Pigeon and Radix Sort
Sorting in the C++ STL

CS 311 Data Structures and Algorithms
Lecture Slides
Monday, March 18, 2013

Chris Hartman
Department of Computer Science
University of Alaska Fairbanks
cmhartman@alaska.edu
Based on material by Glenn G. Chappell

Unit Overview
Algorithmic Efficiency & Sorting

Major Topics

- ✓ Introduction to Analysis of Algorithms
- ✓ Introduction to Sorting
- ✓ Comparison Sorts I
- ✓ More on Big-*O*
- ✓ The Limits of Sorting
- ✓ Divide-and-Conquer
- ✓ Comparison Sorts II
- ✓ Comparison Sorts III
- Radix Sort
- Sorting in the C++ STL

Review
Introduction to Analysis of Algorithms

Efficiency

- General: using few resources (time, space, bandwidth, etc.).
- Specific: fast (time).

Analyzing Efficiency

- Determine how the **size of the input** affects running time, measured in **steps**, in the **worst case**.

Scalable: works well with large problems.

Using Big-*O*	**In Words**
$O(1)$	Constant time
$O(\log n)$	Logarithmic time
$O(n)$	Linear time
$O(n \log n)$	Log-linear time
$O(n^2)$	Quadratic time
$O(b^n)$, for some $b > 1$	Exponential time

Cannot read all of input ↑

Probably not scalable ↓

↑ Faster

Slower ↓

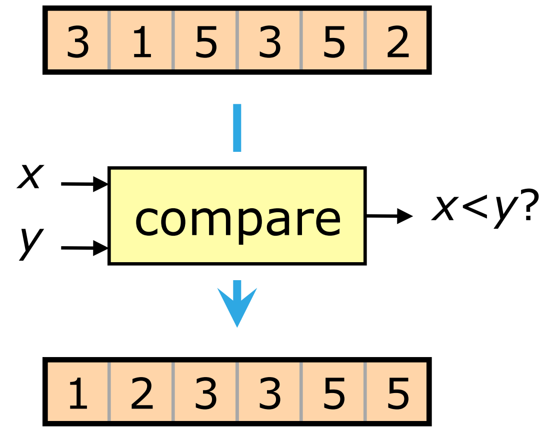
Review
Introduction to Sorting — Basics

Sort: Place a collection of data in order.

Key: The part of the data item used to sort.

Comparison sort: A sorting algorithm that gets its information by comparing items in pairs.

A **general-purpose comparison sort** places no restrictions on the size of the list or the values in it.



Review
Introduction to Sorting — Overview of Algorithms

There is no *known* sorting algorithm that has all the properties we would like one to have.

We will examine a number of sorting algorithms. Most of these fall into two categories: $O(n^2)$ and $O(n \log n)$.

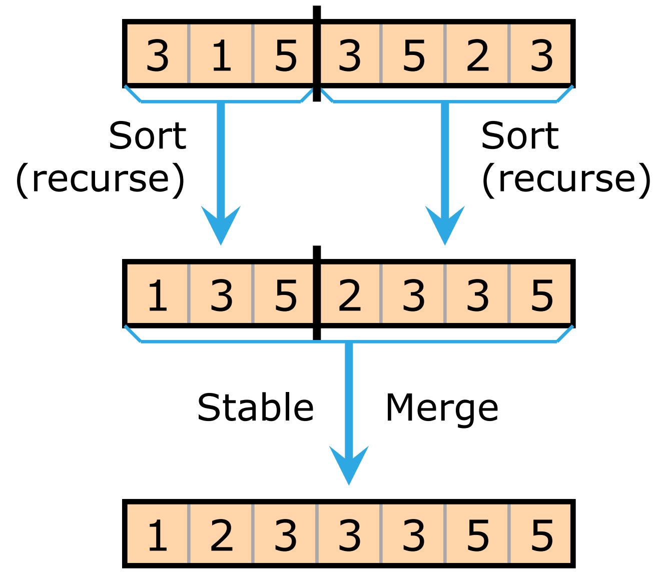
- Quadratic-Time [$O(n^2)$] Algorithms
  - ✓ Bubble Sort
  - ✓ Insertion Sort
  - ✓ Quicksort
  - Treesort (later in semester)
- Log-Linear-Time [$O(n \log n)$] Algorithms
  - ✓ Merge Sort
  - Heap Sort (mostly later in semester)
  - ✓ Introsort
- Special Purpose — Not Comparison Sorts
  - Pigeonhole Sort
  - Radix Sort

Review
Comparison Sorts II — Merge Sort

Merge Sort splits the data in half, recursively sorts each half, and then merges the two.

