Spherical Harmonics and Related Topics

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The spherical surface harmonics are convenient functions for representing the distribution of geophysical quantities over the surface of the spherical Earth.

Consider solutions of Laplace's differential equation

$$\nabla^2 S = 0 , \tag{10.1}$$

in a three-dimensional spherical coordinate system (r, λ, φ), where r is the radial coordinate, λ is longitude and φ is latitude. Here $S(r, \lambda, \varphi)$ is an arbitrary scalar. The ∇^2 operator can be expanded as:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial S}{\partial r}\right) + \frac{1}{r^2\cos\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial^2 S}{\partial\lambda^2} + \frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial S}{\partial\varphi}\right)\right] = 0 . \tag{10.2}$$

We are going to use the familiar "separation of variables" assumption to write $S(r, \lambda, \varphi)$ as the product of a function of r, times a function of λ, times a function of φ. We choose to separate the variables one at a time, for clarity of exposition.

First we separate out the r-dependence. Inspection of (10.2) suggests that $S(r, \lambda, \varphi)$ should be proportional to a power of r. We write

$$S(r, \lambda, \varphi) = r^n Y_n(\varphi, \lambda) . \tag{10.3}$$

The $Y_n(\lambda, \varphi)$ are called *spherical surface harmonics* of order n. The subscript n is attached to $Y_n(\lambda, \varphi)$ to remind us that it corresponds to the function $S(r, \lambda, \varphi)$ whose radial dependence follows r^n. In order for $S(r, \lambda, \varphi)$ to remain finite as $r \to 0$, we need $n \geq 0$. Since $n = 0$ implies that V is independent of radius, we conclude that n must be a positive integer.

Using $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial S}{\partial r}\right) = \frac{n(n+1)}{r^2}S$, which follows immediately from (10.3), we can rewrite (10.2) as

$$\frac{1}{r^2\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial Y_n}{\partial\varphi}\right)+\frac{1}{r^2\cos^2\varphi}\frac{\partial^2 Y_n}{\partial\lambda^2}+\frac{n(n+1)}{r^2}Y_n = 0\,. \tag{10.4}$$

The r's could be cancelled out if we wanted to. Therefore, (10.4) is essentially a problem in two independent variables, namely λ and φ. Note that n, the exponent of r in (10.3), is still visible in (10.4), like the smile of the Cheshire cat.

For later reference, we define the "horizontal Laplacian" of an arbitrary scalar S by

$$\nabla_h{}^2 S = \frac{1}{r^2\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial S}{\partial\varphi}\right)+\frac{1}{r^2\cos^2\varphi}\frac{\partial^2 S}{\partial\lambda^2}, \tag{10.5}$$

so that (10.4) can be rewritten as

$$\nabla_h{}^2 Y_n+\frac{n(n+1)}{r^2}Y_n = 0, \tag{10.6}$$

or

$$\nabla_h{}^2 Y_n = -\frac{n(n+1)}{r^2}Y_n. \tag{10.7}$$

At this point we make an analogy with a trigonometric functions. Suppose that we have a "doubly periodic" function $W(x, y)$ defined on a plane (or a torus) with Cartesian coordinates x and y. As a particular example, let

$$W(x, y) = A\sin(kx)\cos(ly)\,, \tag{10.8}$$

where A is an arbitrary constant. In Cartesian coordinates the two-dimensional Laplacian of W is

$$\begin{aligned}\nabla_h{}^2 W &= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)W \\ &= -(k^2+l^2)W\quad.\end{aligned} \tag{10.9}$$

Compare (10.7) and (10.9). They are closely analogous. Note that $\frac{n(n+1)}{r^2}$ in (10.7) corresponds to $(k^2 + l^2)$ in (10.9). This shows that $n(n+1)$ is proportional to a "total horizontal wave number" on the sphere.

