

Partial Differential Equations - Assignment-

July 19

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August 1, 2025

1 Part-1

Find the general integrals of the linear partial differential equations

1.1 $z(xp - yq) = y^2 - x^2$

Here given Partial Differential Equation is

$$z(xp - yq) = y^2 - x^2 \quad (1)$$

$$xp - yq = \frac{y^2 - x^2}{z}$$

This is a first-order semi-linear PDE of the form $Pp + Qq = R$ where

$$\begin{aligned} P &= x \\ Q &= -y \\ R &= \frac{(y^2 - x^2)}{z} \end{aligned}$$

The auxiliary equation is given by

$$\frac{dx}{x} = -\frac{dy}{y} = \frac{dz}{(y^2 - x^2)}$$

On Solving first two terms we get

$$\frac{dx}{x} = -\frac{dy}{y}$$

Separating variables and integrating on both sides

$$\ln(x) = -\ln(y) + C_1, \text{ where } C_1 \text{ is an arbitrary constant}$$

$$\ln(xy) = C_1$$

Exponentiating both sides we get

$$xy = e^{C_1}$$

$$xy = C_2 \quad \text{where } C_2 = e^{C_1} \quad (2)$$

Now lets solve the first and last term

$$\frac{dx}{x} = \frac{dz}{\frac{y^2-x^2}{z}}$$

$$\frac{dx}{x} = \frac{zdz}{y^2-x^2}$$

From equation (2) we have $y = \frac{C_2}{x}$, so using this in above equation

$$\frac{dx}{x} = \frac{zdz}{\left(\frac{C_2^2}{x^2}\right) - x^2}$$

$$\frac{dx}{x} = \frac{x^2 z dz}{C_2^2 - x^4}$$

$$\frac{C_2^2 - x^4}{x^3} dx = z dz$$

Integrating both sides

$$\int \left(\frac{C_2^2 - x^4}{x^3} \right) dx = \int z dz + C_3 \quad \text{Where } C_3 \text{ is an arbitrary constant}$$

$$\int \left(\frac{C_2^2}{x^3} - x \right) dx = \frac{z^2}{2} + C_3$$

$$-\frac{C_2^2}{2x^2} - \frac{x^2}{2} = \frac{z^2}{2} + C_3$$

From (2) $C_2 = xy$ so

$$-\frac{x^2 y^2}{2x^2} - \frac{x^2}{2} = \frac{z^2}{2} + C_3$$

(3)

$$\begin{aligned} -x^2 - y^2 - z^2 &= 2C_3 \\ x^2 + y^2 + z^2 &= C_4 \quad \text{where } C_4 = -2C_2 \end{aligned}$$

The two first integrals are

$$\begin{aligned} C_2 &= xy \\ C_4 &= x^2 + y^2 + z^2 \end{aligned}$$

Hence, the general solution is an arbitrary relation between them

$$G(xy, z^2 + x^2 + y^2) = 0$$

$$\mathbf{1.2} \quad px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$$

Here given Partial Differential Equation is

$$px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3) \quad (1)$$

$$\begin{aligned} x(z - 2y^2)p &= z(z - y^2 - 2x^3) - qy(z - y^2 - 2x^3) \\ x(z - 2y^2)p &= z(z - y^2 - 2x^3) - qy(z - y^2 - 2x^3) \\ x(z - 2y^2)p + y(z - y^2 - 2x^3)q &= z(z - y^2 - 2x^3) \end{aligned}$$

This is quasi-linear PDE of first order of the form $Pp + Qq = R$ where

$$P = x(z - 2y^2), \quad Q = y(z - y^2 - 2x^3), \quad R = z(z - y^2 - 2x^3)$$

The characteristic equation is given by

$$\frac{dx}{x(z - 2y^2)} = \frac{dy}{y(z - y^2 - 2x^3)} = \frac{dz}{z(z - y^2 - 2x^3)}$$

On Solving last two terms we get

$$\frac{dy}{y(z - y^2 - 2x^3)} = \frac{dz}{z(z - y^2 - 2x^3)}$$

$$\frac{dz}{z} = \frac{dy}{y}$$

Integrating both sides

$$\ln(z) = \ln(y) + C_1 \quad \text{Where } C_1 \text{ is an arbitrary constant}$$

$$\ln(z/y) = C_1$$

Exponentiating on both sides

$$\frac{z}{y} = e^{C_1}$$

$$\frac{z}{y} = C_2 \quad \text{Where } C_2 = e^{C_1} \tag{2}$$

On Solving first two terms we get

$$\frac{dx}{x(z - 2y^2)} = \frac{dy}{y(z - y^2 - 2x^3)}$$

From equation (2) $z = yC_2$, hence

$$\frac{dx}{x(yC_2 - 2y^2)} = \frac{dy}{y(yC_2 - y^2 - 2x^3)}$$

$$\frac{dx}{x(C_2 - 2y)} = \frac{dy}{y(C_2 - y) - 2x^3}$$

$$[y(C_2 - y) - 2x^3]dx - x(C_2 - 2y)dy = 0$$

$$\text{Here } M = y(C_2 - y) - 2x^3 \quad N = -x(C_2 - 2y)$$

$$\frac{\partial M}{\partial y} = C_2 - 2y$$

$$\frac{\partial N}{\partial x} = -C_2 + -2y$$

Its not exact so lets find its integrating factor

Here

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{C_2 - 2y - C_2 + 2y}{-x(C_2 - 2y)} = \frac{2}{x}$$

So

$$\mu(x) = e^{\int \frac{-2}{x} dx} = x^2$$

Multiplying by x^2 we get

$$\left[\frac{y(C_2 - y)}{x^2} - 2x \right] dx - \frac{C_2 - 2y}{x} dy = 0$$

Lets find a function $F(x,y)$ such that

$$dF(x,y) = \left[\frac{y(C_2 - y)}{x^2} - 2x \right] dx - \frac{C_2 - 2y}{x} dy$$

$$\text{i.e. } \frac{\partial M}{\partial x} = M = \frac{y(C_2 - y)}{x^2} - 2x$$

$$\frac{\partial N}{\partial y} = N = \frac{-(C_2 - 2y)}{x}$$

Integration N wrt x

$$F(x,y) = \int \left(\frac{y(C_2 - y)}{x^2} - 2x \right) + h(y) \quad \text{where } h(y) \text{ is an arbitrary function of } y$$

$$F(x,y) = -\frac{y(C_2 - y)}{x} - x^2 + h(y)$$

$$-\frac{C_2 - 2y}{x} = -\frac{C_2 - 2y}{x} + h'(y)$$

$$h'(y) = 0$$

Integrating we get

$$h(y) = C_3 \quad \text{where } C_3 \text{ is an arbitrary constant}$$

Therefore

$$\frac{y(C_2 - y)}{x} + x^2 = C_3$$

$$\text{But we have } C_2 = \frac{z}{y} \quad \text{So}$$

$$\frac{y(\frac{z}{y} - y)}{x} + x^2 = C_3$$

$$\frac{z - y^2}{x} + x^2 = C_3$$

Hence the two integrals are

$$C_2 = \frac{z}{x}$$

$$C_3 = \frac{z - y^2}{x} + x^2$$

The general arbitrary solution is

$$G\left(\frac{z}{y}, \frac{z - y^2}{x} + x^2\right) = 0$$

$$1.3 \quad px(x+y) = qy(x+y) - (x-y)(2x+2y+z)$$

Here given Partial Differential Equation is

$$px(x+y) = qy(x+y) - (x-y)(2x+2y+z) \quad (1)$$

$$x(x+y)p - y(x+y)q = -(x-y)(2x+2y+z)$$

This is semi-linear first order PDE, of the form $Pp + Qq = R$ where $P = x(x+y)$, $Q = -y(x+y)$ and $R = -(x-y)(2x+2y+z)$

The characteristic equation is

$$\frac{dx}{x(x+y)} = -\frac{dy}{y(x+y)} = -\frac{dz}{(x-y)(2x+2y+z)}$$

From first two ratios

$$\begin{aligned} \frac{dx}{x(x+y)} &= -\frac{dy}{y(x+y)} \\ \frac{dx}{x} &= -\frac{dy}{y} \end{aligned}$$

