2-3 Trees continued
Other Balanced Search Trees

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Where Are We? — The Big Problem

Our problem for much of the rest of the semester:

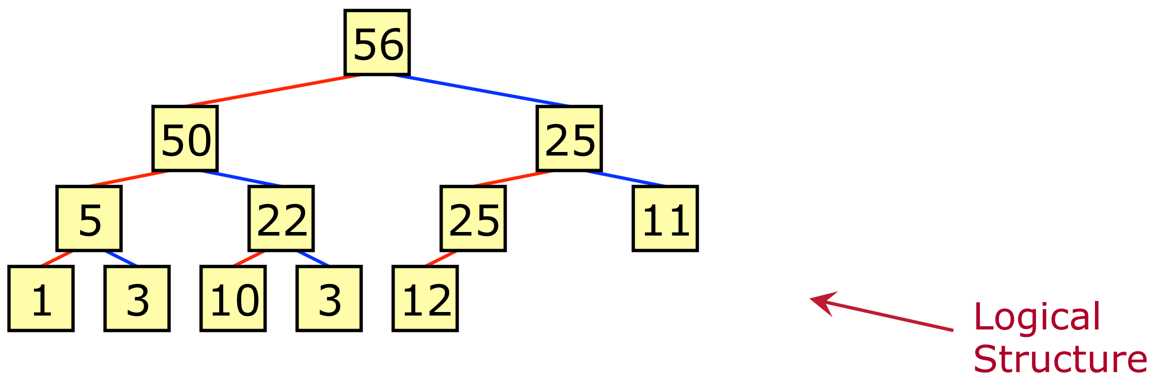
- Store: a collection of data items, all of the same type.
- Operations:
  - Access items [one item: retrieve/find, all items: traverse].
  - Add new item [insert].
  - Eliminate existing item [delete].
- All this needs to be efficient in both time and space.

A solution to this problem is a **container**.

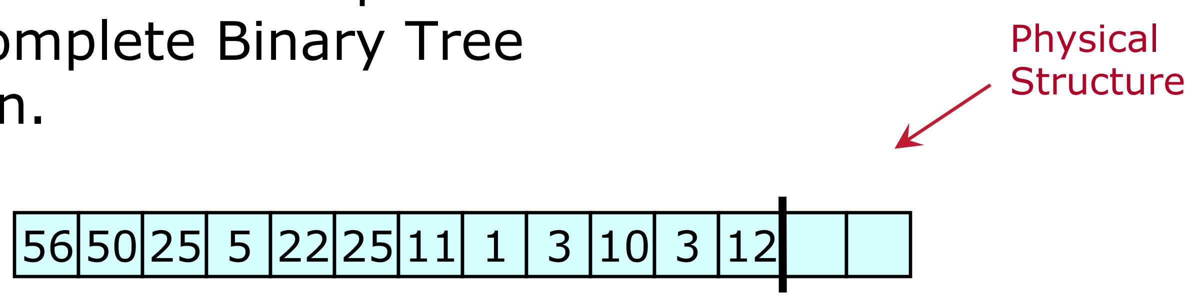
Generic containers: those in which client code can specify the type of data stored.

Review
Binary Heap Algorithms [1/5]

A **Binary Heap** (usually just **Heap**) is a complete Binary Tree in which *no* node has a data item that is less than the data item in either of its children.



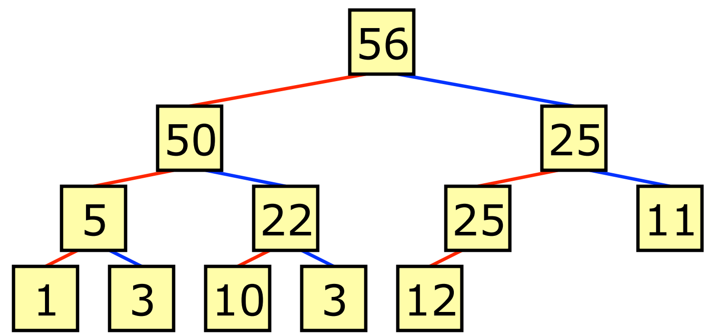
In practice, we often use "Heap" to refer to the array-based complete Binary Tree implementation.



Review
Binary Heap Algorithms [2/5]

We can use a Heap to implement a **Priority Queue**.

- Like a Table, but retrieve/delete only highest key.
  - Retrieve is called “getFront”.
- Key is called “priority”.
- Insert *any* key-data pair.



56	50	25	5	22	25	11	1	3	10	3	12		

Algorithms for the Three Primary Operations

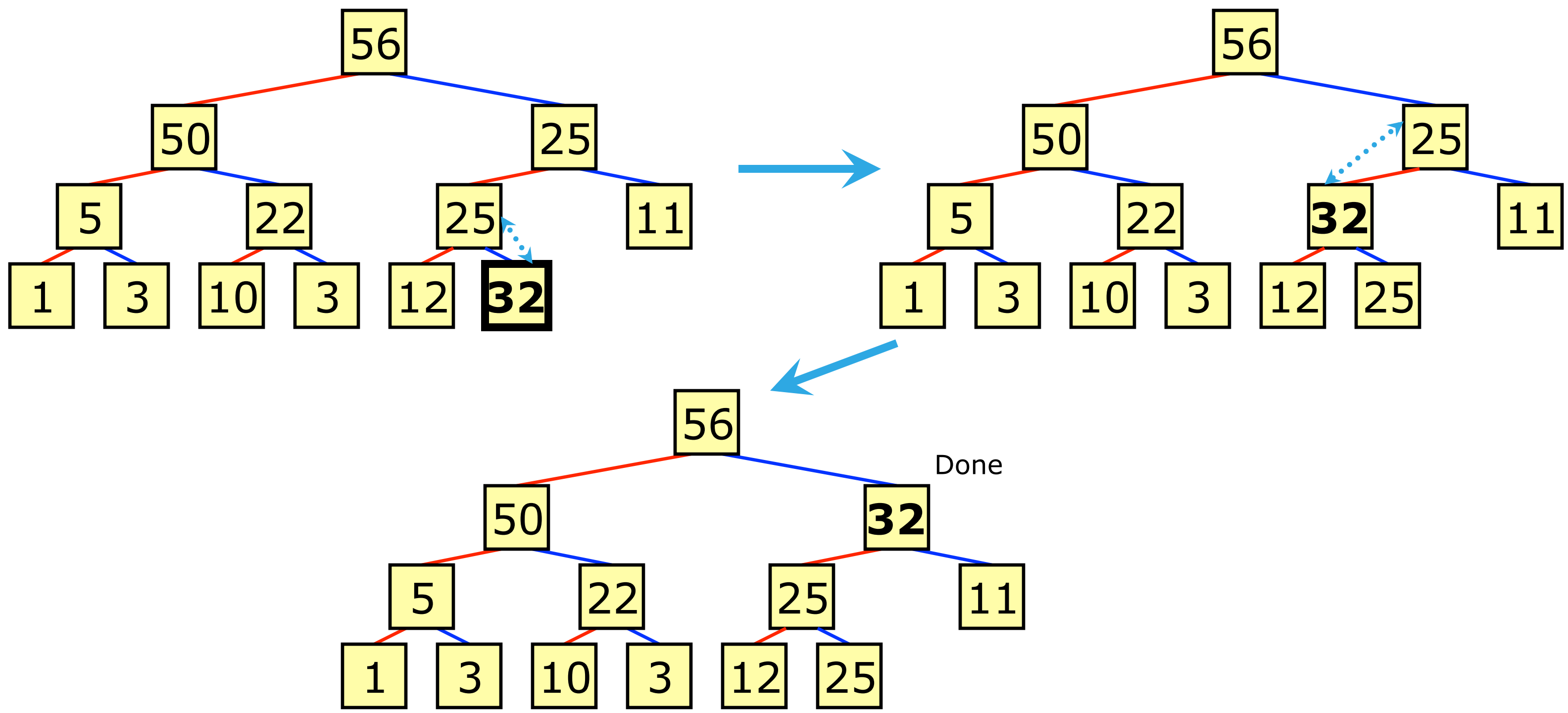
- GetFront
  - Get the root node.
  - Constant time.
- Insert
  - Add new node to end of Heap, “trickle up”.
  - Logarithmic time if no reallocate-and-copy required.
    - Linear time if it may be required. Note: Heaps often do *not* manage their own memory, in which case the reallocation will not be part of the Heap operation.
- Delete
  - Swap first & last items, reduce size of Heap, “trickle down” root.
  - Logarithmic time.

Faster than linear time!

Review
Binary Heap Algorithms [3/5]

To insert into a Heap, add new node at the end. Then “trickle up”.

