Mersenne Twister: A 623-Dimensionally Equidistributed Uniform Pseudo-Random Number Generator

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A new algorithm called Mersenne Twister (MT) is proposed for generating uniform pseudorandom numbers. For a particular choice of parameters, the algorithm provides a super astronomical period of $2^{19937} - 1$ and 623-dimensional equidistribution up to 32-bit accuracy, while using a working area of only 624 words. This is a new variant of the previously proposed generators, TGFSR, modified so as to admit a Mersenne-prime period. The characteristic polynomial has many terms. The distribution up to v bits accuracy for $1 \leq v \leq 32$ is also shown to be good. An algorithm is also given that checks the primitivity of the characteristic polynomial of MT with computational complexity $O(p^2)$ where p is the degree of the polynomial.

We implemented this generator in portable C-code. It passed several stringent statistical tests, including diehard. Its speed is comparable to other modern generators. Its merits are due to the efficient algorithms that are unique to polynomial calculations over the two-element field.

Categories and Subject Descriptors: F.2.1 [**Theory of Computation**]: Numerical Algorithms and Problems—*computations in finite fields*; G.2.1 [**Discrete Mathematics**]: Combinatorics—*recurrences and difference equations*; G.3 [**Probability and Statistics**]: Random Number Generation

General Terms: Algorithms, Experimentation, Theory

Additional Key Words and Phrases: Finite fields, GFSR, incomplete array, inversive-decimation method, k-distribution, Mersenne Primes, Mersenne Twister, m-sequences, MT19937, multiple-recursive matrix method, primitive polynomials, random number generation, tempering, TGFSR

Dedicated to the Memory of Nobuo Yoneda

1. INTRODUCTION

1.1 A Short Summary

We propose a new random number generator, called the *Mersenne Twister*. An implemented C-code, MT19937, has the period $2^{19937} - 1$ and a 623-dimensional equidistribution property, which seem to be the best ever implemented. There are two new ideas added to the previous twisted GFSR [Matsumoto and Kurita 1992; 1994]. One is an *incomplete array* (see Sect. 3.1) to realize a Mersenne-prime period. The other is a fast algorithm to test the primitivity of the characteristic polynomial of a linear recurrence, called the *inversive-decimation method* (see Sect. 4.3). This algorithm does not even require the explicit form of the characteristic polynomial. It needs only (1) the defining recurrence, and (2) some fast algorithm that obtains the present state vector from its 1-bit output stream. The computational complexity of the inversive-decimation method is the order of the algorithm in (2) multiplied by the degree of the characteristic polynomial. To attain higher order equidistribution properties, we used the resolution-wise lattice method (see Tezuka [1990; 1994a]; Couture et al. [1993]), with Lenstra's algorithm [Lenstra 1985; Lenstra et al. 1982] for successive minima.

We stress that these algorithms make full use of the polynomial algebra over the two-element field. There are no corresponding efficient algorithms for integers.

1.2 *k*-Distribution: A Reasonable Measure of Randomness

Many generators of presumably "high quality" have been proposed, but only a few can be used for serious simulations because we lack a decisive definition of good "randomness" for practical pseudorandom number generators, and each researcher concentrates only on his particular set of measures for randomness.

Among many known measures, the tests based on the higher dimensional uniformity, such as the spectral test (c.f., Knuth [1981]) and the *k*-distribution test, described below, are considered to be strongest.[1]

Definition 1.1. A pseudorandom sequence $\mathbf{x}_i$ of w-bit integers of period P, satisfying the following condition, is said to be k-distributed to v-bit accuracy: let $trunc_v(\mathbf{x})$ denote the number formed by the leading v bits of $\mathbf{x}$ and consider P of the kv-bit vectors:

$$(trunc_v(\mathbf{x}_i), trunc_v(\mathbf{x}_{i+1}), \ldots, trunc_v(\mathbf{x}_{i+k-1}))(0 \leq i < P).$$

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[1]For the importance of k-distribution property, see Tootill et al. [1973]; Fushimi and Tezuka [1983]; Couture et al. [1993]; Tezuka [1995; 1994a]; Tezuka and L'Ecuyer [1991]; L'Ecuyer [1996]. A concise description can be seen in L'Ecuyer [1994].

Then, each of the 2^{kv} possible combinations of bits occurs the same number of times in a period, except for the all-zero combination that occurs once less often.

For each $v = 1, 2, \ldots, w$, let $k(v)$ denote the maximum number such that the sequence is $k(v)$-distributed to v-bit accuracy.

Note that the inequality $2^{k(v)v} - 1 \leq P$ holds because at most P patterns can occur in one period, and the number of possible bit patterns in the most significant v bits of the consecutive $k(v)$ words is $2^{k(v)v}$. Since we admit a flaw at zero, we need to add -1. We call this the *trivial upper bound*.

The geometric meaning is as follows. Divide each integer $\mathbf{x}_i$ by 2^w to normalize it into a pseudorandom real number x_i in the [0,1]-interval. Put the P points in the k-dimensional unit cube with coordinates $(x_i, x_{i+1}, \ldots, x_{i+k-1})$ $(i = 0, 1, \ldots, P - 1)$, i.e., the consecutive k tuples, for a whole period (the addition in the suffix is considered modulo P). We divide equally each [0,1] axis into 2^v pieces (in other words, consider only the most significant v bits). Thus, we have partitioned the unit cube into 2^{kv} small cubes. The sequence is k-distributed to v-bit accuracy if each cube contains the same number of points (except for the cube at the origin which contains one less). Consequently, the higher $k(v)$ for each v assures higher-dimensional equidistribution with v-bit precision. By *k-distribution test*, we mean to obtain the values $k(v)$. This test fits the generators based on a linear recursion over the two-element field $\mathbb{F}_2$ (we call these generators $\mathbb{F}_2$-*generators*).

The k-distribution also has a kind of cryptographic interpretation, as follows. Assume that the sequence is k-distributed to v-bit accuracy and that all the bits in the seed are randomly given. Then knowledge of the most significant v bits of the first l words does not allow the user to make any statement about the most significant v bits of the next word if $l < k$. This is because every bit-pattern occurs with equal likelihood in the v bits of the next word, by definition of k-distribution. Thus, if the simulated system is sensitive to the history of the k or fewer previously generated words with v-bit accuracy only, then it is theoretically safe.

1.3 Number of Terms in a Characteristic Polynomial

Another criterion on the randomness of $\mathbb{F}_2$-generators is the number of terms in the characteristic polynomial of the state transition function. Many $\mathbb{F}_2$-generators are based on trinomials, but they show poor randomness (e.g., GFSR rejected in an Ising-Model simulation [Ferrenberg et al. 1992], and a slight modification of trinomials [Fushimi 1990] rejected in Matsumoto and Kurita [1994]). For defects, see Lindholm [1968]; Fredricsson [1975]; Compagner [1991]; Matsumoto and Kurita [1992]; Matsumoto and Kurita [1994]; and Matsumoto and Kurita [1996].

As far as we know, all the known $\mathbb{F}_2$-generators satisfying the following two criteria: (1) high k-distribution properties for each v and (2) characteristic polynomials with many terms (not artificially extracted from a trinomial), are good generators [Tezuka and L'Ecuyer 1991; L'Ecuyer 1996;

Table I.   Cpu Time for 10^7 Generations and Working Area

ID	COMBO	KISS	ran_array	rand	taus88	TT800	MT19937
cpu − time (sec.)	11.14	9.24	23.23	9.64	7.95	9.97	10.18
working area (words)	4	5	1000	1	3	25	624
period	$\sim 2^{61}$	$\sim 2^{127}$	$\sim 2^{129}$	$\sim 2^{31}$	$\sim 2^{88}$	$2^{800} - 1$	$2^{19937} - 1$

Matsumoto and Kurita 1994], according to the stringent tests and results in the actual applications.

1.4 What We Obtained

We introduce an $\mathbb{F}_2$-type generator called the Mersenne Twister (MT) that satisfies the above criteria very well, compared with any previously existing generator. It is a variant of the TGFSR algorithm introduced in Matsumoto and Kurita [1992], improved by Matsumoto and Kurita [1994], and modified here so as to admit a Mersenne-prime period. A set of good parameters is implemented as portable C-code called MT19937 (see Appendix C). This code can be used in any machine with a standard C compiler (including 64-bit machines), as an integer or real number generator. Essentially the same codes are downloadable from the Salzburg University http-site: http://random.mat.sbg.ac.at/news/ and from the MT home page: http://www.math.keio.ac.jp/matumoto/emt.html.

This generator has a tremendously large prime period $2^{19937} - 1$, while consuming a working area of only 624 words. The sequence is 623-distributed to 32 bits accuracy. This has a huge k-distribution property to v-bit accuracy for each v, $v = 1, 2, \ldots, 32$ (see Table II in Sect. 2.2). These values are at least ten times larger than any other implemented generators, and are near the trivial upper bound. Although we do not list them, the k-distributions of the least significant bits are also satisfactorily large. For example, the least significant six bits of MT19937 are 2492-dimensionally equidistributed. The characteristic polynomial has many terms (roughly 100 or more), and has no obvious relation with a trinomial.

MT19937, as a 32-bit random integer generator, passed the diehard tests developed by Marsaglia [1985]. S. Wegenkittl from the PLAB group [Hellekalek et al. 1997] at the University of Salzburg tested MT19937 empirically using Load Tests and Ultimate Load Tests, and reported that MT19937 passed them.

