Spring 2019 Complex Analysis Preliminary Exam

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Where possible, computations have been also done using SageMath code available on GitHub at github.com/tekaysquared/p
(feel free to make pull requests!)

1. Give a conformal mapping from the (open) upper half plane to the slit disk

$$D = \{z \in \mathbb{C} : |z| < 1, z \notin [0,1]\}$$

Proof. Consider the map $z \mapsto \frac{1}{(z+1)^2}$. It is a theorem that a map is conformal if and only if it is holomorphic on the entire domain (in this case the open upper half plane) and nonzero on the entire domain. We see that the provided map is holomorphic except at $z = -1$, which is not in the open upper half plane. Moreover, we can explicitly compute the derivative to be $-2(z+1)^{-3}$ which is nonzero wherever it is defined. Thus, the map is conformal.

We check that it is onto the slit disk D. First, $z + 1$ shifts the half-plane up one unit so that every point has modulus at least one. Then this is composed with the map $z \mapsto z^2$, which wraps the shifted plane around to consist of all points which do not on the positive real line and which have modulus at least one. Finally, inverting the map puts those on the outside of the circle onto the inside of the circle, i.e. D. □

2. Write the first three terms of the Laurent expansion of $f(z) = \dfrac{1}{z^5 - 1}$ centered at 0 and convergent in $|z| < 1$.

Proof. Observe that

$$\frac{1}{z^5 - 1} = \frac{-1}{1 - z^5} = -\sum_{n=0}^{\infty} z^{5n}$$

which converges for $|z^5| < 1$ which is $|z|^5 < 1$ or $|z| < 1$. Thus, the first three nonzero terms of the expansion of f are $a_0 = -1$, $a_5 = -1$, and $a_{10} = -1$. □

5. Determine the radius of convergence for the power series of $\sqrt{z}$ at $z_0 = -3 + 4i$.

Proof. The radius of convergence of the power series of $\sqrt{z}$ is the radius of the largest disk for which there is a holomorphic function which agrees with $\sqrt{z}$. Recall that we define complex exponentiation by $z^\alpha := e^{\alpha \log z}$, so $\sqrt{z} = e^{\log(z)/2}$. Composition of holomorphic functions is holomorphic, so since e^w is entire, the radius of convergence is limited by $\log(z)$.

There is no number $w \in \mathbb{C}$ such that $e^w = 0$, and so there is no possible way to have a holomorphic logarithm at 0. This bounds the radius of convergence by $|-3 + 4i - 0| = 5$.

On the other hand, it is a theorem[1] that if Ω is simply connected and does not contain 0, then there is a branch of the logarithm which is holomorphic on Ω. Consider the open disk D of radius 5 and centered at $-3 + 4i$. Clearly D does not contain 0, and so there is a logarithm (call it $\log_D$) which is holomorphic on D. Thus, we have a disk of radius 5 on which there is a holomorphic function $\log_D$ which agrees with log, and so the radius of convergence is *at least* 5.

[1] Theorem 6.1 in Chapter 3 of Stein and Shakarchi's *Complex Analysis*

Since we know the radius of convergence is both at least 5 and less than or equal to 5, we see that the radius of convergence of the power series for log is in fact 5.

$\square$

8. Define $f(z)$ near 0 by $f(z)^2 = \frac{\sin z}{z}$. What is the radius of convergence of the power series of f at 0.

Proof. Since $f(z)^2 = \frac{\sin z}{z}$, then $f(z) = \frac{(\sin z)^{1/2}}{z^{1/2}}$. Recall that we define exponentiation in $\mathbb{C}$ to be $z^\alpha = e^{\alpha \log z}$, and so we have that

$$f(z) = \frac{e^{\log(\sin(z))/2}}{e^{\log(z)/2}}$$

$\log(0)$ is never defined so we need to look where $\sin(z) = 0$, which is at $z = \pi n$ for $n \in \mathbb{Z}$. Thus, if we call the radius of convergence R, we see that $R \leq \pi$. $\square$

Proof. Since $f(z)^2 = \frac{\sin z}{z}$ then we have $f(z) = \sqrt{\frac{\sin z}{z}}$ which we interpret as

$$f(z) = e^{\log(\sin z/z)/2}$$

Thus, the obstruction to analyticity occurs dependent on the term $\log(\sin z/z)$. First, note that $\sin z/z$ has a removable singularity at $z = 0$ since $\lim_{z\to 0} \frac{\sin z}{z} = \lim_{z \to 0} \frac{\cos z}{1} = 1$. Thus, $\log(\sin z/z)$ can be holomorphically extended in a simply connected disk centered at $z = 0$ provided $\sin z/z$ is nonzero. Thus, an upper bound for the radius of convergence is π since $\sin(\pi)/\pi = 0$. $\square$

Proof. Suppose that there were a holomorphic function satisfying $f(z)^2 = \sin z/z$ $\square$

Hmm. Interesting. Is this well defined? I think so because complex. Maybe its all of C? Theres definitely a pole at 0, but $z^{1/2}$ is entire, $\sin z$ is entire, so $\sqrt{\sin z}$ is entire.

The power series expansion of $1/z$ converges except at 0, and so $1/\sqrt{z}$ is analytic except at 0. Then the product is analytic so $\sqrt{\frac{\sin z}{z}}$ is analytic.

There is a theorem that if f is nonvanishing on Ω then there is a g so that $f = e^g$.

Does the branch cut force the radius of convergence (since centered at 0) to be $R = 0$?

There is a removable discontinuity at $z = 0$, which can be fixed by $f(0) = \pm 1$. Is this function well defined? if not well defined then certainly can't be convergent.