Fall 2019 Manifolds and Topology Preliminary Exam

University of Minnesota

Part A

1. Give a precise definition of the product $\alpha * \beta$ of two paths in a space X, including the conditions under which it is defined.

   Definition. *Let $\alpha, \beta : [0,1] \to X$ be continuous functions into X. If $\alpha(1) = \beta(0)$, then the product $\alpha * \beta : [0,1] \to X$ is defined to be*

   $$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & t \in [0, 1/2] \\ \beta(2t-1) & t \in (1/2, 1] \end{cases}$$

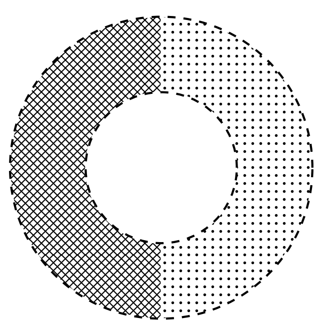
   .

2. Suppose that X is a path connected, semilocally 1-connected space whose fundamental group is $\mathbb{Z}/2 \times \mathbb{Z}/2$. How many isomorphism classes of connected covering spaces does X have?

   Proof. The isomorphism classes of covering spaces correspond to conjugacy classes of the fundamental group. Thus, this question is equivalent to counting the conjugacy classes of $\mathbb{Z}/2 \times \mathbb{Z}/2$. Note that $\mathbb{Z}/2 \times \mathbb{Z}$ is abelian and abelian groups have exactly one conjugacy class for each group element (since $hgh^{-1} = hh^{-1}g = g$ for any g, h). Thus, there are $|\mathbb{Z}/2 \times \mathbb{Z}/2| = 4$ conjugacy classes, and hence there are 4 isomorphism classes of covering spaces of X. □

3. Give an example of a space X with open subsets U, V such that $X = U \cup V$, U is simply connected, V is simply connected, but where X is not simply connected. Then explain why this does not violate the Siefert-van Kampen theorem.

   Proof. Let X be the annulus of inner radius 1 and outer radius 2, shown below,:

   and let U be the left half of the annulus (shown crosshatched) with ϵ extra and let V be the right half of the annulus (shown filled with dots) also with ϵ extra, where both U, V intersect in ϵ-fattenings of the line segments $\ell_+ = [(0,1),(0,2)]$ and $\ell_- = [(0,-1),(0,-2)]$.

   Then $\pi_1(U) = \pi_1(V) = 0$ since both are contractible. Additionally, $\pi_1(U \cap V) = 0$ since both the ϵ-fattenings of line segments ℓ_+, ℓ_- are contractible so regardless of choice of basepoint we are in a contractible connected component. U and V do not violate the hypothesis of the Siefert-van Kampen theorem, because the intersection $U \cap V$ is not path-connected. However if we were to ignore the requirement of path-connectedness, we might conclude that the theorem says $\pi_1(X) = 0/0 \cong 0$.

But we know that $\pi_1(U \cup V) \cong \mathbb{Z}$ because the generators are the trivial loop and the loop which contains the hole in its interior.

Thus, we cannot choose U, V as in this example if we wish to use the Siefert-van Kampen theorem. □

4. Explain why the inclusion $\mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ is not a covering map.

Proof. Suppose $\iota : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ were a covering map.

Then it is a theorem that the induced map on fundamental groups $\iota_* : \pi_1(\mathbb{R}^2 \backslash \{0\}) \to \pi_1(\mathbb{R})$ is an injection.

On the other hand, $\pi_1(\mathbb{R}^2 \backslash \{0\}) \cong \mathbb{Z}$ and $\pi_1(\mathbb{R}) = 0$ and so no injection from $\mathbb{Z} \to 0$ exists, contradicting that ι were a covering map. □

5. Define the *degree* of a map $f : S^n \to S^n$ for $n > 0$. Explain why $n = 0$ is a special case.

Definition. *Let $n > 0$ and $f : S^n \to S^n$ be a map. Then there is an induced homomorphism on homology $f_* : H_n(S^n) \to H_n(S^n)$. Since $H_n(S^n) \cong \mathbb{Z}$, then f_* must be multiplication by an integer (otherwise it would not be a homomorphism). The integer k such that $f_*(x) = kx$ is called the degree of the map.*

Note that $n = 0$ is a special case, since $H_0(S^0) \cong \mathbb{Z}^2$, and so "multiplication by an integer" must be more carefully defined.

