

Simplifying and optimizing the stochastic simulation of rare biochemical events

by

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**A dissertation submitted to The Johns Hopkins University
in conformity with the requirements for the degree of
Doctor of Philosophy**

Baltimore, Maryland

January, 2019

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Abstract

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Acknowledgments

My deepest thanks to my advisor, my committee members, my parents, and all of the others who bore with me during a difficult time. Your support has meant the world to me.

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Chapter 1

Introduction

One of the great arguments of the day was vitalism versus mechanism, a disguised form of the old and continuing debate between those, including the religious, who believe the world has purpose and those who believe it operates automatically and by chance... The German chemist who scoffed in 1895 at the “purely mechanical world” of “scientific materialism” that would allow a butterfly to turn back into a caterpillar was disputing the same issue, an issue as old as Aristotle.

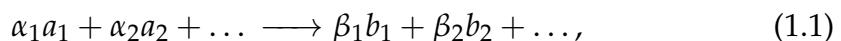
– Richard Rhodes, *Making of the Atomic Bomb*¹

Are living organisms special? Are biological molecules inherently unique, or do they obey the same laws of chemistry and physics as every other molecule? Since the time of Mendel² and Cajal³ the scientific consensus has been converging towards the latter view: that life is made of the same material as the rest of the world. In other words, that the information encoded in an organism’s genotype and (more broadly) biochemical state defines the organism’s phenotype (*i.e.* its appearance and behavior). Over the course

of the 20th century this materialistic viewpoint has become an implicit assumption that underlies all current work done in the life sciences. However, this view is largely taken on faith, as a complete mechanistic link between biochemistry and behavior has been established for only a small subset of cellular systems. Though there is now an extensive (and ever growing) catalog of the biochemical elements from which living systems are formed, the mechanisms by which they interact remain in general unexplained. The aim of this thesis is to present work towards a general method of mechanistically linking biochemistry to phenotype.

1.1 Models of biochemistry

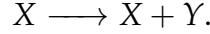
The question at hand is one of dynamics. How does the biochemistry of a cell "unfold" over time to produce a phenotype^{*}? Models, and modeling, form the cornerstone of a workable approach for explaining large-scale biological phenomena in terms of what goes on at the smallest scale. First, we need to establish a theoretical framework in which to reason about biochemistry. Any chemical reaction can be written in the form:



where $\{a_1, a_2, \dots\}$ and $\{b_1, b_2, \dots\}$ are the reactants and products, and $\{\alpha_1, \alpha_2, \dots\}$ and $\{\beta_1, \beta_2, \dots\}$ are their respective stoichiometries. In some cases, a chemical species may appear on both sides of the reaction with the same stoichiometry,

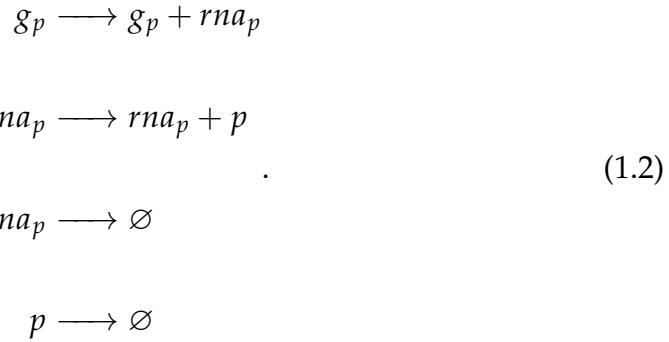
^{*}Equivalently, imagine that you are given a perfect description of the contents of a cell at one moment in time. What can you then say about the contents of that cell at some moment in the future? What can you say about the cell's behavior during all of the moments in between?

as so:



This implies that X is required for the reaction, but is not consumed by it.

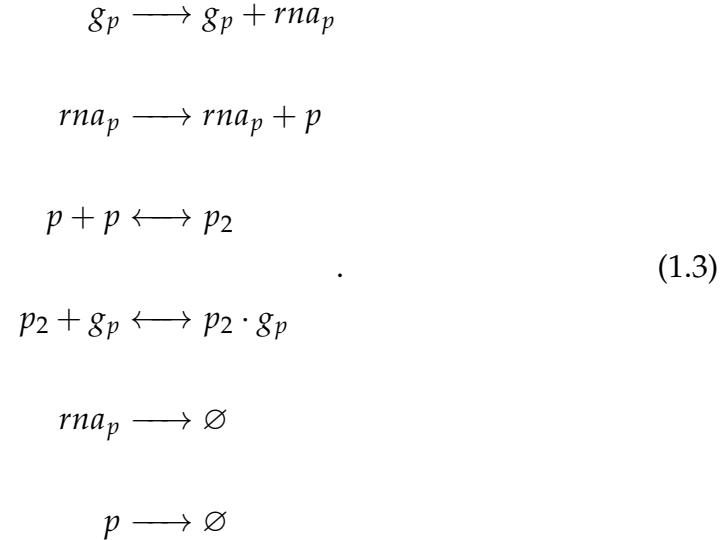
To give a concrete example, consider protein expression from a gene. A simple model of this process can be derived from the central dogma^{4,5}: say there is a segment of DNA that comprises the gene g_p . g_p is transcribed to produce an RNA rna_p . In turn, rna_p is then translated to a protein p . Both rna_p and p also degrade. The reactions that describe this system can be written in terms of Eq 1.1 as so:



In a sense, a list of reactions like Eq 1.2 give a complete description of a system. This level of description can be thought of as forming a qualitative model. Qualitative models can be used to answer questions about what is possible in a system, or how a particular event occurs.

The same approach used to come up with Eq 1.2 can also be used to build more complex, regulated models of expression. For example, say that p is able to dimerize, and that p_2 dimers are able to bind back to g_p and repress further

transcription. The reactions that describe the system would then be:



1.2 Epigenetic landscapes and phenotypes

Low-level models of biochemistry such as Eq 1.2 can be linked to higher-level phenomena via the concept of the epigenetic landscape. Waddington introduced his idea of the epigenetic landscape^{6,7} by first asking the reader to imagine a cell as represented by a high-dimensional space. We call this space the cell's state space. Each dimension of the state space has a one-to-one correspondence with one of the species of metabolites or biomolecules present in the cell. The (non-negative, integer valued) coordinate along each dimension is the count of the corresponding molecule. Thus, each point in the complete state space uniquely defines a possible state of the cell. Just as there can be thousands, or tens of thousands, of distinct chemical species present in a cell, so too can a state space have thousands of dimensions.

The possible states of a cell are combinatorially vast. Simply enumerating

them all would be a herculean computational task. Thus, it is legitimate to ask: what insight about a cell can actually be gained by thinking about it in terms of its state space? An analogy to the study of protein folding is useful here. Levinthal's paradox⁸ says that there are more possible states for a protein to be in than atoms in the universe. Yet somehow proteins still fold. Similarly, cells tend to remain in homeostasis with respect to a particular phenotype (or sometimes to transition from one phenotype to another in an orderly fashion). In both cases, physical forces conspire to limit the occupied states to restricted regions. The epigenetic landscape is a description of the states that tend to be occupied, and of the forces that drive a cell to those states. A single "neighborhood" of states on the landscape constitutes a phenotype.

Waddington originally developed the idea of epigenetic landscapes as a way to explain the constrained diversity he observed in developing organisms. "Diversity" in the sense that embryonic cells have many possible end states (in terms of the cell type of their lineage in the mature organism), and "constrained" in the sense that even harsh chemical perturbations often shift those end states only slightly. As he described it, the landscape of a developing cell is like rough, hilly terrain that is covered in divots connected by valleys. The divots represent the various possible phenotypes, and the valleys the transition pathways in between them. The developing cell, then, is like a ball that starts at a high point on this terrain (see Fig 1.1). The robustness of the development process at any given moment can be thought of as analogous to the steepness of the landscape in the cell's immediate vicinity. As the cell (or its lineage) travels downhill, it proceeds through various phenotypes and

the branching paths in-between. Eventually it reaches the bottom of the hill, along with its terminal cell type.

In the decades since Waddington first proposed it, much work has gone into filling in the details of the theory behind epigenetic landscapes. In the modern view^{9,10,11}, the epigenetic landscape is a representation of the physics of a non-equilibrium system. It can be (approximately) split into two parts: a potential surface (analogous to the potential energy surface of an equilibrium system) and a probability flux. If the potential surface is equivalent to Waddington's rough terrain, then the probability flux is like a strong wind blowing across it. Because of this "wind", a cell's fate is not determined by its potential surface alone. In recent years epigenetic landscapes of various cellular systems, such as the cell cycle^{12,13} and oncogenesis^{14,15}, have been constructed. These high-dimensional landscapes can be plotted (via projection) in terms of one or more species of interest. This gives a quantitative picture of how transitions in the population count of any given species drives transitions between phenotypes (see figure Fig 1.2).

1.3 Quantitative models and networks

Given that an epigenetic landscape can be used to link biochemistry to phenotype, the question then is how to calculate one. A qualitative model is not enough. What is needed is a quantitative model, one that can reproduce the actual species counts. In order to build a quantitative model of biochemical state and behavior, 4 pieces of information are required:

1. The identity of the distinct interacting biochemical elements. These are

often referred to as chemical species, or just species.

2. The list of reactions in which the chemical species participate.
3. The rate laws that describe how quickly each reaction occurs.
4. The quantity of each species present in the system. These are dynamical systems under consideration, so precise quantities can only be spoken of with respect to a particular moment in time (such as an initial time $t = 0$).

Once these 4 pieces have been determined, the next step is to combine them into a cohesive, mathematically tractable model. The standard way to do so is to build a network.

There are many different representations of biochemical networks, each with their own strengths and weaknesses. A simple network representation of our protein expression model can be seen in Fig 1.3. Here, the protein expression model is shown as a digraph (directed graph). Like any network, a digraph consists of a set of nodes and the set of edges which connect them. Additionally, each edge in a digraph encodes a direction, such that they start at a reactant node and end at a product node.

Simple digraph representations are useful since they offer an intuitive picture of the lists of species and reactions that make up a model. Another useful network representation is the Petri net. Originally developed by Petri in the 1960's¹⁶ to model linked chemical reactions, Petri nets offer a natural way to model agent-based processes in general[†]. Petri nets were first applied to

[†]When applying Petri nets to biochemical systems, the chemical species are the agents. This is in the sense that each molecule acts independently.

problems in systems biology around the turn of the century¹⁷, and have since been extensively developed for biological applications^{18,19,20}. Though less easy to understand at a glance than simple digraphs, Petri nets have the virtue of being able to encode unambiguous descriptions of biochemical systems. Due to this formal rigor, once set up they can be mechanically transcribed into various mathematical forms that can then be used as input for many different simulation techniques.

Petri nets are bipartite digraphs. "Bipartite" refers to the fact that every Petri net contains two distinct sets of nodes: one set, called the places, represent chemical species, and the other, called the transitions, represent chemical reactions. For every reaction that a species participates in as a reactant there is an edge that starts at the corresponding place and ends at the corresponding reaction. Similarly, for every reaction product there is an edge that starts at the corresponding transition and ends at the corresponding place. The weight of each edge is the stoichiometry of the attached species (whether as product or reactant) with respect to the attached reaction. Additionally, a Petri net has a marking, a list that contains the initial quantities of each species.

Figure 1.4 shows a Petri net representation of the simple expression model of Eq 1.2. In addition to the graphical representation, a Petri net can be uniquely specified as a set of matrices $\{P, T, M, Pre, Post\}$ ²¹. For the simple

expression model these matrices would be:

$$P = \begin{pmatrix} g_p \\ rna_p \\ p \end{pmatrix}, T = \begin{pmatrix} \text{transcription} \\ \text{translation} \\ rna_p \text{ decay} \\ p \text{ decay} \end{pmatrix}, M = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$Pre = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, Post = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (1.4)$$

where P is the list of places/species, T is the list of transitions/reactions, M is the list of markings corresponding to P , Pre is a $M \times N$ matrix (where M is the count of reactions, and N is the count of unique species) in which Pre_{ij} gives the weight of the edge connecting the i th species to the j th reaction (equivalently, the reactant stoichiometry), and $Post$ is the same thing as Pre except that $Post_{ij}$ gives the weight of the edge connecting the j th reaction to the i th species.

For the more complex regulated expression system described in Eq 1.3, the

equivalent $\{P, T, M, \text{Pre}, \text{Post}\}$ matrices would be:

$$P = \begin{pmatrix} g_p \\ rna_p \\ p \\ p_2 \\ p_2g_p \end{pmatrix}, \quad T = \begin{pmatrix} \text{transcription} \\ \text{translation} \\ rna_p \text{ decay} \\ p \text{ decay} \\ p \text{ dimerization} \\ p \text{ dedimerization} \\ p_2 + g_p \text{ binding} \\ p_2g_p \text{ unbinding} \end{pmatrix}, \quad M = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (1.5)$$

$$\text{Pre} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{Post} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Fig 1.5 shows a plot of the actual Petri net that the above matrices describe.

1.4 Deterministic simulation

1.4.1 ODE models of biochemistry

The traditional way to simulate biochemical systems begins with constructing a set of ordinary differential equations (ODEs). This type of simulation is commonly referred to as deterministic simulation (as opposed to stochastic simulation, which is discussed later in Sec 1.5). If there are N chemical species in a system, for each distinct species X_j there will be an equation of the form:

$$\frac{dX_j}{dt} = f_j(X_1, X_2, \dots, X_N).$$

On the right hand side of the equation is a derivative that represents the rate in change over time in the quantity of X_j . On the left hand side is a function of the quantity of every species (including X_j). The exact form of f_j will depend on the type (and precise mathematical formulation) of the chemical kinetics being modeled.

Given the Petri net representation of a system, it is straightforward to construct the set of ODEs. Say that the system involves N species participating in M reactions. First one constructs the M rate laws $r_i(X)$. The rate law $r_i(X)$ tells you how quickly each reaction i is occurring. Traditionally, the $r_i(X)$ are given a form based on mass action kinetics (which can be traced back to a set of papers published in the 1860s²²). In the mass action view, the rate at which any reaction occurs is directly proportional to the quantity of each reactant. This implies that every reaction has an associated rate law $r_i(X)$ of the form:

$$r_i(X) = k_i X_1^{p_{i1}} X_2^{p_{i2}} \dots X_N^{p_{iN}} = \prod_{j=1}^N X_N^{p_{ij}}, \quad (1.6)$$

where k_i is a rate constant, and p_{ij} is a shorthand for the ij th entry of the *Pre* matrix.

Next, one determines the stoichiometry matrix S . Essentially, the stoichiometry matrix is the difference of the *Post* and *Pre* matrices. The exact definition of S varies in the literature, but for our purposes it will be convenient to define S as the transpose of the difference:

$$S = (\text{Post} - \text{Pre})^\top.$$

Thus, S will be a $N \times M$ matrix. With the rate laws $r_i(X)$ and the stoichiometry

matrix S in hand, the entire set of N ODEs can be written as a single matrix equation:

$$\frac{dP}{dt} = Sr, \quad (1.7)$$

where P is the places matrix, $\frac{dP}{dt}$ is a column vector of the ODEs, and r is a column vector in which the i th entry is the rate law $r_i(X)$.

The nature of the compact form given in Eq 1.7 is most easily explained with an example. For our simple expression system (described in Eq 1.2 and given in Petri net matrix form in Eq 1.4), the column vector of rate laws r can be found from Eq 1.6:

$$r = \begin{pmatrix} k_1 \cdot g_p^1 \cdot rna_p^0 \cdot p^0 \\ k_2 \cdot g_p^0 \cdot rna_p^1 \cdot p^0 \\ k_3 \cdot g_p^0 \cdot rna_p^1 \cdot p^0 \\ k_4 \cdot g_p^0 \cdot rna_p^0 \cdot p^1 \end{pmatrix} = \begin{pmatrix} k_1 \cdot g_p \\ k_2 \cdot rna_p \\ k_3 \cdot rna_p \\ k_4 \cdot p \end{pmatrix}$$

and the stoichiometry matrix S is:

$$S = \left[\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]^\top = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^\top = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

The set of ODEs that describe the system is then:

$$\begin{pmatrix} \frac{d}{dt}g_p \\ \frac{d}{dt}rna_p \\ \frac{d}{dt}p \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} k_1 g_p \\ k_2 rna_p \\ k_3 rna_p \\ k_4 p \end{pmatrix} = \begin{pmatrix} 0 \\ k_1 g_p - k_3 rna_p \\ k_2 rna_p - k_4 p \end{pmatrix}.$$

Solving[†] the above system of ODEs yields:

$$\begin{pmatrix} g_p \\ rna_p \\ p \end{pmatrix} = \begin{pmatrix} M_1 \\ \frac{k_1 M_1}{k_3} + e^{-tk_3} \left(M_2 - \frac{k_1 M_1}{k_3} \right) \\ \frac{k_1 k_2 M_1}{k_3 k_4} + e^{-tk_3} \frac{(k_2(k_1 M_1 - k_3 M_2))}{k_3(k_3 - k_4)} + e^{-tk_4} \left(M_3 + \frac{k_2(k_4 M_2 - k_1 M_1)}{k_4(k_3 - k_4)} \right) \end{pmatrix}, \quad (1.8)$$

where M_j is the initial marking (*i.e.* quantity) of species j . For any given marking M and set of rate constants k_i , the expressions of Eq 1.8 can be used to find the quantity of each species at any given time t .

Often, a modeler is not concerned with the initial behavior of a system[§], but only with the behavior of the system over long periods of time. In these cases, solutions such as those found in Eq 1.8 can be simplified by considering their value in the limit $t \rightarrow \infty$, also called the steady state limit. The only terms in Eq 1.8 that explicitly depend on time are exponential decay terms of the form e^{-tx} . These exponential decay terms quickly approach 0 as t increases, yielding:

$$\begin{pmatrix} g_p \\ rna_p \\ p \end{pmatrix} \approx \begin{pmatrix} M_1 \\ \frac{k_1 M_1}{k_3} \\ \frac{k_1 k_2 M_1}{k_3 k_4} \end{pmatrix}.$$

[†]The complete details of solving a system of ODEs is outside of the scope of this thesis. In general, it suffices that a solution can easily be obtained using a computational algebra system such as Mathematica²³.

[§]As far as reproducing experimental results go, the initial behavior of a system is usually irrelevant. When studying biological systems, it is typical for the system to preexist any experimental observation (though an obvious exception would be any stop-flow experiment).

1.4.2 The benefits and shortcomings of deterministic simulations

Deterministic, ODE based approaches such as the one given in the previous section were first used to model biological systems during the early 1900's^{24,25,26,27}. Since then, ODEs have seen wide use in the simulation of biological systems. Most such systems of ODEs do not have a closed-form solution as in Eq 1.8. Instead, numerical integration²⁸ can be used to determine the quantity of each species up to a finite time. Even when numerical integration is required, ODE based methods tend to be some of the most computationally efficient and easy to implement techniques, contributing to their popularity.

Fig 1.6 shows a realization of an ODE simulation of our simple expression system. The expression in Eq 1.8 was used to find the concentration of protein at every time point up to ten hours. On the right hand side of the figure is a histogram of the count of protein along the time series. This can be thought of as an empirical estimation of the epigenetic landscape of the simple expression system. Thus, the ODE simulation predicts that the landscape of this system effectively consists of a single state at protein count = 50.

If we were to repeat this simulation, the exact same result would be produced. This highlights a key feature of ODE simulations: they're deterministic, in the sense that, given a particular system and set of initial conditions, the method will always produce exactly the same result (at least up to numerical accuracy, in the cases where numerical integration is required). It is because

of this reproducibility of outcome that ODE simulations are also called deterministic simulations.

The theoretical formulation of the ODE approach relies on a number of assumptions²⁹. In particular, every chemical species involved in a reaction is assumed to be distributed across the reaction volume in a completely homogeneous fashion. The concentration of any given species at any point in space is equal to the concentration of that species at any other point. This is in effect a continuum view, in which chemical species are not made of discrete particles but instead can be treated as infinitely divisible interacting fluids.

In the words of van Kampen, at the scale of test tubes the continuum assumptions are "actually better obeyed than might be expected"³⁰. At significantly smaller scales, such as those of a single cell, these assumptions break down. At some point it becomes no longer reasonable to assume that a chemical substance is an infinitely divisible fluid. Instead, one begins to have to account for effects³¹ arising from the influence of the single molecules of which chemical substances are truly made. In addition to small scales, these single molecule effects also become prominent when a substance is present in a very low concentration. In this type of situation it becomes more appropriate (and more accurate) to measure quantities in terms of particle count, also sometimes called copy number.

The assumptions of ODE modeling break down to some extent when applied to any gene expression system. This is due to the fact that the DNA of each gene is typically present at very low copy numbers (just 2-4 for most

genes in a diploid organism) in each cell. Moreover, many of the other components of expression systems, such as specific RNAs and proteins, also have low copy number. Thus ODE models cannot recapitulate the full behavior of gene expression. Even for our simple expression model, deterministic simulation fails to reproduce many of its details (as shown in Fig 1.7 and discussed in Sec 1.6). Alternative methods have to be used in order to capture the full dynamics through which the information encoded in genes becomes proteins.

1.5 Stochastic simulation

The justification for using the stochastic approach, as opposed to the mathematically simpler deterministic approach, was that the former presumably took account of fluctuations and correlations, whereas the latter did not... in the deterministic formulation no distinction is made between the average of a product and the product of the averages; i.e., it is automatically assumed that $\langle x_i x_j \rangle = \langle x_i \rangle \langle x_j \rangle$. For $i \neq j$ this assumption nullifies the effects of correlations, and for $i = j$ it nullifies the effects of fluctuations.

– Daniel T Gillespie, *A General Method for Numerically Simulating the Stochastic Time Evolution of Coupled Chemical Reactions*³²

Stochastic simulation is a particularly robust method for recreating the behavior of interacting biomolecules. The stochastic simulation method stems from work done by Gillespie. In his seminal 1976 paper “A general method for numerically simulating the stochastic time evolution of coupled chemical reactions”³², Gillespie laid out the theoretical framework of stochastic chemical

kinetics (which was still in its infancy^{33,34,35} at that point) and proposed the first practical algorithm for simulating stochastic chemical systems.

1.5.1 The theory of stochastic simulation

The essential assumption of stochastic chemical kinetics is that for every reaction R_i there exists a constant c_i such that:

$$c_i dt \equiv \text{average probability that any set of competent molecules react per } R_i \text{ during the time interval } (t, t + dt]. \quad (1.9)$$

It can be shown³⁶ that for any well stirred system that is also at thermal equilibrium, the condition in Eq 1.9 will be true for any reaction that follows mass-action kinetics.

The continuum chemical kinetics used in deterministic simulations assumes that each chemical species is homogeneously distributed across the reaction volume under study. In this view, each substance is in some sense spread infinitely finely over the volume. In stochastic kinetics, this homogeneity assumption is recast as the well-stirred assumption, in order to account for the existence of discrete molecules. Now it is assumed that only the probability distribution of each species is uniformly distributed over the volume. This means that in the stochastic view, at any given moment each particle of each species in a volume is equally likely to be anywhere within that volume. It is reasonable to assume that a system is well-stirred if it is small enough and if non-reactive collisions greatly outnumber collisions that result in a reaction. This description happens to describe most cellular compartments reasonably

well. The fact that water molecules tend to vastly outnumber all other kinds within a cell is enough to satisfy the "non-reactive collisions" assumption³².

