

A Quick Introduction to Vectors, Vector Calculus, and Coordinate Systems

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4.1 Physical laws and coordinate systems

For the present discussion, we define a "coordinate system" as a system for describing positions in space; we do not worry about whether the coordinate system is in motion or is accelerating. Any physical law should be expressible in a form that is invariant with respect to our choice of coordinate systems; we certainly do not expect that the laws of physics change when we switch from spherical coordinates to cartesian coordinates! This means that we should be able to express physical laws without making reference to any coordinate system. Nevertheless we do often use coordinate systems when discussing physical laws, and so we need to understand how the laws can be expressed in different coordinate systems, and in particular how the laws "transform" as we change from one coordinate system to another.

4.2 Scalars, vectors, and tensors

A tensor is a quantity that is defined without reference to any particular coordinate system. A tensor is simply "out there," and has a meaning that is the same whether we happen to be working in spherical coordinates, or cartesian coordinates, or whatever. Tensors are therefore just what we need to formulate physical laws.

The simplest kind of tensor, called a "tensor of rank 0," is a *scalar*, which is represented by a single number -- essentially a magnitude with no direction. An example of a scalar is the temperature. Not all quantities that are represented by a single number are scalars, because not all of them are defined without reference to any particular coordinate system. An example of a number that is *not* a scalar is the longitudinal component of the wind, which can only be defined through reference to a particular spherical coordinate system, in this case latitude-longitude coordinates.

A scalar is expressed in exactly the same way regardless of what coordinate system is used. For example, if someone tells you the temperature in Fort Collins, you don't need to ask whether the person is using spherical coordinates or some other coordinate system, because it makes no difference at all.

Vectors are tensors that can be represented by a magnitude and a direction. An example is the wind vector. In atmospheric science, vectors are normally either three dimensional or two dimensional, but in principle they have any number of dimensions. A scalar can be considered to be a vector in a one-dimensional space.

A vector can be expressed in a particular coordinate system by an ordered list of numbers, which are called the "components" of the vector in the particular coordinate system. When we change from one coordinate system to another, a vector transforms according to

$$\mathbf{V}' = \{\mathbf{M}\}\mathbf{V}. \tag{4.1}$$

Here $\mathbf{V}$ is the representation of the vector in the first coordinate system (i.e., $\mathbf{V}$ is the list of the components of the vector in the first coordinate system), $\mathbf{V}$ is the representation the vector in the second coordinate system, and $\{\mathbf{M}\}$ is a matrix.

Note that not all ordered lists of numbers are vectors. For example, the list

(mass of the moon, distance from Fort Collins to Denver)

is not a vector, because it does not obey a rule of the form (4.1).

It is also possible to define higher-order tensors. An example of importance in Atmospheric Science is the flux (a vector quantity, which has a direction) of momentum (a second vector quantity, which has a second, generally different direction). The momentum flux, also called a "stress," and equivalent to a force per unit area, thus has a magnitude and "two directions." One of the directions is associated with the force vector, and the other is associated with the normal vector to the unit area in question.

Because the momentum flux is associated with two directions, it is said to be a tensor of rank two. Vectors are considered to be tensors of rank one, and scalars are tensors of rank zero. Like a vector, a tensor of rank 2 can be expressed in a particular coordinate system, i.e., we can define the "components" of the tensor with respect to a particular coordinate system. The components of a tensor of rank 2 can be arranged in the form of a two-dimensional matrix, in contrast to the components of a vector, which form an ordered one-dimensional list.

When we change from one coordinate system to another, a tensor of rank 2 transforms according to

$$\mathbf{T} = \{\mathbf{M}\}\mathbf{T}\{\mathbf{M}\}^{-1}, \tag{4.2}$$

where $\{\mathbf{M}\}$ is the matrix introduced in Eq. (4.1) above, and $\{\mathbf{M}\}^{-1}$ is its inverse.

In atmospheric science, we rarely meet tensors with ranks higher than two.

