## 3.1. Effectively 1D systems

In this chapter, we will discuss properties of *effectively 1D systems* whose position is completely described by just one generalized coordinate $q$ (plus constrains):

$$L = L(q, \dot{q}, t)\,. \tag{3.1}$$

For these systems, the Lagrangian approach gives us one ordinary, second-order differential equation

$$f(\ddot{q}, \dot{q}, q, t) = 0. \tag{3.2}$$

In many systems the Lagrangian may be presented in the following simple form:

$$L = T_{\text{ef}}(\dot{q}) - U_{\text{ef}}(q,t), \quad T_{\text{ef}} = \frac{m_{ef}}{2}\dot{q}^2, \tag{3.3}$$

so that the Lagrangian equation (2.17) is

$$m_{\text{ef}}\ddot{q} = -\frac{\partial U_{\text{ef}}(q,t)}{\partial q}. \tag{3.4}$$

Here $m_{\text{ef}}$ is some constant which may be considered an effective mass of the system, and the function $U_{\text{ef}}$ its effective potential energy.

Of course, Eqs. (3) and (4) are valid for a particle confined to a straight line (say, axis $x$), where Eq. (4) is just the $x$-component of the 2nd Newton law – cf. Eq. (2.20). In this particular case, $T_{\text{ef}} = T = m\dot{x}^2/2$, $U_{\text{ef}} = U(x,t)$, so that $m_{\text{ef}}$ is just the usual mass $m$. For other systems it is not necessarily so, and even the dimensionality of $m_{\text{ef}}$ may be different. For example, Eq. (2.21) shows that for the bead-on-the-rotating ring problem (Fig. 1.6), $m_{\text{ef}} = mR^2$. (The effective mass would be equal $m$ if we selected, as we might, $R\theta$ rather than $\theta$ for the generalized coordinate.)

More substantially, we have already seen that for that problem the genuine kinetic energy $T$, expressed by the first of Eqs. (2.21), is *not* a quadratically-homogeneous function of the generalized velocity. However, its Lagrangian still may be presented in form (3):

$$L = T - U = \frac{m}{2}R^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) - mg(1-\cos\theta) = T_{\text{ef}} - U_{\text{ef}}\,, \tag{3.5}$$

$$T_{\text{ef}} \equiv \frac{m}{2}R^2\dot{\theta}^2, \quad U_{\text{ef}} \equiv -\frac{m}{2}R^2\omega^2\sin^2\theta + mg(1-\cos\theta). \tag{3.6}$$

Note that with this new partitioning of $L$, $T_{\text{ef}}$ includes only a part of the full kinetic energy $T$ of the bead, while $U_{\text{ef}}$ includes not only the real potential energy $U$ in the gravity field, but also a contribution due to the effective centrifugal "force" – see Eq. (2.21). (We will revisit this issue in Chapter 5.)

## 3.2. Equilibrium and Stability

*Autonomous* systems, whose Lagrangian (and hence function $U_{\text{ef}}$) do not depend on time explicitly, at certain initial conditions may stay in equilibrium in one or several *stationary* (also called "fixed") points $q_n$. In such a point, the effective force expressed by the RHP of Eq. (4) should vanish,[1] so that the point corresponds to either the minimum or a maximum of the effective potential energy $U_{\text{ef}}(x)$:

$$\frac{dU_{\text{ef}}}{dq}(q_n) = 0. \tag{3.7}$$

In order to explore the stability of such equilibrium, we should consider possible small deviations $x \equiv q(t) - q_n$ of the coordinate from each fixed point. Now, we can expand function $U_{ef}$ in the Taylor series near the equilibrium:

$$U(q) = U(q_n) + \frac{dU}{dq}(q_n)\,x + \frac{1}{2}\frac{d^2U}{dq^2}(q_n)\,x^2 + \ldots \tag{3.8}$$

