Spring 2019 Complex Analysis Preliminary Exam

University of Minnesota

1. Give a conformal mapping from the (open) upper half plane to the slit disk

$$D = \{z \in \mathbb{C} : |z| < 1, z \notin [0, 1]\}$$

Proof. Consider the map $f(z) = \frac{z-i}{z+1} : \mathbb{C} \to \mathbb{C}$. Note that this is a fractional linear transformation. Let H be the open upper half plane. We first want to show that $f(H) = D$.

First, when we restrict f to the real line, we see $\lim_{x\to\pm\infty} f(x) = 1$. We also see that

$$\begin{aligned} f(1) &= \frac{1-i}{1+i} \\ &= \frac{(1-i)^2}{|1+i|^2} \\ &= \frac{1-2i-1}{(\sqrt{2})^2} \\ &= -2i/2 \\ &= -i \end{aligned}$$

and

$$\begin{aligned} f(-1) &= \frac{-1-i}{-1+i} \\ &= \frac{(-1-i)^2}{|-1+i|^2} \\ &= \frac{1+2i-1}{(\sqrt{2})^2} \\ &= +i \end{aligned}$$

finally

$$f(+i) = \frac{i-i}{i+i} = 0.$$

Thus, because fractional linear transformations preserve circles-and-lines, and $+i \in H$ gets mapped to the interior of D, then $f(H) = D$.

Now we show that $f(z)$ is conformal on the open upper half plane. f is conformal where its derivative is nonzero, and so we compute $f' = \frac{(z+1)-(z-i)}{(z+1)^2} = \frac{1-i}{(z+1)^2}$, which is defined except at $z = -1$ (which is not in the open upper half plane so we need not worry) and is nonzero wherever it is defined. Thus, f is a conformal mapping from H to D.

□

2. Write the first three terms of the Laurent expansion of $f(z) = \dfrac{1}{z^5 - 1}$ centered at 0 and convergent in $|z| < 1$.

Proof. Observe that

$$\frac{1}{z^5-1} = \frac{-1}{1-z^5} = -\sum_{n=0}^{\infty} z^{5n}$$

which converges for $|z^5| < 1$ which is $|z|^5 < 1$ or $|z| < 1$. Thus, the first three nonzero terms of the expansion of f are $a_0 = -1$, $a_5 = -1$, and $a_{10} = -1$. $\square$

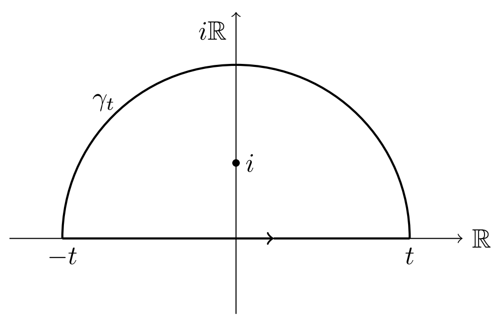
3. Classify entire functions f such that $|f(z)| \leq 1 + \sqrt{|z|}$

4. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x)}{1+x^2} dx$

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = \frac{-ie^{iz}}{1+z^2}$. When we restrict to $x \in \mathbb{R}$, we have $f(x) = \frac{-ie^{ix}}{1+x^2} = \frac{-i(\cos x + i \sin x)}{(z+i)(z-i)}$ and so f has real part $\operatorname{re} f(x) = \frac{\sin(x)}{1+x^2}$. Thus, the real part of the integral $\operatorname{re}\left(\int_{-\infty}^{\infty} f(z)dz\right)$ is the integral we wish to compute.

Note that the numerator of f is entire and the denominator is also entire and is only 0 at $z = \pm i$.

Let $t > 0$ and let γ_t be the curve given by the union of $[-t, t] \subset \mathbb{R} \subset \mathbb{C}$ with the upper half-circle of radius t, with positive orientation. Visually:



Since f is holomorphic on and inside γ_t except at i, which is a simple pole the residue theorem tells us that $\int_{\gamma_t} f(z)dz = 2\pi i \operatorname{res}_i f(z)$.

To compute $\operatorname{res}_i f(z)$, take $\lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{-ie^{iz}}{z+i} = \frac{-i}{e(2i)} = \frac{-1}{2e}$

So then $\int_{\gamma_t} f(z)dz = -\frac{\pi i}{e}$.

