

Spectral Theory

[Eigenvalue Decomposition (EVD)
Singular Value Decomposition (SVD)

Eigenvalue Decomposition

Office Hours (This week only)

- TW 4:45 ~ 6:15

Contents

- ① Complex Numbers and Complex Arrays
- ② Eigenvalue Decomposition (EVD)
- ③ Notes on EVD

Complex Numbers and Complex Arrays

Complex Numbers

In what follows, we assume all scalars, vectors, and matrices may be complex.

Notation.

- \mathbb{R} : the set of all real numbers
- \mathbb{C} : the set of all complex numbers, *i.e.*,

$$\{z = x + iy \mid x, y \in \mathbb{R}\} \quad \text{where } i = \sqrt{-1}.$$

Complex Numbers in MATLAB

Let $z = x + iy \in \mathbb{C}$.

MATLAB	Name	Notation
<code>real(z)</code>	real part of z	$\operatorname{Re} z$
<code>imag(z)</code>	imaginary part of z	$\operatorname{Im} z$
<code>conj(z)</code>	conjugate of z	\bar{z}
<code>abs(z)</code>	modulus of z	$ z $
<code>angle(z)</code>	argument of z	$\arg(z)$

Euler's Formula

- Recall that the Maclaurin series for e^t is

$$e^t = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \quad -\infty < t < \infty.$$

- Replacing t by it and separating real and imaginary parts (using the cyclic behavior of powers of i), we obtain

$$e^{it} = \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!}}_{\cos(t)} + i \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}}_{\sin(t)}$$

- The result is called the **Euler's formula**.

$$\boxed{e^{it} = \cos(t) + i \sin(t).}$$

Polar Representation and Complex Exponential

- **Polar representation:** A complex number $z = x + iy \in \mathbb{C}$ can be written as $z = re^{i\theta}$ where

$$r = |z|, \quad \tan \theta = \frac{y}{x}.$$

- **Complex exponentiation:**

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y).$$

Complex Vectors

Denote by $\mathbb{C}^n = \mathbb{C}^{n \times 1}$ the space of all column vectors of n *complex* elements.

- The **hermitian** or **conjugate transpose** of $\mathbf{u} \in \mathbb{C}^n$ is denoted by \mathbf{u}^* :

$$\mathbf{u}^* \in \mathbb{C}^{1 \times n}.$$

- The inner product of $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ is defined by

$$\mathbf{u}^* \mathbf{v} = \sum_{k=1}^n \overline{u}_k v_k.$$

The 2-norm for complex vectors is defined in terms of this inner product:

$$\|\mathbf{u}\|_2^2 = \mathbf{u}^* \mathbf{u}.$$

Complex Matrices

Denote by $\mathbb{C}^{m \times n}$ the space of all complex matrices with m rows and n columns.

- The **hermitian** or conjugate transpose of $A \in \mathbb{C}^{m \times n}$ is denoted by A^* :

$$A^* = (\overline{A})^T = \overline{(A^T)} \in \mathbb{C}^{n \times m}.$$

- A **unitary** matrix is a complex analogue of an orthogonal matrix. If $U \in \mathbb{C}^{n \times n}$ is unitary, then

$$U^*U = UU^* = I$$

and


$$\|U\mathbf{z}\|_2 = \|\mathbf{z}\|_2, \quad \text{for any } \mathbf{z} \in \mathbb{C}^n.$$

Complex Matrices: Some Analogies

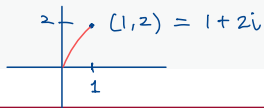
	Real	Complex
Norm	$\ \mathbf{v}\ _2 = \sqrt{\mathbf{v}^T \mathbf{v}}$	$\ \mathbf{u}\ _2 = \sqrt{\mathbf{u}^* \mathbf{u}}$
Symmetry	$S^T = S$ (symmetric matrix)	$S^* = S$ (hermitian matrix)
Orthonormality	$Q^T Q = I$ (orthogonal matrix)	$U^* U = I$ (unitary matrix)
Householder	$H = I - \frac{2}{\mathbf{v}^T \mathbf{v}} \mathbf{v} \mathbf{v}^T$	$H = I - \frac{2}{\mathbf{u}^* \mathbf{u}} \mathbf{u} \mathbf{u}^*$

