CHAPTER 12 Vertical Differencing for Quasi-Static Models

12.1 Introduction

Vertical differencing is a very different problem from horizontal differencing. This may seem odd. The explanation is simply the overwhelming strength of gravitational effects, which act only in the vertical.

The first step in devising a vertical differencing scheme is the choice of a vertical coordinate system. The most obvious choice of vertical coordinate system, and one of the least useful, is height. The equations of motion are frequently expressed using vertical coordinates other than height (e.g. Kasahara, 1974). Logically, the only requirement for a vertical coordinate is that it vary monotonically with height. Even this requirement can be relaxed; e.g. a vertical coordinate can be independent of height over some layer of the atmosphere, provided that the layer is not too deep.

Factors to be weighed in choosing a vertical coordinate system for a particular application include the following:

- the form of the lower boundary condition (simpler is better);
- the form of the continuity equation (simpler is better);
- the form of the horizontal pressure gradient force (simpler is better);
- the form of the hydrostatic equation (simpler is better);
- the "vertical motion" seen in the coordinate system (less vertical motion is simpler and better);
- the method used to compute the vertical motion (simpler is better).

Each of these factors will be discussed below, for specific vertical coordinates. We begin, however, by presenting the basic governing equations, *for quasi-static motions,* using a general vertical coordinate.

12.2 General vertical coordinate

Kasahara (1974) published a detailed discussion of general vertical coordinates; an abbreviated and simplified summary is given here. Konor and Arakawa (1997) presented another very nice discussion of this topic.

In a general vertical coordinate, ζ, the equation expressing conservation of an arbitrary intensive scalar, ψ, can be written as

$$\left(\frac{\partial}{\partial t}m_\zeta\psi\right)_\zeta + \nabla_\zeta \bullet (m_\zeta \boldsymbol{V}\psi) + \frac{\partial}{\partial p}(m_\zeta\dot{\zeta}\psi) = m_\zeta S_\psi . \tag{12.1}$$

Here

$$m_\zeta \equiv \frac{\partial p}{\partial \zeta}, \tag{12.2}$$

$$\dot{\zeta} \equiv \frac{D\zeta}{Dt} \tag{12.3}$$

is the rate of change of ζ following a particle, and S_ψ is the source or sink of ψ, per unit mass. Eq. (12.1) can be derived by adding up the fluxes of ψ across the boundaries of a control volume. We can obtain the continuity equation in s-coordinates from (12.1) by putting $\psi \equiv 1$ and $S_\psi \equiv 0$:

$$\left(\frac{\partial m_\zeta}{\partial t}\right)_\zeta + \nabla_\zeta \bullet (m_\zeta \mathbf{V}) + \frac{\partial}{\partial p}(m_\zeta\dot{\zeta}) = 0 . \tag{12.4}$$

By combining (12.1) and (12.4), we can obtain the advective form:

$$m_\zeta\left[\left(\frac{\partial\psi}{\partial t}\right)_\zeta + \mathbf{V}\bullet\nabla_\zeta\psi + \dot{\zeta}\frac{\partial\psi}{\partial\zeta}\right] = S_\psi \tag{12.5}$$

This shows that the Lagrangian or material time derivative is given by

$$\frac{D}{Dt}(\ ) = \left(\frac{\partial}{\partial t}\right)_\zeta + \mathbf{V}\bullet\nabla_\zeta + \dot{\zeta}\frac{\partial}{\partial\zeta} . \tag{12.6}$$

The lower boundary condition that no mass crosses the Earth's surface is expressed by requiring that a particle which is on the Earth's surface remain there:

$$\frac{\partial\zeta_S}{\partial t} + \mathbf{V}_S \bullet \nabla\zeta_S - \dot{\zeta}_S = 0 . \tag{12.7}$$

Eq. (12.7) can be derived by integration of (12.4) throughout the entire atmospheric column, which gives

$$\frac{\partial}{\partial t}\int_{\zeta_S}^{\zeta_\infty} m_\zeta \, ds + \nabla \bullet \left(\int_{\zeta_S}^{\zeta_\infty} m_\zeta \mathbf{V} \, ds \right) + \left(\frac{\partial \zeta_S}{\partial t} + \mathbf{V}_S \bullet \nabla \zeta_S - \dot{\zeta}_S \right) - \left(\frac{\partial \zeta_\infty}{\partial t} + \mathbf{V}_\infty \bullet \nabla \zeta_\infty - \dot{\zeta}_\infty \right) = 0 \quad . \tag{12.8}$$

Here ζ_∞ is the value of ζ at the top of the atmosphere. The quantity $\frac{\partial \zeta_\infty}{\partial t} + \mathbf{V}_\infty \bullet \nabla \zeta_\infty - \dot{\zeta}_\infty$ represents the mass flux across the top of the atmosphere, which we can assume is equal to zero. The quantity $\frac{\partial \zeta_S}{\partial t} + \mathbf{V}_S \bullet \nabla \zeta_S - \dot{\zeta}_S$, which is identical to the left-hand side of (12.7), represents the mass flux across the Earth's surface, and this we set to zero as our lower boundary condition; in other words, the requirement that no mass cross the Earth's surface is expressed by (12.7). In the special case for which ζ_S is independent of time and the horizontal coordinates, (12.7) reduces to $\dot{\zeta}_S = 0$.

Konor and Arakawa (1997) derive a diagnostic equation that can be used to compute $\dot{\zeta}$ for a large family of vertical coordinates. In these notes, we omit this derivation but show how to compute the vertical velocity for each coordinate system discussed in the following subsections.

The hydrostatic equation is

$$\frac{\partial p}{\partial z} = -\rho g , \tag{12.9}$$

where ρ is the density, and g is the acceleration of gravity. Eq. (12.9) can be flipped over to obtain

$$\frac{\partial \phi}{\partial p} = -\alpha , \tag{12.10}$$

where $\phi \equiv gz$ and $\alpha \equiv 1/\rho$ is the specific volume. Using the chain rule, (12.10) can be rewritten as

$$\frac{\partial \phi}{\partial \zeta} = -\alpha \frac{\partial p}{\partial \zeta} = -\alpha m_\zeta . \tag{12.11}$$

The horizontal pressure gradient force per unit mass (HPGF) can be expressed as

$$\text{HPGF} = -\alpha \nabla_z p . \tag{12.12}$$

The gradient at constant ζ is related to the gradient at constant z by

$$\nabla_z = \nabla_\zeta - (\nabla_\zeta z)\frac{\partial}{\partial z} \quad . \tag{12.13}$$

We can therefore write

$$\begin{aligned} \text{HPGF} &= -\alpha\nabla_\zeta p + \alpha(\nabla_\zeta z)\frac{\partial p}{\partial z} \\ &= -\alpha\nabla_\zeta p - \nabla_\zeta\phi \end{aligned} \tag{12.14}$$

This cannot be made more specific without choosing the definition of ζ. Note, however, that in general the HPGF is not simply a gradient. This means that in general it is not irrotational; it can spin up or spin down a circulation on a ζ surface, except for special choices of ζ which allow the right-hand side of (12.14) to collapse to a gradient. It is clear that one such choice is $\zeta \equiv p$.

We are now in a position to examine various particular choices of ζ.

12.3 Vertical coordinate systems

12.3.1 Height

The continuity equation in height coordinates is complicated; it is nonlinear and involves the time derivative of a quantity that varies with height, namely the density:

$$\left(\frac{\partial \rho}{\partial t}\right)_z + \nabla_z \bullet (\rho \mathbf{V}) + \frac{\partial}{\partial z}(\rho w) = 0 . \tag{12.15}$$

This is a direct transcription of (12.4), with the appropriate changes in notation.

The lower boundary condition is

$$\frac{\partial z_S}{\partial t} + \mathbf{V}_S \bullet \nabla z_S - w_S = 0 . \tag{12.16}$$

Normally we can assume that z_S is independent of time, but (12.16) is able to accommodate the effects of a specified time-dependent value of z_S (e.g. to represent the effects of an earthquake, or a wave on the sea surface). Because height surfaces intersect the Earth's surface, height-coordinates are relatively difficult to implement in numerical models. This complexity is mitigated somewhat by the fact that the horizontal spatial coordinates where the height surfaces meet the Earth's surface are normally independent of time.

The thermodynamic energy equation is

$$c_p\rho\left(\frac{\partial T}{\partial t}\right)_z = -c_p\rho\left(\mathbf{V} \bullet \nabla_z T + w\frac{\partial T}{\partial z}\right) + \omega + Q . \tag{12.17}$$

Here Q is the diabatic heating per unit volume, and

$$\begin{aligned} \omega \equiv \frac{Dp}{Dt} &= \left(\frac{\partial p}{\partial t}\right)_z + \mathbf{V} \bullet \nabla_z p + w\frac{\partial p}{\partial z} \\ &= \left(\frac{\partial p}{\partial t}\right)_z + \mathbf{V} \bullet \nabla_z p - \rho g w \quad . \end{aligned} \tag{12.18}$$

In height coordinates the vertical velocity, w, is computed using "Richardson's equation," which is an expression of the physical fact that hydrostatic balance applies not just at a particular instant, but continuously through time. We therefore begin by taking the time derivative of both sides of the hydrostatic equation, (12.9):

$$\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t}\right) = -\left(\frac{\partial \rho}{\partial t}\right)g \, . \tag{12.19}$$

This is the physical principle that we will use to infer the vertical velocity. The basic idea of the derivation is that we will use the continuity equation and the thermodynamic energy equation, together with the equation of state, to eliminate the time derivatives in (12.19). The resulting equation is purely diagnostic, i.e. it does not involve time derivatives, and it can be solved for $w(z)$.

