Spring 2018 Manifolds and Topology Preliminary Exam

University of Minnesota

Part A

1. Define what it means for two continuous paths $\alpha, \beta : [0,1] \to X$, both starting at the same point p and ending at the same point q, to be homotopic. Define the fundamental group $\pi_1(X,p)$ as a set.

   Definition. *Two paths α, β as given above are said to be homotopic (written $\alpha \sim \beta$), if there exists a continuous map $H(s,t) : [0,1]^2 \to X$ such that $\alpha(t) = H(0,t)$ and $\beta(t) = H(1,t)$.*

   Definition. *Note that the relation $\sim$ defines an equivalence relation on all continuous paths sharing a common starting and ending point. If a continous path starts and ends at the same point p, we call it a loop in X with basepoint p. Let $\mathcal{L}_p = \{$loops in X with basepoint $p\}$. Then as a set, $\pi_1(X,p) = \mathcal{L}_p/\sim$ is the set of equivalence classes of $\mathcal{L}_p$ under $\sim$.*

2. Suppose X is a space. Define what it means for two covering maps $p : Y \to X$ and $q : Z \to X$ to be isomorphic.

   Definition. *Two covering maps $p : Y \to X$, $q : Z \to X$ are isomorphic if $f : Y \to Z$ is a homeomorphism and the following diagram commutes*

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

3. Suppose p is a point in the space X and α is a loop in X starting and ending at a point p. If c is the constant loop at p, show explicitly that the concatenated loop $\alpha * c$ is homotopic to α.

   Proof. Recall that concatenation of loops γ_1, γ_2 is given as

$$\gamma_1 * \gamma_2 = \begin{cases} \gamma_1(2t) & t \in [0,1/2] \\ \gamma_2(2t-1) & t \in (1/2,1] \end{cases},$$

   so

$$\alpha * c = \begin{cases} \alpha(2t) & t \in [0,1/2] \\ p & t \in (1/2,1]. \end{cases}$$

   Define

$$H(s,t) = \begin{cases} \alpha(\frac{2t}{1+s}) & t \in [0, \frac{s+1}{2}] \\ c & t \in (\frac{s+1}{2}, 1]. \end{cases}$$

   Note that $H(0,t) = \alpha * c$ and $H(1,t) = \alpha$ and $H(s,t)$ is continuous in s since $\frac{2}{1+s}$ is continuous. $\square$

5. Suppose X is a space with open subsets U and V such that X is the union $U \cup V$, both U, V are simply-connected, and $H_1(U \cap V) \neq 0$. Show that $H_2(X)$ is nontrivial.

   Proof. We know that given U, V as above, we have the Mayer-Vietoris long exact sequence:

$$\cdots \longrightarrow H_2(X) \longrightarrow H_1(U \cap V) \longrightarrow H_1(U) \otimes H_1(V) \longrightarrow \cdots$$

Since U, V are simply connected, they have trivial pi_1. H_1 is the abelianization of π_1, and so since the trivial group is abelian, we have $H_1(U) \otimes H_1(V) = 0 \otimes 0 \cong 0$.

We are also given that $H_1(U \cap V) \neq 0$.

Thus, the map $H_1(U \cap V) \to 0$ is the zero map. Then exactness of the sequence tells us that $H_2(X) \to H_1(U \cap V) \neq 0$ is a surjection. This shows that $H_2(X)$ is nontrivial, since the map $0 \to H_1(U \cap V) \neq 0$ can never be a surjection. □

6. For $n > 0$, define the degree of a continuous map $f : S^n \to S^n$.

   Definition. *Let f be a continuous map from S^n to itself. Let f_* be the induced map on homology. Then $f_*(x) = kx$ for some integer k. Such k is defined to be the degree of f.*

7. Determine, with proof, the degree of the map $f : S^1 \to S^1$. Given by $(x, y) \mapsto (-x, -y)$.

   Proof. Consider $f = g \circ h$ where h takes $(x, y) \mapsto (-x, y)$ and g takes $(x, y) \mapsto (x, -y)$. Then $f_* : H_\bullet(S^n) \to H_\bullet(S^n)$ is $f_* = (g \circ h)_* = g_* \circ h_*$. Since g, h swap hemispheres, then $\deg(f) = \deg(g)\deg(h) = (-1)^2 = 1$. □

8. Let M be the Möbius band
$$[0, 1]^2 / \{(x, 0) \sim (1 - x, 1)\}$$
   with boundary ∂M. Show that there does not exist a continuous retraction $r : M \to \partial M$.

   Proof. (From StackExchange) Suppose that $\pi_1(\partial M) = \langle \alpha \rangle$ and $\pi_1(M) = \langle \beta \rangle$. The picture

   shows that the inclusion $\iota : \partial M \to M$ induces the homomorphism $\iota_* : \pi_1(\partial M) \to \pi_1(M)$, where $\iota_*(\alpha) = \beta^2$, since the boundary must be traversed twice to have a closed loop.

