Models

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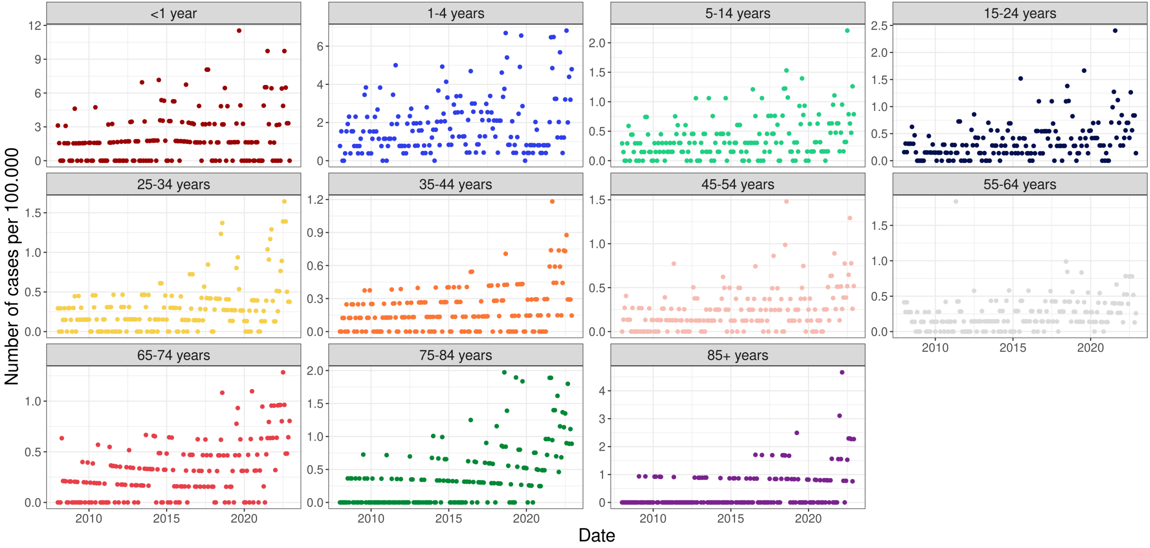
VTEC / STEC

Date	ageGroup	y	n
2008-01-01	<1 year	2	64137
2008-01-01	1-4 years	2	259910
2008-01-01	5-14 years	2	680529
2008-01-01	15-24 years	1	631724
...	...	...	...
2022-12-01	55-64 years	2	773073
2022-12-01	65-74 years	5	621965
2022-12-01	75-84 years	4	449423
2022-12-01	85+ years	3	132181

VTEC / STEC

Shiga- og veratoxin producerende E. coli.



Fomulation

$$Y_{it}|u_{it} \sim \mathrm{Pois}\left(w_{it}\lambda_i \exp(u_{it})\right) \tag{1a}$$

$$u_{it} \sim \mathrm{N}(0, \sigma^2) \tag{1b}$$

Implementation - Objective function in C++

```
#include <TMB.hpp>                // Links in the TMB libraries
template<class Type>
Type objective_function<Type>::operator() ()
{
  DATA_VECTOR(y);                         // Data vector transmitted from R
  DATA_VECTOR(w)                  // Data vector transmitted from R
  DATA_FACTOR(ageGroup);          // Data factor transmitted from R

  PARAMETER_VECTOR(u);                // Random effects

  // Parameters
  PARAMETER_VECTOR(lambda);         // Parameter value transmitted from R
  PARAMETER(log_sigma_u);                 // Parameter value transmitted from R

  Type sigma_u = exp(log_sigma_u);

  int nobs = y.size();
  Type mean_ran = Type(0);

  int j;

  Type f = 0;                 // Declare the "objective function" (neg. log. likelihood)
  for(int i=0; i < nobs; i++){
    f -= dnorm(u[i],mean_ran,sigma_u,true);
    j = ageGroup[i];
    f -= dpois(y[i],exp(log(lambda[j])-log(w[j]))*exp(u[i]),true);
  }

  return f;
}
```