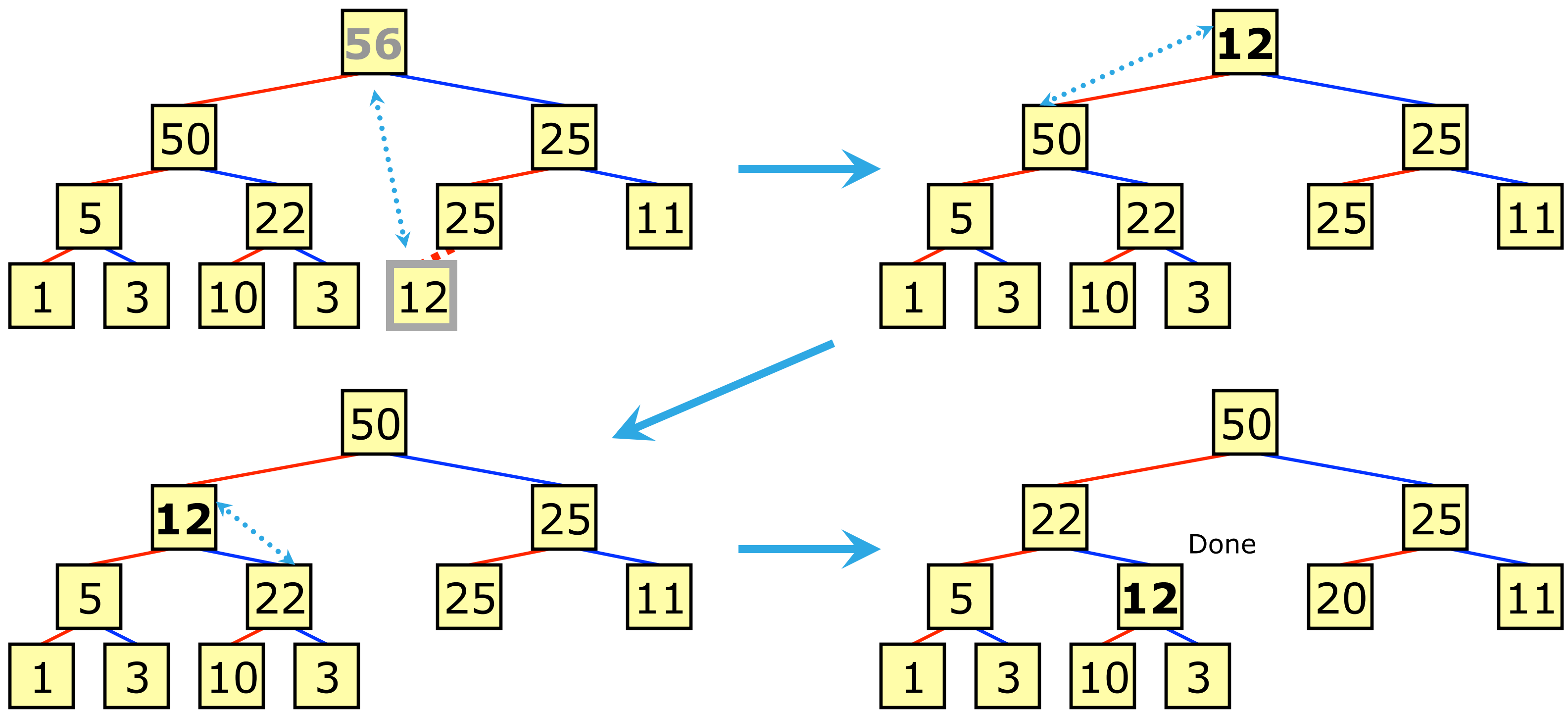
- If new value is greater than its parent, then swap them. Repeat at new position.



Review
Binary Heap Algorithms [4/5]

To delete the root item from a Heap, swap root and last items, and reduce size of Heap by one. Then “trickle down” the new root item.

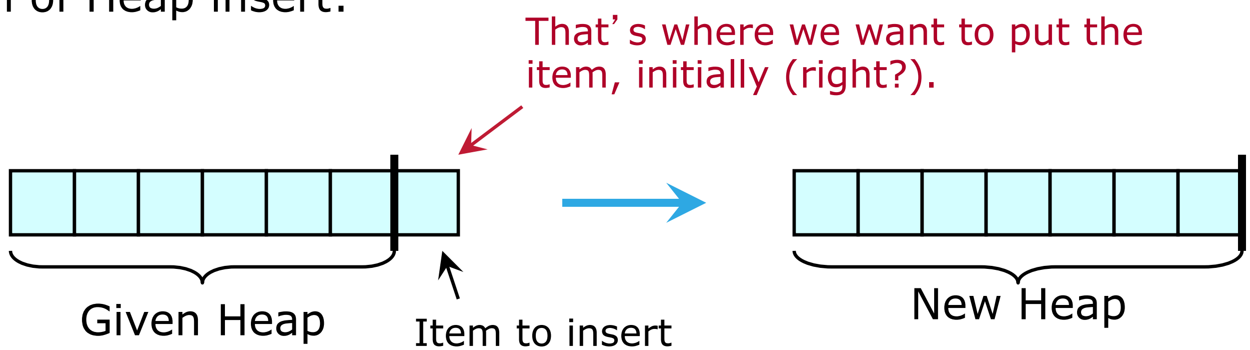
- If the root is ≥ all its children, stop.
- Otherwise, swap the root item with its **largest** child and recursively fix the proper subtree.



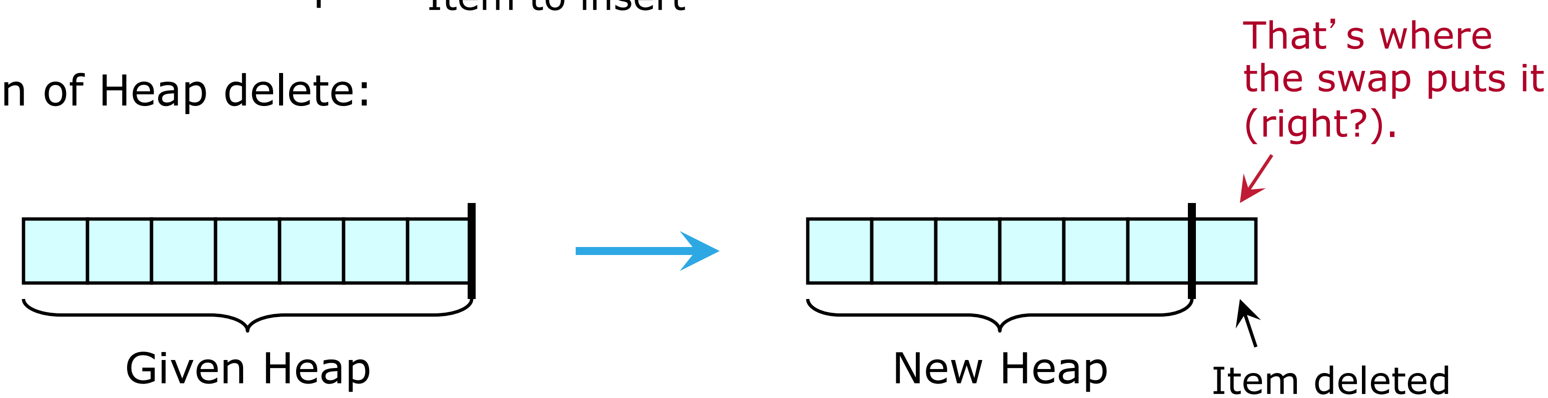
Review
Binary Heap Algorithms [5/5]

Heap insert and delete are usually given a random-access range. The item to insert or delete is last item of the range; the rest is a Heap.

- Action of Heap insert:



- Action of Heap delete:



Note that Heap algorithms can do **all** their work using **swap**.

- This usually allows for both speed and safety.

Review
Binary Search Trees — Efficiency

	B.S.T. (balanced & average case)	Sorted Array	B.S.T. (worst case)
Retrieve	Logarithmic	Logarithmic	Linear
Insert	Logarithmic	Linear	Linear
Delete	Logarithmic	Linear	Linear

Binary Search Trees have poor worst-case performance.

But they have very good performance:

- On average.
- If balanced.
  - But we do not know an efficient way to make them *stay* balanced.

Can we efficiently keep a Binary Search Tree balanced?

- We will look at this question again later.

Overview of Advanced Table Implementations

We will cover the following advanced Table implementations.

