1 Physics of Motion

We will be simulating the motion of spheres. Actually, most of the work will be done for you, and you will just have to deal with the spheres colliding with each other and with the fixed surroundings. A sphere has position c, orientation matrix M (i.e. model2rotated), linear velocity v and angular velocity w. Angular velocity means it is rotating about an axis in the direction of w with a rate, radians per second, equal to the length of w.

Velocity is the derivative of position or position is the integral of velocity. If there is gravity, acceleration is the derivative of velocity or velocity is the integral of acceleration. Gravity is $a = -g$. We can approximate an integral using sums with a small time step Δt:

$$v \leftarrow v + a\Delta t, \qquad c \leftarrow c + v\Delta t.$$

I will put other notes in your box explaining why it is better to update the velocity first.

The update of M is a rotation about the axis $w/|w|$ by angle $|w|\Delta t$. Set $w^z = w/|w|$, w^x equal to $(0,1,0) \times w^z$ unitized, and $w^y = w_z \times w_x$. The matrix $W = [w^x, w^y, w^z]$ takes $(0,0,1)$ to w^z. $W^{-1} = W^t$ takes w^z to $(0,0,1)$. Let R^z be Mat.rotation(2, $|w|$), the rotation about z by the angle we want. The matrix we want is

$$R = WR^zW^{-1}.$$

It takes w^z to $(0,0,1)$, rotates about $(0,0,1)$, and then takes $(0,0,1)$ back to w^z. The update is $M \leftarrow RM$.

This all is the update when the spheres are not touching. If they bump into each other, we have to update v and w.

1.1 Frictionless Elastic Collisions

Two *frictionless* spheres bump into each other. What happens? Since they are frictionless, nothing happens to their angular velocities.

Spheres 1 and 2 have masses m_1 and m_2 and velocities v_1 and v_2. Their centers at the moment of collision are c_1 and c_2. The point of contact is $p = (c_1 + c_2)/2$ and the normal n on sphere 2 at p is $c_1 - c_2$ normalized. n is the inward normal to sphere 1.

The momentum of sphere 1 is m_1v_1 and for sphere 2 it is m_2v_2. Total momentum is conserved, which means that whatever the collision adds to 1's momentum must be subtracted from 2's momentum. Also, since the spheres are frictionless (for now), they can only push each other in the direction of n. Just like when you are standing on something slippery, you can only push downward. Pushing sideways offers no resistance.

The kinetic energy of sphere 1 is $m_1v_1^2/2$ or $(m_1v_1)^2/(2m_1)$, the square of the momentum divided by $2m_1$. Ditto sphere 2.

Energy is conserved, but in reality some of the kinetic energy of the spheres will be converted to heat. We will model a perfect *elastic* collision the conserves the kinetic energy.

After the collision, the momenta are $m_1v_1 + in$ and $m_2v_2 - in$ where i is the magnitude of the *impulse* caused by the collision. We set the energy before equal to the energy after and solve for i:

$$\frac{(m_1v_1)^2}{2m_1} + \frac{(m_2v_2)^2}{2m_2} = \frac{(m_1v_1 + in)^2}{2m_1} + \frac{(m_2v_2 - in)^2}{2m_2},$$

$$\frac{(m_1v_1)^2}{2m_1} + \frac{(m_2v_2)^2}{2m_2} = \frac{(m_1v_1)^2 + 2m_1iv_1 \cdot n + (in)^2}{2m_1} + \frac{(m_2v_2)^2 - 2m_2iv_2 \cdot n + (in)^2}{2m_2},$$

$$\frac{(m_1v_1)^2}{2m_1} + \frac{(m_2v_2)^2}{2m_2} = \frac{(m_1v_1)^2}{2m_1} + iv_1 \cdot n + \frac{i^2}{2m_1} + \frac{(m_2v_2)^2}{2m_2} - iv_2 \cdot n + \frac{i^2}{2m_2},$$

$$0 = iv_1 \cdot n + \frac{i^2}{2m_1} - iv_2 \cdot n + \frac{i^2}{2m_2}.$$

One solution is $i = 0$, meaning the spheres just keep going into each other without bouncing like ghosts. Assuming, $i \neq 0$,

$$\frac{i}{2m_1} + \frac{i}{2m_2} = (v_2 - v_1) \cdot n.$$

If $m_1 = m_2$, the solution is $i = m_1(v_2 - v_1) \cdot n$.

If sphere 2 is a fixed, we can set $v_2 = 0$ and $m_2 = \infty$, and the solution is $i = -2m_1v_1 \cdot n$.

