Spring 2018 Complex Analysis Preliminary Exam

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Where possible, computations have been also done using SageMath code available on GitHub at github.com/tekaysquared/prelims (feel free to make pull requests!)

1. Write the first three terms of the Laurent expansion of $f(z) = \dfrac{1}{z(z-1)(z-2)}$ centered at 0 and convergent in $|1| < z < |2|$

   Proof. The core idea of the computation is to split the function into a product of power series. First, we observe that
$$\frac{1}{z-1} = \frac{1}{z(1-1/z)}$$
and see the geometric series
$$\frac{1}{1-1/z} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n,$$
which converges for $|1/z| < 1$, or equivalently $|z| > 1$. Similarly we see that
$$\frac{1}{z-2} = \frac{-1}{2(1-z/2)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$
for $|z/2| < 1$, which is to say for $|z| < 2$. Thus we have
$$\begin{aligned} f(z) &= \frac{1}{z}\left(\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right)\left(\frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right) \\ &= \frac{-1}{2z}\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right). \end{aligned}$$
Note that the above product converges when each term converges, which is to say on the annulus $1 < |z| < 2$.

   Now note that the coefficient of z^{-1} of the Laurent expansion is
$$\begin{aligned} -\frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) &= \frac{-1}{2}\left[\sum_{n\geq 0}(1/2)^n - 1\right] \\ &= -\frac{1}{2}\left(\frac{1}{1-1/2} - 1\right) \\ &= -\frac{1}{2}. \end{aligned}$$

   The coefficient of z^0 is
$$-\frac{1}{2}\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right) = -\frac{1}{2}\left(2 - 1 - \frac{1}{2}\right) = -1/4$$

The coefficient of z^1 is

$$-\frac{1}{2}\left(\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots\right) = -\frac{1}{2}\left(2-1-\frac{1}{2}-\frac{1}{4}\right)$$
$$= -\frac{1}{8}.$$

Therefore

$$f(z) = \cdots - \frac{1}{2z} - \frac{1}{4} - \frac{z}{8} + \cdots$$

□

Note that there is also a Laurent series which converges for the annulus $0 < |z| < 1$. This can be found by using the geometric series expansion

$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n$$

which of course converges for $|z| < 1$, and using the same expansion of $\frac{1}{z-2}$ as above. The expansion which converges on the punctured unit disk is the one provided by SageMath. For another example of this, see this math StackExchange post.

2. Let f be an entire function so that $\Re f(z)$ is *bounded*. Show f is constant.

Proof. Let f be entire so it admits a power series representation

$$f(z) = \sum_{n\geq 0} \alpha_n z^n.$$

Let γ_R be a circle of radius R centered at the origin. The Cauchy inequality tells us that

$$|\alpha_n| = \frac{|f^{(n)}(0)|}{n!}$$
$$\leq \frac{\max_{z\in\gamma_R}|f|}{R^n}.$$

Now note that for any z we have $|f(z)| \leq \Re f(z) + \Im f(z)$ (the triangle inequality). This implies that, letting B be an explicit bound so that $|\Re f(z)| < B$ for all z.

$$\frac{\max_{z\in\gamma_R}|f|}{R^n} \leq \frac{\max_{z\in\gamma_R}(\Re f(z) + \Im f(z))}{R^n}$$
$$\leq \frac{\max_{z\in\gamma_R}(B + \Im f(z))}{R^n}$$
$$= \frac{B + \max_{z\in\gamma_R}\Im f(z)}{R^n}$$
$$= \frac{B+R}{R^n}.$$

Note that this holds for any R. So for any $n > 1$, we have

$$|\alpha_n| \leq \lim_{R\to\infty} \frac{B+R}{R^n} = 0.$$

This means that the power series representation for f can be simplified to

$$f(z) = \alpha_0 + \alpha_1 z.$$

But linear functions are unbounded, and so we know α_1 must be zero. Hence $f(z) = \alpha_0$ is a constant function. □

3. Evaluate $\displaystyle\int_0^\infty \frac{\cos tx}{1+x^2}dx$ for $t \in \mathbb{R}$.

Proof. Let $R > 1$ be a real number and let γ_R be the curve in the upper half plane defined by

$$[-R, R] \cup \{z \in \mathbb{C} : |z| = R, \Im(z) > 0\}.$$

Let $\gamma'_R := \{z \in \mathbb{C} : |z| = R, \Im(z) > 0\}$. Let

$$f(z) := \frac{e^{itz}}{1+z^2}$$

note that for $z \in \mathbb{R}$, f agrees with the requested integral, since Euler's formula tells us that $e^{itz} = \cos(tz) + i \sin t(z)$ for $z \in \mathbb{R}$.