- Balanced Search Trees
  - Binary Search Trees are hard to keep balanced, so to make things easier we allow more than 2 children:
    - (part) **2-3 Tree**
      - Up to 3 children
    - **2-3-4 Tree**
      - Up to 4 children
    - **Red-Black Tree**
      - Binary-tree representation of a 2-3-4 tree
  - Or back up and try a balanced Binary Tree again:
    - **AVL Tree**
- Alternatively, forget about trees entirely:
  - **Hash Tables**
- Finally, "the Radix Sort of Table implementations":
  - **Prefix Tree**

Review
2-3 Trees — Introduction & Definition [1/2]

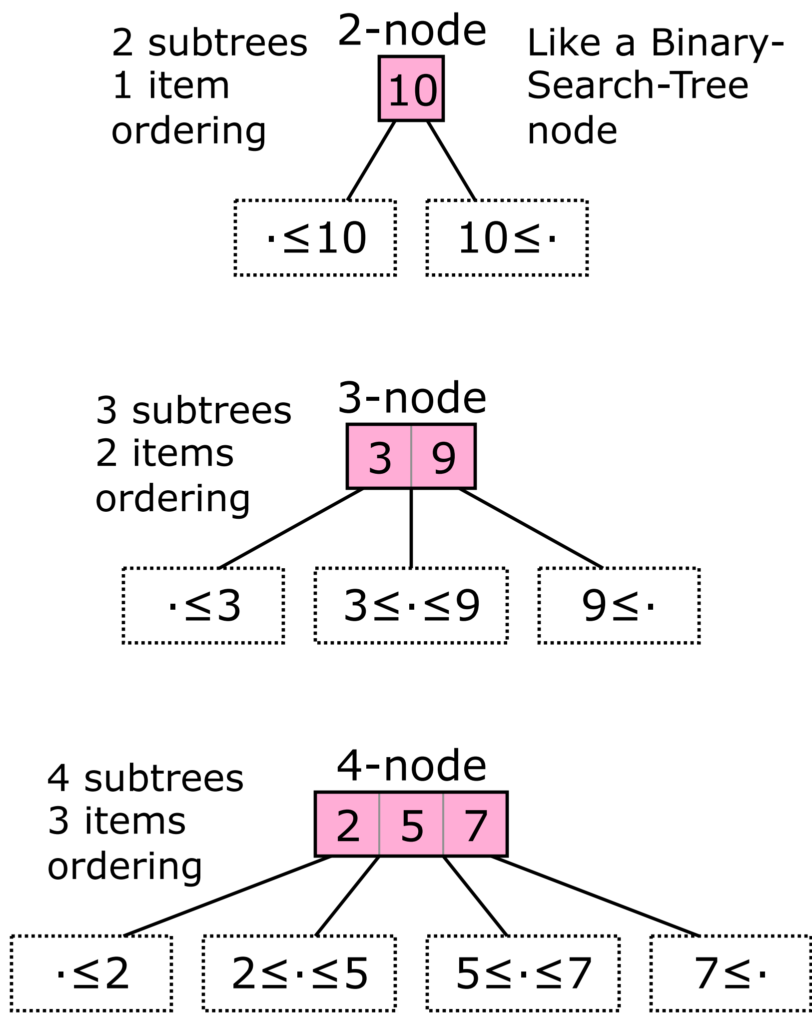
A Binary-Search-Tree style node is a **2-node**.

- This is a node with 2 subtrees and 1 data item.
- The item's value lies between the values in the two subtrees.

In a "2-3 Tree" we also allow a node to be a **3-node**.

- This is a node with 3 subtrees and 2 data items.
- Each of the 2 data items has a value that lies between the values in the corresponding pair of consecutive subtrees.

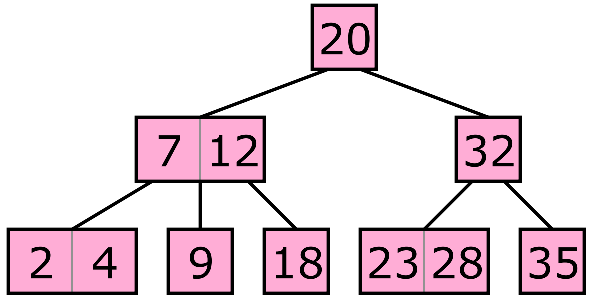
Later, we will look at "2-3-4 trees", which can also have **4-nodes**.



2-3 Trees — Introduction & Definition [2/2]

A **2-3 Search Tree** (generally we just say **2-3 Tree**) is a tree with the following properties.

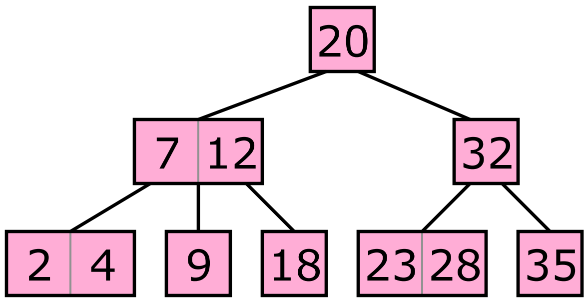
- All nodes contain either 1 or 2 data items.
  - If 2 data items, then the first is ≤ the second.
- All leaves are at the same level.
- All non-leaves are either *2-nodes* or *3-nodes*.
  - They must have the associated order properties.



Review
2-3 Trees — Operations: Traverse & Retrieve

How do we **traverse** a 2-3 Tree?

- We generalize the procedure for doing an **inorder traversal** of a Binary Search Tree.
  - For each leaf, go through the items in it.
  - For each non-leaf 2-node:
    - Traverse subtree 1.
    - Do item.
    - Traverse subtree 2.
  - For each non-leaf 3-node:
    - Traverse subtree 1.
    - Do item 1.
    - Traverse subtree 2.
    - Do item 2.
    - Traverse subtree 3.
- This procedure lists all the items in sorted order.



How do we **retrieve** by key in a 2-3 Tree?

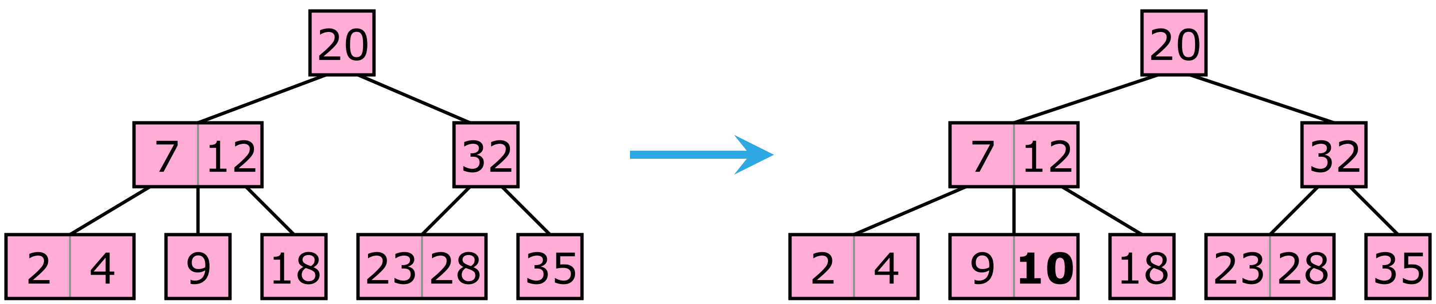
- Start at the root and proceed downward, making comparisons, just as in a Binary Search Tree.
- 3-nodes make the procedure *slightly* more complex.

2-3 Trees — Operations: Insert [1/4]

Ideas in the 2-3 Tree **insert** algorithm:

- Start by adding the item to the appropriate leaf.
- Allow nodes to expand when legal.
- If a node gets too big (3 items), split the subtree rooted at that node and propagate the **middle** item upward.
- If we end up splitting the entire tree, then we create a new root node, and all the leaves advance one level simultaneously.

Example 1: Insert 10.

