Fall 2019 Manifolds and Topology Preliminary Exam

University of Minnesota

Part A

1. Give a precise definition of the product $\alpha * \beta$ of two paths in a space X, including the conditions under which it is defined.

   Definition. *Let $\alpha, \beta : [0,1] \to X$ be continuous functions into X. If $\alpha(1) = \beta(0)$, then the product $\alpha * \beta : [0,1] \to X$ is defined to be*

   $$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & t \in [0,1/2] \\ \beta(2t-1) & t \in (1/2,1] \end{cases}$$

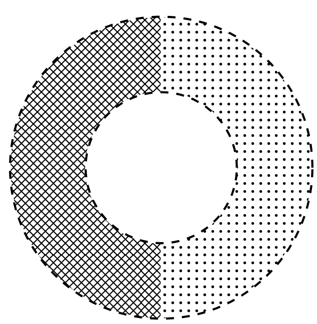
   .

2. Suppose that X is a path connected, semilocally 1-connected space whose fundamental group is $\mathbb{Z}/2 \times \mathbb{Z}/2$. How many isomorphism classes of connected covering spaces does X have?

   Proof. The isomorphism classes of covering spaces correspond to conjugacy classes of the fundamental group. Thus, this question is equivalent to counting the conjugacy classes of $\mathbb{Z}/2 \times \mathbb{Z}/2$. Note that $\mathbb{Z}/2 \times \mathbb{Z}$ is abelian and abelian groups have exactly one conjugacy class for each group element (since $hgh^{-1} = hh^{-1}g = g$ for any g, h). Thus, there are $|\mathbb{Z}/2 \times \mathbb{Z}/2| = 4$ conjugacy classes, and hence there are 4 isomorphism classes of covering spaces of X. □

3. Give an example of a space X with open subsets U, V such that $X = U \cup V$, U is simply connected, V is simply connected, but where X is not simply connected. Then explain why this does not violate the Siefert-van Kampen theorem.

   Proof. Let X be the annulus of inner radius 1 and outer radius 2, shown below,:

   and let U be the left half of the annulus (shown crosshatched) with ϵ extra and let V be the right half of the annulus (shown filled with dots) also with ϵ extra, where both U, V intersect in ϵ-fattenings of the line segments $\ell_+ = [(0,1),(0,2)]$ and $\ell_- = [(0,-1),(0,-2)]$.

   Then $\pi_1(U) = \pi_1(V) = 0$ since both are contractible. Additionally, $\pi_1(U \cap V) = 0$ since both the ϵ-fattenings of line segments ℓ_+, ℓ_- are contractible so regardless of choice of basepoint we are in a contractible connected component. U and V do not violate the hypothesis of the Siefert-van Kampen theorem, because the intersection $U \cap V$ is not path-connected. However if we were to ignore the requirement of path-connectedness, we might conclude that the theorem says $\pi_1(X) = 0/0 \cong 0$.

But we know that $\pi_1(U \cup V) \cong \mathbb{Z}$ because the generators are the trivial loop and the loop which contains the hole in its interior.

Thus, we cannot choose U, V as in this example if we wish to use the Siefert-van Kampen theorem. $\square$

4. Explain why the inclusion $\mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ is not a covering map.

Proof. Suppose $\iota : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ were a covering map.

Then it is a theorem that the induced map on fundamental groups $\iota_* : \pi_1(\mathbb{R}^2 \backslash \{0\}) \to \pi_1(\mathbb{R})$ is an injection.

On the other hand, $\pi_1(\mathbb{R}^2 \backslash \{0\}) \cong \mathbb{Z}$ and $\pi_1(\mathbb{R}) = 0$ and so no injection from $\mathbb{Z} \to 0$ exists, contradicting that ι were a covering map. $\square$

5. Define the *degree* of a map $f : S^n \to S^n$ for $n > 0$. Explain why $n = 0$ is a special case.

Definition. *Let $f : S^n \to S^n$ be a map. Then there is an induced homomorphism on homology $f_* : H_n(S^n) \to H_n(S^n)$. Since $H_n(S^n) \cong \mathbb{Z}$, then f_* must be multiplication by an integer (otherwise it would not be a homomorphism). The integer k such that $f_*(x) = kx$ is called the degree of the map.*

Note that $n = 0$ is a special case, since $H_0(S^0) \cong \mathbb{Z}^2$, and so "multiplication by an integer" must be more carefully defined.

