

The Anelastic and Boussinesq Approximations

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14.1 Introduction

The atmosphere contains sound waves, and the linearized equations which describe the evolution of the atmosphere contain solutions corresponding to sound waves. These solutions are derived in Section 2 below.

Since sound waves have no meteorological significance, it is useful to have a system of equations which has no sound-wave solutions, but is still applicable to the study of turbulence and cumulus convection as well as large-scale motions. The familiar quasi-static approximation filters sound waves, but seriously distorts the small-scale motions. The distortion arises because, for these motions, the perturbation pressure force is not hydrostatically balanced by the perturbation density field.

The anelastic approximation was invented by Ogura and Phillips (1962) in order to filter sound waves without assuming hydrostatic balance. The Boussinesq equations are a simplified subset of the anelastic equations, valid only for relatively shallow motions. Durran (1989) and Bannon (1996) give recent analyses and improved alternatives to the anelastic and Boussinesq approximations. The present note follows the classical approach of Ogura and Phillips.

Although the anelastic and Boussinesq equations are very useful, we shall see that they have certain drawbacks.

14.2 The exact equations

The basic equations in height coordinates, without rotation and friction, are

$$\frac{D\mathbf{V}}{Dt} = -\frac{1}{\rho}\nabla_z p \,, \tag{14.1}$$

$$\frac{Dw}{\mathrm{Dt}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \,, \tag{14.2}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_z + \nabla_z \bullet (\rho \mathbf{V}) + \frac{\partial}{\partial z}(\rho w) = 0 \,, \tag{14.3}$$

$$\dot{\theta} \equiv \frac{D\theta}{Dt} = \frac{Q}{\Pi}. \tag{14.4}$$

Here $\frac{D}{Dt}$ is the Lagrangian time derivative, $\mathbf{V}$ is the horizontal velocity, ρ is density, p is pressure, w is the vertical velocity, z is height, g is the acceleration of gravity, θ is the potential temperature, Q is the heating rate per unit mass, and Π is the Exner function, which satisfies

$$c_p T = \Pi\theta, \tag{14.5}$$

where c_p is the heat capacity of air at constant pressure, and T is temperature. We can also write

$$\Pi = c_p\left(\frac{p}{p_0}\right)^{\kappa}, \tag{14.6}$$

where

$$\kappa \equiv \frac{R}{c_p}, \tag{14.7}$$

and R is the specific gas constant. Finally, we can include the prognostic equation for an arbitrary scalar, which is

$$\left(\frac{\partial}{\partial t}\rho A\right)_z + \nabla_z \bullet (\rho\mathbf{V}A) + \frac{\partial}{\partial z}(\rho w A) = \rho S_A, \tag{14.8}$$

where S_A is the source of A per unit mass.

We will need the ideal gas law, which is

$$p = \rho R T. \tag{14.9}$$

The following relationships can be derived using the ideal gas law:

$$p = p_0\left(\frac{\rho R\theta}{p_0}\right)^{\frac{1}{1-\kappa}}, \tag{14.10}$$

$$\Pi = c_p \left(\frac{\rho R \theta}{p_0} \right)^{\frac{\kappa}{1-\kappa}}, \tag{14.11}$$

$$T = \theta \left(\frac{\rho R \theta}{p_0} \right)^{\frac{\kappa}{1-\kappa}}. \tag{14.12}$$

14.3 Sound-wave solutions

Consider one-dimensional, small-amplitude motions, with no mean flow, no rotation, no stratification, no friction, and no heating. We adopt the following linearized system of equations:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0} \frac{\partial}{\partial x}(\delta p), \tag{14.13}$$

$$\lambda \frac{\partial}{\partial t}(\delta \rho) + \frac{\partial}{\partial x}(\rho_0 u) = 0, \tag{14.14}$$

$$\frac{\partial}{\partial t}\left(\frac{\delta \theta}{\theta_0} \right) = 0 \tag{14.15}$$

$$\frac{\delta \theta}{\theta_0} = \frac{\delta T}{T_0} - \kappa \frac{\delta p}{p_0}, \tag{14.16}$$

$$\frac{\delta p}{p_0} = \frac{\delta \rho}{\rho_0} + \frac{\delta T}{T_0}, \tag{14.17}$$

where λ is normally equal to one but will be set equal to zero to obtain the anelastic system. We can eliminate unknowns so as to obtain a single wave equation in the single unknown δp:

$$\frac{\partial^2}{\partial t^2}(\delta p) - \frac{c_s}{\lambda} \frac{\partial^2}{\partial x^2}(\delta p) = 0. \tag{14.18}$$

Here we have used

$$c_s^2 = \gamma R T \quad , \tag{14.19}$$

where

$$\gamma \equiv \frac{1}{1-\kappa}. \tag{14.20}$$

The solutions of (14.18) are

$$\delta p = P \exp[ik(x \pm c_s t)] \tag{14.21}$$

for $\lambda = 1$, and c_s is seen to be the signal velocity (the speed of sound).