6. Suppose X, Y are spaces that are both abstractly homeomorphic to S^7. Show that the "degree" of the map $f : X \to Y$ is only well-defined up to sign.

Proof. Let ϕ_X be the isomorphism $X \to S^7$ and ϕ_Y be the isomorphism $Y \to S^7$. Note that S^7 is homeomorphic to itself via both the identity map id and the antipodal map a.

Suppose we consider the homeomorphisms

$$g : X \overset{\phi_X}{\to} S^7 \overset{id}{\to} S^7 \overset{\phi_Y^{-1}}{\to} Y$$

and

$$h : X \overset{\phi_X}{\to} S^7 \overset{a}{\to} S^7 \overset{\phi_Y^{-1}}{\to} Y.$$

The antipodal map has degree $(-1)^7$, while the identity has degree 1. Now if we consider the map f given in the statement of the problem, then if we consider the homeomorphism going through id, we could get a positive sign, while if we go through a, we get a negative sign, hence the degree is only well defined up to sign. □

7. Let X be the space of upper triangular invertible 2×2 matrices:

$$X = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in M_2(\mathbb{R}) : a \neq 0, d \neq 0 \right\} \subseteq \mathbb{R}^3$$

Determine the homology groups $H_*(X)$.

Proof. First, note that $X \cong (\mathbb{R}\backslash\{0\}) \times (\mathbb{R}) \times (\mathbb{R}\backslash\{0\}) \subseteq \mathbb{R}^3$ by the map taking $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto (a, b, d)$.

Thus, we compute the homology groups $H_*(\mathbb{R}\backslash\{0\})$ and $H_*(\mathbb{R})$.

Since $X_1 := \mathbb{R}\backslash\{0\}$ is the disjoint union of the strictly positive real axis $\mathbb{R}_{>0}$ and the strictly negative real axis $\mathbb{R}_{<0}$, which are homeomorphic via the map $x \mapsto -x$, we have

$$H_n(\mathbb{R}_{>0}) = H_n(\mathbb{R}_{<0}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{else} \end{cases}$$

Which means that

$$H_n(\mathbb{R}\backslash 0) = H_n(\mathbb{R}_{<0} \sqcup \mathbb{R}_{>0}) = H_n(\mathbb{R}_{<0}) \oplus H_n(\mathbb{R}_{>0}) = \begin{cases} \mathbb{Z}^2 & n = 0 \\ 0 & \text{else} \end{cases}$$

For CW-complexes A, B a Künneth formula tells us that the following is a short exact sequence:

$$0 \to \bigoplus_i (H_i(A) \otimes H_{n-i}(B)) \to H_n(A \times B) \to \bigoplus_i \operatorname{Tor}(H_i(A), H_{n-i-1}(B)) \to 0 \tag{i}$$

We will first compute $H_\bullet(\mathbb{R} \times \mathbb{R}\backslash 0)$, taking $\mathbb{R} = A$ and $\mathbb{R}\backslash 0 = B$.
Rewriting the above sequence for our case, we see

$$0 \to \bigoplus_i (H_i(\mathbb{R}) \otimes H_{n-i}(\mathbb{R}\backslash 0)) \to H_n(\mathbb{R} \times \mathbb{R}\backslash 0) \to \bigoplus_i \operatorname{Tor}(H_i(\mathbb{R}), H_{n-i-1}(\mathbb{R}\backslash 0)) \to 0$$