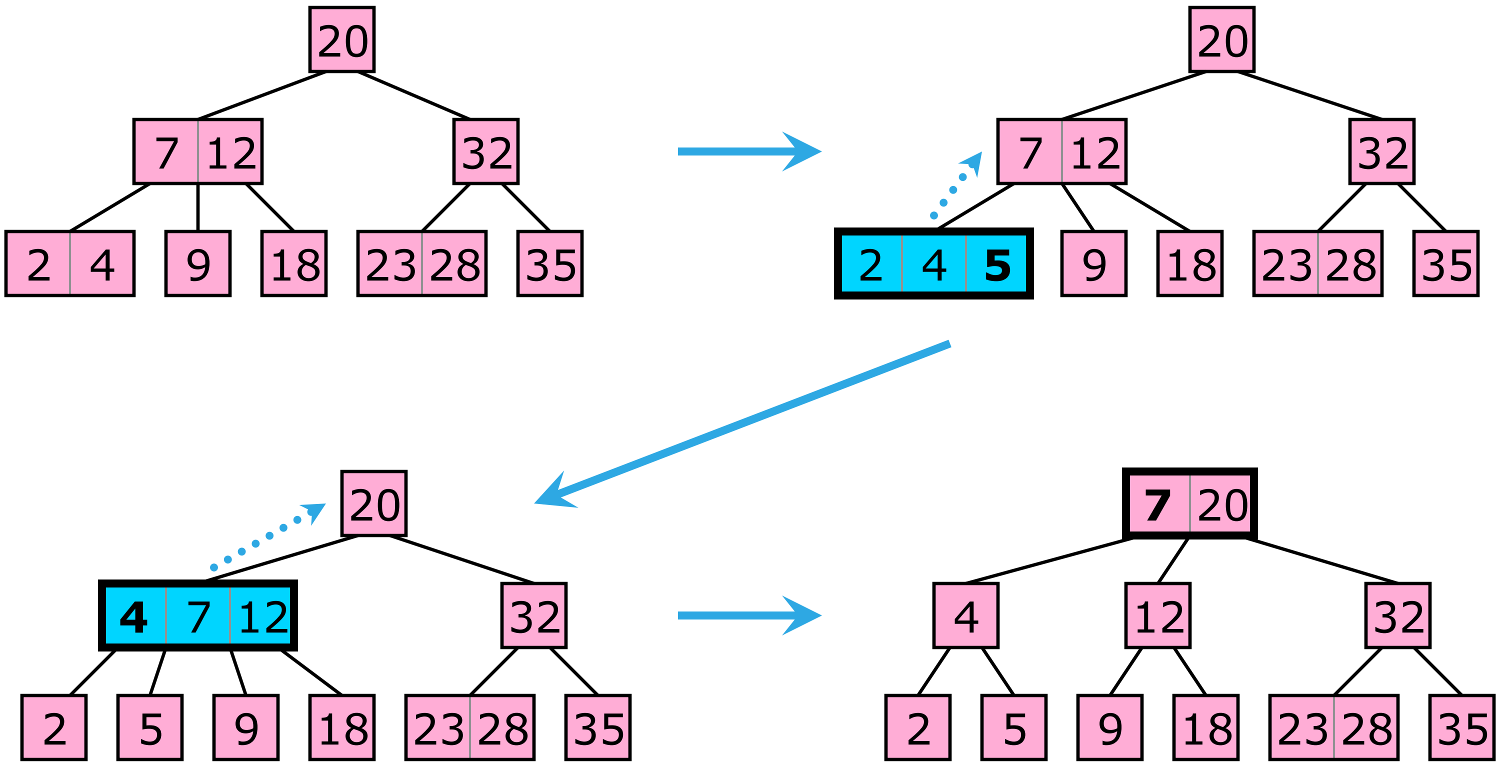


Review

2-3 Trees — Operations: Insert [2/4]

Example 2: Insert 5.

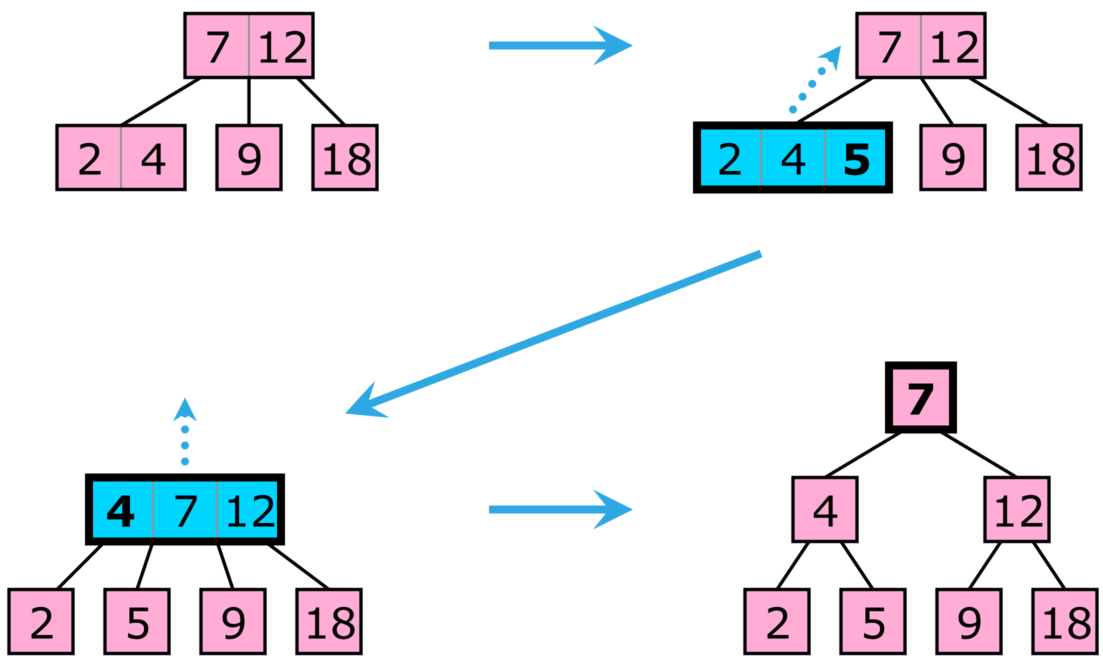
- Over-full nodes are blue.



Review
2-3 Trees — Operations: Insert [3/4]

Example 3: Insert 5.

- Here we see how a 2-3 Tree increases in height.



2-3 Trees — Operations: Insert [4/4]

2-3 Tree **Insert** Algorithm (outline)

- Find the leaf the new item goes in.
  - Note: In the process of finding this leaf, you may determine that the given key is already in the tree. If you do, act accordingly.
- Add the item to the proper node.
- If the node is overfull, then split it (dragging subtrees along, if necessary), and move the middle item up:
  - If there is no parent, then make a new root. Done.
  - Otherwise, add the moved-up item to the parent node. To add the item to the parent, do a recursive call to the insertion procedure.

2-3 Trees — Operations: Delete [1/8]

Deleting from a 2-3 Tree is similar to inserting.

- We will use the recursive-thinking idea to avoid describing every detail of the process.
- We try to delete from a leaf. If it does not work, rearrange.
- If that does not work, bring an item from the parent down. This is deleting from the parent. Recurse (or reduce the height and we are done).
- As with inserting, we start at a leaf and work our way up.

2-3 Trees — Operations: Delete [2/8]

Observation

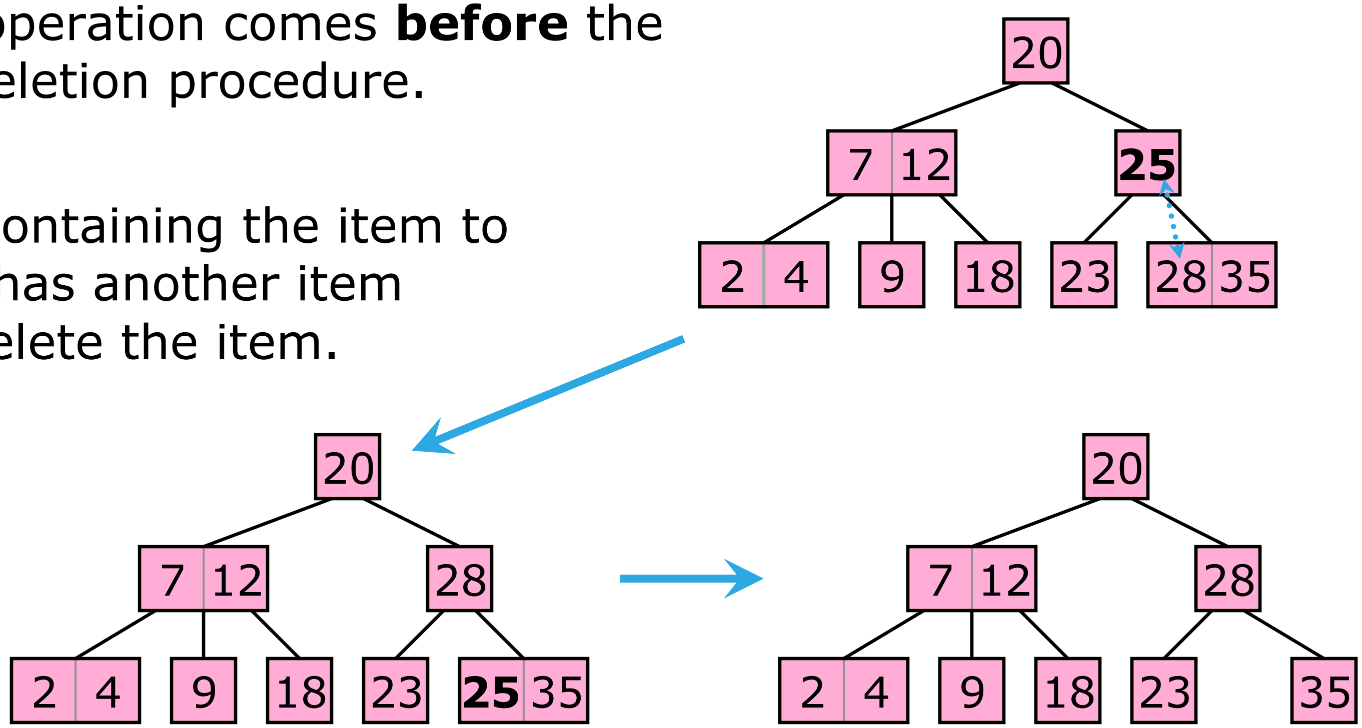
- We can always start our deletion at a leaf.
- If the item to be deleted is not in a leaf, swap it with its "inorder" successor.
  - It must have one. (Why?)
- This swap operation comes **before** the recursive deletion procedure.

