6.3 Cross Spectrum Analysis

Cross spectral analysis allows one to determine the relationship between two time series as a function of frequency. Normally, one supposes that statistically significant peaks at the same frequency have been shown in two time series and that we wish to see if these periodicities are related with each other and, if so, what the phase relationship is between them. One may extend this concept a bit by considering whether it may make sense to do cross spectral analysis even in the absence of peaks in the power spectrum. Suppose we have two time series whose power spectra both are indistinguishable from red noise? Under these circumstances what might cross-spectral analysis still be able to reveal? It might be that within this red noise spectrum there are in fact coherent modes at particular frequencies. We can test for this by looking at the coherency spectrum.

Suppose we have two time series $x(t)$ and $y(t)$ and we want to look for relationships between them in particular frequency bands. First consider harmonic analysis in terms of line spectra:

$$x = \bar{x} + \sum_{k=1}^{n/2-1} \left(A_{xk} \cos\left(\frac{2\pi kt}{T}\right) + B_{xk} \sin\left(\frac{2\pi kt}{T}\right)\right) + A_{xN/2} \cos\left(\frac{\pi Nt}{T}\right) \qquad (6.72a)$$

$$y = \bar{y} + \sum_{k=1}^{n/2-1} \left(A_{yk} \cos\left(\frac{2\pi kt}{T}\right) + B_{yk} \sin\left(\frac{2\pi kt}{T}\right)\right) + A_{yN/2} \cos\left(\frac{\pi Nt}{T}\right) \qquad (6.72b)$$

Because of the orthogonality of the functions for evenly spaced data we can write the covariance between them as a sum of contributions from particular frequencies.

$$\overline{x'\,y'} = \frac{1}{2} \sum_{k=1}^{\frac{N}{2}-1} \left(A_{xk} A_{yk} + B_{xk} B_{yk} \right) + A_{x\,N/2} A_{y\,N/2} \qquad (6.73)$$

$$= \sum_{k=1}^{\frac{N}{2}} \mathrm{CO}(k) = \text{the cospectrum of } x \text{ and } y$$

6.3.1 Complex Exponential Expansion:

Very often it is more convenient to use the complex exponential representation of a Fourier series, rather than the real trigonometric form given above. We can write,

$$x = \bar{x} + \sum_{k=-N/2}^{N/2} \left\{ F_x(k) \right\} \tag{6.74}$$

$$y = \bar{y} + \sum_{k=-N/2}^{N/2} \left\{ F_y(k) \right\}$$

where

$$\begin{aligned} F_x(k) &= C_{xk} e^{i\left(\frac{2\pi kt}{T}\right)} e^{i\theta_{xk}} = \frac{1}{2}\left(A_{xk} - iB_{xk}\right) e^{i\left(\frac{2\pi kt}{T}\right)} \\ F_y(k) &= C_{yk} e^{i\left(\frac{2\pi kt}{T}\right)} e^{i\theta_{xk}} = \frac{1}{2}\left(A_{yk} - iB_{yk}\right) e^{i\left(\frac{2\pi kt}{T}\right)} \end{aligned} \tag{6.75}$$

A, *B*, and *C* are all real quantities.

In terms of these complex *F*'s then we can write the variance (power) of *x* as:

$$\begin{aligned} \overline{x'^2} &= \sum_{k=N/2}^{N/2} F_{xx}(k); \quad F_{xx}(k) = 2F_x(k) \cdot F_x^*(k) = C_{xk}^2 \\ \overline{y'^2} &= \sum_{k=N/2}^{N/2} F_{yy}(k); \quad F_{yy}(k) = 2F_y(k) \cdot F_y^*(k) = C_{yk}^2 \end{aligned} \tag{6.76}$$

where the superscript asterisk indicates the complex conjugate, and the covariance as:

$$\overline{x'\,y'} = \text{Real}\left\{ \sum_{k=-N/2}^{N/2} F_{xy}(k) \right\}; F_{xy}k = F_x(k) \cdot F_y^*(k) = C_{xk} C_{yk} e^{i\left(\theta_{xk} - \theta_{yk}\right)} \tag{6.78}$$

