CHAPTER 4 Conservation of momentum and energy

4.1 Introduction

In this chapter, we systematically explore the conservation of momentum and, especially energy in its various forms. We also introduce some approximations that are appropriate when considering large-scale motions in the Earth's atmosphere, and derive some fundamental secondary relationships that will be used later in the course.

In the first part of the chapter, we write the equations as they apply at a point, and show molecular fluxes rather than turbulent fluxes. Near the end of the chapter we discuss the effects of the turbulent fluxes and also such processes as precipitation.

4.2 Conservation of momentum on a rotating sphere

The length of a day is 86400 s, so the Earth rotates about its axis with an angular velocity of $\frac{2\pi}{(86400\ \mathrm{s})} = 7.29 \times 10^{-5}\ s^{-1}$. This angular velocity can be represented by a vector, $\mathbf{\Omega}$, pointing towards the celestial North Pole. Consider a coordinate system that is rotating with the Earth, and refer to Fig. 4.1. Newton's statement of momentum conservation, as applied in the rotating coordinate system, is

$$\frac{D\mathbf{V}}{Dt} = -2\mathbf{\Omega} \times \mathbf{V} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) - \nabla\phi_a - \alpha\nabla p - \alpha\nabla \bullet \mathbf{F} . \tag{4.1}$$

Here $\mathbf{r}$ is a position vector extending from the center of the Earth to a particle of air whose position is generally changing with time. The gravitational potential is ϕ_a, and $\mathbf{F}$ is the stress tensor associated with molecular effects. Note that $\nabla \bullet \mathbf{F}$ is a vector. The pressure-gradient term is $-\alpha\nabla p$, where α is the specific volume, and p is the pressure.

The term $-2\mathbf{\Omega} \times \mathbf{V}$ represents the Coriolis acceleration, whose direction is perpendicular to $\mathbf{V}$. The term $-\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$ represents the centripetal acceleration. You should be able to show that

$$\mathbf{V}_e = \mathbf{\Omega} \times \mathbf{r} = (\Omega r \cos\varphi)\mathbf{e}_\lambda , \tag{4.2}$$

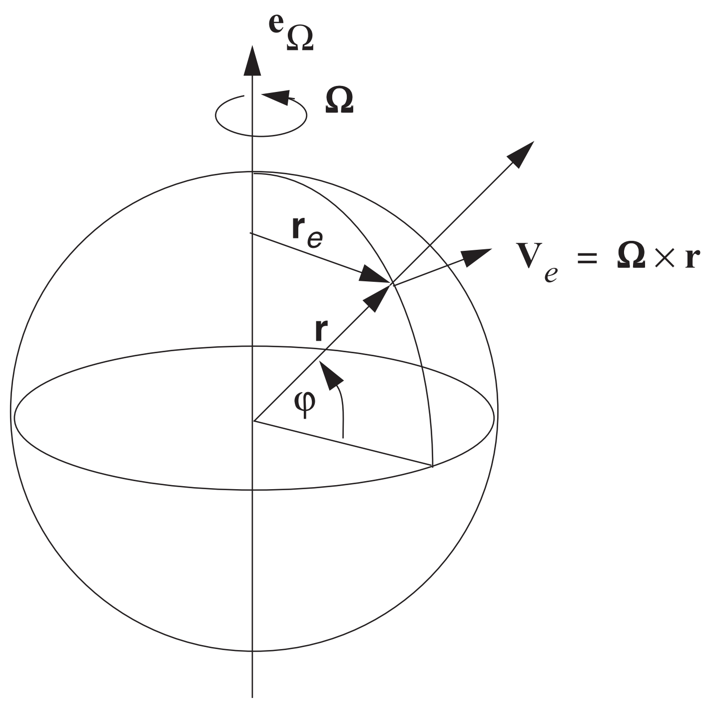

Figure 4.1: Sketch defining notation used in the text.

where $\mathbf{e}_\lambda$ is a unit vector pointing east, φ is latitude, and $\mathbf{V}_e$ is the velocity (as seen in the inertial frame) that a particle at radius r and latitude φ experiences due to the Earth's rotation (refer to Fig. 4.1). With this notation, we find that

$$\begin{aligned} -\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) &= (\Omega^2 r \cos\varphi)\mathbf{e}_\lambda \times \mathbf{e}_\Omega \\ &= \Omega^2 \mathbf{r}_e \quad , \end{aligned} \tag{4.3}$$

where $\mathbf{r}_e$ is the vector shown in Fig. 4.1, and $\mathbf{e}_\Omega$ is a unit vector pointing toward the celestial north pole. This shows that the centripetal acceleration points outward, in the direction of $\mathbf{r}_\mathrm{e}$, which is perpendicular to the axis of the Earth's rotation. It can be shown that

$$\Omega^2 \mathbf{r}_\mathrm{e} = \nabla\left[\frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2\right]. \tag{4.4}$$

According to (4.4), the centripetal acceleration can be regarded as the gradient of a potential, called the "centrifugal potential." The "apparent" gravity, $\mathbf{g}$, due to the combined effects of true gravity and the centripetal acceleration, can be defined as

$$\mathbf{g} = \mathbf{g}_a - \Omega^2 \mathbf{r}_e , \tag{4.5}$$

where $\mathbf{g}_a \equiv \nabla\Phi_a$, and using (4.4) we see that the potential of $\mathbf{g}$ is

$$\phi = \phi_a - \frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2 . \tag{4.6}$$

so that $\mathbf{g} \equiv \nabla\phi$. We refer to ϕ as the "geopotential."

As is well known, the Earth's gravitational potential, ϕ_a, decreases with distance from the center of the Earth. In addition, it varies with latitude and longitude, at a fixed distance from the center of the Earth, due to inhomogeneities in the distribution of mass within the solid Earth and the oceans. This means that a surface of constant ϕ_a is not spherical. Moreover, as can be seen from (4.4) and (4.6), the centripetal acceleration also varies geographically and with distance from the center of the Earth. As a result, surfaces of constant ϕ are only approximately spherical. In particular, the centripetal acceleration causes these surfaces to bulge outward at low latitudes, so that their shapes are well approximated by "oblate spheroids." To the extent that we wish to consider geographical variations of ϕ, the oblateness of the Earth's surface should also be taken into account. For most purposes, however,

$$\mathbf{g} \cong \mathbf{g}_a = -g\mathbf{k} , \tag{4.7}$$

because the centripetal acceleration is small compared to $\mathbf{g}_a$. Here $\mathbf{k}$ is a unit vector pointing upward, away from the center of the Earth. When we use (4.6) with a spatially constant value of g, as is conventional, we must also approximate the shape of the Earth as a sphere, with minor topographical bumps.

Using (4.6) we can now write the equation of motion (4.1) as

$$\frac{D\mathbf{V}}{Dt} = -2\mathbf{\Omega} \times \mathbf{V} - \nabla\phi - \alpha\nabla p - \alpha\nabla \bullet \mathbf{F} . \tag{4.8}$$

Another useful form of this equation is

$$\frac{\partial\mathbf{V}}{\partial t} + [2\mathbf{\Omega} + (\nabla \times \mathbf{V})] \times \mathbf{V} + \nabla(\tfrac{1}{2}\mathbf{V} \bullet \mathbf{V}) = -\nabla\phi - \alpha\nabla p - \alpha\nabla \bullet \mathbf{F} . \tag{4.9}$$

To obtain (4.9) from (4.8) we have used the vector identity

$$(\mathbf{V} \bullet \nabla)\mathbf{V} = (\nabla \times \mathbf{V}) \times \mathbf{V} + \nabla(\tfrac{1}{2}\mathbf{V} \bullet \mathbf{V}) . \tag{4.10}$$

We will have occasion to use both (4.8) and (4.9).

Now consider spherical coordinates, (λ, φ, r). The unit vectors in the (λ, φ, r) coordinates are $\mathbf{e}_\lambda$, $\mathbf{e}_\varphi$, and $\mathbf{e}_r$, respectively. As shown in the Appendix on "Vectors and coordinate systems," the vector operators that will be used in this course, i.e. the gradient, divergence, curl, and Laplacian, can be expressed in spherical coordinates as follows:

$$\nabla A = \left(\frac{1}{r\cos\varphi}\frac{\partial A}{\partial\lambda}, \frac{1}{r}\frac{\partial A}{\partial\varphi}, \frac{\partial A}{\partial r} \right), \tag{4.11}$$

$$\nabla \bullet \mathbf{H} = \frac{1}{r\cos\varphi}\frac{\partial H_\lambda}{\partial\lambda} + \frac{1}{r\cos\varphi}\frac{\partial}{\partial\varphi}(H_\varphi\cos\varphi) + \frac{1}{r^2}\frac{\partial}{\partial r}(H_r r^2), \tag{4.12}$$

$$\nabla \times \mathbf{H} = \left\{ \frac{1}{r}\left[\frac{\partial H_r}{\partial\varphi} - \frac{\partial}{\partial r}(rH_\varphi)\right], \right.$$
$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\lambda) - \frac{1}{r\cos\varphi}\frac{\partial H_r}{\partial\lambda},$$
$$\left. \frac{1}{r\cos\varphi}\left[\frac{\partial H_\varphi}{\partial\lambda} - \frac{\partial}{\partial\varphi}(H_\lambda\cos\varphi)\right] \right\}, \tag{4.13}$$

$$\nabla^2 A = \frac{1}{r^2\cos\varphi}\left[\frac{\partial}{\partial\lambda}\left(\frac{1}{\cos\varphi}\frac{\partial A}{\partial\lambda}\right) + \frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial A}{\partial\varphi}\right) + \frac{\partial}{\partial r}\left(r^2\cos\varphi\frac{\partial A}{\partial r}\right)\right]. \tag{4.14}$$

Here A is an arbitrary scalar, and $\mathbf{H} = (H_\lambda, H_\varphi, H_r)$ is an arbitrary vector.

