Assignment 2

October 25, 2023

Task 2

A is $m \times n$ with SVD $A = U\Sigma V^T$.

First, let's prove that $m \geqslant n$. Let $m < n$, $\operatorname{rank} A \leqslant m < n$ and $\operatorname{rank} A^T = \operatorname{rank} A$. But then $A^T A$ is $n \times n$ and $\operatorname{rank} A^T A \leqslant \operatorname{rank} A < n$, so $A^T A$ is singular and $\left(A^T A\right)^{-1}$ does not exist. Thus $m \leqslant n$.

The matrices U, Σ and V^T have the following properties (full SVD):

- U is $m \times m$, unitary, possibly singular.
- $\Sigma = \left[\frac{S}{O}\right]$, where S is the non-zero diagonal submatrix of Σ and O is the zero submatrix. If $m = n$, then $\Sigma = S$. Let also $\tilde{\Sigma} = \left[S^{-1}|O\right]$ — such matrix that $\tilde{\Sigma}\Sigma = S^{-1}S = \hat{I}$. Note that $\tilde{\Sigma}^T\Sigma^T = \left[S^{-1}|O\right] \cdot \left[\frac{S}{O}\right] = \hat{I}$ and $\tilde{\Sigma}\tilde{\Sigma}^T = \left[S^{-1}|O\right] \cdot \left[\frac{S^{-1}}{O}\right] = S^{-2}$.
- V is $n \times n$, unitary, non-singular.

Having that properties, let's solve the problems:

(i)

$$\left(A^TA\right)^{-1} = \left(\left(U\Sigma V^T\right)^T \cdot U\Sigma V^T\right)^{-1} = \left(V\Sigma^T \cdot U^TU \cdot \Sigma V^T\right)^{-1} =$$

$$V\left(\Sigma^T\Sigma\right)^{-1}V^T = V\left([S|O]\begin{bmatrix} S \\ \hline O \end{bmatrix}\right)^{-1}V^T = V\left(S^2\right)V^T = VS^{-2}V^T.$$

(ii)

$$\left(A^TA\right)^{-1}A^T = VS^{-2}V^T \cdot V\Sigma^TU^T = VS^{-2} \cdot \Sigma^TU^T = V\tilde{\Sigma}\tilde{\Sigma}^T \cdot \Sigma^TU^T = V\tilde{\Sigma}U^T.$$

(iii)

$$A\left(A^TA\right)^{-1} = U\Sigma V^T \cdot VS^{-2}V^T = U\Sigma \cdot \tilde{\Sigma}\tilde{\Sigma}^TV^T = U\tilde{\Sigma}^TV^T.$$

(iv)

$$A\left(A^TA\right)^{-1}A^T = U\Sigma V^T \cdot VS^{-2}V^T \cdot V\Sigma^TU^T = U\Sigma \cdot \tilde{\Sigma}\tilde{\Sigma}^T \cdot \Sigma^TU^T = UU^T = \hat{I} = \hat{I} \cdot \hat{I} \cdot \hat{I}.$$

Task 3

Compute SVD

The given matrix is

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}.$$

First we compute the eigenvalues of AA^T:

$$AA^T = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} = \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix}$$

$$\det\left(AA^T - \lambda I\right) = 25 \cdot \det\begin{bmatrix} 5-\tilde{\lambda} & 3 \\ 3 & 5-\tilde{\lambda}. \end{bmatrix} = 25\left(\tilde{\lambda}^2 - 10\tilde{\lambda} + 16\right) = 25\left(\tilde{\lambda}-2\right)\left(\tilde{\lambda}-8\right),$$

where $\lambda = 25\tilde{\lambda}$. $\lambda_1 = 25 \cdot 8 = 200$, $\lambda_2 = 25 \cdot 2 = 50$. The singular values are $\sigma_1 = 10\sqrt{2}$ and $\sigma_2 = 5\sqrt{2}$. This gives

$$\Sigma = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}.$$

The left singular vectors are the unit eigenvectors of AA^T:

$$\left(A - \hat{I}\lambda_1\right) \cdot s_1 = \begin{bmatrix} -75 & 75 \\ 75 & -75 \end{bmatrix} \cdot \begin{bmatrix} s_1^1 \\ s_1^2 \end{bmatrix} = 75 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} s_1^1 \\ s_1^2 \end{bmatrix} = 0 \;\Rightarrow\; s_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\left(A - \hat{I}\lambda_2\right) \cdot s_2 = \begin{bmatrix} 75 & 75 \\ 75 & 75 \end{bmatrix} \cdot \begin{bmatrix} s_2^1 \\ s_2^2 \end{bmatrix} = 75 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_2^1 \\ s_2^2 \end{bmatrix} = 0 \;\Rightarrow\; s_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This gives

