Math 3607

Final Exam

- Written: 04/26 (T) 12:00 ~ 04/29 (F) 23:59
  (Gradescope)
- Quiz : Entire Friday (3 attempts ; 1 hour each)
  (Carmen)

04/25/2022

Lecture 38 Review for Final Exam

Eigenvalue Decomposition $\quad A \in \mathbb{R}^{n\times n}$ (square)

- General: $AV = VD \longleftarrow A\vec{v}_j = \lambda_j \vec{v}_j$ for $j = 1, \cdots, n$

- EVD: $A = VDV^{-1}$
  (when V is invertible)

- Useful in studying matrix powers

$$A^k = (VDV^{-1})(VDV^{-1})\cdots(VDV^{-1})$$
$$= VD^k V^{-1}$$

$\longleftarrow$ " /V " in MATLAB
$\uparrow$ forward slash

$$V = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

- HW 8: #2, #3

(Recursively defined sequences such as Fibonacci or Pell numbers.)

Singular Value Decomposition $\quad A \in \mathbb{C}^{m\times n} \quad (m \geq n)$

- $A = U \Sigma V^{*}$ ← conjugate transpose (or Hermitian)

U, V unitary → { Thick SVD, Thin SVD }

Σ real diagonal

- $U^* U = UU^* = I_{m\times m}$
  i.e. $U^{-1} = U^*$
- $V^* V = VV^* = I_{n\times n}$
  i.e. $V^{-1} = V^*$

$A = U\Sigma V^*$

$A^* = (U\Sigma V^*)^*$

$= (V^*)^* \Sigma^* U^*$

$= V \Sigma^T U^*$

- analytical properties (2-norm)

- Useful in studying A^T, $A^H = A^*$, A^+, ... → pseudoinverse.

- Low-rank approx. : Image Compression

- HW9: #1

Root finding $f(r) = 0$

- Fixed point iteration (linear)
  Newton's method (quad.)
  Secant method (superlinear)

- Convergence (series analysis)

$$\epsilon_{k+1} \approx C\,\epsilon_k^p$$

HW 9: #5 (b)

- Multidimensional Newton's method.

(Scalar) $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$

(Vector) $\vec{x}_{k+1} = \vec{x}_k - \left[J(\vec{x}_k)\right]^{-1} \vec{f}(\vec{x}_k)$

Jacobian

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots \end{bmatrix}$$

- Lambert's W function

$$y = x e^{x} \iff x = W(y)$$

- HW9 #3
- Week 11 supp.

Piecewise polynomial interpolation and numerical differentiation

- See Supplementary Resources for practice problems

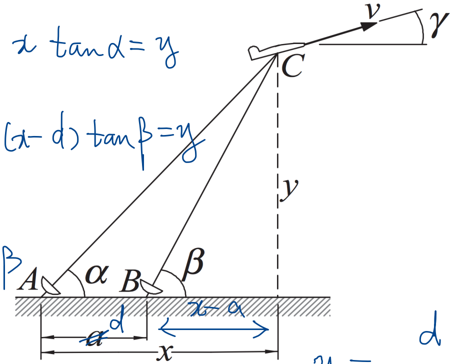
- ☆ Optimal step size

- ☆ Richardson extrapolation

Airplane Velocity

The radar stations A and B, separated by the distance $a = 500$ m, track a plane C by recording the angles α and β at one-second intervals. Your goal, back at air traffic control, is to determine the speed of the plane.

$$\begin{cases} \tan\alpha = \dfrac{y}{x} \longrightarrow x\tan\alpha = y \\ \tan\beta = \dfrac{y}{x-d} \longrightarrow (x-d)\tan\beta = y \end{cases}$$



$$x\tan\alpha = (x-d)\tan\beta$$

$$x = \frac{d\tan\beta}{\tan\beta - \tan\alpha}, \qquad y = \frac{d\tan\alpha\tan\beta}{\tan\beta - \tan\alpha}$$

Data from radar stations

time(s)	α(°)	β(°)
0.00	α_1	β_1
0.01	α_2	β_2
0.02	α_3	β_3
0.03	α_4	β_4
⋮	⋮	⋮
1.00	α_{101}	β_{101}

Conversion

$$\longrightarrow$$

$$x = \frac{d \tan\beta}{\tan\beta - \tan\alpha}$$

$$y = \;...$$

time (s)	x (m)	y (m)
0.00	x_1	y_1
0.01	x_2	y_2
0.02	x_3	y_3
⋮	⋮	⋮
1.00	x_{101}	y_{101}

num. diff $\longrightarrow$

Δt

time	$x'(t)$	$y'(t)$	$\sqrt{(x'(t))^2 + (y'(t))^2}$
0.00	$\frac{x_2 - x_1}{\Delta t}$		
0.01	$\frac{x_3 - x_1}{2\Delta t}$		
0.02	$\vdots$		
0.03			
$\vdots$			
1.00	$\frac{x_{101} - x_{100}}{\Delta t}$		

$\rightarrow$ 2nd ~~1st~~ -order FD using first 3 data

$\rightarrow$ 2nd-order CD

$\rightarrow$ 2nd ~~1st~~ -order BD using last 3 data

finite difference formulas