Exercises: Square Linear Systems

Problems marked with ✏ are to be done by hand; those marked with 💻 are to be solved using a computer.

1. (Interpolation; **FNC** 2.1.1) Suppose you want to interpolate the points $(-1, 0)$, $(0, 1)$, $(2, 0)$, $(3, 1)$, and $(4, 2)$ by a polynomial of as low a degree as possible.
   (a) ✏ What degree should you expect this polynomial to be? (The degree could be lower in special cases where some coefficients are exactly zero.)
   (b) ✏ Write out a linear system of equations for the coefficients of the interpolating polynomial.
   (c) 💻 Use MATLAB to solve the system in (b) numerically.

2. (Hermite interpolant; **FNC** 2.1.4) ✏ Say you want to find a cubic polynomial $p(x)$ such that $p(0) = 0$, $p'(0) = 1$, $p(1) = 2$, and $p'(1) = -1$. (This is known as a *Hermite interpolant.*) Write out a linear system of equations for the coefficients of $p(x)$.

3. (Gaussian transformation matrices; Su20 final exam) ✏ Let $\{\mathbf{e}_j \in \mathbb{R}^n \mid j \in \mathbb{N}[1, n]\}$ be the standard unit basis of $\mathbb{R}^n$, *i.e.*, $\mathbf{e}_1 = (1, 0, 0, \cdots, 0)^{\mathrm{T}}$, $\mathbf{e}_2 = (0, 1, 0, \cdots, 0)^{\mathrm{T}}$, $\ldots$, $\mathbf{e}_n = (0, 0, 0, \cdots, 1)^{\mathrm{T}}$. In this problem, we denote by G_j the Gaussian transformation matrix of the form
$$G_j = I + \sum_{i=j+1}^{n} a_{i,j} \mathbf{e}_i \mathbf{e}_j^{\mathrm{T}}.$$
   In addition, let $P(i, j) \in \mathbb{R}^{n \times n}$ be the elementary permutation matrix obtained by interchanging the i-th and the j-th rows of the same-sized identity matrix.
   (a) Let $1 \leqslant j < k < \ell \leqslant n$. Show that $P(k, \ell) G_j P(k, \ell) = I + \sum_{i=j+1}^{n} b_{i,j} \mathbf{e}_i \mathbf{e}_j^{\mathrm{T}}$, where
$$b_{i,j} = \begin{cases} a_{i,j}, & \text{if } i \neq k \text{ and } i \neq \ell, \\ a_{\ell,j}, & \text{if } i = k, \\ a_{k,j}, & \text{if } i = \ell. \end{cases}$$
   (b) Show that $G_j^{-1} = I - \sum_{i=j+1}^{n} a_{i,j} \mathbf{e}_i \mathbf{e}_j^{\mathrm{T}}$.
   (c) Let $j < k$. Show that $G_j G_k = I + \sum_{i=j+1}^{n} a_{i,j} \mathbf{e}_i \mathbf{e}_j^{\mathrm{T}} + \sum_{i=k+1}^{n} a_{i,k} \mathbf{e}_i \mathbf{e}_k^{\mathrm{T}}$.

(d) Use the previous parts to find PLU factorization, $PA = LU$, by hand.

$$A = \begin{bmatrix} 5 & -5 & -2 \\ 5 & -2 & 7 \\ 10 & -3 & 18 \end{bmatrix}.$$

4. (Permutation matrix; **LM** 10.1–8) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{bmatrix}.$$

Do the following by hand.

(a) Multiply A by a permutation matrix P to interchange the 1^{st} and the 4^{th} rows. Write out P explicitly.

(b) Multiply A by a permutation matrix P to interchange the 1^{st} and the 4^{th} columns. Write out P explicitly.

(c) Multiply A by two permutation matrices P and Q to interchange the 1^{st} and the 4^{th} rows and columns. Write out P and Q explicitly.

(d) Find a permutation matrix (more complicated than those above) which moves columns as described below:

- 2^{nd} to 1^{st};
- 3^{rd} to 2^{nd};
- 4^{th} to 3^{rd};
- 1^{st} to 4^{th};
- 5^{th} to 5^{th} (unmoved).

Show that this permutation matrix is not its own inverse. What is the smallest positive integer k such that $P^k = I$? Write this permutation matrix as a product of *elementary* permutation matrices.

5. (Dramadah; **FNC** 2.4.4) Let D_n be the matrix created using MATLAB's `gallery` function using

```
gallery('dramadah', n)
```

where n is a positive integer. It has interesting properties: the entries of D_n are all 0 or 1, and the entries of D_n^{-1} are all integers. Run an experiment that verifies that if $D_n = LU$ is an LU factorization, then the entries of L, U, L^{-1}, and U^{-1} are all integers for $n = 2, 3, \ldots, 50$. You will have to do something more clever than visual inspection of the matrix entries to determine that they are integers; the `round` and `any` commands may be helpful.

6. (Matrix norms; Sp20 midterm) Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}.$$

(a) Calculate $\|A\|_1$, $\|A\|_2$, $\|A\|_\infty$, and $\|A\|_F$ all by hand.

(b) Imagine that MATLAB does not offer `norm` function and you are writing one for others to use, which begins with

```
function MatrixNorm(A, j)
% MatrixNorm   computes matrix norms
% Usage:
%   mat_norm(A, 1) returns the 1-norm of A
%   mat_norm(A, 2) is the same as mat_norm(A)
%   mat_norm(A, 'inf') returns the infinity-norm of A
%   mat_norm(A, 'fro') returns the Frobenius norm of A
```

Complete the program. (*Hint*: To handle the second input argument properly which can be a number or a character, use `ischaracter` and/or `strcmp`.)

7. (FLOP Counting) Do **LM** 10.1–12(a,b,d). Justify your calculation of p and c for each part.