Since constant $U(q_n)$ is arbitrary and does not affect motion, the first derivative $dU/dq$ vanishes in any fixed point (see Eq. (7)), and the higher terms are small at $|x| \to 0$,[2] stability is determined by the sign of the second derivative in the fixed point, playing the role of the effective spring constant

$$k_{\text{ef}} \equiv \frac{d^2U}{dq^2}(q_n). \tag{3.9}$$

Indeed, neglecting the higher terms, Eq. (4) takes the familiar form

$$m_{\text{ef}}\ddot{x} + k_{\text{ef}}\,x = 0. \tag{3.10}$$

If $k_{\text{ef}} > 0$, i.e. the fixed point corresponds to the minimum of potential energy, this equation describes sinusoidal oscillations of the system,

$$x(t) = c_c \cos\omega_0 t + c_s \sin\omega_0 t, \quad \omega_0 = \sqrt{\frac{k_{\text{ef}}}{m_{\text{ef}}}}, \tag{3.11}$$

about the fixed point which is hereby stable. On the other hand, at the potential energy maximum ($k < 0$), the general solution to Eq. (8),

$$x(t) = c_+ \exp(+\lambda t) + c_- \exp(-\lambda t), \quad \lambda \equiv \sqrt{\frac{|k_{\text{ef}}|}{m_{\text{ef}}}}, \tag{3.12}$$

has an exponentially growing part and shows that the fixed point is unstable.

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[1] Later in this chapter we will consider an interesting example of non-autonomous "parametric" systems which also may have a fixed point.

[2] These terms are only important in the special case $k_{\text{ef}} = 0$.

The same result may be obtained by another way which may be more readily generalized to more complex systems (e.g., those with dissipation). Let us notice that the *quadratic* expansion of $U_{ef}(q)$ given by Eq. (8) is equivalent to a *linear* expansion of its derivative in the equation of motion (4):

$$-\frac{dU_{\text{ef}}}{dq}(q_n) = -k_{ef}x, \tag{3.13}$$

immediately resulting in the linear equation (10). Hence, in order to analyze stability of a fixed point $q_n$, it is sufficient to plug the definition of *x*,

$$q = q_n + x, \tag{3.14}$$

into the equation of motion, "*linearize*" this equation in *x* (meaning expanding it in series in *x* and neglecting all terms above the linear one) and study possible solutions to the resulting linear equation. Later in this course, we will repeatedly use this approach for more complex systems.

As a 1D example, consider again the bead-on-the-rotating-ring problem (Fig. 1.6). Using the new partitioning of its Lagrange function, expressed by Eqs. (5) and (6), we can readily obtain that $dU_{\text{ef}}/d\theta = 0$ in four fixed points:

$$\theta_1 = 0, \qquad \theta_2 = \pi, \qquad \theta_{3,4} = \pm\arccos(\Omega^2/\omega^2), \tag{3.15}$$

where $\Omega$ is the frequency of small oscillations of the system at $\omega = 0$ - see Eq. (2.24). We see that the last two fixed points, corresponding to bead rotating on either side of the ring, exist only if the angular speed $\omega$ of the ring rotation exceeds $\Omega$. (If $\omega >> \Omega$, $\theta_{3,4} \to \pm\pi/2$, in accordance with common sense.)

Now, stability. Expanding nonlinear terms of Eq. (2.23) in Taylor series in small deviations $x \equiv \theta - \theta_n$ from each of the fixed points, and keeping only the terms proportional to $x$, we get linear differential equations similar to Eq. (10):

$$\begin{aligned} &\ddot{x} + \left(\Omega^2 - \omega^2\right)x = 0, \quad \text{for } \theta \approx \theta_1, \\ &\ddot{x} - \left(\Omega^2 + \omega^2\right)x = 0, \quad \text{for } \theta \approx \theta_2, \\ &\ddot{x} + \left(\omega^2 - \frac{\Omega^4}{\omega^2}\right)x = 0, \quad \text{for } \theta \approx \theta_{3,4}. \end{aligned} \tag{3.16}$$

This result shows that the first fixed point (corresponding to the bead position at the bottom of the ring) is stable if $\omega < \Omega$, and unstable beyond this threshold, i.e. as soon as fixed points $\theta_3$ and $\theta_4$ exist. The second fixed point (bead on the top of the ring) is always unstable. Finally, the third and fourth fixed points are always stable when they exist (at $\omega > \Omega$). This fixed-point analysis may be summarized in a simple way: an increase of the ring rotation speed $\omega$ beyond the threshold value $\Omega$ causes the bead to move on one of the ring sides.