Stable Merge

- Linear time, stable.
- In-place for Linked List. Uses buffer [$O(n)$ space] for array.



Analysis

- Efficiency: $O(n \log n)$. Average same. ☺
- Requirements on data: Works for Linked Lists, etc. ☺
- Space Efficiency: $O(\log n)$ space for Linked List. Can eliminate recursion to make this in-place. $O(n)$ space for array. 😐/☺/☹
- Stable: Yes. ☺
- Performance on Nearly Sorted Data: Not better or worse. 😐

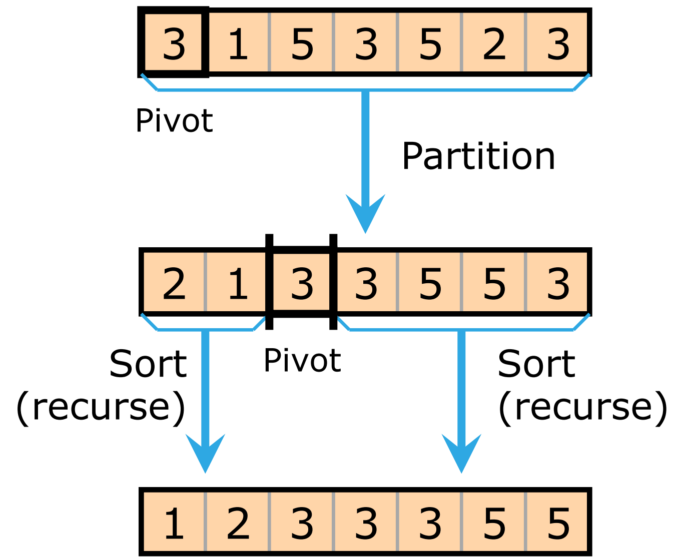
Notes

- Practical & often used.
- Fastest known for (1) stable sort, (2) sorting a Linked List.
- Good standard for judging sorting algorithms

Review
Comparison Sorts III — Quicksort: Introduction, Partition

Quicksort is another divide-and-conquer algorithm. Procedure:

- Choose a list item (the **pivot**).
- Do a **Partition**: put items less than the pivot before it, and items greater than the pivot after it.
- Recursively sort two sublists: items before pivot, items after pivot.



We did a simple pivot choice: the first item. Later, we improve this.

Fast Partition algorithms are in-place, but not stable.

- Note: In-place Partition does not give us an in-place Quicksort. Quicksort uses memory for recursion.

Review
Comparison Sorts III — Better Quicksort: Optimizations

Unoptimized Quicksort is slow (quadratic time) on nearly sorted data and uses a lot of space (linear) for recursion.

We discussed three optimizations:

- Median-of-three pivot selection.
  - Improves performance on most nearly sorted data.
  - Requires random-access data.
- Tail-recursion elimination on the larger recursive call.
  - Reduces space usage to logarithmic.
- Do not sort small sublists; finish with Insertion Sort.
  - General speed up.
  - May adversely affect cache hits.

With these optimizations, Quicksort is still $O(n^2)$ time.

Median-of-three

Initial State:

2	12	9	10	3	1	6

Pivot

After Partition:

2	1	3	6	10	9	12

Review
Comparison Sorts III — Better Quicksort: Analysis of Quicksort

Efficiency ☹

- Quicksort is $O(n^2)$.
- Quicksort has a **very** good $O(n \log n)$ average-case time. ☺☺

Requirements on Data ☹

- Non-trivial pivot-selection algorithms (median-of-3 and others) are only efficient for random-access data.

Space Usage 😐

- Quicksort uses space for recursion.
  - Additional space: $O(\log n)$, if clever tail-recursion elimination is done.
  - Even if **all** recursion is eliminated, $O(\log n)$ additional space is still used.
  - This additional space need not hold any data items.