Next, we separate the longitude and latitude dependence in the $Y_n(\lambda, \varphi)$, i.e.

$$Y_n(\varphi, \lambda) = \Phi(\varphi)\Lambda(\lambda), \tag{10.10}$$

where $\Phi(\varphi)$ and $\Lambda(\lambda)$ are to be determined. By substitution of (10.10) into (10.4), we find that

$$\frac{\cos^2\varphi}{\Phi}\left[\frac{1}{\cos\varphi}\frac{d}{d\varphi}\left(\cos\varphi\frac{d\Phi}{d\varphi}\right) + n(n+1)\Phi\right] = -\frac{1}{\Lambda}\frac{d^2\Lambda}{d\lambda^2}. \tag{10.11}$$

The left-hand side of (10.11) does not contain λ, and the right-hand side does not contain φ, so both sides must be a constant, c. Then the longitudinal structure of the solution is governed by

$$\frac{d^2\Lambda}{d\lambda^2} + c\Lambda = 0. \tag{10.12}$$

It follows that $\Lambda(\lambda)$ must be a trigonometric function of longitude, i.e.

$$\Lambda = A_s\exp(is\lambda), \text{ where } s = \sqrt{c} \tag{10.13}$$

and A_s is an arbitrary complex constant. The cyclic condition $\Lambda(\lambda + 2\pi) = \Lambda(\lambda)$ implies that $\sqrt{c}$ must be an integer, which we denote by s. We refer to s as the zonal wave number. Note that s is non-dimensional, and can be either positive or negative.

The corresponding equation for $\Phi(\varphi)$, which determines the meridional structure of the solution, is

$$\frac{1}{\cos\varphi}\frac{d}{d\varphi}\left(\cos\varphi\frac{d\Phi}{d\varphi}\right) + \left[n(n+1) - \frac{s^2}{\cos^2\varphi}\right]\Phi = 0. \tag{10.14}$$

Note that the *zonal* wave number, s, appears in this *meridional* structure equation, as does n. The longitude and radius dependencies have disappeared, but the zonal wave number and the exponent of the radius are still visible.

For convenience, we define a new independent variable to measure latitude,

$$\mu \equiv \sin\varphi, \tag{10.15}$$

so that $d\mu \equiv \cos\varphi d\varphi$. Then (10.14) becomes

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Phi}{d\mu}\right] + \left[n(n+1) - \frac{s^2}{1-\mu^2}\right]\Phi = 0. \tag{10.16}$$

Eq. (10.16) is simpler than (10.14), in that (10.16) does not involve trigonometric functions of the independent variable. This added simplicity is the motivation for using (10.15). The solutions of (10.16) are called the associated Legendre functions, are denoted by $P_n^s(\mu)$, and are given by

$$P_n^s(\mu) = \frac{(2n)!}{2^n n!(n-s)!}(1-x^2)^{\frac{s}{2}} \cdot \left[\mu^{n-s} - \frac{(n-s)(n-s-1)}{2(2n-1)}\mu^{n-s-2}\right.$$
$$\left. + \frac{(n-s)(n-s-1)(n-s-2)(n-s-3)}{2 \cdot 4(2n-1)(2n-3)}\mu^{n-s-4} - \ldots\right] \quad . \tag{10.17}$$

The subscript n and superscript s are just "markers" to remind us that $P_n^s(\mu)$ is that n and s appear as parameters in (10.16), denoting the radial exponent and zonal wave number of $S(r, \lambda, \varphi)$, respectively. The expansion in (10.17) is continued out as far as necessary to include all non-negative powers of x. The factor in brackets is, therefore, a polynomial of degree $(n-s)$, and so *we must require that*

$$n \geq s. \tag{10.18}$$

Substitution can be used to demonstrate that, for $n \geq s$, the associated Legendre functions are indeed solutions of (10.16).

In view of the leading factor of $(1-x^2)^{\frac{s}{2}}$ in (10.17), the complete function $P_n^s(x)$ is a polynomial in x for even values of s, but not for odd values of s. The functions $P_n^s(x)$ are said to be of "order n" and "rank s."

Here are some examples of associated Legendre functions, which you might want to check for their consistency with (10.17):

Note that we have allowed s to take all non-negative integer values up and including, but not exceeding, the value of n.