Hierachical Poisson-Normal model

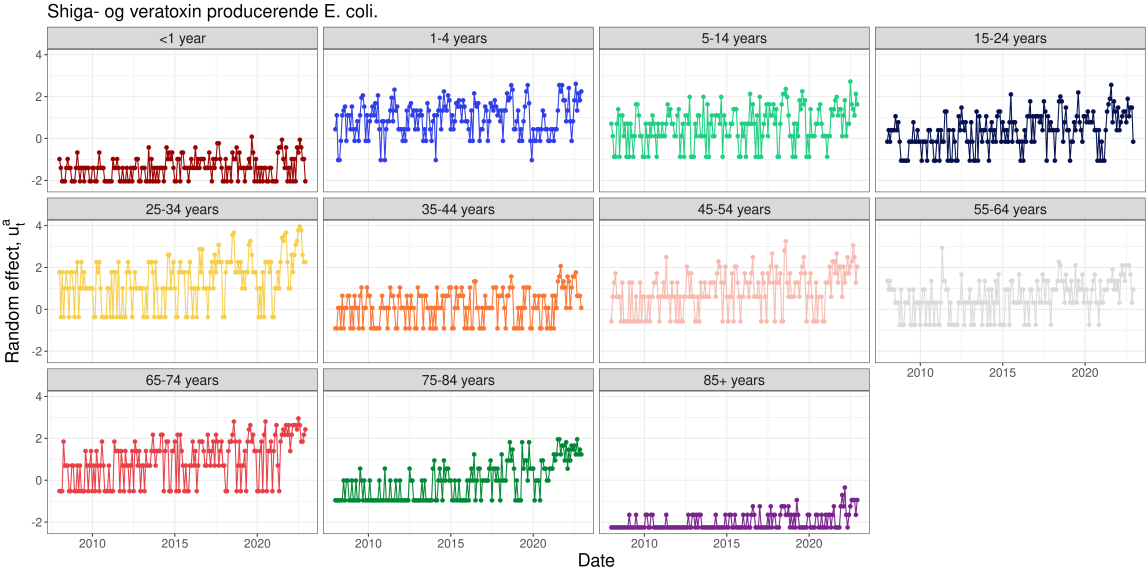
Implementation - Call from R

```r
# Import libraries
library(readr)
library(dplyr)
library(TMB)

# Import the data
dat <- read_rds(file = "../../data/processed/dat.rds")

# Only consider some of the data
y <- dat %>%
  filter(caseDef == "Shiga- og veratoxin producerende E. coli.") %>%
  group_by(Date, ageGroup) %>%
  mutate(y = sum(cases)) %>%
  select(Date, ageGroup, y, n)

compile(file = "PoissonLognormal.cpp")  # Compile the C++ file
dyn.load(dynlib("PoissonLognormal"))    # Dynamically link the C++ code

# Function and derivative
PoisLN <- MakeADFun(
  data = list(y = y$y, ageGroup = y$ageGroup, w = y$n),
  parameters = list(u = rep(1, length(y$y)),
                    lambda = rep(1, nlevels(y$ageGroup)),
                    log_sigma_u = log(1)),
  random = "u",
  DLL = "PoissonLognormal"
)

opt <- nlminb(start = PoisLN$par, PoisLN$fn, PoisLN$gr, lower = c(0.01, 0.01))
```

Results

Parameter	Estimate	Std. Error
$\log(\lambda_{<1year})$	410814.56	...
$\log(\lambda_{1-4years})$	305337.95	28427.34
$\log(\lambda_{5-14years})$	578647.21	64256.49
$\log(\lambda_{15-24years})$	782223.95	96931.77
$\log(\lambda_{25-34years})$	148651.67	13670.29
$\log(\lambda_{35-44years})$	738191.24	113778.60
$\log(\lambda_{45-54years})$	305727.09	31939.10
$\log(\lambda_{55-64years})$	457519.57	53292.63
$\log(\lambda_{65-74years})$	166840.39	16968.32
$\log(\lambda_{75-84years})$	280202.19	54641.59
$\log(\lambda_{85+years})$	917515.87	...
$\log(\sigma_u)$	0.45	0.02

Results



Shiga- og veratoxin producerende E. coli.
<1 year
1-4 years
5-14 years
15-24 years
25-34 years
35-44 years
45-54 years
55-64 years
65-74 years
75-84 years
85+ years
Random effect, u_t^a
Date
4
2
0
-2
2010
2015
2020

