

## Fall 2015 Manifolds and Topology Preliminary Exam

University of Minnesota

## Part A

1. (a) Define what it means for two paths  $\alpha, \beta : [0, 1] \rightarrow X$ , with the same start and end points, to be homotopic. Show that this is an equivalence relation.

*Proof.* Let  $X$  be a topological space and  $\alpha, \beta$  be as in the problem statement. Then  $\alpha, \beta$  are said to be homotopic if there is a function  $H(s, t) : [0, 1]^2 \rightarrow X$ , continuous in both  $s$  and  $t$ , such that  $H(0, t) = \alpha(t)$  and  $H(1, t) = \beta(t)$ . We call such a function a homotopy from  $\alpha$  to  $\beta$ . If there is a homotopy from  $\alpha$  to  $\beta$ , we also say that  $\alpha, \beta$  are homotopic.

We now show that the relation  $\sim$  defined by  $\alpha \sim \beta$  if  $\alpha$  is homotopic to  $\beta$  is reflexive, symmetric, and transitive. To see reflexivity observe that  $H(s, t) = \alpha(t)$  for all  $s$  is a homotopy from  $\alpha$  to itself. To symmetricity, note that if  $H(s, t)$  is a homotopy from  $\alpha$  to  $\beta$ , then  $H(1 - s, t)$  is a homotopy from  $\beta$  to  $\alpha$ . To see transitivity, let  $\alpha \sim \beta$  and  $\beta \sim \gamma$ , and let  $H_1$  be a homotopy from  $\alpha$  to  $\beta$  and  $H_2$  be a homotopy from  $\beta$  to  $\gamma$ . Then

$$H(s, t) := \begin{cases} H_1(2s, t) & 0 \leq s \leq 1/2 \\ H_2(2s - 1, t) & 1/2 \leq s \leq 1 \end{cases}$$

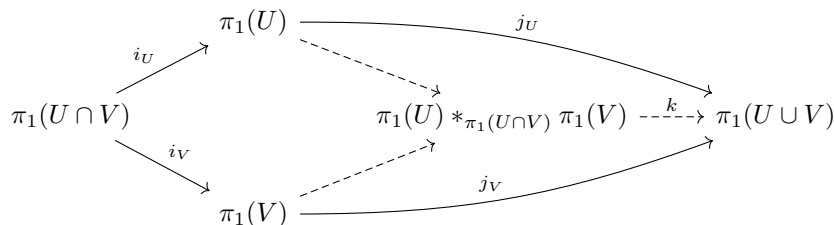
is a homotopy from  $\alpha$  to  $\gamma$ , so  $\alpha \sim \gamma$ .

- (b) Define what it means for a space to be simply connected

**Definition.** Let  $X$  be a topological space. Then  $X$  is simply connected if and only if there is a unique homotopy class of paths connecting any two points in  $X$ . Equivalently, we say  $X$  is simply connected if the fundamental group  $\pi_1(X)$  is the trivial group.

- (c) Give a complete statement of the Siefert-van Kampen theorem relating the fundamental groups of  $U$ ,  $V$ ,  $U \cup V$ ,  $U \cap V$ , including all necessary assumptions.

**Theorem.** *Let  $U, V$  be open, path connected topological spaces such that  $U \cap V$  is nonempty and path connected. The inclusion maps of  $U \hookrightarrow U \cup V$  and  $V \hookrightarrow U \cup V$  induce group homomorphisms  $j_U : \pi_1(U) \rightarrow \pi_1(U \cup V)$  and  $j_V : \pi_1(V) \rightarrow \pi_1(U \cup V)$ . Then  $U \cup V$  is path connected, and  $j_U, j_V$  form a commutative pushout diagram:*



Since this is a pushout diagram, then  $k$  is an isomorphism.

- (d) Describe the fundamental group of  $\mathbb{R}^2 - \{(-1, 0), (1, 0)\}$

*Proof.* Define  $X := \mathbb{R}^2 - \{(-1, 0), (1, 0)\}$ . Consider the sets  $U = \{(x, y) : y > -1/2\} - \{(1, 0)\}$  and  $V = \{(x, y) : x < 1/2\} - \{(-1, 0)\}$ . First, observe that  $U$  and  $V$  both deformation retract to circles, and so  $\pi_1(U) \cong \pi_1(V) \cong \mathbb{Z}$ . Then, observe that  $U, V$  satisfy the hypotheses of the

Siefert-van Kampen theorem, since  $U \cap V = \{(x, y) \in \mathbb{R}^2 : -1/2 < y < 1/2\}$  which is non-empty and path connected.

Then the theorem tells us that  $\pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$ . We see that  $U \cap V$  is contractable, and so  $\pi_1(U \cap V)$  is the trivial group. Then this includes into  $\pi_1(U)$  and  $\pi_1(V)$  by taking the single (identity) element of  $\pi_1(U \cap V)$  to the identities in  $\pi_1(U)$  and  $\pi_1(V)$  respectively. Thus, the amalgamated product  $\pi_1(U) * \pi_1(V) / N$  where  $N = \langle i_V(g)i_U(g)^{-1}, i_U(g)i_V(g)^{-1} : g \in \pi_1(U \cap V) \rangle$ . To compute  $N$ , we note that since  $i_U, i_V$  are group homomorphisms, they take the identity to the identity, and since the only  $g \in \pi_1(U \cap V)$  is exactly the identity, then

$$N = \langle i_U(e)i_V(e)^{-1}, i_V(e)i_U(e)^{-1} \rangle = \langle e \rangle$$

so  $N$  is the trivial group.

Now since  $N$  is trivial,  $\pi_1(U) * \pi_1(V) / N \cong \pi_1(U) * \pi_1(V) \cong \mathbb{Z} * \mathbb{Z}$  □

2. (a) State the classification theorem relating (connected) covering spaces of a (connected) space  $X$  to the fundamental group of  $X$ . (You may take as given the standard assumptions that  $X$  is locally path connected and locally simply connected.)

**Theorem.** *Let  $X$  be connected, locally path connected, and locally simply connected. Then there is a bijection between isomorphism classes of path connected covering spaces and conjugacy classes of  $\pi_1(X, x_0)$ .*

- (b) Suppose  $X$  is as in the previous problem and that the fundamental group of  $X$  is  $\mathbb{Z}/2 \times \mathbb{Z}/4$ . How many isomorphism classes of connected covering space does  $X$  have?

*Proof.* Because of the result of the last theorem, this boils down to counting conjugacy classes of  $\mathbb{Z}/2 \times \mathbb{Z}/4$ . Note that  $\mathbb{Z}/2 \times \mathbb{Z}/4$  is an abelian group and so its conjugacy classes are singletons. Thus, the number of isomorphism classes of connected covering spaces of  $X$  is the size of  $\mathbb{Z}/2 \times \mathbb{Z}/4$ , which is 8. □

- (c) A continuous map  $f : X \rightarrow Y$  is a local homeomorphism if, for every point  $x \in X$ , there exist open neighborhoods  $U$  of  $x$  and  $V$  of  $f(x)$  so that  $f$  restricts to a homeomorphism  $U \rightarrow V$ . Give an example of a local homeomorphism which is not a covering map.

3. (a) Calculate the homology groups of the torus  $S^1 \times S^1$

*Proof.* We know that there is one connected component, one hole (the donut hole), and one void (where the jelly filling would go). Thus, we expect the answer to be  $H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & n = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$  □