It can be shown that the associated Legendre functions are *orthogonal*, i.e.

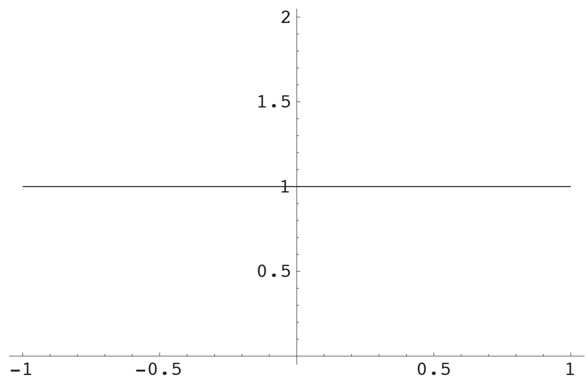
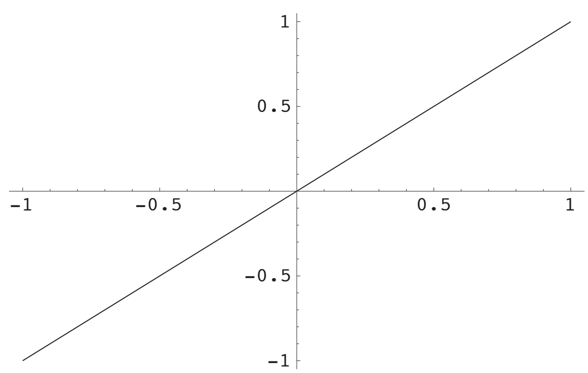
Associated Legendre Function	Plot
$P_0^0(x) = 1$	
$P_1^0(x) = x$	

Table 10.1: Examples of Associated Legendre Functions.

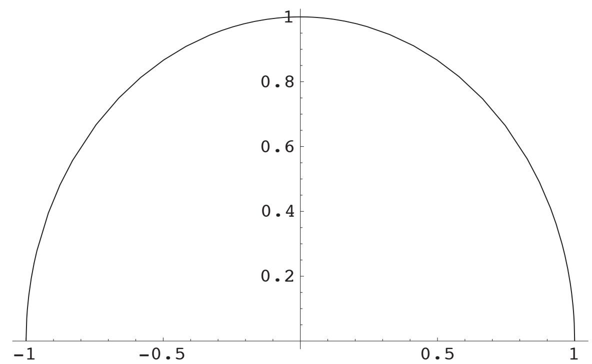
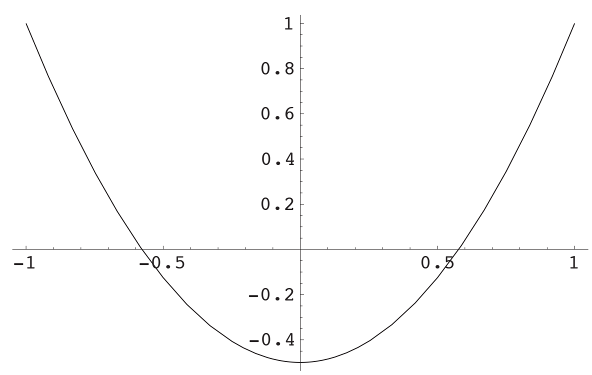
Associated Legendre Function	Plot
$P_1^1(x) = \sqrt{1-x^2}$	
$P_2^0(x) = \frac{3}{2}\left(x^2 - \frac{1}{3}\right)$	

Table 10.1: Examples of Associated Legendre Functions.

