zk_problems

September 13, 2019

[1]:
```
import math
import matplotlib.pyplot as plt
import numpy as np
import scipy.special as sps
```

1 Grid walk

Given an $a \cdot a$ grid, start in the lower left corner. Reach the upper right corner by moving only up or right. How many unique valid walks exist?

Every valid walk has a length of $2x$, and can be encoded as a sequence of that many bits. The valid bit sequences are those of length $2x$ in which exactly x bits are flipped to 1. The total count of such sequences c can be calculated as:

$$c = \binom{2x}{x} \tag{1}$$

1.0.1 basic definitions

[2]:
```
def randomTraj(n, x=5):
    traj = np.zeros((2*x, n), dtype=np.uint8)
    traj[-x:, :] = 1

    for row in traj.T:
        np.random.shuffle(row)

    return traj

def binTraj(traj):
    return np.unique(traj, axis=1, return_counts=True)

def countTraj(traj):
    return binTraj(traj)[0].shape[1]
```

1.0.2 solutions

[3]:
```
def closedForm(x=5):
    return int(sps.binom(2*x, x))

def stochastic(n, x=5):
    traj = randomTraj(n, x=x)
    return countTraj(traj)
```

1.0.3 check closed form vs stochastic solution

[4]:
```
x = 5

det = closedForm(x=x)
stoch = stochastic(5000, x=x)

print(det)
assert det == stoch
```

