Fall 2018 Complex Analysis Preliminary Exam

University of Minnesota

Where possible, computations have been also done using SageMath code available on GitHub at github.com/tekaysquared/prelims (feel free to make pull requests!)

1. Tell the values of i^i.

   Proof. Recall that z^i is given by $e^{i \log z}$ on a suitably defined branch of the logarithm. As long as we choose a branch whose branch-cut is not along the positive imaginary axis, we see $\log i = i\pi/2 + 2\pi ik$ since $e^{\pi i/2}e^{2\pi ik} = e^{i\pi/2} = i$

   Thus, for $k \in \mathbb{Z}$, i^i takes on the values of

   $$e^{i \log i} = e^{i(\pi i/2+2\pi ik)} = e^{-\pi/2+2\pi k}$$

   which each lie on the real axis! *Incroyable!*

   □

2. Write the Laurent expansion of $f(z) = \frac{1}{z^4-1}$ centered at 0 and convergent in $|z| > 1$.

   Proof. Factor out z^{-4} to see that

   $$\begin{aligned} f(z) &= \frac{1}{z^4(1-1/z^4)} \\ &= \frac{1}{z^4}\sum_{n=0}^{\infty}\frac{1}{z^{4n}} \end{aligned}$$

   which converges for $|1/z^4| < 1$ which is to say for $|z| > 1$. So then f has a Laurent expansion

   $$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

   where

   $$a_n = \begin{cases} 1 & n = -4k \text{ for nonzero positive integers } k \\ 0 & \text{otherwise} \end{cases}$$

   □

3. Let f be an entire function so that $\Re f(z)$ is *nonnegative* for all $z \in \mathbb{C}$. Show that f is constant.

   Proof. Picard's little theorem states that if $f : \mathbb{C} \to \mathbb{C}$ is entire and nonconstant then the image of f misses at most one point. If $\Re f(z)$ is nonnegative for all z, then $z = -1, -2$ are two points which are not contained in the image, and so f must be constant. □

4. Evaluate $\displaystyle\int_0^\infty \frac{x^{1/5}}{1+x^2}dx$

Proof. Let $R > 1$, and let C_R^+ be the open semicircle of radius R in the upper half plane. Let $\gamma_R = C_R^+ \cup [-R, R]$. Let the orientation of γ_R be counter clockwise.

Let $f(z) = \frac{z^{1/5}}{1+z^2}$. We will compute the integral

$$\int_{\gamma_R} \frac{z^{1/5}}{1+z^2} dx.$$

First, note that f has simple poles at $z = \pm i$. The only pole contained inside of the closed curve γ_R is the pole at $+i$ (since we took $R > 1$). Then by the residue theorem, we know that

$$\int_{\gamma_R} f(z)dz = 2\pi i \operatorname{res}_i f.$$

Since there is a simple pole at i, we may compute $\operatorname{res}_i f = \lim_{z\to i}(z-i)\frac{z^{1/5}}{(z+i)(z-i)} = \frac{i^{1/5}}{2i}$. Substituting this into the equation above, we obtain

$$\int_{\gamma_R} f(z)dz = \pi i^{1/5},$$

independent of R.

Now consider splitting γ_R up into the integral along C_R^+ and $[-R, R]$. Then

$$\int_{\gamma_R} f(z)dz = \int_{C_R^+} f(z)dz + \int_{[-R,R]} f(z)dz$$

Since the length of C_R^+ is πR, we can bound the magnitude of $\int_{C_R^+} f(z)dz$ using the estimation lemma (also called the ML lemma), by

$$\begin{aligned}\left|\int_{C_R^+} f(z)dz\right| &\leq \pi R \max_{z\in C_R^+} \left|\frac{z^{1/5}}{1+z^2}\right| \\ &= \pi R \max_{z\in C_R^+} \frac{R^{1/5}}{|1+z^2|}\end{aligned}$$

and viewing $|\cdot|$ as distance, we obtain $|1+z^2| > |z^2|$ on the upper half plane. This allows us to bound

$$\left|\int_{C_R^+} f(z)dz\right| < \pi R \max_{z\in C_R^+} \frac{R^{1/5}}{|z^2|} = \pi R \max_{z\in C_R^+} \frac{R^{1/5}}{R^2} = \frac{\pi}{R^{4/5}}.$$

Taking the limit $R \to \infty$, we thus see that the portion of the integral along C_R^+ vanishes.