We compare the speed of MT19937 with other modern generators (Table I) in a Sun workstation. MT is comparable to other generators,[2] which have much shorter periods.

We conclude that MT is one of the most promising pseudorandom number generators at the present time. However, it is desirable to apply other statistical tests, too. Stringent tests to scrutinize MT are welcome.

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[2]Wegenkittl reported that the speed of MT19937 in the Dec-Alpha machine is even faster than rand. This high-speed is probably due to the modern hardware architecture like cache memory and pipeline processing, to which MT19937 fits well.

1.5 Comparison with Other Generators

Table I compares the speed of MT with other generators. The combined generators COMBO and KISS are downloaded from Marsaglia's http-site (http://stat.fsu.edu/~geo/diehard.html). The generator ran_array is Lüscher's method for discarding a lagged-Fibonacci generator, recommended in Knuth [1997]. The rand generator is a C-library standard random number generator. The generator taus88 is a combined Tausworthe generator[3] [L'Ecuyer 1996; c.f., Tezuka and L'Ecuyer 1991]. TT800, a small cousin of MT19937, is a TGFSR generator[4] [Matsumoto and Kurita 1994].

We measured the time consumed in generating 10^7 random numbers on a Sun workstation. Since ran_array discards 90% of the generated sequence, it is much slower than the others.

MT19937 and ran_array consume more memory,[5] but it would not be a major problem in simulations where not that many random number generators run in parallel. MT19937 has the longest period.

1.6 Limitations and Hints for Use

This generator, as it is, does not create cryptographically secure random numbers. For cryptographic purposes, one needs to convert the output with a secure hashing algorithm (see, for example, Rueppel [1986]). Otherwise, by a simple linear transformation (T^{-1} where T is the tempering matrix given by (2.2)–(2.5) in Sect. 2.1), the output of MT19937 becomes a linear recurring sequence given by (2.1) in Sect. 2.1. One can then easily guess the present state from output of a sufficiently large size. (See the conditions in Proposition 4.2, and note that the recurrence satisfies these conditions.)

This generator is developed for generating [0,1]-uniform real random numbers, with special attention paid to the most significant bits. The rejected generators in Ferrenberg et al. [1992] are exactly the generators whose most significant bits have a defect (see Tezuka et al. [1993] for SWB, and the weight distribution test in Matsumoto and Kurita [1992] for the trinomial GFSR). Thus, we think our generator would be the most suitable for a Monte Carlo simulation such as that of Ferrenberg et al. [1992]. If one needs (0,1)-random numbers, simply discard the zeros; if one needs 64-bit integers, simply concatenate two words.

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[3]This is a very fast and economical generator, which has optimized k-distribution.

[4]TT800 in Matsumoto and Kurita [1994] is designed as a real number generator, and has a defect at the least significant bits. The downloadable version of TT800 at Salzburg University ⟨http://random.mat.sbg.ac.at/news/⟩ has been improved in this regard. This generator and taus88 were the two flawless generators in the Load Tests in Hellekalek [1997], in which most short-period linear congruential generators and some of the inversive generators are rejected. The TGFSR were also tested by Matsumoto and Kurita [1992; 1994]. We got many favorable email messages from users of TT800. As far as we know, no test has rejected this generator, due to its good k-distribution property.

[5]Perhaps the figure for the working area of ran_array is a bit misleading. A figure of 1000 words was attained by choosing "the safest method" in Knuth [1997], namely, discarding 90% and using Knuth's code. It is easy to reduce the figure to 100 by rewriting the code, but then it becomes slower.

2. MT ALGORITHM

2.1 Description of MT

Throughout this paper, bold letters, such as $\mathbf{x}$ and $\mathbf{a}$, denote *word vectors*, which are w-dimensional row vectors over the two-element field $\mathbb{F}_2 = \{0, 1\}$, identified with machine words of size w (with the least significant bit at the right).

The MT algorithm generates a sequence of word vectors, which are considered to be uniform pseudorandom integers between 0 and $2^w - 1$. Dividing by $2^w - 1$, we regard each word vector as a real number in $[0,1]$.

The algorithm is based on the following linear recurrence

$$\mathbf{x}_{k+n} := \mathbf{x}_{k+m} \oplus (\mathbf{x}_k^u | \mathbf{x}_{k+1}^l)A, \quad (k = 0, 1, \ldots). \tag{2.1}$$

We explain the notation: We have several constants, an integer n, which is the degree of the recurrence, an integer r (hidden in the definition of $\mathbf{x}_k^u$), $0 \leq r \leq w - 1$, an integer m, $1 \leq m \leq n$, and a constant $w \times w$ matrix A with entries in $\mathbb{F}_2$. We give $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}$ as initial seeds. Then, the generator generates $\mathbf{x}_n$ by the above recurrence with $k = 0$. By putting $k = 1, 2, \ldots$, the generator determines $\mathbf{x}_{n+1}, \mathbf{x}_{n+2}, \ldots$. On the right-hand side of the recurrence, $\mathbf{x}_k^u$ means "the upper $w - r$ bits" of $\mathbf{x}_k$, and $\mathbf{x}_{k+1}^l$ "the lower r bits" of $\mathbf{x}_{k+1}$. Thus, if $\mathbf{x} = (x_{w-1}, x_{w-2}, \ldots, x_0)$, then, by definition, $\mathbf{x}^u$ is the $w - r$ bits vector $(x_{w-1}, \ldots, x_r)$, and $\mathbf{x}^l$ is the r bits vector $(x_{r-1}, \ldots, x_0)$. Thus $(\mathbf{x}_k^u | \mathbf{x}_{k+1}^l)$ is just the concatenation; namely, a word vector obtained by concatenating the upper $w - r$ bits of $\mathbf{x}_k$ and the lower r bits of $\mathbf{x}_{k+1}$ in that order. Then the matrix A is multiplied from the right by this vector. Finally, add $\mathbf{x}_{k+m}$ to this vector ($\oplus$ is bitwise addition modulo two), and then generate the next vector $\mathbf{x}_{k+n}$.

The reason why we chose the complicated recurrence (1) will become clear in Sect. 3.1. Here we note that if $r = 0$, then this recurrence reduces to the previous TGFSR proposed in Matsumoto and Kurita [1992; 1994], and if $r = 0$ and $A = I$, it reduces to GFSR [Lewis and Payne 1973].

We choose a form of the matrix A so that multiplication by A is very fast. Here is a candidate:

$$A = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ a_{w-1} & a_{w-2} & \cdots & \cdots & a_0 \end{pmatrix},$$

then the calculation of $\mathbf{x}A$ can be done using only bit operations:

$$\mathbf{x}A = \begin{cases} \text{shiftright}(\mathbf{x}) & \text{if } x_0 = 0 \\ \text{shiftright}(\mathbf{x}) \oplus \mathbf{a} & \text{if } x_0 = 1 \end{cases},$$

where $\mathbf{a} = (a_{w-1}, a_{w-2}, \ldots, a_0)$, $\mathbf{x} = (x_{w-1}, x_{w-2}, \ldots, x_0)$. Also, $\mathbf{x}_k^u$ and $\mathbf{x}_{k+1}^l$ of the recurrence (2.1) can be calculated with a bitwise AND opera-

tion. Thus the calculation of the recurrence (2.1) is realized with bitshift, bitwise EXCLUSIVE-OR, bitwise OR, and bitwise AND operations.

To improve k-distribution to v-bit accuracy, we multiply each generated word by a suitable $w \times w$ invertible matrix T from the right (called *tempering* in Matsumoto and Kurita [1994]). For the tempering matrix $\mathbf{x} \mapsto \mathbf{z} = \mathbf{x}T$, we chose the following successive transformations:

$$\mathbf{y} := \mathbf{x} \oplus (\mathbf{x} \gg u) \tag{2.2}$$

$$\mathbf{y} := \mathbf{y} \oplus ((\mathbf{y} \ll s) \text{ AND } \mathbf{b}) \tag{2.3}$$

$$\mathbf{y} := \mathbf{y} \oplus ((\mathbf{y} \ll t) \text{ AND } \mathbf{c}) \tag{2.4}$$

$$\mathbf{z} := \mathbf{y} \oplus (\mathbf{y} \gg l), \tag{2.5}$$

where l, s, t, and u are integers, $\mathbf{b}$ and $\mathbf{c}$ are suitable bitmasks of word size, and $(\mathbf{x} \gg u)$ denotes the u-bit shiftright and $(\mathbf{x} \ll u)$ the u-bit shiftleft. The transformations (2.3) and (2.4) are the same as those used in Matsumoto and Kurita [1994]. The transformations (2.2) and (2.5) are added so that MT can improve[6] the least significant bits.

For executing the recurrence (2.1), it is enough to take an array of n words as a working area, as follows. Let $\mathbf{x}[0 : n - 1]$ be an array of n unsigned integers of word size, i be an integer variable, and $\mathbf{u}$, $\mathbf{ll}$, $\mathbf{a}$ be unsigned constant integers of word size.

