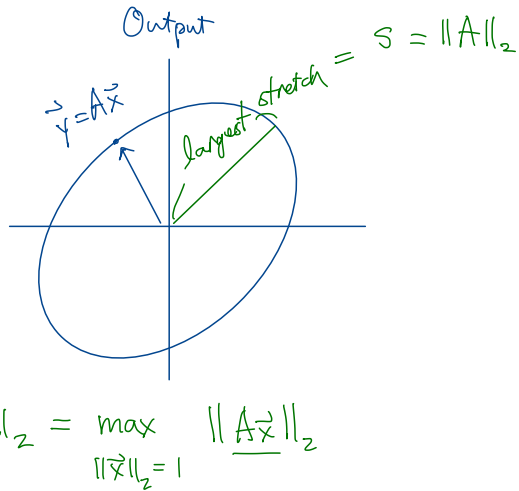
max. over all
unit vectors

- The definition of this particular matrix norm is **induced** from the vector p -norm.
- By construction, matrix p -norm is a compatible norm. It is also consistent.
- Induced norms describe how the matrix stretches unit vectors with respect to the vector norm.

Example $p=2$, $n=2$ (i.e., 2-D)



$$A \in \mathbb{R}^{2 \times 2}$$



Induced Matrix Norms

The commonly used p -norms (for $p = 1, 2, \infty$) can also be calculated by

$$\begin{aligned}\|A\|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|, &>> \max(\text{sum}(\text{abs}(A), 1)) \\ \|A\|_2 &= \sqrt{\lambda_{\max}(A^T A)} = \sigma_{\max}(A), &>> \text{sqrt}(\max(\text{eig}(A' * A))) \\ \|A\|_\infty &= \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|. &>> \max(\text{sum}(\text{abs}(A), 2))\end{aligned}$$

Handwritten notes: "eigenvalue" with an arrow pointing to λ_{\max} ; "singular value" with an arrow pointing to σ_{\max} .

In words,

- The 1-norm of A is the maximum of the 1-norms of all column vectors.
- The 2-norm of A is the square root of the largest eigenvalue of $A^T A$.
- The ∞ -norm of A is the maximum of the 1-norms of all row vectors.

Non-Induced Matrix Norm – Frobenius Norm

Definition 3 (Frobenius Norm of a Matrix)

The Frobenius norm of $A \in \mathbb{R}^{m \times n}$ is given by

$$\|A\|_F = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2}.$$

$$\gg \text{sqrt} (A(:)' * A(:))$$

- This is not induced from a vector p -norm.
- However, both p -norm and the Frobenius norm are consistent and compatible.
- For compatibility of the Frobenius norm, the vector norm must be the 2-norm, that is, $\|Ax\|_2 \leq \|A\|_F \|x\|_2$.

Norms in MATLAB

`norm(A)` : 2-norm by default.

$$\vec{v}^T \vec{v} = [v_1 \dots v_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \underline{v_1^2 + \dots + v_n^2}$$

- Vector p -norms can be easily computed:

```
norm(v, 1)      % = sum(abs(v))  
norm(v, 2)      % = sqrt(v'*v) if v is a column  
norm(v, 'inf')  % = max(abs(v))
```

- The same function `norm` is used to calculate matrix p -norms:

```
norm(A, 1)      % = max(sum(abs(A), 1))  
norm(A, 2)      % = max(sqrt(eig(A'*A)))  
norm(A, Inf)    % = max(sum(abs(A), 2))
```

- To calculate the Frobenius norm:

```
norm(A, 'fro')  % = sqrt(A(:)'*A(:))  
               % = norm(A(:), 2)
```

Conditioning

Conditioning of Solving Linear Systems: Overview

- Analyze how robust (or sensitive) the solutions of $A\mathbf{x} = \mathbf{b}$ are to perturbations of A and \mathbf{b} .
- For simplicity, consider separately the cases where

- \mathbf{b} changes to $\mathbf{b} + \delta\mathbf{b}$, while A remains unchanged, that is

$$A\mathbf{x} = \mathbf{b} \longrightarrow A(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}.$$

- A changes to $A + \delta A$, while \mathbf{b} remains unchanged, that is

$$A\mathbf{x} = \mathbf{b} \longrightarrow (A + \delta A)(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b}.$$

- Assume the matrix norm used is consistent and compatible.