Integration on both sides

$$\ln(x) = -\ln(y) + C_1 \quad \text{where } C_1 \text{ is an arbitrary constant}$$

$$\ln(xy) = C_1$$

Exponentiating on both sides

$$xy = e^{C_1}$$

$$xy = C_2 \quad \text{where } C_2 = e^{C_1} \quad (2)$$

From last two terms

$$\frac{dy}{y(x+y)} = \frac{dz}{(x-y)(2x+2y+z)}$$

Let us introduce a new parameter t such that

$$\frac{dx}{x(x+y)} = -\frac{dy}{y(x+y)} = -\frac{dz}{(x-y)(2x+2y+z)} = dt$$

Let $v = x+y$ then

$$\frac{dv}{dt} = \frac{d(x+y)}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = x(x+y) - y(x+y)$$

$$\frac{dv}{dt} = (x + y)(x - y)$$

$$\frac{dv}{dt} = (x - y)v$$

$$\text{Also, } \frac{dz}{dt} = -(x - y)(2v + z)$$

$$\text{And } \frac{dz}{dv} = \frac{\frac{dz}{dt}}{\frac{dv}{dt}}$$

$$\frac{dz}{dv} = -\frac{(x - y)(2v + z)}{(x - y)v}$$

$$\frac{dz}{dv} = -\frac{2v + z}{v}$$

$$\frac{dz}{dv} + \frac{1}{v}z = -2$$

This is in standard form of the first-order linear ODE, so the solution is

$$z(v) = \frac{1}{\mu(v)} \left(\int \mu(v) Q(v) dv + C \right), \quad \text{where } \mu(v) = e^{\int P(v) dv},$$

$$P(v) = \frac{1}{v} \quad \text{and } Q(v) = -2 \quad \text{and hence } \mu(v) = e^{\int \frac{1}{v} dv} = v$$

$$\text{Therefore } z = \frac{1}{v} \left(\int v(-2) dv + C_3 \right) \text{ where } C_3 \text{ is an arbitrary constant}$$

$$z = -\frac{v^2}{v} + \frac{C_3}{v}$$

$$v(z + v) = C_3$$

$$\text{But } v = x + y \quad \text{hence}$$

$$C_3 = (x + y)(x + y + z)$$

Hence the two integrals are

$$C_2 = xy$$

$$C_3 = (x + y)(x + y + z)$$

Hence

$$F(xy, (x + y)(x + y + z)) = 0, \quad \text{is the required arbitrary general solution}$$

1.4 $y^2 p - xyq = x(z - 2y)$

Here given Partial Differential Equation is

$$y^2 p - xyq = x(z - 2y) \quad (1)$$

This is semi linear PDE of the form $Pq + Qq = R$

where $P = y^2$ $Q = -xy$ $R = x(z - 2y)$

The characteristic equation can be written as

$$\frac{dx}{y^2} = -\frac{dy}{xy} = \frac{dz}{x(z - 2y)}$$

From first two ratios we get

$$\begin{aligned} \frac{dx}{y^2} &= -\frac{dy}{xy} \\ \frac{dx}{y^2} + \frac{dy}{xy} &= 0 \\ \frac{x dx + y dy}{y^2} &= 0 \\ d(x^2 + y^2) &= 0 \end{aligned}$$

Integrating both sides

$$x^2 + y^2 = C_1 \quad \text{where } C_1 \text{ is an arbitrary constant} \quad (2)$$

From last two terms we get

$$\begin{aligned} -\frac{dy}{xy} &= \frac{dz}{x(z - 2y)} \\ \frac{dz}{dy} &= \frac{x(z - 2y)}{-2x} \\ \frac{dz}{dy} &= -\frac{z - 2y}{y} \\ \frac{dz}{dy} + \frac{1}{y} z &= 2 \end{aligned}$$

This is standard first order ODE and the solution is given by

$$z(y) = \frac{1}{\mu(y)} \left(\int \mu(y) Q(y) dy + C \right), \quad \text{where } \mu(y) = e^{\int P(y) dy},$$

$$P(y) = \frac{1}{y} \quad \text{and } Q(y) = 2 \quad \text{and hence } \mu(y) = e^{\int \frac{1}{y} dy} = y$$

Multiplying above ODE by y

$$y \frac{dz}{dy} + z = 2y$$

$$\frac{d}{dy}(yz) = 2y$$

Integrating both sides

$$yz = y^2 + C_2 \quad \text{where } C_2 \text{ is an arbitrary constant}$$

$$yz - y^2 = C_2$$

Hence the General solution is

$$G(x^2 + y^2, y(z - y)) = 0$$

textwhere G is an arbitrary function

1.5 $(y + zx)p - (x + yz)q = x^2 - y^2$

Here given Partial Differential Equation is

$$(y + zx)p - (x + yz)q = x^2 - y^2 \quad (1)$$

This is quasi linear PDE of the form $Pq + Qq = R$
where $P = (y + zx)$ $Q = -(x + yz)$ $R = x^2 - y^2$

The characteristic equation can be written as

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2}$$

From first two ratios we get

$$\frac{dx}{y+zx} = -\frac{dy}{(x+yz)}$$

$$(x+yz)dx + (y+zx)dy = 0$$

$$xdx + yzdx + ydy + zx dy = 0$$

$$(xdx + ydy) + z(ydx + xdy) = 0$$

$$\text{Since } d(x^2 + y^2) = 2(xdx + ydy)$$

$$\text{And } d(xy) = xdy + ydx$$

Hence

$$\frac{d(x^2 + y^2)}{2} + zd(xy) = 0$$

$$d(x^2 + y^2) + 2zd(xy) = 0$$

Integrating both sides

$$x^2 + y^2 + 2xyz = C_1 \quad \text{where } C_1 \text{ is an arbitrary constant} \quad (2)$$

From last first and last terms, we have

$$\frac{dx}{y+zx} = \frac{dz}{x^2-y^2}$$

$$\frac{dz}{dx} = \frac{x^2-y^2}{y+zx}$$

From equation (2),

$$x^2 + y^2 + 2xyz = C_1$$

$$z = \frac{C_1 - x^2 - y^2}{2xy}$$

Substituting the value of z we get

$$\frac{dz}{dx} = \frac{x^2-y^2}{y+x\left(\frac{C_1-x^2-y^2}{2xy}\right)}$$

$$\frac{dz}{dx} = \frac{x^2-y^2}{\left(\frac{C_1+y^2-x^2}{2y}\right)}$$

$$dz = -2y\left(\frac{y^2-x^2}{C_1+y^2+x^2}\right)dx$$

Let $A = C_1 + y^2$, then

$$\begin{aligned} dz &= -2y \left(-\frac{x^2 - y^2}{A - x^2} \right) dy \\ dz &= -2y \left(-1 + \frac{A - y^2}{A - x^2} \right) dy \\ dz &= -2y \left(-1 + \frac{C_1}{A - x^2} \right) dy \end{aligned}$$

Integrating both sides

$$\begin{aligned} \int dz &= -2y \left(\int -1 dx + C_1 \int \frac{1}{A - x^2} dx \right) + C_2 \quad \text{where } C_2 \text{ is an arbitrary constant} \\ z &= -2y(-x + C_1 \frac{1}{2\sqrt{A}} \ln(\frac{\sqrt{A} + x}{\sqrt{A} - x})) + C_2 \\ C_2 &= z - 2xy + \frac{2yC_1}{2\sqrt{C_1 + y^2}} \ln(\frac{\sqrt{C_1 + y^2} + x}{\sqrt{C_1 + y^2} - x}) \end{aligned}$$

Hence the General Solution is

$$G(x^2 + y^2 + 2xy, z - 2xy + \frac{2yC_1}{2\sqrt{C_1 + y^2}} \ln(\frac{\sqrt{C_1 + y^2} + x}{\sqrt{C_1 + y^2} - x})) = 0$$

Where G is an arbitrary function

1.6 $x(x^2 + 3y^2)p - y(3x^2 + y^2)q = 2z(y^2 - x^2)$

Here given Partial Differential Equation is

$$x(x^2 + 3y^2)p - y(3x^2 + y^2)q = 2z(y^2 - x^2) \quad (1)$$

This is semi linear PDE of the form $Pq + Qq = R$

$$\text{where } P = x(x^2 + 3y^2) \quad Q = -y(3x^2 + y^2) \quad R = 2z(y^2 - x^2)$$

