Spectral Theory

Eigenvalue Decomposition (EVD)

Singular Value Decomposition (SVD)

Eigenvalue Decomposition

Office Hours (This week only)

- TW 4:45 ~ 6:15

Contents

Complex Numbers and Complex Arrays

Complex Numbers

In what follows, we assume all scalars, vectors, and matrices may be complex.

Notation.

- $\mathbb{R}$: the set of all real numbers
- $\mathbb{C}$: the set of all complex numbers, *i.e.*,

$$\{z = x + iy \,|\, x, y \in \mathbb{R}\} \quad \text{where } i = \sqrt{-1}.$$

Complex Numbers in MATLAB

Let $z = x + iy \in \mathbb{C}$.

MATLAB	Name	Notation
`real(z)`	real part of z	$\operatorname{Re} z$
`imag(z)`	imaginary part of z	$\operatorname{Im} z$
`conj(z)`	conjugate of z	$\overline{z}$
`abs(z)`	modulus of z	$\lvert z \rvert$
`angle(z)`	argument of z	$\arg(z)$

Euler's Formula

- Recall that the Maclaurin series for e^t is

$$e^t = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \quad -\infty < t < \infty.$$

- Replacing t by it and separating real and imaginary parts (using the cyclic behavior of powers of i), we obtain

$$e^{it} = \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!}}_{\cos(t)} + i \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}}_{\sin(t)}$$

- The result is called the **Euler's formula**.

$$\boxed{e^{it} = \cos(t) + i\sin(t).}$$

Polar Representation and Complex Exponential

- **Polar representation:** A complex number $z = x + iy \in \mathbb{C}$ can be written as $z = re^{i\theta}$ where
$$r = |z|, \quad \tan\theta = \frac{y}{x}.$$
- **Complex exponentiation:**
$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y).$$

Complex Vectors

Denote by $\mathbb{C}^n = \mathbb{C}^{n\times 1}$ the space of all column vectors of n *complex* elements.

- The **hermitian** or **conjugate transpose** of $\mathbf{u} \in \mathbb{C}^n$ is denoted by $\mathbf{u}^*$:

$$\mathbf{u}^* \in \mathbb{C}^{1\times n}.$$

$$\vec{u}^* = (\overline{\vec{u}})^T$$

- The inner product of $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ is defined by

$$\mathbf{u}^*\mathbf{v} = \sum_{k=1}^{n} \overline{u}_k v_k.$$

The 2-norm for complex vectors is defined in terms of this inner product:

$$\|\mathbf{u}\|_2^2 = \mathbf{u}^*\mathbf{u}.$$

Complex Matrices

Denote by $\mathbb{C}^{m\times n}$ the space of all complex matrices with m rows and n columns.

- The **hermitian** or conjugate transpose of $A \in \mathbb{C}^{m\times n}$ is denoted by A^*:

$$A^* = \left(\overline{A}\right)^{\mathrm{T}} = \overline{\left(A^{\mathrm{T}}\right)} \in \mathbb{C}^{n\times m}.$$

- A **unitary** matrix is a complex analogue of an orthogonal matrix. If $U \in \mathbb{C}^{n\times n}$ is unitary, then

$$U^*U = UU^* = I$$

$U^* = U^{-1}$

and

$$\|U\mathbf{z}\|_2 = \|\mathbf{z}\|_2\,, \quad \text{for any } \mathbf{z} \in \mathbb{C}^n.$$

Complex Matrices: Some Analogies

	Real	Complex
Norm	$\lVert \mathbf{v} \rVert_2 = \sqrt{\mathbf{v}^{\mathrm{T}}\mathbf{v}}$	$\lVert \mathbf{u} \rVert_2 = \sqrt{\mathbf{u}^*\mathbf{u}}$
Symmetry	$S^{\mathrm{T}} = S$ (symmetric matrix)	$S^* = S$ (hermitian matrix)
Orthonormality	$Q^{\mathrm{T}}Q = I$ (orthogonal matrix)	$U^*U = I$ (unitary matrix)
Householder	$H = I - \dfrac{2}{\mathbf{v}^{\mathrm{T}}\mathbf{v}}\mathbf{v}\mathbf{v}^{\mathrm{T}}$	$H = I - \dfrac{2}{\mathbf{u}^*\mathbf{u}}\mathbf{u}\mathbf{u}^*$

Eigenvalue Decomposition (EVD)

Key Problems in Linear Algebra

- Given $A \in \mathbb{R}^{n\times n}$ and $\vec{b} \in \mathbb{R}^n$, find $\vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$.

