Spring 2019 Complex Analysis Preliminary Exam

University of Minnesota

1. Give a conformal mapping from the (open) upper half plane to the slit disk

$$D = \{z \in \mathbb{C} : |z| < 1, z \notin [0,1]\}$$

Proof. Consider the map $f(z) = \frac{z-i}{z+1} : \mathbb{C} \to \mathbb{C}$. Note that this is a fractional linear transformation.

Let H be the open upper half plane. We first want to show that $f(H) = D$.

First, when we restrict f to the real line, we see $\lim_{x\to\pm\infty} f(x) = 1$. We also see that

$$\begin{aligned} f(1) &= \frac{1-i}{1+i} \\ &= \frac{(1-i)^2}{|1+i|^2} \\ &= \frac{1-2i-1}{(\sqrt{2})^2} \\ &= -2i/2 \\ &= -i \end{aligned}$$

and

$$\begin{aligned} f(-1) &= \frac{-1-i}{-1+i} \\ &= \frac{(-1-i)^2}{|-1+i|^2} \\ &= \frac{1+2i-1}{(\sqrt{2})^2} \\ &= +i \end{aligned}$$

finally

$$f(+i) = \frac{i-i}{i+i} = 0.$$

Thus, because fractional linear transformations preserve circles-and-lines, and $+i \in H$ gets mapped to the interior of D, then $f(H) = D$.

Now we show that $f(z)$ is conformal on the open upper half plane. f is conformal where its derivative is nonzero, and so we compute $f' = \frac{(z+1)-(z-i)}{(z+1)^2} = \frac{1-i}{(z+1)^2}$, which is defined except at $z = -1$ (which is not in the open upper half plane so we need not worry) and is nonzero wherever it is defined. Thus, f is a conformal mapping from H to D.

□

2. Write the first three terms of the Laurent expansion of $f(z) = \dfrac{1}{z^5 - 1}$ centered at 0 and convergent in $|z| < 1$.

Proof. Observe that

$$\frac{1}{z^5 - 1} = \frac{-1}{1 - z^5} = -\sum_{n=0}^{\infty} z^{5n}$$

which converges for $|z^5| < 1$ which is $|z|^5 < 1$ or $|z| < 1$. Thus, the first three nonzero terms of the expansion of f are $a_0 = -1$, $a_5 = -1$, and $a_{10} = -1$. □

3. Classify entire functions f such that $|f(z)| \leq 1 + \sqrt{|z|}$

Proof. We appeal to Cauchy's inequality which states that if f is entire, and γ_R is any circle of radius R centered at the origin, then

$$|f^{(n)}(0)| \leq \frac{n! \sup_{z \in \gamma_R} f(z)}{R^n}.$$

Since f is entire, pick a power series representation centered at 0, and call it $f(z) =: \sum_{n \geq 0} \alpha_n z^n$. Then $\alpha_n = \frac{f^{(n)}(0)}{n!}$, and so

$$\begin{aligned} |\alpha_n| &\leq \frac{\sup_{z \in \gamma_R} f(z)}{R^n} && \text{(By Cauchy's ineq.)} \\ &\leq \frac{\sup_{z \in \gamma_R} (1 + \sqrt{|z|})}{R^n} && \text{(Bound provided)} \\ &= \frac{1 + \sqrt{R}}{R^n} && (|z| \text{ is constant on } \gamma_R). \end{aligned}$$

Since this must hold for all R and all n, we take the limit

$$|\alpha_n| \leq \lim_{R \to \infty} \frac{1 + \sqrt{R}}{R^n} = \begin{cases} 0 & n \geq 1 \\ \infty & n = 0 \end{cases}$$

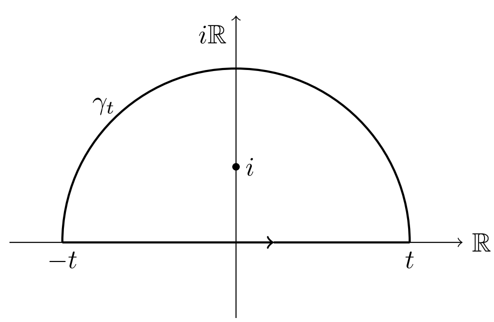
which forces f to be a constant function. Note now that if f is constant, then $f(z) = f(0)$ for all z, and so the provided bound that $|f(z)| \leq 1 + \sqrt{|z|}$ tells us that $|f(z)| \leq 1$. So we conclude that $f(z)$ is a constant function taking a value inside the unit disk. □

4. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x)}{1 + x^2} dx$

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = \frac{-ie^{iz}}{1+z^2}$. When we restrict to $x \in \mathbb{R}$, we have $f(x) = \frac{-ie^{ix}}{1+z^2} = \frac{-i(\cos x + i \sin x)}{(z+i)(z-i)}$ and so f has real part $\operatorname{re} f(x) = \frac{\sin(x)}{1+x^2}$. Thus, the real part of the integral $\operatorname{re}\left(\int_{-\infty}^{\infty} f(z) dz\right)$ is the integral we wish to compute.

Note that the numerator of f is entire and the denominator is also entire and is only 0 at $z = \pm i$.

Let $t > 0$ and let γ_t be the curve given by the union of $[-t, t] \subset \mathbb{R} \subset \mathbb{C}$ with the upper half-circle of radius t, with positive orientation. Visually:



Since f is holomorphic on and inside γ_t except at i, which is a simple pole the residue theorem tells us that $\int_{\gamma_t} f(z)dz = 2\pi i \operatorname{res}_i f(z)$.

To compute $\operatorname{res}_i f(z)$, take $\lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{-ie^{iz}}{z+i} = \frac{-i}{e(2i)} = \frac{-1}{2e}$

So then $\int_{\gamma_t} f(z)dz = -\frac{\pi i}{e}$.

We can split the integral into the part along $[-t,t]$ and along C_t, the upper half-circle of radius t, as

$$\int_{\gamma_t} f(z)dz = \int_{[-t,t]} f(z)dz + \int_{C_t} f(z)dz.$$

We can use the estimation lemma to bound the magnitude of integral over the half-circle as

$$|\int_{C_t} f(z)dz| \leq \pi t \sup_{z\in C_t} |f(z)|.$$

To compute

$$\begin{aligned}\sup_{z\in C_t} |f(z)| &= \sup_{z\in C_t} \left|\frac{-ie^{iz}}{1+z^2}\right| \\ &= \sup_{z\in C_t} \frac{|e^{iz}|}{|1+z^2|}\end{aligned}$$

When we consider $z = x+iy \in C_t$, we see that $|e^{iz}| = |e^{i(x+iy)}| = |e^{ix-y}| = e^{-y} \leq 1$ so

$$\begin{aligned}\sup_{z\in C_t} \frac{|e^{iz}|}{|1+z^2|} &\leq \sup_{z\in C_t} \frac{1}{|1+z^2|} \\ &\leq \sup_{z\in C_t} \frac{1}{||z^2|-|-1||} && \text{(reverse triangle inequality)} \\ &= \sup_{z\in C_t} \frac{1}{||z^2|-1|} \\ &= \sup_{z\in C_t} \frac{1}{|t^2-1|} \\ &= \frac{1}{|t^2-1|}.\end{aligned}$$

Thus, we have

$$\left|\int_{C_t} f(z)dz\right| \leq \pi t \frac{1}{|t^2-1|}$$

and taking the limit $t\to\infty$ we see

$$\lim_{t\to\infty} \int_{C_t} f(z)dz = 0.$$

Thus, in the limit

$$\lim_{t\to\infty} \int_{\gamma_t} f(z)dz = \lim_{t\to\infty} \int_{[-t,t]} f(z)dz = \int_{\mathbb{R}} f(z)dz.$$

The integral over γ_t was independent of t (thanks residue theorem ☺), so we see

$$\int_{\mathbb{R}} f(z)dz = \frac{-\pi i}{e}.$$

We wanted to compute the real part

$$\int_{\mathbb{R}} \frac{\sin x}{1+x^2} dx = 0.$$

We could also see that $1+x^2$ is an even function and $\sin(x)$ is an odd function so $\sin(x)/(1+x^2)$ is an odd function, so its integral must be 0, but why do that when we could use the residue theorem ☺. □

5. Determine the radius of convergence for the power series of $\sqrt{z}$ at $z_0 = -3+4i$.

Proof. The radius of convergence of the power series of $\sqrt{z}$ is the radius of the largest disk for which there is a holomorphic function which agrees with $\sqrt{z}$. Recall that we define complex exponentiation by $z^\alpha := e^{\alpha \log z}$, so $\sqrt{z} = e^{\log(z)/2}$. Composition of holomorphic functions is holomorphic, so since e^w is entire, the radius of convergence is limited by $\log(z)$.

