

Fall 2019 Manifolds and Topology Preliminary Exam

University of Minnesota

Part A

1. Give a precise definition of the product $\alpha * \beta$ of two paths in a space X , including the conditions under which it is defined.

Definition. Let $\alpha, \beta : [0, 1] \rightarrow X$ be continuous functions into X . If $\alpha(1) = \beta(0)$, then the product $\alpha * \beta : [0, 1] \rightarrow X$ is defined to be

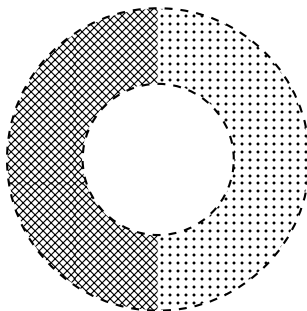
$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & t \in [0, 1/2] \\ \beta(2t - 1) & t \in (1/2, 1] \end{cases}$$

2. Suppose that X is a path connected, semilocally 1-connected space whose fundamental group is $\mathbb{Z}/2 \times \mathbb{Z}/2$. How many isomorphism classes of connected covering spaces does X have?

Proof. The isomorphism classes of covering spaces correspond to conjugacy classes of the fundamental group. Thus, this question is equivalent to counting the conjugacy classes of $\mathbb{Z}/2 \times \mathbb{Z}/2$. Note that $\mathbb{Z}/2 \times \mathbb{Z}$ is abelian and abelian groups have exactly one conjugacy class for each group element (since $hgh^{-1} = hh^{-1}g = g$ for any g, h). Thus, there are $|\mathbb{Z}/2 \times \mathbb{Z}/2| = 4$ conjugacy classes, and hence there are 4 isomorphism classes of covering spaces of X . \square

3. Give an example of a space X with open subsets U, V such that $X = U \cup V$, U is simply connected, V is simply connected, but where X is not simply connected. Then explain why this does not violate the Siefert-van Kampen theorem.

Proof. Let X be the annulus of inner radius 1 and outer radius 2, shown below,:



and let U be the left half of the annulus (shown crosshatched) with ϵ extra and let V be the right half of the annulus (shown filled with dots) also with ϵ extra, where both U, V intersect in ϵ -fattenings of the line segments $\ell_+ = [(0, 1), (0, 2)]$ and $\ell_- = [(0, -1), (0, -2)]$.

Then $\pi_1(U) = \pi_1(V) = 0$ since both are contractible. Additionally, $\pi_1(U \cap V) = 0$ since both the ϵ -fattenings of line segments ℓ_+, ℓ_- are contractible so regardless of choice of basepoint we are in a contractible connected component. U and V do not violate the hypothesis of the Siefert-van Kampen

theorem, because the intersection $U \cap V$ is not path-connected. However if we were to ignore the requirement of path-connectedness, we might conclude that the theorem says $\pi_1(X) = 0/0 \cong 0$.

But we know that $\pi_1(U \cup V) \cong \mathbb{Z}$ because the generators are the trivial loop and the loop which contains the hole in its interior.

Thus, we cannot choose U, V as in this example if we wish to use the Siefert-van Kampen theorem. \square

4. Explain why the inclusion $\mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ is not a covering map.

Proof. Suppose $\iota : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ were a covering map.

Then it is a theorem that the induced map on fundamental groups $\iota_* : \pi_1(\mathbb{R}^2 \setminus \{0\}) \rightarrow \pi_1(\mathbb{R})$ is an injection.

On the other hand, $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$ and $\pi_1(\mathbb{R}) = 0$ and so no injection from $\mathbb{Z} \rightarrow 0$ exists, contradicting that ι were a covering map. \square

5. Define the *degree* of a map $f : S^n \rightarrow S^n$ for $n > 0$. Explain why $n = 0$ is a special case.

Definition. Let $n > 0$ and $f : S^n \rightarrow S^n$ be a map. Then there is an induced homomorphism on homology $f_* : H_n(S^n) \rightarrow H_n(S^n)$. Since $H_n(S^n) \cong \mathbb{Z}$, then f_* must be multiplication by an integer (otherwise it would not be a homomorphism). The integer k such that $f_*(x) = kx$ is called the *degree* of the map.

Note that $n = 0$ is a special case, since $H_0(S^0) \cong \mathbb{Z}^2$, and so “multiplication by an integer” must be more carefully defined.

6. Suppose X, Y are spaces that are both abstractly homeomorphic to S^7 . Show that the “degree” of the map $f : X \rightarrow Y$ is only well-defined up to sign.
7. Let X be the space of upper triangular invertible 2×2 matrices:

$$X = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in M_2(\mathbb{R}) : a \neq 0, d \neq 0 \right\} \subseteq \mathbb{R}^3$$

Determine the homology groups $H_*(X)$.