 

## 3.3. Hamiltonian systems: Types of motion

Autonomous systems with $U_{\text{ef}} = U_{\text{ef}}(q)$, described by Lagrangian (3), are also called *Hamiltonian systems,* because their Hamiltonian function $H$ (again, not necessarily the mechanical energy $E$!) is conserved:

$$H = \frac{m_{\text{ef}}}{2}\dot{q}^2 + U_{\text{ef}}(q) = \text{const}, \tag{3.17}$$

This equation gives us the first integral of motion. Solving it for $\dot{q}$, we get the first-order differential equation

$$\frac{dq}{dt} = \pm\left\{\frac{2}{m_{\text{ef}}}[H - U_{\text{ef}}(q)]\right\}^{1/2} \tag{3.18}$$

which may be further integrated for arbitrary $U_{\text{ef}}(q)$:

$$\pm\left(\frac{m_{\text{ef}}}{2}\right)^{1/2}\int\frac{dq}{[H - U_{\text{ef}}(q)]^{1/2}} = t - t_0, \tag{3.19}$$

This result gives the reciprocal form, $t(q)$, of the desired law of the motion $q(t)$. (Two constants, $H$ and $t_0$, participating in this result, as well as the proper choice of the sign before the integral, are determined by initial conditions.) Of course, for that the integral has to be first taken, either analytically or numerically, but even the latter procedure is much easier than the numerical integration of the initial, second-order differential equation of motion.

Equation (14) also allows a general classification of 1D system motion:

(a) If $H > U_{\text{ef}}(x)$ in the whole range of interest, the effective kinetic energy $T_{\text{ef}} = (m_{\text{ef}}/2)\dot{q}^2$ is always positive. Hence the motion velocity $\dot{q}$ cannot change sign, and retains the sign it had initially. This is the unbound motion in one direction (Fig. 1a).

(b) Now let the particle approach a "*classical turning point*" A where $H = U_{\text{ef}}$ (Fig. 1b).[3] According to Eq. (17), (18), in that point the particle velocity vanishes. This corresponds to the particle reflection from the "potential wall", with the change of velocity sign.

(c) If, after the reflection from point A, the particle runs into another classical turning point B (Fig. 1c), the reflection process is repeated again and again, so that the particle is bound to a periodic motion between two turning points.

[3] This terminology comes from quantum mechanics which shows that actually a particle (or rather its wavefunction) can to some extent penetrate the "classically forbidden range" where $H < U_{\text{ef}}(x)$.

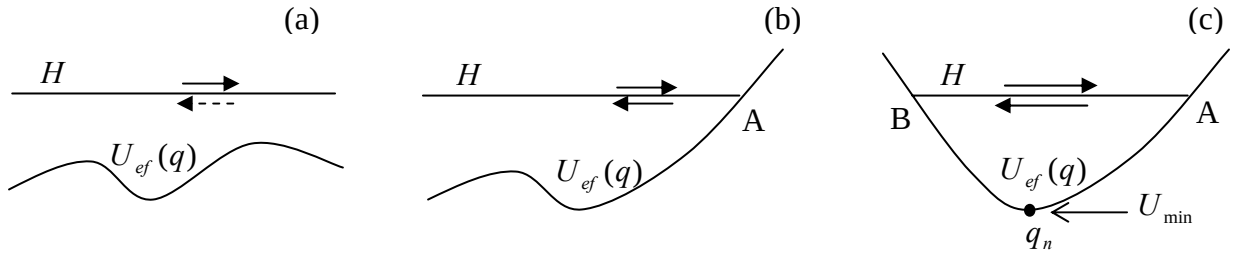


Fig. 3.1. The energy diagram for three different cases of 1D motion: (a) unbounded motion, with the initial velocity sign conserved, (b) reflection from the "classical tuning point", accompanied with the velocity sign change, and (c) bound, periodic motion between two classical turning points.