For $\lambda = 0$, we get

$$\frac{\partial^2}{\partial x^2}(\delta p) = 0, \tag{14.22}$$

which has the solutions $\delta p = A + Bx$. In this case, the effective "signal velocity" is infinite. There are no wave solutions.

This analysis shows that if we can justify neglect of the density-tendency term in the continuity equation, we can filter out sound waves. The next section presents a scale analysis designed to determine when such an approximation can be justified. The scale analysis is also used to introduce some additional simplifying approximations.

14.4 Scale analysis

Consider a *hydrostatically balanced reference state for which the thermodynamic state variables may be functions of height* z, *but are independent of time and the horizontal coordinates. We denote this reference state by subscript 0, and departures from it by* $\delta(\ )$, *i.e.*

$$(\ ) \equiv (\ )_0 + \delta(\ ). \tag{14.23}$$

The reference state need not necessarily be identical to the "*initial" state,* or to a *"basic" state* upon which perturbations are to be imposed. Nevertheless, we assume that the actual range of variation of the thermodynamic variables is no greater, in order of magnitude, than the departure from the reference state, and that for each of the thermodynamic variables, this departure from the reference state is fractionally small.

For our scale analysis, we need thermodynamic height scales, defined as follows:

$$H_\rho \equiv \left|\frac{1}{\rho_0}\frac{d\rho_0}{dz}\right|^{-1} \sim 10 \text{ km}, \tag{14.24}$$

$$H_p \equiv \left|\frac{1}{p_0}\frac{dp_0}{dz}\right|^{-1} \equiv \frac{RT_0}{g} \sim 8 \text{ km}, \tag{14.25}$$

$$H_\theta \equiv \left|\frac{1}{\theta_0}\frac{d\theta_0}{dz}\right|^{-1} \sim 70 \text{ km}, \tag{14.26}$$

$$H_T \equiv \left|\frac{1}{T_0}\frac{dT_0}{dz}\right|^{-1} \sim 40 \text{ km}, \tag{14.27}$$

Note that Eq. (14.25) gives the definition of T_0. The numerical values given in (14.24)-(14.27) are for typical tropospheric soundings.

We first analyze the continuity equation, which can be written as

$$\begin{aligned}\frac{\partial}{\partial t}(\delta\rho) + \rho_0\left(1 + \frac{\delta\rho}{\rho_0}\right)\left(\nabla_H \cdot \mathbf{V}_H + \frac{\partial w}{\partial z}\right) + \\ \mathbf{V}_H \cdot \nabla_H(\delta\rho) + w\frac{\partial}{\partial z}(\delta\rho) + w\frac{\partial \rho_0}{\partial z} = 0\end{aligned} \tag{14.28}$$

We have used the fact that ρ_0 is horizontally homogeneous. Below we assume that $(|\delta\rho|/\rho_0) \ll 1$. We proceed by comparing $\rho_0\frac{\partial w}{\partial z}$ with each of the remaining terms. First, note that

$$\frac{\left|\frac{\partial}{\partial t}(\delta\rho)\right|}{\left|\rho_0\frac{\partial w}{\partial z}\right|} \sim \frac{|\delta\rho|/\tau}{|w|\rho_0/D} = \frac{|\delta\rho|}{\rho_0}\frac{D}{|w|\tau} = \frac{g(|\delta\rho|/\rho_0)}{\left(\frac{g}{D}\right)\tau^2\frac{w}{\tau}} \tag{14.29}$$

where D is the depth of the motions, and τ *is the time scale of interest*. Since we are interested in boundary-layer eddies, we will choose τ in the range 100 s to 1000 s. If we were interested in sound waves, we would choose τ on the order of 10^{-3} s. Since

$$\left|\frac{w}{\tau}\right| \sim g\frac{|\delta\rho|}{\rho_0}, \tag{14.30}$$

we can neglect $\frac{\partial}{\partial t}(\delta\rho)$ if

$$\tau^2 \gg D/g \le (30 \text{ seconds})^2. \tag{14.31}$$

For large boundary-layer eddies and cumulus clouds, the term can safely be neglected.