Proof. First, note that $X \cong (\mathbb{R} \setminus \{0\}) \times (\mathbb{R}) \times (\mathbb{R} \setminus \{0\}) \subseteq \mathbb{R}^3$ by the map taking $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto (a, b, d)$.

Thus, we compute the homology groups $H_*(\mathbb{R} \setminus \{0\})$ and $H_*(\mathbb{R})$.

Since $X_1 := \mathbb{R} \setminus \{0\}$ is the disjoint union of the strictly positive real axis $\mathbb{R}_{>0}$ and the strictly negative real axis $\mathbb{R}_{<0}$, which are homeomorphic via the map $x \mapsto -x$, we have

$$H_n(\mathbb{R}_{>0}) = H_n(\mathbb{R}_{<0}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{else} \end{cases}$$

Which means that

$$H_n(\mathbb{R} \setminus \{0\}) = H_n(\mathbb{R}_{<0} \sqcup \mathbb{R}_{>0}) = H_n(\mathbb{R}_{<0}) \oplus H_n(\mathbb{R}_{>0}) = \begin{cases} \mathbb{Z}^2 & n = 0 \\ 0 & \text{else} \end{cases}$$

For CW-complexes A, B a Künneth formula tells us that the following is a short exact sequence:

$$0 \rightarrow \bigoplus_i (H_i(A) \otimes H_{n-i}(B)) \rightarrow H_n(A \times B) \rightarrow \bigoplus_i \text{Tor}(H_i(A), H_{n-i-1}(B)) \rightarrow 0 \quad (\text{i})$$

We will first compute $H_\bullet(\mathbb{R} \times \mathbb{R} \setminus \{0\})$, taking $\mathbb{R} = A$ and $\mathbb{R} \setminus \{0\} = B$.

Rewriting the above sequence for our case, we see

$$0 \rightarrow \bigoplus_i (H_i(\mathbb{R}) \otimes H_{n-i}(\mathbb{R} \setminus 0)) \rightarrow H_n(\mathbb{R} \times \mathbb{R} \setminus 0) \rightarrow \bigoplus_i \text{Tor}(H_i(\mathbb{R}), H_{n-i-1}(\mathbb{R} \setminus 0)) \rightarrow 0$$

First, note that for every i we have $H_i(\mathbb{R})$ is torsion-free, so $\text{Tor}(H_i(\mathbb{R}), H_{n-i-1}(\mathbb{R} \setminus 0)) = 0$, hence

$$\bigoplus_i (H_i(\mathbb{R}) \otimes H_{n-i}(\mathbb{R} \setminus 0)) \cong H_n(\mathbb{R} \times \mathbb{R} \setminus 0)$$

When $n = 0$ we have

$$H_0(\mathbb{R}) \otimes H_0(\mathbb{R} \setminus 0) = \mathbb{Z} \otimes \mathbb{Z}^2 \cong \mathbb{Z}^2$$

When $n > 0$ we have

$$H_i(\mathbb{R}) \otimes H_{n-i}(\mathbb{R} \setminus 0) = \begin{cases} \mathbb{Z} \otimes 0 & i = 0 \\ 0 \otimes 0 & i \neq 0, n \\ 0 \otimes \mathbb{Z}^2 & i = n \end{cases} = 0,$$

and so

$$H_n(\mathbb{R} \times \mathbb{R} \setminus 0) = \begin{cases} \mathbb{Z}^2 & n = 0 \\ 0 & \text{else} \end{cases}.$$

We now use (i) with $A = \mathbb{R} \setminus 0$ and $B = \mathbb{R} \times \mathbb{R} \setminus 0$ and the fact that every homology group of A is torsion-free to see

$$\bigoplus_i (H_i(\mathbb{R} \setminus 0) \otimes H_{n-i}(\mathbb{R} \times \mathbb{R} \setminus 0)) \cong H_n(\mathbb{R} \setminus 0 \times \mathbb{R} \times \mathbb{R} \setminus 0) \cong H_n(X).$$

Arguing as above we see when $n = 0$ that

$$H_0(\mathbb{R} \setminus 0) \otimes H_0(\mathbb{R} \times \mathbb{R} \setminus 0) = \mathbb{Z}^2 \otimes \mathbb{Z}^2 \cong \mathbb{Z}^4,$$

and for $n > 0$

$$H_i(\mathbb{R} \setminus 0) \otimes H_{n-i}(\mathbb{R} \times \mathbb{R} \setminus 0) = \begin{cases} \mathbb{Z}^2 \otimes 0 & i = 0 \\ 0 \otimes 0 & i \neq 0, n \\ 0 \otimes \mathbb{Z}^2 & i = n \end{cases} = 0,$$

and so

$$H_n(X) = \begin{cases} \mathbb{Z}^4 & n = 0 \\ 0 & \text{else} \end{cases}$$

□

Part B

1. Give an example of a compact 2-dimensional manifold M for which there exists an embedding of $M \rightarrow \mathbb{R}^n$ into Euclidean space of strictly smaller dimension than that given by the Whitney embedding theorem.