6.3.2 Complex Cross-Spectrum

We can write $F_{xy}(k)$ as

$$F_x(k)F_y^*(k) = \frac{1}{2}\left(A_{xk} - iB_{xk}\right)e^{i\left(\frac{2\pi kt}{T}\right)}\frac{1}{2}\left(A_{yk} - iB_{yk}\right)e^{-i\left(\frac{2\pi kt}{T}\right)} \tag{6.79}$$

$$= \frac{1}{4}\left\{A_{xk}A_{yk} + B_{xk}B_{yk} + i\left(A_{xk}B_{yk} - A_{yk}B_{xk}\right)\right\}$$

This is the complex cross spectrum between the two times series $x(t)$ and $y(t)$.

For real input series $x(t)$, $y(t)$

$$A_k = A_{-k} \quad \text{and} \quad B_k = -B_{-k} \text{ so that}$$

$$F_x(k)\, F_y^*(k) = F_x(-k)\, F_y^*(-k)$$

so that

$$F_{xy}(k) + F_{xy}(-k) = \frac{1}{2}\left\{A_{xk}A_{yk} + B_{xk}B_{yk} + i\left(A_{xk}B_{yk} - A_{yk}B_{xk}\right)\right\} \tag{6.80}$$

This is the cross-spectrum of x at y for wave k and its real part = cospectrum = $x'y'$
Its imaginary part = quadrature spectrum.

In complex notation the cross spectrum can be written

$$\begin{aligned} F_{xy}(k) &= C_{xk}C_{yk}\, e^{i\left(\theta_{xk} - \theta_{yk}\right)} \\ &= C_{xk}C_{yk}\left(\cos\left(\theta_{xk} - \theta_{yk}\right) + i\sin\left(\theta_{xk} - \theta_{yk}\right)\right) \\ \theta_{xk} &= \theta_{yk} \text{ real} \\ \theta_{\mathrm{xk}} &\neq \theta_{yk} \pm \frac{\pi}{2} \text{ complex} \end{aligned} \tag{6.81}$$

Thus the cospectrum (the real part) is the in-phase signal and the quadrature spectrum (complex) is the out-of-phase signal.

6.3.3 Spectral Coherence:

For a single line spectrum *k* we must have

$$\mathrm{Coh}^2 = \frac{\left|F_{xy}(k)\right|^2}{F_{xx}F_{yy}} = \frac{\left(C_{xk}C_{yk}\right)^2}{\left(C_{xk}\right)^2\left(C_{yk}\right)^2} = 1 \text{ show} \tag{6.82}$$

Let's consider what happens if we add two wavenumbers together to get power spectra, cospectra, quadrature spectra, and coherence-squared for the combined cross-spectral analysis. This can be accomplished by averaging adjacent wavenumbers within a single spectra, or by averaging the same wavenumber from separate realizations of the spectrum.

Suppose we have two spectral coefficients k=1 and k=2 for two time series x and y, as follows

$$F_x(k) = C_{xk}\exp\left\{i2\pi k\frac{t}{T}\right\}\exp\left(i\theta_{xk}\right); \quad k = 1,2 \tag{6.83a}$$

$$F_y(k) = C_{yk}\exp\left\{i2\pi k\frac{t}{T}\right\}\exp\left(i\theta_{yk}\right); \quad k = 1,2 \tag{6.83b}$$

If we average the power spectra for the two time series over the two wavenumbers, we obtain,

$$\overline{P}_x = C_{x1}^2 + C_{x2}^2; \quad \overline{P}_y = C_{y1}^2 + C_{y2}^2 \tag{6.84}$$

The averaged cross-spectrum is given by

$$\overline{F}_{xy} = C_{x1}C_{y1}\exp\left\{i\left(\theta_{x1} - \theta_{y1}\right)\right\} + C_{x2}C_{y2}\exp\left\{i\left(\theta_{x2} - \theta_{y2}\right)\right\} \tag{6.85}$$

