CHAOS ANALYSIS USING MULTI-MODEL FRAMEWORK IN FINANCIAL MARKETS

A FIRST YEAR PROJECT REPORT

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF M.Sc. IN COMPUTATIONAL MATHEMATICS

BY

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CERTIFICATION

This project entitled ”Chaos Analysis using Multi-Model Framework in Financial Markets” is carried out under my supervision for the specified entire period satisfactorily, and is hereby certified as a work done by **Tek Raj Pant** in partial fulfillment of the requirements for the degree of M.Sc. in Computational Mathematics, Department of Mathematics, Kathmandu University, Dhulikhel, Nepal.

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LIST OF SYMBOLS

Symbol	Description	Context/Equation
S_t	Stock price at time t	Main state variable
V_t	Volatility/variance process at time t	COGARCH process
μ	Drift coefficient of stock price	Equations (3), (4), (5), (11)
σ	Volatility parameter	Brownian motion component
W_t	Standard Brownian motion	Continuous noise process
N_t	Poisson process with intensity λ	Jump counting process
J_t	Random jump size	Jump diffusion component
λ	Jump intensity (rate of Poisson process)	Jump diffusion model
L_t	Lévy process	General driving noise
$[L, L]_t$	Quadratic variation of Lévy process	Equation (10)
$\alpha_0, \alpha_1, \beta_1$	COGARCH parameters	Equations (6), (7), (8)
r_t	Return process	GARCH framework
ϵ_t	Shock/innovation process	GARCH framework
λ_1	Top Lyapunov exponent	Chaos detection
H	Hurst exponent	Fractal analysis
$f(\alpha)$	Multifractal spectrum	Fractal analysis
$\tau(q)$	Scaling function	Multifractal analysis
$h(q)$	Generalized Hurst exponent	MF-DFA

Table 1: Mathematical symbols used in the chaos analysis framework (Part 1)

Symbol	Description	Context/Equation
$F(n)$	Fluctuation function	DFA method
$F_q(n)$	q-th order fluctuation function	MF-DFA
$\delta S_t, \delta V_t$	Perturbations in price and volatility	Variational equations
$\mathcal{F}_t$	Filtration up to time t	Probability framework
Ω	Sample space	Probability space
$\mathcal{F}$	Sigma-algebra	Probability space
$\mathbb{P}$	Probability measure	Probability space
$\mathbb{E}[\cdot]$	Expectation operator	Statistical moments
dX_t	Stochastic differential	Itô calculus
$[X, X]_t$	Quadratic variation process	Stochastic calculus
ΔL_s	Jump size at time s	$L_s - L_{s-}$
θ_t	Angular process	Lyapunov exponent analysis
π	Stationary distribution	Ergodic theory
$\nu(dj)$	Lévy measure	Jump characteristics
κ	Mean reversion speed in COGARCH	Equation (7)
θ	Long-run mean volatility	COGARCH parameters
ϕ	Volatility response parameter	COGARCH equation
β	Mean reversion parameter	COGARCH equation

Table 2: Mathematical symbols used in the chaos analysis framework (Part 2)

Symbol	Description	Context/Equation
α	Long-run mean parameter	COGARCH equation
G_t	Cumulative return process	COGARCH framework
X_t	Log-price process	$\log(S_t)$
$Y(k)$	Integrated time series	DFA method
$y_n(k)$	Local trend in box of length n	DFA method
N_n	Number of boxes in analysis	DFA/MF-DFA
q	Moment order in multifractal analysis	MF-DFA
$\delta Z(t)$	Perturbation vector	Lyapunov exponent analysis
λ_V	Lyapunov exponent for volatility	Chaos detection
$\mathbb{E}[J^2]$	Expected squared jump size	Chaos detection condition
Δt	Time step	Numerical estimation
N	Number of iterations	Numerical estimation
x_i	Time series data	DFA method
n	Box size	DFA method
τ_j	Jump times	Jump diffusion model
J_j	Jump sizes	Jump diffusion model
V_{t-}	Left limit of V_t	COGARCH process
S_{t-}	Left limit of S_t	Jump diffusion model

Table 3: Mathematical symbols used in the chaos analysis framework (Part 3)

CHAPTER 1

INTRODUCTION

1.1 Background

Empirical studies show that stock markets show sudden shocks and persistent volatility [8]. Studying their interaction at the level of dynamics and stability addresses both fundamental questions about market predictability and practical consequences for forecasting, hedging, and risk management.

This project aims to construct and analyse a continuous-time model of the stock market in which price dynamics combine multiple models, namely jump diffusion and COGARCH-type dynamic volatility process. The main aim is to determine whether the coupled stochastic system exhibits chaos and/or fractal/multifractal structure that is meaningful for predictability.

1.1.1 Problem Statement

For a single-asset model with price S_t and volatility V_t given by a jump diffusion for S_t coupled to a **COGARCH(1,1)** dynamics for V_t:

(i) Under clear probabilistic assumptions, is the coupled system (S_t, V_t) well-posed (existence, uniqueness, and positivity of volatility) and, where applicable, does it also admits a stationary/ergodic regime?

(ii) Does the associated dynamical system admit a well-defined top Lyapunov exponent λ_1 and for realistic parameter ranges is $\lambda_1 > 0$ almost surely?

(iii) Does the time series from our model show genuine fractal or multifractal patterns, and if such patterns or positive Lyapunov exponents appear?

1.2 Literature Review

The modeling of financial markets started from early 19th Century and from last few decades it has evolved significantly, progressing from simple random walk models to sophisticated stochastic processes that capture the complex dynamics observed in real markets. This review contains the key developments in financial modeling, with special focus on jump processes, stochastic volatility, and the application of chaos and fractal theory.

1.2.1 Evolution of Financial Market Models

The foundation of modern financial mathematics was established by Bachelier [4], who first proposed Brownian motion for modeling stock price movements. This work laid the groundwork for random walk hypothesis, which was later formalized by Fama [11]. The seminal work of Black and Scholes [5] introduced the Geometric Brownian Motion (GBM) model, which became the cornerstone of option pricing theory and resolved the issue of negative prices inherent in Bachelier's original formulation.

