PHY 501 **Classical Mechanics** Fall 2006

Homework 02 with Solutions

Problem 2.1. Use the general result derived in class,

$$T = \left(2m_{\text{ef}}\right)^{1/2} \int_B^A \frac{dq}{[H - U_{\text{ef}}(q)]^{1/2}}, \tag{1}$$

to find the functional dependence of the period T of oscillations of a 1D particle of mass m in the potential $U(x) = \alpha x^{2n}$ ($\alpha > 0$, n is a positive integer) on energy E. Explore the limit $n \to \infty$.

Solution: Let us reduce the integral to dimensionless form, using the problem symmetry:

$$T = 2\left(2m\right)^{1/2} \int_0^A \frac{dx}{[E - \alpha x^{2n}]^{1/2}} = 2\left(\frac{2m}{E}\right)^{1/2} \int_0^a \frac{dx}{[1-(x/a)^{2n}]^{1/2}} = 2\left(\frac{2m}{E}\right)^{1/2} a \int_0^1 \frac{d\xi}{[1-\xi^{2n}]^{1/2}},$$

where the classical turning point is found from the condition $E - \alpha a^{2n} = 0,$ giving

$$a = \left(\frac{E}{\alpha}\right)^{\frac{1}{2n}}.$$

Combining these two formulas, we get

$$T \propto E^{\frac{1}{2n} - \frac{1}{2}}.$$

For $n = 2$ we recover our class result (energy-independent frequency of a harmonic oscillator), while at $n \to \infty$, the period decreases with energy as $1/\sqrt{E}$. (***Derive this asymptotic result from simple arguments to get an additional 5-point credit.***)

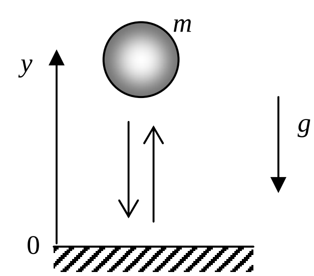
Note that all the analysis did not require the dimensionless integral (which of course does not depend on E), but if you like, here it is:

$$\int_0^1 \frac{d\xi}{[1-\xi^{2n}]^{1/2}} = \frac{\sqrt{\pi}}{2n} \Gamma\left(\frac{1}{2n}\right) / \Gamma\left(\frac{n+1}{2n}\right),$$

where $\Gamma(x)$ is the so-called gamma-function which may be considered just as the generalization of the factorial function from real to continuous arguments: $\Gamma(k) = (k-1)!$ - see, e.g., Chapter 6 of the famous manual by M. Abramowitz and I. Stegun.

Problem 2.2. A small, very stiff ball is bouncing of the floor. Neglecting energy loss,

- integrate the equation of motion directly to find the ball height as a function of time (sketch the result);



- explain how the problem (including bouncing) may be described as a particle motion in a 1D field of potential forces; sketch the corresponding potential energy $U(x)$;

- check that the relation between the bouncing period T and the ball energy E, following from your first result, does satisfy general Eq. (1).

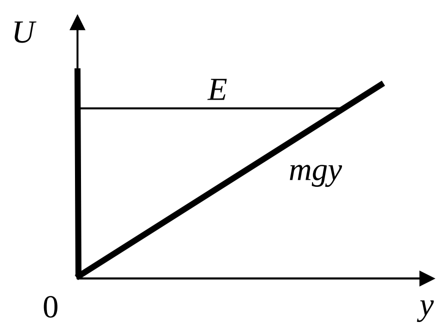
Solution: For a stiff ball, the time of contact with the floor is negligible, so that most of the time it is moving in the gravity field: $\ddot{y} = -g$. An elementary integration of this equation, with the y-origin at the floor (see Fig.) yields

$$y(t) = v_0 t - \frac{g}{2} t^2,$$

where time is counted from the instant of the previous bound, and $v_0 = \sqrt{2E/m}$ is the velocity at the floor level. Requiring the ball to return to the floor at time T, for the bouncing period we get

$$T = \frac{2v_0}{g} = \frac{2}{g}\sqrt{\frac{2E}{m}}.$$

We can get the same result from Eq. (1) by noticing that pressing a stiff ball into the floor requires a very fast rise of potential energy at $y < 0$, so that the potential profile looks as a triangular well:



Integrating Eq. (1) between the classical turning points 0 and $a = E/mg$, we get

$$T = (2m)^{1/2} \int_0^a \frac{dy}{[E - mgy]^{1/2}} = \left(\frac{2m}{E}\right)^{1/2} a \int_0^1 \frac{d\xi}{[1-\xi]^{1/2}} = \left(\frac{2m}{E}\right)^{1/2} \frac{E}{mg} \int_0^1 \frac{d\xi}{[1-\xi]^{1/2}}.$$

The table integral here equals 2, so that the results obtained by both methods coincide, as they should.