Integration of (12.19) with respect to z, from a given level to the top of the atmosphere, and use of the continuity equation, (12.15), gives

$$\frac{\partial}{\partial t}p(z) = -g\nabla_z \bullet \int_z^{\infty} (\rho \mathbf{V}) \, dz + g\rho(z)w(z) \, ; \tag{12.20}$$

this is the "pressure-tendency equation," applied to an arbitrary level inside the column. By using (12.18) in (12.17), we can obtain

$$c_p \rho \left(\frac{\partial T}{\partial t}\right)_z = -c_p \rho \left(\mathbf{V} \bullet \nabla_z T + w\frac{\partial T}{\partial z}\right) + \left[\left(\frac{\partial p}{\partial t}\right)_z + \mathbf{V} \bullet \nabla_z p - \rho g w\right] + Q \ , \tag{12.21}$$

or

$$c_p \rho \left(\frac{\partial T}{\partial t}\right)_z = -c_p \rho \mathbf{V} \bullet \nabla_z T - \rho w c_p (\Gamma_d - \Gamma) + \left[\left(\frac{\partial p}{\partial t}\right)_z + \mathbf{V} \bullet \nabla_z p\right] + Q \ , \tag{12.22}$$

where

$$\Gamma \equiv -\frac{\partial T}{\partial z} \, , \tag{12.23}$$

$$\Gamma_d \equiv \frac{g}{c_p}. \tag{12.24}$$

The equation of state is

$$p = \rho R T. \tag{12.25}$$

Logarithmic differentiation of (12.25) gives

$$\frac{1}{p}\left(\frac{\partial p}{\partial t}\right)_z = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial t}\right)_z + \frac{1}{T}\left(\frac{\partial T}{\partial t}\right)_z, \tag{12.26}$$

which can be rearranged to

$$c_p \rho\left(\frac{\partial T}{\partial t}\right)_z = \frac{c_p}{R}\left(\frac{\partial p}{\partial t}\right)_z - c_p T\left(\frac{\partial \rho}{\partial t}\right)_z. \tag{12.27}$$

Eliminating $c_p \rho\left(\frac{\partial T}{\partial t}\right)_z$ between (12.22) and (12.27) gives

$$-c_p \rho \mathbf{V} \bullet \nabla_z T - \rho w c_p (\Gamma_d - \Gamma) + \left[\left(\frac{\partial p}{\partial t}\right)_z + \mathbf{V} \bullet \nabla_z p\right] + Q = \frac{c_p}{R}\left(\frac{\partial p}{\partial t}\right)_z - c_p T\left(\frac{\partial \rho}{\partial t}\right)_z, \tag{12.28}$$

or

$$(-c_p \rho \mathbf{V} \bullet \nabla_z T + \mathbf{V} \bullet \nabla_z p) - \rho w c_p (\Gamma_d - \Gamma) + Q = \frac{c_v}{R}\left(\frac{\partial p}{\partial t}\right)_z - c_p T\left(\frac{\partial \rho}{\partial t}\right)_z, \tag{12.29}$$

$$c_v \equiv c_p - R. \tag{12.30}$$

Substitution from (12.15) and (12.20) allows us to eliminate the time derivatives from the right-hand side of (12.29). The result is

$$\begin{aligned}
&(-c_p \rho \mathbf{V} \bullet \nabla_z T + \mathbf{V} \bullet \nabla_z p) - \rho w c_p (\Gamma_d - \Gamma) + Q \\
&= \frac{c_v}{R}\left[-g \nabla_z \bullet \int_z^{\infty} (\rho \mathbf{V})\, dz + g \rho w\right] + c_p T\left[\nabla_z \bullet (\rho \mathbf{V}) + \frac{\partial}{\partial z}(\rho w)\right].
\end{aligned} \tag{12.31}$$

This can be rearranged to

$$
\begin{aligned}
& c_p T\frac{\partial}{\partial z}(\rho w) + \rho w\left[g\frac{c_v}{R} + c_p(\Gamma_d - \Gamma)\right] \\
&= (-c_p\rho\mathbf{V}\bullet\nabla_z T + \mathbf{V}\bullet\nabla_z p) - c_p T\nabla_z\bullet(\rho\mathbf{V}) + g\frac{c_v}{R}\nabla_z\bullet\int_z^\infty(\rho\mathbf{V})\,dz + Q \ .
\end{aligned}
\tag{12.32}
$$

This can be simplified considerably as follows. Expand the vertical derivative term using the product rule:

$$
\frac{c_p T}{\rho}\frac{\partial}{\partial z}(\rho w) = c_p T\frac{\partial w}{\partial z} + w\frac{c_p T}{\rho}\frac{\partial \rho}{\partial z} \ . \tag{12.33}
$$

Now use

$$
\frac{1}{p}\frac{\partial p}{\partial z} = \frac{1}{\rho}\frac{\partial \rho}{\partial z} + \frac{1}{T}\frac{\partial T}{\partial z} \ , \tag{12.34}
$$

which is equivalent to

$$
\frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\frac{\rho g}{p} + \frac{\Gamma}{T} = \frac{1}{T}\left(-\frac{g}{R} + \Gamma\right) . \tag{12.35}
$$

Substitute (12.35) into (12.33) to obtain

$$
\frac{c_p T}{\rho}\frac{\partial}{\partial z}(\rho w) = c_p T\frac{\partial w}{\partial z} + w c_p\left(-\frac{g}{R} + \Gamma\right) . \tag{12.36}
$$

Finally, substitute (12.36) into (12.32), and combine terms:

$$
\rho c_p T\frac{\partial w}{\partial z} = (-c_p\rho\mathbf{V}\bullet\nabla_z T + \mathbf{V}\bullet\nabla_z p) - c_p T\nabla_z\bullet(\rho\mathbf{V}) + g\frac{c_v}{R}\nabla_z\bullet\int_z^\infty(\rho\mathbf{V})\,dz + Q \tag{12.37}
$$

Eq. (12.37), which is Richardson's equation, is a partial differential equation which can be integrated vertically for w given a variety of things which involve both the mean horizontal motion and the heating rate. A physical interpretation of (12.37) is that the vertical motion will be whatever it takes to maintain hydrostatic balance through time as various processes represented on the right-hand side of (12.37) attempt to upset that balance. The complexity of Richardson's equation has discouraged the use of height coordinates in quasi-static models; one of the very few exceptions was the early NCAR GCM (Kasahara and Washington, 1967).

As an example to illustrate the implications of (12.37), suppose that we have horizontally uniform heating but no horizontal motion. Then (12.37) drastically simplifies to

$$\rho c_p T \frac{\partial w}{\partial z} = Q . \tag{12.38}$$

If the lower boundary is flat so that

$$w = 0 \quad \text{at } z = 0 , \tag{12.39}$$

then we obtain

$$w(z) = \int_0^z \frac{Q}{\rho c_p T} \, dz , \tag{12.40}$$

i.e. heating (cooling) below a given level induces rising (sinking) motion at that level. The rising motion induced by heating below a given level can be interpreted as a manifestation of the upward movement of air particles as the air below expands.

Finally, (12.14) reduces to

$$\text{HPGF} = -\alpha \nabla_z p , \tag{12.41}$$

which is identical to (12.12).

12.3.2 Pressure

For the case of pressure coordinates, the mass variable is simply unity, as (12.2) reduces to

$$m_{\zeta p} = 1 \tag{12.42}$$

As a result, the continuity equation is relatively simple in pressure coordinates; it is linear and does not involve a time derivative:

$$\nabla_p \bullet \mathbf{V} + \frac{\partial \omega}{\partial p} = 0 . \tag{12.43}$$

On the other hand, the lower boundary condition is complicated:

$$\frac{\partial p_S}{\partial t} + \mathbf{V}_S \bullet \nabla p_S - \omega_S = 0 . \tag{12.44}$$

Note that the surface pressure can be predicted using the surface pressure-tendency equation, which is actually derived in part through the use of (12.44):

$$\frac{\partial p_S}{\partial t} = -\nabla \bullet \int_0^{p_{SS}} \mathbf{V} \; dp\,. \tag{12.45}$$

Nevertheless, the fact that pressure surfaces intersect the ground, and do so at locations which change with time (unlike height coordinates), means that building a model that uses pressure coordinates is very complicated. Largely for this reason pressure coordinates are hardly ever used in numerical models.

As already noted, the hydrostatic equation is

$$\frac{\partial \phi}{\partial p} = -\alpha\,. \tag{12.46}$$

Finally, (12.14) reduces to

$$\text{HPGF} = -\nabla_p \phi\ . \tag{12.47}$$

12.3.3 Log-pressure

Let T_0 be a reference temperature. *Define* the "log-pressure coordinate" z^* by

$$dz^* = -\frac{RT_0}{g} d(lnp) = -\frac{RT_0}{g}\frac{dp}{p}\,, \tag{12.48}$$

Note that $dz^* = dz$ when $T(p) = T_0$. Define

$$\phi^* \equiv gz^*\,. \tag{12.49}$$

Although generally $\phi \neq \phi^*$, we can force $\phi(p = p_S) = \phi^*(p = p_S)$. From (12.48), we see that $d\phi^* = -RT_0 \frac{dp}{p}$.

To derive the form of the hydrostatic equation in z^*-coordinates, start with

$$dz^* = -\frac{RT}{g}\frac{dp}{p}\,, \tag{12.50}$$

which leads to

$$d(\phi - \phi^*) = -R(T - T_0)\frac{dp}{p}\,. \tag{12.51}$$

Since ϕ^* and T_0 are independent of time, we note that

$$\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial p}\right)_p = -\frac{R}{p}\left(\frac{\partial T}{\partial t}\right)_p . \tag{12.52}$$

12.3.4 The σ-coordinate

The σ-coordinate of Phillips (1957) is defined by

$$\sigma \equiv \frac{p}{p_S} . \tag{12.53}$$

Note that

$$dp = \sigma dp_S , \tag{12.54}$$

where the differential can represent either a fluctuation in time or a fluctuation in (horizontal) space, with a fixed value of σ. Also,

$$\frac{\partial}{\partial p}(\ \ ) = \frac{1}{p_S}\frac{\partial}{\partial \sigma}(\ \ ) . \tag{12.55}$$

Here the differentials are evaluated at constant latitude and longitude.

