Useful theorems for Manifolds and Topology Preliminary Exams

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1 Fundamental Group

Theorem (Siefert-van Kampen)**.** *Let U, V be open, path connected topological spaces such that $U \cap V$ is nonempty and path connected. The inclusion maps of $U \hookrightarrow U \cup V$ and $V \hookrightarrow U \cup V$ induce group homomorphisms $j_U : \pi_1(U) \to \pi_1(U \cup V)$ and $j_V : \pi_1(V) \to \pi_1(U \cup V)$. Then $U \cup V$ is path connected, and j_U, j_V form a commutative pushout diagram:*

$$\begin{tikzcd}
 & \pi_1(U) \arrow[dr, dashed] \arrow[drr, bend left, "j_U"] & & \\
\pi_1(U \cap V) \arrow[ur, "i_U"] \arrow[dr, "i_V"'] & & \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \arrow[r, dashed, "k"] & \pi_1(U \cup V) \\
 & \pi_1(V) \arrow[ur, dashed] \arrow[urr, bend right, "j_V"'] & &
\end{tikzcd}$$

Since this is a pushout diagram, then k is an isomorphism.

2 Covering Spaces

Theorem. *Let X be path connected, locally path connected, and semilocally simply connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$ obtained by associating the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ to the covering space $(\tilde{X}, \tilde{x}_0)$. If basepoints are ignored, this gives a bijection between isomorphism classes of path-connected covering spaces $p : \tilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.*

Lemma. *If G is an abelian group, then the conjugacy classes of G are all singletons, so if G is finite, then $|G|$ is the number of conjugacy classes.*

3 Manifolds

Theorem (Sard's theorem)**.** *Let M, N be smooth manifolds with or without boundary, and $F : M \to N$ be a smooth map. Then the set of critical values of F has measure 0 in N.*

Theorem (Whitney embedding theorem)**.** *Every smooth n-manifold with or without boundary admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.*

Theorem (Coordinate formula for the Lie Bracket [?] prop. 8.26 verbaitm)**.** *Let X, Y be smooth vector fields on a smooth manifold M with or without boundary, and let $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ be the coordinate expressions for X and Y in terms of some smooth local coordinates (x^i) for M. Then $[X, Y]$ has the following coordinate expression:*

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

or

$$[X, Y] = (XY^j - YX^j) \frac{\partial}{\partial x^j}$$

To add: Global Rank Theorem p. 83 of Lee