Chapter 1. Mechanics Fundamentals: A Brief Review

This chapter lists the notions and facts of classical mechanics which are supposed to be known to students from their undergraduate studies. Due to this reason, explanations (if any) are brief.

1.1. Kinematics

- Cartesian (orthogonal, linear) coordinates $\{x_1, x_2, x_3\} = \{x, y, z\}$. Unit vectors $\vec{n}_1, \vec{n}_2, \vec{n}_3$ (Fig. 1).[1] Note the "correct" (generally accepted) order: the rotation $\vec{n}_1 \to \vec{n}_2 \to \vec{n}_3$ should be counterclockwise if watched from inside the positive quadrant.

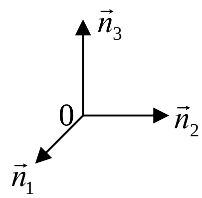

Fig. 1.1. Cartesian coordinates.

- Newtonian (absolute) space and time independent of the mass distribution.

- Space: Euclidian metric

$$r^2 = \sum_{i=1}^{3} x_i^2. \tag{1.1}$$

- Material point ("particle"): body with size/shape negligible (for the given problem). Its position is completely characterized by its radius-vector

$$\vec{r}(t) = \sum_{i=1}^{3} \vec{n}_i x_i \tag{1.2}$$

specific for a particular reference frame. Finding the laws of motion $\vec{r}(t)$ of all particles participating in the given problem may be considered the final goal of classical mechanics.

- (Instant) velocity

$$\vec{v}(t) \equiv \frac{d\vec{r}}{dt} \equiv \dot{\vec{r}}. \tag{1.3}$$

- (Instant) acceleration

$$\vec{a}(t) \equiv \frac{d\vec{v}}{dt} \equiv \dot{\vec{v}} = \ddot{\vec{r}}. \tag{1.4}$$

($\vec{v}$ and $\vec{a}$ are also reference-frame-specific.)

- Frame-to-frame transfer (Fig. 2):

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[1] In references to figures and formulas within the same chapter of these notes, the chapter number is dropped.

$$\vec{r}^{\,B} = \vec{r}^{\,A} - \vec{r}_B^{\,A}, \tag{1.5}$$

where the upper index shows the reference frame. For frame *translation* (no rotation), similar relations are valid for $\vec{v}$ and $\vec{a}$:

$$\vec{v}^{\,B} = \vec{v}^{\,A} - \vec{v}_B^{\,A}, \tag{1.6}$$

$$\vec{a}^{\,B} = \vec{a}^{\,A} - \vec{a}_B^{\,A}, \tag{1.7}$$

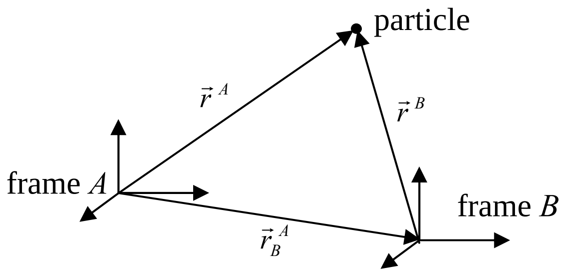

Fig. 1.2. Radius-vector transfer between two reference frames.

but for the case of mutual rotation of the frames the relations are more complex and will be discussed in Chapter 5 below.

1.2. Dynamics: Newton laws

- Generally, classical dynamics is completely described, in addition to the kinematic relations, by three Newton laws. These laws are *experimental* and cannot be theoretically *derived*; they, however, may be *postulated* if all corollaries of such postulates comply with all known experimental results.

- In order for the 1st law not to be just a particular case of the 2nd law (for zero force), it may be postulated as the existence of at least one reference frame (called *inertial*) in which any isolated particle moves with $\vec{v} = \text{const}$, i.e. $\vec{a} = 0$. According to Eq. (7), this postulate implies that there is an infinite number of inertial frames, because all frames moving uniformly without rotation relative to an inertial frame ($\vec{a}_B^{\,A} = 0$) are also inertial.

- A possible joint formulation of the 2nd and 3rd laws postulates that each particle may be characterized by a scalar constant (called *mass*), such that at any interaction of N particles (isolated from the rest of the world), in any inertial system,

$$\vec{P} \equiv \sum_{i=1}^{N} m_i \vec{v}_i = \text{const.} \tag{1.8}$$

(Each component of the sum, $\vec{p}_i \equiv m_i \vec{v}_i$, is called the *momentum* of the corresponding particle, and the whole sum $\vec{P}$, the *full momentum* of the system.) Differentiating Eq. (10) written for a couple of interacting particles, over time we see that

