Math 3607: Homework 8

Wednesday, March 30, 2022

TOTAL: 30 points

- Problems marked with ✎ are to be done by hand; those marked with 💻 are to be solved using a computer.
- **Important note.** Do not use *Symbolic Math Toolbox*. Any work done using `sym` or `syms` will receive NO credit.
- **Another important note.** When asked write a MATLAB function, write one at the end of your live script.

1. (Using `eig`; **FNC** 7.2.3) 💻 Use `eig` to find the EVD of each matrix. Then for each eigenvalue λ, use the `rank` command to verify that $\lambda I - A$ is singular.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -2 & 2 & -1 \\ -1 & -2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -3 & -2 & -1 \\ -2 & 4 & -2 & -1 \\ -1 & -2 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}.$$

2. (Polynomial evaluation of matrices; **FNC** 7.2.5 and Su20 final exam) Let $p(z) = c_1 + c_2 z + \cdots + c_n z^{n-1}$. The value of p for a square matrix input is defined as

$$p(X) = c_1 I + c_2 X + \cdots + c_n X^{n-1}.$$

   (a) ✎ Show that if $X \in \mathbb{R}^{k \times k}$ has an EVD, then $p(X)$ can be found using only evaluations of p at the eigenvalues and two matrix multiplications.

   (b) 💻 Complete the following program which, given coefficients $\mathbf{c} = (c_1, c_2, \dots, c_n)^\mathrm{T}$, evaluates the corresponding polynomial at $\mathbf{x}$, which can be a number, a vector, or a square matrix. If $\mathbf{x}$ is a scalar or a vector, use *Horner's method*[1]; if $\mathbf{x}$ is a square matrix, use the result from the previous part.

```
function y = mypolyval(c, x)
%MYPOLYVAL evaluates a polynomial at points  x  given its coeffs.
% Input:
%   c   coefficient vector (c_1, c_2, ..., c_n)^T
%   x   points of evaluation
%       - if x is a scalar or a vector, use Horner's method
%       - if x is a square matrix, use the result from (a)
%       - otherwise, produce an error message.
```

[1] See Problem 1(b) of HW6 or p. 24 of Lecture 11 slides.

3. (Recursively defined sequences) The Pell numbers $0, 1, 2, 5, 12, 29, 70, 169, 208, 985, \ldots$ are defined recursively by

$$P_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2P_{n-1} + P_{n-2} & \text{otherwise} \end{cases}$$

Using an EVD of a suitable 2-by-2 matrix, find the general formula for the kth Pell number.

4. (Singular values by hand) Calculate the singular values of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

by solving a 2×2 eigenvalue problem. Confirm your answer using MATLAB.

5. (SVD and the 2-norm) Let $A \in \mathbb{R}^{n \times n}$. Show that

   (a) A and A^{T} have the same singular values.

   (b) $\|A\|_2 = \|A^{\mathrm{T}}\|_2$.

6. (Vandermonde matrix, SVD, and rank) Let $\mathbf{x}$ be a vector of 1000 equally spaced points between 0 and 1, and let A_n be the $1000 \times n$ Vandermonde-type matrix whose (i, j) entry is x_i^{j-1} for $j = 1, \ldots, n$.

   (a) Print out the singular values of A_1, A_2, and A_3.

   (b) Make a semi-log plot of the singular values of A_{25}.

   (c) Use `rank` to find the rank of A_{25}. How does this relate to the graph from part (b)? You may want to use the help document for the `rank` command to understand what it does.