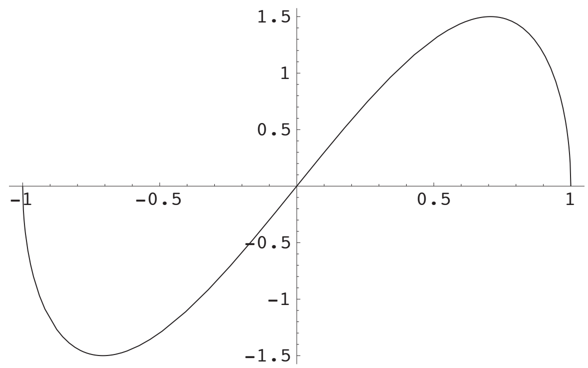
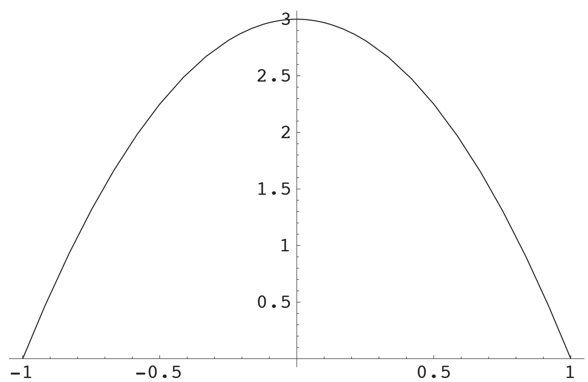
Associated Legendre Function	Plot
$P_2^1(x) = 3x\sqrt{1-x^2}$	
$P_2^2(x) = 3(1-x^2)$	

Table 10.1: Examples of Associated Legendre Functions.

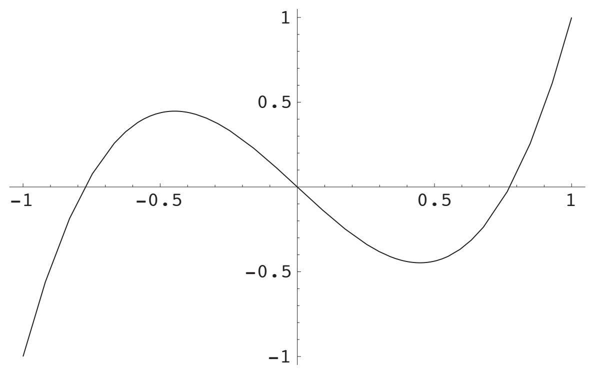
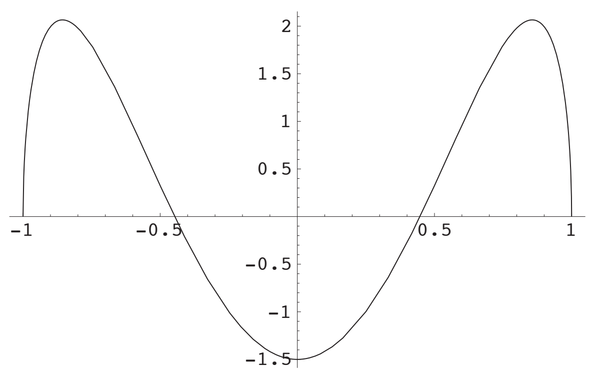
Associated Legendre Function	Plot
$P_3^0(x) = \frac{5}{2}\left(x^3 - \frac{3}{5}x\right)$	
$P_3^1(x) = \frac{15}{2}\sqrt{1-x^2}\left(x^2 - \frac{1}{5}\right)$	

Table 10.1: Examples of Associated Legendre Functions.

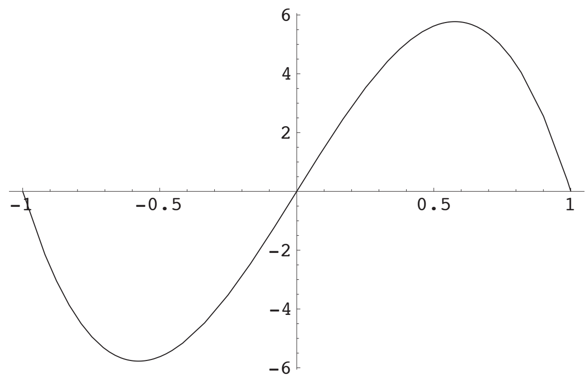
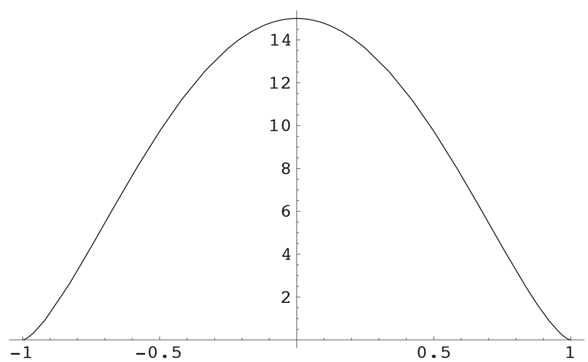
Associated Legendre Function	Plot
$P_3^2(x) = 15(1-x^2)x$	
$P_3^3(x) = 15\sqrt{1-x^2}(1-x^2)$	

