Pfaffian Differential Equations Assignment - Page 33

Tek Raj Pant

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1 Part 2 - Page 33

Verify that the following equations are integrable and find their primitives

1.1 $2y(a-x)dx + [z - y^2 + (a-x)^2]dy - ydz = 0$

We are given the PfDE

$$2y(a-x)dx + [z - y^2 + (a-x)^2]dy - ydz = 0 \tag{1}$$

Let's define the vector $\vec{X} = (P\,, Q\,, R\,)$, where $P \;=\; 2y(a-x), Q \;= z - y^2 + (a-x)^2$, $R = -y$ so $\vec{X} = ((2y(a-x)), (z - y^2 + (a-x)^2), -y)$.
To check the integrability, lets calculate the $\vec{X} \cdot (\nabla \times \vec{X})$

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} 2y(a-x) & (z - y^2 + (a-x)^2 & -y \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y(a-x) & (z - y^2 + (a-x)^2 & -y \end{vmatrix}$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = 2y(a-x)(-1-1) + (z - y^2 + (a-x)^2)(0-0) - y(-2(a-x) - 2a + 2x)$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = -4y(a-x) - y(-2a + 2x - 2a + 2x)$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = -4y(a-x) - y(-4a + 4x)$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = -4ay + 4xy + 4ay - 4xy$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = 0$$

$$(2)$$

Since $\vec{X} \cdot \operatorname{curl} \vec{X} = 0$, the given Pfaffian Differential Equation (1) is integrable. Dividing the given PfDE by y^2 on both sides

$$\begin{aligned} \frac{2}{y}(a-x)\,dx + \frac{z}{y^2}\,dy - dy + \frac{(a-x)^2}{y^2}\,dy - \frac{1}{y}\,dz &= 0 \\ \frac{2}{y}(a-x)\,dx + \frac{(a-x)^2}{y^2}dy + \frac{z}{y^2}dy - dy - \frac{1}{y}dz &= 0 \end{aligned} \tag{3}$$

Here the terms $\frac{2}{y}(a-x)\,dx + \frac{(a-x)^2}{y^2}\,dy$ can be written as the differential of $-\frac{(a-x)^2}{y}$

$$d(-\frac{(a-x)^2}{y}) = -d(\frac{(a-x)^2}{y}) = \frac{2}{y}(a-x)\,dx + \frac{(a-x)^2}{y^2}\,dy$$

Also the remaining terms $\frac{z}{y^2}dy - \frac{1}{y}dz - dy$ can be written as the differential of (-$\frac{z}{y} - y)$

$$d(-\frac{z}{y} - y) = -(\frac{z}{y^2}dy + \frac{1}{y}dz) - dy = \frac{z}{y^2}dy - \frac{1}{y}dz - dy$$

Thus the equation (3) becomes

$$d(-\frac{(a-x)^2}{y}) + d(-\frac{z}{y} - z) = 0$$

Combining the differentials:

$$d(-\frac{(a-x)^2}{y} - \frac{z}{y} - y) = 0$$
$$d(\frac{(a-x)^2}{y} + \frac{z}{y} + y) = 0$$

Integrating on both sides

$$frac(a-x)^2y + \frac{z}{y} + y = C, where C is an arbitrary constant.$$

This is the required primitives of the given PfDE.

1.2 $y(1+z^2)dx - x(1+z^2)dy + (x^2+y^2)dz = 0$

We are given the PfDE

$$y(1+z^2)dx - x(1+z^2)dy + (x^2+y^2)dz = 0 \tag{1}$$

Let's define the vector $\vec{X} = (P\,, Q\,, R\,)$, where $P = y(1+z^2), Q = -x(1+z^2)$, $R = x^2+y^2$ so $\vec{X} = (y(1+z^2),\ x(1+z^2), x^2+y^2)$.
To check the integrability, lets calculate the $\vec{X}\cdot(\nabla\times\vec{X})$

$$\vec{X}\cdot(\nabla\times\vec{X}) = \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$
$$\vec{X}\cdot(\nabla\times\vec{X}) = \begin{vmatrix} y(1+z^2) & -x(1+z^2) & x^2+y^2 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y(1+z^2) & -x(1+z^2) & x^2+y^2 \end{vmatrix}$$
$$\vec{X}\cdot(\nabla\times\vec{X}) = y(1+z^2)(2y+2xz) + x(1+z^2)(2x-2yz) + (x^2+y^2)(-1-z^2-1-z^2)$$
$$\vec{X}\cdot(\nabla\times\vec{X}) = (1+z^2)(2y^2-2xyz+2x^2+2xyz) - 2(x^2+y^2)(1-z^2)$$
$$\vec{X}\cdot(\nabla\times\vec{X}) = (1+z^2)(2y^2+2x^2-2x^2-2y^2)$$
$$\vec{X}\cdot(\nabla\times\vec{X}) = 0$$

