CHAPTER 7 | *The Energy Cycle*

7.1   *Available potential energy*

We have shown that, under dry adiabatic and frictionless processes,

$$\frac{d}{dt}\int_V (c_v T + \phi + K)\rho\, dV = 0 .$$

(7.1)

Here the integral is over the mass of the whole atmosphere. For each vertical column, the vertically integrated sum of the mass-weighted internal and potential energies is equal to the vertical integral of the mass-weighted enthalpy, i.e.,

$$\int_0^{p_S} (c_v T + \phi)\, dp = \int_0^{p_S} c_p T\, dp ,$$

(7.2)

provided that we have hydrostatic balance. This can be demonstrated as follows: Using hydrostatics in the form

$$\frac{\partial p}{\partial z} = -\rho g,$$

(7.3)

we find that the vertically integrated potential energy, P, satisfies

$$P \equiv \int_0^\infty gz\rho\, dz = -\int_0^\infty \left(\frac{\partial p}{\partial z}z\right)dz = -\int_0^\infty \left[\frac{\partial}{\partial z}(pz) - p\right]dz$$

$$= -[(pz)\ \big|_{z=0}^{z=\infty} - \int_0^\infty \rho RT dz] = \int_0^\infty \rho RT dz .$$

(7.4)

Eq. (7.4) says that the total potential energy of the column is proportional to the average temperature of the column. The explanation is that warmer air occupies a larger volume, so that a warmer column is "taller." It follows from (7.4) that

$$P + I = \int_0^\infty \rho R T \, dz + \int_0^\infty c_v T \rho \, dz$$

$$= \int_0^\infty (c_v + R) T \rho \, dz$$

$$= \int_0^\infty c_p T \rho \, dz \ . \tag{7.5}$$

Here I is the vertically integrated internal energy. Note that $c_v T + \phi = c_p T$ *is not true*; this would imply that $\phi = RT$, which is obviously nonsense.

Let H and K be the total (mass integrated) enthalpy and kinetic energy of the atmosphere. It follows from (7.1) and (7.2) that $H + K$ is invariant, under dry-adiabatic frictionless processes, i.e.

$$\frac{d}{dt}(H + K) = 0 \ . \tag{7.6}$$

Imagine that we have the power to spatially rearrange the mass of the atmosphere at will, adiabatically and reversibly. As the parcels move, their entropy and potential temperature do not change, but their enthalpy and temperature do change. Suppose that we are given a state of the atmosphere - a set of maps, if you like. Starting from this given state, we move parcels around adiabatically and without friction until we find the unique state of the system that minimizes H. This means that we have reduced H as much as possible from its value in the given state. Because $H + K$ does not change, K *is maximized* in this special state, which Lorenz (1955) called the "reference state," and which we will call the "A-state."

You should prove for yourself that the mass-integrated potential energy of the entire atmosphere is lower in the A-state than in the given state. This means that the center of gravity of the atmosphere descends as the atmosphere passes from the given state to the A-state.

In passing from the given state to the A-state we have

$$H_{gs} \to H_{min},$$

$$K_{gs} \to K_{gs} + (H_{gs} - H_{min}) = K_{max}, \tag{7.7}$$

where H_{min} is the value of H in the A-state, and subscript gs denotes the given state. The non-negative quantity

$$A \equiv H_{gs} - H_{min} \geq 0 \tag{7.8}$$

is called the "available potential energy," or APE. The APE was first defined by Lorenz (1955; see Fig. 7.1). Eq. (7.8) gives the fundamental definition of the APE.

Figure 7.1: Prof. Edward N. Lorenz, who proposed the concept of available potential energy, and has published a great deal of additional very fundamental research in the atmospheric sciences and the relatively new science of nonlinear dynamical systems.

Notice that the APE is a property of the *entire* atmosphere; it cannot be rigorously defined for a portion of the atmosphere, although the literature does contain studies in which the APE is computed, without rigorous justification, for portions of the atmosphere, e.g. the Northern Hemisphere.

The A-state is invariant under adiabatic processes, because it depends only on the probability distribution of θ over the mass, rather than on any particular spatial arrangement of the air. Therefore

$$\frac{dA}{dt} \;=\; \frac{d}{dt}(H_{gs} - H_{rs}) \;=\; \frac{dH_{gs}}{dt} \,. \tag{7.9}$$

Then (7.6) implies that

$$\frac{d}{dt}(A + K) \;=\; 0 \,. \tag{7.10}$$

The sum of the available potential energy and the kinetic energy is invariant under adiabatic frictionless processes that only convert between A and K.

We can deduce the properties of the A-state as follows: There can be no horizontal pressure gradients in the A-state, because if there were K could increase. It follows that the potential temperature must be constant on isobaric surfaces, which is of course equivalent to the statement that p is constant on θ surfaces. This means that *both* the variance of θ on isobaric surfaces *and* the variance of p on isentropic surfaces are measures of the APE. There

can be no static instability $(\partial\theta/\partial z < 0)$ in the A-state for a similar reason. From these considerations we conclude that, in the A-state, θ and T are uniform on each pressure-surface, or equivalently that p is uniform on each θ-surface, and also that θ does not decrease upward.

It would appear, and to a large extent it is true, that in passing from the given state to the A-state, no mass can cross an isentropic surface, because we allow only dry adiabatic processes for which $\dot{\theta} = 0$. This implies that $\overline{p}^{\,\theta}$, *the average pressure on an isentropic surface, cannot change* as we pass to the A-state. There is an important exception to this rule, however. If $\partial\theta/\partial z < 0$ in the given state, then the average pressure on an isentropic surface will be different in the A-state. This case of static instability is discussed below.

A complication arises: What about θ-surfaces that intersect the ground in the given state? These are treated as though they continue along the ground, as shown in Fig. 7.2,
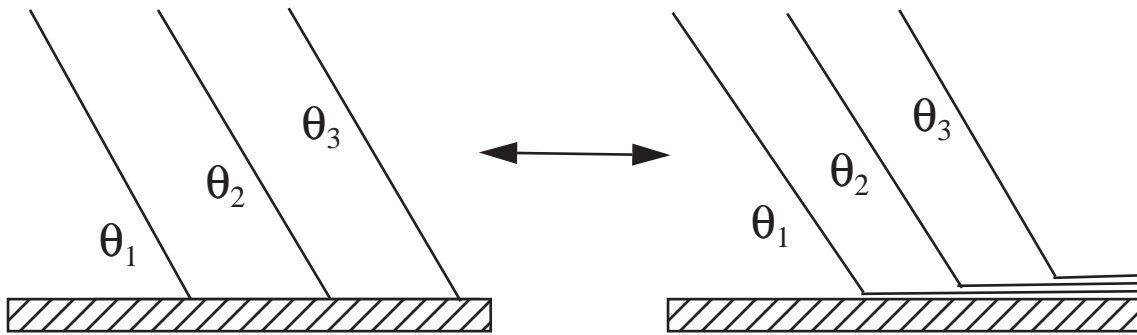
Figure 7.2:   Sketch illustrating the concept of "massless layers."

Because the "layers" between the isentropic surfaces that are following the ground contain no mass, they have no effect on the physics. They are called "massless layers." Where a θ-surface intersects the Earth's surface, the pressure is $p = p_S$.

The concept of massless layers allows us to write

$$\int_0^{p_S} (\ \ )\,dp = \int_0^{\infty} (\ \ )\frac{\partial p}{\partial\theta}\,d\theta, \tag{7.11}$$

where ( ) can be anything. This will be used below. Note that the lower limit of integration on the right-hand side of (7.11) is zero.

A useful expression for the APE can be derived as follows. The total enthalpy is given by

$$H = c_p p_0^{-\kappa} \int_M p^\kappa \theta \, dM$$

$$= \frac{c_p a^2}{g p_0^\kappa} \int_{-\pi/2}^{2\,\pi/2} \int_0^{2\pi} \int_0^{p_S} p^\kappa \theta \cos\varphi \, dp \, d\lambda \, d\varphi. \tag{7.12}$$

Integration by parts gives

$$H = \frac{c_p a^2}{(1+\kappa)g p_0^\kappa} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^\infty p^{1+\kappa} \cos\varphi \, d\theta \, d\lambda \, d\varphi. \tag{7.13}$$

Note that vertical integration is now with respect to θ rather than p, and that the lower limit of integration is $\theta = 0$. Let $\overline{p}^\theta$ be the average pressure on an isentropic surface (taking into account intersections with the ground). Recall that, provided that there are no regions of dry static instability, $\overline{p}^\theta$ is the same in the A-state as in the given state. Then use of (7.13) in (7.8) gives

$$A = \frac{c_p a^2}{(1+\kappa)g p_0^\kappa} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^\infty [p^{1+\kappa}(\theta) - (\overline{p}^\theta)^{1+\kappa}] \cos\varphi \, d\theta \, d\lambda \, d\varphi. \tag{7.14}$$

Note that (7.14) is valid only if $\partial\theta/\partial z \geq 0$ everywhere in the given state, because we have assumed that $\overline{p}^\theta$ is the same in the A-state as in the given state. So long as this assumption is satisfied, (7.14) is exact. *The most general expression for the available potential energy is the definition $A \equiv H_{gs} - H_{min}$.*

Let p' be the departure of p from its average on an isentropic surface, so that $p = \overline{p}^\theta + p'$. Lorenz used the binomial theorem to write

$$\overline{p^{1+\kappa}(\theta)} - (\overline{p}^\theta)^{1+\kappa} \cong (\overline{p}^\theta)^{1+\kappa} \left[\frac{\kappa(1+\kappa)}{2!} \overline{\left(\frac{p'}{\overline{p}^\theta}\right)^2} \right], \tag{7.15}$$

and he showed that this is actually a rather good approximation. In (7.15) the leading term of the expansion (the linear term) vanishes because of the averaging. Substitution of (7.15) in to (7.14) gives

$$A \cong \frac{R a^2}{2 g p_0^\kappa} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^\infty (\overline{p}^\theta)^{1+\kappa} \overline{\left(\frac{p'}{\overline{p}^\theta}\right)^2} \cos\varphi \, d\theta \, d\lambda \, d\varphi. \tag{7.16}$$

Because he wanted to express his results in terms of a vertical integral over pressure rather than potential temperature, Lorenz also used

$$p' \cong -\theta' \frac{\overline{\partial p}^{\theta}}{\partial \theta} = -\theta' \left(\frac{\partial}{\partial p} \bar{\theta} \right)^{-1} , \tag{7.17}$$

where θ' represents the departure of θ from $\bar{\theta}$, its global average on a p-surface. Substitution of (7.17) into (7.16) gives

$$A \cong \frac{Ra^2}{2gp_0^{\kappa}} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^{p_S} \frac{\bar{\theta}^2}{(p^{1-\kappa}) \left(-\frac{\partial}{\partial p} \bar{\theta} \right)} \left(\frac{\theta'}{\bar{\theta}} \right)^2 \cos\varphi \, dp \, d\lambda \, d\varphi . \tag{7.18}$$

Note that in (7.18) the independent variable for vertical integration has been changed from θ *to* p. Eq. (7.18) involves a weighted average of the square of the departure of θ from its mean on the pressure surface. The average of the square of the departure from the mean is called the "variance about the mean," or just the variance. The variance is a measure of how variable a quantity is; if the quantity is constant, and so everywhere equal to its mean, its variance must be zero. If the quantity is not constant, its variance is positive. Because we are interested in variability, variances are quite important in the study of the general circulation.