Next, we analyze the $\partial\rho_0/\partial z$ term:

$$\frac{\left|w\frac{\partial\rho_0}{\partial z}\right|}{\left|\rho_0\frac{\partial w}{\partial z}\right|} \sim \frac{D}{H_\rho}. \tag{14.32}$$

For $D/H_\rho \sim 1$, the two terms are the same size. For motions that are shallow in the sense that $D/H_\rho \ll 1$, we can neglect $\rho_0\frac{\partial w}{\partial z}$. For the present, we allow the possibility that $D/H_\rho \sim 1$.

Now consider the advection of $\delta\rho$:

$$\frac{\left|\mathbf{V}_H \cdot \nabla_H \delta\rho\right|}{\left|\rho_0\frac{\partial w}{\partial z}\right|} \sim \frac{V|\delta\rho|/L}{\rho_0|w|/D} = \frac{|\delta\rho|}{\rho_0}\frac{V}{|w|}\frac{D}{L}. \tag{14.33}$$

Here L is a horizontal length scale. From (14.33), we conclude that $\mathbf{V}_H \cdot \nabla_H \delta\rho$ can be neglected if

$$\frac{V}{|w|}\frac{D}{L} \le 1. \tag{14.34}$$

This condition will be met if the aspect ratio $\frac{D}{L}$ is sufficiently small.

Finally, we note that since

$$\frac{\left|w\frac{\partial}{\partial z}\delta\rho\right|}{\left|\rho_0\frac{\partial w}{\partial z}\right|} \sim \frac{\delta\rho}{\rho_0} \ll 1, \tag{14.35}$$

the vertical advection of $\delta\rho$ is negligible.

Generally speaking, the remaining terms have to be kept. In summary, we have

$$\nabla \cdot (\rho_0 \mathbf{V}) = 0 \text{ for } D/H_\rho \sim 1, \tag{14.36}$$

and

$$\nabla \cdot \mathbf{V} = 0 \text{ for } D/H_\rho \ll 1 . \tag{14.37}$$

Now consider the horizontal pressure gradient force. Since p_0 is horizontally homogeneous, we have

$$-\frac{1}{\rho}\nabla_H p = -\frac{1}{\rho}\nabla_H(\delta p) \cong -\frac{1}{\rho_0}\nabla_H \delta p = -\nabla_H\left(\frac{\delta p}{\rho_0}\right) . \tag{14.38}$$

The vertical pressure gradient force requires somewhat more analysis. Recall that the reference state is in hydrostatic balance, i.e.,

$$\frac{dp_0}{dz} = -\rho_0 g . \tag{14.39}$$

We can then write

$$\begin{aligned} \frac{-1}{\rho}\frac{\partial p}{\delta z} - g &= \frac{-1}{(\rho_0 + \delta\rho)}\frac{\partial}{\partial z}(p_0 + \delta p) - g \\ &= \frac{-1}{(\rho_0 + \delta\rho)}\frac{\partial}{\partial z}(\delta p) + \left(\frac{\rho_0}{\rho_0 + \delta\rho} - 1\right)g \\ &\cong \frac{-1}{\rho_0}\frac{\partial}{\partial z}(\delta p) - \frac{\delta\rho}{\rho_0}g \\ &= -\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) - \frac{\delta p}{\rho_0}\frac{\partial}{\partial z}(ln\rho_o) - \frac{\delta\rho}{\rho_0}g. \end{aligned} \tag{14.40}$$

Note that in the last line of (14.40) the basic state density appears inside the vertical derivative. Eq. (14.40) can be simplified as follows. From the definition of potential temperature,

$$\theta \equiv T\left(\frac{p_0}{p}\right)^\kappa , \tag{14.41}$$

and the ideal gas law, we can show that

$$c_p ln\,\theta = c_v ln\,p - c_p ln\,\rho , \tag{14.42}$$

so that, as an approximation,

$$c_p\frac{\delta\theta}{\theta_0} \cong c_v\frac{\delta p}{p_0} - c_p\frac{\delta\rho}{\rho_0} . \tag{14.43}$$

Of course, we also have

$$c_p ln\,\theta_0 = c_v ln\,p_0 - c_p ln\,\rho_0\,. \tag{14.44}$$

Substituting above, and using $\gamma \equiv c_p / c_v$, we find that

$$\begin{aligned} -\frac{1}{\rho}\frac{\partial p}{\partial z} - g &\cong -\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) + \frac{\delta p}{\rho_0}\frac{\partial}{\partial z}\left(ln\,\theta_0 - \frac{1}{\gamma} ln p_0\right) + \left(\frac{\delta\theta}{\theta_0} - \frac{1}{\gamma}\frac{\delta p}{p_0}\right)g \\ &= -\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) + \frac{\delta p}{\rho_0}\frac{\partial}{\partial z}(ln\theta_o) + g\frac{\delta\theta}{\theta_0}, \end{aligned} \tag{14.45}$$

where we have used (14.39) to obtain the final equality. We now argue that the $ln\theta_0$ term of (14.45) can be neglected.