The continuity equation in σ-coordinates can be written as

$$\frac{\partial p_S}{\partial t} + \nabla_\sigma \bullet (p_S \mathbf{V}) + \frac{\partial}{\partial \sigma}(p_S \dot{\sigma}) = 0 . \tag{12.56}$$

Although this equation does contain a time derivative, the differentiated quantity, p_S, is independent of height, which makes (12.56) considerably simpler than (12.4).

The lower boundary condition in σ-coordinates is very simple:

$$\dot{\sigma} = 0 \text{ at } \sigma = 1 . \tag{12.57}$$

This simplicity was in fact Phillips' motivation for the invention of σ-coordinates. The upper boundary condition is similar:

$$\dot{\sigma} = 0 \text{ at } \sigma = 0 . \tag{12.58}$$

The continuity equation in σ-coordinates plays a dual role. First, it is used to predict the surface pressure. This is done by integrating (12.56) through the depth of the vertical column, to obtain

$$\frac{\partial p_S}{\partial t} = -\nabla \bullet \left(\int_0^1 p_S \mathbf{V} d\sigma \right). \tag{12.59}$$

Here we have used the upper and lower boundary conditions (12.57) and (12.58). Note that (12.59) is equivalent to (12.45).

The continuity equation is also used to determine $p_S \dot{\sigma}$. Once $\frac{\partial p_S}{\partial t}$ has been evaluated using (12.59), which does not involve $p_S \dot{\sigma}$, we can substitute back into (12.56) to obtain

$$\frac{\partial}{\partial \sigma}(p_S \dot{\sigma}) = \nabla \bullet \left(\int_0^1 p_S \mathbf{V} d\sigma \right) - \nabla_\sigma \bullet (p_S \mathbf{V}). \tag{12.60}$$

This can be integrated vertically to obtain $p_S \dot{\sigma}$ as a function of σ, starting from either the Earth's surface or the top of the atmosphere, and using the appropriate boundary condition at the top or bottom. The result obtained is the same regardless of the direction of integration.

The hydrostatic equation is simply

$$\frac{1}{p_S} \frac{\partial \phi}{\partial \sigma} = -\alpha. \tag{12.61}$$

Finally, the horizontal pressure-gradient force takes a relatively complicated form:

$$\text{HPGF} = -\sigma \alpha \nabla p_S - \nabla_\sigma \phi . \tag{12.62}$$

Using the hydrostatic equation, (12.61), we can rewrite this as

$$\text{HPGF} = \sigma \frac{1}{p_S} \frac{\partial \phi}{\partial \sigma} \nabla p_S - \nabla_\sigma \phi . \tag{12.63}$$

Rearranging, we find that

$$
\begin{aligned}
p_S(\mathrm{HPGF}) &= \sigma\frac{\partial\phi}{\partial\sigma}\nabla p_S - p_S\nabla_\sigma\phi \\
&= \left[\frac{\partial}{\partial\sigma}(\sigma\phi) - \phi\right]\nabla p_S - p_S\nabla_\sigma\phi\,. \\
&= \frac{\partial}{\partial\sigma}(\sigma\phi)\nabla p_S - \nabla_\sigma(p_S\phi) \quad .
\end{aligned}
\tag{12.64}
$$

Vertically integrating (12.64) through the entire vertical column, we obtain

$$
\int_0^1 p_S(\mathrm{HPGF})d\sigma = \phi_S\nabla p_S - \nabla_\sigma\left(\int_0^1 p_S\phi d\sigma\right). \tag{12.65}
$$

When we integrate around any closed path, the second term on the right-hand side of (12.64) vanishes because it is the integral of a gradient. The first term also vanishes, unless there is topography along the path of integration. In short, the vertically integrated HPGF vanishes except in the presence of topography, in which case “mountain torque” may result. This conclusion is reached very easily when we start from (12.64).

Consider the two contributions to the HPGF when evaluated near a mountain, as illustrated in Fig. 12.1. Near steep topography, the spatial variations of p_S and the near-

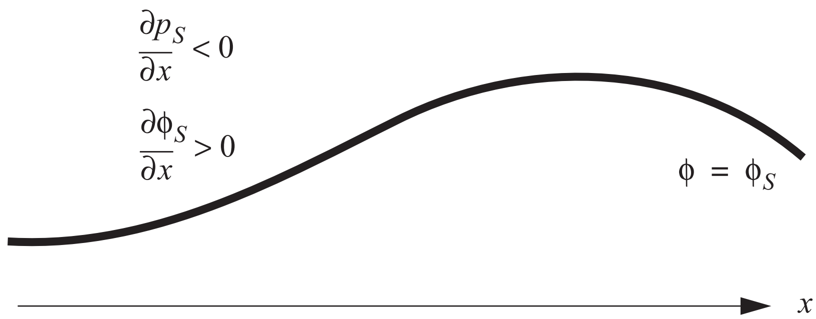

Figure 12.1: A mountain. As we move uphill in the x direction, the surface pressure decreases and the surface geopotential increases.

surface value of ϕ, along a σ-surface, are strong and of opposite sign. For example, moving uphill p_S decreases while ϕ_S increases. As a result, the two terms on the right-hand side of (12.62) are individually large and opposing, and the HPGF is the relatively small difference between them -- a dangerous situation. This can cause problems in numerical models based on the σ- coordinate, in which near steep mountains the relatively small discretization errors in the individual terms of the right-hand side of (12.62) can be as large as the HPGF. This will be discussed further below.

12.4 More on the HPGF in σ -coordinates

Consider Fig. 12.2. At the point O, we have $\sigma = \sigma^*$ and $p = p^*$. We can write

$$-\nabla_p\phi = -\nabla_\sigma\phi + (\nabla_\sigma\phi - \nabla_p\phi) = -\nabla_\sigma\phi + \nabla(\phi_{\sigma=\sigma^*} - \phi_{p=p^*}) . \tag{12.66}$$

Compare with (12.62). Evidently

$$-\sigma\alpha\nabla p_s = \nabla + (\phi_{\sigma=\sigma^*} - \phi_{p=p^k}) . \tag{12.67}$$

The right-hand-side of (12.66) involves the gradient of the difference between ϕ on a σ-surface and ϕ on a p-surface. Computation of this difference in a vertically discrete model

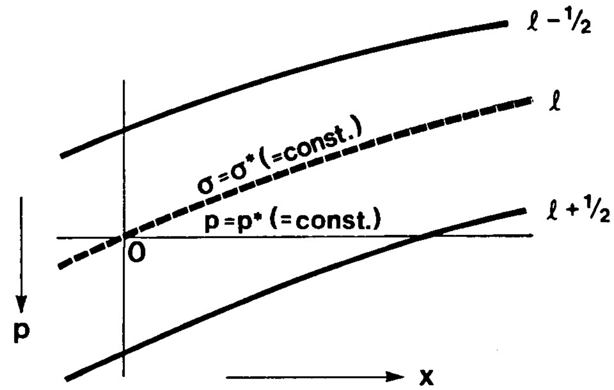

Figure 12.2: Evaluating the horizontal pressure gradient force.

amounts to vertical interpolation of ϕ from a σ-surface to a p-surface, and therefore should depend on the temperature through the hydrostatic equation. For a model that is discrete in both the horizontal and vertical, we must choose $\delta\sigma$ and δx so that

$$\frac{\delta\sigma}{\delta x} \geq \frac{\left|\left(\frac{\delta\phi}{\delta x}\right)_\sigma\right|}{\left|\left(\frac{\delta\phi}{\delta\sigma}\right)_x\right|} \sim \frac{|\text{rate of change of } \phi \text{ along a } \sigma\text{-surface}|}{p_s\alpha} . \tag{12.68}$$

The numerator of the right-hand side of (12.68) increases when the terrain is steep. The denominator increases when T is warm, i.e. near the surface. The inequality (12.68) means that δx must be fine enough for a given $\delta\sigma$; this shows that increasing the vertical resolution of a model σ-coordinate can cause problems unless the horizontal resolution is correspondingly increased. One way to minimize problems is to artificially smooth the topography.

12.4.1 Hybrid sigma-pressure coordinates

The advantage of the sigma coordinate is realized in the lower boundary condition. The disadvantage, in terms of the complicated and poorly behaved pressure-gradient force, is realized at all levels. This has motivated the use of hybrid coordinates that reduce to sigma at the lower boundary, and become pure pressure-coordinates at higher levels. In principle there are many ways of doing this. The most basic reference on this topic is the work of Simmons and Burridge (1981). They recommended the coordinate

$$\xi(\dot{p}, p_S) \equiv \frac{p}{p_S} + \left(\frac{p}{p_S} - 1\right)\left(\frac{p}{p_S} - \frac{p}{p_0}\right), \tag{12.69}$$

where p_0 is specified as 1013.2 mb. Inspection of (12.69) shows that $\xi = 1$ for $p = p_S$, and $\xi = 0$ for $p = 0$. Eq. (12.69) can be expanded and simplified to yield

$$\xi = \frac{p}{p_0} + \left(\frac{p}{p_S}\right)^2\left(1 - \frac{p_S}{p_0}\right). \tag{12.70}$$

Inspection of (12.70) shows that $\xi \to \frac{p}{p_0}$ as $\frac{p}{p_S} \to 0$. With (12.69), the pressure on an η-surface varies by less than one percent near the 10 mb level as the surface pressure varies in the range 1013 mb to 500 mb. When we evaluate the HPGF with the ξ-coordinate, there are still two terms, as with the σ-coordinate, but above the lower troposphere one of the terms is strongly dominant.

