Eddy - Zonal Flow Interaction
and the Eliassen-Palm Flux

References:
- Andrews, D. G., J. R. Holton, and C. B. Leovy, 1987: Middle Atmosphere Dynamics, pp. 123-133.
- James, I., 1994: Introduction to Circulating Atmospheres, pp. 100-107.

We start with the quasi-geostrophic equations in Cartesian coordinates on a mid-latitude beta - plane, using z=H *ln*(p_0/p) as a vertical coordinate:

$$\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + v_g \frac{\partial u_g}{\partial y} - fv = X - \frac{\partial \Phi}{\partial x} \qquad (1)$$

$$\frac{\partial v_g}{\partial t} + u_g \frac{\partial v_g}{\partial x} + v_g \frac{\partial v_g}{\partial y} + fu = Y - \frac{\partial \Phi}{\partial y} \qquad (2)$$

$$\frac{\partial \theta}{\partial t} + u_g \frac{\partial \theta}{\partial x} + v_g \frac{\partial \theta}{\partial y} + w \frac{d\hat{\theta}}{dz} = \tilde{Q} \qquad (3)$$

$$\tilde{Q} = \frac{Q}{C_p} \left(\frac{p_0}{p} \right)^{\kappa}$$

$$f = f_0 + \beta y$$

where X and Y refer to components of friction, and θ the deviation of potential temperature about its horizontal time mean $\hat{\theta}$

Here Q is thermodynamic heating, and the "hat" denotes a time-invariant horizontal mean (basic state). (X,Y) is friction. With some work, it is possible to write the equations for the zonal mean potential temperature and zonal wind as:

$$\frac{\partial[u_g]}{\partial t} = -\frac{\partial}{\partial y}[u_g^* v_g^*] + f[v] + [X] \qquad (4)$$

$$\frac{\partial[\theta]}{\partial t} = -\frac{\partial}{\partial y}[v_g^* \theta^*] - [w]\frac{d\hat{\theta}}{dz} + [\tilde{Q}] \qquad (5)$$

The thermal wind equation becomes:

$$f_0 \frac{\partial[u_g]}{\partial z} = -\frac{R}{H}\left(\frac{p}{p_0}\right)^{\kappa} \frac{\partial[\theta]}{\partial y} \equiv -S_\theta \frac{\partial[\theta]}{\partial y} \qquad (6)$$

Thus taking f_0 ∂\∂z of equation (4), and S_θ ∂/∂y of equation (5), and adding them, the tendency terms on the left hand side cancel. By putting the terms involving [w] and [v] on the left hand side, we can rewrite the resulting equation as:

$$-f^2\frac{\partial[v]}{\partial z}+S_\theta\frac{d\hat{\theta}}{dz}\frac{\partial[w]}{\partial y}=$$

$$-f\frac{\partial^2}{\partial z\partial y}[u_g^* v_g^*]-S_\theta\frac{\partial^2}{\partial y^2}[v_g^*\theta^*]+f_0\frac{\partial[X]}{\partial z}+S_\theta\frac{\partial}{\partial y}[\tilde{Q}] \qquad (7)$$

where derivatives of the mean meridional circulation on the left hand side are expressed in terms of the eddy flux convergences, heating and friction on the right hand side.

The physical meaning of this is that the eddy convergences (and also friction and heating) change the zonal mean u and θ, but in order to maintain thermal wind balance, a mean meridional circulation must be set up.
So the eddies will not only change the zonal flow directly, but they will also induce a mean meridional circulation which alters the zonal flow such that thermal wind balance is maintained.

Question: What if the mean meridional circulation changes the zonal mean flow in a way such as to cancel the effect of the eddy convergences?

To formally set up this problem, we use the continuity equation to set up a mean meridional stream function, as we have done before:

$$\frac{\partial[v]}{\partial y}+\rho_0^{-1}\frac{\partial}{\partial z}(\rho_0[w])=0 \tag{8}$$

$$[v]=-\rho_0^{-1}\frac{\partial}{\partial z}(\rho_0\Psi) \tag{9}$$

$$[w]=\frac{\partial}{\partial y}\Psi \tag{10}$$

where ρ_0 is defined as:

$$\rho_0=\rho_s e^{-z/H}$$

with ρ_s a constant. The sign of the streamfunction Ψ has been chosen so that a direct thermal circulation (rising in low latitudes, sinking in high latitudes) corresponds to a positive maximum in Ψ in the NH.