Easy Case

- If the leaf containing the item to be deleted has another item in it, just delete the item.

Example

- Delete 25.

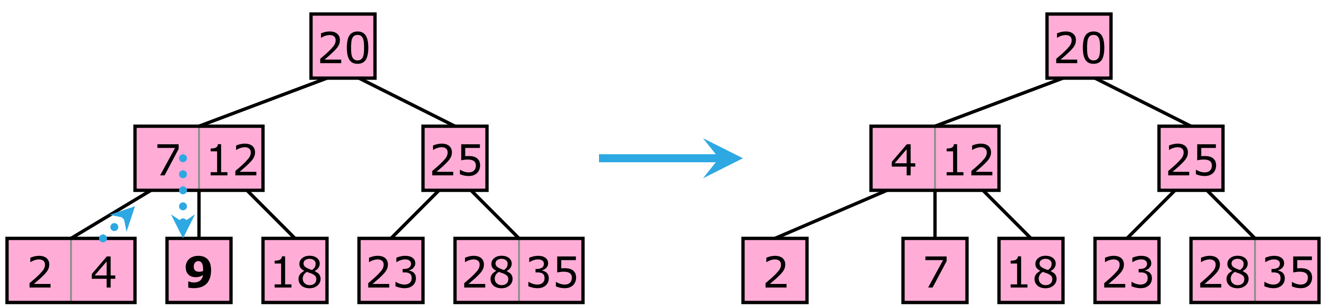


2-3 Trees — Operations: Delete [3/8]

Semi-Easy Case

- Suppose the item to be deleted is in a node that contains no other item.
- If, next to this node, there is a sibling that contains 2 items, we can rearrange (rotate) using the parent.

Example: Delete 9.

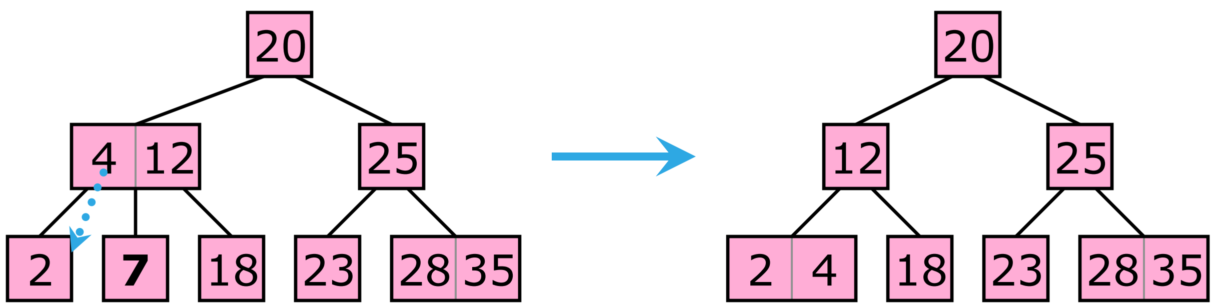


2-3 Trees — Operations: Delete [4/8]

Hard Case

- If the item to be deleted is in a node with no other item, and there are no nearby 2-item siblings, then we must bring down an item from the parent and place it in a nearby sibling node.
- We need to join nodes/subtrees to make the invariants work.

Example: Delete 7.



In the above example, recursively "delete" 4 from the tree consisting of the first two levels. Since 4's node has another item in it, this is the easy case; we simply get rid of 4 (and then put it in the node containing 2).

2-3 Trees — Operations: Delete [5/8]

If we do a recursive delete above the leaf level, where do “orphaned” subtrees go?

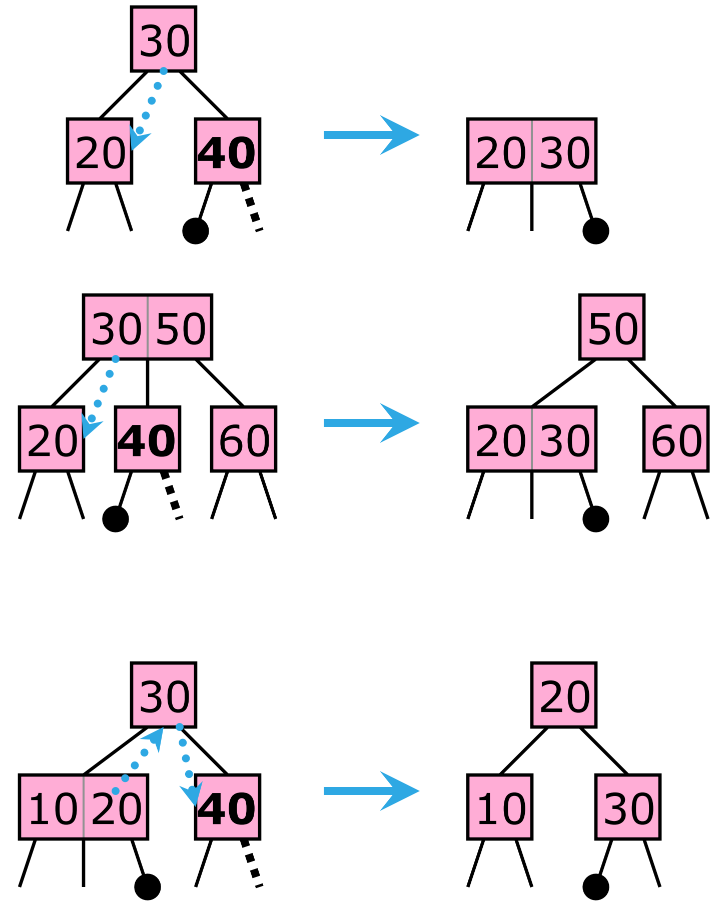
Consider two Hard Case examples.

- We delete 40. Why? Because one of its subtrees is going away. What do we do with the other subtree?
- Answer: Make it a subtree of the item we bring down.

Consider a Semi-Easy Case example.

- Again, we delete 40. One of its subtrees is going away. 30 is coming down to replace it. 20 is going up. What do we do with the right-subtree of 20?
- Answer: Make it the left subtree of 30.

Idea: There is always exactly one spot available for an orphaned subtree. Put it in that spot.



2-3 Trees — Operations: Delete [6/8]

2-3 Tree **Delete** Algorithm (outline)

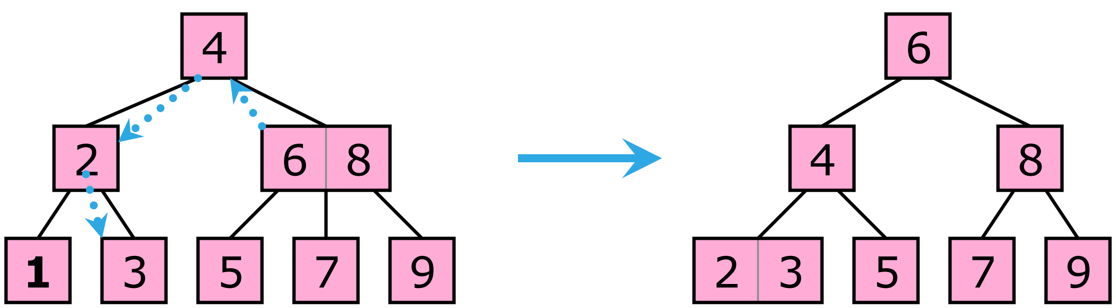
- Find the node holding the given key.
  - Note: In the process of this search, you may determine that the given key is not in the tree. If you do, act accordingly.
- If the above node is not a leaf, then swap its item with its successor in the traversal ordering. Continue with the deletion procedure: delete the given key from its new (leaf) node.
- 3 Cases
  - **Easy Case** (item shares a node with another item). Delete item. Done.
  - **Semi-Easy Case** (otherwise: item has a consecutive sibling holding 2 items). Do rotation: sibling item up, parent down, to replace the item to be deleted. Done.
  - **Hard Case** (otherwise). Eliminate the node holding the item, and move item from the parent down, adding it to consecutive sibling node. Eliminate item from parent using a recursive call to the deletion procedure (dragging subtrees along).

2-3 Trees — Operations: Delete [7/8]

A few more examples.

Example: Delete 1.