12.5 The θ-coordinate

As a solution to the problem with the HPGF in σ-coordinates, Mesinger and Janjic (1985) proposed the η-coordinate, which is being used operationally at NCEP (the National Centers for Environmental Prediction):

$$\eta \equiv \left(\frac{p - p_T}{p_s - p_T}\right)\eta_s, \tag{12.71}$$

where

$$\eta_s = \frac{p_{rf}(z_s) - p_T}{p_{rf}(0) - p_T}. \tag{12.72}$$

In (12.71), the factor in ( ) is a generalized σ, which reduces to Phillips' σ when $p_T = 0$. (In fact, many σ-coordinate models use such a generalized σ, with $p_T \leq 100$ mb .) We can therefore write

$$\eta = \sigma\eta_s, \tag{12.73}$$

where σ refers to the generalized σ-coordinate. Of course, σ=1 at the Earth's surface, but (12.71) shows that $\eta = \eta_s$ at the Earth's surface. According to (12.72), $\eta_s = 1$ (just as $\sigma_s = 1$) if $z_s = 0$. Here $z_s = 0$ is chosen to be at or near "sea level." The function p_{rf} is a pre-specified reference pressure, which is chosen to be a typical surface pressure for $z = z_s$. Because z_s depends on the horizontal coordinates, $p_{rf}(z_s)$ does too, and so, therefore, does η_s. In fact, after choosing $p_{rf}(z_s)$ and $z_s(x, y)$, one can make a map of $\eta_s(x, y)$, and of course this map is independent of time.

When we build a σ-coordinate model, we must specify (i.e., choose) fixed values of σ to serve as layer-edges and/or layer centers. Similarly, when we build an η-coordinate model, we must specify fixed values of η to serve as layer edges and/or layer centers. The values of η to be chosen include the possible values of η_s. This means that only a few discrete choices of η_s are permitted; the number increases as the vertical resolution of the model increases. Mountains must come in a few discrete sizes, like off-the-rack clothing! This is sometimes called the "step-mountain" approach. Fig. 12.3 shows how the η-coordinate works near mountains. Note that, unlike σ-surfaces, η-surfaces are nearly flat in the sense that the pressure is nearly uniform on them. The circled u-points have $u = 0$, as a boundary

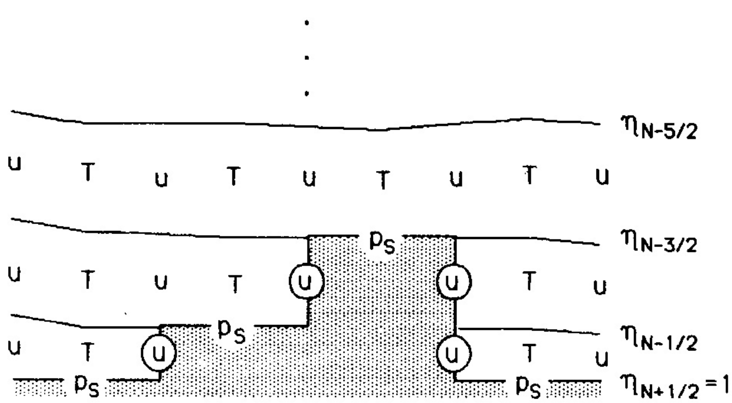

Figure 12.3: A schematic picture of the representation of mountains using the η coordinate.

condition on the sides of the mountains.

In η-coordinates, the HPGF still consists of two terms:

$$-\nabla_p \phi = -\nabla_\eta \phi - \alpha \nabla_\eta p . \tag{12.74}$$

Because the η-surfaces are nearly flat, however, these two terms are each comparable in magnitude to the HPGF itself, even near mountains, so the problem of near-cancellation does not occur.

12.5.1 Potential temperature

The potential temperature is defined by

$$\theta \equiv T\left(\frac{p_0}{p}\right)^{\kappa} . \tag{12.75}$$

The potential temperature increase upwards in a statically stable atmosphere, so that in such a case there is a monotonic relationship between θ and z. Note, however, that potential temperature cannot be used as a vertical coordinate when static instability occurs, and that the vertical resolution of a θ-coordinate model becomes very poor when the atmosphere is close to neutrally stable.

Potential temperature coordinates have particularly useful properties which have been recognized for many years, and which have become more widely appreciated during the past decade or so. In the absence of heating, potential temperature is conserved following a particle. This means that the vertical motion in θ-coordinates is proportional to the heating rate:

$$\dot{\theta} = \frac{\theta}{c_p T} Q ; \tag{12.76}$$

in the absence of heating, there is "no vertical motion," from the point of view of theta coordinates; we can also say that, in the absence of heating, a particle that is on a given theta surface remains on that surface. Eq. (12.76) is in fact the form of the thermodynamic energy equation in θ-coordinates. In fact, θ-coordinates provide an especially simply pathway for the derivation of many important results, including the conservation equation for the Ertel potential vorticity. In addition, θ-coordinates prove to have some important advantages for the design of numerical models (e.g. Eliassen and Raustein, 1968; Bleck, 1973; Johnson and Uccellini, 1983; Hsu and Arakawa, 1990).

The continuity equation in θ-coordinates is given by

$$\left(\frac{\partial m_\theta}{\partial t}\right)_\theta + \nabla_\theta \bullet (m_\theta \mathbf{V}) + \frac{\partial}{\partial p}(m_\theta \dot{\theta}) = 0, \tag{12.77}$$

which is a direct transcription of (12.4). Note, however, that $\dot{\theta} = 0$ in the absence of heating; in such case, (12.77) reduces to

$$\left(\frac{\partial m_\theta}{\partial t}\right)_\theta + \nabla_\theta \bullet (m_\theta \mathbf{V}) = 0 , \tag{12.78}$$

which is analogous to the continuity equation of a shallow-water model.

The lower boundary condition in θ-coordinates is

$$\frac{\partial \theta_S}{\partial t} + \mathbf{V}_S \bullet \nabla \theta_S - \dot{\theta}_S = 0 . \tag{12.79}$$

This equation must be used to predict θ_S. The complexity of the lower boundary condition in θ-coordinates is one of its chief drawbacks.

For the case of θ-coordinates, the hydrostatic equation, (12.11) reduces to

$$\frac{\partial\phi}{\partial\theta} = -\alpha\frac{\partial p}{\partial\theta}. \tag{12.80}$$

"Logarithmic differentiation" of (12.75) gives

$$\frac{d\theta}{\theta} = \frac{dT}{T} - \kappa\frac{dp}{p}. \tag{12.81}$$

It follows that

$$\alpha\frac{\partial p}{\partial\theta} = c_p\frac{\partial T}{\partial\theta} - c_p\frac{T}{\theta}. \tag{12.82}$$

Substitution of (12.82) into (12.80) gives

$$\frac{\partial M}{\partial\theta} = \Pi, \tag{12.83}$$

where

$$M \equiv c_p T + \phi. \tag{12.84}$$

and

$$\Pi = c_p\frac{T}{\theta}. \tag{12.85}$$

The HPGF in θ-coordinates is

$$\text{HPGF} = -\alpha\nabla_\theta p - \nabla_\theta\phi. \tag{12.86}$$

From (12.75), we see that

$$\frac{d\theta}{\theta} = \frac{dT}{T} - \kappa\frac{dp}{p}. \tag{12.87}$$

It follows that

$$\nabla_\theta p = c_p\left(\frac{p}{RT}\right)\nabla_\theta T. \tag{12.88}$$

Substitution of (12.88) into (12.86) gives

$$\text{HPGF} = -\nabla_\theta M. \tag{12.89}$$

Of course, θ-surfaces can intersect the Earth's surface, but we can consider these to follow the Earth's surface, by defining imaginary "massless layers," as shown in Fig. 12.4. Since no mass resides between the θ surfaces in the portion of the domain where they "touch

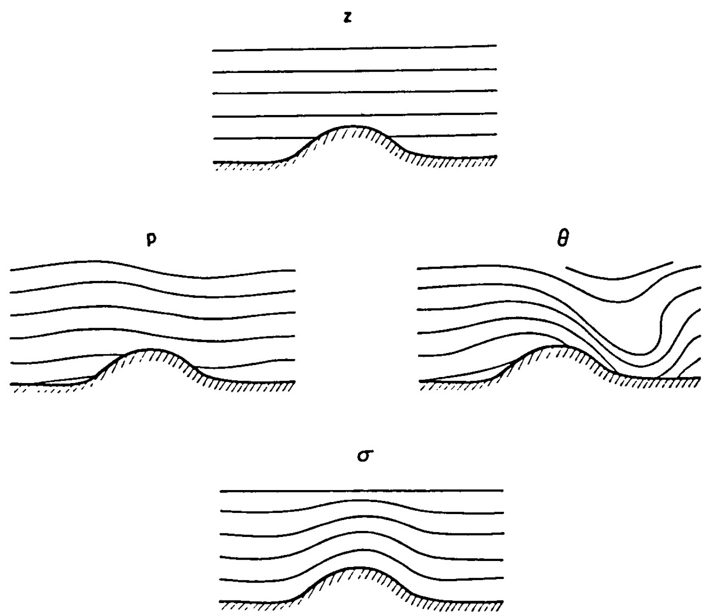

Figure 12.4: Four possible vertical coordinate systems.

the Earth's surface," no harm is done by this fantasy.

Obviously, a model that follows this approach has to be able to deal with massless layers. This represents a practical difficulty which has caused θ-coordinates to be avoided by most modelers up to this time.

12.5.2 Entropy

The entropy coordinate is very similar to the θ-coordinate. We define the entropy by

$$s = c_p \ln\theta, \tag{12.90}$$

so that

$$ds = c_p \frac{d\theta}{\theta}. \tag{12.91}$$

The hydrostatic equation can then be written as

$$\frac{\partial M}{\partial s} = T. \tag{12.92}$$

This is a particularly attractive form because the “thickness” is simply given by the temperature.

