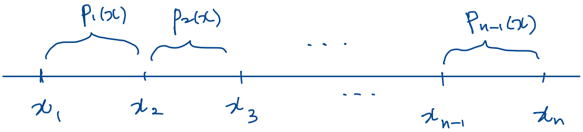
Piecewise Cubic Interpolation

Framework (set-up)

Given nodes $x_1, x_2, \cdots, x_n$



Our interpolant is

the piecewise defined function

$$\mathbb{P}(x) = \begin{cases} p_1(x) & \text{for } x \in [x_1, x_2) \\ p_2(x) & \text{for } x \in [x_2, x_3) \\ \vdots & \\ p_i(x) & \text{for } x \in [x_i, x_{i+1}) \\ \vdots & \\ p_{n-1}(x) & \text{for } x \in [x_{n-1}, x_n] \end{cases}$$

each of $p_1(x), \cdots, p_{n-1}(x)$ is a cubic polynomial.

Interpolation prop

$$\mathbb{P}(x_i) = y_i \quad \text{for } i = 1, \cdots, n$$

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Problem Set-Up: General Piecewise Cubic Interpolation

We now seek a piecewise cubic polynomial which interpolates the data (x_i, y_i) for $i = 1, \ldots, n$, with $x_1 < x_2 < \cdots < x_n$, defined as

$$\mathbb{P}(x) = \begin{cases} p_1(x), & x \in [x_1, x_2) \\ p_2(x), & x \in [x_2, x_3) \\ \vdots & \vdots \\ p_{n-1}(x), & x \in [x_{n-1}, x_n] \end{cases},$$

where the ith *local* cubic polynomial p_i is written in shifted power form as

w/ center at left end point of the i^{th} interval.

$$p_i(x) = c_{i,1} + c_{i,2}(x - x_i) + c_{i,3}(x - x_i)^2 + c_{i,4}(x - x_i)^3.$$

<u>Unknown coeffs</u> : $4(n-1)$

four coeff for each i

$(n-1)$ intervals

Hermite Cubic Interpolation

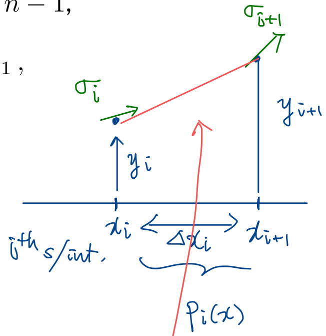
$x_1, x_2, \dots, x_n$ (nodes)

If the slopes at the breakpoints are prescribed, *i.e.*, for each $i = 1, \dots, n-1$,

$$p_i(x_i) = y_i\,, \quad p_i'(x_i) = \sigma_i\,, \quad p_i(x_{i+1}) = y_{i+1}\,, \quad p_i'(x_{i+1}) = \sigma_{i+1}\,,$$

then we can solve for the four unknown coefficients $c_{i,j}$, $j = 1, \dots, 4$:

$$\boxed{\begin{aligned} c_{i,1} &= y_i\,, & c_{i,3} &= \frac{3y[x_i, x_{i+1}] - 2\sigma_i - \sigma_{i+1}}{\Delta x_i}\,, \\ c_{i,2} &= \sigma_i\,, & c_{i,4} &= \frac{\sigma_i + \sigma_{i+1} - 2y[x_i, x_{i+1}]}{(\Delta x_i)^2}\,. \end{aligned}}$$



where $\Delta x_i = x_{i+1} - x_i$ and

$$y[x_i, x_{i+1}] = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}. \qquad \text{(Newton's divided difference)}$$

slope of the line connecting (x_i, y_i) and (x_{i+1}, y_{i+1})

This is called **Hermite cubic interpolation**.

$$p_i(x) = c_{i,1} + c_{i,2}(x - x_i) + c_{i,3}(x - x_i)^2 + c_{i,4}(x - x_i)^3$$

