A Standard Test Set for Numerical Approximations to the Shallow Water Equations in Spherical Geometry

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Abstract

A suite of seven test cases is proposed for the evaluation of numerical methods intended for the solution of the shallow water equations in spherical geometry. The shallow water equations exhibit the major difficulties associated with the horizontal dynamical aspects of atmospheric modeling on the spherical earth. These cases are designed for use in the evaluation of numerical methods proposed for climate modeling and to identify potential trade-offs which must always be made in numerical modeling. Before a proposed scheme is applied to a full baroclinic atmospheric model it should perform well on these problems in comparison with other currently accepted numerical methods. The cases are presented in order of complexity. They consist of advection across the poles, steady state geostrophically balanced flow of both global and local scales, forced nonlinear advection of an isolated low, zonal flow impinging on an isolated mountain, Rossby-Haurwitz waves and observed atmospheric states. One of the cases is also identified as a computer performance/algorithm efficiency benchmark for assessing the performance of algorithms adapted to massively parallel computers.

1 Introduction

The early days of global atmospheric modeling saw significant efforts in adapting then current numerical methods to solving fluid flow on the surface of the sphere. A large component of this effort was directed toward finite difference approaches. The review article by Williamson [31] discusses the many finite difference approaches that were applied to the problem at that time and gives a lengthy list of references. The introduction of the spectral transform method by Orszag [17], and Eliasen, Machenhauer and Rasmussen [7] made the spectral method cost effective in terms of storage and processor time compared with finite difference approaches. The review by Machenhauer [15] discusses the various applications of the spectral method in detail. The spectral method presents a natural solution to problems introduced by spherical geometry in part because it provides an isotropic representation in spectral space even though the commonly adopted underlying Gaussian grid does not. The spectral transform method is widely accepted as the basis of operational numerical weather prediction and global climate models. Although not universally adopted the method has become the rule rather than the exception. As a result little effort has been directed in the last decade toward developing alternative methods of approximation for global atmospheric models.

Currently there is renewed interest in alternative methods for a variety of reasons. The European Centre for Medium Range Weather Forecasts (ECMWF) has reported [6] that at resolutions greater than those currently used in in operational numerical forecast models the computational cost of the Legendre transform associated with the spectral method will become a significant fraction of the total cost of the model. Thus other methods are likely to become economically competitive. The spectral representation contributes to unphysical structures in the predicted fields such as negative water vapor [22]. Traditional finite difference approximations also suffer from this defect. However, recently shape preserving and essentially non-oscillatory schemes have been developed to address this deficiency. Spectral models require a global domain and have thus been based on a normalized vertical coordinate such as pressure divided by surface pressure. Over steep mountains the horizontal pressure gradient force in such systems is a small difference of two large terms and difficult to approximate accurately. Mesh refinement near mountains, or admittance of explicit lateral boundaries where mountains can penetrate the grid, appear as potential alternatives. The spectral method also presents problems with efficient implementation on some of the new computer architectures although these are not necessarily unique to the spectral method. The global communication required by the spectral transform may be difficult to achieve efficiently on massively parallel computers with distributed memory. With grid point based schemes a similar communication problem may arise however associated with the elliptic problem introduced by a semi-implicit time stepping algorithm.

The renewed interest in algorithm development has led to the need to define standard test cases with which potential schemes may be compared. Strict comparisons based on such test cases will aid in rationally choosing the compromises which must be made in numerical modeling. We present a suite of test cases in this report for numerical approximations to the shallow water equations in spherical geometry. The shallow water equations on a rotating sphere serve as a primary test problem for numerical methods used in modeling global atmospheric flows. They describe the behavior of a shallow homogeneous incompressible and inviscid fluid layer. They present the major difficulties found in the horizontal aspects of three dimensional global atmospheric modeling. Thus they provide a first test to weed out potentially non-competitive schemes without the effort of building a complete model. However, because they do not represent the complete atmospheric system, the shallow water equations are only a first test. Ultimately schemes which look attractive based on these tests must be applied to the complete baroclinic problem. We hope that the existence of a standard test set for the shallow water equations will encourage the continued exploration of alternative numerical methods and provide the community with a mechanism for judging the relative merits of numerical schemes and parallel computers for atmospheric flow calculations.

We present here a suite of seven test cases in increasing order of complexity. Several analytic treatments included in the suite provide objective standards for judging the accuracy of numerical schemes and provide quick checks on the validity of code. The first test consists of advection of a structure of compact support by a specified wind field corresponding to solid body rotation whose axis is not necessarily coincident with that of the rotation of the earth. As such this case deals with only a subset of the shallow water equations, namely the continuity equation, but concentrates on a scheme's ability to deal with the poles of the spherical coordinate system.

The second and third cases present steady state, nonlinear zonal geostrophic flow. They are a global form with the wind corresponding to solid body rotation and a local form where the wind field has compact support. In both cases the spherical coordinate poles are not necessarily coincident with the earth's rotation axis. As with the first case these test a scheme's ability to handle the poles, but in addition nonlinearities can come into play.

The succeeding test cases are of increasing complexity and realism, exercising the more subtle aspects of atmospheric flows. One case uses an analytic forcing function to drive a low around the sphere. The case mimics the more complicated local structures observed in the atmosphere. Another case consists of zonal flow impinging on an isolated mountain in which a downstream wavetrain is set up. A Rossby-Haurwitz wave case is also included. Analytic solutions for the Rossby-Haurwitz wave in the shallow water context are not known but this wave has become a standard test case in meteorology. A reference solution is provided by a high resolution spectral transform model integration. Finally, actual weather patterns are presented for initial conditions. Since they obviously have no analytic solution a reference solution is provided again by a high resolution spectral transform model run. As mentioned above, analytic solutions for the last three cases are not known. Reference solutions will be provided by a high resolution spectral transform model. For it to be accepted it must be duplicated by a high resolution solution provided by at least one other different method.

With each test case we ask for a variety of specific measures of the error of the numerical solution. Just as there is no single ideal test case, there is no single measure that determines the quality of a scheme for atmospheric modeling. We include the variety of test cases and error measures to provide as much information as possible to would-be users so they can evaluate the various tradeoffs involved with the schemes.

The second test in the suite is also proposed as a performance benchmarking problem. Such benchmarking is particularly important since the efficiency of schemes must be evaluated considering the computing environment for which they are designed.

As an initial basis of comparison we provide in a companion report [13] solutions to these problems from a spectral transform approximation at resolutions currently used in atmospheric models. The code for this spectral model is documented in Hack and Jakob [8] which also provides details on how to obtain copies of this code. Spectral models are widely but not universally adopted in climate modeling and numerical weather prediction. We encourage centers currently using other methods to run these tests with their schemes and to submit the results for comparison. To facilitate comparison of schemes, a machine readable copy of standard FORTRAN routines which calculate the initial conditions and analytic or reference solutions is available from netlib@ornl.gov. A file summarizing performance statistics contributed by members of the community will also be maintained. In addition a list of corrections to this paper will be maintained along with a bibliography of reports presenting results of tests of numerical schemes based on this test suite and any modifications to the test suite generally agreed upon by the community. Please submit additional performance data and references for the bibliography as they become available to John Drake (bbd@ornl.gov).

2 The Shallow Water Equations on a Sphere

For convenience, we summarize many forms in which the shallow water equations can be written. The reader is referred to standard texts such as Holton [11] and Haltiner and Williams [9] for more general development.

