# Fall 2015 Manifolds and Topology Preliminary Exam

University of Minnesota

**Part A**

1. (a) Define what it means for two paths $\alpha, \beta : [0,1] \to X$, with the same start and end points, to be homotopic. Show that this is an equivalence relation.

*Proof.* Let $X$ be a topological space and $\alpha, \beta$ be as in the problem statement. Then $\alpha, \beta$ are said to be homotopic if there is a function $H(s,t) : [0,1]^2 \to X$, continuous in both $s$ and $t$, such that $H(0,t) = \alpha(t)$ and $H(1,t) = \beta(t)$. We call such a function a homotopy from $\alpha$ to $\beta$. If there is a homotopy from $\alpha$ to $\beta$, we also say that $\alpha, \beta$ are homotopic.

We now show that the relation $\sim$ defined by $\alpha \sim \beta$ if $\alpha$ is homotopic to $\beta$ is reflexive, symmetric, and transitive. To see reflexivity observe that $H(s,t) = \alpha(t)$ for all $s$ is a homotopy from $\alpha$ to itself. To symmetricity, note that if $H(s,t)$ is a homotopy from $\alpha$ to $\beta$, then $H(1-s,t)$ is a homotopy from $\beta$ to $\alpha$. To see transitivity, let $\alpha \sim \beta$ and $\beta \sim \gamma$, and let $H_1$ be a homotopy from $\alpha$ to $\beta$ and $H_2$ be a homotopy from $\beta$ to $\gamma$. Then

$$H(s,t) := \begin{cases} H_1(2s,t) & 0 \le s \le 1/2 \\ H_2(2s-1,t) & 1/2 \le s \le 1 \end{cases}$$

is a homotopy from $\alpha$ to $\gamma$, so $\alpha \sim \gamma$. ☐

(b) Define what it means for a space to be simply connected

**Definition.** *Let $X$ be a topological space. Then $X$ is simply connected if and only if there is a unique homotopy class of paths connecting any two points in $X$. Equivalently, we say $X$ is simply connected if the fundamental group $\pi_1(X)$ is the trivial group.*

(c) Give a complete statement of the Siefert-van Kampen theorem relating the fundamental groups of $U$, $V$, $U \cup V$, $U \cap V$, including all necessary assumptions.

**Theorem.** *Let $U, V$ be open, path connected topological spaces such that $U \cap V$ is nonempty and path connected. The inclusion maps of $U \hookrightarrow U \cup V$ and $V \hookrightarrow U \cup V$ induce group homomorphisms $j_U : \pi_1(U) \to \pi_1(U \cup V)$ and $j_V : \pi_1(V) \to \pi_1(U \cup V)$. Then $U \cup V$ is path connected, and $j_U, j_V$ form a commutative pushout diagram:*

$$\begin{tikzcd}
 & \pi_1(U) \arrow[dr, dashed] \arrow[drr, bend left, "j_U"] & & \\
\pi_1(U \cap V) \arrow[ur, "i_U"] \arrow[dr, "i_V"'] & & \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \arrow[r, dashed, "k"] & \pi_1(U \cup V) \\
 & \pi_1(V) \arrow[ur, dashed] \arrow[urr, bend right, "j_V"'] & &
\end{tikzcd}$$

*Since this is a pushout diagram, then $k$ is an isomorphism.*

(d) Describe the fundamental group of $\mathbb{R}^2 - \{(-1,0),(1,0)\}$

*Proof.* Define $X := \mathbb{R}^2 - \{(-1,0),(1,0)\}$. Consider the sets $U = \{(x,y) : y > -1/2\} - \{(1,0)\}$ and $V = \{(x,y) : x < 1/2\} - \{(-1,0)\}$. First, observe that $U$ and $V$ both deformation retract to circles, and so $\pi_1(U) \cong \pi_1(V) \cong \mathbb{Z}$. Then, observe that $U, V$ satisfy the hypotheses of the

Siefert-van Kampen theorem, since $U \cap V = \{(x, y) \in \mathbb{R}^2 : -1/2 < y < 1/2\}$ which is non-empty and path connected.