Step 0. $\mathbf{u} \leftarrow \underbrace{1 \cdots 1}_{w - r} \underbrace{0 \cdots 0}_{r}$;(bitmask for upper $w - r$ bits)

$\mathbf{ll} \leftarrow \underbrace{0 \cdots 0}_{w - r} \underbrace{1 \cdots 1}_{r}$;(bitmask for lower r bits)

$\mathbf{a} \leftarrow a_{w-1}a_{w-2} \cdots a_1 a_0$;(the last row of the matrix A)

Step 1. $i \leftarrow 0$

$\mathbf{x}[0], \mathbf{x}[1], \ldots, \mathbf{x}[n - 1] \leftarrow$ "any nonzero initial values"

Step 2. $\mathbf{y} \leftarrow (\mathbf{x}[i] \text{ AND } \mathbf{u}) \text{ OR } (\mathbf{x}[(i + 1) \bmod n] \text{ AND } \mathbf{ll})$
  ;(computing $(\mathbf{x}_i^u|\mathbf{x}_{i+1}^l)$)

Step 3. $\mathbf{x}[i] \leftarrow \mathbf{x}[(i + m) \bmod n] \text{ XOR } (y \gg 1)$

$\text{XOR} \begin{cases} 0 & \text{if the least significant bit of } \mathbf{y} = 0 \\ \mathbf{a} & \text{if the least significant bit of } \mathbf{y} = 1 \end{cases}$;(multiplying A)

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[6]These do not exist in the TT800 code in Matsumoto and Kurita [1994]. The code at the Salzburg http was improved in this regard by adding (2.5).

Step 4. ;(calculate $\mathbf{x}[i]T$)

      $\mathbf{y} \leftarrow \mathbf{x}[i]$

      $\mathbf{y} \leftarrow \mathbf{y} \text{ XOR} (\mathbf{y} \gg u)$;(shiftright $\mathbf{y}$ by u bits and add to $\mathbf{y}$)

      $\mathbf{y} \leftarrow \mathbf{y} \text{ XOR} ((\mathbf{y} \ll s) \text{ AND } \mathbf{b})$

      $\mathbf{y} \leftarrow \mathbf{y} \text{ XOR} ((\mathbf{y} \ll t) \text{ AND } \mathbf{c})$

      $\mathbf{y} \leftarrow \mathbf{y} \text{ XOR} (\mathbf{y} \gg l)$

      output $\mathbf{y}$

Step 5. $i \leftarrow (i + 1) \bmod n$

Step 6. Goto Step 2.

By rewriting the whole array at one time, we can dispense with modulo-n operations. Thus, we need only very fast operations (see the code in Appendix C).

We have the following two classes of parameters: (1) *period parameters*, determining the period: integer parameters w (word size), n (degree of recursion), m (middle term), r (separation point of one word), and a vector parameter $\mathbf{a}$ (matrix A); (2) *tempering parameters* for k-distribution to v-bit accuracy: integer parameters l, u, s, t and the vector parameters $\mathbf{b}$, $\mathbf{c}$.

2.2 Good Parameters with Large k-Distributions

Table II lists some period parameters that yield the maximal period $2^{nw-r} - 1$, and tempering parameters with a good k-distribution property. The trivial upper bound $k(v) \leq \lfloor (nw - r)/v \rfloor$ is also shown. The table shows that we could not attain these bounds even after tempering. One sees that $k(v)$ tends to be near a multiple of n. We prove this only for $k(v)$ less than $2(n - 1)$ (see Proposition B.2). This proposition explains why $k(v)$ cannot be near the bound $\lfloor (nw - r)/v \rfloor$ if $\lfloor (nw - r)/v \rfloor < 2(n - 1)$. We conjecture a more general obstruction, as in the case of Matsumoto and Kurita [1994].

One may argue that the gap between the bounds and the attained values is a problem, see Tezuka [1994a]. In our opinion, "to attain a larger $k(v)$" is usually more important than "to attain the upper bound in a limited working area" (although this depends on the memory restriction). The number of terms in a characteristic polynomial is also shown under the ID.

Table II. Parameters and k Distribution of Mersenne Twister

| Generator		The order of equidistribution					
ID	Parameters	$k(1)$	$k(2)$	$k(3)$	$k(4)$	$k(5)$	$k(6)$
		$k(7)$	$k(8)$	$k(9)$	$k(10)$	$k(11)$	$k(12)$
(the number of		$k(13)$	$k(14)$	$k(15)$	$k(16)$	$k(17)$	$k(18)$
terms in the		$k(19)$	$k(20)$	$k(21)$	$k(22)$	$k(23)$	$k(24)$
characteristic		$k(25)$	$k(26)$	$k(27)$	$k(28)$	$k(29)$	$k(30)$
polynomial)		$k(31)$	$k(32)$				
	Upper bounds	11213	5606	3737	2803	2242	1868
	$\lfloor \frac{nw-r}{v} \rfloor$ for	1601	1401	1245	1121	1019	934
	$(w,n,r)=(32,351,19)$	862	800	747	700	659	622
	$1 \le v \le 32$	590	560	533	509	487	467
		448	431	415	400	386	373
		361	350				
MT11213A	$(w,n,m,r)=(32,351,175,19)$	11213	5606	3560	2803	2111	1756
	a = E4BD75F5	1405	1401	1055	1053	709	704
	u = 11	703	702	701	700	356	352
	s = 7,b = 655E5280	351	351	351	350	350	350
(177)	t = 15,c = FFD58000	350	350	350	350	350	350
	l = 17	350	350				
MT11213B	$(w,n,m,r)=(32,351,175,19)$	11213	5606	3565	2803	2113	1759
	a = CCAB8EE7	1408	1401	1056	1053	715	704
	u = 11	702	702	701	700	355	352
	s = 7,b = 31B6AB00	351	351	351	351	350	350
(151)	t = 15,c = FFE50000	350	350	350	350	350	350
	l = 17	350	350				
	Upper bounds	19937	9968	6645	4984	3987	3322
	$\lfloor \frac{nw-r}{v} \rfloor$ for	2848	2492	2215	1993	1812	1661
	$(w,n,r)=(32,624,31)$	1533	1424	1329	1246	1172	1107
	$1 \le v \le 32$	1049	996	949	906	866	830
		797	766	738	712	687	664
		643	623				
MT19937	$(w,n,m,r)=(32,624,397,31)$	19937	9968	6240	4984	3738	3115
	a = 9908B0DF	2493	2492	1869	1869	1248	1246
	u = 11	1246	1246	1246	1246	623	623
	s = 7, b = 9D2C5680	623	623	623	623	623	623
(135)	t = 15, c = EFC60000	623	623	623	623	623	623
	l = 18	623	623				
TT800	$(w,n,m,r)=(32,25,7,0)$	800	400	250	200	150	125
	a = 8EBFD028	100	100	75	75	50	50
	u : not exist	50	50	50	50	25	25
	s = 7, b = 2B5B2500	25	25	25	25	25	25
(93)	t = 15, c = DB8B0000	25	25	25	25	25	25
	l =16	25	25				
ran_array	Knuth's new recommendation.	129	64	43	32	25	21
		18	16	14	12	11	10
	Here we list the trivial	9	9	8	8	7	7
	upper bounds.	6	6	6	5	5	5
		5	4	4	4	4	4

3. WHY MT?

3.1 How We Reached MT

As is the case for any $\mathbb{F}_2$-linear generating method, the MT algorithm is just an iteration of a fixed linear transformation on a fixed vector space. In the

case of MT, the vector space is the set of the following type of arrays of (0,1)-entries, with r bits missing at the upper-right corner.
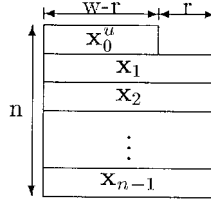
We call this object an $(n \times w - r)$-*array* or an *incomplete array*.

The state transition is given by the following linear transformation B
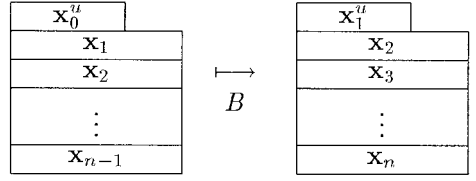
where $\mathbf{x}_n$ is obtained by the defining recurrence (2.1) for $k = 0$. By a general theory of linear recurrence (see Appendix A), each entry of the $(n \times w - r)$-array is a linear recurring sequence satisfying the recurrence corresponding to the characteristic polynomial $\varphi_B(t)$ of the transformation B. The sequence attains the maximal period $2^p - 1 = 2^{nw-r} - 1$, if and only if $\varphi_B(t)$ is primitive, i.e., t generates the multiplicative group $(\mathbb{F}_2[t]/\varphi_B(t))^{\times}$.

Attaining this bound has the great advantage that the state vector assumes every bit-pattern in the $(n \times w - r)$-array once in a period, except for the zero state. Consequently, the sequence $\{\mathbf{x}_n\}$ is $(n - 1)$-distributed. Since any initial seed except zero lies on the same orbit, the choice of an initial seed does not affect the randomness for the whole period. This is very different than the original GFSR, in which the initialization is critical [Fushimi and Tezuka 1983].

Since $(n - 1)$ is the order of equidistribution, we would like to make n as large as the memory restriction permits. We think that in recent computers n up to 1000 is reasonable. On the other hand, one may claim that n up to 10 is enough. However, one SWB [Marsaglia and Zaman 1991] for example, is 43-distributed since the orbit is one; but it failed in the Ising-Model test [Ferrenberg et al. 1992]. The system simulated has a "good memory," remembering a large number of previously generated words. There are applications where the n-dimensional distributions for very large n become important.

In TGFSR, an essential bound on n comes from the difficulty of factorization. We have to certify that the order of t modulo $\varphi_B(t)$ is $2^p - 1$, but then

we need all the proper factors of $2^p - 1$. Even a modern technique can factorize $2^p - 1$ for around $p < 2000$ only (for example, Brillhart et al. [1988]). For TGFSR, $p = nw$ and $2^{nw} - 1$ can never be prime, unless n or w is 1. Thus, we need to factorize it.