The last case of *periodic oscillations* presents large practical interest, and the balance of this chapter will be devoted to a detailed analysis of this phenomenon and associated effects. First, let us notice that Eq. (19) immediately enables us to calculate the oscillation period:

$$T = (2m)^{1/2} \int_B^A \frac{dq}{[H - U_{\text{ef}}(q)]^{1/2}}, \tag{3.20}$$

where the additional factor 2 accounts for two time intervals: for the path from B to A and back (Fig. 1c). Since, according to Eq. (17), in each classically allowed point $q$ the velocity magnitude is the same, these intervals are equal to each other.

Equation (20) may be readily linked to the fixed point analysis carried out in the previous section. Indeed, Fig. 1c shows that the oscillations described by Eq. (20) may be considered as taking place in a "potential well" whose bottom, $U_{\text{ef}}(q_n) = U_{\min}$, corresponds to a stable fixed point $q_n$. Hence, if the oscillation amplitude is small (which happens if $H$ is only slightly above $U_{\min}$) and the potential energy profile smooth, we can use the Taylor expansion (6). Plugging it into Eq. (20), and using the symmetry of the problem $U(q_n + x) = U(q_n - x)$, we get

$$T = 2(2m_{\text{ef}})^{1/2} \int_0^a \frac{dx}{[H - k_{\text{ef}} x^2/2]^{1/2}} = 2\left(\frac{2m_{\text{ef}}}{H}\right)^{1/2} a \int_0^1 \frac{d\xi}{\sqrt{1-\xi^2}}, \tag{3.21}$$

where $q_n \pm a$ are the classical turning points: $k_{\text{ef}} a^2/2 = H$, i.e. $a = \sqrt{2H/k_{\text{ef}}}$. Plugging the last expression into Eq. (21), and taking into account that the table integral[4] in that equation equals $\pi/2$, we finally get the formula

$$T = \frac{2\pi}{\omega_0}, \tag{3.22}$$

[4] Actually, the integral is easy: taking $\xi = \sin\zeta$, we have $d\xi = \cos\zeta\, d\zeta = \sqrt{1-\xi^2}\, d\zeta$, so that the integral turns into just $\int_0^{\pi/2} d\zeta = \frac{\pi}{2}$.

 

where $\omega_0$ is given by Eq. (11), as it should be.

In order to conclude our analysis of free oscillations in 1D quadratic potential (the so-called *harmonic oscillator*), let us emphasize again that their frequency $\omega_0$ does not depend on their amplitude, i.e. on the difference $H - U_{\min}$ (while it is small), and that constants $c_c$ and $c_s$ participating in Eq. (11) (and hence the amplitude and phase of the free oscillations) are determined by initial conditions. Now let us study other types of oscillations in this important system.[5]

## 3.4. Forced oscillations

Additional (external) force $F(t)$ may induce *forced* oscillations in a harmonic oscillator. This process is described by a linear but *inhomogeneous* differential equation

$$m_{\text{ef}}\ddot{x} + k_{\text{ef}}x = F(t). \tag{3.23}$$

For a particle of mass $m$, confined to a straight line, such equation (with $m_{\text{ef}} = m$ and $F(t) = F_x^{\text{ext}}(t)$) is just an expression of the 2nd Newton law. In the more general case, Eq. (23) may be obtained by modifying the potential energy of the system to describe a generalized external force:

$$U(q) \to U(q,t) = U(q) - F(t)q = U(q) - F(t)x + \text{const}. \tag{3.24}$$

Such potential energy is frequently called the *Gibbs potential*, because it is very close to the notion of the Gibbs energy in thermodynamics.

Linear differential equation (23) may be solved by two different methods whose relative convenience depends on the character of function $F(t)$.

1. Frequency domain. Let us present function $F(t)$ as a Fourier sum of sinusoidal harmonics:[6]

$$F(t) = \text{Re}\left[\sum_{\omega} F_{\omega} e^{-i\omega t}\right]. \tag{3.25}$$

Then, due to linearity of Eq. (23), its general solution may be presented as a sum of the free oscillations with the "own" frequency $\omega_0$, described by Eq. (11), and forced oscillations due to each of the Fourier components with frequencies $\omega \neq \omega_0$. (In physics, this mathematical fact is frequently called the *linear superposition principle*.) In order to find the forced oscillations due to one Fourier component of the force, let us keep only this component in Eq. (23):

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[5] The results described in the balance of this chapter are applicable not only to mechanical oscillations of single particles, but to a broad range of oscillators, notably including electromagnetic and optical systems. This is why it belongs not only to mechanics, but also to broader field of *classical dynamics*.

