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Module 4 Practice Problems
(Spectral Theory – EVD and SVD)

Problems marked with are to be done by hand; those marked with are to be solved using a computer.

1. (True/False) Determine whether each of the following is true or false.

   (a) Given a square matrix $A \in \mathbb{R}^{n\times n}$, we can always find an orthogonal matrix $V \in \mathbb{R}^{n\times n}$ and a diagonal matrix $D \in \mathbb{R}^{n\times n}$ such that $AV = VD$.

   (b) If $A \in \mathbb{R}^{5\times 5}$ has 5 distinct eigenvalues, then A has an EVD.

   (c) If $A \in \mathbb{R}^{5\times 5}$ has 3 distinct eigenvalues, then A does not have an EVD.

   (d) A square matrix $A \in \mathbb{R}^{m\times m}$ with $\det(A) = 0$ does not have an SVD.

   (e) A rank deficient matrix $A \in \mathbb{R}^{m\times n}$ has an SVD.

   (f) Let $A \in \mathbb{R}^{m\times n}$. Then $B = AA^{\mathrm{T}} \in \mathbb{R}^{m\times m}$ is a diagonalizable matrix.

2. (Visualization of spectra and pseudospectra) The eigenvalues of *Toeplitz* matrices, which have a constant value on each diagonal, have beautiful connections to complex analysis. Define six 64×64 Toeplitz matrices using

```
z = zeros(1,60);
A{1} = toeplitz( [0,0,0,0,z], [0,1,1,0,z] );
A{2} = toeplitz( [0,1,0,0,z], [0,2i,0,0,z] );
A{3} = toeplitz( [0,2i,0,0,z], [0,0,1,0.7,z] );
A{4} = toeplitz( [0,0,1,0,z], [0,1,0,0,z] );
A{5} = toeplitz( [0,1,2,3,z], [0,-1,-2,0,z] );
A{6} = toeplitz( [0,0,-4,-2i,z], [0,2i,-1,2,z] );
```

   (The variable `A` constructed hereinabove is a *cell array*. Type `doc cell` to learn more about this.) For each of the six matrices, do the following.

   (a) Plot the eigenvalues of `A{#}` as red dots in the complex plane. (Set `'MarkerSize'` to be 3.)

   (b) Let E and F be 64×64 random matrices generated by `randn`. On top of the plot from part (a), plot the eigenvalues of the matrix $A + \varepsilon E + i\varepsilon F$ as blue dots, where $\varepsilon = 10^{-3}$. (Set `'MarkerSize'` to be 1.)

   (c) Repeat part (b) 49 more times (generating a single plot).

   Arrange all six plots in a 3×2 grid using `subplot`. Make sure all figures are drawn in 1:1 aspect ratio.

3. (EVD and powers of a matrix) Let $A \in \mathbb{R}^{n\times n}$ has an EVD $A = VDV^{-1}$ and suppose that all its eigenvalues are either positive or negative ones. Show that $A^2 = I$.

Note. To gain a geometric intuition about this problem, think about the eigenvalue decomposition of a Householder reflector $H = I - 2\mathbf{u}\mathbf{u}^{\mathrm{T}}$.

4. (Rayleigh quotient) ✎ Let
$$A = \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix}.$$
and define a function $R_A : \mathbb{R}^2 \to \mathbb{R}$ by
$$R_A(\mathbf{x}) = \frac{\mathbf{x}^{\mathrm{T}} A \mathbf{x}}{\mathbf{x}^{\mathrm{T}} \mathbf{x}}.$$
   (a) Write out $R_A(\mathbf{x})$ explicitly as a function of x_1 and x_2.
   (b) Find $R_A(\mathbf{x})$ for $x_1 = 1$, $x_2 = 2$.
   (c) Confirm that $\mathbf{x} = (1, 2)^{\mathrm{T}}$ is an eigenvector of A, whose corresponding eigenvalue is equal to the value computed in part (b).
   (d) Find the gradient vector $\nabla R_A(\mathbf{x})$.
   (e) Show that the gradient vector is zero when $x_1 = 1$, $x_2 = 2$.

Note. The map R_A constructed above is known as the *Rayleigh quotient*. As confirmed in part (c), this map is known to send an eigenvector of A to its associated eigenvalue.