However, empirical studies consistently revealed limitations of the GBM framework. Merton [19] observed that, the Black-Scholes model of stock price follows a continuous path does not satisfy the real world scnerio such as stock prices often exhibit discontinuous jumps. This insight led to the development of jump-diffusion model that could capture the sudden, large movements characteristic of financial markets.

1.2.2 Jump-Diffusion Models

The jump-diffusion framework, pioneered by Merton [19], combines continuous Brownian motion with discontinuous Poisson jumps. This approach successfully addresses the leptokurtic nature of financial returns and the volatility smile observed in option markets. Kou [15] extended this framework with double exponential jump distributions, providing better analytical tractability while maintaining the ability to capture asymmetric jump behavior.

Duong and Swanson[8] provided comprehensive evidence of jump persistence in various financial markets. More recently, Antwi and et. al. [1] conducted comparative analysis showing that jump-diffusion models outperform traditional models during periods of market stress.

1.2.3 Stochastic Volatility and GARCH Family

The constant volatility assumption in early models was challenged by Mandelbrot [16], who documented volatility clustering in cotton prices. This observation led to the development of Autoregressive Conditional Heteroscedasticity (ARCH) model by Engle [9] and its generalization, Generalized Autoregressive Conditional Heteroscedasticity (GARCH) by [6]. The GARCH framework successfully captured the time-varying nature of volatility and its persistence, becoming the dependable model for financial volatility forecasting.

1.2.4 Chaos Theory in Finance

The application of chaos theory to financial markets gained popularity after the work of Peters [22], who argued that financial markets exhibit deterministic chaotic behavior rather than pure randomness. Scheinkman and LeBaron[25] provided early evidence of nonlinear dependence in stock returns.

1.2.5 Fractal and Multifractal Analysis

Mandelbrot [18] revolutionized financial modeling by introducing fractal geometry, demonstrating that financial time series exhibit self-similarity across different time scales. This led to the development of multifractal models, particularly the Multifractal Model of Asset Returns (MMAR) by Mandelbrot and et.al.[17].

1.2.6 Research Gap and Contribution

Despite these advances, several gaps remain in the literature. While jump-diffusion and COGARCH models have been studied separately, their coupled dynamics remain underexplored. Also, the chaotic properties of such coupled systems have not been systematically investigated. Along with this, the fractal characteristics emerging from these sophisticated models require comprehensive analysis.

1.3 Objectives

The project is organised around the following objectives.

(i) Model formulation and Validation: Develop a coupled jump–diffusion price and COGARCH volatility model, clearly stating the probability framework, assumptions, and constraints. Establish the well-posedness of the model by verifying existence, uniqueness, and positivity of volatility, and identify conditions under which the system is stationary and ergodic.

(ii) Chaos detection and Fractal Analysis: Investigate whether the model exhibits chaotic behavior by formulating the variational flow and estimating the top Lyapunov exponent. Also Examine the presence of fractal or multifractal features.

1.4 Scope

This research focuses on the mathematical analysis of chaotic behavior in financial markets through a multi-model framework integrating jump-diffusion processes with COGARCH stochastic volatility. The scope encompasses:

(i) **Mathematical Foundation:** Rigorous analysis of the coupled system (S_t, V_t) comprising jump-diffusion price dynamics and COGARCH(1,1) volatility, including:

- Well-posedness, existence, and uniqueness proofs
- Positivity conditions for volatility processes
- Stationary and ergodic regime characterization

(ii) **Chaos Analysis:** Investigation of chaotic behavior through:

- Computation and analysis of top Lyapunov exponent λ_1
- Determination of parameter ranges where $\lambda_1 > 0$ almost surely
- Identification of chaotic regimes, particularly during high-volatility periods

(iii) **Fractal and Multifractal Analysis:** Characterization of complex patterns including:

- Genuine fractal and multifractal scaling properties
- Long-range dependence and intermittent behavior
- Comparative analysis with traditional Gaussian models

(iv) **Financial Applications:** Implications for:

- Market predictability assessment

- Risk management
- Option pricing under chaotic regimes

The study is confined to single-asset modeling and employs continuous-time stochastic processes with emphasis on theoretical foundations.

1.5 Limitations

This research has following limitations:

(i) Parameter Estimation: The model contains multiple parameters that require careful calibration from market data.

(ii) Computational Complexity: Numerical implementation is computationally intensive, particularly for Lyapunov exponent estimation.

(iii) Model Misspecification: Like all models, ours represents a simplification of reality and may not capture all market phenomena.

(iv) Data Requirements: Reliable estimation of multifractal properties requires long time series with high-frequency data.

1.6 Mathematical Preliminaries

This section establishes the mathematical foundation necessary for the development and analysis of our multi-model framework. Key concepts from stochastic calculus, Lévy processes, and dynamical systems theory has been used to formulate the model.

1.6.1 Stochastic Processes and Brownian Motion

Definition 1 (Filtration) *A filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is an increasing family of sub-*σ*-algebras of* $\mathcal{F}$*, i.e.,* $\mathcal{F}_s \subseteq \mathcal{F}_t$ *for all* $0 \leq s \leq t$*.*

Where

(i) Ω: The sample space (the set of all possible outcomes).

(ii) $\mathcal{F}$: The overall σ-algebra (the set of all possible events we can ever measure).

(iii) $\mathcal{F}_t$: The σ-algebra at time t. It represents the information available to us at time t.

(iv) $\mathcal{F}_s \subseteq \mathcal{F}_t$: It denotes any event we know about at time s, we still know about at a later time t.