4.3 Differential operators

Several familiar differential operators can be defined without reference to any coordinate system. These operators are in a sense more fundamental than, for example, $\frac{\partial}{\partial x}$, where x is a particular spatial coordinate. The coordinate-independent operators that we need most often for atmospheric science (and for most other branches of physics too) are:

$$\text{the } \textit{gradient}\text{, denoted by } \nabla A \text{, where } A \text{ is an arbitrary scalar;} \tag{4.3}$$

$$\text{the } \textit{divergence}\text{, denoted by } \nabla \bullet \mathbf{Q} \text{, where } \mathbf{Q} \text{ is an arbitrary vector;} \tag{4.4}$$

$$\text{the } \textit{curl}\text{, denoted by } \nabla \times \mathbf{Q} \text{; and} \tag{4.5}$$

Note that the gradient and curl are vectors, while the divergence is a scalar. The gradient operator accepts scalars as "input," while the divergence and curl operators consume vectors. In discussions of two-dimensional motion, it is often convenient to introduce a further operator called the *Jacobian*, denoted by

$$J(p, q) \equiv \mathbf{k} \bullet (\nabla p \times \nabla q) \ . \tag{4.6}$$

Here the gradient operators are understood to produce vectors in the two-dimensional space, and $\mathbf{k}$ is a unit vector perpendicular to the two-dimensional surface.

A definition of the gradient operator that does not make reference to any coordinate system is:

$$\nabla A \equiv \lim_{S \to 0} \left[\frac{1}{V} \oint_S A \mathbf{n} \ dS \right] , \tag{4.7}$$

where S is the surface bounding a volume V, and $\mathbf{n}$ is the outward normal on S. Here the terms "volume" and "bounding surface" are used in the following generalized sense. In a three-dimensional space, "volume" is literally a volume, and "bounding surface" is literally a surface. In a two-dimensional space, "volume" means an area, and "bounding surface" means the curve bounding the area. In a one-dimensional space, "volume" means a curve, and "bounding surface" means the end points of the curve. The limit in (4.7) is one in which the volume and the area of its bounding surface shrink to zero.

As an example, consider a cartesian coordinate system (x, y), with unit vectors $\mathbf{i}$ and $\mathbf{j}$ in the x and y directions, respectively. Consider a "box" of width Δx and height Δy, as shown in Figure 1.

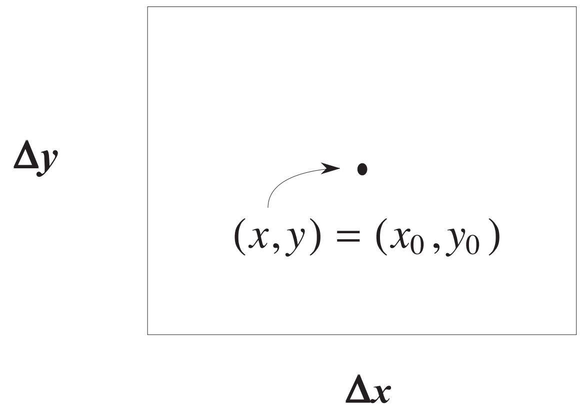

Figure 4.1: Diagram illustrating a box in a two-dimensional space, of width Δx and height Δy, and with center at $(x, y) = (x_0, y_0)$.

We can write

$$\nabla A = \lim_{(\Delta x,\, \Delta y) \to 0} \left(\frac{1}{\Delta x \Delta y} \left[A\left(x_0 + \frac{\Delta x}{2}, y_0 \right) \Delta y \mathbf{i} + A\left(x_0, y_0 + \frac{\Delta y}{2} \right) \Delta x \mathbf{j} \right. \right.$$
$$\left. \left. - A\left(x_0 - \frac{\Delta x}{2}, y_0 \right) \Delta y \mathbf{i} - A\left(x_0, y_0 - \frac{\Delta y}{2} \right) \Delta x \mathbf{j} \right] \right\} \tag{4.8}$$
$$= \frac{\partial A}{\partial x} \mathbf{i} + \frac{\partial A}{\partial y} \mathbf{j} \ .$$

This is the answer that we expect.