6. Suppose X, Y are spaces that are both abstractly homeomorphic to S^7. Show that the "degree" of the map $f : X \to Y$ is only well-defined up to sign.

7. Let X be the space of upper triangular invertible 2×2 matrices:

$$X = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in M_2(\mathbb{R}) : a \neq 0, d \neq 0 \right\} \subseteq \mathbb{R}^3$$

Determine the homology groups $H_*(X)$.

Proof. First, note that $X \cong (\mathbb{R}\backslash\{0\}) \times (\mathbb{R}) \times (\mathbb{R}\backslash\{0\}) \subseteq \mathbb{R}^3$ by the map taking $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto (a, b, d)$.

Thus, we compute the homology groups $H_*(\mathbb{R}\backslash\{0\})$ and $H_*(\mathbb{R})$.

Since $X_1 := \mathbb{R}\backslash\{0\}$ is the disjoint union of the strictly positive real axis $\mathbb{R}_{>0}$ and the strictly negative real axis $\mathbb{R}_{<0}$, which are homeomorphic via the map $x \mapsto -x$, we have

$$H_n(\mathbb{R}_{>0}) = H_n(\mathbb{R}_{<0}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{else} \end{cases}$$

Which means that

$$H_n(\mathbb{R}\backslash 0) = H_n(\mathbb{R}_{<0} \sqcup \mathbb{R}_{>0}) = H_n(\mathbb{R}_{<0}) \oplus H_n(\mathbb{R}_{>0}) = \begin{cases} \mathbb{Z}^2 & n = 0 \\ 0 & \text{else} \end{cases}$$

Cross product on homology is that $H_i(A) \times H_j(B) = H_{i+j}(A \times B)$, and it is easy to see that this extends to $H_i(A) \times H_j(B) \times H_k(C) = H_{i+j+k}(A \times B \times C)$.

So then we have $H_{i+j+k}(X) = H_i(\mathbb{R}\backslash\{0\}) \times H_j(\mathbb{R}) \times H_k(\mathbb{R}\backslash\{0\})$.

This tells us that

$$H_0(X) = H_0(\mathbb{R}\backslash\{0\}) \times H_0(\mathbb{R}) \times H_0(\mathbb{R}\backslash\{0\}) = \mathbb{Z}^2 \times \mathbb{Z} \times \mathbb{Z}^2 = \mathbb{Z}^5.$$

To compute $H_1(X)$ we look at

$$H_1(\mathbb{R}\backslash\{0\}) \times H_0(\mathbb{R}) \times H_0(\mathbb{R}\backslash\{0\}) = H_0(\mathbb{R}\backslash\{0\}) \times H_0(\mathbb{R}) \times H_1(\mathbb{R}\backslash\{0\}) = 0 \times \mathbb{Z} \times \mathbb{Z}^2 = \mathbb{Z}^3$$

and

$$H_0(\mathbb{R}\backslash\{0\}) \times H_1(\mathbb{R}) \times H_0(\mathbb{R}\backslash\{0\}) = 0$$

□

Part B

1. Give an example of a compact 2-dimensional manifold M for which there exists an embedding of $M \to \mathbb{R}^n$ into Euclidean space of strictly smaller dimension than that given by the Whitney embedding theorem.

   Proof. The Whitney embedding theorem states that a d-dimensional manifold with or without boundary can be properly smoothly embedded in $\mathbb{R}^{2d+1}$. Here, we take $d = 2$ and so the embedding certainly exists in $\mathbb{R}^5$.