On the other hand, the test for the primality of an integer is much easier. So there are many Mersenne primes found (i.e., primes of the form $2^p - 1$) that are up to $p = 1398269$ (see http://www.utm.edu:80/research/primes/mersenne.shtml#known).

If we eliminate r bits from the $(n \times w)$-array, as in MT, then the dimension of the state space is $nw - r$. One can attain any number in this form, including Mersenne exponents. Then we do not need factorization. This is the reason we use an $(n \times w - r)$-array.

In determining the next state, each bit of $\mathbf{x}_0^u$ and $\mathbf{x}_1^l$ must be fully reflected, since otherwise the state space is smaller. Thus, the recurrence (2.1).

Knuth [1996] also informed us of the following justification for this recurrence. One might have used $(\mathbf{x}_{k+1}^l | \mathbf{x}_k^u)$ instead of $(\mathbf{x}_k^u | \mathbf{x}_{k+1}^l)$ in the recurrence (2.1). The former seems more natural, since, for example, the matrix S in Appendix A coincides with A. But he noticed that when $r = w - 1$, then the sequence can never have maximal period. Actually, it is easy to check that the most significant bit of each generated word satisfies a trinomial linear recurrence with order n, and this does not satisfy the maximality.

3.2 Primitivity is Easy for MT

Another justification for the recurrence (2.1) is that primitivity can easily be checked by *inversive-decimation methods* (Sect. 4.3). Since we chose a Mersenne exponent $p = nw - r$ as the size of the incomplete array, there is an algorithm to check primitivity with $O(lp^2)$ computations, where l is the number of nonzero terms in the characteristic polynomial. The easiest case is $l = 3$, and accordingly there is a list up to $p = 132049$ for trinomials [Heringa et al. 1992]. One can implement a recurrence with such a characteristic trinomial in an incomplete array.

However, the trinomials and its "slight" modifications always show erroneous weight distributions, as stated in Sect. 1.3. Thus, what we desire is a linear recurrence such that its characteristic polynomial has many terms and is easily checked for primitivity.

The recurrence of MT satisfies this. Its characteristic polynomial has experimentally ~ 100 terms (see Table II), and in spite of the many terms and because of the peculiar form of the recurrence (2.1), primitivity can be checked with $O(p^2)$ computations (see Sect. 4.3).

Note that for large-modulus generators, the primitivity check is a hard number-theoretic task (e.g., Marsaglia and Zaman [1991]). This is an advantage of $\mathbb{F}_2$-generators over integer-operation generators.

3.3 *k*-Distribution is Easy for MT

It was discovered by Tezuka [1994b] that the *k*-distribution to 2-bit accuracy for TGFSR in Matsumoto and Kurita [1992] is very low. A follow up to this failure was satisfactorily completed by Matsumoto and Kurita [1994].

For the same reason, the *k*-distribution property of the raw sequence generated by the recurrence (2.1) is poor, so we need to modify the output by multiplying by a matrix *T* (i.e., *tempering*, see Sect. 5). We then succeed in realizing good *k*-distributions.

We must comment here that spectral tests with dimension more than 100 are almost impossible for computational reasons for any existing generators based on large-modulus calculus. On the other hand, for MT, we can execute *k*-distribution tests to v-bit accuracy for *k* more than 600. This is another advantage of $\mathbb{F}_2$-generators over large-modulus generators.

3.4 MT is One of Multiple-Recursive Matrix Methods (MRMM)

Soon after TGFSR was proposed, Niederreiter developed a general class of random number generators, including TGFSR, *multiple-recursive matrix methods* (MRMM) [Niederreiter 1993; 1995]. MRMM is to generate a random vector sequence over $\mathbb{F}_2$ by the linear recurrence

$$\mathbf{x}_{k+n} := \mathbf{x}_{k+n-1}A_{n-1} + \cdots + \mathbf{x}_{k+1}A_1 + \mathbf{x}_kA_0 \quad (k = 0, 1, \ldots),$$

where $\mathbf{x}_k$ are row vectors and A_i are $w \times w$ matrices. MT belongs to this class because the defining recurrence (2.1) can be written as

$$\mathbf{x}_{k+n} = \mathbf{x}_{k+m} + \mathbf{x}_{k+1}\begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix}A + \mathbf{x}_k\begin{pmatrix} I_{w-r} & 0 \\ 0 & 0 \end{pmatrix}A,$$

where I_r, I_{w-r} is the identity matrix of size r, $w - r$, respectively.

Even after tempering, the generated sequence still belongs to this class. It is easy to see from the definition that a sequence of word vectors belongs to this class if and only if $\mathbf{x}_{k+n}$ is determined by a linear transformation from the preceding *n* vectors $\mathbf{x}_{k+n-1}, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_k$. (This is nothing but a linear recurring sequence of vector values, as stated in Niederreiter [1993]. The important point in Niederreiter [1995] is the analysis of properties such as discrepancy.) Since the tempering matrix is a linear isomorphism, it preserves this property. Thus, MT can be said to be a neat implementation of the general concept MRMM.

Unfortunately, the detailed investigation in Niederreiter [1995] is not applicable as it is, since he mainly considered the case with the maximal period $2^{nw} - 1$ only. Modification of Niederreiter's work to cover MT would be possible and valuable. We guess that MT's performance would not be much different than those in Niederreiter [1995].

4. HOW TO FIND PERIOD PARAMETERS

4.1 Difficulty in Obtaining the Period

Since we have chosen n and r so that $nw - r = p$ is a large Mersenne exponent, primitivity can be checked by (for example, Heringa [1992]):

$$\begin{cases} t^2 \not\equiv t \bmod \varphi_B(t) \\ t^{2^p} \equiv t \bmod \varphi_B(t) \end{cases}.$$

It is possible to calculate this directly, as was done previously (see Appendix A.1 for the explicit form of $\varphi_B(t)$). However, this is an $O(lp^2)$-calculation, where l is the number of terms. To take the square modulo $\varphi_B(t)$, we need to divide a polynomial of degree $2p$ by $\varphi_B(t)$. We need $O(p)$-times subtraction by $\varphi_B(t)$ and each subtraction requires $O(l)$-operations. We iterate this p times, which amounts to $O(lp^2)$. In our case, p is very large (>10000), and according to our experiment, the direct computation may need several years to catch a primitive polynomial.

We contrived an algorithm called the *inversive-decimation method* with $O(p^2)$ operations for the primitivity test for MT, which took only two weeks to find one primitive polynomial for MT with degree 19937. This algorithm may be used for other generators as well if the generator satisfies the condition of Proposition 4.2, below.

4.2 A Criterion for Primitivity

Let $\mathcal{S}^\infty$ denote the $\mathbb{F}_2$-vector space of all infinite sequences of 0,1. That is,

$$\mathcal{S}^\infty := \{\chi = (\dots, x_5, x_4, x_3, x_2, x_1, x_0) | x_i \in \mathbb{F}_2\}.$$

Let D (*delay* operator) and H (*decimation* operator) be linear operators from $\mathcal{S}^\infty$ to $\mathcal{S}^\infty$ defined by

$$D(\dots, x_4, x_3, x_2, x_1, x_0) = (\dots, x_5, x_4, x_3, x_2, x_1),$$

$$H(\dots, x_4, x_3, x_2, x_1, x_0) = (\dots, x_{10}, x_8, x_6, x_4, x_2, x_0).$$

Let $\varphi(t)$ be the characteristic polynomial of a linear recurrence. Then, χ satisfies the recurrence if and only if $\varphi(D)\chi = 0$.

It is easy to check that

$$DH = HD^2.$$

Since the coefficients are in $\mathbb{F}_2$, we have $\varphi(t^2) = \varphi(t)^2$, thus if $\varphi(D)\chi = 0$, then

$$\varphi(D)H\chi = H\varphi(D^2)\chi = H\varphi(D)^2\chi = 0,$$

i.e., $H\chi$ also satisfies the same recurrence.

It is easy to see that if the period of $\chi \in \mathscr{S}^{\infty}$ is $2^p - 1$, then $H^p\chi = \chi$; but the converse may not be true. However, the following theorem holds.

THEOREM 4.1. *Let $\varphi(t)$ be a polynomial over $\mathbb{F}_2$ whose degree p is a Mersenne exponent. Take $\chi \in \mathscr{S}^{\infty}$ such that $\varphi(D)\chi = 0$ and $H\chi \neq \chi$. Then $\varphi(t)$ is primitive if and only if $H^p\chi = \chi$.*

PROOF.   Let V be the p-dimensional linear space

$$V := \{\chi \in \mathscr{S}^{\infty} | \varphi(D)\chi = 0\},$$

and τ be a linear mapping from $\mathscr{S}^{\infty}$ to $\mathbb{F}_2$ defined by

$$\tau(\ldots, x_2, x_1, x_0) = x_0.$$

We consider a bilinear pairing $(a|b)$ defined by

$$\begin{aligned} \mathbb{F}_2[t]/\varphi(t) \times V &\rightarrow &\mathbb{F}_2 \\ (g(t), \chi) &\mapsto &(g(t)|\chi) := \tau(g(D)\chi). \end{aligned}$$

This is well-defined, and nondegenerate because if $\tau(g(D)\chi) = 0$ for all $g(D)$, then $\chi = 0$ follows from $\tau(D^n\chi) = 0$ for all n.