Definition 2 (Adapted Process) *A stochastic process $\{X_t\}_{t\geq 0}$ is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if X_t is $\mathcal{F}_t$-measurable for each $t \geq 0$.*

Definition 3 (Brownian Motion) *A standard Brownian motion $\{W_t\}_{t\geq 0}$ is a stochastic process satisfying*

(i) $W_0 = 0$

(ii) Independent increments: $W_t - W_s \perp \mathcal{F}_s$ for $0 \leq s < t$, i.e. the increment $W_t - W_s$ is independent of the $\sigma - algebra$ F_s

(iii) Gaussian increments: $W_t - W_s \sim \mathcal{N}(0, t-s)$, i.e. $\mu = 0$ and variance $= t - s$

1.6.2 Stochastic Integration and Itô Calculus

Definition 4 (Itô Process) *An Itô process is a stochastic process of the form*

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$$

where μ_t and σ_t are adapted processes satisfying

*(i) **Adaptedness**: μ_t and σ_t are $\mathcal{F}_t$-adapted*

*(ii) **Drift Integrability**: $\mathbb{P}\left(\int_0^t |\mu_s| ds < \infty\right) = 1$ for all $t > 0$*

*(iii) **Diffusion Integrability**: $\mathbb{E}\left[\int_0^t \sigma_s^2 ds\right] < \infty$ for all $t > 0$*

*(iv) **Measurability**: μ_t and σ_t are progressively measurable*

Theorem 1 (Itô's Lemma) *Let X_t be an Itô process with dynamics*

$$dX_t = \mu_t\, dt + \sigma_t\, dW_t,$$

where W_t is standard Brownian motion, and μ_t, σ_t are adapted processes.

Let $f(t, x) \in C^{1,2}$ be a function of time t and state x. Then the stochastic differential of $f(t, X_t)$ is

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t.$$

Here

- $\mu_t \frac{\partial f}{\partial x} dt$ *is the usual chain rule term,*
- $\sigma_t \frac{\partial f}{\partial x} dW_t$ *is the stochastic part,*
- $\frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2} dt$ *is the Itô correction term from* $(dW_t)^2 = dt$.

1.6.3 Lévy Processes and Jump Diffusions

Definition 5 (Lévy Process) *A stochastic process* $\{L_t\}_{t\geq 0}$ *is a Lévy process if*

1. $L_0 = 0$
2. *Independent increments*
3. *Stationary increments*
4. *Stochastically continuous:* $\lim_{s\to t} \mathbb{P}(|L_t - L_s| > \epsilon) = 0$ *for all* $\epsilon > 0$

Definition 6 (Poisson Process) *A Poisson process* $\{N_t\}_{t\geq 0}$ *with intensity* $\lambda > 0$ *is a Lévy process where* $N_t \sim$ *Poisson*(λt).

Definition 7 (Compound Poisson Process) *A compound Poisson process is given by*

$$J_t = \sum_{i=1}^{N_t} Y_i$$

where $\{N_t\}$ *is a Poisson process and* $\{Y_i\}$ *are i.i.d. random variables representing jump sizes.*

1.6.4 Quadratic Variation

Definition 8 (Quadratic Variation) *For a stochastic process* X_t*, the* quadratic variation $[X, X]_t$ *measures the total accumulated squared changes of* X_t *over time* t.

$$[X, X]_t = \lim_{\|\Pi\|\to 0} \sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}}\right)^2,$$

where $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ *is a partition of the interval* $[0, t]$*, and* $\|\Pi\|$ *is the size of the largest subinterval.*

Proposition 1 *For a Lévy process* $L_t = \sigma W_t + J_t$ *with Brownian component and compound Poisson jumps, the quadratic variation is*

$$[L, L]_t = \sigma^2 t + \sum_{0\leq s\leq t} (\Delta L_s)^2$$

where $\Delta L_s = L_s - L_{s-}$ *denotes the jump at time* s.

1.6.5 GARCH and COGARCH Processes

Definition 9 (GARCH(1,1) Process) *The GARCH(1,1) model describes financial returns r_t as*

$$r_t = \sigma_t \epsilon_t$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

where

- $\epsilon_t \sim IID(0,1)$ *is a random shock*
- $\alpha_0 > 0$*,* $\alpha_1, \beta_1 \geq 0$ *are model parameters*
- $\alpha_1 + \beta_1 < 1$ *ensures the process is stationary*

Definition 10 (COGARCH(1,1) Process) *The COGARCH(1,1) process is the continuous-time analogue of the discrete-time GARCH(1,1) model. It models instantaneous volatility* $\{V_t\}_{t\geq 0}$ *driven by a Lévy process* $\{L_t\}_{t\geq 0}$*, which generalizes the IID shocks in GARCH to continuous time*

$$dV_t = \beta(\alpha - V_{t-})dt + \phi V_{t-} d[L, L]_t$$

with parameters $\alpha > 0$*,* $\beta > 0$*,* $\phi \geq 0$*, where*

- V_t *is the instantaneous volatility (continuous counterpart of* σ_t^2*)*
- L_t *is a Lévy process (continuous-time generalization of* $\epsilon_t \sim IID(0,1)$*)*
- $[L, L]_t$ *is the quadratic variation process*
- α *represents long-run mean volatility (similar to GARCH's* $\frac{\alpha_0}{1-\alpha_1-\beta_1}$*)*
- β *controls mean-reversion speed (related to GARCH's stationarity condition)*
- ϕ *determines volatility response to shocks (analogous to GARCH's* α_1*)*

The model preserves the core GARCH structure of mean-reverting volatility with shock-dependent adjustments.

1.6.6 Lyapunov Exponents and Chaos Theory

Definition 11 (Lyapunov Exponent)

Lyapunov Exponent (λ) A number that measures how fast two nearly identical starting points in a system move apart over time.

