Fall 2019 Complex Analysis Preliminary Exam

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Where possible, computations have been also done using SageMath code available on GitHub at github.com/tekaysquared/prelims (feel free to make pull requests!)

1. Give a conformal mapping from the (open) upper half-disk $H = \{z : |z| < 1 \text{ and } \operatorname{im}(z) > 0\}$ to the slit disk
$$D = \{z \in \mathbb{C} : |z| < 1,\ z \notin [0, 1]$$

   Proof. First, let $f(z) = 1/z$. Then $f(H) = \{z : |z| > 1 \text{ and } \operatorname{im}(z) > 0\}$. Since both f and $f'(z) = -z^{-2}$ are never 0, f is a conformal mapping. Now let $g(z) = z^2$. Then $g(f(H)) = \{re^{i\theta} : r > 1 \text{ and } \theta \in (0, 2\pi)\}$. $g = 0$ and $g' = 0$ only at $z = 0$, which is not in $f(H)$ so $g \circ f$ is a conformal mapping from H to $g(f(H))$. Finally, note that $f \circ g \circ f(H) = D$. Again f' is never zero so the composition $f \circ g \circ f : H \to D$ is a conformal mapping. □

2. Write the first three terms of the Laurent expansion of $f(z) = \dfrac{1}{z(z-1)(z-2)}$ centered at 0 and convergent in $|1| < z < |2|$

   Proof. The core idea of the computation is to split the function into a product of power series. First, we observe that
$$\frac{1}{z-1} = \frac{1}{z(1 - 1/z)}$$
   and see the geometric series
$$\frac{1}{1 - 1/z} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n,$$
   which converges for $|1/z| < 1$, or equivalently $|z| > 1$. Similarly we see that
$$\frac{1}{z-2} = \frac{-1}{2(1 - z/2)} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$
   for $|z/2| < 1$, which is to say for $|z| < 2$. Thus we have
$$\begin{aligned} f(z) &= \frac{1}{z}\left(\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right)\left(\frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right) \\ &= \frac{-1}{2z}\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right). \end{aligned}$$
   Note that the above product converges when each term converges, which is to say on the annulus $1 < |z| < 2$.

Now note that the coefficient of z^{-1} of the Laurent expansion is

$$\begin{aligned}
-\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right) &= \frac{-1}{2}\left[\sum_{n\geq 0}(1/2)^n - 1\right] \\
&= -\frac{1}{2}\left(\frac{1}{1-1/2}-1\right) \\
&= -\frac{1}{2}.
\end{aligned}$$

The coefficient of z^0 is

$$-\frac{1}{2}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right) = -\frac{1}{2}\left(2-1-\frac{1}{2}\right) = -1/4$$

The coefficient of z^1 is

$$\begin{aligned}
-\frac{1}{2}\left(\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots\right) &= -\frac{1}{2}\left(2-1-\frac{1}{2}-\frac{1}{4}\right) \\
&= -\frac{1}{8}.
\end{aligned}$$

Therefore

$$f(z) = \cdots - \frac{1}{2z} - \frac{1}{4} - \frac{z}{8} + \cdots$$

□

Note that there is also a Laurent series which converges for the annulus $0 < |z| < 1$. This can be found by using the geometric series expansion

$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n$$

which of course converges for $|z| < 1$, and using the same expansion of $\frac{1}{z-2}$ as above. This is the one provided by SageMath. For another example of this, see this math StackExchange post.

3. Classify entire functions f so that $|f(z)| \leq C|z|$ for some constant C.

   Proof. Liouville's theorem tells us that bounded, entire functions are constant. If $f(z)/z$ is entire, then Liouville's theorem would tell us that $f(z)/z$ is a constant (call it k), and so since $f(z)/z = k$, then $f(z) = kz$. If $f(z)/z$ is not entire, then there is a simple pole at $z = 0$ (since $f(z)$ is entire). This would imply that in (e.g.) the open disk centered at $z_0 = 1$ with radius 1, we would have $f(z)/z$ being unbounded. But this contradicts that $|f(z)/z| \leq C$ is bounded on all of $\mathbb{C} - \{0\}$.

   Thus, the functions f which satisfy $|f(z)| \leq C|z|$ are linear functions with no constant term. □

4. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2} dx$.

Proof. We compute this real integral by passing to complex values and computing.

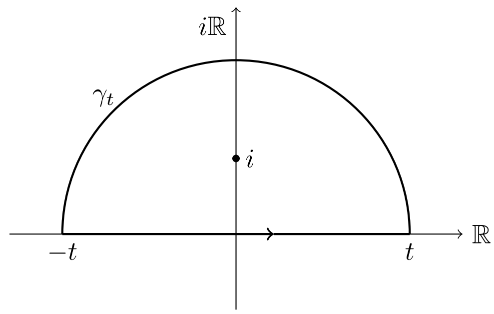
Define

$$f(z) = \frac{e^{iz}}{1+z^2} = \frac{e^{iz}}{(z+i)(z-i)}.$$

We make this choice of f because for x on the real axis, we have $f(x) = \frac{\cos(x)+i\sin(x)}{1+x^2}$, which is to say $\operatorname{re} f(x) = \frac{\cos(x)}{1+x^2}$.

For $t > 1$, let γ_t denote the union of the line from $[-t, t]$ with the semi-circle of radius t in the upper half-plane (call it C_t), with the orientation of γ_t being counter clockwise.