Eigenvalue Decomposition (EVD)

Key Problems in Linear Algebra

- Given $A \in \mathbb{R}^{n \times n}$ and $\vec{b} \in \mathbb{R}^n$, find $\vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$.
- Given $A \in \mathbb{R}^{m \times n}$ and $\vec{b} \in \mathbb{R}^m$, find $\vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$.
($m \geq n$)
- Given $A \in \mathbb{C}^{n \times n}$, find $\lambda \in \mathbb{C}$ and $\vec{v} \in \mathbb{C}^n$ s.t. $A\vec{v} = \lambda\vec{v}$.

n-by-n matrix
with complex entries.

Eigenvalue Decomposition



$$|1 \pm 2i| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Eigenvalue Problem

Find a scalar **eigenvalue** λ and an associated nonzero **eigenvector** \mathbf{v} satisfying

$$A\mathbf{v} = \lambda\mathbf{v}.$$

- The **spectrum** of A is the set of all eigenvalues; the **spectral radius** is $\max_j |\lambda_j|$.

E.g. if the eigenvalues of A are $3, 5, -1 \pm 2i$, then

- Spectrum of $A = \{3, 5, -1+2i, -1-2i\}$
- Spectral radius of $A = \max\{3, 5, \sqrt{5}\}$

$$= \textcircled{5}$$

- The problem is equivalent to

$$\begin{aligned} \vec{0} &= A\vec{v} - \lambda\vec{v} = A\vec{v} - \lambda I\vec{v} \\ &= (A - \lambda I)\vec{v} \end{aligned}$$

- An eigenvalue of A is a root of the **characteristic polynomial**

\vec{v} is nonzero iff $A - \lambda I$ is not invertible
iff $\det(A - \lambda I) = 0$.

Eigenvalue Decomposition (cont')

Let $A \in \mathbb{C}^{n \times n}$ and suppose that $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$ for $k \in \mathbb{N}[1, n]$.

$\lambda_1, \dots, \lambda_n$: eigenvalues of A
 $\vec{v}_1, \dots, \vec{v}_n$: eigenvectors of A
(comes to $\lambda_1, \dots, \lambda_n$ respectively)

- Then

$$\begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{v}_1 & \lambda_2 \mathbf{v}_2 & \cdots & \lambda_n \mathbf{v}_n \end{bmatrix},$$

$$A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = D \text{ (diagonal)}$$

$$\implies AV = VD.$$

(works for any square matrix.)

columns are
e-vec. of A

diagonal entries
are e-val. of A .

- If V is nonsingular, we can further write

all e-vectors are linearly independent $A = VDV^{-1}$

which is called an **eigenvalue decomposition (EVD)** of A . If \mathbf{v} is an eigenvector of A , then so is $c\mathbf{v}$, $c \neq 0$. Thus an EVD is not unique.

Eigenvalue Decomposition (cont')

a sufficient cond. for diagonalizability

If A has an EVD, we say that A is **diagonalizable**; otherwise **nondiagonalizable**.

Theorem 1 (Diagonalizability)

If $A \in \mathbb{C}^{n \times n}$ has n distinct eigenvalues, then A is diagonalizable.

Notes.

- Let $A, B \in \mathbb{C}^{n \times n}$. We say that B is **similar** to A if there exists a nonsingular matrix X such that

$$B = XAX^{-1}.$$

- So diagonalizability is similarity to a diagonal matrix.

★ Similar matrices share the same eigenvalues.

Idea of proof: $\det(A - \lambda I)$
show $\Rightarrow \det(XBX^{-1} - \lambda I)$

$$A = \underbrace{V D V^{-1}}_{\text{nonsingular}}$$

$\Rightarrow A$ is similar to D .

