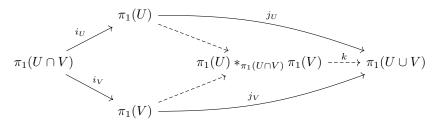
## Useful theorems for Manifolds and Topology Preliminary Exams

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## 1 Fundamental Group

**Theorem** (Siefert-van Kampen). Let U, V be open, path connected topological spaces such that  $U \cap V$  is nonempty and path connected. The inclusion maps of  $U \hookrightarrow U \cup V$  and  $V \hookrightarrow U \cup V$  induce group homomorphisms  $j_U : \pi_1(U) \to \pi_1(U \cup V)$  and  $j_V : \pi_1(V) \to \pi_1(U \cup V)$ . Then  $U \cup V$  is path connected, and  $j_U, j_V$  form a commutative pushout diagram:



Since this is a pushout diagram, then k is an isomorphism.

## 2 Covering Spaces

**Theorem.** Let X be path connected, locally path connected, and semilocally simply connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces  $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$  and the set of subgroups of  $\pi_1(X, x_0)$  obtained by associating the subgroup  $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$  to the covering space  $(\tilde{X}, \tilde{x}_0)$ . If basepoints are ignored, this gives a bijection between isomorphism classes of path-connected covering spaces  $p: \tilde{X} \to X$  and conjugacy classes of subgroups of  $\pi_1(X, x_0)$ .

**Lemma.** If G is an abelian group, then the conjugacy classes of G are all singletons, so if G is finite, then |G| is the number of conjugacy classes.

## 3 Manifolds

**Theorem** (Sard's theorem). Let M, N be smooth manifolds with or without boundary, and  $F: M \to N$  be a smooth map. Then the set of critical values of F has measure 0 in N.

**Theorem** (Whitney embedding theorem). Every smooth n-manifold with or without boundary admits a proper smooth embedding into  $\mathbb{R}^{2n+1}$ .

**Theorem** (Coordinate formula for the Lie Bracket [?] prop. 8.26 verbaitm). Let X, Y be smooth vector fields on a smooth manifold M with or without boundary, and let  $X = X^i \frac{\partial}{\partial x^i}$  and  $Y = Y^j \frac{\partial}{\partial x^j}$  be the coordinate expressions for X and Y in terms of some smooth local coordinates  $(x^i)$  for M. Then [X, Y] has the following coordinate expression:

$$[X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right) \frac{\partial}{\partial x^j}$$

or

$$[X,Y] = (XY^j - YX^j)\frac{\partial}{\partial x^j}$$

To add: Global Rank Theorem p. 83 of Lee