Module 3 Practice Problems
(Overdetermined Linear Systems)

Problems marked with ✎ are to be done by hand; those marked with 💻 are to be solved using a computer.

1. (Periodic fit; **FNC** 3.1.3) 💻 In this problem you are trying to find an approximation to the periodic function $f(t) = e^{\sin(t-1)}$ over one period, $0 \leqslant t \leqslant 2\pi$. In MATLAB, let `t=linspace(0,2*pi,200)'` and let `b` be a column vector of evaluations of f at those points.

   (a) Find the coefficients of the least square fit
   $$f(t) \approx c_1 + c_2 t + \cdots + c_7 t^6.$$

   (b) Find the coefficients of the least squares fit
   $$f(t) \approx d_1 + d_2 \cos(t) + d_3 \sin(t) + d_4 \cos(2t) + d_5 \sin(2t).$$

   (c) Plot the original function $f(t)$ and the two approximations from (a) and (b) together on a well-labeled graph.

2. (More fitting exercise; **FNC** 3.1.4) 💻 Define the following data in MATLAB:

```
t = (0:.5:10)';   y = tanh(t);
```

   (a) Fit the data to a cubic polynomial and plot the data together with the polynomial fit.

   (b) Fit the data to the function $c_1 + c_2 z + c_3 z^2 + c_4 z^3$, where $z = t^2/(1 + t^2)$. Plot the data together with the fit. What feature of z makes this fit much better than the original cubic?

3. (Normal equation; Sp20 final) ✎ The following set of data points is to be fitted to a straight line $p(x) = c_1 + c_2 x$ via linear least square approximation:

x_j	-1	2	5
y_j	3	6	9

   (a) Write out the conditions y_j "=" $p(x_j)$, for $1 \leqslant j \leqslant 3$, and turn them into a matrix *equation* of the form $\mathbf{y}$ "=" $X\mathbf{c}$.

   (b) Write out the squared 2-norm of the residual $\|\mathbf{r}\|_2^2$ where $\mathbf{r} = X\mathbf{c} - \mathbf{y}$; call it $g(c_1, c_2)$. Do not simplify your answer.

   (c) The function g is minimized at $\mathbf{c}$ where $\nabla g = \mathbf{0}$. Turn this condition into a single matrix equation for $\mathbf{c}$.

(d) Verify that the result of the previous part agrees with the *normal equation* $X^T X\mathbf{c} = X^T\mathbf{y}$.

4. (LLS via QR factorization; Au20 midterm) Suppose you have the following functions available in your working directory:

```
function [Q,R] = gs(A)
% GS  Computes thin QR using Gram-Schmidt
    [m,n] = size(A);
    Q = A;
    R = zeros(n);
    for j = 1:n
        if j > 1
            R(1:j-1,j) = Q(:,1:j-1)'*Q(:,j);
            Q(:,j) = Q(:,j) - Q(:,1:j-1)*R(1:j-1,j);
        end
        R(j,j) = norm(Q(:,j));
        Q(:,j) = Q(:,j)/R(j,j);
    end
end
```

```
function x = backsub(U,b)
%BACKSUB  Solves an upper triangular system
    n = length(U);
    x = zeros(n,1);
    for i = n:-1:1
        x(i) = ( b(i) - U(i,i+1:n)*x(i+1:n) ) / U(i,i);
    end
end
```

Using the functions provided, complete the following program which solves the linear least square problem $A\mathbf{x}$ "=" $\mathbf{b}$ using the thin QR factorization.

```
function x = lls_qr(A, b)
%LLS_QR solves linear least squares by (thin) QR factorization.
% Input:
%   A   (m x n) coefficient matrix with m >= n
%   b   (m x 1) right-hand side
% Output:
%   x   minimizer of the 2-norm of residual Ax - b
```

5. (Orthogonal decomposition; **LM** 12.6–11) Let the matrix $A \in \mathbb{R}^{10\times 4}$ be defined by

```
A = reshape(1:40, 10, 4);
```

Write the vector $\mathbf{x} = (1, 4, 9, 16, \dots, 100)^{\mathrm{T}}$ as $\mathbf{u} + \mathbf{z}$ where $\mathbf{u} \in \mathcal{R}(A)$ and $\mathbf{z} \perp \mathcal{R}(A)$ using the orthogonal projection matrix.

Note. Let $A = [\,\mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n\,] \in \mathbb{R}^{m\times n}$ where $\mathbf{a}_j \in \mathbb{R}^m$ is the j-th column vector of A. Recall that $\mathcal{R}(A)$ denotes the *range of* A or the *column space of* A, that is,

$$\mathcal{R}(A) = \mathrm{span}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n),$$

the subspace consisting of all linear combinations of the columns of A.

Note. See also **FNC** 3.3.8; part (c) of the problem may be useful.

6. (Pseudoinverse and Householder matrix by hand) Answer the following questions.

   (a) Find the pseudoinverse A^+ when

   $$A = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

   (b) Find a Householder matrix H such that

   $$H \begin{bmatrix} -6 \\ 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ 0 \end{bmatrix}.$$

7. (Properties of Householder transformation matrices) Prove Theorem 6 in Module 3 lectures (p. 50).

> **Theorem 6.** Let $\mathbf{v} = \|\mathbf{z}\|_2\,\mathbf{e}_1 - \mathbf{z}$ and let H be the *Householder transformation* defined by
>
> $$H = I - 2\frac{\mathbf{v}\mathbf{v}^{\mathrm{T}}}{\mathbf{v}^{\mathrm{T}}\mathbf{v}}.$$
>
> Then
>
> (a) H is symmetric;
>
> (b) H is orthogonal;
>
> (c) $H\mathbf{z} = \|\mathbf{z}\|_2\,\mathbf{e}_1$.

8. (Improving `myqr.m`) Modify `myqr.m` presented on pp. 53-4 of Module 3 lecture slides by following the instructions found in **LM** 12.6–8(b).