Definitions of the divergence and curl operators that do not make reference to any coordinate system are:

$$\nabla \bullet \mathbf{Q} \equiv \lim_{S \to 0} \left[\frac{1}{V} \oint_S \mathbf{Q} \bullet \mathbf{n} \ dS \right] , \tag{4.9}$$

$$\nabla \times \mathbf{Q} \equiv \lim_{S \to 0} \left[\frac{1}{V} \oint_S \mathbf{Q} \times \mathbf{n} \ dS \right] . \tag{4.10}$$

We can work through exercises similar to (4.8) for these operators too. You might want to try this yourself, to see if you understand.

4.4 Vector identities

There are many useful identities that relate the divergence, curl, and gradient operators. Most of the following identities can be found in any mathematics reference manual, e.g. Beyer (1984). Let A and B be arbitrary scalars, and let $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ be arbitrary vectors. Then:

$$\nabla\times\nabla A = 0 , \tag{4.11}$$

$$\nabla\bullet(\nabla\times\mathbf{V}) = 0 , \tag{4.12}$$

$$\mathbf{V_1}\times\mathbf{V_2} = -\mathbf{V_2}\times\mathbf{V_1} , \tag{4.13}$$

$$\nabla\bullet(A\mathbf{V}) = A(\nabla\bullet\mathbf{V}) + \mathbf{V}\bullet\nabla A , \tag{4.14}$$

$$\nabla\times(A\mathbf{V}) = \nabla A\times\mathbf{V} + A(\nabla\times\mathbf{V}) , \tag{4.15}$$

$$\mathbf{V}_1\bullet(\mathbf{V}_2\times\mathbf{V}_3) = (\mathbf{V}_1\times\mathbf{V}_2)\bullet\mathbf{V}_3 = \mathbf{V}_2\bullet(\mathbf{V}_3\times\mathbf{V}_1) , \tag{4.16}$$

$$\mathbf{V}_1\times(\mathbf{V}_2\times\mathbf{V}_3) = \mathbf{V}_2(\mathbf{V}_3\bullet\mathbf{V}_1) - \mathbf{V}_3(\mathbf{V}_1\bullet\mathbf{V}_2) , \tag{4.17}$$

$$\nabla\times(\mathbf{V}_1\times\mathbf{V}_2) = \mathbf{V}_1(\nabla\bullet\mathbf{V}_2) - \mathbf{V}_2(\nabla\bullet\mathbf{V}_1) - (\mathbf{V}_1\bullet\nabla)\mathbf{V}_2 + (\mathbf{V}_2\bullet\nabla)\mathbf{V}_1 , \tag{4.18}$$

$$\begin{aligned} J(A, B) &\equiv \mathbf{k}\bullet(\nabla A\times\nabla B) \\ &= -\mathbf{k}\bullet(\nabla B\times\nabla A) = \mathbf{k}\bullet\nabla\times(A\nabla B) = -\mathbf{k}\bullet\nabla\times(B\nabla A) \quad , \end{aligned} \tag{4.19}$$

$$\nabla^2\mathbf{V} \equiv (\nabla\bullet\nabla)\mathbf{V} = \nabla(\nabla\bullet\mathbf{V}) - \nabla\times(\nabla\times\mathbf{V}) . \tag{4.20}$$

Identity (4.20) is very nice. It says that the Laplacian *of a vector* is the gradient of the divergence of the vector, minus the curl of the curl of the vector. This can be used, for example, in a parameterization of momentum diffusion.

