Spectral Theory

- Eigenvalue decomposition (EVD)
- Singular value decomposition (SVD)

Eigenvalue Decomposition

Office Hours (This week only)

- TW 4:45 ~ 6:15

22:44

[redacted] 15:07

To You

Hi Professor,

Another idea I had for encouraging in-person attendance is bringing snacks/candy to lectures 🙂

Best,

[redacted]

KT You 16:58

[redacted]

Love that idea! 🍰🍬

Tae Eun

Reply to All

Contents

Complex Numbers and Complex Arrays

Complex Numbers

In what follows, we assume all scalars, vectors, and matrices may be complex.

Notation.

- $\mathbb{R}$: the set of all real numbers
- $\mathbb{C}$: the set of all complex numbers, *i.e.*,

$$\{z = x + iy \mid x, y \in \mathbb{R}\} \quad \text{where } i = \sqrt{-1}.$$

Complex Numbers in MATLAB

Let $z = x + iy \in \mathbb{C}$.

MATLAB	Name	Notation
`real(z)`	real part of z	$\operatorname{Re} z$
`imag(z)`	imaginary part of z	$\operatorname{Im} z$
`conj(z)`	conjugate of z	$\overline{z}$
`abs(z)`	modulus of z	$\lvert z \rvert$
`angle(z)`	argument of z	$\arg(z)$

Euler's Formula

- Recall that the Maclaurin series for e^t is

$$e^t = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \qquad -\infty < t < \infty.$$

- Replacing t by it and separating real and imaginary parts (using the cyclic behavior of powers of i), we obtain

$$e^{it} = \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!}}_{\cos(t)} + i \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}}_{\sin(t)}$$

- The result is called the **Euler's formula**.

$$\boxed{e^{it} = \cos(t) + i\sin(t).}$$

Polar Representation and Complex Exponential

- **Polar representation:** A complex number $z = x + iy \in \mathbb{C}$ can be written as $z = re^{i\theta}$ where
$$r = |z|, \quad \tan\theta = \frac{y}{x}.$$
- **Complex exponentiation:**
$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y).$$

Complex Vectors

Denote by $\mathbb{C}^n = \mathbb{C}^{n\times 1}$ the space of all column vectors of n *complex* elements.

- The **hermitian** or **conjugate transpose** of $\mathbf{u} \in \mathbb{C}^n$ is denoted by $\mathbf{u}^*$:

$$\mathbf{u}^* \in \mathbb{C}^{1\times n}.$$

- The inner product of $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ is defined by

$$\mathbf{u}^*\mathbf{v} = \sum_{k=1}^{n} \overline{u}_k v_k.$$

  The 2-norm for complex vectors is defined in terms of this inner product:

$$\|\mathbf{u}\|_2^2 = \mathbf{u}^*\mathbf{u}.$$

Complex Matrices

Denote by $\mathbb{C}^{m\times n}$ the space of all complex matrices with m rows and n columns.

- The **hermitian** or conjugate transpose of $A \in \mathbb{C}^{m\times n}$ is denoted by A^*:

$$A^* = \left(\overline{A}\right)^{\mathrm{T}} = \overline{\left(A^{\mathrm{T}}\right)} \in \mathbb{C}^{n\times m}.$$

- A **unitary** matrix is a complex analogue of an orthogonal matrix. If $U \in \mathbb{C}^{n\times n}$ is unitary, then

$$U^*U = UU^* = I$$

  and

$$\|U\mathbf{z}\|_2 = \|\mathbf{z}\|_2\,, \quad \text{for any } \mathbf{z} \in \mathbb{C}^n.$$

Complex Matrices: Some Analogies

	Real	Complex
Norm	$\|\mathbf{v}\|_2 = \sqrt{\mathbf{v}^{\mathrm{T}}\mathbf{v}}$	$\|\mathbf{u}\|_2 = \sqrt{\mathbf{u}^*\mathbf{u}}$
Symmetry	$S^{\mathrm{T}} = S$ (symmetric matrix)	$S^* = S$ (hermitian matrix)
Orthonormality	$Q^{\mathrm{T}}Q = I$ (orthogonal matrix)	$U^*U = I$ (unitary matrix)
Householder	$H = I - \dfrac{2}{\mathbf{v}^{\mathrm{T}}\mathbf{v}}\mathbf{v}\mathbf{v}^{\mathrm{T}}$	$H = I - \dfrac{2}{\mathbf{u}^*\mathbf{u}}\mathbf{u}\mathbf{u}^*$