Since $\vec{X}\cdot\operatorname{curl}\vec{X} = 0$, the given Pfaffian Differential Equation (1) is integrable.

Dividing the given PfDE by $(x^2+y^2)(1+z^2)$, we get

$$\frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy + \frac{1}{1+z^2}dz = 0$$

The terms $\frac{y}{x^2+y^2} - \frac{x}{x^2+y^2}$ can be expressed as the differential $d(-\tan^{-1}\frac{y}{x})$

$$-d(\tan^{-1}\frac{y}{x}) = -(\frac{xdy-ydx}{x^2+y^2})$$

Also $d(\tan^{-1}z) = \frac{dz}{1+z^2}$
Hence the equation becomes

$$-d(\tan -1\frac{y}{x}) + d(\tan^{-1}z) = 0$$

Integrating both sides we get

$$\tan^{-1} z - \tan^{-1} \frac{y}{x} = C, \text{where C is an arbitrary constant}$$

This is the require primitives

1.3 $(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0$

We are given the PfDE

$$(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0 \tag{1}$$

Let's define the vector $\vec{X} = (P\,, Q\,, R\,)$, where $P\, = \,(y^2 + yz + z^2)$
$Q\, = (z^2 + zx + x^2)$, $R = (x^2 + xy + y^2)$
So $\vec{X} = ((y^2 + yz + z^2), (z^2 + zx + x^2), (x^2 + xy + y^2))$.
To check the integrability, lets calculate the $\vec{X} \cdot (\nabla \times \vec{X})$

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} (y^2 + yz + z^2) & (z^2 + zx + x^2) & (x^2 + xy + y^2) \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 + yz + z^2) & (z^2 + zx + x^2) & (x^2 + xy + y^2) \end{vmatrix}$$

$$\begin{aligned} \vec{X} \cdot (\nabla \times \vec{X}) &= (y^2 + yz + z^2)(2y - 2z) + (z^2 + zx + x^2)(2z - 2x) \\ &\quad + (x^2 + xy + y^2)(2x - 2y) \\ \vec{X} \cdot (\nabla \times \vec{X}) &= 2y^3 + 2y^2z + 2yz^2 - 2y^2z - 2yz^2 - 2z^3 \\ &\quad + 2z^3 + 2z^2x + 2zx^2 - 2zx^2 - 2x^3 - 2zx^2 \\ &\quad + 2x^3 + 2x^2y + 2xy^2 - 2x^2y - 2xy^2 - 2y^3 \\ \vec{X} \cdot (\nabla \times \vec{X}) &= 0 \end{aligned}$$

Since $\vec{X} \cdot \operatorname{curl} \vec{X} = 0$, the given Pfaffian Differential Equation (1) is integrable.

Since the given PfDE is homogeneous of degree 2
Let $y = ux$ and $z = vx$

$$dy = udx + xdu \qquad\qquad dz = vdx + xdv$$

Then the given PfDE becomes

$$
\begin{aligned}
&x^2(u^2+uv+v^2)\,dx + x^2u\,(v^2+v+1)\,dx + x^2v\,(u^2+u+1)\,dx \\
&\quad +\, x^3(v^2+v+1)\,du +\, x^3(u^2+u+1)\,dv = 0 \\
&x^2\Big[(u^2+uv+v^2)+u(v^2+v+1)+v(u^2+u+1)\Big]\,dx \\
&\quad + x^3\,(v^2+v+1)\,du + x^3\,(u^2+u+1)\,dv = 0. \\
&x^2\Big[(u+v+1)(u+uv+v)\Big]dx \\
&\quad + x^3\,(v^2+v+1)\,du \\
&\quad + x^3(u^2+u+1)dv = 0.
\end{aligned}
$$