There is no number $w \in \mathbb{C}$ such that $e^w = 0$, and so there is no possible way to have a holomorphic logarithm at 0. This bounds the radius of convergence from above by $|-3+4i-0| = 5$.

On the other hand, it is a theorem[1] that if Ω is simply connected and does not contain 0, then there is a branch of the logarithm which is holomorphic on Ω. Consider the open disk D of radius 5 and centered at $-3+4i$. Clearly D does not contain 0, and so there is a logarithm (call it $\log_D$) which is holomorphic on D. Thus, we have a disk of radius 5 on which there is a holomorphic function $\log_D$ which agrees with log, and so the radius of convergence is *at least* 5.

Since we know the radius of convergence is both at least 5 and less than or equal to 5, we see that the radius of convergence of the power series for log is in fact 5.

□

6. Let f, g be holomorphic functions on $\{z : |z| < 2\}$ with f nonvanishing on $|z| = 1$. Show that for all sufficiently small $\epsilon > 0$ the function $f + \epsilon g$ has the same number of zeros inside $|z| = 1$ as does f.

Proof. Since f, g are holomorphic on $\{z : |z| < 2\}$, they are holomorphic on the compact sets $D = \{z : |z| \leq 1\}$ and its boundary $\partial D = \{z : |z| = 1\}$.

Rouche's theorem states that if $|\epsilon g| \leq |f|$ on ∂D (which can be thought of as a closed curve) then f and $f + \epsilon g$ have the same number of zeros inside D. Thus our goal will be to find $\epsilon > 0$ which establishes this bound.

Since f, g are holomorphic on ∂D, they are continuous, and since ∂D is a closed subset of $\mathbb{C}$, it is compact. The modulus function is also continuous, and so by composition, $|f|, |g|$ are both continuous real-valued functions and thus achieve a maximum and minimum on ∂D.

Let $m = \min_{z \in \partial D} f$ and $M = \max_{z \in \partial D} g$. Pick $\epsilon < \frac{m}{M}$. Then on ∂D

$$\begin{aligned}
|\epsilon g| &= \epsilon |g| \\
&< \frac{m}{M}|g| \\
&\leq \frac{m}{M} M \\
&= m \leq |f|.
\end{aligned}$$

□

7. Prove that $z^3 + w^3 = 1$ defines an elliptic curve.

We provide two proofs, one using the *genus-degree formula*, a corollary to the *Riemann-Hurwitz theorerm on ramified coverings*, and the other using the Riemann-Hurwitz theorem itself.

Proof. (Genus-degree formula). The genus-degree formula states that the genus of a smooth plane curve is given by $\frac{(d-1)(d-2)}{2}$, where d is the degree of the polynomial defining the curve.

We can homogenize this curve as the locus of points $z^3 + w^3 - u^3 = 0$, and we will specialize $u \to 1$. To check that this is smooth, we compute all partial derivatives, $\frac{\partial}{\partial z} = 3z^2$, $\frac{\partial}{\partial w} = 3w^2$ and $\frac{\partial}{\partial u} = -3u^2$. The curve is not smooth when all three partial derivatives are simultaneously 0, which occurs at the point $[0:0:0]$. But such a point is not on our curve, since we want $u = 1$ always. Thus, we can conclude we are dealing with a smooth curve of degree 3, and so the genus-degree formula tells us that

