Piecewise Linear Interpolation

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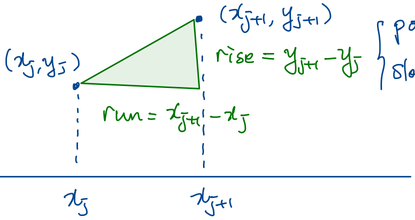
e.g. $n=3$

$$p(x) = \begin{cases} \boxed{}, & x \in [x_1, x_2] \\ \boxed{}, & x \in [x_2, x_3] \end{cases}$$

Assume that $x_1 < x_2 < \cdots < x_n$ are fixed. The function $p(x)$ defined piecewise[1] by

$$p(x) = y_j + \frac{y_{j+1} - y_j}{x_{j+1} - x_j}(x - x_j)\,, \quad \text{for } x \in [x_j, x_{j+1}],\ 1 \leqslant j \leqslant n-1$$

- is linear on each interval $[x_j, x_{j+1}]$; (i.e., is piecewise linear w.r.t. the given nodes)
- connects any two consecutive data points (x_j, y_j) and (x_{j+1}, y_{j+1}) by a straight line.



point: (x_j, y_j)

slope: $\dfrac{y_{j+1} - y_j}{x_{j+1} - x_j}$

[1] Note the formula changes depending on which interval x lies in.

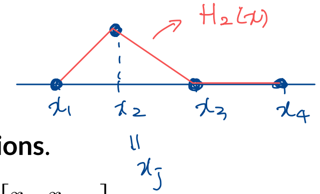
Hat Functions

$\mathcal{I}(\vec{e}_j)$ the piecewise linear interpolant of the j^{th} unit basis vector.

Denote by $H_j(x)$ the jth *piecewise linear* cardinal function:

$$H_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j], \\ \dfrac{x_{j+1} - x}{x_{j+1} - x_j}, & x \in [x_j, x_{j+1}], \\ 0, & \text{otherwise}, \end{cases} \qquad j = 1, 2, \ldots, n.$$

e.g. $n = 4$, $j = 2$



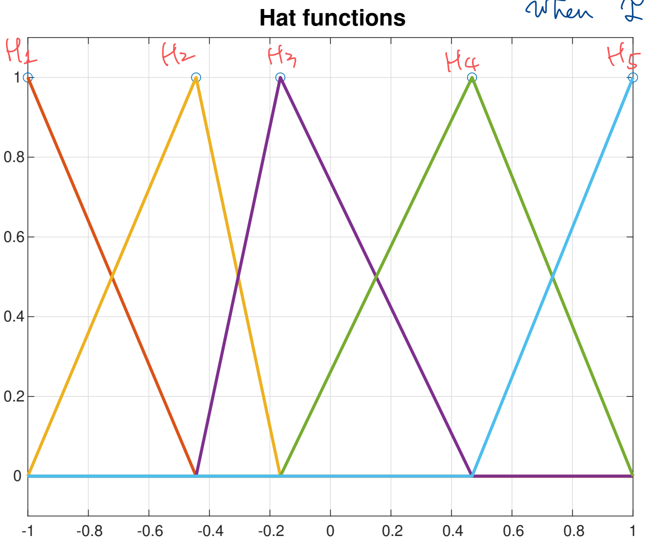
- The functions $H_1, \ldots, H_n$ are called **hat functions** or **tent functions**.
- Each H_j is globally continuous and is linear inside each interval $[x_j, x_{j+1}]$

Note: The definitions of $H_1(x)$ and $H_n(x)$ require additional nodes x_0 and x_{n+1} for x outside of $[x_1, x_n]$, which is not relevant in the discussion of interpolation.

Hat Functions (cont')

$\mathcal{I}(\vec{e}_j) = H_j$

when $\mathcal{I}$ represents the PL interp.



Hat Functions As Basis

- Any linear combination of hat functions is continuous and is linear inside each interval $[x_j, x_{j+1}]$.
- Conversely, *any* such function is expressible as a unique linear combination of hat functions, *i.e.*,

$$\sum_{j=1}^{n} c_j H_j(x), \quad \text{for some choice of } c_1, \ldots, c_n.$$

- No smaller set of functions has the same properties.