Using equations 9 and 10 in equation (7), we obtain:

$$f^2\frac{\partial}{\partial z}\left(\rho_0^{-1}\frac{\partial}{\partial z}(\rho_0\Psi)\right)+\left(S_\theta\frac{d\hat{\theta}}{dz}\right)\frac{\partial^2}{\partial y^2}\Psi=S \tag{11}$$

$$S=-f_0\frac{\partial^2}{\partial z\partial y}[u_g^*v_g^*]-S_\theta\frac{\partial^2}{\partial y^2}[v_g^*\theta^*]+f_0\frac{\partial[F]}{\partial z}+S_\theta\frac{\partial}{\partial y}[\tilde{Q}]$$

This is an elliptical equation in the meridional/height plane of the general form:

$$\nabla^2 \Psi = S$$

This can be solved if suitable boundary conditions on Ψ are specified. We will apply [v] = 0 at bounding values of y (i.e. the ”equator” and “pole”), and [w] = 0 at the bottom and top of the atmosphere. This gives Ψ = constant (which we can take to be 0) along the meridional and vertical boundaries.

A well-known property of such equations is that a ***maximum*** ***(minimum)*** in the source term S is associated with a ***minimum*** ***(maximum)*** in Ψ hence an ***indirect*** ***(direct)*** circulation. (Remember that in one dimension, the second derivative S of a function Ψ which has a minimum at a point is not only positive but has a local maximum at that point.)

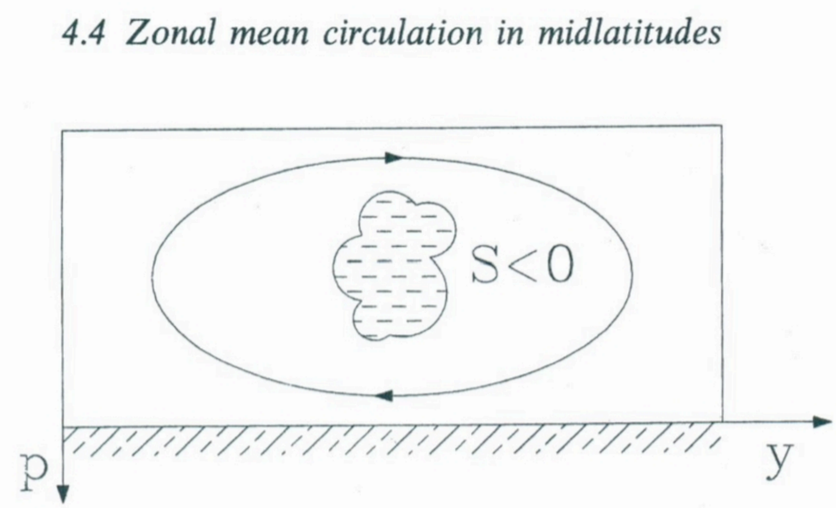

Fig. 4.12. Schematic illustration of the solution of Eq. (4.31).

Thermodynamic heating:

Consider a simple application of equation (11). Take the heating to be positive at lower latitudes and negative at higher latitudes, so that the source term $\partial[Q]/\partial y$ is negative for the most part. Since a negative source S will give a *direct circulation*, we will have rising air at lower latitudes and sinking air at higher latitudes.

An important point to remember is that we are using quasi-geostrophic equations - this argument can not describe the Hadley cell in the tropics.

At low latitudes, the adiabatic cooling of the rising air tends to cool the air, while at higher latitudes the adiabatic warming of the sinking air tends to warm the air, so that in both cases the meridional circulation tends to offset the temperature changes due to heating. With Q positive (negative) at lower (higher) latitudes, the actual T gradient changes by less than we might expect if the atmosphere were unable to circulate.

Role of the Coriolis Force:

The Coriolis force acts on the poleward moving air at upper levels to give a westerly acceleration, while at low levels an easterly acceleration occurs - thus the wind shear is increased to maintain thermal wind balance with the increased meridional temperature gradient.