4.5 Spherical coordinates

Spherical coordinates are obviously of special importance in geophysics. In spherical coordinates, the gradient, divergence, and curl operators can be expressed as follows:

$$\nabla A = \left(\frac{1}{r\cos\varphi}\frac{\partial A}{\partial\lambda}, \frac{1}{r}\frac{\partial A}{\partial\varphi}, \frac{\partial A}{\partial r}\right) , \tag{4.21}$$

$$\nabla \bullet \mathbf{V} = \frac{1}{r\cos\varphi}\frac{\partial V_\lambda}{\partial\lambda} + \frac{1}{r\cos\varphi}\frac{\partial}{\partial\varphi}(V_\varphi\cos\varphi) + \frac{1}{r^2}\frac{\partial}{\partial r}(V_r r^2) \ , \tag{4.22}$$

$$\nabla \times \mathbf{V} = \left\{\frac{1}{r}\left[\frac{\partial V_r}{\partial\varphi} - \frac{\partial}{\partial r}(rV_\varphi)\right],\right.$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_\lambda) - \frac{1}{r\cos\varphi}\frac{\partial V_r}{\partial\lambda},$$

$$\left.\frac{1}{r\cos\varphi}\left[\frac{\partial V_\varphi}{\partial\lambda} - \frac{\partial}{\partial\varphi}(V_\lambda\cos\varphi)\right]\right\}, \tag{4.23}$$

$$\nabla^2 A = \frac{1}{r^2\cos\varphi}\left[\frac{\partial}{\partial\lambda}\left(\frac{1}{\cos\varphi}\frac{\partial A}{\partial\lambda}\right) + \frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial A}{\partial\varphi}\right) + \frac{\partial}{\partial r}\left(r^2\cos\varphi\frac{\partial A}{\partial r}\right)\right] \tag{4.24}$$

$$J(p, q) = \frac{1}{a^2\cos\varphi}\left(\frac{\partial p}{\partial\lambda}\frac{\partial q}{\partial\varphi} - \frac{\partial q}{\partial\lambda}\frac{\partial p}{\partial\varphi}\right). \tag{4.25}$$

Here A is an arbitrary scalar, and $\mathbf{V}$ is an arbitrary vector.

As an example, consider how the two-dimensional version of (4.21) can be derived from (4.7). Fig. 4.2 illustrates the problem. Here we have replaced r by a, the radius of the Earth. The angle θ depicted in the figure arises from the gradual rotation of $\mathbf{e}_\lambda$ and $\mathbf{e}_\varphi$, the unit vectors associated with the spherical coordinates, as the longitude changes; the directions of $\mathbf{e}_\lambda$ and $\mathbf{e}_\varphi$ in the center of the area element, where ∇A is defined, are different from their respective directions on either east-west wall of the area element. Inspection of Fig. 4.2 shows that θ satisfies

$$\tan\theta = \frac{-\frac{1}{2}[a\cos(\varphi + d\varphi) - a\cos\varphi]d\lambda}{ad\varphi} \rightarrow -\frac{1}{2}\left(\frac{\partial}{\partial\varphi}\cos\varphi\right)d\lambda = \frac{1}{2}\sin\varphi d\lambda \cong \sin\theta \ . \tag{4.26}$$

Note that θ is of "differential" or infinitesimal size. Nevertheless, we demonstrate below that it cannot be neglected in the derivation of (4.7). The line integral in (4.7) can be expressed as