Unlike Merge Sort

Stability ☹

- Efficient versions of Quicksort are not stable.

Performance on Nearly Sorted Data 😐

- An unoptimized Quicksort is **slow** on nearly sorted data: $O(n^2)$.
- Quicksort + median-of-3 is $O(n \log n)$ on most nearly sorted data.

Review
Comparison Sorts III: Introsort

Recall: Quicksort does a linear time operation (Partition), then calls itself recursively.

- If the recursion depth is around log n, then it uses $O(n \log n)$ steps.
  - Count both sub-lists as recursive calls. Ignore the tail-recursion trick.
- Thus, Quicksort is slow only **when the recursion gets too deep**.

Apply introspection:

- Do optimized Quicksort, but keep track of the recursion depth.
- If the depth exceeds some threshold ($k \log n$, for some k), switch to Heap Sort for the current sublist being sorted.
  - Musser suggested a threshold of $2 \log_2 n$.

The resulting algorithm is called **Introsort**.

Musser's 1997 paper discusses the speed-ups we have covered:

- Use the median-of-3 rule for pivot selection.
- Stop the recursion prematurely, and finish with Insertion Sort.
  - Maybe. This can adversely affect cache performance.
- However, it is no longer necessary to handle the larger and smaller recursive calls differently, since the recursion-depth limit already makes sure that excessive recursive calls are not made.

Review
Comparison Sorts III Introsort — Analysis

Efficiency ☺☺
- Introsort is $O(n \log n)$.
- Introsort also has an average-case time of $O(n \log n)$ [of course].
  - Its average-case time is just as good as Quicksort. ☺☺

Requirements on Data ☹
- Introsort requires random-access data.

Space Usage 😐
- Introsort uses space for recursion (or simulated recursion).
  - Additional space: $O(\log n)$ — even if all recursion is eliminated.
  - This additional space need not hold any data items.

Stability ☹
- Introsort is not stable.

Performance on Nearly Sorted Data 😐
- Introsort is not significantly faster or slower on nearly sorted data.

Review
Comparison Sorts III - When is it Best?

Algorithm	**When This Algorithm is the *Best* One**
Bubble Sort	Never
Insertion Sort	▪For small lists ▪When you are guaranteed nearly sorted data
Merge Sort	▪When stability is needed ▪For special data types, especially Linked Lists
Heap Sort	In certain special situations: ▪When a list is operated on during the sorting process ▪When you only care about the ordering of part of a list ▪Etc. (more about this later in the semester)
Quicksort	Never
Introsort	Most of the time (if you do not care about stability, data accessed via slow connections, sequential-access data, …)

Now, what if (say) Quicksort is written for you, but nothing else is?

Should you write your own? Maybe. It depends on the situation. **Think!**

Radix Sort
Background

We have looked in detail at five general-purpose comparison sorts.

Now we look at two sorting algorithms that do not use a comparison function:

- Pigeonhole Sort.
- Radix Sort.

Later in the semester, we will look closer at Heap Sort, which *is* a general-purpose comparison sort, but which can also be conveniently modified to handle other situations.

Preliminaries: Pigeonhole Sort — Description

Suppose we have a list to sort, and:

- Keys lie in a small fixed set of values.
- Keys can be used to index an array.
  - E.g., they might be small-ish nonnegative integers.

Not general-purpose

Not even a comparison sort

Procedure

- Make an array of empty lists (**buckets**), one for each possible key.
- Iterate through the given list; insert each item at the end of the bucket corresponding to its value.
- Copy items in each bucket, in order, back to the original list.

Time efficiency: **linear time**, if written properly.

- How is this possible? Answer: We are not doing general-purpose comparison sorting. Our $\Omega(n \log n)$ bound does not apply.

This algorithm is often called **Pigeonhole Sort**.

- Not applicable to many situations; requires a limited set of keys.
- Pigeonhole Sort is stable, and uses linear additional space.

Radix Sort
Preliminaries: Pigeonhole Sort — Write It

TO DO

- Examine code for a function to do Pigeonhole Sort.