Introduce the following shorthand notation to make manipulation less space consuming.

$$\begin{aligned} A_1 &= C_{x1}C_{y1} & A_2 &= C_{x2}C_{y2} \\ \Theta_1 &= \theta_{x1} - \theta_{y1} & \Theta_2 &= \theta_{x2} - \theta_{y2} \end{aligned} \tag{6.86}$$

So that we have,

$$\overline{F}_{xy} = A_1\exp\left\{i\Theta_1\right\} + A_2\exp\left\{i\Theta_2\right\} \tag{6.87}$$

Next compute the amplitude of the cross-spectrum.

$$\left|\overline{F}_{xy}\right|^2 = \overline{F}_{xy}\overline{F}_{xy}^* = \left(A_1 \exp\{i\Theta_1\} + A_2 \exp\{i\Theta_2\}\right)\left(A_1 \exp\{-i\Theta_1\} + A_2 \exp\{-i\Theta_2\}\right) \quad (6.88)$$

$$= A_1^2 + A_1A_2 \exp\{i(\Theta_1 - \Theta_2)\} + A_2A_1 \exp\{i(\Theta_2 - \Theta_1)\} + A_2^2$$

$$= A_1^2 + 2A_1A_2 \cos(\Theta_1 - \Theta_2) + A_2^2$$

Returning now to the original unshortened notation

$$\left|\overline{F}_{xy}\right|^2 = \overline{F}_{xy}\overline{F}_{xy}^* = C_{x1}^2C_{y1}^2 + 2C_{x1}C_{y1}C_{x2}C_{y2}\cos(\Theta_1 - \Theta_2) + C_{x2}^2C_{y2}^2 \quad (6.89)$$

The purpose of this exercise was to calculate the coherence-squared, for which the formula in the present context is,

$$Coh^2 = \frac{\left|\overline{F}_{xy}\right|^2}{\overline{P}_x \cdot \overline{P}_y} \quad (6.90)$$

We have the numerator; we now can write down the denominator

$$\overline{P}_x \cdot \overline{P}_y = C_{x1}^2C_{y1}^2 + C_{x1}^2C_{y2}^2 + C_{x2}^2C_{y1}^2 + C_{x2}^2C_{y2}^2 \quad (6.91)$$

And so putting the whole mess together by substituting (6.28) and (6.30) into (6.29), we have,

$$Coh^2 = \frac{C_{x1}^2C_{y1}^2 + 2C_{x1}C_{y1}C_{x2}C_{y2}\cos(\Theta_1 - \Theta_2) + C_{x2}^2C_{y2}^2}{C_{x1}^2C_{y1}^2 + C_{x1}^2C_{y2}^2 + C_{x2}^2C_{y1}^2 + C_{x2}^2C_{y2}^2} \quad (6.92)$$

First of all, notice that the coherence squared will be largest when the phase differences Q_1 and Θ_2 are equal. This means that the phase difference between *x* and *y* at the frequency in question is the same in the two realizations. If the two time series do not have the same phase relationship for the two wavenumbers we average together, then the coherence will decrease. If we have a real physical relationship, then we expect the phase relationship to remain stable as we average frequencies or realizations together. The coherence statistic tests for this.

Note, however, that this is not all that the coherence tests for. If we suppose that the phase differences are the same, so that the cosine term is unity, we still do not have a coherence of unity. The coherence also depends on the relationship of the amplitudes. The terms on the left and right in the numerator and denominator are identical. It is the central term on the top and the middle two terms on the bottom that are potentially different. If we suppose that the phase difference is constant so that the cosine is unity, then we can use the inequality below to show that the coherency cannot exceed unity.

