Homework 7 (Solution)

Math 3607

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```
clear, close all, format short
```

Problem 1 (FNC 4.1.4)

Let's write a script solving the problem for a single P value:

```
P = 500;
n = 300;
FV = 1e6;   % value at maturity
```

We need to find an interest rate r which satisfies

$$\frac{12P}{r}\left(\left(1+\frac{r}{12}\right)^n - 1\right) = 1,000,000.$$

In other words, r is a root of

$$f(r) = \frac{12P}{r}\left(\left(1+\frac{r}{12}\right)^n - 1\right) - 1,000,000.$$

```
f = @(r) 12*P/r*( (1+r/12)^n - 1 ) - FV; % objective function
r = fzero(f, 0.01)                       % use 0.01 as an initial guess
```

```
r = 0.1235
```

Now we carry out the same computation for $P = 500, 550, ...1000$, while all other parameters are fixed.

```
n = 300;
FV = 1e6;
P = 500:50:1000;
for j = 1:length(P)
    f = @(r) 12*P(j)/r*( (1+r/12)^n - 1 ) - FV;
    r = fzero(f, 0.01);
```

```
    if j == 1
        fprintf(' %4s %8s\n', 'P', 'r')
        fprintf(' %13s\n', repmat('-', 1, 13))
    end
    fprintf(' %4d %8.4f\n', P(j), r)
end
```

```
    P        r
-------------
  500   0.1235
  550   0.1181
  600   0.1132
  650   0.1086
  700   0.1043
  750   0.1003
  800   0.0965
  850   0.0929
  900   0.0895
  950   0.0862
 1000   0.0831
```

Problem 2 (FNC 4.1.6, Lambert's W function)

Since $y = W(x)$ iff $x = ye^y$, y is a root of $f(y) = x - ye^y$ for a given x, which can be found using `fzero` as follows:

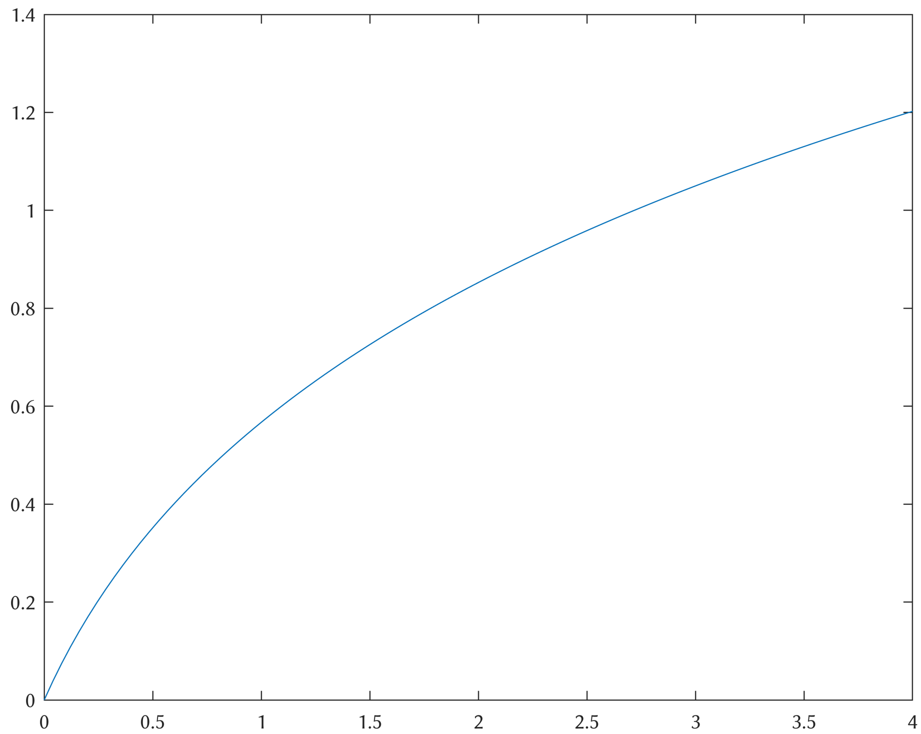
```
% if x is stored
y = fzero(@(y) x - y*exp(y), 1); % use 1 as an initial guess
```

Thus we can write a MATLAB function which computes $W(x)$ by

```
function y = lambertW(x)
% LAMBERT y = lambertW(x)
% Evaluate Lambert's W function y = W(x)
% by solving x = y*exp(y).
%
% Input:
%   x    an array input
% Output:
%   y    W(x)
    y = zeros(size(x));
    for j = 1:numel(x)
        y(j) = fzero( @(y) x(j) - y*exp(y), 1 );
    end
end
```

Note that the function can take an array input `x` and produce corresponding values in an array `y` of the same dimensions as `x`. So we can use this function just as any other built-in mathematical functions, say to plot its graph:

```
x = linspace(0, 4, 100);
plot(x, lambertW(x))
```



Note. MATLAB actually comes with this function; it is named `lambertw`. Let's compare:

```
norm( lambertW(x) - lambertw(x) )
```

```
ans = 7.3983e-16
```

This confirms that our code works very nicely!

Under the hood, MATLAB's `lambertw` uses a very fast rootfinding algorithm called *Halley's method*, which exhibits the cubic convergence! Unlike Newton or secant method which uses a linear model, Halley uses a Padé approximation (linear-over-linear rational function) to generate iterates. Take a look at the source code by:

```
type lambertw.m
```

Problem 3 (FNC 4.2.1 and 2)

(a) For easy distinction, I will denote the three functions by g_1, g_2, and g_3. We confirm that the given r is a fixed point by showing $g(r) = r$.

- $g_1(3) = \frac{1}{2}\left(3 + \frac{9}{3}\right) = 3$.

- $g_2(\pi) = \pi + \frac{1}{4}\sin(\pi) = \pi$.
- $g_3(\pi) = \pi + 1 - \tan(\pi/4) = \pi + 1 - 1 = \pi$.

Fixed point iteration converges when $|g'_j(r)| < 1$.

- $g'_1(x) = \frac{1}{2}\left(1 - \frac{9}{x^2}\right) \implies g'_1(3) = 0$ (converge)
- $g'_2(x) = \frac{1}{4}\cos(x) \implies g'_2(\pi) = -\frac{1}{4}$ (converge)
- $g'_3(x) = 1 - \frac{1}{4}\sec^2(x/4) \implies g'_3(\pi) = 1 - \frac{1}{2} = \frac{1}{2}$ (converge)

(b) Begin by defining g_j as anonymous functions

```
g1 = @(x) (x + 9/x)/2;
g2 = @(x) pi + sin(x)/4;
g3 = @(x) x + 1 - tan(x/4);
```

and the noted fixed points:

```
r1 = 3;
r2 = pi;
r3 = pi;
```

Inspired by the function `fpi` from Lecture 17 and the examples from the accompanying live script, we write the following helper function:

```
function x = myfpi(g, x0, n)
% Generates fixed point iterates x_0, x_1, ..., x_{n-1}.
% All iterates are stored in a single (column) vector x.
    x = zeros(n, 1);
    x(1) = x0;
    for k = 1:n-1
        x(k+1) = g(x(k));
    end
end
```

(This function is also included at the end of this file.)

Let's study the sequence $x_0, x_1, \ldots, x_{14}$ generated by $x_{k+1} = g_1(x_k)$:

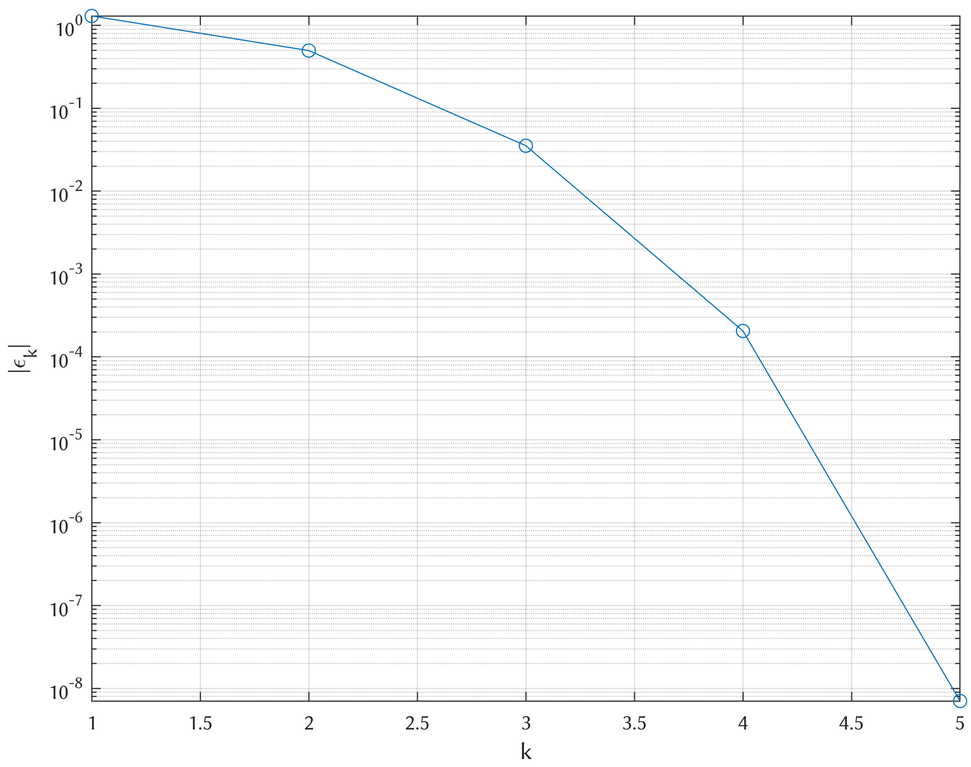
```
format long e
n = 10;
x = myfpi(g1, 1.7, n)
```