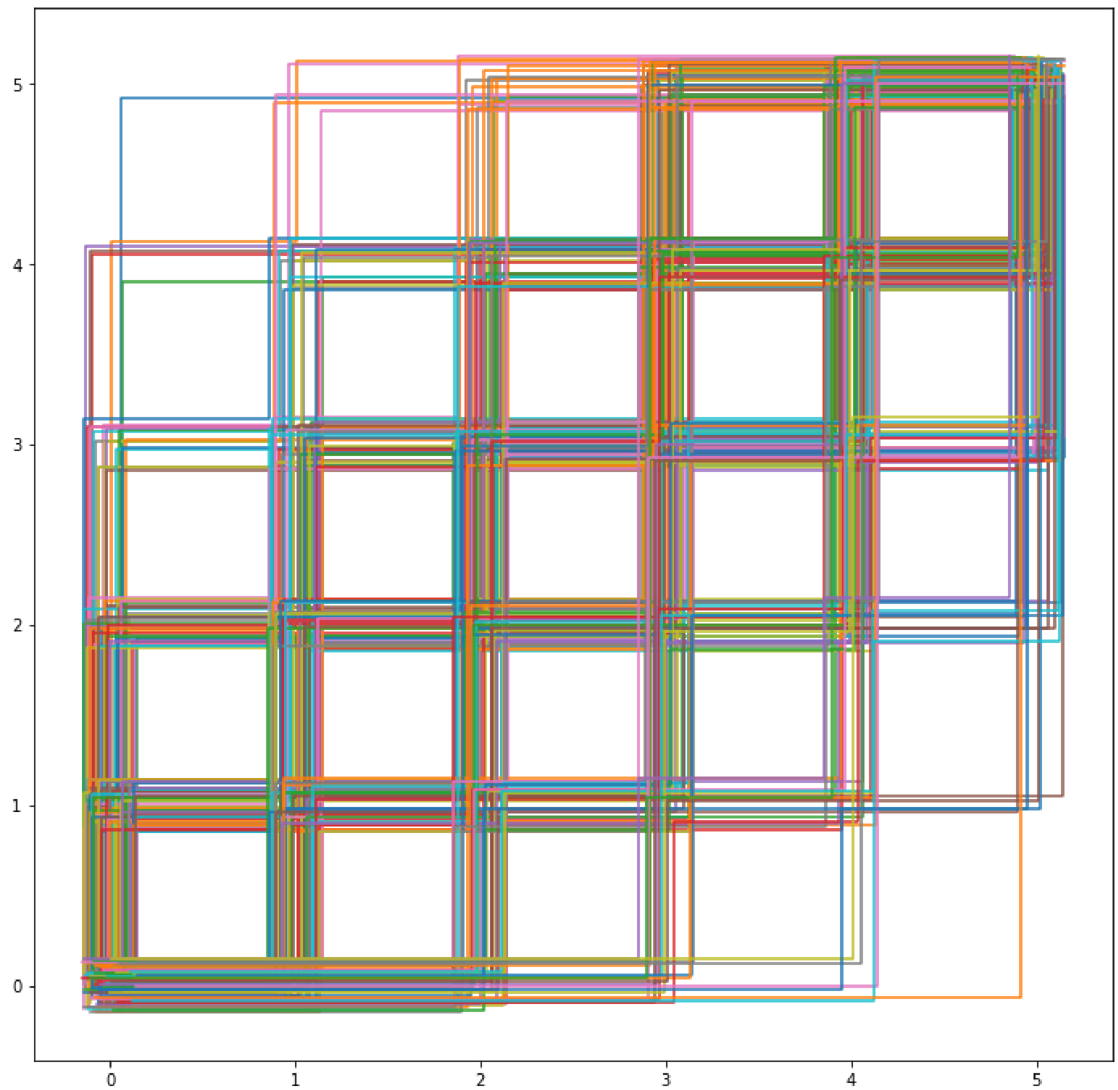
```
252
```

1.0.4 visualize stochastic grid walk

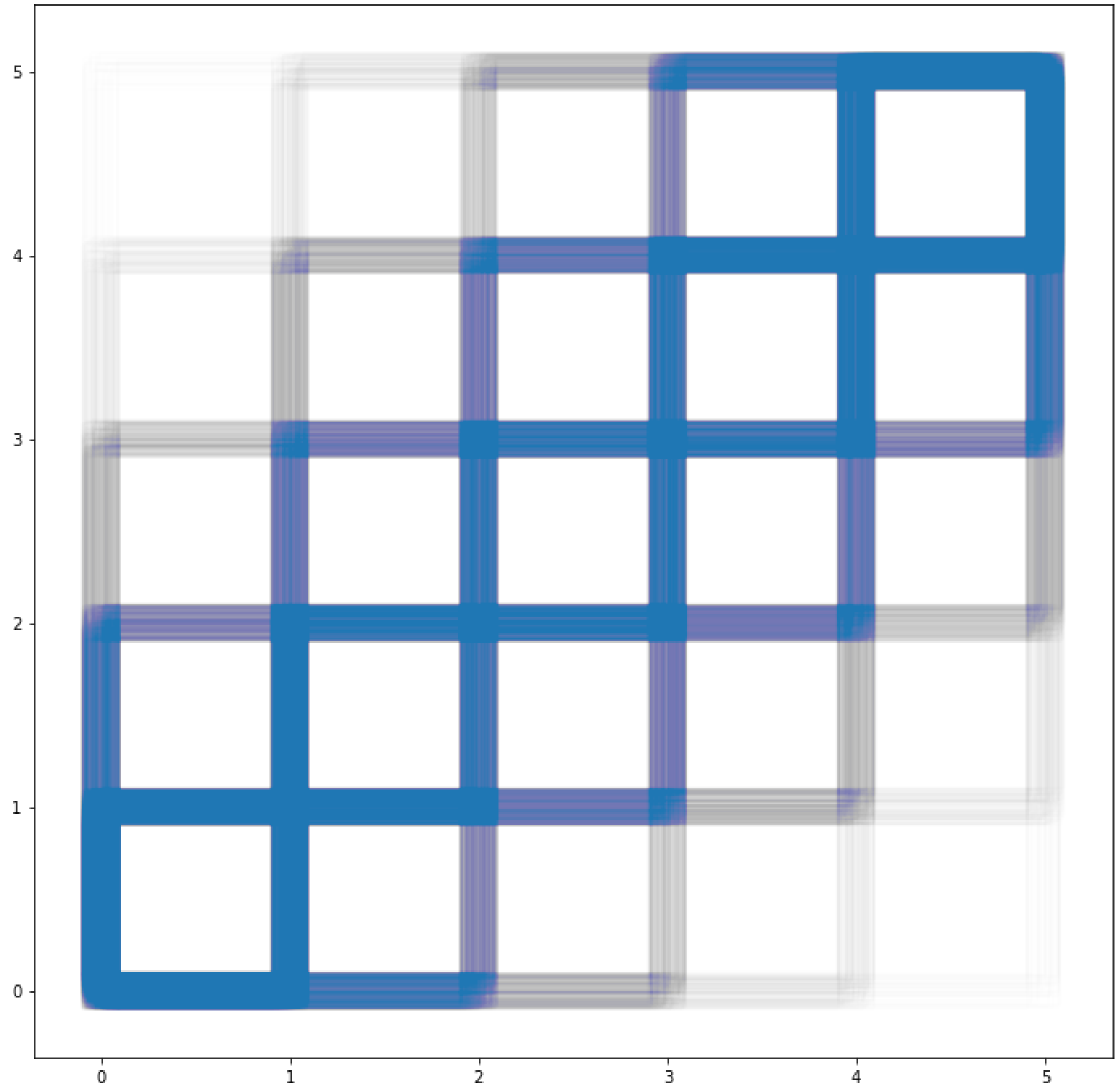
[5]:
```
def trajCoord(traj, uniq=False):
    traj,counts = binTraj(traj) if uniq else (traj, None)

    return np.concatenate([
        # add a (0, 0) start point to all trajectories
        np.zeros((2, 1, traj.shape[1]), dtype=np.uint8),
        (np.arange(2, dtype=np.uint8)[:, None, None] ^ traj[None, ...]).
 ↪cumsum(axis=1)
    ], axis=1), counts

def plotTraj(traj, fuzz=None, prob=False, uniq=False, **kwargs):
    coord,counts = trajCoord(traj, uniq=uniq)

    if fuzz is not None:
        coord = coord + np.random.uniform(-fuzz, fuzz, size=(coord.shape[0],␣
 ↪coord.shape[2]))[:, None, :]

    fig = plt.figure(figsize=(12,12))
    ax = fig.gca()

    if prob and uniq:
        counts = counts/counts.sum()

        cmax = counts.max()
```

```
        counts *= cmax if cmax > .25 else .25/cmax

        for row,a in zip(coord.T, counts):
            pltkwargs = {'lw': 2}
            pltkwargs.update(kwargs)
            ax.plot(*row.T, alpha=a, **pltkwargs)
    else:
        ax.plot(*coord, **kwargs)

    return coord
```