Let F be a mapping from $\mathbb{F}_2[t]/\varphi(t)$ to $\mathbb{F}_2[t]/\varphi(t)$ given by $F(g(t)) = g(t)^2$. Then it is easy to check that F is the adjoint of H, i.e.,

$$(F(g(t))|\chi) = (g(t)|H\chi)$$

holds (it is enough to consider the case of $g(t) = t^i$).

A condition of the theorem is

$$\mathrm{Ker}(H^p - 1) \not\supseteq \mathrm{Ker}(H - 1),$$

which is now equivalent to

$$\mathrm{Ker}(F^p - 1) \not\supseteq \mathrm{Ker}(F - 1).$$

This means the existence of $g(t) \in \mathbb{F}_2[t]/\varphi(t)$ such that $g(t)^{2^p} = g(t)$ and $g(t)^2 \neq g(t)$.

Let l (≥ 1) be the smallest integer such that $g(t)^{l+1} = g(t)$. Then, since $g(t)^{2^p} = g(t)$, it follows that $l | 2^p - 1$, and $l \neq 1$ by assumption. Since p is a Mersenne exponent, l must be at least $2^p - 1$. Since 0 is an orbit, all nonzero elements lie on one orbit, and it is purely periodic. Since this orbit contains 1, $g(t)$ is invertible and it must generate $(\mathbb{F}_2[t]/\varphi(t))^{\times}$. Moreover, the order of $(\mathbb{F}_2[t]/\varphi(t))^{\times}$ must be $2^p - 1$. Then $\mathbb{F}_2[t]/\varphi(t)$ is a field, so $\varphi(t)$ is primitive. $\square$

4.3 Inversive-Decimation for Primitivity Testing

PROPOSITION 4.2 (*Inversive-decimation-method*).   *Let V, the state space of the generator, be a p-dimensional vector space over* $\mathbb{F}_2$*, where p is a Mersenne-exponent. Let* $f : V \rightarrow V$ *be a linear state transition map. Let* $b : V \rightarrow \mathbb{F}_2$ *be a linear map (e.g., looking up one bit from the state). Assume that f and b are computable in O(1)-time. Assume that* $\Phi : V \rightarrow \mathbb{F}_2^p$*, given by*

$$\Phi : S \mapsto (bf^{p-1}(S),\ bf^{p-2}(S),\ \ldots,\ bf(S),\ b(S)),$$

is bijective, and that the inverse map is computable with time complexity $O(p)$.

Then, primitivity of the characteristic polynomial of f can be tested with time complexity $O(p^2)$.

Note that the last condition is essential. The other conditions are automatically satisfied for most efficient $\mathbb{F}_2$-generators. In order to apply this algorithm, finding a good b satisfying the last condition is crucial.

PROOF.   Let χ be the infinite sequence $(\ldots,\ bf^2(S),\ bf(S),\ b(S))$. Since Φ is invertible with order $O(p)$, we can choose an S with $H(\chi) \neq \chi$.

By Theorem 4.1, it is enough to show that the first p bits of $H^{m+1}(\chi)$ can be obtained from the first p bits of $H^m(\chi)$ with an $O(p)$-calculation. From the first p bits of $H^m(\chi)$, we can obtain a state S_m which yields $H^m(\chi)$ by using the inverse of Φ with $O(p)$ computations. From this state, we can generate the first $2p$ bits of $H^m(\chi)$ with an $O(2p)$ calculation, since f and b are of $O(1)$. Then decimate these $2p$ bits. Now we obtain the first p bits of H^{m+1}, with an $O(p)$ calculation.   □

The above proposition can be applied to MT in the following way. For simplicity, assume $r > 0$. Put $S = (\mathbf{x}_{n-1}, \ldots, \mathbf{x}_1, \mathbf{x}_0^u)$, i.e., an initial ($n \times w - r$)-array.

Let b be the map that takes the upper-right corner of this incomplete array. Thus, $b(S) = x_{1,0}$, the least significant bit of $\mathbf{x}_1$. We have to find an inverse morphism to Φ, which calculates from $(x_{p,0}, x_{p-1,0}, \ldots, x_{1,0})$ the state S that produces this p-bit stream at the least significant bit, with only $O(p)$-calculation.

If $x_{p-n+1,0}$, $x_{p-n+m,0}$ and $x_{p,0}$ are known, we can calculate $x_{p-n+1,1}$ if $r > 1$, or $x_{p-n,1}$ if $r \leq 1$. This is because by step 2 and step 3 of the algorithm in Sect. 2.1, the following equation holds between $x_{p-n+1,0}$, $x_{p-n+m,0}$, $x_{p,0}$ and $x_{p-n+1,1}$.

$$x_{p-n+1,1} = \begin{cases} x_{p-n+m,0} \oplus x_{p,0} & \text{if } x_{p-n+1,0} = 0 \\ x_{p-n+m,0} \oplus x_{p,0} \oplus a_0 & \text{if } x_{p-n+1,0} = 1. \end{cases}$$

It is clear that the same relation holds between $x_{i-n+1,0}$, $x_{i-n+m,0}$, $x_{i,0}$ and $x_{i-n+1,1}$ for $i = n,\ n+1,\ \ldots,\ p$. Thus from $x_{1,0},\ \ldots,\ x_{p,0}$, we can

calculate $x_{1,1}, \ldots, x_{p-n+1,1}$. In general, for $i = n, n + 1, \ldots, p, j = 1, 2, \ldots, w - 1$, the following equations hold:

$$x_{i-n+1,j} = \begin{cases} x_{i-n+m,j-1} \oplus x_{i,j-1} & \text{if } x_{i-n+1,0} = 0 \\ x_{i-n+m,j-1} \oplus x_{i,j-1} \oplus a_{j-1} & \text{if } x_{i-n+1,0} = 1 \end{cases} (j < r)$$
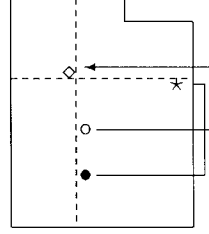
$$x_{i-n,j} = \begin{cases} x_{i-n+m,j-1} \oplus x_{i,j-1} & \text{if } x_{i-n+1,0} = 0 \\ x_{i-n+m,j-1} \oplus x_{i,j-1} \oplus a_{j-1} & \text{if } x_{i-n+1,0} = 1 \end{cases} (j \geq r).$$
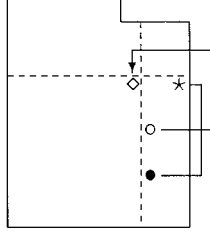
$$j < r$$

$$\begin{cases} \diamond = & x_{i-n+1,j} \\ \bullet = & x_{i,j-1} \\ \circ = & x_{i-n+m,j-1} \\ \star = & x_{i-n+1,0} \end{cases}$$

$$j \geq r$$

$$\begin{cases} \diamond = & x_{i-n,j} \\ \bullet = & x_{i,j-1} \\ \circ = & x_{i-n+m,j-1} \\ \star = & x_{i-n+1,0} \end{cases}$$

If $x_{1,0}, \ldots, x_{k,0}$ and $x_{m,j}, x_{m+1,j}, \ldots, x_{k+n-1,j}$ $(n \leq k + n - 1 \leq p)$ are known, then $x_{1,j+1}, x_{2,j+1}, \ldots, x_{k,j+1}$ (if $j < r - 1$), or $x_{0,j+1}, x_{1,j+1}, \ldots, x_{k-1,j+1}$ (if $j \geq r - 1$) can be calculated. Furthermore, from $(x_{i,j-1}, \ldots, x_{i,0})$, $(x_{i-n+m,j-1}, \ldots, x_{i-n+m,0})$, and $x_{i-n+1,0}$, we can calculate $(x_{i-n,j}, \ldots, x_{i-n,r}, x_{i-n+1,r-1}, \ldots, x_{i-n+1,1}, x_{i-n+1,0})$, i.e., the lower $(j + 1)$ bits of $(\mathbf{x}^u_{i-n} | \mathbf{x}^l_{i-n+1})$ at the same time, by

$$(x_{i-n,j}, \ldots, x_{i-n,r}, x_{i-n+1,r-1}, \ldots, x_{i-n+1,1}, x_{i-n+1,0}) =$$

$$(0, 0, \ldots, 0, x_{i-n+1,0})$$

$$+ (x_{i,j-1}, \ldots, x_{i,1}, x_{i,0}, 0)$$

$$+ (x_{i-n+m,j-1}, \ldots, x_{i-n+m,1}, x_{i-n+m,0}, 0)$$

$$+ \begin{cases} 0 & \text{if } x_{i-n+1,0} = 0 \\ (a_{j-1} \ldots, a_1, a_0, 0) & \text{if } x_{i-n+1,0} = 1 \end{cases}.$$

So, by setting $\mathbf{y}_0 = 0$, $\mathbf{y}_i = (0, \ldots, 0, x_{i,0})$ $(i = 1, \ldots, p)$ and repeating the following recurrence from $i = p$ until $i = n$,

$$\mathbf{y} \leftarrow \mathbf{y}_i + \mathbf{y}_{i-n+m} + \begin{cases} 0 & \text{if the least significant bit of } \mathbf{y}_{i-n+1} = 0 \\ \mathbf{a} & \text{if the least significant bit of } \mathbf{y}_{i-n+1} = 1 \end{cases}$$

$$(\mathbf{y}^u_{i-n} | \mathbf{y}^l_{i-n+1}) \leftarrow \text{shiftleft}(\mathbf{y}) + (0, \cdots, 0, x_{i-n+1,0})$$

we get $S = (\mathbf{y}_{n-1}, \ldots, \mathbf{y}_0)$. Now Proposition 4.2 can be applied.

