The Limits of Sorting
Divide-and-Conquer
Comparison Sorts II

CS 311 Data Structures and Algorithms
Lecture Slides
Friday, March 1, 2013

Chris Hartman
Department of Computer Science
University of Alaska Fairbanks
cmhartman@alaska.edu
Based on material by Glenn G. Chappell

Unit Overview
Algorithmic Efficiency & Sorting

Major Topics

- ✓ Introduction to Analysis of Algorithms
- ✓ Introduction to Sorting
- Comparison Sorts I
- More on Big-*O*
- The Limits of Sorting
- Divide-and-Conquer
- Comparison Sorts II
- Comparison Sorts III
- Radix Sort
- Sorting in the C++ STL

Review
Introduction to Analysis of Algorithms [1/2]

Efficiency

- General: using few resources (time, space, bandwidth, etc.).
- Specific: fast (time).
  - Also can be qualified, e.g., space efficiency.

Analyzing Efficiency

- Measure running time in **steps**.
- Determine how the **size of the input** affects running time.
- **Worst case**: max steps for given input size.

Scalable: works well with large problems. Also “**scales well**”.

Review
Introduction to Analysis of Algorithms [2/2]

We say $g(n)$ is $O(f(n))$ if

- There exist constants k and n_0 such that
- $g(n) \le k \times f(n)$, whenever $n \ge n_0$.

Useful: $g(n)$ = max steps used by an algorithm for input of size n.

Efficiency categories we will use.

Using Big-O	In Words
$O(1)$	Constant time
$O(\log_b n)$, for some $b > 1$	Logarithmic time
$O(n)$	Linear time
$O(n \log_b n)$, for some $b > 1$	Log-linear time
$O(n^2)$	Quadratic time
$O(b^n)$, for some $b > 1$	Exponential time

Cannot read all of input ↑

Probably not scalable ↓

↑ Faster

Slower ↓

I will also allow $O(n^3)$, $O(n^4)$, etc.

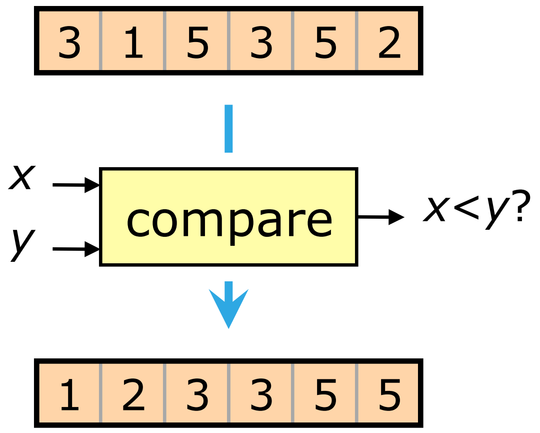
Review
Introduction to Sorting — Basics, Analyzing

Sort: Place a collection of data in order.

Key: The part of the data item used to sort.

Comparison sort: A sorting algorithm that gets its information by comparing items in pairs.

A **general-purpose comparison sort** places no restrictions on the size of the list or the values in it.



Five criteria for analyzing a general-purpose comparison sort:

- (Time) Efficiency
- Requirements on Data
- Space Efficiency
- Stability
- Performance on Nearly Sorted Data

In-place = no large additional space required.

Stable = never changes the order of equivalent items.

1. All items close to proper places, OR
2. few items out of order.

Review
Introduction to Sorting — Overview of Algorithms

There is no *known* sorting algorithm that has all the properties we would like one to have.

We will examine a number of sorting algorithms. Most of these fall into two categories: $O(n^2)$ and $O(n \log n)$.

- Quadratic-Time [$O(n^2)$] Algorithms
  - ✓ Bubble Sort
  - Insertion Sort
  - Quicksort
  - Treesort (later in semester)
- Log-Linear-Time [$O(n \log n)$] Algorithms
  - Merge Sort
  - Heap Sort (mostly later in semester)
  - Introsort (not in the text)
- Special Purpose — Not Comparison Sorts
  - Pigeonhole Sort
  - Radix Sort

Review
Comparison Sorts I — Bubble Sort: Analysis

(Time) Efficiency ☹

- Bubble Sort is $O(n^2)$.
- Bubble Sort also has an average-case time of $O(n^2)$. ☹

Requirements on Data ☺

- Bubble Sort does not require random-access data.
- It works on Linked Lists.

Space Efficiency ☺

- Bubble Sort can be done in-place.

Stability ☺

- Bubble Sort is stable.

