Updated on July 15, 2021

Module 2 Practice Problems (Solutions)

1. Recall that the (relative) condition number of the function f at x is given by

$$\kappa_f(x) = \left| \frac{x f'(x)}{f(x)} \right|.$$

   (a) Calculation shows that

$$\kappa_f(x) = \frac{x}{\cosh x \sinh x} \text{ or } \frac{4xe^{2x}}{e^{4x} - 1}.$$

   Using the exponential form, one can confirm that $\kappa_f(x)$ does not blow up anywhere.

   (b) When simplified,

$$\kappa_f(x) = \left| \frac{xe^x}{e^x - 1} - 1 \right|$$

   Some notable limits are

$$\begin{aligned} \lim_{x \to 0} \kappa_f(x) &= 0, \\ \lim_{x \to \infty} \kappa_f(x) &= \infty, \\ \lim_{x \to -\infty} \kappa_f(x) &= 1. \end{aligned}$$

   Note that $\kappa_f(x)$ blows up as $x \to \infty$.

   (c) Another calculus exercise leads to

$$\kappa_f(x) = \left| 1 - \frac{x \sin(x)}{1 - \cos(x)} \right|.$$

   It is not very difficult to see that $\lim_{x \to 0} \kappa_f(x) = 1$, even though the denominator of the second term approaches 0 as $x \to 0$. However, there are infinitely many nonzero x at which $1 - \cos(x)$ vanishes while $x \sin(x)$ does not, namely, at $x = 2n\pi$, where n is a nonzero integer. Therefore, $\kappa_f(x) \to \infty$ as $x \to 2n\pi$, where $n \in \mathbb{Z} \setminus \{0\}$.

7. (a) Since $P(k, \ell)^2 = I$, observe that

$$\begin{aligned} P(k,\ell) G_j P(k,\ell) &= P(k,\ell) I P(k,\ell) + P(k,\ell) \left(\sum_{i=j+1}^{n} a_{i,j} \mathbf{e}_i \mathbf{e}_j^{\mathrm{T}} \right) P(k,\ell) \\ &= I + \sum_{i=j+1}^{n} a_{i,j} \underbrace{P(k,\ell) \mathbf{e}_i \mathbf{e}_j^{\mathrm{T}} P(k,\ell)}_{\text{☺}}, \end{aligned}$$

and so we just need to figure out the behavior of $P(k,\ell)\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}P(k,\ell)$. Because $\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}$ is a matrix with 1 on the (i,j)-position and zeros elsewhere, and since $j<k<\ell$, the right-multiplication by $P(k,\ell)$ swaps two columns that consist only of zeros, *i.e.*, $\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}P(k,l)=\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}$.

$$☺ = P(k,\ell)\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}P(k,\ell) = P(k,\ell)\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}.$$

Since $\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}$ has only one nontrivial element on the i^{th} row, if i is neither k nor ℓ, the left-multiplication by $P(k,\ell)$ swaps two rows with all zeros and so $P(k,\ell)\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}=\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}$. However, if i equals either k or ℓ, the left-multiplication creates a meaningful effect. Precisely, if $i=k$, $P(k,\ell)\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}=P(k,\ell)\mathbf{e}_k\mathbf{e}_j^{\mathrm{T}}=\mathbf{e}_\ell\mathbf{e}_j^{\mathrm{T}}$; if $i=\ell$, $P(k,\ell)\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}=P(k,\ell)\mathbf{e}_\ell\mathbf{e}_j^{\mathrm{T}}=\mathbf{e}_k\mathbf{e}_j^{\mathrm{T}}$. In sum,

$$\begin{aligned}
P(k,\ell)G_jP(k,\ell) &= I+\sum_{i=j+1}^{n}a_{i,j}P(k,\ell)\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}P(k,\ell)\\
&= I+\sum_{i\neq k,\ell}a_{i,j}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}+\underbrace{a_{k,j}\mathbf{e}_\ell\mathbf{e}_j^{\mathrm{T}}}_{i=k}+\underbrace{a_{\ell,j}\mathbf{e}_k\mathbf{e}_j^{\mathrm{T}}}_{i=\ell}\\
&= I+\sum_{i\neq k,\ell}b_{i,j}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}+b_{\ell,j}\mathbf{e}_\ell\mathbf{e}_j^{\mathrm{T}}+b_{k,j}\mathbf{e}_k\mathbf{e}_j^{\mathrm{T}} = I+\sum_{i=j+1}^{n}b_{i,j}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}.
\end{aligned}$$