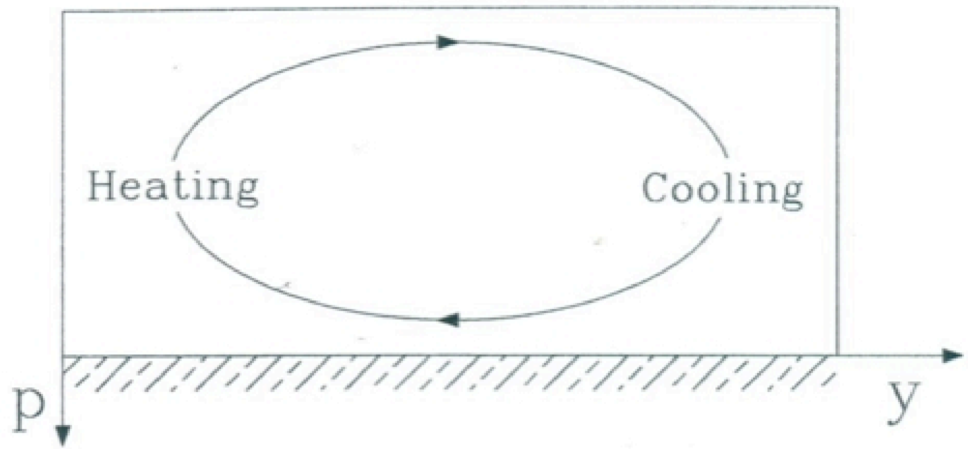

Fig. 4.13. The meridional circulation generated in response to a gradient of heating.

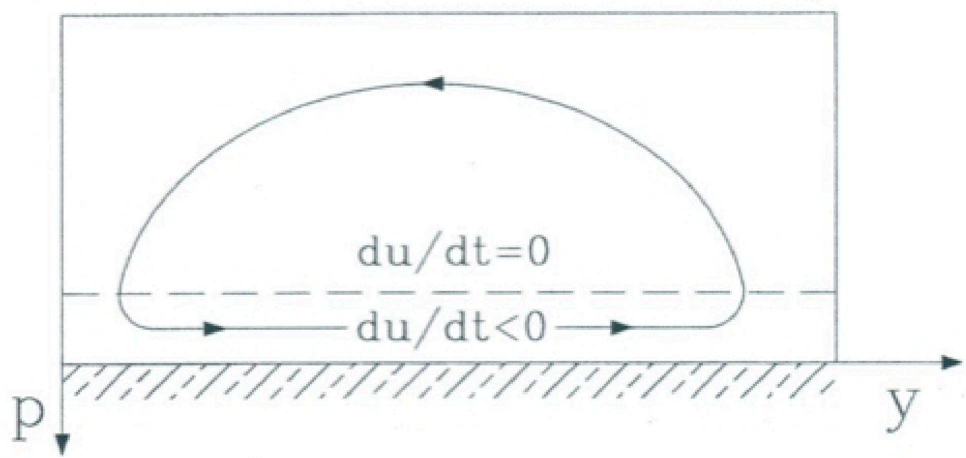

Fig. 4.14. The meridional circulation induced by surface friction.

Effect of Friction:

In mid-latitudes, surface winds are observed to be westerly. We expect therefore that surface friction will decelerate the westerlies, so that $F < 0$. Since surface friction is generally strong compared to friction further up, we have a positive source term $S = \partial F/\partial z$. Thus an *indirect* circulation will be forced (see picture on previous page). The Coriolis force acting on the low-level poleward flow gives westerly acceleration, thereby compensating for the effect of friction.

Also, the descending air causes adiabatic warming in the tropics, and the adiabatic ascent causes cooling at high latitudes, so that the meridional temperature gradient increases, keeping the atmosphere in thermal wind balance with the increased wind shear.

Transient Eddy heat flux:

We have found that the transients eddy heat flux [v*T*] is largest at low levels in mid-latitudes. At the latitude of maximum eddy heat flux, the second derivative is negative. Since the source term in equation 11 is proportional to $- \partial^2[v^*T^*]/\partial y$, the heat flux is associated with a *positive* source term. Hence it will induce an *indirect circulation.*

Again, while the eddies try to transport heat poleward, this effect is partially offset by the induced mean meridional circulation, which leads to adiabatic wamring (due to descent) at low latitudes, and adiabatic cooling due to ascent at high latitudes.

The lower level westerlies are accelerated (upper level westerlies are decelerated) by the Coriolis force. Thus the vertical wind shear decreases, in order to keep the circulation in thermal wind balance with the reduction in meridional temperature gradient caused by the heat flux.