Building on Eq 1.9, a propensity function (referred to by some sources as a hazard equation²¹) can be defined for each reaction R_i :

$$a_i(X) dt \equiv \text{probability that during the time interval } (t, t + dt] \text{ an } R_i \text{ reaction will take place,} \quad (1.10)$$

where X is a vector of the species counts (X_1, X_2, \dots, X_N) . The propensity function of any reaction following mass action kinetics can be defined as:

$$a_i(X) = c_i h_i(X), \quad (1.11)$$

where $h_i(X)$ is the count of sets of molecules that can participate in reaction R_i . Given the relevant Petri net matrices, a propensity function from the family defined by Eq 1.11 can be constructed for each of the M reactions of a system with N species by means of a general formula²¹:

$$a_i(X) = c_i \prod_{j=1}^N \binom{X_j}{p_{ij}}, \quad (1.12)$$

where p_{ij} is the ij th element of the Petri net matrix Pre .

In theory, all systems can be modeled using the propensity functions for

these four elementary reaction types:

$$\begin{aligned}
 a_i(X) &= c_i, & \emptyset \xrightarrow{c_i} \text{product} & \quad (\text{zeroth order}), \\
 a_i(X) &= c_i X_j, & X_j \xrightarrow{c_i} \text{product} & \quad (\text{first order}), \\
 a_i(X) &= c_i X_j X_k, & X_j + X_k \xrightarrow{c_i} \text{product} & \quad (\text{second order}), \\
 a_i(X) &= c_i \frac{X_j(X_j - 1)}{2}, & 2X_j \xrightarrow{c_i} \text{product} & \quad (\text{second order self}),
 \end{aligned} \tag{1.13}$$

where the actual formulae in the left column were derived from Eq 1.12. Other reactions, such as those of order three or above, are considered nonphysical, since an instantaneous collision of any three molecules is extremely rare. Given a unit volume ¹¹, the first three propensity functions above are identical to the corresponding deterministic rate laws, *i.e.* $a_i(X) = r_i(X)$ and $c_i = k_i$. There is no deterministic rate law equivalent to the second order self propensity function. For large enough X_j there is a reasonable approximation:

$$r_i(X) \approx \frac{c_i}{2} X_j^2 = k_i X_j^2, \quad (\text{deterministic second order self}), \tag{1.14}$$

where we've set $k_i = \frac{c_i}{2}$ above.

Technically, any reaction not in Eq 1.13 should be considered a compound reaction, and modeled by decomposition into 2 or more elementary reactions. In practice, however, it is not uncommon for modelers to use propensity

¹¹ see Wilkinson²¹, chapter 6, for full details on converting between deterministic rate laws and stochastic propensities when dealing with real volumes.

functions with a wide variety of forms, such as Hill equations:

$$a_i(X) = \frac{X_j^h}{\kappa^h + X_j^h}, \quad X_j \longrightarrow \text{product} \quad (\text{Hill equation}),$$

where h and κ are arbitrary constants. The use of such alternative propensity functions are then justified on an empirical basis (e.g. they reproduce a particular feature of the modeled system), rather than on the basis of stochastic kinetics per se³⁷.

1.5.2 The stochastic simulation algorithm

The Gillespie algorithm, or (as Gillespie himself refers to it³⁸) the stochastic simulation algorithm (SSA), was first proposed by Gillespie in 1976³². Though there have been many improvements developed for the implementation^{39,40,41,42} of the algorithm, the basic algorithm (specifically the direct method (DM) variant) itself remains 40 plus years later the gold standard of stochastic simulation.

SSA treats the underlying chemical system as a Markov process. This means that the next state that a system occupies is determined (in a probabilistic sense) solely by its present state**. The probability of each possible next state can be calculated by means of probability expressions of the form:

$$p(t', i|X) dt \equiv \text{probability that the next reaction in the system takes place in the instantaneous interval } (t + t', t + t' + dt] \text{ and is of the type } R_i. \quad (1.15)$$

**Although the framework is also flexible enough to allow factors that are explicitly time-dependent, such as externally applied chemical driving forces

The goal of SSA is to generate trajectories (*i.e.* time series) of the underlying stochastic chemical system by means of successively calculating and then drawing a random sample from the probability distribution $p(t', i|X)$ described in Eq 1.15.

The DM variant of SSA is based on the fact that $p(t', i|X)$ can be split into two independent distributions:

$$p(t', i|X) = p_1(t'|X) \cdot p_2(i|t', X)$$

Stochastic kinetic theory and basic probability can be used to derive expressions³² for p_1 and p_2 :

$$\begin{aligned} p_1(t'|X) &= Ae^{-At} \\ p_2(i|t', X) &= \frac{a_i}{A}, \end{aligned} \tag{1.16}$$

where

$$a_i = a_i(X)$$

$$A = \sum_{i=1}^M a_i(X),$$

The actual DM algorithm can be thought of as a "kinetic" Monte Carlo method²⁹ that works based on the generation of two uniform random numbers in each loop:

1. Calculate the current value of the propensity functions $a_i(X)$ and their sum $A(X)$, given the current state (t, X) where t is the current time and X is a vector of the current species counts.

2. Sample p_1 and p_2 by drawing two uniform random numbers u_1 and u_2 from the unit interval:

(a) Calculate t' as:

$$t' = \frac{1}{A} \ln \left(\frac{1}{u_1} \right)$$

(b) Calculate i as:

$$i = \text{the first value of } i \text{ such that } \sum_{j=1}^i a_j > u_2 A$$

3. Update the current time as $t + t'$, update the current species counts as $X + S_{i*}$, where S_{i*} is the i th column of the stoichiometry matrix S (see Secs 1.3 and 1.4.1).

4. Write out any desired information about the state, then either terminate the simulation (given an appropriate condition has been met), or continue by returning to step 1.

After running (and recording) many such iterations, the data collected can be thought of as an exact realization of one replicate from the ensemble of the modeled system. In real computational experiments, typically many such replicates are run and the data from them are combined to produce a final analysis. For example, the epigenetic landscape of a system can be calculated by simply binning (*i.e.* into a histogram) the species count data of one or more replicates.

1.6 Deterministic vs stochastic

In general, deterministic simulation requires less computational effort than stochastic. So why should a computational scientist bother using stochastic simulation? The theoretical justifications discussed in the previous few sections can be put into concrete terms by comparing deterministic and stochastic simulations performed on the same model. Even for very simple gene regulatory networks (GRNs) the difference in the level of detail captured by each simulation type is clear.

A comparison of deterministic and stochastic simulations of our simple expression system (see Eq 1.2) can be seen in Fig 1.7. The top panel shows results from a version of the simple expression system with a relatively high average protein expression level (around $\sim 5 \cdot 10^3$). Under these conditions, the time series produced by the deterministic and stochastic simulations (shown in the top left panel) are in reasonably good agreement. As well, the epigenetic landscape predicted by stochastic simulation (shown in the top right panel) is in good agreement with the landscape predicted by deterministic simulation (which will just be a single peak around $\sim 5 \cdot 10^3$).

On the other hand, the deterministic and stochastic simulations diverge in the limit of low protein expression. The bottom two panels of Fig 1.7 show simulations of low expression variants of the simple expression system. Though the mean expression levels of the different simulation methods match (all are around ~ 50 counts), the moment-to-moment behavior of the simulated systems do not. The deterministic simulations predict that the protein count will be constant. On the other hand, the stochastic simulations

predict (correctly) that the protein count will fluctuate significantly about the mean. Further, the deterministic simulations predict that there is no difference between the systems depicted in the middle and bottom panels, while the stochastic simulations again correctly show that the fluctuations in the bottom variant will be significantly larger. This agrees with the analysis of Ozbudak and coworkers⁴³. They defined the fluctuation strength in terms of the Fano factor $\frac{\sigma_p^2}{\langle p \rangle}$, then showed that it will be equal to:

$$\frac{\sigma_p^2}{\langle p \rangle} = 1 + \frac{k_2}{k_3 + k_4} \quad (1.17)$$

The simulations in Fig 1.7 are clear examples of one of the key weaknesses of deterministic simulation: it does not capture fluctuations correctly. These kinds of species count fluctuations are often referred to in the literature as “noise”⁴⁴. This is something of a misnomer, as cellular noise has been found to play an important role in (and sometimes be a primary driver of^{45,46}) a wide variety of decision making^{47,48} and developmental processes⁴⁹. On a more immediately visible note, the bottom two panels of Fig 1.7 show the dominant role that noise can play in moulding the epigenetic landscape. Most of the landscapes of the simple expression system variants are roughly symmetrical. Alternatively, the large fluctuations present in the noisiest variant (at the bottom of the figure) skew its landscape towards larger counts, giving it a long tail. This long tail is a nice illustration of the complex, non-symmetrical behaviors that can emerge in the limit of low copy number, demonstrating the need for a stochastic approach to the simulation of even the simplest GRNs.

If a system includes at least one nonlinear reaction (*i.e.* a reaction for which

the corresponding rate law/propensity is nonlinear), it becomes possible for the results of deterministic and stochastic simulation to diverge completely. When working with nonlinear systems, deterministic simulation is in general unable to reproduce even the correct mean behavior⁵⁰. Both the p dimerization and the $p_2 + g_p$ binding reactions of our self regulating expression system (see Eq 1.3) are second order, and thus nonlinear. Fig 1.8 shows the results from a deterministic and a stochastic simulation of this system. The deterministic and stochastic means do indeed differ significantly, by $\sim 15\%$.

Regardless of the ability of deterministic simulation to reproduce the correct mean behavior, a larger issue remains. The self regulating expression system depicted in Fig 1.8 has more than one metastable (*i.e.* long-lived) state. Evidence for this can be seen in both the time series (*e.g.* the long stretch around hour 6 during which the protein count stays fixed near 0) and via the fact that its epigenetic landscape has more than one mode/peak. Deterministic simulation cannot be used effectively to map the various states of this kind of multi-stable system, nor to calculate the dynamics of the transitions in between them. Thus, in order to construct the epigenetic landscape of a complex, nonlinear system with multiple possible states, stochastic simulation is required.

1.7 Rare events, statistics, and the limits of stochastic simulation

With respect to a particular system, an event is rare⁵¹ if it occurs at a much slower rate than the fastest event^{††}. In a multi-stable system, switching between states tends to be a rare event since it often requires the co-occurrence of specific fluctuations in two or more noisy reaction channels. For example, in order for the self regulating expression system shown in Fig 1.8 to switch from active to repressed, first a p dimerization reaction (itself a rare event) must occur. Then the $p_2 + g_p$ binding reaction must happen before the p_2 dedimerization reaction has a chance to fire.

The existence of a rare event implies a large separation between the significant timescales of a system. This timescale separation can prove challenging^{52,53} for many different simulation techniques. Theoretically, SSA simulation can exactly reproduce the behavior of any system, regardless of the presence of rare events. In practice however, it can be difficult, or even impossible, to produce an accurate simulation of a rare event systems using SSA. This apparent paradox can be understood by considering the error statistics of SSA.

When studying a rare event, it is standard practice to first determine its mean first passage time (*MFPT*), the mean waiting time before the event occurs. Given a rare event, stochastic simulation can be used to determine its *MFPT*. The procedure is equivalent to estimating the mean of a random

^{††}How much slower? There is no formal definition of a rare event, but as a rule of thumb assume that in order to qualify as rare, an event must occur at least 3 orders of magnitude less often than the system's most frequent event

variable \mathfrak{w} (in this case, \mathfrak{w} is the waiting time in between occurrences of the event). A single sample \mathfrak{w}_i is drawn from \mathfrak{w} by running a trajectory until the first occurrence of the rare event, then recording the final simulation time. n samples are drawn by running n such replicates. The mean of these samples:

$$\frac{1}{n} \sum_{i=1}^n \mathfrak{w}_i,$$

is then an estimator of $MFPT$.

The accuracy of the $MFPT$ calculated by SSA depends on the size of the sample count n . The exact form of the dependence depends in turn on the exact form of \mathfrak{w} (though it can in general be said that the accuracy increases along with the n). For a rare switching event (*e.g.* a transition that takes a system between two well separated states \mathcal{A} and \mathcal{B}), \mathfrak{w} will tend towards an exponential distribution⁵⁴, assuming that there are no intervening states. In the limit of large sample sizes, the margin of error of the stochastic $MFPT$ calculation is then given by a simple formula (see Sec 2.2.4 for derivations and a complete discussion):

$$\frac{z_\alpha}{\sqrt{n}},$$

where z_α is the z score for confidence level α (*i.e.* $z_{.95} \approx 1.96$). Fig 1.10 shows the margins of error for a wide range of sample count values. In particular, it shows that a very large (38415) sample count is required in order to ensure that a simulation produces an accurate (*i.e.* no more than 1% error) $MFPT$ value.

For simple enough systems with few enough components, the computational cost (in terms of CPU time) required for accurate stochastic simulation

is merely an inconvenience. For complex systems involving rare events, the computational cost can be prohibitive. Very roughly, the cost of sampling a rare event using standard SSA will scale as:

$$\frac{\Phi_{\text{fastest}}}{\Phi_{\text{rare}}}, \quad (1.18)$$

the quotient of the fluxes of the fastest event and the rare event. This means that for a rare enough event, the computational requirements of accurate simulation will easily eclipse the resources provided by any single computer. When a single computer is insufficient, one approach is to scale up your simulations to make use of HPC/cluster resources (an example of this is shown in Fig 1.9). A better approach is to make use of enhanced sampling. Enhanced sampling is a family of simulation methods that scale more efficiently than Eq 1.18 with respect to rare events.

1.8 Enhanced sampling

We are here concerned with essentially the same problem that some of the other speakers have spoken about. We wish to estimate the probability that a particle is transmitted through a shield, when this probability is of the order of 10^{-6} to 10^8 , and we wish to do this by sampling about a thousand life histories. It's clear that a straightforward approach will not give the results desired.

– Kahn and Harris, *Estimation of particle transmission by random sampling*⁵⁵

So begins the first paper ever written on the topic of enhanced sampling^{‡‡}. The family of enhanced sampling methods can be traced back to work done by von Neumann⁵⁶ and a few others^{57,58} in the 1950s, during the early days of Monte Carlo nuclear physics simulations.

Given that a researcher is interested in a rare event, stochastic simulation, though potentially more rigorous and accurate than the alternatives, is grossly inefficient. The trajectories of stochastic simulations are bound to follow the probability distributions of their underlying systems, and so by definition waste all but a slim minority of their time fluctuating around the high-likelihood regions of state space. Many researchers (both in systems biology and in many related fields^{vanErp:2005jua, 59,60,61}) over the years have asked themselves, “why can’t I just confine my simulations to the part of state space I actually want to see?”

The typical goal of running a stochastic simulation is to collect samples, and then use those samples to estimate the value of some system property that cannot be calculated directly. Certainly it is technically feasible to simply bias or confine a simulation so that it remains within a region of interest, producing more relevant samples more quickly. However, any artificial bias added to the simulation will also bias the samples, and thereby distort the statistics of the estimation process. It is possible to both have your cake (confine your simulation to a region) and eat it too (produce an unbiased sample) by applying bias in a controlled fashion, while at the same time keeping track

^{‡‡}Technically, Kahn et al., 1951⁵⁵ is one of two papers about enhanced sampling that were published in the same conference proceedings. However, Kahn starts on an earlier page than von Neumann, 1951⁵⁶, so I’ll call Kahn the first.

of any distortion you cause via bookkeeping. With the right information the collected samples can then be unbiased, producing the desired result. This is the essence of the enhanced sampling methods.

In the broadest terms, all enhanced sampling methods follow the same basic script:

1. Generate a biased version of the original system.
2. Draw samples from the biased system.
3. Based on bookkeeping information recorded during the preceding steps, recover an unbiased sample by applying statistical methods to the biased sample.

There are now dozens (if not hundreds) of different enhanced sampling methods, and while some of them may not strictly follow these steps, each and every one of them performs the 3 actions listed above in some fashion.

1.9 Enhanced sampling methods for stochastic simulation of biochemistry

Beginning with the proposal of FFS in 2005⁶², a number of different enhanced sampling methods have been developed for use in stochastic simulations of biochemical systems. Along with FFS, nonequilibrium umbrella sampling (NEUS) and weighted ensemble (WE) are also highly cited methods.

FFS, NEUS and WE have a number of features in common. Each method requires the user to specify a function of their system's complete state to use

as an order parameter:

$$\mathcal{O}(t, X) = (o_0, o_1, \dots)$$

Additionally, each requires the user to specify a region of interest in terms of a set of bins that span an interval along the order parameter. Taken together, the order parameter and the tiling define a constraint that will be used to bias the user’s system. The order parameter defines the degrees of freedom of the constraint, while the tiling defines one or more bounded regions along those degrees of freedom. Ultimately this constraint is used to guide the system into and along the region of interest, though the actual implementation of the constraint varies from method to method.

1.9.1 Forward flux sampling

The first step of FFS is to define two points of interest in your system’s state space (we’ll call these \mathcal{A} and \mathcal{B}), a 1D order parameter that can unambiguously differentiate those two points, and a 1D sequence $(\lambda_0, \lambda_1, \dots, \lambda_N)$ of “interfaces” (*i.e.* bins) that lie along the order parameter in between \mathcal{A} and \mathcal{B} . If a trajectory passes below $\lambda_{\mathcal{A}} = \lambda_0$ it is said to be in the \mathcal{A} state, and if it passes above $\lambda_{\mathcal{B}} = \lambda_N$ it is said to be in the \mathcal{B} state. FFS also requires the user to manually specify a maximum simulation time T and a set of trajectory counts (n_1, n_2, \dots, n_N) , the significance of which are discussed in the following paragraphs.

Given the above setup, the goal of an FFS simulation is to calculate $\Phi_{\mathcal{A}, \mathcal{B}}$, the flux from state \mathcal{A} into \mathcal{B} . $\Phi_{\mathcal{A}, \mathcal{B}}$ can be decomposed into a smaller flux and

a probability term:

$$\Phi_{\mathcal{A},\mathcal{B}} = \Phi_{\mathcal{A},0} P(\lambda_{\mathcal{B}} | \lambda_0),$$

where $\Phi_{\mathcal{A},0}$ is the flux from the vicinity of \mathcal{A} to the initial interface λ_0 , and $P(\lambda_{\mathcal{B}} | \lambda_0)$ is the probability that a trajectory will reach \mathcal{B} before it falls below λ_0 , given that the trajectory is currently at λ_0 . The probability can be further decomposed into a sequence of probabilities along the interfaces:

$$\Phi_{\mathcal{A},\mathcal{B}} = \Phi_{\mathcal{A},0} \prod_{i=1}^N P(\lambda_i | \lambda_{i-1}). \quad (1.19)$$

The actual FFS simulation consists of an ordered sequence of $N + 1$ distinct phases. A schematic illustration of these phases is shown in Fig 1.11. Each phase produces an estimate of one of the terms on the left hand side of Eq 1.19.

The simulation begins with phase 0, at the start of which a single unbiased trajectory is initialized at \mathcal{A} and allowed to run until the user-specified simulation time T . Whenever this trajectory crosses λ_0 while traveling forward (*i.e.* towards \mathcal{B}) the species counts at the point of crossing are recorded in a list C_0 . If the trajectory ever leaves state \mathcal{A} (*e.g.* by crossing into \mathcal{B}), the simulation time is paused, and doesn't resume incrementing until the trajectory reenters \mathcal{A} . The phase 0 trajectory is terminated once it reaches time T , and the flux term in Eq 1.19 is then estimated as:

$$\Phi_{\mathcal{A},0} = \frac{n_0^s}{T},$$

where n_0^s is the count of entries in C_0 . Phase 0 then ends, and phase 1 begins.

During each phase $i > 0$, a user-defined number n_i of trajectories are run.

Each trajectory is initialized at a randomly chosen point from C_{i-1} , the list of crossing points from the previous phase. The trajectory is run until it either falls below λ_0 or passes above λ_i . If the trajectory passed above λ_i its species count at the point of crossing is recorded in C_i , and in either case the trajectory is terminated. Once n_i trajectories have run, the corresponding probability term in Eq 1.19 is then estimated as:

$$P(\lambda_i | \lambda_{i-1}) = \frac{n_i^s}{n_i}.$$

Phase i then ends, and $i + 1$ begins. Once phase N finishes, the FFS simulation is over. The flux $\Phi_{\mathcal{A},\mathcal{B}}$ can be calculated from Eq 1.19, and the *MFPT* can be calculated as the inverse flux, $\Phi_{\mathcal{A},\mathcal{B}}^{-1} = MFPT$.

Various extensions for the FFS method have been developed. One of the more useful ones is from a follow-up paper⁶³ that showed how FFS can be used to calculate a wide variety of values beyond flux and *MFPT*. They define a set of weights that can be used to combine biased data collected during the individual phases into a single set of unbiased results, equivalent to those that could be gained from an unconstrained SSA simulation. Among other things, this enables the use of FFS to rapidly construct the epigenetic landscape of a system.

A more complete description of FFS (including a including a step-by-step listing of the algorithm) is given later in this thesis in Sec 2.2.1. The version given in Sec 2.2.1 differs somewhat from the original implementation of FFS, with each change made either for the sake of parallelizing the algorithm, or to make one of the method's outputs more amenable to statistical analysis. On

the other hand, the description given above is faithful to the version from the originating papers^{62,64,65}.

1.9.2 Nonequilibrium umbrella sampling

NEUS was first proposed⁶⁶ in 2007, a couple of years after FFS was introduced, and bears it a number of similarities.^{67,68}

Fig 1.12

1.9.3 Weighted ensemble

WE is a method that originated⁵⁹ in the molecular dynamics community. More than a decade after it was first proposed, a paper⁶⁹ was published that showed how WE could be adapted for use with stochastic biochemical simulations.^{70,71,72}

Fig 1.13

Interestingly, WE turns out to be nearly identical to the “splitting technique” attributed to von Neumann in the first enhanced sampling paper⁵⁵. Thus, WE can be thought of as an accidental rediscovery of the splitting method⁷³.

1.10 Simplifying and optimizing forward flux

The original formulation of FFS⁶² left a great of room for improvement. Specifically, FFS adds a large number of unspecified degrees of freedom to a system,

in the form of a large set of extra simulation parameters. When a computational scientist sets up a FFS simulation, she must specify two points of interest, a 1D order parameter, a set of interfaces λ_i , the phase 0 maximum simulation time T , and the set of phase $i > 0$ trajectory counts n_i . These simulation parameters are required in addition to all of the model data required for a standard SSA simulation. In theory, the choice of FFS parameters will not have an effect on simulation outcome⁶⁵, only on simulation efficiency.

The additional simulation parameters present several challenges to users. For new users, the FFS parameters can be an imposing hurdle. They are likely one of the stumbling blocks preventing wider adoption of FFS methods. In particular, the order parameter and the interfaces are conceptually complicated. It takes time for a beginner to learn how to find reasonable values for these parameters. Even for an experienced user it can take a significant amount of trial and error⁷⁴ to find good values when working with a novel system. Further, for most systems there exists an optimal⁷⁵ set of FFS parameters that will maximize their efficiency and minimize their runtime.