We now turn our attention to the portion of the integral along the real axis.

$$\begin{aligned}\int_{[-R,R]} f(z)dz &= \int_{-R}^0 f(z)dz + \int_0^R f(z)dz \\ &= \int_R^0 f(-z)dz + \int_0^R f(z)dz \\ &= \int_R^0 \frac{(-z)^{1/5}}{1+(-z)^2}dz + \int_0^R f(z)dz \\ &= -\int_0^R \frac{(-z)^{1/5}}{1+z^2}dz + \int_0^R f(z)dz \\ &= -(-1)^{1/5}\int_0^R f(z) + \int_0^R f(z)dz \\ &= \left(-(-1)^{1/5}+1\right)\int_0^R f(z)dz \\ &= \left(-e^{i\pi/5}+1\right)\int_0^R f(z)dz\end{aligned}$$

so taking $R \to \infty$ we see that

$$\pi i^{1/5} = \left(1 - e^{i\pi/5}\right) \int_0^\infty f(z)dz$$

or

$$\frac{\pi e^{\pi i/10}}{1 - e^{i\pi/5}} = \int_0^\infty f(z)dz$$

□

5. Determine the radius of convergence of the power series for $\log z$ at $z_0 = -4 + 3i$.

Proof. Let R denote the radius of convergence of the power series of $\log z$ centered at z_0. Now, note that there is no logarithm which takes a value at 0, since $e^w = 0$ is never true for $w \in \mathbb{C}$. Thus, the power series expansion can converge for a disk of radius at most $|-4 + 3i - 0| = 5$, and so

$$R \leq 5.$$

On the other hand, it is a theorem that if Ω is a simply connected subset of $\mathbb{C}$ which does not contain 0 then there is a branch of the logarithm which is holomorphic on Ω. Observe that the open disk $D_5(-4+3i) := \{z \in \mathbb{C} : |-4+3i-z| < 5\}$ is simply connected and does not contain zero. Thus, there is a branch of the logarithm (call it $\log_{D_5}$) which is holomorphic on D_5. Since we have constructed a disk of radius 5 on which there is a holomorphic logarithm, we see that

$$R \geq 5.$$

Since we have bounded R both above and below by 5, we see that $R = 5$. □

6. By a suitable change of coordinates, write $w^2 = z^4 + 1$ in the Weierstrass form $y^2 = x^3 + ax + b$.

7. Try to define w locally as a holomorphic function of z by the relation $w^5 - 5zw + 1 = 0$. For which z does this fail to some extent.[1]

Proof. The holomorphic inverse theorem states that if $F(z, w)$ is a polynomial and $F(z_0, w_0) = 0$ and $\frac{\partial F}{\partial w}(z_0, w_0) \neq 0$, then there is a holomorphic expression for w in terms of z in a sufficiently small radius of z_0. Thus, we want to find all the points where $F(z, w)$ and $\frac{\partial F}{\partial w}(z, w)$ are simultaneously 0.

We now consider $F(z, w) =: P(w), P'(w) \in \mathbb{C}(z)[w]$. We want to find nontrivial points where they are simultaneously 0. They are certainly simultaneously 0 when their GCD is 0, so we compute it. First,

$$P(w) = \frac{w}{5}P'(w) - 4zw + 1$$

and then

$$P'(w) = q(w)(-4zw + 1) -$$

□

[1] Follow the procedure in section three of P.G.'s notes: http://www-users.math.umn.edu/ garrett/m/complex/notes_2014-15/ERNPRHH.pdf, Example 3.0.1.