Finally, Lorenz used the hydrostatic equation in the form

$$-\frac{\partial \theta}{\partial p} = \frac{\kappa \theta}{p} \left(\frac{\Gamma_d - \Gamma}{\Gamma_d} \right), \tag{7.19}$$
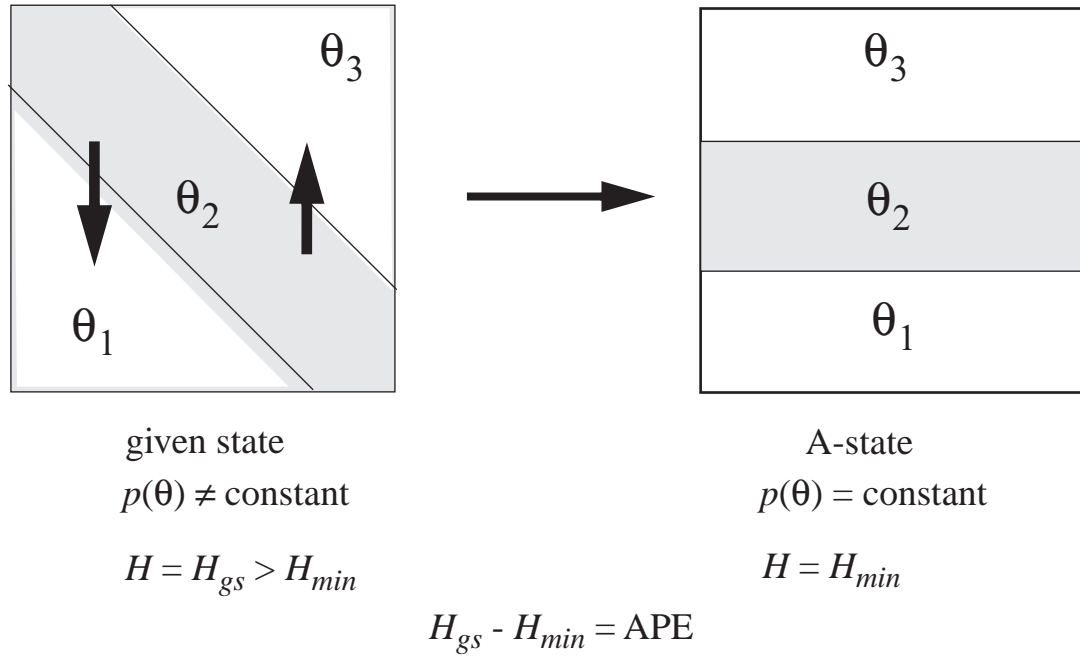
where $\Gamma \equiv -\frac{\partial T}{\partial z}$ is the lapse rate of temperature and $\Gamma_d \equiv \frac{g}{c_p}$ is the dry adiabatic lapse rate, as well as

$$\frac{\theta'}{\bar{\theta}} = \frac{T}{\bar{T}}, \tag{7.20}$$

to rewrite (7.18) as

$$A = \frac{a^2}{2} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^{p_S} \frac{\bar{T}}{(\Gamma_d - \Gamma)} \left(\frac{T}{\bar{T}} \right)^2 \cos\varphi \, dp \, d\lambda \, d\varphi . \tag{7.21}$$

This result shows that the available potential energy is closely related to the variance of temperature on isobaric surfaces. It also increases as the lapse rate of temperature increases, i.e. as the atmosphere becomes less stable in the dry static sense.

given state   A-state

$p(\theta) \neq$ constant $p(\theta) =$ constant

$H = H_{gs} > H_{min}$ $H = H_{min}$

$$H_{gs} - H_{min} = \text{APE}$$

Figure 7.3:   Sketch illustrating the transition from a given state to the A-state. Here p
increases downward and we assume that $\theta_3 > \theta_2 > \theta_1$, which means that the
atmosphere is statically stable.

The APE is only a tiny fraction of the total PE, in the real atmosphere. The APE is comparable in magnitude to the total kinetic energy. Both are on the order of 10^6 -10^7 J m^{-2}.

As shown by the heavy arrows in Fig. 7.3, warm air must rise (move to lower pressure) and cold air must sink (move to higher pressure) to pass from the given state to the A-state. This is what happens as APE is released; the APE of the A-state itself is obviously zero.

7.2   The gross static stability

The A-state used in the definition of A defines a correspondence or mapping between θ and p. For each p there is one possible value of θ (the converse is not necessarily true). We can say that in the A-state, p and θ are perfectly correlated, (or, more correctly, perfectly anti-correlated).

Consider the opposite limit, in which θ and p are *completely uncorrelated*. In this "S-state," all possible values of θ occur, with equal probability, for any given p. This will be the case if the θ surfaces are vertical, so that $\partial\theta/\partial p = 0$, and if the surface pressure is globally uniform. See Fig. 7.4.

To see why the surface pressure must be globally uniform, suppose that there are variations in the surface pressure in the S-state (independent of height at each location).

Therefore, if the surface pressure varies geographically, we can say that the surface pressure varies with θ. As and example, suppose that, in the S-state, for $\theta = \theta_1$ the surface pressure is 900 mb. Then if i tell you that i am a pressure 1000 mb, you know that my θ cannot be θ_1, i.e., you have a clue that will help you guess my θ. This shows that variations of the surface pressure n the S-state would violate the rule that θ and p are completely uncorrelated in the S-state. Therefore, the surface pressure must be uniform in the S-state.

$$S \equiv H_{\text{S-state}} - H_{gs}. \tag{7.22}$$

The total enthalpy of the S-state is given by

$$
\begin{aligned}
H_{\text{S-state}} &= c_p p_0^{-\kappa} \int_M p^\kappa \theta \, dM \\
&= \frac{c_p a^2}{g p_0^\kappa} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^{p_S} p^\kappa \theta \cos\varphi \, dp \, d\lambda \, d\varphi \\
&= \frac{c_p a^2}{g p_0^\kappa} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \theta \left(\int_0^{p_S} p^\kappa dp \right) d\lambda \cos\varphi \, d\varphi \\
&= \frac{c_p a^2}{g p_0^\kappa} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \theta \left(\frac{p_S^{1+\kappa}}{1+\kappa} \right) d\lambda \cos\varphi \, d\varphi \\
&= \frac{c_p a^2}{g p_0^\kappa} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \theta \left[\frac{(\overline{p_S})^{1+\kappa}}{1+\kappa} \right] d\lambda \cos\varphi \, d\varphi \\
&= \frac{c_p a^2}{g p_0^\kappa} \left[\frac{(\overline{p_S})^{1+\kappa}}{1+\kappa} \right] \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \theta \, d\lambda \cos\varphi \, d\varphi \quad .
\end{aligned}
\tag{7.23}
$$

In (7.23), $\overline{p_S}$ is the globally averaged surface pressure, which is the same in the given state and the S-state. Eq. (7.23) can be rewritten as

$$H_{\text{S-state}} = \frac{c_p a^2}{g} \left(\frac{\tilde{p}^\kappa}{p_0^\kappa} \right) \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \left(\int_0^{\overline{p_S}} \theta \, dp \right) \cos\varphi \, dp \, d\lambda \, d\varphi, \tag{7.24}$$

where

$$\tilde{p}^\kappa \equiv \frac{(\overline{p_S})^\kappa}{1+\kappa}. \tag{7.25}$$

In (7.24), the integral over pressure simply amounts to multiplication by $\overline{p_S}$, because θ is independent of height in the S-state.
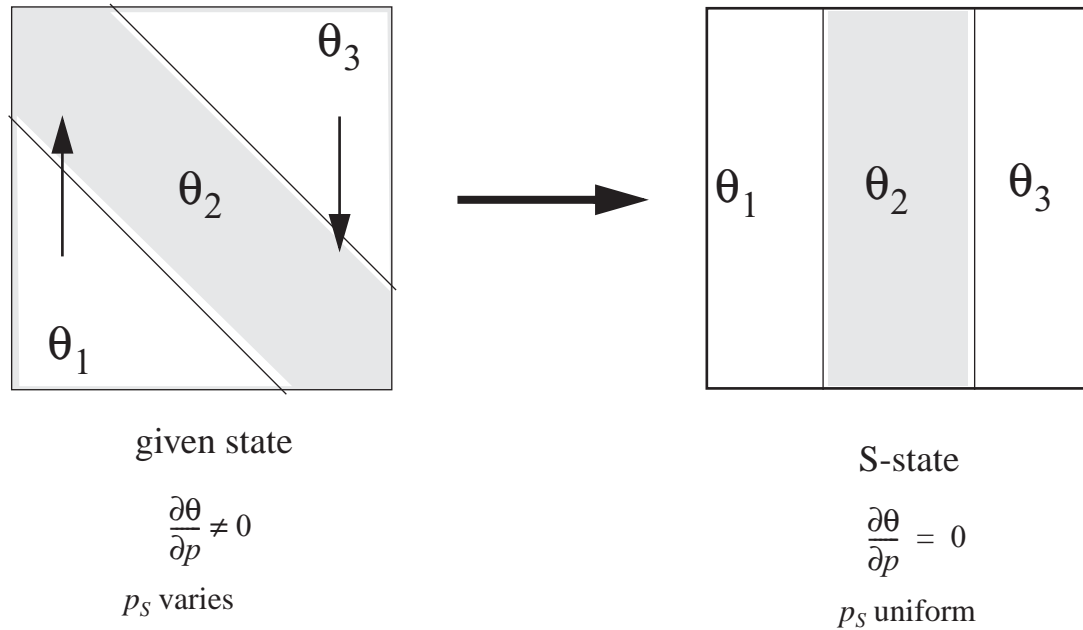
The integral of the potential temperature over the entire mass of the atmosphere must be exactly the same for the given state and the S-state, i.e.,

$$\frac{a^2}{g}\left[\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\left(\int_0^{p_S}\theta\,dp\right)\cos\varphi\,d\lambda\,d\varphi\right]_{\text{given state}}$$
$$= \frac{a^2}{g}\left[\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\left(\int_0^{\overline{p_S}}\theta\,dp\right)\cos\varphi\,d\lambda\,d\varphi\right]_{\text{S-state}} \quad . \tag{7.26}$$

This allows us to rewrite (7.24) as

$$H_{\text{S-state}} = \frac{c_p a^2}{g}\left(\frac{\tilde{p}^\kappa}{p_0^\kappa}\right)\left[\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\left(\int_0^{p_S}\theta\,dp\right)\cos\varphi\,d\lambda\,d\varphi\right]_{\text{given state}} \quad . \tag{7.27}$$

Now substitute (7.27) into (7.22), to obtain

Figure 7.4: Sketch illustrating the transition from a given state to the S state used to define the gross static stability.

$$S \equiv \frac{c_p a^2}{g} \left(\frac{\tilde{p}^{\kappa}}{p_0^{\kappa}} \right) \left[\int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \left(\int_0^{p_S} \theta dp \right) \cos\varphi d\lambda d\varphi \right]_{\text{given state}}$$

$$- \frac{c_p a^2}{g p_0^{\kappa}} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^{p_S} p^{\kappa} \theta \cos\varphi dp d\lambda d\varphi \quad . \tag{7.28}$$

This can be rearranged to

$$S \equiv \frac{c_p a^2}{g p_0^{\kappa}} \left\{ \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \left[\int_0^{p_S} (\tilde{p}^{\kappa} - p^{\kappa}) \theta dp \right] \cos\varphi d\lambda d\varphi \right\}_{\text{given state}} . \tag{7.29}$$

Integration by parts gives

$$S \equiv \frac{c_p a^2}{g p_0^{\kappa}(1+\kappa)} \left\{ \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \left[\int_0^{p_S} [(\overline{p_S})^{\kappa} p - p^{1+\kappa}] \left(-\frac{\partial\theta}{\partial p} \right) dp \right] \cos\varphi d\lambda d\varphi \right\}_{\text{given state}} , \tag{7.30}$$

which shows that S is a weighted average of $(-\partial\theta/\partial p)$; it is, therefore, a measure of the static stability, and this accounts for its name. Like the available potential energy, the gross static stability is defined only for the atmosphere as a whole.

The globally averaged surface pressure and the probability distribution of θ are invariant under adiabatic processes. This means that the S-state is invariant as well. Because S is defined as the difference between the total enthalpy of the S-state and the total enthalpy of the given state, we can write

$$\frac{dS}{dt} = -\frac{dH}{dt} . \tag{7.31}$$

Since $\frac{d}{dt}(H + K) = 0$, it follows from (7.31) that

$$\frac{dS}{dt} = \frac{dK}{dt} . \tag{7.32}$$

This shows that when K is produced by conversion from APE, the gross static stability increases. Isn't that what you would expect?