```
x = 10×1
     1.700000000000000e+00
     3.497058823529412e+00
     3.035325038341661e+00
```

```
3.000205555964860e+00
3.000000007041726e+00
3.000000000000000e+00
3.000000000000000e+00
3.000000000000000e+00
3.000000000000000e+00
3.000000000000000e+00
```

This looks good. Let's analyze the errors.

```
err = abs(x - r1);
clf
semilogy(err, 'o-'), axis tight, grid on
xlabel('k'), ylabel('|\epsilon_k|')
```



Wait, this is faster than linear convergence? Yes, but for a good reason. We discovered in the previous part that $g_1'(3) = 0$ which, according to the FPI convergence theorem, implies that

$$\lim_{k\to\infty} \frac{|\epsilon_{k+1}|}{|\epsilon_k|} = |g'(3)| = 0,$$ superlinear convergence!

Unfortunately, this is difficult to confirm as the denominator quickly underflows (to zero):

```
err(2:end) ./ err(1:end-1)
```

```
ans = 9×1
      3.823529411764706e-01
```

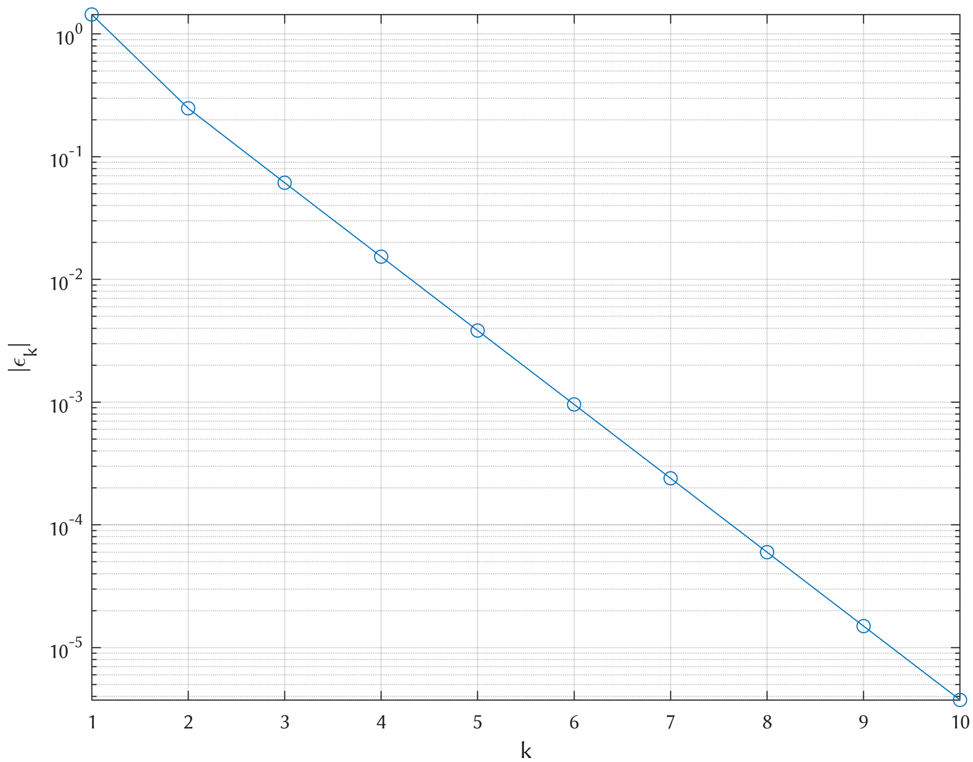
```
     7.106812447434803e-02
     5.818987735329192e-03
     3.425697871600165e-05
                         0
                       NaN
                       NaN
                       NaN
                       NaN
```

Moving onto g_2:

```
format long e
n = 10;
x = myfpi(g2, 1.7, n)
```

```
x = 10×1
     1.700000000000000e+00
     3.389508856202910e+00
     3.080246551988983e+00
     3.156919561362624e+00
     3.137761076665939e+00
     3.142550545476954e+00
     3.141353180654625e+00
     3.141652521823013e+00
     3.141577686531497e+00
     3.141596395354367e+00
```

```
err = abs(x - r2);
clf
semilogy(err, 'o-'), axis tight, grid on
xlabel('k'), ylabel('|\epsilon_k|')
```



In this case, we see that the errors draw a nice straight line on the log-linear graph. Furthermore, the ratios of errors converge beautifully to the expected $\sigma = |g_2'(\pi)| = 1/4$:

```
err(2:end) ./ err(1:end-1)
```

```
ans = 9×1
      1.719738249191035e-01
      2.474469234128392e-01
      2.498432234955372e-01
      2.499902120273960e-01
      2.499993882928684e-01
      2.499999617684353e-01
      2.499999976100850e-01
      2.499999998479356e-01
      2.499999999851645e-01
```

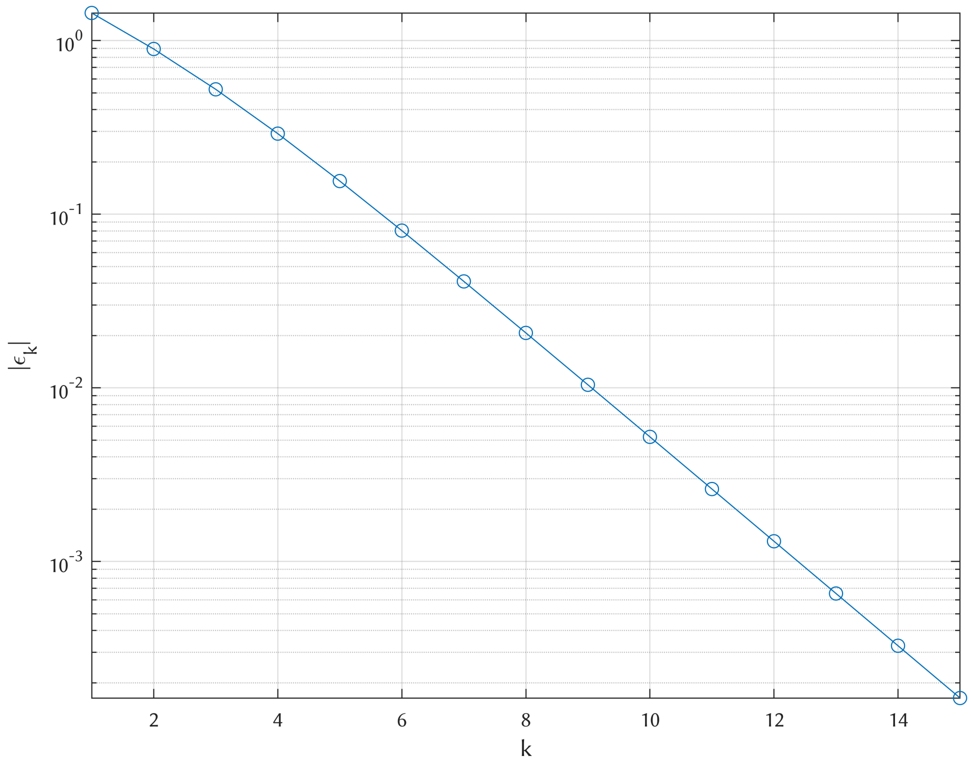
Lastly for g_3:

```
format long e
n = 15;
x = myfpi(g3, 1.7, n)
```

```
x = 15×1
      1.700000000000000e+00
      2.247416847573013e+00
      2.617881264257347e+00
      2.850599005225683e+00
      2.986452451970267e+00
      3.061162389406461e+00
```

```
        3.100590041247924e+00
        3.120884031572326e+00
        3.131185104358556e+00
        3.136375386158261e+00
        :
```

```
err = abs(x - r3);
clf
semilogy(err, 'o-'), axis tight, grid on
xlabel('k'), ylabel('|\epsilon_k|')
```



We also see that the errors draw a nice straight line on the log-linear graph. Furthermore, the ratios of errors converge beautifully to the expected $\sigma = |g_3'(\pi)| = 1/2$:

```
err(7:end) ./ err(6:end-1)
```

```
ans = 9×1
        5.097908450034160e-01
        5.050561619051057e-01
        5.025708240006618e-01
        5.012964450721191e-01
        5.006510261133111e-01
        5.003262197391994e-01
        5.001632872715480e-01
        5.000816880784664e-01
        5.000408551610744e-01
```

Note that convergence is not as fast as in the previous case (even though both converge linearly) because the convergence rate $\sigma = 1/2$ for this case is larger than the previous one.

Problem 4

(a)

Simple Calculus 1 exercise shows that

$$f'(x) = \frac{1}{2\sqrt{|x|}}, \text{ for all } x \in \mathbb{R}.$$

One Newton iteration takes any nonzero initial iterate $x_0 \neq 0$ to

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - 2\text{sign}(x_0)|x_0| = -x_0, \text{ regardless of the sign of } x_0.$$

If you are unsure, examine two cases, $x_0 > 0$ and $x_0 < 0$, separately. Repeating the same computation with x_0 replaced by $-x_0$, we find that

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -x_0 - 2\text{sign}(-x_0)|-x_0| = x_0.$$

Back to the starting point! This means that the iterates generated by the Newton's iteration formula will just rock back and forth between x_0 and $-x_0$:

$$x_0, -x_0, x_0, -x_0, \ldots$$

a hopeless divergence scenario.

The following (minimal) Newton iteration code confirms our prediction.

```
f = @(x) sign(x).*sqrt(abs(x));
fprime = @(x) 1./(2*sqrt(abs(x)));
x = 1;
for k = 1:10
    x = x - f(x)/fprime(x)
end
```

```
x =
    -1
x =
     1
x =
    -1
x =
     1
x =
    -1
x =
     1
x =
```

```
x =
    -1
x =
     1
x =
    -1
x =
     1
```

Ponder. Why do you think it is happening? Is it violating the convergence theorem for Newton's method?

(b)

Set - up

Let $\{x_k\}$ be iterates generated by Newton's iteration formula

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad (*)$$

Let r be a double root of f, that is,

$$f(r) = f'(r) = 0 \quad \text{but } f''(r) \neq 0.$$

Let $\epsilon_k = x_k - r$ as in lecture.

Substitute $x_k = r + \epsilon_k$ into $(*)$

$$\epsilon_{k+1} = \epsilon_k - \frac{f(r+\epsilon_k)}{f'(r+\epsilon_k)}$$

Taylor-expand at r

$$\epsilon_{k+1} = \epsilon_k - \frac{\cancel{f(r)} + \cancel{f'(r)}\epsilon_k + \frac{f''(r)}{2}\epsilon_k^2 + \frac{f'''(r)}{6}\epsilon_k^3 + O(\epsilon_k^4)}{\cancel{f'(r)} + f''(r)\epsilon_k + \frac{f'''(r)}{2}\epsilon_k^2 + O(\epsilon_k^3)}$$

by assumption

$$= \epsilon_k - \frac{\frac{\cancel{f''(r)}}{2}\epsilon_k^{\cancel{2}}\left(1 + \frac{1}{3}\frac{f'''(r)}{f''(r)}\epsilon_k + O(\epsilon_k^2)\right)}{\cancel{f''(r)}\,\cancel{\epsilon_k}\left(1 + \frac{1}{2}\frac{f'''(r)}{f''(r)}\epsilon_k + O(\epsilon_k^2)\right)}$$

Geometric series

$$= \epsilon_k - \frac{1}{2}\epsilon_k\left(1 + \frac{1}{3}\frac{f'''(r)}{f''(r)}\epsilon_k + O(\epsilon_k^2)\right)\left(1 - \frac{1}{2}\frac{f'''(r)}{f''(r)}\epsilon_k + O(\epsilon_k^2)\right)$$

$$= \frac{1}{2}\epsilon_k + \frac{1}{12}\frac{f'''(r)}{f''(r)}\epsilon_k^2 + O(\epsilon_k^3) = \frac{1}{2}\epsilon_k + O(\epsilon_k^2)$$

Conclusion $\epsilon_{k+1} \simeq \frac{1}{2}\epsilon_k$, linear convergence!

Functions Used

Lambert's W function

```
function y = lambertW(x)
% LAMBERT y = lambertW(x)
% Evaluate Lambert's W function y = W(x)
% by solving x = y*exp(y).
%
% Input:
%   x    an array input
% Output:
%   y    W(x)
    y = zeros(size(x));
    for j = 1:numel(x)
        y(j) = fzero( @(y) x(j) - y*exp(y), 1 );
    end
end
```

Fixed Point Iteration

```
function x = myfpi(g, x0, n)
% Generates fixed point iterates x_0, x_1, ..., x_{n-1}.
% All iterates are stored in a single (column) vector x.
    x = zeros(n, 1);
    x(1) = x0;
    for k = 1:n-1
        x(k+1) = g(x(k));
```

```
    end
end
```