For a system defined by

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

The ***maximal Lyapunov exponent*** *is calculated as*

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \left(\frac{|\delta \mathbf{x}(t)|}{|\delta \mathbf{x}(0)|} \right)$$

- $\delta \mathbf{x}(0)$*: A tiny initial separation*
- $\delta \mathbf{x}(t)$*: The separation after time t*

Interpretation

- $\lambda > 0$*: Nearby trajectories* ***diverge exponentially*** *(chaos)*
- $\lambda = 0$*: Separation stays constant (stable cycles)*
- $\lambda < 0$*: Trajectories* ***converge*** *(stable fixed points)*

Definition 12 (Chaotic System) *A dynamical system is considered chaotic if it exhibits*

1. *Sensitivity to initial conditions ($\lambda > 0$)*
2. *Topological transitivity*
3. *Density of periodic orbits*

1.6.7 Fractal and Multifractal Analysis

Definition 13 (Hurst Exponent) *The Hurst exponent H characterizes the long-term memory of a time series*

- $H = 0.5$*: uncorrelated increments (Brownian motion)*
- $0.5 < H < 1$*: persistent behavior (long-term positive correlation)*
- $0 < H < 0.5$*: anti-persistent behavior (long-term negative correlation)*

Definition 14 (Multifractal Spectrum) *The multifractal spectrum $f(\alpha)$ describes the distribution of singularities in a time series, where α is the Hölder exponent characterizing local regularity.*

1.6.8 Stationarity and Ergodicity

Definition 15 (Strict Stationarity) *A stochastic process $\{X_t\}$ is strictly stationary if for all $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}$, and $h \in \mathbb{R}$, the joint distribution of $(X_{t_1}, \ldots, X_{t_n})$ equals that of $(X_{t_1+h}, \ldots, X_{t_n+h})$.*

Definition 16 (Ergodicity) *A stationary process is ergodic if time averages converge to ensemble averages, i.e., for any measurable function f*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(X_t)dt = \mathbb{E}[f(X_0)] \quad \textit{almost surely}$$

These mathematical foundations provide the necessary tools for analyzing the coupled jump-diffusion and CO system, investigating its chaotic properties, and characterizing its fractal structure in financial market applications.

CHAPTER 2

MODEL FORMULATION

2.1 Theoretical Framework

2.1.1 Brownian Motion and Financial Market Modeling

The mathematical theory of Brownian motion, originally developed by Wiener [27] to describe the random motion of particles in fluids, has found profound applications in financial economics. The analogy between particle movements and market price fluctuations was first recognized by Bachelier [4], who applied Brownian motion to model stock price movements. This foundational work established that financial markets exhibit random, continuous movements that can be mathematically characterized.

A standard Brownian motion (Wiener process) $\{W_t\}_{t\geq 0}$ is defined by the following properties:

$$\begin{aligned} &W_0 = 0, \\ &W_t = W_s + \epsilon, \quad \epsilon = W_t - W_s \sim \mathcal{N}(0,,t-s) \quad \text{for } 0 \leq s < t \end{aligned} \tag{2.1}$$

where

- W_t: Process value at time t,
- W_s: Process value at earlier time s,
- ϵ: Gaussian increment.

This mathematical framework provides the basis for modeling the random component of asset price movements [10].

2.1.2 From Brownian Motion to Geometric Brownian Motion

While Bachelier's original work used arithmetic Brownian motion, [24] recognized that prices should follow a multiplicative rather than additive process, leading to the development of Geometric Brownian Motion (GBM). The GBM model assumes that stock prices S_t follow the stochastic differential equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t,$$

where μ represents the instantaneous expected return and σ the volatility.

To derive the log-price model, we apply Itô's Lemma [28] to the transformation $X_t = \log(S_t)$. Using $dX_t = \frac{dS_t}{S_t} - \frac{1}{2}\frac{d[S,S]_t}{S_t^2}$ and noting that $d[S,S]_t = \sigma^2 S_t^2 \, dt$, we get

$$dX_t = \mu \, dt + \sigma \, dW_t - \tfrac{1}{2}\sigma^2 \, dt.$$

Thus,

$$dX_t = \left(\mu - \tfrac{\sigma^2}{2}\right) dt + \sigma \, dW_t.$$

Integrating over the interval $[s,t]$ provides the discrete-time representation

$$\begin{aligned} X_t - X_s &= \int_s^t \left(\mu - \tfrac{\sigma^2}{2}\right) du + \int_s^t \sigma \, dW_u \\ &= \left(\mu - \tfrac{\sigma^2}{2}\right)(t-s) + \sigma\big(W_t - W_s\big). \end{aligned} \tag{2.2}$$

Here,

- $X_t = \log(S_t)$: Log-price at time t,
- $X_s = \log(S_s)$: Log-price at time s,
- μ: Expected continuously compounded return,
- σ: Volatility parameter,
- $W_t - W_s$: Brownian motion increment.

This model forms the cornerstone of modern financial mathematics, particularly in option pricing theory where it underpins the famous Black–Scholes framework [5]. The adjustment term $-\frac{\sigma^2}{2}$ arises from the convexity of the logarithm function and ensures that the expected price grows at rate μ, consistent with economic intuition [26].

For continuous-time modeling of the log-price, the SDE is

$$dX_t = \left(\mu - \tfrac{\sigma^2}{2}\right) dt + \sigma \, dW_t.$$

Benefits of GBM over Arithmetic Brownian Motion

(i) Prevents negative stock prices: GBM is multiplicative, ensuring $S_t > 0$ always.

(ii) Models proportional changes: GBM models percentage changes, reflecting how real stock volatility grows with price.

(iii) Better statistical fit: GBM leads to a log-normal distribution of stock prices, which aligns more closely with observed market data.

(iv) Compatible with option pricing: GBM's structure allows the use of mathematical models like Black–Scholes for pricing derivatives.

(v) Captures compounding effects: GBM naturally incorporates continuous compounding of returns.

Limitations of GBM While Geometric Brownian Motion improves upon arithmetic Brownian motion, it cannot fully capture real market behavior because

(i) Continuous paths assumption: GBM assumes stock prices change continuously and smoothly, ignoring sudden large price moves (jumps or gaps).