Radix Sort
Description

Based on Pigeonhole Sort, we can design a useful algorithm: **Radix Sort**.

Suppose we want to sort a list of **strings** (in some sense):

- Character strings.
- Numbers, considered as strings of digits.
- Short-ish sequences of some other kind.

Call the entries in a string "**characters**".

- These need to be valid keys for Pigeonhole Sort.
- In particular, we must be able to use them as array indices.

The algorithm will arrange the list in **lexicographic order**.

- This means sort first by first character, then by second, etc.
- For strings of letters, this is alphabetical order.
- For positive integers (padded with leading zeroes), this is numerical order.

Radix Sort Procedure

- Pigeonhole Sort the list using the **last** character as the key.
- Take the list resulting from the previous step and Pigeonhole Sort it, using the **next-to-last** character as the key.
- Continue …
- After re-sorting by **first** character, the list is sorted.

Radix Sort Example

Here is the list to be sorted.

- **583 508 183 90 223 236 924 4 426 106 624**

We first sort them by the units digit, using Pigeonhole Sort.

Nonempty buckets are underlined

- **90 583 183 223 924 4 624 236 426 106 508**

Then Pigeonhole Sort again, based on the tens digit, in a stable manner (note that the tens digit of 4 is 0).

- **4 106 508 223 924 624 426 236 583 183 90**

Again, using the hundreds digit.

- **4 90 106 183 223 236 426 508 583 624 924**

And now the list is sorted.

Radix Sort
Write It, Comments

TO DO

- Write Radix Sort for small-ish positive integers.

Comments

- Radix Sort makes very strong assumptions about the values in the list to be sorted.
- It requires linear additional space (but not for data items).
- It is stable.
- It does not perform especially well or badly on nearly sorted data.
- Of course, what we really care about is speed. *See the next slide.*

Radix Sort
Efficiency [1/2]

How Fast is Radix Sort?

- Fix the number of characters and the character set.
- Then each sorting pass can be done in linear time.
  - Pigeonhole Sort with one bucket for each possible character.
- And there are a fixed number of passes.
- Thus, Radix Sort is $O(n)$: **linear time**.

How is this possible?

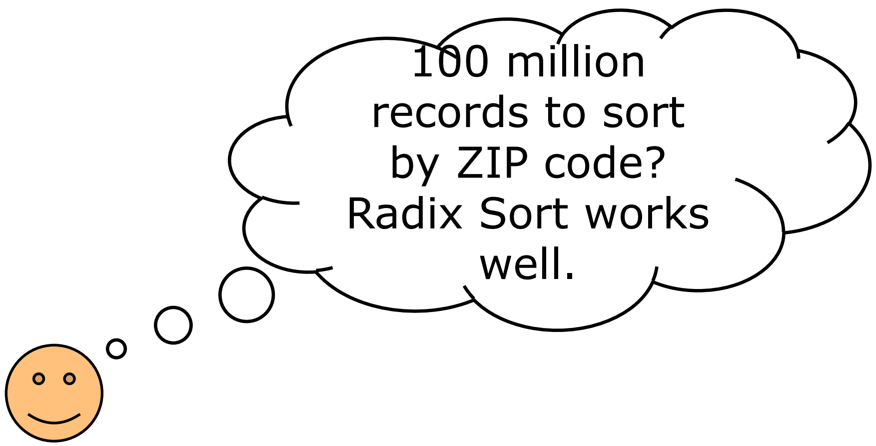
- Radix Sort is a sorting algorithm. However, again, it is neither general-purpose nor a comparison sort.
  - It places restrictions on the values to be sorted: not general-purpose.
  - It gets information about values in ways other than making a comparison: not a comparison sort.
- Thus, our argument showing that $\Omega(n \log n)$ comparisons were required in the worst case, does not apply.

Radix Sort
Efficiency [2/2]

In practice, Radix Sort is not really as fast as it might seem.

- There is a hidden logarithm. The number of passes required is equal to the length of a string, which is something like the logarithm of the number of possible values.
- The number of passes is fixed, since we limit the length of a string. This limits the number of possible values in the list.
- However, if we consider Radix Sort applied to a list in which *all the values might be different*, then it is in the same efficiency class as normal sorting algorithms.

