Chapter 2. Lagrangian Formalism

The goal of this chapter is to describe the Lagrangian approach to classical mechanics, which is extremely useful for obtaining the differential equations of motion (and sometimes their first integrals) for system with holonomic constraints.

2.1. Lagrange equation

In many cases, constraints imposed on the motion of a system of N particles may be described by $3N$ equations[1]

$$\vec{r}_i = \vec{r}_i(q_1, q_2, ..., q_J, t), \quad 1 \le i \le N, \tag{2.1}$$

where q_j are *generalized* coordinates chosen to completely define the system position, and $J \le 3N$ is the number of real degrees of freedom, taking constraints into account.

For example, for our bead on rotating ring (Fig. 1.6), $J = 1$, and it may be described by one generalized coordinate – e.g., polar angle θ. With the reference frame centered at the bottom of the ring, Eq. (2.1) has the form

$$\vec{r} = \{R\sin\theta\cos\omega t, R\sin\theta\sin\omega t, R(1-\cos\theta)\}. \tag{2.2}$$

We will revisit this example later in this section.

Now, let us consider a set of small *virtual displacement* (a.k.a. *variation*) δq_j allowed by constraints. (Variations differ from real displacements dq_j by describing a possible variation of the motion, in particular do not involve the change of time – see Fig. 2.1)

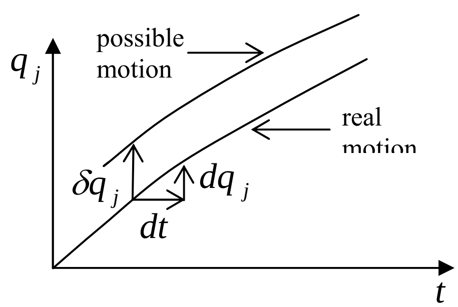

Fig. 2.1. Displacement dq_j and variation δq_j.

As a result, if we calculate the variation of Eq. (1) as a function of several arguments, we should leave time unchanged (formally, taking $\delta t = 0$):

$$\delta\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j. \tag{2.3}$$

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[1] Such constraints are called *holonomic*. Possibly, the simplest example of non-holonomic constraint are hard walls constraining motion of gas molecules. Such non-holonomic constraints are better dealt with other methods, e.g. by imposing boundary conditions on (otherwise unconstrained) motion.

Now let us break the force acting upon each particle into two parts: the frictionless part $\vec{N}_i$ of the constraining force and the "external" force $\vec{F}_i$ (possibly including the frictional part of the constraining force):

$$m_i\dot{\vec{v}}_i - \vec{F}_i = \vec{N}_i. \tag{2.4}$$

Let us scalar-multiply both sides of this equation by the virtual displacement $\delta\vec{r}_i$ and sum the result over all particles (index *i*). Since such displacement has to be allowed by the constraints, its 3*N*- dimensional vector with 3D components $\delta\vec{r}_i$ has to be perpendicular to the 3*N*-dimensional vector of force with 3D components $\vec{N}_i$. (For example, for the problem shown in Fig. 1.5, the virtual displacement may be only along the ring, while force $\vec{N}$ exerted by the ring is perpendicular to that direction.) This condition may be expressed as

$$\sum_i \vec{N}_i \cdot \delta\vec{r}_i = 0. \tag{2.5}$$

As a result, the RHP of the sum is zero and we get the so-called *D'Alambert principle* (actually, first obtained by J. Bernoulli):

$$\sum_i (m_i\dot{\vec{v}}_i - \vec{F}_i)\cdot\delta\vec{r}_i = 0. \tag{2.6}$$

Plugging Eq. (3) into Eq. (6), we get

$$\sum_{i,j} m_i\dot{\vec{v}}_i \cdot \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j - \sum_j Q_j\delta q_j = 0, \tag{2.7}$$

where we have introduced a convenient notion of *generalized forces*[2]

$$Q_j \equiv \sum_i \vec{F}_i \cdot \frac{\partial\vec{r}_i}{\partial q_j}. \tag{2.8}$$

Let us present the expression under the first sum over *i* in Eq. (7) as a full derivative of time minus the extra part:

$$\dot{\vec{v}}_i \cdot \frac{\partial\vec{r}_i}{\partial q_j} = \frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial\vec{r}_i}{\partial q_j}\right) - \dot{\vec{r}}_i\,\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_j}\right). \tag{2.9}$$

Now, since for any function the variation (∂) and differentiation (d) are interchangeable operations (Fig. 2), we can write

$$\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_j}\right) = \frac{\partial\vec{v}_i}{\partial q_j}. \tag{2.10}$$

[2] Note that since the dimensionality of generalized coordinates is arbitrary (in our bead-on-a-ring problem, it may be angle θ), that of generalized forces may also differ from [*F*].