7.3   Examples: The available potential energies of three simple systems

7.3.1   The APE associated with static instability

As an example, consider a simple system containing two parcels of equal mass. In the

given state, parcels with potential temperature θ_1 and θ_2 reside at pressures p_1 and p_2, respectively. We assume that $\theta_1 < \theta_2$ and $p_1 < p_2$, so that the given state is statically unstable. The enthalpy per unit mass of parcel i is $c_p \theta_i \left(\dfrac{p_i}{p_0} \right)^{\kappa}$. If the parcels are interchange ("swapped") so that parcel number 2 goes to pressure p_1 and vice versa, the change in the total enthalpy per unit mass is

$$\Delta H = c_p(\theta_1 - \theta_2)\left[\left(\frac{p_2}{p_0} \right)^{\kappa} - \left(\frac{p_1}{p_0} \right)^{\kappa} \right], \tag{7.33}$$

which is negative. This implies that the total enthalpy has been reduced and so is minimized by the swap; the final state is the A-state, and the change in enthalpy given by (7.33) is the available potential energy of the system per unit mass.

Lorenz (1978) generalized the concept of APE for a moist atmosphere, in which moist adiabatic processes are acknowledged to be, well, adiabatic. Randall and Wang (1992) showed that this moist APE can be used to define a generalized CAPE that represents the potential energy available for conversion into the kinetic energy of cumulus convection.

7.3.2   The APE associated with meridional temperature gradients

As a second example consider an idealized planet, with no orography and a uniform surface pressure in the given state. Suppose that the potential temperature of the given state is a function of latitude only:

$$\theta_{gs}(\mu) = \theta_0(1 - \Delta_H \mu^2), \tag{7.34}$$

Here Δ_H is a constant, and $\mu \equiv \sin\varphi$. The subscript "gs" denotes the given state. Eq. (7.34) is the same meridional distribution of θ as used for θ_E in Chapter 4. Recall that for realistic states $0 < \Delta_H < 1$. For realistic states, $\Delta_H > 0$. What is the available potential energy of this idealized atmospheric state? To answer this question, we need to complete the total enthalpies of the given state and the A-state, and subtract them. The first step is to find the A-state.

The mass in a latitude belt of width $d\varphi$ is:

$$dm = 2(2\pi a \cos\varphi)\left(\frac{p_S}{g} \right)(a d\varphi)$$

$$= \frac{4\pi a^2}{g} p_S d\mu . \tag{7.35}$$

where $\mu \equiv \sin\varphi$, and $d\mu = \cos\varphi d\varphi$. In (7.35) the leading factor of two is included because we have symmetry across the equator, so when we increment latitude in one hemisphere we actually pick up mass from two "rings" of air, one in each hemisphere. The rate of change of

θ as we add mass is:

$$\frac{d\theta}{dm} = \left(\frac{d\theta}{d\mu}\right)_{gs}\left(\frac{dm}{d\mu}\right)^{-1}. \tag{7.36}$$

The subscript "gs" denotes the given state. Combining (7.35) through (7.37), we get

$$\frac{d\theta}{dm} = \frac{g}{4\pi a^2 p_S}\left(\frac{d\theta}{d\mu}\right)_{gs}. \tag{7.37}$$

In the A-state, the θ-surfaces are flat, and the increment of mass between two θ-surfaces is

$$dm = \frac{4\pi a^2}{g}dp, \tag{7.38}$$

or

$$dm = \frac{4\pi a^2}{g}\left(\frac{dp}{d\theta}\right)_{rs}d\theta, \tag{7.39}$$

so that

$$\left(\frac{d\theta}{dp}\right)_{rs} = \frac{d\theta}{dm}\frac{4\pi a^2}{g}. \tag{7.40}$$

The subscript "*rs*" denotes the A-state.

Now substitute $\frac{d\theta}{dm}$ from (7.37) into (7.40). We can do this because the distribution of θ over the mass must be the same in the A-state as in the given state. The result is

$$\left(\frac{d\theta}{dp_*}\right)_{rs} = \left(\frac{d\theta}{d\mu}\right)_{gs}. \tag{7.41}$$

where

$$p_* \equiv \frac{p}{p_S}. \tag{7.42}$$

We have to be careful when we look at this equation. The left-hand-side refers to the distribution of θ with pressure *in the A-state*. The right-hand-side refers to the distribution of

θ with μ *in the given state*. Keep in mind that, in this idealized example, θ does not vary with pressure in the given state, and it does not vary with μ in the A-state.

From (7.41), we see that p_* *plays the same role in the A-state as* μ *plays in the given state.* Also recall that the potential temperature must increase upward in the A-state. Referring back to (7.34), we conclude that

$$\theta_{rs} = \theta_0(1 - \Delta_H p_*^2),\qquad(7.43)$$

which is the desired formula for the distribution of potential temperature in the A-state. It can be verified that (7.43) gives the "right" values of θ at $p = 0$ and $p = p_S$.

Now all we have to do to find the APE is work out the enthalpy in the given state and the A-state, and subtract. In the given state, the potential temperature is independent of height, so (7.34) gives

$$
\begin{aligned}
H_{gs} &= \int_{-1}^{1}\int_{0}^{p_S} 2\pi a^2 c_p \theta_0 (1 - \Delta_H \mu^2)\left(\frac{p}{p_0}\right)^{\kappa}\frac{dp}{g}\,d\mu \\
&= 4\pi a^2 c_p \theta_0 \left(\frac{p_S}{g}\right)\left(\frac{p_S}{p_0}\right)^{\kappa}\left(\frac{1 - \frac{1}{3}\Delta_H}{1 + \kappa}\right).
\end{aligned}
\qquad(7.44)
$$

Similarly, the total enthalpy of the A-state is

$$
\begin{aligned}
H_{min} &= \int_{-1}^{1}\int_{0}^{p_S} 2\pi a^2 c_p \theta_0 \left[1 - \Delta_H\left(\frac{p}{p_S}\right)^2\right]\left(\frac{p}{p_0}\right)^{\kappa}\frac{dp}{g}\,d\mu \\
&= 4\pi a^2 c_p \theta_0 \left(\frac{p_S}{g}\right)\left(\frac{p_S}{p_0}\right)^{\kappa}\left[\left(\frac{1}{1 + \kappa}\right) - \left(\frac{\Delta_H}{3 + \kappa}\right)\right].
\end{aligned}
\qquad(7.45)
$$

Finally, we obtain

$$
A = H_{gs} - H_{min} = 4\pi a^2 c_p \theta_0 \Delta_H \left(\frac{p_S}{g}\right)\left(\frac{p_S}{p_0}\right)^{\kappa}\left[\frac{2\kappa}{3(3 + \kappa)(1 + \kappa)}\right].\qquad(7.46)
$$

Note that A is proportional to Δ_H, as might be expected.

Now consider a meridional transport process:

$$\frac{\partial \theta_{gs}}{\partial t} = -\frac{1}{a\cos\varphi\,\partial\varphi}(F_\theta\cos\varphi)$$

$$= -\frac{1}{a\,\partial\mu}(F_\theta\sqrt{1-\mu^2}).$$

(7.47)

Here F_θ is a meridional flux of potential temperature, which we regard as given, and assume to be independent of height and longitude and symmetrical about the Equator. Note that

$$F_\theta = 0 \text{ at both poles.}$$

(7.48)

Assume that the surface pressure does not change with time at any latitude. What is the time rate of change of the APE associated with this meridional redistribution of potential temperature?

To answer this question, first note that the time rate of change of the total enthalpy of the given state is:

$$\frac{\partial H_{gs}}{\partial t} = \int_{-1}^{1}\int_{0}^{p_S} 2\pi a^2 c_p\left(\frac{\partial\theta}{\partial t}\right)\left(\frac{p}{p_0}\right)^\kappa\frac{dp}{g}d\mu$$

$$= \frac{2\pi a^2 c_p}{(1+\kappa)}\left(\frac{p_S}{g}\right)\left(\frac{p_S}{p_0}\right)^\kappa\int_{-1}^{1}\frac{\partial\theta}{\partial t}d\mu$$

$$= -\frac{2\pi a^2 c_p}{(1+\kappa)}\left(\frac{p_S}{g}\right)\left(\frac{p_S}{p_0}\right)^\kappa\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{a\cos\varphi\,\partial\varphi}(F_\theta a\cos\varphi)\cos\varphi\,d\varphi$$

$$= 0 .$$

(7.49)

Here we have used the facts that $\partial\theta/\partial t$ is independent of height and that p_S is independent of latitude, and we have substituted from (7.47). According to (7.49), the specified transport process has no effect on the total enthalpy of the given state. The reason is that the average θ on each pressure surface is unchanged, and it follows that the average temperature on each pressure surface is unchanged.

The meridional transport process can, however, alter the total enthalpy of the A-state. Here again there is a possibility of confusion. As already mentioned, the specified meridional transport process does not alter the average value of θ on a pressure surface. This statement seems to imply that the process is isentropic, and we already know that isentropic processes do not alter the A-state. The entropy is proportional to $ln(\theta)$, however, and the average value of $ln(\theta)$ *is* altered by the transport process. Another point of view is that generally

speaking the specified transport process is *not reversible*. For example, if F_θ is a downgradient flux due to diffusive mixing, then it could, in principle, homogenize θ throughout the entire atmosphere. This process would clearly be irreversible. Following such homogenization, the A-state would be the same as the (homogenized) given state and would, therefore, be different from the A-state found above.

From (7.49), it follows that

$$\frac{dA}{dt} = -\frac{dH_{min}}{dt} \ .$$
(7.50)

Now recall that, based on comparison of (7.34) and (7.43), p_* plays the same role in the A-state as μ plays in the given state. In particular, $p_* = 1$ (the surface) corresponds to $\mu = 1$ (the pole), and $p_* = 0$ (the "top of the atmosphere") corresponds to $\mu = 0$ (the Equator). Our goal is to determine the time-rate of change of θ in the A-state *at a particular instant*, namely, the time when the distribution of θ satisfies (7.34) [and (7.43)], and the time-rate-of-change of θ in the given state satisfies (7.47). We can find the time-rate-of-change of θ in the A-state at this instant by going to our expression for the time-rate-of-change of θ in the given state, and simply replacing μ by p_*, everywhere. The time-rate-of-change of θ in the A-state thus satisfies

$$\frac{\partial \theta_{rs}}{\partial t} = -\frac{1}{a}\frac{\partial}{\partial p_*}[(F_\theta \sqrt{1 - p_*^2})] \ .$$
(7.51)

Again there is a possibility of confusion. We have specified that F_θ is not a function of height, although of course it does depend on latitude. It would thus appear that we can pull F_θ out of the derivative in (7.51), but this is not correct. The reason is that, when μ was replaced by p_* we also replaced the μ-dependence of F_θ by a corresponding p_*-dependence. Thus, in (7.51), F_θ should be regarded as a function of p_*, but *not* as a function of latitude! This is understandable, because F_θ is acting to change $\theta_{rs}(p_*)$. As an example, suppose that F_θ is symmetric about the Equator, and poleward in both hemispheres, and that $\Delta_H > 0$ so that the poles are in fact colder than the tropics. Then F_θ tends to warm the poles and cool the tropics, reducing Δ_H and, we expect, reducing A. As the tropics cool and the poles warm in the given state, the A-state evolves in a corresponding way, so that θ_{rs} cools aloft and warms at the lower levels.

We now write the time-rate-of-change of the total enthalpy in the A-state as

$$\frac{\partial H_{min}}{\partial t} = \int_{-1}^{1}\int_{0}^{p_S} 2\pi a^2 c_p \left(\frac{\partial\theta}{\partial t}\right)\left(\frac{p}{p_0}\right)^{\kappa}\frac{dp}{g}d\mu$$

$$= \frac{4\pi a^2 c_p p_S}{g}\left(\frac{p_S}{p_0}\right)^{\kappa}\int_{0}^{1}\left(\frac{\partial\theta}{\partial t}\right)(p_*)^{\kappa}dp_* \tag{7.52}$$

$$= \frac{4\pi a^2 c_p p_S}{ga}\left(\frac{p_S}{p_0}\right)^{\kappa}\int_{0}^{1}\left[-\frac{\partial}{\partial p_*}(F_{\theta}\sqrt{1-p_*^2})\right]p_*^{\kappa}dp_* \ .$$

We cannot do the integral on the last line of (7.52), because the form of F_{θ} has not been specified. The last step would be to substitute (7.52) into (7.50).