$$C_{x1}^2C_{y2}^2 + C_{x2}^2C_{y1}^2 \geq 2C_{x1}C_{y1}C_{x2}C \quad (6.93)$$

Under what conditions could the equality hold in the above expression? If the equality holds, and the phase differences are the same, then the coherence is one, its maximum possible value. The equality would hold if all of the *C*'s were equal, of course, but it would also hold if the amplitude ratios between *x* and *y* were the same in the two realizations, as follows.

$$\frac{C_{x1}}{C_{y1}} = \frac{C_{x2}}{C_{y2}} \tag{6.94}$$

So the coherence will be unity when realizations are averaged, only under the conditions that the two realizations averaged together show the same phase difference between the two variables and the same amplitude ratio between the two variables. In short, the two variables are linearly related to each other. The coherence thus shows how well these two conditions are satisfied.

For a continuous cross spectrum:

$$\Phi_{xy}(k) = CO(k) + iQ(k) = \left|\Phi_{xy}\right| \exp\{i\theta_x\} \tag{6.95}$$

$$coh^2(k) = \frac{\left|\Phi_{xy}(k)\right|^2}{\Phi_{xx}(k)\Phi_{yy}(k)} = \frac{CO^2(k) + Q^2(k)}{\Phi_{xx}(k)\Phi_{yy}(k)} \tag{6.96}$$

$\Phi_{xy}(k)$ continuous cross power spectrum between *x* and *y*

$\Phi_{xx}(k)$ continuous cross power spectrum of *x*

$\Phi_{yy}(k)$ continuous cross power spectrum of *y*

• For two unrelated (linearly) time series *x* and *y* the coherency coh^2 (k) decreases rapidly with the number of degrees of freedom in the spectral estimate since the phases between the two time series are essentially random as a function of frequency. For two linearly related time series the phase difference and amplitude ratios will remain more constant and the coherency will drop off more slowly as we add realizations and thereby significance to our estimates of coherence.

• Note the similarity between the coherence and the correlation coefficient.

$$r^2 = \frac{\left(\overline{x'y'}\right)^2}{\overline{x'^2}\,\overline{y'^2}} \quad R^2(k) = coh^2(k) = \frac{\left|CO(k) + iQ(k)\right|^2}{\Phi_{xx} \cdot \Phi_{yy}} \tag{6.97}$$

The coherence, however, takes into account out-of-phase relationships and can examine the variance of two signals in a selected frequency range.

In the lag correlation method we compute the cross spectrum by performing the Fourier transform of the cross-correlation function

$$\Phi_{xy}(k) = CO(k) + iQ(k) = \int_{-\tau_L}^{\tau_L} r_{xy}(\tau) e^{-ik\tau} d\tau \tag{6.98}$$

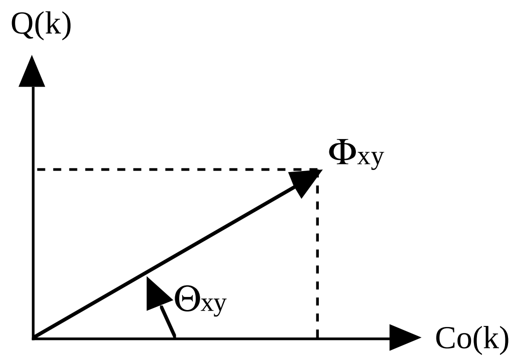
If the maximum lag L is less than $N/2$ then the cross-spectrum will be smoothed over and above the smoothing inherent in the data window and the number of degrees of freedom can be increased accordingly.

Since $r_{xy}(\tau)$ is not necessarily symmetric (there may be a phase lag between the time series), Φ_{xy} will have real, CO_{xy}, and imaginary, Q_{xy}, parts.

The coherence-squared has a probability distribution as shown in the table on the page 108. We can test whether two time series have any relation to each other by testing whether we can reject a null hypothesis that there is no relation between the two time series.

