Derivation of the Navier-Stokes Equation

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25.1 Analysis of the relative motion near a point

Suppose that the velocity of the fluid at position $\mathbf{r}_0(x, y, z)$ and time t is $\mathbf{V}(x, y, z, t)$, and that the simultaneous velocity at a neighboring position $\mathbf{r}_0 + \mathbf{r}$ is $\mathbf{V} + \delta\mathbf{V}$. Then the velocity at the neighboring position $\mathbf{r}_0 + \mathbf{r}$ relative to the velocity of the reference position $\mathbf{r}_0$ is

$$\begin{pmatrix} \delta u \\ \delta v \\ \delta w \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial z} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \tag{25.1}$$

In vector form, this can be written as

$$\delta\mathbf{V} = \mathbf{D} \cdot \mathbf{r} , \tag{25.2}$$

where $\mathbf{D}$ is called the deformation tensor because it describes how the fluid element is being deformed by the non-uniform motion field. Introducing Cartesian tensor notation, we can alternatively express (25.1) as

$$\delta v_i = \frac{\partial v_i}{\partial x_j} x_j \text{ , where } i = 1, 2, 3 . \tag{25.3}$$

We can decompose $\partial v_i / \partial x_j$ into two parts which are symmetric and anti-symmetric in the suffices i and j. They are

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) , \tag{25.4}$$

and

$$\xi_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right), \tag{25.5}$$

respectively. The deformation $\mathbf{D}$ can now be expressed by

$$\mathbf{D} = \varepsilon + \Xi, \tag{25.6}$$

where we define two "parts" of $\mathbf{D}$:

$$\varepsilon \equiv \begin{pmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{pmatrix}, \Xi \equiv \begin{pmatrix} 0 & \xi_{xy} & \xi_{xz} \\ \xi_{yx} & 0 & \xi_{yz} \\ \xi_{zx} & \xi_{zy} & 0 \end{pmatrix}. \tag{25.7}$$

We show below that ε is related to the divergence, and Ξ is related to the vorticity. Correspondingly, δV can be divided into two parts, i.e.

$$\delta V = \delta V^{(s)} + \delta V^{(a)}, \qquad \text{where} \qquad \delta V_i^{(s)} = e_{ij}X_j, \delta V_i^{(a)} = \xi_{ij}X_j . \tag{25.8}$$

Here the superscripts s and a denote "symmetric" and "antisymmetric," respectively.

We refer to ε as the *rate of strain tensor* (or *strain tensor*). The diagonal elements of ε, i.e. $e_{xx} = \frac{\partial u}{\partial x}, e_{yy} = \frac{\partial v}{\partial y}, e_{zz} = \frac{\partial w}{\partial z}$, are called the *normal strains*, and represent the rate of volume expansion, as illustrated in Fig. 25.1 a. The sum of the normal strains is the divergence of $\mathbf{V}$, i.e.

$$e_{xx} + e_{yy} + e_{zz} = e_{ii} = \nabla \cdot \mathbf{V} . \tag{25.9}$$

The off-diagonal elements of ε, namely e_{xy}, e_{yz}, and e_{zx}, are called the *shearing strains*, and express the rate of shearing deformation, as illustrated in Fig. 25.1 b. They can be written as

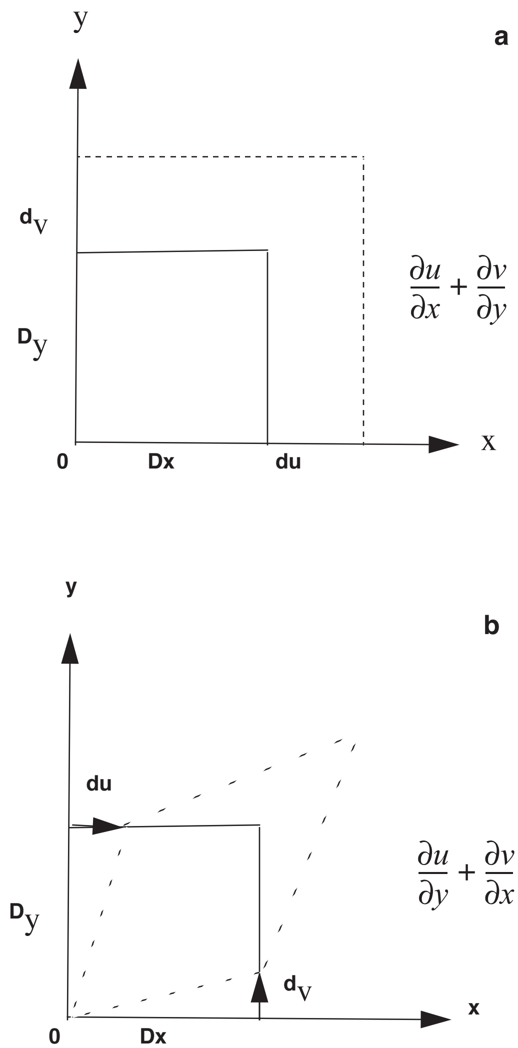


Figure 25.1: a) The normal strains, representing the rate of volume expansion. b) The shearing strains, which represent the rate of shearing deformation.