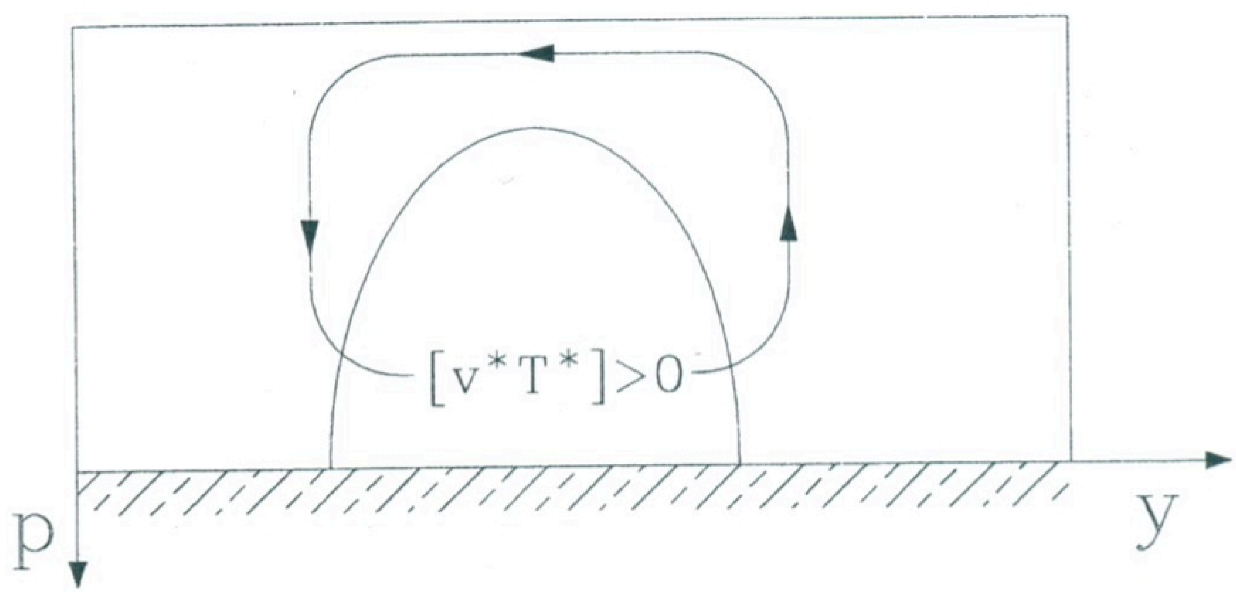

Fig. 4.15. The meridional circulation induced by poleward eddy temperature fluxes.

Transient Eddy Momentum Fluxes

The transient eddy momentum flux increases with z, reaching a maximum near the tropopause. At this level, there is a general convergence towards latitudes near 50N, with poleward flux to the south and equatorward flux further north. (In other words, a convergence of flux near 50N). Since the source term here is essentially the vertical derivative of the momentum flux convergence, it is positive for most of the troposphere.

As with the transient eddy heat flux, this leads to an indirect circulation. The Coriolis force acting on the low level poleward flow and upper level equatorward flow again tends to reduce the vertical shear, or to make the flow more barotropic. This can be shown to be the active process during the occlusion of cyclones.

A Better Summary of Eddy - Zonal Flow Interactions

The eddy fluxes induce a mean meridional circulation [v], [w] that partially cancels the effects of the eddy fluxes. Is it possible to re-write the zonal mean flow / eddy equations in a way which reflects the true (total) effects of the eddy fluxes?

A considerable body of theory exists which addresses this question. For quasi-geostrophic dynamics, we start by defining a new meridional circulation, replacing [v] and [w] in equations (4) and (5) by $\left[\tilde{v}\right]$ and $\left[\tilde{w}\right]$:

$$[\tilde{v}] \equiv [v] - \rho_0^{-1} \frac{\partial}{\partial z}\left(\rho_0 [v^* \theta^*] / \frac{d\hat{\theta}}{dz}\right) \tag{13}$$

$$[\tilde{w}] \equiv [w] + \frac{\partial}{\partial y}\left([v^* \theta^*] / \frac{d\hat{\theta}}{dz}\right) \tag{14}$$

