Module 1 Practice Problems

This problem set consists of three sections:

- The first section contains problems on loops, arrays, and vectorization techniques.
- The second section is all about drawing 2-D or 3-D graphics.
- Additional practice problems in the last section resemble exam problems.

Loops, Arrays, and Vectorization

1. Using a for-loop, demonstrate the convergence of the series expansions of the following functions. Evaluate each function at the indicated value.

   (a) $\cos(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ at $x = 5.4$.

   (b) $\exp(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ at $x = 0.2$.

   (c) $\ln(1+x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^n}{n}$ at $x = -0.5$.

2. Do the following problems on construction and manipulation of 1-D and 2-D arrays.

   - **LM** 3.1–3(b,c,e,g)
   - **LM** 3.1–4(c,e)
   - **LM** 3.1–5(d,f)
   - **LM** 3.1–16.
   - **LM** 3.2–7.

3. Compute the value of each mathematical expression by two different ways: using a single MATLAB statement and using a for-loop. Confirm that two methods yield the same result for each case. To begin, let $n = 2021$ and create a (random) vector $\mathbf{x} =\in \mathbb{R}^n$ by

```
n = 2021; x = rand(n, 1);
```

   Use this array for each of the following parts.

   (a) $\displaystyle\sum_{k=1}^{n} x_k$

   (b) $\displaystyle\prod_{k=1}^{n} x_k$

(c) $\min_{1 \leqslant k \leqslant n} x_k$

(d) $\max_{1 \leqslant k \leqslant n} x_k$

(e) $\frac{1}{n} \sum_{k=1}^{n} x_k$

(f) $\mathbf{s} \in \mathbb{R}^n$ where $s_j = \sum_{k=1}^{j} x_k$, for $j = 1, 2, \ldots, n$

(g) $\mathbf{p} \in \mathbb{R}^n$ where $p_j = \prod_{k=1}^{j} x_k$, for $j = 1, 2, \ldots, n$

Hint. An example answer to part (a):

```
n = 2021; x = rand(n, 1);
s = 0;
for j = 1:n
    s = s + x(j);
end
fprintf('Result using ''sum'' function: %24.16f\n', sum(x));
fprintf('Result using a ''for'' loop  : %24.16f\n', s);
```

4. In the following example, a for-loop is replaced by a simpler vectorized code. Note that print statement (using `fprintf`) is readily vectorized as well.

```
%% Example: Calculate y = 10*x + 1 where x is a random vector
% for-loop
x = rand(1,5);
y = zeros(size(x));                          % preallocation
for i = 1:5
    y(i) = 10*x(i) + 1;
    fprintf('x(%d) = %8.4f; y(%d) = %8.4f\n', i, x(i), i, y(i));
end

% clear y
clear y

% vectorized equivalent
y = 10*x + 1;
fprintf('x(%d) = %8.4f; y(%d) = %8.4f\n', [1:5; x; 1:5; y]);
```

In a similar fashion, replace the following for-loops with vectorized statements.

(a) (Evaluation on equispaced points) $\mathbf{y} = \sin \mathbf{x}$, where $\mathbf{x}$ is the vector of n equispaced points on $[0, 2\pi]$.

```
n = 11;
for j = 1:n
    x(j) = 2*pi*(j-1)/n;
    y(j) = sin(x(j));
end
```

(b) (Cumulative summation) $\mathbf{y} = (y_k)$, where $y_k = \sum_{j=1}^{k} j$ for $k = 1, 2, \ldots, n$.

```
n = 10;
s = 0;
y = zeros(1,n);
for k = 1:10
    s = s + k;
    y(k) = s;
    fprintf('Sum of integers 1 to %2d: %5d\n', k, y(k));
end
```

Note: Loop v.s. Vectorization – Timing Comparison

Sometimes, but not always, a vectorized code will be much faster than an equivalent loop. For example:

```
N = 1e7;
theta = linspace(0, 2*pi, N);
nRepeat = 5;

% loop
tic
for j = 1:nRepeat
    for i = 1:N
        y(i) = sin(theta(i));
    end
end
t1 = toc/nRepeat;

% vectorized
clear y

tic
for j = 1:nRepeat
    y = sin(theta);
end
t2 = toc/nRepeat;

fprintf('Time in loop            : %8.4f\n', t1);
fprintf('Time in vectorized code : %8.4f\n', t2);
fprintf('Vectorized code is %6.2f times faster.\n', t1/t2);
```

The result:

```
Time in loop            :   0.2045
Time in vectorized code :   0.0536
Vectorized code is   3.82 times faster.
```

Graphics

1. On a single graph, make a plot of the functions $\sinh, \cosh$, and $\tanh$ for $-1 \leqslant x \leqslant 1$. Give each curve a different color and add a legend.

2. Create anonymous functions for each of the following functions, using the "dot" operator in your function definition where necessary.

$$f(x) = \tan^{-1}(x), \quad g(x) = \sqrt[3]{x}, \quad h(x) = x^3 + (5-x)^2 - 7.$$

   Then, for each of the following parts, plot the requested expressions over the interval $[-5, 5]$.

   (a) Plot $y = f(x)$, $y = f(x/10)$, and $y = f(10x)$ on a single graph.

   (b) Plot $y = g(f(x))$.

   (c) Plot $y = g(x)f(10h(x))$.

3. Recall the identity

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

   Make a standard and a log-log plot[1]. of $e - r_n$ for $n = 5, 10, 15, \ldots, 500$. What does the log-log plot reveal about the asymptotic behavior of $e - r_n$ as $n \to \infty$?