2.1 Flux Form

The shallow water equations on a rotating sphere can be written in flux form as

$$\frac{\partial h^*\mathbf{v}}{\partial t} + \nabla\cdot(\mathbf{v}h^*\mathbf{v}) = -f\hat{\mathbf{k}}\times h^*\mathbf{v} - gh^*\nabla h \tag{1}$$

and

$$\frac{\partial h^*}{\partial t} + \nabla\cdot(h^*\mathbf{v}) = 0, \tag{2}$$

where h^* is the depth of the fluid and h is the height of the free surface above a reference sphere (sea level). If h_s denotes the height of the underlying mountains, $h = h^* + h_s$. The horizontal (on the sphere) vector velocity is denoted $\mathbf{v}$ with components u and v in the longitudinal (λ) and latitudinal (θ) directions respectively. The ∇ operator is the spherical horizontal gradient operator given by

$$\nabla(\ ) \equiv \frac{\hat{\mathbf{i}}}{a\cos\theta}\frac{\partial}{\partial\lambda}(\ ) + \frac{\hat{\mathbf{j}}}{a}\frac{\partial}{\partial\theta}(\ ) \tag{3}$$

and $\nabla\cdot$ is the spherical horizontal divergence operator given by

$$\nabla\cdot\mathbf{v} \equiv \frac{1}{a\cos\theta}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial(v\cos\theta)}{\partial\theta}\right] \tag{4}$$

The longitudinal, latitudinal and outward radial unit vectors are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, respectively, f is the Coriolis parameter, g is the gravitational constant and a is the radius of the earth. The Coriolis parameter is given by $2\Omega \sin\theta$ where Ω is the rotation rate of the earth.

The equations for the spherical components can be derived by writing $\mathbf{v} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}}$ and using

$$\frac{dh^*\mathbf{v}}{dt} = \hat{\mathbf{i}}\frac{dh^*u}{dt} + \hat{\mathbf{j}}\frac{dh^*v}{dt} + h^*u\frac{d\hat{\mathbf{i}}}{dt} + h^*v\frac{d\hat{\mathbf{j}}}{dt}. \tag{5}$$

Equation (1) in terms of spherical components is then

$$\frac{\partial h^*u}{\partial t} + \nabla \cdot (h^*u\mathbf{v}) - \left(f + \frac{u}{a}\tan\theta\right) h^*v + \frac{gh^*}{a\cos\theta}\frac{\partial h}{\partial \lambda} = 0, \tag{6}$$

$$\frac{\partial h^*v}{\partial t} + \nabla \cdot (h^*v\mathbf{v}) + \left(f + \frac{u}{a}\tan\theta\right) h^*u + \frac{gh^*}{a}\frac{\partial h}{\partial \theta} = 0. \tag{7}$$

2.2 Advective Form

The advective form of the horizontal momentum and mass continuity equations can be written

$$\frac{d\mathbf{v}}{dt} = -f\hat{\mathbf{k}} \times \mathbf{v} - g\nabla h. \tag{8}$$

and

$$\frac{dh^*}{dt} + h^*\nabla \cdot \mathbf{v} = 0 \tag{9}$$

where the substantial derivative is given by

$$\frac{d}{dt}(\ ) \equiv \frac{\partial}{\partial t}(\ ) + (\mathbf{v}\cdot\nabla)(\ ). \tag{10}$$

The equations for the spherical components are

$$\frac{\partial u}{\partial t} + \mathbf{v}\cdot\nabla u - \left(f + \frac{u}{a}\tan\theta\right) v + \frac{g}{a\cos\theta}\frac{\partial h}{\partial \lambda} = 0, \tag{11}$$

$$\frac{\partial v}{\partial t} + \mathbf{v}\cdot\nabla v + \left(f + \frac{u}{a}\tan\theta\right) u + \frac{g}{a}\frac{\partial h}{\partial \theta} = 0, \tag{12}$$

and

$$\frac{\partial h^*}{\partial t} + \mathbf{v}\cdot\nabla h^* + \frac{h^*}{a\cos\theta}\left(\frac{\partial u}{\partial \lambda} + \frac{\partial v\cos\theta}{\partial \theta}\right) = 0. \tag{13}$$

2.3 Vorticity Divergence Form

The horizontal momentum can also be specified in terms of relative vorticity,

$$\zeta \equiv \hat{\mathbf{k}} \cdot (\nabla \times \mathbf{v}), \tag{14}$$

and horizontal divergence,

$$\delta \equiv \nabla \cdot \mathbf{v}. \tag{15}$$

The curl operator is given by

$$\hat{\mathbf{k}} \cdot (\nabla \times \mathbf{v}) = \frac{1}{a \cos\theta} \left[\frac{\partial v}{\partial \lambda} - \frac{\partial u \cos\theta}{\partial \theta} \right]. \tag{16}$$

Using the vector identity

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla \left(\frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) + \zeta \hat{\mathbf{k}} \times \mathbf{v}, \tag{17}$$

the horizontal momentum equation becomes

$$\frac{\partial \mathbf{v}}{\partial \mathbf{t}} = -(\zeta + f)\hat{\mathbf{k}} \times \mathbf{v} - \nabla(gh + \frac{\mathbf{v} \cdot \mathbf{v}}{2}), \tag{18}$$

or in spherical component form

$$\frac{\partial u}{\partial t} = (\zeta + f)v - \frac{1}{a \cos\theta} \frac{\partial}{\partial \lambda} \left[gh + \frac{1}{2}(u^2 + v^2) \right] \tag{19}$$

$$\frac{\partial v}{\partial t} = -(\zeta + f)u - \frac{1}{a} \frac{\partial}{\partial \theta} \left[gh + \frac{1}{2}(u^2 + v^2) \right]. \tag{20}$$

Applying the curl and divergence operators $\hat{\mathbf{k}} \cdot \nabla \times (\ )$ and $\nabla \cdot (\ )$ to the momentum equation yields

$$\frac{\partial \zeta}{\partial t} = -\nabla \cdot (\zeta + f)\mathbf{v} \tag{21}$$

$$\frac{\partial \delta}{\partial t} = \hat{\mathbf{k}} \cdot \nabla \times (\zeta + f)\mathbf{v} - \nabla^2 \left(gh + \frac{\mathbf{v} \cdot \mathbf{v}}{2} \right), \tag{22}$$

or in terms of spherical components

$$\begin{aligned} \frac{\partial \zeta}{\partial t} \quad = \quad & -\frac{1}{a \cos\theta} \frac{\partial}{\partial \lambda}[(\zeta + f)u] \\ & - \frac{1}{a \cos\theta} \frac{\partial}{\partial \theta}[(\zeta + f)v \cos\theta] \end{aligned} \tag{23}$$

$$\begin{aligned}\frac{\partial \delta}{\partial t} &= \frac{1}{a\cos\theta}\frac{\partial}{\partial \lambda}[(\zeta+f)v] \\ &- \frac{1}{a\cos\theta}\frac{\partial}{\partial \theta}[(\zeta+f)u\cos\theta] \\ &- \nabla^2\left[gh+\frac{1}{2}(u^2+v^2)\right]\end{aligned} \tag{24}$$

where

$$\nabla^2(\ ) = \frac{1}{a^2\cos^2\theta}\frac{\partial^2(\ )}{\partial\lambda^2} + \frac{1}{a^2\cos\theta}\frac{\partial}{\partial\theta}\left(\frac{\cos\theta\partial(\ )}{\partial\theta}\right). \tag{25}$$