[7]:
```
traj = randomTraj(int(1e4), x=5)
coord = plotTraj(traj, fuzz=.15, uniq=True)
```



[8]:
```
traj = randomTraj(int(1e4), x=5)
coord = plotTraj(traj, fuzz=.1, color='C0', alpha=.005)
```



2 Expected length of coin flip sequence terminating in two consequetive heads

2.0.1 stochastic brute force solution for $E(x)$

[2]:
```
def flips(n=int(1e6)):
    flip = ''.join(np.random.choice(('0', '1'), size=n).tolist()).split('11')

    # re-append the '11' and censor the final sequence
```

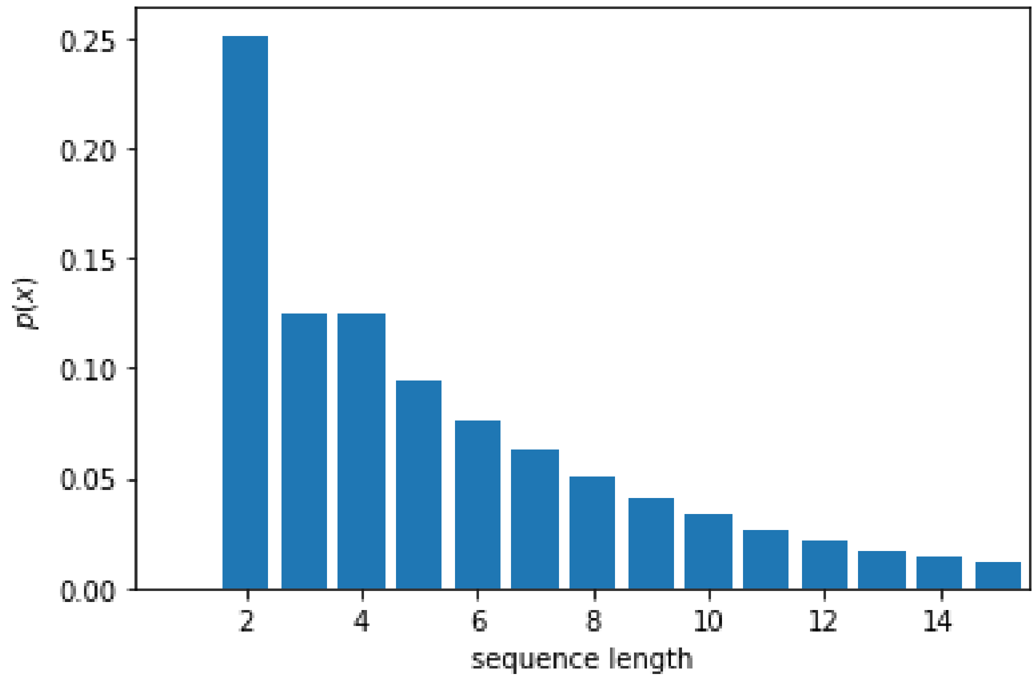
```python
    return ['11' + f for f in flip[:-1]]

def binLen(flip, density=False):
    count = np.bincount([len(f) for f in flip])

    return count/len(flip) if density else count

def stochastic(n=int(1e7)):
    lens = binLen(flips(n=n))
    return lens @ np.arange(lens.size)/lens.sum()
```