Inputs: A, \vec{b}

Outputs: \vec{x}

Sensitivity to Perturbation of RHS

Case 1. $A\mathbf{x} = \mathbf{b} \rightarrow A(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}$

- Bound $\|\delta\mathbf{x}\|$ in terms of $\|\delta\mathbf{b}\|$:

$$A\mathbf{x} + A\delta\mathbf{x} = \mathbf{b} + \delta\mathbf{b}$$

$$A\delta\mathbf{x} = \delta\mathbf{b} \quad \implies \quad \|\delta\mathbf{x}\| \leq \|A^{-1}\| \|\delta\mathbf{b}\|.$$

$$\delta\mathbf{x} = A^{-1}\delta\mathbf{b}$$

- Sensitivity in terms of relative errors:

$$\frac{\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|}}{\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}} = \frac{\|\delta\mathbf{x}\| \|\mathbf{b}\|}{\|\delta\mathbf{b}\| \|\mathbf{x}\|} \leq \frac{\|A^{-1}\| \|\delta\mathbf{b}\| \cdot \|A\| \|\mathbf{x}\|}{\|\delta\mathbf{b}\| \|\mathbf{x}\|} = \|A^{-1}\| \|A\|.$$

Sensitivity to Perturbation of Matrix

Case 2. $A\mathbf{x} = \mathbf{b} \rightarrow (A + \delta A)(\mathbf{x} + \delta \mathbf{x}) = \mathbf{b}$

- Bound $\|\delta \mathbf{x}\|$ now in terms of $\|\delta A\|$:

$$\begin{aligned} A\mathbf{x} + A\delta \mathbf{x} + (\delta A)\mathbf{x} + (\delta A)\delta \mathbf{x} &= \mathbf{b} \\ A\delta \mathbf{x} &= -(\delta A)\mathbf{x} - (\delta A)\delta \mathbf{x} \\ \delta \mathbf{x} &= -A^{-1}(\delta A)\mathbf{x} - A^{-1}(\delta A)\delta \mathbf{x} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \|\delta \mathbf{x}\| &\lesssim \|A^{-1}\| \|\delta A\| \|\mathbf{x}\|. \\ &\text{(first-order truncation)} \end{aligned}$$

- Sensitivity in terms of relative errors:

$$\frac{\frac{\|\delta \mathbf{x}\|}{\|\mathbf{x}\|}}{\frac{\|\delta A\|}{\|A\|}} = \frac{\|\delta \mathbf{x}\| \|A\|}{\|\delta A\| \|\mathbf{x}\|} \lesssim \frac{\|A^{-1}\| \|\delta A\| \|\mathbf{x}\| \cdot \|A\|}{\|\delta A\| \|\mathbf{x}\|} = \|A^{-1}\| \|A\|.$$

Matrix Condition Number

- Motivated by the previous estimations, we define the **matrix condition number** by

$$\kappa(A) = \|A^{-1}\| \|A\|,$$

where the norms can be any p -norm or the Frobenius norm.

- A subscript on κ such as 1, 2, ∞ , or F(robenius) is used if clarification is needed.

Matrix Condition Number (cont')

- We can write

$$\frac{\|\delta \mathbf{x}\|}{\|\mathbf{x}\|} \leq \kappa(A) \frac{\|\delta \mathbf{b}\|}{\|\mathbf{b}\|}, \quad \frac{\|\delta \mathbf{x}\|}{\|\mathbf{x}\|} \leq \kappa(A) \frac{\|\delta A\|}{\|A\|},$$

where the second inequality is true only in the limit of infinitesimal perturbations δA .

- The matrix condition number $\kappa(A)$ is equal to the condition number of solving a linear system of equation $A\mathbf{x} = \mathbf{b}$.
- The exponent of $\kappa(A)$ in scientific notation determines the approximate number of digits of accuracy that will be lost in calculation of \mathbf{x} .
- Since $1 = \|I\| = \|A^{-1}A\| \leq \|A^{-1}\|\|A\| = \kappa(A)$, a condition number of 1 is the best we can hope for.
- If $\kappa(A) > \boxed{\text{eps}}^{-1}$, then for computational purposes the matrix is singular.

Condition Numbers in MATLAB

- Use `cond` to calculate various condition numbers:

```
cond(A)           % the 2-norm; or  cond(A, 2)
cond(A, 1)        % the 1-norm
cond(A, Inf)      % the infinity-norm
cond(A, 'fro')    % the Frobenius norm
```

- A condition number estimator (in 1-norm)

```
condest(A)        % faster than cond
```

- The fastest method to estimate the condition number is to use `linsolve` function as below:

```
[x, inv_condest] = linsolve(A, b);
fast_condest = 1/inv_condest;
```