Now note that f has simple poles at $z = \pm i$. Thus, the residue theorem tells us that

$$\int_{\gamma_R} f(z)dz = 2\pi i \operatorname{res}_i f$$

since $-i$ is not inside γ_R. Since the pole at i is simple, we can compute

$$\operatorname{res}_i f = \lim_{z \to i}(z-i)f(z) = \frac{e^{iti}}{2i} = \frac{e^{-t}}{2i}$$

so we get that

$$\int_{\gamma_R} f(z)dz = \pi e^{-t}.$$

Since γ_R is the union of $[-R, 0) \cup [0, R] \cup \gamma'$, we may split the integral

$$\int_{\gamma_R} f(z)dz = \int_{[-R,0)} f(z)dz + \int_{[0,R]} f(z)dz + \int_{\gamma'_R} f(z)dz.$$

By the estimation lemma, we know that

$$\left| \int_{\gamma'_R} f(z)dz \right| \le L \max_{z \in \gamma'_R} |f(z)|$$

where $L = \pi R$ is the length of γ'_R. We now compute

$$\begin{aligned}
\max_{z \in \gamma'_R} |f(z)| &= \max_{z \in \gamma'_R} \frac{|e^{itz}|}{|1+z^2|} \\
&= \max_{z \in \gamma'_R} \frac{e^{-t\Im z}}{|1+z^2|} \\
&= \max_{z \in \gamma'_R} \frac{e^{-t\Im z}}{|z+i||z-i|}
\end{aligned}$$

Now we collect the following inequalities:

$$e^{-t\Im z} \le 1 \forall z \in \gamma'_R$$

and

$$|z+i| \ge R \; \forall z \in \gamma'_R$$

which can be seen by considering $|z+i|$ as the distance between $z \in \gamma'_R$ and $-i$ which is clearly at least R (the distance is minimized when $z = R$). Similarly

$$|z-i| \ge R \; \forall z \in \gamma'_R$$

which is the distance between z and $+i$, minimized when $z = iR$. Combining these inequalities, we see that

$$\max_{z\in\gamma'_R} f(z) \le \frac{1}{R^2}.$$

Thus, we can bound the value of the integral on the upper half circle as

$$|\int_{\gamma'_R} f(z)| \le \frac{\pi}{R}.$$

So as the radius tends to infinity, the integral vanishes.

Now, since along the real axis, $f(z) = \frac{\cos tz}{1+z^2}$ we have that it is an even function so

$$2\int_{[0,R]} f(z)dz = \int_{[-R,0)} f(z)dz + \int_{[0,R]} f(z)dz.$$

Thus, we can rewrite the integral which we had previously broken up as

$$\pi e^{-t} = \int_{\gamma_R} f(z)dz = 2\int_{[0,R]} f(z)dz + \int_{\gamma'_R} f(z)dz \le 2\int_{[0,R]} f(z)dz + \frac{\pi}{R}.$$

Taking the limit as $R \to \infty$, we see that

$$\pi e^{-t} = 2\int_{[0,\infty]} f(z)dz + 0.$$

or that

$$\frac{\pi e^{-t}}{2} = \int_0^\infty \frac{\cos(tx)}{1+x^2}dx$$

□

4. Determine the radius of convergence of the power series for $\frac{z}{1-e^z}$ at $z = 0$.

Proof. Let R denote the radius of convergence of $f(z) := \frac{z}{1-e^z}$ at $z = 0$.

R is the largest value such that there is a holomorphic function on $D_R = \{z : |z| < R\}$ which agrees with of f on $D_R\backslash\{0\}$ Observe that by L'Hôpitals rule

$$\lim_{z\to 0}\frac{z}{1-e^z} \overset{L'H}{=} \lim_{z\to 0}\frac{1}{-e^z} = -1$$

and so f is continuously extendable at $z = 0$.