5. (2-norm and principal singular value) ✎ Let $A \in \mathbb{C}^{m\times n}$ have an SVD $A = USV^*$. The following problem walks you through the proof of the fact that $\|A\|_2 = \sigma_1$.
   (a) Use the technique of Lagrange multipliers to show that among vectors that satisfy $\|\mathbf{x}\|_2^2 = 1$, any vector that maximizes $\|A\mathbf{x}\|_2^2$ must be an eigenvector of A^*A. (*Hint.* If B is any hermitian matrix, *i.e.,* $B^* = B$, the gradient of the scalar function $\mathbf{x}^* B\mathbf{x}$ with respect to $\mathbf{x}$ is $2B\mathbf{x}$.)
   (b) Use the result of part (a) to prove that $\|A\|_2 = \sigma_1$, the *principal singular value* of A.

6. (LLS via SVD) ✎ Recall that the linear least square (LLS) problem $A\mathbf{x}$ "=" $\mathbf{b}$ can be rewritten as
$$\widehat{R}\mathbf{x} = \widehat{Q}^{\mathrm{T}}\mathbf{b}, \tag{1}$$
using the thin QR factorization $A = \widehat{Q}\widehat{R}$. Multiplying (1) by $\widehat{R}^{-1}$ gives a formula for the pseudoinverse
$$A^+ = \widehat{R}^{-1}\widehat{Q}^{\mathrm{T}}. \tag{$\star\star$}$$
The whole process can be turned into the following *QR-based algorithm for LLS problem*:
   i. Factor $A = \widehat{Q}\widehat{R}$.
   ii. Let $\mathbf{z} = \widehat{Q}^{\mathrm{T}}\mathbf{b}$.
   iii. Solve $\widehat{R}\mathbf{x} = \mathbf{z}$ for $\mathbf{x}$ using backward substitution.

Now assuming that $A \in \mathbb{R}^{m\times n}$ with $m \geq n$, establish analogous results using the thin SVD,
$$A = \widehat{U}\widehat{\Sigma}V^T. \tag{2}$$
That is:

(a) Derive an equation similar to (1) by substituting (2) into the associated normal equation.

(b) Find an alternate formula for A^+ using the result of part (a).

(c) Write down an *SVD-based algorithm for LLS problem* using the result of part (a).

7. (Low-rank approximation) Find the rank-1 matrix closest to A as measured in the 2-norm.

   (a) $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

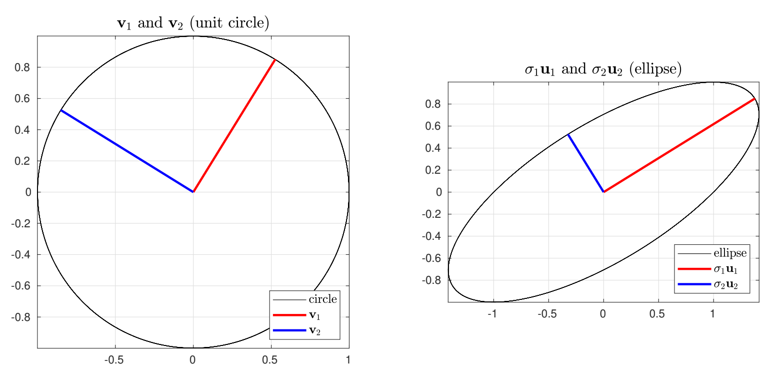
   (b) $A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$, where $b > 0$.

8. (Visualizing SVD) Write a MATLAB function which, given $A \in \mathbb{R}^{2\times 2}$, plots the right singular vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in the unit circle and also the scaled left singular vectors $\sigma_1\mathbf{u}_1$ and $\sigma_2\mathbf{u_2}$ in the appropriate ellipse. Apply your program to the matrices

$$A_1 = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Example Output.



Considerations.

- Use `subplot` to produce two plots side by side. Make sure the unit circle looks like a circle.
- Draw corresponding vectors $\mathbf{v}_j$ and $\sigma_j\mathbf{u}_j$ in the same color. In the example figure above, the first singular vectors are drawn in red, the second ones in blue.
- Use `legend` to create legends.