

Applications of LU Factorization

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Solving a Square System

Multiplying $A\mathbf{x} = \mathbf{b}$ on the left by P we obtain

$$\underbrace{PA}_{=LU} \mathbf{x} = \underbrace{P\mathbf{b}}_{=: \boldsymbol{\beta}} \longrightarrow LU\mathbf{x} = \boldsymbol{\beta},$$

which can be solved in two steps:

- Define $U\mathbf{x} = \mathbf{y}$ and solve for \mathbf{y} in the equation

$$L\mathbf{y} = \boldsymbol{\beta}. \quad \text{(forward elimination)}$$

- Having calculated \mathbf{y} , solve for \mathbf{x} in the equation

$$U\mathbf{x} = \mathbf{y}. \quad \text{(backward substitution)}$$

Solving a Square System (cont')

- Using the instructional codes (`backsub`, `forelim`, `myplu`):

```
[L,U,P] = myplu(A);  
x = backsub( U, forelim(L, P*b) );
```

- Using MATLAB's built-in functions:

```
[L,U,P] = lu(A);  
x = U \ (L \ (P*b));
```

- The backslash is designed so that triangular systems are solved with the appropriate substitution.
- The most compact way:

```
x = A \ b;
```

- The backslash does partial pivoting and triangular substitutions silently and automatically.

Computing Inverses

Observe that

$$(PA)^{-1} = (LU)^{-1} \longrightarrow A^{-1}P^{-1} = U^{-1}L^{-1} \longrightarrow LUA^{-1} = P$$

So solve $LU\mathbf{a}_i = \mathbf{p}_i$ with forward and backward substitution for each column \mathbf{p}_i of P . Then

$$A^{-1} = \left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array} \right].$$

Computing Determinants

Observe that

$$\det(A) = \det(P^{-1}LU) = \det(P^{-1}) \det(L) \det(U) = \frac{\det(L) \det(U)}{\det(P)}.$$

Useful facts.

- The determinant of a triangular matrix is the product of its diagonal entries. (What are diagonal entries of L ?)
- P is a row permutation of the identity matrix (which has determinant 1), and each row swap negates the determinant. So if s is the number of row swaps, then $\det(P) = (-1)^s$.

It follows that

$$\det(A) = (-1)^s \prod_{i=1}^n u_{ii}.$$