

# The Gent-McWilliams Theory of Eddy Fluxes Along Isentropic Surfaces

David A. Randall

*Department of Atmospheric Science  
Colorado State University, Fort Collins, Colorado 80523*

---



---

The continuity equation in isentropic coordinates is

$$\left(\frac{\partial}{\partial t}\right)_\theta m + \nabla_\theta \bullet (m\mathbf{V}) + \frac{\partial}{\partial \theta}(m\dot{\theta}) = 0, \quad (12.1)$$

where

$$m \equiv -\frac{\partial p}{\partial \theta} \quad (12.2)$$

is the pseudo-density. When divided by  $g$ ,  $m$  gives the mass per unit vertical increment of  $\theta$ . In the absence of heating, (12.1) reduces to

$$\left(\frac{\partial}{\partial t}\right)_\theta m + \nabla_\theta \bullet (m\mathbf{V}) = 0. \quad (12.3)$$

Introducing the isentropic (two-dimensional) Lagrangian derivative,

$$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t}\right)_\theta + \mathbf{V} \bullet \nabla_\theta \quad (12.4)$$

we can rewrite (12.1) as

$$\frac{Dm}{Dt} = -m \nabla_\theta \bullet \mathbf{V} - \frac{\partial}{\partial \theta}(m\dot{\theta}). \quad (12.5)$$

Conservation of an arbitrary tracer of mixing ratio  $\tau$  is expressed by

$$\left(\frac{\partial}{\partial t}\right)_\theta (m\tau) + \nabla_\theta \bullet (m\mathbf{V}\tau) + \frac{\partial}{\partial \theta}(m\dot{\theta}\tau) = mR(\tau), \quad (12.6)$$

where

$$mR(\tau) \equiv \nabla_{\theta} \bullet (\mu m \nabla_{\theta} \tau) \quad (12.7)$$

represents isentropic mixing with a mixing coefficient  $\mu$ . Note that

$$R(\theta) \equiv 0, \quad (12.8)$$

because  $\nabla_{\theta} \theta \equiv 0$ .

In the absence of heating, (12.6) reduces to

$$\left( \frac{\partial}{\partial t} \right)_{\theta} (m\tau) + \nabla_{\theta} \bullet (m \mathbf{V} \tau) = \nabla_{\theta} \bullet (\mu m \nabla_{\theta} \tau). \quad (12.9)$$

Combining (12.1) with (12.6) and using (12.4) gives the advective form

$$m \frac{D\tau}{Dt} + m \dot{\theta} \frac{\partial \tau}{\partial \theta} = mR(\tau), \quad (12.10)$$

which reduces, in the absence of heating, to

$$m \frac{D\tau}{Dt} = mR(\tau). \quad (12.11)$$

Gent and McWilliams (1990; hereafter GM90) observe that (12.3) and (12.11) ensure that, in the absence of heating, the following three conditions are satisfied:

- A All domain-averaged moments of  $\theta$  are conserved, and the mass between any two isentropic surfaces is conserved.
- B With insulating boundary conditions, the domain-average of  $\tau$  is conserved between any two isentropic surfaces, and higher moments of  $\tau$  decrease in time if  $\tau$  has gradients on the isentropic surfaces.
- C In view of (12.8), Eq. (12.11) reduces to  $\frac{D\theta}{Dt} = 0$  when we replace  $\tau$  by  $\theta$ .

The dependent variables introduced so far consist of  $m$ ,  $\tau$ ,  $\mathbf{V}$ ,  $\dot{\theta}$ , and  $R(\tau)$ . Suppose that each of these quantities is decomposed into a mean (perhaps an ensemble mean), denoted by an overbar, and a departure from the mean, denoted by a prime, i.e.

$$(\quad) = (\overline{\quad}) + (\quad'). \quad (12.12)$$

We want to understand how  $\bar{\tau}$  evolves with time. We can approach this through the usual Reynolds averaging procedure. If we were working in height coordinates, so that  $m$  denoted the ordinary density, the effects arising from fluctuations of  $m$  would be neglected (except for buoyancy terms). The reason is that, on height surfaces, the fractional changes in the density are tiny compared to the mean density. In  $\theta$ -coordinates, however, the fractional changes of  $m$  can be quite comparable to the mean, and as a result fluctuations of  $m$  can be quite important. GM90 wrote

$$\nabla_{\theta} \bullet (\overline{m\mathbf{V}}) = \nabla_{\theta} \bullet (\bar{m}\bar{\mathbf{V}}) + \nabla_{\theta} \bullet (\overline{m'\mathbf{V}'}). \quad (12.13)$$

According to (12.13), the mass transport by the mean flow is supplemented by a kind of “fluid-dynamical peristalsis,” in which velocity fluctuations are correlated with variations in the pseudo-density, leading to a net eddy mass flux along (or “between”) isentropic surfaces. It is quite possible for  $\overline{m'\mathbf{V}'}$  to be comparable in magnitude to  $\bar{m}\bar{\mathbf{V}}$ . A parameterization will be needed to determine the eddy mass flux  $\overline{m'\mathbf{V}'}$ .