Eq. (4.12) can be expanded as

$$\nabla \bullet \mathbf{H} = \frac{1}{r\cos\varphi}\frac{\partial H_\lambda}{\partial\lambda} + \frac{1}{r\cos\varphi}\frac{\partial}{\partial\varphi}(H_\varphi\cos\varphi) + \frac{\partial}{\partial r}H_r + \frac{2H_r}{r}. \tag{4.15}$$

Because the Earth's atmosphere is very thin compared to the radius of the Earth, the last term is negligible, and we can approximate the divergence operator by

$$\nabla \bullet \mathbf{H} \cong \frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial\lambda}H_\lambda + \frac{\partial}{\partial\varphi}(H_\varphi\cos\varphi)\right] + \frac{\partial}{\partial r}H_r . \tag{4.16}$$

Note that r has been replaced by a in the first two terms. In this course, we normally use (4.16) rather than (4.12), largely because it is traditional to do so. It is not at all clear, however, that the approximation (4.16) actually makes our work simpler. Note that the approximation would not be applicable to a deep atmosphere, such as that of a star or of Jupiter.

We can represent the velocity vector in terms of zonal, meridional, and radial components, as

$$\mathbf{V} \equiv u\mathbf{e}_\lambda + v\mathbf{e}_\varphi + w\mathbf{e}_r , \tag{4.17}$$

where

$$u \equiv r\cos\varphi\frac{D\lambda}{Dt} ,\ v \equiv r\frac{D\varphi}{Dt} ,\ \ w \equiv \frac{Dr}{Dt} , \tag{4.18}$$

and

$$\begin{aligned}\frac{D}{Dt} &\equiv \frac{\partial}{\partial t} + \frac{D\lambda}{Dt}\frac{\partial}{\partial\lambda} + \frac{D\varphi}{Dt}\frac{\partial}{\partial\varphi} + \frac{Dr}{Dt}\frac{\partial}{\partial r} \\ &= \frac{\partial}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\varphi} + w\frac{\partial}{\partial r}.\end{aligned} \tag{4.19}$$

The directions in which the unit vectors $\mathbf{e}_\lambda$, $\mathbf{e}_\varphi$, and $\mathbf{e}_r$ actually point depend on where you are. Therefore, as an air particle moves from place to place, the directions of the unit vectors change. Because of this, we must expand (4.17), taking into account a total of six terms:

$$\frac{D\mathbf{V}}{Dt} = \left(\frac{Du}{Dt}\mathbf{e}_\lambda + u\frac{D\mathbf{e}_\lambda}{Dt}\right) + \left(\frac{Dv}{Dt}\mathbf{e}_\varphi + v\frac{D\mathbf{e}_\varphi}{Dt}\right) + \left(\frac{Dw}{Dt}\mathbf{e}_r + w\frac{D\mathbf{e}_r}{Dt}\right). \tag{4.20}$$

Simple geometrical reasoning leads to the following formulae:

$$\begin{aligned}\frac{D\mathbf{e}_\lambda}{Dt} &= \frac{D\lambda}{Dt}\sin\varphi\mathbf{e}_\varphi - \cos\varphi\frac{D\lambda}{Dt}\mathbf{e}_r \\ &= \left(\frac{u\tan\varphi}{a}\right)\mathbf{e}_\varphi - \frac{u}{a}\mathbf{e}_r ,\end{aligned} \tag{4.21}$$

$$\begin{aligned}\frac{D\mathbf{e}_\varphi}{Dt} &= \frac{D\lambda}{Dt}\sin\varphi\mathbf{e}_\lambda - \frac{D\varphi}{Dt}\mathbf{e}_r \\ &= \left(\frac{u\tan\varphi}{a}\right)\mathbf{e}_\lambda - \frac{v}{a}\mathbf{e}_r ,\end{aligned} \tag{4.22}$$

$$\begin{aligned}\frac{D\mathbf{e}_r}{Dt} &= \cos\varphi\frac{D\lambda}{Dt}\mathbf{e}_\lambda + \frac{D\varphi}{Dt}\mathbf{e}_\varphi \\ &= \frac{u}{a}\mathbf{e}_\lambda + \frac{v}{a}\mathbf{e}_\varphi .\end{aligned} \tag{4.23}$$

When (4.21)-(4.23) are taken into account, (4.8) can be written as

$$\frac{Du}{Dt} + \frac{uw}{r} - \frac{uv\tan\varphi}{r} = fv - \bar{f}w - \frac{\alpha}{r\cos\varphi}\frac{\partial p}{\partial\lambda} - \alpha(\nabla \bullet \mathbf{F})_\lambda ,$$
$$\frac{Dv}{Dt} + \frac{vw}{r} + \frac{u^2\tan\varphi}{r} = -fu - \frac{\alpha}{r}\frac{\partial p}{\partial\varphi} - \alpha(\nabla \bullet \mathbf{F})_\varphi , \quad (4.24)$$
$$\frac{Dw}{Dt} - \left(\frac{u^2+v^2}{r}\right) = \bar{f}u - \alpha\frac{\partial p}{\partial r} - \alpha(\nabla \bullet \mathbf{F})_r - g .$$

Here

$$f \equiv 2\Omega\sin\varphi \text{ and } \bar{f} \equiv 2\Omega\cos\varphi . \quad (4.25)$$

Eqs. (4.24) are the components of the equation of motion in spherical coordinates Further explanation is given in the Appendix.

By using the continuity equation in spherical coordinates, we can rewrite (4.24) in flux form:

$$\frac{\partial}{\partial t}(\rho u) + \nabla \bullet (\rho\mathbf{V}u) + \rho\frac{uw}{r} - \rho uv\frac{\tan\varphi}{r} = \rho fv - \rho\bar{f}w - \frac{1}{r\cos\varphi}\frac{\partial p}{\partial\lambda} - (\nabla \bullet \mathbf{F})_\lambda ,$$

$$\frac{\partial}{\partial t}(\rho v) + \nabla \bullet (\rho\mathbf{V}v) + \rho\frac{vw}{r} + \rho u^2\frac{\tan\varphi}{r} = -\rho fu - \frac{1}{r}\frac{\partial p}{\partial\varphi} - (\nabla \bullet \mathbf{F})_\varphi ,$$

$$\frac{\partial}{\partial t}(\rho w) + \nabla \bullet (\rho\mathbf{V}w) - \rho\left(\frac{u^2+v^2}{r}\right) = \rho\bar{f}u - \frac{\partial p}{\partial r} - \rho g - (\nabla \bullet \mathbf{F})_r .$$
$$(4.26)$$

These equations are fairly exact. Various approximations will be introduced later.

4.3 Conservation of kinetic energy and potential energy

As you probably already know, the kinetic energy equation can be derived from the equation of motion. To do this, start with the full three-dimensional equation of motion in the form

$$\frac{\partial\mathbf{V}}{\partial t} + (\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \times \mathbf{V} + \nabla(K + \phi) = -\alpha\nabla p - \alpha\nabla \bullet \mathbf{F} . \quad (4.27)$$

Here we are using height coordinates. Dotting (4.27) with $\mathbf{V}$, we find that

$$\frac{DK}{Dt} + \mathbf{V} \bullet \nabla\phi = -\alpha\mathbf{V} \bullet \nabla p - \alpha\mathbf{V} \bullet (\nabla \bullet \mathbf{F}) , \quad (4.28)$$

where

$$K \equiv \frac{1}{2}\mathbf{V} \bullet \mathbf{V} \tag{4.29}$$

is the kinetic energy per unit mass.

Note that K and, therefore, the total energy depend on the choice of coordinate system. For example, if we compare a coordinate system that is rotating with the Earth to an inertial coordinate system, the actual value of the kinetic energy at a given place in the atmosphere will differ by a large amount. The Lagrangian time rate of change of the kinetic energy will be the same, however, and this is what matters, because it is $\frac{DK}{Dt}$, rather than K itself, that appears in the energy conservation equation (4.28).

Since ϕ is independent of time in height coordinates, we see that

$$\frac{D\phi}{Dt} = \mathbf{V} \bullet \nabla\phi = wg\,. \tag{4.30}$$

This can be called "the potential energy equation." Use of (4.30) and (4.31) allows us to rewrite (4.28) as

$$\frac{D(K+\phi)}{Dt} = -\alpha\mathbf{V} \bullet \nabla p - \alpha\mathbf{V} \bullet (\nabla \bullet \mathbf{F})\,. \tag{4.31}$$

We refer to $K+\phi$ as the mechanical energy per unit mass. Eq. (4.31) is sometimes called the mechanical energy equation.