12.5.3 Hybrid $\sigma-\theta$ coordinates

Konor and Arakawa (1997) discuss a hybrid vertical coordinate which reduces to θ away from the surface, and to σ near the surface. This hybrid coordinate is designed to combine the strengths of θ and σ coordinates, while avoiding their weaknesses. Similar efforts have been reported by other authors, e.g. Johnson and Uccellini (1983) and Zhu et al. (1992). For further discussion, see the paper of Konor and Arakawa (1997).

12.5.4 Summary of vertical coordinate systems

The table on the following page summarizes key properties of some important vertical coordinate systems. All of the systems discussed here (with the exception of the entropy coordinate) have been used in many theoretical and numerical studies. Each system has its advantages and disadvantages, which must be weighed with a particular application in mind. At present, there seems to be a movement away from σ coordinates and towards θ or hybrid θ-σ coordinates.

The θ-coordinate has many advantages. In the absence of heating, $\frac{D\theta}{Dt} \equiv \dot{\theta} = 0$, so there is “no vertical velocity.” This helps to minimize, for example, the problems associated with, e.g., the vertical advection of moisture. The HPGF has the simple form

$$-\nabla_p \phi = -\nabla_\theta (c_p T + gz), \tag{12.93}$$

i.e. it is a gradient. The quantity

$$M \equiv c_p T + \phi \tag{12.94}$$

is called the “Montgomery potential” or “Montgomery stream function.” It satisfies a form of the hydrostatic equation which is natural for use with θ-coordinates:

$$\frac{\partial M}{\partial \theta} = c_p \left(\frac{p}{p_0}\right)^{\kappa}, \tag{12.95}$$

where $\kappa \equiv R/c_p$.

The dynamically important isentropic potential vorticity, q, is easily constructed in θ-coordinates, since it involves the curl of $\mathbf{V}$ on a θ-surface:

Coordinate	Hydrostatics	HPGF	Vertical Velocity	Continuity	LBC
z	$\frac{\partial p}{\partial z} = -\rho g$	$-\alpha \nabla_z p$	$w \equiv \frac{Dz}{Dt}$	$\frac{\partial \rho}{\partial t} + \nabla_z \bullet (\rho \boldsymbol{V}_H) + \frac{\partial}{\partial z}(\rho w) = 0$	$\boldsymbol{V}_S \bullet \nabla z_S - w_S = 0$
p	$\frac{\partial \phi}{\partial p} = -\alpha$	$-\nabla_p \phi$	$\omega \equiv \frac{Dp}{Dt}$	$\nabla_p \bullet \boldsymbol{V}_H + \frac{\partial \omega}{\partial p} = 0$	$\frac{\partial p_S}{\partial t} + \boldsymbol{V}_S \bullet \nabla p_S - \omega_S = 0$
$z^* \equiv -\frac{RT_0}{g} ln\left(\frac{p}{p_0}\right)$	$\frac{\partial z}{\partial z_*} = -\frac{T}{T_0}$	$-\nabla_{z^*} \phi$	$w^* \equiv \frac{Dz^*}{Dt}$ $\equiv \frac{-H\omega}{p}$	$\nabla_{Z^*} \bullet \boldsymbol{V}_H + \frac{\partial w^*}{\partial z_*} - \frac{w^*}{H} = 0$	$\frac{\partial z^*_S}{\partial t} + \boldsymbol{V}_S \cdot \nabla z^*_S - w^*_S = 0$
$\sigma \equiv \frac{p - p_T}{\pi}$	$\frac{1}{\pi}\frac{\partial \phi}{\partial \sigma} = -\alpha$	$-\nabla_\sigma \phi - \sigma \alpha \nabla \pi$	$\dot{\sigma} \equiv \frac{D\sigma}{Dt}$	$\frac{\partial \pi}{\partial t} + \nabla_\sigma \bullet (\pi \boldsymbol{V_H}) + \frac{\partial}{\partial \sigma}(\pi \dot{\sigma}) = 0$	$-\dot{\sigma}_S = 0$
q	$\frac{\partial \psi}{\partial \theta} = \Pi$	$-\nabla_\theta \psi$	$\dot{\theta} \equiv \frac{D\theta}{Dt}$	$\frac{\partial m}{\partial t} + \nabla_\theta \bullet (m \boldsymbol{V}_H) + \frac{\partial}{\partial \theta}(m\dot{\theta}) = 0$	$\frac{\partial \theta_S}{\partial t} + \boldsymbol{V}_S \bullet \nabla \theta_S - \dot{\theta}_S = 0$
s	$\frac{\partial \psi}{\partial s} = T$	$-\nabla_S \psi$	$\dot{s} \equiv \frac{Ds}{Dt}$	$\frac{\partial \mu}{\partial t} + \nabla_S \bullet (\mu \boldsymbol{V}_H) + \frac{\partial}{\partial s}(\mu \dot{s}) = 0$	$\frac{\partial s_S}{\partial t} + \boldsymbol{V}_S \bullet \nabla s_S - \dot{s}_S = 0$

$$q \equiv (\mathbf{k} \bullet \nabla_\theta \times \mathbf{V} + f)\frac{\partial\theta}{\partial p} . \tag{12.96}$$

The available potential energy is also easily obtained, since it involves the distribution of p on θ-surfaces.

12.6 Vertical staggering

After the choice of vertical coordinate system, the next issue is the choice of vertical staggering. Two possibilities are discussed here, and are illustrated in Fig. 12.5. These are the "Lorenz" or "L" grid, and the "Charney-Phillips" or "C-P" grid. Suppose that both grids have N wind-levels. The L-grid also has N θ-levels, while the C-P grid has $N+1$ θ-levels. On both grids, ϕ is hydrostatically determined on the wind-levels, and

$$\phi_l - \phi_{l+1} \sim \theta_{l+\frac{1}{2}} . \tag{12.97}$$

(Exercise: Show that $\partial\phi\Delta\Pi = -\theta$, where $\Pi \equiv c_p(p/p_0)^\kappa$.)

On the C-P grid, θ is located between ϕ-levels, so (12.97) is convenient. With the L-grid, θ must be interpolated, e.g.

$$\phi_l - \phi_{l+1} \sim \frac{1}{2}(\theta_l + \theta_{l+1}) . \tag{12.98}$$

Because (12.98) involves averaging, an oscillation in θ is not "felt" by ϕ, and so has no effect on the winds. This allows the possibility of a computational mode in the vertical. No such problem occurs with the C-P grid.

There is a second, less obvious problem with the L grid. The vertically discrete potential vorticity corresponding to (12.96) is

$$q_l \equiv (\mathbf{k} \bullet \nabla_\theta \times \mathbf{V}_l + f)\left(\frac{\partial\theta}{\partial p}\right)_l . \tag{12.99}$$

It is obvious that (12.99) "wants" the potential temperature to be defined at levels "in between" the wind levels, as they are on the C-P grid. In contrast, on the L grid the potential temperature and wind are defined at the same level. Suppose that we have N wind levels. Then with the C-P grid we will have $N+1$ potential temperature levels and N potential vorticities. This is nice. With the L grid, on the other hand, it can be shown that we effectively have $N+1$ potential vorticities. The "extra" degree of freedom in the potential vorticity is spurious, and allows a kind of computational baroclinic instability (Arakawa and Moorthi, 1988). This is a drawback of the L grid.

As Lorenz (1955) pointed out, however, the L-grid is very convenient for maintaining total energy conservation, because the kinetic and thermodynamic energies are defined at the same levels. Today, almost all models use the L-grid. This may change.

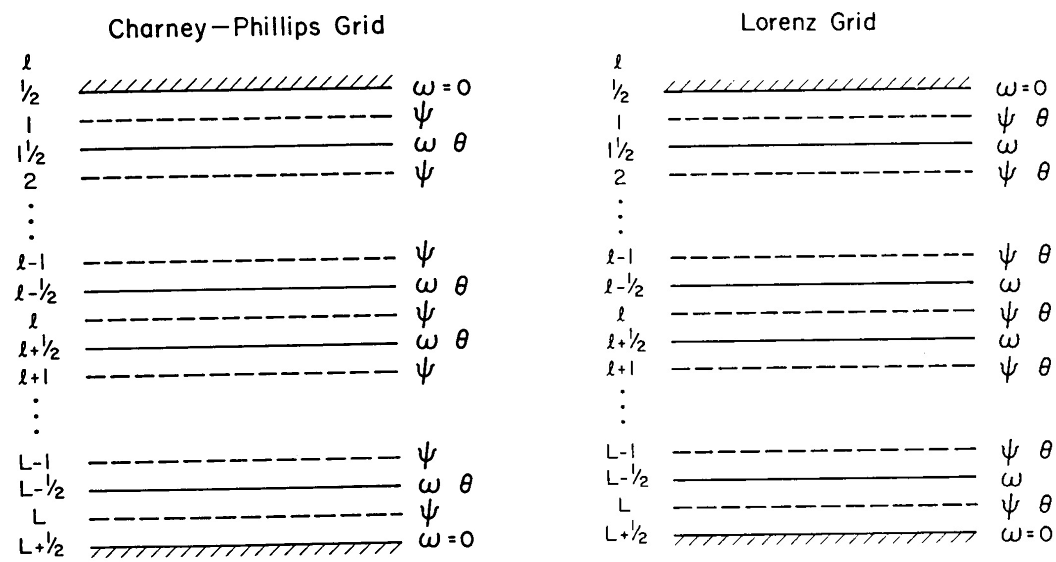

Figure 12.5: Schematic illustration of the Charney – Phillips grid and the Lorenz grid.