7.3.3   The APE associated with surface pressure variations

APE can occur in the presence of surface pressure gradients, even in the absence of potential temperature gradients; this is analogous to the APE of shallow water with a non-uniform free-surface height. To explore this possibility in a simple framework, consider a planet with an atmosphere of uniform potential temperature, θ_0. The surface pressure, $p_S(\lambda, \varphi)$, is given as a function of longitude, λ and latitude φ. For simplicity we assume that the Earth's surface is flat, although a similar but more complicated analysis can be developed for the case of arbitrary surface topography.

We begin with (7.8), the basic definition of the APE. The total enthalpy satisfies

$$H = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{2\pi}\int_{0}^{p_S} c_p \theta\left(\frac{p}{p_0}\right)^{\kappa}\frac{dp}{g}a\cos\varphi d\lambda a d\varphi \ . \tag{7.53}$$

Because $\theta = \theta_0$ everywhere in this problem, we can simplify (7.53) considerably:

$$H = \frac{c_p \theta_0 a^2}{g(1+\kappa)p_0^{\kappa}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{2\pi} p_S^{1+\kappa}\cos\varphi d\lambda d\varphi \ . \tag{7.54}$$

Eq. (7.54) can be applied to both the given state and the A-state, so that the APE is given by

$$A = \frac{c_p \theta_0 a^2}{g(1+\kappa)p_0^{\kappa}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{2\pi} [(p_S)_{gs}^{1+\kappa} - (p_S)_{rs}^{1+\kappa}]\cos\varphi d\lambda d\varphi \ . \tag{7.55}$$

Before we can evaluate the double integrals in (7.55), we must substitute for $[p_S(\lambda, \varphi)]_{gs}$ and $[p_S(\lambda, \varphi)]_{rs}$. The former is assumed to be known. Our problem thus reduces to finding

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$[p_S(\lambda, \varphi)]_{rs}$. This is very simple, because (since by assumption there are no mountains) the surface pressure is globally uniform in the A-state, and equal to the globally averaged surface pressure in the given state. We denote this globally averaged surface pressure by $\bar{p}_S$, and rewrite (7.55) as

$$A = \frac{c_p \theta_0 a^2}{g(1 + \kappa)p_0^{\kappa}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} [(p_S)_{gs}^{1 + \kappa} - (\bar{p}_S)^{1 + \kappa}] \cos \varphi \, d\lambda \, d\varphi \; . \tag{7.56}$$

As an exercise, prove that (7.56) gives $A \geq 0$.

7.4   *Variance budgets*

From (7.21) we see that the available potential energy is closely related to the variance of temperature or potential temperature on pressure surfaces.

We now examine a conversion process that couples the variance associated with the meridional gradient of the zonally averaged potential temperature with the eddy variance of potential temperature. This same process is closely related to the conversion between the zonal available potential energy, A_Z, and eddy available potential energy, A_E. The eddy potential temperature variance interacts with the meridional gradient of the zonally averaged potential temperature through

$$\frac{\partial}{\partial t}[\theta_*^2] \sim \frac{-[\theta_* v_*]}{a} \frac{\partial}{\partial \varphi}[\theta] \; . \tag{7.57}$$

The term shown in the right-hand side of (7.57) is called the "meridional gradient-production term." There are actually several additional terms; the others are discussed below. To gain an intuitive understanding of the gradient production terms, consider the simple example illustrated in Fig. 7.5. State A consists of two latitude belts of equal mass, each with uniform θ. State B is obtained by homogenizing State A. Consider the average of the square of θ, for each state. For State A,

$$\overline{\theta^2} = \frac{1}{2}(\theta_1^2 + \theta_2^2) \; . \tag{7.58}$$

Here the overbar denotes an average over both latitude belts. For State B,

$$\overline{\theta}^2 = \left[\frac{1}{2}(\theta_1 + \theta_2)\right]^2 = \frac{1}{4}(\theta_1^2 + 2\theta_1\theta_2 + \theta_2^2)$$

$$= \frac{1}{2}(\theta_1^2 + \theta_2^2) - \frac{1}{4}(\theta_1^2 - 2\theta_1\theta_2 + \theta_2^2)$$

$$= \frac{1}{2}(\theta_1^2 + \theta_2^2) - \frac{1}{4}(\theta_1 - \theta_2)^2$$

$$\leq \frac{1}{2}(\theta_1^2 + \theta_2^2) \ .$$

$$(7.59)$$

This shows that down-gradient transport (in this case, mixing) reduces the square of the mean state. It correspondingly increases the square of the fluctuations. In other words, it leads to a conversion of mean-state variance to eddy variance. This appears as a "dissipation" of the
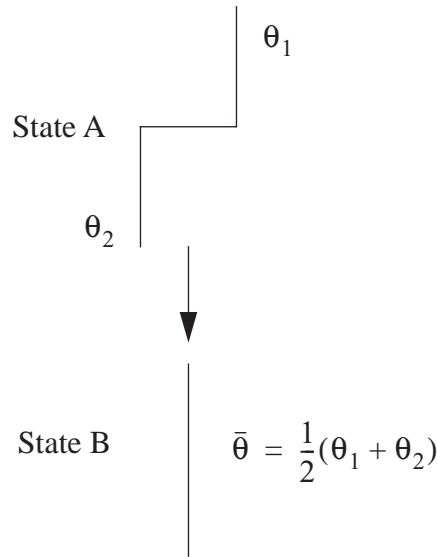
Figure 7.5: A simple example used to explain the idea of gradient production. State B is obtained by homogenizing State A. In both State A and State B, the average of θ is $\bar{\theta} = \frac{1}{2}(\theta_1 + \theta_2)$. Variance seems to "disappear" in passing from State A to State B. In reality it is converted from the variance of the mean to an eddy variance.

mean-state variance. Further discussion is given later.

To see where (7.57) comes from, start from the conservation equation for potential temperature, i.e.,

$$\frac{\partial \theta}{\partial t} + \frac{1}{a\cos\varphi\partial\lambda}(u\theta) + \frac{1}{a\cos\varphi\partial\varphi}(v\theta\cos\varphi) + \frac{\partial}{\partial p}(\omega\theta) = \dot{\theta},$$ (7.60)

and also mass continuity:

$$\frac{1}{a\cos\varphi\partial\lambda}\frac{\partial u}{} + \frac{1}{a\cos\varphi\partial\varphi}\frac{\partial}{}(v\cos\varphi) + \frac{\partial\omega}{\partial p} = 0.$$ (7.61)

Zonally average (7.60) and (7.61), to obtain

$$\frac{\partial}{\partial t}[\theta] + \frac{1}{a\cos\varphi\partial\varphi}\frac{\partial}{}([v\theta]\cos\varphi) + \frac{\partial}{\partial p}[\omega\theta] = [\dot{\theta}],$$ (7.62)

$$\frac{1}{a\cos\varphi\partial\varphi}\frac{\partial}{}([v]\cos\varphi) + \frac{\partial}{\partial p}[\omega] = 0.$$ (7.63)

Subtracting (7.62) and (7.63) from (7.60) and (7.61), respectively, we find that

$$\frac{\partial\theta_*}{\partial t} + \frac{1}{a\cos\varphi\partial\lambda}\frac{\partial}{}(u\theta) + \frac{1}{a\cos\varphi\partial\varphi}\frac{\partial}{}\{(v\theta - [v\theta])\cos\varphi\} + \frac{\partial}{\partial p}(\omega\theta - [\omega\theta])$$

$$= \frac{\partial\theta_*}{\partial t} + \frac{1}{a\cos\varphi\partial\lambda}\frac{\partial}{}(u_*[\theta] + [u]\theta_* + u_*\theta_*)$$

$$+ \frac{1}{a\cos\varphi\partial\varphi}\frac{\partial}{}\{(v_*[\theta] + [v]\theta_* + v_*\theta_* - [v_*\theta_*])\cos\varphi\}$$

$$+ \frac{\partial}{\partial p}(\omega_*[\theta] + [\omega]\theta_* + \omega_*\theta_* - [\omega_*\theta_*])$$

$$= \frac{\partial\theta_*}{\partial t} + \frac{1}{a\cos\varphi\partial\lambda}\frac{\partial}{}([u]\theta_*) + \frac{1}{a\cos\varphi\partial\varphi}\frac{\partial}{}\{([v]\theta_*)\cos\varphi\} + \frac{\partial}{\partial p}([\omega]\theta_*)$$ (7.64)

$$+ \frac{1}{a\cos\varphi\partial\lambda}\frac{\partial}{}(u_*\theta_*) + \frac{1}{a\cos\varphi\partial\varphi}\frac{\partial}{}(v_*\theta_*\cos\varphi) + \frac{\partial}{\partial p}(\omega_*\theta_*)$$

$$+ \frac{1}{a\cos\varphi\partial\lambda}\frac{\partial}{}(u_*[\theta]) + \frac{1}{a\cos\varphi\partial\varphi}\frac{\partial}{}\{(v_*[\theta])\cos\varphi\} + \frac{\partial}{\partial p}(\omega_*[\theta])$$

$$- \frac{1}{a\cos\varphi\partial\varphi}\frac{\partial}{}([v_*\theta_*]\cos\varphi) - \frac{\partial}{\partial p}([\omega_*\theta_*])$$

$$= \dot{\theta}_*,$$

$$\frac{1}{a\cos\varphi\partial\lambda}\frac{\partial u_*}{} + \frac{1}{a\cos\varphi\partial\varphi}\frac{\partial}{}(v_*\cos\varphi) + \frac{\partial\omega_*}{\partial p} = 0.$$ (7.65)

To obtain the first equality of (7.64), we have used

$$v\theta = ([v] + v_*)([\theta] + \theta_*) = [v][\theta] + v_*[\theta] + [v]\theta_* + v_*\theta_*, \tag{7.66}$$

$$[v\theta] = [v][\theta] + [v_*\theta_*], \tag{7.67}$$

$$v\theta - [v\theta] = v_*[\theta] + [v]\theta_* + v_*\theta_* - [v_*\theta_*], \tag{7.68}$$

and so on. We can use (7.63) and (7.65) to rewrite (7.64) as follows:

$$\left(\frac{\partial}{\partial t} + \frac{[u]}{a\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{[v]}{a}\frac{\partial}{\partial\varphi} + [\omega]\frac{\partial}{\partial p}\right)\theta_* + \left(\frac{u_*}{a\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{v_*}{a}\frac{\partial}{\partial\varphi} + \omega_*\frac{\partial}{\partial p}\right)\theta_*$$

$$+ \frac{v_*}{a}\frac{\partial}{\partial\varphi}[\theta] + \omega_*\frac{\partial}{\partial p}[\theta] = \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v_*\theta_*]\cos\varphi) + \frac{\partial}{\partial p}[\omega_*\theta_*] + \dot{\theta}_*. \tag{7.69}$$

Multiplying (7.69) by θ_*, and using (7.65) again, we obtain:

$$\left(\frac{\partial}{\partial t} + \frac{[u]}{a\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{[v]}{a}\frac{\partial}{\partial\varphi} + [\omega]\frac{\partial}{\partial p}\right)\left(\frac{1}{2}\theta_*^2\right)$$

$$+ \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left\{u_*\left(\frac{1}{2}\theta_*^2\right)\right\} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{v_*\left(\frac{1}{2}\theta_*^2\right)\cos\varphi\right\} + \frac{\partial}{\partial p}\left\{\omega_*\left(\frac{1}{2}\theta_*^2\right)\right\}$$

$$= \theta_*\left\{\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v_*\theta_*]\cos\varphi) + \frac{\partial}{\partial p}[\omega_*\theta_*]\right\} - \frac{v_*\theta_*}{a}\frac{\partial}{\partial\varphi}[\theta] - \omega_*\theta_*\frac{\partial}{\partial p}[\theta] + \dot{\theta}_*\theta_*.$$

$$\tag{7.70}$$

Zonally averaging gives

$$\left(\frac{\partial}{\partial t} + \frac{[v]}{a}\frac{\partial}{\partial\varphi} + [\omega]\frac{\partial}{\partial p}\right)\left(\frac{1}{2}[\theta_*^2]\right)$$

$$+ \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left[v_*\left(\frac{1}{2}\theta_*^2\right)\right]\cos\varphi\right\} + \frac{\partial}{\partial p}\left\{\left[\omega_*\left(\frac{1}{2}\theta_*^2\right)\right]\right\} \tag{7.71}$$

$$= -\frac{[v_*\theta_*]}{a}\frac{\partial}{\partial\varphi}[\theta] - [\omega_*\theta_*]\frac{\partial}{\partial p}[\theta] + [\dot{\theta}_*\theta_*].$$

Finally, we can use (7.63) to rewrite (7.71) in flux form:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}[\theta_*^2]\right) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left([v]\frac{1}{2}[\theta_*^2]\right)\cos\varphi\right\} + \frac{\partial}{\partial p}\left([\omega]\frac{1}{2}[\theta_*^2]\right)$$

$$+ \quad \underbrace{\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left[v_*\left(\frac{1}{2}\theta_*^2\right)\right]\cos\varphi\right\} + \frac{\partial}{\partial p}\left\{\left[\omega_*\left(\frac{1}{2}\theta_*^2\right)\right]\right\}}_{\text{eddy transport}}$$

(7.72)

$$= \quad \underbrace{-\frac{[v_*\theta_*]}{a}\frac{\partial}{\partial\varphi}[\theta] - [\omega_*\theta_*]\frac{\partial}{\partial p}[\theta]}_{\text{gradient production}} + [\dot\theta_*\theta_*] \; .$$

According to (7.72), $\frac{1}{2}[\theta_*^2]$ can change due to advection by the mean meridional circulation, or due to transport by the eddies themselves, or due to "gradient production."

