

Spring 2018 Complex Analysis Preliminary Exam

University of Minnesota

Where possible, computations have been also done using SageMath code available on GitHub at github.com/tekaysquared/prelims (feel free to make pull requests!)

1. Write the first three terms of the Laurent expansion of $f(z) = \frac{1}{z(z-1)(z-2)}$ centered at 0 and convergent in $|1| < z < |2|$

Proof. The core idea of the computation is to split the function into a product of power series. First, we observe that

$$\frac{1}{z-1} = \frac{1}{z(1-1/z)}$$

and see the geometric series

$$\frac{1}{1-1/z} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n,$$

which converges for $|1/z| < 1$, or equivalently $|z| > 1$. Similarly we see that

$$\frac{1}{z-2} = \frac{-1}{2(1-z/2)} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

for $|z/2| < 1$, which is to say for $|z| < 2$. Thus we have

$$\begin{aligned} f(z) &= \frac{1}{z} \left(\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n \right) \left(\frac{-1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \right) \\ &= \frac{-1}{2z} \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \right) \left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots \right). \end{aligned}$$

Note that the above product converges when each term converges, which is to say on the annulus $1 < |z| < 2$.

Now note that the coefficient of z^{-1} of the Laurent expansion is

$$\begin{aligned} -\frac{1}{2} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \right) &= \frac{-1}{2} \left[\sum_{n \geq 0} (1/2)^n - 1 \right] \\ &= -\frac{1}{2} \left(\frac{1}{1-1/2} - 1 \right) \\ &= -\frac{1}{2}. \end{aligned}$$

The coefficient of z^0 is

$$-\frac{1}{2} \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \right) = -\frac{1}{2} \left(2 - 1 - \frac{1}{2} \right) = -1/4$$

The coefficient of z^1 is

$$\begin{aligned} -\frac{1}{2} \left(\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \right) &= -\frac{1}{2} \left(2 - 1 - \frac{1}{2} - \frac{1}{4} \right) \\ &= -\frac{1}{8}. \end{aligned}$$

Therefore

$$f(z) = \cdots - \frac{1}{2z} - \frac{1}{4} - \frac{z}{8} + \cdots$$

□

Note that there is also a Laurent series which converges for the annulus $0 < |z| < 1$. This can be found by using the geometric series expansion

$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n$$

which of course converges for $|z| < 1$, and using the same expansion of $\frac{1}{z-2}$ as above. The expansion which converges on the punctured unit disk is the one provided by SageMath. For another example of this, see this [math StackExchange](#) post.