$$
\begin{aligned}
e_{xy} &= e_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \\
e_{yz} &= e_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right), \\
e_{zx} &= e_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right).
\end{aligned}
\tag{25.10}
$$

The elements of the anti-symmetric tensor, Ξ, are

$$\xi_{zy} = -\xi_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \xi_{xz} = -\xi_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \xi_{yx} = -\xi_{xy} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \quad (25.11)$$

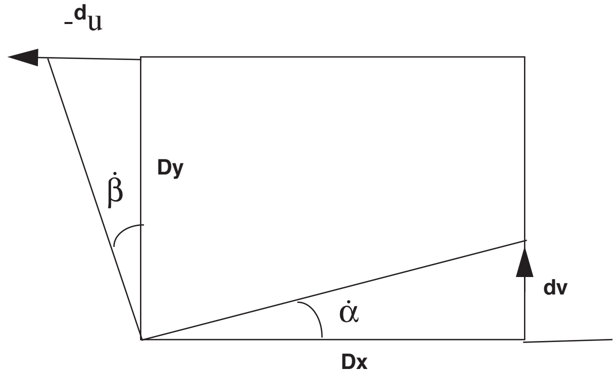
Introducing the notation $\omega_x = -2\xi_{yz}, \omega_y = -2\xi_{zx}, \omega_z = -2\xi_{zx}$, we see that $(\omega_x, \omega_y, \omega_z)$ is the *vorticity vector*

$$\omega = \nabla \times \mathbf{V} . \quad (25.12)$$

In tensor notation, (25.12) may be simply written

$$\omega_i = \varepsilon_{ijk}\frac{\partial v_k}{\partial x_j} , \quad (25.13)$$

where $\varepsilon_{ijk} = 1$ when i, j, k are in even permutation, and $\varepsilon_{ijk} = -1$ when i, j, k are in odd permutation. For example, $\omega_z = \omega_3 = \varepsilon_{321}\frac{\partial v_2}{\partial x_1} + \varepsilon_{312}\frac{\partial v_1}{\partial x_2} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, which gives twice the angular velocity about the z-axis, as illustrated in Fig. 25.2. Thus



$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \dot{\alpha} + \dot{\beta}$$

Figure 25.2: The component of the vorticity in the z direction.

$$\begin{pmatrix} \delta u^{(a)} \\ \delta v^{(a)} \\ \delta w^{(a)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_z & -\omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} , \quad (25.14)$$

or

$$\delta \mathbf{V}^{(a)} = \frac{1}{2}(\underset{\sim}{\omega} \times \mathbf{r}) \,. \tag{25.15}$$

25.2 Invariants of the strain tensor

Consider the scalar product

$$\begin{aligned}\mathbf{r} \cdot \varepsilon \cdot \mathbf{r} &= \mathbf{r} \cdot (\varepsilon \cdot \mathbf{r}) \\ &= e_{xx}x^2 + e_{yy}y^2 + e_{zz}z^2 + 2(e_{xy}xy + e_{yz}yz + e_{zx}zx) \\ &= f(x, y, z) \,.\end{aligned} \tag{25.16}$$

By putting $f(x, y, z) = \text{constant}$, we obtain a surface of second order, the *strain quadratic*, also called the *ellipsoid of strain*.

We refer the strain quadratic to its principal axes (X_1, X_2, X_3). Then (25.16) takes the form

$$e_1X_1^2 + e_2X_2^2 + e_3X_3^2 = F(X_1, X_2, X_3) = \text{constant} \,. \tag{25.17}$$

Here e_1, e_2, and e_3 are the *principal values* of the strain tensor, and are called the *principal normal strains*.