Dividing both sides by $\frac{1}{x^3\Big[(u+v+1)(u+uv+v)\Big]}$

$$
\frac{1}{x}dx + \frac{v^2+v+1}{(u+v+1)(u+uv+v)}du + \frac{u^2+u+1}{(u+v+1)(u+uv+v)}dv = 0
$$

Using Partial Fraction

$$
\begin{aligned}
\frac{v^2+v+1}{(u+v+1)(u+uv+v)} &= \frac{v+1}{u+uv+v} - \frac{1}{u+v+1} \\
\frac{u^2+u+1}{(u+v+1)(u+uv+v)} &= \frac{u+1}{u+uv+v} - \frac{1}{u+v+1}
\end{aligned}
$$

So,

$$
\frac{1}{x}dx + \frac{v+1}{u+uv+v}du - \frac{1}{u+uv+1}du + \frac{u+1}{u+uv+v}dv - \frac{1}{u+v+1}dv = 0
$$

Integrating we get

$\ln(x) + \ln(u + uv + v) - \ln(u + v + 1) + \ln(u + uv + v) - \ln(u + v + 1) = C$,
where C is an arbitrary constant

$$\ln\left(x\left(\frac{u + uv + v}{u + v + 1}\right)^2\right) = C$$

Exponentiating both sides:

$$x\left(\frac{u + uv + v}{u + v + 1}\right)^2 = e^C = C_1$$

Substituting $u = \dfrac{y}{x},\ v = \dfrac{z}{x}$:

$$x\left(\frac{\frac{y}{x} + \frac{y}{x} \cdot \frac{z}{x} + \frac{z}{x}}{\frac{y}{x} + \frac{z}{x} + 1}\right)^2 = C_1$$

$$= x\left(\frac{\frac{y+z}{x} + \frac{yz}{x^2}}{\frac{y+z+x}{x}}\right)^2$$

$$= x\left(\frac{y + z + \frac{yz}{x}}{y + z + x} \cdot \frac{1}{x}\right)^2$$

$$= x\left(\frac{yx + zx + yz}{x(y + z + x)}\right)^2$$

$$= x \cdot \left(\frac{yx + yz + zx}{x(y + z + x)}\right)^2$$

$$= \frac{(yx + yz + zx)^2}{x(y + z + x)^2} = C_1$$

$$\Rightarrow \left(\frac{yx + yz + zx}{y + z + x}\right)^2 = C_1 x$$

Hence, the required primitive is:

$$\boxed{(\frac{yx + yz + zx}{y + z + x})^2 = C_1 x}$$

1.4 $yz\,dx + xz\,dy + xy\,dz = 0$

We are given the Pfaffian differential equation:

$$yz\,dx + xz\,dy + xy\,dz = 0 \tag{1}$$

Define the vector $\vec{X} = (P, Q, R)$, where:

$$P = yz, \quad Q = xz, \quad R = xy$$

Let's compute $\vec{X} \cdot (\nabla \times \vec{X})$:

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Expanding the determinant:

$$\vec{X} \cdot (\nabla \times \vec{X}) = yz \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz) \right) - xz \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz) \right) + xy \left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz) \right)$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = yz(x - x) - xz(y - y) + xy(z - z) = 0$$

Since $\vec{X} \cdot (\nabla \times \vec{X}) = 0$, the given Pfaffian differential equation is **integrable**.

The left-hand side expression can be written as:

$$yz\,dx + xz\,dy + xy\,dz = d(xyz)$$

Hence, the equation becomes:

$$d(xyz) = 0$$

Integrating both sides:

$$xyz = C, \quad \text{where } C \text{ is an arbitrary constant.}$$

This is the general solution of the given PfDE.

1.5 $(1 + yz)dx + x(z - x)dy - (1 + xy)dz = 0$

We are given the Pfaffian differential equation:

$$(1 + yz)\,dx + x(z - x)\,dy - (1 + xy)\,dz = 0 \tag{1}$$

Lets define the vector $\vec{X} = (P, Q, R)$, where:

$$P = 1 + yz, \quad Q = x(z - x), \quad R = -(1 + xy)$$

Lets check integrability by computing $\vec{X} \cdot (\nabla \times \vec{X})$

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} 1+yz & x(z-x) & -(1+xy) \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1+yz & x(z-x) & -(1+xy) \end{vmatrix}$$

$$\begin{aligned}
&= (1+yz)(-x-x) - x(z-x)(-y-y) - (1+xy)(z-2x-z) \\
&= -2x(1+yz) + 2xy(z-x) + 2x(1+xy) \\
&= (-2x - 2xyz) + 2xyz - 2x^2y + 2x + 2x^2y \\
&\vec{X} \cdot (\nabla \times \vec{X}) = 0
\end{aligned}$$

Since $\vec{X} \cdot (\nabla \times \vec{X}) = 0$, the Pfaffian differential equation is **integrable**.