We now summarize the algorithm. Let $\mathbf{x}[0 : 2p - 1]$, $\mathbf{initial}[0 : n - 1]$ be arrays of unsigned w-bit integers, i, j, k be integer variables, and $\mathbf{u}$, $\mathbf{ll}$, $\mathbf{a}$, unsigned w-bit integers.

Step 1. $\mathbf{u} \leftarrow \underbrace{1 \cdots 1}_{w-r} \underbrace{0 \cdots 0}_{r}$

$\mathbf{ll} \leftarrow \underbrace{0 \cdots 0}_{w-r} \underbrace{1 \cdots 1}_{r}$

$\mathbf{a} \leftarrow a_{w-1}a_{w-2} \cdots a_1a_0$

for $j \leftarrow 0$ **to** $n-1$ **do**
  begin
    $\mathbf{x}[j] \leftarrow$ some initial value such that $\mathbf{x}[2] \neq \mathbf{x}[3]$
    $\mathbf{initial}[j] \leftarrow \mathbf{x}[j]$
  end

Step 2. **for** $i \leftarrow 0$ **to** $p-1$ **do** **begin**
    Generate $2p-n$ times
    $\mathbf{x}[j] \leftarrow \mathbf{x}[2j-1] (j=1, 2, 3, \ldots, p)$
    for $k \leftarrow p$ **to** n **do**
    begin
      $\mathbf{y} \leftarrow \mathbf{x}[k] \oplus \mathbf{x}[k - n + m]$
      $\oplus \begin{cases} 0 & \text{if the least significant bit of } \mathbf{x}[k - n + 1] = 0 \\ \mathbf{a} & \text{if the least significant bit of } \mathbf{x}[k - n + 1] = 1 \end{cases}$
      $\mathbf{y} \leftarrow$ shiftleft($\mathbf{y}$).
      Set the least significant bit of $\mathbf{y}$ to that of $\mathbf{x}[k - n = 1]$
      $\mathbf{x}[k - n + 1] \leftarrow (\mathbf{u} \text{ AND } \mathbf{x}[k - n + 1]) \text{ OR } (\mathbf{ll} \text{ AND } \mathbf{y})$
      $\mathbf{x}[k - n] \leftarrow (\mathbf{u} \text{ AND } \mathbf{y}) \text{ OR } (\mathbf{ll} \text{ AND } \mathbf{x}[k - n])$
      $k \leftarrow k-1$
    end
    $i \leftarrow i + 1$
    end

Step 3. if $(\mathbf{x}[0] \text{ AND } \mathbf{u}) = (\mathbf{initial}[0] \text{ AND } \mathbf{u})$ and $\mathbf{initial}[j] = \mathbf{x}[j]$ $(j + 1, 2, \ldots, n - 1)$ then the period is $2^p - 1$ else the period is not $2^p - 1$.

5. HOW TO FIND TEMPERING PARAMETERS

5.1 Lattice Methods

To compute $k(v)$ we use the *lattice* method developed by Tezuka [1990]; Couture et al. [1993]; Tezuka [1994a] with the algorithm by Lenstra [1985] to find the successive minima in the formal power series over $\mathbb{F}_2$. In Matsumoto and Kurita [1994] we computed the k-distribution by obtaining the rank of a matrix, but this time we could not do so because of the computational complexity $O(p^3)$.

Here, we briefly recall the method to obtain $k(v)$ by using the lattice. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_i, \ldots$ be a sequence in which each bit satisfies one common linear recurrence with a primitive characteristic polynomial $\varphi(t)$. Thus, if we put $\mathbf{x}_i = (x_{i,w-1}, \ldots, x_{i,0})$, then the infinite sequences $\chi_{w-1} = (x_{0,w-1}, x_{1,w-1}, \ldots, x_{i,w-1}, \ldots), \ldots, \chi_0 = (x_{0,0}, x_{1,0}, \ldots, x_{i,0}, \ldots)$ are

subject to the recurrence given by $\varphi(t)$ (MT satisfies this, see Appendix A.2).

Now, the l-th k-tuple up to v-bit accuracy is $(\text{first}_k(D^l \chi_{w-1}), \ldots, \text{first}_k(D^l \chi_{w-v}))$, where first_k denotes the first k bits of the sequence. Hence, k-distribution to v-bit accuracy is equivalent to the surjectivity of

$$l \mapsto (\text{first}_k(D^l(\chi_{w-1})), \ldots, \text{first}_k(D^l \chi_{w-v}))),$$

as a map from the integers to the nonzero vectors in the $(v \times k)$-dimensional space over $\mathbb{F}_2$. (Then the multiplicity in one small cube would be 2's power to the difference between the dimension of the state space and $v \times k$.) To obtain the maximal $k =: k(v)$ so that the above map is surjective, we use the lattice structure. Let K be the field of Laurent power series:

$$K = \left\{ \sum_{j=-n}^{\infty} \alpha_j t^{-j} \middle| \alpha_j \in \mathbb{F}_2, n \in \mathbb{Z} \right\}.$$

We identify each χ_i with a Laurent power series by

$$\chi_i := \sum_{j=0}^{\infty} x_{j,i} t^{-j}.$$

Let A be the polynomial ring $\mathbb{F}_2[t] \subset K$, and consider the sub A-module L of K^v spanned by the $(v + 1)$ vectors

$$(\chi_{w-1}, \chi_{w-2}, \ldots, \chi_{w-v}), (1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \cdots, (0, \ldots, 0, 1).$$

This can be proved to be a v-dimensional free A-submodule, i.e., a lattice. We define the successive minima of L as follows. Define a nonArchimedean valuation to $x \in K$ by

$$|x| = \begin{cases} 0 & \text{if } x = 0 \\ 2^k & \text{if } x \neq 0 \text{ and } k \text{ is the largest exponent of nonzero terms.} \end{cases}$$

For each v-dimensional vector $X = (x_1, \ldots, x_v) \in K^v$, define its norm by

$$\|X\| = \max_{1 \le i \le v} |x_i|.$$

Definition 5.1   Let $X_1, \ldots, X_v$ be points in a lattice $L \in K^v$ of rank v. We call $X_1, \ldots, X_v$ a reduced basis of L over $\mathbb{F}_2$ if the following properties hold:

(1) X_1 is a shortest nonzero vector in L.
(2) X_i is a shortest vector among the vectors in L but outside the K-span $\langle X_1, \ldots, X_{i-1} \rangle_K$, for each $1 \le i \le v$.

The numbers $\sigma_i = \|X_i\|$ are uniquely determined by the lattice, and $s_i = \log_2 \sigma_i$ $i = 1, \ldots, v$ are called its successive minima.

THEOREM (Couture et al. 1993; Tezuka 1994a).   *The sequence* $\mathbf{x}_1, \mathbf{x}_2, \ldots,$ $\mathbf{x}_i, \ldots$ *is* $(-s_v)$-*distributed to* v-*bit accuracy, where* s_v *is the* v-*th successive minimum of the lattice* L *in* K^v *associated with the sequence.*

Thus the calculation of $k(v)$ is reduced to obtaining the successive minima. There is an efficient algorithm for this [Lenstra 1985]. Since the dimension of the state space is large for MT, we need several programming techniques for an efficient implementation.

We make only one comment: we adopted "lazy evaluation." We keep only one $(n \times w - r)$-array for describing one vector in K^v, and calculate the coefficients of t^{-k} by generating k words. Here again, we depend on the ease of generating the MT sequence.

The time complexity of Lenstra's algorithm is $O(v^4 p^2)$ (see Lenstra [1985]; Tezuka [1994a]), which might be larger than time complexity $O(p^3)$ in obtaining the rank of a $p \times p$ matrix for large v. However, according to our experiments, Lenstra's algorithm is much faster. This could be because (1) we need $p^2 \sim 4 \times 10^8$ bits of memory for the rank of matrix, which may invoke swapping between memory and disk and (2) $O(v^4 p^2)$ is the worst case, and on average the order seems to be less.

5.2 Tempering

To attain $k(v)$ near the trivial bound, we multiply the output vector by a *tempering* matrix T. We could not make the realized values meet the trivial bound. We show a tighter bound (Appendix B), but we could not attain that bound either. In addition, we have no efficient algorithm corresponding to the one in Matsumoto and Kurita [1994]. So, using the same backtracking technique, accelerated by Tezuka's resolution-wise lattice, we searched for parameters with $k(v)$ as near to $\lfloor (nw - r)/v \rfloor$ as possible.

Let $\{\mathbf{x}_i\}$ be an MT sequence, and then define a sequence $\{\mathbf{z}_i\}$ by

$$\mathbf{z}_i := \mathbf{x}_i T,$$

where T is a regular $\mathbb{F}_2$-matrix representing the composition of the transformations (2.2), (2.3), (2.4), and (2.5) described in Sect. 2.1. Since T is regular, the period of $\{\mathbf{z}_i\}$ is the same as that of $\{\mathbf{x}_i\}$. About the peculiar form of tempering and how to search the tempering parameters, please refer to Matsumoto and Kurita [1994]. The parameter in (2.5) is chosen so that the least significant bits have a satisfactory k-distribution.