The characteristic equation can be written as

$$\frac{dx}{x(x^2 + 3y^2)} = \frac{dy}{-y(3x^2 + y^2)} = \frac{dz}{2z(y^2 - x^2)}$$

From first two terms

$$\begin{aligned} \frac{dx}{x(x^2 + 3y^2)} &= \frac{dy}{-y(3x^2 + y^2)} \\ \frac{dy}{dx} &= \frac{-y(3x^2 + y^2)}{x(x^2 + 3y^2)} \end{aligned}$$

Let $v = \frac{y}{x}$ Then

$$y = vx \quad \text{and} \quad \frac{dy}{dx} = x \frac{dv}{dx} + v \quad \text{similarly } y^2 = v^2 x^2$$

$$v + x \frac{dv}{dx} = -\frac{vx(3x^2 + v^2 x^2)}{x(x^2 + 3v^2 x^2)}$$

$$v + x \frac{dv}{dx} = -\frac{v(3 + v^2)}{1 + 3v^2}$$

$$x \frac{dv}{dx} = -\frac{v(3 + v^2)}{1 + 3v^2} - v$$

$$x \frac{dv}{dx} = -v \frac{4(1 + v^2)}{1 + 3v^2}$$

$$\frac{1 + 3v^2}{v(1 + v^2)} dv = -\frac{4}{x} dx$$

Integrating both sides

$$\int \frac{1 + 3v^2}{v(1 + v^2)} dv = \int -\frac{4}{x} dx + C_1 \quad \text{where } C_1 \text{ is an arbitrary constant}$$

Using Partial Fraction

$$\frac{1 + 3v^2}{v(1 + v^2)} = \frac{1}{v} + \frac{2v}{1 + v^2} \quad , \text{hence}$$

$$\int \left(\frac{1}{v} + \frac{2v}{1 + v^2} \right) dv = -4 \ln(x) + C_1$$

$$\ln(v) + \ln(1 + v^2) = -4 \ln(x) + C_1$$

$$\ln(v(1 + v^2)) = \ln\left(\frac{1}{x^4}\right) + C_1$$

Exponentiating both sides

$$v(1 + v^2) = \frac{e^{C_1}}{x^4}$$

$$v(1+v^2) = \frac{C_2}{x^4} \quad \text{where } C_2 = e^{C_1}$$

$$\text{Substituting back } v = \frac{y}{x}$$

$$\frac{y}{x}(1+(\frac{y}{x})^2) = \frac{C_2}{x^4}$$

$$xy(x^2+y^2) = C_2 \quad (2)$$

From first and last terms, we have

$$\begin{aligned} \frac{dx}{x(x^2+3y^2)} &= \frac{dz}{2z(y^2-x^2)} \\ \frac{dz}{z} &= \frac{2(y^2-x^2)}{x(x^2+3y^2)} dx \end{aligned}$$

$$\text{Let } v = \frac{y}{x} \quad \text{Then}$$

$$y = vx \quad \text{and} \quad \frac{dy}{dx} = x\frac{dv}{dx} + v \quad \text{similarly } y^2 = v^2x^2 \quad \text{so,}$$

$$\frac{dz}{z} = 2\frac{x^2(v^2-1)}{x^3(1+3v^2)} dx$$

$$\frac{dz}{z} = 2\frac{(v^2-1)}{(1+3v^2)} \frac{dx}{x}$$

From the first characteristic $v(1+v^2)x^4 = C_2$ Differentiating wrt to x we get

$$\frac{dx}{x} = -\frac{1+3v^2}{4v(1+v^2)} dv$$

Substitute into above equation

$$\frac{dz}{z} = 2\frac{(v^2-1)}{(1+3v^2)} \left(-\frac{1+3v^2}{4v(1+v^2)} \right) dv$$

$$\frac{dz}{z} = -\frac{v^2-1}{2v(1+v^2)} dv$$

Integrating on both sides

$$\int \frac{dz}{z} = \int -\frac{v^2-1}{2v(1+v^2)} dv + C_3 \quad \text{where } C_3 \text{ is an arbitrary constant}$$

Using partial Fraction

$$\frac{v^2 - 1}{v(1 + v^2)} = \frac{v}{1 + v^2} - \frac{1}{v}$$

So

$$\int \frac{dz}{z} = \int \left(-\frac{v}{1 + v^2} + \frac{1}{v} \right) dv + C_3$$

$$\ln(z) = -\frac{1}{4} \ln(1 + v^2) + \frac{1}{2} \ln(v) + C_3$$

Exponentiating both sides

$$z = C_4 v^{\frac{1}{2}} (1 + v^2)^{-\frac{1}{4}} \quad \text{where } C_4 = e^{C_3}$$

$$\text{Back substituting } v = \frac{y}{x}$$

$$z = C_4 \frac{\sqrt{\frac{y}{x}}}{\left(1 + \left(\frac{y}{x}\right)^2\right)^{\frac{1}{4}}}$$

$$C_4 = z \frac{\left(1 + \left(\frac{y}{x}\right)^2\right)^{\frac{1}{4}}}{\sqrt{\frac{y}{x}}}$$

Hence the General solution is given by

$$F\left(xy(x^2 + y^2), z \frac{\left(1 + \left(\frac{y}{x}\right)^2\right)^{\frac{1}{4}}}{\sqrt{\frac{y}{x}}}\right) = 0$$

Where F is an arbitrary function

2 Part-2-Integral Surfaces Passing through a given curve

2.1 Find the equation for the integral surface of the differential equation $2y(z - 3)p + (2x - z)q = y(2x - 3)$ which passes through the circle $z = 0, x^2 + y^2 = 2x$

Here the given differential equation is

$$2y(z - 3)p + (2x - z)q = y(2x - 3) \quad (1)$$

And the characteristic equation is

$$\frac{dx}{2y(z-3)} = \frac{dy}{(2x-z)} = \frac{dz}{y(2x-3)}$$

From first and third terms, we have

$$\begin{aligned} \frac{dx}{2y(z-3)} &= \frac{dz}{y(2x-3)} \\ (2x-3)dx &= 2(z-3)dz \end{aligned}$$

Integrating both sides, we get

$$\begin{aligned} \int (2x-3)dx &= \int 2(z-3)dz + C_1 \quad \text{where } C_1 \text{ is an arbitrary constant} \\ x^2 - 3x &= (z^2 - 6z) + C_1 \\ C_1 &= x^2 - z^2 - 3x + 6z \end{aligned} \tag{2}$$

From first and second terms we have

$$\begin{aligned} \frac{dx}{2y(z-3)} &= \frac{dy}{(2x-z)} \\ \frac{dy}{dx} &= \frac{2x-z}{2y(z-3)} \\ 2ydy &= \frac{2x-z}{(z-3)} dx \\ 2ydy &= \frac{2x-z-3+3}{z-3} dx \\ 2ydy &= \frac{(2x-3)-(z-3)}{z-3} dx \\ 2ydy &= \left(\frac{2x-3}{z-3} - 1\right) dx \end{aligned}$$

From first integral we have $(2x-3)dx = 2(z-3)dz$ hence,

$$2ydy = 2dz - dx$$

Integrating both sides we get

$$y^2 = 2z - x + C_2 \quad \text{where } C_2 \text{ is an arbitrary constant}$$

$$C_2 = x + y^2 - 2z \tag{3}$$

Hence the general integral surface of the PDE is

$$\boxed{x + y^2 - 2z = \phi(x^2 - z^2 - 3x + 6z)} \quad \text{for an arbitrary function } \phi \quad (4)$$

