

Fall 2015 Manifolds and Topology Preliminary Exam

University of Minnesota

Part A

1. (a) Define what it means for two paths $\alpha, \beta : [0, 1] \rightarrow X$, with the same start and end points, to be homotopic. Show that this is an equivalence relation.

Proof. Let X be a topological space and α, β be as in the problem statement. Then α, β are said to be homotopic if there is a function $H(s, t) : [0, 1]^2 \rightarrow X$, continuous in both s and t , such that $H(0, t) = \alpha(t)$ and $H(1, t) = \beta(t)$. We call such a function a homotopy from α to β . If there is a homotopy from α to β , we also say that α, β are homotopic.

We now show that the relation \sim defined by $\alpha \sim \beta$ if α is homotopic to β is reflexive, symmetric, and transitive. To see reflexivity observe that $H(s, t) = \alpha(t)$ for all s is a homotopy from α to itself. To symmetricity, note that if $H(s, t)$ is a homotopy from α to β , then $H(1 - s, t)$ is a homotopy from β to α . To see transitivity, let $\alpha \sim \beta$ and $\beta \sim \gamma$, and let H_1 be a homotopy from α to β and H_2 be a homotopy from β to γ . Then

$$H(s, t) := \begin{cases} H_1(2s, t) & 0 \leq s \leq 1/2 \\ H_2(2s - 1, t) & 1/2 \leq s \leq 1 \end{cases}$$

is a homotopy from α to γ , so $\alpha \sim \gamma$. □

- (b) Define what it means for a space to be simply connected

Definition. Let X be a topological space. Then X is simply connected if and only if there is a unique homotopy class of paths connecting any two points in X . Equivalently, we say X is simply connected if the fundamental group $\pi_1(X)$ is the trivial group.

- (c) Give a complete statement of the Siefert-van Kampen theorem relating the fundamental groups of $U, V, U \cup V, U \cap V$, including all necessary assumptions.

Theorem. Let U, V be open, path connected topological spaces such that $U \cap V$ is nonempty and path connected. The inclusion maps of $U \hookrightarrow U \cup V$ and $V \hookrightarrow U \cup V$ induce group homomorphisms $j_U : \pi_1(U) \rightarrow \pi_1(U \cup V)$ and $j_V : \pi_1(V) \rightarrow \pi_1(U \cup V)$. Then $U \cup V$ is path connected, and j_U, j_V form a commutative pushout diagram:

$$\begin{array}{ccccc}
 & & \pi_1(U) & & \\
 & \nearrow i_U & & \searrow j_U & \\
 \pi_1(U \cap V) & & & & \pi_1(U \cup V) \\
 & \searrow i_V & & \nearrow j_V & \\
 & & \pi_1(V) & &
 \end{array}$$

$\pi_1(U) \xrightarrow{j_U} \pi_1(U \cup V)$
 $\pi_1(U \cap V) \xrightarrow{i_U} \pi_1(U) \xrightarrow{j_U} \pi_1(U \cup V)$
 $\pi_1(U \cap V) \xrightarrow{i_V} \pi_1(V) \xrightarrow{j_V} \pi_1(U \cup V)$
 $\pi_1(U) \xrightarrow{j_U} \pi_1(U \cup V)$

Since this is a pushout diagram, then k is an isomorphism.

- (d) Describe the fundamental group of $\mathbb{R}^2 - \{(-1, 0), (1, 0)\}$

Proof. Define $X := \mathbb{R}^2 - \{(-1, 0), (1, 0)\}$. Consider the sets $U = \{(x, y) : y > -1/2\} - \{(1, 0)\}$ and $V = \{(x, y) : x < 1/2\} - \{(-1, 0)\}$. First, observe that U and V both deformation retract to circles, and so $\pi_1(U) \cong \pi_1(V) \cong \mathbb{Z}$. Then, observe that U, V satisfy the hypotheses of the

Siefert-van Kampen theorem, since $U \cap V = \{(x, y) \in \mathbb{R}^2 : -1/2 < y < 1/2\}$ which is non-empty and path connected.