In certain special cases (e.g., big lists of small numbers) Radix Sort can be useful.



Sorting in the C++ STL
Specifying the Interface

Iterator-based sorting functions can be specified two ways:

- Given a range
  - "**`last`**" is actually just past the end, as usual.

```
template<typename Iterator>
void sortIt(Iterator first, Iterator last);
```

- Given a range and a comparison.

```
template<typename Iterator, typename Ordering>
void sortIt(Iterator first, Iterator last, Ordering compare);
```

"**`compare`**", above, should be something you can use to compare two values.

- "**`compare(val1, val2)`**" should be a legal expression, and should return a **`bool`**: true if **`val1`** comes before **`val2`** (think "less-than").
- So **`compare`** can be a function (passed as a function pointer).
- It can also be an object with **`operator()`** defined: a **function object**.

Sorting in the C++ STL
Overview of the Algorithms [1/4]

The C++ Standard Template Library has six sorting algorithms:

- Global function **`std::sort`**
- Global function **`std::stable_sort`**
- Member function **`std::list<T>::sort`**
- Global functions **`std::partial_sort`** and **`partial_sort_copy`**.
- Combination of two global functions: **`std::make_heap`** & **`std::sort_heap`**

We now look briefly at each of these.

Sorting in the C++ STL
Overview of the Algorithms [2/4]

Function **`std::sort`**, in **`<algorithm>`**
- Global function.
- Takes two random-access iterators and an optional comparison.
- $O(n^2)$, but has $O(n \log n)$ average-case.
  - This became $O(n \log n)$ in the new C++11 standard.
  - It is currently $O(n \log n)$ in good STL implementations.
- Not stable. $O(\log n)$ additional space used.
- Algorithm used:
  - Quicksort is what the standards committee was thinking.
  - Introsort is what good implementations now use.
  - Other algorithms (Heap Sort?) are possible, but unlikely.

Function **`std::stable_sort`**, in **`<algorithm>`**
- Global function.
- Takes two random-access iterators and an optional comparison.
- $O(n \log n)$.
- Stable. $O(n)$ additional space used.
- Algorithm used: probably Merge Sort, general sequence version.

Sorting in the C++ STL
Overview of the Algorithms [3/4]

Function **`std::list<T>::sort`**, in **`<list>`**

- Member function. Sorts only objects of type **`std::list<T>`**.
- Takes either no parameters or a comparison.
- $O(n \log n)$. Stable.
- Algorithm used: probably Merge Sort, Linked-List version.

Sorting in the C++ STL
Overview of the Algorithms [4/4]

We will look at the last two STL algorithms in more detail later in the semester, when we cover Priority Queues and Heaps:

- Functions **`std::partial_sort`** and **`std::partial_sort_copy`**, in **`<algorithm>`**
  - Global functions.
  - Take three random-access iterators and an optional comparison.
  - $O(n \log n)$. Not stable.
  - Solve a more general problem than comparison sorting.
  - Algorithm used: probably Heap Sort.
- Combination: **`std::make_heap`** & **`std::sort_heap`**, in **`<algorithm>`**
  - Both Global functions.
  - Both take two random-access iterators and an optional comparison.
  - Combination is $O(n \log n)$. Not stable.
  - Solves a more general problem than comparison sorting.
  - Algorithm used: Heap Sort.

Sorting in the C++ STL
Using the Algorithms [1/2]

Algorithm **`std::sort`** is declared in the header **`<algorithm>`**.

Call it with two iterators:

```
vector<int> v;
std::sort(v.begin(), v.end());
    // Ascending order
```

Default constructor call. We can only pass an **object**, not a **type**.

Or use two iterators and a comparison:

```
std::sort(v.begin(), v.end(), std::greater<int>());
    // Descending order
```

- Class template **`std::greater`** is defined in **`<functional>`**.

Use **`std::stable_sort`** similarly to **`std::sort`**.

Sorting in the C++ STL
Using the Algorithms [2/2]

When sorting a **`std::list`**, use the **`sort`** member function:

```
#include <list>

std::list<int> myList;
myList.sort();                         // Ascending order
myList.sort(std::greater<int>());      // Descending order
```