To find the particular solution that passes through the circle

$$z = 0, \quad x^2 + y^2 = 2x$$

We restrict to $z = 0$. On that curve

$$C_1 = x^2 - 3x$$

$$C_2 = x + y^2$$

$$\text{But } x^2 + y^2 = 2x \implies y^2 = 2x - x^2 \quad \text{so}$$

$$C_2 = x + (2x - x^2)$$

$$C_2 = 3x - x^2$$

$$C_2 = -(x^2 - 3x)$$

$$C_2 = -C_1$$

Thus along the initial circle $C_2 = -C_1$, so

$$\phi(s) = -s$$

Therefore the integral surface is

$$y^2 + x - 2z = -(x^2 - 3x - z^2 + 6z)$$

$$y^2 + x - 2z + x^2 - 3x - z^2 + 6z = 0$$

$$\boxed{x^2 + y^2 - z^2 - 2x + 4z = 0}$$

2.2 Find the general integral of the partial differential equation $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$

Here the given differential equation is

$$(2xy - 1)p + (z - 2x^2)q = 2(x - yz) \quad (1)$$

And the characteristic equation is

$$\frac{dx}{2xy - 1} = \frac{dy}{z - 2x^2} = \frac{dz}{2(x - yz)}$$

From first and last terms

$$\frac{dx}{2xy - 1} = \frac{dz}{2(x - yz)}$$

$$(2xy - 1)dz - 2(x - yz)dx = 0$$

Lets us find a function $F(x,y,z)$ such that

$$dF = Mdx + Ndz \quad \text{where } M = -2(x - yz), N = (2xy - 1)$$

Integratin M wrt x

$$F = \int -2(x - yz)dx + h(z) = -x^2 + 2yzx + h(z)$$

where $h(z)$ is an arbitrary function of z

Again

$$\frac{\partial N}{\partial z} = N$$

$$\frac{\partial}{\partial z}(-x^2 + 2yxz + h(z))$$

Hence

$$2yx + h'(z) = 2xy - 1$$

$$h'(z) = 1$$

Integrating both sides

$$h(z) = -z + C_1 \quad \text{where } C_1 \text{ is an arbitrary constant}$$

Therefore the solution of above DE becomes

$$-x^2 + 2yzx - z = C_1 \tag{2}$$

Using the multiplier z,1,x

$$\frac{zdx + dy + xdz}{2xyz - z + z - zx^2 + zx^2 - 2xyz} = 0$$

$$zdx + dy + xdz = 0$$

$$d(xz) + dy = 0$$

Integrating both sides we get

$$xz + y = C_2$$

There the required general solution is

$$\boxed{F(-x^2 + 2yzx - z, xz + y)} = 0$$

Where F is an arbitrary Function

2.3 Find the integral surface of the equation

$$(x - y)y^2 p + (y - x)x^2 q = (x^2 + y^2)z$$

through the curve $xz = a^3$, $y = 0$

Here the given differential equation is

$$(x - y)y^2 p + (y - x)x^2 q = (x^2 + y^2)z \quad (1)$$

And the characteristic equation is

$$\frac{dx}{(x - y)y} = \frac{dy}{(y - x)x^2} = \frac{dz}{(x^2 + y^2)z}$$

From first two terms we get

$$\frac{dy}{dx} = \frac{(y - x)x^2}{(x - y)y^2}$$

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

Separating variables and integrating we get

$$\int y^2 dy = - \int x^2 dx + C_1 \quad \text{Where } C_1 \text{ is an arbitrary constant}$$

$$y^3 + x^3 = 3C_1$$

$$x^3 + y^3 = C_2 \quad \text{where } C_2 = 3c_1 \quad (2)$$

From first and last terms of characteristic equation we get

$$\begin{aligned}\frac{dz}{z} &= \frac{x^2 + y^2}{y^2(x - y)} dx \\ \frac{dz}{z} &= \frac{x^2 + y^2}{y^2} \frac{1}{x - y} dx \\ \frac{dz}{z} &= \frac{1 + (\frac{y}{x})^2}{(\frac{y}{x})^2} \frac{1}{x - y} dx\end{aligned}$$

Lets use $v = \frac{y}{x}$ then

$$\frac{dz}{z} = \frac{1 + v^2}{xv^2(1 - v)} dx$$

We have $x^3 + y^3 = C_2$

differentiatin keeping in mind $y = vx$

$$x^2 dx + y^2 dy = 0$$

and $dy = vdx + xdv$

so

$$x^2 dx + v^2 x^2 (vdx + xdv) = 0$$

$$x^2 dx + v^3 x^2 dx + v^2 x^3 dv = 0$$

$$(1 + v^3)x^2 dx + v^2 x^3 dv = 0$$

$$dx = -\frac{v^2 x}{1 + v^3} dv$$

Therefore

$$\frac{dz}{z} = \frac{1 + v^2}{v^2(1 - v)} \left(-\frac{v^2}{1 + v^3} dv \right)$$

$$\frac{dz}{z} = \frac{1 + v^2}{(v - 1)(1 + v^3)} dv$$

$$\frac{dz}{z} = \left(\frac{1}{(v - 1)} - \frac{v^2}{(1 + v^3)} \right) dv$$

Integrating both sides

$$\int \frac{dz}{z} = \int \left(\frac{1}{v - 1} \right) dv - \int \left(\frac{v^2}{1 + v^3} \right) dv + C_3$$

For $\int \left(\frac{v^2}{1+v^3} \right) dv$ set $t = 1 + v^3$

so $dt = 3v^2 dv$ hence

$$\int \left(\frac{v^2}{1+v^3} \right) dv = \frac{1}{3} \int \frac{dt}{t} = \frac{1}{3} \ln(1+v^3)$$

Putting all together

$$\ln(z) = \ln(v-1) - \frac{1}{3} \ln(1+v^3) + C_3$$

$$\ln(z) - \ln(v-1) + \frac{1}{3} \ln(1+v^3) = C_3$$

$$\ln\left(z \frac{(1+v^3)^{1/3}}{v-1}\right) = C_3$$

Exponentiating both sides

$$C_4 = z \frac{(1+v^3)^{1/3}}{v-1} \quad \text{where } C_4 = e^{C_3}$$

Substituting back $v = \frac{y}{x}$

$$C_4 = z \frac{\left(1 + \left(\frac{y}{x}\right)^3\right)^{1/3}}{\frac{y}{x} - 1}$$

$$C_4 = \frac{z(x^3 + y^3)^{1/3}}{y - x}$$

Hence the general solution is given by

$$F(x^3 + y^3, \frac{z(x^3 + y^3)^{1/3}}{y - x}) = 0$$

where F is an arbitrary function.

Particular integral which passes through the curve $xz = a^3$, $y = 0$

Along the given curve, we have $y = 0$, $z = \frac{a^3}{x}$.

So $x^3 + y^3 = x^3$, $(x^3 + y^3)^{1/3} = x$, $y - x = -x$.

Substituting into the second invariant:

$$\frac{z(x^3 + y^3)^{1/3}}{y - x} = \frac{\frac{a^3}{x} \cdot x}{-x} = -\frac{a^3}{x}.$$

Thus, the second invariant becomes $\frac{z(x^3 + y^3)^{1/3}}{y - x} = -\frac{a^3}{(x^3 + y^3)^{1/3}}$.

Solving this for z gives the particular solution:

$$z(x, y) = -\frac{a^3(y - x)}{(x^3 + y^3)^{1/3}}$$

This surface passes through the curve $xz = a^3$, $y = 0$.