Also note that the velocity and the pseudodensity will in general be imperfectly correlated. Uncorrelated fluctuations of  $m$  and  $\mathbf{V}$  produce no net mass flux, but they can produce diffusive transports of intensive properties. In general, therefore, we expect joint fluctuations of  $m$  and  $\mathbf{V}$  to produce both a net mass flux and diffusion.

From (12.13), we see that the average of (12.3) is

$$\left(\frac{\partial}{\partial t}\right)_{\theta} \bar{m} + \nabla_{\theta} \bullet (\bar{m}\bar{\mathbf{V}} + \mathbf{F}) = 0, \quad (12.14)$$

where

$$\mathbf{F} \equiv \overline{m'\mathbf{V}'} . \quad (12.15)$$

Much of the following discussion deals with the question: *How does the eddy mass flux  $\mathbf{F}$  enter into the averaged tracer transport equation?* A comparison of (12.14) and (12.1) shows that the eddy mass flux along isentropic surfaces acts as a “pseudo-heating” in the averaged continuity equation. Adopting this point of view, we can define  $Q$  by

$$\nabla_{\theta} \bullet \mathbf{F} \equiv \frac{\partial}{\partial \theta} (\bar{m}Q), \quad (12.16)$$

so that  $Q$  corresponds to a pseudo- $\dot{\theta}$ . Integrating (12.16), we see that

$$\bar{m}Q = \int_{\theta} (\nabla_{\theta} \bullet \mathbf{F}) \, d\theta. \quad (12.17)$$

This means that  $mQ$  is an “apparent vertical mass flux” as seen in the equations for the averaged quantities. Use of (12.16) allows us to rewrite (12.14) as

$$\left(\frac{\partial}{\partial t}\right)_\theta \bar{m} + \nabla_\theta \bullet (\bar{m}\bar{\mathbf{V}}) + \frac{\partial}{\partial \theta}(\bar{m}Q) = 0. \quad (12.18)$$

Correspondingly, we redefine the Lagrangian time derivative as

$$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t}\right)_\theta + \bar{\mathbf{V}} \bullet \nabla_\theta + Q \frac{\partial}{\partial \theta}. \quad (12.19)$$

According to (12.19), there is a pseudo-vertical advection in the equations for the averaged quantities, even when the microscale flow is adiabatic. We also note that

$$\begin{aligned} \frac{\partial}{\partial \theta}(\bar{m}Q\bar{\tau}) &= \bar{\tau} \frac{\partial}{\partial \theta}(\bar{m}Q) + \bar{m}Q \frac{\partial}{\partial \theta} \bar{\tau} \\ &= \bar{\tau} \nabla_\theta \bullet \mathbf{F} + \bar{m}Q \frac{\partial}{\partial \theta} \bar{\tau}. \end{aligned} \quad (12.20)$$

Here we have used (12.16). Equation (12.20) is used below.

*GM90 assume that properties A, B, and C, listed above, apply to the averaged continuity and tracer equations.* They then analyze the implications of these assumptions, as follows:

Property A (mass conservation) can be ensured if

$$\mathbf{F} \bullet \mathbf{n} = 0 \text{ on all lateral boundaries,} \quad (12.21)$$

where  $\mathbf{n}$  is a horizontal vector normal to the lateral boundary, and if

$$Q = 0 \text{ on the top and bottom boundaries.} \quad (12.22)$$

To find the conditions necessary to ensure property B (conservation of  $\tau$  and decreasing higher moments of  $\tau$ ), GM90 rewrite (12.11) as

$$\bar{m} \frac{D\bar{\tau}}{Dt} = \bar{m}R(\bar{\tau}) + E(\bar{\tau}), \quad (12.23)$$

where  $E(\bar{\tau})$  is undefined at this point, and we interpret the Lagrangian derivative according to (12.19). Expanding (12.23), we obtain

$$\bar{m} \left[ \left(\frac{\partial}{\partial t}\right)_\theta + \bar{\mathbf{V}} \bullet \nabla_\theta + Q \frac{\partial}{\partial \theta} \right] \bar{\tau} = \bar{m}R(\bar{\tau}) + E(\bar{\tau}). \quad (12.24)$$

Combining this with (12.14), we find that

$$\left(\frac{\partial}{\partial t}\right)_{\theta}(\bar{m}\bar{\tau}) + \nabla_{\theta} \bullet (\bar{m}\bar{\mathbf{V}}\bar{\tau}) + \frac{\partial}{\partial \theta}(\bar{m}Q\bar{\tau}) = \bar{m}R(\bar{\tau}) + E(\bar{\tau}) \quad (12.25)$$

We can maintain property B by ensuring that

$$E(\bar{\tau}) = \frac{\partial}{\partial \theta}(\bar{m}Q\bar{\tau}) + \nabla_{\theta} \bullet \mathbf{G}(\bar{\tau}), \quad (12.26)$$

where  $\mathbf{G}(\bar{\tau})$  is an unknown vector. The form of Eq. (12.26) ensures that the integral of  $E(\bar{\tau})$  over the whole domain vanishes.

