

Useful theorems for Manifolds and Topology Preliminary Exams

University of Minnesota

March 9, 2020

1 Fundamental Group

Theorem (Siefert-van Kampen). *Let U, V be open, path connected topological spaces such that $U \cap V$ is nonempty and path connected. The inclusion maps of $U \hookrightarrow U \cup V$ and $V \hookrightarrow U \cup V$ induce group homomorphisms $j_U : \pi_1(U) \rightarrow \pi_1(U \cup V)$ and $j_V : \pi_1(V) \rightarrow \pi_1(U \cup V)$. Then $U \cup V$ is path connected, and j_U, j_V form a commutative pushout diagram:*

$$\begin{array}{ccccc}
 & & \pi_1(U) & \xrightarrow{j_U} & \pi_1(U \cup V) \\
 & \nearrow i_U & \searrow & \nearrow & \\
 \pi_1(U \cap V) & & & \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) & \xrightarrow{k} \pi_1(U \cup V) \\
 & \searrow i_V & \nearrow & \nwarrow & \\
 & & \pi_1(V) & \xrightarrow{j_V} &
 \end{array}$$

Since this is a pushout diagram, then k is an isomorphism.

2 Covering Spaces

Theorem. *Let X be path connected, locally path connected, and semilocally simply connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$ obtained by associating the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ to the covering space (\tilde{X}, \tilde{x}_0) . If basepoints are ignored, this gives a bijection between isomorphism classes of path-connected covering spaces $p : \tilde{X} \rightarrow X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.*

Lemma. *If G is an abelian group, then the conjugacy classes of G are all singletons, so if G is finite, then $|G|$ is the number of conjugacy classes.*

3 Manifolds

Theorem (Sard's theorem). *Let M, N be smooth manifolds with or without boundary, and $F : M \rightarrow N$ be a smooth map. Then the set of critical values of F has measure 0 in N .*

Theorem (Whitney embedding theorem). *Every smooth n -manifold with or without boundary admits a proper smooth embedding into \mathbb{R}^{2n+1} .*

To add: Global Rank Theorem p. 83 of Lee