Eq. (7.72) governs the "eddy variance."(7.62) There is also a contribution to the global variance of θ that comes from the meridional and vertical gradients of $[\theta]$. To derive an equation for this part of the θ global variance of θ, start by rewriting (7.62) as

$$\left(\frac{\partial}{\partial t} + \frac{[v]}{a}\frac{\partial}{\partial\varphi} + [\omega]\frac{\partial}{\partial p}\right)[\theta]$$

$$= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v_*\theta_*]\cos\varphi) - \frac{\partial}{\partial p}[\omega_*\theta_*] + [\dot\theta] \; .$$

(7.73)

Multiplication by $[\theta]$ gives

$$\left(\frac{\partial}{\partial t} + \frac{[v]}{a}\frac{\partial}{\partial\varphi} + [\omega]\frac{\partial}{\partial p}\right)\frac{1}{2}[\theta]^2$$

$$= -\frac{[\theta]}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v_*\theta_*]\cos\varphi) - [\theta]\frac{\partial}{\partial p}([\omega_*\theta_*]) + [\theta][\dot\theta] \; .$$

(7.74)

This can be rearranged to

$$\left(\frac{\partial}{\partial t} + \frac{[v]}{a}\frac{\partial}{\partial \varphi} + [\omega]\frac{\partial}{\partial p}\right)\frac{1}{2}[\theta]^2$$

$$= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}([\theta][v_*\theta_*]\cos\varphi) - \frac{\partial}{\partial p}[\theta][\omega_*\theta_*] \tag{7.75}$$

$$+ \frac{[v_*\theta_*]}{a}\frac{\partial}{\partial \varphi}[\theta] + [\omega_*\theta_*]\frac{\partial}{\partial p}[\theta] + [\theta][\dot\theta] \ .$$

Converting back to flux form, we find that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}[\theta]^2\right) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left\{\left([v]\frac{1}{2}[\theta]^2\right)\cos\varphi\right\} + \frac{\partial}{\partial p}\left([\omega]\frac{1}{2}[\theta]^2\right)$$

$$= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}([\theta][v_*\theta_*]\cos\varphi) - \frac{\partial}{\partial p}([\theta][\omega_*\theta_*]) \tag{7.76}$$

$$+ \frac{[v_*\theta_*]}{a}\frac{\partial}{\partial \varphi}[\theta] + [\omega_*\theta_*]\frac{\partial}{\partial p}[\theta] + [\theta][\dot\theta] \ .$$

When we add (7.76) and (7.72), the gradient production terms cancel. This shows that the gradient production terms represent a "conversion" between $\frac{1}{2}[\theta]^2$ and $[\theta_*^2]$. We obtain:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}[\theta]^2 + \frac{1}{2}[\theta_*^2]\right) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left\{[v]\left(\frac{1}{2}[\theta]^2 + \frac{1}{2}[\theta_*^2]\right)\cos\varphi\right\} + \frac{\partial}{\partial p}\left\{[\omega]\left(\frac{1}{2}[\theta]^2 + \frac{1}{2}[\theta_*^2]\right)\right\}$$

$$+ \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left\{\left[v_*\left(\frac{1}{2}\theta_*^2\right)\right]\cos\varphi\right\} + \frac{\partial}{\partial p}\left\{\left[\omega_*\left(\frac{1}{2}\theta_*^2\right)\right]\right\}$$

$$= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}([\theta][v_*\theta_*]\cos\varphi) - \frac{\partial}{\partial p}([\theta][\omega_*\theta_*]) + [\theta][\dot\theta] + [\dot\theta_*\theta_*] \ .$$

$$\tag{7.77}$$

Finally, integration of (7.77) over the entire atmosphere gives

$$\frac{d}{dt}\int_M\left(\frac{1}{2}[\theta]^2 + \frac{1}{2}[\theta_*^2]\right)dM = \int_M([\theta][\dot\theta] + [\dot\theta_*\theta_*])dM \ . \tag{7.78}$$

This shows that, in the absence of heating, the sum of the eddy variance and the variance of the zonal mean is a constant. Note also that the potential temperature variance increases with time if we "heat where it's hot and cool where it's cold."

7.5 *Generation of available potential energy, and its conversion into kinetic energy*

Earlier we derived (7.21), Lorenz's approximate expression for the available potential energy of a statically stable atmosphere, which is repeated here for convenience:

$$A = \frac{a^2}{2} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^{p_S} \frac{\bar{T}}{(\Gamma_d - \Gamma)} \left(\frac{T'}{\bar{T}} \right)^2 \cos\varphi \, dp \, d\lambda \, d\varphi . \tag{7.79}$$

Recall that in this equation an overbar represents a global mean on a pressure surface, and a prime denotes a departure from the global mean. The APE is an integral of the variance of the temperature about its global mean on pressure surfaces. In the last section, we derived an equation for the time rate of change of the potential temperature variance. We now work out an approximate equation for the time rate of change of A due to generation and conversion to or from kinetic energy.

Let the subscript "GM" denote a global mean on an isobaric surface, i.e.,

$$(\ )_{GM} \equiv \frac{1}{4\pi a^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} (\ ) a^2 \cos\varphi \ d\lambda \, d\varphi . \tag{7.80}$$

We can show that, for any two quantities A and B,

$$(AB)_{GM} = ([A][B] + [A_* B_*])_{GM} , \tag{7.81}$$

and

$$\begin{aligned} \{(A - A_{GM})(B - B_{GM})\}_{GM} &= (AB)_{GM} - A_{GM} B_{GM} \\ &= ([A][B] + [A_* B_*])_{GM} - A_{GM} B_{GM} . \end{aligned} \tag{7.82}$$

As a special case of (7.82), the variance of an arbitrary quantity A about its global mean is given by

$$\begin{aligned} \Xi_A &\equiv (A - A_{GM})_{GM}^2 \\ &= ([A]^2 + [A_*^2])_{GM} - A_{GM}^2 . \end{aligned} \tag{7.83}$$

Using the notation introduced above, we can approximate (7.75) by

$$A \cong 2\pi a^2 \int_0^{p_S} \frac{T_{GM}}{(\Gamma_d - \Gamma_{GM})} \left(\frac{\Xi_T}{T_{GM}^2} \right) dp$$

$$= 2\pi a^2 \int_0^{p_S} \frac{T_{GM}}{(\Gamma_d - \Gamma_{GM})} \left(\frac{\Xi_\theta}{\theta_{GM}^2} \right) dp.$$

(7.84)

We now work out an equation for $\dfrac{dA}{dt}$, based on (7.80). The global means of (7.60) and (7.61) are

$$\frac{\partial \theta_{GM}}{\partial t} + \frac{\partial (\omega\theta)_{GM}}{\partial p} = \dot{\theta}_{GM},$$

(7.85)

and

$$\frac{\partial \omega_{GM}}{\partial p} = 0,$$

(7.86)

respectively. Since $\omega = 0$ at $p = 0$, it follows from (7.86) that

$$\omega_{GM} = 0 \text{ for all } p.$$

(7.87)

This allows us to write

$$(\omega\theta)_{GM} = \omega_{GM}\omega_{GM} + \{(\omega - \omega_{GM})(\theta - \theta_{GM})\}_{GM}$$

$$= \{\omega(\theta - \theta_{GM})\}_{GM}.$$

(7.88)

Area-weighted integration of (7.77) over all latitudes, at a given pressure-level, leads to

$$\frac{\partial}{\partial t}\left(\frac{1}{2}[\theta]^2 + \frac{1}{2}[\theta_*^2]\right)_{GM} + \frac{\partial}{\partial p}\left\{[\omega]\left(\frac{1}{2}[\theta]^2 + \frac{1}{2}[\theta_*^2]\right) + \left[\omega_*\left(\frac{1}{2}\theta_*^2\right)\right]\right\}_{GM}$$

$$= -\frac{\partial}{\partial p}([\theta][\omega_*\theta_*])_{GM} + ([\theta][\dot{\theta}] + [\dot{\theta}_*\theta_*])_{GM}.$$

(7.89)

Here we ignore, as usual, the complications arising from the fact that some pressure surfaces intersect the Earth's surface. From (7.79), we see that

$$\frac{\overline{\partial \Xi_\theta}}{\partial t} = \frac{\partial([\theta]^2 + [\theta_*^2])_{GM}}{\partial t} - 2\theta_{GM}\frac{\overline{\partial \theta}_{GM}}{\partial t}. \tag{7.90}$$

Substituting into (7.86) from (7.85) and (7.89), we find that

$$\frac{\partial}{\partial t}\left(\frac{\Xi_\theta}{2}\right) = -\frac{\partial}{\partial p}\left\{[\omega]\left(\frac{1}{2}[\theta]^2 + \frac{1}{2}[\theta_*^2]\right) + \left[\omega_*\left(\frac{1}{2}\theta_*^2\right)\right] + ([\theta][\omega_*\theta_*])_{GM}\right\}_{GM}$$

$$+ \theta_{GM}\frac{\partial(\omega\theta)_{GM}}{\partial p} + \{([\theta][\dot\theta] + [\dot\theta_*\theta_*])_{GM} - \theta_{GM}\dot\theta_{GM}\}$$

$$= -\frac{\partial}{\partial p}\left\{[\omega]\left(\frac{1}{2}[\theta]^2 + \frac{1}{2}[\theta_*^2]\right) + \left[\omega_*\left(\frac{1}{2}\theta_*^2\right)\right] + ([\theta][\omega_*\theta_*])_{GM} - \theta_{GM}(\omega\theta)_{GM}\right\}_{GM}$$

$$- (\omega\theta)_{GM}\frac{\partial}{\partial p}\theta_{GM} + \{(\theta - \theta_{GM})(\dot\theta - \dot\theta_{GM})\}_{GM}$$

$$= -\frac{\partial}{\partial p}\left\{[\omega]\left(\frac{1}{2}[\theta]^2 + \frac{1}{2}[\theta_*^2]\right) + \left[\omega_*\left(\frac{1}{2}\theta_*^2\right)\right] + ([\theta][\omega_*\theta_*])_{GM} - \theta_{GM}(\omega\theta)_{GM}\right\}_{GM}$$

$$- \{\omega(\theta - \theta_{GM})\}_{GM}\frac{\partial}{\partial p}\theta_{GM} + \{(\theta - \theta_{GM})(\dot\theta - \dot\theta_{GM})\}_{GM}.$$

Here we can recognize what is going on. Vertical transport is clearly visible, as are gradient production and "Heat where it's hot, cool where it's cold." Now we apply (7.19) in the form

$$-\frac{\partial \theta_{GM}}{\partial p} \cong \frac{\kappa\theta_{GM}}{p}\left(\frac{\Gamma_d - \Gamma_{GM}}{\Gamma_d}\right). \tag{7.91}$$