[4]:
```python
stochastic()
```

[4]: 6.00276607571412

[39]:
```python
xmax = 15
x = np.arange(1, xmax + 1)
y = binLen(flips(), density=True)[1:xmax + 1]
print(y)

fig = plt.figure()
ax = fig.gca()
ax.bar(x, y)
ax.set_xlim(.1, xmax + .55)
ax.set_xlabel('sequence length')
ax.set_ylabel(r'$p(x)$')
pass
```

```
[0.         0.25127611 0.12469034 0.12500225 0.09408759 0.07677082
 0.0632389  0.05078666 0.04123755 0.03325396 0.02627207 0.02177942
 0.01693289 0.01434767 0.01140256]
```



2.0.2 probability of some sequence lengths

$$p(1) = 0 \tag{2}$$
$$p(2) = \frac{1}{2}\frac{1}{2} = \frac{1}{4} \tag{3}$$
$$p(3) = \frac{1}{2}\frac{1}{4} = \frac{1}{8} \tag{4}$$
$$p(4) = 2\frac{1}{2}\frac{1}{8} = \frac{1}{8} \tag{5}$$
$$p(5) = [1 - p(2)]\frac{1}{8} = \frac{3}{4}\frac{1}{8} = \frac{3}{32} \tag{6}$$
$$p(6) = [1 - (p(2) + p(3))]\frac{1}{8} = \frac{5}{8}\frac{1}{8} = \frac{5}{64} \tag{7}$$
$$\tag{8}$$

2.0.3 recursive probability formula for $x > 3$

$$p(x) = \frac{1}{8}[1 - \sum_{y=1}^{x-3} p(y)], \quad x > 3 \tag{9}$$

2.0.4 numerical approximation to the recursive formula

[23]:
```
_probCache = {1: 0, 2: 1/4, 3: 1/8}
def prob(x):
    if x < 1:
        return 0
    if x in _probCache:
        return _probCache[x]

    p = (1/8)*(1 - sum(prob(y) for y in range(2, x - 2)))

    _probCache[x] = p
    return p

def numerical(X=int(1e4)):
    return sum(x * prob(x) for x in range(1, X))
```

[8]:
```
[prob(x) for x in np.arange(1, 15)]
```

[8]:
```
[0,
 0.25,
 0.125,
 0.125,
 0.09375,
 0.078125,
 0.0625,
 0.05078125,
 0.041015625,
 0.033203125,
 0.02685546875,
 0.021728515625,
 0.017578125,
 0.01422119140625]
```