Then the theorem tells us that $\pi_1(X) = \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V)$. We see that $U \cap V$ is contractable, and so $\pi_1(U \cap V)$ is the trivial group. Then this includes into $\pi_1(U)$ and $\pi_1(V)$ by taking the single (identity) element of $\pi_1(U\cap V)$ to the identities in $\pi_1(U)$ and $\pi_1(V)$ respectively. Thus, the amalgamated product $\pi_1(U)*\pi_1(V)/N$ where $N = \langle i_V(g)i_U(g)^{-1}, i_U(g)i_V(g)^{-1} : g \in \pi_1(U\cap V)\rangle$. To compute $N$, we note that since $i_U, i_V$ are group homomorphisms, they take the identity to the identity, and since the only $g \in \pi_1(U \cap V)$ is exactly the identity, then

$$N = \langle i_U(e)i_V(e)^{-1}, i_V(e)i_U(e)^{-1}\rangle = \langle e\rangle$$

so $N$ is the trivial group.

Now since $N$ is trivial, $\pi_1(U) * \pi_1(V)/N \cong \pi_1(U) * \pi_1(V) \cong \mathbb{Z} * \mathbb{Z}$ □

2. (a) State the classification theorem relating (connected) covering spaces of a (connected) space $X$ to the fundamental group of $X$. (You may take as given the standard assumptions that $X$ is locally path connected and locally simply connected.)

   **Theorem.** *Let $X$ be connected, locally path connected, and locally simply connected. Then there is a bijection between isomorphism classes of path connected covering spaces and conjugacy classes of $\pi_1(X, x_0)$.*

   (b) Suppose $X$ is as in the previous problem and that the fundamental group of $X$ is $\mathbb{Z}/2\times\mathbb{Z}/4$. How many isomorphism classes of connecting covering space does $X$ have?

   *Proof.* Because of the result of the last theorem, this boils down to counting conjugacy classes of $\mathbb{Z}/2 \times \mathbb{Z}/4$. Note that $\mathbb{Z}/2 \times \mathbb{Z}/4$ is an abelian group and so its conjugacy classes are singletons. Thus, the number of isomorphism classes of connected covering spaces of $X$ is the size of $\mathbb{Z}/2\times\mathbb{Z}/4$, which is 8. □

   (c) A continuous map $f : X \to Y$ is a local homeomorphism if, for every point $x \in X$, there exist open neighborhoods $U$ of $x$ and $V$ of $f(x)$ so that $f$ restricts to a homeomorphism $U \to V$. Give an example of a local homeomorphism which is not a covering map.

3. (a) Calculate the homology groups of the torus $S^1 \times S^1$

   *Proof.* First, we recall the fact that $H_n(S^1) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{else} \end{cases}$, and note that it is torsion-free. Since we can put a CW structure on $S^1$, a Künneth formula gives the short exact sequence

$$0 \to \bigoplus_i H_i(S^1) \otimes H_{n-i}(S^1) \to H_n(S^1 \times S^1) \to \bigoplus_i \operatorname{Tor}(H_i(S^1), H_{n-i-1}) \to 0.$$

   But since the homology groups are always torsion-free we obtain the isomorphism

$$\bigoplus_i H_i(S^1) \otimes H_{n-i}(S^1) \cong H_n(S^1 \times S^1).$$

   When $n = 0$ we have

$$H_0(S^1) \otimes H_0(S^1) = \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}.$$

   When $n = 1$ we have

$$\big(H_0(S^1) \otimes H_1(S^1)\big) \oplus \big(H_1(S^1) \otimes H_0(S^1)\big) = (\mathbb{Z} \otimes \mathbb{Z}) \oplus (\mathbb{Z} \otimes \mathbb{Z}) \cong \mathbb{Z}^2.$$

   When $n = 2$ we have

$$\big(H_0(S^1) \otimes H_2(S^1)\big) \oplus \big(H_1(S^1) \otimes H_1(S^1)\big) \oplus \big(H_2(S^1) \otimes H_0(S^1)\big) = (\mathbb{Z}\otimes 0) \oplus (\mathbb{Z}\otimes\mathbb{Z}) \oplus (0\otimes\mathbb{Z}) \cong \mathbb{Z}$$

For $n > 2$ we have

$$H_i(S^1) \otimes H_{n-i}(S^1) = \left( \begin{cases} \mathbb{Z} \otimes 0 & i = 0, 1 \\ 0 \otimes 0 & 1 < i < n-1 \\ 0 \otimes \mathbb{Z} & i = n-1, n \end{cases} \right) \cong 0.$$

Thus, we can conclude

$$H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & n = 0, 1, 2 \\ 0 & \text{else} \end{cases}$$

□