In (4.31), the rate at which work is done by the pressure force, per unit mass, is represented by $-\alpha\mathbf{V} \bullet \nabla p$. This expression can be rewritten as follows:

$$\begin{aligned} -\alpha\mathbf{V} \bullet \nabla p &= -\alpha\nabla \bullet (p\mathbf{V}) + \alpha(p\nabla \bullet \mathbf{V}) \\ &= -\alpha\nabla \bullet (p\mathbf{V}) + p\frac{D\alpha}{Dt}\,. \end{aligned} \tag{4.32}$$

Here we have used the continuity equation to eliminate $\alpha\nabla \bullet \mathbf{V}$. The $\nabla \bullet (p\mathbf{V})$ term on the second line of (4.32) has the form of a flux divergence, and so represents a spatial redistribution of energy by the pressure force. The $p\frac{D\alpha}{Dt}$ term represents the work done by volume expansion (analogous to the work done in inflating a balloon). We refer to $p\frac{D\alpha}{Dt}$ as the "expansion-work" term[1].

Similarly, the friction term of (4.31) can be expanded to reveal two physically distinct parts, as follows:

$$-\alpha\mathbf{V} \bullet (\nabla \bullet \mathbf{F}) = -\alpha\nabla \bullet (\mathbf{F} \bullet \mathbf{V}) - \delta, \tag{4.33}$$

where

$$\delta \equiv -\alpha(\mathbf{F} \bullet \nabla) \bullet \mathbf{V} \tag{4.34}$$

is the rate of kinetic energy dissipation per unit volume. Note that $\mathbf{F} \bullet \mathbf{V}$ is a vector. The quantity $\nabla \bullet (\mathbf{F} \bullet \mathbf{V})$ in (4.33) has the form of a flux divergence, and so represents a *spatial redistribution* of kinetic energy as friction (represented by $\mathbf{F}$) causes air parcels to do work on each other. Because this is just a spatial redistribution of energy, it does not change the total amount of kinetic energy in the atmosphere, except where friction does work on the lower boundary.

In contrast, kinetic energy dissipation, represented in (4.33) by δ, represents a true sink of kinetic energy for the atmosphere. It can be shown that dissipation is always a sink of kinetic energy, i.e.

$$\delta \geq 0. \tag{4.35}$$

As discussed later, the dissipation of kinetic energy appears as a source of thermodynamic energy, i.e. as "frictional heating." It is a weak but persistent source of internal energy for the atmosphere.

Substitution of (4.32) and (4.33) into (4.31) gives

$$\frac{D(K+\phi)}{Dt} = -\alpha\nabla \bullet (p\mathbf{V} + \mathbf{F} \bullet \mathbf{V}) + p\frac{D\alpha}{Dt} - \delta. \tag{4.36}$$

Using continuity, (4.36) can be rewritten in flux form:

$$\frac{\partial}{\partial t}(\rho K) + \nabla \bullet [\rho\mathbf{V}(K+\phi) + p\mathbf{V} + \mathbf{F} \bullet \mathbf{V}] = \rho p\frac{D\alpha}{Dt} - \rho\delta. \tag{4.37}$$

All contributions to $\frac{\partial}{\partial t}(\rho K)$ on the left-hand side of (4.37) represent transport processes, which merely redistribute energy in space. In contrast, the expansion-work term, $p\frac{D\alpha}{Dt}$, need not integrate to zero. It can be either positive or negative at a given place and time. Recall, however, that the dissipation term is always a sink. It follows that *in an average over the*

[1] Because α is a constant for a fluid of constant density (e.g., "shallow water"), the internal energy of a constant-density fluid is nonconvertible (like the ruble), although it is not zero (unlike the ruble). Because it is nonconvertible, the internal energy of a constant-density fluid plays no role in the energy cycle; we can just ignore it.

whole atmosphere, and over time, the $p\frac{D\alpha}{Dt}$ *term must be positive, i.e. it must act as a source of mechanical energy:*

$$\int_V \overline{p\frac{D\alpha}{Dt}}\rho dV = \int_V \overline{\delta\rho} dV \geq 0 \ . \tag{4.38}$$

Here the integral is taken over the entire mass of the atmosphere and the overbars represent a time average. Eq. (4.38) is a very fundamental result. It means that *on the average* the pressure force must do positive expansion work to compensate for the dissipation of kinetic energy. In order that $\int_V p\frac{D\alpha}{Dt}\rho dV$ be positive, *expansion must take place, in an average sense, at a higher pressure than compression*. For example, we can have expansion in the lower troposphere and compression in the upper troposphere.

Given that in an average sense the expansion work term of (4.36) must act as a source of mechanical energy, we have to ask where this energy comes from. The answer is that it comes from the thermodynamic energy of the atmosphere. This will be demonstrated later. Expansion work represents an energy *conversion* process, which can have either sign locally but is positive when averaged over the whole atmosphere and over time.

Similarly, given that the dissipation term of (4.36) represents a sink of mechanical energy, we have to ask where the energy goes. The answer is that it appears as a source of thermodynamic energy. Dissipation is, therefore, another energy conversion process -- a conversion that runs in only one direction.

Eq. (4.38) simply means that the rate of kinetic energy dissipation must be equal, on the average, to the rate of kinetic energy generation. For the whole atmosphere, this rate has been estimated to be on the order of 5 W m^{-2}. For comparison, recall that the solar radiation absorbed by the Earth-atmosphere system is about 240 W m^{-2}. Evidently the climate system is not very efficient at converting the absorbed solar energy into atmospheric kinetic energy.

The mechanical energy generation term can be written in another form. To see this, note that

$$\begin{aligned} p\frac{D\alpha}{Dt} &= \frac{D}{Dt}(p\alpha) - \alpha\frac{Dp}{Dt} \\ &= \frac{D}{Dt}(RT) - \omega\alpha \ . \end{aligned} \tag{4.39}$$

This shows that in a time average over the whole atmosphere the expansion-work term is closely related to the product $\omega\alpha$. We can interpret $\omega\alpha$ as a rate of conversion between mechanical and thermodynamic energy.

Substituting (4.39) into (4.36), we obtain

$$\frac{D(K+\phi-RT)}{Dt} = -\alpha\nabla\bullet(p\mathbf{V}+\mathbf{F}\bullet\mathbf{V})-\omega\alpha-\delta\,. \tag{4.40}$$

This is somewhat easier to interpret when we convert to flux form:

$$\frac{\partial}{\partial t}[\rho(K+\phi)]+\nabla\bullet[\rho\mathbf{V}(K+\phi)] = -\nabla\bullet(\mathbf{F}\bullet\mathbf{V})-\rho(\omega\alpha+\delta)+\frac{\partial p}{\partial t}\,. \tag{4.41}$$

Note that the pressure-work term of (4.40), involving $\nabla\bullet(p\mathbf{V})$, has disappeared via a cancellation, but "in its place" we pick up a new term involving the local time-rate-of-change of the pressure, $\frac{\partial p}{\partial t}$. From (4.41) we see that *in an average over the whole atmosphere, and over time, the* $-\omega\alpha$ *term of (4.41) must be positive, i.e. it must act as a source of mechanical energy:*

$$-\int_V \overline{\omega\alpha\rho}\,dV = \int_V \overline{\delta\rho}\,dV \ge 0\,. \tag{4.42}$$

Comparing (4.38) and (4.42), we see that $-\int_V \overline{\omega\alpha\rho}\,dV = \int_V \overline{p\frac{D\alpha}{Dt}\rho}\,dV$.

4.4 Conservation of thermodynamic energy

The internal energy of a perfect gas is given by

$$e = c_v T\,, \tag{4.43}$$

where c_v, the heat capacity at constant volume, is a constant. For dry air, $c_v = \frac{5R}{2} \cong 713$ J kg^{-1} K^{-1}.

More generally, the internal energy also includes the latent heat associated with the potential condensation of water vapor[2], and we find that for moist air

$$e \cong c_v T + Lq_v\,. \tag{4.44}$$

This equation is approximate because we have neglected the heat capacity of the water vapor, as well as the heat capacity of any liquid (or ice) that might be present. In atmospheric science we frequently define the internal energy as the internal energy of dry air, and treat the latent heat as an "external" source or sink of internal energy. This is not strictly correct, but

[2]. We could also add the latent heats of other atmospheric constituents, e.g., nitrogen, oxygen, and carbon dioxide, to represent the effects of their potential condensation. We do not bother to do so because those constituents do not condense under conditions realized in the Earth's atmosphere.

does no harm in most applications. We will follow this convention in this course, i.e., we will define the internal energy per unit mass by (4.43).

When thermodynamic energy is added to a system, the energy input equals the sum of the work done and the change in the internal energy:

$$\frac{De}{Dt} + p\frac{D\alpha}{Dt} = -\alpha\nabla \bullet (\mathbf{R} + \mathbf{F}_s) + LC + \delta . \tag{4.45}$$

Here e is given by (4.43); $\mathbf{F}_s$ is the vector flux of internal energy due to molecular diffusion; $\mathbf{R}$ is the vector flux of energy due to radiation[3]. Note that the dissipation rate appears here as a source of internal energy. Equation (4.45) is a statement of the conservation of thermodynamic energy[4], applied to a moving particle.

