PHY 501 **Classical Mechanics** Fall 2006

Homework 01 with Solutions

For each of the systems listed below:

(a) introduce convenient generalized coordinate(s) q_j,

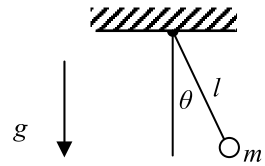
(b) write down the Lagrangian L as a function of $q_j, \dot{q}_j$ and (if appropriate) time,

(c) write down the Lagrangian equation(s) of motion,

(d) calculate the Hamiltonian H; find whether it is conserved,

(e) calculate energy E; is $E = H$?; is energy conserved?

Problem 1.1. Stretchable pendulum (i.e. a mass on a spring which exerts force $F = -k(l - l_0)$, where k and l_0 are positive constants) confined to a vertical plane:



Solution:

$$L = T - U = \frac{m}{2}\left(\dot{l}^2 + l^2\dot{\theta}^2\right) + mgl\cos\theta - \frac{k}{2}(l - l_0)^2 + \text{const.}$$

From here, the Lagrangian equations of motion are:

$$\ddot{l} + \omega^2(l - l_0) - l\dot{\theta}^2 - g\cos\theta = 0,$$

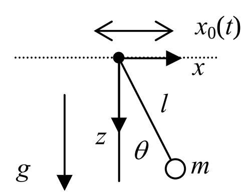
$$l\ddot{\theta} + 2\dot{l}\dot{\theta} + g\sin\theta = 0, \quad \omega^2 \equiv k/m.$$

Since T is a quadratic-homogeneous function of $\dot{l}$ and $\dot{\theta}$, H equals the total energy E:

$$E = T + U = \frac{m}{2}\left(\dot{l}^2 + l^2\dot{\theta}^2\right) - mgl\cos\theta + \frac{k}{2}(l - l_0)^2 + \text{const.}$$

and since $\partial L/\partial t = 0,$ both are conserved.

Problem 1.2. Fixed-length pendulum hanging from a horizontal support whose motion law $x_0(t)$ is fixed. (No vertical plane constraint here.)



Solution: The Lagrangian of this system is

$$L = T - U = \frac{m}{2}\left[(\dot{x} + \dot{x}_0)^2 + \dot{y}^2 + \dot{z}^2\right] - mgz\,,$$

where x, y, and z are the pendulum coordinates in the (non-inertial) system of the moving support (see Fig.). Introducing spherical coordinates

$$x = l\sin\theta\cos\varphi,$$
$$y = l\sin\theta\sin\varphi,$$
$$z = l\cos\theta,$$

we can use angles θ and φ as the generalized coordinates, in which

$$L = T - U = \frac{m}{2}\left[l^2\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\varphi}^2 + \dot{x}_0^2(t) + 2l\dot{x}_0(t)(\dot{\theta}\cos\theta\cos\varphi - \dot{\varphi}\sin\theta\sin\varphi)\right] + mgl\cos\theta.$$

From here the Lagrangian equations of motion are

$$\ddot{\theta} - \dot{\varphi}^2 \sin\theta \cos\theta + \Omega^2 \sin\theta + \frac{\ddot{x}_0(t)}{l} \cos\theta \cos\varphi = 0,$$

$$\ddot{\varphi} \sin^2 \varphi + \dot{\theta}\dot{\varphi} \sin 2\theta - \frac{\ddot{x}_0(t)}{l} \sin\theta \sin\varphi = 0.$$

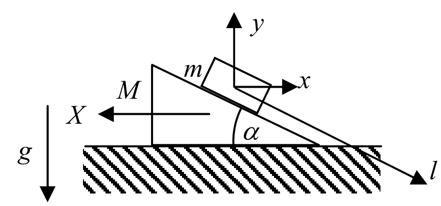
where $\Omega^2 \equiv g/l.$ We see that the equations of motion depend only on the acceleration of the support point, as it should be.

The Hamiltonian function is

$$H = \frac{m}{2}\left[l^2\dot{\theta}^2 + l^2 \sin^2\theta\, \dot{\varphi}^2 - \dot{x}_0^2\right] - mgl\cos\theta.$$

In this problem, kinetic energy T is not a quadratic-homogeneous function of the generalized velocities (because of terms with $\dot{x}_0(t)$ which is a fixed function of time and does not qualify as a generalized velocity), so that the $H \neq E,$ and since $\partial L/\partial t \neq 0,$ neither of these functions is conserved.

Problem 1.3. A block of mass *m* that can slide, without friction, along the inclined surface of a heavy wedge (mass *M*). The wedge is free to move, also without friction, along a horizontal surface. (Both motions are within the vertical plane.)



Solution: Cartesian coordinates *x* and *y* of the block (see Figure above) and their time derivatives may be readily expressed via the horizontal coordinate *X* of the wedge and shift *l* of the block along the wedge:

$$x = l\cos\alpha - X, \quad y = -l\sin\alpha,$$
$$\dot{x} = \dot{l}\cos\alpha - \dot{X}, \quad \dot{y} = -\dot{l}\sin\alpha,$$

so that *X* and *l* may be used as the generalized coordinates for this problem. (Many other choices of generalized coordinates are possible here.) Using these relations, the Lagrangian

$$L = T - U = \frac{M}{2}\dot{X}^2 + \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy$$

may be re-written as

$$L = \frac{M+m}{2}\dot{X}^2 + \frac{m}{2}\dot{l}^2 - m\dot{X}\dot{l}\cos\alpha + mgl\sin\alpha.$$

This gives us the following Lagrangian equations of motion:

$$\frac{d}{dt}\left[(M+m)\dot{X} - m\dot{l}\cos\alpha\right] = 0,$$
$$\frac{d}{dt}\left[m\dot{l} - m\dot{X}\cos\alpha\right] - mg\sin\alpha = 0,$$

Since the kinetic energy is a quadratic-homogeneous function of the generalized velocities $\dot{X}$ and $\dot{l}$, the Hamiltonian function is equal to the total energy *E*:

$$H = E = \frac{M+m}{2}\dot{X}^2 + \frac{m}{2}\dot{l}^2 - m\dot{X}\dot{l}\cos\alpha - mgl\sin\alpha.$$

and since $\partial L/\partial t = 0,$ both are conserved.