(b) It is sufficient to show that $G_jG_j^{-1}=G_j^{-1}G_j=I$. Using the given alternate expressions, we write $G_jG_j^{-1}$ as

$$(I+a_{i,j}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}})(I-a_{i,j}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}) = I+\cancel{a_{i,j}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}}-\cancel{a_{i,j}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}}-a_{i,j}^2\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}} = I-a_{i,j}^2\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}.$$

Note that the last term vanishes:

$$\mathbf{e}_i\left(\mathbf{e}_j^{\mathrm{T}}\mathbf{e}_i\right)\mathbf{e}_j^{\mathrm{T}}=\mathbf{e}_i\delta_{j,i}\mathbf{e}_j^{\mathrm{T}}=0$$

since $j\neq i$. So $G_jG_j^{-1}=I$. We can show similarly that $G_j^{-1}G_j=I$, which completes the proof.

(c) Let's write out

$$\begin{aligned}
G_jG_k &= \left(I+\sum_{i=j+1}^{n}a_{i,j}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}\right)\left(I+\sum_{i=k+1}^{n}a_{i,k}\mathbf{e}_i\mathbf{e}_k^{\mathrm{T}}\right)\\
&= I+\sum_{i=j+1}^{n}a_{i,j}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}+\sum_{i=k+1}^{n}a_{i,k}\mathbf{e}_i\mathbf{e}_k^{\mathrm{T}}+\left(\sum_{i=j+1}^{n}a_{i,j}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}\right)\left(\sum_{i=k+1}^{n}a_{i,k}\mathbf{e}_i\mathbf{e}_k^{\mathrm{T}}\right).
\end{aligned}$$

So we will be done once we show that the last term is zero (as a matrix). To properly expand the last term, we need to change one of the two summation indices, say i to i' for the second summation. Then observe that

$$\left(\sum_{i=j+1}^{n}a_{i,j}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}\right)\left(\sum_{i'=k+1}^{n}a_{i',k}\mathbf{e}_{i'}\mathbf{e}_k^{\mathrm{T}}\right)=\sum_{i=j+1}^{n}\sum_{i'=k+1}^{n}a_{i,j}a_{i',k}\mathbf{e}_i\underbrace{\mathbf{e}_j^{\mathrm{T}}\mathbf{e}_{i'}}_{=0}\mathbf{e}_k^{\mathrm{T}}=\mathbf{0}\text{ (zero matrix).}$$

(Since $j<k$ and $i'\geq k+1$, $j<k<i'$. In particular, j never equals i' and so the inner product $\mathbf{e}_j^{\mathrm{T}}\mathbf{e}_{i'}=0$.) This shows the desired equality.

8. Let

$$P = P(1,4) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since left-multiplication and right-multiplication (by a permutation matrix) acts on rows and columns, respectively,

(a) PA

(b) AP

(c) PAP (In other words, $P = Q$.)

(d) Let

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that we obtain this by permuting rows of the identity matrix I as described. Then left-multiplication by P results in the desired effect. Observe that this action shifts the first four rows cyclically, leaving the fifth unchanged. Thus, after four multiplication, all rows are back to their original positions. Thus $P^k = I$ for any $k = 4n$ with $n \in \mathbb{N}$ and so $k = 4$ is the smallest such positive integer. It is clear that this cyclic permutation can be broken into a series of simple swaps:

$$\begin{aligned}(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4) &\xrightarrow{P(1,2)} (\mathcal{R}_2, \mathcal{R}_1, \mathcal{R}_3, \mathcal{R}_4) \\ &\xrightarrow{P(2,3)} (\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_1, \mathcal{R}_4) \xrightarrow{P(3,4)} (\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_1).\end{aligned}$$

Hence,

$$PA = P(3,4)P(2,3)P(1,2)A.$$

In MATLAB:

```
A([1 4], :) = A([4 1], :);                                        % (a)
A(:, [1 4]) = A(:, [4 1]);                                        % (b)
A(:, [1 4]) = A(:, [4 1]); A([1 4], :) = A([4 1], :);             % (c)
A(:,[1,2,3,4]) = A(:,[2,3,4,1])                                   % (d)
```