An alternative statement of the conservation of thermodynamic energy, obtained using (4.39) in (4.45), is

$$c_p\frac{DT}{Dt} = \omega\alpha - \alpha\nabla \bullet (\mathbf{R} + \mathbf{F}_s) + LC + \delta , \tag{4.46}$$

where

$$c_p = R + c_v \cong 1000 \text{ J kg}^{-1}\text{ K}^{-1}. \tag{4.47}$$

Eq. (4.46) shows that, for an adiabatic, isobaric process, the enthalpy, η, is a conserved variable. For an ideal gas, the enthalpy is given by

$$\eta = c_p T . \tag{4.48}$$

More generally, the enthalpy can be written as

$$\eta = e + p\alpha . \tag{4.49}$$

For saturated air containing liquid, it turns out that

$$\eta \cong c_p T - Ll , \tag{4.50}$$

where L is the latent heat of condensation (e.g., Lorenz 1979; Emanuel, 1994) and l is the liquid water mixing ratio.

It is also possible to express the conservation of thermodynamic energy in terms of the

3. This notation conflicts with that used for the gas constant, but there should be little chance of confusion.

4. Also called the "First Law of Thermodynamics," although that terminology seems rather medieval.

potential temperature. By using , we can show that

$$c_p \frac{D\theta}{Dt} = \left(\frac{\theta}{T}\right)[-\alpha\nabla \bullet (\mathbf{R} + \mathbf{F}_s) + LC + \delta] . \tag{4.51}$$

In the absence of heating and dissipation, $\frac{D\theta}{Dt} = 0$, i.e. θ is conserved following a particle in the absence of heating. This is one of the reasons that θ is a particularly useful quantity.

In summary, the thermodynamic energy equation can be expressed in the three equivalent forms (4.45), (4.46), and (4.51). Each of these forms will be used later.

4.5 Conservation of total energy

Adding (4.36) and (4.45) gives

$$\frac{D(K + \phi + e)}{Dt} = -\alpha\nabla \bullet (p\mathbf{V} + \mathbf{F} \bullet \mathbf{V}) - \alpha\nabla \bullet (\mathbf{R} + \mathbf{F}_s) + LC . \tag{4.52}$$

Note that the $p\frac{D\alpha}{Dt}$ terms have cancelled, as have the dissipation terms. The cancellations occur because these terms represent conversions between thermodynamic and mechanical energy. Alternatively, we can add (4.40) and (4.46) to obtain

$$\frac{D(K + \phi + e)}{Dt} = -\alpha\nabla \bullet (p\mathbf{V} + \mathbf{F} \bullet \mathbf{V}) - \alpha\nabla \bullet (\mathbf{R} + \mathbf{F}_s) + LC . \tag{4.53}$$

Here we have used $c_p T - p\alpha = e$, and the $\omega\alpha$ and dissipation terms have cancelled, as before, because they represent energy conversions. As should be expected, (4.52) and (4.53) are identical. You can think of (4.36) and (4.45) as a "matched set," and (4.40) and (4.46) as a second, equivalent, matched set.

Although total energy conservation is expressed by (4.52) [and (4.53)], the energy conversions associated with latent heat release are not shown explicitly. To remedy this, we use the water vapor conservation equation in the form

$$\frac{Dq_v}{Dt} = -\alpha\nabla \bullet (\mathbf{F}_{q_v}) - C , \tag{4.54}$$

where $\mathbf{F}_{q_v}$ is the flux of water vapor due to molecular diffusion, and C is the rate of condensation. Next, we multiply (4.54) by the latent heat of condensation, L, neglect variations of L in time and space, and add the result to (4.52). This gives

$$\frac{D}{Dt}(K + \phi + e + Lq_v) = -\alpha\nabla \bullet (p\mathbf{V} + \mathbf{F} \bullet \mathbf{V} + \mathbf{R} + \mathbf{F}_h) . \tag{4.55}$$

Here $\mathbf{F}_h \equiv \mathbf{F}_s + L\mathbf{F}_{q_v}$ is the sum of the molecular fluxes of sensible and latent heat; for reasons explained later, this sum will be called the molecular flux of moist static energy. From (4.55) we see that the total energy per unit mass is given by the sum of the kinetic, potential, internal, and latent energies:

$$e_t = K + \phi + e + Lq_v \ . \tag{4.56}$$

Every term on the right-hand side of (4.55) is the divergence of a flux, i.e., each term represents a spatial redistribution of energy. This shows that the total energy of the atmosphere is conserved apart from exchanges across its upper and lower boundaries. To see how this works, start by using the continuity equation to convert (4.55) to flux form:

$$\frac{\partial}{\partial t}[\rho(K + \phi + e + Lq_v)] + \nabla \bullet [\rho \mathbf{V}(K + \phi + e + Lq_v) + p\mathbf{V} + \mathbf{F} \bullet \mathbf{V} + \mathbf{R} + \mathbf{F}_h] = 0 \ . \tag{4.57}$$

Next, distinguish between horizontal and vertical fluxes of energy, as follows:

$$\begin{aligned} &\frac{\partial}{\partial t}[\rho(K + \phi + e + Lq_v)] \\ &+ \nabla_H \bullet [\rho \mathbf{V}_H(K + \phi + e + Lq_v) + p\mathbf{V}_H + (\mathbf{F} \bullet \mathbf{V})_H + \mathbf{R}_h + (\mathbf{F}_h)_H] \\ &+ \frac{\partial}{\partial z}[\rho w(K + \phi + e + Lq_v) + pw + (\mathbf{F} \bullet \mathbf{V})_z + R_z + (F_h)_z] = 0 \quad . \end{aligned} \tag{4.58}$$

Here the subscripts H and z denote the "horizontal part" of a vector (i.e., a vector in the horizontal plane), and the (positive upward) vertical component of a vector, respectively. Next, we vertically integrate through the entire atmospheric column, using Leibniz' Rule to take the integrals inside the derivatives. The result can be written as

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\int_{z_S}^{\infty} \rho(K + \phi + e + Lq_v)dz\right] + \nabla_H \bullet \left\{\int_{z_S}^{\infty} [\rho \mathbf{V}(K + \phi + e + Lq_v)]dz\right\} \\ &+ [\rho(K + \phi + e + Lq_v)]_S\left(\frac{\partial z_S}{\partial t} + \mathbf{V}_H \bullet \nabla z_S - w_S\right) \\ &+ \nabla_H \bullet \left[\int_{z_S}^{\infty} (p\mathbf{V}_H + (\mathbf{F} \bullet \mathbf{V})_H + \mathbf{R}_h + (\mathbf{F}_h)_H)dz\right] \\ &= -p_S[(\mathbf{V}_H)_S \bullet \nabla z_S - w_S] - \{[(\mathbf{F} \bullet \mathbf{V})_H]_S \bullet \nabla z_S - [(\mathbf{F} \bullet \mathbf{V})_z]_S\} \\ &- \{[(\mathbf{F}_h)_H]_S \bullet \nabla z_S - [(F_h)_z]_S\} - (\mathbf{R}_z)_\infty - \{[\mathbf{R}_h]_S \bullet \nabla z_S - (R_z)_S\} \quad . \end{aligned} \tag{4.59}$$

Recall that the condition that no mass crosses the Earth's surface can be written as

$$\frac{\partial z_S}{\partial t} + \mathbf{V}_H \bullet \nabla z_S - w_S = 0 . \tag{4.60}$$

With the use of (4.60), we can rewrite (4.59) as

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\int_{z_S}^{\infty} \rho(K + \phi + e + Lq_v)dz\right] \\ &+ \nabla_H \bullet \left[\int_{z_S}^{\infty} (\rho \mathbf{V}(K + \phi + e + Lq_v) + p\mathbf{V}_H + (\mathbf{F} \bullet \mathbf{V})_H + \mathbf{R}_h + (\mathbf{F}_h)_H)dz\right] \\ &= p_S \frac{\partial z_S}{\partial t} - \{[(\mathbf{F} \bullet \mathbf{V})_H]_S \bullet \nabla z_S - [(\mathbf{F} \bullet \mathbf{V})_z]_S\} \\ &- \{[(\mathbf{F}_h)_H]_S \bullet \nabla z_S - [(F_h)_z]_S\} - (\mathbf{R}_z)_\infty - \{[\mathbf{R}_h]_S \bullet \nabla z_S - (R_z)_S\} \quad . \end{aligned} \tag{4.61}$$

The terms on the right-hand side of (4.59) represent the effects of fluxes at the upper and lower boundaries. As pointed out earlier, the only flux of energy at the upper boundary is that due to radiation, denoted by $-(\mathbf{R}_z)_\infty$. At the lower boundary there are energy fluxes due to pressure-work, frictional work, the molecular flux of moist static energy, and radiation.

When $\frac{\partial z_S}{\partial t} = 0$, the pressure-work term vanishes. Over the ocean, however, the surface height fluctuates due to the passage of waves, and so in the presence of such waves the atmosphere and ocean can exchange energy due to pressure work. It is interesting that one effect of such an energy exchange can be to add energy to the waves that make the energy exchange possible.

Even over land, the vegetation moves as the wind blows through it, so the pressure-work term can be non-zero. In addition, an earthquake can impart energy to the atmosphere through the pressure-work term.

