Fall 2019 Complex Analysis Preliminary Exam

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Where possible, computations have been also done using SageMath code available on GitHub at github.com/tekaysquared/prelims (feel free to make pull requests!)

1. Give a conformal mapping from the (open) upper half-disk $H = \{z : |z| < 1 \text{ and } \operatorname{im}(z) > 0\}$ to the slit disk
$$D = \{z \in \mathbb{C} : |z| < 1,\ z \notin [0,1]$$

   Proof. First, let $f(z) = 1/z$. Then $f(H) = \{z : |z| > 1 \text{ and } \operatorname{im}(z) > 0\}$. Since both f and $f'(z) = -z^{-2}$ are never 0, f is a conformal mapping. Now let $g(z) = z^2$. Then $g(f(H)) = \{re^{i\theta} : r > 1 \text{ and } \theta \in (0, 2\pi)\}$. $g = 0$ and $g' = 0$ only at $z = 0$, which is not in $f(H)$ so $g \circ f$ is a conformal mapping from H to $g(f(H))$. Finally, note that $f \circ g \circ f(H) = D$. Again f' is never zero so the composition $f \circ g \circ f : H \to D$ is a conformal mapping. □

2. Write the first three terms of the Laurent expansion of $f(z) = \dfrac{1}{z(z-1)(z-2)}$ centered at 0 and convergent in $|1| < z < |2|$

   Proof. The core idea of the computation is to split the function into a product of power series. First, we observe that
$$\frac{1}{z-1} = \frac{1}{z(1-1/z)}$$
and see the geometric series
$$\frac{1}{1-1/z} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n,$$
which converges for $|1/z| < 1$, or equivalently $|z| > 1$. Similarly we see that
$$\frac{1}{z-2} = \frac{-1}{2(1-z/2)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$
for $|z/2| < 1$, which is to say for $|z| < 2$. Thus we have
$$\begin{aligned} f(z) &= \frac{1}{z}\left(\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right)\left(\frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right) \\ &= \frac{-1}{2z}\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right). \end{aligned}$$
Note that the above product converges when each term converges, which is to say on the annulus $1 < |z| < 2$.

Now note that the coefficient of z^{-1} of the Laurent expansion is

$$
\begin{aligned}
-\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right) &= \frac{-1}{2}\left[\sum_{n\geq 0}(1/2)^n - 1\right] \\
&= -\frac{1}{2}\left(\frac{1}{1-1/2}-1\right) \\
&= -\frac{1}{2}.
\end{aligned}
$$

The coefficient of z^0 is

$$
-\frac{1}{2}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right) = -\frac{1}{2}\left(2-1-\frac{1}{2}\right) = -1/4
$$

The coefficient of z^1 is

$$
\begin{aligned}
-\frac{1}{2}\left(\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots\right) &= -\frac{1}{2}\left(2-1-\frac{1}{2}-\frac{1}{4}\right) \\
&= -\frac{1}{8}.
\end{aligned}
$$

Therefore

$$
f(z) = \cdots - \frac{1}{2z} - \frac{1}{4} - \frac{z}{8} + \cdots
$$

□

Note that there is also a Laurent series which converges for the annulus $0 < |z| < 1$. This can be found by using the geometric series expansion

$$
\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n
$$

which of course converges for $|z| < 1$, and using the same expansion of $\frac{1}{z-2}$ as above. This is the one provided by SageMath. For another example of this, see this math StackExchange post.

5. Determine the radius of convergence of the power series for $z \log z$ at $z_0 = -3 + 4i$.

Proof. We will look for the largest R for which there is a disk D_R of radius R centered at z_0 on which there is a holomorphic function agreeing with $z \log z$. The product of holomorphic functions is holomorphic, so because $g(z) = z$ is entire, the radius of convergence of $z \log z$ is limited by $f(z) = \log z$.

To find the radius of convergence of f at z_0, observe that there is no number $w \in \mathbb{C}$ such that $e^w = 0$, and so the R is bounded above by $|-3+4i-0| = 5$.

On the other hand recall that it is a theorem[1] that if D is a simply connected region which does not contain 0, then there is a branch of the logarithm (call it $\log_D$) which is holomorphic on D. Consider the (open) disk D_5 of radius 5 centered at z_0. Clearly this does not contain 0, and so there is a holomorphic $\log_{D_5}$. Thus, we see $R \geq 5$.

Since $R \leq 5$ and $R \geq 5$, we have $R = 5$. □

[1] Theorem 6.1 in Chapter 3 of Stein and Shakarchi's *Complex Analysis*