   If there were a retraction $r : M \to \partial M$, then the induced homomorphism r_* on fundamental groups would satisfy $r_* \circ \iota_*(\alpha) = \alpha$. But $\iota_*(\alpha) = \beta^2$, so then $r_*(\beta^2) = r_*(\beta)^2 = \alpha$, or $r_*(\beta) = \sqrt{\alpha} \notin \pi_1(\partial M)$. □

9. Suppose X is a (connected, locally contractible) space whose fundamental group is the group $\mathbb{Z}/2 \times \mathbb{Z}/4$. How many isomorphism classes of covering maps $Y \to X$ are there with Y path-connected.

   Proof. Isomorphism classes of path connected covering spaces are in bijection with conjugacy classes of subgroups of the fundamental group. When the fundamental group is abelian, the conjugacy classes of subgroups are simply the subgroups (since $ghg^{-1} = gg^{-1}h = h$) of the group. Note that $\mathbb{Z}/2 \times \mathbb{Z}/4$ is abelian, and has 8 subgroups.

   Namely, Goursat's lemma tells us the subgroups are in bijection with 5-tuples $(G_1, G_2, H_1, H_2, \varphi)$ where $G_1 \trianglelefteq G_2 \leq Z/2$ and $H_1 \trianglelefteq H_2 \leq \mathbb{Z}/4$, and $\varphi : G_2/G_1 \to H_2/H_1$ is an isomorphism. The possible pairs of G_1, G_2 are:
$$(\{0\}, \{0\}), (\{0\}, Z/2), (\mathbb{Z}/2, \mathbb{Z}/2)$$
   yielding quotients
$$0 \cong 0/0, \mathbb{Z}/2 \cong \mathbb{Z}/2/0, 0 \cong \mathbb{Z}/2/\mathbb{Z}/2$$
   and the possible pairs H_1, H_2 are:
$$(\{0\}, \{0\}), (\{0\}, \mathbb{Z}/2), (\{0\}, \mathbb{Z}/4), (\mathbb{Z}/2, \mathbb{Z}/2), (\mathbb{Z}/2, \mathbb{Z}/4), (\mathbb{Z}/4, \mathbb{Z}/4)$$

which yield quotients:

$$0, \mathbb{Z}/2, \mathbb{Z}/4, 0, \mathbb{Z}/2, 0$$

.

There are 6 different 4-tuples of quotients who yield the trivial group. The only isomorphism of the trivial group is the trivial map, so these contribute +6 to our count of of subgroups. There are 2 different 4-tuples of quotients who yield the group $\mathbb{Z}/2$. Similarly, there is only one isomorphism from $\mathbb{Z}/2$ to itself. Thus, there are 8 subgroups of $\mathbb{Z}/2 \times \mathbb{Z}/4$ and so there are 8 isomorphism classes of covering maps $Y \to X$ with Y being path connected.

Note: 6 of the subgroups are of the form $G \times H$ where $G \leq \mathbb{Z}/2$ and $H \leq \mathbb{Z}/4$. The two remaining subgrops are

$$\langle (1,0) + (1,3) \rangle$$

□

Alternate proof.

10. Suppose X is a path connected space, A is a path-connected subspace, $p \in A$ and C is the mapping cone

$$(X \times \{1\} \cup A \times [0,1])/\{(a,0) \sim (a',0)\}$$

use the Seifert-van Kapen theorem to express the fundamental group $\pi_1(C, [(p, 1/2)])$ in terms of the map $\pi_1(A, p) \to \pi_1(X, p)$.

Part B

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7. Calculate the Lie bracket $[X, Y]$ of the vector fields $X = xy\frac{\partial}{\partial x}$ and $Y = x^3 \cos(x)\frac{\partial}{\partial x}$ on $\mathbb{R}^2$.

Proof. First, we recall the following theorem:

Theorem (Coordinate formula for the Lie Bracket)**.** *Let X, Y be smooth vector fields on a smooth manifold M with or without boundary, and let $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ be the coordinate expressions for X and Y in terms of some smooth local coordinates (x^i) for M. Then $[X, Y]$ has the following coordinate expression:*

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

Since we are working in $\mathbb{R}^2$ we have coordinates $x^1 = x$ and $x^2 = y$. We have $X = xy\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y}$, $Y = x^3 \cos x \frac{\partial}{\partial x} + 0\frac{\partial}{\partial y}$. So then since $\frac{\partial 0}{\partial x^i} = X^2 = Y^2 = 0$ we have

$$\begin{aligned}
[X, Y] &= \left(xy\frac{\partial Y^1}{\partial x} - x^3 \cos x \frac{\partial X^1}{\partial x} \right) \frac{\partial}{\partial x} \\
&= \big(xy(3x^2 \cos x - x^3 \sin x) - x^3 y \cos(x) \big) \frac{\partial}{\partial x} \\
&= \big(2x^3 y \cos x - x^4 y \sin x \big) \frac{\partial}{\partial x}
\end{aligned}$$

□

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