The friction terms of (4.59) represent the work done by surface drag. This will be discussed further near the end of this Chapter.

As already discussed, the surface moist static energy flux is a very important energy source for the atmosphere.

Finally, the radiative energy flux is quite important at both the upper and lower boundaries of the atmosphere.

When we integrate (4.59) horizontally over the entire sphere, the horizontal flux divergence term integrates to zero, and we get

$$\frac{d}{dt}\left[\int_A\left[\int_{z_S}^{\infty}\rho(K+\phi+e+Lq_v)dz\right]dA\right]$$

$$= \int_A p_S\frac{\partial z_S}{\partial t}dA - \int_A\{[(\mathbf{F}\bullet\mathbf{V})_H]_S\bullet\nabla z_S - [(\mathbf{F}\bullet\mathbf{V})_z]_S\}dA \tag{4.62}$$

$$-\int_A\{[(\mathbf{F}_h)_H]_S\bullet\nabla z_S-[(F_h)_z]_S\}dA-\int_A(\mathbf{R}_z)_\infty dA-\int_A\{[\mathbf{R}_h]_S\bullet\nabla z_S-(R_z)_S\}dA.$$

Here $\int_A(\ \ )dA$ denotes the integral over the sphere. Eq. (4.62) shows that, in the absence of heating and friction, and when the height of the Earth's surface is independent of time,

$$\frac{d}{dt}\left[\int_A\left[\int_{z_S}^{\infty}\rho(K+\phi+e+Lq_v)dz\right]dA\right] = 0, \tag{4.63}$$

i.e., the total energy of the atmosphere is invariant. Eq. (4.63) is a very important conclusion, which will be used later in the definition of available potential energy.

The combined effect of the pressure-work and frictional work terms of (4.62) is to remove energy from the global atmosphere. It follows that, in a time average, the radiative and molecular flux terms must act as an energy source for the atmosphere. As discussed later, the rate at which the atmosphere does the frictional work on the lower boundary is on the order of tenths of a W m^{-2}. The pressure-work term is typically even smaller. As discussed in Chapter 1, the remaining individual terms on the right-hand side of (4.62) are typically larger by several orders of magnitude. It follows that these remaining terms must very nearly cancel in a time average, i.e.,

$$\begin{aligned}0 \cong &-\int_A\{[(\mathbf{F}_h)_H]_S\bullet\nabla z_S-[(F_h)_z]_S\}dA\\ &-\int_A(\mathbf{R}_z)_\infty dA-\int_A\{[\mathbf{R}_h]_S\bullet\nabla z_S-(R_z)_S\}dA\quad.\end{aligned} \tag{4.64}$$

In this sense, the total "heating" of the atmosphere is very nearly zero. Eq. (4.64) can also be written as

$$\int_V[-\alpha\nabla\bullet(\mathbf{R}+\mathbf{F}_s)+LC]\rho dV\cong 0\ , \tag{4.65}$$

where $\int_V(\ \ )\rho dV$ denotes a mass-weighted integral over the entire atmosphere.

4.6 Static energies

By using the equation of state and (4.47), we can rewrite the total energy equation, (4.57), as

$$\frac{\partial}{\partial t}[\rho(K+\phi+c_pT+Lq_v)]+\nabla\bullet[\rho\mathbf{V}(K+\phi+c_pT+Lq_v)+\mathbf{R}+\mathbf{F}_h] = -\nabla\bullet(\mathbf{F}\bullet\mathbf{V})+\frac{\partial p}{\partial t} \quad . \tag{4.66}$$

Here the enthalpy appears in place of the internal energy in the time derivative and flux divergence terms. A key difference between (4.57) and (4.66) is that there is no pressure-work term in the latter. A price that we pay for this simplification is the appearance of the $\frac{\partial p}{\partial t}$ term on the right-hand side of (4.66). Note, however, that this term drops out in a time average.

The $\frac{\partial p}{\partial t}$ term of (4.66) looks funny. As an aid in its interpretation, write

$$\begin{aligned}\frac{\partial p}{\partial t} &= \frac{\partial}{\partial z}\left(z\frac{\partial p}{\partial t}\right)-z\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t}\right)\\ &= \frac{\partial}{\partial z}\left(z\frac{\partial p}{\partial t}\right)-z\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial z}\right)\\ &= \frac{\partial}{\partial z}\left(z\frac{\partial p}{\partial t}\right)-\frac{\partial}{\partial t}\left(z\frac{\partial p}{\partial z}\right) \quad .\end{aligned} \tag{4.67}$$

This can be substituted into (4.66) to obtain

$$\frac{\partial}{\partial t}\left[\rho(K+\phi+c_pT+Lq_v)+z\frac{\partial p}{\partial z}\right]+\nabla\bullet[\rho\mathbf{V}(K+\phi+c_pT+Lq_v)+\mathbf{R}+\mathbf{F}_h] = -\nabla\bullet(\mathbf{F}\bullet\mathbf{V})+\frac{\partial}{\partial z}\left(z\frac{\partial p}{\partial t}\right) \quad . \tag{4.68}$$

In the hydrostatic limit the $\rho\phi$ and $z\frac{\partial p}{\partial z}$ terms inside the time-rate-of-change operator cancel. The second term on the right-hand side of (4.68) represents a vertical flux of energy associated with the time-rate-of-change of the pressure.

The contribution of the kinetic energy to the total energy is typically quite negligible. For example, an air parcel zipping along at a rather extreme 100 m s^{-1} has a kinetic energy per unit mass of 5 x 10^3 J kg^{-1}. (Keep in mind that the kinetic energy is proportional to the square of the wind speed, so that a parcel moving at a more typical 10 m s^{-1} has a kinetic energy 100 times smaller.) If a parcel traveling at 100 m s^{-1} resides on the 200 mb surface, its potential energy per unit mass (relative to sea level) is about 1.2 x 10^5 J kg^{-1}, or about 24 times greater than its kinetic energy. If the temperature of the fast parcel is a mere 200 K, which is if anything a little too cold for the 200 mb surface, its internal energy per unit mass is about 1.5 x 10^5 J kg^{-1}, about 30 times greater than its kinetic energy. For these reasons, we can usually neglect K in (4.66).

In addition, the friction and pressure-tendency terms of (4.66) can often be neglected.

With these simplifying approximations, (4.66) reduces to

$$\frac{\partial}{\partial t}(\rho h) + \nabla \bullet (\rho \mathbf{V} h + \mathbf{R} + \mathbf{F}_h) = 0 , \tag{4.69}$$

where

$$h \equiv c_p T + \phi + L q_v \tag{4.70}$$

is the *moist static energy*, whose latitude-height distribution was discussed in Chapter 3. According to (4.69), the moist static energy is approximately conserved under both moist adiabatic and dry adiabatic processes. Since precipitation does not affect the water vapor mixing ratio, temperature, or geopotential height, the moist static energy is conserved even for pseudoadiabatic processes, in which condensed water is assumed to precipitate out immediately. For many practical purposes, conservation of total energy is (approximately) equivalent to conservation of moist static energy.

We did not have to use the hydrostatic approximation to derive (4.69); this is important, because it means that (4.69) can be used in the analysis of non-hydrostatic processes, e.g. cumulus convection. We will do just that in Chapter 5.

Note that *conservation of moist static energy is an approximation to the total energy equation, rather than the thermodynamic energy equation*. This is why there is no dissipation term in (4.69); such a term would of course appear in any version of the thermodynamic energy equation (although we might justify neglecting it under some conditions).

Because the water vapor mixing ratio, q_v, is conserved under dry adiabatic processes, conservation of moist static energy implies that the dry static energy,

$$s \equiv c_p T + \phi , \tag{4.71}$$

is approximately conserved under dry adiabatic processes.

4.7 Entropy

For any gas or liquid, the entropy per unit mass, s, satisfies

$$T\frac{Ds}{Dt} = \frac{De}{Dt} + p\frac{D\alpha}{Dt} . \tag{4.72}$$

From (4.45), this implies that

$$T\frac{Ds}{Dt} = -\alpha\nabla \bullet (\mathbf{R} + \mathbf{F}_s) + LC + \delta . \tag{4.73}$$

Using the equation of state with (4.72), we can show that

$$s = c_p ln\left(\frac{T}{T_0}\right) - R ln\left(\frac{p}{p_0}\right) . \tag{4.74}$$

In (4.74), T_0, and p_0 are suitable reference values that arise here as "constants of integration." There is a simple relationship between the entropy and the potential temperature, i.e.,

$$s \equiv c_p ln\left(\frac{\theta}{\theta_0}\right) . \tag{4.75}$$

Here θ_0 is the potential temperature corresponding to T_0, and p_0.

For saturated air containing liquid water, (4.75) must be replaced by

$$s \cong c_p ln\left(\frac{\theta}{\theta_0}\right) - \frac{Ll}{T} \tag{4.76}$$

(e.g. Lorenz, 1979; Emanuel, 1994).

Eq. (4.73) can be rearranged as

$$\begin{aligned} \frac{Ds}{Dt} &= \frac{[-\alpha\nabla \bullet (\mathbf{R} + \mathbf{F}_s) + LC + \delta]}{T} \\ &\geq \frac{[-\alpha\nabla \bullet (\mathbf{R} + \mathbf{F}_s) + LC]}{T} . \end{aligned} \tag{4.77}$$

According to (4.77), dissipation and heating can change the entropy; because $\delta \geq 0$, dissipation never decreases and normally increases the entropy.