Performance on Nearly Sorted Data ☺/☹

- (1) We can write Bubble Sort to be $O(n)$ if no item is far out of place. ☺
- (2) Bubble Sort is $O(n^2)$ even if only one item is far out of place. ☹

Comparison Sorts I
Insertion Sort — Description

Bubble Sort essentially constructs a sorted sequence in (backwards) order:

- Find the greatest item (by “bubbling”), then find the next greatest, etc.
- So for each **position**, starting with the last, it finds the **item** that belongs there.

Suppose we “flip” this idea.

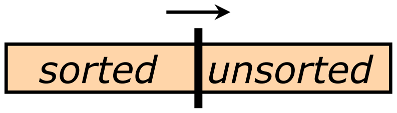
- Instead of looking through the positions and determining what item belongs in each, look through the given **items**, determine in which **position** each belongs, and then insert it in that position.

This idea leads to an algorithm called **Insertion Sort**.

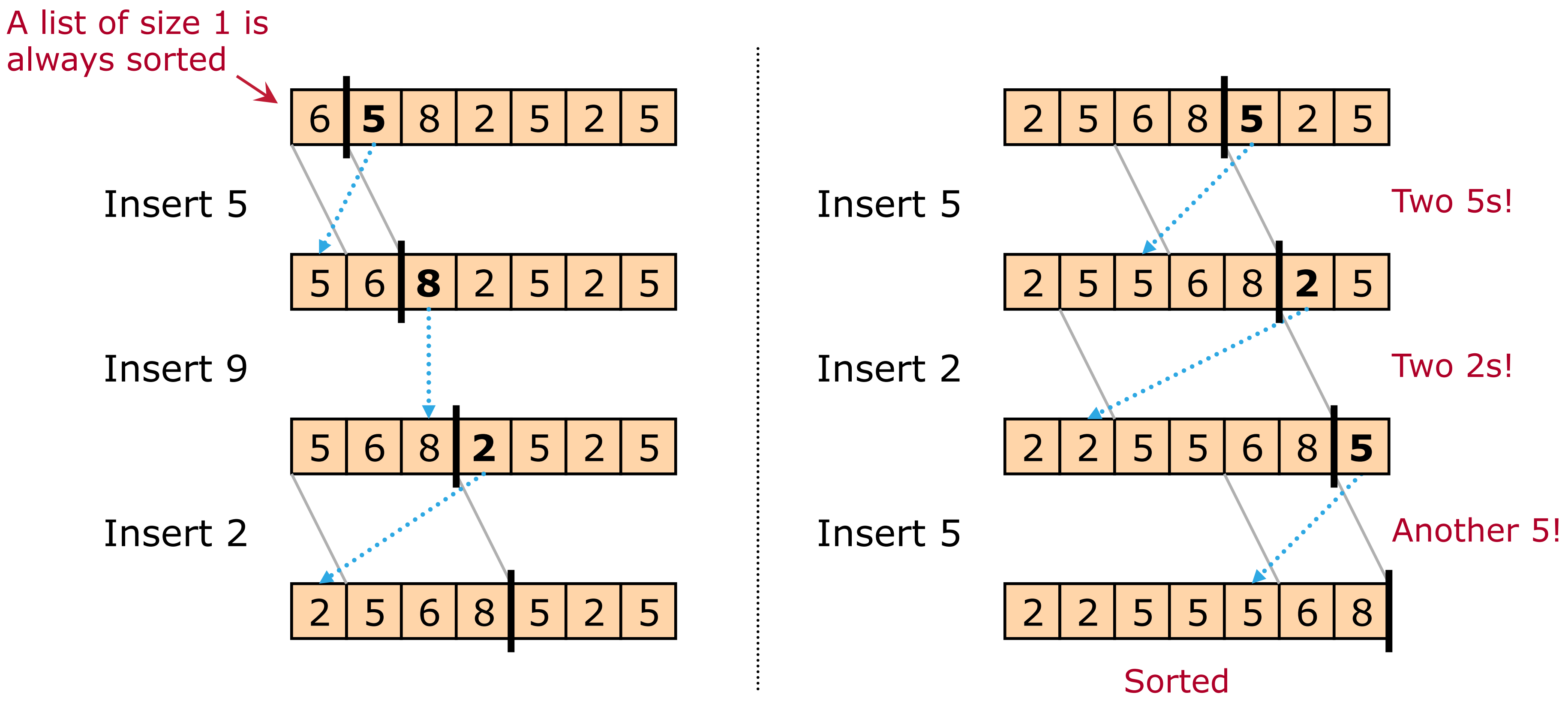
- Iterate through the items in the sequence.
- For each, insert it in the proper place among the preceding items.
- Thus, when we are processing item k, we have items 0 .. $k–1$ already in sorted order.