Formulation

$$Y_{it}|u_{it} \sim \mathrm{Pois}(\lambda_i u_{it}) \tag{2a}$$

$$u_{it} \sim \mathrm{G}(1/\beta, \beta) \tag{2b}$$

Probability function for Y

$$\begin{aligned} P[Y = y] &= g_Y(y; \lambda, \beta) \\ &= \frac{\lambda^y}{y!\Gamma(1/\beta)\beta^{1/\beta}} \frac{\beta^{y+1/\beta}\Gamma(y + 1/\beta)}{(\lambda\beta + 1)^{y+1/\beta}} \\ &= \frac{\Gamma(y + 1/\beta)}{\Gamma(1/\beta)y!} \frac{1}{(\lambda\beta + 1)^{1/\beta}} \left(\frac{\lambda\beta}{\lambda\beta + 1}\right)^y \\ &= \binom{y + 1/\beta - 1}{y} \frac{1}{(\lambda\beta + 1)^{1/\beta}} \left(\frac{\lambda\beta}{\lambda\beta + 1}\right)^y \ , \ for\ y = 0, 1, 2, \ldots \end{aligned} \tag{3}$$

where we have used the convention

$$\binom{z}{y} = \frac{\Gamma(z + 1)}{\Gamma(z + 1 - y)y!} \tag{4}$$

The marginal distribution of Y is a negative binomial distribution, $Y \sim \mathrm{NB}\left(1/\beta, 1/(\lambda\beta + 1)\right)$

Proof

The probability function for the conditional distribution of Y for given u

$$f_{Y|u}(y;\lambda,u) = \frac{(\lambda u)^y}{y!}\exp(-\lambda u) \tag{5}$$

and the probability density function for the distribution of u is

$$f_u(u;\beta) = \frac{1}{\beta\Gamma(1/\beta)}\left(\frac{u}{\beta}\right)^{1/\beta-1}\exp(-u/\beta) \tag{6}$$

Proof

Given (5) and (6), the probability function for the marginal distribution of Y is determined from

$$
\begin{aligned}
g_Y(y;\lambda,\beta) &= \int_{u=0}^{\infty} f_{Y|u}(y;\lambda,u) f_u(u;\beta)\,du \\
&= \int_{u=0}^{\infty} \frac{(\lambda u)^y}{y!}\exp(-\lambda u)\frac{1}{\beta\Gamma(1/\beta)}\left(\frac{u}{\beta}\right)^{1/\beta-1}\exp(-u/\beta)\,du \\
&= \frac{\lambda^y}{y!\Gamma(1/\beta)\beta^{1/\beta}}\int_{u=0}^{\infty} u^{y+1/\beta}\exp\big(-u(\lambda\beta+1)/\beta\big)\,du
\end{aligned}
\tag{7}
$$

Proof

In (7) it is noted that the integrand is the *kernel* in the probability density function for a Gamma distribution, $\mathrm{G}\left(y+1/\beta, \beta/(\lambda\beta+1)\right)$. As the integral of the density shall equal one, we find by adjusting the norming constant that

$$\int_{u=0}^{\infty} u^{y+1/\beta} \exp\left(-u/\Big(\beta/(\lambda\beta+1)\Big)\right) du = \frac{\beta^{y+1/\beta}\Gamma(y+1/\beta)}{(\lambda\beta+1)^{y+1/\beta}} \tag{8}$$

and then (3) follows

Inference on individual group means

Consider the hierarchical Poisson-Gamma model in (2), and assume that a value $Y = y$ has been observed. Then the conditional distribution of u for given $Y = y$ is a Gamma distribution,

$$u|Y = y \sim \mathrm{G}\left(y + 1/\beta, \beta/(\lambda\beta + 1)\right) \tag{9}$$

with mean

$$\mathrm{E}[u|Y = y] = \frac{y\beta + 1}{\lambda\beta + 1} \tag{10}$$

and variance

$$\mathrm{V}[u|Y = y] = \frac{(y\beta^2 + \beta)}{(\lambda\beta + 1)^2} \tag{11}$$

Proof

The conditional distribution is found using Bayes Theorem

$$\begin{aligned}
g_u(u|Y=y) &= \frac{f_{y,u}(y,u)}{g_Y(y;\lambda,\phi)} \\
&= \frac{f_{y|u}(y;u)g_u(u)}{g_Y(y;\lambda,\beta)} \\
&= \frac{1}{g_Y(y;\lambda,\beta)}\left(\frac{(\lambda u)^y}{y!}\exp(-\lambda u)\frac{1}{\beta\Gamma(1/\beta)}\left(\frac{u}{\beta}\right)^{1/\beta-1}\exp(-u/\beta)\right) \\
&\propto u^{y+1/\beta-1}\exp\big(-u(\lambda\beta+1)/\beta\big)
\end{aligned} \tag{12}$$