Using these definitions in equations (4) and (5):

$$\frac{\partial [u_g]}{\partial t} - f[\tilde{v}] - [X] = \rho_0^{-1} \vec{\nabla} \cdot \vec{F} \tag{15}$$

$$\frac{\partial [\theta]}{\partial t} + [\tilde{w}] \frac{d\hat{\theta}}{dz} - [\tilde{Q}] = 0 \tag{16}$$

where F is a vector in the meridional (y,z) plane called the Eliassen-Palm flux vector:

$$\vec{F} = (F^{(y)}, F^{(z)}) = \left(\rho_0 [u^* v^*], \rho_0 [v^* \theta^*] / \frac{d\hat{\theta}}{dz}\right) \tag{17}$$

and the divergence is just given by:

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial y} F^{(y)} + \frac{\partial}{\partial z} F^{(z)}$$

The continuity equation is not changed in form:

$$\frac{\partial}{\partial y}[\tilde{v}] + \rho_0^{-1}\frac{\partial}{\partial z}(\rho_0[\tilde{w}]) = 0 \tag{18}$$

Note that in equations (15) and (16) the eddy heat and momentum fluxes do not enter separately - but only in the combination given by F.

It can be proven that for steady, small amplitude, eddies in adiabatic flow, that the mean flow is not affected by the eddies, so that

$$\frac{\partial}{\partial t}[u_g] = \frac{\partial}{\partial t}[\theta] = \vec{\nabla}\cdot\vec{F} = 0 \tag{19}$$

This is called a "non-acceleration theorem."

Note that adiabatic here means no friction or heating. The meaning of *steady waves* we will explain. (Note that steady waves are NOT the same as stationary waves)

A generalization of these statements holds for the primitive equations, although the definition of the Eliassen-Palm flux F changes.

To demonstrate the non-acceleration theorem in the quasi-geostrophic system, we first rewrite equations (1)-(3) in potential vorticity form:

$$\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)q= -\frac{\partial X}{\partial y}+\frac{\partial Y}{\partial x}+f_0\rho_0^{-1}\left(\rho_0\tilde{Q}/\frac{d\hat{\theta}}{dz}\right) \qquad (20)$$

where the quasi-geostrophic potential vorticity *q* is given by:

$$q=\zeta_g+f_0\rho^{-1}\frac{\partial}{\partial z}\left(\rho_0\theta/\frac{d\hat{\theta}}{dz}\right) \qquad (21)$$

In the adiabatic case (no friction and heating), we can write equation (20) as:

$$\left(\frac{\partial}{\partial t}+v_g^*\frac{\partial}{\partial y}\right)[q]+\frac{d}{dt}q^*+\left(u_g^*\frac{\partial q^*}{\partial x}+v_g^*\frac{\partial q^*}{\partial y}\right)=0 \qquad (22)$$

$$\left(\frac{\partial}{\partial t}+v_g^*\frac{\partial}{\partial y}\right)[q]+\frac{d}{dt}q^*+\frac{\partial}{\partial x}\left(u_g^*q^*\right)+\frac{\partial}{\partial y}\left(v_g^*q^*\right)=0 \qquad (23)$$

where here the linear operator d/dt is defined as:

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t} + [u_g] \frac{\partial}{\partial x} \right)$$

The zonal mean of equation (23) can be written as:

$$\frac{\partial}{\partial t}[q] = -\frac{\partial}{\partial y}\left[v_g^* q^*\right] \tag{24}$$

but it can be shown that:

$$\left[v_g^* q^*\right] = \rho_0^{-1}\vec{\nabla}\cdot\vec{F} \tag{25}$$

so that the vanishing of the divergence of the "E-P" flux leads to the vanishing of the eddy forcing and non-acceleration conditions.

Related to the EP-flux, we can derive a conservation law for the wave (eddy) disturbance. To see this we start with the tendency equation for the eddy potential vorticity, which we obtain by subtracting equation (24) from equation (23):

$$\frac{\partial}{\partial t}q^* + [u_g]\frac{\partial}{\partial x}q^* + v_g^*\frac{\partial}{\partial y}[q] = 0 \tag{26}$$

where we have assumed small amplitude eddies, and so have neglected the terms of second order in the eddy amplitude. Multiplying equation (26) by q*,

$$\left(\frac{\partial}{\partial t}+[u_g]\frac{\partial}{\partial x}\right)(\frac{1}{2}q^{*2})+(q^*v_g^*)\frac{\partial}{\partial y}[q]=0 \qquad (27)$$