[6] Using complex functions substantially simplifies formulas which otherwise have a mix of sin's and cos's. Note, however, that this convenience is mostly limited to linear systems.

 

$$\ddot{x} + \omega_0^2 x = \mathrm{Re}\left[\frac{F_\omega}{m_{\mathrm{ef}}} e^{-i\omega t}\right]. \tag{3.26}$$

Let us look for the solution to this equation in the natural form

$$x(t) = \mathrm{Re}\left[a e^{-i(\omega t - \varphi)}\right] = \mathrm{Re}\left[A_\omega e^{-i\omega t}\right], \tag{3.27}$$

where $A_\omega \equiv a\exp(i\varphi)$ is the *complex amplitude* of the oscillations, which carries information about their both (real) amplitude and phase. Plugging Eq. (27) into Eq. (26), we readily get the well-known expression

$$A_\omega = \frac{F_\omega}{m_{\mathrm{ef}}} \frac{1}{\left(\omega_0^2 - \omega^2\right)} \tag{3.28}$$

which describes in particular a sharp increase of the oscillation amplitude $a_\omega = \left|A_\omega\right|$ (the *resonance*) at $\omega \to \omega_0$ - see the black line in Fig. 2.

Before moving further, let us note just one fact whose physical significance is not too high, but which is very useful for the mathematical analysis of more complex cases (see Sec. 3.5 below): solution (27) is incorrect at the exact resonance ($\omega = \omega_0$) where Eq. (26) cannot be only satisfied with a "*secular*" term:

$$x_{\mathrm{sec}}(t) = \mathrm{Re}\left[\frac{F_\omega}{2i\omega m_{\mathrm{ef}}} t e^{-i\omega t}\right], \quad \text{for } \omega = \omega_0. \tag{3.29}$$

which may be interpreted as harmonic oscillations with linearly (e.g., indefinitely) growing amplitude.

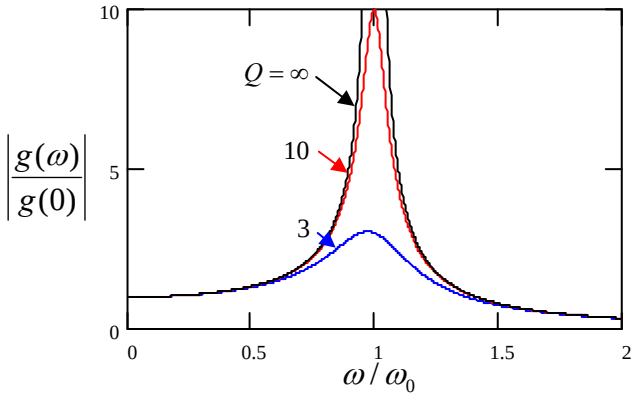


Fig. 3.2. Linear resonance in a harmonic oscillator for several values of the $Q$-factor.

The linear resonance analysis may be readily generalized to the case of an oscillator with linear *damping* (dissipation), described by equation[7]

[7] Since the added term describes energy dissipation, such equation may be obtained from the general Lagrange equation (2.14) with a non-conservative force $Q_j \propto q_j$, but not from the canonical equation (2.17). In classical mechanics the introduction of such linear dissipative force is phenomenological, but in statistical mechanics it may be derived, under rather broad assumptions, as an *average* force exerted on a

$$\ddot{x} + 2\delta\dot{x} + \omega_0 x = \frac{F(t)}{m_{\text{ef}}}. \tag{3.30}$$