Using (25.16), we see that

$$\left.\begin{aligned}\delta u^{(s)} &= \frac{1}{2}\frac{\partial f}{\partial x} = e_{xx}x + e_{xy}y + e_{xz}z \\ \delta v^{(s)} &= \frac{1}{2}\frac{\partial f}{\partial y} = e_{yx}x + e_{yy}y + e_{yz}z \\ \delta w^{(s)} &= \frac{1}{2}\frac{\partial f}{\partial z} = e_{zx}x + e_{zy}y + e_{zz}z\end{aligned}\right\} . \tag{25.18}$$

In terms of the principal axes (X_1, X_2, X_3),

$$\delta v_1^{(s)} = \frac{1}{2}\frac{\partial F}{\partial X_1} = e_1X_1, \delta v_2^{(s)} = e_2X_2, \delta v_3^{(s)} = e_3X_3 \,. \tag{25.19}$$

$(\delta v_1^{(s)}, \delta v_2^{(s)}, \delta v_3^{(s)})$ are the projections of $\delta v^{(s)}$ upon the principal axes. Therefore, the symmetric part of $\delta \mathbf{V}$ is composed of three expansions (or contractions) in the mutually orthogonal directions of the principal axes of the strain quadratic.

We next consider the sum of the diagonal elements of **e**, i.e.

$$e_{xx} + e_{yy} + e_{zz} = e_{ii} = \nabla \cdot \mathbf{V} . \tag{25.20}$$

This quantity is an invariant under the rotation of the axes of reference. Consider the transformation by which the surface $f(x, y, z) =$ the constant of (25.16) is transformed into $\boldsymbol{F}(X_1, X_2, X_3) =$ the constant of (25.17). We can write the transformation between the two Cartesian coordinates in the schematic, tabular form shown in Table 25.1

	x	y	z
X1	a1	b1	g1
X2	a2	b2	g2
X3	a3	b3	g3

Table 25.1: The transformation between two Cartesian coordinate systems. Here ai, bi, gi, are the direction cosines between the axes.

Here ai, bi, gi, are the *direction cosines* between the axes. We find that

$$\left. \begin{array}{c} \sum_{i=1}^{3} \alpha_i^2 = 1, \ \sum_{i=1}^{3} \alpha_i \beta_i = 0 \\ \alpha_i \alpha_j + \beta_i \beta_j + \gamma_i \gamma_j = \delta_{ij} \end{array} \right\} , \tag{25.21}$$

where δ_{ij} is *Kronecker's delta* $(\delta_{ij} = 1 \text{ when } i = j, \delta_{ij} = 0 \text{ when } i \neq j)$.

Substituting the expression

$$X_i = \alpha_i x + \beta_i y + \gamma_i z \text{ where } i = 1, 2, 3 \tag{25.22}$$

into (25.17), we obtain

$$\left. \begin{array}{c} \sum_{i=1}^{3} \alpha_i^2 = 1, \ \sum_{i=1}^{3} \alpha_i \beta_i = 0 \\ \alpha_i \alpha_j + \beta_i \beta_j + \gamma_i \gamma_j = \delta_{ij} \end{array} \right\} . \tag{25.23}$$

This expression must be identical with (25.16), so that

$$\left.\begin{aligned} e_{xx} &= \alpha_1^2 e_1 + \alpha_2^2 e_2 + \alpha_3^2 e_3,\ e_{xy} = \alpha_1\beta_1 e_1 + \alpha_2\beta_2 e_2 + \alpha_3\beta_3 e_3 \ , \\ e_{yy} &= \beta_1^2 e_1 + \beta_2^2 e_2 + \beta_3^2 e_3,\ e_{yz} = \beta_1\gamma_1 e_1 + \beta_2\gamma_2 e_2 + \beta_3\gamma_3 e_3 \ , \\ e_{zz} &= \gamma_1^2 e_1 + \gamma_2^2 e_2 + \gamma_3^2 e_3,\ e_{zy} = \gamma_1\alpha_1 e_1 + \gamma_2\alpha_2 e_2 + \gamma_3\alpha_3 e_3 \ , \end{aligned}\right\} . \tag{25.24}$$

It follows that

$$\Delta = e_{xx} + e_{yy} + e_{zz} = \sum_{i=1}^{3} (\alpha_i^2 + \beta_i^2 + \gamma_i^2) e_i = e_1 + e_2 + e_3 \tag{25.25}$$

i.e., the sum $e_{xx} + e_{yy} + e_{zz} = \nabla \cdot \mathbf{V}$ is independent of the choice of reference. Of course, this quantity is called the divergence, and we "knew already" that it is independent of the choice of reference.