2.4 Bounded Differential Expression Form

The spherical vector component forms of the equations contain individual unbounded differential expressions approaching the poles. Swarztrauber [3] has developed a form for the equations containing only bounded differential expressions

$$\frac{\partial u}{\partial t} = -\frac{u}{a}\delta - \frac{v}{a}\frac{\partial u}{\partial \theta} + \frac{u}{a}\frac{\partial v}{\partial \theta} + fv - \frac{g}{a\cos\theta}\frac{\partial h}{\partial\lambda} \tag{26}$$

$$\frac{\partial v}{\partial t} = -\frac{u}{a}\zeta - \frac{u}{a}\frac{\partial u}{\partial \theta} - \frac{v}{a}\frac{\partial v}{\partial \theta} - fu - \frac{g}{a}\frac{\partial h}{\partial\theta}. \tag{27}$$

2.5 Stream Function, Velocity Potential Form

The spherical velocity components can be avoided by the introduction of a horizontal stream function, ψ, and velocity potential, χ. The equation relating horizontal velocity and these two scalar quantities is

$$\mathbf{v} = \hat{\mathbf{k}} \times \nabla\psi + \nabla\chi. \tag{28}$$

The spherical wind components are related to the stream function and velocity potential by

$$u = -\frac{1}{a}\frac{\partial\psi}{\partial\theta} + \frac{1}{a\cos\theta}\frac{\partial\chi}{\partial\lambda} \tag{29}$$

$$v = \frac{1}{a\cos\theta}\frac{\partial\psi}{\partial\lambda} + \frac{1}{a}\frac{\partial\chi}{\partial\theta}. \tag{30}$$

The application of the curl and divergence operators to (29) and (30) gives the absolute vorticity

$$\eta \equiv \zeta + f = \nabla^2\psi + f \tag{31}$$

and divergence

$$\delta = \nabla^2 \chi. \tag{32}$$

In terms of the stream function and velocity potential the horizontal momentum and mass continuity equations can be written [16]

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \nabla \chi) - J(\eta, \psi) = 0, \tag{33}$$

$$\frac{\partial \delta}{\partial t} - \nabla \cdot (\eta \nabla \psi) - J(\eta, \chi) = -\nabla^2 (K + gh), \tag{34}$$

and

$$\frac{\partial h^*}{\partial t} + \nabla \cdot (h^* \nabla \chi) - J(h^*, \psi) = 0. \tag{35}$$

In spherical coordinates the Jacobian operator is defined by

$$J(\alpha, \beta) = \frac{1}{a^2 \cos^2 \theta} \left(\frac{\partial \alpha}{\partial \lambda} \frac{\partial \beta}{\partial \theta} - \frac{\partial \alpha}{\partial \theta} \frac{\partial \beta}{\partial \lambda} \right). \tag{36}$$

Here K is the kinetic energy $\frac{1}{2}(u^2 + v^2)$ and can be expressed in terms of stream function and velocity potential as

$$K = \frac{1}{2} [\nabla \cdot (\psi \nabla \psi) - \psi \nabla^2 \psi + \nabla \cdot (\chi \nabla \chi) - \chi \nabla^2 \chi] + J(\psi, \chi). \tag{37}$$

The J and $\nabla \cdot$ operators, curl and divergence, have the following integral properties according to Gauss's theorem,

$$\int \int_A J(\alpha, \beta) dA = \int_C \alpha \frac{\partial \beta}{\partial s} ds \tag{38}$$

and

$$\int \int_A \nabla \cdot (\alpha \nabla \beta) dA = \int_C \alpha \frac{\partial \beta}{\partial n} ds \tag{39}$$

where $\frac{\partial}{\partial s}$ is the derivative along C and $\frac{\partial}{\partial n}$ is the derivative normal to the curve.

2.6 General Orthogonal Coordinates

The general orthogonal coordinate form is useful when considering approximations based on various map projections. Let (x, y) be the orthogonal coordinates and m_x and m_y be the metric coefficients so the distance increment $(d\ell)$ satisfies

$$(d\ell)^2 = m_x^2 dx^2 + m_y^2 dy^2. \tag{40}$$

The velocity vector $\mathbf{v}$ has components

$$U = m_x \frac{dx}{dt} \tag{41}$$

$$V = m_y \frac{dy}{dt} \tag{42}$$

in the x and y directions, respectively. The equations of motion are

$$\frac{dU}{dt} - \left[f + \frac{1}{m_x m_y} \left(V \frac{\partial m_y}{\partial x} - U \frac{\partial m_x}{\partial y} \right) \right] V + \frac{g}{m_x} \frac{\partial h}{\partial x} = 0 \tag{43}$$

$$\frac{dV}{dt} + \left[f + \frac{1}{m_x m_y} \left(V \frac{\partial m_y}{\partial x} - U \frac{\partial m_x}{\partial y} \right) \right] U + \frac{g}{m_y} \frac{\partial h}{\partial y} = 0 \tag{44}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{U}{m_x} \frac{\partial}{\partial x} + \frac{V}{m_y} \frac{\partial}{\partial y}. \tag{45}$$

The continuity equation is

$$\frac{dh^*}{dt} + \frac{h^*}{m_x m_y} \left[\frac{\partial}{\partial x}(m_y U) + \frac{\partial}{\partial y}(m_x V) \right] = 0. \tag{46}$$

Note, for spherical coordinates

$$x = \lambda \quad , \quad y = \theta \tag{47}$$

$$m_x = a \cos\theta \quad , \quad m_y = a \tag{48}$$

$$u = m_x \frac{dx}{dt} = a \cos\theta \frac{d\lambda}{dt} \tag{49}$$

$$v = m_y \frac{dy}{dt} = a \frac{d\theta}{dt}. \tag{50}$$

Commonly used map projections are north and south polar stereographic

$$m_x = m_y = \frac{1}{2}(1 \pm \sin\theta) = \frac{4a^2}{x^2 + y^2 + 4a^2} \tag{51}$$

and Mercator's

$$m_x = m_y = \cos\theta. \tag{52}$$

All major map projections are described from a geographical point of view by Steers [25]. Saucier [24] discusses the common projections used in meteorology. More recently Pearson [19] has summarized the field.

2.7 Three-Dimensional, Constrained Form

Côté [4] developed a three-dimensional vector form for the horizontal momentum equations using the undetermined Lagrange multiplier method to constrain the motion to be on the surface of the sphere.

$$\frac{d\mathbf{V}}{dt} = \mathbf{F} + \mu\mathbf{r} \tag{53}$$

where

$$\mathbf{V} = \frac{d\mathbf{r}}{dt} \tag{54}$$

is the three-dimensional velocity vector in a rotating frame,

$$\mathbf{F} = -f\mathbf{r} \times \mathbf{V} - g\nabla h \tag{55}$$

and μ is the Lagrange multipler determined by requiring

$$\mathbf{r} \cdot \mathbf{r} = a^2 \tag{56}$$

be satisfied for all time; $\mathbf{r}$ is the position vector. Evaluation of the Lagrange multiplier for the continuous equations gives $\mu = -\mathbf{V} \cdot \mathbf{V}$ and leads to the usual Eulerian form. There are advantages, however, in determining the Lagrange multiplier after the time discretization [5]. In this approach the three-dimensional equation is solved rather than the usual two-dimensional and $\mu\mathbf{r}$ represents a supplementary force which keeps fluid elements on the surface of the sphere. After discretization, however, the calculation can be carried out in two-dimensional space.