12.7 Conservation properties of vertically discrete models using σ-coordinates

We now investigate conservation properties of the vertically discretized equations, using σ-coordinates, and using the L-grid. The discussion follows Arakawa and Lamb (1977), although some of the ideas originated with Lorenz (1960).

For simplicity, we consider only vertical discretization, and keep the temporal and horizontal derivatives in continuous form.

Conservation of mass is expressed, in the vertically discrete system, by

$$\frac{\partial p_S}{\partial t} + \nabla_\sigma \bullet (p_S \mathbf{V}_l) + \left[\frac{\delta(p_S\dot{\sigma})}{\delta\sigma}\right]_l = 0 , \tag{12.100}$$

where

$$[\delta(\ )]_l \equiv (\ )_{l+\frac{1}{2}} - (\ )_{l-\frac{1}{2}} . \tag{12.101}$$

Summing (12.100) over all levels, and using the boundary conditions

$$\dot{\sigma}_{\frac{1}{2}} = \dot{\sigma}_{L+\frac{1}{2}} = 0 , \tag{12.102}$$

and

$$\sum_{l=1}^{L} \delta\sigma_l = 1 , \tag{12.103}$$

we obtain

$$\frac{\partial p_S}{\partial t} + \nabla \bullet \sum_{l=1}^{L} [(p_S \mathbf{V}_l)(\delta\sigma_l)] = 0 , \tag{12.104}$$

which is the vertically discrete form of the surface pressure tendency equation. From (12.104), we see that mass is, in fact, conserved, i.e. the vertical mass fluxes do not produce any net source or sink of mass.

We use

$$p_{l+\frac{1}{2}} = p_S \sigma_{l+\frac{1}{2}} + p_T , \tag{12.105}$$

where p_T is a constant, and the constant values of $\sigma_{l+\frac{1}{2}}$ are prescribed for each layer edge when the model is started up. Eq. (12.105) tells how to compute layer-edge pressures. The method to discuss layer-center pressures will be discussed later.

Similarly, we can conserve an intensive scalar, such as the potential temperature θ, by using

$$\frac{\partial}{\partial t}(p_S \theta_l) + \nabla \bullet (p_S \mathbf{V}_l \theta_l) + \left[\frac{\delta(p_S \dot{\sigma} \theta)}{\delta\sigma}\right]_l = 0 . \tag{12.106}$$

In order to use (12.106) it is necessary to define values of θ at the layer edges, via an interpolation. We have already discussed the interpolation issue in the context of advection, and that discussion applies to vertical advection as well as horizontal advection. In particular, the interpolation methods that allow conservation of an arbitrary function of the advected quantity can be used for vertical advection.

Now refer back to the discussion of the horizontal pressure-gradient force, in connection with (12.64) and (12.65). A finite-difference analog of (12.64) is

$$p_S(\mathrm{HPGF})_l = \left[\frac{\delta(\sigma\phi)}{\delta\sigma}\right]_l \nabla p_S - \nabla(p_S \phi_l) . \tag{12.107}$$

Multiplying (12.107) by $\delta\sigma_l$, and summing over all layers, we obtain a finite-difference analog of (12.65). This means that if we use the form of the HPGF given by (12.107), the vertically integrated HPGF cannot generate a circulation inside a closed path, in the absence of topography (Arakawa and Lamb, 1977). *This "principle" provides a rational way to choose which of the many possible forms of the HPGF should be used in the model*. At this point, of course, the form is not fully determined, because we do not yet have a method to compute either ϕ_l or the layer-edge values of ϕ that appear in (12.107). A suitable method is derived below.

Eq. (12.107) is equivalent to

$$p_S(\mathrm{HPGF})_l = \left\{\left[\frac{\delta(\sigma\phi)}{\delta\sigma}\right]_l - \phi_l\right\}\nabla p_S - p_S\nabla\phi_l . \tag{12.108}$$

By comparison with (12.62), we identify

$$p_S(\sigma\alpha)_l = \phi_l - \left[\frac{\delta(\sigma\phi)}{\delta\sigma}\right]_l . \tag{12.109}$$

This will be used later.

Next consider total energy conservation. We begin by reviewing the continuous case. Potential temperature conservation is expressed by

$$\frac{\partial}{\partial t}(p_S\theta) + \nabla \bullet (p_S\mathbf{V}\theta) + \frac{\partial}{\partial\sigma}(p_S\dot{\sigma}\theta) = 0 \tag{12.110}$$

Here we assume no heating for simplicity. Using continuity this can be expressed in advective form:

$$\frac{\partial\theta}{\partial t} + \mathbf{V} \bullet \nabla\theta + \dot{\sigma}\frac{\partial\theta}{\partial\sigma} = 0 . \tag{12.111}$$

With the use of the definition of θ ,i.e.,

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa} , \tag{12.112}$$

and the equation of state, (12.111) can be used to derive the thermodynamic energy equation in the form

$$c_p\left(\frac{\partial T}{\partial t} + \mathbf{V} \bullet \nabla T + \dot{\sigma}\frac{\partial T}{\partial\sigma}\right) = \omega\alpha . \tag{12.113}$$

Here

$$\begin{aligned}\omega &\equiv \left(\frac{\partial p}{\partial t}\right)_\sigma + \mathbf{V} \bullet \nabla_\sigma p + \dot{\sigma}\frac{\partial p}{\partial \sigma} \\ &= \sigma\left(\frac{\partial p_S}{\partial t} + \mathbf{V} \bullet \nabla p_S\right) + p_S\dot{\sigma} \quad .\end{aligned} \tag{12.114}$$

Continuity then allows us to transform (12.113) to the flux form:

$$\frac{\partial}{\partial t}(p_S c_p T) + \nabla \bullet (p_S \mathbf{V} c_p T) + \frac{\partial}{\partial \sigma}(p_S \dot{\sigma} c_p T) = p_S \omega \alpha . \tag{12.115}$$

The potential temperature equation, (12.110), is approximated by (12.106), which has already been discussed. Suppose that the model explicitly predicts θ_l. Adopting the definition

$$\theta_l = \frac{T_l}{P_l}, \tag{12.116}$$

where for convenience we define

$$P_l \equiv \left(\frac{p_l}{p_0}\right)^\kappa, \tag{12.117}$$

we will now derive a finite-difference analog of (12.110), by starting from (12.106). Recall that the method to determine p_l has not been specified yet. The advective form corresponding to (12.106) is

$$p_S\left(\frac{\partial \theta_l}{\partial t} + \mathbf{V}_l \bullet \nabla \theta_l\right) + \frac{1}{(\delta\sigma)_l}\left[(p_S\dot{\sigma})_{l+\frac{1}{2}}\left(\theta_{l+\frac{1}{2}} - \theta_l\right) + (p_S\dot{\sigma})_{l-\frac{1}{2}}\left(\theta_l - \theta_{l-\frac{1}{2}}\right)\right] = 0 . \tag{12.118}$$

Substitute (12.116) into (12.118), to obtain the corresponding prediction equation for T_l:

$$\begin{aligned}&p_S\left(\frac{\partial T_l}{\partial t} + \mathbf{V}_l \bullet \nabla T_l\right) - p_S \frac{T_l}{P_l}\frac{\partial P_l}{\partial p_S}\left(\frac{\partial p_S}{\partial t} + \mathbf{V}_l \bullet \nabla p_S\right) \\ &+ \frac{1}{(\delta\sigma)_l}\left[(p_S\dot{\sigma})_{l+\frac{1}{2}}\left(P_l\theta_{l+\frac{1}{2}} - T_l\right) + (p_S\dot{\sigma})_{l-\frac{1}{2}}\left(T_l - P_l\theta_{l-\frac{1}{2}}\right)\right] = 0 \quad .\end{aligned} \tag{12.119}$$

The derivative $\frac{\partial P_l}{\partial p_S}$ cannot be evaluated until we specify the form of P_l. We now introduce the layer-edge temperatures, i.e., $T_{l+\frac{1}{2}}$ and $T_{l-\frac{1}{2}}$, although the method to determine them has not yet been specified. We rewrite (12.119) as

$$p_S\left(\frac{\partial T_l}{\partial t} + \mathbf{V}_l \bullet \nabla T_l\right) + \frac{1}{(\delta\sigma)_l}\left[(p_S\dot{\sigma})_{l+\frac{1}{2}}\left(T_{l+\frac{1}{2}} - T_l\right) + (p_S\dot{\sigma})_{l-\frac{1}{2}}\left(T_l - T_{l-\frac{1}{2}}\right)\right]$$
$$= p_S\frac{T_l \partial P_l}{P_l \partial p_S}\left(\frac{\partial p_S}{\partial t} + \mathbf{V}_l \bullet \nabla p_S\right) \tag{12.120}$$
$$+ \frac{1}{(\delta\sigma)_l}\left[(p_S\dot{\sigma})_{l+\frac{1}{2}}\left(T_{l+\frac{1}{2}} - P_l\theta_{l+\frac{1}{2}}\right) + (p_S\dot{\sigma})_{l-\frac{1}{2}}\left(P_l\theta_{l-\frac{1}{2}} - T_{l-\frac{1}{2}}\right)\right] .$$

Obviously the left-hand side of (12.120) can be rewritten in flux form through the use of the vertically discrete continuity equation:

$$\frac{\partial}{\partial t}(p_S T_l) + \nabla \bullet (p_S \mathbf{V}_l T_l) + \left[\frac{\delta(p_S\dot{\sigma}T)}{\delta\sigma}\right]_l = p_S\frac{T_l \partial P_l}{P_l \partial p_S}\left(\frac{\partial p_S}{\partial t} + \mathbf{V}_l \bullet \nabla p_S\right)$$
$$+ \frac{1}{(\delta\sigma)_l}\left[(p_S\dot{\sigma})_{l+\frac{1}{2}}\left(T_{l+\frac{1}{2}} - P_l\theta_{l+\frac{1}{2}}\right) + (p_S\dot{\sigma})_{l-\frac{1}{2}}\left(P_l\theta_{l-\frac{1}{2}} - T_{l-\frac{1}{2}}\right)\right] . \tag{12.121}$$

By comparison of (12.115) with (12.121), we identify

$$\frac{p_S(\omega\alpha)_l}{c_p} = p_S\frac{T_l \partial P_l}{P_l \partial p_S}\left(\frac{\partial p_S}{\partial t} + \mathbf{V}_l \bullet \nabla p_S\right)$$
$$+ \frac{1}{(\delta\sigma)_l}\left[(p_S\dot{\sigma})_{l+\frac{1}{2}}\left(T_{l+\frac{1}{2}} - P_l\theta_{l+\frac{1}{2}}\right) + (p_S\dot{\sigma})_{l-\frac{1}{2}}\left(P_l\theta_{l-\frac{1}{2}} - T_{l-\frac{1}{2}}\right)\right] . \tag{12.122}$$

This result will be used below.