To satisfy property C (Lagrangian derivative of  $\theta$  is  $Q$ ), we see from (12.23) that

$$mQ = E(\theta) \quad (12.27)$$

is required. This motivates us to write out (12.26) for the case  $\tau = \theta$ ; the result is

$$\begin{aligned} E(\theta) &= \frac{\partial}{\partial \theta}(\bar{m}Q\theta) + \nabla_{\theta} \bullet \mathbf{G}(\theta) \\ &= \theta \frac{\partial}{\partial \theta}(\bar{m}Q) + \bar{m}Q + \nabla_{\theta} \bullet \mathbf{G}(\theta). \end{aligned} \quad (12.28)$$

With the use of (12.16), this can be rewritten as

$$E(\theta) = \theta \nabla_{\theta} \bullet \mathbf{F} + mQ + \nabla_{\theta} \bullet \mathbf{G}(\theta). \quad (12.29)$$

Now simply compare (12.27) and (12.29), to obtain

$$\nabla_{\theta} \bullet [\theta \mathbf{F} + \mathbf{G}(\theta)] = 0. \quad (12.30)$$

GM90 leap from (12.30) to

$$\nabla_{\theta} \bullet [\bar{\tau} \mathbf{F} + \mathbf{G}(\bar{\tau})] = 0. \quad (12.31)$$

This *assumption* allows us to rewrite (12.23) as

$$\begin{aligned}
\overline{m} \frac{D\bar{\tau}}{Dt} &= \overline{m} R(\bar{\tau}) + E(\bar{\tau}) \\
&= \overline{m} R(\bar{\tau}) + \frac{\partial}{\partial \theta} (\overline{m} Q \bar{\tau}) + \nabla_{\theta} \bullet \mathbf{G}(\bar{\tau}) \\
&= \overline{m} R(\bar{\tau}) + \frac{\partial}{\partial \theta} (\overline{m} Q \bar{\tau}) - \nabla_{\theta} \bullet (\bar{\tau} \mathbf{F}) \\
&= \overline{m} R(\bar{\tau}) + \bar{\tau} \nabla_{\theta} \bullet \mathbf{F} + \overline{m} Q \frac{\partial}{\partial \theta} \bar{\tau} - \nabla_{\theta} \bullet (\bar{\tau} \mathbf{F}) \\
&= \overline{m} R(\bar{\tau}) - \mathbf{F} \bullet \nabla_{\theta} \bar{\tau} + \overline{m} Q \frac{\partial}{\partial \theta} \bar{\tau} .
\end{aligned} \tag{12.32}$$

Here (12.20) has been used. Expanding  $\frac{D\bar{\tau}}{Dt}$  in (12.32) using (12.19), we see that (12.32) reduces to

$$\overline{m} \left[ \left( \frac{\partial}{\partial t} \right)_{\theta} + \left( \overline{\mathbf{V}} + \frac{\mathbf{F}}{m} \right) \bullet \nabla_{\theta} \right] \bar{\tau} = \overline{m} R(\bar{\tau}) . \tag{12.33}$$

This shows that the averaged scalar experiences a horizontal advection by the velocity  $\frac{\mathbf{F}}{m}$ , in addition to the advection by  $\overline{\mathbf{V}}$ . *The eddies advect the tracer.* This is an extremely simple and appealing result, which we might be tempted to guess at, given (12.14). Note, however, that the eddies can diffuse as well as advect. GM90 note that, as a result, the diffusion term on the right-hand side of (12.33) will incorporate a much larger mixing coefficient than would appear in the unaveraged equations.

In order to make use of (12.33), we have to determine  $\mathbf{F}$ . GM90 suggested that

$$\mathbf{F} = -\frac{\partial}{\partial \theta} (\kappa \nabla_{\theta} \bar{p}) , \tag{12.34}$$

where  $\kappa$  is a non-negative parameter. Substitution of (12.34) into (12.14) gives

$$\left( \frac{\partial}{\partial t} \right)_{\theta} \bar{m} + \nabla_{\theta} \bullet (\bar{m} \overline{\mathbf{V}}) = -\nabla_{\theta} \bullet \left[ \frac{\partial}{\partial \theta} (\kappa \nabla_{\theta} \bar{p}) \right] \tag{12.35}$$

