

Notes on SVD

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Properties of SVD

SVD and the 2-Norm

Theorem 1

Let $A \in \mathbb{C}^{m \times n}$ have an SVD $A = U\Sigma V^*$. Then

- 1 $\|A\|_2 = \sigma_1$ and $\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}$.
- 2 The rank of A is the number of nonzero singular values.
- 3 Let $r = \min\{m, n\}$. Then

$$\kappa_2(A) = \|A\|_2 \|A^+\|_2 = \frac{\sigma_1}{\sigma_r}.$$

Connection to EVD

Let $A = U\Sigma V^* \in \mathbb{C}^{m \times n}$ and $B = A^*A$. Observe that

- $B \in \mathbb{C}^{n \times n}$ is a *hermitian matrix*¹, i.e., $B^* = B$.
- B has an EVD:

$$B = (V\Sigma^*U^*) (U\Sigma V^*) = V\Sigma^*\Sigma V^* = V (\Sigma^*\Sigma) V^{-1}.$$

- The squares of singular values of A are eigenvalues of B .
- An EVD of $B = A^*A$ reveals the singular values and a set of right singular vectors of A .

¹This is the \mathbb{C} -extension of real symmetric matrices.

Theorem 2

*The nonzero singular values of $A \in \mathbb{C}^{m \times n}$ are the square roots of the nonzero eigenvalues of A^*A or AA^* .*

Unitary Diagonalization and SVD

Unitary Diagonalization of Hermitian Matrices

The previous discussion is relevant to hermitian matrices constructed in a specific manner. For a generic hermitian matrix, we have the following result.

Theorem 3 (Spectral Decomposition)

Let $A \in \mathbb{C}^{n \times n}$ be hermitian. Then A has a unitary diagonalization

$$A = VDV^{-1},$$

where $V \in \mathbb{C}^{n \times n}$ is unitary and $D \in \mathbb{R}^{n \times n}$ is diagonal.

In words, a hermitian matrix (or symmetric matrix) has a complete set of orthonormal eigenvectors and all its eigenvalues are real.

Notes on Unitary Diagonalization and Normal Matrices

- A unitarily diagonalizable matrix $A = VDV^{-1}$ with $D \in \mathbb{C}^{n \times n}$, is called a **normal matrix**². All hermitian matrices are normal.
- Let $A = VDV^{-1} \in \mathbb{C}^{n \times n}$ be normal. Since $\kappa_2(V) = 1$ (why?), Bauer-Fike implies that eigenvalues of A can be changed by no more than $\|\delta A\|_2$.

²Usual definition: $A \in \mathbb{C}^{n \times n}$ is normal if $AA^* = A^*A$.

Unitary Diagonalization and SVD

Theorem 4

Let $A \in \mathbb{C}^{n \times n}$ be hermitian. Then the singular values of A are the absolute values of the eigenvalues of A .

Precisely, if $A = VDV^{-1}$ is a unitary diagonalization of A , then

$$A = (V \operatorname{sign}(D)) |D| V^*$$

is an SVD, where

$$\operatorname{sign}(D) = \begin{bmatrix} \operatorname{sign}(d_1) & & \\ & \ddots & \\ & & \operatorname{sign}(d_n) \end{bmatrix}, \quad |D| = \begin{bmatrix} |d_1| & & \\ & \ddots & \\ & & |d_n| \end{bmatrix}.$$

When Do Unitary EVD and SVD Coincide?

Theorem 5

If $A = A^*$, then the following statements are equivalent:

- ① Any unitary EVD of A is also an SVD of A .
- ② The eigenvalues of A are positive numbers.
- ③ $\mathbf{x}^* A \mathbf{x} > 0$ for all nonzero $\mathbf{x} \in \mathbb{C}^n$. (HPD)

- The equivalence of 1 and 2 is immediate from Theorem 4
- The property in 3 is called the **hermitian positive definiteness**, c.f., symmetric positive definiteness.
- The equivalence of 2 and 3 can be shown conveniently using **Rayleigh quotient**; see next slide.

Note: Rayleigh Quotient

Let $A \in \mathbb{R}^{n \times n}$ be fixed. The **Rayleigh quotient** is the map $R_A : \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$R_A(\mathbf{x}) = \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

- R_A maps an eigenvector of A into its associated eigenvalue, *i.e.*, if $A\mathbf{v} = \lambda\mathbf{v}$, then $R_A(\mathbf{v}) = \lambda$.
- If $A = A^T$, then $\nabla R_A(\mathbf{v}) = \mathbf{0}$ for an eigenvector \mathbf{v} , and so

$$R_A(\mathbf{v} + \epsilon \mathbf{z}) = R_A(\mathbf{v}) + 0 + O(\epsilon^2) = \lambda + O(\epsilon^2), \quad \text{as } \epsilon \rightarrow 0.$$

The Rayleigh quotient is a quadratic approximation of an eigenvalue.