It is possible to devise automated methods that can find and set the optimal FFS parameters values without user input. Several examples are discussed in the paragraphs below. These optimization routines kill two birds with one stone: they simplify the setup of an FFS simulation and make the method easier to use, while also speeding the simulation up. Between the published optimization methods and the work presented in this thesis (see Sec 1.11), it is possible to find optimized values of every FFS parameter (even the order parameter⁷⁶). Thus, it should be possible to devise a method that can

automatically set an optimal method of every parameter all at once, though none has yet been published.

Along this vein, Borrero and coworkers devised a number of different optimization routines⁷⁷ for FFS. These routines are designed around the concept that the user's computational resources are limited. Thus, they require the user to input a quantity of computational effort, given in terms of the total number of trajectories that will be run during a single FFS simulation. The routines then find an optimal set of parameters that minimizes overall simulation error while holding the computational effort fixed at the user-specified level. For standard FFS, they devised one routine that can be used to find the optimal choice of n_i (the number of trajectories to run in each phase $i > 0$), and another that can be used to find the optimal choice of λ_i (the placement of the interfaces along the 1D order parameter). Since computational effort is held fixed, Borrero's optimization methods will not speed a simulation up, but will instead decrease the level of error in the simulation's results. Indeed, they found that their optimization methods could reduce simulation error by as much as 40%.

Borrero's interface optimization method generates an initial rough guess of λ_i and then using iterative rounds of FFS simulation in order to refine it. Kratzer and coworkers developed a more elegant alternative approach⁷⁴ to optimizing λ_i that avoids the need to perform entire extra simulations. Instead, they figured out a way to dynamically generate an optimal placement of interfaces, one at a time, during an otherwise standard FFS simulation. Kratzer describes two different variants of his method, but both follow roughly

the same script: at the start of a simulation only the first and last interfaces, λ_0 and λ_N , are defined. At the start of each phase $i > 0$, a fixed number of trial trajectories are launched from λ_{i-1} under controlled conditions. Based on the outcomes of these trial trajectories, a location for λ_i is chosen, and the FFS simulation proceeds as normal (until the start of the next phase). Kratzer found that his dynamic interface optimization methods were able to speed simulations up by as much as 2X, relative to a simulation run with manually placed interfaces.

The optimization schemes of both Borrero and Kratzer ignore the contribution of phase 0 to the simulation error. Technically, they both treat the outcome of phase 0 as deterministic, causing it to “fall out” of their analyses of the simulation variances and errors. This is standard practice⁷⁸ in analyses of the error statistics of FFS. We devised a novel approach to the analysis of FFS error that allows the contribution of phase 0 to be calculated directly (see Sec 2.3.1.1). Contrary to previous claims⁶⁵ in the literature, it can be shown that phase 0 can be a large contributor to overall simulation error (see Secs 2.3.4 and 2.3.5).

1.11 Forward flux pilot sampling

In many ways enhanced sampling seems too good to be true. It seems kind of like something for nothing, or a free lunch. Though the developers of various enhanced sampling methods have claimed that they are faster than direct sampling while introducing no extra approximations or errors^{79,78}, it remains to be definitely proven one way or the other. Thus it is reasonable to ask

“does enhanced sampling actually work? Will it actually produce an answer of equivalent accuracy in less time than the established methods?” More prosaically, when performing FFS simulations, a researcher might wonder “how many trajectories should I run in each phase in order to get a result without too much error?” All of these are questions that the work presented in the subsequent chapters of this thesis attempts to answer.

In the next chapter we present a new analysis of the error in the output of FFS simulations. Using a novel derivation, we find a more general form of the FFS error relation (*i.e.* simulation error as a function of the simulation parameters) than has been presented in the literature^{78,77,65}. In particular, this allows us to calculate the previously unconsidered error arising from phase 0. We show that this phase 0 error can indeed be a significant contribution to the overall simulation error.

Next, we derive an equation that gives the optimal (in terms of computational cost) number of trajectories to run in each phase, given a user-defined desired maximum level of error in the simulation results, which we call an error goal. Based on this optimizing equation, we develop a novel variant of the FFS enhanced sampling method which we call forward flux pilot sampling (FFPilot). FFPilot replaces the T and the entire set of n_i parameters of standard FFS (*i.e.* the parameters that determine the computational effort expended during each phase) with a single error goal parameter. Given that error goal, FFPilot will plan out and run the fastest possible FFS simulation. Thus, FFPilot is both a simplification and an optimization of the FFS approach. We wrote an optimized, fully parallelized implementation of the FFPilot algorithm in

C++. This implementation of FFPilot was added to the Lattice Microbes⁸⁰, a stochastic simulation software package published by the Roberts Lab.

The remainder of the chapter is dedicated to a thorough validation and exploration of FFPilot. In simulations of 1D biochemical systems (*i.e.* systems with only a single chemical species), we show that FFPilot is indeed able to control error in the final simulation results as expected. In fact, FFPilot controls error so well that it is able to uncover a previously invisible problem in all existing analyses of FFS error, including my own. The published error analyses^{78,77,65} all assume that sampling error (*i.e.* error due to running too few trajectories) is the sole source of simulation error. However, when using FFS to simulate a complex system with a rough, multidimensional epigenetic landscape, it turns out that other sources of error can become significant. Results from FFPilot simulations of multidimensional systems show that while sampling error is still the dominant source of error, there is an anomalous extra error that FFPilot is unable to control. We demonstrate that the anomalous error is due to correlations between the trajectory starting point distributions along different interfaces.

The final chapter of the thesis is a tutorial that explains in detail how to use the FFPilot implementation in Lattice Microbes. Complete examples are given that show how to use FFPilot to calculate both the *MFPT* and the epigenetic landscape of a system.

Figures

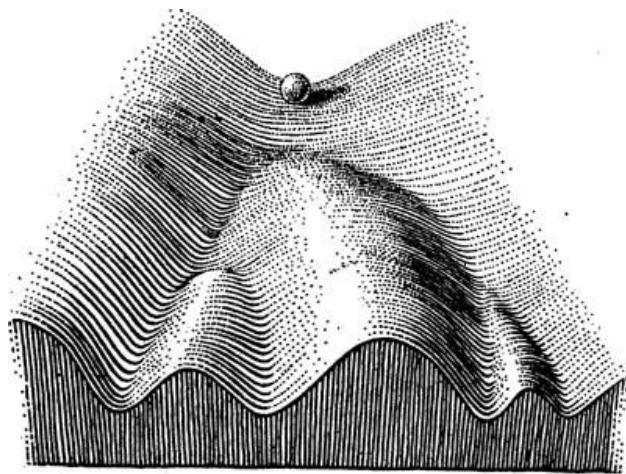


Figure 1.1: The epigenetic landscape of a developing cell, as envisioned by Waddington. The ball represents a cell at the beginning of development. Eventually the ball/cell will reach the bottom of the hill, ending up in one of several possible terminal cell types. The steep terrain counteracts most perturbations by forcing the ball back onto the path, making the whole process robust. Reprinted from Waddington, 1957⁷.

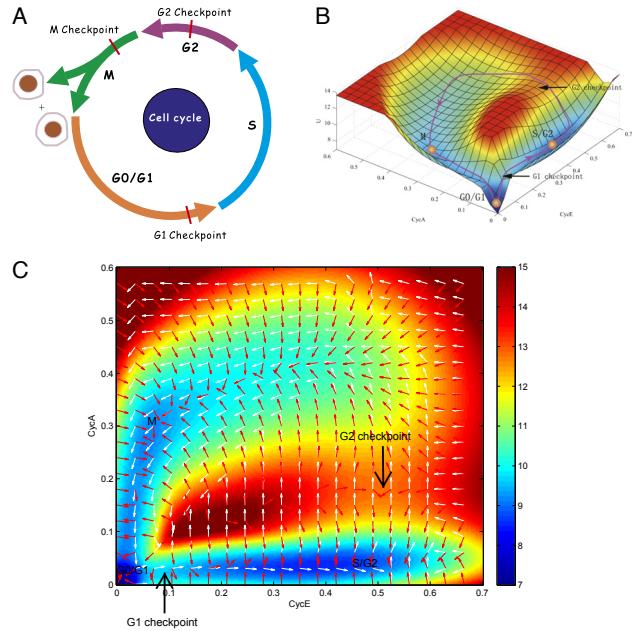


Figure 1.2: Three views of the mammalian cell cycle. A) The classical schematic view. B) The potential surface of the cell cycle. Data was taken from a 44 dimensional model and projected onto the concentrations of two cyclins, CycA and CycE (arbitrary units). The low points on the surface correspond to the various phenotypes along the cell cycle, and the transition path between them is shown. C) A 2D rendition of the complete epigenetic landscape of the model shown in B. The red arrows show the gradient of the potential, whereas the white arrows show the probability flux. Note that they never exactly coincide. Reprinted from Li et al., 2014¹².

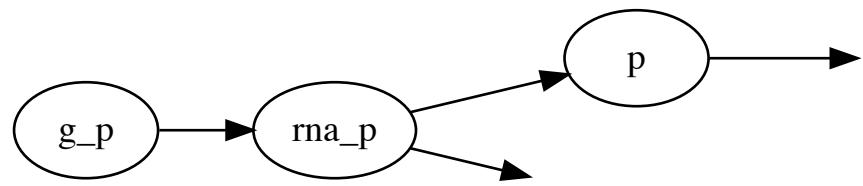


Figure 1.3: A simple digraph representation of the simple gene expression model from Eq 1.2. An arrow from one species to another implies that the first species is involved in the production of the second. Arrows to empty space imply a decay reaction.

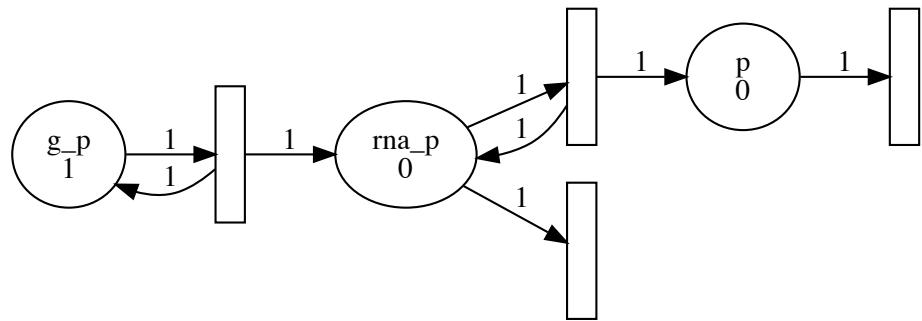


Figure 1.4: A petri net representation of the simple genetic expression model from Eq 1.2. Arrows starting at places (round nodes) and ending at transitions (rectangular nodes) imply that the connected species is a reactant in the connected reaction. Arrows starting at transitions and ending at places imply that the connected reaction produces the connected species. Above each arrow is a weight that gives its stoichiometry. The marking (the count of each species) is shown underneath the name of each species. Transitions that have no arrows leaving them imply a decay reaction.

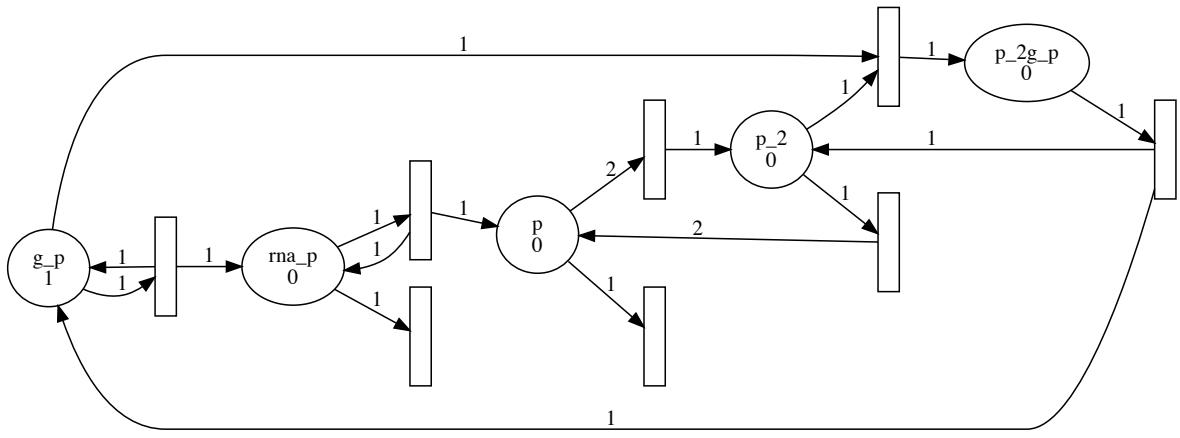


Figure 1.5: A petri net representation of the self regulating genetic expression model from Eq 1.3. The network is represented as described in Fig 1.4.

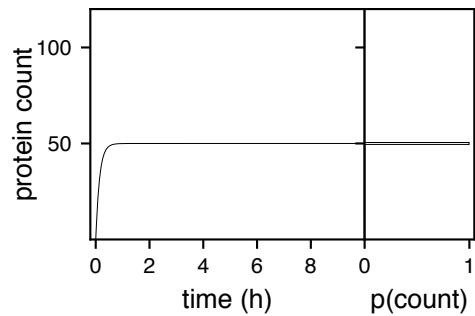


Figure 1.6: Time series from a deterministic simulation of the simple expression system (see Eq 1.2) with $k_1 = .1, k_2 = .1, k_3 = .1, k_4 = 2 \cdot 10^{-3}$. The subplot along the left side shows the epigenetic landscape (as calculated via a histogram of the adjacent time series).

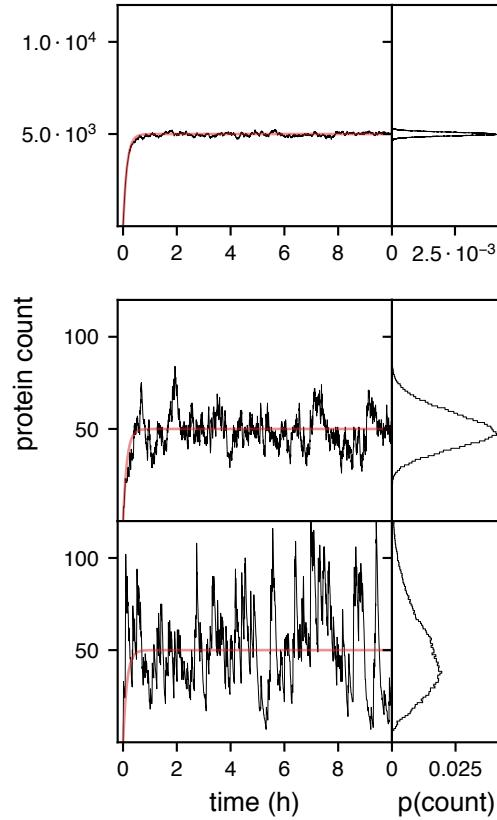


Figure 1.7: (Black lines) time series from stochastic simulations of the simple expression system (see Eq 1.2). The subplots along the left side show the epigenetic landscape of each variant (as calculated via a histogram of data from 10 simulation days, including the adjacent stochastic time series). (Red lines) the deterministic simulation of the same system, for comparison purposes. (Top panel) simple expression system with $k_1 = 10, k_2 = .1, k_3 = .1, k_4 = 2 \cdot 10^{-3}$ (Middle panel) same, with $k_1 = .1, k_2 = .1, k_3 = .1, k_4 = 2 \cdot 10^{-3}$. (Bottom panel) same, with $k_1 = .01, k_2 = 1, k_3 = .1, k_4 = 2 \cdot 10^{-3}$. The same constants were used in the deterministic and stochastic simulations (*i.e.* $c_i = k_i$). The mean expression level of the system simulated in the top panel is 100x that of the systems in the bottom two panels. From top to bottom the noise strengths are 1.98, 1.98, and 10.8039 (see Eq 1.17).

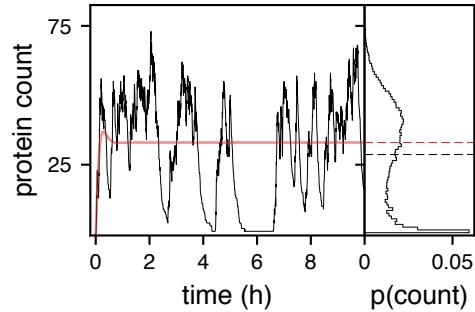


Figure 1.8: (Black line) time series from a stochastic simulation of the self regulating expression system (see Eq 1.3). (Red line) the deterministic simulation of the same system. The subplot along the left side shows the epigenetic landscape. (Black and red dashed lines) the mean protein expression value as calculated by stochastic and deterministic simulation, at 28.7 and 33.0 respectively (15% divergence). The deterministic rate constants used were $k_1 = .1, k_2 = .1, k_3 = .1, k_4 = 2 \cdot 10^{-3}, k_5 = 10^{-6}, k_6 = 1, k_7 = 1, k_8 = 2 \cdot 10^{-3}$. The stochastic rate constants were the same (*i.e.* $c_i = k_i$), except for $c_5 = 2 \cdot k_5 = 2 \cdot 10^{-6}$ (as per Eq 1.14).

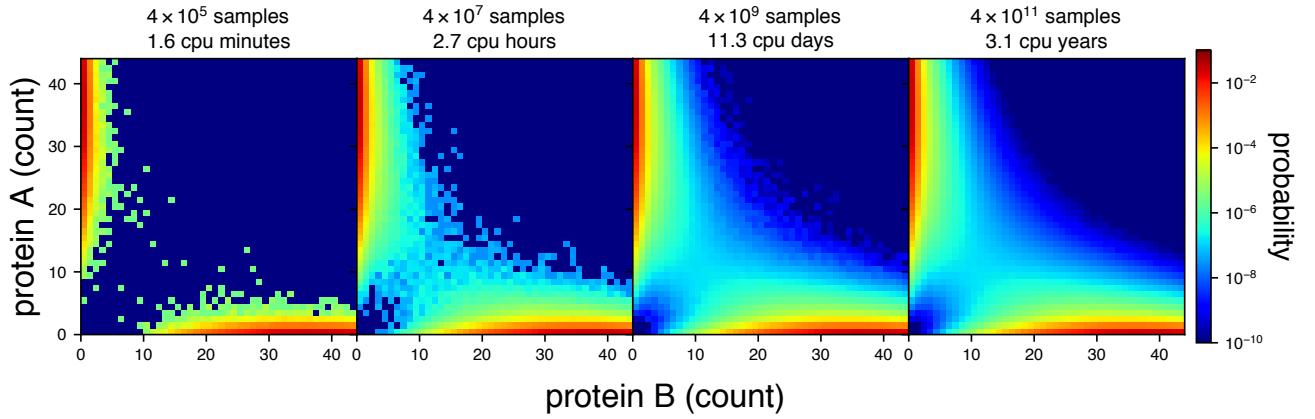


Figure 1.9: Two dimensional epigenetic landscapes generated from the $\text{GTS}_{\theta=1}$ system (described in Sec 2.3.5). From left to right, the subplots show versions of the same landscape calculated using an increasing number of species count samples. The total sample count used in each landscape, and the CPU time used to collect those samples, is shown above each subplot. The samples were taken from 20 trajectories, each of which was generated using several thousand CPU cores running a parallelized implementation of SSA⁸⁰.

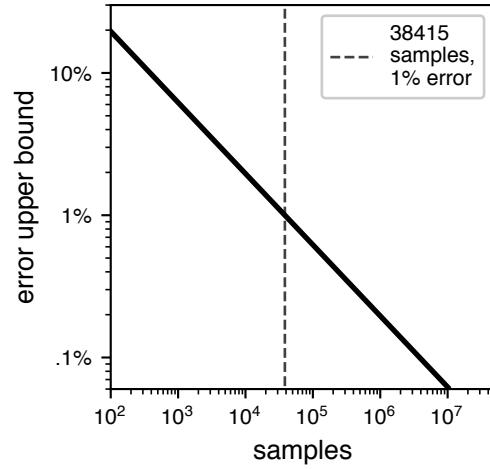


Figure 1.10: The margin of error (95% confidence) vs sample count when using stochastic simulation to calculate the *MFPT* of a rare state switching event. The x-axis begins at 100 in order to emphasize that the relationship is only valid in the limit of large sample size. See Sec 2.2.4 for derivation and details.

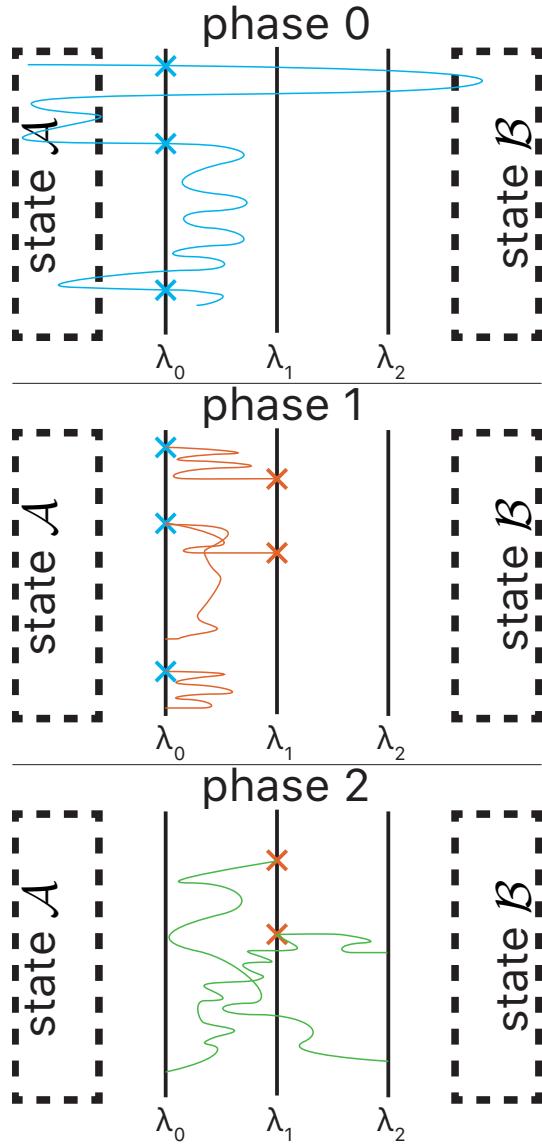


Figure 1.11: A schematic representation of an FFS simulation run from $\mathcal{A} \rightarrow \mathcal{B}$. The simulation shown has 3 phases, one for each interface λ_i that is defined. During each phase $i > 0$, trajectories are initialized from one of the points (chosen at random) where a trajectory in the previous phase crossed λ_{i-1} while traveling in the forward direction (*i.e.* towards \mathcal{B}). During phase 2 (the final phase), trajectories that cross λ_2 are considered to have completed the transition into \mathcal{B} .

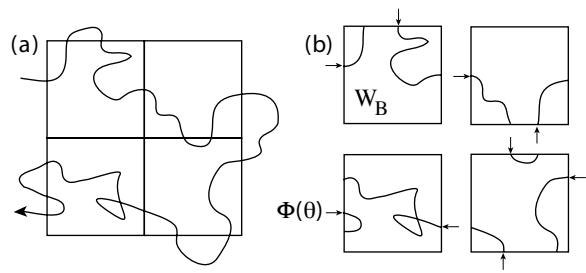


Figure 1.12: Reprinted from Dickson et al., 2004⁶⁷.

