Fall 2018 Complex Analysis Preliminary Exam

University of Minnesota

Where possible, computations have been also done using SageMath code available on GitHub at github.com/tekaysquared/prelims (feel free to make pull requests!)

2. Write the Laurent expansion of $f(z) = \frac{1}{z^4-1}$ centered at 0 and convergent in $|z| > 1$.

Proof. Factor out z^{-4} to see that

$$\begin{aligned} f(z) &= \frac{1}{z^4(1-1/z^4)} \\ &= \frac{1}{z^4}\sum_{n=0}^{\infty} \frac{1}{z^{4n}} \end{aligned}$$

which converges for $|1/z^4| < 1$ which is to say for $|z| > 1$. So then f has a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

where

$$a_n = \begin{cases} 1 & n = -4k \text{ for nonzero positive integers } k \\ 0 & \text{otherwise} \end{cases}$$

□

6. Determine the radius of convergence of the power series for $\log z$ at $z_0 = -4+3i$.

Proof. Let R denote the radius of convergence of the power series of $\log z$ centered at z_0. Now, note that there is no logarithm which takes a value at 0, since $e^w = 0$ is never true for $w \in \mathbb{C}$. Thus, the power series expansion can converge for a disk of radius at most $|-4+3i-0| = 5$, and so

$$R \leq 5.$$

On the other hand, it is a theorem that if Ω is a simply connected subset of $\mathbb{C}$ which does not contain 0 then there is a branch of the logarithm which is holomorphic on Ω. Observe that the open disk $D_5(-4+3i) := \{z \in \mathbb{C} : |-4+3i-z| < 5\}$ is simply connected and does not contain zero. Thus, there is a branch of the logarithm (call it $\log_{D_5}$) which is holomorphic on D_5. Since we have constructed a disk of radius 5 on which there is a holomorphic logarithm, we see that

$$R \geq 5.$$

Since we have bounded R both above and below by 5, we see that $R = 5$. □