Module 5 Problem Solving Sessions

Contents

Rootfinding

FZERO to Solve Complex Problem

- **FNC** 4.1.5 (Kepler's Law)

Lambert W-Function

- **FNC** 4.1.6

More With Lambert W-Function

Question. Show that solutions of the equation $2^x = 5x$

$$r = -\frac{W\left(-\log(2)/5\right)}{\log 2}.$$

(Here, as usual in this class, $\log(\,\cdot\,) = \ln(\,\cdot\,)$ is the natural logarithmic function.)
Then numerically verify the result using `fzero`[1]

[1] Two real-valued solutions, $r_1 \approx 0.2355$ and $r_2 \approx 4.488$.

FPI: When Convergence Is Faster Than Expected

- **FNC** 4.2.6

FPI: Conditions for Convergence

- **FNC** 4.2.7

Stopping Criteria

- **FNC** 4.3.8

Exercise with Series Analysis

Linear Convergence of Newton's Method

Newton's Method for Multiple Roots

Assume that $f \in C^{m+1}[a, b]$ has a root r of multiplicity m. Then Newton's method is locally convergent to r, and the error ϵ_k at step k satisfies

$$\lim_{k\to\infty} \frac{\epsilon_{k+1}}{\epsilon_k} = \frac{m-1}{m} \qquad \text{(linear convergence)}$$

- See Problem 4 of HW07 (**FNC** 4.3.7)
- Remedy: Modify the iteration formula

$$x_{k+1} = x_k - \frac{mf(x_k)}{f'(x_k)}$$

Calculating nth Roots

Question. Let n be a positive integer. Use Newton's method to produce a quadratically convergent method for calculating the nth root of a positive number a. Prove quadratic convergence.

Predicting Next Error

Question. Let $f(x) = x^3 - 4x$.

- The function $f(x)$ has a root at $r = 2$. If the error $\epsilon_k = x_k - r$ after four steps of Newton's method is $\epsilon_4 = 10^{-6}$, estimate ϵ_5.
- Do the same to the root $r = 0$.

Secant Method

Assume that iterates $x_1, x_2, \ldots$ generated by the secant method converge to a root r and $f'(r) \neq 0$. Let $\epsilon_k = x_k - r$.

Exercise. Show that

1. The error ϵ_k satisfies the approximate equation

$$\left|\epsilon_{k+1}\right| \approx \left|\frac{f''(r)}{2f'(r)}\right| \left|\epsilon_k\right| \left|\epsilon_{k-1}\right| .$$

2. If in addition $\lim_{k\to\infty} \left|\epsilon_{k+1}\right| / \left|\epsilon_k\right|^{\alpha}$ exists and is nonzero for some $\alpha > 0$, then

$$\left|\epsilon_{k+1}\right| \approx \left|\frac{f''(r)}{2f'(r)}\right|^{\alpha-1} \left|\epsilon_k\right|^{\alpha}, \quad \text{where } \alpha = \frac{1+\sqrt{5}}{2}.$$