Now use continuity to rewrite (4.77) in flux form, then pass to an integral over a closed system to obtain:

$$\frac{dS}{dt} = \int_V \left[\frac{-\alpha\nabla \bullet (\mathbf{R} + \mathbf{F}_s) + LC + \delta}{T}\right]\rho dV , \tag{4.78}$$

where

$$S \equiv \int_V \rho s dV \tag{4.79}$$

is the total entropy of the entire atmosphere. In an average over a sufficiently long time, (4.78) reduces to

$$\int_V \overline{\left[\frac{-\alpha\nabla \bullet (\mathbf{R} + \mathbf{F}_s) + LC + \delta}{T}\right]} \rho dV = 0, \tag{4.80}$$

from which it follows that

$$\int_V \overline{\left[\frac{-\alpha\nabla \bullet (\mathbf{R} + \mathbf{F}_s) + LC}{T}\right]} \rho dV < 0. \tag{4.81}$$

This is an important result. It means that, for the atmosphere as a whole, heating acts to decrease the entropy. In order for (4.81) to be satisfied, *heating must occur, on the average, where the temperature is high, and cooling must occur, on the average, where the temperature is low*. This implies that heating and cooling try to make temperature contrasts increase with time. One way to arrange this is to have heating in the tropics and near the surface, where the air is warm, and cooling near the poles and up high, where the air is cold.

To see that this conclusion can be drawn from (4.81), let

$$Q \equiv -\alpha\nabla \bullet (\mathbf{R} + \mathbf{F}_s) + LC \tag{4.82}$$

denote the heating rate, and divide Q and T into averages, denoted by overbars, and the corresponding local departures from the means, denoted by primes:

$$Q = \overline{Q} + Q', T = \overline{T} + T'. \tag{4.83}$$

Then we see that

$$\overline{\left(\frac{Q}{T}\right)} = \overline{\left(\frac{\overline{Q}+Q'}{\overline{T}+T'}\right)} = \overline{\frac{\overline{Q}(1+Q'/\overline{Q})}{\overline{T}(1+T'/\overline{T})}}$$

$$\cong \frac{\overline{Q}}{\overline{T}}\overline{(1+Q'/\overline{Q})(1-T'/\overline{T})}$$

$$= \frac{\overline{Q}}{\overline{T}}\overline{\left[1+\frac{Q'}{\overline{Q}}-\frac{T'}{\overline{T}}-\frac{Q'T'}{\overline{Q}\,\overline{T}}\right]}$$

$$= \frac{\overline{Q}}{\overline{T}}\left(1-\frac{\overline{Q'T'}}{\overline{Q}\,\overline{T}}\right)$$

$$= \frac{\overline{Q}}{\overline{T}}-\frac{\overline{Q'T'}}{\overline{T}^2}\,. \tag{4.84}$$

For $\overline{Q} = 0$ [see (4.65)], we get

$$\overline{\left(\frac{Q}{T}\right)} = -\frac{\overline{Q'T'}}{\overline{T}^2}\,. \tag{4.85}$$

If we add energy where the temperature is already warm, and remove energy where the temperature is already cool, then $\overline{Q'T'} > 0$, so $\overline{(Q/T)}$ will be negative, and (4.81) will be satisfied. Such a process tends to increase the variability of temperature. We show later that this implies generation of available potential energy.

We have concluded that heating tends to increase the temperature contrasts within the atmosphere. In order for the system to achieve a steady state, some other process must oppose this effect of the heating. That process is energy transport, which on the average carries energy from warm regions to cool regions, e.g. from the tropics towards the polar regions, and from the warm surface towards the cold upper atmosphere. The energy transports by the general circulation tend to cool where the temperature is high (e.g. the tropical lower troposphere), and tend to warm where the temperature is low (e.g. the polar troposphere).

A final important point is that the global entropy budget is fundamentally different from the global energy budget. Energy is conserved, which means that in a time average the fluxes of energy into the system must be balanced by fluxes out. This is not true for entropy. Entropy is generated by a wide variety of dissipative processes; the Earth makes entropy. For this reason, there is a net entropy flux out of the Earth system, via radiation, and in a time average this flux has to be equal to the Earth's entropy production rate. The Earth's entropy production rate can, therefore, be measured using satellite data. This has been done. For further discussion, see Stephens and O'Brien (1992).

4.8 Approximations

Up to here the discussion has been fairly exact. In this and the following section, we introduce some very useful approximations. In the present section we focus on the equations of motion. The relevant approximations are:

(i) Replace r by a everywhere, where a is the radius of the Earth. An approximation of this form can be justified for an atmosphere that is thin compared to the radius of the planet, and so it is called the "thin atmosphere approximation." It is a good approximation for Earth, but would not apply, e.g., to Jupiter.

(ii) Drop the terms containing $\bar{f}$. This means that the horizontal component of $\mathbf{\Omega}$ disappears from the equations. This is often called "the traditional approximation." There is an ongoing discussion as to whether or not this is a good idea.

(iii) Neglect $\frac{uw}{r}$ and $\frac{vw}{r}$, the curvature terms involving w, in the equations for u and v, respectively, neglect $\frac{u^2+v^2}{r}$ in the equation of vertical motion.

Next, we introduce a fourth, very familiar approximation, called the quasi-static approximation. For resting air, the vertical component of (4.26) reduces to the "hydrostatic equation:"

$$\frac{\partial p}{\partial z} = -\rho g \,. \tag{4.86}$$

With an appropriate boundary condition, (4.86) allows us to compute $p(z)$ from $\rho(z)$. Even when the air is moving, (4.86) gives a good *approximation* to $p(z)$, simply because $\frac{Dw}{Dt}$ and the vertical component of the friction force are small compared to g. Eq. (4.86) as applied to moving air is called the hydrostatic approximation, and it is applicable to virtually all meteorological phenomena, including violent thunderstorms.

For large-scale circulations, the approximate $p(z)$ determined through the use of (4.86) can be used to compute the pressure gradient force in the equation of horizontal motion. To do so is to use the *quasi-static approximation*. The quasi-static approximation applies very well for large-scale motions, but it is not applicable to many small-scale motions, such as thunderstorms. When the quasi-static approximation is made, the effective kinetic energy is due entirely to the horizontal motion; the contribution of the vertical component, w, is neglected. For large-scale motions, $w \ll (u, v)$, so that this quasistatic kinetic energy is very close to the true kinetic energy. Further discussion is given in a hand-out on the quasi-static approximation, available from the instructor.

With these approximations, (4.24) is replaced by

$$\begin{aligned}\frac{Du}{Dt} - \frac{uv\tan\varphi}{a} &= fv - \frac{\alpha}{a\cos\varphi}\frac{\partial p}{\partial\lambda} - \alpha(\nabla \bullet \mathbf{F})_\lambda\ ,\\ \frac{Dv}{Dt} + \frac{u^2\tan\varphi}{a} &= -fu - \frac{\alpha}{a}\frac{\partial p}{\partial\varphi} - \alpha(\nabla \bullet \mathbf{F})_\varphi\ ,\\ 0 &= -g - \alpha\frac{\partial p}{\partial z}\ .\end{aligned} \tag{4.87}$$

4.9 The mechanical energy equation in other vertical coordinate systems

Before leaving this Chapter, we return briefly to the mechanical energy equation. This time we derive it in pressure coordinates, using the quasi-static approximation for the pressure-gradient term. We start from

$$\frac{\partial \mathbf{V}}{\partial t} + (2\mathbf{\Omega} + \nabla_p \times \mathbf{V}) \times \mathbf{V} + \nabla_p(K + \phi) + \omega\frac{\partial \mathbf{V}}{\partial p} = -\alpha\nabla \bullet \mathbf{F}\ . \tag{4.88}$$

Recall that in pressure coordinates the vector $\mathbf{V}$ represents the horizontal motion, and the kinetic energy per unit mass, K, is that part associated with the horizontal motion only. Taking the dot product of (4.88) with $\mathbf{V}$, we obtain

$$\frac{DK}{Dt} + \mathbf{V} \bullet \nabla_p\phi + \alpha\nabla \bullet (\mathbf{F} \bullet \mathbf{V}) = -\delta\ . \tag{4.89}$$

Using the continuity and hydrostatic equations in pressure coordinates, this can be rewritten as

$$\frac{\partial K}{\partial t} + \nabla_p \bullet [\mathbf{V}(K + \phi)] + \frac{\partial}{\partial p}[\omega(K + \phi)] + \alpha\nabla_p \bullet (\mathbf{F} \bullet \mathbf{V}) = -\omega\alpha - \delta\ . \tag{4.90}$$

