Fall 2018 Complex Analysis Preliminary Exam

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Where possible, computations have been also done using SageMath code available on GitHub at github.com/tekaysquared/prelims (feel free to make pull requests!)

1. Tell the values of i^i.

   Proof. Recall that z^i is given by $e^{i \log z}$ on a suitably defined branch of the logarithm. As long as we choose a branch whose branch-cut is not along the positive imaginary axis, we see $\log i = i\pi/2 + 2\pi ik$ since $e^{\pi i/2}e^{2\pi ik} = e^{i\pi/2} = i$

   Thus, for $k \in \mathbb{Z}$, i^i takes on the values of

   $$e^{i \log i} = e^{i(\pi i/2 + 2\pi ik)} = e^{-\pi/2 + 2\pi k}$$

   on the positive real axis.

   □

2. Write the Laurent expansion of $f(z) = \frac{1}{z^4 - 1}$ centered at 0 and convergent in $|z| > 1$.

   Proof. Factor out z^{-4} to see that

   $$\begin{aligned} f(z) &= \frac{1}{z^4(1 - 1/z^4)} \\ &= \frac{1}{z^4} \sum_{n=0}^{\infty} \frac{1}{z^{4n}} \end{aligned}$$

   which converges for $|1/z^4| < 1$ which is to say for $|z| > 1$. So then f has a Laurent expansion

   $$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

   where

   $$a_n = \begin{cases} 1 & n = -4k \text{ for nonzero positive integers } k \\ 0 & \text{otherwise} \end{cases}$$

   □

3. Let f be an entire function so that $\Re f(z)$ is *nonnegative* for all $z \in \mathbb{C}$. Show that f is constant.

   Proof. Denote $z = x + iy$ where x, y are real, and let $f(z) = f(x, y) = u(x, y) + iv(x, y)$.

   We know that if $z = x + iy$ and $f(z) = f(x, y) = u(x, y) + iv(x, y)$, where u, v are real-valued functions, then in order for f to be entire it must satisfy the Cauchy-Riemann equations

   $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

   □

6. Determine the radius of convergence of the power series for $\log z$ at $z_0 = -4 + 3i$.

Proof. Let R denote the radius of convergence of the power series of $\log z$ centered at z_0. Now, note that there is no logarithm which takes a value at 0, since $e^w = 0$ is never true for $w \in \mathbb{C}$. Thus, the power series expansion can converge for a disk of radius at most $|-4 + 3i - 0| = 5$, and so

$$R \leq 5.$$

On the other hand, it is a theorem that if Ω is a simply connected subset of $\mathbb{C}$ which does not contain 0 then there is a branch of the logarithm which is holomorphic on Ω. Observe that the open disk $D_5(-4+3i) := \{z \in \mathbb{C} : |-4+3i-z| < 5\}$ is simply connected and does not contain zero. Thus, there is a branch of the logarithm (call it $\log_{D_5}$) which is holomorphic on D_5. Since we have constructed a disk of radius 5 on which there is a holomorphic logarithm, we see that

$$R \geq 5.$$

Since we have bounded R both above and below by 5, we see that $R = 5$. $\square$