Exercises: Preliminaries (Solutions)

3. Recall that the (relative) condition number of the function f at x is given by

$$\kappa_f(x) = \left|\frac{xf'(x)}{f(x)}\right|.$$

(a) Calculation shows that

$$\kappa_f(x) = \frac{x}{\cosh x \sinh x} \quad \text{or} \quad \frac{4xe^{2x}}{e^{4x}-1}.$$

Using the exponential form, one can confirm that $\kappa_f(x)$ does not blow up anywhere.

(b) When simplified,

$$\kappa_f(x) = \left|\frac{xe^x}{e^x-1} - 1\right|$$

Some notable limits are

$$\lim_{x\to 0} \kappa_f(x) = 0,$$
$$\lim_{x\to \infty} \kappa_f(x) = \infty,$$
$$\lim_{x\to -\infty} \kappa_f(x) = 1.$$

Note that $\kappa_f(x)$ blows up as $x \to \infty$.

(c) Another calculus exercise leads to

$$\kappa_f(x) = \left|1 - \frac{x\sin(x)}{1-\cos(x)}\right|.$$

It is not very difficult to see that $\lim_{x\to 0} \kappa_f(x) = 1$, even though the denominator of the second term approaches 0 as $x \to 0$. However, there are infinitely many nonzero x at which $1-\cos(x)$ vanishes while $x\sin(x)$ does not, namely, at $x = 2n\pi$, where n is a nonzero integer. Therefore, $\kappa_f(x) \to \infty$ as $x \to 2n\pi$, where $n \in \mathbb{Z}\backslash\{0\}$.