Spring 2018 Complex Analysis Preliminary Exam

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Where possible, computations have been also done using SageMath code available on GitHub at github.com/tekaysquared/prelims (feel free to make pull requests!)

1. Write the first three terms of the Laurent expansion of $f(z) = \dfrac{1}{z(z-1)(z-2)}$ centered at 0 and convergent in $|1| < z < |2|$

Proof. The core idea of the computation is to split the function into a product of power series. First, we observe that

$$\frac{1}{z-1} = \frac{1}{z(1-1/z)}$$

and see the geometric series

$$\frac{1}{1-1/z} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n,$$

which converges for $|1/z| < 1$, or equivalently $|z| > 1$. Similarly we see that

$$\frac{1}{z-2} = \frac{-1}{2(1-z/2)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

for $|z/2| < 1$, which is to say for $|z| < 2$. Thus we have

$$\begin{aligned} f(z) &= \frac{1}{z}\left(\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right)\left(\frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right) \\ &= \frac{-1}{2z}\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right). \end{aligned}$$

Note that the above product converges when each term converges, which is to say on the annulus $1 < |z| < 2$.

Now note that the coefficient of z^{-1} of the Laurent expansion is

$$\begin{aligned} -\frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) &= \frac{-1}{2}\left[\sum_{n\geq 0}(1/2)^n - 1\right] \\ &= -\frac{1}{2}\left(\frac{1}{1-1/2} - 1\right) \\ &= -\frac{1}{2}. \end{aligned}$$

The coefficient of z^0 is

$$-\frac{1}{2}\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right) = -\frac{1}{2}\left(2 - 1 - \frac{1}{2}\right) = -1/4$$

The coefficient of z^1 is

$$\begin{aligned} -\frac{1}{2}\left(\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots\right) &= -\frac{1}{2}\left(2-1-\frac{1}{2}-\frac{1}{4}\right) \\ &= -\frac{1}{8}. \end{aligned}$$

Therefore

$$f(z) = \cdots - \frac{1}{2z} - \frac{1}{4} - \frac{z}{8} + \cdots$$

□

Note that there is also a Laurent series which converges for the annulus $0 < |z| < 1$. This can be found by using the geometric series expansion

$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n$$

which of course converges for $|z| < 1$, and using the same expansion of $\frac{1}{z-2}$ as above. The expansion which converges on the punctured unit disk is the one provided by SageMath. For another example of this, see this math StackExchange post.

4. Determine the radius of convergence of the power series for $\frac{z}{1-e^z}$ at $z = 0$.

Proof. Let R denote the radius of convergence of $f(z) := \frac{z}{1-e^z}$ at $z = 0$.

R is the largest value such that there is a holomorphic function on $D_R = \{z : |z| < R\}$ which agrees with of f on $D_R \backslash \{0\}$ Observe that by L'Hôpitals rule

$$\lim_{z\to 0} \frac{z}{1-e^z} \overset{L'H}{=} \lim_{z\to 0} \frac{1}{-e^z} = -1$$

and so f is continuously extendable at $z = 0$.

Further observe that $f(z)$ is holomorphic in the punctured open disk $D = \{z : 0 < |z| < 2\pi\}$.

Since we can extend f to be continuous at $z = 0$ by taking $f(0) := -1$, Riemann's theorem on removable singularities tells us that in fact this extension is holomorphic on $D \cup \{0\}$. Thus, since the radius of $D \cup \{0\}$ is 2π, then

$$R \geq 2\pi. \tag{1}$$

On the other hand, $\lim_{z\to 2\pi i} f(z) = \infty$ and so there is no *holomorphic* function which agrees with $f(z)$ at $2\pi i$. Thus,

$$R \leq 2\pi. \tag{2}$$

Combining (1) and (2) we see that $R = 2\pi$. □