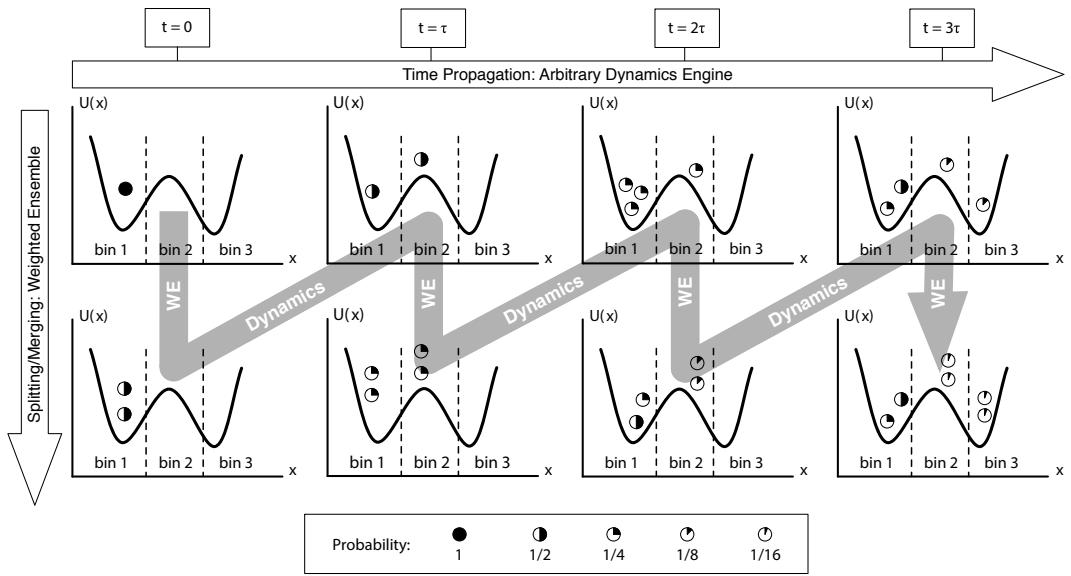


Figure 1.13: Reprinted from Donovan et al., 2013⁷⁰.

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Chapter 2

Automatic error control during forward flux sampling of rare events in master equation models

2.1 Introduction

Understanding how inanimate biochemical molecules come together and interact to form a living cell is one of the fundamental goals of biology¹. The cell's state can be described as a point in a high dimensional space, where each dimension corresponds to the concentration of a different molecule. Complex, nonequilibrium regulatory and signaling networks connect these molecules through positive and negative interactions, naturally resulting in a large number of metastable states within the space², representing different cellular phenotypes. Projected into two dimensions, the phase space appears as a rough landscape, sometimes referred to as a quasipotential², epigenetic^{Waddington:1957ub}, or phenotypic³ landscape. Stochastic fluctuations cause the cell's state to move along the landscape and occasionally jump between metastable states.

Metastable systems, because they depend upon random fluctuations, are typically modeled using a formulation of the chemical master equation^{4,5} or using stochastic differential equations⁶. In the former case, the models are often numerically studied using the stochastic simulation algorithm^{7,8} (SSA) or one of its many varieties^{9,10,11}. Such simulations have provided insight into diverse biological processes, including: the lysis/lysogeny decision in bacteriophage λ ¹², the *lac* operon in *Escherichia coli*¹³, check-pointing during the cell cycle¹⁴, differentiation of stem cells¹⁵, the binding of an intrinsically disordered peptide to a protein¹⁶, macrophage regulation¹⁷, and gradient detection during yeast mating¹⁸.

Transitions between metastable states in stochastic biochemical systems are infrequent in that one must wait a long time to observe a large fluctuation that causes the system to switch states¹⁹. The time spent in the transition region is also very short relative to the waiting time²⁰. Such dynamics are known as rare events, and are expensive to simulate as most of the computational effort is spent simply simulating the waiting state. Numerical methods for improving the efficiency of simulating rare events, generally known as enhanced sampling (ES) techniques, have a long history. The earliest work in the field is typically credited to Kahn *et al.*²¹, but has since been applied to the study of many systems in molecular mechanics. The key assumption is that metastable systems exhibit a large barrier separating the states on some free energy landscape. By biasing the simulation toward the transition path between the states one can more efficiently recover its free energy profile and,

thus, the transition rates. For example: umbrella sampling^{22,23} uses overlapping biasing potentials to confine multiple simulations to narrow windows along the path; metadynamics^{24,25} gradually adds a repulsive potential to low points on the free energy surface causing the system to explore low probability regions of phase space and to eventually cross the barrier; transition path sampling^{26,27,28} generates a statistically correct set of transition paths starting from an initial trial path using acceptance and rejection criteria; weighted ensemble^{29,30} runs multiple independent trajectories while dynamically splitting and merging them, with careful accounting of the trajectory weights, to balance simulations along the transition path. In general, a final unbiasing and/or recombination step is always needed to calculate unbiased statistics.

Although stochastic biochemical systems are described by quasipotential rather than free energy landscapes, the underlying physics is compatible with ES. Consequently, many varieties of ES can be applied to stochastic biochemical systems with rare event dynamics. Three of the most well known candidates are FFS^{31,32,33,34}, nonequilibrium umbrella sampling^{35,36,37}, and weighted ensemble^{38,39,40,41,42}. Although much discussion has taken place regarding the strengths and weaknesses of each of these methods^{43,44} there is no consensus as to an optimal approach. In the case of a transition between only two metastable states along a single order parameter, FFS appears to be a reasonable choice and is the focus of this work.

The FFS method can be used to calculate both the transition rate, *i.e.*, the inverse of the *MFPT*, between two metastable states and the probability

distribution along the transition path³³. Although it can be applied to non-stationary processes⁴⁵, here we investigate only stationary processes. The fundamental operation of FFS is to partition the phase space along an order parameter using a series of non-intersecting interfaces that track progress over the transition barrier. Many trial trajectories are started successively from each interface in order to compute the probability of advancing to the next interface versus returning to the initial state. The product of the advancement probabilities and the probability flux out of the initial state gives the mean transition rate.

The generation of many trial trajectories at each of the interfaces is a large part of the computational work in FFS. It is not surprising, then, that a number of authors have tried to optimize performance of FFS by studying the interplay between the number of trajectories sampled and the statistical error. Allen *et al.*⁴⁶ first introduced a framework for studying the relationship between error in the estimated transition rate constant and error in the estimated interface trial probabilities. They also studied the computational efficiency of FFS, taken to be the inverse of cost times error, and showed that it was relatively insensitive to the choice of interface parameters. Borrero and Escobedo⁴⁷ presented two techniques to minimize FFS error for a fixed cost either by optimizing the number of trial runs at each interface or by iteratively refining the placement of the interfaces. Kratzer *et al.*⁴⁸ extended this idea with an algorithm to automatically define interface number and position on-the-fly by constraining the number of successful trials to be the same for each interface. However, none of these previous works included a systematic treatment of the

error arising from the estimate of the flux out of the initial state and they all assume that computational cost per unit distance along the order parameter is fixed. Jian *et al.*⁴⁹ later showed that interface placement along a complex transition landscape, *e.g.* one with a metastable intermediate state, cannot be correctly optimized by methods that assume cost is proportional to interface distance.

Here we present a method to optimally perform an FFS simulation of a nonequilibrium stationary process at a given margin of error and confidence interval. The key advance of our method is to estimate all of the interface probabilities and costs from a short pilot simulation and then use those parameters to optimize the configuration of a full FFS simulation. We minimize the total computational cost to achieve a user specified statistical error, which greatly reduces the number of choices that need to be made by the user. Our method includes a new formal treatment of the sampling error arising in the calculation of the flux out of the initial state, which is shown to have a significant influence on total cost. Unlike previous efforts, the cost in our method varies by interface, which enables it to account for changes in computational efficiency along the order parameter. We evaluate the capability of our method to control sampling error on three different models exhibiting rare event dynamics. In two one-dimensional models, sampling error dominates and is controlled precisely while in a multidimensional model landscape error becomes significant and requires oversampling. Finally, we derive an expression for the speedup of FFS relative to direct simulation for the same level of error and show that the advantage of FFS is substantial and increases

with the rarity of the event.

The remainder of this work is organized as follows: In Sec 2.2 we introduce our theoretical framework for estimating the sampling error in a FFS simulation. In Secs 2.3.1 and 2.3.2 we derive the optimizing equation used by our method to control sampling error and then describe the FFPilot algorithm. In Secs 2.3.3-2.3.5 we evaluate the accuracy and performance of our method using three increasingly complex models. In Secs 2.3.6 and 2.3.7 we analyze sources of error outside of sampling error. Finally, in Secs 2.3.8 and 2.3.9 we evaluate the theoretical efficiency and measure the performance of FFPilot.

2.2 Theory and Methods

2.2.1 Simulation of Rare Events in Stochastic Processes

The standard stochastic simulation protocol, here referred to as direct sampling (DS), proceeds in two iterated steps: 1) increment the system time with the time of the next stochastic event, 2) update the system state according to the event. These two steps are repeated until some predetermined stopping condition (simulation steps, simulation time, etc.) is reached, at which point the simulation is terminated. Repeated DS simulation produces a data set from which ensemble average quantities can be estimated by straightforward averaging. Running more DS replicates leads to increased accuracy.

As discussed above, DS has the disadvantage that it requires a long simulation time to sample each rare event. Therefore, calculating rare event statistics is computationally expensive. Enhanced sampling (ES) methods use

a combination of constraints on simulation trajectories and statistical unbiasing methods in order to enrich the sampling of rare events. Forward flux sampling (FFS) is a popular ES method that was initially proposed by Allen and coworkers³¹. We implemented the FFS algorithm as follows:

1. Choose two points of interest in the state space of the model system, \mathcal{A} and \mathcal{B} . These will serve as an initial and a final state for the simulation.
2. Choose a one dimensional parameter \mathcal{O} for which $\mathcal{O}(\mathcal{A}) \neq \mathcal{O}(\mathcal{B})$. If $\mathcal{O}(\mathcal{A}) > \mathcal{O}(\mathcal{B})$, swap $\mathcal{A} \Leftrightarrow \mathcal{B}$.
3. Choose a set $\{\lambda_0, \dots, \lambda_N\}$ of interface values of \mathcal{O} such that $\mathcal{O}(\mathcal{A}) < \lambda_i < \mathcal{O}(\mathcal{B})$ for every λ_i . We say that a trajectory has fluxed forward with respect to a λ_i when it crosses the interface traveling in the direction of increasing \mathcal{O} , and that it has fluxed backward when it crosses traveling in the direction of decreasing \mathcal{O} .
4. Begin FFS phase 0:
 - (a) Execute a DS trajectory initialized at \mathcal{A} . If the trajectory fluxes forward across λ_0 , record the elapsed simulation time τ since the previous crossing event, and also record the state \mathcal{X} . Multiple phase 0 trajectories can be executed in parallel.
 - (b) If the trajectory ever crosses the final interface λ_N , pause the tracking of the waiting time until the trajectory moves backward across λ_0 .
 - (c) Once n_0 samples of τ have been collected, terminate the trajectories and move to the next phase.

5. Begin FFS phase $i > 0$:
 - (a) Randomly choose a state \mathcal{X} from the collection of states at which any trajectory in the previous phase crossed λ_{i-1} .
 - (b) Execute a DS trajectory initialized at \mathcal{X} . Allow the trajectory to run until it either crosses λ_0 back into \mathcal{A} or moves forward across λ_i . Terminate the trajectory and record a trajectory outcome value, either 0 or 1, depending on whether the trajectory moved backward or forward, respectively. If the trajectory moved forward, add the endpoint, which will lie along λ_i , to the set of states that will be used to initialize trajectories during the next phase.
 - (c) Repeat steps 5a-b.
 - (d) Once n_i trajectory outcomes have been collected terminate the phase. Every phase i trajectory can be executed in parallel.
6. The procedure for FFS phase i is then repeated for phase $i + 1$ (during which trajectories are launched from λ_i) until the final phase, phase N , is reached. Trajectories in phase N begin at λ_{N-1} and may move forward across λ_N and into \mathcal{B} .

The overall aim of FFS is to ratchet a simulation from $\mathcal{A} \rightarrow \mathcal{B}$ across state space. The immediate goal of each phase is to estimate a phase weight. During phase i , samples are taken from an observable w_i that behaves as a random variable. w_0 is the waiting time in between forward flux events across λ_0 , and each $w_{i>0}$ is the final outcome (0 for fall back or 1 for flux forward) of each trajectory launched in that i . The phase weight w_i is defined as the true

expected value of the random variable w_i :

$$w_i = E [w_i] = \begin{cases} \tau_{\mathcal{A}} & \text{if } i = 0, \\ P(\lambda_i | \lambda_{i-1}) & \text{otherwise,} \end{cases} \quad (2.1)$$

where $\tau_{\mathcal{A}}$ is the expected waiting time in between λ_0 crossing events, and $P(\lambda_i | \lambda_{i-1})$ is the probability that a trajectory launched from λ_{i-1} (*i.e.* launched during phase i) crosses forward past λ_i before it falls back behind the starting interface λ_0 . The phase weights can be used to reweight the output of an FFS simulation so as to calculate unbiased statistics.

2.2.2 Estimating Values of Interest from DS and ES Stochastic Simulations

As mentioned in the previous section, estimating values of interest from DS simulation is straightforward. One valid estimator of any ensemble average quantity is simply the arithmetic mean taken across an appropriate set of direct observations. For example, in order to calculate the *MFPT* of the switching process that takes some multistate system from $\mathcal{A} \rightarrow \mathcal{B}$, n DS simulations are initialized in \mathcal{A} . Each of these n replicate simulations is allowed to run until the first time it enters \mathcal{B} , at which point it is terminated. The time that each simulation i ran for is then a single sample FPT_i of the first passage time of the switching process. The mean of these first passage time samples is an estimate of *MFPT*⁵⁰:

$$\widehat{MFPT}_{ds} = \frac{1}{n} \sum_{i=1}^n FPT_i, \quad (2.2)$$

where \widehat{MFPT}_{ds} is the estimator (*i.e.* estimation function) of *MFPT* specific to DS simulation. Throughout this paper, we place a $\widehat{}$ above a symbol to

indicate that we are referring to an estimator of a value rather than to the value itself.

Equivalent ensemble average estimators can be calculated using FFS. Although the results of FFS simulations are biased and partitioned, statistical protocols allow FFS results to be recombined and reweighted so as to recapitulate the results of the unbiased ensemble. In order to perform these reweightings, we need estimates of the phase weights. During phase i of FFS, n_i samples $\{\mathfrak{w}_{i1}, \dots, \mathfrak{w}_{in_i}\}$ of an underlying random process \mathfrak{w}_i are taken. The mean of this sample can be used as an estimator \hat{w}_i of phase weight i :

$$\hat{w}_i = \frac{\sum_{j=1}^{n_i} \mathfrak{w}_{ij}}{n_i} = \begin{cases} \frac{\sum_{j=1}^{n_0} \tau_j}{n_0} & \text{if } i = 0, \\ \frac{n_i^s}{n_i} & \text{otherwise,} \end{cases} \quad (2.3)$$

where each τ_i is a sample of the waiting time in between phase 0 forward flux events and n_i^s is the total count of trajectories that successfully fluxed forward during phase $i > 0$.

In an idealized situation in which it were possible to know the exact values of the phase weights w_i , the exact value of the *MFPT* could be found via a simple combination W of the phase weights:

$$MFPT = \frac{\tau_A}{\prod_{i>0} P(\lambda_i | \lambda_{i-1})} = \frac{w_0}{\prod_{i>0} w_i} = W. \quad (2.4)$$

In reality, we only know the phase weight estimators \hat{w}_i , so we must instead estimate the value of the *MFPT* by way of the combination of estimators \hat{W} :

$$\widehat{MFPT}_{\text{ffs}} = \frac{\hat{w}_0}{\prod_{i>0} \hat{w}_i} = \hat{W}. \quad (2.5)$$

where $\widehat{MFPT}_{\text{ffs}}$ is the estimator of *MFPT* specific to FFS simulation. Other

quantities of interest, such as the stationary PDF, can also be calculated using the phase weights³³.

2.2.3 Predicting Simulation Error in Terms of Margin of Error

Stochastic simulation results are not single valued, but are instead estimators, random variables with associated distributions. For example, when attempting to calculate $MFPT$, each complete round of simulations can be thought of as performing a single draw from the distribution of possible \widehat{MFPT} values. A prediction about the likely level of error in any given set of simulations can be made based on the characteristics of this estimate distribution. One way to quantify confidence in this prediction is in terms of the margin of error. The margin of error of a random variable X is defined as the ratio of half the width of a specified confidence interval and its expected value:

$$\zeta [X, \alpha] = \frac{ub_\alpha [X] - lb_\alpha [X]}{2E [X]}, \quad (2.6)$$

where $\zeta [X, \alpha]$ is the margin of error of X at confidence level α , $E [X]$ is the expected value of X , and $lb_\alpha [X]$ and $ub_\alpha [X]$ are the lower and upper confidence bounds, respectively.

For many values of interest $\zeta [X, \alpha]$ is dependent on simulation parameters that are set by the user. For example, when calculating $MFPT$, the margin of error $\zeta [\widehat{MFPT}, \alpha]$ is dependent upon the total number of replicate simulations (when using DS), or upon the count of trajectories run in each phase (when using FFS).

2.2.4 Determining Margin of Error from Simulation Parameters

It is straightforward to determine the margin of error of an estimator that depends on a single underlying observable. For example, consider the margin of error of an $MFPT$ estimate as determined via DS simulation, $\zeta \left[\widehat{MFPT}_{ds}, \alpha \right]$. It has been shown that observations of first passage times between two well separated states follow an exponential distribution during DS simulations²⁰. Given this distribution of first passage times, the central limit theorem⁵¹ gives the distribution of $MFPT$ estimates in the limit of large sample size:

$$\frac{\sqrt{n} \left(\widehat{MFPT}_{ds} - E [FPT] \right)}{\sqrt{V [FPT]}} \xrightarrow{D} \mathcal{N} (0, 1), \quad (2.7)$$

where n is the count of first passage time observations, $E [FPT] = MFPT$ is the expected value of the first passage time, $V [FPT]$ is the variance, $\mathcal{N} (0, 1)$ is the standard normal distribution (*i.e.* mean 0 and variance 1), and \xrightarrow{D} signifies that the distribution on the left converges to the one on the right. Eq 2.7 implies⁵¹ that an estimate of $MFPT$ calculated using n observations will follow a normal distribution such that:

$$\begin{aligned} E \left[\widehat{MFPT}_{ds} \right] &= MFPT, \\ V \left[\widehat{MFPT}_{ds} \right] &= \frac{V [FPT]}{n} = \frac{MFPT^2}{n}, \end{aligned} \quad (2.8)$$

where the fact that $V = E^2$ for an exponential distribution was used to factor out $V [FPT]$. The lower and upper bounds of the confidence interval of any normally distributed random variable X can be found using a standard

formula⁵² that depends on the first two moments of X :

$$\begin{aligned} lb_\alpha [X] &= E[X] - z_\alpha \sqrt{V[X]}, \\ ub_\alpha [X] &= E[X] + z_\alpha \sqrt{V[X]}, \end{aligned} \tag{2.9}$$

where z_α is the z score associated with the confidence level α (e.g. $z_{.95} \approx 1.96$)^{*}.

Plugging Eqs 2.8 and 2.9 into Eq 2.6 yields the margin of error of the DS *MFPT* estimator:

$$\zeta \left[\widehat{MFPT}_{ds}, \alpha \right] = \frac{z_\alpha}{\sqrt{n}}. \tag{2.10}$$

Thus, if say, 10^5 replicate trajectories are produced during a DS simulation, the resultant estimate of *MFPT* will have no more than $\frac{z_{.95}}{\sqrt{10^5}} \approx \frac{1.96}{\sqrt{316}} = 0.62\%$ difference from the true value 95% of the time.

2.2.5 Minimizing the Computational Cost Required to Achieve a Desired Error Goal

We define the computational cost \mathcal{C} of a simulation to be equivalent to the average in-simulation time required to complete it (alternatively, one may use the count of simulation steps). When estimating *MFPT* using DS simulation, the computational cost is:

$$\mathcal{C}_{ds} = n \cdot MFPT, \tag{2.11}$$

where n is the total number of replicate trajectories. Eq 2.10 illustrates the direct tradeoff between computational cost and simulation accuracy. In order

^{*}The z score z_α for any confidence level α can be calculated as $z_\alpha = \sqrt{2}\text{Erfinv}(\alpha)$, where Erfinv is the Inverse Error Function⁵³, and α is expressed as a fraction. Some authors write the z score as $z_{(1-\alpha)/2}$ instead of z_α ⁵².

to find the minimum number of replicate trajectories that are required to achieve a particular error goal in DS simulations, Eq 2.10 can be solved for n :

$$n = \left(\frac{z_\alpha}{\zeta} \right)^2 \quad (2.12)$$

2.3 Results

2.3.1 Derivation of the FFPilot optimizing equation

In this subsection we find the choice of number trajectories to launch in each FFS phase (which we will also refer to as sample count or n_i) that will minimize simulation run time while fixing the margin of error ζ of the estimator of *MFPT*. We will refer to the *MFPT* estimator in this subsection as \hat{W} . The derivation of the optimal choice of n_i starts with the characterization of the moments and distribution of \hat{W} . We then use the moments and distribution of \hat{W} to find ζ as a function of the per-phase sample counts n_i . We then use the method of Lagrange multipliers to find the optimal choice of n_i that will keep ζ fixed while minimizing simulation run time.

2.3.1.1 The Moments and Distribution of the FFS *MFPT* Estimator \hat{W}

Each individual FFS phase weight w_i can be thought of as the first moment (*i.e.* the mean) of an observable \mathfrak{w}_i of a random process that is specific to phase i . By the end of every FFS phase i , n_i samples have been drawn from \mathfrak{w}_i (see Sec 2.2.1 for a more concrete description), at which point the phase weight w_i is estimated as:

$$\hat{w}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} \mathfrak{w}_{ij}$$

Given the above form of \hat{w}_i , and given that the observable w_i meets certain regularity conditions⁵⁴, the asymptotic distribution of the individual \hat{w}_i terms can be determined from the central limit theorem:

$$\frac{\sqrt{n_i}(\hat{w}_i - w_i)}{\sqrt{V[w_i]}} \xrightarrow{D} \mathcal{N}(0, 1). \quad (2.13)$$

The moments of each \hat{w}_i can be determined by the appropriate interpretation of Eq 2.13:

$$\begin{aligned} E[\hat{w}_i] &= w_i, \\ V[\hat{w}_i] &= \frac{V[w_i]}{n_i}. \end{aligned} \quad (2.14)$$

An estimator with this type of convergence behavior is said to be \sqrt{n} -consistent.

