Numerical Integration

Contents

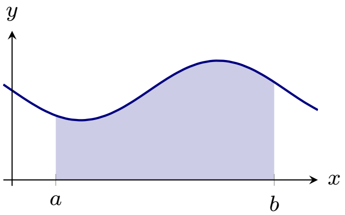
Introduction

Quadrature Problem

Consider the definite integral

$$I\{f\} = \int_a^b f(x)\, dx$$

which represents the *net area* of the region between the curve $y = f(x)$ and the x-axis on $[a, b]$.



We seek to approximate it numerically by a weighted sum

$$I\{f\} \approx \sum_{i=1}^{n} \omega_i f(x_i).$$

- ω_i's are called the *weights*;
- x_i's are called the *nodes* for the particular numerical method used.

Some Questions

Q1. Why do we care?

- An exact antiderivative of f is not accessible
- f may be known at limited points

Q2. How do we do?

- Replace $f(x)$ by an approximate function $p(x)$ and integrate it instead.

Q3. What are good candidates for $p(x)$?

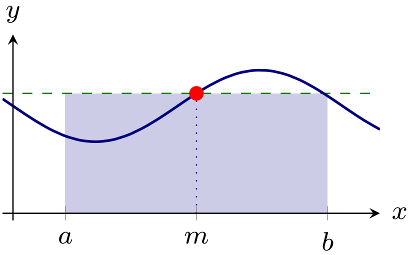
- In choosing $p(x)$, we require that (a) $f(x) - p(x)$ is not too large and (b) $p(x)$ can be exactly integrated (by hand).
  - (piecewise) constant $\longrightarrow$ (composite) midpoint rule
  - (piecewise) linear $\longrightarrow$ (composite) trapezoidal rule
  - (piecewise) quadratic $\longrightarrow$ (composite) Simpson's rule

Newton-Cotes Formulas

Newton-Cotes methods are a collection of numerical integration methods in which nodes are equally spaced in $[a, b]$. Let $m = (a + b)/2$.

- **Midpoint Method:**

$$I \approx f(m)(b - a)$$



Midpoint method: one node

Newton-Cotes Formulas

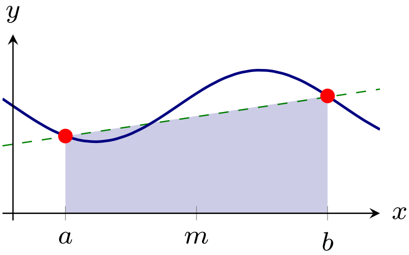
Newton-Cotes methods are a collection of numerical integration methods in which nodes are equally spaced in $[a, b]$. Let $m = (a + b)/2$.

- **Midpoint Method:**

$$I \approx f(m)(b - a)$$

- **Trapezoidal Method:**

$$I \approx \frac{1}{2}\big(f(a) + f(b)\big)\,(b - a)$$



Trapezoid method: two nodes

Newton-Cotes Formulas

Newton-Cotes methods are a collection of numerical integration methods in which nodes are equally spaced in $[a, b]$. Let $m = (a + b)/2$.

- **Midpoint Method:**

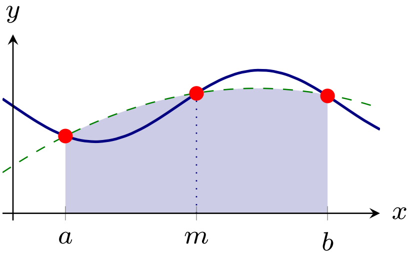
$$I \approx f(m)(b - a)$$

- **Trapezoidal Method:**

$$I \approx \frac{1}{2}\big(f(a) + f(b)\big)\,(b - a)$$

- **Simpson's Method:**

$$I \approx \frac{1}{6}\big(f(a) + 4f(m) + f(b)\big)\,(b - a)$$



Simpson's method: three node

Convergence of Midpoint Method

Convergence of Trapezoidal Method

Derivation of Simpson's Method via Extrapolation

Composite Methods

Composite Methods

Errors for Composite Methods

MATLAB Implementation

MATLAB has four functions that calculate 1-D definite integrals:

- `quad`: adaptive Simpson's method
- `quadl`: Gauss-Lobatto quadrature
- `quadgk`: Gauss-Kronrod quadrature
- `integral`

```
a = 2; b = 5;
f = @(x) x.*exp(x);
f_int = @(x) (x - 1).*exp(x);
exact_area = f_int(b) - f_int(a);
int1 = quad(f, a, b);
int2 = quadl(f, a, b);
int3 = quadgk(f, a, b);
int4 = integral(f, a, b);
```

Composite Trapezoidal and Midpoint Methods

For better accuracy, we can subdivide the interval $[a, b]$ into equispaced subintervals

$$a = x_1 < x_2 < \cdots < x_n = b \quad \text{with } x_i = a + (i-1)h \text{ and } h = \frac{b-a}{n-1}.$$

- **Composite Trapezoidal Method**:

$$\sum_{i=1}^{n-1} \frac{1}{2}\left(f(x_i) + f(x_{i+1})\right)h = \frac{1}{2}\left(f(x_1) + f(x_n)\right)h + \sum_{i=2}^{n-1} f(x_i)h.$$

- **Composite Midpoint Method**:

$$\sum_{i=1}^{n-1} f(x_{i+1/2})h,$$

where $x_{i+1/2} = (x_i + x_{i+1})/2$.