Now we divide by $\partial[q]/\partial y$ **to obtain:**

$$\left(\frac{\partial}{\partial t}+[u_g]\frac{\partial}{\partial x}\right)\left(\frac{\frac{1}{2}q^{*2}}{\frac{\partial[q]}{\partial y}}\right)+(q^*v_g^*)=0 \qquad (28)$$

Please note that we have neglected correction terms that arise because we have moved $\partial[q]/\partial y$ **within the t-derivative (moving it within the x-derivative is OK because it has no x-dependence). However, from equation (24) it is clear that the correction term, which involve** q^{*2} $\partial^2[q]/\partial t\partial y$ **will be of fourth order in the eddy amplitude, and can be neglected compared to the all the second order terms in equation (28). Multiplying by** ρ_0 **, zonally averaging and using equation (25), we obtain:**

$$\frac{\partial}{\partial t}\left(\frac{\frac{1}{2}\rho_0\left[q^{*2}\right]}{\frac{\partial[q]}{\partial y}}\right)+\vec{\nabla}\cdot\vec{F}=0$$

$$\frac{\partial}{\partial t}A+\vec{\nabla}\cdot\vec{F}=0 \qquad (29)$$

where equation (29) defines the *wave action A*. As long as $\partial[q]/\partial y$ is positive definite, the wave action is a positive definite measure of eddy strength, and equation (29) can be slightly rewritten in the form a conservation law, which states that the time derivative of a quantity is changed only by the divergence of an appropriate flux.

Note that the definition of *"steady waves"* is that $[q^{*2}]$ does not change in time, or alternatively that A does not change in time. (Within the small eddy approximation, these two definitions are equivalent.)

Equation (29) is valid when the eddies are small amplitude and the flow is adiabatic, and its interpretation as a conservation law depends on $\partial[q]/\partial y$ being positive definite.

This may be violated, that is $\partial[q]/\partial y$ may change sign, in one case of interest, namely *barotropic/baroclinic instability.* However, since this equation is local in the (y,z) plane, away from the regions where $\partial[q]/\partial y$ changes sign, the interpretation of (29) as a conservation law for wave action is still valid.

Reference: Edmon, H. J., B. J. Hoskins, and M. E. McIntyre, 1980. "Eliassen-Palm Cross Sections for the Troposphere", J. Atmos. Sci. 37, 2600-2616.

Schematic Depiction of the Eliassen-Palm flux vector F in the pure Eady and Charney instability problems (vertical shear of [u] only).

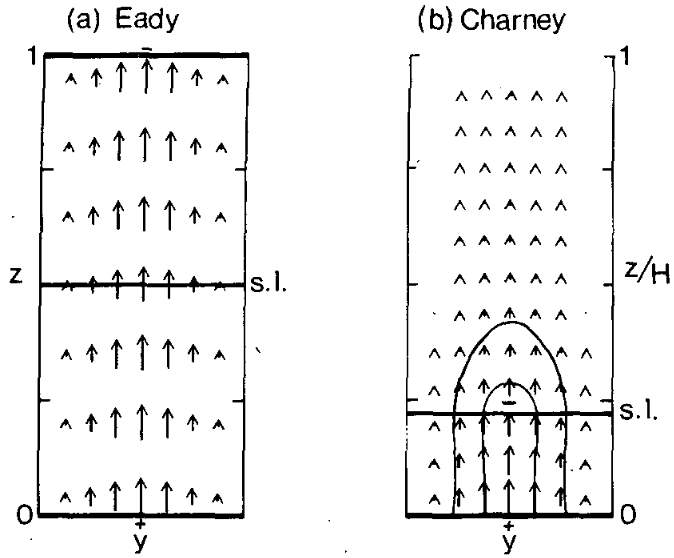

FIG. 2. Examples of Eliassen-Palm cross sections for linear baroclinic instabilities, from (a) Eady's theory and (b) Charney's theory. Steering levels are marked s.l. In (b), H is the density scale height, and the contours for $\nabla \cdot \mathbf{F}$ have values -0.4 and -0.8 of the maximum. The thick horizontal boundaries are meant to suggest infinite crowding of $\nabla \cdot \mathbf{F}$ contours representing concentrated divergence at the boundaries, positive at each lower boundary and negative in (a) at the upper boundary; see Eq. (4.1).