Given that the *MFPT* estimator \hat{W} is defined in Eq 2.5 as a function $g[\hat{w}]$ of a vector of \sqrt{n} -consistent estimators $\hat{w} = \{\hat{w}_0, \hat{w}_1, \dots, \hat{w}_N\}$, we can apply the multivariate delta method⁵⁵ to determine the moments and distribution of \hat{W} . From the multivariate delta method, we know that:

$$\frac{\hat{W} - w_0 \prod_{i>0} w_i^{-1}}{\sqrt{\nabla_{\mathbf{w}}^T \Sigma \nabla_{\mathbf{w}}}} \xrightarrow{D} \mathcal{N}(0, 1), \quad (2.15)$$

where $\nabla_{\mathbf{w}}$ is the gradient of $g(\hat{w})$ evaluated at \mathbf{w} , $\nabla_{\mathbf{w}}^T$ is the transpose of $\nabla_{\mathbf{w}}$, and Σ is the covariance matrix of the phase weight estimators \hat{w}_i . We can determine the moments of \hat{W} by interpreting Eq 2.15:

$$\begin{aligned} E[\hat{W}] &= \frac{w_0}{\prod_{i>0} w_i} = MFPT, \\ V[\hat{W}] &= \nabla_{\mathbf{w}}^T \Sigma \nabla_{\mathbf{w}}. \end{aligned} \quad (2.16)$$

A simpler form of $V[\hat{W}]$ can be found. To begin with, we find $\nabla \mathbf{w}$. In column vector form it is:

$$\nabla \mathbf{w} = \begin{pmatrix} w_0^{-1} w_0 \prod_{j>0} w_j^{-1} \\ -w_1^{-1} w_0 \prod_{j>0} w_j^{-1} \\ \vdots \\ -w_N^{-1} w_0 \prod_{j>0} w_j^{-1} \end{pmatrix} = \begin{pmatrix} s_0 \\ -s_1 \\ \vdots \\ -s_N \end{pmatrix}, \quad (2.17)$$

where we have substituted s_i for $w_i^{-1} w_0 \prod_{j>0} w_j^{-1}$ for the sake of brevity. If the covariance of \hat{w}_i and \hat{w}_j is written as σ_{ij} , then the covariance matrix Σ is:

$$\Sigma = \begin{pmatrix} \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} \\ \sigma_{10} & \sigma_{11} & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N0} & \sigma_{N1} & \cdots & \sigma_{NN} \end{pmatrix},$$

and the variance of \hat{W} can be written as:

$$V[\hat{W}] = (s_0 \ -s_1 \ \cdots \ -s_N) \begin{pmatrix} \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} \\ \sigma_{10} & \sigma_{11} & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N0} & \sigma_{N1} & \cdots & \sigma_{NN} \end{pmatrix} \begin{pmatrix} s_0 \\ -s_1 \\ \vdots \\ -s_N \end{pmatrix}.$$

If we impose the assumption of independence on all of the phase weight estimators \hat{w}_i , then only the diagonal elements of the covariance are non-zero:

$$V[\hat{W}] = (s_0 \ -s_1 \ \cdots \ -s_N) \begin{pmatrix} \sigma_{00} & 0 & \cdots & 0 \\ 0 & \sigma_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{NN} \end{pmatrix} \begin{pmatrix} s_0 \\ -s_1 \\ \vdots \\ -s_N \end{pmatrix}.$$

Under this condition of independence the form of $V[\hat{W}]$ can be simplified

considerably:

$$V[\widehat{W}] = \sum_i s_i^2 \sigma_{ii},$$

$$V[\widehat{W}] = \sum_i w_i^{-2} \sigma_{ii} \frac{w_0^2}{\prod_{j>0} w_j^2},$$

and since $\sigma_{ii} = V[\widehat{w}_i]$:

$$V[\widehat{W}] = \frac{w_0^2}{\prod_{j>0} w_j^2} \sum_i \frac{V[\widehat{w}_i]}{w_i^2}.$$

Finally, plugging in substitutions from Eq 2.4 and from Eq 2.14, the variance of \widehat{W} is:

$$V[\widehat{W}] = MFPT^2 \sum_i \frac{V[\mathfrak{w}_i]}{w_i^2 n_i}. \quad (2.18)$$

Alternatively, the variance of \widehat{W} can be derived from the formula for the variance of a product of random variables (see supplemental text). The expressions derived from each technique agree in the high sample count limit.

The result in Eq 2.18 agrees with and is similar to the established result of Allen *et al.*⁴⁶ concerning the variance of estimates produced by a complete FFS simulation. Unlike earlier work, however, we have imposed no particular form on \mathfrak{w}_i , the random process underlying each phase i . As will be seen in Sec 2.3.1.4, this generalization allows us to study the contributions of phase 0 to the overall error of an FFS simulation for the first time.

2.3.1.2 Margin of Error of \widehat{W}

Now we derive a formula for $\zeta[\widehat{W}, \alpha]$, the margin of error of the *MFPT* estimator \widehat{W} . From Eq 2.15 we know that \widehat{W} follows a normal distribution.

The lower and upper confidence bounds of \hat{W} are found by plugging the the moments of \hat{W} (given by Eqs 2.16 and 2.18) into the bounds formulas for a normally distributed random variable (given by Eq 2.9):

$$\begin{aligned} ub_{\alpha} [\hat{W}] &= MFPT \left(1 + z_{\alpha} \sqrt{\sum_i \frac{V[\mathfrak{w}_i]}{w_i^2 n_i}} \right), \\ lb_{\alpha} [\hat{W}] &= MFPT \left(1 - z_{\alpha} \sqrt{\sum_i \frac{V[\mathfrak{w}_i]}{w_i^2 n_i}} \right). \end{aligned} \quad (2.19)$$

Plugging Eq 2.19 into the margin of error definition (given by Eq 2.6) yields the desired margin of error:

$$\zeta [\hat{W}, \alpha] = z_{\alpha} \sqrt{\sum_i \frac{V[\mathfrak{w}_i]}{w_i^2 n_i}}. \quad (2.20)$$

2.3.1.3 Derivation of the General Optimizing Equation

As we can see from Eq 2.20, there are many different choices of n_i that will give the same value of $\zeta [\hat{W}, \alpha]$. What we really want to find is the optimal choice of n_i that will minimize simulation run time while keeping $\zeta [\hat{W}, \alpha]$ fixed. We can find a formula for this optimal choice using the method of Lagrange multipliers⁵⁶.

For the method of Lagrange multipliers, we need a function to minimize, the target function $f[x]$, and a function to hold constant, the constraint equation $g[x]$. We use the total computational cost \mathcal{C} as the target function, which for FFS is:

$$f [n_i] = \mathcal{C}_{\text{ffs}} = \sum_{i=0}^N n_i \mathfrak{c}_i \quad (2.21)$$

where \mathfrak{c}_i is the average computational cost per sample. For the constraint

equation, we square both sides of Eq 2.20 and set it equal to zero:

$$g[n_i] = \sum_i \frac{k_i}{n_i} - \frac{\zeta^2}{z_\alpha^2} = 0, \quad (2.22)$$

where

$$k_i = \frac{V[\mathfrak{w}_i]}{w_i^2}.$$

Now that we've chosen a target and a constraint function, the next step of the method is to combine Eqs 2.21 and 2.22 in order to write out the Lagrangian:

$$\mathcal{L} = \sum_i n_i \mathfrak{c}_i + \lambda \left(\sum_i \frac{k_i}{n_i} - \frac{\zeta^2}{z_\alpha^2} \right), \quad (2.23)$$

and then find its gradient $\nabla \mathcal{L}$:

$$\nabla \mathcal{L} = \{\partial_\lambda \mathcal{L}, \partial_{n_0} \mathcal{L}, \dots, \partial_{n_N} \mathcal{L}\},$$

We find the values of each of the partial derivatives individually. Calculating $\partial_\lambda \mathcal{L}$ is trivial, and when finding each separate $\partial_{n_i} \mathcal{L}$ we can eliminate all but one term from both sums, since terms that don't depend on n_i will vanish:

$$\partial_\lambda \mathcal{L} = \sum_i \frac{k_i}{n_i} - \frac{\zeta^2}{z_\alpha^2}.$$

$$\partial_{n_i} \mathcal{L} = \mathfrak{c}_i - \frac{\lambda k_i}{n_i^2},$$

Now we can write out the actual gradient

$$\nabla \mathcal{L} = \left\{ \sum_i \frac{k_i}{n_i} - \frac{\zeta^2}{z_\alpha^2}, \dots, \mathfrak{c}_i - \frac{\lambda k_i}{n_i^2}, \dots \right\}. \quad (2.24)$$

The next step is to set each component of the gradient Eq 2.24 equal to zero and solve the resulting set of $N + 2$ equations for λ and each n_i . We begin by

solving $\partial_{n_i} \mathcal{L} = 0$ for n_i in terms of λ :

$$\mathfrak{c}_i - \frac{\lambda k_i}{n_i^2} = 0,$$

$$n_i = \left\{ -\frac{\sqrt{\lambda} \sqrt{k_i}}{\sqrt{\mathfrak{c}_i}}, \frac{\sqrt{\lambda} \sqrt{k_i}}{\sqrt{\mathfrak{c}_i}} \right\}. \quad (2.25)$$

Next, we solve $\partial_\lambda \mathcal{L} = 0$ for $\sqrt{\lambda}$ by substituting in the positive expression for n_i found in Eq 2.25:

$$\sum_i \frac{k_i}{\frac{\sqrt{\lambda} \sqrt{k_i}}{\sqrt{\mathfrak{c}_i}}} - \frac{\zeta^2}{z_\alpha^2} = 0, \\ \sqrt{\lambda} = \frac{z_\alpha^2}{\zeta^2} \sum_i \sqrt{\mathfrak{c}_i k_i}. \quad (2.26)$$

Now we eliminate λ from our expression for n_i in Eq 2.25 by substituting in the expression for $\sqrt{\lambda}$ we found in Eq 2.26:

$$n_i = \frac{z_\alpha^2}{\zeta^2} \sqrt{\frac{k_i}{\mathfrak{c}_i} \sum_j \sqrt{\mathfrak{c}_j k_j}}. \quad (2.27)$$

2.3.1.4 The Optimizing Equation for FFS

A form of the general optimizing equation given in Eq 2.27 that is more specific to FFS can be found by considering the properties of the observable \mathfrak{w}_i in each phase. During a phase $i > 0$, each trajectory launched and finished is equivalent to a single sample taken from $\mathfrak{w}_{i>0}$. The j th trajectory of a phase will either succeed (*i.e.* cross forward to the next interface) with probability p_{ij} , or it will fail (*i.e.* fall back into its starting basin), with probability $1 - p_{ij}$. The sample taken from $\mathfrak{w}_{i>0}$ is 1 if the trajectory succeeds, and 0 otherwise. Thus,

the outcome of the j th trajectory of phase i is a Bernoulli random variable with probability parameter p_{ij} .

For multidimensional systems, p_{ij} is dependent upon the starting point of a trajectory. Since the starting point of each trajectory is chosen at random, p_{ij} in general varies from trajectory to trajectory. Thus, each $\mathfrak{w}_{i>0}$ is technically a mixture of Bernoulli random variables. However, for the purposes of the optimizing equation, it can be shown that no accuracy is lost if each $\mathfrak{w}_{i>0}$ is treated as a single Bernoulli random variable (see Appendix .1) with moments:

$$\begin{aligned} E[\mathfrak{w}_{i>0}] &= w_i = p_i, \\ V[\mathfrak{w}_{i>0}] &= p_i(1-p_i), \end{aligned} \tag{2.28}$$

where p_i is the crossing probability in phase i (*i.e.* $P(\lambda_i | \lambda_{i-1})$).

The k_i terms in Eq 2.27 can be expanded to yield:

$$n_i = \frac{z_\alpha^2}{\zeta^2} \sqrt{\frac{V[\mathfrak{w}_i]}{w_i^2 \mathfrak{c}_i}} \sum_j \sqrt{\frac{V[\mathfrak{w}_j] \mathfrak{c}_j}{w_j^2}}. \tag{2.29}$$

The moments from Eq 2.28 can then be plugged into Eq 2.29 to yield the FFS specific form of the optimizing equation:

$$n_i = \begin{cases} \frac{z_\alpha^2}{\zeta^2} \sqrt{\frac{V[\mathfrak{w}_0]}{w_0^2 \mathfrak{c}_0}} \left(\sqrt{\frac{V[\mathfrak{w}_0] \mathfrak{c}_0}{w_0^2}} + \sum_{j=1}^N \sqrt{\frac{(1-p_j) \mathfrak{c}_j}{p_j}} \right) & \text{if } i = 0, \\ \frac{z_\alpha^2}{\zeta^2} \sqrt{\frac{1-p_i}{p_i \mathfrak{c}_i}} \left(\sqrt{\frac{V[\mathfrak{w}_0] \mathfrak{c}_0}{w_0^2}} + \sum_{j=1}^N \sqrt{\frac{(1-p_j) \mathfrak{c}_j}{p_j}} \right) & \text{otherwise.} \end{cases} \tag{2.30}$$

We call this form the FFPILOT optimizing equation.

In regards to phase 0, the precise forms of $E[\mathfrak{w}_0] = w_0^2$ and $V[\mathfrak{w}_0]$ are unknown. We have found that \mathfrak{w}_0 , the waiting time in between phase 0 forward flux events, does not in general follow an exponential distribution

(see supplemental Fig S2). In fact, the distribution of w_0 seems to be highly model dependent. Thus, in order to avoid any assumptions about w_0 , we leave the ratio $\frac{V[w_0]}{w_0^2}$ unexpanded in Eq 2.30.

Of the assumptions made in deriving Eq 2.30, two of the most significant are the assumption of large sample size, and the assumption of the uncorrelatedness of the phases during an FFS simulation. The large sample size assumption underlies the validity of Eqs 2.14 and 2.15. In general this assumption can be satisfied by setting a minimum floor on the number of samples n_i taken in each phase (an implementation of this sample size floor is discussed in the next section).

Ensuring that the uncorrelatedness assumption is satisfied is altogether trickier. Most of the preexisting FFS literature takes the uncorrelatedness of the phases as a given^{46,47,45}, but in practice we have found this to not always be the case. This issue is discussed in detail in Sec 2.3.6. In brief, systems with complex, high dimensional state spaces tend to have correlations between the outcomes of trajectories across the different phases. This results in non-zero covariance between the different phase weights, which effectively like adding an extra term to Eq 2.18. In other words, when the phases are correlated, our approach will somewhat underestimate the actual variance, and Eq 2.30 will somewhat underestimate the number of samples required to achieve a particular error goal.

2.3.2 FFPilot: A Sampling Algorithm Designed to Take Advantage of the Optimization Equation

We wanted to be able to apply the optimizing equation to real biochemical networks, but to do so we need some prior knowledge of the system under study. As shown in Eq 2.30, two global and $2 \cdot (N + 1)$ phase-specific parameters are required in order to apply the optimization equation and thereby calculate the optimal value of n_i . The two global parameters, the margin of error ζ and the confidence interval z score z_α , are independent of the model being studied and are set according to the desired error goal. The other parameters, the ratio $\frac{V[\mathfrak{w}]}{w_0^2}$ (from phase 0), the successful crossing probabilities p_i (from phases $i > 0$), and the per-sample computational costs c_i , are model dependent and vary for each different combination of model, order parameter, and interface placement.

In general the exact values of the model-dependent parameters are unknown and estimates must be used instead. Rather than simplifying using assumptions such as constant cost^{46,47}, we produce rough but conservative estimates of the necessary parameters using a pilot simulation. By conservative, we mean that the estimates, when plugged into the optimization equation, will be likely to give values of n_i that are at least as large as the true values. This condition ensures that simulations run with n_i trajectories per phase will produce results that are at least as accurate as the specified error goal.

We call our new enhanced sampling protocol FFPilot. The basic concept of FFPilot is to break an FFS simulation up into two stages. First a pilot stage is executed (see supplemental Fig S1), from which the parameters required

for the optimization equation are estimated. Then, based on the results of the optimization equation, a production stage is planned and executed, from which the actual simulation output is calculated.

The FFPilot algorithm proceeds as follows:

1. Specify an error goal in terms of a target margin of error. Optionally, the confidence interval (which defaults to 95%) can be specified as well. As with standard FFS, the user must specify an order parameter and interface placements.
2. Begin FFPilot pilot stage:
 - (a) Set the pilot stage sample count n_{pilot} to a single fixed value. Throughout this paper we used a value of $n_{\text{pilot}} = 10^4$.
 - (b) Run a complete FFS simulation, following the algorithm described in Sec 2.2.1. Unlike standard FFS, the number of samples to collect in each phase is determined by a blind optimization method.
 - i. Phase 0 proceeds the same as in standard FFS, using $n_0 = n_{\text{pilot}}$ as the sample count.
 - ii. In phases $i > 0$, trajectories are run until n_{pilot} successful forward flux events are observed. It can be shown that, for a relatively modest number of successes, this method constrains error to within 2% when estimating the individual phase weights (see Appendix .2 for complete details).
3. When the pilot stage is finished, estimate the values required for the optimization equation from the results of the pilot simulation. Use

confidence intervals to form conservative estimates that, when plugged into the optimization equation, are likely to yield values of n_i that are as large or larger than those required for the error goal.

4. Begin FFPilot production stage:
 - (a) Determine n_i , the number of samples to collect in each phase, based on the error goal and Eq 2.30, the FFPilot optimizing equation (as parameterized in step 3).
 - (b) Run another complete FFS simulation using the values of n_i calculated in step 4a.
5. Collect results from the production stage simulation and use/analyze them in the same way as would be done for standard FFS simulation. The results from the pilot stage are ignored for the purposes of calculating the final simulation results as sampling differences in states at the interfaces would introduce additional error.

In terms of the error in the final simulation results, the optimization equation is inaccurate in the low sample count limit. Therefore, we enforce a minimum floor (10^3) on the count of samples taken in each phase of the production stage.

The effectiveness of the pilot stage blind optimization method can be related to an earlier finding of Borerro *et al.*⁴⁷. They showed that a fixed quantity of computational effort is optimally spent during an FFS simulation when the interfaces and the trajectory counts per phase are arranged in such a way that each interface encounters an equal flux of trajectories crossing them.

Although we do not constrain the computational effort spent during our blind optimization, our approach produces equal flux across each interface as well.

2.3.3 Rare Event Model

We began our testing of FFPilot with a toy model of a barrier crossing process, which we refer to as the rare event model (REM). REM models a particle in a discrete potential field in which there are two metastable states, \mathcal{A} and \mathcal{B} , connected by a transition path (see top of Fig 2.1). Particles in \mathcal{A} have a constant propensity to initiate a transition by entering the transition path. Because of the constant propensity the waiting times in between transition attempts are exponentially distributed.

The transition path itself is composed of a sequence of N steps; particles enter the path at step 1. At each successive step, a particle will instantaneously either proceed to the next step with probability p_i , or fall back into \mathcal{A} with probability $1 - p_i$. If the particle successfully passes the final step it enters \mathcal{B} . In effect, the particle's fate once it enters the transition path can be thought of as the outcome of N weighted coin flips. If all N coins land heads up, the particle completes the transition $\mathcal{A} \rightarrow \mathcal{B}$. Otherwise, the particle falls back into \mathcal{A} .

We designed REM to map precisely onto FFPilot sampling in order to directly test the validity of the assumptions and simplifications that were made in the derivation of the FFPilot optimizing equation (Eq 2.27). Simulations of REM can be carried out using either DS or FFPilot. To simulate a single replicate of a particle starting in \mathcal{A} using DS, first the time until the particle

leaves \mathcal{A} is randomly selected from an exponential random variable according to the propensity of entering the transition path. The particle's behavior at each step is then randomly chosen, either falling back to \mathcal{A} or proceeding to the next step according to the appropriate p_i . If the particle falls back into \mathcal{A} the process is repeated until it successfully passes to \mathcal{B} . The total time the particle took to transition to \mathcal{B} is the $\mathcal{A} \rightarrow \mathcal{B}$ first passage time for that trajectory.

To simulate a single replicate of a particle starting in \mathcal{A} using FFPilot, first an order parameter and the interface positions must be chosen (see section Sec 2.2.1). We chose the step number i as the order parameter, and placed the interfaces between each step i . The phase 0 weight is calculated by first drawing many samples of the \mathcal{A} leaving time according to the propensity, and then taking the mean of those samples. The remaining phase weights are determined by repeatedly starting a particle at step i , randomly selecting if it continues on to the next step according to p_i , and then calculating the average probability of success from the observations. The pilot stage of FFPilot is accomplished by first running the phase 0 weight calculation n_{pilot} times, then running each phase $i > 0$ weight calculation until n_{pilot} success events are observed. The outcome of the pilot stage is then fed into the FFPilot optimizing equation Eq 2.30, and the results are used to determine how many samples to take during the FFPilot production stage. For the purposes of parameterizing Eq 2.30, the phase 0 relative variance and the per-phase costs c_i are all set equal to 1 (see Table 2.1 for complete parameters). MFPT is then estimated as the product of the production stage phase weights.

We first studied the distribution of the *MFPT* estimators. Taken together, Eqs 2.15, 2.16, and 2.18 describe the normal distribution that repeated estimation of *MFPT* is expected to produce. In our derivation we have assumed that we are working in the high sampling limit for values of w_i and n_i of interest. To test this assumption, we performed $1.6 \cdot 10^5$ independent simulations of REM using both DS and FFPilot using a 1% error goal. Fig 2.2 shows the distributions from our simulations. The black line in each figure is the normal distribution with mean and variance given by Eqs 2.16 and 2.18. The binned *MFPT* estimates from both the DS and the FFPilot REM simulations are in excellent agreement with the predicted normal distribution.

Next, we tested how well the FFPilot approach was able to control sampling error in simulations of REM. If the method works as expected, 95% of simulations should have errors at or below the FFPilot error goal. We executed 1000 FFPilot simulations of REM at 3 different error goals (10%, 3.2%, and 1%). We used the full FFPilot algorithm to determine how many trajectories to start at each interface.

The percent errors of the *MFPT* calculated in each of these simulations are shown in Fig 2.3. The percent errors were calculated relative to the analytically determined *MFPT*. As can be seen, the 95th error percentiles (marked by the red lines) are located along $x = y$, indicating that overall error in the *MFPT* estimates was constrained to the error goal. REM has no source of error aside from sampling error, and under these conditions FFPilot precisely controls the total simulation error.

2.3.4 Self Regulating Gene Model

We next tested FFPilot with a relatively simple biochemical network, the self regulating gene model (SRG)⁵. SRG models expression of a single protein A . A is produced through autocatalysis, and decays via a first order process (see Fig 2.1).

In the deterministic formulation of SRG, the rate of change in the quantity of protein A is:

$$\frac{dA}{dt} = k_{low} + (k_{high} - k_{low}) \frac{A^h}{k_{50}^h + A^h} - A \quad (2.31)$$

For a given set of parameters, the fixed points of the state space of SRG can be found by setting Eq 2.31 equal to 0 and solving for A . For all of the parameters we used in our simulations there are three fixed points, two stable and one unstable. One of the stable fixed points corresponds to a state with a low count of A , and the other corresponds to a state with high count of A .

