Complex Analysis Preliminary Written Exam Prep

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The goal of this document is to give a general outline of how to solve problems for the Complex Analysis prelim. I hope that this document contains all of the theorems/definitions required to solve a prelim, so it is just up to you to practice solving the problems on old exams using these theorems/techniques. If I missed anything, please email karnx018[at]umn.edu, or check out the GitHub repository and make a pull request. See the study log for specific old prelim questions relevant to each section.

1 Laurent Series

Suppose we are given a function and are asked to find a Laurent series expansion (or some terms of it) centered at 0 and convergent on some region.

1. Factor the function into pieces which can be easily rewritten as an infinite series, for example a geometric series
$$\frac{1}{1-a} = \sum_{n\geq 0} a^n,$$
or
$$e^z = \sum_{n\geq 0} \frac{z^n}{n!}.$$
Historically, sin and cos have not been included, but I'll record them here just in case. If you are given something like $\frac{1}{z-1}$ there are two things you could do. More on this to follow.

2. Rewrite the function as a product of infinite series

3. Check the regions on which the series converges.[1] For example if $f(z) = \frac{1}{z-1}$ we could rewrite as either $f = \frac{-1}{1-z}$ or $f = \frac{1}{z(1-1/z)}$. The first way converges for $|z| < 1$. The second way converges for $|1/z| < 1$ or $|z| > 1$. Also remember that region of convergence of a product of series is the intersection of their individual regions of convergence, so if $\sum_{n\geq 0} a_n z^n$ converges for $|z| > 1$ and $\sum_{n\geq 0} b_n z^n$ converges for $|z| < 2$, then the product
$$\left(\sum_{n\geq 0} a_n z^n\right)\left(\sum_{n\geq 0} b_n z^n\right)$$
converges for $1 < |z| < 2$.

4. If the region is the one which was requested, write the series. If you are only asked for a few terms, it can be helpful to write out the terms as a sum, for example
$$\frac{1}{z}(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots)$$
or
$$(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots).$$

[1] Checking convergence is important because in the past different regions have been requested. For example Spring 2019 #2 and Fall 2018 #2 gave functions of the form $\frac{1}{z^n-1}$ asked for convergence on $|z| < 1$ and $|z| > 1$ respectively. The different regions of convergence changes the answer significantly. For more see StackExchange.

In the first case, recall multiplication of power series given by

$$\sum a_n z^n \sum b_n z^n = \sum (\sum_{k=0}^{n} a_n b_{k-n}) z^n.$$

Also, don't forget to include the factor of $\frac{1}{z}$ out front.

For the second case, each coefficient will be itself an infinite sum. For this example, the coefficient of $\frac{1}{z}$ is $\sum a_n b_{n+1}$, the constant coefficient is $\sum a_n b_n$, and the coefficient of z will be $\sum a_{n+1} b_n$. Historically, the coefficients have been a geometric series, and so can be actually computed using $\sum r^n = \frac{1}{1-r}$. An older example also had a copy of $e = \sum \frac{1}{n!}$ hidden in there.

2 Power series

Suppose we are given a function $f(z)$ and asked to find the radius of convergence for its power series centered at some given point z_0. Some good facts to know are:

a) The radius of convergence is the radius of the largest disk centered at z_0 on which there is a holomorphic function agreeing with the given function.

b) If $\Omega \subset \mathbb{C}$ such that Ω is simply connected (homotopic to a point) and $0 \notin \Omega$ then there is a branch of the logarithm (call it $\log_\Omega$) which is holomorphic on Ω.[2]

c) **Riemann's theorem on removable discontinuities** states (among other things) that if $D \subseteq \mathbb{C}$ is open and $a \in D$, then if f is a function which is holomorphic on $D - \{a\}$, then f is continuously extendable if and only if it is holomorphically extendable. There are also other conclusions of the theorem, but the two above conclusions are the ones which are relevant to this type of problem.

The general idea is to first bound the size of the disk from above, then from below. One way to bound from below is to explicitly construct a holomorphic function on a disk of the desired size which agrees with given function, by using (b) and (c) above. One way to bound from above is to find a pole or essential singularity, and compute the distance between a pole or essential singularity and the given point z_0.

3 Conformal mapping

The important theorem for a map to be conformal is that it f conformal when f' is nonzero. For a great lecture about this, see the Herb Gross video Complex Variables: Lec 3. Conformal Mappings. Professor Gross goes into great depth about the intuition of why angles are preserved. It is a bit hand-wavy but very digestible.

A really useful conformal map is the so-called "Cayley map" which is a conformal mapping between the open upper half plane H and the unit disk D. There are many maps which go by the name "Cayley map," but we will use the one given on Wolfram MathWorld.