In case  $\kappa$  is spatially constant, this reduces to

$$\left( \frac{\partial}{\partial t} \right)_{\theta} \bar{m} + \nabla_{\theta} \bullet (\bar{m} \overline{\mathbf{V}}) = \kappa \nabla_{\theta}^2 \bar{m} . \tag{12.36}$$

This shows that the effect of (12.34) is to drive  $m$  towards uniformity on  $\theta$ -surfaces, or in other words to drive  $p$  towards uniformity on  $\theta$ -surfaces. A state in which  $p$  is uniform along  $\theta$ -surfaces is one in which the available potential energy is zero (Lorenz, 1955). This means that the effect of (12.34) is to remove available potential energy from the system. An interpretation is that the available potential energy has been converted into subgrid-scale kinetic energy, through subgrid-scale baroclinic instability.

Comparison of (12.34) with (12.16) shows that

$$\frac{\partial}{\partial \theta}(\bar{m}Q) = -\nabla_{\theta} \bullet \left[ \frac{\partial}{\partial \theta}(\kappa \nabla_{\theta} \bar{p}) \right], \quad (12.37)$$

from which we can conclude that

$$mQ = -\nabla_{\theta} \bullet (\kappa \nabla_{\theta} p), \quad (12.38)$$

provided that  $Q$  is zero on the boundaries.

## References and Bibliography

- Basdevant, C., B. Legras, R. Sadourny, and Me Béland, 1981: A study of barotropic model flows: Intermittency, waves, and predictability. *J. Atmos. Sci.*, **38**, 2305-2326.
- Charney, J. G., 1971: Geostrophic turbulence. *J. Atmos. Sci.*, **28**, 1087-1095.
- Danabasoglu, G., J. C. McWilliams, and P. R. Gent, 1994: The role of mesoscale tracer transports in the global ocean circulation. *Science*, **264**, 1123-1126.
- Fjortoft, R., 1953: On the changes in the spectral distribution of kinetic energy for a two-dimensional non-divergent flow. *Tellus*, **5**, 225-230.
- Gent, P. R., and J. C. McWilliams, 1990: Isopycnal mixing in ocean circulation models. *J. Phys. Oceanogr.*, **20**, 150-155.
- Gent, P. R., and J. C. McWilliams, 1996: Eliassen-Palm fluxes and the momentum equation in non-eddy-resolving ocean circulation models. *J. Phys. Oceanogr.*, **26**, 2539-2546.
- Green, J. S. A., 1970: Transfer properties of the large-scale eddies and the general circulation of the atmosphere. *Quart. J. Roy. Meteor. Soc.*, **96**, 157-185.
- Hoskins, B. J., M. E. McIntyre, and A. W. Robertson, 1985: On the use and significance of isentropic potential vorticity maps. *Quart. J. Roy. Meteor. Soc.*, **111**, 877-946.

- Leith, C. E., 1968: Diffusion approximation for two-dimensional turbulence. *Phys. Fluids*, **11**, 671-673.
- Lilly, D. K., 1972: Numerical simulation studies of two-dimensional turbulence. *Geophys. Fluid Mech.*, **3**, 289-319; **4**, 1-28.
- Lilly, D. K., 1983: Stratified turbulence and the mesoscale variability of the atmosphere. *J. Atmos. Sci.*, **40**, 749-761.
- Lorenz, E. N., 1955: Available potential energy and the maintenance of the general circulation. *Tellus*, **7**, 157-167.
- McWilliams, J. C., 1984: The emergence of isolated coherent vortices in turbulent flow. *J. Fluid Mech.*, **146**, 21-43.
- McWilliams, J. C., J. B. Weiss, and I. Yavneh, 1994: Anisotropy and coherent vortex structures in planetary turbulence. *Science*, **264**, 410-413.
- Merilees, P. E., and T. Warn, 1972: The resolution implications of geostrophic turbulence. *J. Atmos. Sci.*, **29**, 990-991.
- Merilees, P. E., and T. Warn, 1975: On energy and enstrophy exchanges in two-dimensional non-divergent flow. *J. Fluid Dyn.*, **69**, 625-630.
- Rhines, P., 1975: Waves and turbulence on a  $\beta$ -plane. *J. Fluid Mech.*, **69**, 417-443.
- Sadourny, R., and C. Basdevant, 1985: Parameterization of subgrid scale barotropic and baroclinic eddies in quasi-geostrophic models: Anticipated potential vorticity method. *J. Atmos. Sci.*, **42**, 1353-1363.
- Saltzman, B., 1970: Large-scale atmospheric energetics in the wavenumber domain. *Rev. Geophys. Space Phys.*, **8**, 289-302.
- Sun, D.-Z., and R. S. Lindzen, 1994: A PV view of the zonal mean distribution of temperature and wind in the extratropical troposphere. *J. Atmos. Sci.*, **51**, 757-772.