To formulate SRG as a stochastic model, we use the chemical master equation (CME) (see Table 2.2 for complete reaction list). The CME models the probability for the system to be in any state. Additionally, fluctuations due to population noise can cause the system to transition back and forth between the low and high states. The *MFPT* between the states is related to the entropic barrier separating them.

We wanted to study how the height of the barrier between the low A and high A states affects the accuracy of FFPilot. Towards this end, we parameterized 3 different variants of SRG with different barrier heights, and thus different *MFPT* values. We call these three variants $\text{SRG}_{h=2.4}$, $\text{SRG}_{h=2.3}$,

and $\text{SRG}_{h=2.2}$, after the Hill coefficient used in the protein A production rate law. We tuned the other parameters in the model in order to approximately balance the occupancy of the low A and high A states in each of the variants (see Table 2.3 for parameter values).

For all FFPilot simulations of SRG we used the count of protein A as the order parameter. We determined the positions of the interfaces by first placing λ_0 a quarter of the distance (in terms of the order parameter) from the lower fixed point to the intermediate fixed point, then λ_N three quarters of the distance from the lower fixed point to the upper fixed point. We then placed 11 more interfaces spaced evenly between λ_0 and λ_N .

Unlike REM, trajectories in the low A state do not cross λ_0 and enter the transition pathway with a fixed propensity. Instead, the propensity changes dynamically with the system state, giving rise to a complex distribution of waiting times in between crossing events. In deriving the FFPilot optimizing equation (more specifically, when deriving Eq 2.18), we assumed that the first two central moments of the phase 0 waiting time distribution (w_0 and $V[\mathfrak{w}_0]$) exist. In order to test this assumption we executed simulations in which we only performed phase 0, collecting 10^6 crossing events for λ_0 .

Fig 2.4 shows the phase 0 inter-event time distributions. The tail of each distribution is fit well by a single exponential distribution, but the distribution near 0 is not. We estimated the value of the relative variance, $\frac{V[\mathfrak{w}_0]}{w_0^2}$, used in the phase 0 terms of the FFPilot optimizing equation (Eq 2.30) to be 6.80, 6.43, and 5.96 for $\text{SRG}_{h=2.4}$, $\text{SRG}_{h=2.3}$, and $\text{SRG}_{h=2.2}$, respectively.

Next, we examined how well the FFPilot approach was able to control

sampling error with respect to $MFPT$. We executed 1000 FFPilot simulations of $SRG_{h=2.4}$, $SRG_{h=2.3}$, and $SRG_{h=2.2}$ using error goals 1%, 3.2%, and 10%. We estimated $MFPT$ from each simulation, then found the percent error relative to the value estimated from a DS simulation executed with an error goal of 0.62%.

The errors are shown in Fig 2.5. The 95th error percentiles are again located precisely along $x = y$. The accuracy of the estimates show that FFPilot is able to control error in both phase 0 (regardless of the exact distribution of the inter-event times) and the remaining phases for SRG.

We also looked at the contribution of phase 0 to the overall cost of the pilot stage. Applying Eq 2.21 to values from Table 2.3, we found that for all variants of SRG phase 0 required around $\sim 35\%$ of the total simulation time. This finding is in contrast to the longstanding assumption in the FFS literature that phase 0 does not significantly contribute to the cost of a simulation and should therefore be extensively sampled.

2.3.5 Genetic Toggle Switch Model

The last model we investigated using FFPilot was a more complex gene regulatory network, one of a family of systems commonly referred to as a genetic toggle switch (GTS)⁵⁷. Our GTS has seven species that interact with one another via fourteen reactions, all of which are first or second order (see Table 2.4). GTS consists of a single piece of operator DNA, O . When O is not bound to anything it can produce either of two proteins, A and B . A and B can both decay, they can both form homodimers, and those dimers can both

bind back to O . Only one dimer can bind to O at any given time. When O is bound to a dimer of either protein, it can only produce more of that same protein (see Fig 2.1).

The combination of positive feedback (of monomer production on dimer/operator binding) and negative feedback (of dimer/operator binding on production of the competing monomer) gives GTS bistable dynamics⁵⁸. The system as a whole switches between a state with high levels of the various forms of A and low levels of B , and a state with low levels of A and high levels of B .

For GTS we defined three order parameters. One, which we called Δ , is the difference between the total count of protein B and the total count of protein A . Another, which we called Σ , is the sum of the total count of protein A and the total count of protein B . The last, Ω , takes a value from [-1,1] based solely on the state of the single operator. In terms of the underlying species counts, the order parameters can be written as:

$$\Delta = B + 2B_2 + 2OB_2 - (A + 2A_2 + 2OA_2)$$

$$\Sigma = A + 2A_2 + 2OA_2 + B + 2B_2 + 2OB_2$$

$$\Omega = -OA_2 + OB_2$$

where A and B are the monomer counts, A_2 and B_2 are the dimer counts, and OA_2 and OB_2 are the dimer-operator complex counts. Equivalently, Ω can be said to have one of three categorical values:

$$\Omega \rightarrow \{OA_2, O, OB_2\}$$

Although all of our GTS simulations are based on a stochastic master equation formulation of the system, it is helpful to consider the more straightforward deterministic formulation when trying to understand the system's overall behavior (see supplemental Table S1 for the deterministic rate equations). For a given set of parameters, the fixed points of the deterministic formulation can be found. For all of the parameter sets we used in our simulations there are three fixed points in terms of Δ , two stable fixed points and one unstable fixed point. One of the stable fixed points corresponds to the state with a high level of A and a low level of B , and the other stable fixed point corresponds to the state with a low level of A and a high level of B . We refer to these two states as \mathcal{A} and \mathcal{B} , respectively.

We wanted to be able to tune the rarity of the $\mathcal{A} \rightarrow \mathcal{B}$ event without disrupting the overall dynamics of GTS. To do so, we added a relative protein turnover parameter θ . The rate constants of all of the expression and decay reactions for both A and B are multiplied by θ . Since θ does not affect the birth/death ratio of each protein, the steady state levels of both A and B are constant with respect to θ . However, θ does have a large effect on the rate of $\mathcal{A} \rightarrow \mathcal{B}$ switching. We used three different variants of GTS in our simulations, $\text{GTS}_{\theta=1}$, $\text{GTS}_{\theta=1}$, and $\text{GTS}_{\theta=10}$, see Table 2.5 for complete parameters.

For all FFPilot simulations we used Δ as the order parameter. We tiled the state space in terms of Δ by placing λ_0 at $\Delta = -27$, λ_N at $\Delta = 27$, and then placing 11 more interfaces evenly spaced in between.

As with SRG, we were interested in the distribution of phase 0 inter-event times of GTS in order to establish the validity of Eq 2.30 with respect to

GTS. Fig 2.6 shows the results of our phase 0 inter-event time distribution simulations. The phase 0 distributions of the different GTS variants differ a great deal, but interestingly their $\frac{V[w_0]}{w_0^2}$ values (the ratio of moments required for Eq 2.30) are very similar. $\frac{V[w_0]}{w_0^2}$ was found to be 8.15, 8.15, and 8.41 for $\text{GTS}_{\theta=.1}$, $\text{GTS}_{\theta=1}$, and $\text{GTS}_{\theta=10}$, respectively.

We next ran a test to examine how well the full FFPilot protocol was able to control sampling error in estimations of *MFPT* of the $\mathcal{A} \rightarrow \mathcal{B}$ switching process. We executed 1000 FFPilot simulations of $\text{GTS}_{\theta=.1}$, $\text{GTS}_{\theta=1}$, and $\text{GTS}_{\theta=10}$ using error goals 1%, 3.2%, and 10%. We estimated *MFPT* for each simulation, then found the percent errors relative to the results from high accuracy DS simulations of equivalent models, which were run with a 0.62% error goal.

The percent errors of the *MFPT* estimates are shown in Fig 2.7. As can be seen in the figure, the 95th error percentiles (marked by the red lines) are located somewhat above $x = y$, indicating that the overall errors in the estimated *MFPT* values are above the desired errors. The 95th percentile lines do decrease along with error goal, implying that FFPilot partially but not completely controls error in simulations of GTS. The anomalous dispersion decreases as the height of the barrier between \mathcal{A} and \mathcal{B} in probability space decreases. This implies that the extra error is caused by a system dependent property and is not directly related to undersampling.

We again found that phase 0 contributed significantly to the cost of GTS simulations. From parameters listed in Table 2.5 and Eq 2.21, we calculated the share of total simulation time consumed by phase 0, which was found to be 19%, 23%, and 33% for $\text{GTS}_{\theta=.1}$, $\text{GTS}_{\theta=1}$, and $\text{GTS}_{\theta=10}$, respectively.

2.3.6 Interface Landscape Error in Genetic Toggle Switch

We sought to understand the causes of the extra error in the GTS simulations. We chose the condition with the largest anomalous errors, $\text{GTS}_{\theta=10}$ executed with an error goal of 10%, and examined the phase weight estimates produced by each of the 1000 replicate simulations we had run with that condition. These phase weights are shown in the top half of Fig 2.8. The phase weights estimated by an equivalent FFPilot simulation run with an error goal of 0.1% are shown as dashed lines, and serve as a point of reference (there is no exact method for extracting the phase weights from a DS simulation). The dispersion of phase weight estimates around the reference weight is much greater in certain phases, especially phases 4 and 5. By itself, this is not an indication that FFPilot is failing to correctly estimate sampling counts for these phases. By design, FFPilot allows for different levels of dispersion in different phases when it is determining the optimal simulation plan.

In order to determine how much of the phase weight dispersion represents FFPilot functioning as intended and how much of the dispersion is truly anomalous, we calculated an optimal set of error goal targets for each simulation phase. From the FFPilot optimizing equation (Eq 2.30) we derived

analytic expressions for the per-phase error goals:

$$\zeta_{i=0} = \zeta \sqrt{\frac{\sqrt{\frac{V[\mathbf{w}_0]\mathbf{c}_0}{w_i^2}}}{\sqrt{\frac{V[\mathbf{w}_0]\mathbf{c}_0}{w_i^2}} + \sum_{j=1}^N \sqrt{\frac{(1-p_j)\mathbf{c}_j}{p_j}}}}, \quad (2.32)$$

$$\zeta_{i>0} = \zeta \sqrt{\frac{\sqrt{\frac{(1-p_i)\mathbf{c}_i}{p_i}}}{\sqrt{\frac{V[\mathbf{w}_0]\mathbf{c}_0}{w_i^2}} + \sum_{j=1}^N \sqrt{\frac{(1-p_j)\mathbf{c}_j}{p_j}}}}.$$

Just as with the overall error goal, in any given phase $100 \cdot \alpha\%$ percent of simulations will have a level of sampling error in the phase weight estimate at or below the relevant per-phase error goal ζ_i . We parameterized Eq 2.32 using the phase weight and phase cost estimates from the very high accuracy (0.1% error goal) FFPilot simulation of $\text{GTS}_{\theta=10}$ mentioned above.

The error goal targets we calculated are shown as the dashed lines in the bottom half of Fig 2.8. The phase weight percent errors (calculated relative to the estimates from the 0.1% error goal simulation) are shown as dots, and the red lines mark the 95th percentiles of the errors. If the red line and the dashed line overlap for a particular phase, it means that the error in this phase is dominated by sampling error, which FFPilot is able to completely account for. If the red line is above the dashed line, then there is more error occurring in that phase than was predicted by FFPilot, and the magnitude of the separation of the two lines represents the magnitude of the anomalous (as opposed to the predicted) error. Interestingly, FFPilot estimates the majority of phase weights to within the desired error goal. The extra error in the *MFPT* estimate appears to be due primarily to extra error in only three of the phase

weight estimates, those from phases 3-5. Further, the bulk of the extra error is concentrated in just two of those phase weight estimates, those from phases 4 and 5. Interfaces λ_4 and λ_5 also happen to be the interfaces immediately preceding the transition midpoint.

We hypothesized that there must be some particular feature of the state space landscape of GTS that the simulations are exploring during phases 4 and 5 that is responsible for the anomalous error. We further reasoned that the same features that are responsible for what Allen and coworkers call landscape variance⁴⁶ could be related to the increased error. Here we define a new source of error, which we call landscape error, that is due to two factors: (1) misrepresentation of some regions of the state space in the set of trajectory starting points at λ_i , and (2) significant differences in $P(\lambda_i|\lambda_{i-1})$ as a function of trajectory starting state. The total probability factor $P(\lambda_i|\lambda_{i-1})$ that is measured in each phase $i > 0$ can be thought of as a mixture of many independent probabilities, one for each state along λ_{i-1} , weighted by the (normalized) count of times the state is used as a starting point for a phase i trajectory. If either of factors (1) or (2) occurs alone, $P(\lambda_i|\lambda_{i-1})$ will still be correctly estimated. However, if the factors occur together they can lead to significant errors. In other words, if the landscapes assembled at λ_3 and λ_4 are heterogeneous across replicate simulations, and if differences in those landscapes can lead to differences in the effective value of $P(\lambda_i|\lambda_{i-1})$, then landscape error may be the cause of the anomalous simulation error we observe in our simulations of GTS. Of particular importance is the fact that the landscape error in phase i is due to errors in the landscape assembled from

the endpoints of successful trajectories launched during phase $i - 1$. Thus, no amount of extra sampling performed during phase i alone can completely abolish landscape error.

We wanted to test if the conditions for landscape error were present in FFPilot simulations of $\text{GTS}_{\theta=10}$. To this end, we chose two simulations, which we will call replicate 6 and replicate 8, that had a large divergence in their *MFPT* estimates. Looking at the phase weights estimates produced by these two simulations (Fig 2.9), the divergence in the *MFPT* estimates can be seen to be mostly due to divergence in the phase weight 4 and 5 estimates (as expected). Exploring phase 5 in greater depth, we calculated the landscape occupancy along λ_4 in terms of the orthogonal order parameter Σ . The λ_4 occupancies for replicates 6 and 8 $P(\Sigma|\Omega, \lambda_4)$ (which are binned by Σ and operator state Ω), are shown in the lower left-hand subplots of Fig 2.10 as lines colored blue or gold, respectively. Replicate 6 has somewhat higher occupancy in the OA_2 and O states, and replicate 8 has higher occupancy in the OB_2 states. Thus, condition (1) for landscape error in phase 5, heterogeneous occupancies along λ_4 , is indeed satisfied.

Next, we launched 10^6 independent trajectories from each starting state along λ_4 that had non-zero occupancy in either replicate 6 or 8. Just as in a normal FFPilot simulation, we stopped each trajectory when it either reached the next interface or fell back into the initial basin, and we took note of the stopping states. This gave us a highly accurate estimate of $P(\lambda_5|\lambda_4)$ as a function of trajectory starting state. These state-dependent probability values $P(\lambda_5|\Sigma, \Omega, \lambda_4)$, (which are also binned by Σ and operator state Ω), are shown

as filled circles in Fig 2.10. $P(\lambda_5|\Sigma, \Omega, \lambda_4)$ varies a great deal across both Σ and Ω , meaning that condition (2) is also satisfied for our simulations of GTS $_{\theta=10}$, and that landscape error is indeed a possible explanation for the anomalous error.

In order to test if landscape error alone is a sufficient explanation for the observed anomalous error, we recalculated the phase 4 and 5 weights of replicates 6 and 8 from their landscapes alone, according to:

$$P(\lambda_5|\lambda_4) = \sum_{\Sigma} \sum_{\Omega} [P(\lambda_5|\Sigma, \Omega, \lambda_4) \times P(\Sigma, \Omega|\lambda_4)]. \quad (2.33)$$

If landscape error is indeed the sole cause of the anomalous error, we expected that the phase weights derived from Eq 2.33 would closely match those originally estimated by replicates 6 and 8, even though these original estimates vary greatly between the replicates. In this view, the phase $i > 0$ weight estimate produced by each simulation converges (with increasing trajectory count) to a unique value of $P(\lambda_i|\lambda_{i-1})$, as determined by their heterogeneous samples of the landscape along λ_{i-1} . The recalculated phase weight values are plotted as empty circles in Fig 2.9, and they do indeed closely agree with the original estimates. Thus, we conclude that sampling error is indeed being handled correctly by FFPilot, and the anomalous error in our GTS simulation results is due to landscape error.

2.3.7 Eliminating Landscape Error in GTS via Oversampling

Although a complete mathematical treatment of landscape error is beyond the scope of this paper, we wanted to investigate strategies for eliminating

landscape error within the limited context of GTS. To this end we ran FFpilot simulations of $GTS_{\theta=10}$ under a variety of different oversampling schemes. As can be seen in Fig 2.8, for $GTS_{\theta=10}$ landscape error is mainly an issue in phases 3-5. Based on this, we initially hypothesized that increasing sampling by 10X in phases 2-4 (*i.e.* increasing sampling in each of the phases preceding the problematic phases) would abolish landscape error.

The results from 1000 simulations of $GTS_{\theta=10}$ with 10X phase 2-4 sampling at an error goal of 10% are shown in the second column of Fig 2.11. Oversampling in these phases alone has only a minor effect on simulation error. Under these conditions the 95th percentile of error was 38%, whereas the error without oversampling is 47%. Based on these results, we tried out a more expansive oversampling strategy. In addition to oversampling by 10X in phases 2-4, we oversampled by 20X in phase 0 and 10X in phase 1 as well. The results from 1000 simulations run with 20X phase 0, 10X phase 1-4 oversampling are shown in the last column of Fig 2.11. Error is reduced dramatically under these conditions, to just under 10%. We also tried 10X phase 0 oversampling, but found that it was not quite sufficient to eliminate landscape error (see supplemental Fig S5).

Thus, oversampling in phases 2-4 alone had almost no effect, but oversampling in phases 0-4 was enough to eliminate the landscape error. In order to understand this difference, we eliminated oversampling in each phase individually. The results from these simulations are shown in supplemental Fig S6. The effect of skipping oversampling in phase 0 is particularly dramatic, leading to an increase in simulation error of nearly 25%. So long as phase 0

is being oversampled, the increase in simulation error from skipping oversampling in any of phases 1-4 is less dramatic (2%-7%) but still significant. This implies that landscape errors are correlated. Effectively, defects in the sampled landscape distribution at any λ_i may carry over to λ_{i+1} .

2.3.8 Theoretical Efficiency of DS, FFS, and FFPilot Simulations

Enhanced sampling is commonly assumed to be more efficient than DS. By controlling for simulation error, a direct comparison can be made between DS and FFPilot simulations, and the speedup of one simulation method versus the other can be assessed.

It is straightforward to derive an expression for the cost of a DS simulation \mathcal{C}_{ds} as a function of the error goal. Plugging Eq 2.12 into Eq 2.11 yields:

$$\mathcal{C}_{\text{ds}} = MFPT \frac{z_\alpha^2}{\zeta^2}. \quad (2.34)$$

The cost of an FFS simulation \mathcal{C}_{ffs} is given by Eq 2.21: $\mathcal{C}_{\text{ffs}} = \sum_{i=0}^N n_i \mathfrak{c}_i$. We can plug the FFPilot optimizing equation (Eq 2.30) into Eq 2.21 in order to expand the n_i values. This yields an expression for $\mathcal{C}_{\text{ffs-opt}}$, the cost of an optimized FFS simulation given *a priori* knowledge of the parameters required for the optimizing equation:

$$\mathcal{C}_{\text{ffs-opt}} = \frac{z_\alpha^2}{\zeta^2} \left(\sqrt{\frac{\mathfrak{c}_0 V [\mathfrak{w}_0]}{w_0^2}} + \sum_{i=1}^N \sqrt{\frac{\mathfrak{c}_i (1 - p_i)}{p_i}} \right)^2. \quad (2.35)$$

Next, we examined the theoretical efficiency of the FFPilot approach. Due

to the FFPilot optimizing equation, the production stage of an FFPilot simulation can be thought of the most computationally efficient FFS simulation possible with respect to a given error goal. However, if the pilot stage is too expensive it may be possible that in general FFPilot is inefficient relative to the traditional FFS algorithm. Thus we wanted to determine if FFPilot simulation is reasonably efficient, and, if so, under what conditions.

The total cost of an FFPilot simulation $\mathcal{C}_{\text{ffpilot}}$ can be found by adding a second term to the RHS of Eq 2.35 that specifically accounts for the extra runs performed during the pilot stage:

$$\mathcal{C}_{\text{ffpilot}} = \frac{z_\alpha^2}{\zeta^2} \left(\sqrt{\frac{c_0 V [\mathfrak{w}_0]}{w_0^2}} + \sum_{i=1}^N \sqrt{\frac{c_i (1 - p_i)}{p_i}} \right)^2 + n_{\text{pilot}} \left(c_0 + \sum_{j=1}^N \frac{c_j}{p_j} \right). \quad (2.36)$$

Eq 2.36 gives $\mathcal{C}_{\text{ffpilot}}$ as a function of error goal. The production stage cost increases with error goal $\frac{\zeta^2}{z_\alpha^2}$ whereas the pilot stage cost remains fixed. For a low enough error goal, the pilot stage term in $\mathcal{C}_{\text{ffpilot}}$ will be negligible compared to the overall simulation cost. For simulations run with $n_{\text{pilot}} = 10^4$ the approximation $\mathcal{C}_{\text{ffpilot}} \approx \mathcal{C}_{\text{ffs-opt}}$ holds when the error goal was <5% (see Fig 2.12).

2.3.9 Speedup of FFPilot vs DS in Simulations with Equivalent Error

We determined the speedup of FFPilot vs DS both as a function of simulation error and as a function of switching event rarity. We also examined the model dependence of the speedup. For SRG the speedup of FFPilot over DS is

considerable and scales directly with switching event rarity (see top of Fig 2.12 and supplemental Fig S9). Given a 1% error goal, for $\text{SRG}_{h=2.2}$ the speedup of FFPilot over DS is 44X, whereas for $\text{SRG}_{h=2.4}$ the speedup increases to 323X.

For GTS, the FFPilot vs DS speedup results are somewhat more complicated to interpret, due to the extra landscape error in FFPilot. In order to make a fair comparison between FFPilot and DS simulations of GTS, we oversampled our FFPilot simulations (see Sec 2.3.7) such that the observed error matched the error goal. The extra computational cost imposed by the oversampling must be taken into account when considering the speedup. The cost of an oversampled FFPilot simulation can be found by multiplying each sample count by its oversampling factor.

As with SRG, the simulation times of GTS simulations, both FFPilot and DS, were found to scale with switching event rarity roughly as a power law (see bottom of Fig 2.12 and Fig 2.13). If the effects of landscape error are ignored, then the FFPilot over DS speedups are as considerable for GTS as for SRG. Given a 1% error goal, the speedup for $\text{GTS}_{\theta=10}$ is 21X, while the speedup for $\text{GTS}_{\theta=.1}$ is 647X. However, if the oversampling required to correct for landscape error in GTS simulations is taken into account, the FFPilot speedup shrinks. With oversampling, given a 1% error goal the speedup for $\text{GTS}_{\theta=10}$ is 2X, while the speedup for $\text{GTS}_{\theta=.1}$ is 80X.

