Math 3607: Homework 7

Wednesday, March 9, 2022

TOTAL: 30 points

- Problems marked with ✎ are to be done by hand; those marked with 💻 are to be solved using a computer.
- **Important note.** Do not use *Symbolic Math Toolbox*. Any work done using `sym` or `syms` will receive NO credit.
- **Another important note.** When asked write a MATLAB function, write one at the end of your live script.

1. (Orthogonal decomposition; **FNC** 3.3.8) ✎ The matrix $P = \widehat{Q}\widehat{Q}^{\mathrm{T}}$ derived from the thin QR factorization has some interesting and important properties.

   (a) Show that $P = AA^{+}$.

   (b) Prove that $P^2 = P$. (This is a defining property for a *projection matrix*.)

   (c) Clearly, any vector $\mathbf{x}$ may be written as $\mathbf{x} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} = P\mathbf{x}$ and $\mathbf{v} = (I - P)\mathbf{x}$. Prove that for $P = \widehat{Q}\widehat{Q}^{\mathrm{T}}$, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. (Together with part (b), this means that P is an *orthogonal projector*.)

2. (Pseudoinverse and Householder matrix by hand) ✎ Answer the following questions.

   (a) Find the pseudoinverse A^{+} when

   $$A = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

   (b) Find a Householder matrix H such that

   $$H \begin{bmatrix} -6 \\ 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ 0 \end{bmatrix}.$$

3. (Properties of Householder transformation matrices) ✎ Let $\mathbf{v} = \|\mathbf{z}\|_2\, \mathbf{e}_1 - \mathbf{z}$ and let H be the *Householder transformation* defined by

   $$H = I - 2\frac{\mathbf{v}\mathbf{v}^{\mathrm{T}}}{\mathbf{v}^{\mathrm{T}}\mathbf{v}}.$$

   Show that

   (a) H is symmetric;

(b) H is orthogonal;

(c) $H\mathbf{z} = \|\mathbf{z}\|_2 \, \mathbf{e}_1$.

4. (Adapted from **FNC** 3.3.3.) Let $x_1, x_2, \ldots, x_m$ be m equally spaced points in $[-1, 1]$ and V be the Vandermonde-type matrix

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{bmatrix},$$

where $m = 400$ and $n = 5$. Find the thin QR factorization of $V = \widehat{Q}\widehat{R}$, and, on a single graph, plot every column of $\widehat{Q}$ against the vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)^{\mathrm{T}}$.

5. (Gram-Schmidt in MATLAB; **LM** 12.6–2) Do **LM** 12.6–2. Write a MATLAB function to factor a matrix $A \in \mathbb{R}^{m \times n}$ for $m \geqslant n$ by the Gram-Schmidt method (**not** the modified Gram-Schmidt method). Check these numerically against the MATLAB function `qr` which calculates thin QR. Analytically, $Q^{\mathrm{T}}Q - I = O$ and $QR - A = O$, so check how accurately these two equations hold using your function for large values of m and n.