Let $y =$ constant, $\Rightarrow dy = 0$
Then the given PfDE reduces to:

$$(1+yz)\,dx - (1+xy)\,dz = 0$$

$$\Rightarrow \frac{dx}{1+xy} = \frac{dz}{1+yz}$$

Integrating while treating y as constant:

$$\int \frac{dx}{1+xy} = \int \frac{dz}{1+yz}$$

$$\Rightarrow \frac{1}{y}\ln(1+xy) = \frac{1}{y}\ln(1+yz) + C$$

$$\Rightarrow \frac{1+xy}{1+yz} = \phi(y) \tag{2}$$

Now set $z = \alpha = 1$, hence $dz = 0$
So the original PfDE becomes:

$$(1+y)\,dx + x(1-x)\,dy = 0$$

$$\Rightarrow \frac{dx}{x(1-x)} = -\frac{dy}{1+y}$$

Integrating on both sides, we get

$$\ln(x) - \ln(1-x) = -\ln(1+y) + C_2$$

$$\Rightarrow \ln\left(\frac{x(1+y)}{(1-x)}\right) = C_2$$

Exponentially both sides

$$\frac{x(1+y)}{1-x} = C, \quad \text{where} \quad C = e^{C_2} \tag{3}$$

Substituting z = 1 in equation (2)

$$\frac{1+xy}{1+y} = \phi(y) \tag{4}$$

Also from equation (3)

$$x(1+y) = C - Cx$$
$$\Rightarrow x = \frac{C}{1+y+C}$$

Using this value of x in equation (4), we get

$$\frac{1}{1+y} + \frac{Cy}{1+y+C} = \phi(y)$$
$$\Phi(y) = \frac{1+C}{1+y+C}$$

Substituting $\phi(y) inequation$ (2) we get

$\frac{1+xy}{1+yz} = \frac{1+C}{1+y+C}$

This is the required general solution of given PfDE

1.6 $y(x+4)(y+z)dx - x(y+3z)dy + 2xydz = 0$

We are given the Pfaffian differential equation:

$$y(x+4)(y+z)\,dx - x(y+3z)\,dy + 2xy\,dz = 0 \tag{1}$$

Let us define the vector $\vec{X} = (P, Q, R)$, where:

$$P = y(x+4)(y+z), \quad Q = -x(y+3z), \quad R = 2xy$$

Let us check integrability by computing $\vec{X} \cdot (\nabla \times \vec{X})$:

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} y(x+4)(y+z) & -x(y+3z) & 2xy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y(x+4)(y+z) & -x(y+3z) & 2xy \end{vmatrix}$$

$$
\begin{aligned}
&= y(x+4)(y+z)(2x+3x) - (-x(y+3z))\,(2y - y(x+4)) \\
&\quad + 2xy\,(-(y+3z) - (x+4)(2y+z)) \\
&= 5xy(x+4)(y+z) - xy(y+3z)(x+2) + 2xy(-9y - 7z - 2xy - xz) \\
&= xy\big[5xy + 5xz + 20y + 20z - xy - 2y - 3xz - 6z - 18y - 14z - 4xy - 2xz\big] \\
&\vec{X}\cdot(\nabla\times\vec{X}) = 0
\end{aligned}
$$

Since $\vec{X}\cdot(\nabla\times\vec{X}) = 0$, the Pfaffian differential equation is **integrable**.
Let x = constants this implies dx = 0
Then the given PfDE reduced to

$$
\begin{aligned}
-x(y+3z)\,dy + 2xy\,dz &= 0 \\
2xy\,dz &= x(y+3z)\,dy \\
\frac{dz}{dy} &= \frac{y+3z}{2y} \\
\frac{dz}{dy} - \frac{3z}{2y} &= \frac{1}{2}
\end{aligned}
$$