Indeed, plugging Eq. (27) into Eq. (28), we get

$$A_\omega = \frac{F_\omega}{m_{\text{ef}}} g(\omega), \qquad g(\omega) = \frac{1}{\left(\omega_0^2 - \omega^2\right) - 2i\omega\delta}, \tag{3.31}$$

so that[8]

$$a_\omega = |A_\omega| = \frac{|F_\omega|}{m_{\text{ef}}} |g(\omega)|, \qquad |g(\omega)| = \frac{1}{\left[(\omega_0^2 - \omega^2)^2 + (2\delta\omega)^2\right]^{1/2}}. \tag{3.32}$$

This result shows that dissipation smoothes the resonance peak – see Fig. 2 which is plotted for three values of the so-called *Q-factor*

$$Q \equiv \frac{\omega_0}{2\delta}. \tag{3.33}$$

The dimensionless factor $Q$ allows for several interpretations.

(a) As an elementary analysis of Eq. (32) shows, $Q$ gives the ratio of the oscillator response magnitude $|g(\omega)|$ at $\omega = \omega_0$ and $\omega = 0$ - see Fig. 2.

(b) For the most interesting case of weak damping ($Q >> 1$), its reciprocal value, $Q^{-1}$, gives the normalized value of the so-called *Full Width on Half Maximum* (FWHM) of the resonance curve:

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q}. \tag{3.34}$$

Indeed, $\Delta\omega$ is defined as the distance $\omega_2 - \omega_1$ between the two values of $\omega$ at which the square of oscillator response, $|g(\omega)|^2$ is a half of its resonance value. In the weak damping limit, both these points are very close to $\omega_0$, so that with good accuracy $(\omega_0^2 - \omega^2) \approx -2\omega\xi$, where

$$\xi \equiv \omega - \omega_0 \tag{3.35}$$

is a very convenient parameter called *detuning*. With this simplification, Eq. (32) yields

$$|g(\omega)|^2 = \frac{1}{4\omega^2\left(\delta^2 + \xi^2\right)}, \tag{3.36}$$

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body by an environment whose numerous parts (degrees of freedom) are in random states. Since such force also has a random component, dissipation is fundamentally related to fluctuations, and the latter effects may be neglected (as they are in this course) only if the oscillation energy is much higher than the energy scale of random fluctuations of the environment (in the thermal equilibrium, $k_B T$).

[8] Note that as soon as $\delta \neq 0$, the special character of point $\omega = \omega_0$ disappears.

 

so that points $\omega_{1,2}$ correspond to $\xi^2 = \delta^2$, i.e. $\omega_{1,2} = \omega_0 \mp \delta = \omega_0(1 \mp 1/2Q)$, which means that $\omega_2 - \omega_1 = \omega_0 / Q$, thus proving Eq. (34).

(c) Finally, one more convenient definition of $Q$ comes from the decay of free oscillations. Let us find the law of that decay from Eq. (30) with $F = 0$. The general theory of linear, ordinary differential equations tells us that the general solution of a homogeneous second-order equation has the form

$$x_{\text{free}}(t) = \sum_{k=1,2} c_k e^{\lambda_k t}, \tag{3.37}$$

where constants $c_k$ are determined by initial conditions. Since the conditions may be always selected so that one of these two coefficients vanish, each of the two terms has to satisfy our differential equation of motion independently. Plugging such term into Eq. (28), we see that each of exponents $\lambda$ has to satisfy the following "characteristic" equation:

$$\lambda^2 + 2\delta\lambda + \omega_0^2 = 0. \tag{3.38}$$

Solving it, and plugging the result into Eq. (37), for not very high damping ($\delta < \omega_0$), i.e. we get

$$x_{\text{free}}(t) = e^{-\delta t}\left(c_c \cos \omega_0' t + c_s \sin \omega_0' t\right), \quad \omega_0' \equiv \sqrt{\omega_0^2 - \delta^2}. \tag{3.39}$$

The result shows that, besides some re-normalization of the free oscillation frequency, damping leads to an exponential decay of oscillation amplitude with time constant $1/\delta$. For the oscillation energy, which scales as amplitude squared, the time constant is twice shorter:

$$\tau = \frac{1}{2\delta} = \frac{Q}{\omega_0}. \tag{3.40}$$

This equation gives one more interpretation, and simultaneously one more effective means for experimental measurement of the $Q$-factor.[9]

