The goal of this document is to give a general outline of how to solve problems for the Complex Analysis prelim.

1 Laurent Series

See the study log for specific old prelim questions.

Suppose we are given a function and are asked to find a Laurent series expansion (or some terms of it) centered at 0 and convergent on some region.

1. Factor the function into pieces which can be easily rewritten as an infinite series, for example a geometric series
$$\frac{1}{1-a} = \sum_{n\geq 0} a^n,$$
or
$$e^z = \sum_{n\geq 0} \frac{z^n}{n!}.$$
Historically, sin and cos have not been included, but I'll record them here just in case. If you are given something like $\frac{1}{z-1}$ there are two things you could do. More on this to follow.

2. Rewrite the function as a product of infinite series

3. Check the regions on which the series converges.[1] For example if $f(z) = \frac{1}{z-1}$ we could rewrite as either $f = \frac{-1}{1-z}$ or $f = \frac{1}{z(1-1/z)}$. The first way converges for $|z| < 1$. The second way converges for $|1/z| < 1$ or $|z| > 1$. Also remember that region of convergence of a product of series is the intersection of their individual regions of convergence, so if $\sum_{n\geq 0} a_n z^n$ converges for $|z| > 1$ and $\sum_{n\geq 0} b_n z^n$ converges for $|z| < 2$, then the product
$$\left(\sum_{n\geq 0} a_n z^n\right)\left(\sum_{n\geq 0} b_n z^n\right)$$
converges for $1 < |z| < 2$.

4. If the region is the one which was requested, write the series. If you are only asked for a few terms, it can be helpful to write out the terms as a sum, for example
$$\frac{1}{z}(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots)$$
or
$$(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots).$$
In the first case, recall multiplication of power series given by
$$\sum a_n z^n \sum b_n z^n = \sum (\sum_{k=0}^{n} a_n b_{k-n}) z^n.$$
Also, don't forget to include the factor of $\frac{1}{z}$ out front.

   For the second case, each coefficient will be itself an infinite sum. For this example, the coefficient of $\frac{1}{z}$ is $\sum a_n b_{n+1}$, the constant coefficient is $\sum a_n b_n$, and the coefficient of z will be $\sum a_{n+1} b_n$. Historically, the coefficients have been a geometric series, and so can be actually computed using $\sum r^n = \frac{1}{1-r}$. An older example also had a copy of $e = \sum \frac{1}{n!}$ hidden in there.

[1] Checking convergence is important because in the past different regions have been requested. For example Spring 2019 #2 and Fall 2018 #2 gave functions of the form $\frac{1}{z^n-1}$ asked for convergence on $|z| < 1$ and $|z| > 1$ respectively. The different regions of convergence changes the answer significantly. For more see StackExchange.