$$\dot{\vec{p}}_1 = -\dot{\vec{p}}_2. \tag{1.9}$$

Naming the derivative $\dot{\vec{p}}_1$ the *force* $\vec{F}_{12}$ with which particle 2 acts on particle 1 (and similarly $\dot{\vec{p}}_2$). Now, returning to the general case of several interacting particles, and postulating that $\vec{F}_{ij}$ do not depend on the presence of other particles, we get both the 2nd and the 3rd Newton laws:[2]

$$\dot{\vec{p}}_i \equiv m\vec{a}_i = \sum_{j\neq i} \vec{F}_{ij} \equiv \vec{F}_i, \tag{1.10}$$

$$\vec{F}_{ij} = -\vec{F}_{ji}. \tag{1.11}$$

1.3. OK, can we go home now?

Not yet. As a matter of principle, if the dependence of all pair forces $\vec{F}_{ij}$ of particle positions (and maybe time) is known, the Newton laws, augmented by equations of kinetics, allow the calculation of the laws of motion $\vec{r}_i(t)$, and hence trajectories, for all particles. For example, for one particle the 2nd law (10) gives the second order differential equation

$$m\ddot{\vec{r}} = \vec{F}(\vec{r},t) \tag{1.12}$$

which may integrated – either analytically or at least numerically. For certain cases, this is simple. For example, for the motion in gravitational field of the Earth near its surface, an acceptable approximation is

$$\vec{F} = m\vec{g}, \tag{1.13}$$

with $\vec{g}$ constant and directed down, so that Eq. (12) becomes just $\ddot{\vec{r}} = \vec{g}$ and may be easily integrated:

$$\begin{aligned} \dot{\vec{r}}(t) &\equiv \vec{v}(t) = \int^{t} \vec{g}dt = \int_0^{t} \vec{g}dt + \vec{v}(0) = \vec{g}t + \vec{v}(0), \\ \vec{r}(t) &= \int_0^{t} \vec{v}(t')dt' + \vec{r}(0) = \frac{\vec{g}}{2}\, t^2 + \vec{v}(0)t + \vec{r}(0). \end{aligned} \tag{1.14}$$

Each of these equations gives time evolution of three scalar components of vectors $\vec{v}$ and $\vec{r}$.

This looks (and indeed is) simple, but in most cases, especially multi-particle dynamics, integration of the equation of motion is very complex, and any help the general theory may provide is highly valuable. In many cases, such help is given by

1.4. Conservation laws

(i) Momentum. The first such law, the conservation of the full momentum of any system of particles isolated from the rest of the world, has already been discussed – see Eq. (8). In the

[2] Note that for bodies of varying mass (e.g., rockets), $\dot{\vec{p}} \neq m\vec{a}$, and the 2nd law is only valid for $\dot{\vec{p}}$.

case of one free particle the law is reduced to trivial $\vec{p} = \text{const.}$ Reminder: all this is only valid in an inertial system.

(ii) Angular momentum of a particle is defined as

$$\vec{L} \equiv \vec{r} \times \vec{p}, \tag{1. 15}$$

where $\vec{a} \times \vec{b}$ means the *vector* product of the operands. Now, differentiating Eq. (1.15) over time, we get

$$\dot{\vec{L}} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}}. \tag{1.16}$$

In the first product, $\dot{\vec{r}}$ is just the velocity vector $\vec{v}$ which is parallel to the particle momentum $m\vec{v}$, so that this product vanishes, since the vector product of parallel vectors is zero. In the second product, $\dot{\vec{p}}$ is just the full force $\vec{F}$ acting on the particle, so that Eq. (16) is reduced to

$$\dot{\vec{L}} = \vec{\tau} \equiv \vec{r} \times \vec{F}. \tag{1.17}$$

where $\vec{\tau}$ is called *torque*. (It is clearly reference-frame specific! And again, the frame has to be inertial.) Now, for a *central* force $\vec{F} \| \vec{r}$ the torque vanishes, so that the angular momentum is conserved:

$$\vec{L} = \text{const.} \tag{1.18}$$

For a system of N particles, the full angular momentum is naturally defined as

$$\vec{L} \equiv \sum_{i=1}^{N} \vec{L}_i. \tag{1.19}$$

Using Eq. (17) for each $\vec{L}_i$, and dividing each force $\vec{F}_i$ into two parts (Fig. 2): the internal force due to interactions within the system

$$\vec{F}_i^{(\text{int})} = \sum_{j \neq i} \vec{F}_{ij} \tag{1.20}$$

and the external force $\vec{F}_i^{(\text{ext})}$, we get

$$\dot{\vec{L}} = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \vec{r}_i \times \vec{F}_{ij} + \vec{\tau}^{(\text{ext})}, \qquad \vec{\tau}^{(\text{ext})} \equiv \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i^{(\text{ext})}. \tag{1.21}$$