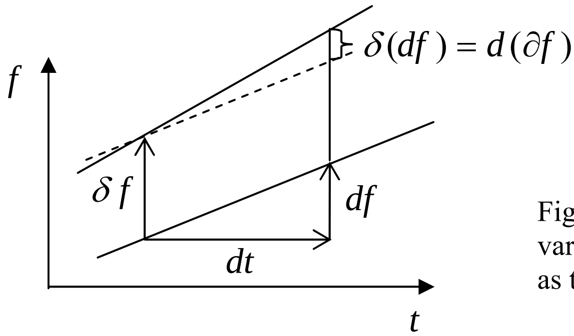

Fig. 2.2. To the first order, a variation of a differential is the same as the differential of the variation.

Also, direct differentiation of Eq. (1),

$$\vec{v}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}, \tag{2.11}$$

shows that particle velocities are functions of not only generalized coordinates and time (in Eq. (11) these dependencies are hidden in the partial derivatives), but also linear functions of generalized velocities $\dot{q}_j$, with coefficients

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}. \tag{2.12}$$

(The last expression may be also viewed as a peculiar form of swapping derivatives and variations.) With the account of Eqs. (9), (10), and (12), Eq. (7) turns into

$$\sum_j \left\{ \frac{d}{dt}\left[\sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right] - \sum_i m_i \vec{v}_i \frac{\partial \vec{v}_i}{\partial q_j} - Q_j \right\} \delta q_j = 0. \tag{2.13}$$

This result may be further simplified using the kinetic energy of the system

$$T = \frac{1}{2}\sum_i m_i \vec{v}_i \cdot \vec{v}_i. \tag{2.14}$$

Let us commit ourselves, from this point on, to considering T as a function of not only the generalized coordinates q_i and time, but also of the generalized velocities $\dot{q}_i$ as *independent variables* rather than derivatives of q_i. Then the partial derivatives

$$\frac{\partial T}{\partial q_j} = \sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j}, \quad \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}, \tag{2.13}$$

are exactly the sums over *i* in Eq. (13). In order for that expression to be zero for an arbitrary set of variations δq_j, the expression in the curly brackets should be zeros for each *j*, so that finally we get a system of *J Lagrange equations*

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j = 0. \tag{2.14}$$

This is as far as we can go for arbitrary forces. However, if all the forces may be expressed in the form similar to Eq. (1.26),

$$\vec{F}_i = -\vec{\nabla} U(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t) = -\left\{ \frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}, \frac{\partial U}{\partial z_i} \right\}, \tag{2.15}$$

where U is the potential energy of the system, we can write

$$Q_j \equiv \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \left(\frac{\partial U}{\partial x_i} \cdot \frac{\partial x_i}{\partial q_j} + \frac{\partial U}{\partial y_i} \cdot \frac{\partial y_i}{\partial q_j} + \frac{\partial U}{\partial z_i} \cdot \frac{\partial z_i}{\partial q_j} \right) = -\frac{\partial U}{\partial q_j}. \tag{2.16}$$

Taking into account that since the potential energy depends only on coordinates but not velocities, $\partial U / \partial \dot{q}_j = 0,$ and the Lagrange equation (16) may be presented in its canonical form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad L \equiv T - U. \tag{2.17}$$

Using this equation, remember that:

(i) Each system has only *one* Lagrange function L, but is described by J Lagrange equations of motion for $j = 1, 2, \ldots, J$.

(ii) Differentiating T, do not forget to consider generalized velocities as independent variables.

2.2. Examples

As the simplest example, consider a particle constrained to move along x axis:

$$T = \frac{m}{2} v^2, \quad U = U(x). \tag{2.18}$$

In this case, it is natural consider x as the (only) generalized coordinate, so that $q = x, \dot{q} = v,$ and

$$L \equiv T - U = \frac{m}{2} v^2 - U(x). \tag{2.19}$$

Considering $v = \dot{x}$ as an independent variable, we get $\partial L / \partial v = mv,$ so that the Lagrange equation of motion (only one equation in this case of the single degree of freedom!) yields

$$\frac{d}{dt}(mv) = -\frac{\partial U}{\partial x}, \tag{2.20}$$

evidently the same result as the x-component of the 2nd Newton law.

