The Quasi-Static Approximation

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For resting air, the equation of vertical motion reduces to

$$\frac{\partial p}{\partial z} = -\rho g . \tag{32.1}$$

This is called the hydrostatic approximation. With appropriate boundary conditions, (32.1) allows us to compute p from $\rho(z)$. Even when the air is moving, (32.1) gives a good approximation to $p(z)$, simply because $\frac{Dw}{Dt}$ is small. *Under what conditions can this approximate $p(z)$ be used to compute the pressure gradient force in the equation of horizontal motion?* To do so is to use the *quasi-static approximation*.

To investigate the range of validity of the quasistatic approximation, let

$$p = p_s(z) + \delta p ,$$

$$\alpha = \alpha_s(z) + \delta\alpha , \tag{32.2}$$

where subscript *s* denotes a "reference sounding" that varies with height only. Assume that the reference state is hydrostatically balanced, i.e.

$$\alpha_s \frac{\partial p_s}{\partial z} = -g . \tag{32.3}$$

The vertical momentum equation can be written as

$$\frac{Dw}{Dt} = -\alpha_s \frac{\partial p_s}{\partial z} - \delta\alpha \frac{\partial p_s}{\partial z} - \alpha_s \frac{\partial \delta p}{\partial z} - \delta\alpha \frac{\partial \delta p}{\partial z} - g \ . \tag{32.4}$$

Here we neglect friction. The first and last terms on the right-hand-side of (32.4) cancel, because of (32.3). The second-to-last term on the right-hand side of (32.4) is dropped because it contains the product of two δ's. This is justified if $\alpha_s \gg |\delta\alpha|$, and $p_s \gg |\delta p|$.

Again neglecting friction, the horizontal momentum equation is

$$\frac{D\mathbf{V}_H}{Dt} = -\alpha\nabla_H(\delta p) - f\mathbf{k}\times\mathbf{V}_H .$$

$$NV \qquad \alpha_s\frac{\delta p}{L} \qquad fV \tag{32.5}$$

Here N^{-1} is a time scale, L is a horizontal length scale, and V is a horizontal velocity scale. We scale the remaining terms of (32.4) as follows:

$$\frac{Dw}{Dt} \cong \frac{\delta\alpha}{\alpha_s}g - \alpha_s\frac{\partial}{\partial z}(\delta p) .$$

$$NW \qquad \alpha_s\frac{\delta p}{D} \tag{32.6}$$

Here W is a vertical velocity scale, and D is a depth scale.

Consider two cases:

1) $N \geq f$ (short time scale)

This is the case in which the advective time scale, N^{-1}, is small compared to an inertial period, which is usually true for *small-scale motions*. If $N \sim \frac{V}{L}$, then $N \geq f$ is equivalent to $Ro \geq 1$, where $Ro \equiv \frac{V}{fL}$. For this case, we find from (32.5) that

$$\alpha_s\delta p \sim NLV . \tag{32.7}$$

To have $\left|\frac{Dw}{Dt}\right| \ll \left|\alpha\frac{\partial}{\partial z}(\delta p)\right|$ we need

$$NW \ll \frac{NLV}{D} , \tag{32.8}$$

or

$$\left(\frac{W}{D} \cdot \frac{L}{V}\right)\left(\frac{D}{L}\right)^2 \ll 1 \ . \tag{32.9}$$

Normally (from continuity), we have

$$\frac{W}{D} \cdot \frac{L}{V} \le 1 \ , \tag{32.10}$$

provided that the stratification is stable. Then (32.9) shows that "fast" motions can be quasistatic if they are "shallow," but not if they are "deep."

2) $N \le f$ (long time scale)

This is the case in which the advective time scale is relatively long, which is usually true for *large-scale motions*. If we assume that $N \sim \frac{V}{L}$, then $N \le f$ is equivalent to $Ro \le 1$. For this case, (32.5) leads to

$$\alpha_s \delta p \sim fLV \ , \tag{32.11}$$

which is essentially an expression of geostrophic balance. To have $\left|\frac{Dw}{Dt}\right| \ll \left|\alpha_s \frac{\partial}{\partial z}(\delta p)\right|$ we need

$$NW \ll fL\frac{V}{D} \ , \tag{32.12}$$

or

$$\left(\frac{W}{D} \cdot \frac{L}{V}\right)\left(\frac{D}{L}\right)^2 \frac{N}{f} \ll 1 \ . \tag{32.13}$$

Compare this with (32.9). The only difference is the factor of $\frac{N}{f}$. From (32.9), we see that

$$\left(\frac{D}{L}\right)^2 \ll 1 \ , \tag{32.14}$$

i.e. "shallow motion," is sufficient for quasistatic balance. When $N \ll f$ (long time scale), we can have quasistatic balance even for $\left(\frac{D}{L}\right)^2 \sim 1$.

When the quasistatic approximation is made, the effective kinetic energy is due entirely to the horizontal motion; the vertical component, w, makes no contribution. For large-scale motions, $w \ll (u, v)$, so that this quasistatic kinetic energy is very close to the true kinetic energy.