4. Here are two different ways of plotting a sawtooth wave. Study the code.

```
x = [0:7; 1:8];
y = [zeros(1,8); ones(1,8)];
subplot(121)
plot(x, y, 'b'), axis equal
subplot(122)
plot(x(:), y(:), 'b'), axis equal
```

5. In MATLAB, the eigenvalues of a square matrix `A` are computed using `eig` function; type `help eig` in the Command Window.

   (a) Generate a hundred random matrices using `randn(100)`, and plot all of their eigenvalues as dots in the complex plane on one graph. (Thus, you should see $100 \times 100 = 10,000$ dots.) Use `axis equal` to make the aspect ratio one-to-one. What you do observe?

   (b) Repeat the experiment with a hundred random complex matrices generated by `complex(randn(100), randn(100))`. You should be able to see one very clear qualitative difference between the previous case and this one.

   Hint. If `z` is complex, then `plot(z)` is equivalent to `plot(real(z), imag(z))`.

6. Make surface plots of the following functions over the given ranges:

   (a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$, for $|x| \leqslant 3, \ |y| \leqslant 3$.

   (b) $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$, for $|x| \leqslant 2, \ |y| \leqslant 4$.

[1] Type `help loglog` in the command window. Similar functions are `semilogx` and `semilogy`, which draw log-linear plots.

(c) $f(x, y) = |x| + |y|$, for $|x| \leqslant 1$, $|y| \leqslant 1$.

7. Plot the surface represented by

$$\begin{aligned} x &= u(3 + \cos(v)) \cos(2u), \\ y &= u(3 + \cos(v)) \sin(2u), \\ z &= u \sin(v) - 3u, \end{aligned}$$

for $u \in [0, 2\pi]$, $v \in [0, 2\pi]$.

More Problems

Below are some more practice problems which resemble the style of exam problems.

1. (Guess-The-Number) Write the following game in which a user is to guess the integer randomly generated by the computer. In the program:

   - User inputs the lower and the upper bounds of the range.
   - The program generates a random integer within the specified range and stores it in a variable.
   - Use a `while`-loop for repeated guessing.
     - If the user guessed a number larger than the generated number, print out "Your guess is too high. Try again!".
     - If the user guessed a number smaller than the generated number, print out "Your guess is too low. Try again!".
     - If the user guessed the number correctly, print out "Congratulations!" and terminate the program.

   Below is an example run of the program.

   ```
   >> guess
      Enter the lower bound: 1
      Enter the upper bound: 100
      Guess a number: 50
      Your guess is too low. Try again!
      Guess a number: 75
      Your guess is too low. Try again!
      Guess a number: 87
      Your guess is too high. Try again!
      Guess a number: 81
      Your guess is too low. Try again!
      Guess a number: 84
      Your guess is too high. Try again!
      Guess a number: 82
      Congratulations!
   ```

2. (Handling large numbers and scientific notation; Adapted from **LM** 5.6) A product of terms can grow or decay much faster than a sum of terms, leading to an *overflow* or an *uderflow* in a floating-point architecture. This difficulty can usually be avoided by replacing

   $$P_n = \prod_{i=1}^{n} a_i \quad \text{by} \quad \log|P_n| = \sum_{i=1}^{n} \log|a_i| \tag{$\spadesuit$}$$

   as long as none of the terms are 0; if one or more terms are 0 the product is immediate. The result is then $P_n = f \times 10^m$ in (base-10) scientific notation where $|f| \in [1, 10)$ and f can be positive or negative, and where m is an integer.

   Write a MATLAB function which calculates the product of all the elements of an input vector `a` by using ($\spadesuit$).

- The name of the function should be `logprod.m` and must output f and m.
- $f = 0$ if one of the elements of `a` is 0.
- The code must check each element to determine if it is positive, negative, or zero, and also keep track of the overall sign of the product.
- If a zero element is found, the function must exit immediately with $f = m = 0$.

3. (Continued fraction revisited) A *continued fraction* is an infinite expression of the form

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}.$$

If all the a_k's are equal to 1, the continued fraction is equal to ϕ, the golden ratio:

$$\phi = 1 + \cfrac{1}{1 + \frac{1}{1 + \frac{1}{1+\cdots}}} = \frac{1+\sqrt{5}}{2}.$$

Denote by ϕ_n the n-term truncation of the continued fraction representation of the above, that is,

$$\begin{aligned}
\phi_0 &= 1, \\
\phi_1 &= 1 + \frac{1}{1} = 1 + \frac{1}{\phi_0}, \\
\phi_2 &= 1 + \cfrac{1}{1 + \frac{1}{1}} = 1 + \frac{1}{\phi_1}, \\
\phi_3 &= 1 + \cfrac{1}{1 + \frac{1}{1+\frac{1}{1}}} = 1 + \frac{1}{\phi_2}, \\
&\vdots \\
\phi_n &= 1 + \frac{1}{\phi_{n-1}}, \quad \text{for any } n \geqslant 1.
\end{aligned}$$

They can be used to approximate the golden ratio, *i.e.*, $\phi_n \to \phi$ as $n \to \infty$.

Use a loop to evaluate ϕ_j for $j = 1, 2, \ldots$ until you get 16 correct digits after the decimal place. Use `fprintf` in each iteration to show that the iterates are converging to the correct value. Your first few iterates and errors should look like this:

```
n   phi_n               error
1   2.0000000000000000  3.8197e-01
2   1.5000000000000000  1.1803e-01
3   1.6666666666666665  4.8633e-02
4   1.6000000000000001  1.8034e-02
5   1.6250000000000000  6.9660e-03
6   1.6153846153846154  2.6494e-03
..................................
```