As the second example, let us return to the bead on the rotating ring (Fig. 1.6). Associating q with the polar angle θ, we see that in this case the kinetic energy depends not only on the generalized velocity, but also on the generalized coordinate:[3]

$$T = \frac{m}{2}R^2\left(\dot{\theta}^2 + \omega^2 \sin^2\theta\right), \quad U = mg(1-\cos\theta) + \text{const.} \tag{2.21}$$

Here it is especially important to remember that at composing the Lagrange equation, θ and $\dot{\theta}$ have to be treated as independent variables:

$$\frac{\partial T}{\partial \dot{\theta}} = mR^2\dot{\theta}, \quad \frac{\partial T}{\partial \theta} = mR^2\omega^2 \sin\theta\cos\theta, \quad \frac{\partial U}{\partial \theta} = mg\sin\theta, \tag{2.22}$$

so that the equation of motion is

$$\frac{d}{dt}\left(mR^2\dot{\theta}\right) - \left(mR^2\omega^2\sin\theta\cos\theta - mg\sin\theta\right) = 0. \tag{2.23}$$

As a sanity check, at $\omega \to 0$ we get equation of the usual pendulum:

$$\ddot{\theta} + \Omega^2 \sin\theta = 0, \quad \Omega^2 \equiv g/R. \tag{2.24}$$

We will explore the full equation (23) in more detail later.

2.3. Hamiltonian function and energy

Does the fact that the Lagrange equation (17) has been derived using Eq. (15) mean that it always conserves energy? Not quite. Let us start with the introduction of two new (and very important notions: the *generalized momentum*

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j} \tag{2.25}$$

and the *Hamiltonian function*

$$H \equiv \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L = \sum_j p_j \dot{q}_j - L \tag{2.26}$$

Let us differentiate the definition of H over time:

$$\frac{dH}{dt} = \sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\frac{d}{dt}\dot{q}_j\right] - \frac{dL}{dt}. \tag{2.27}$$

The last derivative has to be calculated considering L as a function of q_j, $\dot{q}_j$, and t:

[3] This expression for $T = (m/2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ may be readily obtained either by differentiating Eqs. (2) over time, or just by noticing that the velocity vector has two perpendicular components: one along the ring (with magnitude $R\dot{\theta}$) and another one normal to the ring plane (with magnitude $\omega\rho = \omega R\sin\theta$ - see Fig. 1.6).

$$\frac{dL}{dt} = \sum_j \left(\frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} \dot{q}_j + \frac{\partial L}{\partial q_j} \frac{d}{dt} q_j \right) + \frac{\partial L}{\partial t}. \tag{2.28}$$

Using the Lagrange equation (17), we see that the first term in the RHP of Eq. (27) cancels with the second term of Eq. (28), while the second term of Eq. (27) cancels with the first term of Eq. (28). Thus we arrive at a very simple result:

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}. \tag{2.29}$$

Its most important corollary is that if the Lagrangian function does not depend on time explicitly ($\partial L / \partial t = 0$), the Hamiltonian an integral of motion:

$$H = \text{const.} \tag{2.30}$$

Let us see how it works on the two simple examples discussed in the previous section. For the first example of a 1D particle, the generalized momentum

$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{2.31}$$

coincides with the usual momentum (or rather its *x*-component), and the Hamiltonian

$$H \equiv p\dot{x} - L = m\dot{x}^2 - \left(\frac{m}{2}\dot{x}^2 - U \right) = \frac{m}{2}\dot{x}^2 + U \tag{2.32}$$

coincides with particle's energy $E = T + U$. Since the Lagrangian does not depend on time explicitly, both *H* and *E* are conserved.

However, it is not always that simple! Indeed, let us consider again the bead-on-the-rotating-ring problem (Fig. 1.6). In this case,

$$p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}, \tag{2.33}$$

and

$$\begin{aligned} H \equiv p_\theta \dot{\theta} - L &= mR^2\dot{\theta}^2 - \left[\frac{m}{2} R^2 \left(\dot{\theta}^2 + \omega^2 \sin^2\theta \right) - mgR(1 - \cos\theta) \right] \\ &= \frac{m}{2} R^2 \left(\dot{\theta}^2 - \omega^2 \sin^2\theta \right) + mgR(1 - \cos\theta), \end{aligned} \tag{2.34}$$

so that, as soon as $\omega \neq 0$, the Hamiltonian *differs* from the mechanical energy

$$E \equiv T + U = \frac{m}{2} R^2 \left(\dot{\theta}^2 + \omega^2 \sin^2\theta \right) + mgR(1 - \cos\theta). \tag{2.35}$$

The difference, $E - H = mR^2\omega^2 \sin^2\theta$, evidently changes at the bead motion along the ring, so that although *H* is an integral of motion (since $\partial L / \partial t = 0$), energy *E* is *not* conserved.