We can split the integral into the part along $[-t, t]$ and along C_t, the upper half-circle of radius t, as

$$\int_{\gamma_t} f(z)dz = \int_{[-t,t]} f(z)dz + \int_{C_t} f(z)dz.$$

We can use the estimation lemma to bound the magnitude of integral over the half-circle as

$$|\int_{C_t} f(z)dz| \leq \pi t \sup_{z\in C_t} |f(z)|.$$

To compute

$$\begin{aligned}\sup_{z\in C_t} |f(z)| &= \sup_{z\in C_t} \left|\frac{-ie^{iz}}{1+z^2}\right| \\ &= \sup_{z\in C_t} \frac{|e^{iz}|}{|1+z^2|}\end{aligned}$$

When we consider $z = x + iy \in C_t$, we see that $|e^{iz}| = |e^{i(x+iy)}| = |e^{ix-y}| = e^{-y} \leq 1$ so

$$\begin{aligned}
\sup_{z \in C_t} \frac{|e^{iz}|}{|1+z^2|} &\leq \sup_{z \in C_t} \frac{1}{|1+z^2|} \\
&\leq \sup_{z \in C_t} \frac{1}{||z^2| - |-1||} && \text{(reverse triangle inequality)} \\
&= \sup_{z \in C_t} \frac{1}{||z^2| - 1|} \\
&= \sup_{z \in C_t} \frac{1}{|t^2 - 1|} \\
&= \frac{1}{|t^2-1|}.
\end{aligned}$$

Thus, we have

$$\left| \int_{C_t} f(z)dz \right| \leq \pi t \frac{1}{|t^2-1|}$$

and taking the limit $t \to \infty$ we see

$$\lim_{t \to \infty} \int_{C_t} f(z)dz = 0.$$

Thus, in the limit

$$\lim_{t \to \infty} \int_{\gamma_t} f(z)dz = \lim_{t \to \infty} \int_{[-t,t]} f(z)dz = \int_{\mathbb{R}} f(z)dz.$$

The integral over γ_t was independent of t (thanks residue theorem ☺), so we see

$$\int_{\mathbb{R}} f(z)dz = \frac{-\pi i}{e}.$$

We wanted to compute the real part

$$\int_{\mathbb{R}} \frac{\sin x}{1+x^2} dx = 0.$$

We could also see that $1 + x^2$ is an even function and $\sin(x)$ is an odd function so $\sin(x)/(1+x^2)$ is an odd function, so its integral must be 0, but why do that when we could use the residue theorem ☺. □

5. Determine the radius of convergence for the power series of $\sqrt{z}$ at $z_0 = -3 + 4i$.

Proof. The radius of convergence of the power series of $\sqrt{z}$ is the radius of the largest disk for which there is a holomorphic function which agrees with $\sqrt{z}$. Recall that we define complex exponentiation by $z^\alpha := e^{\alpha \log z}$, so $\sqrt{z} = e^{\log(z)/2}$. Composition of holomorphic functions is holomorphic, so since e^w is entire, the radius of convergence is limited by $\log(z)$.

There is no number $w \in \mathbb{C}$ such that $e^w = 0$, and so there is no possible way to have a holomorphic logarithm at 0. This bounds the radius of convergence by $|-3 + 4i - 0| = 5$.

On the other hand, it is a theorem[1] that if Ω is simply connected and does not contain 0, then there is a branch of the logarithm which is holomorphic on Ω. Consider the open disk D of radius 5 and centered at $-3 + 4i$. Clearly D does not contain 0, and so there is a logarithm (call it $\log_D$) which is holomorphic on D. Thus, we have a disk of radius 5 on which there is a holomorphic function $\log_D$ which agrees with log, and so the radius of convergence is *at least* 5.

Since we know the radius of convergence is both at least 5 and less than or equal to 5, we see that the radius of convergence of the power series for log is in fact 5.

□

8. Define $f(z)$ near 0 by $f(z)^2 = \frac{\sin z}{z}$. What is the radius of convergence of the power series of f at 0.

[1] Theorem 6.1 in Chapter 3 of Stein and Shakarchi's *Complex Analysis*