2.8 Cartesian form

It may be advantageous to evaluate the surface derivatives using a Cartesian form. By extending the surface vector $\mathbf{v} = (u, v)^T$ to the three-dimensional $\mathbf{v}_s = (w, v, u)^T$ the shallow water equations can be embedded in the system

$$\frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{S}(\mathbf{v}_s)\mathbf{v}_s + \boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\delta} = 0, \tag{57}$$

where

$$\mathbf{S}(\mathbf{v}_s) = \begin{pmatrix} \frac{\partial w}{\partial r} & \frac{1}{a}(\frac{\partial w}{\partial \theta} - v) & \frac{1}{a\cos\theta}(\frac{\partial w}{\partial \lambda} - u\cos\theta) \\ \frac{\partial v}{\partial r} & \frac{1}{a}(\frac{\partial v}{\partial \theta} + w) & \frac{1}{a\cos\theta}(\frac{\partial v}{\partial \lambda} - u\sin\theta) \\ \frac{\partial u}{\partial r} & \frac{1}{a}\frac{\partial u}{\partial \theta} & \frac{1}{a\cos\theta}(\frac{\partial u}{\partial \lambda} - v\sin\theta + w\cos\theta) \end{pmatrix}, \tag{58}$$

r is the radial coordinate ($r = a$ at the earth's surface) and

$$\boldsymbol{\alpha} = \begin{pmatrix} \frac{u^2+v^2}{a} \\ 0 \\ 0 \end{pmatrix}, \tag{59}$$

$$\boldsymbol{\beta} = \begin{pmatrix} 0 \\ \frac{g}{a}\frac{\partial h}{\partial \theta} \\ \frac{g}{a\cos\theta}\frac{\partial h}{\partial \lambda} \end{pmatrix}, \tag{60}$$

and

$$\boldsymbol{\delta} = \begin{pmatrix} 0 \\ fu \\ -fv \end{pmatrix}. \tag{61}$$

If we define $\mathbf{V} = (X, Y, Z)^T$ as the velocity in Cartesian coordinates (x, y, z) then

$$\mathbf{v}_s = \mathbf{QV} \tag{62}$$

where

$$\mathbf{Q} = \begin{pmatrix} \cos\theta\cos\lambda & \cos\theta\sin\lambda & \sin\theta \\ -\sin\theta\cos\lambda & -\sin\theta\sin\lambda & \cos\theta \\ -\sin\lambda & \cos\lambda & 0 \end{pmatrix}. \tag{63}$$

Substituting (62) into (57) and multiplying by $\mathbf{Q}^T$ we obtain the Cartesian form

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{CV} + \mathbf{Q}^T(\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\delta}) = 0. \tag{64}$$

In this equation

$$\mathbf{C} = \mathbf{Q}^T\mathbf{SQ} = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial X}{\partial z} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial z} \\ \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z} \end{pmatrix}, \tag{65}$$

$$\mathbf{Q}^T\boldsymbol{\alpha} = \frac{1}{a^2}\begin{pmatrix} x(X^2+Y^2+Z^2) \\ y(X^2+Y^2+Z^2) \\ z(X^2+Y^2+Z^2) \end{pmatrix}, \tag{66}$$

$$\mathbf{Q}^T\boldsymbol{\delta} = \frac{2\Omega z}{a^2}\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \tag{67}$$

and

$$\mathbf{Q}^T\boldsymbol{\beta} = \mathbf{P}\nabla_c h \tag{68}$$

where

$$\mathbf{P} = \frac{g}{a^2}\begin{pmatrix} a^2-x^2 & -xy & -xz \\ -xy & a^2-y^2 & -yz \\ -xz & -yz & a^2-z^2 \end{pmatrix}, \tag{69}$$

and

$$\nabla_c h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}\right)^T. \tag{70}$$

Similarly the continuity equation in Cartesian form is

$$\frac{\partial h^*}{\partial t} + \mathbf{V}^T\mathbf{P}\nabla_c h^* + h^*\nabla_c \cdot \mathbf{V} = 0. \tag{71}$$

The matrix P projects an arbitrary Cartesian vector onto a plane that is tangent to the sphere at the point (x, y, z). For methods of evaluating the $\mathbf{C}$ matrix and the Cartesian gradient the reader is referred to [26].

3 Test Cases

The following test cases are proposed to evaluate and compare numerical schemes intended for global atmospheric models. The cases in the series increase in complexity. We suggest the tests be run in order without proceeding to the next until the numerical scheme is reasonably successful on the current one. For some schemes some of the requested parameter settings define trivial tests and realistically provide only a superficial check of the code rather than a useful measure of the quality of the scheme. These situations should be identified so that no false conclusions are drawn. Ideally the full set should be reported for each proposed scheme and trivial cases for that scheme acknowledged.

In the test cases that contain significant energy or enstropy cascade into the model truncation ranges, the addition of an explicit diffusion term may be desirable and may lead to improvement in some of the error measures. In fact, in practical applications in atmospheric modeling such terms are almost always included. Therefore, they should be included in cases 5 through 7 of this suite with the form and coefficients chosen to be appropriate to the scheme being tested.

Case 2 also provides a benchmark for timing implementations on various machines. It exercises the complete set of equations and since it is a steady state solution no extra computations are required during the integration. For timing purposes an integration should be performed with all extra output processes removed after it has been demonstrated that the scheme and codes solve the problem properly.

These tests represent necessary conditions only, i.e. any scheme must do well in these tests compared to currently acceptable schemes. Any scheme that performs well in these tests can then be incorporated in a global baroclinic general circulation model with state-of-the-art physics and definitive tests can be conducted.

Parameters relevant to the earth and all test cases are

$$a = 6.37122 \times 10^6 \text{m} \tag{72}$$

$$\Omega = 7.292 \times 10^{-5} \text{s}^{-1} \tag{73}$$

$$g = 9.80616 \text{ m s}^{-2}. \tag{74}$$

Unless specifically mentioned, the height of the mountains is taken to be zero ($h_s = 0$) and $h^* = h$.

1. Advection of Cosine Bell over the Pole

   This is the only case of the suite that does not deal with the complete shallow water equations. It tests the advective component in isolation. Many shallow water codes can be easily changed for this test by overwriting the predicted wind field every time step with the analytically specified advecting wind. Since this wind field is nondivergent the equation for the height of the free surface reduces to the advection equation. For some methods, semi-implicit for example, some additional changes may be required to isolate the height forecast from the wind forecast.

A cosine bell is advected once around the sphere. Several orientations of the advecting wind are specified including around the equator, directly over the poles and minor shifts from these two orientations to avoid symmetries. This case is specified in eqns. (4.2)-(4.5) of Williamson and Rasch [34]. The advecting wind is given by

$$u = u_0(\cos\theta\cos\alpha + \sin\theta\cos\lambda\sin\alpha) \tag{75}$$

$$v = -u_0\sin\lambda\sin\alpha. \tag{76}$$

In terms of stream function and velocity potential this is

$$\psi = -au_0(\sin\theta\cos\alpha - \cos\lambda\cos\theta\sin\alpha) \tag{77}$$

$$\chi = 0. \tag{78}$$

The parameter α is the angle between the axis of solid body rotation and the polar axis of the spherical coordinate system. Tests should be run with $\alpha = 0.0, 0.05, \pi/2 - 0.05$ and $\pi/2$.