First, note that for every i we have $H_i(\mathbb{R})$ is torsion-free, so $\operatorname{Tor}(H_i(\mathbb{R}), H_{n-i-1}(\mathbb{R}\backslash 0)) = 0$, hence

$$\bigoplus_i (H_i(\mathbb{R}) \otimes H_{n-i}(\mathbb{R}\backslash 0)) \cong H_n(\mathbb{R} \times \mathbb{R}\backslash 0)$$

When $n = 0$ we have

$$H_0(\mathbb{R}) \otimes H_0(\mathbb{R}\backslash 0) = \mathbb{Z} \otimes \mathbb{Z}^2 \cong \mathbb{Z}^2$$

When $n > 0$ we have

$$H_i(\mathbb{R}) \otimes H_{n-i}(\mathbb{R}\backslash 0) = \left(\begin{cases} \mathbb{Z} \otimes 0 & i = 0 \\ 0 \otimes 0 & i \neq 0, n \\ 0 \otimes \mathbb{Z}^2 & i = n \end{cases} \right) = 0,$$

and so

$$H_n(\mathbb{R} \times \mathbb{R}\backslash 0) = \begin{cases} \mathbb{Z}^2 & n = 0 \\ 0 & \text{else} \end{cases}.$$

We now use (i) with $A = \mathbb{R}\backslash 0$ and $B = \mathbb{R} \times \mathbb{R}\backslash 0$ and the fact that every homology group of A is torsion-free to see

$$\bigoplus_i (H_i(\mathbb{R}\backslash 0) \otimes H_{n-i}(\mathbb{R} \times \mathbb{R}\backslash 0)) \cong H_n(\mathbb{R}\backslash 0 \times \mathbb{R} \times \mathbb{R}\backslash 0) \cong H_n(X).$$

Arguing as above we see when $n = 0$ that

$$H_0(\mathbb{R}\backslash 0) \otimes H_0(\mathbb{R} \times \mathbb{R}\backslash 0) = \mathbb{Z}^2 \otimes \mathbb{Z}^2 \cong \mathbb{Z}^4,$$

and for $n > 0$

$$H_i(\mathbb{R}\backslash 0) \otimes H_{n-i}(\mathbb{R} \times \mathbb{R}\backslash 0) = \left(\begin{cases} \mathbb{Z}^2 \otimes 0 & i = 0 \\ 0 \otimes 0 & i \neq 0, n \\ 0 \otimes \mathbb{Z}^2 & i = n \end{cases} \right) = 0,$$

and so

$$H_n(X) = \begin{cases} \mathbb{Z}^4 & n = 0 \\ 0 & \text{else} \end{cases}$$

$\square$

8. Suppose X is a space with open subsets U, V such that $X = U \cup V$ and both U and V are contractable. What is the relationship between the homology groups of X and $U \cap V$

Proof. The Mayer-Vietoris sequence tells us that the following sequence is exact:

$$\cdots \longrightarrow \tilde{H}_n(U \cap V) \longrightarrow \tilde{H}_n(U) \oplus \tilde{H}_n(V) \longrightarrow \tilde{H}_n(U \cup V) \longrightarrow \tilde{H}_{n-1}(U \cap V) \longrightarrow \cdots$$

Since U, V are contractable, $\tilde{H}_n(U) = \tilde{H}_n(V) = 0$ for all n. Thus, we have the exactness of the sequence

$$\cdots \longrightarrow 0 \longrightarrow \tilde{H}_n(U \cup V) \longrightarrow \tilde{H}_{n-1}(U \cap V) \longrightarrow 0 \longrightarrow \cdots$$

and so we have that $\tilde{H}_n(X) \cong \tilde{H}_{n-1}(U \cap V)$ for all n.

□

9. Suppose that X is a path connected space, A is a path connected subspace, $p \in A$, and C is the mapping cone

$$(X \times \{1\} \cup A \times [0,1]) / \{(a,0) \sim (a',0)\}.$$

Use the Seifert-Van Kampen theorem to express the fundamental group $\pi_1(C, [(p, 1/2)])$ in terms of the map $\pi_1(A, p) \to \pi_1(X, p)$.