Problem 2.3. Use the time-domain approach (i.e. the Green's function method) to find dynamics in a linear oscillator with damping, $x(t)$ induced by a resonant force suddenly turned on:

$$F(t) = \begin{cases} 0, & \text{for } t < 0, \\ F_0 \cos\omega_0 t, & \text{for } t > 0. \end{cases}$$

Sketch (or plot) the resulting function $x(t)$, and give its physical interpretation. Explore the trend at $\delta \to 0$.

Notice: The frequency in Eq. (2) is the real own frequency ω_0 of the oscillator, rather than its re-normalized value $\omega_0' = \sqrt{\omega_0^2 - \delta^2}$.

Solution: Using our general result for the Green's function approach, we get

$$x(t) = \int_0^\infty \frac{F(t-\tau)}{m_{\text{ef}}} G(\tau) d\tau = \int_0^t \frac{F_0}{m_{\text{ef}}} \cos\omega_0 (t-\tau) \frac{1}{\omega_0'} e^{-\delta\tau} \sin\omega_0' \tau d\tau.$$

Using the elementary formula for the multiplication of two trigonometric functions, we reduce the expression to there integrals of which one (of the bare exponent) is elementary and two other are of the type

$$\int_0^t \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \omega\tau \, e^{-\delta\tau} d\tau.$$

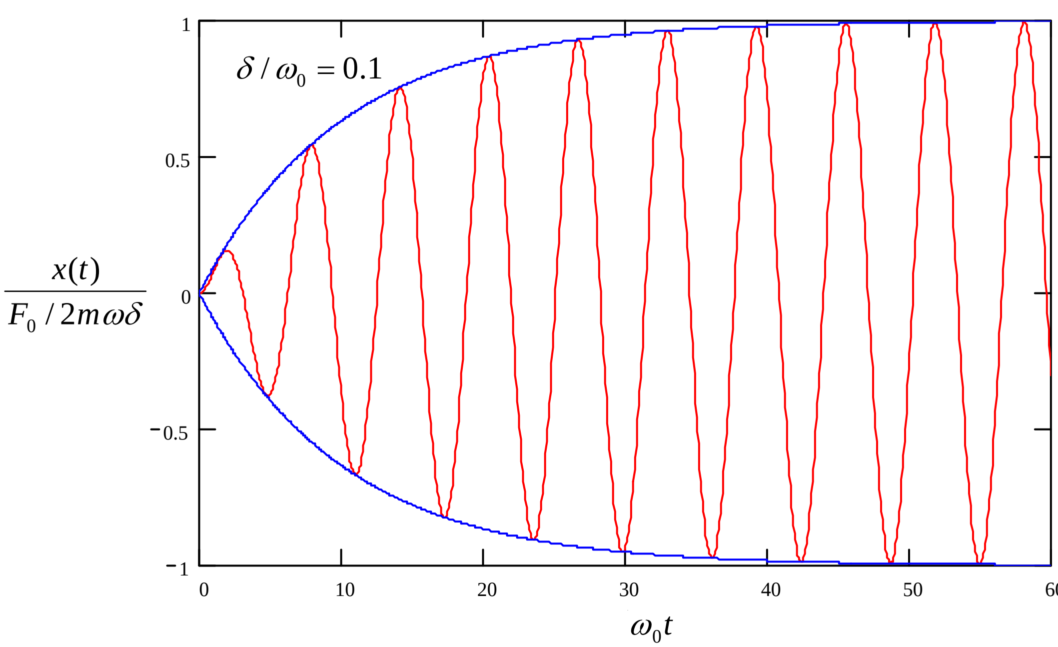
Such integrals calculated as, respectively, the real and imaginary parts of

$$\int_0^t e^{i\omega\tau} e^{-\delta\tau} d\tau = \int_0^t e^{(i\omega - \delta)\tau} d\tau = \frac{e^{i\omega\tau} e^{-\delta\tau}}{i\omega - \delta}\Bigg|_0^t.$$

The final answer,

$$x(t) = \frac{F_0}{2m\delta} \left[\frac{1}{\omega_0} \sin\omega_0 t - \frac{e^{-\delta t}}{\omega_0'} \sin\omega_0' t \right],$$

may be interpreted as a sum of the stationary oscillations with external force frequency ω_0 (and the due phase shift $\Delta\varphi = \pi/2$ relative the force) and decaying oscillations at the re-normalized frequency ω_0'. On the other hand, looking at the plot of the whole process,



one may legitimately think of it (especially at $\delta \to 0$) as a single-frequency oscillation with the time-dependent amplitude

$$a = a(t) = \frac{F_0}{2m\omega\delta}(1 - e^{-\delta t}).$$

(This "envelope" is plotted above with dashed lines.) The envelope law may be readily obtained using the Van der Pol method.

At $\delta \to 0,$ the transient time and the finite amplitude of oscillations both tend to infinity as $1/\delta,$ and the initial part of the transient becomes independent of damping:

$$x(t) \to \frac{F_0}{2m\omega} t \sin \omega t.$$

This is of course just the secular term describing the solution at $\delta = 0$ - cf. Eq. (3.29) of the lecture notes.