The initial cosine bell test pattern that is to be advected is given by

$$h(\lambda,\theta) = \begin{cases} (h_0/2)(1 + cos\frac{\pi r}{R}) & \text{if } r < R \\ 0 & \text{if } r \geq R \end{cases} \tag{79}$$

where $h_0 = 1000$ m and r is the great circle distance between (λ,θ) and the center, initially taken as $(\lambda_c,\theta_c) = \left(\frac{3\pi}{2}, 0\right)$.

$$r = a\arccos\left[\sin\theta_c\sin\theta + \cos\theta_c\cos\theta\cos(\lambda-\lambda_c)\right]. \tag{80}$$

The radius $R = \frac{a}{3}$ and the advecting wind velocity $u_0 = 2\pi a/(12 \text{ days})$, which is equivalent to about 40 m/sec. This solution translates without any change of shape.

Error measures: Plots of contour lines (interval = 100 m with zero contour) on orthographic projection with perspective centered over the true solution. True solution should also be contoured on the same plot with dashes but without the zero contour. Plot after one rotation. Contour maps of the error should also be provided after one rotation.

Some global measures of the error are also desirable. Define I to be a discrete approximation to the global integral

$$I(h) = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} h(\lambda, \theta) \cos\theta d\theta d\lambda, \tag{81}$$

which is consistent with the numerical approximations being tested; for example, Gaussian quadrature would be selected for the spectral transform method. The following normalized global errors should be graphed as a function of time sampled each time step where h_T is the true solution.

$$\ell_1(h) = I[|h(\lambda,\theta) - h_T(\lambda,\theta)|]/I[|h_T(\lambda,\theta)|] \tag{82}$$

$$\ell_2(h) = \{I[(h(\lambda,\theta) - h_T(\lambda,\theta))^2]\}^{\frac{1}{2}}/\{I[h_T(\lambda,\theta)^2]\}^{\frac{1}{2}} \tag{83}$$

$$\ell_\infty(h) = \max_{\text{all}\lambda,\theta} |h(\lambda,\theta) - h_T(\lambda,\theta)| / \max_{\text{all}\lambda,\theta} |h_T(\lambda,\theta)|. \tag{84}$$

In addition, the normalized mean, variance, minimum and maximum values should be graphed as a function of time sampled each time step. Let $\bar{h}$ denote the mean

$$\bar{h} = I[h(\lambda,\theta)]/4\pi, \tag{85}$$

then the normalized mean and variance are written

$$M = (\bar{h} - \bar{h}_T)/\bar{h}_0 \tag{86}$$

$$V = \{I[(h-\bar{h})^2] - I[(h_T - \bar{h}_T)^2]\}/I[(h_0 - \bar{h}_0)^2] \tag{87}$$

and the minimum and maximum

$$h_{\max} = (\max_{\text{all}\lambda,\theta} h(\lambda,\theta) - \max_{\text{all}\lambda,\theta} h_T(\lambda,\theta))/\Delta h \tag{88}$$

$$h_{\min} = (\min_{\text{all}\lambda,\theta} h(\lambda,\theta) - \min_{\text{all}\lambda,\theta} h_T(\lambda,\theta))/\Delta h \tag{89}$$

where Δh is the difference between the maximum and minimum values of the true solution initially and h_T and h_0 are the true solution and initial field respectively.

2. Global Steady State Nonlinear Zonal Geostrophic Flow

This case is a steady state solution to the non-linear shallow water equations. It consists of solid body rotation or zonal flow with the corresponding geostrophic height field. The Coriolis parameter is a function of latitude and longitude so the flow can be specified with the spherical coordinate poles not necessarily coincident with earth's rotation axis. Again several orientations are specified.

The velocity field from eqns. (4.1)–(4.2) of Williamson and Browning [32] is initially (and for all time)

$$u = u_0(\cos\theta\cos\alpha + \cos\lambda\sin\theta\sin\alpha) \tag{90}$$

$$v = -u_0\sin\lambda\sin\alpha. \tag{91}$$

In this report the angle α has the opposite sign as that in Williamson and Browning [32] but the same sign as that in Williamson and Rasch [34]. In terms of stream function and velocity potential, the velocity field is

$$\psi = -au_0(\sin\theta\cos\alpha - \cos\lambda\cos\theta\sin\alpha) \tag{92}$$

$$\chi = 0. \tag{93}$$

The absolute vorticity is

$$\eta = \left(\frac{2u_0}{a} + 2\Omega\right)(-\cos\lambda\cos\theta\sin\alpha + \sin\theta\cos\alpha). \tag{94}$$

The analytic h field is given by

$$gh = gh_0 - \left(a\Omega u_0 + \frac{u_0^2}{2}\right)(-\cos\lambda\cos\theta\sin\alpha + \sin\theta\cos\alpha)^2. \tag{95}$$

It may be desirable to modify the initial wind and height fields so they satisfy a discrete nonlinear geostrophic relationship consistent with the scheme being tested. This could prevent spurious gravity waves from contaminating the numerical solution. The discrete balance may also be used to define the true solution for the purposes of calculating the error diagnostics.

These changes are allowed but must be reported with the results along with the error comparing the discrete initial state to the analytic. The Coriolis parameter associated with this solution is

$$f = 2\Omega(-\cos\lambda\cos\theta\sin\alpha + \sin\theta\cos\alpha). \tag{96}$$

The parameter values used should be $u_0 = 2\pi a/(12 \text{ days})$ as in case 1 and $gh_0 = 2.94 \times 10^4$ m^2/s^2.

Tests should be run with $\alpha = 0.0, 0.05, \pi/2 - 0.05$, and $\pi/2$.

Error measures: Contour maps of h field and error after five days on a stereographic projection centered over the axis of the flow. Graphs of the ℓ_1, ℓ_2, and ℓ_∞ errors of h and $\mathbf{v}$ versus time. The h errors are computed as in (82) - (84). The $\mathbf{v}$ errors are given by

$$\begin{aligned}
\ell_1(\mathbf{v}) &= I[\{(u(\lambda,\theta) - u_T(\lambda,\theta))^2 + (v(\lambda,\theta) - v_T(\lambda,\theta))^2\}^{\frac{1}{2}}] \\
&/ \quad I[\{u_T(\lambda,\theta)^2 + v_T(\lambda,\theta)^2\}^{\frac{1}{2}}] && (97) \\
\ell_2(\mathbf{v}) &= \{I[(u(\lambda,\theta) - u_T(\lambda,\theta))^2 + (v(\lambda,\theta) - v_T(\lambda,\theta))^2]\}^{\frac{1}{2}} \\
&/ \quad \{I[u_T(\lambda,\theta)^2 + v_T(\lambda,\theta)^2]\}^{\frac{1}{2}} && (98) \\
\ell_\infty(\mathbf{v}) &= \max_{\text{all}\lambda,\theta}[\{(u(\lambda,\theta) - u_T(\lambda,\theta))^2 + (v(\lambda,\theta) - v_T(\lambda,\theta))^2\}^{\frac{1}{2}}] \\
&/ \quad \max_{\text{all}\lambda,\theta}[\{u_T(\lambda,\theta)^2 + v_T(\lambda,\theta)^2\}^{\frac{1}{2}}] && (99)
\end{aligned}$$

In addition to these graphs a mesh convergence study should be performed. The $\ell_2(h)$ and $\ell_2(\mathbf{v})$ errors at five days for three different resolutions should be shown and a rate of convergence for the method estimated.