A nice way to remember the map was shown to me by Libby Farrell. We can think of the upper half plane as all points which are closer to i than to $-i$. In other words, the upper half plane satisfies $|z-i| < |z-(-i)|$, which would imply $\frac{|z-i|}{|z+i|} = \left|\frac{z-i}{z+i}\right| < 1$. This reminds us that $f(z) := \frac{z-i}{z+i}$ takes the upper half plane conformally to the unit disk. Then we can remember that the formula for the inverse of a linear fractional transform $\frac{az+b}{cz+d}$ looks like the way to take the inverse of a 2×2 matrix namely $\frac{dz-b}{-cz+a}$.[3] Thus, the inverse of $f(z) = \frac{z-i}{z+i}$ is $f^{-1}(z) = \frac{iz+i}{-z+1}$, which would thus be a conformal mapping from the disk to the upper half plane. We could check that (for example) $\frac{0+i}{-0+1}$ takes $0 \mapsto i$, and reminding us that the interior of the disk maps to the interior of the upper half plane.

There are some useful facts about the Cayley map:

[2] Theorem 6.1.i in Chapter 3 of Stein and Shakarchi's *Complex Analysis*. Note that the theorem also includes the assumption that $1 \in \Omega$, but this assumption is used to prove parts ii, iii of the theorem, not part i.

[3] This is not a coincidence! See PG's notes for further detail.

1. In general, a certain half-disk will be in correspondence with quadrant under some version of the Cayley map.
2. For our f^{-1}, the upper-half disk will be taken to the second quadrant. If you forget, just plug in $x+iy$ and compute $f^{-1}(x+iy)$, and conditions on x, y will tell you what goes where.

Finally, remember that multiplication by a constant is a conformal mapping. Thus, since multiplication by i corresponds to a 90-degree rotation of the complex plane, it is useful to remember you can shift things using that multiplication. Finally, the map $z \mapsto z^2$ is a conformal mapping (as long as 0 isn't in the domain of the function) which doubles angles so, for example, the upper half-unit-disk gets mapped to the slit unit-disk under this map.

4 Show a function is constant

There are a few different tools to show an entire function is constant based on the information you are given.

1. We know that an entire function $f(z) = f(x, y) = u(x, y) + iv(x, y)$ satisfies the **Cauchy-Riemann equations**,
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$
Thus, if you can show some of these are 0, you know the other one is zero.
2. **Liouville's theorem** states that every bounded entire function is constant.
3. The **open mapping theorem** states that non-constant holomorphic functions are open maps (i.e. they send open sets to open sets). If you can show that the image under a holomorphic function of an open set is closed, then the function must be constant.
4. **Cauchy's inequality** states that if C is a circular curve of radius R centered at z_0 and f is holomorphic on a region enclosing C, then
$$|f^{(n)}(z_0)| \leq n! \frac{\max_{z \in C} f(z)}{R^n}.$$
The way this will typically show up is given an entire function so that you can take $R \to \infty$ without changing the value, and then (roughly speaking) that sends all the non-constant coefficients to 0. The connection is that the way to extract coefficients is by taking derivatives and evaluating at 0. This also will usually end up cancelling the factor of $n!$.
5. **Picard's little theorem** states that an entire function f which is non-constant misses at most one value in the complex plane. So if you can produce two values which are not in the image of f, then f must be constant.

If you are given some bound on f and show that f is constant, you might be able to be more specific about the values of f, namely that they lie within a disk of some radius from the origin.

5 Compute an integral

A classic question is to compute a real integral by passing to complex integrals first. There are a few different ways to compute such an integral.

1. The **residue theorem** states that if f is a holomorphic in an open set containing a circle C and its interior, except for a pole at z_0 in C then
$$\int_C f(z)dz = 2\pi i \operatorname{res}_{z_0} f.$$
In the case that there is a simple pole at z_0 we can compute
$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z).$$

In the case that there is a pole of order n at z_0, we can compute

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz} \right)^{n-1} (z - z_0)^n f(z).$$

You can remember the second formula from the first by considering

$$(z - z_0)^n f(z) = a_{-n} + \cdots + a_{-1}(z - z_0)^{n-1} + G(z)(z - z_0)^n,$$

and iterating the formal derivative for power series.

2. The **estimation lemma** states that if γ is a contour of length L, then the integral

$$\left| \int_\gamma f(z) dz \right| \le L \sup_{z \in \gamma} |f(z)|.$$

This is especially helpful to show that an integral (or part of an integral) vanishes.

3. As a general technique, compute a finite contour integral, split the contour into a few different pieces (usually one of which is a finite line segment $[-t, t] \subseteq \mathbb{R}$) then take the limit as $t \to \infty$. You can often start out with a closed contour and use the resuidue theorem to get a value, and show that part of the integral vanishes using the estimation lemma. If you set up the integral right, then the part which is *not* on the real line should vanish and so you are left with the $2\pi i \operatorname{res}_{z_0} f$.

One thing to be careful of is that if you have a fractional power then you might not be able to define it at 0, so you need to have another semi-circle of radius epsilon, and then you would take $\epsilon \to 0$. Probably use the estimation lemma again, since the radius is $\epsilon\pi$.