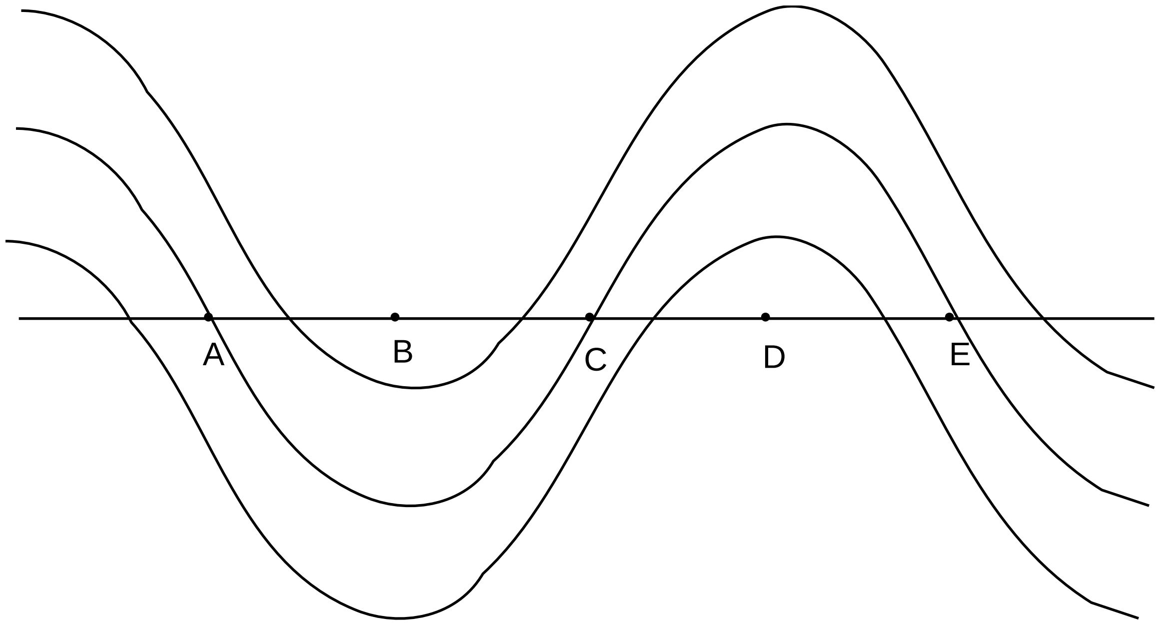
This table gives the upper limits on the coherence squared for a pair of random variables and is intended for *a priori* usage.

An illustrative diagram:

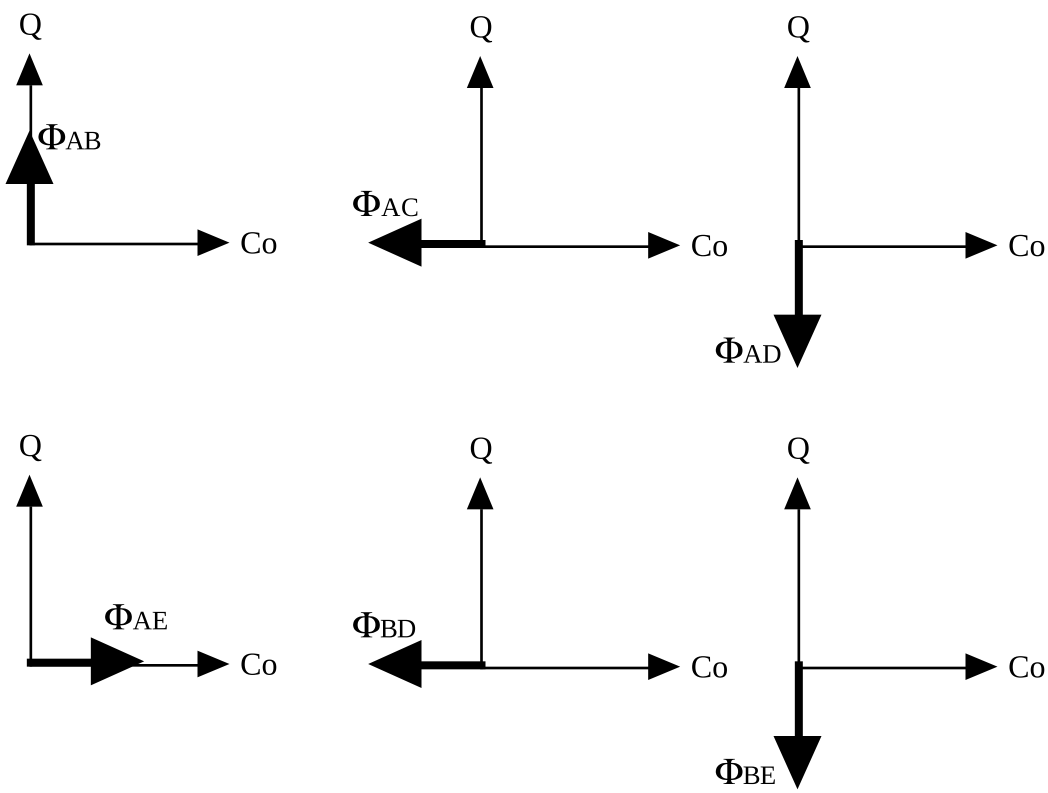


Recall that the phase has meaning only if the coh^2 is significant and that the phase error increases as both the coh^2 and d.o.f. decrease.

Consider the following: Suppose we have a wave moving westward past a line of stations. How will the cross-spectrum vary as a function of station spacing?



We can draw a diagram of the complex cospectra evaluated for each pair of points.



Probability Points of Distribution of Spectral Coherence
squared coherence R^2

	Probability point in per cent				
n	**50**	**90**	**95**	**99**	**99.9**
2	.500	.901	.951	.990	.998
3	.293	.684	.776	.901	.968
4	.206	.539	.632	.785	.901
5	.159	.437	.527	.684	.823
6	.130	.370	.450	.602	.748
7	.109	.319	.393	.536	.684
8	.094	.280	.348	.482	.627
9	.083	.250	.312	.438	.578
10	.074	.226	.283	.401	.536
11	.067	.206	.259	.370	.500
12	.061	.189	.238	.342	.466
13	.056	.175	.221	.319	.441
14	.052	.162	.206	.298	.412
15	.048	.151	.193	.280	.389
16	.045	.142	.181	.264	.370
17	.042	.134	.171	.250	.350
18	.040	.127	.162	.237	.334
19	.038	.120	.154	.226	.319
20	.036	.112	.146	.215	.305
25	.029	.091	.118	.175	.250
30	.024	.076	.098	.147	.212
35	.020	.066	.084	.127	.185
40	.018	.057	.074	.112	.162
45	.016	.051	.066	.100	.145
50	.014	.046	.060	.090	.132
60	.012	.038	.050	.075	.111
70	.010	.033	.042	.065	.096
80	.009	.029	.037	.057	.084
90	.008	.026	.033	.052	.075
100	.007	.023	.030	.045	.068
125	.006	.018	.024	.036	.054
150	.005	.015	.020	.031	.045
175	.004	.013	.017	.026	.039
200	.003	.011	.015	.023	.034

6.4 Mixed Space-Time Spectral Analysis

Mixed space-time spectral analysis is a straightforward extension of harmonic analysis to two dimensions. It is most convenient if the spatial dimension is cyclically continuous, such as in the case of latitude circles, or at least that the spatial dimension has fixed boundaries, like an ocean basin. In such cases we can look for modes of variability in which spatial scales have particular temporal scales. If the behavior is indeed harmonic (wavelike), then we expect mixed space-time spectral analysis to isolate any such modes that are present. For example, if one did mixed space-time spectral analysis of a stringed instrument, one would definitely expect to find a definite relationship between the length scales and the time scales of the oscillations.

Suppose we have a function of longitude, λ, and time, t. We can write:

$$x(\lambda,t) = \sum_{k} \sum_{\pm\omega} W_{k,\pm\omega} \cos\left(k\lambda \pm \omega t + \Phi_{k,\pm\omega}\right) \tag{6.99}$$

where + and – correspond to westward- and eastward-moving waves, respectively.