6. CONCLUSION

We propose a pseudorandom number generator called Mersenne Twister. A portable C-code MT19937 attains a far longer period and far larger k-distributions than any previously existing generator (see Table II). The form of recurrence is cleverly selected so that both the generation and the

parameter search are efficient (Sect. 3). The initialization is care-free. This generator is as fast as other common generators, such as the standard ANSI-C rand, and it passed several statistical tests including diehard. Thus we can say that this is one of the most promising generators at the present time.

As a final remark, we stress that we used efficient algorithms unique to $\mathbb{F}_2[t]$. These algorithms enabled us to obtain better performance than integer-large-modulus generators, from the viewpoint of longer periods (Proposition 4.2) and higher k-distribution property (lattice methods in Sect. 5).

APPENDIX A

A.1 Explicit Form of Matrix B

The explicit form of the $((nw - r) \times (nw - r))$-matrix B in Sect. 3.1 is as follows:

$$
B = \begin{pmatrix}
0 & I_w & 0 & 0 & & & \\
0 & 0 & I_w & 0 & & & \\
\vdots & & & \ddots & & & \\
0 & & & & & & \\
I_w & & & & & & \\
0 & & & & & & \\
\vdots & & & & \ddots & & \\
0 & & & & 0 & I_w & 0 \\
0 & & & & 0 & 0 & I_{w-r} \\
S & & & & 0 & 0 & 0
\end{pmatrix}, \quad
S := \begin{pmatrix} 0 & I_r \\ I_{w-r} & 0 \end{pmatrix} A.
$$

For $l = 0, 1, \ldots,$

$$
(\mathbf{x}_{l+n}, \mathbf{x}_{l+n-1}, \cdots, \mathbf{x}_{l+1}^u) = (\mathbf{x}_{l+n-1}, \mathbf{x}_{l+n-2}, \cdots, \mathbf{x}_l^u) B,
$$

holds; see the recurrence (2.1).

It is not hard to see that

$$
\varphi_B(t) = (t^n + t^m)^{w-r}(t^{n-1} + t^{m-1})^r + a_0(t^n + t^m)^{w-r}(t^{n-1} + t^{m-1})^{r-1}
$$

$$
+ \cdots + a_{r-2}(t^n + t^m)^{w-r}(t^{n-1} + t^{m-1}) + a_{r-1}(t^n + t^m)^{w-r}
$$

$$
+ a_r(t^n + t^m)^{w-r-1} + \cdots + a_{w-2}(t^n + t^m) + a_{w-1},
$$

where the a_is are as in Sect. 2.1.

A.2 Relation between an MT Sequence and its Subsequences

Let $\varphi_B(t)$ be the characteristic polynomial B, and S be a state. Since $\varphi_B(B)S = 0$, each entry of the incomplete array (i.e., an entry in S, BS, B^2S, ...) constitutes a bit-stream that is a solution to the linear recurrence associated with $\varphi_B(t)$. Thus we get the following proposition.

PROPOSITION A.1. *For an MT sequence* $(\ldots, \mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_0^u)$, *its ith-bit subsequence* $(\ldots, x_{3,i}, x_{2,i}, x_{1,i})$ *satisfies the same linear recurring equation corresponding to* $\varphi_B(t)$ *independently of the choice of i.*

Thus, an MT sequence $(\ldots, \mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_0^u)$ *attains the maximal period* $2^p - 1$ *if and only if its i-th bit subsequence* $(\ldots, x_{3,i}, x_{2,i}, x_{1,i})$ *attains the maximal period* $2^p - 1$.

Appendix B. Obstructions to Optimal Distribution

We show here some obstructions preventing MT from achieving the trivial bound on $k(v)$.

PROPOSITION B.1 *Let $j < n$ be an integer. If*

$$\frac{j(j + 3)}{2} v > (j + 1)w - r$$

holds, then the order of equidistribution $k(v)$ to v-bit accuracy of the MT sequence is at most $jn - 1$.

This says that if $v > w - (r/2)$ then $k(v)$ is at most $n - 1$, and that if $v > (3w - r)/5$ then $k(v)$ is at most $2n - 1$. The former is much more restrictive than $k(v) \leq \lfloor (nw - r)/v \rfloor$, for large r, such as MT19937 in Table II.

PROOF. k-distribution to v-bit accuracy is equivalent to the surjectivity of the linear mapping:

$$(\mathbf{x}_0^u, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) \xrightarrow{f} (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) \begin{pmatrix} TQ & & & \\ & TQ & & \\ & & \ddots & \\ & & & TQ \end{pmatrix},$$

where T denotes the $(w \times w)$-tempering matrix and Q denotes the matrix taking the upper v bits, i.e., $Q = \begin{pmatrix} I_v \\ \hline 0 \end{pmatrix}$. Although we used the recurrence (2.1), as far as the surjectivity of this mapping is concerned, we get the same result even if we use the recurrence

$$\mathbf{x}_{k+n} := (\mathbf{x}_k^u | \mathbf{x}_{k+1}^l)A, \quad (k = 0, 1, \cdots),$$

by the same argument as in the proof of Theorem 2.1 in Matsumoto and
Kurita [1994]. So, from now on, we use this recurrence. Now, the dependen-
cies of the $\mathbf{x}_i$s are described by the following diagram.

$$
\begin{array}{ccccccccccc}
\mathbf{x}_0^u & & \mathbf{x}_1 & & \mathbf{x}_2 & & \mathbf{x}_3 & \cdots & & \mathbf{x}_{n-1} & \\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \cdots & \swarrow & \downarrow & \\
\leftarrow & \mathbf{x}_n & & \mathbf{x}_{n+1} & & \mathbf{x}_{n+2} & & \mathbf{x}_{n+3} & \cdots & & \mathbf{x}_{2n-1} & \leftarrow \\
& \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \cdots & \swarrow & \downarrow \\
\leftarrow & \mathbf{x}_{2n} & & \mathbf{x}_{2n+1} & & \mathbf{x}_{2n+2} & & \mathbf{x}_{2n+3} & \cdots & & \mathbf{x}_{3n-1} & \leftarrow \\
& \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \cdots & \swarrow & \downarrow \\
\leftarrow & \mathbf{x}_{3n} & & \mathbf{x}_{3n+1} & & \mathbf{x}_{3n+2} & & \mathbf{x}_{3n+3} & \cdots & & \mathbf{x}_{4n-1} & \leftarrow \\
& \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \cdots & \swarrow & \downarrow \\
\end{array}
$$
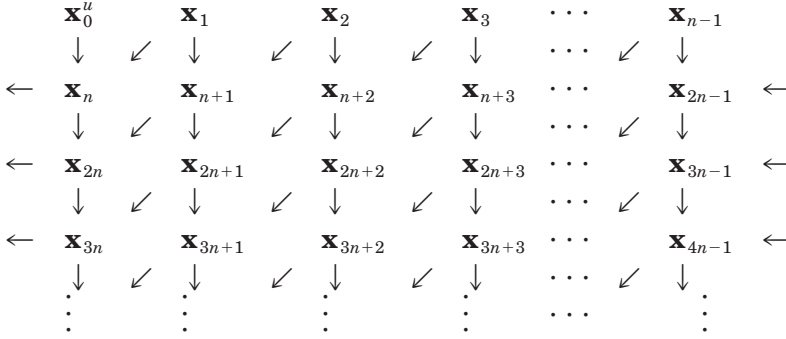
Then, for an integer $j < n$, the part $(\mathbf{x}_0^u, \mathbf{x}_1, \ldots, \mathbf{x}_j)$ of the initial values
determines the left-upper triangular region in the diagram:

$$
(\mathbf{x}_0^u, \mathbf{x}_1, \ldots \quad \ldots \quad \ldots, \mathbf{x}_j,
$$

$$
\mathbf{x}_n, \mathbf{x}_{n+1}, \ldots, \mathbf{x}_{n+j-1},
$$

$$
\vdots
$$

$$
\mathbf{x}_{jn}).
$$

Then, if $k(v) \geq jn$, the multiplication by TQ on each vector must be
surjective as a whole. Thus the dimension of the domain $(j + 1)w - r$ is at
least that of the target $[j(j + 3)/2]v$. This shows that if $k(v) \geq jn$ then
$[j(j + 3)/2]v \leq (j + 1)w - r$. $\square$

PROPOSITION B.2   *If $k(v) \geq n + max\{r, w - r\} - 1$, then $k(v) \geq 2(n -$
1).*

This shows that $k(v)$ tends to be near to n (at most $max\{r, w - r\}$ distance)
if it is less than $2(n - 1)$. This partly explains the absence of intermediate
values of $k(v)$ in Table II.

PROOF.   We apply the simplification of the recurrence as in the above
proof. Moreover, instead of the initial vector $(\mathbf{x}_0^u, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1})$, we use
$(\mathbf{x}_0^u, \tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_{n-1})$, where ˜ denotes the mapping $(\mathbf{x}^u | \mathbf{x}^l) \mapsto (\mathbf{x}^l | \mathbf{x}^u)$, i.e., the
multiplication of

$$
R := \begin{pmatrix} 0 & \mathrm{I}_{w-r} \\ \mathrm{I}_r & 0 \end{pmatrix}.
$$

Then, when we explicitly write down the matrix that gives the upper v bits

of the first $n - 1 + l$ values from the initial value, it becomes

$$
M_l := \left(\begin{array}{ccccccccc}
& & & & A_1 & & & \\
A_2 & & & & A_3 & & & \\
A_4 & & & & & A_1 & & \\
& A_2 & & & & A_3 & & \\
& A_4 & & & & & \ddots & \\
& & \ddots & & & & & A_1 \\
& & & A_2 & & & & A_3 \\
& & & A_4 & & & &
\end{array} \right) \left.\rule{0cm}{2.2cm}\right\} l \text{ pairs,}
$$

where $\binom{A_2}{A_4}$ is the matrix RTQ (T: tempering matrix), partitioned into the first r rows and the last $w - r$ rows, and $\binom{A_3}{A_3}$ is the matrix $RATQ$, partitioned to $w - r$ and r. Let us assume that $k(v) = n + j$ for $n + j < 2(n - 1)$. This is equivalent to saying that the rank of the matrix M_l with $l = j$ is equal to its width, but the rank with $l = j + 1$ is not.