This leads to

$$\frac{\partial}{\partial t}\left(\frac{\Xi_\theta}{2}\right) =$$

$$-\frac{\partial}{\partial p}\left\{[\omega]\left(\frac{1}{2}[\theta]^2 + \frac{1}{2}[\theta_*^2]\right) + \left[\omega_*\left(\frac{1}{2}\theta_*^2\right)\right] + ([\theta][\omega_*\theta_*])_{GM} - \theta_{GM}(\omega\theta)_{GM}\right\}_{GM} \tag{7.92}$$

$$+ \{\omega(\theta - \theta_{GM})\}_{GM}\frac{\kappa\theta_{GM}}{p}\left(\frac{\Gamma_d - \Gamma_{GM}}{\Gamma_d}\right) + \{(\theta - \theta_{GM})(\dot\theta - \dot\theta_{GM})\}_{GM} \quad ,$$

or

$$\frac{\partial}{\partial t}(\Xi_\theta) =$$

$$-\frac{\partial}{\partial p}\left\{[\omega]([\theta]^2 + [\theta_*^2]) + [\omega_*(\theta_*^2)] + 2([\theta][\omega_*\theta_*])_{GM} - 2\theta_{GM}(\omega\theta)_{GM}\right\}_{GM} \quad (7.93)$$

$$+ 2\{\omega(\theta - \theta_{GM})\}_{GM}\frac{\kappa\theta_{GM}}{p}\left(\frac{\Gamma_d - \Gamma_{GM}}{\Gamma_d}\right) + 2\{(\theta - \theta_{GM})(\dot{\theta} - \dot{\theta}_{GM})\}_{GM}.$$

Vertical integration of (7.93) gives

$$\frac{d}{dt}\left(\int_0^{p_S} \Xi_\theta dp\right) = \int_0^{p_S} 2\{\omega(\theta - \theta_{GM})\}_{GM}\frac{\kappa\theta_{GM}}{p}\left(\frac{\Gamma_d - \Gamma_{GM}}{\Gamma_d}\right)dp$$

$$+ 2\{(\theta - \theta_{GM})(\dot{\theta} - \dot{\theta}_{GM})\}_{GM}. \quad (7.94)$$

This can be approximated by

$$\frac{d}{dt}\left(\int_0^{p_S} \frac{T_{GM}}{(\Gamma_d - \Gamma_{GM})}\left(\frac{\Xi_\theta}{\theta_{GM}^2}\right)dp\right) = \int_0^{p_S} 2\left\{\omega\left(\frac{\theta - \theta_{GM}}{\theta_{GM}}\right)\right\}_{GM}\frac{\kappa T_{GM}}{p\Gamma_d}dp$$

$$+ 2\int_0^{p_S} \frac{\{(\theta - \theta_{GM})(\dot{\theta} - \dot{\theta}_{GM})\}_{GM}T_{GM}}{\theta_{GM}^2(\Gamma_d - \Gamma_{GM})}dp. \quad (7.95)$$

Finally, note that

$$\left(\frac{\theta - \theta_{GM}}{\theta_{GM}}\right) = \left(\frac{\alpha - \alpha_{GM}}{\alpha_{GM}}\right) = \left(\frac{T - T_{GM}}{T_{GM}}\right). \quad (7.96)$$

so that

$$\frac{dA}{dt} = 4\pi a^2 \int_0^{p_S} \left\{ \omega\left(\frac{\alpha - \alpha_{GM}}{\alpha_{GM}}\right) \right\}_{GM} \frac{\kappa T_{GM}}{p\Gamma_d} dp$$

$$+ 4\pi a^2 \int_0^{p_S} \frac{\{(\theta - \theta_{GM})(\dot{\theta} - \dot{\theta}_{GM})\}_{GM} T_{GM}}{\theta_{GM}^2(\Gamma_d - \Gamma_{GM})} dp$$

$$= \frac{4\pi a^2}{g} \int_0^{p_S} \{\omega(\alpha - \alpha_{GM})\}_{GM} dp \qquad (7.97)$$

$$+ 4\pi a^2 \int_0^{p_S} \frac{\left\{\left(\frac{\theta - \theta_{GM}}{\theta_{GM}}\right)\left(\frac{T_{GM}}{\theta_{GM}}\right)(\dot{\theta} - \dot{\theta}_{GM})\right\}_{GM}}{(\Gamma_d - \Gamma_{GM})} dp.$$

Recall that $\omega\alpha$ represents conversion between kinetic and non-kinetic energy, so by inspection of (7.97) we can identify rate of conversion of KE into APE as

$$C \cong \frac{4\pi a^2}{g} \int_0^{p_S} \{\omega(\alpha - \alpha_{GM})\}_{GM} dp \qquad . \qquad (7.98)$$

The rate of generation or destruction of APE by heating is

$$G \cong 4\pi a^2 \int_0^{p_S} \frac{\{(\theta - \theta_{GM})(\dot{\theta} - \dot{\theta}_{GM})\}_{GM} T_{GM}}{\theta_{GM}^2(\Gamma_d - \Gamma_{GM})} dp \; . \qquad (7.99)$$

Note that $G > 0$ if we "heat where it's hot and cool where it's cold." This should sound familiar. A heating field that generates APE destroys entropy.
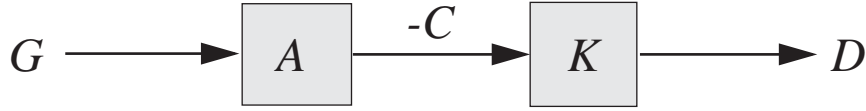
We can summarize (7.97) as

$$\frac{dA}{dt} = C + G. \qquad (7.100)$$

Similarly, we can show that

$$\frac{dK}{dt} = -C - D, \qquad (7.101)$$

Where D is the globally intgrated dissipation rate. We can also conclude that C is negative from the fact that $D > 0$. There must be a net conversion of *APE* into *KE*, in order to supply

Figure 7.6: Sketch illustrating the flow of energy through the atmospheric general circulation. Generation produces APE, which is converted to KE, which in turn is dissipated.

the *KE* that is destroyed by dissipation. Because this conversion depletes the *APE*, the generation of APE must be positive. In other words, energy must flow as shown in Fig. 7.6.

7.6   The governing equations for the eddy kinetic energy, zonal kinetic energy, and total kinetic energy

We now present a discussion of the eddy kinetic energy, zonal kinetic energy, and total kinetic energy equations. The derivations of these equations follow methods similar to those used to derive the conservation equation for the potential energy variance, and so will be omitted here for brevity. A handout showing the details of the derivations is available on request.

We define the eddy kinetic energy per unit mass by

$$KE \equiv \frac{1}{2}[(u^*)^2 + (v^*)^2].$$ (7.102)

It satisfies the following equation:

$$\frac{\partial}{\partial t}KE + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left([v]KE + \frac{1}{2}[v^*u^*u^*] + \frac{1}{2}[v^*v^*v^*]\right)\cos\varphi\right\}$$

$$+ \frac{\partial}{\partial p}\left([\omega]KE + \frac{1}{2}[\omega^*u^*u^*] + \frac{1}{2}[\omega^*v^*v^*]\right) - \left\{\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v^*\phi^*]\cos\varphi) + \frac{\partial}{\partial p}[\omega^*\phi^*]\right\}$$ (7.103)

$$= \frac{-[v^*u^*]}{a}\frac{\partial}{\partial\varphi}[u] - \frac{[v^*v^*]}{a}\frac{\partial}{\partial\varphi}[v] - [\omega^*u^*]\frac{\partial}{\partial p}[u] - [\omega^*v^*]\frac{\partial}{\partial p}[v]$$

$$- [\omega^*\alpha^*] + ([u^*u^*][v] - [u^*v^*][u])\frac{\tan\varphi}{a} + \left[u^*g\frac{\partial}{\partial p}\tau_\lambda{}^*\right] + \left[v^*g\frac{\partial}{\partial p}\tau_\varphi{}^*\right].$$

The terms on the first line of the right-hand side represent gradient production, i.e. the conversion between the kinetic energy of the mean flow and that of the eddies. This conversion is in the sense of increasing the eddy kinetic energy when the eddy momentum flux is "down the gradient," i.e. when it is from higher mean momentum to lower mean

momentum. The $\omega^*\alpha^*$ term represents eddy kinetic energy generation from eddy available potential energy, while the terms involving ϕ^* represent the effects of "pressure work."

The appearance of the metric terms in (7.103) may be somewhat surprising. They arise because we have defined "eddies" in terms of departures from the zonal mean, so that a particular latitude-longitude coordinate system is implicit in the very definition of *KE*. Obviously there cannot be any metric terms in the equation for the *total* kinetic energy per unit mass, which we denote by K.

Define the zonal kinetic energy by

$$KZ \equiv \frac{1}{2}([u]^2 + [v]^2),\tag{7.104}$$

and note that

$$[K] = KZ + KE.\tag{7.105}$$

All three quantities in Eq. (7.105) are independent of longitude. The zonal kinetic energy satisfies

$$
\begin{aligned}
\frac{\partial}{\partial t}KZ &+ \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v]KZ\cos\varphi) + \frac{\partial}{\partial p}([\omega]KZ)\\
&= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v][\phi]\cos\varphi) - \frac{\partial}{\partial p}([\phi][\omega]) - [\omega][\alpha]\\
&\quad -\frac{[u]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v^*u^*]\cos\varphi\} - [u]\frac{\partial}{\partial p}([\omega^*u^*])\\
&\quad -\frac{[v]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v^*v^*]\cos\varphi\} - [v]\frac{\partial}{\partial p}([\omega^*v^*])\\
&\quad + ([u][v^*u^*] - [v][u^*u^*])\frac{\tan\varphi}{a}\\
&\quad + [u]g\frac{\partial}{\partial p}[\tau_\lambda] + [v]g\frac{\partial}{\partial p}[\tau_\varphi].
\end{aligned}\tag{7.106}
$$

The terms on the third and fourth lines of the right-hand side of (7.106) can be interpreted as representing the work done by the mean flow against the "forces" exerted on the mean flow by the eddies, through eddy momentum transport.

Comparison with the eddy kinetic energy equation shows that, as expected, the metric terms do not affect the zonally averaged total kinetic energy. Adding the equations for *KZ* and *KE* gives the equation for the zonally averaged total kinetic energy, $[K]$:

$$
\frac{\partial}{\partial t}[K] + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{ \left([v][K] + \frac{1}{2}[v^*u^*u^*] + \frac{1}{2}[v^*v^*v^*] \right)\cos\varphi \right\}
$$

$$
+ \frac{\partial}{\partial p}\left([\omega][K] + \frac{1}{2}[\omega^*u^*u^*] + \frac{1}{2}[\omega^*v^*v^*] \right)
$$

$$
= \frac{-[v^*u^*]}{a}\frac{\partial}{\partial\varphi}[u] - \frac{[v^*v^*]}{a}\frac{\partial}{\partial\varphi}[v] - [\omega^*u^*]\frac{\partial}{\partial p}[u] - [\omega^*v^*]\frac{\partial}{\partial p}[v]
$$

$$
- \frac{[u]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v^*u^*]\cos\varphi\} - [u]\frac{\partial}{\partial p}([\omega^*u^*])
$$

$$
- \frac{[v]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v^*v^*]\cos\varphi\} - [v]\frac{\partial}{\partial p}([\omega^*v^*])
$$

$$
- \left\{ \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v][\phi]\cos\varphi) + \frac{\partial}{\partial p}([\phi][\omega]) \right\} - [\omega][\alpha]
$$

$$
- \left\{ \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v^*\phi^*]\cos\varphi) + \frac{\partial}{\partial p}[\omega^*\phi^*] \right\} - [\omega^*\alpha^*]
$$

$$
+ \left[u^*g\frac{\partial}{\partial p}\tau_\lambda^* \right] + \left[v^*g\frac{\partial}{\partial p}\tau_\varphi^* \right] + [u]g\frac{\partial}{\partial p}[\tau_\lambda] + [v]g\frac{\partial}{\partial p}[\tau_\varphi].
$$

(7.107)