In the first case, sphere 1 has new velocity $v_1+((v_2-v_1)\cdot n)n$. In the second case, $v_1-2(v_1\cdot n)n$.

1.2 Angular Velocity, Momentum, and Kinetic Energy

A sphere can also be spinning. The angular velocity is represented by w that points along the axis of rotation and whose magnitude is the rate of rotation (in radians per second, for example). If the sphere has center c, a point p on the sphere moves with the linear velocity v of the sphere plus an angular component $w \times (p - c)$. Points on the equator where $p - c$ is perpendicular to w move the fastest. Points near the poles where $p - c$ is nearly collinear with w move the slowest.

A sphere has an angular "mass" I called its moment of inertia. I is actually a 3 by 3 matrix, but for a spherically symmetric object, it is a scalar times the identity matrix, so we will just treat I as a scalar. In this special case, the angular momentum is Iw and the angular kinetic energy is $Iw^2/2$ (in general $w^tIw/2$). If all the mass m is concentrated at the center, the sphere has no "resistance to turning" so I is zero. If m is uniformly distributed in the sphere, I is larger ($\frac{2}{5}mr^2$). It is largest when all the mass of the sphere is in its surface ($\frac{2}{3}mr^2$).

If sphere 1 and sphere 2 are "rough" and spinning, there will be a second impulse perpendicular to n at the point of contact p. The direction u of the impulse will be opposite to the relative velocity of p on spheres 1 and 2. So for example if 1 is spinning with p on the equator, p will have a large tangential velocity. At the moment of contact, point p on sphere 1 will push on sphere 2 in the direction of motion and sphere 2 will "resist". Sphere 1 will feel an impulse in the opposite direction.

The velocity of p on sphere 1 is $v_1 + w \times (p - c_1)$ and the relative velocity of the two ps is

$$U = v_2 + w_2 \times (p - c_2) - v_1 - w_1 \times (p - c_1).$$

u is perpendicular to n so it is $U - (U \cdot n)n$ normalized. We apply impulse ju to sphere 1 at p and $-ju$ to sphere 2 at p and choose j to conserve energy.

As before, ju changes the linear moment of sphere 1 to $m_1 v_1 + ju$, but it also changes the angular momentum. Applying an impulse ju at p "spins up" the sphere along an axis perpendicular to u and to $p - c$. The new angular momentum is $I_1 w_1 + (p - c_1) \times ju$. We set $z_1 = (p - c_1) \times u$ and $z_2 = (p - c_2) \times u$. (z_1 doesn't generally point in the z direction, but it points in the z direction if $p - c_1$ points along x and u points along y.) So the new momenta are $I_1 w_1 + jz_1$ and $I_1 w_1 - jz_2$ and the new angular kinetic energy is momentum squared over $2I_1$ or $2I_2$.

If we choose the impulse to conserve energy,

$$\frac{(m_1 v_1)^2}{2m_1} + \frac{(I_1 w_1)^2}{2I_1} + \frac{(m_2 v_2)^2}{2m_2} + \frac{(I_2 w_2)^2}{2I_2} =$$

$$\frac{(m_1 v_1 + ju)^2}{2m_1} + \frac{(I_1 w_1 + jz_1)^2}{2I_1} + \frac{(m_2 v_2 - ju)^2}{2m_2} + \frac{(I_2 w_2 - jz_2)^2}{2I_2}$$

Assuming $j \neq 0$,

$$v_1 \cdot u + \frac{j}{2m_1} + w_1 \cdot z_1 + \frac{jz_1^2}{2I_1} - v_2 \cdot u + \frac{j}{2m_2} - w_2 \cdot z_2 + \frac{jz_2^2}{2I_2} = 0.$$

$$\frac{j}{2}\left(\frac{1}{m_1} + \frac{z_1^2}{I_1} + \frac{1}{m_2} + \frac{z_2^2}{I_2}\right) = (v_2 - v_1) \cdot u + w_2 \cdot z_2 - w_1 \cdot z_1.$$

If $m_1 = m_2$ and $I_1 = I_2$,

$$j = \left(\frac{1}{m_1} + \frac{z_1^2 + z_2^2}{2I_1}\right)^{-1} ((v_2 - v_1) \cdot u + w_2 \cdot z_2 - w_1 \cdot z_1).$$

If sphere 2 is fixed and $m_2 = I_2 = \infty$ and $v_2 = w_2 = 0$,

$$j = -2\left(\frac{1}{m_1} + \frac{z_1^2}{I_1}\right)^{-1} (v_1 \cdot u + w_1 \cdot z_1).$$