Thus, Eq. (30) expresses a new conservation law, generally different of that of energy conservation. Let us examine when do these laws coincide. In mathematics, there is a notion of a *homogeneous function* $f(x_1, x_2,...)$ *of degree* λ, defined in the following way: for an arbitrary constant a,

$$f(ax_1, ax_2,...) = a^{\lambda} f(x_1, x_2,...). \tag{2.36}$$

Such functions obey *an* (one of many!) *Euler theorem*:

$$\sum_j x_j \frac{\partial f}{\partial x_j} = \lambda f, \tag{2.37}$$

which may be readily proven by the differentiation of both parts of Eq. (36) over a and then taking $a = 1$. Now, consider the case when the kinetic energy is a quadratic function of all generalized velocities $\dot{q}_j$:

$$T = \sum_j f_j(q_1, q_2,...,t)\dot{q}_j^2, \tag{2.38}$$

with no other terms. It is evident that such T satisfies the definition of a homogeneous function (of the velocities) of degree 2, so that the Euler theorem (37) gives

$$\sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = 2T. \tag{2.39}$$

But since $\partial L / \partial \dot{q}_j = \partial T / \partial \dot{q}_j$, the left-hand part of Eq. (39) is exactly the first term in the definition (26) of the Hamiltonian function, so that in this case

$$L = 2T - (T - U) = T + U = E. \tag{2.40}$$

Thus for the kinetic energy of the type (38), for example a free particle with the kinetic energy considered as a function of its Cartesian velocities,

$$T = \frac{m}{2}\left(v_x^2 + v_y^2 + v_z^2\right) \tag{2.41}$$

the notions of the Hamiltonian function and mechanical energy are identical. (Actually, some textbooks do not distinguish these notions at all!) However, as we have seen from our bead-on-the-rotating-ring example, it is not always true. For that problem, the kinetic energy, in addition to the term proportional to $\dot{\theta}^2$, has another, velocity-independent term and hence is not a homogeneous function of the angular velocity – see Eq. (21).

2.4. Other Conservation Laws

Looking at the Lagrange equation (17), we immediately see that if L is independent of some generalized coordinate,[4] $\partial L / \partial q_j = 0$, the corresponding generalized momentum is an integral of motion:

[4] Such coordinates are frequently called *cyclic*, because in some cases (like the second example considered below) they represent periodic coordinates such as angles. However, this terminology may be misleading, because some "cyclic" coordinates may have nothing to do with rotation.

$$p_j \equiv \frac{\partial L}{\partial q_j} = \text{const.} \tag{2.42}$$

For example, for a 1D particle with Lagrangian (19), momentum p_x is conserved if the potential energy is constant (the *x*-component of force is zero).

As another example, let us consider a 2D particle in the field of central forces, $U = U(r)$ - see Fig. 3.

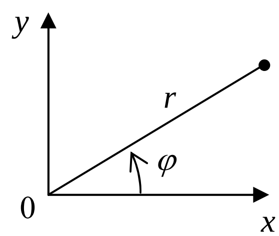

Fig. 2.3. 2D particle in the field of central forces.

If we use the polar coordinates r and φ as the generalized coordinates, the Lagrangian function becomes

$$L \equiv T - U = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - U(r) \tag{2.43}$$

is independent of φ and hence the corresponding generalized momentum

$$p_\varphi \equiv \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} \tag{2.44}$$

is conserved. This is of course a particular case of the angular momentum conservation – see Eq. (1.18). Indeed for 2D motion the angular momentum vector

$$\vec{L} \equiv \vec{r} \times \vec{p} = \begin{vmatrix} \vec{n}_x & \vec{n}_y & \vec{n}_z \\ x & y & z \\ m\dot{x} & m\dot{y} & m\dot{z} \end{vmatrix} \tag{2.45}$$

has only one nonvanishing component, perpendicular to the motion plane:

$$L_z = x(m\dot{y}) - y(m\dot{x}). \tag{2.46}$$

Differentiating the equations of transfer between the polar and Cartesian coordinates (see Fig. 3)

$$x = r\cos\varphi, \quad y = r\sin\varphi, \tag{2.47}$$

and plugging the result into Eq. (46), we see that $L_z = mr^2\dot{\varphi} = p_\varphi$.

Thus the Lagrangian formalism is a good way of searching for integrals of motion. On the other hand, if such the conserved quantity is known or evident *a priori*, it is helpful for the selection of the most appropriate generalized coordinates. For example, in the last problem, if we already know that p_φ is conserved, this provides a good justification for including the corresponding coordinate φ into the list of generalized coordinates.