Proof. The Seifert-Van Kampen theorem tells us the following commutative diagram:

$$\begin{tikzcd}
 & \pi_1(X \times \{1\}) \arrow[dr, "\Phi"] & \\
\pi_1(A \times \{1\}) \arrow[ur] \arrow[dr] & & \pi_1(C) \\
 & \pi_1(A \times [0,1]/\{(a,0) \sim (a',0)\}) \arrow[ur] &
\end{tikzcd}$$

and in particular that Φ is surjective.

Noether's first isomorphism theorem tells us that $\pi_1(C) = \operatorname{im} \Phi \cong \pi_1(X)/\ker \Phi$. Since $A \times [0,1]/\sim$ is contractable (e.g. onto the point $(a, 0)$) we know that $\pi_1(A \times \{1\})$ is contained in the kernel of $\pi_1(X)$. On the other hand, the surjection Φ is constructed by a quotient of the free product $\pi_1(X \times \{1\}) * 0$ by the normal subgroup generated by loops in $\pi_1(A \times \{1\})$, and so the kernel contains *only* points in $\pi_1(A \times \{1\})$. Thus, $\pi_1(C) \cong \pi_1(X \times \{1\})/\pi_1(A \times \{1\}) \cong \pi_1(X)/\pi_1(A)$.

Strictly speaking, we have computed $\pi_1(C, (p, 1))$. However, since everything we have dealt with is path connected, we obtain the isomorphism $\pi_1(C, (p, 1)) \cong \pi_1(C, (p, 1/2))$.

□

10. State the *unique path lifting* theorem for covering maps.[1]

Theorem. *Let $p : \tilde{X} \to X$ be a covering map, and let $\gamma : [0,1] \to X$ be a path in X. Let $\gamma_1 : [0,1] \to \tilde{X}$ and $\gamma_2 : [0,1] \to \tilde{X}$ be two lifts of γ. If there is any $t_0 \in [0,1]$ such that $\gamma_1(t_0) = \gamma_2(t_0)$, then $\gamma_1(t) = \gamma_2(t)$ for all $t \in [0,1]$.*

[1] I think they are asking for thm. 1.34 of Hatcher, but I am not 100% sure.

Part B

1. Give an example of a compact 2-dimensional manifold M for which there exists an embedding of $M \to \mathbb{R}^n$ into Euclidean space of strictly smaller dimension than that given by the Whitney embedding theorem.

   Proof. The Whitney embedding theorem states that a d-dimensional manifold with or without boundary can be properly smoothly embedded in $\mathbb{R}^{2d+1}$. Here, we take $d = 2$ and so the embedding certainly exists in $\mathbb{R}^5$.

   Consider the subset of $\mathbb{R}^2$ given by $I^2 := [0,1] \times [0,1]$. As a closed and bounded subset of $\mathbb{R}^2$ the Heine-Borel theorem tells us this is certainly compact.

   Now consider the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ taking $(x, y) \mapsto (x, y, 0)$. This is a smooth, proper embedding of I^2 into $\mathbb{R}^3$ a Euclidean space of dimension strictly smaller than 5. To see that the map is smooth, note that $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 1$ and $\frac{\partial f}{\partial z} = 0$. □

2. The cylindrical coordinate change is given by

$$(x, y) = (r\cos(\theta), r\sin(\theta)).$$

   Express the vector field

$$(x^2 + y^2)^{-2/3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

   in (r, θ) coordinates.

