# Module 4 Practice Problems (Solutions)

1. (a) **False.** For $AV = VD$ with a diagonal $D$, the columns of $V$ must be eigenvectors of $A$. In general, a matrix $A$ need not have an orthonormal set of eigenvectors.

   **Note.** A real symmetric matrix or, more generally, a complex hermitian matrix is known to have an orthonormal set of eigenvectors by the *spectral theorem*.

   (b) **True.** By Theorem 1 of Module 4, $A$ is diagonalizable, *i.e.,* $A$ has an EVD.

   (c) **False.** As long as $A$ has 5 linearly independent eigenvectors, it has an EVD.

   (d) **False.** Any matrix has an SVD; in particular, any square matrix has an SVD regardless of invertibility.

   (e) **True.** Any matrix has an SVD; in particular, any rectangular matrix has an SVD regardless of its rank.

   (f) **True.** $B$ is a symmetric matrix and so it has an EVD; see p. 35 of Module 4.

2. Download `evd_spectra.mlx` and play with it.

3. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be eigenvalues of $A$, so that the matrix $D$, up to re-ordering, can be written as

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

   Since $A = VDV^{-1}$,

$$A^2 = VD\left(V^{-1}V\right)DV^{-1} = VD^2V^{-1}.$$

   Note that

$$D^2 = \begin{bmatrix} \lambda_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix} = I,$$

   because all eigenvalues are assumed to be either $+1$ or $-1$. It follows that $A^2 = VIV^{-1} = VV^{-1} = I$, as desired.

4. (a) Let $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$. Then

$$\begin{aligned} \mathbf{x}^{\mathrm{T}} A \mathbf{x} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 3x_1^2 - x_1 x_2, \end{aligned} \qquad \text{and} \qquad \mathbf{x}^{\mathrm{T}}\mathbf{x} = x_1^2 + x_2^2.$$

Thus,

$$R_A(\mathbf{x}) = \frac{3x_1^2 - 4x_1x_2}{x_1^2 + x_2^2}.$$

(b) Let $\mathbf{x} = (1,2)^{\mathrm{T}}$. Then by the expression found above,

$$R_A(\mathbf{x}) = \frac{3(1)^2 - 4(1)(2)}{1^2 + 2^2} = -1.$$

(c) It follows immediately from the following calculation

$$A\mathbf{x} = \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\mathbf{x},$$

that $\lambda = -1$ is an eigenvalue of $A$ and $\mathbf{x} = (1,2)^{\mathrm{T}}$ is a corresponding eigenvector of $A$.

(d) Upon partial differentiation, we find that

$$\begin{aligned} \frac{\partial R_A}{\partial x_1} &= \frac{6x_1 - 4x_2}{x_1^2 + x_2^2} - 2x_1 \frac{3x_1^2 - 4x_1x_2}{\left(x_1^2 + x_2^2\right)^2} \\ &= \frac{(6x_1 - 4x_2)(x_1^2 + x_2^2) - 2x_1(3x_1^2 - 4x_1x_2)}{(x_1^2 + x_2^2)^2} \\ &= \frac{4x_1^2x_2 + 6x_1x_2^2 - 4x_2^3}{(x_1^2 + x_2^2)^2} \\ &= 2x_2 \frac{2x_1^2 + 3x_1x_2 - 2x_2^2}{(x_1^2 + x_2^2)^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial R_A}{\partial x_2} &= \frac{-4x_1}{x_1^2 + x_2^2} - 2x_2 \frac{3x_1^2 - 4x_1x_2}{(x_1^2 + x_2^2)^2} \\ &= \frac{-4x_1(x_1^2 + x_2^2) - 2x_2(3x_1^2 - 4x_1x_2)}{(x_1^2 + x_2^2)^2} \\ &= \frac{-4x_1^3 - 6x_1^2x_2 + 4x_1x_2^2}{(x_1^2 + x_2^2)^2} \\ &= -2x_1 \frac{2x_1^2 + 3x_1x_2 - 2x_2^2}{(x_1^2 + x_2^2)^2} \end{aligned}$$

So

$$\nabla R_A(\mathbf{x}) = \frac{2(2x_1^2 + 3x_1x_2 - 2x_2^2)}{(x_1^2 + x_2^2)^2} \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}.$$

(e) Note that the common factor $2x_1^2 + 3x_1x_2 - 2x_2^2$ vanishes when $x_1 = 1$ and $x_2 = 2$. It follows that the gradient $\nabla R_A$ is zero with those values.

7. (c) To find the best rank-1 approximation of $A$, we need the first singular value $\sigma_1$ and the corresponding left and right singular vectors $\mathbf{u}_1$ and $\mathbf{v}_1$. In general, one must work out the eigenvalue problem for $A^{\mathrm{T}}A$ and $AA^{\mathrm{T}}$ to compute SVD by hand; see pp. 35-36 of Module 4. However, since $A$ is symmetric in this problem, it can be done more conveniently.

Note, by the spectral theorem from linear algebra[1], that we can find orthonormal eigenvectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^2$ for $A$ because of the symmetry. In terms of EVD, it means that we can write

$$A = XDX^{-1} = XDX^{\mathrm{T}},$$

where $V = [\mathbf{x}_1\ \mathbf{x}_2] \in \mathbb{R}^{2\times 2}$ is orthogonal and $D = \mathrm{diag}(\lambda_1, \lambda_2) \in \mathbb{R}^{2\times 2}$ is diagonal. This is almost an SVD! It is one if the eigenvalues are all nonnegative; if not, we can tweak things to make it one. Below is how:

It can be readily verified that the eigenvalues of $A$ are $\lambda_1 = 1 + b$ and $\lambda_2 = 1 - b$ and the corresponding eigenvectors are

$$\mathbf{x}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{x}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

[1]See Theorem 6 of Module 4. If you still have the textbook from OSU Math 2568, see Section 4.7 of Johnson, Riess, and Anold.

Note that these vectors are orthonormal, that is, unit and perpendicular, as expected from the spectral theorem. Thus, we can write $A = XDX^{\mathrm{T}}$ where

$$X = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1+b & 0 \\ 0 & 1-b \end{bmatrix}.$$

Now, since $b > 0$ by assumption, $\lambda_1$ is always positive but $\lambda_2$ can be negative if $b > 1$. So $A = XDX^{\mathrm{T}}$ is not quite an SVD. However, with $D$ written as

$$D = \begin{bmatrix} 1 & 0 \\ 0 & \operatorname{sign}(1-b) \end{bmatrix} \begin{bmatrix} 1+b & 0 \\ 0 & |1-b| \end{bmatrix},$$

we can write $A = XDX^{\mathrm{T}}$ as

$$A = \left( \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \operatorname{sign}(1-b) \end{bmatrix}}_{U} \right) \underbrace{\begin{bmatrix} 1+b & 0 \\ 0 & |1-b| \end{bmatrix}}_{\Sigma} \left( \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{V^{\mathrm{T}}} \right) = U\Sigma V^{\mathrm{T}},$$

which is an SVD! From this, we can read off the necessary information:

$$\sigma_1 = 1 + b, \quad \mathbf{u}_1 = \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

It follows that the best rank-1 approximation $A_1$ of $A$ is given by

$$A \approx A_1 = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^{\mathrm{T}} = \frac{1+b}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

**Note.** The solution above is an illustration of Theorem 7 of Module 4. Though we only needed $\sigma_1, \mathbf{u}_1, \mathbf{v}_1$ for this problem, I showed the full details of converting a *unitary diagonalization* into an SVD.

**Note.** Use this result to confirm your answer to part (a).