Table 10.1: Examples of Associated Legendre Functions.

$$\int_{-1}^{1} P_n^s(x) \cdot P_l^s(x)dx = 0, (n \neq l) ,$$
$$\int_{-1}^{1} [P_n^s(x)]^2 dx = \left(\frac{2}{2n+1}\right)\frac{(n+s)!}{(n-s)!} . \tag{10.19}$$

It follows that the functions

$$\sqrt{\left(\frac{2n+1}{2}\right)\frac{(n-s)!}{(n+s)!}}P_n^s(x), n = s, s+1, s+2, \ldots \tag{10.20}$$

are mutually *orthonormal* for $-1 \leq x \leq 1$.

Referring back to (10.10), we see that a particular spherical surface harmonic can be written as

$$Y_n^s(\mu, \lambda) = P_n^s(\mu) exp(is\lambda) \quad . \tag{10.21}$$

It is the product of an associated Legendre function of μ with a trigonometric function of longitude. Note that the arbitrary constant has been set to unity.

Fig. 10.1 shows examples of spherical harmonics of low order, as mapped out

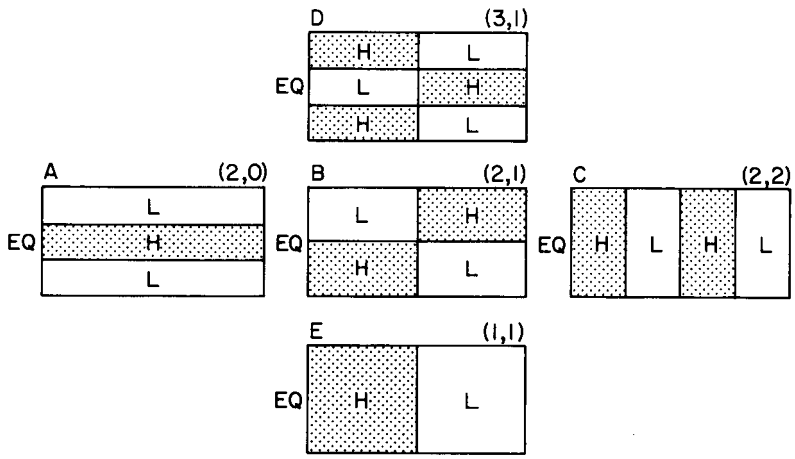


Figure 10.1: Examples of low-resolution spherical harmonics, mapped out onto the plane. The horizontal direction in each panel represents longitude, and the vertical direction represents latitude. The numbers in parentheses in each panel are the appropriate values of *n* and *s*, in that order. Recall that the number of nodes in the meridional direction is *n* – *s*. The shading in each panel represents the sign of the field (and all signs can be flipped arbitrarily). You may think of "white" as negative and "stippled" as positive. From Washington and

onto the longitude-latitude plane. Fig. 10.2 gives similar diagrams for $n = 5$ and $s = 0, 1, 2, ..., 5$, plotted out onto stretched spheres. Fig. 10.3 shows some low-order spherical harmonics mapped onto three-dimensional pseudo-spheres, in which the local radius of the surface of the pseudo-sphere is one plus a constant times the local value of the spherical harmonic.