6 Rouche's theorem

Rouche's theorem is a counting result that under the right conditions two functions have the same number of zeros. Lets state the theorem fully: suppose that Ω is an open subset of $\mathbb{C}$ which contains the closed disk D, and that f, g are holomorphic on Ω. Then if $|f| > |g|$ on ∂D (the boundary of D, aka the circle), then f and $f + g$ have the same number of zeros in the interior of C.

Some ways that this theorem has come up on past exams:

- Show that given $f \neq 0$ on ∂D then f and $f + \epsilon g$ have the same number of zeros for some ϵ. If you get this, compute the ϵ using the fact that ∂D is compact so it attains a max/min, then you can express ϵ directly in terms of the max and min. (See Fall 2019 exam).
- Show that some polynomial has only a certain number of zeros in a given disk or annulus.[4]

7 Holomorphic inverse-function theorem

One example that has not shown up in a few years (as of 2020) but could is the question: Show that some function $f(x)^n = g(x)$ is defined except at a few points. The **holomorphic inverse function theorem** states that: if $F(x, y)$ is a polynomial in x, y at x_0, y_0 satisfying $F(x_0, y_0) = 0$ and $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$, then there is a holomorphic function $y = f(x)$ near x_0 such that $f(x_0) = y_0$ and $F(x, f(x)) = 0$ for x near x_0. So to solve this problem, just check that the partial derivative is 0 at some points, and those are the points where you *cannot* define a holomorphic function in the manner given.

Don't get this one confused with $f(x)^n = g(x)$ where $g(z)$ is not a polynomial. Then you need to appeal to the theorem that n-th roots are well defined in any region not containing 0, and so you find the points where $g(z)$ is 0, and those are the places where it breaks.

[4] See this StackExchange post for an annulus. It includes in the solution counting the number of zeros in a disk.

8 Elliptic curves

There are two common questions which are essentially the same, but are phrased differently. The two questions are

- Compute the genus of a curve $z^n = p(w)$ where p is a polynomial of degree w.
- Show that a curve $z^n = p(w)$ is/is not an elliptic curve.

The second question is a special case of the first, because an elliptic curve is a plane curve with genus 1.

The genus-degree formula[5] tells us that the genus of a smooth irreducible plane curve given by a homogeneous polynomial is

$$\frac{(d-1)(d-2)}{2}$$

where d is the degree of the polynomial.

So we need to check that it is (a) smooth, (b) irreducible. First, we homogenize the polynomial with a variable u which we will eventually specialize to 1. Then we can write the curve as the zero locus of $u^{d-n}z^n - p'(w,u)$, where p' is homogenous of degree d. Compute the partial derivative of the expression with respect to z, w, u, and if they are *ever* simultaneously zero when $u = 1$, then the curve is *not smooth*. Otherwise, it is smooth. Then check that it is irreducible as a polynomial, and you can apply the formula above.

The "right" way to do this in general can be found in Ryan Matske's notes, starting on page 17, or Paul Garrett's notes, but the "right" way is too much work for the particularly nice cases which typically show up on the prelims.

There is an especially nice case for *hyper elliptic curves* Let π be a hyper elliptic curve with $y^2 = f(x)$ where f is a square-free polynomial in x. Then the Riemann-Hurwitz formula[6] tells us that the genus g_Y is

$$g_Y = \begin{cases} \frac{\deg(f)-1}{2} & \text{if } \deg(f) \text{ odd} \\ \frac{\deg f - 2}{2} & \text{if } \deg(f) \text{ even} \end{cases}.$$

To solve these problems rewrite the given function as $y^2 =$(a polynomial) and then use the formula.

Another question which has come up recently is to write it in Weierstrass form.

9 Miscellany

Just some other things that could be helpful at some point:

- **Cauchy integral formula:** if C is a circle containing a then

  $$f^{(n)}(a) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}dz.$$

  The Cauchy inequality is a direct consequence of this. The inequality has shown up on the prelims, but the integral formula itself has not since Paul Garrett started writing the exams.

- The **Argument principle** states that if γ is a curve, and N is the number of zeros of a function f inside of γ and M is the number of poles of f inside γ, then the integral

  $$\int_\gamma \frac{f'(z)}{f(z)}dz = 2\pi i(N-M).$$

  You can remember this because the poles are coefficients of negative-degree terms in the Laurent series expansion of f, and so the number of them contributes negatively to the integral. (This is not a mathematical explanation, just a mnemonic.)

[5]See Griffiths and Harris Ch. 2.1 or Hartshorne Ch 4.

[6]For further detail, see http://www-users.math.umn.edu/ garrett/m/complex/notes_2014-15/ERNPRHH.pdf

- The **Reflection principle** states that if Ω is a region symmetric about the $\mathbb{R}$-axis, and f is a holomorphic function on $\Omega \cap \{z : \operatorname{im} z > 0\}$ which can be continuously extended to $\Omega \cap \mathbb{R}$ in a manner such that the extension is real-valued on $\Omega \cap \mathbb{R}$, then there is a function F holomorphic on Ω which agrees with f on $\Omega \cap \{z : \operatorname{im} z > 0\}$. Moreover, we have the expression that

$$F(z) = \overline{f(\overline{z})}$$