2.4 Discussion

Above, we presented our FFPilot approach for automatically parameterizing a FFS simulation to both minimize simulation time and constrain sampling

error at a user specified margin of error and confidence interval. By performing an inexpensive pilot simulation we obtain estimates of the weights and computational costs at each FFS interface, which are used to optimize a more thorough production simulation. Our pilot simulation approach also provides the advantage that individual interface placement, weights, and costs can be quickly evaluated before a potentially costly simulation is performed, if desired.

Unlike previous FFS optimization techniques, our method accounts for the statistics of phase 0 and also for varying computational cost along the order parameter. Both of these features are important for ensuring that error is controlled while minimizing simulation time. Particularly, optimizing the time spent in phase 0 is important as our results indicate that these calculations can consume a significant fraction of the total simulation time.

Our results show that FFPilot correctly controls the sampling error in FFS simulations. For one-dimensional systems, the *MFPT* estimates fall precisely within the specified error bounds. For higher dimensional systems, our testing reveals that while sampling error is well controlled by FFPilot, error due to the system dependent landscape pushes the total error outside of the specified bounds.

In the genetic toggle switch (GTS), substantial oversampling (from 10X–20X) in some phases is required to achieve the desired margin of error. As revealed by a detailed analysis, the anomalous error is due to underrepresentation of some parts of phase space in the crossing sets of early interfaces,

especially phase 0. In the case of GTS, the system has a much higher probability of switching $\mathcal{A} \rightarrow \mathcal{B}$ from the $\Omega = \text{OB}_2$ state, but this state is rarely occupied during early phases and is thus subject to greater statistical variation. Additionally, the system relaxes much more slowly in the Ω dimension than in the other degrees of freedom, causing the crossing probability distributions at successive interfaces to be correlated in Ω . Errors in the estimation of early crossing distributions therefore lock-in and cannot relax during later phases. These combined effects cause greater variability in the estimation of the phase weights than predicted and an increase in the total error.

Since all multi-dimensional systems of significant complexity likely relax more quickly in some degrees of freedom than others, a general approach is needed to control for landscape error in FFS while minimizing simulation time. If the relationship between convergence of the crossing distributions and error were known these factors could be included in our optimization equation.

Studying stochastic biochemical systems with metastable states will become increasingly important as more regulatory and developmental networks are elucidated in sufficient detail to permit quantitative modeling. Our findings raise the important issue of what other obstacles exist for efficient rare event sampling of these systems using FFS. For example, how to control landscape error in systems with many metastable states, with multiple transition paths, or with metastable intermediates? One possibility would be to study each transition path separately, using a branching approach, and then recombine the results based on the branching probabilities.

In summary, our FFPilot method provides a significant speedup compared to direct simulation of systems with rare event dynamics. Even with oversampling to control landscape error, speedups on the order of 100X for systems with long first passage times can be expected. The automatic optimization of simulation parameters to achieve a desired level of sampling error through the use of our optimization equation makes FFS simulations more robust and efficient.

Supplementary Material

See supplementary material for additional derivations and figures.

Acknowledgments

The authors thank the members of Roberts lab for discussions. This work was supported by the National Science Foundation under grant number PHY-1707961, and by the National Institutes of Health under grant T32 GM008403.

Appendices

.1 The Upper Bound of the Variance of a Sum of Bernoulli Distributions

The overall process of launching trajectories in order to determine the phase weight during each phase of FFS simulation can be thought of as a single draw from a sum of many independent Bernoulli distributions, also called a Poisson binomial distribution (PBD). This viewpoint emphasizes the statistical equivalence between the outcome of the j th trajectory in an FFS phase, which will either fall back into its starting basin or flux forward to the next interface, and the outcome of the j th Bernoulli random variable in a PBD, which will take on a value of either 0 or 1. In both cases “success” (fluxing forward, drawing a 1) occurs with some probability p_j inherent to the individual process, while “failure” (returning to basin, drawing a 0) occurs with probability $(1 - p_j)$. The expected value and the variance of a draw from a PBD of n terms (*i.e.* the sum over an independent draw from each of its constituent Bernoulli random variables) is:

$$\begin{aligned}\mu &= \sum_{j=1}^n p_j, \\ V &= \sum_{j=1}^n (1 - p_j) p_j.\end{aligned}\tag{37}$$

Equivalently, Eq 37 is also the mean and variance of the total count of successful trajectories n_i^s in FFS phase $i > 0$, given that $n = n_i$ trajectories were run.

The variance of a PBD is maximized when the probability parameter of

each of its Bernoulli random variables are all the same p :

$$p_1 = p_2 = \dots = p_n = p,$$

and so the upper bound on the variance of a sum of Bernoulli random variables is:

$$V \leq np(1 - p). \quad (38)$$

Proof. For a given value of μ , the method of Lagrange Multipliers can be used to maximize V with respect to the choice of particular values of the p_i terms. Appropriate constraint and target equations can be taken from Eq 37:

$$\begin{aligned} g [p_j] &= \sum_{j=1}^n p_j - \mu = 0 \\ f [p_j] &= V = \sum_{j=1}^n (1 - p_j) p_j. \end{aligned} \quad (39)$$

A Lagrangian can be formed from Eq 39:

$$\mathcal{L} = \sum_{j=1}^n ((1 - p_j) p_j) - \lambda \left(\mu - \sum_{k=1}^n p_k \right).$$

Next we find the gradient of the Lagrangian:

$$\partial_\lambda \mathcal{L} = \sum_{j=1}^n p_j - \mu$$

$$\partial_{p_j} \mathcal{L} = -2p_j + \lambda + 1.$$

Now we set each part of the gradient to zero and solve the resulting set of

equations. We start by solving for p_i in terms of λ :

$$p_j = \frac{\lambda + 1}{2}.$$

Next we solve for λ alone:

$$\lambda = \frac{2\mu}{n} - 1.$$

Then finally we plug the solution for λ into the gradient of p_i and solve for p_i alone:

$$p_j = \frac{\mu}{n} = p. \quad (40)$$

Thus, the spot at which every p_i is equal to $\frac{\mu}{n}$ is a critical point for the variance of a Bernoulli mixture distribution. It can further be shown that the above critical point is a maximum for the constrained variance using the Bordered Hessian variation of the classical second derivative test. The Bordered Hessian⁵⁹ of a Lagrangian can be defined as:

$$H \equiv \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial \lambda^2} & \frac{\partial \mathcal{L}}{\partial \lambda p_1} & \frac{\partial \mathcal{L}}{\partial \lambda p_2} & \cdots & \frac{\partial \mathcal{L}}{\partial \lambda p_N} \\ \frac{\partial \mathcal{L}}{\partial \lambda p_1} & \frac{\partial \mathcal{L}}{\partial p_1^2} & \frac{\partial \mathcal{L}}{\partial p_1 p_2} & \cdots & \frac{\partial \mathcal{L}}{\partial p_1 p_N} \\ \frac{\partial \mathcal{L}}{\partial \lambda p_2} & \frac{\partial \mathcal{L}}{\partial p_1 p_2} & \frac{\partial \mathcal{L}}{\partial p_2^2} & \cdots & \frac{\partial \mathcal{L}}{\partial p_2 p_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{L}}{\partial \lambda p_N} & \frac{\partial \mathcal{L}}{\partial p_1 p_N} & \frac{\partial \mathcal{L}}{\partial p_2 p_N} & \cdots & \frac{\partial \mathcal{L}}{\partial p_N^2} \end{pmatrix},$$

which for our Lagrangian works out to:

$$H = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & -2 & 0 & \cdots & 0 \\ 1 & 0 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -2 \end{pmatrix}. \quad (41)$$

$\frac{\mu}{n}$ is a maximum if and only if H is negative definite. The negative definiteness of H can be demonstrated by showing that the signs of its leading principal minors demonstrate the appropriate alternating pattern⁵⁹. For any finite value of n , this Hessian can be diagonalized using the Gaussian Elimination technique. This makes it easy to calculate the determinants of the various $x \times x$ upper-left submatrices and to show that the signs of said determinants do indeed follow the pattern of $(-1)^{x-1}$ for $x \geq 3$, satisfying the condition for negative definiteness. This conclusion can be generalized to arbitrary values of n using a proof by induction (the details of which are omitted for brevity). Thus, H is always a negative definite matrix, and so setting each $p_j = \frac{\mu}{n} = p$ does indeed maximize V for a given μ .

From Eq 38, the upper bound on the variance of the count of successful trajectories n_i^s in FFS phase $i > 0$ is:

$$V[n_i^s] \leq n_i p_i (1 - p_i).$$

From Eq 2.3, the phase weight estimator $\hat{w}_{i>0}$ can be given in terms of the success count n_i^s :

$$\hat{w}_{i>0} = \frac{n_i^s}{n_i}.$$

The variance of a quotient $V[x/y]$ is $V[x]/y^2$ given that y is a constant⁵⁶. Thus, the upper bound on the variance of $\hat{w}_{i>0}$ is:

$$V[\hat{w}_{i>0}] = V\left[\frac{n_i^s}{n_i}\right] = \frac{V[n_i^s]}{n_i^2} \leq \frac{1}{n_i} p_i (1 - p_i).$$

Finally, Eq 2.14 can be used to derive an upper bound on $V[\mathfrak{w}_{i>0}]$ from the

bound on $V[\hat{w}_{i>0}]$:

$$V[\mathfrak{w}_{i>0}] = n_i V[\hat{w}_{i>0}] \leq p_i(1 - p_i). \quad (42)$$

Parameterizing the FFPilot optimizing equation (Eq 2.30) with a value of $V[\mathfrak{w}_{i>0}]$ that is at least as large as the true value helps to ensure that the calculated n_i is at least sufficient to achieve the given error goal. The upper bound on the variance given in Eq 42 is equivalent to the variance of a single Bernoulli random variable with parameter p_i . Thus, for the purposes of parameterizing the FFPilot optimizing equation we treat each $\mathfrak{w}_{i>0}$ as a single Bernoulli random variable without any loss in accuracy.

.2 Blind Optimization Method

Taken as a whole, the FFPilot approach to optimizing simulations can function reliably only if the results of the pilot stage are highly accurate (at least in terms of the individual parameter estimates) and computationally inexpensive (relative to the production stage). No prior knowledge of the system under study is used in the setup of the pilot stage. Thus, a “blind” optimization method, one that uses no information about the current phase or any other, must be used during this initial stage.

The blind optimization method that we use during the FFPilot pilot stage works by altering the conditions under which a simulation phase is terminated. During a standard FFS simulation, simulation phase $i > 0$ is terminated once a fixed number of trajectories n_i have launched from λ_{i-1} and have ended, regardless of where (in state space) those trajectories have ended. During

a pilot stage, we instead terminate phase $i > 0$ only once a fixed number $n_i^s = n_{\text{pilot}}$ of *successful* trajectories (*i.e.* the ones that reach λ_i) have been observed.

The advantage of using our blind optimization method is that it is able to produce estimates of the phase weight $w_{i>0} = p_i$ with constrained maximum error. The margin of error for a single phase $i > 0$ can be calculated from Eqs 2.20 and 2.28:

$$\zeta [\hat{p}_i] = z_\alpha \sqrt{\frac{1 - p_i}{p_i n_i}}.$$

The total count of trajectories n_i required to produce n_{pilot} successful trajectories in phase $i > 0$ converges to $\frac{n_{\text{pilot}}}{p_i}$. Thus, with respect to the pilot stage the above equation can be rewritten as:

$$\zeta [\hat{p}_i] = z_\alpha \sqrt{\frac{1 - p_i}{n_{\text{pilot}}}}. \quad (43)$$

The error increases as p_i becomes smaller, but it remains within a finite bound even as p_i goes to zero (see supplemental Fig S10). By default and throughout this paper we use the fixed values of $n_{\text{pilot}} = 10^4$ and $z_\alpha = z_{.95} \approx 1.96$ for all phases of the pilot stage. These values of n_{pilot} and z_α give a maximum \hat{p}_i margin of error of 2%.

During phase 0, the distribution of samples taken from the underlying random variable w_0 (*i.e.* the set of observed waiting times in between λ_0 forward crossing events) is model dependent. This means that no formulation equivalent to Eq 43 is possible for the phase weight $w_0 = \tau_{\mathcal{A}}$. However, the estimator $\hat{\tau}_{\mathcal{A}}$ can be in general assumed to be \sqrt{n} -consistent (as described in Secs 2.2.4 and 2.3.1.1). Plugging Eqs 2.9 and 2.14 into Eq 2.6 gives the

asymptotic margin of error for phase 0 of the pilot stage:

$$\zeta [\hat{\tau}_{\mathcal{A}}] = \frac{z_{\alpha}}{\sqrt{n_{\text{pilot}}}} \frac{\sqrt{V [\mathfrak{w}_0]}}{\tau_{\mathcal{A}}}. \quad (44)$$

It can be ensured that $\tau_{\mathcal{A}} \approx \sqrt{V [\mathfrak{w}_0]}$ through appropriate placement of \mathfrak{w}_0 (*i.e.* away from a basin of attraction). Given that $\tau_{\mathcal{A}}$ and $V [\mathfrak{w}_0]$ are appropriately matched, the margin of error of phase 0 will be roughly proportional to $\frac{z_{\alpha}}{\sqrt{n_{\text{pilot}}}}$. For $n_{\text{pilot}} = 10^4$ and $z_{\alpha} \approx 1.96$, this also works out to a $\hat{\tau}_{\mathcal{A}}$ margin of error of about 2%. Thus, the blind optimization approach used in the FFPilot pilot stage controls error in $\hat{\tau}_{\mathcal{A}}$, though not in such a conveniently bounded fashion as \hat{p}_i .

Tables

Table 2.1: Parameterization and other simulation data from our simplified Rare Event Model (REM). The phase weights w_i were copied from those of GTS $_{\theta=1}$ (see Table 2.5). All sample costs c were set to 1.

phase	$*w_i$	$\dagger c_i^f$	$\ddagger c_i^s$	$\S c_i$	$\P n_i \times 10^{-3}$
0	45.3	—	—	1	1834
1	.092	1	1	1	2020
2	.274	1	1	1	1047
3	.136	1	1	1	1621
4	.150	1	1	1	1527
5	.242	1	1	1	1138
6	.636	1	1	1	486
7	.711	1	1	1	410
8	.858	1	1	1	261
9	.913	1	1	1	199
10	.973	1	1	1	107
11	.989	1	1	1	67
12	.998	1	1	1	30
MFPT	$1.074 \cdot 10^6$				

^{*}The phase weight of phase i .

[†]The expected cost of a failed phase i trajectory.

[‡]The expected cost of a successful phase i trajectory.

[§]The expected cost of a phase i sample.

[¶]The phase i sample count (in thousands) for a 1% error goal with 95% confidence.

Table 2.2: Reaction scheme for our Self Regulating Gene (SRG) models.

Reactions	Rates	Description
$\emptyset \longrightarrow A$	$k_{low} + (k_{high} - k_{low}) \frac{A^h}{k_{50}^h + A^h}$	expression
$A \longrightarrow \emptyset$	βA	decay

Table 2.3: Parameterizations and other simulation data from our SRG models. Per-phase weights, costs, and sample counts were estimated from the average results of 1000 FFPilot simulations (1% error goal). *MFPT* values were estimated from the results of 10^5 DS simulations, plus or minus the 95% confidence interval bounds.

parameter	SRC _{h=2.4}						SRC _{h=2.3}						SRG _{h=2.2}					
	h	2.4	10	200	91	1	2.3	10	200	92.5	1	2.2	10	200	94	1		
phase	${}^*\lambda_i$	${}^+w_i$	${}^\ddagger c_i^f$	${}^\$ c_i^s$	${}^\P c_i$	${}^{\parallel\parallel} n_i \times 10^{-3}$	λ_i	w_i	c_i^f	c_i^s	c_i	$n_i \times 10^{-3}$	λ_i	w_i	c_i^f	c_i^s	c_i	$n_i \times 10^{-3}$
0	23.0	9.66	-	-	9.66	641	23.0	6.89	-	6.89	703	23.0	4.78	-	-	4.78	762	
1	33.6	.007	.130	.913	.135	26395	34.1	.010	.137	.936	.150	18975	34.6	.018	.138	.846	.151	12067
2	44.2	.071	1.56	1.21	1.53	2383	45.2	.112	1.65	1.20	1.60	1620	46.2	.140	1.64	1.30	1.60	1253
3	54.8	.277	2.89	1.14	2.41	850	56.2	.296	2.99	1.24	2.47	714	57.8	.354	3.05	1.23	2.41	556
4	65.3	.565	4.22	1.30	2.57	447	67.3	.580	4.35	1.28	2.57	387	69.3	.572	4.45	1.39	2.70	337
5	75.9	.855	5.45	1.12	1.75	254	78.4	.825	5.63	1.23	2.00	237	80.9	.814	5.78	1.23	2.08	212
6	86.5	.962	6.48	1.03	1.24	147	89.5	.946	6.75	1.08	1.39	147	92.5	.924	7.01	1.24	1.67	141
7	97.1	.993	7.35	.844	.890	73	101	.986	7.71	.926	1.02	85	104	.977	8.12	1.09	1.25	88
8	108	.999	8.11	.700	.709	34	112	.997	8.56	.836	.862	46	116	.993	9.07	.905	.959	53
9	118	1.00	8.83	.767	.769	16	123	.999	9.42	.825	.833	25	127	.997	10.0	.981	1.00	32
10	129	1.00	-	.779	.779	8	134	1.00	10.4	.894	.897	14	139	.999	11.0	.955	.965	20
11	139	1.00	-	1.05	1.05	5	145	1.00	-	1.07	1.07	9	150	.999	12.3	1.24	1.25	14
12	150	1.00	-	1.33	1.33	3	156	1.00	-	1.44	1.44	6	162	1.00	13.8	1.48	1.49	10
MFPT	$(1.521 \pm .0094) \cdot 10^5$						$(4.661 \pm .029) \cdot 10^4$						$(1.271 \pm .0079) \cdot 10^4$					

*The position of interface i in terms of order parameter A .

† The phase weight of phase i .

‡ The expected cost of a failed phase i trajectory.

${}^\$$ The expected cost of a successful phase i trajectory.

¶ The expected cost of a phase i sample.

${}^{\parallel\parallel}$ The phase i sample count (in thousands) for a 1% error goal with 95% confidence.

Table 2.4: Reaction scheme for our Genetic Toggle Switch (GTS) models.

<u>Reactions</u>		<u>Rate Constants</u>		<u>Description</u>
<i>Protein A</i>	<i>Protein B</i>	<i>forward</i>	<i>reverse</i>	
$O \longrightarrow O + A$	$O \longrightarrow O + B$	θ		expression
$A \longrightarrow \emptyset$	$B \longrightarrow \emptyset$	$\theta/4$		decay
$2A \rightleftharpoons A_2$	$2B \rightleftharpoons B_2$	5	5	dimerization
$O + A_2 \rightleftharpoons OA_2$	$O + B_2 \rightleftharpoons OB_2$	5	1	operator binding
$OA_2 \longrightarrow OA_2 + A$	$OB_2 \longrightarrow OB_2 + B$	θ		bound expression

Table 2.5: Parameterizations and other simulation data from our GTS models. Per-phase weights, costs, and sample counts were estimated from the average results of 1000 FFPilot simulations (1% error goal). *MFPT* values were estimated from the results of 10^5 DS simulations, plus or minus the 95% confidence interval bounds.

parameter θ	GTS $_{\theta=1}$					GTS $_{\theta=1}$					GTS $_{\theta=10}$				
	w_i^*	$\mathbb{E} c_i^f$	$\mathbb{E} c_i^s$	$\mathbb{E} c_i$	$n_i \times 10^{-3}$	w_i	c_i^f	c_i^s	c_i	$n_i \times 10^{-3}$	w_i	c_i^f	c_i^s	c_i	$n_i \times 10^{-3}$
phase	${}^*\lambda_i$														
0	-27.0	453	-	-	453	1664	45.3	-	45.3	1337	4.52	-	-	4.52	985
1	-22.5	.092	19.7	63.5	23.7	8010	.092	1.97	6.35	2.37	6436	.090	.197	.643	.238
2	-18.0	.271	148	113	138	1728	.274	14.8	11.2	13.8	1382	.281	1.51	1.12	1.40
3	-13.5	.128	307	176	290	1897	.136	30.7	17.1	28.8	1482	.170	3.13	1.46	2.85
4	-9.0	.111	442	120	406	1738	.150	44.0	10.5	38.9	1201	.339	4.46	.597	3.15
5	-4.5	.082	536	131	503	1849	.242	52.6	10.1	42.3	859	.676	4.98	.529	356
6	0.0	.437	659	118	423	684	.636	61.4	8.93	28.0	451	.872	5.14	.463	1.06
7	4.5	.632	830	167	411	467	.711	73.6	12.7	30.3	365	.873	5.83	.657	1.32
8	9.0	.861	994	140	259	310	.858	85.1	11.1	21.6	276	.918	6.43	.599	1.08
9	13.5	.940	1153	165	225	209	.913	97.3	14.4	21.6	210	.925	7.08	.861	1.33
10	18.0	.988	1283	122	135	116	.973	108	11.5	14.1	140	.960	7.58	.809	1.08
11	22.5	.997	1397	156	159	53	.989	117	15.4	16.5	81	.972	8.14	1.27	1.46
12	27.0	1.00	1493	150	150	19	.998	126	14.9	15.1	37	.989	8.61	1.36	1.44
MFPT						$(7.015 \pm .043) \cdot 10^7$				$(1.074 \pm .0067) \cdot 10^6$					$(7.701 \pm .048) \cdot 10^3$

*The position of interface i in terms of order parameter Δ .

${}^t w_i$ The phase weight of phase i .

$\mathbb{E} c_i^f$ The expected cost of a failed phase i trajectory.

$\mathbb{E} c_i^s$ The expected cost of a successful phase i trajectory.

$\mathbb{E} c_i$ The expected cost of a phase i sample.

$n_i \times 10^{-3}$ The phase i sample count (in thousands) predicted by FFPilot to achieve a 1% error goal, 95% confidence.