This is linear in z, and the integrating factor is

$$\mu(y) = \exp\Big(-\int \frac{3}{2y}\,dy\Big) = \exp\big(-\tfrac{3}{2}\ln y\big) = y^{-\frac{3}{2}}.$$

Multiplying the above ODE by μ:

$$
\begin{aligned}
y^{-\frac{3}{2}}\frac{dz}{dy} - \frac{3}{2y}\,y^{-\frac{3}{2}}\,z &= \tfrac{1}{2}\,y^{-\frac{3}{2}} \\
\frac{d}{dy}\big(z\,y^{-\frac{3}{2}}\big) &= \tfrac{1}{2}\,y^{-\frac{3}{2}}
\end{aligned}
$$

Integrating both sides:

$$
\begin{aligned}
z\,y^{-\frac{3}{2}} &= \tfrac{1}{2}\int y^{-\frac{3}{2}}\,dy + C(x) \\
&= \tfrac{1}{2}\big(\tfrac{y^{-\frac{1}{2}}}{-\frac{1}{2}}\big) + \phi(x)
\end{aligned}
$$

$$\frac{z+y}{y^{-\frac{3}{2}}} = \phi(x) \tag{2}$$

Now set $y = \alpha = 1$ this gives $dy = 0$
Then the given PfDE becomes

$$\begin{aligned}
(x+4)(1+z)\,dx + 2x\,dz = 0 \\
\frac{x+4}{2x}\,dx = -\frac{dz}{1+z} \\
\left(\tfrac{1}{2} + \frac{2}{x}\right)dx = -\frac{dz}{1+z}
\end{aligned}$$

Integrating we get

$$\tfrac{1}{2}\,x + 2\ln(x) = -\ln(1+z) + C_1, \quad C_1 \text{ is an arbitrary constant}$$

Exponentiating both sides

$$x^2(1+z) = C\,\mathrm{e}^{-\frac{1}{2}}, \quad C = \mathrm{e}^{C_1}$$

$$x^2(1+z) = C \tag{3}$$

Substituting $y = 1$ in (2) we get

$$\phi(x) = 1 + z \tag{4}$$

From (3):

$$z = \frac{C}{x^2}\,\mathrm{e}^{-\frac{x}{2}} - 1$$

Substituting this into (4) gives

$$\phi(x) = \frac{C}{x^2}\,\mathrm{e}^{-\frac{x}{2}}$$

Hence equation (2) becomes

$$\begin{aligned}
\frac{y+z}{y^{-\frac{3}{2}}} = \frac{C}{x^2}\,\mathrm{e}^{-\frac{x}{2}}, \\
x^2\,\frac{y+z}{y^{-\frac{3}{2}}\,\mathrm{e}^{-\frac{x}{2}}} = C.
\end{aligned}$$

This is the required first integrals of the given PfDE.

1.7 $yzdx + (x^2y - zx)dy + (x^2z - xy)dz = 0$

We are given the Pfaffian differential equation:

$$yzdx + (x^2y - zx)dy + (x^2z - xy)dz = 0 \tag{1}$$

Let us define the vector $\vec{X} = (P, Q, R)$, where:

$$P = yz, \quad Q = x^2y - zx, \quad R = x^2z - xy$$

Let us check integrability by computing $\vec{X} \cdot (\nabla \times \vec{X})$:

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} yz & x^2y - xz & x^2z - xy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & x^2y - xz & x^2z - xy \end{vmatrix}$$

$$\begin{aligned} &= yz\big((-x) - (-x)\big) - (x^2y - xz)\big((2xz - y) - y\big) + (x^2z - xy)\big((2xy - z) - z\big) \\ &= yz \cdot 0 \; - \; (x^2y - xz)\,2(xz - y) \; + \; (x^2z - xy)\,2(xy - z) \\ &= 2\Big[-(x^2y - xz)(xz - y) + (x^2z - xy)(xy - z)\Big] \\ &= 0 \end{aligned}$$

Since $\vec{X} \cdot (\nabla \times \vec{X}) = 0$, so the given PfDE is integrable.

