Spring 2019 Complex Analysis Preliminary Exam

University of Minnesota

1. Give a conformal mapping from the (open) upper half plane to the slit disk

$$D = \{z \in \mathbb{C} : |z| < 1, z \notin [0,1]\}$$

Proof. Consider the map $f(z) = \frac{z-i}{z+1} : \mathbb{C} \to \mathbb{C}$. Note that this is a fractional linear transformation.

Let H be the open upper half plane. We first want to show that $f(H) = D$.

First, when we restrict f to the real line, we see $\lim_{x \to \pm\infty} f(x) = 1$. We also see that

$$\begin{aligned} f(1) &= \frac{1-i}{1+i} \\ &= \frac{(1-i)^2}{|1+i|^2} \\ &= \frac{1-2i-1}{(\sqrt{2})^2} \\ &= -2i/2 \\ &= -i \end{aligned}$$

and

$$\begin{aligned} f(-1) &= \frac{-1-i}{-1+i} \\ &= \frac{(-1-i)^2}{|-1+i|^2} \\ &= \frac{1+2i-1}{(\sqrt{2})^2} \\ &= +i \end{aligned}$$

finally

$$f(+i) = \frac{i-i}{i+i} = 0.$$

Thus, because fractional linear transformations preserve circles-and-lines, and $+i \in H$ gets mapped to the interior of D, then $f(H) = D$.

Now we show that $f(z)$ is conformal on the open upper half plane. f is conformal where its derivative is nonzero, and so we compute $f' = \frac{(z+1)-(z-i)}{(z+1)^2} = \frac{1-i}{(z+1)^2}$, which is defined except at $z = -1$ (which is not in the open upper half plane so we need not worry) and is nonzero wherever it is defined. Thus, f is a conformal mapping from H to D.

$\square$

2. Write the first three terms of the Laurent expansion of $f(z) = \dfrac{1}{z^5 - 1}$ centered at 0 and convergent in $|z| < 1$.

   Proof. Observe that
$$\frac{1}{z^5 - 1} = \frac{-1}{1 - z^5} = -\sum_{n=0}^{\infty} z^{5n}$$
   which converges for $|z^5| < 1$ which is $|z|^5 < 1$ or $|z| < 1$. Thus, the first three nonzero terms of the expansion of f are $a_0 = -1$, $a_5 = -1$, and $a_{10} = -1$. □

5. Determine the radius of convergence for the power series of $\sqrt{z}$ at $z_0 = -3 + 4i$.

   Proof. The radius of convergence of the power series of $\sqrt{z}$ is the radius of the largest disk for which there is a holomorphic function which agrees with $\sqrt{z}$. Recall that we define complex exponentiation by $z^\alpha := e^{\alpha \log z}$, so $\sqrt{z} = e^{\log(z)/2}$. Composition of holomorphic functions is holomorphic, so since e^w is entire, the radius of convergence is limited by $\log(z)$.

   There is no number $w \in \mathbb{C}$ such that $e^w = 0$, and so there is no possible way to have a holomorphic logarithm at 0. This bounds the radius of convergence by $|-3 + 4i - 0| = 5$.

   On the other hand, it is a theorem[1] that if Ω is simply connected and does not contain 0, then there is a branch of the logarithm which is holomorphic on Ω. Consider the open disk D of radius 5 and centered at $-3 + 4i$. Clearly D does not contain 0, and so there is a logarithm (call it $\log_D$) which is holomorphic on D. Thus, we have a disk of radius 5 on which there is a holomorphic function $\log_D$ which agrees with log, and so the radius of convergence is *at least* 5.

   Since we know the radius of convergence is both at least 5 and less than or equal to 5, we see that the radius of convergence of the power series for log is in fact 5.

   □

8. Define $f(z)$ near 0 by $f(z)^2 = \frac{\sin z}{z}$. What is the radius of convergence of the power series of f at 0.

[1] Theorem 6.1 in Chapter 3 of Stein and Shakarchi's *Complex Analysis*