Figures

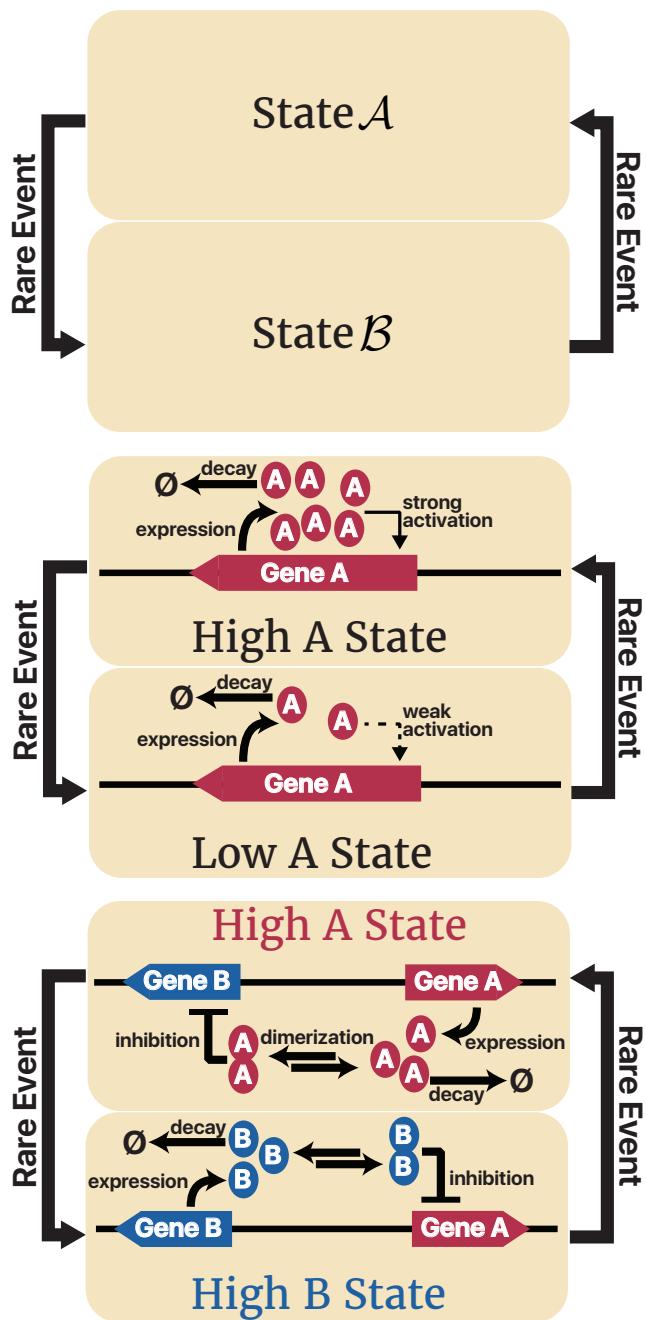


Figure 2.1: Schematics of the model systems investigated. (top) The Rare Event Model (REM, see Sec 2.3.3 and Table 2.1 for complete details). (middle) Self Regulating Gene (SRG, see Sec 2.3.4 and Tables 2.2 and 2.3). (bottom) Genetic Toggle Switch (GTS, see Sec 2.3.5 and supplemental Table S1).

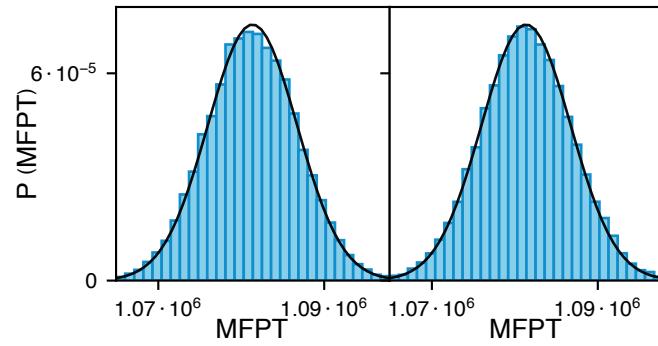


Figure 2.2: $MFPT$ estimates for REM calculated using (left) DS and (right) FFPilot. Shown are (blue bars) binned $MFPT$ estimates from $1.6 \cdot 10^5$ simulations with a 1% error goal and (black lines) $MFPT$ estimate distributions from Eq 2.7 and Eq 2.15, respectively.

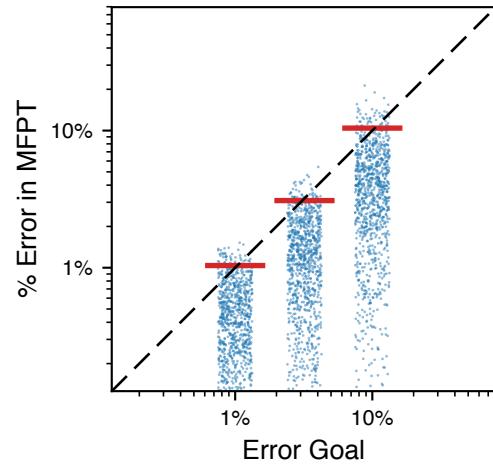


Figure 2.3: Actual error vs error goal of FFPilot simulations of the REM. Each strip shows the *MFPT* estimation error from (blue dots) 1000 independent FFPilot simulations and (red lines) the 95th percentiles of the errors. Jitter was added to the x-position of the dots for visualization.

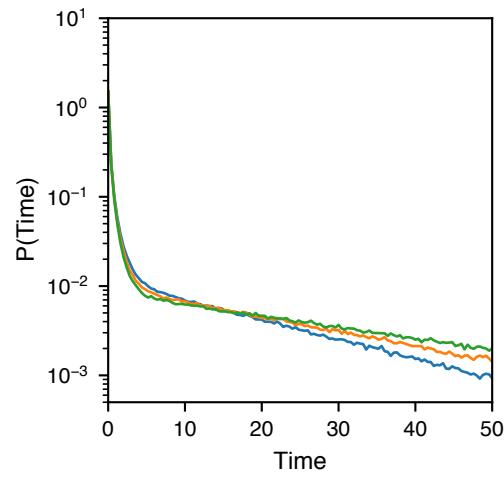


Figure 2.4: Waiting times between forward crossing events during phase 0 of forward flux simulations of the SRG models. Each line shows a distribution calculated from 10^6 samples from a single phase 0 trajectory of (blue) SRG_{h=2.4}, (orange) SRG_{h=2.3}, and (green) SRG_{h=2.2}.

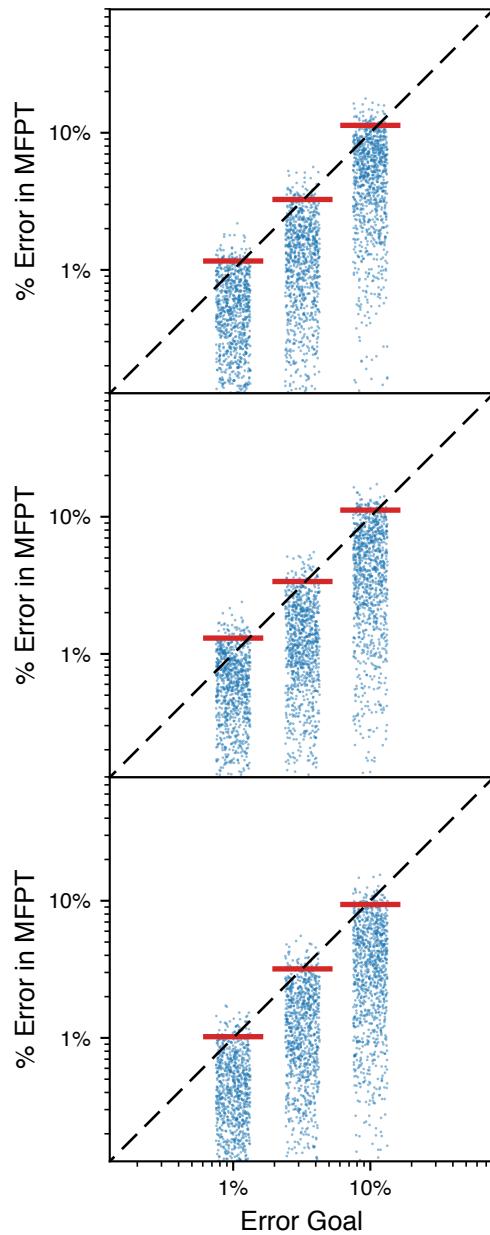


Figure 2.5: Actual error vs error goal of FFPilot simulations for (top) $\text{SRG}_{h=2.4}$, (middle) $\text{SRG}_{h=2.3}$, and (bottom) $\text{SRG}_{h=2.2}$. Each strip shows (blue dots) 1000 independent FFPilot simulations and (red lines) the 95th percentiles of the errors.

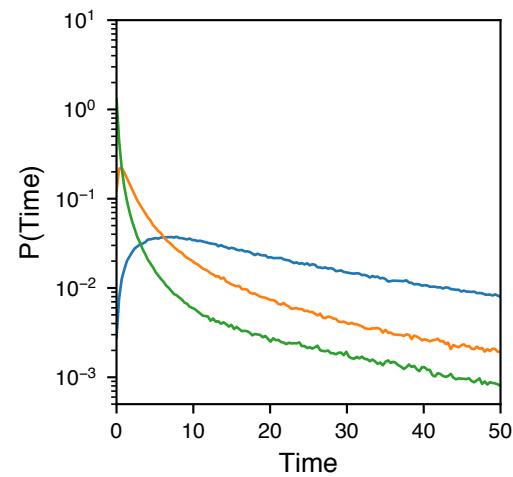


Figure 2.6: Waiting times between forward crossing events during phase 0 of FFS simulations of the GTS models. Each line shows a distribution calculated from 10^6 samples from a single phase 0 trajectory of (blue) GTS $_{\theta=.1}$, (orange) GTS $_{\theta=1}$, and (green) GTS $_{\theta=10}$.

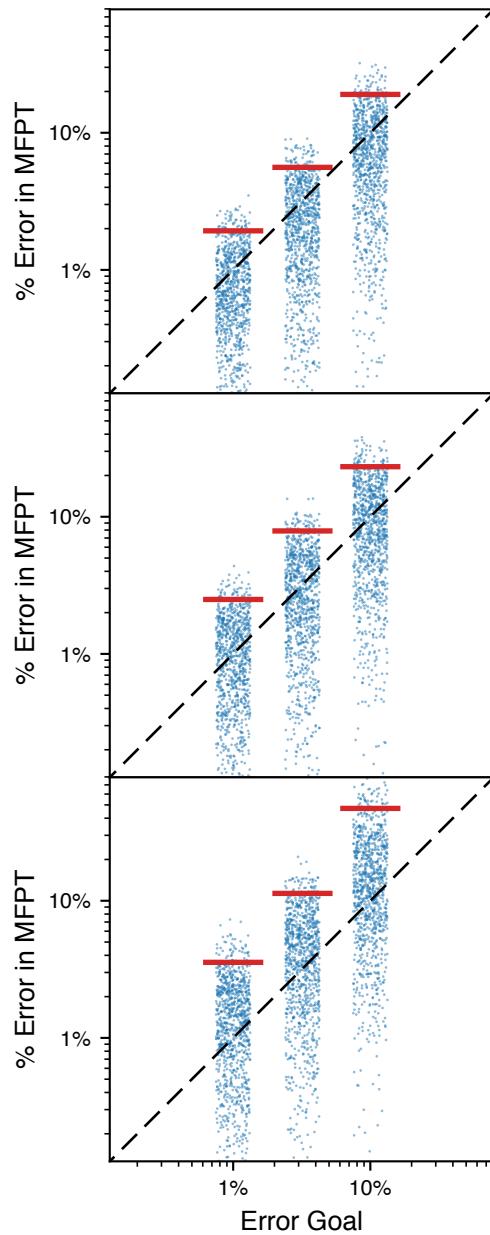


Figure 2.7: Actual error vs error goal of FFPilot simulations for (top) $GTS_{\theta=.1}$, (middle) $GTS_{\theta=1}$, and (bottom) $GTS_{\theta=10}$. Each strip shows (blue dots) 1000 independent FFPilot simulations and (red lines) the 95th percentiles of the errors.

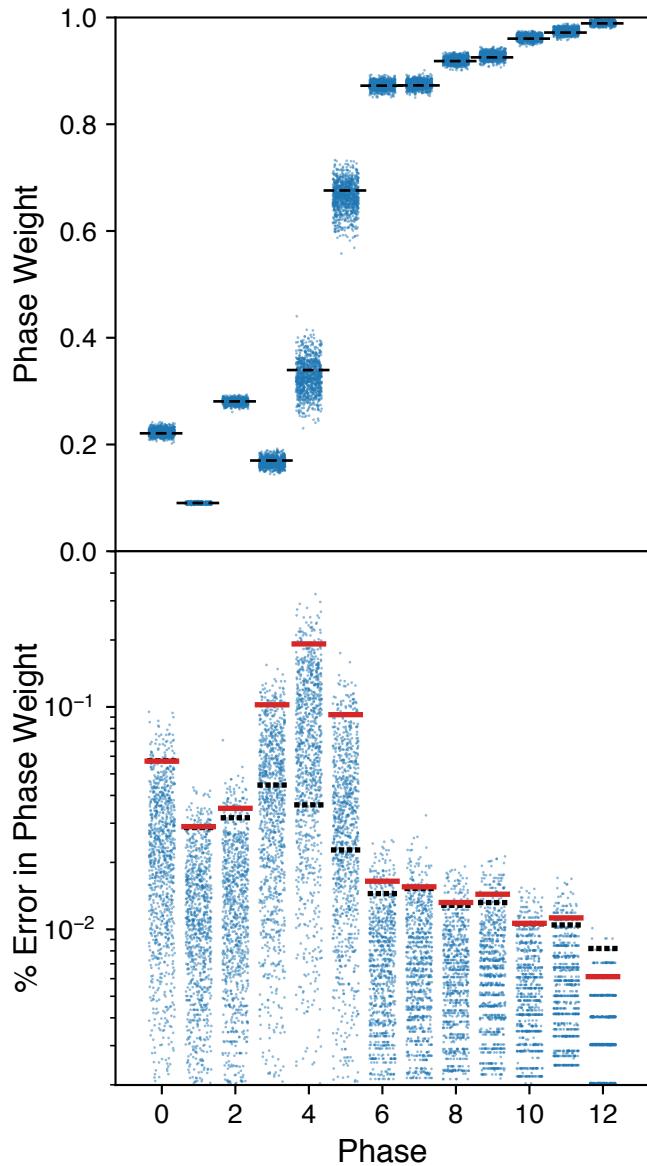


Figure 2.8: $\text{GTS}_{\theta=10}$ phase weight estimates and errors. (top) The phase weights as estimated by (blue dots) 1000 independent FFPilot simulations run to a 10% error goal. The dashed black lines show the phase weights from an FFPilot simulation with a 0.1% error goal. (bottom) The blue dots show the percent error of each phase weight estimate given in the top subplot, relative to the 0.1% error goal simulation. Also shown are (red lines) the observed 95th error percentiles, and (dashed black lines) the expected 95th error percentiles.

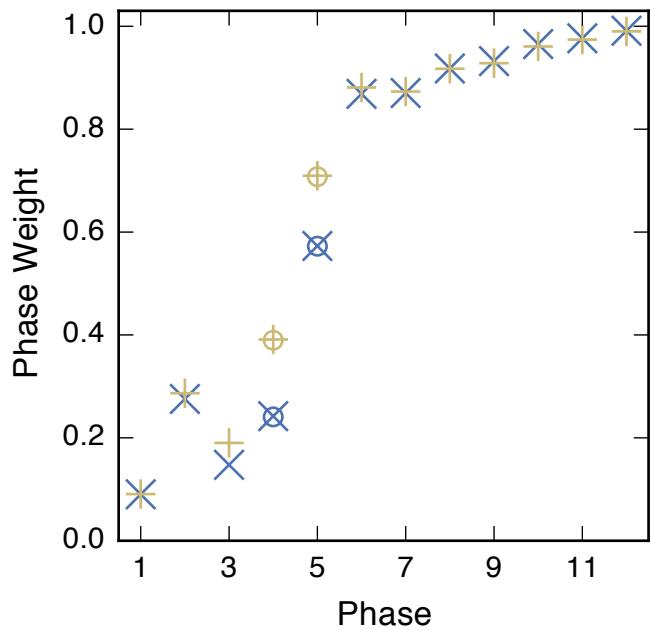


Figure 2.9: The phase weights of two replicates: (blue x) Replicate 6 and (gold +) Replicate 8, from a $\text{GTS}_{\theta=10}$ FFPilot simulation run to an error goal of 10%. Circles show weights for phases 4 and 5 calculated via an alternative method using Eq 2.33.

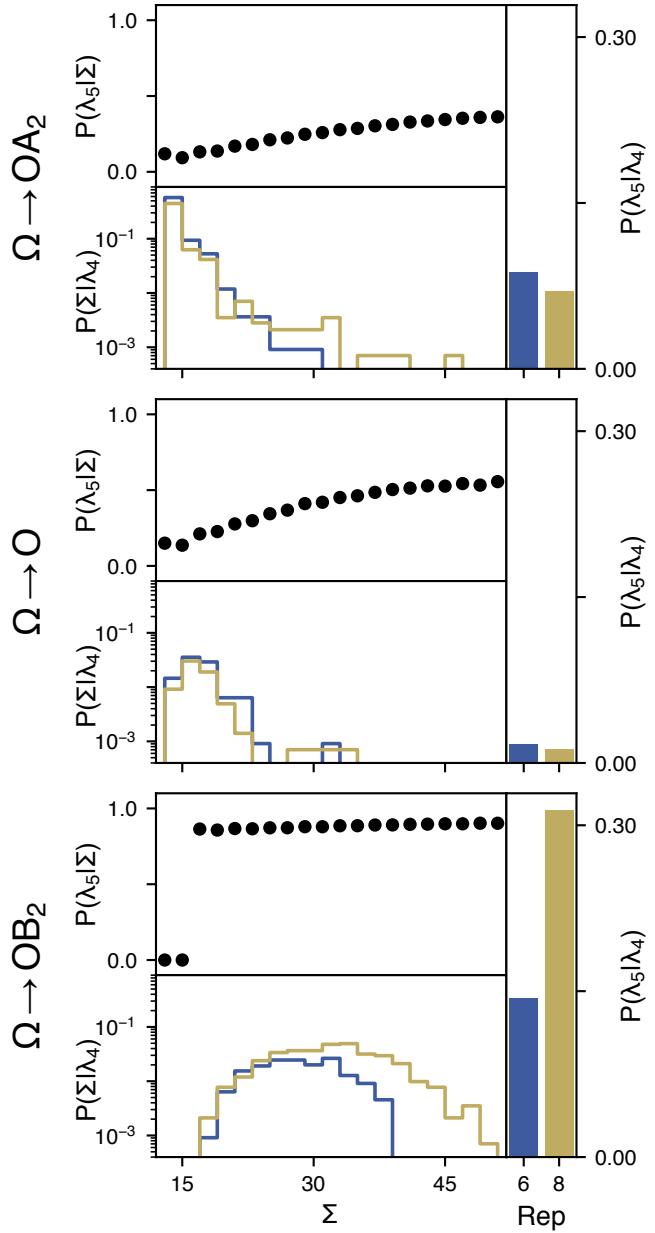


Figure 2.10: Breakout of the calculation of phase weight 5 for replicates 6 and 8. Each group of three plots show data from a slice of GTS state space with a different Ω , the state of the operator DNA. Each subplot within a group shows different probability data. (dots) The probability of a trajectory launched from a point on λ_4 fluxing forward to λ_5 vs the orthogonal order parameter Σ . (lines) The occupancy along Σ at interface λ_4 for (blue) replicate 6 and (gold) replicate 8. (bars) The cumulative probability of fluxing forward across λ_5 when starting at the specified Ω for (blue) replicate 6 and (gold) replicate 8.

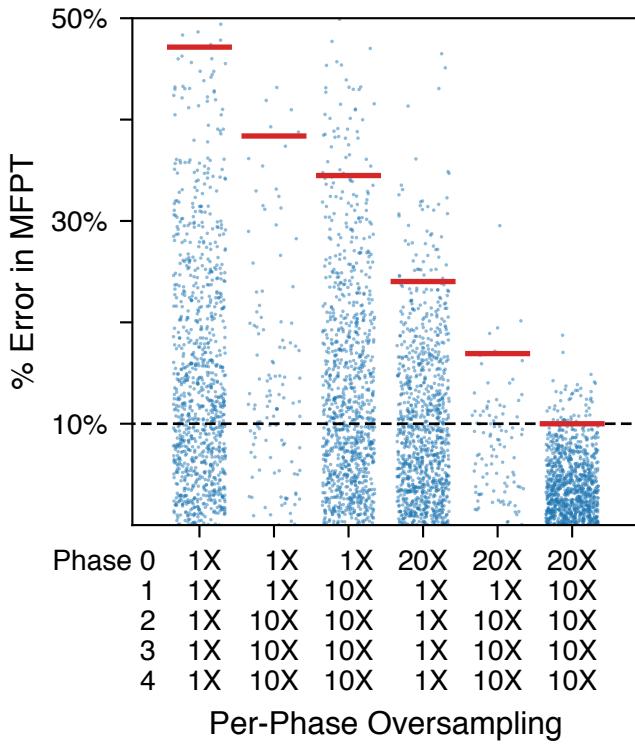


Figure 2.11: Errors from simulations executed with various oversampling schemes. The extent of oversampling in each phase for the separate schemes is indicated beneath each column of results. All simulations were of $GTS_{\theta=10}$ run to an error goal of 10%. Red lines show the 95th error percentiles.

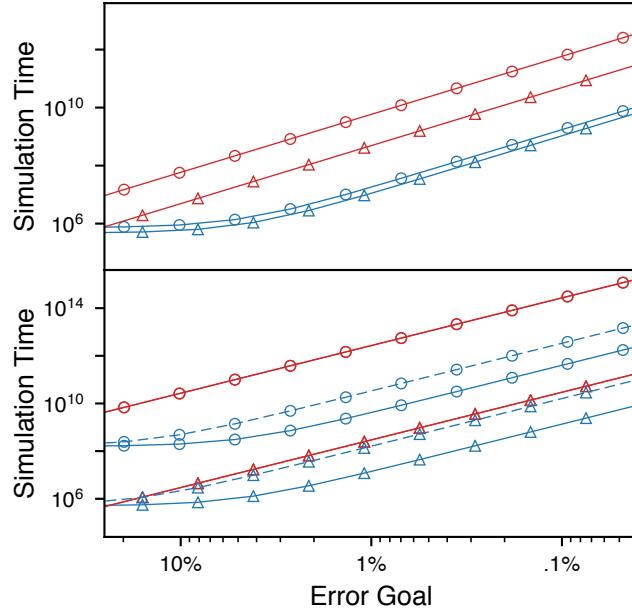


Figure 2.12: Simulation time vs error goal for several SRG and GTS models. Simulation time was calculated using Eq 2.34 for (red lines) DS, and Eq 2.36 for (blue lines) FFPilot and (dashed line) FFPilot with oversampling to correct the landscape error. Parameters were taken from Tables 2.3 and 2.5. (top) Data from (circles) SRG_{h=2.4} and (triangles) SRG_{h=2.2}. (bottom) Data from (circles) GTS _{$\theta=1$} and (triangles) GTS _{$\theta=10$} .

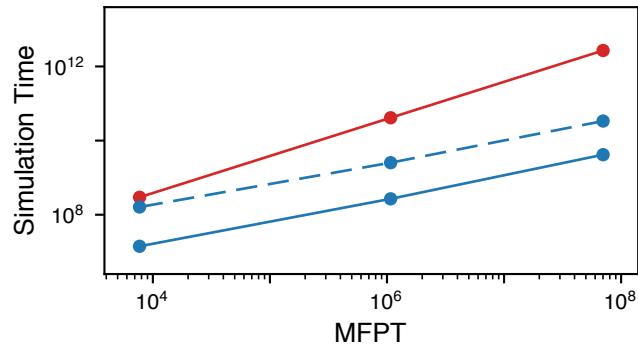


Figure 2.13: Simulation time vs $MFPT$ for several GTS models. Simulation times are shown for (red dots) DS, (blue dots) FFPilot, and (blue dots with dashed line) FFPilot with oversampling to correct for landscape error. The simulation times shown are averaged across 1000 simulations. Connecting lines added for illustration.

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Chapter 3

Discussion and Conclusion

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