Assignment 2

November 12, 2023

Task 1

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Projectors

$$\left(A^T A\right)^{-1} = \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$P_A = A\left(A^T A\right) A^T = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$\left(B^T B\right) = \left(\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}.$$

$$P_B = B\left(B^T B\right) B^T = \frac{1}{6} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 2 \\ -2 & 2 \\ 5 & -2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix}.$$

QR Decompositon

- Matrix A:

$$r_{11} = \|a_1\|_2 = \sqrt{2},\ q_1 = \frac{1}{\sqrt{2}} a_1.$$

$$r_{12} = q_1^T \cdot a_2 = 0,\ r_{22} = \|a_2 - r_{12} q_1\|_2 = \|a_2\|_2 = 1,\ q_2 = a_2.$$

$$B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

- Matrix B:

$$r_{11} = \|b_1\| = \sqrt{2},\ q_1 = \frac{1}{\sqrt{2}} b_1.$$

$$r_{12} = q_1^T \cdot b_2 = \sqrt{2},\ r_{22} = \|b_2 - r_{12} q_1\|_2 = \left\| \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T \right\|_2 = \sqrt{3}.$$

$$q_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T.$$

$$B = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{3} & \sqrt{2} \\ 0 & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}.$$

Task 3

- Equation (2):

$$A = \frac{1}{2} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \\ -1 & -1 \end{bmatrix},\ b = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T.$$

$A = U\Sigma V^T$, where

$$U = \frac{1}{2\sqrt{2}} \begin{bmatrix} -2 & 2 & 0 \\ -\sqrt{2} & -\sqrt{2} & 2 \\ \sqrt{2} & \sqrt{2} & 2 \end{bmatrix},\ \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ V = -\hat{I}.$$

Let $V^T x = \tilde{x},\ U^T b = \tilde{b}$, then $\Sigma \tilde{x} = \tilde{b}$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \end{bmatrix}$$

Thus, $\tilde{x} = \Sigma^T \tilde{b}$ and

$$x = V\tilde{x} = V\Sigma^T U^T b = \frac{1}{2\sqrt{2}} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -2 & -\sqrt{2} & \sqrt{2} \\ 2 & -\sqrt{2} & \sqrt{2} \\ 0 & 2 & 2 \end{bmatrix} \cdot b =$$

$$-\frac{1}{2\sqrt{2}} \begin{bmatrix} -2 & -\sqrt{2} & \sqrt{2} \\ 2 & -\sqrt{2} & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = -\frac{1}{2\sqrt{2}} \begin{bmatrix} -2+\sqrt{2} \\ 2+\sqrt{2} \end{bmatrix}.$$

- Equation (3):

$$A = \frac{1}{2} \begin{bmatrix} \sqrt{2} & 1 & -1 \\ -\sqrt{2} & 1 & -1 \end{bmatrix} = U\Sigma V^T,\ b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$U = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ V = \frac{1}{2} \begin{bmatrix} -\sqrt{2} & -\sqrt{2} & 0 \\ -1 & 1 & \sqrt{2} \\ 1 & -1 & \sqrt{2} \end{bmatrix}.$$

Let $V^T x = \tilde{x},\ U^T b = \tilde{b}$, then $\Sigma \tilde{x} = \tilde{b}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{bmatrix}.$$

Thus, $\tilde{x} = \Sigma^T \tilde{b}$. $x = V\tilde{x} = V\Sigma^T U^T b$:

$$x = \frac{1}{2} \begin{bmatrix} -\sqrt{2} & -\sqrt{2} & 0 \\ -1 & 1 & \sqrt{2} \\ 1 & -1 & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

$$-\frac{1}{2} \begin{bmatrix} -\sqrt{2} & -\sqrt{2} & 0 \\ -1 & 1 & \sqrt{2} \\ 1 & -1 & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}.$$