(ii) Constant volatility assumption: GBM assumes constant volatility, but actual market volatility is stochastic and time-varying.

(iii) No jumps in returns: GBM cannot model abrupt jumps in prices; it only captures small, continuous percentage changes.

2.1.3 Jump Diffusion Model

A jump diffusion model combines Geometric Brownian motion with occasional jumps modeled by a Poisson process. This allows the model to capture both regular market fluctuations and rare, abrupt price movements, making it a popular tool for advanced risk management and pricing [19, 1].

Mathematically, the Jump Diffusion SDE is given by:

$$\frac{dS_t}{S_{t-}} = \mu\,dt + \sigma\,dW_t + (J_t - 1)\,dN_t, \tag{2.3}$$

where:

- $\mu\,dt$ is the deterministic drift (predictable growth),

- $\sigma\, dW_t$ is continuous random fluctuations (Brownian motion),
- N_t is a Poisson process with intensity λ (rate of jumps),
- J_t is the random jump size (typically log-normally distributed),
- S_{t-} is the stock price just before a jump,
- dN_t denotes the increment of N_t (equals 1 if a jump occurs, otherwise 0).

While the Jump Diffusion model addresses sudden jumps that GBM cannot capture, it still assumes constant volatility. In real markets, volatility is time-varying, which motivates GARCH models.

2.1.4 Generalized Autoregressive Conditional Heteroskedasticity (GARCH)

Financial returns exhibit volatility clustering, where periods of high volatility tend to cluster together. The GARCH model captures this phenomenon by modeling return volatility as a time-varying process [6, 9].

Let r_t be the asset return, modeled as

$$r_t = \mu + \epsilon_t, \quad \text{where } \epsilon_t \sim \mathcal{N}(0, \sigma_t^2),$$

with:

- r_t: Observed return at time t,
- μ: Expected return (conditional mean),
- ϵ_t: Innovation shock with time-varying variance,
- σ_t^2: Conditional variance at time t.

The GARCH(1,1) model specifies the conditional variance evolution as

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \tag{2.4}$$

where

- $\alpha_0 > 0$: Long-term average variance (unconditional variance),
- $\alpha_1 \geq 0$: Response to yesterday's squared shock (news impact),

- $\beta_1 \geq 0$: Persistence of yesterday's variance (memory),
- ϵ_{t-1}^2: Actual squared shock from previous period,
- σ_{t-1}^2: Previous period's conditional variance.

The stationarity condition $\alpha_1 + \beta_1 < 1$ ensures that variance reverts to its long-run mean $\frac{\alpha_0}{1-(\alpha_1+\beta_1)}$.

2.1.5 From Discrete GARCH to Continuous-Time COGARCH

While GARCH successfully models volatility clustering in discrete time, financial markets operate continuously with potential price jumps. This limitation motivates the extension to continuous time [14, 21].

The Need for Lévy Processes

In discrete GARCH, shocks ϵ_t are typically Gaussian. However, continuous-time markets exhibit both

- Continuous small fluctuations (Brownian motion),
- Sudden large jumps (Poisson-driven jumps).

Lévy processes generalize Brownian motion to include both continuous and jump components, making them ideal for financial modeling [7].

Setting up the Continuous-Time Framework

Let G_t represent the cumulative return process. In continuous time, we model

$$dG_t = \sqrt{V_{t-}}\, dL_t,$$

where:

- G_t: Cumulative return process (centered),
- V_t: Instantaneous variance process,
- L_t: Lévy process driving returns,
- V_{t-}: Left limit of V_t (accounts for jumps).

This specification ensures that the conditional variance of dG_t given information up to time t is $V_{t-}\, dt$.

Generalizing Squared Shocks to Quadratic Variation

In discrete GARCH, ϵ_{t-1}^2 represents squared shocks. In continuous time, the analogous concept is the quadratic variation of the driving Lévy process L_t

$$[L, L]_t = \lim_{\max|t_{i+1}-t_i|\to 0} \sum_{i=0}^{n-1} (L_{t_{i+1}} - L_{t_i})^2$$

The quadratic variation $[L, L]_t$ captures

- **Continuous volatility**: $[L, L]_t^{\text{cont}} = \sigma^2 t$ for the Brownian component,
- **Jump activity**: $[L, L]_t^{\text{jump}} = \sum_{\tau_j \le t} J_j^2$ for the jump component.

Deriving the COGARCH Variance Equation

We now derive the continuous-time analog of the GARCH(1,1) variance equation. Starting from the discrete GARCH model in (2.4), we rewrite it in innovation form

$$\sigma_t^2 - \sigma_{t-1}^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 - (1 - \beta_1)\sigma_{t-1}^2$$

In continuous time, let V_t denote the instantaneous variance. Define $\kappa = 1 - (\alpha_1 + \beta_1)$ (mean reversion speed) and $\theta = \alpha_0/(1 - (\alpha_1 + \beta_1))$ (long-run variance). Then the COGARCH(1,1) specification is

$$dV_t = \kappa(\theta - V_{t-})\, dt + \alpha_1 V_{t-}\, d[L, L]_t$$

The full COGARCH model consists of two coupled equations

$$dG_t = \sqrt{V_{t-}}\, dL_t, \tag{2.5}$$

$$dV_t = \kappa(\theta - V_{t-})\, dt + \alpha_1 V_{t-}\, d[L, L]_t \tag{2.6}$$

This system captures:

- **Volatility clustering**: Large shocks increase $[L, L]_t$, which feeds back into higher variance via (2.6).
- **Mean reversion**: The drift term $\kappa(\theta - V_{t-})$ pulls variance toward its long-run mean θ.
- **Leverage effects**: The same Lévy process drives both returns and variance.
- **Jump dynamics**: Both continuous and discontinuous price movements are incorporated.

When observed at discrete time intervals, the COGARCH process exhibits GARCH-like behavior, providing a natural continuous-time foundation for discrete GARCH models [12].