- Given $A \in \mathbb{R}^{m\times n}$ and $\vec{b} \in \mathbb{R}^m$, find $\vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x}$ "$=$" $\vec{b}$.
  $(m \geq n)$

- Given $A \in \mathbb{C}^{n\times n}$, find $\lambda \in \mathbb{C}$ and $\vec{v} \in \mathbb{C}^n$ s.t. $A\vec{v} = \lambda \vec{v}$.

n-by-n matrix
with complex entries.

Eigenvalue Decomposition

$(1,2) = 1+2i$

$|1 \pm 2i| = \sqrt{1^2 + 2^2}$

$= \sqrt{5}$

Eigenvalue Problem

Find a scalar **eigenvalue** λ and an associated nonzero **eigenvector** $\mathbf{v}$ satisfying

$$A\mathbf{v} = \lambda\mathbf{v}.$$

- The **spectrum** of A is the set of all eigenvalues; the **spectral radius** is $\max_j |\lambda_j|$.

E.g. if the eigenvalues of A are $3, 5, -1 \pm 2i$, then

- Spectrum of $A = \{3, 5, -1+2i, -1-2i\}$
- Spectral radius of $A = \max\{3, 5, \sqrt{5}\} = 5$

- The problem is equivalent to

$\vec{0} = A\vec{v} - \lambda\vec{v} = A\vec{v} - \lambda I\vec{v}$

$= (A - \lambda I)\vec{v}$

- An eigenvalue of A is a root of the **characteristic polynomial**

$\vec{v}$ is nonzero iff $A - \lambda I$ is not invertible

iff $\boxed{\det(A - \lambda I) = 0}$.

Eigenvalue Decomposition (cont')

$\lambda_1, \cdots, \lambda_n$: eigenvalues of A

$\vec{v}_1, \cdots, \vec{v}_n$: eigenvectors of A

(corres. to $\lambda_1, \cdots, \lambda_n$, respectively)

Let $A \in \mathbb{C}^{n\times n}$ and suppose that $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$ for $k \in \mathbb{N}[1, n]$.

- Then

$$\begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{bmatrix},$$

$$A \underbrace{\left[\begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{array}\right]}_{=V} = \underbrace{\left[\begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{array}\right]}_{=V} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{= D \text{ (diagonal)}}$$

$$\implies AV = VD.$$

(works for any square matrix.)

columns are e-vec. of A

diagonal entries are e-val. of A.

- If V is nonsingular, we can further write

all e-vectors are linearly independent

$$A = VDV^{-1},$$

which is called an **eigenvalue decomposition (EVD)** of A. If $\mathbf{v}$ is an eigenvector of A, then so is $c\mathbf{v}$, $c \neq 0$. Thus an EVD is not unique.

Eigenvalue Decomposition (cont')

a sufficient cond. for diagonalizability

If A has an EVD, we say that A is **diagonalizable**; otherwise **nondiagonalizable**.

Theorem 1 (Diagonalizability)

If $A \in \mathbb{C}^{n\times n}$ has n distinct eigenvalues, then A is diagonalizable.

Notes.

- Let $A, B \in \mathbb{C}^{n\times n}$. We say that B is **similar** to A if there exists a nonsingular matrix X such that

$$B = XAX^{-1}.$$

$$A = \underbrace{V}_{\text{nonsingular}} D \boxed{V^{-1}}$$

- So *diagonalizability* is *similarity to a diagonal matrix*.

