

In two dimensions, a point or a vector is an ordered pair (x, y) . If you think of it as a point, you know how to plot it. If you think of it as a vector, it is a displacement arrow. Let's all do $(3, -4)$.

In three dimensions, you have an additional coordinate (x, y, z) . If x is right on the whiteboard and y is up, then z is coming out at you. That's "world coordinates" in graphics.

Points and vectors can be multiplied by scalars:

$$(x, y, z)s = (xs, ys, zs).$$

What does this mean geometrically? If $s = 2$? If $s = -1$?

You can add two vectors or a point and a vector:

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

What does this mean?

The length of a vector is

$$|(x, y, z)| = \sqrt{x^2 + y^2 + z^2}.$$

Can you multiply vectors? Yes, there are two ways: the **dot** product and the **cross** product:

$$\begin{aligned}(x_1, y_1) \cdot (x_2, y_2) &= x_1x_2 + y_1y_2, \\(x_1, y_1) \times (x_2, y_2) &= x_1y_2 - y_1x_2, \\(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) &= x_1x_2 + y_1y_2 + z_1z_2, \\(x_1, y_1, z_1) \times (x_2, y_2, z_2) &= (y_1z_2 - z_1y_2, z_1x_2 - x_1z_2, x_1y_2 - y_1x_2)\end{aligned}$$

In 2D (two dimensions), both products are scalars. In 3D, the dot product is still a scalar, but the cross product is a vector.

We can see that a vector dot itself is equal to the square of its length:

$$v \cdot v = |v|^2.$$

In general, let θ be the angle between vectors u and v . Then the dot product is

$$u \cdot v = |u||v| \cos \theta.$$

This formula is not how you should *implement* the dot product; it is a *mathematical fact* about it.

Notice that if $u = v$, then $\theta = 0$ and $\cos \theta = 1$.

The dot product is zero when $\theta = \pi/2$ radians or 90 degrees. That means the two vectors are **perpendicular**. In general, the dot product is the length of the projection of u onto v times the length of v or *vice versa*. If the vectors are collinear, this is the product of their lengths. If the vectors are perpendicular, this is zero.

Can the dot product be negative? If it can, when is it the most negative?

In 2D, the cross product has a similar formula:

$$u \times v = |u||v| \sin \theta,$$

where θ is measured counterclockwise from u to v . This is also the **area** of the parallelogram formed by $o = (0, 0)$, u , $u + v$, v or twice the area of the triangle ouv . Can it be negative?

In 3D, the cross product is a vector. Its length is $|u||v|\sin\theta$ and it is perpendicular to both u and v . Which way does it point? The cross product follows the **right hand rule**: if u is your thumb and v your index finger, then $u \times v$ points along your middle finger.

How come the dot and cross product have this formula with the sine and cosine? Well, what *is* the sine and cosine? If a right triangle has hypotenuse of length 1, then the adjacent side is $\cos\theta$ and the opposite side is $\sin\theta$. That's the definition. That means if you start with the vector $u = (1, 0)$ and rotate by θ about the origin, then you must get $v = (\cos\theta, \sin\theta)$. So using the original formulas for dot and cross,

$$\begin{aligned} u \cdot v &= 1 \cdot \cos\theta + 0 \cdot \sin\theta, \\ &= \cos\theta, \\ &= |u||v|\cos\theta, \\ u \times v &= 1 \cdot \sin\theta - 0 \cdot \cos\theta, \\ &= \sin\theta, \\ &= |u||v|\sin\theta. \end{aligned}$$

Since it is true for this u and v , it is true in general by linearity.

Now that we know how to rotate $u_x = (1, 0)$, we want to rotate an arbitrary vector or point. A little drawing shows that rotating $u_y = (0, 1)$ by θ gives $(-\sin\theta, \cos\theta)$. Let's consider a point to be a 2 by 1 matrix:

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

Do you remember how to do matrix multiplication? Verify that,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

So multiplying by this matrix rotates u_x and u_y correctly. By linearity,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

is how you rotate an arbitrary (x, y) . Look at vertex shader in [rotatingSquare3.html](#). See how it uses this formula to rotate each point it gets from [rotatingSquare.js](#).

In 3D, to rotate by θ about the z , x , or y axis.

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

If you want to do more than one rotation, multiply matrices! But what order? What matrix rotates an object about (the world's) x -axis by θ and then rotates the object about (the world's) y -axis by ϕ ? (Basically, you'll be spending about half the course answering the question!)