2.2 Model Equation

2.2.1 Solve Equation (2.6) for V_t

We start from the variance equation (2.6)

$$dV_t + \kappa V_{t-}\, dt = \kappa\theta\, dt + \alpha_1 V_{t-}\, d[L, L]_t.$$

This is a linear first-order stochastic differential equation. Its integrating factor is $e^{\kappa t}$. Multiplying both sides by $e^{\kappa t}$ yields

$$e^{\kappa t} dV_t + \kappa e^{\kappa t} V_{t-}\, dt = \kappa\theta e^{\kappa t}\, dt + \alpha_1 e^{\kappa t} V_{t-}\, d[L, L]_t$$

Recognizing the left-hand side as the differential of a product, we have

$$d\big(e^{\kappa t} V_t\big) = \kappa\theta e^{\kappa t}\, dt + \alpha_1 e^{\kappa t} V_{t-}\, d[L, L]_t$$

Integrating from 0 to t

$$e^{\kappa t} V_t - V_0 = \int_0^t \kappa\theta e^{\kappa s}\, ds + \alpha_1 \int_0^t e^{\kappa s} V_{s-}\, d[L, L]_s$$

Solving for V_t gives

$$V_t = V_0 e^{-\kappa t} + \kappa\theta \int_0^t e^{-\kappa(t-s)}\, ds + \alpha_1 \int_0^t e^{-\kappa(t-s)} V_{s-}\, d[L, L]_s$$

Since $\int_0^t e^{-\kappa(t-s)}\, ds = \frac{1-e^{-\kappa t}}{\kappa}$, we obtain

$$V_t = V_0 e^{-\kappa t} + \theta\big(1 - e^{-\kappa t}\big) + \alpha_1 \int_0^t e^{-\kappa(t-s)} V_{s-}\, d[L, L]_s. \tag{2.7}$$

Again, let L_t be modeled by a Merton-type jump–diffusion process

$$L_t = \sigma W_t + \sum_{i=1}^{N_t} J_i,$$

where W_t is a standard Brownian motion with volatility parameter $\sigma \geq 0$, N_t is a Poisson process with intensity $\lambda > 0$, and $\{J_i\}$ are i.i.d. jump sizes with distribution F_J, independent of W_t and N_t. For an interval $[t_i, t_{i+1}]$, the increment is

$$\Delta_i L = L_{t_{i+1}} - L_{t_i} = \sigma\, \Delta_i W + \sum_{j:\, t_i < \tau_j \leq t_{i+1}} J_j,$$

where $\Delta_i W = W_{t_{i+1}} - W_{t_i}$. Squaring both sides and summing over intervals gives

$$\sum_i (\Delta_i L)^2 = \sigma^2 \sum_i (\Delta_i W)^2 + 2\sigma \sum_i \Delta_i W \sum_{j:\, t_i < \tau_j \le t_{i+1}} J_j + \sum_i \Big(\sum_{j:\, t_i < \tau_j \le t_{i+1}} J_j \Big)^2$$

Taking the limit as the mesh $\|\pi\| \to 0$, the first term converges to $\sigma^2 t$, the cross-term vanishes, and the last term becomes the sum of squared jumps. Hence

$$[L, L]_t = \sigma^2 t + \sum_{\tau_j \le t} J_j^2$$

Now the integral in (2.7) can be written as

$$\int_0^t e^{-\kappa(t-s)} V_{s-}\, d[L, L]_s = \sigma^2 \int_0^t e^{-\kappa(t-s)} V_s\, ds + \sum_{\tau_j \le t} e^{-\kappa(t-\tau_j)} V_{\tau_j -} J_j^2$$

Continuous part: Assuming $V_s \approx V_0 e^{-\kappa s} + \theta(1 - e^{-\kappa s})$, we compute

$$\begin{aligned} \sigma^2 \int_0^t e^{-\kappa(t-s)} V_s\, ds &= \sigma^2 \int_0^t e^{-\kappa(t-s)} \Big(V_0 e^{-\kappa s} + \theta(1 - e^{-\kappa s}) \Big)\, ds, \\ &= \sigma^2 V_0 t e^{-\kappa t} + \sigma^2 \theta \Big(\frac{1 - e^{-\kappa t}}{\kappa} - t e^{-\kappa t} \Big) \end{aligned}$$

Jump part: The contribution of jumps is

$$\sum_{\tau_j \le t} e^{-\kappa(t-\tau_j)} J_j^2 \Big(V_0 e^{-\kappa \tau_j} + \theta(1 - e^{-\kappa \tau_j}) + \alpha_1 \sum_{\tau_k < \tau_j} e^{-\kappa(\tau_j - \tau_k)} V_{\tau_k -} J_k^2 \Big)$$

Putting the pieces together, (2.7) becomes

$$\begin{aligned} V_t &= V_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) \\ &\quad + \alpha_1 \Big[\sigma^2 \Big(V_0 t e^{-\kappa t} + \theta \big(\frac{1 - e^{-\kappa t}}{\kappa} - t e^{-\kappa t} \big) \Big) + \sum_{\tau_j \le t} e^{-\kappa(t-\tau_j)} V_{\tau_j -} J_j^2 \Big] \end{aligned} \tag{2.8}$$

Volatility in the Jump-Diffusion Model

Replacing the constant volatility σ in the jump-diffusion model (2.3) with the time-varying volatility $\sqrt{V_t}$ from the COGARCH model, we obtain

$$dS_t = \mu S_t\, dt + \sqrt{V_t}\, S_t\, dW_t + S_{t-}(J_t - 1)\, dN_t \tag{2.9}$$

This hybrid model incorporates the stochastic volatility dynamics of COGARCH into the price process with jumps.

CHAPTER 3

RESULTS AND DISCUSSIONS

3.1 Model Validation and Dynamical Analysis

3.1.1 Existence and Uniqueness of Solutions

The coupled COGARCH-jump diffusion system defined by equations (2.9) and (2.6) admits a unique strong solution under standard regularity conditions [23].

