Fall 2017 Manifolds and Topology Preliminary Exam

University of Minnesota

Part A

1. Suppose we have two paths α and β from $[0,1]$ to a space X, starting at the same point p and ending at the same point q. Define what it means for α, β to be homotopic, and show that this relation is symmetric.

   Proof. $\alpha, \beta : [0,1] \to X$ with $\alpha(0) = \beta(0) = p$ and $\alpha(1) = \beta(1) = q$ are homotopic if there is a continuous map $h : [0,1]^2 \to X$ such that $h(0,t) = \alpha(t)$ and $h(1,t) = \beta(t)$. Such a map h is called a homotopy. If there is a homotopy from α to β, we say they are homotopic, and write $\alpha \sim \beta$. To see that it is symmetric, consider the map $\phi : [0,1]^2 \to [0,1]^2$ given by $(x,t) \mapsto (1-x,t)$ then $h \circ \phi(t)$ is a homotopy from β to α, so the relation $\sim$ is symmetric. $\square$

2. If X and Y are based spaces determine (with proof) the fundamental group $\pi_1(X \times Y, (x,y))$ in terms of $\pi_1(X,x)$ and $\pi_1(Y,y)$.

   Proof. Let $g \in \pi_1(X \times Y, (x,y))$. Then there is a loop γ in $X \times Y$ so that the homotopy class $[\gamma] = g$. Consider the projection $p_1 : X \times Y \to X$ onto the first coordinate and the projection $p_2 : X \times Y \to Y$ onto the second coordinate. Then since $\gamma(0) = \gamma(1)$, we see that $p_i(\gamma(0)) = p_i(\gamma(1))$ and so $p_i(\gamma)$ are loops in X, Y respectively. Thus the projections induce maps on fundamental groups $(p_i)_*$. What else do I need to show?? $\square$

3. Give an example of a map that is a covering map but is not a homeomorphism.

   Proof. Let $f : \mathbb{R} \to S^1$ be the map taking $x \mapsto (\cos x, \sin x)$. This is a covering map since explain. This is not a homeomorphism since $\pi_1(\mathbb{R}) = 0 \ncong \mathbb{Z} \cong \pi_1(\mathbb{R})$ and the fundamental group is a homeomorphism invariant. $\square$

4. Let X be the space $\mathbb{RP}^2 \times \mathbb{RP}^2$. How many isomorphism classes of covering maps $Y \to X$ are there with Y path-connected?

   Proof. isomorphism classes of covering maps correspond to conjugacy classes of the fundamental group so first we need to find the fundamental group of $\mathbb{RP}^2$ and then count its conjugacy classes. $\square$

5. Prove that the projection map $S^2 \to \mathbb{RP}^2$ is a universal cover.

   Proof. Something with simply connected covering spaces $\square$

6. Suppose X is a path-connected space whose fundamental group $\pi_1(X,x)$ is the symmetric group Σ_3 on three letters. Determine the first homology group $H_1(X)$.

   Proof. The first homology group is the abelianization of the fundamental group. The abelianization of the symmetric group is the trivial group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ prove this, and so $H_1(X) \cong \mathbb{Z}/2\mathbb{Z}$. $\square$

7. Suppose X is a space with open subsets U and V such that X is the union $U \cup V$, both U and V are path-connected, and $U \cap V$ is not path-connected (and nonempty). Show that $H_1(X)$ is nontrivial.

Proof. □

8. Suppose X is a space and $i : A \to X$ is a map with a retraction $r : X \to A$ such that $r \circ i = id$. Show that, for all n, $H_*(A)$ is a direct summand of $H_*(X)$.

9. Define the degree of a continuous map $f : S^n \to S^n$ for $n > 0$.

   Definition. *If $f : S^n \to S^n$, for $n > 0$, then the induced map on homology f_* is multiplication by a constant, since $H_i(S^n)$ is $\mathbb{Z}$ when $i = 0, n$ and 1 otherwise, and the only homomorphism from $\mathbb{Z} \to \mathbb{Z}$ is multiplication by an integer. So if $f_* : H_\bullet(S^n) \to H_\bullet(S^n)$ is $n \mapsto kn$, then we call the integer k the degree of f.*

10. A weak form of the *Lefshetz fixed point theorem* states the following. Suppose that X is a (sufficiently nice) space and $f : X \to X$ is a continuous map such that $f(x) \neq x$ for all $x \in X$. Then the Lefschetz number

$$\sum_k (-1)^k Trace(f_* : H_k(X) \otimes \mathbb{R} \to H_k(X) \otimes \mathbb{R})$$

   is 0. If X is a sphere (spheres are sufficiently nice), what can one conclude about the degree of a map $S^k \to S^k$ that has no fixed points.

Part B

1. State the Whitney embedding theorem on embeddings and immersions of m-dimensional smooth manifolds M into $\mathbb{R}^k$.

2. Give an example (with proof) of a diffeomorphism between $\mathbb{R}$ and $(0, 1)$.

   Proof. $x \mapsto \frac{1}{\pi}\arctan(x) + \frac{1}{2}$. A diffeomorphism is a differentiable homeomorphism with a differentiable inverse. So we need to show our given map is (i) a homeomorphism (ii) differentiable, and (iii) has $\tan(\pi(x - \frac{1}{2}))$ differentiable.

   □

3. Define $f(x, y) = (x + x^4)(y^2 + y)$ a smooth function from $\mathbb{R}^2$ to $\mathbb{R}$. For this function, determine the: singular points; regular points; singular values; regular values.

   Proof. □

4. Suppose $1 < m$. Show that there are no smooth space-filling curves: if f is a smooth function from $(0, 1)$ to $\mathbb{R}^m$, show that the image of f cannot contain the ball of radius 1 around the origin. (Hint: Sard's theorem).

5. Suppose (x, y) are Cartesian coordinates on $\mathbb{R}^2$ and (u, v) are new coordinates given by

$$u = 3x + y - 2$$
$$v = -x + y + 5$$

   Express the vector field $x\frac{\partial}{\partial x}$ in terms of (u, v)-coordinates. Your answer should take the form $P(u, v)\frac{\partial}{\partial u} + Q(u, v)\frac{\partial}{\partial v}$.

6. Calculate the exterior derivative $d\omega$ where ω is the differential form

$$e^{xyz}dx + e^{yz}dy - \cos(xz)dz$$

   on $\mathbb{R}^3$.