Math 3607: Homework 7

(Solutions to By-Hand Problems)

1. (a) Let $A = \widehat{Q}\widehat{R}$ be a thin QR factorization of A. Since $A^+ = \widehat{R}^{-1}\widehat{Q}^{\mathrm{T}}$, we have

$$\begin{aligned} AA^+ &= \left(\widehat{Q}\widehat{R}\right)\left(\widehat{R}^{-1}\widehat{Q}^{\mathrm{T}}\right) \\ &= \widehat{Q}\underbrace{\left(\widehat{R}\widehat{R}^{-1}\right)}_{=I}\widehat{Q}^{\mathrm{T}} = \widehat{Q}\widehat{Q}^{\mathrm{T}} = P. \end{aligned}$$

This shows the desired equality $P = AA^+$. □

(b) Substituting $P = \widehat{Q}\widehat{Q}^{\mathrm{T}}$ into P^2 and simplifying, we have

$$P^2 = \left(\widehat{Q}\widehat{Q}^{\mathrm{T}}\right)\left(\widehat{Q}\widehat{Q}^{\mathrm{T}}\right) = \widehat{Q}\underbrace{\left(\widehat{Q}^{\mathrm{T}}\widehat{Q}\right)}_{=I}\widehat{Q}^{\mathrm{T}} = \widehat{Q}\widehat{Q}^{\mathrm{T}} = P,$$

where $\widehat{Q}^{\mathrm{T}}\widehat{Q} = I$ used in the third equality follows from the fact that $\widehat{Q}$ is a matrix with orthonormal columns. This shows that $P^2 = P$. □

(c) Let $\mathbf{x}$ be any vector and let $P = \widehat{Q}\widehat{Q}^{\mathrm{T}}$. To prove that $\mathbf{u} = P\mathbf{x}$ and $\mathbf{v} = (I-P)\mathbf{x}$ are orthogonal, we shall show that $\mathbf{u}^{\mathrm{T}}\mathbf{v} = 0$. Before showing this, we note that P is symmetric, that is, $P^{\mathrm{T}} = P$ because

$$P^{\mathrm{T}} = \left(\widehat{Q}\widehat{Q}^{\mathrm{T}}\right)^{\mathrm{T}} = \left(\widehat{Q}^{\mathrm{T}}\right)^{\mathrm{T}}\widehat{Q}^{\mathrm{T}} = \widehat{Q}\widehat{Q}^{\mathrm{T}} = P.$$

Now we compute the inner product:

$$\begin{aligned} \mathbf{u}^{\mathrm{T}}\mathbf{v} &= (P\mathbf{u})^{\mathrm{T}}(I-P)\mathbf{x} \\ &= \mathbf{x}^{\mathrm{T}}P^{\mathrm{T}}(I-P)\mathbf{x} \\ &= \mathbf{x}^{\mathrm{T}}(P^{\mathrm{T}} - P^{\mathrm{T}}P)\mathbf{x} \\ &= \mathbf{x}^{\mathrm{T}}(P - P^2)\mathbf{x} && \text{since } P^{\mathrm{T}} = P \\ &= \mathbf{x}^{\mathrm{T}}O\mathbf{x} && \text{since } P^2 = P \\ &= 0, \end{aligned}$$

where O denotes the zero matrix of the same size as P. This shows that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. □

2. (a) Note that $A^{\mathrm{T}}A = 1^2 + (-2)^2 + 3^2 = 14$, a scalar, so the inverse $(A^{\mathrm{T}}A)^{-1}$ is simply the reciprocal $1/14$. It follows that

$$A^+ = \left(A^{\mathrm{T}}A\right)^{-1}A^{\mathrm{T}} = \frac{1}{14}\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{14} & -\frac{1}{7} & \frac{3}{14} \end{bmatrix}.$$

Note that the answer is a row vector.

(b) Let $\mathbf{z} = (-6, 2, 9)^{\mathrm{T}}$. Note that $\|\mathbf{z}\|_2 = \sqrt{36+4+81} = 11$, which confirms that the vector on the right-hand side is indeed $\|\mathbf{z}\|_2\, \mathbf{e}_1$. Thus the Householder matrix H we seek is the reflection operator across $\langle \mathbf{v} \rangle^{\perp}$ where

$$\mathbf{v} = \mathbf{z} - \|\mathbf{z}\|_2 \mathbf{e}_1 = \begin{bmatrix} -6 \\ 2 \\ 9 \end{bmatrix} - \begin{bmatrix} 11 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -17 \\ 2 \\ 9 \end{bmatrix},$$

In other words,

$$H = I - \frac{2}{\mathbf{v}^{\mathrm{T}}\mathbf{v}}\mathbf{v}\mathbf{v}^{\mathrm{T}} = \begin{bmatrix} -6/11 & 2/11 & 9/11 \\ 2/11 & 183/187 & -18/187 \\ 9/11 & -18/187 & 106/187 \end{bmatrix}.$$

3. (a) Done in class.

(b) Done in class.

(c) Note that

$$\mathbf{v}^{\mathrm{T}}\mathbf{v} = \|\mathbf{z}\|_2^2 - 2\|\mathbf{z}\|_2\, \mathbf{e}_1^{\mathrm{T}}\mathbf{z} + \|\mathbf{z}\|_2^2 = 2\|\mathbf{z}\|_2 \left(\|\mathbf{z}\|_2 - \mathbf{e}_1^{\mathrm{T}}\mathbf{z} \right) \tag{1}$$

and

$$\begin{aligned} \mathbf{v}^{\mathrm{T}}\mathbf{z} &= \left(\|\mathbf{z}\|_2\, \mathbf{e}_1 - \mathbf{z} \right)^{\mathrm{T}} \mathbf{z} = \|\mathbf{z}\|_2\, \mathbf{e}_1^{\mathrm{T}}\mathbf{z} - \mathbf{z}^{\mathrm{T}}\mathbf{z} \\ &= \|\mathbf{z}\|_2\, \mathbf{e}_1^{\mathrm{T}}\mathbf{z} - \|\mathbf{z}\|_2^2 = -\|\mathbf{z}\|_2 \left(\|\mathbf{z}\|_2 - \mathbf{e}_1^{\mathrm{T}}\mathbf{z} \right). \end{aligned} \tag{2}$$

Now we have

$$H\mathbf{z} = \left(I - 2\frac{\mathbf{v}\mathbf{v}^{\mathrm{T}}}{\mathbf{v}^{\mathrm{T}}\mathbf{v}} \right)\mathbf{z} = \mathbf{z} - 2\frac{\mathbf{v}^{\mathrm{T}}\mathbf{z}}{\mathbf{v}^{\mathrm{T}}\mathbf{v}}\mathbf{v}. \tag{3}$$

Substituting (1) and (2) into (3), we obtain

$$H\mathbf{z} = \mathbf{z} + \mathbf{v} = \mathbf{z} + \left(\|\mathbf{z}\|_2\, \mathbf{e}_1 - \mathbf{z} \right) = \|\mathbf{z}\|_2\, \mathbf{e}_1.$$