Another invariant of the strain tensor is

$$e_{xx}e_{yy} + e_{yy}e_{zz} + e_{zz}e_{xx} - e_{xy}^2 - e_{yz}^2 - e_{zx}^2 = \text{constant} . \tag{25.26}$$

The proof is left as an exercise.

25.3 Stresses

In a moving viscous fluid, forces act not only normal to a surface but also tangential to it. In Fig. 25.3, **tx**, **ty** and **tz** denote respectively the forces per unit area acting upon the *surfaces normal to the positive x, y, and z directions*, respectively. In general, each force has 3 components, so that we can express them as

$$\left.\begin{aligned} \tau_x &= \tau_{xx}\mathbf{i} + \tau_{yx}\mathbf{j} + \tau_{zx}\mathbf{k} \ , \\ \tau_y &= \tau_{xy}\mathbf{i} + \tau_{yy}\mathbf{j} + \tau_{zy}\mathbf{k} \ , \\ \tau_z &= \tau_{xz}\mathbf{i} + \tau_{yz}\mathbf{j} + \tau_{zz}\mathbf{k} \ . \end{aligned}\right\} . \tag{25.27}$$

Symbolically we can write an element of the stresses as $\tau_{ij}(i = 1, 2, 3; j = 1, 2, 3)$. Here τ_{ij} is *the i-component of the force per unit area exerted across a plane surface element normal to the j-direction*. The stresses thus have nine elements, and can be represented by the matrix

$$\mathbf{T} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} . \tag{25.28}$$

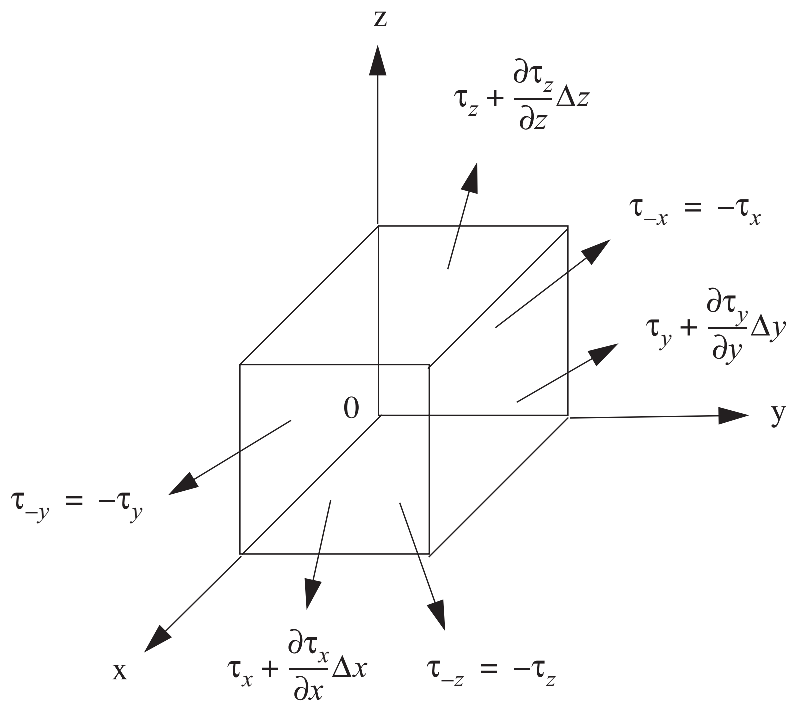


Figure 25.3: Sketch illustrating the normal and tangential forces acting on an element of fluid.

For an arbitrarily chosen plane surface, whose normal vector is $\mathbf{n}$, the force acting per unit area is τ_n. See Fig. 25.4. The equilibrium condition is

$$\tau_{-y} = -\tau_y$$
$$\tau_n \Delta\sigma - \tau_x \Delta\sigma_x - \tau_y \Delta\sigma_y - \tau_z \Delta\sigma_z = 0 \ , \tag{25.29}$$

where $\Delta\sigma$ is the area of the oblique surface, $\Delta\sigma_x$, $\Delta\sigma_y$, $\Delta\sigma_z$, are the areas of the faces lying in the planes $x = 0, y = 0$ and $z = 0$. Denoting the direction cosines of $\mathbf{n}$ by **a**, **b** and **g**, we have $\frac{\Delta\sigma_x}{\Delta\sigma} = \alpha$, $\frac{\Delta\sigma_y}{\Delta\sigma} = \beta$, $\frac{\Delta\sigma_z}{\Delta\sigma} = \gamma$. Therefore (25.29) can be written as

