Exercises: Nonlinear Rootfinding

Problems marked with ✏ are to be done by hand; those marked with 💻 are to be solved using a computer.

1. (Kepler's law; **FNC** 4.1.5) 💻 The most easily observed properties of the orbit of a celestial body around the sun are the period τ and the elliptical eccentricity ϵ. (A circle has $\epsilon = 0$.) From these it is possible to find at any time t the angle $\theta(t)$ made between the body's position and the major axis of the ellipse. This is done through

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\tan\frac{\psi}{2} \tag{1}$$

   where the eccentric anomaly ψ satisfies Kepler's equation:

$$\psi - \epsilon\sin\psi - \frac{2\pi t}{\tau} = 0. \tag{2}$$

   Equation (2) must be solved numerically to find $\psi(t)$, and then (1) can be solved analytically to find $\theta(t)$.

   The asteroid Eros has $\tau = 1.7610$ years and $\epsilon = 0.2230$. Using `fzero` for (2), make a plot of $\theta(t)$ for 100 values of t between 0 and τ (one full period).

2. (Lambert W-function)

   (a) ✏ Show that

$$r = -\frac{W\left(-\log(2)/5\right)}{\log 2}.$$

   is a solution of the equation $2^x = 5x$. (Here, as usual in this class, $\log(\,\cdot\,) = \ln(\,\cdot\,)$ is the natural logarithmic function.)

   (b) 💻 Numerically verify the result using `fzero`[1]

3. (Fast convergence of FPI; **FNC** 4.2.6) ✏ Let $g(x) = 2x - 3x^2$.

   (a) Show that $r = 1/3$ is a fixed point of g.

   (b) Find $g'(1/3)$. How does this affect the error analysis; see p. 24 of Module 5.

4. (Conditions for convergence; **FNC** 4.2.7) ✏ Consider the iteration

$$x_{k+1} = x_k - \frac{f(x_k)}{c}, \quad k = 0, 1, \ldots.$$

   Suppose $f(r) = 0$ and $f'(r)$ exits. Find one or more conditions on c such that the iteration converges to r.

[1] Two real-valued solutions are $r_1 \approx 0.2355$ and $r_2 \approx 4.488$.

5. (Stopping criteria; **FNC** 4.3.8) In rootfinding codes, since the actual error is not available, we use $|x_k - x_{k-1}|$ as an approximate indicator of error to determine when to stop the iteration. Find an example that foils this indicator, that is, a sequence $\{x_k\}$ such that

$$\lim_{k\to\infty} (x_k - x_{k-1}) = 0,$$

but $\{x_k\}$ diverges. (Note: You have seen such sequences in calculus.) Hence the need for residual tolerances and escape hatches in the code!

6. (Calculating n-th roots) Let n be a positive integer. Use Newton's method to produce a quadratically convergent method for calculating the n-th root of a positive number a. Prove quadratic convergence.

7. (Predicting next error) Let $f(x) = x^3 - 4x$.

   (a) The function $f(x)$ has a root at $r = 2$. If the error $\epsilon_k = x_k - r$ after four steps of Newton's method is $\epsilon_4 = 10^{-6}$, estimate ϵ_5.

   (b) Do the same to the root $r = 0$.

8. (Secant method; exercise from lecture) Assume that iterates $x_1, x_2, \ldots$ generated by the secant method converge to a root r and $f'(r) \neq 0$. Let $\epsilon_k = x_k - r$. Show that

   (a) The error ϵ_k satisfies the approximate equation

   $$|\epsilon_{k+1}| \approx \left|\frac{f''(r)}{2f'(r)}\right| |\epsilon_k| \, |\epsilon_{k-1}| \, .$$

   (b) If in addition $\lim_{k\to\infty} |\epsilon_{k+1}| / |\epsilon_k|^\alpha$ exists and is nonzero for some $\alpha > 0$, then

   $$|\epsilon_{k+1}| \approx \left|\frac{f''(r)}{2f'(r)}\right|^{\alpha-1} |\epsilon_k|^\alpha, \quad \text{where } \alpha = \frac{1+\sqrt{5}}{2}.$$

9. (Multidimensional Newton's method; **FNC** 4.5.5) Suppose one wants to find the points on the ellipsoid $x^2/25 + y^2/16 + z^2/9 = 1$ that are closest to and farthest from the point $(5, 4, 3)$. The method of Lagrange multipliers implies that any such point satisfies

$$\begin{cases} x - 5 = \dfrac{\lambda x}{25}, \\ y - 4 = \dfrac{\lambda y}{16}, \\ z - 3 = \dfrac{\lambda z}{9}, \\ \quad\;\; 1 = \dfrac{1}{25}x^2 + \dfrac{1}{16}y^2 + \dfrac{1}{9}z^2 \end{cases}$$

for an unknown value of λ.

   (a) Write out this system in the form $\mathbf{f}(\mathbf{u}) = \mathbf{0}$.

   (b) Write out the Jacobian matrix of this system.

   (c) Use `newtonsys` from class with different initial guesses to find the two roots of this system. Which is the closest point to $(5, 4, 3)$ and which is the farthest?

10. (Optional project: basin of attraction) Do **LM** 13.1–33 and 34. This is a long problem and it is not very likely that it will appear on your exam. However, if you want extra challenge or are interested in generating cool figures using MATLAB skills acquired and your understanding of rootfinding, give it a go. The final outcome of the lengthy process is the colorful representation of so-called the *basin of attraction* as shown below.