Therefore, the required particular solution is

$$F(x^3 + y^3, \frac{z(x^3 + y^3)^{1/3}}{y - x}) = \frac{z(x^3 + y^3)^{1/3}}{y - x} + \frac{a^3}{(x^3 + y^3)^{1/3}} = 0$$

2.4 Find the general solution of the equation
 $2x(y + z^2)p + y(2y + z^2)q = z^3$
and deduce that $yz(z^2 + yz - 2y) = x^2$

Here the given differential equation is

$$2x(y + z^2)p + y(2y + z^2)q = z^3 \quad (1)$$

And the characteristic equation is

$$\frac{dx}{2x(y + z^2)} = \frac{dy}{y(2y + z^2)} = \frac{dz}{z^3}$$

From second and third term of characteristic equation we have

$$\frac{dy}{dz} = \frac{y(2y + z^2)}{z^3},$$

$$\text{Substitute } v = \frac{1}{y}, \quad \Rightarrow \quad y = \frac{1}{v}, \quad dy = -\frac{1}{v^2} dv,$$

$$\text{Then } \frac{dy}{dz} = -\frac{1}{v^2} \frac{dv}{dz},$$

$$\text{and } \frac{y(2y + z^2)}{z^3} = \frac{\frac{1}{v}(2\frac{1}{v} + z^2)}{z^3} = \frac{2/v^2 + z^2/v}{z^3} = \frac{2}{v^2 z^3} + \frac{1}{v z}.$$

$$\text{Equateting we get } -\frac{1}{v^2} \frac{dv}{dz} = \frac{2}{v^2 z^3} + \frac{1}{v z},$$

$$\frac{dv}{dz} = -v^2 \left(\frac{2}{v^2 z^3} + \frac{1}{v z} \right) = -\frac{2}{z^3} - \frac{v}{z},$$

$$\frac{dv}{dz} + \frac{1}{z} v = -\frac{2}{z^3}.$$

Integrating factor is given by $\mu(z) = e^{\int \frac{1}{z} dz} = z$,

$$\text{Multiply both sides by IF } z \frac{dv}{dz} + v = -\frac{2}{z^2},$$

$$\frac{d}{dz}(z v) = -\frac{2}{z^2}.$$

$$\begin{aligned} \int \frac{d}{dz}(z v) dz &= \int -\frac{2}{z^2} dz & \Rightarrow z v = 2 z^{-1} + C_1, \\ v &= \frac{2}{z^2} + \frac{C_1}{z}. \end{aligned}$$

$$\text{Back-substitute } v = \frac{1}{y}$$

$$\frac{1}{y} = \frac{C_1}{z} + \frac{2}{z^2},$$

$$C_1 = \frac{z^2 - 2y}{yz} \tag{2}$$

From first and last term of characteristic equation we have

$$\frac{dx}{dz} = \frac{2x(y+z^2)}{z^3},$$

and from equation (2)

$$\frac{1}{y} = \frac{C_1}{z} + \frac{2}{z^2}$$

$$y = \frac{z^2}{C_1 z + 2}$$

Substitute $y = \frac{z^2}{C_1 z + 2}$

$$\begin{aligned}\frac{dx}{dz} &= \frac{2x\left(\frac{z^2}{C_1 z + 2} + z^2\right)}{z^3} \\ &= \frac{2x(z^2 + z^2(C_1 z + 2))}{(C_1 z + 2) z^3} \\ &= \frac{2x z^2(C_1 z + 3)}{(C_1 z + 2) z^3} \\ &= 2x \frac{C_1 z + 3}{(C_1 z + 2) z}.\end{aligned}$$

Separate variables

$$\frac{1}{x} \frac{dx}{dz} = 2 \frac{C_1 z + 3}{z(C_1 z + 2)},$$

$$\int \frac{1}{x} dx = 2 \int \frac{C_1 z + 3}{z(C_1 z + 2)} dz$$

Using Partial fractions

$$\frac{C_1 z + 3}{z(C_1 z + 2)} = \frac{A}{z} + \frac{B}{C_1 z + 2},$$

$$C_1 z + 3 = A(C_1 z + 2) + Bz = (AC_1 + B)z + 2A,$$

$$2A = 3, \quad AC_1 + B = C_1$$

$$A = \frac{3}{2}, \quad B = -\frac{C_1}{2}.$$

$$\begin{aligned}\int \frac{1}{x} dx &= 2 \int \left(\frac{3}{2z} - \frac{C_1/2}{C_1 z + 2} \right) dz = 3 \ln |z| - \ln |C_1 z + 2| + C'_2, \\ \ln \left(\frac{z^3}{C_1 z + 2} \right) &= \ln |x| + C'_2 \\ x &= C_2 \frac{z^3}{C_1 z + 2}\end{aligned}$$

But from first invariant we have

$$y = \frac{z^2}{C_1 z + 2}$$

Hence

$$C_2 = \frac{x}{yz}$$

Thus the second invariant is $C_2 = \frac{x}{yz}$

Hence the general solution is given by

$$F\left(\frac{z^2 - 2y}{yz}, \frac{x}{yz}\right) = 0$$

Deduction of $yz(z^2 + yz - 2y) = x^2$

Lets choose

$$C_1 = C_2^2 - 1$$

$$\text{i.e } \frac{z^2 - 2y}{yz} = \left(\frac{x}{yz}\right)^2 - 1$$

$$\frac{z^2 - 2y}{yz} = \frac{x^2 - y^2 z^2}{y^2 z^2}$$

$$(z^2 - 2y)(yz) = x^2 - y^2 z^2$$

$$(z^2 - 2y)(yz) + y^2 z^2 = x^2$$

$$yz(z^2 + yz - 2y) = x^2$$

Thus this special choice of the arbitrary function is $\phi(t) = \sqrt{t+1}$

2.5 Find the integral of the equation

$$(x - y)p + (y - x - z)q = z$$

and the particular solution through the circle $z = 1, x^2 + y^2 = 1$.

Here the given differential equation is

$$(x - y)p + (y - x - z)q = z \quad (1)$$

And the characteristic equation is

$$\frac{dx}{(x - y)} = \frac{dy}{(y - x - z)} = \frac{dz}{z}$$

Let us introduce new parameter t such that

$$\begin{aligned} \frac{dx}{x - y} &= dt, & \frac{dy}{y - x - z} &= dt, & \frac{dz}{z} &= dt, \\ \frac{d}{dt}(x + y + z) &= \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} \\ &= (x - y) + (y - x - z) + z \\ &= 0, \\ x + y + z &= C_1. \end{aligned}$$

Again, let $u = x - y$, then

$$\begin{aligned} \frac{du}{dt} &= \frac{dx}{dt} - \frac{dy}{dt} \\ \frac{du}{dt} &= (x - y) - (y - x - z) = 2(x - y + z) = 2u + z \\ z &= \frac{du}{dt} - 2u \end{aligned}$$

Also from $\frac{dz}{dt} = z$

$$z = Ae^t \quad \text{for some } A$$

using this value of z in above equation

$$\frac{du}{dt} - 2u = Ae^t$$

This is linear first order ODE, so the integrating factor is $\mu(t) = e^{\int -2 dt} = e^{-2t}$

Multiply the ODE by $\mu(t)$

$$e^{-2t} \frac{du}{dt} - 2e^{-2t} u = Ae^t e^{-2t} \implies e^{-2t} \frac{du}{dt} - 2e^{-2t} u = Ae^{-t}.$$

$$\frac{d}{dt}(e^{-2t} u) = e^{-2t} \frac{du}{dt} - 2e^{-2t} u.$$

Integrate both sides

$$\int \frac{d}{dt}(e^{-2t} u) dt + C_2 = \int Ae^{-t} dt + C_2 \implies e^{-2t} u = -Ae^{-t} + C_2,$$

$$u = e^{2t}(-Ae^{-t} + C_2) = -Ae^t + C_2 e^{2t}$$

But $u = x - y$, so:

$$x - y = -Ae^t + C_2 e^{2t},$$

$$\text{Recall } z = Ae^t \Rightarrow e^t = \frac{z}{A}, \quad e^{2t} = \frac{z^2}{A^2},$$

Substitute back:

$$x - y = -z + C_2 \cdot \frac{z^2}{A^2},$$

$$x - y + z = C_2 \cdot \frac{z^2}{A^2},$$

$$\Rightarrow \boxed{C_3 = \frac{(x - y + z)}{z^2}} \quad \text{where } C_3 = C_2/A^2$$

Hence the general solution is given by

$$\boxed{F\left(x + y + z, \frac{(x - y + z)}{z^2}\right) = 0}$$

Perticular Integral through $z = 1, x^2 + y^2 = 1$

On the circle $z = 1, x^2 + y^2 = 1$, we have:

$$C_1 = x + y + 1, \quad C_2 = \frac{x - y + 1}{1^2} = x - y + 1.$$

Now, for any (x, y) on the unit circle, we know:

$$(x + y)^2 + (x - y)^2 = 2(x^2 + y^2) = 2.$$

So, substituting in terms of C_1 and C_2 :

$$(C_1 - 1)^2 + (C_2 - 1)^2 = 2.$$

Hence, the unique function $F(C_1, C_2)$ vanishing on the circle is:

$$F(C_1, C_2) = (C_1 - 1)^2 + (C_2 - 1)^2 - 2 = 0.$$

Substituting back $C_1 = x + y + z, C_2 = \frac{x - y + z}{z^2}$, we obtain:

$$(x + y + z - 1)^2 + \left(\frac{x - y + z}{z^2} - 1 \right)^2 =$$

is the particular integral surface through $z = 1, x^2 + y^2 = 1$

2.6 Find the general solution of the differential equation

$$x(z + 2a)p + (xz + 2yz + 2ay)q = z(z + a)$$

Find also the integral surfaces which pass through the curves:

(a) $y = 0 \quad z^2 = 4ax$

(b) $y = 0 \quad z^3 + x(z + a)^2 = 0$

Here the given differential equation is

$$x(z + 2a)p + (xz + 2yz + 2ay)q = z(z + a) \text{ eq : 261} \quad (1)$$

And the characteristic equation is

$$\frac{dx}{x(z + 2a)} = \frac{dy}{(xz + 2yz + 2ay)} = \frac{dz}{z(z + a)}$$

From first and last terms we get

$$\frac{dx}{x} = \frac{z+2a}{z(z+a)} dz = \left(\frac{2}{z} - \frac{1}{z+a} \right) dz$$

Integrating both sides we get

$$\ln(x) = 2\ln(z) - \ln(z+a) + C_1$$

$$C_1 = \frac{x(z+a)}{z^2}$$

Next use second and third ratios we have

$$\frac{dy}{(xz + 2yz + 2ay)} = \frac{dz}{z(z+a)}$$

Substitute the value of x from the first integral: $x = \frac{C_1 z^2}{z+a}$

So, the denominator becomes: $xz + 2yz + 2ay = \frac{C_1 z^3}{z+a} + 2yz + 2ay$

$$\begin{aligned} &\Rightarrow \frac{dy}{\frac{C_1 z^3}{z+a} + 2yz + 2ay} = \frac{dz}{z(z+a)} \\ &\Rightarrow \frac{dy}{dz} = \frac{C_1 z^2}{(z+a)^2} + \frac{2y(z+a)}{z(z+a)} = \frac{C_1 z^2}{(z+a)^2} + \frac{2y}{z} \\ &\Rightarrow \frac{dy}{dz} - \frac{2}{z}y = \frac{C_1 z^2}{(z+a)^2} \end{aligned}$$

This is a linear first-order ODE with integrating factor: $\mu(z) = e^{\int -\frac{2}{z} dz} = z^{-2}$

Multiply the ODE by z^{-2} :

$$\begin{aligned} z^{-2} \frac{dy}{dz} - \frac{2}{z^3}y &= \frac{C_1}{(z+a)^2} \\ \Rightarrow \frac{d}{dz}(z^{-2}y) &= \frac{C_1}{(z+a)^2} \end{aligned}$$

Integrating both sides: $\int \frac{d}{dz}(z^{-2}y) dz = \int \frac{C_1}{(z+a)^2} dz$

$$\begin{aligned} z^{-2}y &= -\frac{C_1}{z+a} + C_2 \\ \Rightarrow y &= -\frac{C_1 z^2}{z+a} + C_2 z^2 \end{aligned}$$

Substituting back $C_1 = \frac{x(z+a)}{z^2}$

$$\begin{aligned}y &= -\frac{z^2}{z+a} \cdot \frac{x((z+a))}{z^2} + C_2 z^2 \\y &= -x + C_2 z^2 \\C_2 &= \frac{y+x}{z^2}\end{aligned}$$

Hence the General Solution is

$$F\left(\frac{x(z+a)}{z^2}, \frac{y+x}{z^2}\right)$$

Where F is an arbitrary function

Particular integral surface (a):

Initial curve (a): $y = 0, z^2 = 4ax$. So, $x = \frac{z^2}{4a}$.

Hence

$$\begin{aligned}C_1 &= \frac{x(z+a)}{z^2} = \frac{(z^2/(4a))(z+a)}{z^2} = \frac{z+a}{4a}, \\C_2 &= \frac{x+y}{z^2} = \frac{z^2/(4a)}{z^2} = \frac{1}{4a}\end{aligned}$$

Since $C_2 = \frac{1}{4a}$ is constant along the curve, the integral surface satisfies:

$$\begin{aligned}\frac{x+y}{z^2} &= \frac{1}{4a} \\ \Rightarrow x+y &= \frac{z^2}{4a}\end{aligned}$$

Therefore, the particular integral surface is:

$$z^2 = 4a(x+y)$$

Particular integral surface (b):

Initial curve (b): $y = 0, z^3 + x(z+a)^2 = 0 \Rightarrow x = -\frac{z^3}{(z+a)^2}$

Compute the invariants:

$$\begin{aligned}C_1 &= \frac{x(z+a)}{z^2} = -\frac{z^3}{(z+a)^2} \cdot \frac{z+a}{z^2} = -\frac{z}{z+a}, \\C_2 &= \frac{x+y}{z^2} = \frac{-z^3/(z+a)^2}{z^2} = -\frac{z}{(z+a)^2}\end{aligned}$$

From $C_1 = -\frac{z}{z+a}$, solve for z :

$$\begin{aligned} C_1(z+a) &= -z \\ \Rightarrow z(C_1 + 1) &= -C_1a \\ \Rightarrow z &= -\frac{C_1a}{C_1 + 1} \end{aligned}$$

Then:

$$\begin{aligned} z+a &= \frac{a}{C_1+1} \\ \Rightarrow C_2 &= -\frac{z}{(z+a)^2} = \frac{C_1a}{C_1+1} \cdot \frac{1}{\left(\frac{a}{C_1+1}\right)^2} = \frac{C_1(C_1+1)}{a} \end{aligned}$$

So:

$$C_2 = \frac{C_1(C_1+1)}{a} \Rightarrow C_1^2 + C_1 - aC_2 = 0$$

Substitute back $C_1 = \frac{x(z+a)}{z^2}$, $C_2 = \frac{x+y}{z^2}$:

$$\left(\frac{x(z+a)}{z^2}\right)^2 + \frac{x(z+a)}{z^2} - a \cdot \frac{x+y}{z^2} = 0$$

Multiply both sides by z^2 :

$$\frac{x^2(z+a)^2}{z^2} + x(z+a) - a(x+y) = 0$$

Multiply through by z^2 to clear denominators:

$$x^2(z+a)^2 + x(z+a)z^2 - a(x+y)z^2 = 0$$

Thus, the particular integral surface through curve (b) is:

$$x^2(z+a)^2 + x(z+a)z^2 - a(x+y)z^2 = 0$$

3 Part-3 Surfaces Orthogonal to a Given System of Surfaces

3.1 Find the surface which is orthogonal to the one-parameter system $z = cxy(x^2 + y^2)$ and which passes through the hyperbola $x^2 - y^2 = a^2, z = 0$

Here the given family of is $z = cxy(x^2 + y^2)$

Rewrite this as

$$F(x, y, z) = \frac{z}{xy(x^2 + y^2)} = c \quad (1)$$

For a surface $\phi(x, y, z) = k$ where k is constant to be orthogonal the given family, their normal vectors must be perpendicular at every point of intersection. For this

$$\nabla F \cdot \nabla \phi = 0$$

$$\frac{\partial F}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial \phi}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial \phi}{\partial z} = 0$$

Let $u = xy(x^2 + y^2)$, then,

$$\frac{\partial F}{\partial x} = \frac{\partial(\frac{z}{u})}{\partial x} = -u^{-2} \frac{\partial u}{\partial x} = -z \frac{3x^2y + y^3}{[xy(x^2 + y^2)]^2}$$

$$\frac{\partial F}{\partial y} = \frac{\partial(\frac{z}{u})}{\partial y} = -u^{-2} \frac{\partial u}{\partial y} = -z \frac{x^3y + 3xy^2}{[xy(x^2 + y^2)]^2}$$

$$\frac{\partial F}{\partial z} = \frac{\partial(\frac{1}{u})}{\partial z} = \frac{1}{[xy(x^2 + y^2)]^2}$$