Proof. The Whitney embedding theorem states that a d -dimensional manifold with or without boundary can be properly smoothly embedded in \mathbb{R}^{2d+1} . Here, we take $d = 2$ and so the embedding certainly exists in \mathbb{R}^5 .

Consider the subset of \mathbb{R}^2 given by $I^2 := [0, 1] \times [0, 1]$. As a closed and bounded subset of \mathbb{R}^2 the Heine-Borel theorem tells us this is certainly compact.

Now consider the map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ taking $(x, y) \mapsto (x, y, 0)$. This is a smooth, proper embedding of I^2 into \mathbb{R}^3 a Euclidean space of dimension strictly smaller than 5. To see that the map is smooth, note that $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 1$ and $\frac{\partial f}{\partial z} = 0$. □

2. The cylindrical coordinate change is given by

$$(x, y) = (r \cos(\theta), r \sin(\theta)).$$

Express the vector field

$$(x^2 + y^2)^{-2/3} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)$$

in (r, θ) coordinates.

Proof. Now recall that the change of coordinate map F which sends (x, y) coordinates to (r, θ) coordinates are $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(\frac{y}{x})$, and so

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{1}{2\sqrt{x^2 + y^2}} \frac{\partial(x^2 + y^2)}{\partial x} \\ &= \frac{1}{2\sqrt{x^2 + y^2}} (2x) \\ &= \frac{x}{\sqrt{x^2 + y^2}} \end{aligned}$$

and by symmetry $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$. Now we compute $\frac{\partial \theta}{\partial x}$ and $\frac{\partial \theta}{\partial y}$. First,

$$\begin{aligned} \frac{\partial \theta}{\partial x} &= \frac{1}{1 + (y/x)^2} \frac{\partial(y/x)}{\partial x} \\ &= \frac{1}{1 + (y/x)^2} \cdot \frac{-y}{x^2} \\ &= \frac{-y}{x^2 + y^2}. \end{aligned}$$

Next,

$$\begin{aligned} \frac{\partial \theta}{\partial y} &= \frac{1}{1 + (y/x)^2} \frac{\partial(y/x)}{\partial y} \\ &= \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} \\ &= \frac{1}{1 + (y/x)^2} \cdot \frac{x}{x^2} \\ &= \frac{x}{x^2 + y^2}. \end{aligned}$$

The chain rule tells us that

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta} \frac{\partial \theta}{\partial x} \\ &= \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial r} - \frac{y}{x^2 + y^2} \frac{\partial}{\partial \theta} \\ &= \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial y} &= \frac{\partial}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta} \frac{\partial \theta}{\partial y} \\
&= \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial r} + \frac{x}{x^2 + y^2} \frac{\partial}{\partial \theta} \\
&= \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}
\end{aligned}$$

Finally, we change coordinates on x, y , leaving their differentials unchanged, then we substitute using the expressions for $\partial/\partial x$ and $\partial/\partial y$ which we found above and reduce using the Pythagorean trigonometric identity to see that the vector field in terms of cylindrical coordinates is:

$$\begin{aligned}
(x^2 + y^2)^{-2/3} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) &= ((r \cos \theta)^2 + (r \sin \theta)^2)^{-2/3} \left((r \cos \theta) \frac{\partial}{\partial x} + (r \sin \theta) \frac{\partial}{\partial y} \right) \\
&= r^{-4/3} \left((r \cos \theta) \frac{\partial}{\partial x} + (r \sin \theta) \frac{\partial}{\partial y} \right) \\
&= r^{-4/3} \left((r \cos \theta) \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) + (r \sin \theta) \left(\sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \right) \right) \\
&= r^{-4/3} \left((r \cos^2 \theta + r \sin^2 \theta) \frac{\partial}{\partial r} + (-\sin \theta \cos \theta + \cos \theta \sin \theta) \frac{\partial}{\partial \theta} \right) \\
&= r^{-4/3} \left(r \frac{\partial}{\partial r} \right) \\
&= \frac{1}{\sqrt[3]{r}} \frac{\partial}{\partial r}
\end{aligned}$$

□

3. Determine the Lie bracket of the vector fields $x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ and $y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$ on \mathbb{R}^2 .

Proof.

□

4. Give an example of a surjection $f : M \rightarrow N$ of manifolds that is not a submersion.

Proof. Consider the projection map $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ taking $(x, y) \mapsto x$. There is no right inverse to the projection map, since π is a many-to-one map. Hence it is not a submersion. □