Spring 2016 Manifolds and Topology Preliminary Exam

University of Minnesota

Part A

1.

2.

3. (a) For $n > 0$ define the degree of a continuous map $S^n \to S^n$.

Definition. *If $n > 0$ and $f : S^n \to S^n$ is continuous map, then the induced map on homology $f_* : H_n(S^n) \to H_n(S^n)$ is a homomorphism $f_* : \mathbb{Z} \to \mathbb{Z}$. The only homomorphism from $\mathbb{Z} \to \mathbb{Z}$ is multiplication by a constant integer, so $f_*(x) = dx$ for $x \in H_n(S^n)$ and $d \in \mathbb{Z}$. Then d is the degree of f.*

(b) If M is an n-dimensional manifold with $n > 0$ and $p \in M$, show that there is an isomorphism of homology groups

$$H_k(M, M\backslash\{p\}) \cong \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

Proof. Since M is a manifold, we may take an open neighborhood $U_p \cong \mathbb{R}^n$. Define $C = M - U_p$. Then $M - C = U_p$ and $(M - p) - C = U_p - p$. The excision theorem tells us that

$$H_*(M, M - p) \cong H_*(M - C, (M - p) - C) = H_*(U_p, U_p - p)$$

The long exact sequence on relative homology tells us

$$\cdots \longrightarrow \tilde{H}_k(U_p - p) \longrightarrow \tilde{H}_k(U_p) \longrightarrow \tilde{H}_k(U_p, U_p - p) \longrightarrow \tilde{H}_{k-1}(U_p - p) \longrightarrow \cdots.$$

We know that $U_p - p$ is homotopic to the $n-1$ sphere and since homology is a homotopy invariant, we get that

$$\tilde{H}_k(U_p - p) = \tilde{H}_k(S^{n-1}) = \begin{cases} \mathbb{Z} & k = n-1 \\ 0 & \text{otherwise} \end{cases}$$

and so when $k \neq n-1$, we have that $\tilde{H}_k(U_p) \cong \tilde{H}_k(U_p, U_p - p)$. Moreover $\tilde{H}_k(U_p) = 0$ for all k, and reduced homology agrees with singular homology for $k > 0$, so we have so far that

$$H_k(U_p, U_p - p) = \begin{cases} 0 & k \neq 0, n-1 \\ ?? & \text{otherwise} \end{cases}$$

Now, focusing near $k = n-1$ we have the exactness of the sequence

$$\cdots \longrightarrow H_n(U_p) \longrightarrow H_n(U_p, U_p - p) \longrightarrow H_{n-1}(U_p - p) \longrightarrow H_{n-1}(U_p) \longrightarrow \cdots.$$

Since $H_n(U_p) = H_{n-1}(U_p) = 0$ the above sequence induces the isomorphism $H_n(U_p, U_p - p) \cong H_{n-1}(U_p - p) \cong \mathbb{Z}$. Finally, near 0 we have

$$\cdots \longrightarrow H_0(U_p) \longrightarrow H_0(U_p, U_p - p) \longrightarrow 0$$

which is exactly

$$\cdots \longrightarrow 0 \longrightarrow H_0(U_p, U_p - p) \longrightarrow 0$$

and so $H_0(U_p, U_p - p) = 0$ □

(c) Show that the subspace

$$\{(x, y, z) | \text{either } (x = 0) \text{ or } (y = z = 0)\ \} \subset \mathbb{R}^3$$

is not a manifold.

Proof. Suppose the subspace *were* a manifold. Then in the neighborhood of $(0,0,0)$, we can find $U_0 \cong \mathbb{R}^n$ for some n (perhaps the manifold is impure). Then $U_0 - (0,0,0)$ is homotopic to an $n-1$ sphere. When $n \geq 1$, this means that $U_0 - (0,0,0)$ has one connected component. On the other hand when $n = 0$, $U_0 - (0,0,0)$ has two connected components.

The subspace is the union of the x axis with the $y-z$-plane. But we see that any neighborhood of $(0,0,0)$ in the subspace given can be broken into three disconnected components: the intersection with the positive x-axis, the negative x-axis, and the punctured $y-z$-plane missing the origin. Thus every neighborhood of the origin is *not* homeomorphic to Euclidean space, and so the subspace is not a manifold. □

(d) For $n \in \mathbb{Z}$ give an example of a continuous map $S^1 \to S^1$ of degree n.

Proof. Let $f : S^1 \to S^1$ be given by $z \mapsto z^n$, where we implicitly think of $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. We claim that the degree of f is n. Let $z = e^{i\theta}$ where $\theta \in \mathbb{R}$. The fiber $f^{-1}(z) = \{z \in S^1 : z = e^{i(\theta+2\pi k)/n}, k \in \mathbb{Z}\}$ consists of n points $Z = \{e^{i\theta/n}, e^{i\theta/n+i2\pi 1/n}, ..., e^{i\theta/n+i2\pi(n-1)/n}\}$. f is orientation preserving, and so $\deg f|_z = 1$ for all $z \in Z$. Then $\deg f = \sum_{z\in Z} \deg f|_z = n$. □