Compare (4.90) with (4.36) and (4.41). The terms $\nabla_p \bullet (\mathbf{V}\phi) + \frac{\partial}{\partial p}(\omega\phi)$ in (4.90) are analogous to $\nabla_z \bullet (p\mathbf{V})$ in (4.36). We can also write (4.90) as

$$\begin{aligned}&\frac{\partial}{\partial t}(K + \phi) + \nabla_p \bullet [\mathbf{V}(K + \phi)] + \frac{\partial}{\partial p}[\omega(K + \phi)] + \alpha\nabla_p \bullet (\mathbf{F} \bullet \mathbf{V})\\ &\qquad = -\omega\alpha - \delta + \frac{\partial\phi}{\partial t}\ ,\end{aligned} \tag{4.91}$$

which is more directly analogous to (4.41). The manipulation

$$\begin{aligned}\frac{\partial\phi}{\partial t} &= \frac{\partial}{\partial p}\left(p\frac{\partial\phi}{\partial t}\right) - p\frac{\partial}{\partial p}\left(\frac{\partial\phi}{\partial t}\right)\\ &= \frac{\partial}{\partial p}\left(p\frac{\partial\phi}{\partial t}\right) - p\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial p}\right)\\ &= \frac{\partial}{\partial p}\left(p\frac{\partial\phi}{\partial t}\right) - \frac{\partial}{\partial t}\left(p\frac{\partial\phi}{\partial p}\right)\\ &= \frac{\partial}{\partial p}\left(p\frac{\partial\phi}{\partial t}\right) + \frac{\partial}{\partial t}(p\alpha)\\ &= \frac{\partial}{\partial p}\left(p\frac{\partial\phi}{\partial t}\right) + \frac{\partial}{\partial t}(RT)\end{aligned} \tag{4.92}$$

allows us to rewrite (4.91) as

$$\begin{aligned}&\frac{\partial}{\partial t}(K+\phi-RT) + \nabla_p \bullet [\mathbf{V}(K+\phi)] + \frac{\partial}{\partial p}[\omega(K+\phi)] + \alpha\nabla_p \bullet (\mathbf{F} \bullet \mathbf{V})\\ &\quad = -\omega\alpha - \delta + \frac{\partial}{\partial p}\left(p\frac{\partial\phi}{\partial t}\right) \quad .\end{aligned} \tag{4.93}$$

The total energy equation corresponding to (4.93) is

$$\begin{aligned}&\frac{\partial}{\partial t}(K+c_vT+\phi) + \nabla_p \bullet [\mathbf{V}(K+c_pT+\phi)] + \frac{\partial}{\partial p}[\omega(K+c_pT+\phi)] + \alpha\nabla_p \bullet (\mathbf{F} \bullet \mathbf{V})\\ &\quad = \frac{\partial}{\partial p}\left(p\frac{\partial\phi}{\partial t}\right) \quad .\end{aligned} \tag{4.94}$$

Compare with (4.68).

Finally, we state the mechanical energy equation in potential temperature coordinates:

$$\begin{aligned}&\left[\frac{\partial}{\partial t}(mK)\right]_\theta + \nabla_\theta \bullet [m\mathbf{V}(K+\phi)] + \frac{\partial}{\partial\theta}[m\dot{\theta}(K+\phi)] + m\alpha\nabla \bullet (\mathbf{F} \bullet \mathbf{V})\\ &\quad = -m\omega\alpha - m\delta + \frac{\partial}{\partial\theta}\left[z\left(\frac{\partial p}{\partial t}\right)_\theta\right] \quad .\end{aligned} \tag{4.95}$$

4.10 The effects of turbulence

In practice, the molecular fluxes of momentum, temperature, and moisture are quite negligible compared with the corresponding fluxes due to turbulence. The large scales of atmospheric motion feel the turbulent fluxes directly, and in effect contribute some of their large-scale energy to maintain the energy of the turbulent scales. This will be discussed in detail in a later chapter. For now, it suffices to say that there is a “flow” of energy from large scales to smaller scales, and from the smaller scales to molecular dissipation.

When $\mathbf{F}$ is dominated by the turbulent momentum flux, we can use the approximation

$$\nabla \bullet (\mathbf{F} \bullet \mathbf{V}) \cong -\frac{\partial}{\partial z}(\mathbf{V} \bullet \rho \overline{w'\mathbf{V}'}) \ . \tag{4.96}$$

This frictional work term is small throughout most of the atmosphere; it matters most of all in the turbulent boundary layer near the surface. In particular, when suitably integrated this term expresses the energy loss by the atmosphere due to work done on the ocean by the surface wind stress.[5] The rate at which the atmosphere does work on the oceans can be roughly estimated as follows: As discussed later, the surface frictional stress is typically less than or on the order of 0.1 Pa. With a few exceptions, the ocean currents have speeds on the order of 0.1 m s^{-1} or slower. The rate at which the ocean gains energy due to the stress applied by the atmosphere is given by the product of the stress with the current speed, which, using the values given above, is roughly 10^{-2} W m^{-2}. The energy that the atmosphere imparts to the wind-driven ocean circulation is obviously quite important for the ocean and for the climate system as a whole. Nevertheless, from the point of view of the atmospheric energy budget, the rate at which energy is lost through work done on the ocean is utterly negligible, compared for example with the net surface radiation.

In a similar way, we can approximate the dissipation rate by

$$\delta \cong -\alpha(\rho \overline{w'\mathbf{V}'}) \bullet \frac{\partial \mathbf{V}}{\partial z} \ . \tag{4.97}$$

As will be discussed later, this is an example of what is called a “gradient production” term. Specifically it represents the rate of production of turbulence kinetic energy (TKE) by conversion from the kinetic energy of the mean flow. The physical picture is that the kinetic energy of the mean flow is converted to TKE, which is then dissipated, i.e. the actual dissipation occurs on small, “turbulent” scales.

We can apply (4.97) to estimate the rate of dissipation in the planetary boundary layer (PBL), as follows: Most of the vertical shear of the horizontal wind typically occurs in the lower part of the PBL, where the momentum flux is fairly close to its surface value. We can therefore approximate the integral of (4.97) through the depth of the PBL by

$$\int_{\mathrm{PBL}} \rho \delta dz \cong -(\rho \overline{w'\mathbf{V}'})_S \mathbf{V}_M , \tag{4.98}$$

where $\mathbf{V}_M$ is the horizontal wind in the upper part of the PBL, near the top of the shear layer.

5. It can also represent the rate at which work is done by the atmosphere as the wind disturbs the vegetation on the land surface, or as dust is lofted into the air.

The bulk aerodynamic formula tells us that

$$(\rho \overline{w'\mathbf{V}'})_S = -\rho_S C_D |\mathbf{V}_M| \mathbf{V}_M . \tag{4.99}$$

Substitution of (4.99) into (4.98) gives

$$\int_{\mathrm{PBL}} \rho \delta dz \cong \rho_S C_D |\mathbf{V}_M|^3 . \tag{4.100}$$

This shows that the rate of dissipation in the PBL increases very strongly as the wind speed increases. For $\mathbf{V}_M = 10 \text{ m s}^{-1}$, we get $\int_{\mathrm{PBL}} \rho \delta dz \cong 1 \text{ W m}^{-2}$. The rate of dissipation of atmospheric kinetic energy in the PBL is thus considerably larger than the rate at which the atmosphere does work on the ocean through the surface wind stress.

4.11 Summary

In this chapter we have discussed the conservation principles for momentum, kinetic energy, potential energy, thermodynamic energy, total energy, and entropy. We have also explored the conversion processes that connect these conservation principles with one another. Useful approximations have been introduced. The results obtained will be used extensively in later chapters.

Problems

1. Show that for an isentropic process

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} = \left(\frac{T}{T_0}\right)^{\frac{\gamma}{\gamma - 1}} , \tag{4.101}$$

where subscript "0" denotes a reference state.

2. The "Exner function" is defined by

$$\pi \equiv c_p \frac{T}{\theta} = c_p \left(\frac{p}{p_0}\right)^{\frac{\gamma - 1}{\gamma}} . \tag{4.102}$$

Show that, for an arbitrary process,

$$\begin{aligned} \alpha dp &= \theta d\pi , \\ Tds &= \pi d\theta , \\ h &= \pi\theta + \text{constant} . \end{aligned} \tag{4.103}$$

Here s is the entropy per unit mass.

3. Show that a process that mixes (i.e., homogenizes) potential temperature increases entropy.

4. Starting, from

$$\int_V p \frac{D\alpha}{Dt} \rho dV > 0 , \tag{4.104}$$

show that

$$\begin{aligned} &\int_V T \frac{Ds}{Dt} \rho dV > 0 , \\ &\int_V \pi \frac{D\theta}{Dt} \rho dV > 0 . \end{aligned} \tag{4.105}$$

Discuss the physical meaning of these inequalities.

5. The velocity associated with the Earth's rotation is

$$\mathbf{V}_e = \Omega r \cos\varphi \mathbf{e}_{\boldsymbol{\lambda}} , \tag{4.106}$$

where $\mathbf{e}_{\boldsymbol{\lambda}}$ is a unit vector pointing east. Show that

$$\mathbf{k} \bullet \nabla \times \mathbf{V}_e = 2\Omega \sin\varphi \tag{4.107}$$

6. Using (4.5), calculate the radius of a geostationary orbit. You will have to take into account the variation of g_a with distance from the center of the Earth.

7. Prove that

$$\Omega^2 \mathbf{r}_e = \nabla\left(\frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2\right) . \tag{4.108}$$

You may adopt a coordinate system for this purpose if you feel that you need one.