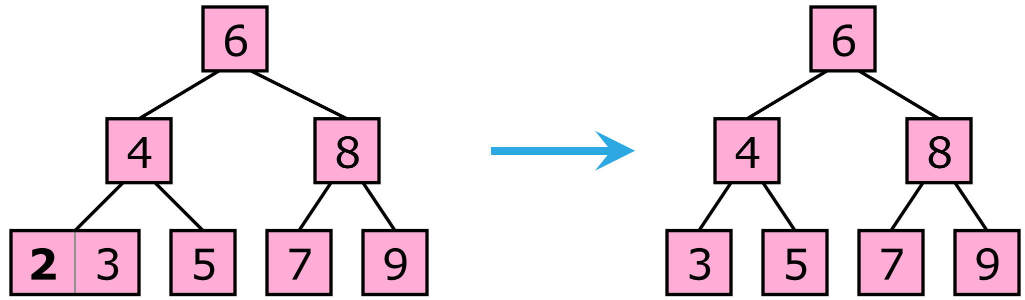
- 1 is “Hard Case”, so we bring down the parent (recursively “delete” 2) and join it with 3 in a single node.
- 2 is “Semi-Easy Case”, so rotate (6 to 4 to 2).
- The 5 is orphaned. We make it the right child of 4.



2-3 Trees — Operations: Delete [8/8]

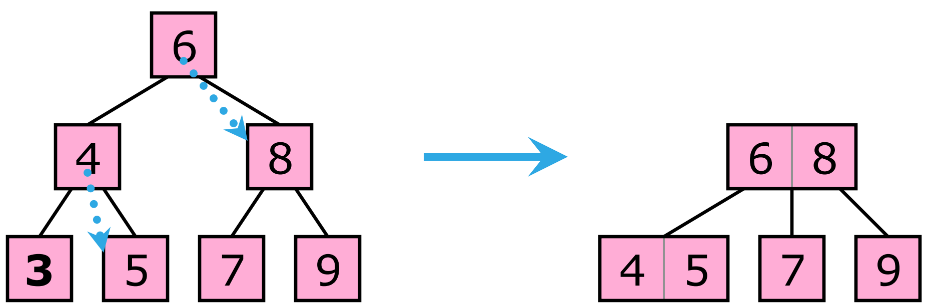
Example: Delete 2.

- This is “Easy Case”.



Example: Delete 3.

- This is “Hard Case”. We need to bring down 4 and join it with 5.
- 4 is “Hard Case”. We need to bring down 6 and join it with 8.
- 6 is the root. We reduce the height of the tree.



2-3 Trees continued
Efficiency

What is the order of the following operations for a 2-3 Tree?

- Traverse
  - $O(n)$ [as usual].
- Retrieve
  - $O(\log n)$.
  - The number of steps is roughly proportional to the height of the tree.
- Insert
  - $O(\log n)$.
  - Comments as for Retrieve.
- Delete
  - $O(\log n)$.
  - Comments as for Retrieve.

This is the **first** time we have seen a delete-by-key that handles *any* given key and is faster than linear time.

This is what we have been looking for.

A 2-3 Tree is a good basis for an implementation of a Table.

However, there are better bases.

- Not necessarily a **lot** better, but better.

Other Balanced Search Trees Better Than a 2-3 Tree?

Again, the Table operations retrieve, insert, and delete are all $O(\log n)$ for a 2-3 Tree implementation.

We do not know any structure in which all operations are $O(\log n)$ [worst case], and at least one is faster.

- Of course, we **can** make some operations better & some worse. For example, for an unsorted Linked List implementation, Table insert is $O(1)$, while Table retrieve & delete are $O(n)$.

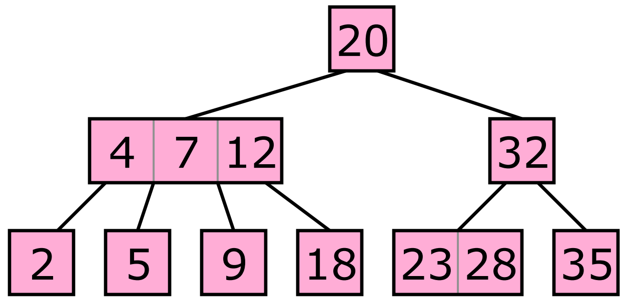
However, we can make everything a **little** faster than a 2-3 Tree, although still $O(\log n)$.

We do this with other kinds of balanced search trees.

- These are all similar to a 2-3 Tree.
- Thus, we will not look at them in great detail.

Other Balanced Search Trees
2-3-4 Trees

In a **2-3-4 Tree**, we also allow 4-nodes.



The insert and delete algorithms are not terribly different from those of a 2-3 Tree.

- They are a little more complex.
- And they tend to be a little faster.

Why not also allow 5-nodes (a “2-3-4-5 Tree”)?

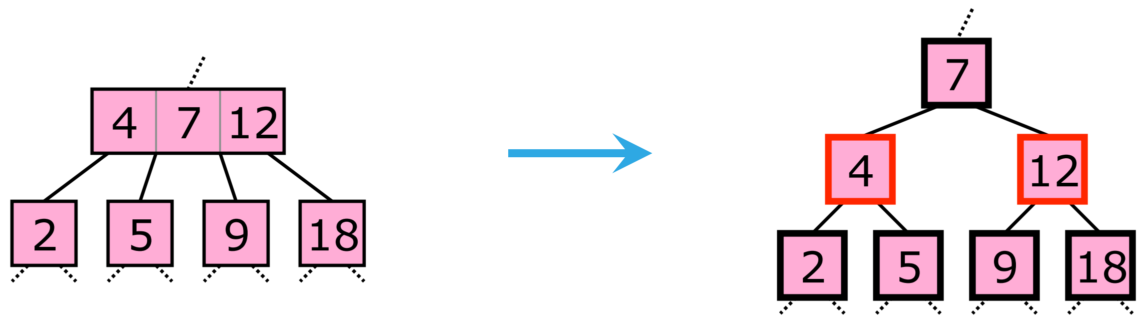
- Because the algorithms tend to be a little slower.

Other Balanced Search Trees
Red-Black Trees — Idea [1/3]

It turns out that we can increase the efficiency of 2-3-4 Tree operations by representing the tree using a Binary Search Tree plus a little more information.

- The representation we will discuss is called a **Red-Black Tree**.

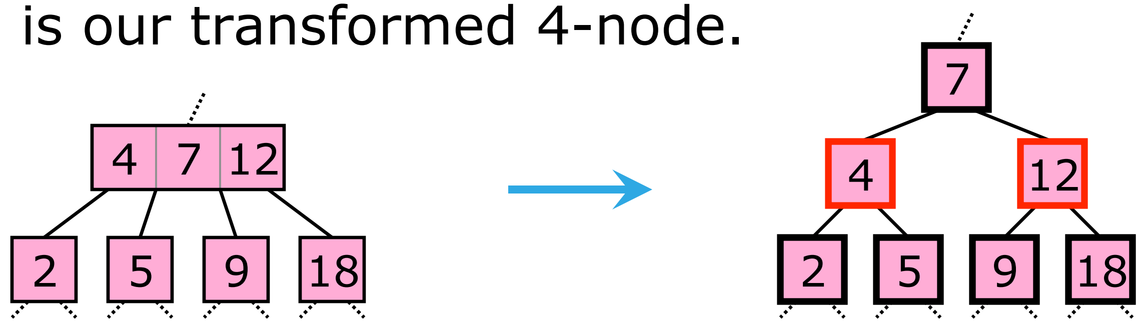
Consider the 4-node below. We can represent this part of the 2-3-4 Tree using only 2-nodes if we add two new nodes (shown in red).



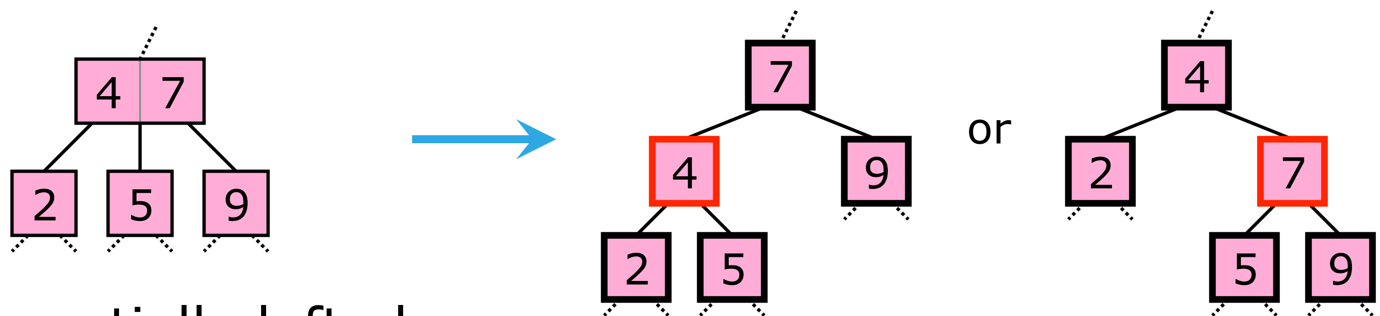
Note that the ordering property of the 2-3-4 Tree translates into the ordering property of a Binary Search Tree.