2. Time domain. Returning to the forced oscillations, it may seem that Eqs.(27) and (32) provide an easy analytical solution to this problem, especially at $t >> \tau$ when the free oscillations have decayed. For a sinusoidal external force, this is true, but let us not forget that real forces may be much more complex functions of time. In this case, we should first expand the function $F(t)$ into the series (25) (which, in the case of non-periodic $F(t)$, is actually an integral) then calculate the oscillator response for each harmonic, and then sum up the results. Taking into account the well-known expression for the reciprocal Fourier transform,

$$F_\omega = \frac{1}{2\pi}\int_{-\infty}^{+\omega} F(t')e^{i\omega t'}dt', \tag{3.41}$$

the result may be summarized in the form[10]

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9 The range of $Q$ in important oscillating systems is very broad, ranging from $Q \sim 10$ for a human leg (with relaxed muscles) to $Q \sim 10^{12}$ for microwave cavities with superconducting walls.

10 The Re operation may now be dropped, because for any physical (real) force the imaginary components should compensate each other.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, g(\omega) \int_{-\infty}^{+\infty} dt' \frac{F(t')}{m_{\text{ef}}} e^{i\omega(t-t')}, \tag{3.42}$$

with the response function $r(\omega)$ given by Eq. (32). Besides requiring two integrations, Eq. (41) is conceptually uncomforting: it seems to indicate that the oscillator position at time $t$ depends not only on the external force exerted at earlier times $t' < t$, but also in future. This would contradict one of the most fundamental principles of physics (and indeed, science as a whole), *causality*: effect cannot precede the cause.

Fortunately, the response functions of all physical systems satisfy the following rule:[11]

$$G(\tau) \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, g(\omega) e^{i\omega\tau} = 0, \quad \text{for } \tau < 0. \tag{3.43}$$

This allows us, after changing the order of integration in Eq. (42), to rewrite it as

$$x(t) = \int_{-\infty}^{t} \frac{F(t')}{m_{\text{ef}}} G(t-t')dt' = \int_{0}^{\infty} \frac{F(t-\tau)}{m_{\text{ef}}} G(\tau)d\tau. \tag{3.44}$$

In is evident that for most problems the "time-domain" result (44) is much easier to use than the frequency-domain approach (while mathematically they are of course equivalent). Moreover, Eq. (44) has a very simple physical sense: it expresses the linear superposition principle in time domain. Indeed, this result may be interpreted as follows: the full effect of force $F(t)$ on the oscillator (actually, at any linear system) may be described as a sum of effects of short pulses of duration $dt'$ and amplitude $F(t')$ – see Fig. 3. The *Green's function* $G(\tau)$ thus describes the oscillator response to the unit pulse, in time $\tau$ after its arrival.

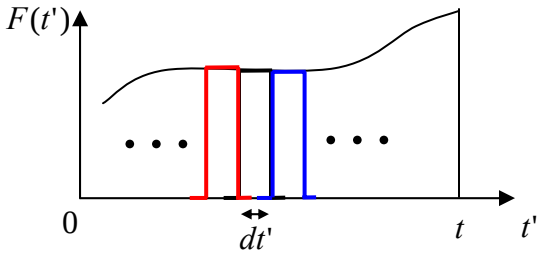


Fig. 3.3. Presentation of a function as a sum of short pulses.

Mathematically, it is more convenient to go to the limit $dt' \to 0$ and describe the elementary short pulse by Dirac's $\delta$-function.[12] Indeed, for the particular case

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[11] Its proof, as well as the frequency-domain expression (the so-called Kramers-Kronig relations) will be, in accordance with tradition, discussed in the E&M course.