Further observe that $f(z)$ is holomorphic in the punctured open disk $D = \{z : 0 < |z| < 2\pi\}$.

Since we can extend f to be continuous at $z = 0$ by taking $f(0) := -1$, Riemann's theorem on removable singularities tells us that in fact this extension is holomorphic on $D \cup \{0\}$. Thus, since the radius of $D \cup \{0\}$ is 2π, then

$$R \ge 2\pi. \tag{1}$$

On the other hand, $\lim_{z\to 2\pi i} f(z) = \infty$ and so there is no *holomorphic* function which agrees with $f(z)$ at $2\pi i$. Thus,

$$R \le 2\pi. \tag{2}$$

Combining (1) and (2) we see that $R = 2\pi$. □

5. Show that there is a holomorphic function f on the region $1 < |z| < 2$ such that $f(z)^2 = (z^2-1)(z^2-4)$.

6. Prove that $w^2 = z^4 + 1$ defines an elliptic curve.

Proof. It will suffice to show that the same curve in $\mathbb{CP}^1$ has genus one. There is a natural inclusion of $\mathbb{C} \hookrightarrow \mathbb{CP}^1$, given by $z \mapsto \mathbb{C}\begin{pmatrix} z \\ 0 \end{pmatrix}$, and we can consider the point at infinity to be written $\begin{pmatrix} z \\ 0 \end{pmatrix}$. In this manner, we can identify points and write the elements of $\mathbb{CP}^1$ as $\mathbb{C} \cup \{\infty\}$. Now consider the covering of $\mathbb{CP}^1$ given by $(z, w) \mapsto z$ where the pair (z, w) satisfies $w^2 - z^4 - 1 = 0$. This cover has degree 2, since the highest exponent of w is 2. There are two distinct square roots of $z^4 + 1$ except in a neighborhood of roots of $z^4 + 1$, which are $\xi = e^{i\pi/2}, \xi^2, \xi^3, \xi^4$.

We now consider $w^2 - z^4 - 1$ as a polynomial in $(\mathbb{C}[z])[w]$. We now compute the order of vanishing at each coefficient at ξ, ξ^2, ξ^3, ξ^4. Since each root of $z^4 + 1$ has multiplicity one, the order of vanishing of the constant coefficient $z^4 + 1$ is one for all powers of ξ. The coefficient of w is 0 and so its order of vanishing is ∞, and it does not depend on ξ. Finally, the order of vanishing of the coefficient $1 \in \mathbb{C}[z]$ of w^2 is 0, which does not depend on ξ.

Thus, we obtain a newton polytope with vertices at $(0, 0), (1, \infty), (2, 1)$ for each. To check if there is a ramification at ∞, we invert coordinates, so

$$\begin{aligned} \frac{1}{w^2} &= \frac{1}{z^4} + 1 \\ z^4 &= w^2 + w^2 z^4 \\ \frac{z^4}{1 + z^4} &= w^2. \end{aligned}$$

There are two distinct solutions, namely $\frac{\pm z^4}{1+z^4}$ so there is not a ramification at ∞.

Away from ∞ we have a slope of $\frac{1}{2}$ which tells us the ramification index $e_\xi = e_{\xi^2} = e_{\xi^3} = e_{\xi^4} = 2$.

Then the Riemann-Hurwitz formula relates the genus g_Y of $Y := \{(z, w) : w^2 = z^4 + 1\}$ to the genus $g_{\mathbb{CP}^1}$ of $\mathbb{CP}^1$ (which is what is being covered) by

$$2g_Y - 2 = n(2g_{\mathbb{CP}^1} - 2) + \sum_{z \in \{\xi, \xi^2, \xi^3, \xi^4\}} (e_z - 1)$$

where $n = 2$ is the degree of the covering map and $e_z = 2$ is the index of ramification. Thus,

$$\begin{aligned} 2g_Y - 2 &= 2(2 \cdot 0 - 2) + 4(2 - 1) \\ &= -4 + 4 = 0 \\ 2g_Y &= 2 \\ \Rightarrow g_Y &= 1. \end{aligned}$$

Thus, the locus Y is an elliptic curve. □

7. Try to define w *locally* as a holomorphic function of z, defined by the relation $w^5 - 5zw + 1 = 0$. What are the branch points?