$$g = \frac{(3-1)(3-2)}{2} = 1.$$

Since the curve has genus 1, it is an elliptic curve. □

[1]Theorem 6.1 in Chapter 3 of Stein and Shakarchi's *Complex Analysis*

Proof. (Riemann-Hurwitz theorem). We will show the equivalent statement that the same curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ has genus 1. We will thus abuse notation and identify points $z \in C$ with their line through the origin, and write the point at infinity as ∞. Consider the ramified covering given by $(z, w) \mapsto z$, where (z, w) satisfy $z^3 + w^3 - 1 = 0$. Since w^3 is the highest power of w, the degree of the ramified covering is 3. There are three distinct holomorphic cube root functions via which we can express z except when $w^3 - 1 = 0$, since the cube root is not holomorphic at the origin. This occurs at $1, \xi = e^{i2\pi/3}, \xi^2$. We now want to write $z^3 + w^3 - 1$ as a monic polynomial in $\mathbb{C}[w][z]$. We can write $z^3 + w^3 - 1 = \sum_{i=0}^{3} c_i(w) z^i$, and so $c_0 = w^3 - 1$, $c_1 = c_2 = 0$, and $z^3 = 1$. We can thus compute the order of vanishing of each of them at $w = 1$. The order of vanishing of c_0 is 1. The order of vanishing of $c_1 = c_2 = \infty$, and the order of vanishing of c_3 at $w = 1$ is 0. The same computation holds for $w = \xi, \xi^2$. Thus, to compute the ramification index, we can construct the Newton polytope with vertices

$$(0,0), (1,\infty), (2,\infty), (3,1).$$

The three lines we care about are the ones connecting $(0,0) \leftrightarrow (1,\infty)$, $(2,\infty) \leftrightarrow (3,1)$, and $(0,0) \leftrightarrow (3,1)$. First we check if there is a ramification at ∞ by inverting coordinates and looking near 0. Inverting coordinates yeilds

$$\begin{aligned}
\frac{1}{z^3} + \frac{1}{w^3} &= 1 \\
w^3 + z^3 &= w^3 z^3 \\
z^3 &= w^3(z^3 - 1) \\
\frac{z^3}{z^3 - 1} &= w^3.
\end{aligned}$$

In a neighbor of zero there are three distinct cube root functions we could use to solve for w, and so there is no ramification at ∞. Then we are only left with the line $(0,0) \leftrightarrow (3,1)$, which has slope $1/3$ and so the ramification index is 3. Thus, we have a set of ramification points given by $\{1, \xi, \xi^2\}$, and each of them has ramification index 3. The Riemann-Hurwitz formula relates the genus of $Y = \{(z, w) : z^3 + w^3 = 1\}$ to the genus of $\mathbb{CP}^1$ by

$$2g_Y - 2 = n(2g_{\mathbb{CP}^1} - 2) + \sum_{z \in \{1, \xi, \xi^2\}} (e_z - 1)$$

where n is the degree of the covering, so we get that

$$\begin{aligned}
2g_Y - 2 &= 3(2 \cdot 0 - 2) + 3 \cdot (3 - 1) \\
&= -6 + 6 = 0 \\
\Rightarrow g_y &= 1
\end{aligned}$$

Since the genus is 1, Y is an elliptic curve. □

8. Define $f(z)$ near 0 by $f(z)^2 = \frac{\sin z}{z}$. What is the radius of convergence of the power series of f at 0.

Proof. Recall that we define exponentiation in $\mathbb{C}$ to be $z^\alpha = e^{\alpha \log z}$, and so we have that

$$f(z) = \left(\frac{\sin z}{z}\right)^{1/2} = e^{\frac{1}{2} \log \frac{\sin z}{z}}.$$

So since e^z is entire, the radius of convergence is the same radius of convergence of $\log \frac{\sin z}{z}$. There is a branch of log which is holomorphic on any simply-connected domain not containing 0, so consider the domain $\Omega = \{z \in \mathbb{C} : |z| < \pi\}$. Although it is morally reprehensible,[2] the radius of convergence does not include the point at which the series is centered, and so the fact that $\sin z \neq 0$ for all $z \in \Omega$ means that the logarithm is well defined on Ω, and thus radius of convergence (call it R) is at least π. On the other hand, any disc of radius larger than π which is centered at 0 contains π, and since $\sin(\pi) = 0$, then the logarithm is not well defined on that disk, and so the radius of convergence is π. □

[2]Perhaps "morally reprehensible" is a bit extreme in this case, since it is easy to compute an analytic continuation of $\sin(z)/z$ by squinting at the power series.