Set $y = \text{constant} \Rightarrow dy = 0$
Then the original equation simplifies to

$$\begin{aligned} yz\,dx + (x^2z - xy)\,dz &= 0 \\ \frac{z}{x}dx + \left(\frac{xz}{y} - 1\right)dz &= 0 \\ \frac{z}{x}dx &= \left(1 - \frac{xz}{y}\right)dz \end{aligned}$$

So,

$$\frac{dx}{dz} = \frac{x}{z} - \frac{x^2}{y} \quad \text{(Bernoulli equation in } x\text{)}$$

Let $u = \frac{1}{x}$, so

$$\frac{du}{dz} = -\frac{1}{x^2}\frac{dx}{dz} \Rightarrow -\frac{1}{u^2}\frac{du}{dz} = \frac{1}{uz} - \frac{1}{y}$$

Multiplying both sides by $-u^2$:

$$\frac{du}{dz} + \frac{u}{z} = \frac{1}{y} \quad \text{(Linear ODE)}$$

Integrating factor:

$$\mu = e^{\int \frac{1}{z} dz} = z$$

Multiply through by z:

$$z\frac{du}{dz} + u = \frac{z}{y} \Rightarrow \frac{d}{dz}(uz) = \frac{z}{y}$$

Integrate:

$$uz = \frac{1}{y}\int z\,dz = \frac{z^2}{2y} + \phi(y), \phi(y) \text{ is function of y since y is held constant}$$

Substitute $u = \frac{1}{x}$:

$$\phi(x) = \frac{z}{x} - \frac{z^2}{2y} \tag{2}$$

Again Set $z = 1 \Rightarrow dz = 0$
The equation becomes:

$$y\,dx + (x^2y - x)\,dy = 0$$
$$dx + (x^2 - \frac{x}{y})\,dy = 0$$

$$\frac{dx}{dy} = -x^2 + \frac{x}{y} \quad \text{(Bernoulli equation in } x\text{)}$$

Let $v = \frac{1}{x}$, so

$$\frac{dv}{dy} = -\frac{1}{x^2}\frac{dx}{dy} \Rightarrow -\frac{1}{v^2}\frac{dv}{dy} = -v^{-2} + \frac{1}{vy}$$

Multiply both sides by $-v^2$:

$$\frac{dv}{dy} + \frac{v}{y} = 1 \quad \text{(Linear ODE)}$$

Integrating factor:

$$\mu = e^{\int \frac{1}{y} dy} = y$$

Multiply above ODE by y:

$$y\frac{dv}{dy} + v = y$$
$$\frac{d}{dy}(vy) = y$$

Integrate:

$$vy = \frac{y^2}{2} + C, \text{ where C is an arbitrary constant}$$

Substituting $v = \frac{1}{x}$

$$\frac{y}{x} - \frac{y^2}{2} = C \tag{3}$$

Using z = 1 in equation (2) we get

$$\phi(x) = \frac{1}{x} - \frac{1}{2y} \tag{4}$$

From equation (3) $\frac{1}{x} = \frac{y}{2} + \frac{C}{y}$ and using this value in equation (4)

$$\phi(y) = (\frac{y}{2} + \frac{C}{y}) - \frac{1}{2y}$$
$$\phi(x) = \frac{y}{2} + \frac{2C-1}{2y}$$

Hence Equation (2) becomes

$$\frac{z}{x} - \frac{z^2}{2y} = \frac{y}{2} + \frac{2C-1}{2y}$$
$$\frac{2yz}{x} - z^2 = y^2 + 2C - 1$$
$$\frac{2yz}{x} = y^2 + z^2 + 2C - 1$$
$$2xyz = x(y^2 + z^2 + C_1) \text{ ,where C1= 2C-1}$$
$$x(y^2 + z^2) - 2yz = -C_1x$$
$$y^2 + z^2 - \frac{2yz}{x} = C2 \text{ ,where C2 = -C1}$$

This is the require primitives of given PfDE

1.8 $2yzdx - 2xzdy - (x^2 - y^2)(z - 1)dz = 0$

We are given the Pfaffian differential equation:

$$2yz\,dx - 2xz\,dy - (x^2 - y^2)(z-1)\,dz = 0 \tag{1}$$

Let us define the vector $\vec{X} = (P, Q, R)$, where:

$$P = 2yz, \quad Q = -2xz, \quad R = -(x^2 - y^2)(z-1)$$

Let us check integrability by computing $\vec{X} \cdot (\nabla \times \vec{X})$:

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} 2yz & -2xz & -(x^2 - y^2)(z-1) \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz & -2xz & -(x^2 - y^2)(z-1) \end{vmatrix}$$

$$\begin{aligned}
&= 2yz\left(\frac{\partial}{\partial y}R - \frac{\partial}{\partial z}Q\right) - (-2xz)\left(\frac{\partial}{\partial x}R - \frac{\partial}{\partial z}P\right) + \left[-(x^2 - y^2)(z-1)\right]\left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P\right) \\
&= 2yz\left[2y(z-1) - (-2x)\right] + 2xz\left[2x(z-1) - 2z\right] - (x^2 - y^2)(z-1)\left[-2z - 2z\right] \\
&= 2yz(2y(z-1) + 2x) + 2xz(2x(z-1) - 2z) + 4z(x^2 - y^2)(z-1) \\
&= \left[4xyz + 4y^2z(z-1)\right] + \left[-4xyz - 4x^2z(z-1)\right] + \left[4z(x^2 - y^2)(z-1)\right] \\
&= 4y^2z(z-1) - 4x^2z(z-1) + 4z(x^2 - y^2)(z-1) \\
&= 0
\end{aligned}$$

Since $\vec{X} \cdot (\nabla \times \vec{X}) = 0$, the given PfDE is integrable.

Divide the original Pfaffian by $x^2 + y^2$:

$$\frac{2yz}{x^2 + y^2}\,dx \;-\; \frac{2xz}{x^2 + y^2}\,dy \;-\; \frac{(x^2 - y^2)(z-1)}{x^2 + y^2}\,dz \;=\; 0.$$

Lets introduce the cylindrical coordinates $x = r\cos\theta,\ y = r\sin\theta,\ x^2 + y^2 = r^2$
Then

$$\begin{aligned}
&x^2 + y^2 = r^2 \\
&x^2 - y^2 = r^2\cos(2\theta) \\
&dx = \cos(\theta)\,dr - r\sin(\theta)\,d\theta \\
&dy = \sin(\theta)\,dr + r\cos(\theta)\,d\theta
\end{aligned}$$

Substituting these values, we get

$$\frac{2z\left(r\sin(\theta)(\cos(\theta)\,dr - r\sin(\theta)\,d\theta) - r\cos(\theta)(\sin(\theta)\,dr + r\cos(\theta)\,d\theta)\right)}{r^2}$$
$$-\ \frac{r^2\cos(2\theta)(z-1)}{r^2}\,dz = 0$$
$$\frac{2zr\left(\sin\theta\cos\theta\,dr - r\sin^2\theta\,d\theta - \sin\theta\cos\theta\,dr - r\cos^2\theta\,d\theta\right)}{r^2} - \cos(2\theta)(z-1)\,dz = 0$$
$$\frac{2zr\left(-r(\sin^2\theta + \cos^2\theta)\,d\theta\right)}{r^2} - \cos(2\theta)(z-1)\,dz = 0$$
$$-\ 2z\,d\theta - \cos(2\theta)(z-1)\,dz = 0$$
$$-\ 2zd\theta = \cos 2\theta(z-1)dz$$

Separating variables

$$\frac{d\theta}{\cos 2\theta} = -\frac{z-1}{2z}dz$$

Integrating both sides:

$$\int \sec 2\theta d\theta = -\frac{1}{2}\int \frac{z-1}{z}dz + C \quad \text{where } C \text{ is an arbitrary constant}$$
$$\frac{1}{2}\ln(\sec 2\theta + \tan 2\theta) = -\frac{1}{2}\int \big(1 - \frac{1}{z}\big)dz + C$$
$$\ln(\sec 2\theta + \tan 2\theta) = z - \ln(z) + C$$

Substituting back the Cartesian coordinates:

$$\sec 2\theta = \frac{1}{\cos 2\theta} = \frac{r^2}{x^2+y^2} \text{ and } \tan 2\theta = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$
$$\text{Hence, } \sec 2\theta + \tan 2\theta = \frac{r^2+2xy}{x^2-y^2} = \frac{x^2+2xy+y^2}{x^2-y^2} = \frac{(x+y)^2}{x^2-y^2}$$

So the solution becomes

$$\ln\big(\frac{(x+y)^2}{x^2-y^2}\big) = z - \ln(z) + C$$

Exponentiating both sides

$$\frac{(x+y)^2}{x^2-y^2} = ze^{-z}e^{C}$$
$$\boxed{\frac{x+y}{z(x-y)} = e^{-z}K \text{ where K} = e^{C}}$$

This is the required primitive.