Assignment 1

M.Sc. in Computational Mathematics

1. Solve the following tridiagonal system of equations using Thomas rule/ algorithm
   $2x - y = 0, -x + 2y - z = 0, -y + 2z - u = 0, -z + 2u = 1$
   Ans: 0.2000 0.4000 0.6000 0.8000
   a. Manually solve this correctly to three decimal places.
   b. Show answer-generating Python code.
2. Solve the linear system $Ax = b$ given by $4x_1 + 3x_2 = 24, 3x_1 + 4x_2 - x_3 = 30, -x_2 + 4x_3 = -24$ has the solution $(3,4,-5)$. With $x^{(0)} = (0;\ 0;\ 0)^t$ in both Gauss-Seidel and Relaxation methods, and $\omega = 1.25$ in relaxation method, Gauss-Seidel method requires 32 iterations and relaxation method requires 14 iterations to accurate to seven decimal places.

a. Show answer generating through Python
b. Solve this from Gauss-Seidel method and Relaxation method accurate to 3 decimal places.

3. How the appropriate value of ω is to choose?
   a. You can follow the following:

Answer: No complete answer to this question is known for general $n \times n$ linear system. Although, the following results can be used in certain situations.

Theorem (Kahan)

If $a_{ii} \neq 0$, for each $i = 1, 2, \cdots, n$, then $\rho(T_\omega) \geq |\omega - 1|$. This implies that the SOR method can converge only if $0 < \omega < 2$.

Theorem (Ostrowski-Reich)

If A is positive definite matrix and $0 < \omega < 2$, then the SOR method converges for any choice of initial approximate vector $\mathbf{x}^{(0)}$.

Theorem

If A is positive definite and tridiagonal, then $\rho(T_g) = [\rho(T_j)]^2 < 1$, and the optimal choice of ω for the SOR method is

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(T_j)]^2}}$$

With this choice of ω, we have $\rho(T_\omega) = \omega - 1$.

Example

The matrix $\begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ *is positive definite and tridiagonal, since eigenvalues of A are* $0.8377, 4, 7.1623$.

$$T_j = -D^{-1}(L+U) = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & -3 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -0.75 & 0 \\ -0.75 & 0 & 0.25 \\ 0 & 0.25 & 0 \end{bmatrix}$$

The eigenvalues of T_j are $-0.7906, 0, 0.7906$. Thus $\rho(T_j) = 0.7906$, and

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(T_j)]^2}} = \frac{2}{1 + \sqrt{1 - (0.7906)^2}} \approx 1.24$$

This explains the rapid convergence of relaxation method in above Example when using $\omega = 1.25$

4. Write the theory for the Relaxation, Guss Seidel, and Jacobian iterative methods' general formulas. Create Python programs using a suitable example. You can follow the lecture that is attached to this assignment.
5. Find the real root of the equations: $f(x,y) = 1 + y^2 - 4x^2 = 0, g(x,y) = 3 + 2x - x^2 - y^2 = 0$ correct to 3 decimal places by using fixed point iterative method where $x_0 = 1$ and $y_0 = 2$.
6. Find a real root of the equations $x^2 - y^2 = 3$ and $x^2 + y^2 = 13$ correct to 3 decimal places by using Newton Raphson Method where $x_0 = y_0 = 2.54951$.

The End