Comparison Sorts I
Insertion Sort — Illustration

Items to left of bold bar are sorted.



Bold item = item to be inserted into sorted section.



Comparison Sorts I
Insertion Sort — Issues

When we insert an item into the list of already sorted items, three operations are needed:

- **Find** (the proper location).
- **Remove** (the old copy).
- **Insert** (into the proper location).

If we have **random-access** data (an array?), then

- Find *could* be fast [Binary Search: $O(\log n)$].
  - In practice, we do not actually use Binary Search, as we will see.
- Remove + Insert is slow-ish [move items up: $O(n)$].
- Find + Remove + Insert is $O(n)$.

If we have a **Linked List**, then

- Find is slow-ish [Sequential Search: $O(n)$].
- Remove is fast [Linked-List removal: $O(1)$].
- Insert is fast [Linked-List insertion: $O(1)$].
- Find + Remove + Insert is $O(n)$.

In both cases, Find + Remove + Insert is $O(n)$.

Comparison Sorts I
Insertion Sort — Write It

TO DO

- Examine implementation of Insertion Sort.
- Analyze, as before.
  - *See the next slide.*

Comparison Sorts I
Insertion Sort — Analysis

(Time) Efficiency ☹

- Insertion Sort is O(n^2).
- Insertion Sort also has an average-case time of O(n^2). ☹

Requirements on Data ☺

- Insertion Sort does not require random-access data.
- It works on Linked Lists.*

Space Efficiency ☺

- Insertion Sort can be done in-place.*

Stability ☺

- Insertion Sort is stable.

Performance on Nearly Sorted Data ☺

- (1) Insertion Sort can be written to be $O(n)$ if each item is at most some constant distance from its proper place.*
- (2) Insertion Sort can be written to be $O(n)$ if only a constant number of items are out of place.

*For one-way sequential-access data (e.g., Linked Lists) we give up *EITHER* in-place *OR* $O(n)$ on type (1) nearly sorted data.

Comparison Sorts I
Insertion Sort — Comments

Insertion Sort is too slow for general-purpose use.

However, Insertion Sort is useful in certain special cases.

- Insertion Sort is fast (linear time) for **nearly sorted data**.
- Insertion Sort is also considered fast for **small lists**.

Insertion Sort often appears as part of another algorithm.

- Most good sorting methods call Insertion Sort for small lists.
- Some sorting methods get the data nearly sorted, and then finish with a call to Insertion Sort. (More on this later.)

Implementation Issue

- For random-access data, Insertion Sort *could* use Binary Search in the "find" step. However, this is not the best method for nearly sorted data. Since that is when Insertion Sort is actually useful, we generally do the "find" step of Insertion Sort using a backwards Sequential Search.

More on Big-*O*
Big-*O* More Generally — Introduction

Recall our definition of big-*O*:

- Algorithm *A* is *order* $f(n)$ [written $O(f(n))$] if
  - There exist constants k and n_0 such that
  - *A* requires **no more than** $k \times f(n)$ time units to solve a problem of size $n \geq n_0$.

The fundamental idea here actually has very little to do with algorithms.

- Rather, this is a method for talking about **how quickly a function grows** (a *mathematical* function, that is).
- We have applied this idea to the (mathematical) function that tells us the maximum number of steps an algorithm takes for input of a given size.
- But we could apply it to other things, too.

More on Big-*O*
Big-*O* More Generally — Definition

Say we have nonnegative real-valued functions *f* and *g* on the nonnegative integers.

- So, for each nonnegative integer *n*, *f*(*n*) and *g*(*n*) are nonnegative real numbers.

We say *g*(*n*) is *O*(*f*(*n*)) if

- There exist constants *k* and n_0 such that
- $g(n) \le k \times f(n)$, whenever $n \ge n_0$.

We get our previous definition of big-*O* if we let *g*(*n*) be the maximum number of steps required to execute algorithm *A* for input of size *n*.

More on Big-*O*
Big-*O* More Generally — Applications

We can now use big-*O* for more general concepts.

For example, space efficiency:

- We have defined "in-place" to be the same as $O(1)$ additional space.
  - "Additional" = beyond the space required by its input.
  - Thus: *Constant* additional space.
- So Bubble Sort and Insertion Sort use $O(1)$, that is, constant, additional space.
  - Our next sorting algorithm will use more than this.

More on Big-*O*
Related Concepts

Related to big-*O* are Ω (omega) and Θ (theta).

