HW05 Hints

1. This problem is to be done purely by hand and most of your answers will involve some sort of summation. Below are some similar questions and answers.

Example 1. What is the j^{th} column of $\mathbf{r}\mathbf{w}^{\mathrm{T}}$, where $\mathbf{r} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^n$? What is the size of the resulting matrix?

Answer. The expression $\mathbf{r}\mathbf{w}^{\mathrm{T}}$, as an outer product, produces an $m \times n$ matrix

$$\begin{aligned}\mathbf{r}\mathbf{w}^{\mathrm{T}} &= \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \\ &= \begin{bmatrix} r_1w_1 & r_1w_2 & \cdots & r_1w_j & \cdots & r_1w_n \\ r_2w_1 & r_2w_2 & \cdots & r_2w_j & \cdots & r_2w_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r_mw_1 & r_mw_2 & \cdots & r_mw_j & \cdots & r_mw_n \end{bmatrix},\end{aligned}$$

whose j^{th} column is

$$\begin{bmatrix} r_1w_j \\ r_2w_j \\ \vdots \\ r_mw_j \end{bmatrix}.$$

Notation. For a matrix M, denote by $[M]_{ij} = m_{ij}$ the (i, j)-element of M.

Observation. Since rows of M becomes columns of M^{T}, and vice versa, the (i, j)-element of M^{T} is the same as the (j, i)-element of M, *i.e.*,

$$[M^{\mathrm{T}}]_{ij} = [M]_{ji} = m_{ji}.$$

This is going to be useful in the following example.

Example 2. What is the j^{th} element of the column vector $A^{\mathrm{T}}\mathbf{x}$, where $A \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^m$? What is the size of the resulting vector?

Answer. Since $A \in \mathbb{R}^{m\times n}$, $A^{\mathrm{T}} \in \mathbb{R}^{n\times m}$ and so $A^{\mathrm{T}}\mathbf{x} \in \mathbb{R}^{n}$, a column vector with n elements. The j^{th} element of $A^{\mathrm{T}}\mathbf{x}$ is

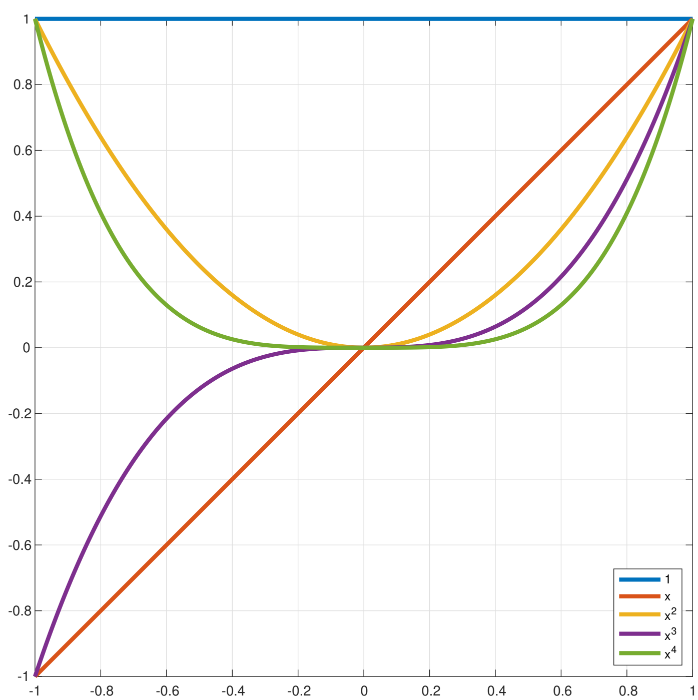
$$\sum_{i=1}^{m}[A^{\mathrm{T}}]_{ji}x_i = \sum_{i=1}^{m}[A]_{ij}x_i = \sum_{i=1}^{m}a_{ij}x_i.$$

2. The symbol $\mathcal{O}$ in the third line of the problem stands for the zero matrix, with suitable dimensions according to the context.

3. Just to clarify what it means to plot a column of a matrix, consider the Vandermonde matrix

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{bmatrix},$$

where $-1 = x_1 < x_2 < \cdots < x_m = 1$ are equally spaced. So each column of V is a set of evenly distributed samples of a monomial function x^j, $j = 0, \ldots, n-1$. That is, the first column is a discrete representation of the constant function $f_1(x) = 1$, the second column $f_2(x) = x$, and the general j^{th} column $f_j(x) = x^{j-1}$. Thus if every column of V is plotted against $\mathbf{x} = (x_1, x_2, \cdots, x_m)^{\mathrm{T}}$, one obtains something like



Now, thin QR factorization of V will find the orthonormal set of vectors whose span coincides with the one of V and put them in $\widehat{Q}$ (with $\widehat{R}$ being the *recovery formula*). When the columns of $\widehat{Q}$ are plotted against $\mathbf{x}$, you will see a visualization of so-called orthogonal polynomials.

4. The key modification must take place in the lines where `X` is defined and `norm_Y` is calculated. The rest can be simply re-used in your function with minimal modification.