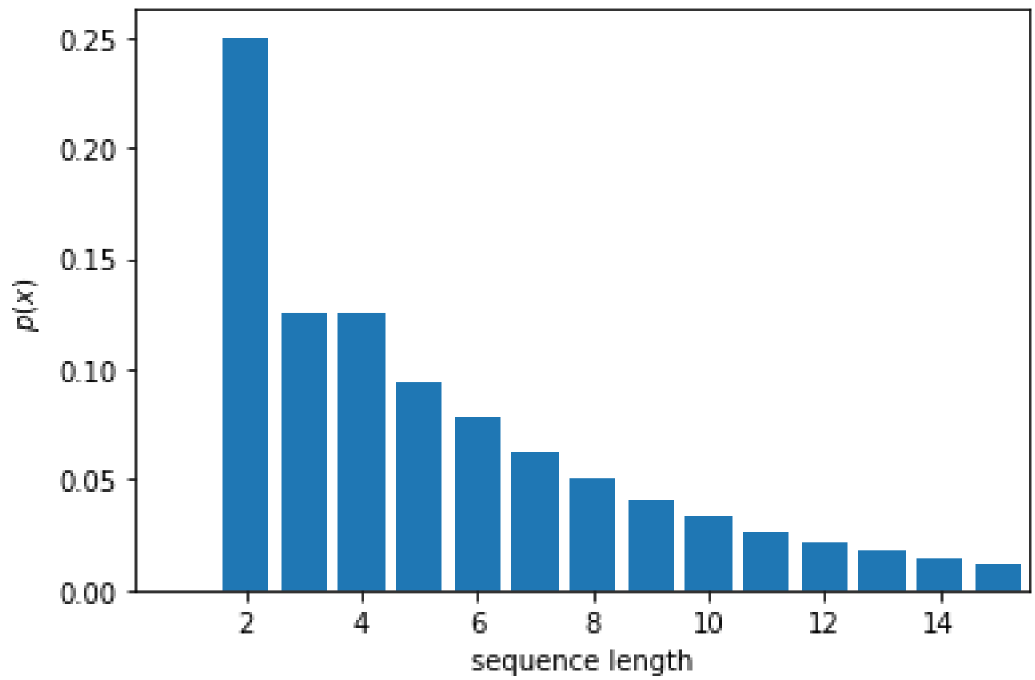
[9]:
```
numerical()
```

[9]:
```
6.000000001387008
```

2.0.5 numerical vs stochastic

[37]:
```
xmax = 15
x = np.arange(1, xmax + 1)
y0 = [prob(var) for var in x]
y1 = binLen(flips(), density=True)[1:xmax + 1]

fig = plt.figure()
ax = fig.gca()
ax.bar(x, y0)
# ax.plot(x, y1, c='C1')
```

```
ax.set_xlim(.1, xmax + .55)
ax.set_xlabel('sequence length')
ax.set_ylabel(r'$p(x)$')
pass
```



3 Scratch

[ ]:
```
def trajCoord(traj):
#     coord = np.zeros((2, traj.shape[0] + 1, traj.shape[1]), dtype=np.uint8)
#     coord[:, 1:, :] = (np.arange(2, dtype=np.uint8)[:, None, None] ^␣
↪bit[None, ...]).cumsum(axis=1)

#     return coord

#     conv = np.zeros((2, bit.shape[0]), dtype=np.uint8)
#     conv[1, 1:] = 1

#     return (conv[..., None] ^ bit[None, ...]).cumsum(axis=1)

#     return (np.arange(2, dtype=np.uint8)[:, None, None] ^ bit[None, ...]).
↪cumsum(axis=1)
```

[53]:
```
arr = np.arange(2)[:, None, None] - gridtraj(100)
print(arr.shape)
```

```
arr
```

```
(2, 100, 10)
```

[53]:
```
array([[[ 0, -1,  0, ..., -1, -1,  0],
        [-1,  0, -1, ...,  0,  0, -1],
        [ 0,  0, -1, ..., -1,  0, -1],
        ...,
        [ 0, -1,  0, ..., -1,  0, -1],
        [ 0, -1, -1, ..., -1,  0,  0],
        [ 0, -1,  0, ..., -1,  0, -1]],

       [[ 1,  0,  1, ...,  0,  0,  1],
        [ 0,  1,  0, ...,  1,  1,  0],
        [ 1,  1,  0, ...,  0,  1,  0],
        ...,
        [ 1,  0,  1, ...,  0,  1,  0],
        [ 1,  0,  0, ...,  0,  1,  1],
        [ 1,  0,  1, ...,  0,  1,  0]]])
```