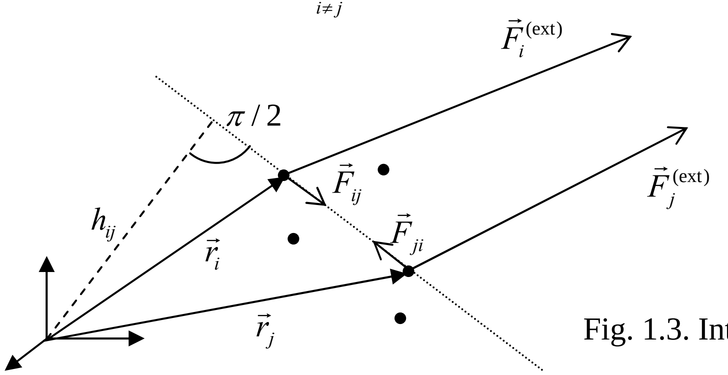

Fig. 1.3. Internal and external forces

The first (double) sum may be always broken into pairs of the type $\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji}$. Each of these pairs is zero because of the 3rd Newton law and the fact that the modulus of each vector product of the pair may be presented as $|F_{ij}|h_{ij}$ - see Fig. 3. As a result, the whole sum vanishes and we are left with

$$\dot{\vec{L}} = \vec{\tau}^{\,(\mathrm{ext})}. \tag{1.22}$$

Hence, if the full external torque $\vec{\tau}^{\,(\mathrm{ext})}$ by some reason vanishes (e.g., the system of particles is isolated from the rest of the world), the conservation law (18) is valid for the full angular momentum, even if individual components $\vec{L}_i$ are not conserved.

(iii) Energy. Energy conservation is arguably the most general physics law, but in mechanics it takes a more modest form of *mechanical energy* conservation, and has limited applicability (just as the two conservation laws discussed above). To derive it, we first define *kinetic energy* of a particle as

$$T \equiv \frac{m}{2} v^2, \tag{1.23}$$

and then notice that its change at a path from point A to point B always obeys the following relation:

$$\Delta T \equiv T(\vec{r}_B) - T(\vec{r}_A) = \int_A^B \vec{F} \cdot d\vec{r}, \tag{1.24}$$

where $\vec{F}$ is the full force acting on the particle, and symbol $\vec{a} \cdot \vec{b}$ denotes the scalar product of vectors $\vec{a}$ and $\vec{b}$. In order to prove Eq. (24), it is sufficient to consider the differential

$$dT \equiv d\left(\frac{m}{2} v^2\right) = m\vec{v} \cdot d\vec{v} = m\frac{d\vec{r}}{dt} \cdot d\vec{v} = m\frac{d\vec{r} \cdot d\vec{v}}{dt} = \dot{\vec{p}} \cdot d\vec{r}. \tag{1.25}$$

Now, using the 2nd Newton Law, $\dot{\vec{p}} = \vec{F}$, we get relation $dT = \vec{F} \cdot d\vec{r}$ whose integration from A to B gives Eq. (24). The integral in the RHP of that equation is called *work* of force $\vec{F}$ on the path from A to B.

The further step may be made only for *potential* (also called *conservative*) forces which may be presented as gradients of some scalar function $U(\vec{r})$ called *potential energy*:[3]

$$\vec{F} = -\vec{\nabla} U, \tag{1.26}$$

For example, for the uniform gravity field (13),

$$U = mgh + \text{const}. \tag{1.27}$$

Integrating Eq. (1.26) along the path from A to B, we get

[3] Note that because of definition (26), the potential energy is only defined to an arbitrary constant.

$$\int_A^B \vec{F} \cdot d\vec{r} = U(\vec{r}_A) - U(\vec{r}_B), \qquad (1.28)$$

i.e. work of potential forces may be presented as the difference of values of $U(\vec{r})$ in the initial and final point of the path. According to Eq. (28), work of such a force on any closed trajectory is zero, because for any choice of points A and B on such path (Fig. 4), [4]

$$\oint \vec{F} \cdot d\vec{r} = \int_A^B \vec{F} \cdot d\vec{r}\,\Big|_{\text{path 1}} + \int_A^B \vec{F} \cdot d\vec{r}\,\Big|_{\text{path 2}} = \left[U(\vec{r}_A) - U(\vec{r}_B)\right] + \left[U(\vec{r}_B) - U(\vec{r}_A)\right] = 0. \qquad (1.29)$$

Now, combining Eqs. (24) and (28) we see that for potential forces

$$T(\vec{r}_B) - T(\vec{r}_A) = U(\vec{r}_A) - U(\vec{r}_B) \qquad (1.30)$$

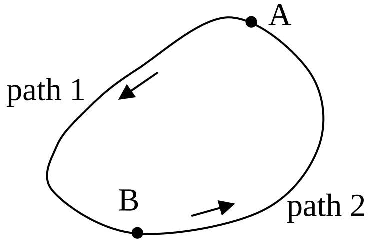

Fig. 1.4. Work on a closed path.

so that the mechanical energy defined as

$$E \equiv T + U \qquad (1.31)$$

is indeed conserved.