We say $g(n)$ is $\Omega(f(n))$ if

- There exist constants k and n_0 such that
- $g(n) \geq k \times f(n)$, whenever $n \geq n_0$.

Our big-*O* definition has "≤" here.

If we say an *algorithm* is $\Omega(f(n))$, then we mean that its *worst-case* number of steps is at least $k \times f(n)$, for some k. (Its best-case may be smaller.)

We say $g(n)$ is $\Theta(f(n))$ if

- $g(n)$ is $O(f(n))$, and
- $g(n)$ is $\Omega(f(n))$.

The "*k*" values used above may be different.

- For example, a function would be $\Theta(n^2)$ if it always lies between (say) $3n^2$ and $5n^2$, for sufficiently large n.

More on Big-O
Summary

Three ways to talk about how fast a function grows.

$g(n)$ is:

- $O(f(n))$ if $g(n) \le k \times f(n)$ …
- $\Omega(f(n))$ if $g(n) \ge k \times f(n)$ …
- $\Theta(f(n))$ if both of the above are true.
  - Possibly with different values of k *(… and n_0)*.

Useful: Let $g(n)$ be the max number of steps required by some algorithm when given input of size n.

Or: Let $g(n)$ be the max amount of additional space required when given input of size n.

The Limits of Sorting
Introduction

We have mentioned that most sorting algorithms fall into one of two categories:

- Slow: $O(n^2)$.
- Fast: $O(n \log n)$.
  - We have not discussed any of these fast algorithms yet, however.

Can we do even better?

- No, not with a general purpose comparison sort.
- Writing a general purpose comparison sort that lies in any time-efficiency category faster than $O(n \log n)$ is **impossible**.
  - Remember: worst-case analysis.
- More precisely: We can **prove** that the **worst-case number of comparisons** performed by a general purpose comparison sort must be $\Omega(n \log_2 n)$.

The Limits of Sorting
Thinking about Sorting

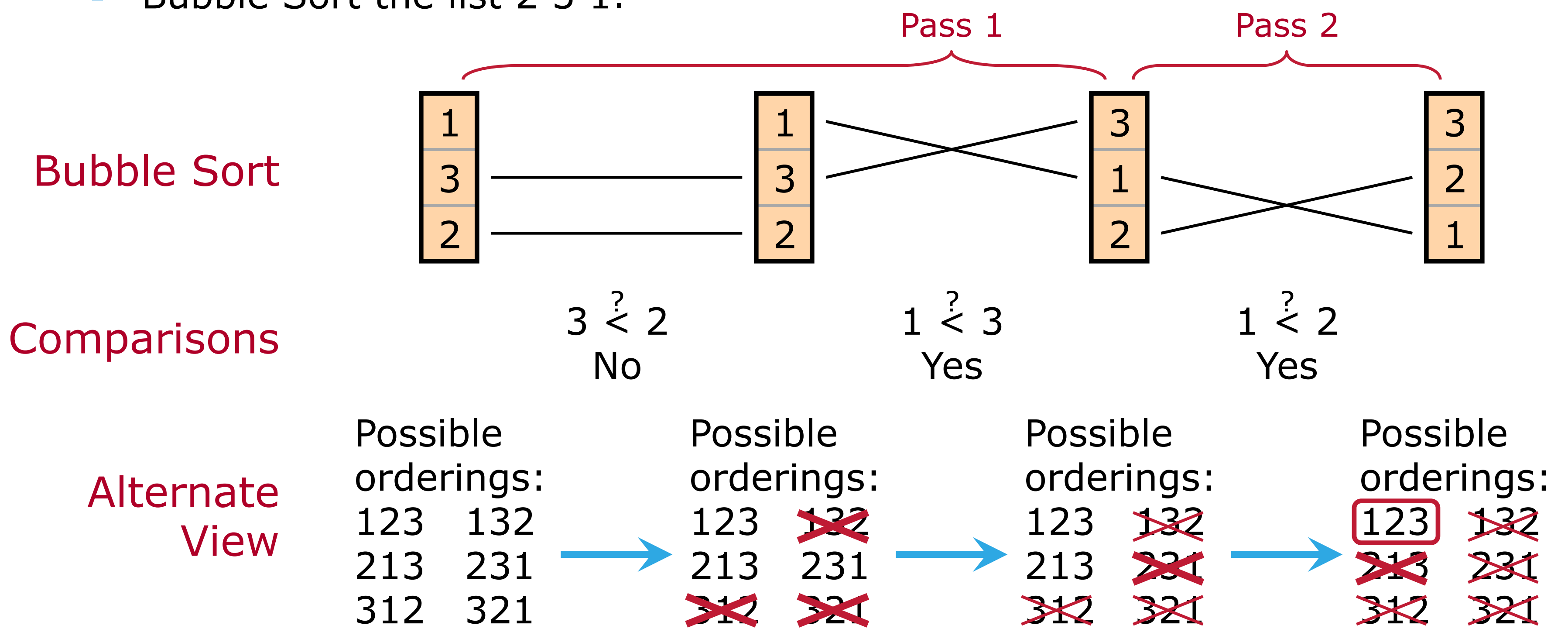
Sorting is determining the ordering of a list. Many orderings are possible.