[59]:
```
traj = gridTraj(int(3), x=5)
print(traj.shape)
print(trajCoord(traj).shape)
```

```
(10, 3)
(2, 11, 3)
```

[80]:
```
fuzz = .2

traj = gridTraj(int(100), x=5)
coord = trajCoord(traj, fuzz=0)

fuzzed = coord + np.random.uniform(-fuzz, fuzz, size=(coord.shape[0], coord.
 ↪shape[2]))[:, None, :]
fuzzed[:, :, :3]
```

[80]:
```
array([[[ 0.1447698 , -0.14971525, -0.12135943],
        [ 0.1447698 ,  0.85028475, -0.12135943],
        [ 1.1447698 ,  0.85028475, -0.12135943],
        [ 1.1447698 ,  1.85028475, -0.12135943],
        [ 2.1447698 ,  2.85028475,  0.87864057],
        [ 2.1447698 ,  3.85028475,  1.87864057],
        [ 2.1447698 ,  3.85028475,  1.87864057],
        [ 3.1447698 ,  4.85028475,  1.87864057],
        [ 4.1447698 ,  4.85028475,  2.87864057],
        [ 5.1447698 ,  4.85028475,  3.87864057],
        [ 5.1447698 ,  4.85028475,  4.87864057]],
```

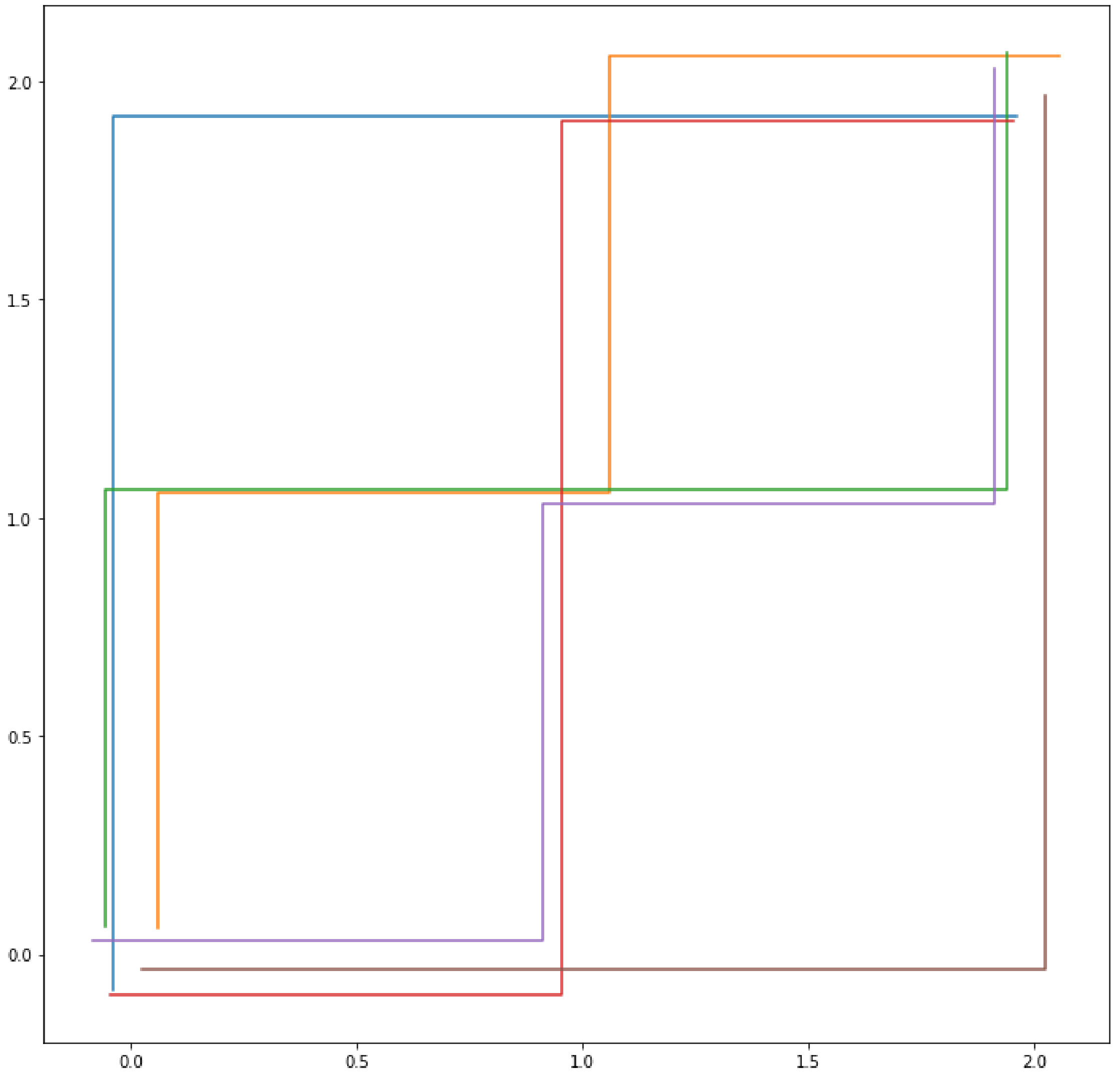
```
[[-0.00635294, -0.19915433,  0.15598262],
 [ 0.99364706, -0.19915433,  1.15598262],
 [ 0.99364706,  0.80084567,  2.15598262],
 [ 1.99364706,  0.80084567,  3.15598262],
 [ 1.99364706,  0.80084567,  3.15598262],
 [ 2.99364706,  0.80084567,  3.15598262],
 [ 3.99364706,  1.80084567,  4.15598262],
 [ 3.99364706,  1.80084567,  5.15598262],
 [ 3.99364706,  2.80084567,  5.15598262],
 [ 3.99364706,  3.80084567,  5.15598262],
 [ 4.99364706,  4.80084567,  5.15598262]]])
```