> The hat functions form a **basis** of the set of functions that are continuous and piecewise linear relative to $\mathbf{x}$ (the vector of nodes).

Abstract note

Polynom. interp.

- Basis : $\{1, x, x^2, \cdots, x^{n-1}\}$ (monomials)
- Matrix : V Vandermond matrix.
- $p(x) = \sum_{j=1}^{n} c_j x^{j-1}$ where

  c_j is obtained by solving

  $V\vec{c} = \vec{y}$

PL interp.

- Basis : $\{H_1, \cdots, H_n\}$ (hat fncs)
- Matrix : I
- $p(x) = \sum_{j=1}^{n} c_j H_j(x)$ where

  c_j is obtained by solving

  $I\vec{c} = \vec{y}$.

Cardinality Conditions

- By construction, the hat functions are cardinal functions for piecewise linear (PL) interpolation, *i.e.*, they satisfy

"Kronecker delta"

$$H_j(x_k) = \delta_{j,k}. = \begin{cases} 1, & j=k \\ 0, & j \neq k \end{cases} \quad \text{(cardinality condition)}$$

- Key consequence of this property is that the piecewise linear interpolant $p(x)$ for the data values in $\mathbf{y}$ is trivially expressed by

$$p(x) = \sum_{j=1}^{n} y_j H_j(x).$$

- Interpolation property:

$$p(x_k) = y_k, \quad k=1,2,\cdots,n$$

- Since $H_1, \cdots, H_n$ form a basis of piecewise linear funcs (w.r.t. nodes),

$$p(x) = \sum_{j=1}^{n} c_j H_j(x)$$

For any $k=1,\cdots,n$,

$$p(x_k) = \sum_{j=1}^{n} c_j H_j(x_k) = \sum_{j=1}^{n} c_j \delta_{j,k} = \boxed{c_k = y_k}$$

Recipe for PL Interpolant

Piecewise Linear Interpolant

The piecewise linear polynomial

$$p(x) = \sum_{j=1}^{n} y_j H_j(x)$$

is the unique such function which passes through all the data points.

Proof: It is easy to check the interpolating property:

$$p(x_k) = \sum_{j=1}^{n} y_j H_j(x_k) = \sum_{j=1}^{n} y_j \delta_{j,k} = y_k \quad \text{for every } k \in \mathbb{N}[1,n].$$

To show uniqueness, suppose $\tilde{p}$ is another such function in the form

$$\tilde{p}(x) = \sum_{j=1}^{n} c_j H_j(x) .$$

Then $p(x_k) - \tilde{p}(x_k) = 0$ for all $k \in \mathbb{N}[1,n]$. This implies that $c_k = y_k$ for all k $\square$

Analysis

Conditioning

Conclusion : PL interp. is well-conditioned.

Lemma

Let $\mathcal{I}$ is the piecewise linear interpolation operator and $\mathbf{z} \in \mathbb{R}^n$. Then

$$\|\mathcal{I}(\mathbf{z})\|_\infty = \|\mathbf{z}\|_\infty .$$

fnc. vec.

- It follows from the lemma that the absolute condition number of piecewise linear interpolation in the infinity norm equals one.

$$\kappa(\vec{y}) = \max_{\Delta\vec{y} \neq \vec{0}} \frac{\|\mathcal{I}(\vec{y}+\Delta\vec{y}) - \mathcal{I}(\vec{y})\|_\infty}{\|\Delta\vec{y}\|_\infty} \quad \text{(def'n)}$$

$$= \max_{\Delta\vec{y} \neq \vec{0}} \frac{\|\mathcal{I}(\Delta\vec{y})\|_\infty}{\|\Delta\vec{y}\|_\infty} \quad \text{(by linearity of } \mathcal{I}\text{)}$$

$$= \max_{\Delta\vec{y} \neq \vec{0}} \frac{\|\Delta\vec{y}\|_\infty}{\|\Delta\vec{y}\|_\infty} = 1 .$$

Proof of lemma. Let

$$p(x) = \mathcal{I}(\mathbf{z}) = \sum_{j=1}^{n} z_j H_j(x).$$

Let k be the index corresponding to the element of $\mathbf{z}$ with the largest absolute value, that is, $z_k = \|\mathbf{z}\|_\infty$. Since $z_k = p(x_k)$, it follows that $\left|p(x_k)\right| = \|\mathbf{z}\|_\infty$ and so $\|p\|_\infty \geqslant \|\mathbf{z}\|_\infty$.