$\Rightarrow$ A is similar to D.

$\Rightarrow$ A is similar to a diag. matrix.

- Similar matrices share the same eigenvalues.

Idea of proof: Show $\det(A - \lambda I)$

$= \det(XBX^{-1} - \lambda I)$

Calculating EVD in MATLAB

- `E = eig(A)`
  produces a column vector `E` containing the eigenvalues of `A`.
- `[V, D] = eig(A)`
  produces V and D in an EVD of A, $A = VDV^{-1}$.

Notes on EVD

Understanding EVD: Change of Basis

Given Given

$X\vec{u} = \vec{z}$

Let $X \in \mathbb{C}^{n\times n}$ be a nonsingular matrix.

- The columns $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ of X form a basis of $\mathbb{C}^n$.
- Any $\mathbf{z} \in \mathbb{C}^n$ is uniquely written as

$$\mathbf{z} = X\mathbf{u} = u_1\mathbf{x}_1 + u_2\mathbf{x}_2 + \cdots + u_n\mathbf{x}_n.$$

  - The scalars $u_1, \ldots, u_n$ are called the **coordinates** of $\mathbf{z}$ with respect to the columns of X.
  - The vector $\mathbf{u} = X^{-1}\mathbf{z}$ is the representation of $\mathbf{z}$ with respect to the basis consisting of the columns of X.

Upshot

Left-multiplication by X^{-1} performs a **change of basis** into the coordinates associated with the columns of X.

output · input

Suppose $A \in \mathbb{C}^{n \times n}$ has an EVD $\underline{A = VDV^{-1}}$. Then, for any $\mathbf{z} \in \mathbb{C}^n$, $\mathbf{y} = A\mathbf{z}$ can be written as

$$\underbrace{V^{-1}\mathbf{y}}_{\text{rep'n of } \vec{y} \text{ w.r.t. } V\text{-basis}} = \underbrace{DV^{-1}\mathbf{z}}_{\text{rep'n of } \vec{z} \text{ w.r.t. } V\text{-basis}}.$$

$\vec{y} = VDV^{-1}\vec{z}$

$V^{-1}\vec{y} = DV^{-1}\vec{z}$

left-multiply by V^{-1}

Interpretation

The matrix A is a diagonal transformation in the coordinates with respect to the V-basis.

What Is EVD Good For?

Suppose $A \in \mathbb{C}^{n \times n}$ has an EVD $A = VDV^{-1}$.

- Economical computation of powers A^k:

$$A^k = V \underbrace{D^k}_{=\begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}} V^{-1}.$$

When $k=2$:

$$A^2 = (VDV^{-1})(VDV^{-1})$$

$$= V D \underbrace{(V^{-1}V)}_{= I} D V^{-1}$$

$$= V D^2 V^{-1}$$

- Analyzing convergence of iterates $(\mathbf{x}_1, \mathbf{x}_2, \ldots)$ constructed by

$$\mathbf{x}_{j+1} = A\mathbf{x}_j, \quad j = 1, 2, \ldots$$

If $\mathbf{x}_1$ is an eigenvector associated to eigenvalue λ, then

$$\mathbf{x}_1 \longrightarrow \lambda\mathbf{x}_1 \longrightarrow \lambda^2\mathbf{x}_1 \longrightarrow \cdots \longrightarrow \lambda^{k-1}\mathbf{x}_1 \longrightarrow \cdots$$

Conditioning of Eigenvalues

Theorem 2 (Bauer-Fike)

Let $A \in \mathbb{C}^{n \times n}$ *be diagonalizable,* $A = VDV^{-1}$*, with eigenvalues* $\lambda_1, \ldots, \lambda_n$*. If* μ *is an eigenvalue of* $A + \delta A$ *for a complex matrix* δA*, then*

$$\min_{1 \leqslant j \leqslant n} \left| \mu - \lambda_j \right| \leqslant \kappa_2(V) \left\| \delta A \right\|_2 .$$