Defining the Brunt-Vaisalla frequency, N, by

$$N^2 \equiv g\frac{\partial}{\partial z}(ln\theta_0) = \frac{g}{H_\theta}, \tag{14.46}$$

we can write

$$\begin{aligned} \frac{\left|\frac{\delta p}{\rho_0}\frac{\partial}{\partial z}(ln\,\theta_0)\right|}{\left|\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right)\right|} &\sim \frac{\left|\frac{\delta p}{\rho_0}\right|\frac{N^2}{g}}{\left|\frac{\delta p}{\rho_0}\right|\frac{1}{D}} \\ &= \frac{N^2 D}{g} = \frac{N^2}{\gamma RT}\left(\frac{RT}{g}\right)\gamma D \\ &= \frac{N^2}{(c_s/H_p)^2}\left(\frac{\gamma D}{H_p}\right)\quad, \end{aligned} \tag{14.47}$$

where c_s is the isentropic sound speed, introduced earlier. From (14.47) we see that, provided that $\gamma D/H_\rho$ is *no greater than order 1*, we can neglect the $ln\theta_0$ term when the frequency of sound waves greatly exceeds the frequency of pure gravity waves. Our conclusion is, then, that

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} - g \cong -\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) + g\frac{\delta\theta}{\theta_0}\,. \tag{14.48}$$

This is the origin of the familiar “buoyancy” term of the equation of vertical motion.

We now turn to the first law of thermodynamics, which can be written as

$$\frac{D(ln\theta)}{Dt} = \frac{Q}{c_p T}, \tag{14.49}$$

where Q is the heating rate per unit mass. Using our assumptions that $|\delta\theta/\theta_0| \ll 1$ and $|\delta T/T_0| \ll 1$, we can immediately rewrite (14.49) as

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_H \cdot \nabla\right)\frac{\delta\theta}{\theta_0} + w\frac{d}{dz}(ln\theta_0) = \frac{Q}{c_p T_0}, \tag{14.50}$$

where we have used the two-dimensional velocity vector $\mathbf{V}_H$.

For shallow motions, a further simplification is possible whenever

$$\left|\frac{1}{\rho_0}\frac{\partial}{\partial z}(\delta p)\right| \leq g\left|\frac{\delta\rho}{\rho_0}\right|. \tag{14.51}$$

This does not mean that the perturbations are in hydrostatic balance, but only that they are close to balance. This assumption is particularly appropriate whenever the buoyancy force plays a key role in the fluid motions, as in convection and gravity waves. Rewriting (14.51) as

$$\left|\frac{\delta p}{p_0}\right| \leq \frac{D\rho_0 g}{p_0}\left|\frac{\delta\rho}{\rho_0}\right|, \tag{14.52}$$

and recognizing that

$$\frac{\rho_0 g}{p_0} = -\left(\frac{1}{p_0}\frac{dp_0}{dz}\right) \equiv -\left(\frac{1}{H_p}\right), \tag{14.53}$$

we see that

$$\left|\frac{\delta p}{p_0}\right| \leq \frac{DH}{H_p} \sim \left|\frac{\delta\rho}{\rho_0}\right|. \tag{14.54}$$

For shallow convection, i.e., $D/H_p \ll 1$, we can neglect $|\delta p/p_0|$ in comparison with $|\delta\rho/\rho_0|$. Then from the state equation and the definition of θ we obtain

$$-\left(\frac{\delta\rho}{\rho_0}\right) \cong \frac{\delta T}{T_0} \cong \frac{\delta\theta}{\theta_0}. \tag{14.55}$$

From (14.48), the vertical pressure gradient force becomes

$$-\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial z} - g \cong -\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) + g\frac{\delta T}{T_0}, \tag{14.56}$$

and the first law of thermodynamics becomes

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_H \cdot \nabla\right)\left(\frac{\delta T}{T_0}\right) + \frac{w}{T_0}\left(\frac{dT_0}{dz} + \frac{g}{c_p}\right) \cong \frac{Q}{c_p T_0}, \tag{14.57}$$

where we have invoked the hydrostaticity of the reference state to write the vertical advection term in terms of T_0 rather than θ_0.