$$\tau_n = \alpha\tau_x + \beta\tau_y + \gamma\tau_z \ , \tag{25.30}$$

or

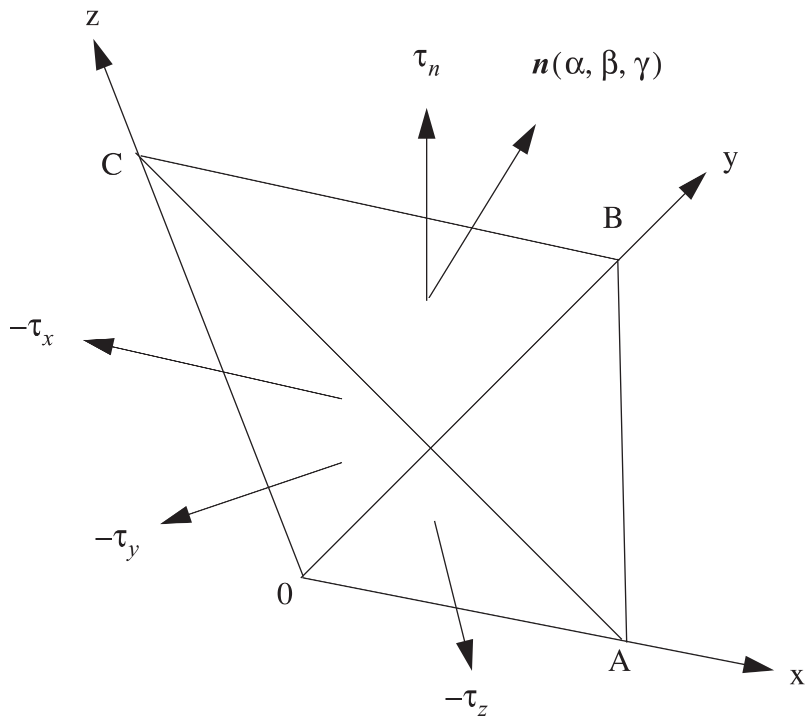


Figure 25.4: Sketch illustrating the normal force acting on a plane surface.

$$\begin{pmatrix} \tau_{xn} \\ \tau_{yn} \\ \tau_{zn} \end{pmatrix} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \tag{25.31}$$

or

$$\tau_{\boldsymbol{n}} = \mathbf{T} \cdot \mathbf{n} . \tag{25.32}$$

Equation (25.32) shows that matrix $\mathbf{T}$ transforms a vector $\mathbf{n}$ into another vector τ_n, so that $\mathbf{T}$ is a tensor, called the *stress tensor*. See Fig. 25.5.

An important property of $\mathbf{T}$ is its *symmetry*, i.e. $\tau_{ij} = \tau_{ji}$. This symmetry can be inferred from the condition of moment equilibrium. Consider for example a moment about the z-axis due to τ_{yx} and τ_{xy}:

$$(\tau_{yx}\Delta y\Delta z)\frac{1}{2}\Delta x - (\tau_{xy}\Delta x\Delta z)\frac{1}{2}\Delta y = \frac{1}{2}(\tau_{yx} - \tau_{xy})\Delta x\Delta y\Delta z . \tag{25.33}$$

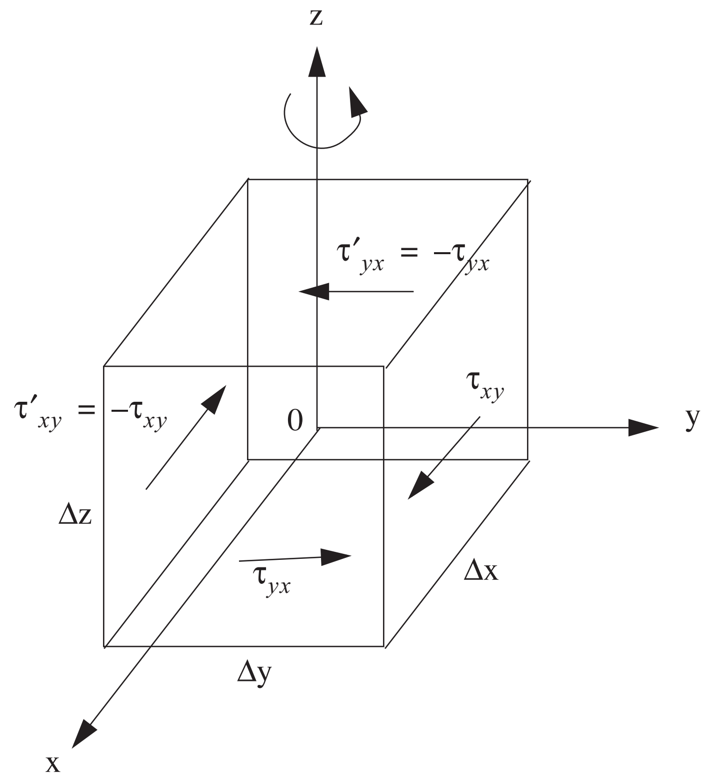


Figure 25.5: Sketch illustrating the components of the stress tensor.