Other Balanced Search Trees
Red-Black Trees — Idea [2/3]

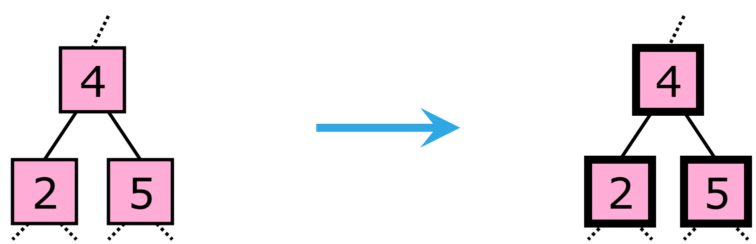
Here again is our transformed 4-node.



We can also apply this process to a 3-node (in two different ways).



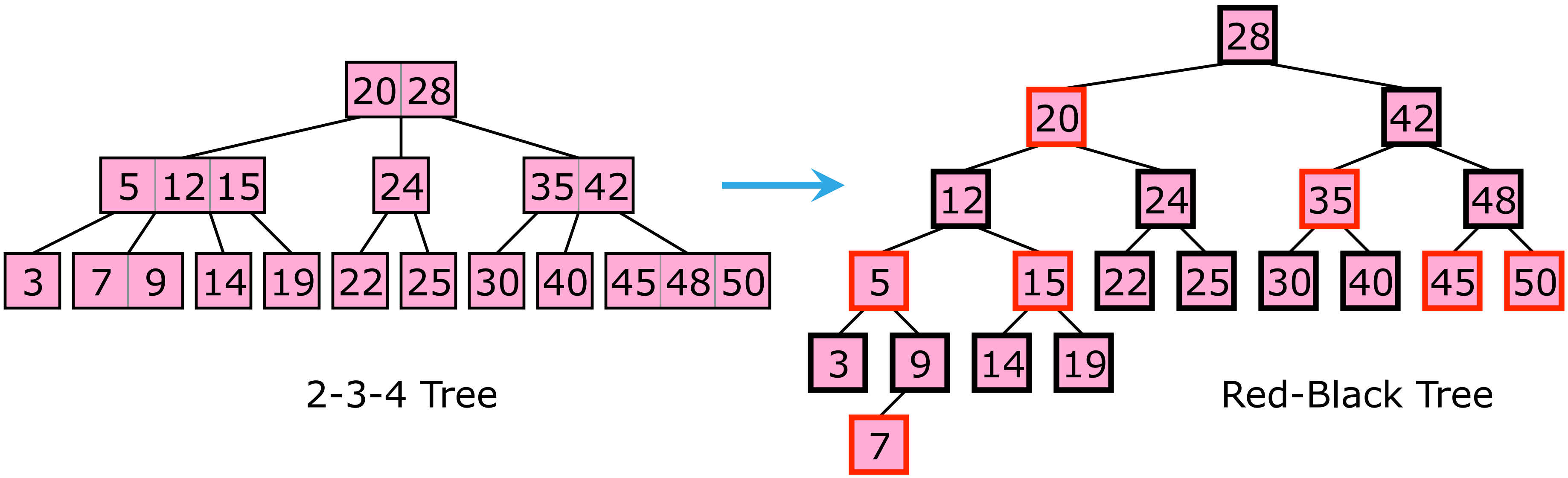
2-nodes are essentially left alone.



Other Balanced Search Trees
Red-Black Trees — Idea [3/3]

A **Red-Black Tree** is a Binary-Tree representation of a 2-3-4 Tree.

- A R.B.T. is a Binary Search Tree in which each node is “**red**” or “**black**”.
- Think of **black** nodes as representing 2-3-4 Tree nodes.
- Think of **red** nodes as being the extra ones required to make a Binary Tree out of the 2-3-4 Tree.



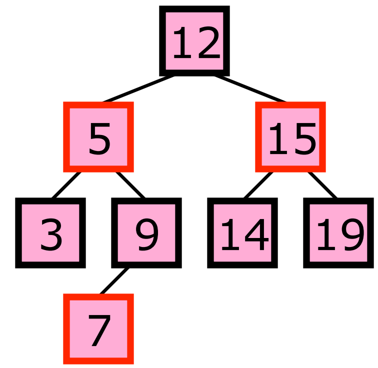
2-3-4 Tree

Red-Black Tree

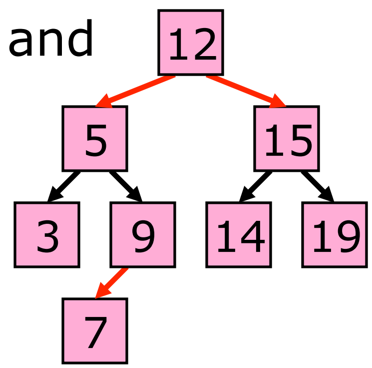
- It is no longer true that every leaf is at the same level. However, given a node, every path from it down to a leaf goes through the same number of **black** nodes.

Other Balanced Search Trees
Red-Black Trees — Variations

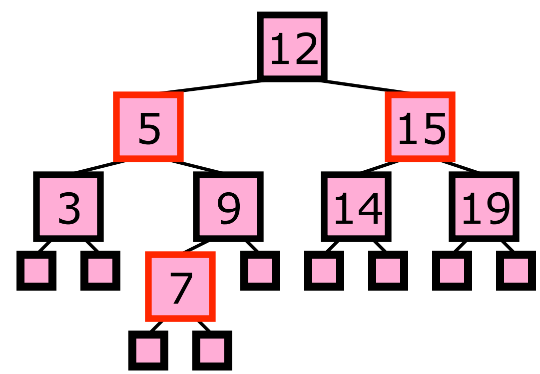
Implementations of Red-Black Trees vary.



- I have presented them as having red and black **nodes**.



- Some descriptions talks about red and black **pointers**.
  - Note that the root is always black, so it does not matter whether the root's color is stored somewhere.



- Some (most?) versions add "**null nodes**" at the bottom.
  - Null nodes are black and have no data.
  - All leaves are null nodes, and all null nodes are leaves.

Other Balanced Search Trees
Red-Black Trees — Usage

How do we use Red-Black Trees?

- Retrieve and traverse are exactly the same as for Binary Search Trees. Just ignore the color.
- Insert and delete algorithms are complicated (and will not be covered). They are based on *rotations*, which we will see when we cover AVL Trees (shortly).

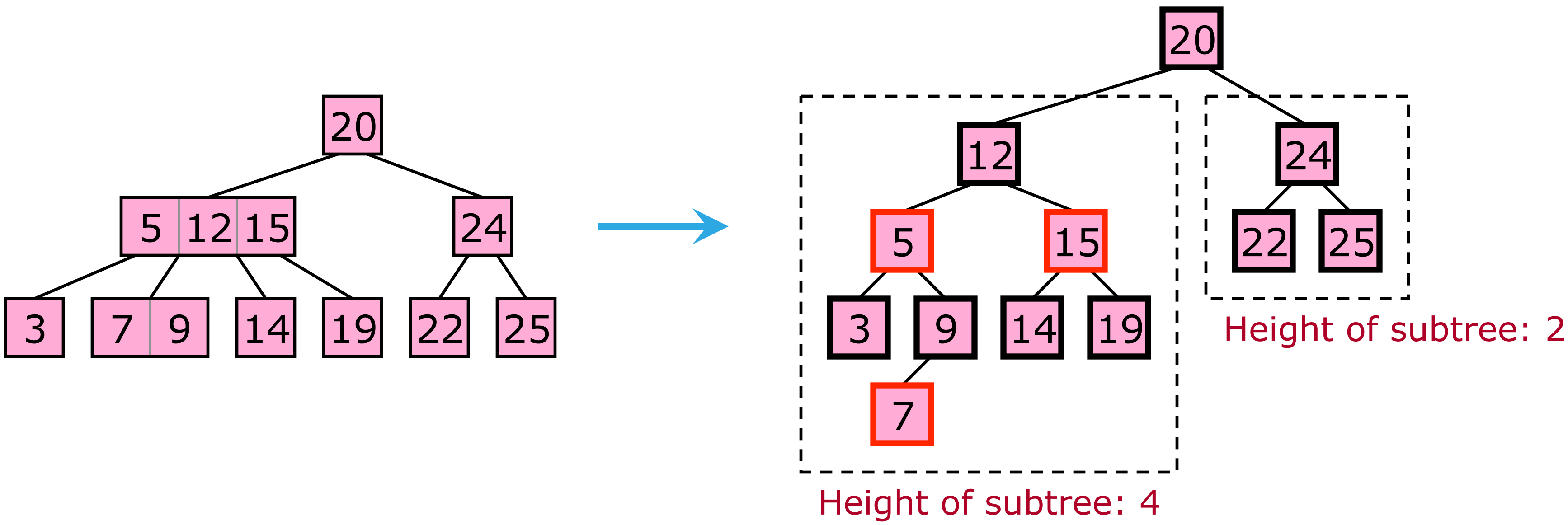
Why do we use Red-Black Trees?