3. Steady State Nonlinear Zonal Geostrophic Flow with Compact Support

This case is similar to the previous except the wind field is nonzero in a limited region. It was introduced by Browning et al. [3]. In the editorial process for that paper some terms were dropped from the last equation in the first column on page 1068. It should read

$$\begin{aligned}
\tilde{v} &= u(\cos\alpha\cos\tilde{\lambda}\sin\lambda\sin\tilde{\theta} \\
&- \cos\lambda\sin\tilde{\lambda}\sin\tilde{\theta} + \sin\alpha\sin\lambda\cos\tilde{\theta}). && (100)
\end{aligned}$$

This case is easiest to write first in a coordinate system (λ', θ') whose poles are coincident with the Earth's rotation axis, followed by a rotation through an angle α to the system (λ, θ) in which the jet is not parallel to the coordinate lines. This is essentially the process used to derive the equations above for solid body rotation, however, in the case with compact support it is more difficult to write the equations in closed form in the (λ, θ) system. Therefore, we present the equations in a series of steps. The velocities components (u', v') in the (λ', θ') system are given by

$$u' = u_0 b(x) b(x_e - x) e^{4/x_e} \tag{101}$$

$$v' = 0 \tag{102}$$

where

$$b(x) = \begin{cases} 0, & x \leq 0 \\ e^{-x^{-1}}, & 0 < x \end{cases}$$

and

$$x = x_e(\theta' - \theta_b)(\theta_e - \theta_b)^{-1}. \tag{103}$$

The parameters are $u_0 = 2\pi a/(12 \text{ days})$, $\theta_b = -\pi/6$, $\theta_e = \pi/2$ and $x_e = 0.3$. Note that u' is infinitely differentiable and has compact support. The stream function and velocity potential are given by

$$\psi'(x) = -a \frac{\theta_e - \theta_b}{x_e} u_0 e^{4/x_e} \int_{x_e}^{x} e^{-\frac{x_e}{x'(x_e - x')}} \, dx', \tag{104}$$

and

$$\chi' = 0. \tag{105}$$

Following (5.13) and (5.15) of [34] and equations for u and v derived in a similar manner to (5.16) and (5.17) of [34] (with $\lambda_A = 0$ and $\theta_A = \alpha$) the rotated form can be written

$$v \cos\theta = -u' \sin\alpha \sin\lambda' \tag{106}$$

$$u \cos\lambda = v \sin\theta \sin\lambda + u' \cos\lambda' \tag{107}$$

with the coordinates related by

$$\sin\theta' = \sin\theta \cos\alpha - \cos\theta \cos\lambda \sin\alpha \tag{108}$$

$$\sin\lambda' \cos\theta' = \sin\lambda \cos\theta. \tag{109}$$

The quadrant in which λ' falls can be determined by insuring that

$$\sin\theta = \sin\theta' \cos\alpha + \cos\theta' \sin\alpha \cos\lambda' \tag{110}$$

is also satisfied. Equation (110) may suffer from precision problems because of the nesting of trigonometric and inverse trigonometric functions. A more stable test is that the principal value (λ'_p) is used for λ' when

$$\cos\alpha\cos\lambda\cos\theta + \sin\alpha\sin\theta \geq 0 \tag{111}$$

otherwise $\lambda' = \pi - \lambda'_p$. This relationship can be obtained by transforming to Cartesian coordinates, rotating the Cartesian coordinates and noting that the principal value is needed in the primed system when $x' \geq 0$. (The x and z coordinates are chosen to go through $(\lambda, \theta) = (0, 0)$ and $(0, \frac{\pi}{2})$, respectively, and the y coordinate can be ignored.) The Coriolis parameter in the two systems is

$$f = 2\Omega\sin\theta' \tag{112}$$

$$f = 2\Omega(-\cos\lambda\cos\theta\sin\alpha + \sin\theta\cos\alpha). \tag{113}$$

For a steady state solution h' must satisfy

$$\frac{(u')^2\tan\theta'}{a} + \frac{g}{a}\frac{\partial h'}{\partial\theta'} + fu' = 0. \tag{114}$$

For the general case the height is difficult to obtain analytically. Therefore, we integrate the form in the prime system

$$h = h_0 - \frac{a}{g}\int_{-\frac{\pi}{2}}^{\theta'}\left(2\Omega\sin\tau + \frac{u'(\tau)\tan\tau}{a}\right)u'(\tau)d\tau \tag{115}$$

numerically to obtain a highly accurate h. The background height, h_0, is given by $gh_0 = 2.94 \times 10^4\ \mathrm{m}^2/s^2$ as in Case 2 and the limit θ' is related to (λ, θ), the point at which the geopotential is desired, by (108).

Tests should be run with $\alpha = 0.0$, and $\pi/3$.

Error measures: Contour maps of field and error after five days on an orthographic projection centered on $(3\pi/2, \pi/4)$. Graphs of the ℓ_1, ℓ_2 and ℓ_∞ errors of h and $\mathbf{v}$ as functions of time. In addition to these graphs a mesh convergence study should be performed. The $\ell_2(h)$ and $\ell_2(\mathbf{v})$ errors at five days for three different resolutions should be shown for the $\alpha = \frac{\pi}{3}$ case and a rate of convergence for the method estimated.

4. Forced Nonlinear System with a Translating Low

The nonlinear steady state tests presented in the previous sections are the simplest measure of the adequacy of a particular numerical method. The performance of a scheme on the nonlinear unsteady equations is also desirable, but analytic solutions are all but nonexistent. Thus, we take the approach followed by Browning et al. [3] who choose a flow $\tilde{u}, \tilde{v}$, and $\tilde{h}$ that is similar in structure to flows observed in the atmosphere. This flow is a solution to the forced shallow water system which can be written in advective form as

$$\frac{du}{dt} - \frac{uv\tan\theta}{a} + \frac{g}{a\cos\theta}\frac{\partial h}{\partial\lambda} - fv = F_u, \tag{116}$$

$$\frac{dv}{dt} + \frac{uu\tan\theta}{a} + \frac{g}{a}\frac{\partial h}{\partial\theta} + fu = F_v, \tag{117}$$

$$\frac{dh}{dt} + \frac{h}{a\cos\theta}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial v\cos\theta}{\partial\theta}\right] = F_h, \tag{118}$$

where the height of the mountains h_s is taken to be zero and the substantial derivative is defined as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{a\cos\theta}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\theta}, \tag{119}$$

and the forcing terms are defined as

$$F_u = \frac{d\tilde{u}}{dt} - \frac{\tilde{u}\tilde{v}\tan\theta}{a} + \frac{g}{a\cos\theta}\frac{\partial\tilde{h}}{\partial\lambda} - f\tilde{v}, \tag{120}$$

$$F_v = \frac{d\tilde{v}}{dt} + \frac{\tilde{u}\tilde{u}\tan\theta}{a} + \frac{g}{a}\frac{\partial\tilde{h}}{\partial\theta} + f\tilde{u}, \tag{121}$$

$$F_h = \frac{d\tilde{h}}{dt} + \frac{\tilde{h}}{a\cos\theta}\left[\frac{\partial\tilde{u}}{\partial\lambda} + \frac{\partial\tilde{v}\cos\theta}{\partial\theta}\right]. \tag{122}$$