The first three lines on the right-hand side of (7.107) come from the "gradient production" terms of the eddy kinetic energy equation and the terms of the zonal kinetic energy equation that represent the work done by the mean flow against the forces exerted on the mean flow by the eddies. Inspection shows that these terms can be combined, as shown in Eq. (7.108):

$$\frac{\partial}{\partial t}[K] + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left([v][K] + \frac{1}{2}[v*u*u*] + \frac{1}{2}[v*v*v*]\right)\cos\varphi\right\}$$

$$+ \frac{\partial}{\partial p}\left([\omega][K] + \frac{1}{2}[\omega*u*u*] + \frac{1}{2}[\omega*v*v*]\right)$$

$$= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{([u][v^*u^*] + [v][v^*v^*])\cos\varphi\}$$

$$-\frac{\partial}{\partial p}([u][\omega^*u^*] + [v][\omega^*v^*]) \tag{7.108}$$

$$-\left\{\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v][\phi]\cos\varphi) + \frac{\partial}{\partial p}([\phi][\omega])\right\} - [\omega][\alpha]$$

$$-\left\{\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v*\phi*]\cos\varphi) + \frac{\partial}{\partial p}[\omega*\phi*]\right\} - [\omega*\alpha*]$$

$$+\left[u*g\frac{\partial}{\partial p}\tau_\lambda*\right] + \left[v*g\frac{\partial}{\partial p}\tau_\varphi*\right] + [u]g\frac{\partial}{\partial p}[\tau_\lambda] + [v]g\frac{\partial}{\partial p}[\tau_\lambda].$$

Here the terms mentioned above have been combined on the first two lines of the right-hand side, and it is apparent that they take the form of a divergence. This implies, of course, that the two terms together integrate to zero over the whole atmosphere. The interpretation of this result is that the gradient production terms represent conversion between the kinetic energy of the mean flow and the kinetic energy of the eddies. We do not see simple cancellation between the corresponding terms of the *KE* and *KZ* equations, because the conversion process is not local; it occurs over a region that is extended in the meridional and vertical directions. Cancellation occurs only when we integrate over the whole atmosphere that.

7.7   Observations of the energy cycle

Arpé et al. (1986) discussed the observed energy cycle of the atmosphere, based on ECMWF analyses. They wrote the following equations for the energy cycle:

$$\frac{d}{dt}KZ = -\sum_m CK(m) + CZ - DZ, \tag{7.109}$$

$$\frac{d}{dt}AZ = -\sum_m CA(m) - CZ + GZ, \tag{7.110}$$

$$\frac{\partial}{\partial t}KE(m) = CK(m) + LK(m) + CE(m) - DE(m), \tag{7.111}$$

$$\frac{\partial}{\partial t}AE(m) = CA(m) + LA(m) - CE(m) + GE(m). \tag{7.112}$$

Here *m* is the zonal wave number. The eddy kinetic energy and eddy available potential energy are defined as functions of the zonal wave number, and the contributions for the individual waves have been worked out. The terms "*LK(m)*" and "*LA(m)*" represent wave-wave interactions, due to nonlinear processes. For example, if we have a "kinetic energy cascade" from lower wave numbers to higher wave numbers, then *LK(m)* will represent a flow of energy from larger scales to smaller scales. If (7.109)-(7.112) are added together, all terms on the right-hand side of the result cancel, except for *GZ, GE, DZ,* and *DE*.

Baroclinic instability of the zonal flow would be represented by the combination of positive *CA* and positive *CE*; the first of these represents the conversion of *AZ* to *AE*, and the second represents the conversion of *AE* to *KE*; the net effect is thus conversion of *AZ* to *KE*.

A tendency for eddies to pump momentum into the jet, increasing its strength, would be represented by negative values of *CK*.

A direct mean meridional circulation, such as the Hadley Circulation, converts *AZ* into *KZ*, and so would be associated with positive values of *CZ*.

It is important to understand the meaning of each term of (7.109) through (7.112), as outlined above. It is also important, however, to notice that certain terms are *not* present, i.e. certain processes do not exist. For example, there is no process that *directly* converts *AZ* into $KE(m)$. Such conversion can occur only indirectly, via a two-step process, e.g. first $AZ \rightarrow AE$ and then $AE \rightarrow KE$.

Fig. 7.7 shows the observed energy cycle for (northern) winter and summer, and for the two hemispheres separately. The figure is arranged so that the "summer hemispheres" are on the right (for both seasons) and the "winter hemispheres" are on the left. The numbers in the boxes represent amounts of energy, and the numbers on the arrows between boxes represent energy conversions, or processes that generate or destroy energy. Note that Arpé et al. have defined the APE for the Northern and Southern Hemispheres separately; as discussed earlier, this is not strictly kosher.

The arrows leading into *AZ* and *AE* from the left represent generation (an arrow leading out simply represents negative generation), and the arrows leading out of *KE* and *KZ* to the right represent dissipation. These arrows can also represent interactions between the hemispheres, however, so that for example we see an arrow leading into *KZ* from the right for the Northern Hemisphere in winter, apparently indicating a physically impossible negative dissipation rate, but actually representing a gain of *KZ* in the Northern Hemisphere via energy exchanges with the Southern Hemisphere.

It is obvious from Fig. 7.7 that, especially in the Northern Hemisphere, the energy flows of the atmosphere are much more vigorous in winter than in summer. Note that *AZ* is several times larger than *AE* or *KZ* or *KE*. For the Northern Hemisphere in winter, *AZ* is strongly generated; this energy is converted to *AE*, leading to a production of KE via baroclinic instability; the eddies act to increase *KZ* by pumping angular momentum into the jet stream, and *KZ* is converted back to *AZ*. Meanwhile *KE* and *KZ* are both dissipated. This suggests that the mean meridional circulation is overall "indirect," i.e. we do not see a net conversion of *AZ* into *KZ* as we would if the direct Hadley circulation were dominating the energetics.

Note, however, that in the summer hemisphere (in both DJF and JJA) the mean meridional circulation is direct, and the eddies are much less active. From the point of view
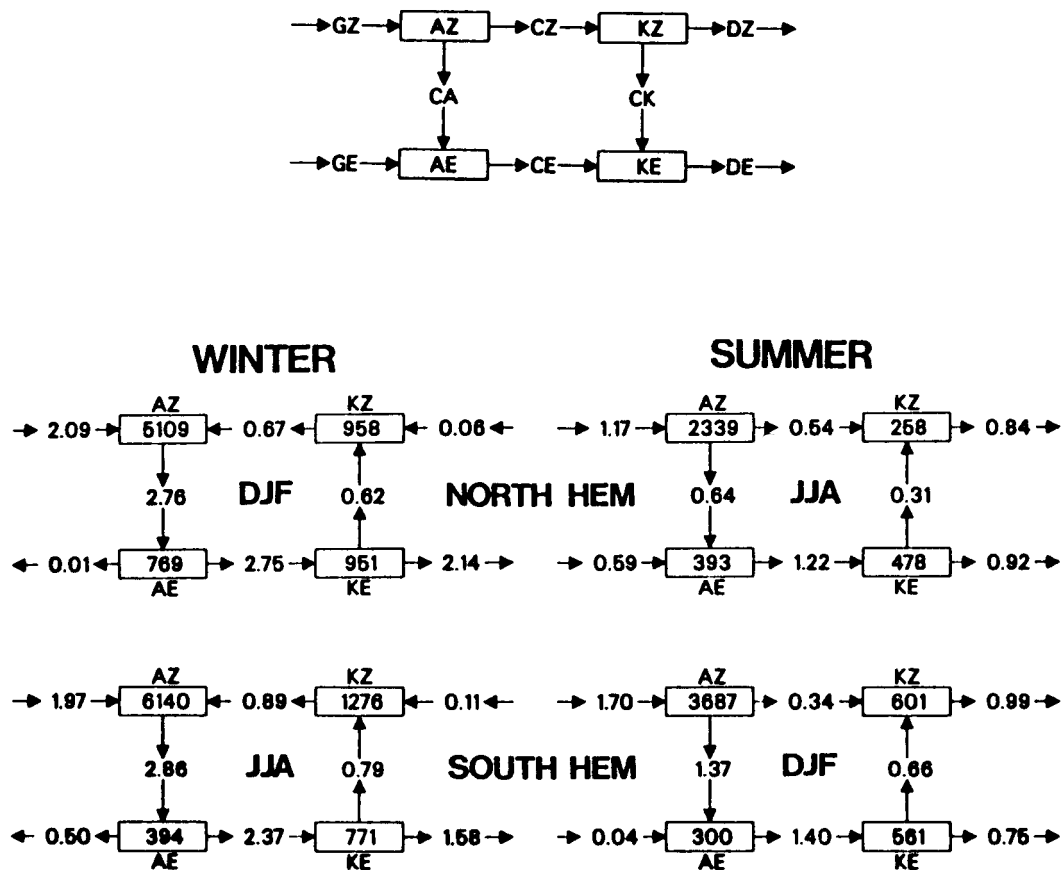
Figure 7.7: Energy cycle during winter (left) and summer (right) for both hemispheres. Integrals between 1000 and 50 mb are presented. Data are calculated from 00 GMT initialized analyses. Energy amounts are given in kJ m^{-2} and conversion rates in W m^{-2}. From Arpé et al. (1986).

of energetics, then, the winter hemisphere is dominated by eddy processes, while the summer hemisphere is dominated by the mean meridional circulation. Of the four conversion processes *CA*, *CZ*, *CK*, and *CE*, only the hemispheric values of *CZ* change sign seasonally; the others fluctuate in magnitude but not in sign.

In all cases, *KE* is supplied from *AE*; baroclinic instability is the dominant mechanism for generating eddies. Fig. 7.8 shows the annual cycles of energy conversions and amounts. The various panels show the zonal means and the Northern and Southern Hemispheres separately. The global means are relatively constant throughout the year, while the individual hemispheres show large seasonal cycles. Within each hemisphere, the winter is much more active, in all respects, than the summer. *CZ* changes sign seasonally. Recall that when *CZ* is positive the MMC is direct overall, and that when *CK* is positive the eddies are deriving kinetic energy from the jet, thus tending to weaken it (rather than acting to increase the kinetic energy of the jet). The figure makes it clear that the energetics of the summer and winter hemispheres are drastically different. The global *AZ* is a maximum shortly after the two
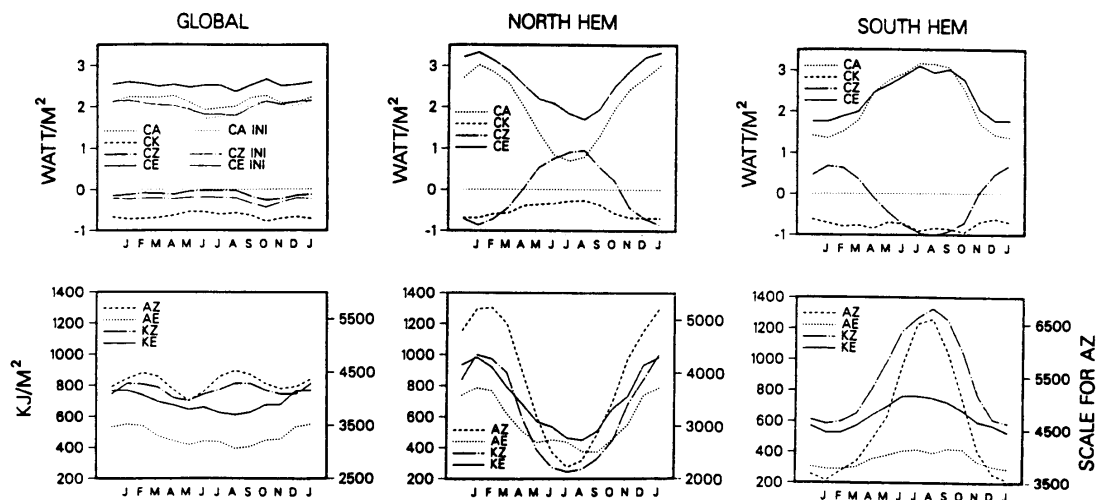
$$CA: AZ \rightarrow AE$$

$$CZ: AZ \rightarrow KZ$$

$$CK: KZ \rightarrow KE$$

$$CE: AE \rightarrow KE$$

Figure 7.8: **Mean annual cycles of energy conversions and amounts for global and hemispheric averages. Data are calculated from 12 hour forecasts and for global averages. Data from initialized analyses are given as well (thinner curves). In the top three panels, the zero line is drawn for reference only. Note that the lower panels have two vertical scales. From Arpé et al. (1986).**

solstices and a minimum shortly after the two equinoxes.