The former condition is equivalent to the triviality of the kernel of M_l, when applied to column vectors from the left. So we now obtain the kernel of these matrices for $l = 1, 2, \ldots$.

Let V_1, V_2 denote the v-dimensional vector space of row vectors, and W, W' denote the row vector space of dimension $w - r$, r, respectively. We first consider the kernel of

$$
\left(\begin{array}{cc}
& A_1 \\
A_2 & A_3
\end{array} \right) : V_2 \oplus V_1 \rightarrow W \oplus W'.
$$

The existence of A_1 implies that the V_1 component is in $A_1^{-1}(0)$, where $^{-1}$ means the inverse image. Then the V_1 component should be mapped by A_3 to an image of A_2, so the projection to the V_2 component of the kernel of M_1 is

$$
A_2^{-1} A_3 A_1^{-1}(0).
$$

The sequence satisfies $k(v) = n - 1$ if and only if M_2 has nontrivial kernel, i.e., the kernel of A_4 has nontrivial intersection with $A_2^{-1} A_3 A_1^{-1}(0)$. If we denote by

$$
\Phi := A_2^{-1} A_3 A_1^{-1} A_4
$$

the corresponding map from the set of subspaces of V_1 to itself, then this can be stated as $\mathrm{Ker}(A_4) \cap \Phi(0) \neq 0$.

It is not difficult to check that the kernel of M_l is nontrivial if and only if $\mathrm{Ker}(A_4) \cap \Phi^l(0) \neq 0$, so the smallest such $l \leq 2(n - 1)$ gives $k(v) = n + l - 2$, if it exists. Thus, to prove the proposition, it is enough to show that $\Phi^{l+1}(0) = \Phi^l(0)$ for $l \geq \max\{r, w - r\} + 1$, since then the smallest l with

$\mathrm{Ker}(A_4) \cap \Phi^l(0) \neq 0$ should satisfy $l \leq \max\{r, w - r\} + 1$, and then $k(v) = n + l - 2$ implies the proposition. To prove the above stability, we note that Φ is a monotonic function with respect to the inclusion. So

$$0 \subset \Phi(0) \subset \Phi^2(0) \subset \cdots .$$

Now the corresponding subspaces in W, W' are increasing, but the dimensions of W, W' are $w - r$, r, respectively. So after applying $\max\{w - r, r\}$ iterations of Φ, they will be stable. By returning to V_1, we know that one more application of Φ stabilizes the space. $\square$

Appendix C. C Program

The C-code for MT19937 follows on the next page. This code works both in 32-bit and 64-bit machines. The function genrand( ) returns a uniformly distributed real pseudorandom number (of type double, with 32-bit precision) in the closed interval [0, 1]. The function sgenrand( ) sets initial values to the array mt[N]. Before using genrand( ), sgenrand( ) must be called with a nonzero unsigned long integer as a seed.

The generator can be modified to a 32-bit unsigned long integer generator by changing two lines, namely, the type of the function genrand( ) and the output scheme. See the comment inside.

The magic numbers are put in the macros so that one can easily change them according to Table II. Essentially the same code is downloadable from the http-site, Salzburg University (see http://random.mat.sbg.ac.at/news/).

Topher Cooper kindly enhanced the robustness in the initialization scheme. Marc Rieffel (marc@scp.syr.edu), who uses MT19937 in a plasma simulation, reported that by replacing the function calls by the macros, the runtime could be reduced by 37%. His code is available from ftp.scp.syr.edu/pub/hawk/mt19937b-macro.c, which also improves several other points.

```
/* A C-program for MT19937: Real number version           */
/* genrand() generates one pseudorandom real number (double)  */
/* which is uniformly distributed on [0,1]-interval, for each  */
/* call. sgenrand(seed) set initial values to the working area */
/* of 624 words. Before genrand(), sgenrand(seed) must be     */
/* called once. (seed is any 32-bit integer except for 0).    */
/* Integer generator is obtained by modifying two lines.      */
/*   Coded by Takuji Nishimura, considering the suggestions by */
/* Topher Cooper and Marc Rieffel in July-Aug. 1997. Comments  */
/* should be addressed to: matumoto@math.keio.ac.jp          */


#include<stdio.h>

/* Period parameters */
#define N 624
#define M 397
#define MATRIX_A 0x9908b0df   /* constant vector a */
#define UPPER_MASK 0x80000000 /* most significant w-r bits */
#define LOWER_MASK 0x7fffffff /* least significant r bits */
/* Tempering parameters */
#define TEMPERING_MASK_B 0x9d2c5680
#define TEMPERING_MASK_C 0xefc60000
#define TEMPERING_SHIFT_U(y)  (y >> 11)
#define TEMPERING_SHIFT_S(y)  (y << 7)
#define TEMPERING_SHIFT_T(y)  (y << 15)
#define TEMPERING_SHIFT_L(y)  (y >> 18)

static unsigned long mt[N]; /* the array for the state vector  */
static int mti=N+1; /* mti==N+1 means mt[N] is not initialized */

/* initializing the array with a NONZERO seed */
void
sgenrand(seed)
    unsigned long seed;
{
    /* setting initial seeds to mt[N] using       */
    /* the generator Line 25 of Table 1 in        */
    /* [KNUTH 1981, The Art of Computer Programming */
    /*    Vol. 2 (2nd Ed.), pp102]                 */
    mt[0]= seed & 0xffffffff;
    for (mti=1; mti<N; mti++)
        mt[mti] = (69069 * mt[mti-1]) & 0xffffffff;
}

double /* generating reals */
/* unsigned long */ /* for integer generation */
genrand()
```

```c
{
    unsigned long y;
    static unsigned long mag01[2]={0x0, MATRIX_A};
    /* mag01[x] = x * MATRIX_A  for x=0,1 */

    if (mti >= N) { /* generate N words at one time */
        int kk;

        if (mti == N+1)   /* if sgenrand() has not been called, */
            sgenrand(4357); /* a default initial seed is used   */

        for (kk=0;kk<N-M;kk++) {
            y = (mt[kk]&UPPER_MASK)|(mt[kk+1]&LOWER_MASK);
            mt[kk] = mt[kk+M] ^ (y >> 1) ^ mag01[y & 0x1];
        }
        for (;kk<N-1;kk++) {
            y = (mt[kk]&UPPER_MASK)|(mt[kk+1]&LOWER_MASK);
            mt[kk] = mt[kk+(M-N)] ^ (y >> 1) ^ mag01[y & 0x1];
        }
        y = (mt[N-1]&UPPER_MASK)|(mt[0]&LOWER_MASK);
        mt[N-1] = mt[M-1] ^ (y >> 1) ^ mag01[y & 0x1];

        mti = 0;
    }

    y = mt[mti++];
    y ^= TEMPERING_SHIFT_U(y);
    y ^= TEMPERING_SHIFT_S(y) & TEMPERING_MASK_B;
    y ^= TEMPERING_SHIFT_T(y) & TEMPERING_MASK_C;
    y ^= TEMPERING_SHIFT_L(y);

    return ( (double)y / (unsigned long)0xffffffff ); /* reals */
    /* return y; */ /* for integer generation */
}

/* this main() outputs first 1000 generated numbers  */
main()
{
    int j;

    sgenrand(4357); /* any nonzero integer can be used as a seed */
    for (j=0; j<1000; j++) {
        printf("%5f ", genrand());
        if (j%8==7) printf("\n");
    }
    printf("\n");
}
```

work was done during the first author's visit to CRM, Montreal University. He is thankful to P. L'Ecuyer and R. Couture for their hospitality and invaluable discussions. In particular, A. Compagner, P. Hellekalek and H. Leeb contributed a number of valuable comments. H. Leeb kindly included MT19937 on the http-site at Salzburg University (see http://random.mat.s-bg.ac.at/news/). We are thankful to H. Niederreiter for information on his general and detailed framework of the multiple-recursive matrix method, and to M. Fushimi for comments on auto-correlations.

We are deeply grateful to D. E. Knuth for very useful discussions and correspondence, including a theoretical justification of recurrence (2.1) in Sect. 3.1, and comments on the least significant bits.

We are thankful to G. Marsaglia and B. Narasimhan for helping us in executing their diehard test.

Part of this study was done during the first author's stay at Max-Planck-Institut für Mathematik, Bonn, and Salzburg University. The discussions with the pLab group at Salzburg greatly helped to improve this paper. Wegenkittl kindly tested MT19937. We thank Topher Cooper and Marc Rieffel for their helpful suggestions about the C-code implementation (see comments in Appendix C).

We are deeply thankful to P. L'Ecuyer for his kind and careful comments on the whole manuscript. Thanks are also due to the anonymous referees.

A. Compagner died while we were revising this paper. We hope he rests in peace.

The first author would like to acknowledge J.

The second author would like to acknowledge his parents, who kindly encouraged him to continue his studies.

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