The goal of this document is to give a general outline of how to solve problems for the Complex Analysis prelim. See the study log for specific old prelim questions relevant to each section.

1 Laurent Series

Suppose we are given a function and are asked to find a Laurent series expansion (or some terms of it) centered at 0 and convergent on some region.

1. Factor the function into pieces which can be easily rewritten as an infinite series, for example a geometric series
$$\frac{1}{1-a}=\sum_{n\geq 0}a^n,$$
or
$$e^z=\sum_{n\geq 0}\frac{z^n}{n!}.$$
Historically, sin and cos have not been included, but I'll record them here just in case. If you are given something like $\frac{1}{z-1}$ there are two things you could do. More on this to follow.

2. Rewrite the function as a product of infinite series

3. Check the regions on which the series converges.[1] For example if $f(z)=\frac{1}{z-1}$ we could rewrite as either $f=\frac{-1}{1-z}$ or $f=\frac{1}{z(1-1/z)}$. The first way converges for $|z|<1$. The second way converges for $|1/z|<1$ or $|z|>1$. Also remember that region of convergence of a product of series is the intersection of their individual regions of convergence, so if $\sum_{n\geq 0}a_nz^n$ converges for $|z|>1$ and $\sum_{n\geq 0}b_nz^n$ converges for $|z|<2$, then the product
$$\left(\sum_{n\geq 0}a_nz^n\right)\left(\sum_{n\geq 0}b_nz^n\right)$$
converges for $1<|z|<2$.

4. If the region is the one which was requested, write the series. If you are only asked for a few terms, it can be helpful to write out the terms as a sum, for example
$$\frac{1}{z}(a_0+a_1z+a_2z^2+\cdots)(b_0+b_1z+b_2z^2+\cdots)$$
or
$$(a_0+a_1z+a_2z^2+\cdots)(b_0+\frac{b_1}{z}+\frac{b_2}{z^2}+\cdots).$$
In the first case, recall multiplication of power series given by
$$\sum a_nz^n\sum b_nz^n=\sum(\sum_{k=0}^{n}a_nb_{k-n})z^n.$$
Also, don't forget to include the factor of $\frac{1}{z}$ out front.
For the second case, each coefficient will be itself an infinite sum. For this example, the coefficient of $\frac{1}{z}$ is $\sum a_nb_{n+1}$, the constant coefficient is $\sum a_nb_n$, and the coefficient of z will be $\sum a_{n+1}b_n$. Historically, the coefficients have been a geometric series, and so can be actually computed using $\sum r^n=\frac{1}{1-r}$. An older example also had a copy of $e=\sum\frac{1}{n!}$ hidden in there.

[1] Checking convergence is important because in the past different regions have been requested. For example Spring 2019 #2 and Fall 2018 #2 gave functions of the form $\frac{1}{z^n-1}$ asked for convergence on $|z|<1$ and $|z|>1$ respectively. The different regions of convergence changes the answer significantly. For more see StackExchange.

2 Power series

Suppose we are given a function $f(z)$ and asked to find the radius of convergence for its power series centered at some given point z_0. Some good facts to know are:

a) The radius of convergence is the radius of the largest disk centered at z_0 on which there is a holomorphic function agreeing with the given function.

b) If $\Omega \subset \mathbb{C}$ such that Ω is simply connected (homotopic to a point) and $0 \notin \Omega$ then there is a branch of the logarithm (call it $\log_\Omega$) which is holomorphic on Ω.[2]

c) **Riemann's theorem on removable discontinuities** states (among other things) that if $D \subseteq \mathbb{C}$ is open and $a \in D$, then if f is a function which is holomorphic on $D - \{a\}$, then f is continuously extendable if and only if it is holomorphically extendable. There are also other conclusions of the theorem, but the two above conclusions are the ones which are relevant to this type of problem.

The general idea is to first bound the size of the disk from above, then from below. One way to bound from below is to explicitly construct a holomorphic function on a disk of the desired size which agrees with given function, by using (b) and (c) above. One way to bound from above is to find a pole or essential singularity, and compute the distance between a pole or essential singularity and the given point z_0.

3 Complex nth roots

4 Conformal mapping

The important theorem for something to be conformal is that it f conformal when f' is nonzero. For a great lecture about this, see the Herb Gross video Complex Variables: Lec 3. Conformal Mappings. Professor Gross goes into great depth about the intuition of why angles are preserved. It is a bit hand-wavy but very digestible.

It appears that a lately a PG favorite is the so-called "Cayley map" which is a conformal mapping between the open upper half plane H and the unit disk D. The Cayley map defined by $f(z) = \frac{z+i}{iz+1}$ gives a conformal mapping from D to H and $g(z) = \frac{z-i}{z+i}$ gives a conformal mapping from H to D.

Probably don't try to recreate the Cayley map, just memorize it.

5 Show a function is constant

There are a few different tools to show an entire function is constant based on the information you are given.

1. We know that an entire function $f(z) = f(x, y) = u(x, y) + iv(x, y)$ satisfies the **Cauchy-Riemann equations**,
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

2. **Liouville's theorem** states that every bounded entire function is constant.

3. The **open mapping theorem** states that non-constant holomorphic functions are open maps (i.e. they send open sets to open sets). If you can show that the image under a holomorphic function of an open set is closed, then the function must be constant.

[2]Theorem 6.1.i in Chapter 3 of Stein and Shakarchi's *Complex Analysis*. Note that the theorem also includes the assumption that $1 \in \Omega$, but this assumption is used to prove parts ii, iii of the theorem, not part i.