The flow is given by

$$\tilde{u} = \bar{u} - \frac{\bar{\psi}_\theta}{a} \tag{123}$$

$$\tilde{v} = \frac{\bar{\psi}_\lambda}{a\cos\theta} \tag{124}$$

$$g\tilde{h} = g\bar{h} + f\bar{\psi}, \tag{125}$$

where

$$\bar{u} = u_0 \sin^{14}(2\theta), \tag{126}$$

$$g\bar{h} = gh_0 - \int_{-\frac{\pi}{2}}^{\theta} [af(\tau) + \bar{u}(\tau)\tan\tau]\bar{u}(\tau)d\tau, \tag{127}$$

$$\bar{\psi}(\lambda, \theta, t) = \psi_0 e^{-\sigma \frac{1-C}{1+C}} \tag{128}$$

with $\psi_0 = -0.03\frac{gh_0}{f_0}$, $\sigma = (12.74244)^2$, $gh_0 = 10^5 m^2/s^2$, $f_0 = 2\Omega \sin\frac{\pi}{4}$ and

$$C = \sin\theta_0 \sin\theta + \cos\theta_0 \cos\theta \cos(\lambda - \frac{u_0}{a}t - \lambda_0). \tag{129}$$

The center of the low is initially located at $(\lambda_0, \theta_0) = (0, \frac{\pi}{4})$. The velocity potential, χ, is zero while the stream function is given by

$$\psi(\lambda, \theta, t) = -\int_{-\frac{\pi}{2}}^{\theta} a\bar{u}(\tau)d\tau + \bar{\psi}(\lambda, \theta, t). \tag{130}$$

The flow is a translating low pressure center superimposed on a jet stream which is symmetrical about the equator. Figure 5 of [3] illustrates the initial height field. This field exhibits some of the properties of middle level tropospheric flow (i.e., a short-wave trough embedded in a westerly jet).

The analytic expressions for the forcing are presented above for momentum. Schemes predicting other variables such as vorticity/divergence or stream function/velocity potential must be able to accept the forcing in terms of momentum as that is what is provided from the parameterizations in atmospheric models (see for example [33].) Thus solutions should be provided using the momentum forcing as prescribed. However, for the purpose of comparison with other schemes it may be advantageous to specify the forcing analytically in terms of the predicted variables if other than momentum. This approach is also allowed for these tests, but if it is chosen, then results with momentum forcing should also be presented.

Tests should be run with $u_0 = 20$ and 40 m/s.

Error measures: Contour maps of solution and error after day 5 on an orthographic projection centered on $(\lambda_c, \pi/4)$, (λ_c is the longitude of the center of the cell). The $\ell_1, \ell_2, \ell_\infty$ errors of

h' and $\mathbf{v}'$ should be plotted as a function of time. Here h' and $\mathbf{v}'$ are the perturbation fields obtained by subtracting the background zonal flow

$$h' = h - \bar{h} \tag{131}$$

$$u' = u - \bar{u} \tag{132}$$

$$v' = v \tag{133}$$

where $\bar{u}$ and $\bar{h}$ are given by (126) and (127) respectively. The true solution is modified in the same way for the error calculation. The mean zonal component is removed so that the error primarily represents that associated with the cell. The graphs should include data sampled every time step so that any oscillatory behavior can be seen.

5. Zonal Flow Over an Isolated Mountain

This case was used by Takacs to study the effect of *a posteriori* methods for conservation of integral invariants [27]. It consists of zonal flow as in case 2 impinging on a mountain. The wind and height field are as in case 2, with $\alpha = 0$, but the mean height is changed to $h_0 = 5400$ m. The surface or mountain height is given by

$$h_s = h_{s_0}(1 - r/R) \tag{134}$$

where $h_{s_0} = 2000$ m, $R = \pi/9$ and $r^2 = \min[R^2, (\lambda - \lambda_c)^2 + (\theta - \theta_c)^2]$.

The center is taken as $\lambda_c = 3\pi/2$ and $\theta_c = \pi/6$. As no analytical solution is known, a reference solution will be provided by a high resolution spectral transform model integration. This will be provided as spectral coefficients at 5-day intervals and a routine to generate point values at arbitrary points. Agreement must be found with at least one other high resolution solution provided by a different numerical scheme in order to have confidence in the error measures. As mentioned earlier, an explicit diffusion should be added to the equations to maintain a realistic kinetic energy spectrum. Details of the coefficients and form chosen should be presented.

Error measures: Contour maps on a rectangular latitude/longitude projection ($\Delta\lambda/\Delta x = \Delta\theta/\Delta y$) of the h field and error at days 5, 10 and 15. Graphs of the ℓ_1, ℓ_2 and ℓ_∞ errors of h and $\mathbf{v}$ calculated vs. the high resolution solution plotted as a function of time sampled daily.

Various normalized global invariants of the continuous equations should also be plotted as a function of time. Define the normalized integral

$$I_i(\xi(t)) = \{I[\xi(\lambda, \theta, t)] - I[\xi(\lambda, \theta, 0)]\}/I[\xi(\lambda, \theta, 0)] \tag{135}$$

where the discrete integral operator I is defined as (81). The following invariants should be presented:

mass (i=1)

$$\xi = h^* \tag{136}$$

total energy (i=2)

$$\xi = \frac{1}{2}h^*\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}(h^2 - h_s^2) \tag{137}$$

potential enstrophy (i=3)

$$\xi = 0.5(\zeta + f)^2/h^* \tag{138}$$

The unnormalized integrals of vorticity and divergence should be presented since their initial values are zero.

vorticity (i=4)

$$\xi = \zeta \tag{139}$$

divergence (i=5)

$$\xi = \delta \tag{140}$$

6. Rossby-Haurwitz Wave

Rossby-Haurwitz waves are analytic solutions of the nonlinear barotropic vorticity equation on the sphere [10]. Although they are not analytic solutions of the shallow water equations they have been used so frequently for meteorological tests that since Phillips' [21] first tests they have become *de facto* standard test cases although generally with different parameters selected by each investigator.