   Proof. Now recall that the change of coordinate map F which sends (x, y) coordinates to (r, θ) coordinates are $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(\frac{y}{x})$, and so

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{1}{2\sqrt{x^2+y^2}}\frac{\partial(x^2+y^2)}{\partial x}\\ &= \frac{1}{2\sqrt{x^2+y^2}}(2x)\\ &= \frac{x}{\sqrt{x^2+y^2}}\end{aligned}$$

   and by symmetry $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$. Now we compute $\frac{\partial \theta}{\partial x}$ and $\frac{\partial \theta}{\partial y}$. First,

$$\begin{aligned}\frac{\partial \theta}{\partial x} &= \frac{1}{1+(y/x)^2}\frac{\partial(y/x)}{\partial x}\\ &= \frac{1}{1+(y/x)^2}\cdot\frac{-y}{x^2}\\ &= \frac{-y}{x^2+y^2}.\end{aligned}$$

   Next,

$$\begin{aligned}\frac{\partial \theta}{\partial y} &= \frac{1}{1+(y/x)^2}\frac{\partial(y/x)}{\partial y}\\ &= \frac{1}{1+(y/x)^2}\cdot\frac{1}{x}\\ &= \frac{1}{1+(y/x)^2}\cdot\frac{x}{x^2}\\ &= \frac{x}{x^2+y^2}.\end{aligned}$$

The chain rule tells us that

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} \\ &= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} - \frac{y}{x^2+y^2}\frac{\partial}{\partial \theta} \\ &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial y} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} \\ &= \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} + \frac{x}{x^2+y^2}\frac{\partial}{\partial \theta} \\ &= \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\end{aligned}$$

Finally, we change coordinates on x, y, leaving their differentials unchanged, then we substitute using the expressions for $\partial/\partial x$ and $\partial/\partial y$ which we found above and reduce using the Pythagorean trigonometric identity to see that the vector field in terms of cylindrical coordinates is:

$$\begin{aligned}(x^2+y^2)^{-2/3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) &= ((r\cos\theta)^2 + (r\sin\theta)^2)^{-2/3}\left((r\cos\theta)\frac{\partial}{\partial x} + (r\sin\theta)\frac{\partial}{\partial y}\right) \\ &= r^{-4/3}\left((r\cos\theta)\frac{\partial}{\partial x} + (r\sin\theta)\frac{\partial}{\partial y}\right) \\ &= r^{-4/3}\left((r\cos\theta)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) + (r\sin\theta)\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\right) \\ &= r^{-4/3}\left((r\cos^2\theta + r\sin^2\theta)\frac{\partial}{\partial r} + (-\sin\theta\cos\theta + \cos\theta\sin\theta)\frac{\partial}{\partial\theta}\right) \\ &= r^{-4/3}\left(r\frac{\partial}{\partial r}\right) \\ &= \frac{1}{\sqrt[3]{r}}\frac{\partial}{\partial r}\end{aligned}$$

□

3. Determine the Lie bracket of the vector fields $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

   Proof. If M is a smooth manifold with or without boundary and $X = X^i\frac{\partial}{\partial x^i}$ and $Y = Y^j\frac{\partial}{\partial x^j}$ are vector fields, then the Lie bracket is

   $$[X, Y] = \left(X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}.$$

   Since we are working in $\mathbb{R}^2$, then $x^1 = x$ and $x^2 = y$. Define $X := x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ so $X^1 = x$ and $X^2 = y$. Similarly $Y := y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ has $Y^1 = y$ and $Y^2 = -x$. So applying the formula for the Lie bracket, we get that the term in front of $\partial/\partial x$ should be

   $$\left(x\frac{\partial y}{\partial x} - y\frac{\partial x}{\partial x}\right) + \left(y\frac{\partial y}{\partial y} + x\frac{\partial x}{\partial y}\right)$$

which simplifies to

$$0 - y + y + 0 = 0$$

and the term in front of $\partial/\partial y$ should be

$$\left(x\frac{\partial(-x)}{\partial x} - y\frac{\partial y}{\partial x}\right) + \left(y\frac{\partial(-x)}{\partial y} + x\frac{\partial y}{\partial y}\right)$$

which simplifies to

$$-x - 0 + 0 + x = 0.$$

Thus

$$[X, Y] = 0.$$

$\square$

4. Give an example of a surjection $f : M \to N$ of manifolds that is not a submersion.

   Proof. Consider the projection map $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ taking $(x, y) \mapsto x$. There is no right inverse to the projection map, since π is a many-to-one map. Hence it is not a submersion. $\square$