By using the orthogonality condition (10.19) for the associated Legendre functions, and also the orthogonality properties of the trigonometric functions, we can show that

$$\int_{-1}^{1}\int_{0}^{2\pi} P_n^s(\mu) exp(is\lambda) P_l^{s'}(\mu) exp(is'\lambda) d\mu d\lambda = 0 \tag{10.22}$$

unless $n = l$ and $s = s'$. The mean value over the surface of a sphere of the square of a spherical surface harmonic is given (for $s \neq 0$), by

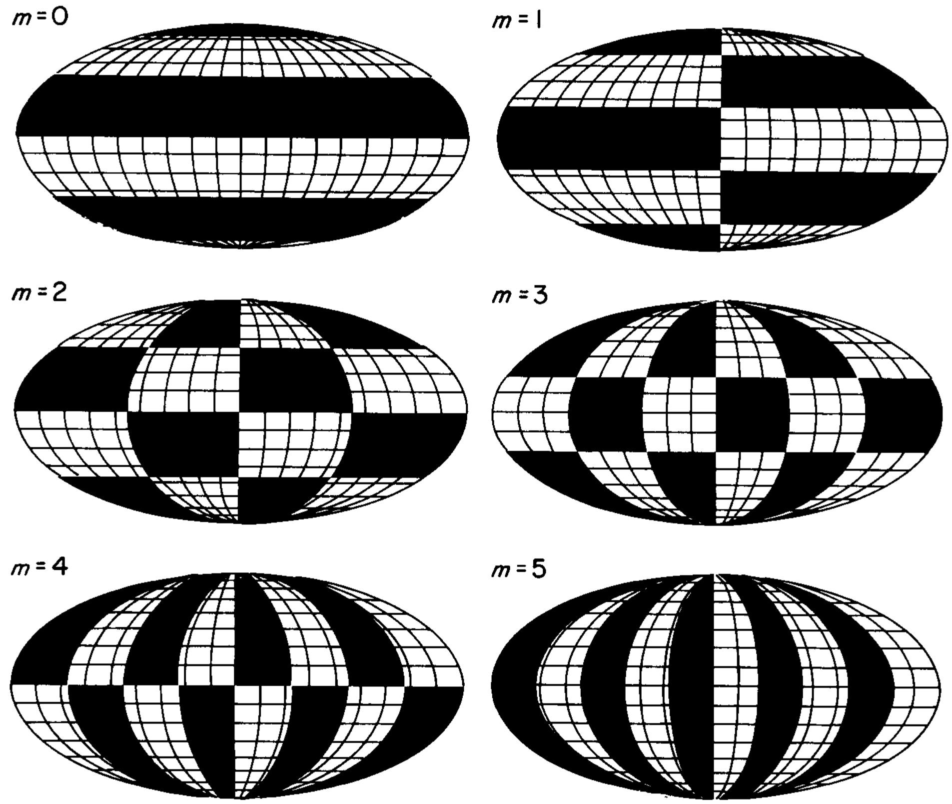


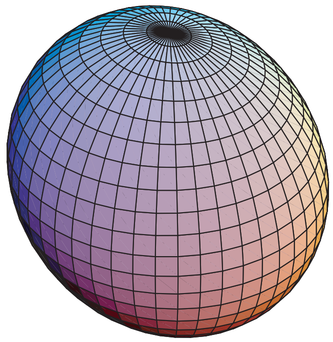
Figure 10.2: Alternating patters of positives and negatives for spherical harmonics with n = 5 and s = 0, 1, 2,...,5. From Baer (1972).

$$\frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}[P_n^s(\mu)exp(is\lambda)]^2 d\mu d\lambda = \frac{1}{2(2n+1)}\frac{(n+s)!}{(n-s)!} . \qquad (10.23)$$