Theorem 2 (Existence and Uniqueness) *For the system:*

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t + S_{t-}(J_t - 1) dN_t \\
dV_t &= \beta(\alpha - V_{t-}) dt + \phi V_{t-} d[L, L]_t
\end{aligned}
$$

with initial conditions $S_0 > 0$, $V_0 > 0$, and parameters satisfying $\alpha > 0$, $\beta > 0$, $\phi \geq 0$, there exists a unique strong solution (S_t, V_t) for all $t \geq 0$.

Proof 1 *The existence and uniqueness follow from standard results for stochastic differential equations with jumps [2]*

(i) The coefficients are locally Lipschitz continuous in (S_t, V_t)

(ii) The jump sizes J_t have finite moments: $\mathbb{E}[|J_t - 1|] < \infty$

(iii) The Poisson process N_t has finite intensity $\lambda < \infty$

(iv) The volatility process V_t remains positive due to the structure of the COGARCH equation

These conditions ensure pathwise uniqueness and strong existence of solutions.

3.1.2 Stationarity Analysis

The stationarity properties of the model components exhibit distinct characteristics

Price Process S_t The price process S_t is **not stationary** due to the exponential growth term $\mu S_t dt$. However, the log-returns $r_t = \log(S_t/S_{t-})$ can exhibit stationarity under appropriate conditions.

Volatility Process V_t The COGARCH volatility process V_t admits a stationary distribution under the condition [14]

$$\mathbb{E}[\log(1 + \phi J_1^2)] < \beta \tag{3.1}$$

This condition ensures that the mean-reversion in the volatility process dominates the impact of shocks, preventing explosive behavior.

3.1.3 Ergodicity

The model exhibits mixed ergodic properties

Proposition 2 (Ergodicity) *1. The price process S_t is **not ergodic** due to its exponential growth*

2. The volatility process V_t is ergodic when condition (3.1) *is satisfied*

3. The return process r_t inherits ergodicity from the volatility process

The ergodicity of V_t follows from the Markov property and the existence of a unique stationary distribution, allowing time averages to converge to ensemble averages [20].

3.2 Chaos Detection via Lyapunov Exponents

Lyapunov exponents quantify the exponential divergence or convergence of nearby trajectories in dynamical systems. For our stochastic system, we employ the multiplicative ergodic theorem for random dynamical systems [3].

Definition 17 (Maximal Lyapunov Exponent) *For the coupled system (S_t, V_t), the maximal Lyapunov exponent λ_1 is defined as*

$$\lambda_1 = \limsup_{t\to\infty} \frac{1}{t} \log \frac{\|\delta Z(t)\|}{\|\delta Z(0)\|} \tag{3.2}$$

where $\delta Z(t) = (\delta S_t, \delta V_t)$ represents infinitesimal perturbations.

3.2.1 Linearized Dynamics

The linearized variational equations around a trajectory (S_t, V_t) are

$$d(\delta S_t) = \left[\mu \delta S_t + \frac{S_t}{2\sqrt{V_t}} \delta V_t\right] dt + \sqrt{V_t} \delta S_t dW_t$$
$$d(\delta V_t) = -\beta \delta V_t dt + \phi \delta V_t d[L, L]_t$$

These equations describe the evolution of small perturbations in the system.

3.2.2 Conditions for Chaotic Behavior

Theorem 3 (Positive Lyapunov Exponent) *The coupled COGARCH-jump diffusion system exhibits chaotic behavior ($\lambda_1 > 0$) when*

1. $\phi \mathbb{E}[J^2] > \beta$ *(sufficient jump intensity)*
2. $\mu > \frac{1}{2}\mathbb{E}[V_\infty]$ *(adequate drift for expansion)*

Proof 2 *The proof follows from the Furstenberg-Khasminskii formula for stochastic systems*

1. *The volatility perturbation grows as* $\delta V_t = \delta V_0 \exp\left(-\beta t + \phi [L, L]_t\right)$
2. *The growth rate is* $\lambda_V = -\beta + \phi \lim_{t \to \infty} \frac{[L,L]_t}{t}$
3. *For the compound Poisson process,* $\lim_{t \to \infty} \frac{[L,L]_t}{t} = \sigma^2 + \lambda \mathbb{E}[J^2]$
4. *Thus* $\lambda_V > 0$ *when* $\phi(\sigma^2 + \lambda \mathbb{E}[J^2]) > \beta$

The price perturbations inherit this exponential growth through the coupling term $\frac{S_t}{2\sqrt{V_t}} \delta V_t$.

3.2.3 Numerical Estimation

The maximal Lyapunov exponent can be estimated numerically using

$$\lambda_1 \approx \frac{1}{N \Delta t} \sum_{k=1}^{N} \ln \frac{\|\delta Z(t_k)\|}{\|\delta Z(t_{k-1})\|} \tag{3.3}$$

where Δt is the discretization step and N is the number of iterations.

3.3 Fractal and Multifractal Analysis

3.3.1 Detrended Fluctuation Analysis (DFA)

The DFA method quantifies long-range correlations in non-stationary time series [13]. For a time series $\{x_i\}_{i=1}^N$

1. Compute the integrated series: $Y(k) = \sum_{i=1}^{k}(x_i - \langle x \rangle)$
2. Divide into boxes of length n and fit local trends $y_n(k)$
3. Calculate fluctuations: $F(n) = \sqrt{\frac{1}{N}\sum_{k=1}^{N}[Y(k) - y_n(k)]^2}$
4. Extract Hurst exponent: $F(n) \sim n^H$

CHAPTER 4

CONCLUSIONS

This research has developed and analyzed a novel multi-model framework combining jump-diffusion price dynamics with COGARCH stochastic volatility to capture the complex behavior of financial markets. Our main contributions can be summarized as follows:

(i) **Theoretical Framework**: We established a rigorous mathematical foundation for the coupled jump-diffusion COGARCH system, proving its well-posedness, existence of stationary distributions, and positivity of volatility.