Visually:



By the residue theorem, we know that as long as $t > 1$

$$\int_{\gamma_t} f(z)dz = 2\pi i \operatorname{res}_i f(z).$$

Since i is a simple pole of $f(z)$ (as seen by the factorization $1 + z^2 = (z+i)(z-i)$), we can compute

$$\begin{aligned}
\operatorname{res}_i f &= \lim_{z\to i}(z-i)f(z) \\
&= \lim_{z\to i}(z-i)\frac{e^{iz}}{(z+i)(z-i)} \\
&= \lim_{z\to i}\frac{e^{iz}}{(z+i)} \\
&= \frac{e^{-1}}{2i} = \frac{1}{2ie}
\end{aligned}$$

and so

$$\int_{\gamma_t} f(z)dz = 2\pi i \frac{1}{2ie} = \frac{\pi}{e}.$$

We can split up the integral of $f(z)$ over γ_t along the real axis and the semi-circle as

$$\int_{\gamma_t} f(z)dz = \int_{[-t,t]} f(z)dz + \int_{C_t} f(z)dz.$$

The estimation lemma or "ML inequality" tells us that since the length of a semicircle of radius r is πr, then

$$\left|\int_{C_t} f(z)dz\right| \le \pi t \max_{z\in C_t} |f(z)|$$

We now compute an upper bound for $\max_{z\in C_t} |f(z)|$,

$$\left|\frac{e^{iz}}{1+z^2}\right| = \frac{|e^{iz}|}{|1+z^2|}$$

Note that C_t only contains points with $\operatorname{im} z > 0$, and so if $z = x+iy$, then $|e^{iz}| = |e^{ix-y}| = e^{-y} < 1$ thus in C_t

$$\left|\frac{e^{iz}}{1+z^2}\right| \leq \frac{1}{|1+z^2|}$$

The reverse triangle inequality tells us (again only considering $z \in C_t$) that

$$|z^2 - (-1)| > \big||z^2| - |-1|\big| = \big||z|^2 - 1\big| = |t^2 - 1|$$

And so thus in C_t

$$\left|\frac{e^{iz}}{1+z^2}\right| \leq \frac{1}{|t^2-1|}$$

.

Taking the limit as $t \to \infty$ we see that the integral over C_t goes to zero, and so

$$\lim_{t\to\infty} \int_{\gamma_t} f(z)dz = \lim_{t\to\infty} \int_{[-t,t]} f(z)dz = \int_{(-\infty,\infty)} f(z)dz.$$

But the integral over γ_t is independent of t, and so

$$\int_{(-\infty,\infty)} f(z)dz = \pi/e.$$

Recall that the integral we *wanted* to compute was the real part of the above integral, but since the integral is real, the real part is the whole integral, and so

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{e}$$

□

5. Determine the radius of convergence of the power series for $z \log z$ at $z_0 = -3 + 4i$.

   Proof. We will look for the largest R for which there is a disk D_R of radius R centered at z_0 on which there is a holomorphic function agreeing with $z \log z$. The product of holomorphic functions is holomorphic, so because $g(z) = z$ is entire, the radius of convergence of $z \log z$ is limited by $f(z) = \log z$.

   To find the radius of convergence of f at z_0, observe that there is no number $w \in \mathbb{C}$ such that $e^w = 0$, and so the R is bounded above by $|-3+4i-0| = 5$.

   On the other hand recall that it is a theorem[1] that if D is a simply connected region which does not contain 0, then there is a branch of the logarithm (call it $\log_D$) which is holomorphic on D. Consider the (open) disk D_5 of radius 5 centered at z_0. Clearly this does not contain 0, and so there is a holomorphic $\log_{D_5}$. Thus, we see $R \geq 5$.

   Since $R \leq 5$ and $R \geq 5$, we have $R = 5$. □

6. Let f, g be holomorphic functions on $\{z : |z| < 2\}$ with f nonvanishing on $|z| = 1$. Show that for all sufficiently small $\varepsilon > 0$ the function $f + \varepsilon g$ has the same number of zeros inside $|z| = 1$ as does f.

[1] Theorem 6.1 in Chapter 3 of Stein and Shakarchi's *Complex Analysis*

Proof. Since f, g are holomorphic on $\{z : |z| < 2\}$, they are holomorphic on the compact sets $D = \{z : |z| \leq 1\}$ and its boundary $\partial D = \{z : |z| = 1\}$.

Rouche's theorem states that if $|\varepsilon g| \leq |f|$ on ∂D (which can be thought of as a closed curve) then f and $f + \varepsilon g$ have the same number of zeros inside D. Thus our goal will be to find $\varepsilon > 0$ which establishes this bound.

Since f, g are holomorphic on ∂D, they are continuous, and since ∂D is a closed subset of $\mathbb{C}$, it is compact. The modulus function is also continuous, and so by composition, $|f|, |g|$ are both continuous real-valued functions and thus achieve a maximum and minimum on ∂D.

Let $m = \min_{z \in \partial D} f$ and $M = \max_{z \in \partial D} g$. Pick $\varepsilon < \frac{m}{M}$. Then on ∂D

$$\begin{aligned} |\varepsilon g| &= \varepsilon |g| \\ &< \frac{m}{M} |g| \\ &\leq \frac{m}{M} M \\ &= m \leq |f|. \end{aligned}$$

$\square$