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Finally, V^T can be found using the following formula:

$$V^T = (U\Sigma)^{-1} A.$$

Compute $(U\Sigma)^{-1}$:

$$(U\Sigma)^{-1} = \begin{bmatrix} 10 & 5 \\ 10 & -5 \end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}^{-1}.$$

Using Gaussian elimination:

$$\begin{bmatrix} 2 & 1 & \vdots & 1 & 0 \\ 2 & -1 & \vdots & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & \vdots & 1 & 0 \\ 0 & -2 & \vdots & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & \vdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \vdots & \frac{1}{2} & -\frac{1}{2} \end{bmatrix},$$

$$\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}.$$

This gives us

$$(U\Sigma)^{-1} = \frac{1}{20} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

and the last term of our SVD,

$$V^T = \frac{1}{20}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix},$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

2D Plots

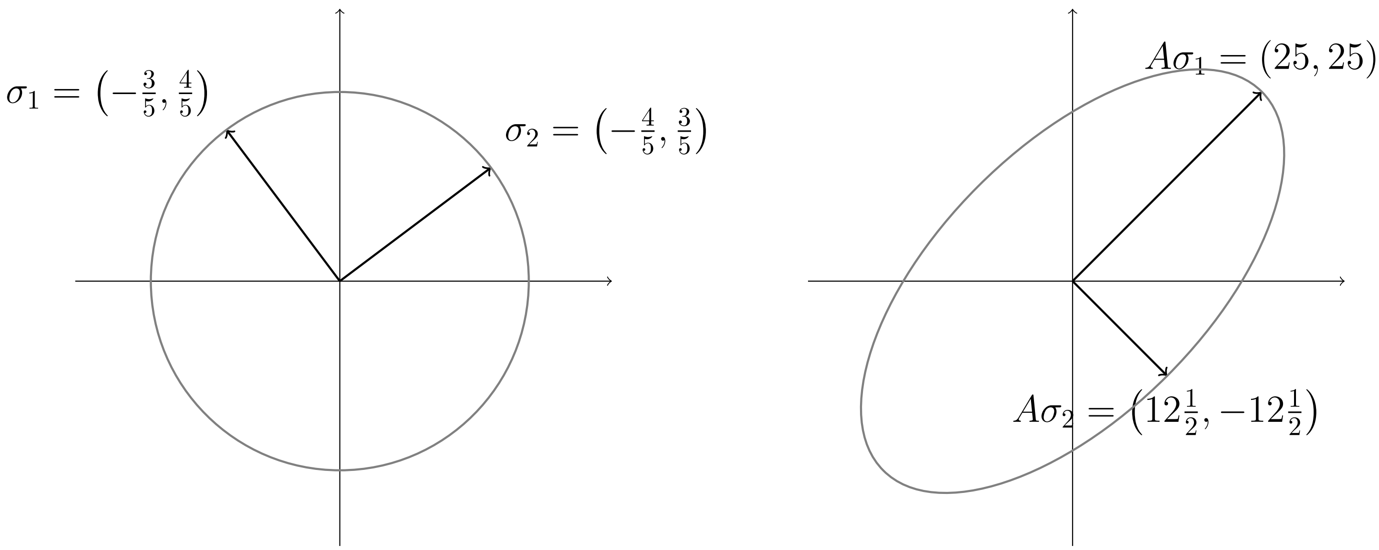

Figure 1: Singular vectors and their images under A (not to scale).

Norm of A

$$\|A\|_2 = \max\{\sigma_1, \sigma_2\} = \sigma_1 = 10\sqrt{2}.$$

$$\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2} = \sqrt{4 + 100 + 121 + 25} = \sqrt{250} = 5\sqrt{10}.$$

Eigenvalues of A

$$\det\left(A - \lambda\hat{I}\right) = \det\begin{bmatrix} -2-\lambda & 11 \\ -10 & 5-\lambda \end{bmatrix} = \lambda^2 - 3\lambda + 100.$$

$$\lambda_{1,2} = \frac{3}{2} + \frac{1}{2}\sqrt{-391};$$

$$\lambda_1 = \frac{3}{2} + \frac{i}{2}\sqrt{391}, \ \lambda_2 = \frac{3}{2} - \frac{i}{2}\sqrt{391}.$$