Returning to the continuous case, we now derive the continuous mechanical energy equation, starting from the continuous momentum equation in the form

$$\left(\frac{\partial \mathbf{V}}{\partial t}\right)_\sigma + [f + \mathbf{k} \bullet (\nabla_\sigma \times \mathbf{V})]\mathbf{k} \times \mathbf{V} + \dot{\sigma}\frac{\partial \mathbf{V}}{\partial \sigma} + \nabla_\sigma K = -\nabla_\sigma \phi - \sigma\alpha\nabla p_S . \tag{12.123}$$

Here $K \equiv \frac{1}{2}\mathbf{V} \bullet \mathbf{V}$ is the kinetic energy per unit mass. Dotting (12.123) with $\mathbf{V}$ gives the mechanical energy equation in the form

$$\left(\frac{\partial K}{\partial t}\right)_\sigma + \mathbf{V} \bullet \nabla_\sigma K + \dot{\sigma}\frac{\partial K}{\partial \sigma} = -\mathbf{V} \bullet (\nabla_\sigma \phi + \sigma\alpha\nabla p_S)\,. \tag{12.124}$$

The corresponding flux form is

$$\frac{\partial}{\partial t}(p_S K) + \nabla \bullet (p_S \mathbf{V} K) + \frac{\partial}{\partial \sigma}(p_S \dot{\sigma} K) = -p_S \mathbf{V} \bullet (\nabla_\sigma \phi + \sigma\alpha\nabla p_S)\,. \tag{12.125}$$

The pressure-work term on the right-hand side of (12.125) has to be manipulated to facilitate comparison with (12.115). Begin as follows:

$$\begin{aligned} -p_S \mathbf{V} \bullet (\nabla_\sigma \phi + \sigma\alpha\nabla p_S) &= -\nabla_\sigma \bullet (p_S \mathbf{V}\phi) + \phi\nabla_\sigma \bullet (p_S \mathbf{V}) - p_S \sigma\alpha \mathbf{V} \bullet \nabla p_S \\ &= -\nabla_\sigma \bullet (p_S \mathbf{V}\phi) - \phi\left[\frac{\partial p_S}{\partial t} + \frac{\partial}{\partial \sigma}(p_S \dot{\sigma})\right] - p_S \sigma\alpha \mathbf{V} \bullet \nabla p_S \\ &= -\nabla_\sigma \bullet (p_S \mathbf{V}\phi) - \frac{\partial}{\partial \sigma}(p_S \dot{\sigma}\phi) + p_S \dot{\sigma}\frac{\partial}{\partial \sigma}\phi - \phi\frac{\partial p_S}{\partial t} - p_S \sigma\alpha \mathbf{V} \bullet \nabla p_S \\ &= -\nabla_\sigma \bullet (p_S \mathbf{V}\phi) - \frac{\partial}{\partial \sigma}(p_S \dot{\sigma}\phi) - p_S \dot{\sigma}\alpha p_S - \phi\frac{\partial p_S}{\partial t} - p_S \sigma\alpha \mathbf{V} \bullet \nabla p_S\,. \end{aligned} \tag{12.126}$$

In the final line of (12.126) we have used hydrostatics. Referring back to (12.114), we can write

$$p_S \dot{\sigma}\alpha p_S + \phi\frac{\partial p_S}{\partial t} + p_S \sigma\alpha \mathbf{V} \bullet \nabla p_S = p_S \omega\alpha + \frac{\partial}{\partial \sigma}\left(\phi\sigma\frac{\partial p_S}{\partial t}\right). \tag{12.127}$$

Substitution of (12.127) into (12.126) gives

$$-p_S \mathbf{V} \bullet (\nabla_\sigma \phi + \sigma\alpha\nabla p_S) = -\nabla_\sigma \bullet (p_S \mathbf{V}\phi) - \frac{\partial}{\partial \sigma}\left(p_S \dot{\sigma}\phi + \phi\sigma\frac{\partial p_S}{\partial t}\right) - p_S \omega\alpha\,. \tag{12.128}$$

Finally, plugging (12.128) back into (12.125), and collecting terms, gives the mechanical energy equation in the form

$$\frac{\partial}{\partial t}(p_S K) + \nabla \bullet [p_S \mathbf{V}(K + \phi)] + \frac{\partial}{\partial \sigma}\left[p_S \dot{\sigma}(K + \phi) + \phi\sigma\frac{\partial p_S}{\partial t}\right] = -p_S \omega\alpha\,. \tag{12.129}$$

Adding (12.115) and (12.129) gives a statement of the conservation of total energy:

$$\frac{\partial}{\partial t}[p_S(K+c_pT)] + \nabla \bullet [\mathbf{V}p_S(K+\phi+c_pT)] + \frac{\partial}{\partial \sigma}\left[p_S\dot{\sigma}(K+\phi+c_pT) + \phi\sigma\frac{\partial p_S}{\partial t}\right] = 0\,. \tag{12.130}$$

Integrating this through the depth of an atmospheric column, we find that

$$\frac{\partial}{\partial t}\left[\int_0^1 p_S(K+c_pT)d\sigma\right] + \nabla \bullet \left[\int_0^1 \mathbf{V}p_S(K+\phi+c_pT)d\sigma\right] + \phi_S\frac{\partial p_S}{\partial t} = 0\,, \tag{12.131}$$

which can also be written as

$$\frac{\partial}{\partial t}\left[\int_0^1 p_S(K+c_pT+\phi_S)d\sigma\right] + \nabla \bullet \left[\int_0^1 \mathbf{V}p_S(K+\phi+c_pT)d\sigma\right] = p_S\frac{\partial \phi_S}{\partial t}\,. \tag{12.132}$$

The right-hand side of (12.132) represents the work done on the atmosphere if the lower boundary is moving with time, e.g., in an earthquake.

We now carry out essentially the same derivation using the vertically discrete system. Taking the dot product of $p_S\mathbf{V}_l$ with the HPGF for layer l, we write, closely following (12.126)-(12.128),

$$\begin{aligned}
-p_S\mathbf{V}_l \bullet [\nabla\phi_l + (\sigma\alpha)_l\nabla p_S] &= -\nabla\bullet(p_S\mathbf{V}_l\phi_l) + \phi_l\nabla\bullet(p_S\mathbf{V}_l) - p_S(\sigma\alpha)_l\mathbf{V}_l \bullet \nabla p_S \\
&= -\nabla\bullet(p_S\mathbf{V}_l\phi_l) - \phi_l\left\{\frac{\partial p_S}{\partial t} + \left[\frac{\delta(p_S\dot{\sigma})}{\delta\sigma}\right]_l\right\} - p_S(\sigma\alpha)_l\mathbf{V}_l \bullet \nabla p_S \\
&= -\nabla\bullet(p_S\mathbf{V}_l\phi_l) - \left[\frac{\delta(p_S\dot{\sigma}\phi)}{\delta\sigma}\right]_l \\
&\quad + \frac{1}{(\delta\sigma)_l}\left[(p_S\dot{\sigma})_{l+\frac{1}{2}}\left(\phi_{l+\frac{1}{2}} - \phi_l\right) + (p_S\dot{\sigma})_{l-\frac{1}{2}}\left(\phi_l - \phi_{l-\frac{1}{2}}\right)\right] \\
&\quad - \phi_l\frac{\partial p_S}{\partial t} - p_S(\sigma\alpha)_l\mathbf{V}_l \bullet \nabla p_S\,.
\end{aligned} \tag{12.133}$$

Continuing down this path, we construct the terms that we need by adding and subtracting

$$
\begin{aligned}
-p_S\mathbf{V}_l \bullet [\nabla\phi_l + (\sigma\alpha)_l\nabla p_S] \;=\; & -\nabla\bullet(p_S\mathbf{V}_l\phi_l) - \left[\frac{\delta(p_S\dot{\sigma}\phi)}{\delta\sigma}\right]_l \\
& + [p_S(\sigma\alpha)_l - \phi_l]\frac{\partial p_S}{\partial t} - p_S\Bigg\{(\sigma\alpha)_l\left(\frac{\partial p_S}{\partial t} + \mathbf{V}_l \bullet \nabla p_S\right) \\
& - \frac{1}{p_S(\delta\sigma)_l}\left[(p_S\dot{\sigma})_{l+\frac{1}{2}}\left(\phi_{l+\frac{1}{2}} - \phi_l\right) + (p_S\dot{\sigma})_{l-\frac{1}{2}}\left(\phi_l - \phi_{l-\frac{1}{2}}\right)\right]\Bigg\} \; .
\end{aligned}
\tag{12.134}
$$