1.5. Why do we need conservation laws?

Let me emphasize that on the top of the Newton laws, the conservation laws discussed above give us *no new information*. Then, why should they be studied? Let us discuss this question in detail, because the motivation for main topic of this course, analytical mechanics, is very much similar.

As a trivial but convincing example of the help provided by the conservation laws, let us consider a bead of mass m sliding, without friction, along a round ring of radius R in the Earth gravity field (13) – see Fig. 5. Say we are only interested in bead's velocity v in the lowest point, after it has been dropped from the rest at the rightmost position.

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[4] This result may be also obtained in another way. Vector analysis says that for any vector-gradient, $\vec{\nabla} \times \vec{F} = 0$. But, according to the Stocke's theorem, for any continuous vector

$$\oint \vec{F} \times d\vec{r} = \int_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS.$$

where S is any surface bound by the closed path we are considering, and $\vec{n}$ is a unit vector normal to this surface. Thus we immediately arrive at Eq. (29).

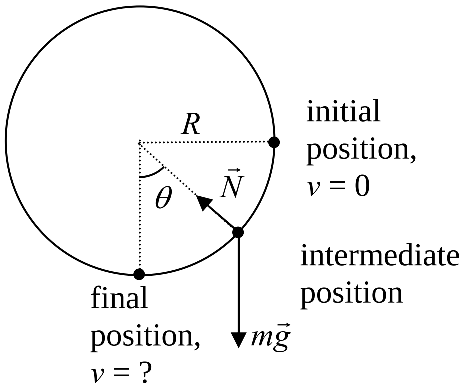

Fig. 1.5. Bead sliding down a vertical ring.

If we want to solve this problem while using only the Newton laws, we have to do pass through the following steps:

(i) consider the bead in an arbitrary intermediate position on a ring,

(ii) draw all the forces acting on it, in our case the gravity force $m\vec{g}$ and force $\vec{N}$ exerted by the ring (Fig. 5),

(iii) write the 2nd Newton law for two nonvanishing components of the bead acceleration, say for its vertical and horizontal components a_x and a_y,

(iv) comprehend that in the absence of friction, force $\vec{N}$ should be normal to the ring, so that we can use two additional equations $N_x = -N\sin\theta$, $N = N_y \cos\theta$,

(v) exclude unknown variables N, N_x and N_y from the resulting system of four equations, thus getting a single differential equation for one variable, the deviation angle θ,

(vi) integrate this equation once to get the expression relating velocity $\dot{\theta}$ and angle θ, and finally

(vii) using our specific initial condition ($\dot{\theta} = 0$ at $\theta = \pi/2$), find the final velocity as $v = -R\dot{\theta}$ at $\theta = 0$.

This is all very much doable, but long. On the other hand, the energy conservation allows us to solve this problem in one shot by writing its balance in the initial and final points:[5]

$$0 + mgR = \frac{m}{2}v^2 + 0. \tag{1.32}$$

Solving Eq. (32) for v immediately gives as the final answer. You have to agree this way is much more effective.

From the mathematical point of view, conserved quantities present *integrals of motion* and liberate us from the necessity to integrate the differential equations of motion, following from the Newton laws.

1.6. Is that enough?

5 Here the constant in Eq. (27) is chosen so that the potential energy is zero in the finite point.

Still, in many cases the conservation laws discussed above provide little help. Consider for example a generalization of the bead-on-the-ring problem, in which the ring is rotated, with a constant angular velocity ω, about its vertical diameter (Fig. 6).

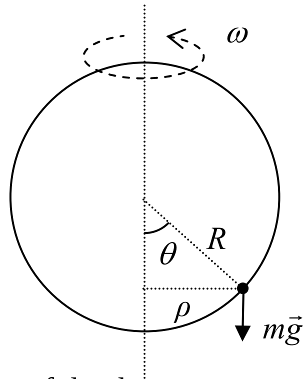

Fig. 1.6. Bead on a rotating ring.

In this situation, none of the three conservation laws holds. In particular, bead's energy

$$E = mgh + \frac{m}{2}v^2 \tag{1.33}$$

is *not* constant, because the external forces rotating the ring may change E. Of course, we still can solve the problem using the Newton laws, but this is even more complex than for the case of ring at rest, in particular because the force $\vec{N}$ exerted on the bead by the ring now may have three rather than two components which are not easily related.

One can readily see that if we could exclude such forces, which ensure external *constraints* of the particle motion, in advance, that would help a lot. Such an exclusion is indeed provided by analytical mechanics, in particular its Lagrangian formulation.

