

## Notes on SVD

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# Properties of SVD

# SVD and the 2-Norm

## Theorem 1

Let  $A \in \mathbb{C}^{m \times n}$  have an SVD  $A = U\Sigma V^*$ . Then

- 1  $\|A\|_2 = \sigma_1$  and  $\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}$ .
- 2 The rank of  $A$  is the number of nonzero singular values.
- 3 Let  $r = \min\{m, n\}$ . Then

$$\kappa_2(A) = \|A\|_2 \|A^+\|_2 = \frac{\sigma_1}{\sigma_r}.$$

# Connection to EVD

Let  $A = U\Sigma V^* \in \mathbb{C}^{m \times n}$  and  $B = A^*A$ . Observe that

- $B \in \mathbb{C}^{n \times n}$  is a *hermitian matrix*<sup>1</sup>, i.e.,  $B^* = B$ .
- $B$  has an EVD:

$$B = (V\Sigma^*U^*)(U\Sigma V^*) = V\Sigma^*\Sigma V^* = V(\Sigma^*\Sigma)V^{-1}.$$

- The squares of singular values of  $A$  are eigenvalues of  $B$ .
- An EVD of  $B = A^*A$  reveals the singular values and a set of right singular vectors of  $A$ .

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<sup>1</sup>This is the  $\mathbb{C}$ -extension of real symmetric matrices.

### Theorem 2

*The nonzero singular values of  $A \in \mathbb{C}^{m \times n}$  are the square roots of the nonzero eigenvalues of  $A^*A$  or  $AA^*$ .*

# Reduction of Dimensions

# Low-Rank Approximations

Let  $A \in \mathbb{C}^{m \times n}$  with  $m \geq n$ . Its thin SVD  $A = \hat{U} \hat{\Sigma} V^*$  can be written as

$$\begin{aligned} A &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_n & \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^* \\ \vdots \\ \mathbf{v}_n^* \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1 \mathbf{u}_1 & \cdots & \sigma_n \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^* \\ \vdots \\ \mathbf{v}_n^* \end{bmatrix} = \sum_{j=1}^r \sigma_j \mathbf{u}_j \mathbf{v}_j^*, \end{aligned}$$

where  $r$  is the rank of  $A$ .

- Each outer product  $\mathbf{u}_j \mathbf{v}_j^*$  is a rank-1 matrix.
- Since  $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ , important contributions to  $A$  come from terms with small  $j$ .



## Low-Rank Approximations (cont')

For  $1 \leq k \leq r$ , define

$$A_k = \sum_{j=1}^k \sigma_j \mathbf{u}_j \mathbf{v}_j^* = U_k \Sigma_k V_k^*,$$

where

- $U_k$  is the first  $k$  columns of  $U$ ;
- $V_k$  is the first  $k$  columns of  $V$ ;
- $\Sigma_k$  is the upper-left  $k \times k$  submatrix of  $\Sigma$ .

This is a rank- $k$  approximation of  $A$ .

# Best Rank- $k$ Approximation

## Theorem 3 (Eckart-Young)

Let  $A \in \mathbb{C}^{m \times n}$ . Suppose  $A$  has rank  $r$  and let  $A = U\Sigma V^*$  be an SVD. Then

- $\|A - A_k\|_2 = \sigma_{k+1}$ , for  $k = 1, \dots, r - 1$ .
- For any matrix  $B$  with  $\text{rank}(B) \leq k$ ,  $\|A - B\|_2 \geq \sigma_{k+1}$ .

# Appendix: Unitary Diagonalization and SVD

# Unitary Diagonalization of Hermitian Matrices

The previous discussion is relevant to hermitian matrices constructed in a specific manner. For a generic hermitian matrix, we have the following result.

## Theorem 4 (Spectral Decomposition)

*Let  $A \in \mathbb{C}^{n \times n}$  be hermitian. Then  $A$  has a unitary diagonalization*

$$A = VDV^{-1},$$

*where  $V \in \mathbb{C}^{n \times n}$  is unitary and  $D \in \mathbb{R}^{n \times n}$  is diagonal.*

In words, a hermitian matrix (or symmetric matrix) has a complete set of orthonormal eigenvectors and all its eigenvalues are real.

# Notes on Unitary Diagonalization and Normal Matrices

- A unitarily diagonalizable matrix  $A = VDV^{-1}$  with  $D \in \mathbb{C}^{n \times n}$ , is called a **normal matrix**<sup>2</sup>. All hermitian matrices are normal.
- Let  $A = VDV^{-1} \in \mathbb{C}^{n \times n}$  be normal. Since  $\kappa_2(V) = 1$  (why?), Bauer-Fike implies that eigenvalues of  $A$  can be changed by no more than  $\|\delta A\|_2$ .

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<sup>2</sup>Usual definition:  $A \in \mathbb{C}^{n \times n}$  is normal if  $AA^* = A^*A$ .

# Unitary Diagonalization and SVD

## Theorem 5

*Let  $A \in \mathbb{C}^{n \times n}$  be hermitian. Then the singular values of  $A$  are the absolute values of the eigenvalues of  $A$ .*

*Precisely, if  $A = VDV^{-1}$  is a unitary diagonalization of  $A$ , then*

$$A = (V \operatorname{sign}(D)) |D| V^*$$

*is an SVD, where*

$$\operatorname{sign}(D) = \begin{bmatrix} \operatorname{sign}(d_1) & & \\ & \ddots & \\ & & \operatorname{sign}(d_n) \end{bmatrix}, \quad |D| = \begin{bmatrix} |d_1| & & \\ & \ddots & \\ & & |d_n| \end{bmatrix}.$$

# When Do Unitary EVD and SVD Coincide?

## Theorem 6

If  $A = A^*$ , then the following statements are equivalent:

- ① Any unitary EVD of  $A$  is also an SVD of  $A$ .
- ② The eigenvalues of  $A$  are positive numbers.
- ③  $\mathbf{x}^* A \mathbf{x} > 0$  for all nonzero  $\mathbf{x} \in \mathbb{C}^n$ . (HPD)

- The equivalence of 1 and 2 is immediate from Theorem 5
- The property in 3 is called the **hermitian positive definiteness**, c.f., symmetric positive definiteness.
- The equivalence of 2 and 3 can be shown conveniently using **Rayleigh quotient**; see next slide.

## Note: Rayleigh Quotient

Let  $A \in \mathbb{R}^{n \times n}$  be fixed. The **Rayleigh quotient** is the map  $R_A : \mathbb{R}^n \rightarrow \mathbb{R}$  given by

$$R_A(\mathbf{x}) = \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

- $R_A$  maps an eigenvector of  $A$  into its associated eigenvalue, *i.e.*, if  $A\mathbf{v} = \lambda\mathbf{v}$ , then  $R_A(\mathbf{v}) = \lambda$ .
- If  $A = A^T$ , then  $\nabla R_A(\mathbf{v}) = \mathbf{0}$  for an eigenvector  $\mathbf{v}$ , and so

$$R_A(\mathbf{v} + \epsilon \mathbf{z}) = R_A(\mathbf{v}) + 0 + O(\epsilon^2) = \lambda + O(\epsilon^2), \quad \text{as } \epsilon \rightarrow 0.$$

The Rayleigh quotient is a quadratic approximation of an eigenvalue.