8. Suppose that you can bench press 100 kg at the North Pole. Assume that the Earth is a perfect sphere and that the acceleration due to the Earth's gravity is horizontally uniform. How much can you lift at the Equator?

9. For a two-dimensional spherical coordinate system, (λ, φ), prove by direct calculation that

$$\frac{D\mathbf{V}}{Dt} + 2\mathbf{\Omega} \times \mathbf{V} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) =$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \times \mathbf{V} + \nabla\left(\frac{1}{2}\mathbf{V} \bullet \mathbf{V}\right) - \nabla\left[\frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2\right] . \tag{4.109}$$

10. Derive (4.24) from (4.9).

11. Consider a spherical coordinate system (λ', φ', r'), which differs from (λ, φ, r) in that the "pole" of the (λ', φ', r') system is *tilted away from the pole of the Earth's rotation axis*, by an angle α, as illustrated in Fig. 4.2. Rewrite the horizontal momentum equation in a the λ', φ', r' coordinate system.

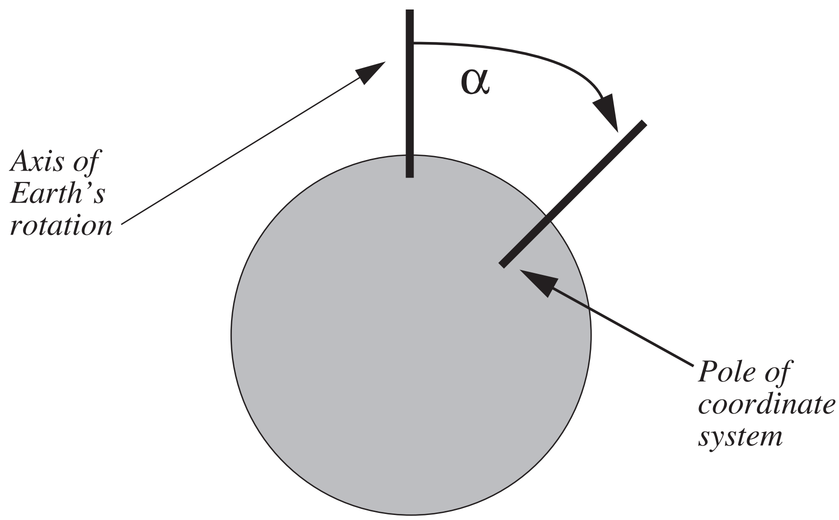

Figure 4.2: Definition of a spherical coordinate system whose pole is at an angle α with respect to the axis of the Earth's rotation. See Problem 11.

$$\frac{\partial \mathbf{V}_H}{\partial t} + (2\mathbf{\Omega} + \nabla \times \mathbf{V}_H) \times \mathbf{V}_H + \nabla(\tfrac{1}{2}\mathbf{V}_H \bullet \mathbf{V}_H) + w\frac{\partial \mathbf{V}_H}{\partial z} = -\alpha\nabla p - \alpha\frac{\partial \mathbf{F}_V}{\partial z} \quad (4.110)$$

12. Consider a planet that is shaped like a doughnut, as illustrated in the sketch given in Fig. 4.3. The planet is spinning about an axis through the hole in the doughnut. It has an "outside equator" and an "inside equator." Define coordinates λ and φ as shown in Fig. 4.3. These are orthogonal curvilinear coordinates. Let $\varphi = 0$ on the outside equator. Note that both φ and λ range between 0 and 2π. Assume that gravity acts perpendicular to the surface of the planet locally, everywhere. (This would not really be true.)

As can be deduced from Fig. 4.3, an increment of distance in the φ direction is

$$dy = r d\varphi , \quad (4.111)$$

and an increment of distance in the λ direction is

$$dx = [b - r(1 - \cos\varphi)]d\lambda \equiv c(\varphi)d\lambda . \quad (4.112)$$

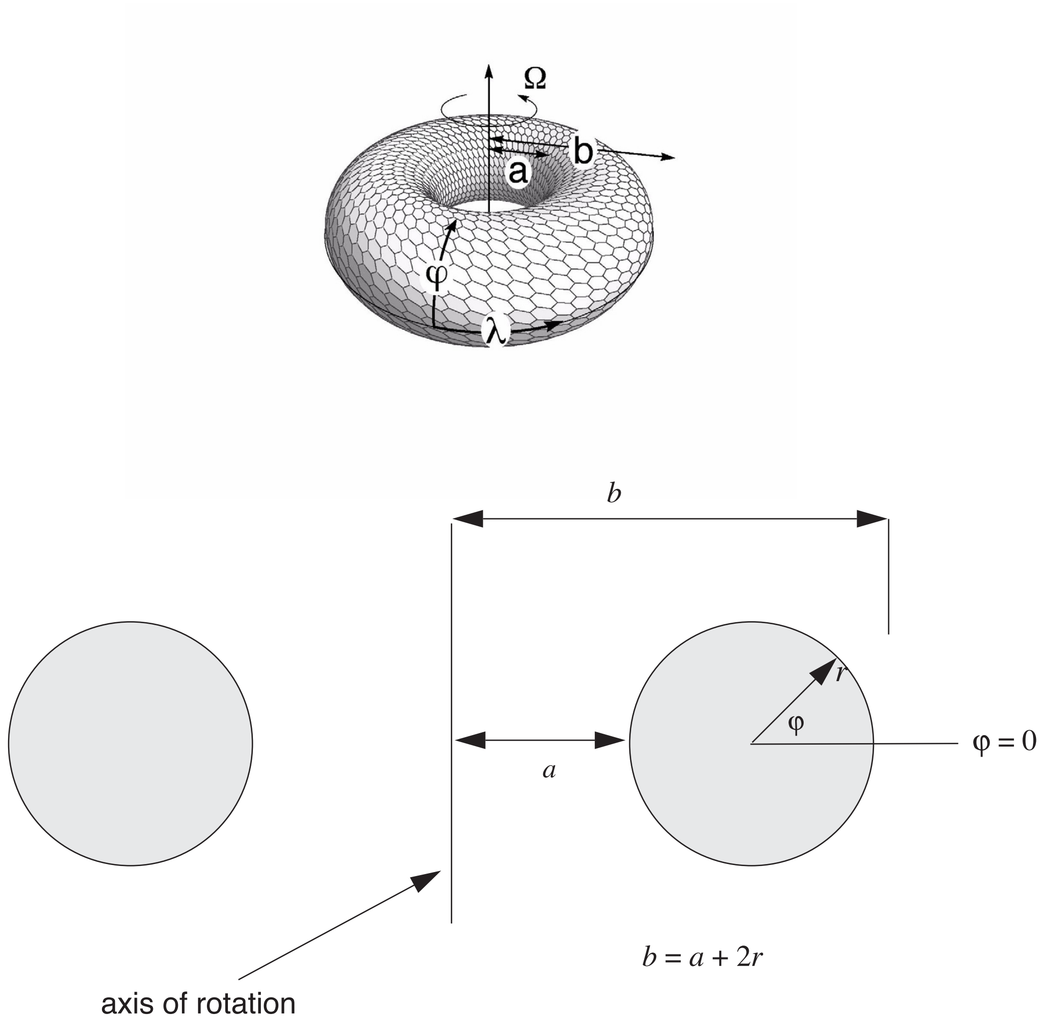

Figure 4.3: A planet shaped like a doughnut. The inner radius is a and the outer radius is b. Latitude is measured from the outer Equator. See Problem 12.

a) Work out the two-dimensional gradient and curl operators in the (λ, φ) system. Do not worry about the third coordinate, which measures height above the surface of the planet.

b) Define velocity components in the λ and φ directions. Call these u and v, respectively. They should have dimensions of length per unit time (e.g. m s^{-1}). Give formulae for u and v in terms of $\frac{D\lambda}{Dt}$ and $\frac{D\varphi}{Dt}$. Write down the two component equations of motion as they apply to u and v. Use Eq. (4.9) as your starting point. Ignore the "vertical" component of the motion. Ignore friction.

c) Derive a form of angular momentum conservation for this planet. Consider only the component of the angular momentum with respect to the planet's axis of rotation. Hint: Start by writing down a suitable definition for the angular momentum per unit mass.

d) Push a particle due "north" (i.e. toward larger φ) starting from the outer equator at $(\lambda = 0, \varphi = 0)$. Note that the initial value of u is zero. Ignore the pressure gradient force and friction. What is the minimum initial v to make the particle travel "all the way around" to $\varphi = 2\pi$, with φ increasing monotonically en route?

e) When this minimum initial v is used, what will be the longitude of the particle when it arrives at $\varphi = 2\pi$? Note: This longitude can be expressed in terms of an integral. You are not required to evaluate the integral; just set it up.

13. Prove that for a closed volume

$$\int_{\mathbf{V}} \frac{DA}{Dt} \rho \, d\mathbf{V} = \frac{d}{dt} \int_{\mathbf{V}} A \rho \, d\mathbf{V} \; . \tag{4.113}$$

14. Prove that in a hydrostatic atmosphere

$$\frac{\partial s}{\partial z} \cong c_p \frac{T}{\theta} \frac{\partial \theta}{\partial z} \; . \tag{4.114}$$

Here s is the dry static energy.

15. Derive (4.95).