To show the other inequality, note that

$$\left|p(x)\right| = \left|\sum_{j=1}^{n} z_j H_j(x)\right| \leqslant \sum_{j=1}^{n} |z_j|\, H_j(x) \leqslant \|\mathbf{z}\|_\infty \sum_{j=1}^{n} H_j(x) = \|\mathbf{z}\|_\infty\,,$$

where the final step uses the fact[2] that $\sum_{j=1}^{n} H_j(x) = 1$. It implies that $\|p\|_\infty \leqslant \|\mathbf{z}\|_\infty$. Therefore, $\|p\|_\infty = \|\mathbf{z}\|_\infty$. □

[2]This property is called the *partition of unity*. Confirm it!

Convergence: Error Analysis

Set-up for analysis.

- Generate a set of data points using a “nice” function f on an interval containing all nodes, *i.e.*, $y_j = f(x_j)$. (The *niceness* of a function is described in precise terms below.)
- Then perform PL interpolation of the data to obtain the interpolant p.
- **Question.** How close is p to f?

Notation (Space of Differentiable Functions)

Let $C^n[a,b]$ denote the set of all functions that are n-times continuously differentiable on $[a,b]$. That is, if $f \in C^n[a,b]$, then $f^{(n)}$ exists and is continuous on $[a,b]$, where derivatives at the end points are taken to be one-sided derivatives.

Convergence: Error Analysis (cont')

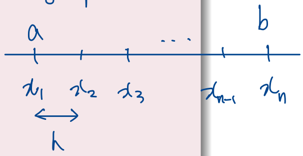
f is twice continuously differentiable

Theorem 1 (Error Theorem for PL Interpolation)

Suppose that $f \in C^2[a, b]$. *Let* p_n *be the piecewise linear interpolant of* $(x_j, f(x_j))$ *for* $j = 1, \ldots, n$, *where*

$$x_j = a + (j-1)h \quad \textit{and} \quad h = \frac{b-a}{n-1}.$$

equispaced nodes



Then

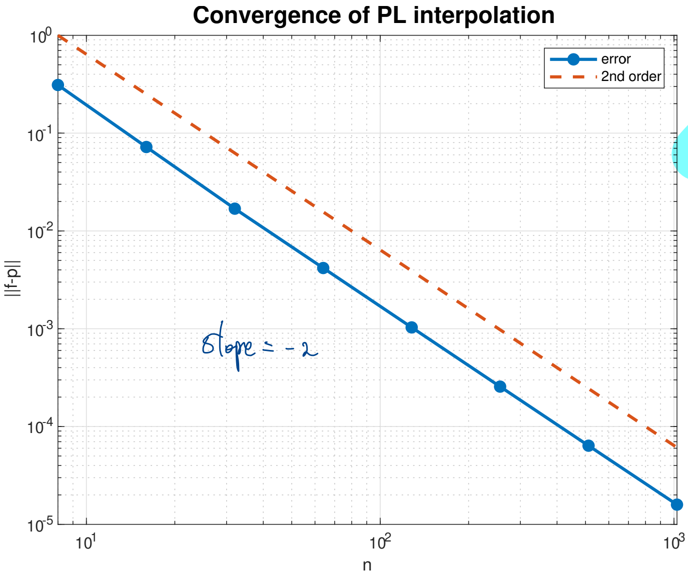
$$\|f - p_n\|_\infty \leqslant \|f''\|_\infty \, h^2.$$

- The theorem pertains to the interpolation on equispaced nodes.
- The significance of the theorem is that the error in the interpolant is $O(h^2)$ as $h \to 0$. (We say that PL interpolation is *second-order accurate*.)
- **Practical implication:** If n is doubled, the PL interpolant becomes about four times more accurate. A log-log graph (`loglog`) of error against n is a straight line.

Convergence: Error Analysis (cont')

$h = \frac{b-a}{n-1}, \quad h \propto \frac{1}{n}$

log – log



$\|f - p_n\|_\infty \lesssim \frac{M}{n^2}$

$\log \|f - p_n\|_\infty$

$\lesssim \boxed{-2} \log(n) + C.$