[82]:
```
binned,counts = binTraj(traj)
```

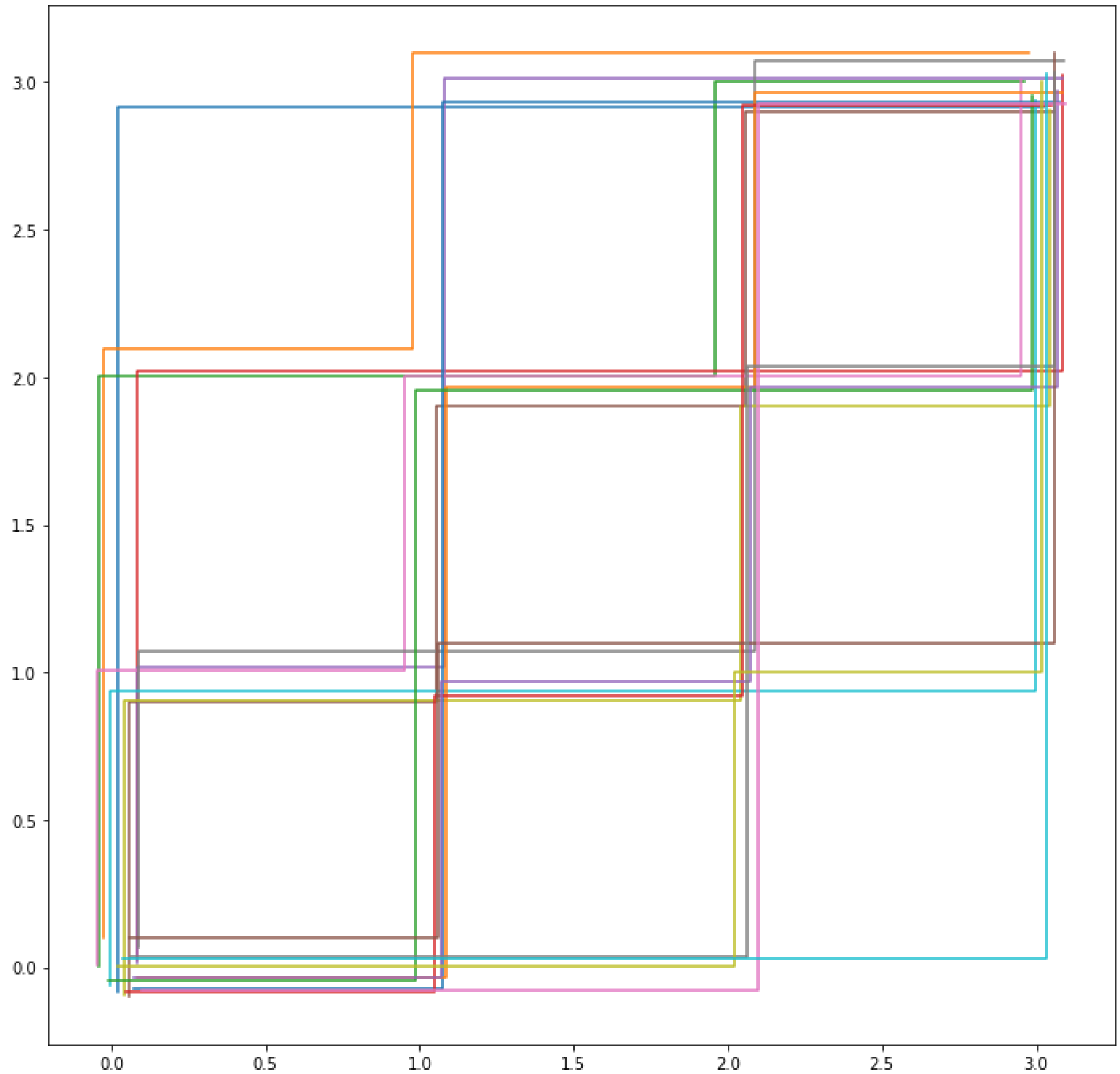
[83]:
```
counts
```

[83]:
```
array([2, 1, 2, 1, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 1, 1, 2, 1, 2, 1,
       1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 2, 1, 1, 1, 1, 1, 1, 2,
       1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1,
       1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1])
```

[155]:
```
traj = gridTraj(20, x=2)
coord = plotTraj(traj, fuzz=.1, uniq=True)
```



[156]:
```
traj = gridTraj(100, x=3)
coord = plotTraj(traj, fuzz=.1, uniq=True)
```



3.0.1 expected value (recursive formula)

$$E(x) = \sum xp(x) \tag{10}$$

$$E(x) = 1 + \frac{2}{4} + \frac{3}{8} + \sum_{x>3} xp(x) \tag{11}$$

$$E(x) = \frac{15}{8} + \sum_{x>3} xp(x) \tag{12}$$

$$E(x) = \frac{15}{8} + \frac{1}{8} \sum_{x>3} [x - xp(x-3)] \tag{13}$$

$$E(x) = \frac{1}{8} \sum_{x>3} [x - xp(x-3)] - \frac{15}{8} \tag{14}$$

$$\tag{15}$$

[231]:
```
flip = np.random.choice(('0', '1'), size=int(1e6))
print(flip)
arr = np.array([len(s) for s in ''.join(flip.tolist()).split('11')])
arr.mean()
```

```
['0' '1' '0' ... '0' '0' '0']
```

[231]: 3.982590695893558

[192]:
```
parr = np.zeros((arr.shape[0] + 2,), dtype=arr.dtype)
parr[2:] = arr[:]
```

[222]:
```
~np.diff(arr, append=0) & arr
```

[222]: array([1, 1, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0])

[220]:
```
(np.diff(~np.diff(arr, append=0) & arr, prepend=0) > 0).nonzero()[0]
```

[220]: array([0, 5, 9])

[207]:
```
np.diff(arr, append=0)
```

[207]: array([-1, 1, -1, 0, 1, -1, 1, 0, -1, 0])

[198]:
```
parr
```

[198]: array([0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0], dtype=int8)

[248]:
```
def flips(n=int(1e6)):
    flip = []
    acc = ''
    for x in np.random.choice(('0', '1'), size=n):
        acc += x
        if acc[-2:] == '11':
            flip.append(acc)
            acc = ''

    return flip

def lenbin(flip, density=False):
    count = np.bincount([len(f) for f in flip])

    return count/len(flip) if density else count

def stochastic(n=int(1e6)):
    lens = lenbin(flips(n=n))
    return lens @ np.arange(lens.size)/lens.sum()
```

[249]:
```
stochastic()
```

[249]: 6.009525125899929

[ ]:
```

```