<u>To see why</u> Say r is a root w/ multiplicity m. of $f(x)$:

$$f(r) = f'(r) = \cdots = f^{(m-1)}(r) = 0 \,, \quad f^{(m)}(r) \neq 0 \qquad \Big| \quad \text{(from the def'n.)}$$

Consequently:

- $$f(r+\epsilon_k) = \cancel{f(r)} + \cancel{f'(r)\epsilon_k} + \cancel{\frac{f''(r)}{2!}\epsilon_k^2} + \cdots + \cancel{\frac{f^{(m-1)}(r)}{(m-1)!}\epsilon_k^{m-1}} + \frac{f^{(m)}(r)}{m!}\epsilon_k^m + O(\epsilon_k^{m+1})$$
$$= \frac{f^{(m)}(r)}{m!}\epsilon_k^m + \frac{f^{(m+1)}(r)}{(m+1)!}\epsilon_k^{m+1} + O(\epsilon_k^{m+2})$$

- $$f'(r+\epsilon_k) = \cancel{f'(r)} + \cancel{f''(r)\epsilon_k} + \cdots + \cancel{\frac{f^{(m-1)}(r)}{(m-2)!}\epsilon_k^{m-2}} + \frac{f^{(m)}(r)}{(m-1)!}\epsilon_k^{m-1} + O(\epsilon_k^m)$$
$$= \frac{f^{(m)}(r)}{(m-1)!}\epsilon_k^{m-1} + \frac{f^{(m+1)}(r)}{m!}\epsilon_k^m + O(\epsilon_k^{m+1})$$

So the Newton's iteration formula (in terms of ϵ_k's) is now written as:

$$\epsilon_{k+1} = \epsilon_k - \frac{m f(r+\epsilon_k)}{f'(r+\epsilon_k)}$$

$$= \epsilon_k - \frac{m\left[\frac{f^{(m)}(r)}{m!}\epsilon_k^m + \frac{f^{(m+1)}(r)}{(m+1)!}\epsilon_k^{m+1} + O(\epsilon_k^{m+2})\right]}{\frac{f^{(m)}(r)}{(m-1)!}\epsilon_k^{m-1} + \frac{f^{(m+1)}(r)}{m!}\epsilon_k^m + O(\epsilon_k^{m+1})}$$

confirm (fill in the details)

$\frac{m}{m}\epsilon_k = \epsilon_k$

$$= \epsilon_k - \frac{1}{m}\epsilon_k + O(\epsilon_k^2) = \frac{m-1}{m}\epsilon_k + O(\epsilon_k^2)$$

$\therefore \frac{\epsilon_{k+1}}{\epsilon_k^1} \to \frac{m-1}{m} \in (0,1)$

(linear convergence)

$\epsilon_{k+1} = * \epsilon_k^2 + O(\epsilon_k^3)$

$\therefore \frac{\epsilon_{k+1}}{\epsilon_k^2} \to *$ as $k \to \infty$

(quad. conv.)

Calculating nth Roots

Question. Let n be a positive integer. Use Newton's method to produce a quadratically convergent method for calculating the nth root of a positive number a. Prove quadratic convergence.

Given $n \in \mathbb{N}$, $a > 0$, calculate $\sqrt[n]{a}$ using Newton.

Note that $\sqrt[n]{a}$ is a root of

$$x^n = a$$

So, it is a root of

$$f(x) = x^n - a$$

$$\boxed{x_{k+1} = \frac{n-1}{n} x_k + \frac{a}{n x_k^{n-1}}}$$

e.g. $(n=2)$

$$x_{k+1} = \frac{1}{2} x_k + \frac{a}{2 x_k}$$

Newton

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^n - a}{n x_k^{n-1}} = \frac{n-1}{n} x_k + \frac{a}{n x_k^{n-1}}$$

Predicting Next Error

$x(x^2-4) = x(x-2)(x+2)$

Question. Let $f(x) = x^3 - 4x$.

(a) - The function $f(x)$ has a root at $r = 2$. If the error $\epsilon_k = x_k - r$ after four steps of Newton's method is $\epsilon_4 = 10^{-6}$, estimate ϵ_5.

(b) - Do the same to the root $r = 0$. (Exercise)

Recall: Newton's method

$$\boxed{\epsilon_{k+1} = \frac{f'(r)}{2f''(r)}\epsilon_k^2 + O(\epsilon_k^3)}$$

(a) $r = 2$,

$$\begin{cases} f'(x) = 3x^2 - 4 \\ f''(x) = 6x \end{cases} \Rightarrow \begin{cases} f'(2) = 8 \\ f''(2) = 12 \end{cases}$$

So $\underline{\text{at } r=2}$

$$\epsilon_{k+1} = \frac{8}{24}\epsilon_k^2 + O(\epsilon_k^3)$$

$k=4$, $\epsilon_k = 10^{-6}$

$$\epsilon_5 = \frac{1}{3}10^{-12} + O(10^{-18}) \approx \boxed{3.33 \times 10^{-11}}$$

Secant Method

Assume that iterates $x_1, x_2, \ldots$ generated by the secant method converges to a root r and $f'(r) \neq 0$. Let $\epsilon_k = x_k - r$.

Exercise.[1] Show that

1. The error ϵ_k satisfies the approximate equation

$$\left|\epsilon_{k+1}\right| \approx \left|\frac{f''(r)}{2f'(r)}\right| \left|\epsilon_k\right| \left|\epsilon_{k-1}\right| .$$

2. If in addition $\lim_{k\to\infty} \left|\epsilon_{k+1}\right| / \left|\epsilon_k\right|^\alpha$ exists and is nonzero for some $\alpha > 0$, then

$$\left|\epsilon_{k+1}\right| \approx \left|\frac{f''(r)}{2f'(r)}\right|^{\alpha-1} \left|\epsilon_k\right|^\alpha , \quad \text{where } \alpha = \frac{1+\sqrt{5}}{2}.$$

[1] This exercise is from Lecture 22.

Hints Error analysis for Secant Method.

- $x_k = r + \epsilon_k$. ($\epsilon_k \to 0$ as $k \to \infty$)
- Taylor expansion
- big-O notation for simplification

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \approx f'(x_k)$$

Recall: iter. form. for secant method

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1}) f(x_k)}{f(x_k) - f(x_{k-1})}$$