   Consider the subset of $\mathbb{R}^2$ given by $I^2 := [0,1] \times [0,1]$. As a closed and bounded subset of $\mathbb{R}^2$ the Heine-Borel theorem tells us this is certainly compact.

   Now consider the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ taking $(x, y) \mapsto (x, y, 0)$. This is a smooth, proper embedding of I^2 into $\mathbb{R}^3$ a Euclidean space of dimension strictly smaller than 5. To see that the map is smooth, note that $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 1$ and $\frac{\partial f}{\partial z} = 0$. □

2. The cylindrical coordinate change is given by

   $$(x, y) = (r\cos(\theta), r\sin(\theta)).$$

   Express the vector field

   $$(x^2 + y^2)^{-2/3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

   in (r, θ) coordinates.

   Proof. Now recall that the change of coordinate map F which sends (x, y) coordinates to (r, θ) coordinates are $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(\frac{y}{x})$, and so

   $$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{1}{2\sqrt{x^2+y^2}}\frac{\partial(x^2+y^2)}{\partial x} \\ &= \frac{1}{2\sqrt{x^2+y^2}}(2x) \\ &= \frac{x}{\sqrt{x^2+y^2}}\end{aligned}$$

   and by symmetry $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$. Now we compute $\frac{\partial \theta}{\partial x}$ and $\frac{\partial \theta}{\partial y}$. First,

   $$\begin{aligned}\frac{\partial \theta}{\partial x} &= \frac{1}{1+(y/x)^2}\frac{\partial(y/x)}{\partial x} \\ &= \frac{1}{1+(y/x)^2} \cdot \frac{-y}{x^2} \\ &= \frac{-y}{x^2+y^2}.\end{aligned}$$

Next,

$$\begin{aligned}
\frac{\partial\theta}{\partial y} &= \frac{1}{1+(y/x)^2}\frac{\partial(y/x)}{\partial y} \\
&= \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} \\
&= \frac{1}{1+(y/x)^2}\cdot\frac{x}{x^2} \\
&= \frac{x}{x^2+y^2}.
\end{aligned}$$

The chain rule tells us that

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial x} \\
&= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} - \frac{y}{x^2+y^2}\frac{\partial}{\partial\theta} \\
&= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial y} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial y} \\
&= \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} + \frac{x}{x^2+y^2}\frac{\partial}{\partial\theta} \\
&= \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}
\end{aligned}$$

Finally, we change coordinates on x, y, leaving their differentials unchanged, then we substitute using the expressions for $\partial/\partial x$ and $\partial/\partial y$ which we found above and reduce using the Pythagorean trigonometric identity to see that the vector field in terms of cylindrical coordinates is:

$$\begin{aligned}
(x^2+y^2)^{-2/3}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right) &= ((r\cos\theta)^2+(r\sin\theta)^2)^{-2/3}\left((r\cos\theta)\frac{\partial}{\partial x}+(r\sin\theta)\frac{\partial}{\partial y}\right) \\
&= r^{-4/3}\left((r\cos\theta)\frac{\partial}{\partial x}+(r\sin\theta)\frac{\partial}{\partial y}\right) \\
&= r^{-4/3}\left((r\cos\theta)\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)+(r\sin\theta)\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\right) \\
&= r^{-4/3}\left((r\cos^2\theta+r\sin^2\theta)\frac{\partial}{\partial r}+(-\sin\theta\cos\theta+\cos\theta\sin\theta)\frac{\partial}{\partial\theta}\right) \\
&= r^{-4/3}\left(r\frac{\partial}{\partial r}\right) \\
&= \frac{1}{\sqrt[3]{r}}\frac{\partial}{\partial r}
\end{aligned}$$

□

3. Determine the Lie bracket of the vector fields $x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ and $y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

   Proof. □

4. Give an example of a surjection $f : M \to N$ of manifolds that is not a submersion.

Proof. Consider the projection map $\pi_1 : \mathbb{R}^2 \to R$ taking $(x, y) \mapsto x$. There is no right inverse to the projection map, since π is a many-to-one map. Hence it is not a submersion. $\square$