[12] As a reminder, this function may be formally defined by the relation

$$\int_{\Delta x} f(x)\delta(x-a)dx = \begin{cases} f(a), & \text{for } a \in \Delta x, \\ 0, & \text{otherwise,} \end{cases}$$

where $f(x)$ is an arbitrary smooth function.

$$\frac{F(t)}{m_{\text{ef}}} = \delta(t - t_0),$$

Equation (44) yields $x(t) = G(t - t_0)$. Hence the Green's function may be calculated as a solution of the differential equation of motion of the system (in our case, Eq. (30)) with the $\delta$-functional right-hand part:

$$\frac{d^2G(\tau)}{d\tau^2} + 2\delta\frac{dG(\tau)}{d\tau} + \omega_0^2 G(\tau) = \delta(\tau), \tag{3.45}$$

with zero initial conditions:

$$G(-0) = \frac{dG}{d\tau}(-0) = 0. \tag{3.46}$$

This equation may be simplified even further. Integrating both sides of Eq. (45) over a infinitesimal interval including the origin (e.g., $[-d\tau/2, +d\tau/2]$), and using Eq. (46), we see that we can find $G(\tau)$ for $\tau > 0$ (i.e. for all substantial times) solving the *homogeneous* equation of motion of the system under analysis, with the following initial conditions:

$$G(+0) = 0, \quad \frac{dG}{d\tau}(+0) = 1. \tag{3.47}$$

This result gives us a convenient way for calculation of the Green's function of any linear system. In particular for the oscillator with damping, imposing boundary conditions (47) on the general solution (39), we immediately get

$$G(\tau) = \frac{1}{\omega_0'} e^{-\delta\tau} \sin \omega_0' \tau. \tag{3.48}$$

We could of course get the same expression as the Fourier transform of the response function $g(\omega)$ (see the first part of Eq. (43)), but that way is much more cumbersome. Equations (44) and (48) provide a very convenient recipe for solving most forced oscillations problems.

As a simple example, let us calculate the transient in the damped oscillator under the effect of a constant force being turned on at $t = 0$:

$$F(t) = \begin{cases} 0, & t < 0, \\ F_0, & t > 0. \end{cases} \tag{3.49}$$

Then Eq. (44) yields

$$x(t) = \int_0^\infty \frac{F(t-\tau)}{m_{\text{ef}}} G(\tau) d\tau = \int_0^t \frac{F_0}{m_{\text{ef}}} \frac{1}{\omega_0'} e^{-\delta\tau} \sin \omega_0' \tau d\tau. \tag{3.50}$$

The simplest way to calculate this integral is to present the sinus as an imaginary part of the exponent:

 

$$x(t) = -\frac{F_0}{m_{\text{ef}}}\frac{1}{\omega_0'}\,\text{Im}\int_0^t e^{-\delta\tau}e^{-i\omega_0'\tau}\,d\tau = \frac{F_0}{m_{\text{ef}}}\frac{1}{\omega_0'}\,\text{Im}\left[\frac{1}{\delta + i\omega_0'}e^{-\delta\tau - i\omega\tau}\right]_0^t$$
$$= \frac{F_0}{k_{\text{ef}}}\left[1 - e^{-\delta t}\left(\cos\omega_0' t + \frac{\delta}{\omega_0'}\sin\omega_0' t\right)\right]. \qquad (3.51)$$

This result (Fig. 4) is natural: it describes nothing more than the transient from the initial equilibrium position $x = 0$ to the new equilibrium point $x = F_0 / k_{\text{ef}}$, accompanied by decaying oscillations.

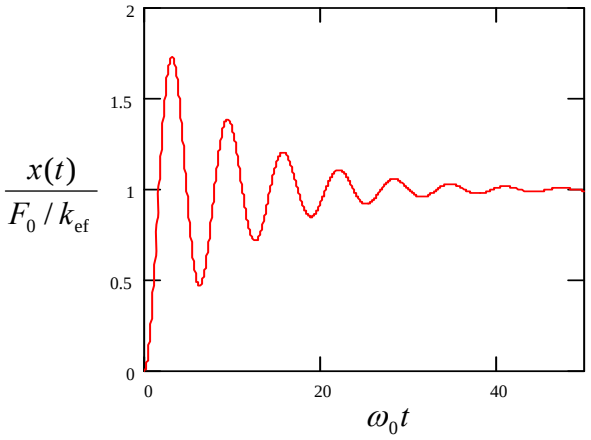


Fig. 3.4. Transient in a linear oscillator, induced by a step-like force $F(t)$ – see Eq. (49), for the particular case $\delta = 0.1\omega_0$.