Each time we do a comparison, we find the relative order of two items.
    Say x < y; we can throw out all orderings in which y comes before x.

We cannot stop until only one possible ordering is left.

Example

- Bubble Sort the list 2 3 1.


Pass 1
Pass 2
Bubble Sort
1 3 2
1 3 2
3 1 2
3 2 1
Comparisons
$3 \overset{?}{<} 2$ No
$1 \overset{?}{<} 3$ Yes
$1 \overset{?}{<} 2$ Yes
Alternate View
Possible orderings: 123 132 213 231 312 321
Possible orderings: 123 ~~132~~ 213 231 ~~312~~ ~~321~~
Possible orderings: 123 ~~132~~ 213 ~~231~~ ~~312~~ ~~321~~
Possible orderings: $\boxed{123}$ ~~132~~ ~~213~~ ~~231~~ ~~312~~ ~~321~~

The Limits of Sorting
Proof Idea

Based on the idea in the previous slide, we can **prove** that the worst-case number of comparisons done by a general purpose comparison sort must be $\Omega(n \log_2 n)$.

Outline of the Proof

- We are given a list of n items to be sorted.
- There are $n! = n \times (n-1) \times \ldots \times 3 \times 2 \times 1$ orderings of n items.
- We start with all $n!$ orderings. We do comparisons, throwing out orderings that do not match our new information.
- With each comparison, we cannot **force** more than **half** of the orderings to be thrown out. (Remember: worst case.)
- How many times must we cut $n!$ in half, to get 1? Answer: $\log_2(n!)$, or a little more, since we deal only with integers. The worst case uses at least that many comparisons.
- And $\log_2(n!)$ turns out to be close to $n(\log_2 n - \log_2 e) + 1/2 \log_2(2\pi n)$.
  - *We will not verify this step. Look up "Stirling's Approximation" for info.*

Divide-and-Conquer
Introduction

The “obvious” way to search is Sequential Search. But we have seen how to do better on sorted data: Binary Search.

Binary Search splits its input into parts and handles them recursively.

This last idea is called **divide-and-conquer**.

- A common way to design fast algorithms.

Questions

- How do we analyze the efficiency of algorithms that use divide-and-conquer?
- Can we use divide-and-conquer to come up with an improved sorting algorithm? One that is $O(n \log n)$? (We have not seen any, yet.)

Divide-and-Conquer
The Master Theorem [1/3]

Suppose we are analyzing an algorithm.

- It takes input of size n.
- The number of steps it requires is at most $T(n)$.
  - T is a (mathematical) function.
- We want to know what $T(n)$ is, roughly.

Suppose our algorithm uses divide-and-conquer:

- It splits the input into $\boxed{b}$ $\boxed{\text{nearly equal-sized}}$ parts.
- It makes a recursive calls each taking one of the parts as input.
  - Write $a = b^k$, for some $\boxed{k} \geq 0$.
- It does some other work requiring $\boxed{f(n)}$ operations.

Important, for our analysis

This gives us a recurrence relation:

- $T(n) = b^k\, T(n/b) + f(n)$.

Given such a recurrence, we can often use the **Master Theorem**.

Divide-and-Conquer
The Master Theorem [2/3]

The Master Theorem*

- Suppose $T(n) = b^k\, T(n/b) + f(n)$, where $b > 1$ and $k \geq 0$.
  - "n/b" means the next integer going up or down, as appropriate.
- Case 1
  - If $f(n)$ is $O(n^{k-\varepsilon})$, for some $\varepsilon > 0$, then $T(n)$ is $\Theta(n^k)$.
- Case 2
  - If $f(n)$ is $\Theta(n^k)$, then $T(n)$ is $\Theta(n^k \log n)$.

*The Master Theorem, as it is usually stated, actually says a little more than this.

- There is a Case 3, which is more complex, but essentially says that, if $f(n)$ is large, then $T(n)$ is of the same order as $f(n)$.
- However, Case 1 and Case 2 are all we will use in this class.