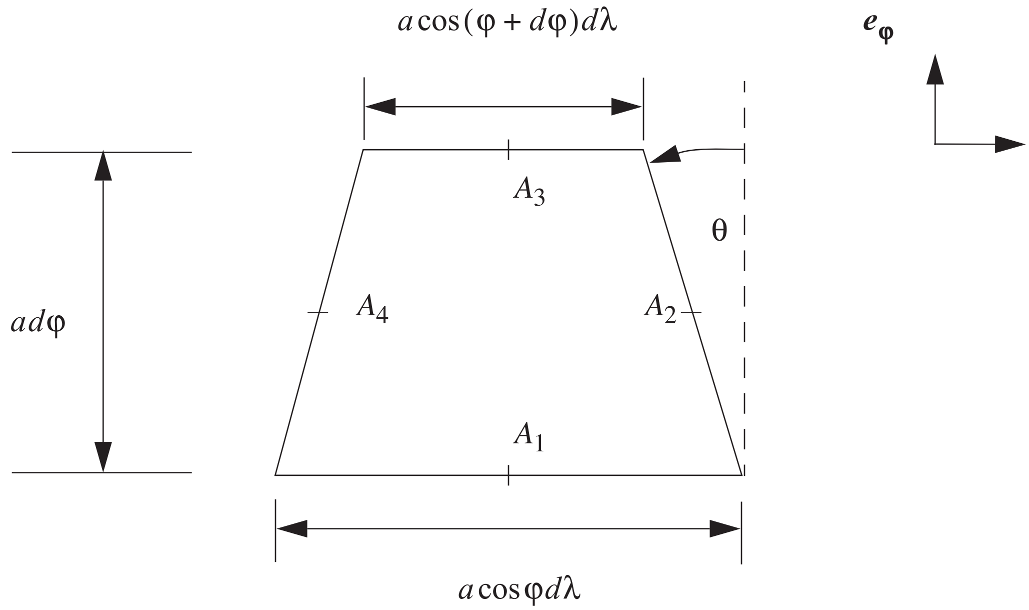

Figure 4.2: Sketch depicting an area element on the sphere, with longitudinal width $d\lambda$, and latitudinal height $d\varphi$.

$$
\begin{aligned}
\frac{1}{Area}\oint A\mathbf{n}\,dl &= [-\mathbf{e}_\varphi A_1 \alpha\cos\varphi\; d\lambda + \mathbf{e}_\lambda A_2 a d\varphi\cos\theta + \mathbf{e}_\varphi A_2 a d\varphi\sin\theta \\
&\quad + \mathbf{e}_\varphi A_3\cos(\varphi + d\varphi)d\lambda - \mathbf{e}_\lambda A_4 a d\varphi\cos\theta + \mathbf{e}_\varphi A_4 a d\varphi\sin\theta]/(a^2\cos\varphi d\lambda d\varphi) \qquad (4.27) \\
&= \mathbf{e}_\lambda\frac{(A_2 - A_4)\cos\theta}{a\cos\varphi d\lambda} + \mathbf{e}_\varphi\frac{\{[A_3 a\cos(\varphi + d\varphi) - A_1 a\cos\varphi]d\lambda + (A_2 + A_4)\sin\theta a d\varphi\}}{a^2\cos\varphi d\lambda d\varphi} .
\end{aligned}
$$

Note how the angle θ has entered here. Put $\cos\theta \to 1$ and $\sin\theta \to \frac{1}{2}\sin\varphi d\lambda$, to obtain

$$
\begin{aligned}
\frac{1}{Area}\oint A\boldsymbol{n}\,dl &= \mathbf{e}_\lambda\frac{(A_2 - A_4)}{a\cos\phi d\lambda} + \mathbf{e}_\varphi\left\{\left[\frac{A_3\cos(\varphi + d\varphi) - A_1\cos\varphi}{a\cos\varphi d\varphi}\right] + \left(\frac{A_2 + A_4}{2}\right)\frac{\sin\varphi}{a\cos\varphi}\right\} \\
&\to \mathbf{e}_\lambda\frac{1}{a\cos\varphi}\frac{\partial A}{\partial\lambda} + \mathbf{e}_\varphi\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(A\cos\varphi) + \frac{A\sin\varphi}{a\cos\varphi}\right] \qquad (4.28) \\
&= \mathbf{e}_\lambda\frac{1}{a\cos\varphi}\frac{\partial A}{\partial\lambda} + \mathbf{e}_\varphi\frac{1}{a}\frac{\partial A}{\partial\varphi} ,
\end{aligned}
$$

which agrees with the two-dimensional version of (4.7).

Similar derivations can be given for (4.22) and (4.23).

4.6 Conclusions

This brief discussion is intended mainly as a refresher, for students who learned these concepts once upon a time, but perhaps have not thought about them for awhile. The references below provide much more information. You can also find lots of resources on the Web.

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