$\Rightarrow A$ is similar to a diag. matrix.

Calculating EVD in MATLAB

- $E = \text{eig}(A)$
produces a column vector E containing the eigenvalues of A .
- $[V, D] = \text{eig}(A)$
produces V and D in an EVD of A , $A = VDV^{-1}$.

Notes on EVD

Understanding EVD: Change of Basis

Let $X \in \mathbb{C}^{n \times n}$ be a nonsingular matrix.

- The columns $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ of X form a basis of \mathbb{C}^n .
- Any $\mathbf{z} \in \mathbb{C}^n$ is uniquely written as

$$\mathbf{z} = X\mathbf{u} = u_1\mathbf{x}_1 + u_2\mathbf{x}_2 + \dots + u_n\mathbf{x}_n.$$

- The scalars u_1, \dots, u_n are called the **coordinates** of \mathbf{z} with respect to the columns of X .
- The vector $\mathbf{u} = X^{-1}\mathbf{z}$ is the representation of \mathbf{z} with respect to the basis consisting of the columns of X .

Given \downarrow $X\mathbf{u} = \mathbf{z}$ \downarrow Given

Upshot

Left-multiplication by X^{-1} performs a **change of basis** into the coordinates associated with the columns of X .

Understanding EVD: Change of Basis (cont')

Suppose $A \in \mathbb{C}^{n \times n}$ has an EVD $A = VDV^{-1}$. Then, for any $\mathbf{z} \in \mathbb{C}^n$, $\mathbf{y} = A\mathbf{z}$ can be written as

$$\underbrace{V^{-1}\mathbf{y}}_{\text{rep'n of } \vec{y} \text{ w.r.t. } V\text{-basis}} = D \underbrace{V^{-1}\mathbf{z}}_{\text{rep'n of } \vec{z} \text{ w.r.t. } V\text{-basis}}$$

rep'n of \vec{y} w.r.t. V -basis rep'n of \vec{z} w.r.t. V -basis

$$\vec{y} = VDV^{-1}\vec{z} \\ V^{-1}\vec{y} = DV^{-1}\vec{z}$$

left-multiply
by V^{-1}

Interpretation

The matrix A is a diagonal transformation in the coordinates with respect to the V -basis.

What Is EVD Good For?

Suppose $A \in \mathbb{C}^{n \times n}$ has an EVD $A = VDV^{-1}$.

- Economical computation of powers A^k :

$$A^k = VD^kV^{-1}.$$

$$\begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \ddots \\ & & & \lambda_n^k \end{bmatrix}$$

- Analyzing convergence of iterates $(\mathbf{x}_1, \mathbf{x}_2, \dots)$ constructed by

$$\mathbf{x}_{j+1} = A\mathbf{x}_j, \quad j = 1, 2, \dots$$

If \mathbf{x}_1 is an eigenvector associated to eigenvalue λ , then

$$\mathbf{x}_1 \longrightarrow \lambda\mathbf{x}_1 \longrightarrow \lambda^2\mathbf{x}_1 \longrightarrow \dots \longrightarrow \lambda^{k-1}\mathbf{x}_1 \longrightarrow \dots$$

When $k=2$:

$$\begin{aligned} A^2 &= (VDV^{-1})(VDV^{-1}) \\ &= V D \underbrace{(V^{-1}V)}_I D V^{-1} \\ &= V D^2 V^{-1} \end{aligned}$$

Conditioning of Eigenvalues

Theorem 2 (Bauer-Fike)

Let $A \in \mathbb{C}^{n \times n}$ be diagonalizable, $A = VDV^{-1}$, with eigenvalues $\lambda_1, \dots, \lambda_n$. If μ is an eigenvalue of $A + \delta A$ for a complex matrix δA , then

$$\min_{1 \leq j \leq n} |\mu - \lambda_j| \leq \kappa_2(V) \|\delta A\|_2.$$