Divide-and-Conquer
The Master Theorem [3/3]

How the Master Theorem is applied to analyze a recursive algorithm:

- An algorithm takes input of size *n*. It splits its input into nearly equal-sized parts, and makes recursive calls, each call handling one of the parts.
  - *b* is the number of nearly equal-sized parts.
  - b^k is the number of recursive calls. Find *k*.
  - $f(n)$ is the amount of extra work done, in each function call.
- Case I: $f(n)$ is $O(n^{[\text{less than } k]}) \rightarrow$ Algorithm is $\Theta(n^k)$.
- Case II: $f(n)$ is $\Theta(n^k) \rightarrow$ Algorithm is $\Theta(n^k \log n)$.

Find $f(n)$, hopefully involving a power of *n*.

Here, the exponent of *n* is *k*.

Here, the exponent of *n* is less than *k*.

Divide-and-Conquer
Efficiency of Searching

Sequential Search is easily seen to be $O(n)$.

Analyze Binary Search

- Find b, k, f
  - Binary Search splits its input into 2 nearly-equal-sized parts.
    - Thus $b = 2$.
  - Binary Search makes 1 recursive call.
    - Thus $b^k = 1$, and so $k = 0$.
  - In addition, Binary Search does a comparison: constant time.
    - So $f(n)$ is $O(1)$, and, in fact, $\Theta(1)$.
- Which Case?
  - $1 = n^0 = n^k$. The exponent of n is k, and so we are in Case 2.
- Conclusion
  - By the Master Theorem, Case 2, $T(n)$ is $\Theta(n^k \log n)$.
  - Simplify: Binary Search is $\Theta(\log n)$, and therefore $O(\log n)$.

Divide-and-Conquer Logarithmic Time

Divide and Conquer is a common way to get algorithms with order $O(\log n)$ or $O(n \log n)$.

Earlier we said that the base of the logarithm does not matter. Why is this?

- Suppose (for example) that an algorithm takes $3 \log_2 n$ steps.
- Clearly, this algorithm is $O(\log_2 n)$.
- Is it also $O(\log_{10} n)$?
  - Yes!
  - $3 \log_2 n = 3(\log_2 10 \times \log_{10} n) = (3 \log_2 10) \times \log_{10} n$.
- Fact: If a and b are greater than 1, $O(\log_a n)$ and $O(\log_b n)$ mean the same thing.
- Thus we generally leave off the base and say "$O(\log n)$".
- Similarly, we say "$O(n \log n)$", etc.

Divide-and-Conquer Thoughts

We can use the Master Theorem "backwards".

- We have been saying, "Here is the order of $f(n)$; what is the order of the algorithm as a whole?"
- Instead, we can say, "We want an algorithm with a certain order; how large is $f(n)$ allowed to be?"

How it works: Suppose we use divide-and-conquer.

- We split our input into b nearly equal-sized parts.
- We make b^k recursive calls.
- The Master Theorem says: To be efficient [$O(n^k \log n)$], we can only do additional work requiring $O(n^k)$ steps.

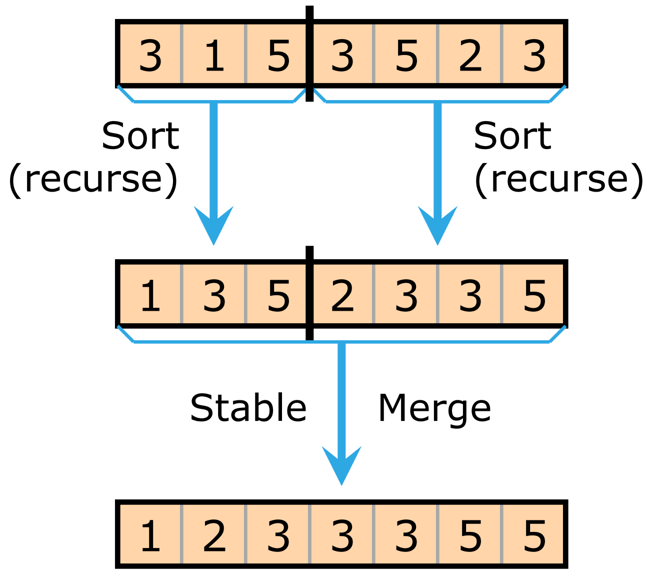
Getting Logarithmic and Log-Linear Time

- If an algorithm is $O(\log n)$ or $O(n \log n)$, then it there is a good chance that it uses some form of divide-and-conquer.
- So divide-and-conquer might get us a fast sorting algorithm.

Comparison Sorts II
Merge Sort — Introduction

How can we use divide-and-conquer to build a better sorting algorithm?