We identify the *kernel* of the probability density function

$$u^{y+1/\beta-1}\exp(-u(\lambda\beta+1)/\beta) \tag{13}$$

as the kernel of a Gamma distribution, $\mathrm{G}(y+1/\beta,\beta/(\lambda\beta+1))$

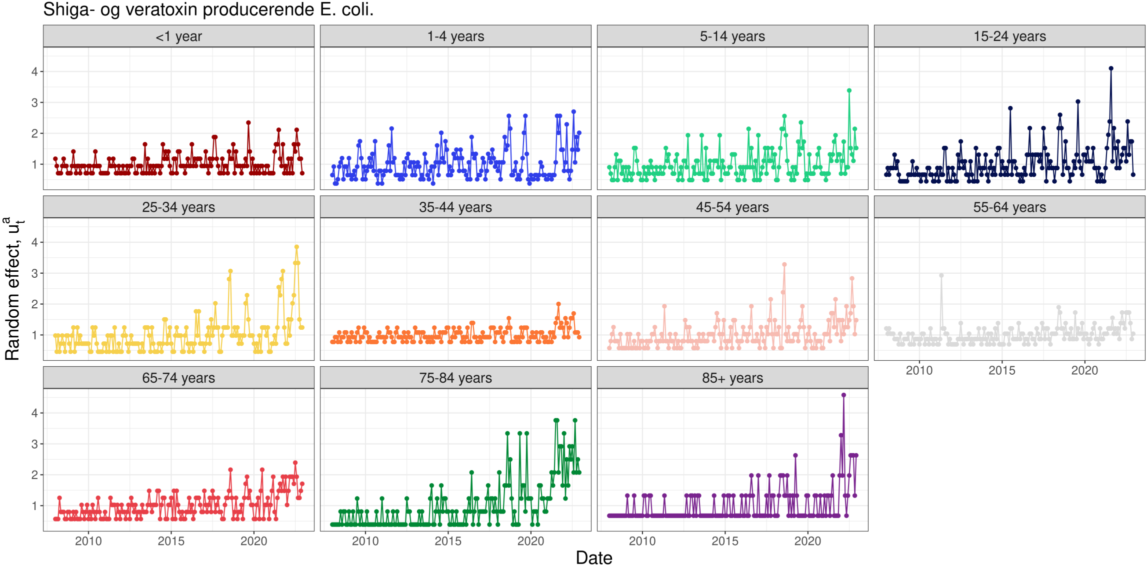
Implementation - Define negative likelihood function in R

```r
# Define the negative likelihood function for the marginal distribution of Y
nll.age <- function(theta, data){
  # Extract counts
  y <- data$y
  # Extract agegroups
  ageGroup <- data$ageGroup
  # Extract number of agegroups
  n.ageGroup <- n_distinct(data$ageGroup)

  # Define parameters
  lambda <- theta[1:n.ageGroup]
  beta <- theta[(n.ageGroup+1):(n.ageGroup*2)]

  # Construct the size and probability for the negative binomial distribution
  r <- 1/beta
  p <- 1/(lambda*beta+1)

  # Initilize the log-likelihood
  ll <- 0
  for(i in 1:nrow(data)){
    ll = ll + dnbinom(x = y[i],
                      size = r[ageGroup[i]],
                      prob = p[ageGroup[i]],
                      log = TRUE)
  }

  # Return the negative log-likelihood
  -ll
}
```

Results

ageGroup	λ	β
<1 year	1.24	0.33
1-4 years	4.57	0.36
5-14 years	2.47	0.42
15-24 years	2.57	0.48
25-34 years	2.08	0.57
35-44 years	1.47	0.20
45-54 years	1.88	0.39
55-64 years	1.77	0.25
65-74 years	1.89	0.40
75-84 years	1.45	1.08
85+ years	0.50	0.97

Results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75-84 years
85+ years
Random effect, u_t^a
Date
1
2
3
4
2010
2015
2020