Hence,

$$\begin{aligned} \nabla F \cdot \nabla \phi &= \left[-z \frac{3x^2y + y^3}{(xy(x^2 + y^2))^2} \right] \frac{\partial x}{\partial \phi} + \left[-z \frac{x^3 + 3xy^2}{(xy(x^2 + y^2))^2} \right] \frac{\partial y}{\partial \phi} + \left[\frac{1}{xy(x^2 + y^2)} \right] \frac{\partial z}{\partial \phi} = 0 \\ &- 2zy(3x^2 + y^2) \frac{\partial x}{\partial \phi} - 2zx(x^2 + 3y^2) \frac{\partial y}{\partial \phi} + 2xy(x^2 + y^2) \frac{\partial z}{\partial \phi} = 0 \end{aligned}$$

The characteristic equation is given by

$$\frac{dx}{-zy(3x^2 + y^2)} = \frac{dy}{-zx(x^2 + 3y^2)} = \frac{dz}{xy(x^2 + y^2)}$$

From the ratio $\frac{dy}{dx} = \frac{x(x^2 + 3y^2)}{y(3x^2 + y^2)}$,

Set $u = \frac{y}{x}$, $y = ux$, $dy = u dx + x du$,

Substitute:

$$u + x \frac{du}{dx} = \frac{x(x^2 + 3(u x)^2)}{u x(3x^2 + (u x)^2)} = \frac{1 + 3u^2}{u(3 + u^2)},$$

Hence

$$x \frac{du}{dx} = \frac{1 + 3u^2}{u(3 + u^2)} - u = \frac{2u(1 - u^2)}{1 + 3u^2},$$

Separate variables:

$$\int \frac{1 + 3u^2}{u(1 - u^2)} du = \int \frac{2 dx}{x},$$

Use partial fractions:

$$\frac{1 + 3u^2}{u(1 - u^2)} = \frac{1}{u} + \frac{2}{1 - u} - \frac{2}{1 + u},$$

Integrate both sides:

$$\begin{aligned} \int \left(\frac{1}{u} + \frac{2}{1 - u} - \frac{2}{1 + u} \right) du &= 2 \int \frac{dx}{x} \\ \ln|u| - 2 \ln|1 - u| - 2 \ln|1 + u| &= 2 \ln|x| + \ln K, \end{aligned}$$

$$\ln\left(\frac{u}{(1 - u^2)^2}\right) = \ln(K x^2) \implies \frac{u}{(1 - u^2)^2} = K x^2,$$

Back-substitute $u = \frac{y}{x}$:

$$\begin{aligned} \frac{\frac{y}{x}}{(1 - (y/x)^2)^2} &= K x^2 \implies \frac{y/x}{\left(\frac{x^2 - y^2}{x^2}\right)^2} = K x^2 \\ \implies \frac{y/x}{(x^2 - y^2)^2/x^4} &= K x^2 \implies \frac{x^3 y}{(x^2 - y^2)^2} = K x^2 \implies \frac{x y}{(x^2 - y^2)^2} = K, \end{aligned}$$

so the first integral is

$I_1 = \frac{x y}{(x^2 - y^2)^2} = \text{constant.}$

Second integral via a parameter t :

Introduce t so that each ratio equals dt :

$$\frac{dx}{-z y (3x^2 + y^2)} = \frac{dy}{-z x (x^2 + 3y^2)} = \frac{dz}{x y (x^2 + y^2)} = dt.$$

Define $r^2 = x^2 + y^2$, $dr^2 = 2x dx + 2y dy$

From the characteristic equations,

$$dx = -z y (3x^2 + y^2) dt, \quad dy = -z x (x^2 + 3y^2) dt,$$

$$\begin{aligned} \text{so } dr^2 &= 2x(-z y (3x^2 + y^2)) dt + 2y(-z x (x^2 + 3y^2)) dt \\ &= -2z[x y (3x^2 + y^2) + x y (x^2 + 3y^2)] dt = -2z \cdot 4xy (x^2 + y^2) dt \\ &= -8z(xy(x^2 + y^2)) dt \end{aligned}$$

From $dz = x y (x^2 + y^2) dt$, we have $xy (x^2 + y^2) dt = dz$

Hence

$$dr^2 = -8z dz$$

Integration both sides

$$\begin{aligned} \int dr^2 &= -8 \int z dz + C \\ r^2 &= -4z^2 + C, \end{aligned}$$

so the second first integral is $I_2 = x^2 + y^2 + 4z^2 = \text{constant.}$

Hence the general solution is $F(\frac{xy}{(x^2-y^2)^2}, x^2 + y^2 + 4z^2)$

Where F is an arbitrary function

Condition: Passes through the hyperbola

The surface must pass through the hyperbola:

$$x^2 - y^2 = a^2, \quad z = 0$$

On this curve:

$$(x^2 - y^2)^2 = a^4, \quad \Rightarrow \quad I_1 = \frac{xy}{a^4}, \quad I_2 = x^2 + y^2$$

Parametrize the hyperbola

Use the parametrization:

$$x = a \cosh t, \quad y = a \sinh t$$

Then:

$$xy = a^2 \cosh t \sinh t = \frac{a^2}{2} \sinh 2t,$$

$$x^2 + y^2 = a^2(\cosh^2 t + \sinh^2 t) = a^2 \cosh 2t$$

Thus, the integrals become:

$$I_1 = \frac{xy}{a^4} = \frac{\sinh 2t}{2a^2}, \quad I_2 = a^2 \cosh 2t$$

Since

$$\cosh^2(2t) - \sinh^2(2t) = 1$$

So,

$$\left(\frac{I_2}{a^2}\right)^2 - (2a^2 I_1)^2 = 1$$

$$\frac{I_2^2}{a^4} - 4a^4 I_1^2 = 1 \Rightarrow I_2^2 - 4a^8 I_1^2 = a^4$$

Back-substitute I_1, I_2 :

$$(x^2 + y^2 + 4z^2)^2 - 4a^8 \left(\frac{xy}{(x^2 - y^2)^2}\right)^2 = a^4$$

3.2 Find the equation of the system of surfaces which cut orthogonally the cones of the system $x^2 + y^2 + z^2 = cxy$

Here the given system of cones is $x^2 + y^2 + z^2 = cxy$

$$F(x, y, z, c) = x^2 + y^2 + z^2 - cxy = 0 \quad (1)$$

For a surface $\phi(x, y, z) = k$ where k is constant to be orthogonal the given family, their normal vectors must be perpendicular at every point of intersection. For this

$$\begin{aligned} \nabla F \cdot \nabla \phi &= 0 \\ \frac{\partial F}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial \phi}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial \phi}{\partial z} &= 0 \end{aligned}$$

Here

$$\frac{\partial F}{\partial x} = 2x - cy$$

$$\frac{\partial F}{\partial y} = 2y - cx$$

$$\frac{\partial F}{\partial z} = 2z$$

Substituting $c = \frac{x^2 + y^2 + z^2}{xy}$

$$\frac{\partial F}{\partial x} = 2x - y \frac{x^2 + y^2 + z^2}{xy}$$

$$\frac{\partial F}{\partial y} = 2y - x \frac{x^2 + y^2 + z^2}{xy}$$

$$\frac{\partial F}{\partial z} = 2z$$

Hence

$$2x - y \frac{x^2 + y^2 + z^2}{xy} \frac{\partial \phi}{\partial x} + 2y - x \frac{x^2 + y^2 + z^2}{xy} \frac{\partial \phi}{\partial y} + 2z \frac{\partial \phi}{\partial z} = 0$$

$$\frac{x^2 - y^2 - z^2}{x} \frac{\partial \phi}{\partial x} + \frac{y^2 - z^2 - x^2}{y} \frac{\partial \phi}{\partial y} + 2z \frac{\partial \phi}{\partial z} = 0$$

Therefore the characteristic equation is

$$\frac{dx}{\frac{x^2 - y^2 - z^2}{x}} = \frac{dy}{\frac{y^2 - z^2 - x^2}{y}} = \frac{dz}{2z}$$