- Because they tend to be just a little more efficient than 2-3-4 Trees, which are just a little more efficient than 2-3 Trees.
- All three have $O(\log n)$ insert, delete, and retrieve.

Red-Black Trees are the most common basis for implementations of C++ STL Tables (**`std::set`**, **`std::map`**, etc.).

Other Balanced Search Trees
Red-Black Trees — Notes [1/2]

A Red-Black Tree is not necessarily a balanced Binary Tree, as we defined "balanced" earlier.



However, a Red-Black Tree with *n* nodes cannot have height more than 2 $\log_2(n + 1)$.

Thus, the height is $O(\log n)$, which makes the retrieve, insert, and delete operations $O(\log n)$.

Other Balanced Search Trees
Red-Black Trees — Notes [2/2]

In practice, we *never* do the 2-3-4 Tree to Red-Black Tree conversion. Rather, we implement only a Red-Black Tree.

- The conversion was illustrated here in order to explain where Red-Black Trees come from and how they work.

If you need an efficient balanced search tree for in-memory data, use a Red-Black Tree.

- The insert & delete algorithms get rather complex. Look up the details.

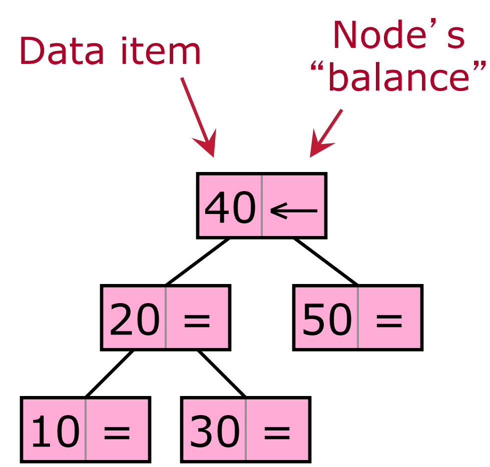
Other Balanced Search Trees
AVL Trees — Definition

The first kind of self-balancing search tree was the “AVL Tree”.

- AVL trees are named after the authors of a 1962 paper describing them: Georgy Maximovich **A**delson-**V**elsky and Yevgeniy Mikhailovich **L**andis.
- These days, AVL Trees are mostly a historical curiosity.

An **AVL Tree** is a balanced (in our original, strict sense) Binary Search Tree in which each node has an extra piece of data: its “balance”: left high [←], right high [→], or even [=].

- Recall: a Binary Tree is *balanced*, if, for each node in the tree, its two subtrees have heights differing by at most 1.

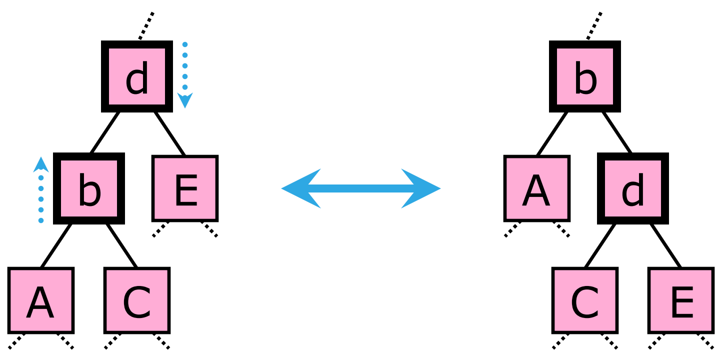


A-V & L discovered logarithmic-time algorithms to do insert and delete while maintaining the balanced property.

Other Balanced Search Trees
AVL Trees — Rotation

We will not cover all of the details of the AVL Tree algorithms.

- We note that they rest on an operation known as **rotation**.
- Rotation is pictured below. For nodes labeled A, C, E, the subtrees of which they are the roots are moved along with them.
- Note that we have seen something (roughly) like this before, in the "semi-easy case" of 2-3 Tree deletion.



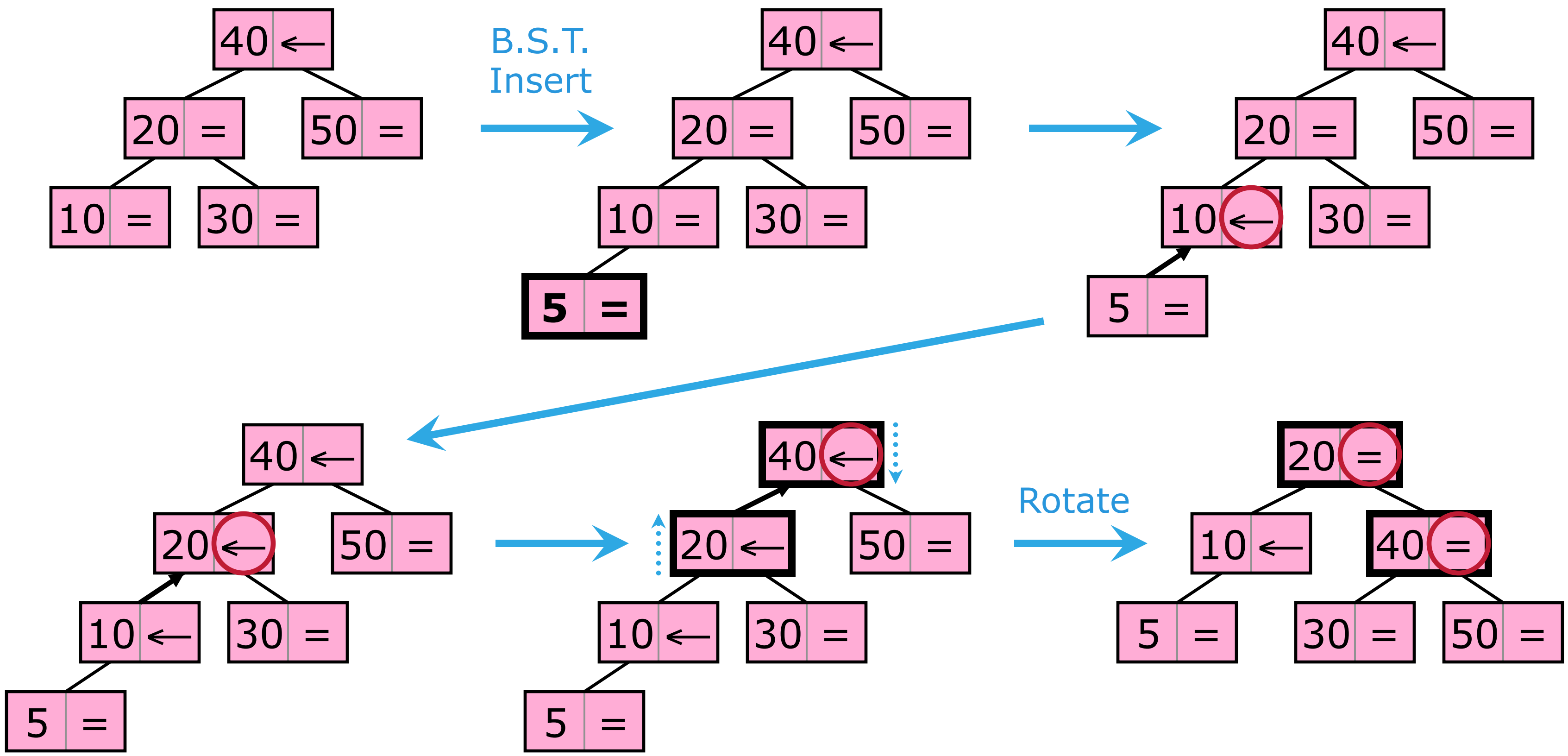
When we allow rotations, we can insert or delete using at most $O(\log n)$ operations, while maintaining the balanced property.

- Thus, insert and delete (and, by the balanced property, retrieve) are $O(\log n)$ operations for an AVL Tree.

Other Balanced Search Trees
AVL Trees — Example

Quick example of AVL Tree insert: Do Binary Search Tree insert, then proceed up to the root, adjusting “balances” and, if needed, rotating.

- Below we illustrate Insert 5.



Other Balanced Search Trees
Wrap-Up

All balanced search trees offer an implementation of the Table ADT in which the insert, delete, and retrieve operations are $O(\log n)$.

Generally, the Red-Black Tree is agreed to have best **overall** performance.

- It is the one that tends to be used to implement things like **`std::map`**.
- The word "overall" is important. For example, an AVL Tree has a faster retrieve operation than a Red-Black Tree.
  - But a sorted array has an even faster retrieve; no one uses AVL Trees.

Implementation details may be changed due to various trade-off's.

- Space vs. time, etc.