Pfaffian Differential Equations Assignment - Page 26

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1 Part 1 Page 26

Determine which of the following equations are integrable, and find the solution of those which one

1.1 $y\,dx + x\,dy + 2z\,dz = 0$

We are given the differential equations

$$y\,dx + x\,dy + 2z\,dz = 0 \tag{1}$$

Let's define the vector field $\vec{X} = (P\,, Q\,, R\,)$, where $P = y, Q = x$, $R = 2z$ so $\vec{X} = (y, x, 2z)$ To check the integrability, lets calculate the $\vec{X} \cdot (\nabla \times \vec{X})$

$$\begin{aligned}
\vec{X} \cdot (\nabla \times \vec{X}) &= \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
\vec{X} \cdot (\nabla \times \vec{X}) &= \begin{vmatrix} y & x & 2z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 2z \end{vmatrix} \\
\vec{X} \cdot (\nabla \times \vec{X}) &= y(0-0) \,-\, x(0-0) +\, 2z(1-1) \\
\vec{X} \cdot (\nabla \times \vec{X}) &= 0
\end{aligned} \tag{2}$$

Since $\vec{X} \cdot \operatorname{curl} \vec{X} = 0$, the given Pfaffian Differential Equation (1) is integrable.

So lets find a function $F\,(x,\,y,\,z)$**, such that** $\quad dF = y\,dx + x\,dy + 2z\,dz$ i.e. $\frac{\partial F}{\partial x} = y$, $\frac{\partial F}{\partial y} = x$ and $\frac{\partial F}{\partial z} = 2z$ Treat y and z as constants and integrate $\frac{\partial F}{\partial x} = y$ wrt x we get,

$$F(x,y,z) = x\,y \; + \; h(y,z) \tag{3}$$

, here $h(y,z)$ is an unknown function of y and z only.

Differentiate equation (3) wrt y, we get $\frac{\partial F}{\partial y} = x + \frac{\partial h}{\partial y}$
Since, we have $\frac{\partial F}{\partial y} = x$, so equation both

$$x + \frac{\partial h}{\partial y} = x \qquad\qquad => \frac{\partial h}{\partial y} = 0 \tag{4}$$

Integrating above equation wrt y we get $h(y,z) = g(z)$, where g(z) is unknown function of z
Now equation (3) becomes
$F(x,y,z) = xy + g(z)$
Differentiation wrt z we get
$\frac{\partial F}{\partial z} = \frac{dg}{dz}$
Again we also have $\frac{\partial F}{\partial z} = 2z$
Hence $\frac{dg}{dz} = 2z$
Integration both sides wrt z, we get
$g(z) = \int 2zdz + C$ where C is an arbitrary constant
$g(z) = z^2 + C$
Using this value of g(z), we get
$F(x,y,z) = xy + z^2 + C$
This is the required general solution fo given PfDE

1.2 $z(z+y)dx + z(z+x)dy - 2xydz = 0$

Given Pfaffian Differential equation is

$$z(z+y)dx + z(z+x)dy - 2xydz = 0 \tag{1}$$

Lets define a vector $\vec{X} = (P,Q,R)$, where
$P = z(z+y)$
$Q = z(z+x)$
$R = -2xy$
Hence $\vec{X} = (z(z+y), z(z+x), -2xy)$

Now lets compute $\vec{X} \cdot (\nabla \times \vec{X})$

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} z(z+y) & z(z+x) & -2xy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z(z+y) & z(z+x) & -2xy \end{vmatrix}$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = z(-3x(z+y) - 2z(z+y)) + 3yz(z+x) + 2z^2(z+x)$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = -3xz(z+y) - 2z^2(z+y) + 3yz(z+x) + 2z^2(z+x)$$

Here the $\vec{X} \cdot (\nabla \times \vec{X})$ is $\neq 0$, so the given PfDE is not integrable.

1.3 $yzdx + 2xzdy - 3xydz = 0$

Given Pfaffian Differential equation is

$$yzdx + 2xzdy - 3xydz = 0 \tag{1}$$

Lets define a vector $\vec{X} = (P, Q, R)$, where
$P = yz$
$Q = 2xz$
$R = -3xy$
Hence $\vec{X} = (yz, 2xz, -3xy)$
Now lets compute $\vec{X} \cdot (\nabla \times \vec{X})$

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} yz & 2xz & -3xy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2xz & -3xy \end{vmatrix}$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = yz(-5x) + (2xz)(4y) - 3xyz$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = -5xyz + 8xyz - 3xyz$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = 0$$