If we have such an expansion then we can write

Power Spectrum

$$P_{k,\pm\omega}(x) = \frac{1}{2} W_{k,\pm\omega}^2 \tag{6.100}$$

If we have two time series $x(\lambda, t)$ and $x^*(\lambda, t)$ we can write the cospectra between x and x^* as (note that for the next few lines we have dispensed with the convention used heretofore that a starred quantity is a complex conjugate)

$$K_{k_1\pm\omega}\left(x, x^*\right) = \frac{1}{2} W_{k_1\pm\omega} W_{k_1\pm\omega}^* \cos\left(\Phi_{k_1\pm\omega}^* - \Phi_{k_1\pm\omega}\right) \tag{6.101}$$

and the quadrature spectrum as

$$Q_{k,\pm\omega}\left(x, x^*\right) = \frac{1}{2} W_{k,\pm\omega} W_{k,\pm\omega}^* \sin\left(\Phi_{k,\pm\omega}^* - \Phi_{k,\pm\omega}\right) \tag{6.102}$$

So that the coherency is written

$$Coh_{k,\pm\omega}^2\left(x, x^*\right) = \frac{K_{k,\pm\omega}^2\left(x, x^*\right) + Q_{k,\pm\omega}^2\left(x, x^*\right)}{P_{k\pm\omega}(x) \cdot P_{k\pm\omega}\left(x^*\right)} \tag{6.103}$$

How do we obtain the expansion (6.99) in practice?

1.) First perform a zonal Fourier Transform to obtain sine and cosine coefficients at each time.

$$x(\lambda,t)=\sum_k C_k(t)\cos k\lambda + S_k(t)\sin k\lambda \tag{6.104}$$

Then Fourier transfer these sine and cosine coefficients in time.

$$C_k(t)=\sum_\omega A_{k,\omega}\cos\omega t + B_{k,\omega}\sin\omega t \tag{6.105}$$

$$S_k(t)=\sum_\omega a_{k,\omega}\cos\omega t + b_{k,\omega}\sin\omega t \tag{6.106}$$

These *A*, *B*, *a*, *b* can re-related to

$$W_{k_1-w}$$

through a simple manipulation

$$4W_{k,\pm\omega}^2=(A\mp b)^2+(\mp B-a)^2 \tag{6.107}$$

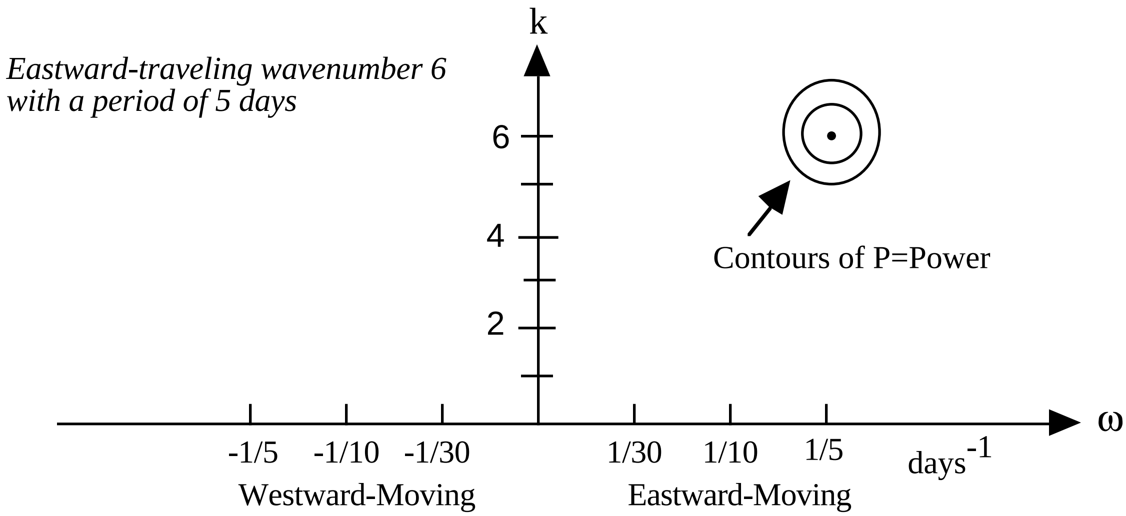
$$\phi_{k,\pm\omega}=\tan^{-1}\left\{\frac{(\mp B-a)}{(A\mp b)}\right\} \tag{6.108}$$