Fig. 7.9 shows the vertical and meridional distributions of the zonally averaged eddy kinetic energy for January and August. The Northern Hemisphere shows a strong seasonal cycle, while the Southern Hemisphere does not. Wave numbers 10 and above play little role.

Fig. 7.10 again shows the mean annual cycles of the energy amounts and conversions for the two hemispheres, but this time the information is given for several distinct wave number groups. Wave numbers 10-15 are quite unimportant, while wave numbers 4-9 tend to be the most active energetically, as would be expected from the theory of baroclinic instability.

7.8   The role of heating

We have seen that heating generates APE only when it occurs where the temperature is warm. Heating at cold temperatures actually destroys APE, because it reduces temperature
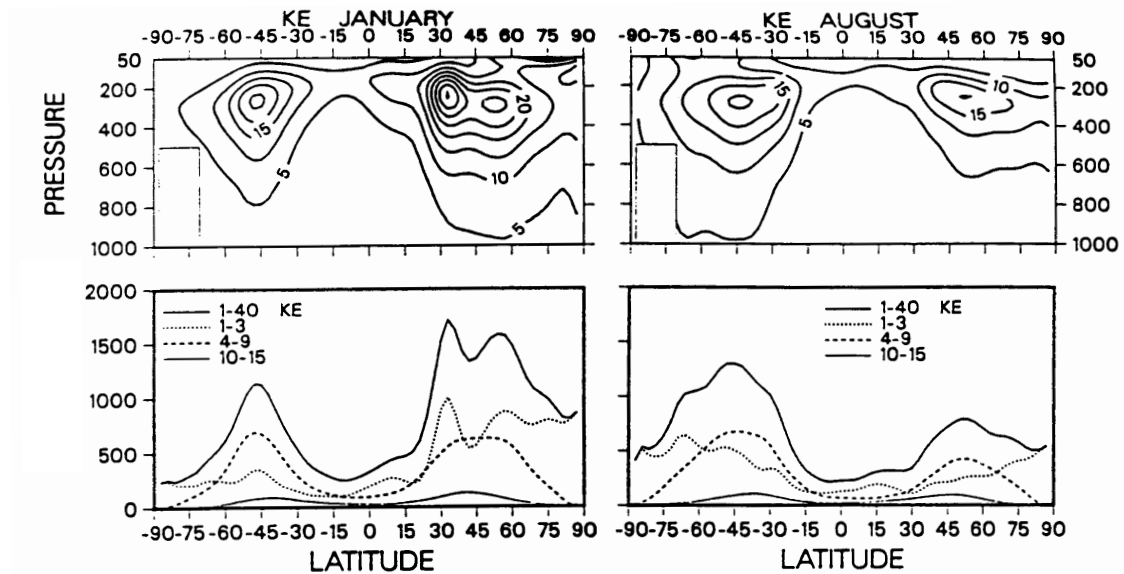
Figure 7.9: Vertical and meridional distribution of zonal mean eddy kinetic energy KE together with contributions by wavenumber groups to vertical integrals in January and August. Units in the cross–sections are $J\,(m^2 Pa)^{-1} = 100\,kJ\,(m^2 Bar)^{-1}$. From Arpé et al. (1986).

contrasts in the atmosphere. In many cases, heating is in fact a response to a lowering of the temperature. For example, large surface sensible heat fluxes heat the cold air rushing from the continents out over warm ocean currents in the winter. Similarly, deep moist convection heats the upper troposphere when cooling occurs aloft, e.g. because of large-scale lifting or the horizontal advection of cold air aloft. Such examples illustrate that heating does not necessarily promote a more vigorous circulation.

7.9 Summary

We defined the available potential energy and the gross static stability, and studied the generation and conversion of available potential energy. Finally we presented observations of the atmospheric energy cycle.

The observations of Arpé et al. remind us again of the wide range of eddy scales that are simultaneously at work in the general circulation the atmosphere. All of these eddies undergo their life cycles in the presence of the same mean meridional circulation. The complicated nonlinear interactions among the eddies are the subject of a later chapter, in which we view the general circulation as a kind of large-scale turbulence.
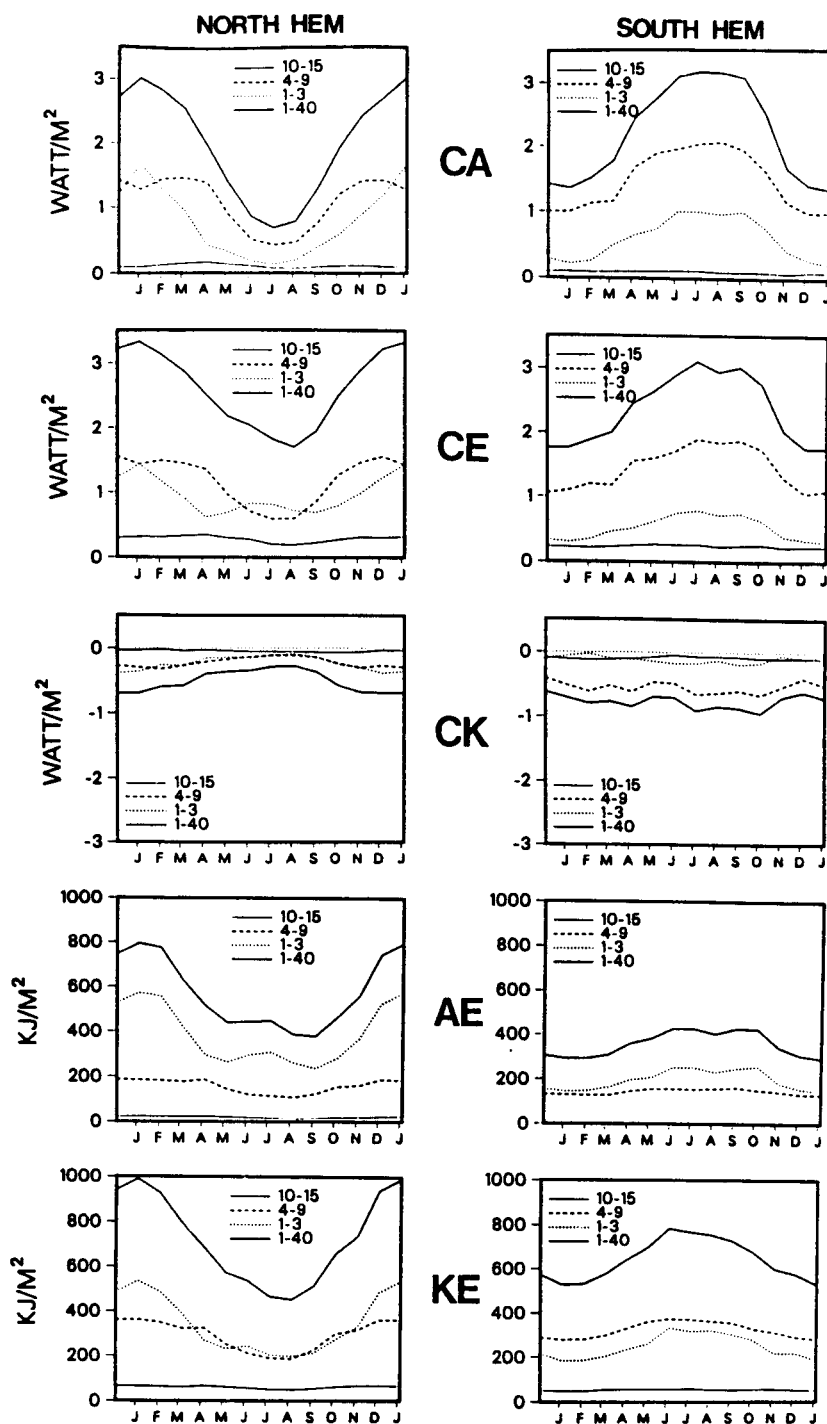
Problems

1. Derive (7.19).

2. Prove that

Figure 7.10: Mean annual cycles of energy amounts and conversions in both hemispher
and contributions from wavenumber groups calculated from 12 hour
forecasts. From Arpé et al. (1986).

$$\int_0^{p_S} p^\kappa \theta \, dp = \frac{1}{1+\kappa} \int_0^\infty p^{1+\kappa} d\theta \qquad (7.113)$$

Note that on the right-hand side the lower limit of integration is zero. This result is used to derive (7.13) from (7.12).

3. Prove that the mass-integrated gravitational potential energy of the entire atmosphere is smaller in the A-state than in the given state.

4. Prove that $S \geq 0$, where S is given by (7.30). State any assumptions.

5. The shallow water equations are:

$$\frac{\partial h}{\partial t} + \nabla \bullet (h\mathbf{V}) = 0, \qquad (7.114)$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\zeta + f)\mathbf{k} \times \mathbf{V} + \nabla[K + g(h + h_S)] = 0, \qquad (7.115)$$

where $\zeta \equiv \mathbf{k} \bullet (\nabla \times \mathbf{V})$, $K \equiv \frac{1}{2}(\mathbf{V} \bullet \mathbf{V})$, h is the depth of the water, and h_S is the height of the lower boundary (see the sketch below).
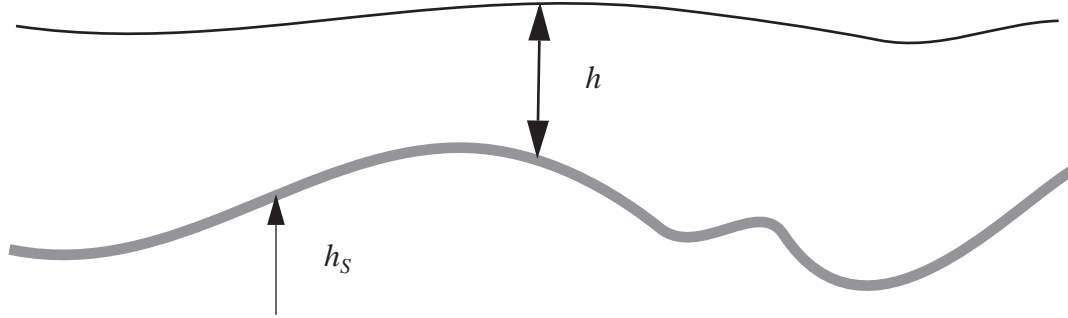
Show that the available potential energy of the system, per unit area, is $A = \frac{1}{2}g(\overline{H^2} - \overline{H}^2)$, where $H \equiv h + h_S$, and the bar represents an average over the whole domain. Prove that $\frac{d}{dt}(A + \overline{hK}) = 0$.

6. a) For the example given in the discussion beginning with Eq. (7.34), calculate the variance of θ for both the given state and the A-state, and demonstrate that the two variances are equal.

b) Continuing the example, assume that

$$F_\theta = -\frac{D}{a}\frac{\partial\theta}{\partial\varphi},$$ (7.116)

where D is a positive diffusion coefficient. Derive expressions for the time rates of change of the APE and the potential temperature variance, *valid at the instant when* $\theta(\varphi)$ *satisfies Eq. (7.34).*

c) Suppose that we add a "heating" term to (7.47), so that it becomes

$$\frac{\partial\theta}{\partial t} = -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(F_\theta\cos\varphi) + Q(\varphi).$$ (7.117)

Continue to use the form of F_θ given in part b) above. Find the form of $Q(\varphi)$ needed to maintain a steady state. Plot $Q(\varphi)$. Find the global mean of $Q(\varphi)$. Find the covariance of $Q(\varphi)$ and θ. Discuss.

7.   Suppose that

$$\frac{d}{dt}AZ = GZ - \frac{AZ}{\tau_Z} \cong 0,$$

$$\frac{d}{dt}KE = CE - \frac{KE}{\tau_E} \cong 0,$$

Estimate the values of the time scales τ_Z and τ_E, by using the numerical values given in Fig. 7.7. Compare *GZ* and *GE* with the actual rates of change of *AZ* and *KE* as shown in Fig. 7.8.