- Suppose we split a list into two equal-sized (or nearly so) pieces, and sort each piece recursively.
- Then **merge** the two parts into a single sorted list.
    - Do this in a stable manner: **Stable Merge**.
- The resulting sorting algorithm is called **Merge Sort**.
    - John von Neumann, 1945.

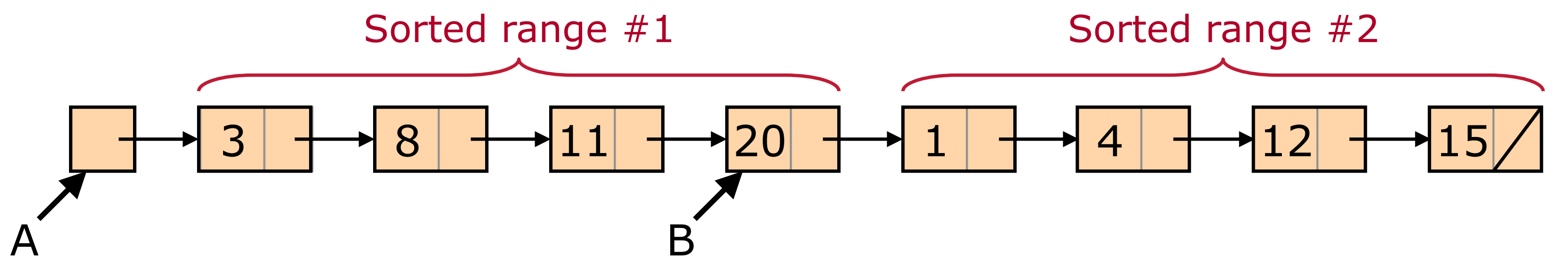


If we want an $O(n \log n)$ sort, how long can a Stable Merge operation take?

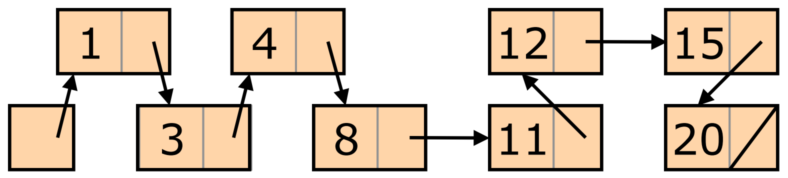
- 2 nearly equal-sized parts ($b = 2$).
- 2 recursive calls ($b^k = 2$, and so $k = 1$).
- Case 2. Stable Merge is allowed $O(n^k) = O(n^1) = O(n)$: linear time.

Comparison Sorts II
Merge Sort — Merging in a Linked List

We can do an efficient Stable Merge of a **Linked List** in-place.



- To merge two sorted ranges within a Linked List:
  - Keep two pointers, A and B. A starts at the head, B at the end of range #1.
  - Check whether the item after B's node is less than the item after A's node. If so, remove the item after B's node and re-insert it after A.
    - This requires only pointer operations. We do not actually move any nodes.
  - Advance A and B as appropriate and repeat.

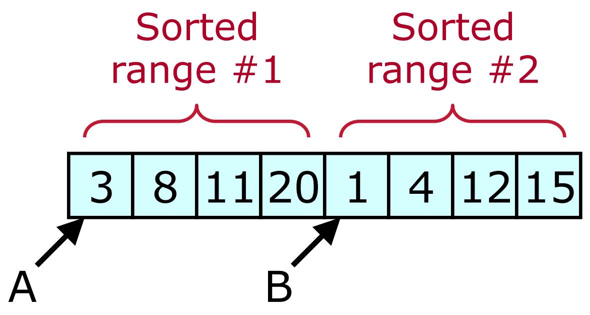


Result of Merge operation

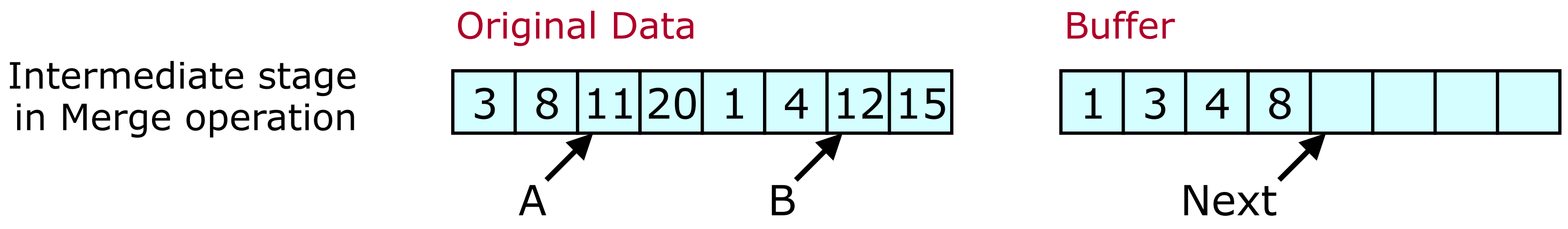
Comparison Sorts II
Merge Sort — General-Purpose Merge

Efficient Stable Merge in an **array** generally uses a separate buffer.