This is all straightforward mathematics. The important question is how to interpret the spectra obtained.

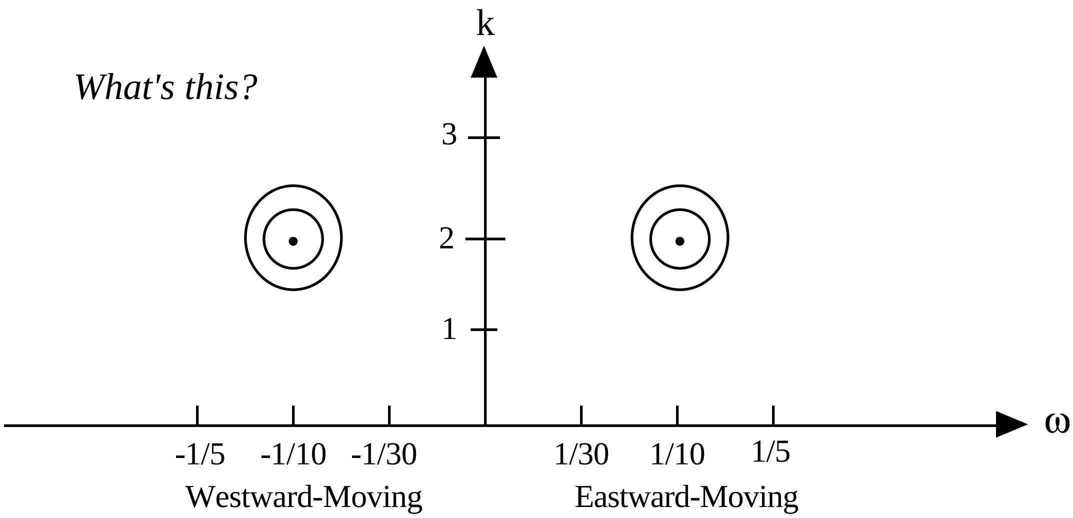
A useful plot of the power spectrum

$$P_{k,\pm\omega}(x)$$

can be made by contouring it in wavenumber-frequency coordinates. Although we have drawn smooth contours across wavenumber, for zonal wavenumber analysis only discrete wavenumbers exist. If the variation of amplitude is rather smooth across wavenumber, this won’t be a disaster, but it should be kept in mind that only integral wavenumbers exist.



This contour plot is very simple to interpret. It is an eastward-moving wave 6 with a period of 5 days. But how do you interpret the spectrum contoured below?



The plot above could be either of two possibilities.

a.) Eastward- and westward-moving waves, each with a 10-day period.
b.) A stationary wave 2 with amplitude oscillating with a period of 10 days.

How do you distinguish these two possibilities? One way to approach this problem is to ask if the eastward and westward waves are related, are they coherent with each other, do they bear a constant phase relationship to each other (standing wave), or are the eastward and westward waves linearly independent? Two somewhat different approaches to this question have been presented. One way to judge this is to formulate a coherence-squared between the eastward and westward waves Pratt (1976), and Hayashi (1977), *JAM*, **16**, 368-73. Another method is to look at the coherence in time between the sine and cosine coefficients of a particular wavenumber. Schafer, (1979), *JAS*, **36**, 1117-1123 uses the coherence in time of the sine and cosine coefficients to ask whether what is seen are "waves" or "noise".