9. (a) Following the instruction of the problem, the loop simply changes to

```
for j = 1:n-1
    i = j+1:n;
    L(i,j) = A(i,j) / A(j,j);    % row multiplier
    A(i,j:n) = A(i,j:n) - L(i,j)*A(j,j:n);
end
```

(b) For $n = 5$, iteration with $j = 3$ first defines

$$L_{4:5,3} = \begin{bmatrix} \lambda_{4,3} \\ \lambda_{5,3} \end{bmatrix} \equiv \begin{bmatrix} a_{4,3}/a_{3,3} \\ a_{5,3}/a_{3,3} \end{bmatrix},$$

and then replaces the 2×3 block $A_{4:5,3:5}$ by

$$\begin{aligned} A_{4:5,3:5} - L_{4:5,3}A_{3,3:5} &= \begin{bmatrix} a_{4,3} & a_{4,4} & a_{4,5} \\ a_{5,3} & a_{5,4} & a_{5,5} \end{bmatrix} - \begin{bmatrix} \lambda_{4,3} \\ \lambda_{5,3} \end{bmatrix} \begin{bmatrix} a_{3,3} & a_{3,4} & a_{3,5} \end{bmatrix} \\ &= \begin{bmatrix} a_{4,3} & a_{4,4} & a_{4,5} \\ a_{5,3} & a_{5,4} & a_{5,5} \end{bmatrix} - \begin{bmatrix} \lambda_{4,3}a_{3,3} & \lambda_{4,3}a_{3,4} & \lambda_{4,3}a_{3,5} \\ \lambda_{5,3}a_{3,3} & \lambda_{5,3}a_{3,4} & \lambda_{5,3}a_{3,5} \end{bmatrix} \\ &= \begin{bmatrix} a_{4,3} - \lambda_{4,3}a_{3,3} & a_{4,4} - \lambda_{4,3}a_{3,4} & a_{4,5} - \lambda_{4,3}a_{3,5} \\ a_{5,3} - \lambda_{5,3}a_{3,3} & a_{5,4} - \lambda_{5,3}a_{3,4} & a_{5,5} - \lambda_{5,3}a_{3,5} \end{bmatrix}. \end{aligned}$$

11. (a) We use the following convenient formulas for matrix $$-norms for $p = 1, \infty$:

$$\|A\|_1 = \max_{1\le j\le 2} \left\{ \sum_{i=1}^{2} |A_{i,j}| \right\} = \max\{1, 5\} = 5,$$
$$\|A\|_\infty = \max_{1\le i\le 2} \left\{ \sum_{j=1}^{2} |A_{i,j}| \right\} = \max\{3, 3\} = 3.$$

The calculation of Frobenius norm is very straightfoward:

$$\|A\|_F = \sqrt{\sum_{i=1}^{2}\sum_{j=1}^{2} |A_{i,j}|} = \sqrt{1^2 + 2^2 + 3^3} = \sqrt{14}.$$

To compute the 2-norm of A. We first find that

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$$

has two real positive eigenvalues:

$$\left\{ \begin{aligned} \det\left(\lambda I - A^{\mathrm{T}}A\right) &= \det\begin{pmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 13 \end{pmatrix} \\ &= (\lambda-1)(\lambda-13) - 4 \\ &= \lambda^2 - 14\lambda + 9. \end{aligned} \right\} \implies \lambda = 7 \pm 2\sqrt{10}.$$

Therefore,

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^{\mathrm{T}}A)} = \sqrt{7 + 2\sqrt{10}}.$$

(b) Below is an implementation of the alternate and convenient reformulation of $\|\cdot\|_p$ for $p = 1, 2, \infty$ or Frobenius case.

```
function y = MatrixNorm(A, j)
% mat_norm   computes matrix norms
% Usage:
%   MatrixNorm(A, 1) returns the 1-norm of A
```

```
%   MatrixNorm(A, 2) is the same as mat_norm(A)
%   MatrixNorm(A, 'inf') returns the infinity-norm of A
%   MatrixNorm(A, 'fro') returns the Frobenius norm of A
    if j == 1
        y = max(sum(abs(A), 1));
    elseif j == 2
        y = sqrt(max(eig(A'*A)));
    elseif strcmp(j, 'inf')
        y = max(sum(abs(A), 2));
    elseif strcmp(j, 'fro')
        y = sqrt(sum(sum(abs(A).^2)));
    else
        error('Invalid second input.');
    end
end
```