The initial velocity field is nondivergent and given by the stream function,

$$\psi = -a^2\omega \sin\theta + a^2 K \cos^R\theta \sin\theta \cos R\lambda, \tag{141}$$

where ω, K and R are constants. Haurwitz [10] showed that this pattern moves from west to east without change of shape in a nondivergent barotropic model with angular velocity ν given by

$$\nu = \frac{R(3+R)\omega - 2\Omega}{(1+R)(2+R)}. \tag{142}$$

The velocity components are given by

$$u = a\omega\cos\theta + aK\cos^{R-1}\theta(R\sin^2\theta - \cos^2\theta)\cos R\lambda \tag{143}$$

$$v = -aKR\cos^{R-1}\theta\sin\theta\sin R\lambda \tag{144}$$

and the vorticity by

$$\zeta = 2\omega\sin\theta - K\sin\theta\cos^R\theta(R^2+3R+2)\cos R\lambda. \tag{145}$$

The height is obtained from the stream function by solving the balance equation so the initial tendency of the divergence is zero [21].

$$gh = gh_0 + a^2A(\theta) + a^2B(\theta)\cos R\lambda + a^2C(\theta)\cos 2R\lambda \tag{146}$$

$$A(\theta) = \frac{\omega}{2}(2\Omega+\omega)\cos^2\theta + \frac{1}{4}K^2\cos^{2R}\theta[(R+1)\cos^2\theta + (2R^2-R-2) - 2R^2\cos^{-2}\theta] \tag{147}$$

$$B(\theta) = \frac{2(\Omega+\omega)K}{(R+1)(R+2)}\cos^R\theta[(R^2+2R+2) - (R+1)^2\cos^2\theta] \tag{148}$$

$$C(\theta) = \frac{1}{4}K^2\cos^{2R}\theta[(R+1)\cos^2\theta - (R+2)]. \tag{149}$$

In the past the qualitative aspects of the solutions have generally been examined. To compliment the qualitative aspects we provide a reference solution from a high resolution spectral transform model integration. This will be provided as daily spectral coefficients and a routine to generate point values at an arbitrary point. The parameters are $\omega = K = 7.848 \times 10^{-6}\mathrm{s}^{-1}$ and $h_0 = 8 \times 10^3$ m. Only a wave number $R = 4$ is chosen for the initial condition. Unstable waves [12] are not chosen since slightly different perturbations may lead to growth of different unstable modes as might be indicated in Kreiss and Oliger [14].

Error measures: Contour maps on a rectangular latitude/longitude projection ($\Delta\lambda/\Delta x = \Delta\theta/\Delta y$) of the h field and error at day 0, 7 and 14. The $\ell_1, \ell_2, \ell_\infty$ errors of h and $\mathbf{v}$ calculated vs. the high resolution solution plotted as a function of time sampled daily. The five global invariants (Eqs. 136–140) listed with the flow over an isolated mountain (Case 5) should also be graphed as a function of time.

7. Analyzed 500 mb Height and Wind Field Initial Conditions

The last case consists of atmospheric initial conditions of the 500 mb height and winds from several atmospheric states. The first is for 0000 GMT 21 December 1978, which Ritchie [23] used to test his semi-Lagrangian scheme. This case, with strong flow over the North Pole, has pointed out shortcomings of schemes in the past. A second case is 0000 GMT 16 January 1979. This case is characterized initially by two cut-off lows. The flow pattern develops into a typical blocking situation. It has been studied extensively by Bengtsson [2]. The third case is 0000 GMT 9 January 1979, which initially has strong zonal flow. The last two cases are from the FGGE case studies selected by WGNE and discussed by Baumhefner and Bettge [1]. The shallow water equations should not necessarily be expected to predict the atmosphere well in these cases. The variety is chosen to illustrate any variability in the characteristics of schemes depending on atmospheric state.

The initial data are truncated to T63 spectral resolution, which includes all scales available in the analyses as provided in Trenberth and Olson [29]. Ideally, nonlinear normal mode initialization consistent with the scheme being tested should be applied to the initial data to prevent gravity waves from contaminating the solution. The changes made by the initialization scheme should be submitted along with the error summary. However, because of the extra work necessary to develop the initialization codes, an initialized data set is also provided which is obtained *via* nonlinear normal mode initialization with a high resolution spectral transform model. Although it may be advantageous to use an initialization procedure consistent with the scheme being tested, the choice is left to the scheme's proponents. As mentioned earlier, an explicit diffusion should be added to the equations to maintain a realistic kinetic energy spectrum. Details of the coefficients and form chosen should be presented.

Error measures: The 'true' or reference solution will be obtained initially with the spectral transformation method applied to the finest resolution possible. Agreement must be found between at least two different schemes at high resolution to have confidence. The reference solution will be provided in terms of spherical harmonic coefficients so that it can be reconstituted on any computational grid. The ℓ_1, ℓ_2 and ℓ_∞ errors of h and $\mathbf{v}$ should be plotted daily from 5-day forecasts. In addition, plots on north and south polar stereographic projections of the forecast and forecast error should be provided for day 1 and day 5. The five global invariants (Eqs. 136–140) listed with the flow over an isolated mountain (Case 5) should also be graphed as a function of time. A graph of the height field at every time step at the grid point closest to 40N and 105W should be provided to indicate any temporal noise or residual gravity waves in the forecasts.

4 Performance Benchmark

To exhibit the performance of a numerical scheme on a given computer system, the computer CPU time and storage requirements for a 5-day run of case 2 with $\alpha = \pi/4$ (to avoid most symmetries) should be reported for various resolutions. Results from Case 2 need only be presented for schemes whose computational characteristics are independent of the solution. For other methods such as adaptive or iterative ones the results should be presented for all tests. The number of time steps taken and the errors in h and $\mathbf{v}$ at 5 days, as in (82)-(84) and (97)-(99) should be given for each resolution. Any time step restrictions or special cases should be recorded so that the computational effort corresponding to a climate simulation can be judged. Enough data should be provided so that comparisons can be made between schemes based on the computational resources required to achieve a given level of accuracy. These should include the total CPU time required, the number of operations required for the calculation, a measure of the sustained computational rate in gigaflops, and the data space storage required for each resolution. The machine, compiler and precision used should also be documented.

For parallel computers the wall clock time, as measured on the host computer, should be reported as well as the maximum time spent on any one processor. The maximum size of the data space required on any processor should also be reported. Execution times for a given resolution with the use of increasing numbers of processors should be given to indicate how the algorithm scales . The speedup and parallel efficiency for a each resolution should be given as a function of the number of processors. The parallel speedup is defined as $S_p = T_1/T_p$, where T_1 is the time required to execute the sequential algorithm on a single processor and T_p is the execution time for the parallel algorithm using p processors. The parallel efficiency is given by $E_p = S_p/p$. These measures may require an approximation of T_1 due to memory constraints in the single processor case. The method and assumptions used to approximate T_1 should be clearly stated. No output or unnecessary computation should be performed during the 5-day simulation.

5 General Comments

Ideally, all contouring should be via linear interpolation on the original computational grids without smoothing or additional interpolation to an intermediate grid in order to provide an indication of any noise in the solution. The utility of the various tests included in this suite will become apparent as more investigators apply their schemes to them. We hope investigators will use all the tests and publish in refereed journals selected results that illustrate both the strengths and weaknesses of the schemes. In-house technical reports containing the results from all the tests could provide the complete documentation of a scheme. We expect the suite will evolve informally with time as investigators point out weaknesses in the tests and suggest alternatives with arguments as to why they are good test cases. Several other cases are currently under consideration for inclusion. These consist of Thompson's nonlinear series solution to the equations [28] and modons in spherical geometry [30]. The latter do not have an analytical solution for the shallow water equations and a high resolution numerical solution will be required for a reference solution. J. Côté (personal communication) is developing a test case following the recent studies of inertial motion on the sphere [18] , [20]. This case will complement the pure advection Case 1 and deal only with the momentum equations.

The test suite will only become standard to the extent the community finds it useful. This suite is fairly large but contains a variety of test cases and error measures. This variety is needed in order to provide as much information as possible to would-be users so they can evaluate the importance of the various tradeoffs required in their applications.

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