For simplicity, we have not considered the virtual temperature effect in the preceding analysis. It can be included simply by replacing θ by θ_v and T by T_v in the respective buoyancy terms of the anelastic and Boussinesq equations of motion.

14.5 Summary of the anelastic and Boussinesq systems

The anelastic system of equations is collected below.

Continuity:

$$\nabla \cdot (\rho_0 \mathbf{V}) = 0 \tag{14.58}$$

Equation of motion:

$$\frac{\partial \mathbf{V}}{\partial t} + (2\mathbf{\Omega} + \nabla\times\mathbf{V}) \times \mathbf{V} + \nabla K = -\nabla\left(\frac{\delta p}{\rho_0}\right) + g\mathbf{k}\frac{\delta\theta_v}{\theta_{v_0}} - \frac{\mathbf{F}}{\rho_0} \tag{14.59}$$

First Law of Thermodynamics:

$$\frac{D}{Dt}\left(\frac{\delta\theta}{\theta_0}\right) + w\frac{d}{dz}(ln\theta_0) = \frac{Q}{c_p T_0} \tag{14.60}$$

Here $\mathbf{\Omega}$ is the angular velocity of the earth's rotation, $K \equiv \frac{1}{2}|\mathbf{V}|^2$ is the kinetic energy per unit mass, and $\mathbf{F}$ is the frictional force, per unit mass. These equations are valid provided that the following conditions are met:

1. All thermodynamic variables depart only slightly from their reference distributions.

2. The time-scale of the motions is a few minutes or longer [see (14.31)].

3. The aspect ratio of the motions is not too large [see (14.34)].

4. The frequency of the motions is much less than the frequency of sound waves [see (14.47)]. This condition overlaps somewhat with the second condition above.

If, in addition, the depth of the motions is much less than H_p, and if the motions are strongly influenced by the buoyancy force (see (14.51)), we can further simplify to obtain the Boussinesq equations, which are collected below.

Continuity:

$$\nabla \cdot \mathbf{V} = 0 \tag{14.61}$$

Equation of Motion:

$$\frac{\partial \mathbf{V}}{\partial t} + (2\boldsymbol{\Omega} + \nabla \times \mathbf{V}) \times \mathbf{V} + \nabla K = -\nabla\left(\frac{\delta p}{\rho_0}\right) + g\mathbf{k}\frac{\delta T_v}{T_{v_o}} - \frac{\mathbf{F}}{\rho_0} \tag{14.62}$$

First Law of Thermodynamics:

$$\frac{D}{Dt}\left(\frac{\delta T}{T_0}\right) + \frac{w}{T_0}\left(\frac{dT_0}{dz} + \frac{g}{c_p}\right) = \frac{Q}{c_p T_0} . \tag{14.63}$$

14.6 The anelastic pressure equation

One of the rewards of our scale analysis is the relatively simple *diagnostic* form of the anelastic continuity equation, which lacks a time derivative term. With the full system of equations (before simplification by our scale analysis), we must predict *two* of the three independent thermodynamic state variables, ρ and θ, (as in (14.28) and (14.49)). The third variable, p, is then determined by the equation of state. But in the anelastic system, only *one* thermodynamic variable, θ, is to be predicted. The other two, ρ and p, are determined diagnostically by the equation of state, and by our requirement that the three-dimensional mass flux be nondivergent, i.e. by the anelastic continuity equation.

To derive this anelastic pressure equation, we first use the continuity equation, (14.58), to write the equation of motion, (14.59), as

$$\frac{\partial}{\partial t}(\rho_0 \mathbf{V}) + \mathbf{A} = -\rho_0 \nabla\left(\frac{\delta p}{\rho_0}\right), \tag{14.64}$$

where, for convenience, we define

$$\mathbf{A} \equiv -\{\rho_0[(2\boldsymbol{\Omega} + \nabla\times\mathbf{V}) \times \mathbf{V} + \nabla K] + \mathbf{V}[\nabla \cdot (\rho_0 \mathbf{V})]\} \\ -2\boldsymbol{\Omega} \times \rho_0 \mathbf{V} + g\mathbf{k}\left(\frac{\delta\theta_v}{\theta_{v_0}}\right) - \mathbf{F} \quad . \tag{14.65}$$

Taking the divergence of (14.65), and using (14.58), we obtain

$$\nabla \cdot \left[\rho_0 \nabla\left(\frac{\delta p}{\rho_0}\right)\right] = -\nabla \cdot \mathbf{A}\,, \tag{14.66}$$

which is the anelastic pressure equation. The physical meaning of (14.66) is simply that the pressure field must be whatever it takes to keep the three-dimensional mass flux non-divergent. The pressure field does "air traffic control."