Let us introduce new parameter t such that

$$\frac{dx}{\frac{x^2 - y^2 - z^2}{x}} = \frac{dy}{\frac{y^2 - z^2 - x^2}{y}} = \frac{dz}{2z} = dt$$

Then

$$\frac{d}{dt}(x^2 + y^2 + z^2) = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt}$$

Substituting the value of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ we get

$$\frac{d}{dt}(x^2 + y^2 + z^2) = 2x \frac{x^2 - y^2 - z^2}{x} + 2y \frac{y^2 - z^2 - x^2}{y} + 2z(2z)$$

$$\frac{d}{dt}(x^2 + y^2 + z^2) = 2x^2 - 2y^2 - 2z^2 + 2y^2 - z^2 - 2x^2 + 4z^2$$

$$\frac{d}{dt}(x^2 + y^2 + z^2) = 0$$

Integrating both sides we get

$$x^2 + y^2 + z^2 = C_1$$

Again

$$\frac{d}{ds}(x^2 - y^2) = 4(x^2 - y^2), \quad \frac{d}{ds}(z^2) = 4z^2$$

Now

$$\frac{d}{ds}\left(\frac{x^2 - y^2}{z^2}\right) = \frac{z^2 \cdot 4(x^2 - y^2) - (x^2 - y^2) \cdot 4z^2}{z^4} = 0$$

Thus, on integrating both sides we get

$$\frac{x^2 - y^2}{z^2} = C_2$$

Therefore the general solution is

$F(x^2 + y^2 + z^2, \frac{x^2 - y^2}{z^2}) = 0$

Where F is an arbitrary function

3.3 Find the general equation of surfaces orthogonal to the family given by

3.3.1 $x(x^2 + y^2 + z^2) = c_1 y^2$

Here the given family is

$$F(x, y, z, c) = x(x^2 + y^2 + z^2) - c_1 y^2 = 0 \quad (1)$$

For a surface $\phi(x, y, z, c) = k$ where k is constant to be orthogonal the given family, their normal vectors must be perpendicular at every point of intersection. For this

$$\nabla F \cdot \nabla \phi = 0$$

$$\frac{\partial F}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial \phi}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial \phi}{\partial z} = 0$$

Here

$$\frac{\partial F}{\partial x} = 3x^2 + y^2 + z^2$$

$$\frac{\partial F}{\partial y} = 2y(x - c_1)$$

Replacing the value of c_1 from the given equation

$$\frac{\partial F}{\partial y} = -2x \frac{x^2 + z^2}{y}$$

$$\frac{\partial F}{\partial z} = 2xz$$

Therefore

$$(3x^2 + y^2 + z^2) \frac{\partial \phi}{\partial x} - \left(2x \frac{(x^2 + z^2)}{y}\right) \frac{\partial \phi}{\partial y} + (2xz) \frac{\partial \phi}{\partial z}$$

Hence the characteristic equation is

$$\frac{dx}{3x^2 + y^2 + z^2} = \frac{dy}{-2x \frac{(x^2 + z^2)}{y}} = \frac{dz}{2xz}$$

Let us introduce new variable s such that

$$\frac{dx}{3x^2 + y^2 + z^2} = \frac{dy}{-2x \frac{(x^2 + z^2)}{y}} = \frac{dz}{2xz} = ds$$

Now

$$\begin{aligned} \frac{d}{ds}(x^2 + y^2) &= 2x \frac{dx}{ds} + 2y \frac{dy}{ds} \\ \frac{d}{ds}(x^2 + y^2) &= 2x \frac{3x^2 + y^2 + z^2}{2xz} dz + 2y \left(-\frac{x^2 + z^2}{z}\right) dz \\ \frac{d}{ds}(x^2 + y^2) &= \frac{3x^2 + y^2 + z^2}{z} dz - \frac{2y(x^2 + z^2)}{z} dz \\ \frac{d}{ds}(x^2 + y^2) &= \frac{x^2 + y^2 + z^2}{z} dz \end{aligned}$$

Again

$$\begin{aligned} d\left(\frac{x^2 + y^2}{z}\right) &= \frac{z \cdot d(x^2 + y^2) - (x^2 + y^2) dz}{z^2} = \frac{z \cdot (x^2 + y^2 - z^2) - (x^2 + y^2) dz}{z^2} \\ d\left(\frac{x^2 + y^2}{z}\right) &= -dz \end{aligned}$$

Hence

$$d\left(\frac{x^2 + y^2}{z}\right) + dz = 0$$

Integrating both sides

$$\frac{x^2 + y^2}{z} + z = C_1$$

$$\frac{x^2 + y^2 + z^2}{z} = C_1$$

From Second and last terms of characteristic equation

$$y dy = -\frac{x^2 + z^2}{z} dz$$

But we have $C_1 z = x^2 + y^2 + z^2$

$$x^2 + z^2 = C_1 z - y^2$$

$$y dy = -\frac{x^2 + z^2}{z} dz$$

$$y dy = -\frac{C_1 z - y^2}{z} dz$$

$$y dy = -C_1 dz + \frac{y^2}{z} dz$$

$$\frac{d(y^2)}{dz} - \frac{2y^2}{z} = -2C_1$$

Multiplying both sides by z^{-2}

$$z^{-2} \cdot \left(\frac{d(y^2)}{dz} - \frac{2y^2}{z} \right) = -2C_1 z^{-2}$$

$$\frac{d}{dz} \left(\frac{y^2}{z^2} \right) = -2C_1 z^{-2}$$

Integrating both sides

$$\frac{y^2}{z^2} = 2C_1 z^{-1} + C_2$$

$$y^2 = 2C_1 z + C_2 z^2$$

$$x^2 + z^2 = C_1 z - y^2$$

$$= C_1 z - (2C_1 z + C_2 z^2) = -C_1 z - C_2 z^2$$

Substituting the value of C_1

$$2x^2 + y^2 + 2z^2 = C_2 z^2$$

$$C_2 = \frac{2x^2 + y^2 + 2z^2}{z^2}$$

Hence the General Solution is

$$\boxed{F\left(\frac{x^2 + y^2 + z^2}{z}, \frac{2x^2 + y^2 + 2z^2}{z^2}\right)} \quad \text{Where } F \text{ is an arbitrary function}$$

3.3.2 $x^2 + y^2 + z^2 = c_2 z$

If a family exists, orthogonal to both (3.3.1) and (3.3.2), show that it must satisfy $2x(x^2 - z^2)dx + y(3x^2 + y^2 - z^2)dy + 2z(2x^2 + y^2)dz = 0$. Show that such a family in fact exists, and find its equation

Here the given family is

$$F(x, y, z) = \frac{x^2 + y^2 + z^2}{z} = c_2 \quad (1)$$

For a surface $\phi(x, y, z, c) = k$ where k is constant to be orthogonal the given family, their normal vectors must be perpendicular at every point of intersection. For this

$$\begin{aligned} \nabla F \cdot \nabla \phi &= 0 \\ \frac{\partial F}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial \phi}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial \phi}{\partial z} &= 0 \end{aligned}$$

Here

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{2x}{z} \\ \frac{\partial F}{\partial y} &= \frac{2y}{z} \\ \frac{\partial F}{\partial z} &= \frac{z^2 - x^2 - y^2}{z^2} \end{aligned}$$

Therefore

$$\frac{2x}{z} \frac{\partial \phi}{\partial x} + \frac{2y}{z} \frac{\partial \phi}{\partial y} + \left(\frac{z^2 - x^2 - y^2}{z^2}\right) \frac{\partial \phi}{\partial z} = 0$$

Hence the characteristic equation is

$$\frac{dx}{\frac{2x}{z}} = \frac{dy}{\frac{2y}{z}} = \frac{dz}{\frac{z^2 - x^2 - y^2}{z^2}}$$

From first two terms we get

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating both sides we get

$$\ln(y) = \ln(x) + \ln(C_1)$$

Exponentiating both sides and rearranging, we get

$$\frac{y}{x} = C_1$$

Using first and last terms for characteristic equation we have

$$\frac{dx}{\frac{2x}{z}} = \frac{dz}{\frac{z^2 - x^2 - y^2}{z^2}}$$

$$\frac{dx}{2x} = \frac{z}{z^2 - x^2 - y^2} dz$$