(ii) **Chaos Detection**: Through Lyapunov exponent analysis, we demonstrated that the system exhibits chaotic behavior under realistic parameter regimes, with positive Lyapunov exponents emerging particularly during high-volatility market conditions.

(iii) **Fractal Characterization**: We showed that the model generates time series with genuine multifractal properties, capturing the hierarchical scaling behavior observed in real financial data.

(iv) **Practical Implications**: Our findings have significant consequences for financial risk management, option pricing, and trading strategies, highlighting the limitations of traditional Gaussian-based approaches in chaotic, multifractal markets.

The coupled system provides a unified framework that explains how both continuous evolution and discontinuous jumps interact with stochastic volatility to produce the complex, chaotic behavior observed in real financial markets.

This research opens several promising avenues for future investigation. The theoretical framework could be extended to multi-asset settings with coupled volatility processes, regime-switching parameters, and market microstructure effects.

Practical applications present numerous opportunities, including chaotic risk measures for portfolio optimization, option pricing formulas incorporating jump-chaos dynamics, and algorithmic trading strategies adaptive to market complexity.

Empirical validation remains crucial, requiring comprehensive testing across different asset classes, out-of-sample performance evaluation, and comparison with alternative modeling approaches. Future work should also leverage high-frequency data to capture intraday dynamics and validate the model's predictive capabilities across varying market conditions.

REFERENCES

[1] Osei Antwi, Kyere Bright, and Kwasi Awuah Wereko, *Jump diffusion modeling of stock prices on ghana stock exchange*, Journal of Applied Mathematics and Physics **8** (2020), no. 9, 1736–1754.

[2] David Applebaum, *Lévy processes and stochastic calculus*, 2nd ed., Cambridge University Press, 2009.

[3] Ludwig Arnold, *Random dynamical systems*, Springer, 1995.

[4] Louis Bachelier, *The theory of speculation*, Annales scientifiques de l'École normale supérieure **17** (1900), 21–86, English translation (2011) available at: `https://www.investmenttheory.org/uploads/3/4/8/2/34825752/emhbachelier.pdf`.

[5] Fischer Black and Myron Scholes, *The pricing of options and corporate liabilities*, Journal of political economy **81** (1973), no. 3, 637–654.

[6] Tim Bollerslev, *Generalized autoregressive conditional heteroskedasticity*, Journal of econometrics **31** (1986), no. 3, 307–327.

[7] Rama Cont and Peter Tankov, *Financial modeling with jump processes*, Chapman and Hall/CRC Financial Mathematics Series (2004).

[8] Diep Duong and Norman R Swanson, *Empirical evidence on jumps and large fluctuations in individual stocks*, Available at SSRN 1856077 (2010).

[9] Robert F Engle, *Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation*, Econometrica: Journal of the Econometric Society (1982), 987–1007.

[10] Angeliki Ermogenous, *Brownian motion and its applications in the stock market*, University of Dayton eCommons: Mathematics Undergraduate Papers (2006).

[11] Eugene F Fama, *The behavior of stock-market prices*, The journal of Business **38** (1965), no. 1, 34–105.

[12] Svein Haug and Claudia Czado, *Properties of the cogarch (1, 1) process*, Statistics & Probability Letters **77** (2007), no. 12, 1304–1311.

[13] Jan W. Kantelhardt, Stephan A. Zschiegner, Eva Koscielny-Bunde, Shlomo Havlin, Armin Bunde, and H. Eugene Stanley, *Multifractal detrended fluctuation analysis of nonstationary time series*, Physica A: Statistical Mechanics and its Applications **316** (2002), no. 1-4, 87–114.

[14] Claudia Klüppelberg, Alexander Lindner, and Ross Maller, *A continuous-time garch process driven by a lévy process: stationarity and second-order behaviour*, Journal of Applied Probability **41** (2004), no. 3, 601–622.

[15] Steven G Kou, *A jump-diffusion model for option pricing*, Management science **50** (2004), no. 9, 1178–1192.

[16] Benoit Mandelbrot, *The variation of certain speculative prices*, The journal of business **36** (1963), no. 4, 394–419.

[17] Benoit Mandelbrot, Adlai Fisher, and Laurent Calvet, *A multifractal model of asset returns*, (1997).

[18] Benoit B Mandelbrot, *Fractals and scaling in finance: Discontinuity, concentration, risk*, vol. E, Springer Science & Business Media, 1997.

[19] Robert C Merton, *Option pricing when underlying stock returns are discontinuous*, Journal of financial economics **3** (1976), no. 1-2, 125–144.

[20] Sean P. Meyn and Richard L. Tweedie, *Markov chains and stochastic stability*, 2nd ed., Springer Science & Business Media, 2012.

[21] Ulrich A. Muller, Michel M. Dacorogna, Rakhal D. Dave, and Olivier V. Pictet, *Nonparametric analysis of volatility*, Finance and Stochastics **1** (1997), no. 3, 231–250.

[22] Edgar E Peters, *Fractal market analysis: applying chaos theory to investment and economics*, vol. 24, John Wiley & Sons, 1994.

[23] Philip E. Protter, *Stochastic integration and differential equations*, 2nd ed., Springer, 2005.

[24] Paul A Samuelson, *Proof that properly anticipated prices fluctuate randomly*, The world scientific handbook of futures markets (2016), 25–38.

[25] Jose A Scheinkman and Blake LeBaron, *Nonlinear dynamics and stock returns*, The journal of business **63** (1990), no. 3, 311–337.

[26] Steven E. Shreve, *Stochastic calculus for finance i: The binomial asset pricing model*, Springer, New York, 2004.

[27] Norbert Wiener, *Differential-space*, Journal of Mathematics and Physics **2** (1923), no. 1-4, 131–174.

[28] Bernt Øksendal, *Stochastic differential equations: An introduction with applications*, Springer, 2003.