The shear stresses acting on the negative x- and y-surfaces contribute also, so that the total moment about the z-axis is

$$(\tau_{yx} - \tau_{xy})\Delta x \Delta y \Delta z \ . \tag{25.34}$$

Euler's equation (for a rotating rigid body) states that

$$I_z \frac{d\Omega_z}{dt} = (\tau_{yx} - \tau_{xy})\Delta x \Delta y \Delta z \ , \tag{25.35}$$

where $I_z = \frac{(\Delta x)^2 + (\Delta y)^2}{12} = \frac{\rho}{12}\Delta x \Delta y \Delta z[(\Delta x)^2 + (\Delta y)^2]$ is the *moment of inertia* of the fluid element $\rho \Delta x \Delta y \Delta z$, and Ω_z is the angular velocity about the positive z-axis. Then (25.35) reduces to

$$\frac{\rho}{12}\{(\Delta x)^2 + (\Delta y)^2\}\frac{d\Omega_z}{dt} = \tau_{yx} - \tau_{xy} \ . \tag{25.36}$$

In order that $\frac{d\Omega_z}{dt}$ remains finite, we need $(\tau_{yx} - \tau_{xy}) \to 0$ when $\Delta x, \Delta y \to 0$. It follows that $\tau_{yx} = \tau_{xy}$. Similar considerations establish that

$$\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz} . \tag{25.37}$$

The three diagonal elements, txx, tyy, tzz, are normal stresses. The six off-diagonal elements are *tangential stresses*.

25.4 Invariants of the stress tensor

As in the case of the strain tensor, we have a scalar product

$$\mathbf{r} \cdot \mathsf{T} \cdot \mathbf{r} = \tau_1 X_1^2 + \tau_2 X_2^2 + \tau_3 X_3^2 = \text{constant} , \tag{25.38}$$

where t1, t2, t3 are the principal stresses acting on the surface elements normal to the principal axes (X^1, X^2, X^3) of the *ellipsoid of stress*. It can be shown that $\mathbf{r} \cdot \mathsf{T} \cdot \mathbf{r}$ is an invariant under the rotation of the axes of reference. The sum of the normal stresses

$$\tau_{xx} + \tau_{yy} + \tau_{zz} = \tau_{ii} = \tau_1 + \tau_2 + \tau_3 = -3p , \tag{25.39}$$

is also an invariant. For a moving fluid, (25.39) is the definition of pressure p, which reduces to the static pressure when the fluid is at rest.

25.5 The resultant forces due to the spatial variation of the stresses

The resultant force acting in the x-direction on a volume element of fluid $\Delta x \Delta y \Delta z$ is

$$\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}\Delta x\right)\Delta y \Delta z - \tau_{xx}\Delta y \Delta z + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}\Delta y\right)\Delta x \Delta z$$

$$-\tau_{xy}\Delta y \Delta z + \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z}\Delta z\right)\Delta x \Delta y - \tau_{xz}\Delta x \Delta y = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right)\Delta x \Delta y \Delta z . \tag{25.40}$$

Therefore the three components of the resultant force (vector) per unit volume are

$$\left.\begin{aligned} X &= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \frac{\partial \tau_{xj}}{\partial x_j} \\ Y &= \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = \frac{\partial \tau_{yj}}{\partial x_j} \\ Z &= \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \frac{\partial \tau_{zj}}{\partial x_j} \end{aligned}\right\} . \tag{25.41}$$

Equation (25.41) can be compactly written as

$$\begin{aligned} Div\mathbf{T} &= \frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y} + \frac{\partial \tau_z}{\partial z} \\ &= \mathbf{i}\frac{\partial \tau_{xj}}{\partial x_j} + \mathbf{j}\frac{\partial \tau_{yj}}{\partial y_j} + \mathbf{k}\frac{\partial \tau_{zj}}{\partial zj} . \end{aligned} \tag{25.42}$$

$Div\mathbf{T}$ is a *tensor divergence* and, therefore, is a vector.