Eigenvalue Decomposition (EVD)

Key Problems in Linear Algebra

- Square linear system: $\left[\begin{array}{l}\text{Given: } A \in \mathbb{R}^{n\times n} \text{ and } \vec{b} \in \mathbb{R}^n \\ \text{Want: } \vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} = \vec{b}\end{array}\right.$

- Overdetermined linear system: $\left[\begin{array}{l}\text{Given: } A \in \mathbb{R}^{m\times n} \text{ and } \vec{b} \in \mathbb{R}^m \\ \text{Want: } \vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} \text{ ``="} \vec{b}\end{array}\right.$

- Eigenvalue Problem: $\left[\begin{array}{l}\text{Given: } A \in \mathbb{C}^{n\times n} \\ \text{Want: } \lambda \in \mathbb{C} \text{ and } \underset{\text{nonzero}}{\vec{v} \in \mathbb{C}^n} \text{ s.t. } A\vec{v} = \lambda\vec{v}\end{array}\right.$

Eigenvalue Decomposition

n-by-n matrix w/ complex entries

Eigenvalue Problem : Given $A \in \mathbb{C}^{n \times n}$

Find a scalar **eigenvalue** λ and an associated nonzero **eigenvector** $\mathbf{v}$ satisfying

$$A\mathbf{v} = \lambda \mathbf{v}.$$

stretch or compress

- The **spectrum** of A is the set of all eigenvalues; the **spectral radius** is $\max_j |\lambda_j|$.
- The problem is equivalent to

$$\vec{0} = A\vec{v} - \lambda\vec{v} = A\vec{v} - \lambda I\vec{v} = (A - \lambda I)\vec{v}$$

This equation has a nontrivial solution $\vec{v}$ iff $A - \lambda I$ is singular.

$$\vec{0} = B\vec{v}$$

$$B^{-1}\vec{0} = B^{-1}B\vec{v}$$

$$\vec{0} = \vec{v}$$

- An eigenvalue of A is a root of the **characteristic polynomial**

$$\underbrace{\det(A - \lambda I)}_{\text{characteristic poly.}} = 0$$

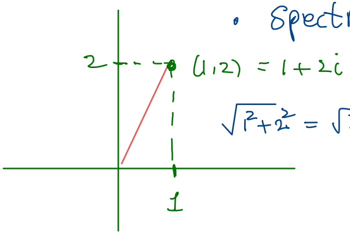
Example If the eigenvalues of $A \in \mathbb{R}^{4\times 4}$ are

$$-3, \quad 5, \quad 1 \pm 2i.$$

Then

- Spectrum of $A = \{-3, 5, 1+2i, 1-2i\}$

- Spectral radius of A



$\sqrt{1^2+2^2} = \sqrt{5}$

$$= \max\{|-3|, |5|, |1+2i|, |1-2i|\}$$

$$= \max\{3, 5, \sqrt{5}, \sqrt{5}\}$$

$$= 5$$

Eigenvalue Decomposition (cont')

λ_k is an e-value of A and $\vec{v}_k$ is an e-vec. of A corresp. to λ_k.

Let $A \in \mathbb{C}^{n\times n}$ and suppose that $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$ for $k \in \mathbb{N}[1,n]$.

- Then

$$\begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{bmatrix},$$

$$A \underbrace{\left[\begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{array}\right]}_{=V} = \underbrace{\left[\begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{array}\right]}_{=V} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{=D}$$

$$\Longrightarrow \quad AV = VD.$$

(works for any square matrix.)

Columns of V are e-vectors of A. Diag. entries are e-values of A.