Then the theorem tells us that $\pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$. We see that $U \cap V$ is contractable, and so $\pi_1(U \cap V)$ is the trivial group. Then this includes into $\pi_1(U)$ and $\pi_1(V)$ by taking the single (identity) element of $\pi_1(U \cap V)$ to the identities in $\pi_1(U)$ and $\pi_1(V)$ respectively. Thus, the amalgamated product $\pi_1(U) * \pi_1(V) / N$ where $N = \langle i_V(g)i_U(g)^{-1}, i_U(g)i_V(g)^{-1} : g \in \pi_1(U \cap V) \rangle$. To compute N , we note that since i_U, i_V are group homomorphisms, they take the identity to the identity, and since the only $g \in \pi_1(U \cap V)$ is exactly the identity, then

$$N = \langle i_U(e)i_V(e)^{-1}, i_V(e)i_U(e)^{-1} \rangle = \langle e \rangle$$

so N is the trivial group.

Now since N is trivial, $\pi_1(U) * \pi_1(V) / N \cong \pi_1(U) * \pi_1(V) \cong \mathbb{Z} * \mathbb{Z}$ □

2. (a) State the classification theorem relating (connected) covering spaces of a (connected) space X to the fundamental group of X . (You may take as given the standard assumptions that X is locally path connected and locally simply connected.)

Theorem. *Let X be connected, locally path connected, and locally simply connected. Then there is a bijection between isomorphism classes of path connected covering spaces and conjugacy classes of $\pi_1(X, x_0)$.*

- (b) Suppose X is as in the previous problem and that the fundamental group of X is $\mathbb{Z}/2 \times \mathbb{Z}/4$. How many isomorphism classes of connected covering space does X have?

Proof. Because of the result of the last theorem, this boils down to counting conjugacy classes of $\mathbb{Z}/2 \times \mathbb{Z}/4$. Note that $\mathbb{Z}/2 \times \mathbb{Z}/4$ is an abelian group and so its conjugacy classes are singletons. Thus, the number of isomorphism classes of connected covering spaces of X is the size of $\mathbb{Z}/2 \times \mathbb{Z}/4$, which is 8. □

- (c) A continuous map $f : X \rightarrow Y$ is a local homeomorphism if, for every point $x \in X$, there exist open neighborhoods U of x and V of $f(x)$ so that f restricts to a homeomorphism $U \rightarrow V$. Give an example of a local homeomorphism which is not a covering map.
3. (a) Calculate the homology groups of the torus $S^1 \times S^1$

Proof. First, we recall the fact that $H_n(S^1) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{else} \end{cases}$, and note that it is torsion-free.

Since we can put a CW structure on S^1 , a Künneth formula gives the short exact sequence

$$0 \rightarrow \bigoplus_i H_i(S^1) \otimes H_{n-i}(S^1) \rightarrow H_n(S^1 \times S^1) \rightarrow \bigoplus_i \text{Tor}(H_i(S^1), H_{n-i-1}) \rightarrow 0.$$

But since the homology groups are always torsion-free we obtain the isomorphism

$$\bigoplus_i H_i(S^1) \otimes H_{n-i}(S^1) \cong H_n(S^1 \times S^1).$$

When $n = 0$ we have

$$H_0(S^1) \otimes H_0(S^1) = \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}.$$

When $n = 1$ we have

$$(H_0(S^1) \otimes H_1(S^1)) \oplus (H_1(S^1) \otimes H_0(S^1)) = (\mathbb{Z} \otimes \mathbb{Z}) \oplus (\mathbb{Z} \otimes \mathbb{Z}) \cong \mathbb{Z}^2.$$

When $n = 2$ we have

$$(H_0(S^1) \otimes H_2(S^1)) \oplus (H_1(S^1) \otimes H_1(S^1)) \oplus (H_2(S^1) \otimes H_0(S^1)) = (\mathbb{Z} \otimes 0) \oplus (\mathbb{Z} \otimes \mathbb{Z}) \oplus (0 \otimes \mathbb{Z}) \cong \mathbb{Z}$$

For $n > 2$ we have

$$H_i(S^1) \otimes H_{n-i}(S^1) = \begin{pmatrix} \mathbb{Z} \otimes 0 & i = 0, 1 \\ 0 \otimes 0 & 1 < i < n-1 \\ 0 \otimes \mathbb{Z} & i = n-1, n \end{pmatrix} \cong 0.$$

Thus, we can conclude

$$H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & n = 0, 1, 2 \\ 0 & \text{else} \end{cases}$$

□