25.6 Physical laws connecting the stress and strain tensors

The relation between the stress tensor and the strain tensor may be established by the following assumptions.

i)

$$\tau_{ij} = -p\delta_{ij} + F_{ij} , \tag{25.43}$$

where F_{ij} is the *frictional stress tensor* which depends on the space derivatives of the velocity, i.e. the strain tensor.

ii) The relationship between F_{ij} and $e^{\mathbf{ij}}$ is linear.

iii) The relation does not depend on the frame of reference.

We have already shown that, with reference to the respective principal axes,

$$\mathbf{T} = \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix}, \varepsilon = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix} . \tag{25.44}$$

Because $\mathbf{t1}$, $\mathbf{t2}$, $\mathbf{t3}$ are normal stresses and $e^{\mathbf{1}}$, $e^{\mathbf{2}}$, $e^{\mathbf{3}}$ are normal strains, it is natural to assume that the principal axes of both tensors are the same. We therefore assume that

$$\left.\begin{aligned}\tau_1 &= -p + \lambda\nabla\cdot\mathbf{V} + 2\mu e_1\\ \tau_2 &= -p + \lambda\nabla\cdot\mathbf{V} + 2\mu e_2\\ \tau_3 &= -p + \lambda\nabla\cdot\mathbf{V} + 2\mu e_3\end{aligned}\right\}, \tag{25.45}$$

where p satisfies (25.39).

Writing direction cosines between the principal axes (X^1, X^2, X^3) and axes of a Cartesian coordinate system (x, y, z) as shown in Table 25.2,

	x	y	z
X^1	a1	b1	g1
X^2	a2	b2	g2
X^3	a3	b3	g3

Table 25.2: Direction cosines between principle axes.

we find that

$$\begin{aligned}\tau_{xx} &= \alpha_1^2\tau_1 + \alpha_2^2\tau_2 + \alpha_3^2\tau_3\\ \tau_{xy} &= \alpha_1\beta_1\tau_1 + \alpha_2\beta_2\tau_2 + \alpha_3\beta_3\tau_3\end{aligned}, \tag{25.46}$$

etc. Substituting (25.45) into (25.46), and using (25.24), we obtain

$$\left(\begin{aligned}\tau_{xx} &= [-p + \lambda(\nabla\cdot\mathbf{V}) + 2\mu e_1]\alpha_1^2 + [-p + \lambda(\nabla\cdot\mathbf{V}) + 2\mu e_2]\alpha_2^2\\ &\quad + [-p + \lambda(\nabla\cdot\mathbf{V}) + 2\mu e_3]\alpha_3^2 - p + \lambda\Delta + 2\mu e_{xx},\\ \tau_{xy} &= [-p + \lambda(\nabla\cdot\mathbf{V}) + 2\mu e_1]\alpha_1\beta_1 + [-p + \lambda(\nabla\cdot\mathbf{V}) + 2\mu e_2]\alpha_2\beta_2\\ &\quad + [-p + \lambda(\nabla\cdot\mathbf{V}) + 2\mu e_3]\alpha_3\beta_3 = 2\mu e_{xy},\end{aligned}\right. \tag{25.47}$$

etc. In general we have

$$\tau_{ij} = -p\delta_{ij} + \lambda\nabla\cdot\mathbf{V}\delta_{ij} + 2\mu e_{ij}. \tag{25.48}$$

This expression gives

$$\tau_{ii} = -3p + \lambda\nabla\cdot\mathbf{V} + 2\mu\nabla\cdot\mathbf{V}$$

$$= -3p + (3\lambda + 2\mu)\nabla\cdot\mathbf{V}\ . \tag{25.49}$$

Because we have defined $\tau_{ij} = -3p$, we conclude that

$$3\lambda + 2\mu + 0\ . \tag{25.50}$$

Thus we have the final expression

$$\tau_{ij} = -p\delta_{ij} + 2\mu\left(e_{ij} - \frac{1}{3}\nabla\cdot\mathbf{V}\delta_{ij}\right)$$

$$= -p\delta_{ij} + F_{ij}\ . \tag{25.51}$$

25.7 The Navier-Stokes equation

Substitution of (25.51) into (25.42) gives the resultant forces per unit volume, as expressed by the divergence of the stress tensor:

$$Div\mathbf{T} = -\nabla p + Div\mathbf{F} \tag{25.52}$$

The equation of motion in the inertial frame (with subscript *a* omitted) is

$$\frac{D\mathbf{V}}{Dt} = -\nabla\phi - \alpha\nabla p + \alpha Div\mathbf{F} \tag{25.53}$$

or, in Cartesian tensor notation,

$$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial\phi}{\partial x_i} - \alpha\frac{\partial p}{\partial x_i} + \alpha\frac{\partial}{\partial x_j}\left\{2\mu\left[e_{ij} - \frac{1}{3}(\nabla\cdot\mathbf{V})\delta_{ij}\right]\right\}$$

$$= -\frac{\partial\phi}{\partial x_i} - \alpha\frac{\partial p}{\partial x_i} + \alpha\frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\right\}\ . \tag{25.54}$$

This is called the *Navier-Stokes equation*.