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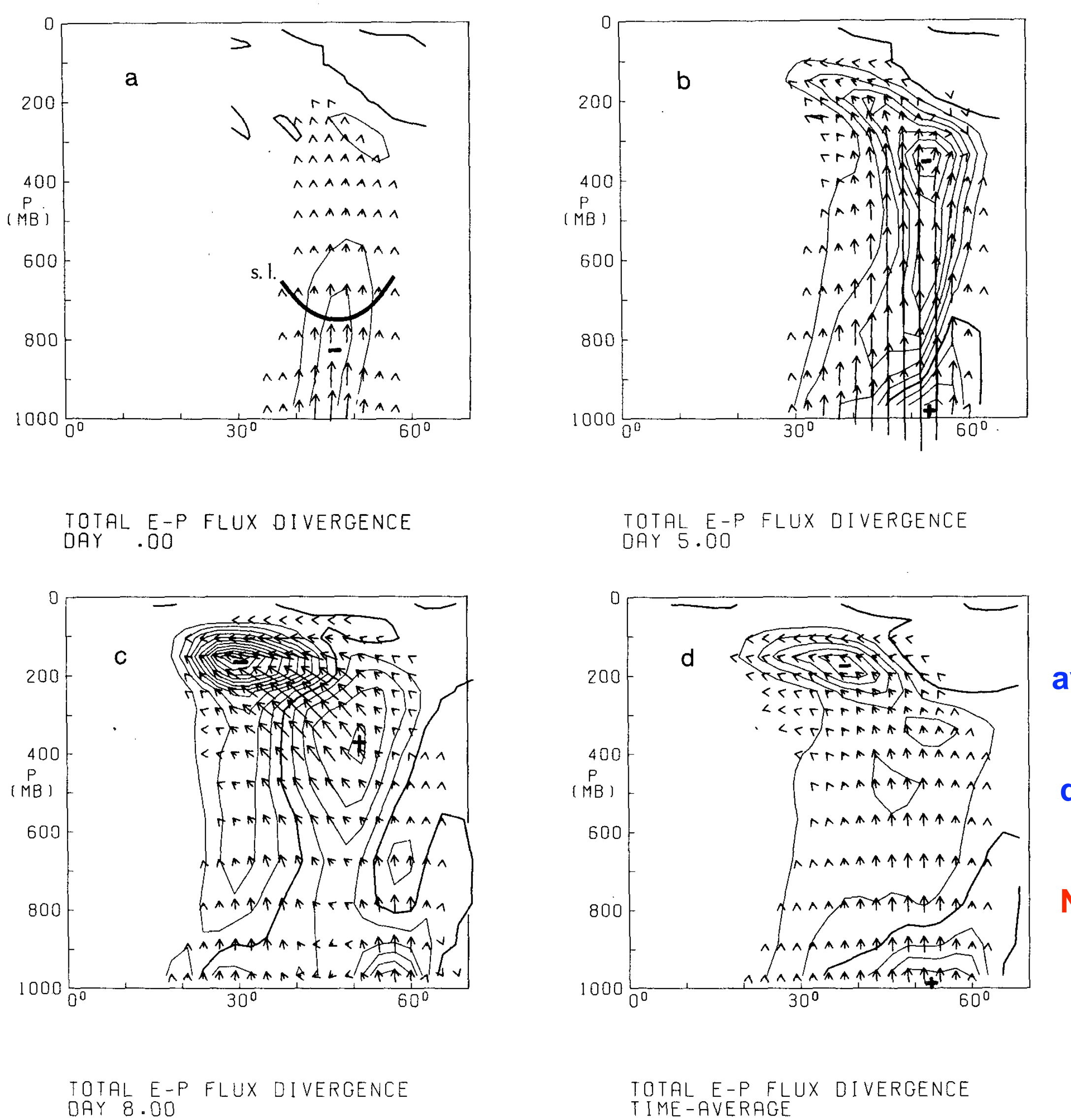
Eliassen-Palm flux for a linear, growing normal mode on a realistic basic state

Eliassen-Palm flux for day 5 of non-linear baroclinic life cycle of same disturbance as in panel (a)



Eliassen-Palm flux for day 10 of non-linear baroclinic life cycle of same disturbance as in panel (a)

Eliassen-Palm flux for average over entire non-linear baroclinic life cycle of same disturbance as in panel (a)

Note acceleration of [u] in lower troposphere, deceleration of [u] in upper troposphere