Since $\vec{X} \cdot (\nabla \times \vec{X}) = 0$, hence given PfDE (1) is integrable.
Now lets check if its exact or not

We have $P = yz, Q = 2xz, R = -3xy$
Now

$$\frac{\partial P}{\partial y} = z$$
$$\frac{\partial Q}{\partial x} = 2z$$

Here $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, hence the given PfDE is not exact.
Let $\mu(xyz)$ be the integrating factor, and since all terms involves the product of xyz, so lets try
$\mu(x, y, z) = \frac{1}{xyz}$
Multiplying (1) by μ on both sides we get

$$\frac{1}{xyz}(yzdx + 2xzdy - 3xydz) = 0$$
$$\frac{1}{x}dx + \frac{2}{y}dy - \frac{3}{z}dz = 0$$

Integrating both sides we get

$$\int \frac{1}{x}dx + \int \frac{2}{y}dy - \int \frac{3}{z}dz = 0$$
$$\ln(x) + 2\ln(y) - 3\ln(z) = C$$

, where C is an arbitrary constant.

$\ln(\frac{xy^2}{z^3}) = C$
Exponentiating on both sides we get
$\frac{xy^2}{z^3} = e^C$
$\frac{xy^2}{z^3} = C_1$, where $C_1 = e^C$
is the required General Solution

1.4 $2xzdx + zdy - dz = 0$

Given Pfaffian Differential equation is

$$2xzdx + zdy - dz \tag{1}$$

Lets define a vector $\vec{X} = (P, Q, R)$, where
$P = 2xz$

$Q = z$
$R = -1$
Hence $\vec{X} = (2xz, z, -1)$
Now lets compute $\vec{X} \cdot (\nabla \times \vec{X})$

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = \begin{vmatrix} 2xz & z & -1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz & z & -1 \end{vmatrix}$$

$$\vec{X} \cdot (\nabla \times \vec{X}) = 2xz(-1) + z(2x) + 0$$
$$\vec{X} \cdot (\nabla \times \vec{X}) = -2xz + 2xz$$
$$\vec{X} \cdot (\nabla \times \vec{X}) = 0$$

Here the $\vec{X} \cdot (\nabla \times \vec{X}) = 0$, so the given PfDE is integrable.
Now check if the equation is exact or not.

$$\frac{\partial P}{\partial y} = 0, \frac{\partial Q}{\partial x} = 0$$
$$\frac{\partial Q}{\partial z} = 1, \frac{\partial R}{\partial y} = 0$$
$$\implies \frac{\partial Q}{\partial z} \neq \frac{\partial R}{\partial y}$$

Since the given PfDE is not exact, so lets say μ is the integrating factor.
Since all terms of the (1) contains z so lets try $\mu(x) = \frac{1}{z}$
Multiplying (1) by μ we get

$$\frac{1}{z}(2xzdx + zdy - dz) = 0$$
$$2xdx + dy - \frac{1}{z}dz = 0$$

Integrating both sides
$x^2 + y - \ln(z) = C$, where C is arbitrary constant
This is the required general solution of the given PfDE.

1.5 $(y^2+xz)dx+(x^2+y^2)dy+3z^2dz=0$

Given Pfaffian Differential equation is

$$(y^2+xz)dx+(x^2+y^2)dy+3z^2dz=0 \tag{1}$$

Lets define a vector $\vec{X}=(P,Q,R)$, where
$P=y^2+xz$
$Q=x^2+y^2$
$R=3z^2$
Hence $\vec{X}=((y^2+xz),(x^2+y^2),3z^2)$
Now lets compute $\vec{X}\cdot(\nabla\times\vec{X})$

$$\vec{X}\cdot(\nabla\times\vec{X})=\begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\vec{X}\cdot(\nabla\times\vec{X})=\begin{vmatrix} (y^2+xz) & (x^2+yz) & 3z^2 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2+xz) & (x^2+yz) & 3z^2 \end{vmatrix}$$

$$\vec{X}\cdot(\nabla\times\vec{X})=(y^2+xz)(-y)+(x^2+yz)(x)+3z^2(2x-2y)$$

$$\vec{X}\cdot(\nabla\times\vec{X})=-y^3-xyz+x^3+xyz+6xz^2-6yz^2$$

$$\vec{X}\cdot(\nabla\times\vec{X})=x^3-y^3+6xz^2-6yz^2$$

Since $\vec{X}\cdot(\nabla\times\vec{X})\neq 0$, for all values of x,y and z i.e. is not identically zero, hence the given PfDE is not integrable.