

Piecewise Cubic Interpolation

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Hermite Cubic Interpolation

Problem Set-Up: General Piecewise Cubic Interpolation

We now seek a piecewise cubic polynomial which interpolates the data (x_i, y_i) for $i = 1, \dots, n$, with $x_1 < x_2 < \dots < x_n$, defined as

$$(x) = \begin{cases} p_1(x), & x \in [x_1, x_2) \\ p_2(x), & x \in [x_2, x_3) \\ \vdots & \vdots \\ p_{n-1}(x), & x \in [x_{n-1}, x_n] \end{cases},$$

where the i th *local* cubic polynomial p_i is written in shifted power form as

$$p_i(x) = c_{i,1} + c_{i,2}(x - x_i) + c_{i,3}(x - x_i)^2 + c_{i,4}(x - x_i)^3.$$

Hermite Cubic Interpolation

If the slopes at the breakpoints are prescribed, *i.e.*, for each $i = 1, \dots, n - 1$,

$$p_i(x_i) = y_i, \quad p'_i(x_i) = \sigma_i, \quad p_i(x_{i+1}) = y_{i+1}, \quad p'_i(x_{i+1}) = \sigma_{i+1},$$

then we can solve for the four unknown coefficients $c_{i,j}$, $j = 1, \dots, 4$:

$$\begin{aligned} c_{i,1} &= y_i, & c_{i,3} &= \frac{3y[x_i, x_{i+1}] - 2\sigma_i - \sigma_{i+1}}{\Delta x_i}, \\ c_{i,2} &= \sigma_i, & c_{i,4} &= \frac{\sigma_i + \sigma_{i+1} - 2y[x_i, x_{i+1}]}{(\Delta x_i)^2}. \end{aligned}$$

where $\Delta x_i = x_{i+1} - x_i$ and

$$y[x_i, x_{i+1}] = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}. \quad (\text{Newton's divided difference})$$

This is called **Hermite cubic interpolation**.

Implementation

```
function c = hermiteCoeff(x,y,s)
% Input:
%   x,y,s   data points and slopes
% Output:
%   c       coefficients in matrix form
n = length(x);
c = zeros(n-1, 4);
dx = diff(x);
dy = diff(y);
dydx = dy./dx;
c(:,1) = y;
c(:,2) = s;
c(:,3) = (3*dydx - 2*s(1:n-1) - s(2:n))./dx;
c(:,4) = (s(1:n-1) + s(2:n-1) - 2*dydx)./(dx.^2);
end
```

Convergence: Error Analysis

Theorem 1 (Error Theorem for Hermite Cubic Interpolation)

Let $f \in C^4[a, b]$ and let (x) be the Hermite cubic interpolant of

$$(x_i, f(x_i), f'(x_i)), \quad \text{for } i = 1, \dots, n,$$

where

$$x_j = a + (j - 1)h \quad \text{and} \quad h = \frac{b - a}{n - 1}.$$

Then

$$\|f - \|_{\infty} \leq \frac{1}{384} \|f^{(4)}\|_{\infty} h^4.$$

Drawbacks of Hermite Cubic Interpolation

- The interpolant (x) is in C^1 and so its display may be too crude in graphical applications.
- In other applications, there may be difficulties if $''(x)$ is discontinuous.
- In experimental settings where y_i are measurements of some sort, we may not have the first derivative information required for the cubic Hermite process.

Cubic Splines

Cubic Splines

Idea: Put together cubic polynomials to make the result as smooth as possible.

- At interior breakpoints: for $j = 2, 3, \dots, n - 1$
 - matching values: $p_{j-1}(x_j) = p_j(x_j)$ [($n - 2$) eqns]
 - matching first derivatives: $p'_{j-1}(x_j) = p'_j(x_j)$ [($n - 2$) eqns]
 - matching second derivative: $p''_{j-1}(x_j) = p''_j(x_j)$ [($n - 2$) eqns]
- So, together with the n interpolating conditions, we have total of $(4n - 6)$ equations.
- To match up with the number of unknowns $(4n - 4)$, we need to impose two more conditions on the boundary:
 - 1 slopes at each end (clamped cubic spline)
 - 2 second derivatives at the endpoints (natural cubic spline)
 - 3 periodic boundary condition
 - 4 *not-a-knot* boundary condition: $p_1(x) \equiv p_2(x)$ and $p_{n-2}(x) \equiv p_{n-1}(x)$.

Convergence: Error Analysis

Theorem 2 (Error Theorem for Clamped Cubic Splines)

Let $f \in C^4[a, b]$ and let (x) be the cubic spline interpolant of

$$(x_i, f(x_i)), \quad \text{for } i = 1, \dots, n,$$

with the exact boundary conditions

$$\sigma_1 = f'(x_1) \quad \text{and} \quad \sigma_n = f'(x_n),$$

in which

$$x_j = a + (j - 1)h \quad \text{and} \quad h = \frac{b - a}{n - 1}.$$

Then

$$\|f - \sigma\|_{\infty} \leq \frac{5}{384} \|f^{(4)}\|_{\infty} h^4.$$

Remarks

- Hermite cubic interpolation is about five times as accurate as cubic spline interpolation, yet both have *fourth-order accuracy*.
- Unlike the former, the latter does not require first derivatives.