- If V is nonsingular, we can further write

(i.e., all e-vectors are linearly independent)

$$\boxed{A = VDV^{-1},}$$

which is called an **eigenvalue decomposition (EVD)** of A. If $\mathbf{v}$ is an eigenvector of A, then so is $c\mathbf{v}$, $c \neq 0$. Thus an EVD is not unique.

Eigenvalue Decomposition (cont')

a sufficient cond. for diagonalizability

If A has an EVD, we say that A is **diagonalizable**; otherwise **nondiagonalizable**.

Theorem 1 (Diagonalizability)

If $A \in \mathbb{C}^{n\times n}$ has n distinct eigenvalues, then A is diagonalizable.

Notes.

- Let $A, B \in \mathbb{C}^{n\times n}$. We say that B is **similar** to A if there exists a nonsingular matrix X such that

$$B = XAX^{-1}. \quad (A \sim B)$$

E.g. If A has an EVD, then $A = VDV^{-1}$

nonsingular.

So A is similar to D.

- So *diagonalizability* is *similarity to a diagonal matrix*.
- Similar matrices share the same eigenvalues.

Calculating EVD in MATLAB

- `E = eig(A)`
  produces a column vector `E` containing the eigenvalues of `A`.
- `[V, D] = eig(A)`
  produces V and D in an EVD of A, $A = VDV^{-1}$.

Notes on EVD

Understanding EVD: Change of Basis

Let $X \in \mathbb{C}^{n\times n}$ be a nonsingular matrix.

- The columns $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ of X form a basis of $\mathbb{C}^n$.
- Any $\mathbf{z} \in \mathbb{C}^n$ is uniquely written as

$$\mathbf{z} = X\mathbf{u} = u_1\mathbf{x}_1 + u_2\mathbf{x}_2 + \cdots + u_n\mathbf{x}_n.$$

  - The scalars $u_1, \ldots, u_n$ are called the **coordinates** of $\mathbf{z}$ with respect to the columns of X.
  - The vector $\mathbf{u} = X^{-1}\mathbf{z}$ is the representation of $\mathbf{z}$ with respect to the basis consisting of the columns of X.

Upshot

Left-multiplication by X^{-1} performs a **change of basis** into the coordinates associated with the columns of X.

Understanding EVD: Change of Basis (cont')

Suppose $A \in \mathbb{C}^{n\times n}$ has an EVD $A = VDV^{-1}$. Then, for any $\mathbf{z} \in \mathbb{C}^n$, $\mathbf{y} = A\mathbf{z}$ can be written as

$$V^{-1}\mathbf{y} = DV^{-1}\mathbf{z}.$$

Interpretation

The matrix A is a diagonal transformation in the coordinates with respect to the V-basis.

What Is EVD Good For?

Suppose $A \in \mathbb{C}^{n \times n}$ has an EVD $A = VDV^{-1}$.

- Economical computation of powers A^k:

$$\boxed{A^k = VD^kV^{-1}.}$$

- Analyzing convergence of iterates $(\mathbf{x}_1, \mathbf{x}_2, \ldots)$ constructed by

$$\mathbf{x}_{j+1} = A\mathbf{x}_j, \quad j = 1, 2, \ldots$$

  If $\mathbf{x}_1$ is an eigenvector associated to eigenvalue λ, then

$$\mathbf{x}_1 \longrightarrow \lambda\mathbf{x}_1 \longrightarrow \lambda^2\mathbf{x}_1 \longrightarrow \cdots \longrightarrow \lambda^{k-1}\mathbf{x}_1 \longrightarrow \cdots$$

Conditioning of Eigenvalues

Theorem 2 (Bauer-Fike)

Let $A \in \mathbb{C}^{n\times n}$ *be diagonalizable,* $A = VDV^{-1}$*, with eigenvalues* $\lambda_1, \ldots, \lambda_n$*. If* μ *is an eigenvalue of* $A + \delta A$ *for a complex matrix* δA*, then*

$$\min_{1\leqslant j\leqslant n} \left|\mu - \lambda_j\right| \leqslant \kappa_2(V) \left\|\delta A\right\|_2 .$$