

Piecewise Linear Interpolation

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Assume that $x_1 < x_2 < \dots < x_n$ are fixed. The function $p(x)$ defined piecewise¹ by

$$p(x) = y_j + \frac{y_{j+1} - y_j}{x_{j+1} - x_j}(x - x_j), \quad \text{for } x \in [x_j, x_{j+1}], 1 \leq j \leq n - 1$$

- is linear on each interval $[x_j, x_{j+1}]$;
- connects any two consecutive data points (x_j, y_j) and (x_{j+1}, y_{j+1}) by a straight line.

¹Note the formula changes depending on which interval x lies in.

Hat Functions

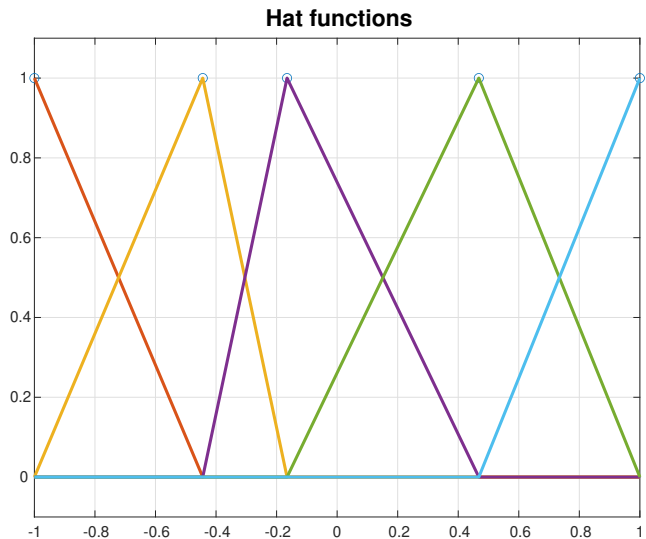
Denote by $H_j(x)$ the j th *piecewise linear* cardinal function:

$$H_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j], \\ \frac{x_{j+1} - x}{x_{j+1} - x_j}, & x \in [x_j, x_{j+1}], \\ 0, & \text{otherwise,} \end{cases} \quad j = 1, 2, \dots, n.$$

- The functions H_1, \dots, H_n are called **hat functions** or **tent functions**.
- Each H_j is globally continuous and is linear inside each interval $[x_j, x_{j+1}]$

Note: The definitions of $H_1(x)$ and $H_n(x)$ require additional nodes x_0 and x_{n+1} for x outside of $[x_1, x_n]$, which is not relevant in the discussion of interpolation.

Hat Functions (cont')



Hat Functions As Basis

- Any linear combination of hat functions is continuous and is linear inside each interval $[x_j, x_{j+1}]$.
- Conversely, *any* such function is expressible as a unique linear combination of hat functions, *i.e.*,

$$\sum_{j=1}^n c_j H_j(x), \quad \text{for some choice of } c_1, \dots, c_n.$$

- No smaller set of functions has the same properties.

The hat functions form a **basis** of the set of functions that are continuous and piecewise linear relative to \mathbf{x} (the vector of nodes).

Cardinality Conditions

- By construction, the hat functions are cardinal functions for piecewise linear (PL) interpolation, *i.e.*, they satisfy

$$H_j(x_k) = \delta_{j,k}. \quad (\text{cardinality condition})$$

- Key consequence of this property is that the piecewise linear interpolant $p(x)$ for the data values in \mathbf{y} is trivially expressed by

$$p(x) = \sum_{j=1}^n y_j H_j(x).$$

Recipe for PL Interpolant

Piecewise Linear Interpolant

The piecewise linear polynomial

$$p(x) = \sum_{j=1}^n y_j H_j(x)$$

is the unique such function which passes through all the data points.

Proof: It is easy to check the interpolating property:

$$p(x_k) = \sum_{j=1}^n y_j H_j(x_k) = \sum_{j=1}^n y_j \delta_{j,k} = y_k \quad \text{for every } k \in \mathbb{N}[1, n].$$

To show uniqueness, suppose \tilde{p} is another such function in the form

$$\tilde{p}(x) = \sum_{j=1}^n c_j H_j(x).$$

Then $p(x_k) - \tilde{p}(x_k) = 0$ for all $k \in \mathbb{N}[1, n]$. This implies that $c_k = y_k$ for all k



Analysis

Lemma

Let \mathcal{I} is the piecewise linear interpolation operator and $\mathbf{z} \in \mathbb{R}^n$. Then

$$\|\mathcal{I}(\mathbf{z})\|_{\infty} = \|\mathbf{z}\|_{\infty}.$$

- It follows from the lemma that the absolute condition number of piecewise linear interpolation in the infinity norm equals one.

Conditioning (cont')

Proof of lemma. Let

$$p(x) = \mathcal{I}(\mathbf{z}) = \sum_{j=1}^n z_j H_j(x).$$

Let k be the index corresponding to the element of \mathbf{z} with the largest absolute value, that is, $z_k = \|\mathbf{z}\|_\infty$. Since $z_k = p(x_k)$, it follows that $|p(x_k)| = \|\mathbf{z}\|_\infty$ and so $\|p\|_\infty \geq \|\mathbf{z}\|_\infty$.

To show the other inequality, note that

$$|p(x)| = \left| \sum_{j=1}^n z_j H_j(x) \right| \leq \sum_{j=1}^n |z_j| H_j(x) \leq \|\mathbf{z}\|_\infty \sum_{j=1}^n H_j(x) = \|\mathbf{z}\|_\infty,$$

where the final step uses the fact² that $\sum_{j=1}^n H_j(x) = 1$. It implies that $\|p\|_\infty \leq \|\mathbf{z}\|_\infty$.
Therefore, $\|p\|_\infty = \|\mathbf{z}\|_\infty$. □

²This property is called the *partition of unity*. Confirm it!

Convergence: Error Analysis

Set-up for analysis.

- Generate a set of data points using a “nice” function f on an interval containing all nodes, i.e., $y_j = f(x_j)$. (The *niceness* of a function is described in precise terms below.)
- Then perform PL interpolation of the data to obtain the interpolant p .
- **Question.** How close is p to f ?

Notation (Space of Differentiable Functions)

Let $C^n[a, b]$ denote the set of all functions that are n -times continuously differentiable on $[a, b]$. That is, if $f \in C^n[a, b]$, then $f^{(n)}$ exists and is continuous on $[a, b]$, where derivatives at the end points are taken to be one-sided derivatives.

Theorem 1 (Error Theorem for PL Interpolation)

Suppose that $f \in C^2[a, b]$. Let p_n be the piecewise linear interpolant of $(x_j, f(x_j))$ for $j = 1, \dots, n$, where

$$x_j = a + (j - 1)h \quad \text{and} \quad h = \frac{b - a}{n - 1}.$$

Then

$$\|f - p_n\|_{\infty} \leq \|f''\|_{\infty} h^2.$$

- The theorem pertains to the interpolation on equispaced nodes.
- The significance of the theorem is that the error in the interpolant is $O(h^2)$ as $h \rightarrow 0$. (We say that PL interpolation is *second-order accurate*.)
- **Practical implication:** If n is doubled, the PL interpolant becomes about four times more accurate. A log-log graph ($\log \log$) of error against n is a straight line.

Convergence: Error Analysis (cont')