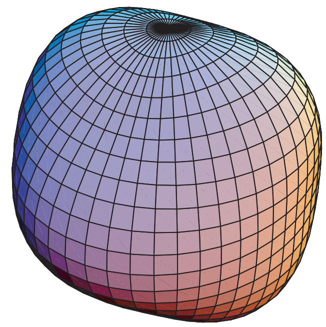
For the special case $s = 0$, the corresponding value is $1/(2n+1)$. The mean values given by (10.23), for a given n, vary greatly with s, which is inconvenient for the interpretation of data. For this reason, it is customary to use, instead of $P_n^s(\mu)$, the *semi-normalized associated Legendre functions*. These functions are identical with $P_n^s(\mu)$ when $s = 0$. For $s > 0$, the semi-normalized functions are defined by

$$P_{n,s}(\mu) = \sqrt{2\frac{(n-s)!}{(n+s)!}} \cdot P_n^s(\mu) . \qquad (10.24)$$

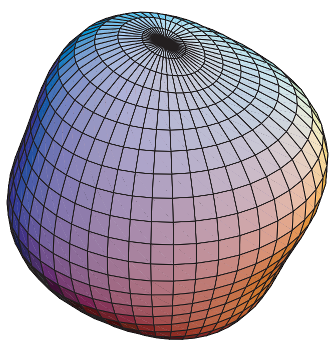
The mean value over the sphere of the square of $P_{n,s}(\mu)exp(is\lambda)$ is then $(2n+1)^{-1}$, for any n and s.



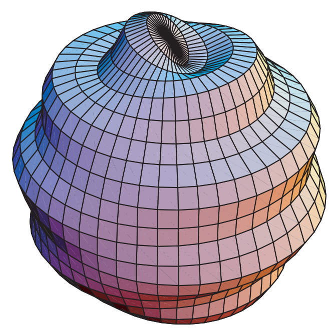
Two-Dimensional Index = 2, Zonal wave number = 1



Two-Dimensional Index = 4, Zonal wave number = 2



Two-Dimensional Index = 4, Zonal wave number = 1



Two-Dimensional Index = 10, Zonal wave number = 1

Figure 10.3: Selected spherical harmonics mapped onto three-dimensional pseudo-spheres, in which the local radius of the surface of the pseudo-sphere is one plus a constant times the local value of the spherical harmonic.

The spherical harmonics can be shown to form a complete orthonormal basis, and so can be used to represent an arbitrary function, $T(\lambda, \varphi)$ of latitude and longitude:

$$T(\lambda, \varphi) = \sum_{s=-\infty}^{\infty} \sum_{n=|s|}^{\infty} T_n^s Y_n^s(\lambda, \varphi) \tag{10.25}$$

Here the T_n^s are the expansion coefficients. Note that the sum over s ranges over both positive and negative values, and that the sum over n is taken so that $n - |s| \geq 0$.

The sums in (10.25) range over an infinity of terms. In practice, of course, we must truncate after a finite number of terms, so that (10.25) is replaced by

$$T(\lambda, \varphi) \cong \sum_{s=-M}^{M} \sum_{n=|s|}^{N(s)} T_n^s Y_n^s \tag{10.26}$$

The sum over n ranges up to $N(s)$, which has to be specified somehow. The sum over s ranges from $-M$ to M. It can be shown that *this ensures that the final result is real*; this is an important result which you should try to prove to yourself.

The choice of $N(s)$ fixes what is called the "truncation procedure." There are two commonly used truncation procedures. The first, called "rhomboidal," takes

$$N(s) = M + |s| . \tag{10.27}$$

The second, called "triangular," takes

$$N(s) = M . \tag{10.28}$$

Triangular truncation has the following beautiful property. In order to actually perform a spherical harmonic transform, it is necessary to adopt a spherical coordinate system (λ, φ). There are of course infinitely many such systems. There is no reason in principle that the coordinates have to be chosen in the conventional way, so that the poles of the coordinate system coincide with the Earth's poles of rotation. The choice of a particular spherical coordinate system is, therefore, somewhat arbitrary. Suppose that we choose two different spherical coordinate systems (tilted with respect to one another in an arbitrary way), perform a triangularly truncated expansion in both, then plot the results. It can be shown that the two maps will be identical. This means that the arbitrary orientations of the spherical coordinate systems used had no effect whatsoever on the results obtained. The coordinate system used "disappears" at the end. Triangular truncation is very widely used today, in part because of this nice property.