- $p_i(x_i) = \boxed{c_{i,1} = y_i}$

- $p_i'(x) = c_{i,2} + 2c_{i,3}(x - x_i) + 3c_{i,4}(x - x_i)^2$

$$p_i'(x_i) = \boxed{c_{i,2} = \sigma_i}$$

Implementation

```
function c = hermiteCoeff(x,y,s)
% Input:
%   x,y,s   data points and slopes
% Ouput:
%   c       coefficients in matrix form
    n = length(x);
    c = zeros(n-1, 4);
    dx = diff(x);
    dy = diff(y);
    dydx = dy./dx;
    c(:,1) = y;
    c(:,2) = s;
    c(:,3) = (3*dydx - 2*s(1:n-1) - s(2:n))./dx;
    c(:,4) = (s(1:n-1) + s(2:n-1) - 2*dydx))./(dx.^2);
end
```

Convergence: Error Analysis

the collection of all four times continuously differentiable functions on $[a, b]$

Theorem 1 (Error Theorem for Hermite Cubic Interpolation)

Let $f \in C^4[a,b]$ *and let* $p(x)$ *be the Hermite cubic interpolant of*

$$\left(x_i, f(x_i), f'(x_i)\right), \quad \text{for } i = 1, \ldots, n,$$

x-coord, y-coord, slope

where

$$x_j = a + (j-1)h \quad \text{and} \quad h = \frac{b-a}{n-1}.$$

Then

$$\|f - p\|_\infty \quad \cancel{\|f\|_\infty} \leqslant \frac{1}{384}\left\|f^{(4)}\right\|_\infty h^4. \qquad (4^{\text{th}}\text{-order accurate})$$

E.g. Suppose $\|f - p\|_\infty \le 10^{-4}$ when $h = 0.2$

What is an upper bound on $\|f - p\|_\infty$ when $h = 0.1$?

Ans. $10^{-4}/16$.

Drawbacks of Hermite Cubic Interpolation

i.e., P is continuously differentiable

- The interpolant $P(x)$ is in C^1 and so its display may be too crude in graphical applications.
- In other applications, there may be difficulties if $P''(x)$ is discontinuous.
- In experimental settings where y_i are measurements of some sort, we may not have the first derivative information required for the cubic Hermite process.

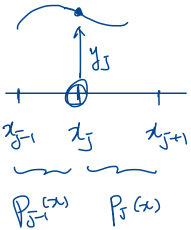
Cubic Splines

Cubic Splines

In fancy terms, we seek $P \in C^2[a,b]$, i.e., twice continuously differentiable interpolant.

Idea: Put together cubic polynomials to make the result as smooth as possible.

- At interior breakpoints: for $j = 2, 3, \cdots, n-1$
  - matching values: $p_{j-1}(x_j) = p_j(x_j)$ $[(n-2)$ eqns$]$
  - matching first derivatives: $p'_{j-1}(x_j) = p'_j(x_j)$ $[(n-2)$ eqns$]$
  - matching second derivative: $p''_{j-1}(x_j) = p''_j(x_j)$ $[(n-2)$ eqns$]$



- So, together with the n interpolating conditions, we have total of $(4n-6)$ equations.
- To match up with the number of unknowns $(4n-4)$, we need to impose two more conditions on the boundary:

# of eqns : $4n-6$

# of unk. : $4n-4$

1. slopes at each end (clamped cubic spline)
2. second derivatives at the endpoints (natural cubic spline)
3. periodic boundary condition
4. *not-a-knot* boundary condition: $p_1(x) \equiv p_2(x)$ and $p_{n-2}(x) \equiv p_{n-1}(x)$.

MATLAB's default

" breakpoint = node

Convergence: Error Analysis

Theorem 2 (Error Theorem for Clamped Cubic Splines)

Let $f \in C^4[a,b]$ and let $P(x)$ be the cubic spline interpolant of

$$\left(x_i, f(x_i)\right), \quad \text{for } i = 1, \dots, n,$$

with the exact boundary conditions

$$\sigma_1 = f'(x_1) \quad \text{and} \quad \sigma_n = f'(x_n),$$

in which

$$x_j = a + (j-1)h \quad \text{and} \quad h = \frac{b-a}{n-1}.$$

Then

$$\|f - P\|_\infty$$

$$\cancel{\|f - \|_\infty} \leqslant \frac{5}{384} \left\|f^{(4)}\right\|_\infty h^4. \qquad (4^{th}\text{-order accurate})$$

Remarks

- Hermite cubic interpolation is about five times as accurate as cubic spline interpolation, yet both have *fourth-order accuracy*.
- Unlike the former, the latter does not require first derivatives.