Using (12.109) in the form

$$
p_S(\sigma\alpha)_l - \phi_l \;=\; -\left[\frac{\delta(\sigma\phi)}{\delta\sigma}\right]_l
\tag{12.135}
$$

we can rewrite this as

$$
\begin{aligned}
-p_S\mathbf{V}_l \bullet [\nabla\phi_l + (\sigma\alpha)_l\nabla p_S] \;=\; & -\nabla\bullet(p_S\mathbf{V}_l\phi_l) - \left\{\frac{\delta\left[\left(p_S\dot{\sigma} + \sigma\frac{\partial p_S}{\partial t}\right)\phi\right]}{\delta\sigma}\right\}_l \\
-p_S\Bigg\{(\sigma\alpha)_l\left(\frac{\partial p_S}{\partial t} + \mathbf{V}_l \bullet \nabla p_S\right) - & \left[\frac{(p_S\dot{\sigma})_{l+\frac{1}{2}}\left(\phi_{l+\frac{1}{2}} - \phi_l\right) + (p_S\dot{\sigma})_{l-\frac{1}{2}}\left(\phi_l - \phi_{l-\frac{1}{2}}\right)}{p_S(\delta\sigma)_l}\right]\Bigg\} \; .
\end{aligned}
\tag{12.136}
$$

By comparing with (12.128), we infer that

$$
\begin{aligned}
p_S(\omega\alpha)_l \;=\; & p_S(\sigma\alpha)_l\left(\frac{\partial p_S}{\partial t} + \mathbf{V}_l \bullet \nabla p_S\right) \\
& - \left[\frac{(p_S\dot{\sigma})_{l+\frac{1}{2}}\left(\phi_{l+\frac{1}{2}} - \phi_l\right) + (p_S\dot{\sigma})_{l-\frac{1}{2}}\left(\phi_l - \phi_{l-\frac{1}{2}}\right)}{(\delta\sigma)_l}\right] \; .
\end{aligned}
\tag{12.137}
$$

We have now reached the crux of the problem. In order to ensure total energy conservation, the form of $p_S(\omega\alpha)_l$ given by (12.137) must match that given by (12.122). In order for this to happen, we need the following conditions to be satisfied:

$$
(\sigma\alpha)_l \;=\; c_p\frac{T_l\partial P_l}{P_l\partial p_S},
\tag{12.138}
$$

$$\phi_l - \phi_{l+\frac{1}{2}} = c_p\left(T_{l+\frac{1}{2}} - P_l\theta_{l+\frac{1}{2}}\right), \tag{12.139}$$

$$\phi_{l-\frac{1}{2}} - \phi_l = c_p\left(P_l\theta_{l-\frac{1}{2}} - T_{l-\frac{1}{2}}\right). \tag{12.140}$$

Eq. (12.138) gives an expression for $(\sigma\alpha)_l$. We already had one, though, in Eq. (12.109). Requiring that these two formulae agree, we obtain

$$\phi_l - \left[\frac{\delta(\sigma\phi)}{\delta\sigma}\right]_l = c_p p_S \frac{T_l \partial P_l}{P_l \partial p_S}. \tag{12.141}$$

This is a finite-difference form of the hydrostatic equation.

With the use of Eq. (12.116), Eqs. (12.139)-(12.140) can be rewritten as

$$\left(c_p T_{l+\frac{1}{2}} + \phi_{l+\frac{1}{2}}\right) - (c_p T_l + \phi_l) = P_l c_p\left(\theta_{l+\frac{1}{2}} - \theta_l\right), \tag{12.142}$$

and

$$(c_p T_l + \phi_l) - \left(c_p T_{l-\frac{1}{2}} + \phi_{l-\frac{1}{2}}\right) = P_l c_p\left(\theta_l - \theta_{l-\frac{1}{2}}\right), \tag{12.143}$$

respectively. These are also finite-difference analogs of the hydrostatic equation. The subscripts in these equations are arbitrary. Add one to each subscript in (12.143), and add the result to (12.142). This yields

$$\phi_l - \phi_{l+1} = c_p(P_{l+1} - P_l)\theta_{l+\frac{1}{2}}. \tag{12.144}$$

If the forms of P_l and $\theta_{l+\frac{1}{2}}$ are specified, we can use (12.144) to integrate the hydrostatic equation upward from level $l+1$ to level l.

It is still necessary, however, to determine the value of ϕ_L, i.e., the layer-center geopotential for the lowest model layer. This can be done by first summing $(\delta\sigma)_l$ times (12.141) over all layers:

$$\sum_{l=1}^{L} \phi_l(\delta\sigma)_l - \phi_S = \sum_{l=1}^{L} p_S c_p \frac{T_l \partial P_l}{P_l \partial p_S}(\delta\sigma)_l . \qquad (12.145)$$

But

$$\begin{aligned} \sum_{l=1}^{L} \phi_l(\delta\sigma)_l &= \sum_{l=1}^{L} \phi_l\left(\sigma_{l+\frac{1}{2}} - \sigma_{l-\frac{1}{2}}\right) \\ &= \phi_L + \sum_{l=1}^{L-1} \sigma_{l+\frac{1}{2}}(\phi_l - \phi_{l+1}) \\ &= \phi_L + \sum_{l=1}^{L-1} \sigma_{l+\frac{1}{2}} c_p (P_{l+1} - P_l)\theta_{l+\frac{1}{2}} . \end{aligned} \qquad (12.146)$$

This is an identity. We can therefore write

$$\phi_L = \phi_S + \sum_{l=1}^{L} p_S c_p \frac{T_l \partial P_l}{P_l \partial p_S}(\delta\sigma)_l - \sum_{l=1}^{L-1} \sigma_{l+\frac{1}{2}} c_p (P_{l+1} - P_l)\theta_{l+\frac{1}{2}} . \qquad (12.147)$$

This is a bit odd, because it says that the thickness between the Earth's surface and the middle of the lowest model layer depends on θ at all levels in the entire column. From a mathematical point of view there is nothing wrong with that. In effect, all values of θ are being used to estimate the effective value of θ between the surface and level L. From a physical point of view, however, it is better for the thickness between the surface and level L to depend only on the lowest-level value of θ. Arakawa and Suarez (1983) showed that under some conditions the form (12.147) can lead to large errors in the horizontal pressure-gradient force. We return to this point below.

It remains to specify the forms of P_l and $\theta_{l+\frac{1}{2}}$. Phillips (1974) suggested

$$P_l = \left(\frac{1}{1+\kappa}\right)\left[\frac{\delta(Pp)}{\delta p}\right]_l , \qquad (12.148)$$

on the grounds that this helps to give a good simulation of vertical wave propagation. The form of $\theta_{l+\frac{1}{2}}$ can be chosen to permit conservation of some function of θ.

Arakawa and Suarez (1983) proposed a modified version of the scheme, in which (12.147) is replaced by

$$\phi_L = \phi_S + A_{L+\frac{1}{2}} c_p \theta_L \,, \tag{12.149}$$

where $A_{L+\frac{1}{2}}$ is a nondimensional parameter discussed below. The point of (12.149) is that only θ_L influences the thickness between the surface and the middle of the bottom layer; the remaining values of θ do not enter. This makes the hydrostatic equation "local." Arakawa and Suarez showed how this can be done with only minimal modifications to the derivation given above. The starting point is to replace (12.144) by

$$\phi_l - \phi_{l+1} = c_p\left(A_{l+\frac{1}{2}}\theta_l + B_{l+\frac{1}{2}}\theta_{l+1}\right), \tag{12.150}$$

and

where, again, $A_{l+\frac{1}{2}}$ and $B_{l+\frac{1}{2}}$ are to be determined. Substituting (12.150) and (12.141) into the identity

$$\phi_L - \phi_S = \sum_{l=1}^{L} [\phi\delta\sigma - \delta(\sigma\phi)]_l - \sum_{l=1}^{L-1} \sigma_{l+\frac{1}{2}}(\phi_l - \phi_{l+1}) \,, \tag{12.151}$$

we obtain

$$\phi_L - \phi_S = \sum_{l=1}^{L} c_p p_S \frac{T_l \partial P_l}{P_l \partial p_S}(\delta\sigma)_l - \sum_{l=1}^{L-1} \sigma_{l+\frac{1}{2}} c_p\left(A_{l+\frac{1}{2}}\theta_l + B_{l+\frac{1}{2}}\theta_{l+1}\right). \tag{12.152}$$

With the use of (12.116), we can write this as

$$\phi_L - \phi_S = \sum_{l=1}^{L} c_p p_S \theta_l \frac{\partial P_l}{\partial p_S}(\delta\sigma)_l - \sum_{l=1}^{L-1} \sigma_{l+\frac{1}{2}} c_p\left(A_{l+\frac{1}{2}}\theta_l + B_{l+\frac{1}{2}}\theta_{l+1}\right). \tag{12.153}$$

Every term on the right-hand-side of (12.153) involves a layer-center value of θ. We "collect terms" around individual values of θ_l and force the coefficients to vanish for $l < L$. This gives

$$p_S \frac{\partial P_l}{\partial p_S}(\delta\sigma)_l = \sigma_{l+\frac{1}{2}} A_{l+\frac{1}{2}} + \sigma_{l-\frac{1}{2}} B_{l+\frac{1}{2}} \,. \tag{12.154}$$

With the use of (12.154), (12.153) reduces to

$$\phi_L - \phi_S = \left[p_S \frac{\partial P_L}{\partial p_S} (\delta\sigma)_L - \sigma_{L-\frac{1}{2}} B_{L-\frac{1}{2}} \right] c_p \theta_L \,, \tag{12.155}$$

which has the form of (12.149).

12.8 Summary and conclusions

The representation of the vertical structure of the atmosphere in numerical models is a problem which is receiving a lot of attention at present. Among the most promising of the current approaches are those based on isentropic or quasi-isentropic coordinate systems. Similar methods are being used in ocean models.

At the same time, models are more commonly being extended through the stratosphere and beyond, while vertical resolutions are increasing; the era of hundred-layer models appears to be upon us.