Fig. 10.4 shows an example based on 500 mb height data, provided originally on a 2.5° longitude-latitude grid. The figure shows how the data look when represented by just a few spherical harmonics (top left), a few more (top right), a moderate number (bottom left) and at full 2.5° resolution. The smoothing effect of severe truncation is clearly visible.

References and Bibliography

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Jarraud, M., and A. J. Simmons, 1983: The spectral technique. *Seminar on Numerical Methods for Weather Prediction*, European Centre for Medium Range Weather

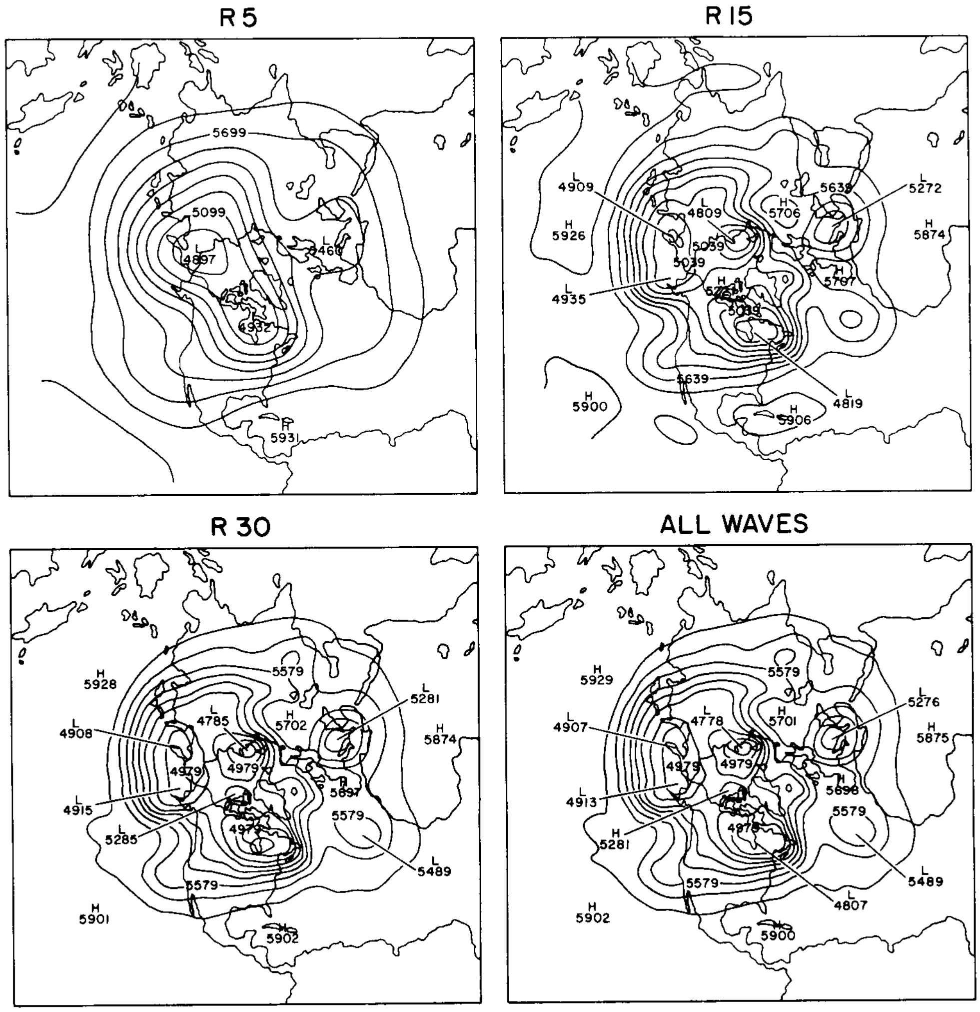

Figure 10.4: Demonstration of the effects of various horizontal truncations of 500 mb patterns of geopotential height (m) of data provided originally on a 2.5° longitude-latitude grid. From Washington and Parkinson (1986).

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