- Note, this Stable Merge algorithm does not *require* an array; it works with just about any kind of data.



As before, use two pointers. Check which item comes first, and copy that to the buffer. Advance the pointers as appropriate.



At the end, we *may* wish to copy the buffer back to the original array.

Comparison Sorts II
Merge Sort — Putting It All Together

Conclusion: For both Linked Lists and arrays, a Stable Merge can be done in linear time.

How do we write the sort itself?

- We find the middle of the list, recurse twice, and Merge.
  - Finding the middle of an array: $O(1)$.
  - Finding the middle of a Linked List: $O(n)$.
- However, we already do the linear-time Merge operation at each step. Adding $O(1)$ or $O(n)$ additional steps only makes this into a slightly slower linear-time operation.
  - $O(1) + O(n) = O(n) + O(n) = O(n)$.
- Conclusion: Merge Sort might be **written differently for different types of data**. However, it is always $O(n \log n)$.

Now let's write Merge Sort, and analyze it in detail.

Comparison Sorts II
Merge Sort — Do It

When writing Merge Sort, the only part that takes any work is the Stable Merge operation. Once Stable Merge is done, writing the sort is very easy.

TO DO

- Examine implementation of Merge Sort.
  - Use the general-purpose Stable Merge algorithm, and make it a separate function.
- Analyze.
  - *See the next slide.*

Notes

- We allocated a buffer every time a Stable Merge was done. It would be more efficient to allocate once, in a wrapper function, and then pass a pointer when calling each function.
- We merged to the buffer and then copied the buffer back. This copy-back can often be avoided, but it adds complexity to the code.

Comparison Sorts II
Merge Sort — Analysis

Efficiency ☺
- Merge Sort is $O(n \log n)$.
- Merge Sort also has an average-case time of $O(n \log n)$.

Requirements on Data ☺
- Merge Sort does not require random-access data.
- Operations needed. General: copy. Linked List: NONE (compare).

Space Usage 😐/☺/☹
- Recursive Merge Sort uses stack space: recursion depth $\approx \log_2 n$.
  - An iterative version can avoid this (small) memory requirement.
- For a Linked List, no more is needed: $O(\log n)$ additional space. 😐
  - Or $O(1)$ additional space, for an iterative version. ☺
- General-purpose Merge Sort uses a buffer: $O(n)$ additional space. ☹

Stability ☺
- Merge Sort is stable.

Performance on Nearly Sorted Data 😐
- Merge Sort is still log-linear time on nearly sorted data.

Comparison Sorts II
Merge Sort — Comments

Merge Sort is very practical and is often used.

- Merge Sort is considered to be the **fastest known** algorithm:
  - When a stable sort is required.
  - When sorting a Linked List.
- Merge Sort is the usual way to implement two of the six sorting algorithms in the C++ Standard Template Library.

Stable Merge is done differently for different kinds of data.

- Thus, while the overall structure is the same, different versions of Merge Sort can differ greatly in lower-level details.
- Merge Sort is *almost* two different algorithms.

I have seen research indicating that one can do a linear-time Stable Merge in an array without an extra buffer.

- However, even highly regarded Merge Sort implementations still allocate the buffer.
- C++ Standard Library algorithm **`std::stable_sort`** tries to allocate a buffer. If this fails, then it is allowed to be $O(n\,[\log n]^2)$.
- I have not quite figured out this issue.

Comparison Sorts II
Merge Sort — Comparing Algorithms

Merge Sort does essentially everything we would like a sorting algorithm to do:

- It runs in $O(n \log n)$ time.
- It is stable.
- It works well with various kinds of data — especially Linked Lists.

Thus, Merge Sort is a good standard by which to judge sorting algorithms.

When evaluating some other sorting algorithm, ask:

- How is this algorithm better than Merge Sort?
  - If it is not better in any way, then use Merge Sort.
- How is this algorithm worse than Merge Sort?
  - If it is better than Merge Sort in some way, then it must also be worse in some way.
- In this application, are the advantages worth the disadvantages?