The fact that the pressure is determined diagnostically in this way means that the pressure field plays only a *passive* role in the dynamics of the motions we are considering. The distribution of the pressure at a given instant is completely determined by the distributions of the other variables; the past history of the pressure itself is irrelevant.

Although the anelastic pressure equation simplifies things insofar as we have gone from two prognostic equations and one diagnostic equation to one prognostic equation and two diagnostic equations, the diagnostic equation for the pressure field must be solved as a second-order three-dimensional boundary value problem, for which appropriate boundary conditions must be supplied.

14.7 A comparison with the quasi-static system

As mentioned in Section 1, the quasi-static approximation also filters sound waves, but it cannot be used to study PBL turbulence or cumulus convection because for these motions the perturbation pressure and the perturbation density are not quasi-statically balanced. If we tried to use the quasi-static system, serious errors would be introduced.

14.8 Some drawbacks

There are several problems with the anelastic and Boussinesq equations. The most fundamental weakness is that the equations have intrinsic errors on the order of a few percent for most motions, simply as a consequence of the various approximations made. We must always ask whether these errors are acceptable.

There is no guarantee that the solutions obtained with the anelastic system will actually be consistent with the assumptions made in their derivation. For example, it would be possible to obtain solutions in which the departures of the thermodynamic variables from the reference state were not fractionally small.

Lilly (1996) points out that the Boussinesq system conserves volume rather than mass. Giving up exact mass conservation should be enough to make anyone nervous.

A more subtle problem is that the classical anelastic system "leaks" energy, i.e., it does not have a conservation of energy theorem. To show this, we first dot the equation of motion (14.59) with the momentum vector $\rho_o \mathbf{V}$, and use the continuity equation (14.58) to obtain the kinetic energy equation:

$$\frac{\partial(\rho_0 K)}{\partial t} + \nabla \cdot [\mathbf{V}(\rho_0 K + \delta p)] = g\rho_0 w \frac{\delta\theta}{\theta_0}, \tag{14.67}$$

For simplicity, we have neglected friction and the virtual temperature correction, which are irrelevant to the present discussion. By combining (14.58) and (14.60), we can derive

$$\frac{\partial}{\partial t}\left(P\frac{\delta\theta}{\theta_0}\right) + \nabla \cdot \left(\mathbf{V}P\frac{\delta\theta}{\theta_0}\right) = g\rho_o w\frac{\delta\theta}{\theta_0} - Pw\frac{d}{dz}(ln\theta_0), \tag{14.68}$$

where we have neglected heating, for simplicity, and where

$$P \equiv \rho_0 g z \tag{14.69}$$

is the potential energy per unit volume. Subtracting (14.68) from (14.67), we obtain.

$$\frac{\partial}{\partial t}\left(\rho_0 K - P\frac{\delta\theta}{\theta_0}\right) + \nabla \cdot \left[\mathbf{V}\left(\rho_0 K - P\frac{\delta\theta}{\theta_0} - \delta p\right)\right] = Pw\frac{d}{dz}(ln\theta_0). \tag{14.70}$$

The term on the right-hand side of (14.70) is spurious; it is replaced by zero in a derivation that proceeds from the exact equations. It represents an infinite reservoir of energy associated with the stratification of the reference state. It can be forced to vanish by taking the reference state to be isentropic, but often this is unacceptable because it makes the departures from the reference state large.

See Durran (1989), Lilly (1996), and Bannon (1996) for a discussion of refined anelastic systems that do conserve energy.

A final drawback to the anelastic system is that the anelastic pressure equation can only be solved through the imposition of boundary conditions which must sometimes be specified rather arbitrarily. Also, the numerical algorithms usually employed to solve the pressure equation are expensive and cumbersome.

14.9 Conclusions

The anelastic and Boussinesq equations have some useful properties, and they have been employed in many studies of PBL turbulence and cumulus convection. Their intrinsic errors, and particularly their failure to conserve total energy, make it important to proceed with caution in any application.

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