When the fluid is *incompressible* $(\nabla\cdot\mathbf{V} = 0)$, and m is constant, (25.54) reduces to

$$\frac{Dv_i}{Dt} = -\frac{\partial\phi}{\partial x} - \alpha\frac{\partial p}{\partial x_i} + \alpha\mu\left(\frac{\partial^2 v_i}{\partial x_j^2}\right), \tag{25.55}$$

or, in vector form,

$$\frac{D\mathbf{V}}{Dt} = -\nabla\phi - \alpha\nabla p + \alpha\mu\nabla^2\mathbf{V}\,. \tag{25.56}$$

Here we see the Laplacian of the vector $\mathbf{V}$.

It should be noted that $e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ vanishes when $\mathbf{V} = \Omega \times \mathbf{r}$ (rigid rotation), and that $\nabla \cdot (\Omega \times \mathbf{r}) = 0$. This means that we do not have to worry about the distinction between $\mathbf{V}_a = \mathbf{V} + \Omega \times \mathbf{r}$ and $\mathbf{V}$ in the expression $\mathbf{F} = Div\mathsf{F}$ in the equation of relative motion. The frictional force is the same in the rotating frame as in the inertial frame. This should be intuitively obvious.

25.8 A proof that the dissipation is non-negative

The dissipation rate is given by

$$\begin{aligned}(\mathsf{F}\cdot\nabla)\cdot\mathbf{V} &= F_{ij}\frac{\partial v_i}{\partial x_j} \equiv \delta \\ &= F_{11}\frac{\partial u}{\partial x} + F_{12}\frac{\partial u}{\partial y} + F_{13}\frac{\partial u}{\partial z} \\ &+ F_{21}\frac{\partial v}{\partial x} + F_{22}\frac{\partial v}{\partial y} + F_{23}\frac{\partial v}{\partial z} \\ &+ F_{31}\frac{\partial w}{\partial x} + F_{32}\frac{\partial w}{\partial y} + F_{33}\frac{\partial w}{\partial z}\,.\end{aligned} \tag{25.57}$$

Since $F_{ij} = F_{ji}$, we have

$$\delta = F_{ij}\frac{\partial v_i}{\partial x_j} = F_{ji}\frac{\partial v_i}{\partial x_j} = F_{ij}\frac{\partial v_j}{\partial x_i}, \tag{25.58}$$

so we can write

$$2\delta = F_{ij}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{25.59}$$

But

$$F_{ij} = 2\mu\left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial v_i}{\partial x_i}\delta_{ij}\right]$$

$$= 2\mu\left\{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)(1-\delta_{ij}) + \left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial v_i}{\partial x_i}\right]\delta_{ij}\right\}, \tag{25.60}$$

or

$$F_{ij} = 2\mu\left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)(1-\delta_{ij}) + \frac{2}{3}\frac{\partial v_i}{\partial x_j}\delta_{ij}\right]. \tag{25.61}$$

Substituting (25.61) into (25.59), we find that

$$\delta = \mu\left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)(1-\delta_{ij}) + \frac{2}{3}\frac{\partial v_i}{\partial x_j}\delta_{ij}\right]\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$$= \mu\left[2e_{ij}e_{ij}(1-\delta_{ij}) + \frac{4}{3}(\nabla\cdot\mathbf{V})^2\delta_{ij}\right]. \tag{25.62}$$

This expression is obviously non-negative, since $(1-\delta_{ij}) \geq 0$ and $\delta_{ij} \geq 0$. For a nondivergent flow, the dissipation can be expanded, using Cartesian coordinates, as

$$\delta = 2\mu e_{ij}e_{ij}$$

$$= \frac{1}{2}\mu\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right.$$

$$+ \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2$$

$$\left. + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z} + \frac{\partial v}{\partial z}\right)^2\right]$$

$$= \mu\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2\right]. \tag{25.63}$$

This shows that **d** is the sum of squares; again, it cannot be negative.