PHY 501 **Classical Mechanics** Fall 2006

Homework 03 with Solution

Problem 3.1. Use the Van der Pol method to analyze forced oscillations in an oscillator with weak nonlinear damping, described by equation

$$\ddot{x} + 2\delta\dot{x} + \omega_0{}^2 x + \beta\dot{x}^3 = \frac{F_0}{m}\cos\omega t,$$

($\omega \approx \omega_0,\ \beta,\delta > 0$). In particular, find the stationary amplitude of forced oscillations and analyze their stability. Discuss the effect(s) of the nonlinear term on the resonance.

Solution: Using the van der Pol method in the regular way, we get the following reduced equations:

$$\begin{aligned} \dot{a} &= -\delta(a)a - \frac{F_0}{2m\omega}\sin\varphi, \\ a\dot{\varphi} &= -\xi a - \frac{F_0}{2m\omega}\cos\varphi, \end{aligned} \tag{1}$$

where $\delta(a) \equiv \delta + (3/8)\beta\omega^2 a^2$ is the nonlinear damping coefficient which grows with the oscillation amplitude. Using this notation, the only fixed point of Eqs. (1) satisfies formally the same equations as those describing the linear oscillator:

$$\begin{aligned} \delta(a_0)a_0 &= -\frac{F_0}{2m\omega}\sin\varphi_0, \\ \xi a_0 &= -\frac{F_0}{2m\omega}\cos\varphi_0. \end{aligned} \tag{2}$$

For the oscillation amplitude, Eqs. (2) give

$$\frac{F_0}{2m\omega} = a_0\sqrt{\delta^2(a_0) + \xi^2}. \tag{3}$$

This equation, giving the reciprocal dependence $F_0(a_0)$, shows that the system exhibits the usual symmetric resonance at $\xi \approx 0$ (i.e., at $\omega \approx \omega_0$), but for larger F_0 the resonance curves $a_0(\xi)$ are somewhat flattened on the top.

In order to analyze stability of the fixed point, we can, as usual, plug into Eqs. (1) the following solution: $a = a_0 + \tilde{a}(t), \varphi = \varphi_0 + \tilde{\varphi}(t),$ and then linearize the resulting differential equations with respect to small variations $\tilde{a}$ and $\tilde{\varphi}$. The resulting linear equations

$$\dot{\tilde{a}} = -\delta'(a_0)a_0\tilde{a} - \delta(a_0)\tilde{a} - \frac{F_0}{2m\omega}\cos\varphi_0\,\tilde{\varphi},$$
$$a_0\dot{\tilde{\varphi}} = \frac{F_0}{2m\omega}\sin\varphi_0\tilde{\varphi} + \frac{F_0}{2m\omega}\cos\varphi_0\tilde{a},$$

may be substantially simplified using Eqs.(2):

$$\dot{\tilde{a}} = -[\delta'(a_0)a_0 + \delta(a_0)]\tilde{a} + \xi a_0\,\tilde{\varphi},$$
$$a_0\dot{\tilde{\varphi}} = -\xi\tilde{a} - \delta(a_0)a_0\,\tilde{\varphi}$$

As the result, the characteristic equation has a simple form:

$$\begin{vmatrix} -\delta'(a_0)a_0 - \delta(a_0) - \lambda & \xi \\ -\xi & -\delta(a_0) - \lambda \end{vmatrix} = 0,$$

and its solution may be presented as

$$\lambda_{\pm} = -\delta(a_0) - \frac{1}{2}\delta'(a_0)a_0 \pm \sqrt{\left(\frac{1}{2}\delta'(a_0)a_0\right)^2 - \xi^2}. \qquad (4)$$

There can be two cases here. If the detuning is small, $|\xi| < \delta'(a_0)a_0/2,$ i.e. we are at or very close to resonance, than the expression under the square root of Eq. (4) is positive, and both roots $\lambda_{\pm}$ are real. Note, however, that because of the subtraction of ξ^2, the square root's magnitude is always less than $\delta'(a_0)a_0/2$. Since the latter term is positive (differentiating $\delta(a)$, we get $\delta'(a_0)a_0 = (3/8)\beta\omega^2 a_0^2 > 0$), both roots $\lambda_{\pm}$ are negative, so that the fixed point is a stable node.

For larger values of detuning, $|\xi| > \delta'(a_0)a_0/2,$ both roots $\lambda_{\pm}$ are complex, with equal and negative real parts, so that the fixed point is a stable focus. (Note that for a passive linear oscillator, $\lambda_{\pm} = -\delta \pm i|\xi|,$ so that the fixed point is a stable focus for any ξ.)

Thus for an oscillator with weak nonlinearity of *damping*, the fixed point corresponding to forced oscillations is stable for any values of ξ and F_0, i.e. for any frequency and amplitude of the driving force. As we have seen in class, for nonlinearity of the *detuning* (e.g., the amplitude-dependent frequency ω_0), the situation may be very different.