

Outline

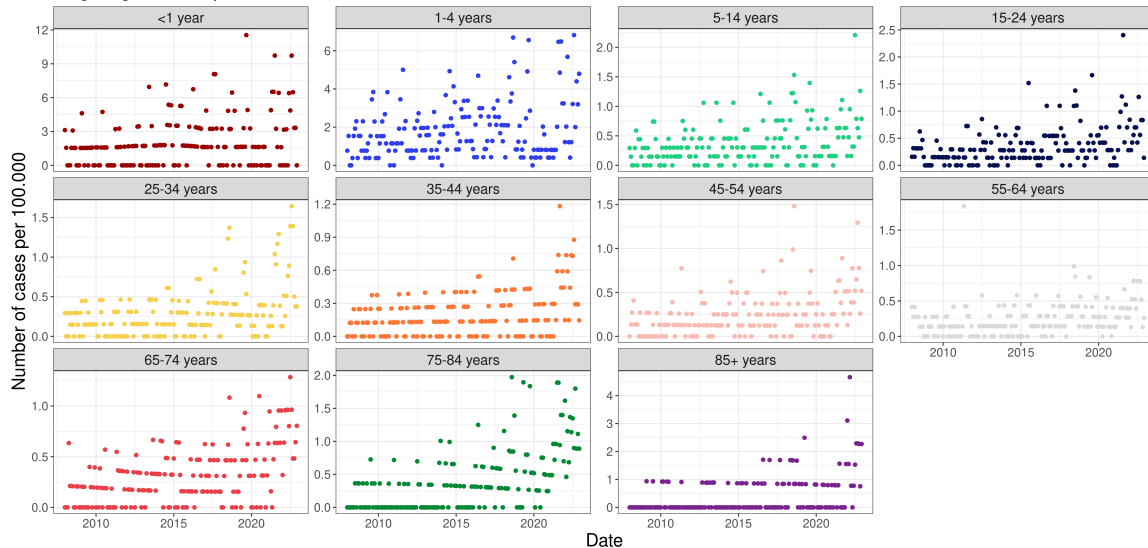
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- Hierarchical Poisson-Gamma model
 - Formulation
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Date	ageGroup	y	n
2008-01-01	<1 year	2	64137
2008-01-01	1-4 years	2	259910
2008-01-01	5-14 years	2	680529
2008-01-01	15-24 years	1	631724
...
2022-12-01	55-64 years	2	773073
2022-12-01	65-74 years	5	621965
2022-12-01	75-84 years	4	449423
2022-12-01	85+ years	3	132181

Data exploration

VTEC / STEC

Shiga- og veratoxin producerende E. coli.



Hierarchical Poisson-Normal model

Fomulation



$$Y_{it}|u_{it} \sim \text{Pois}(w_{it}\lambda_i \exp(u_{it})) \quad (1a)$$

$$u_{it} \sim \text{N}(0, \sigma_i^2) \quad (1b)$$

Implementation - Objective function in C++

```

#include <TMB.hpp>           // Links in the TMB libraries
template<class Type>
Type objective_function<Type>::operator() ()
{
    DATA_VECTOR(y);          // Data vector transmitted from R
    DATA_VECTOR(w);          // Data vector transmitted from R
    DATA_FACTOR(ageGroup);   // Data factor transmitted from R

    PARAMETER_VECTOR(u);      // Random effects

    // Parameters
    PARAMETER_VECTOR(lambda); // Parameter value transmitted from R
    PARAMETER(log_sigma_u);   // Parameter value transmitted from R

    Type sigma_u = exp(log_sigma_u);

    int nobs = y.size();
    Type mean_ran = Type(0);

    int j;

    Type f = 0;               // Declare the "objective function" (neg. log. likelihood)
    for(int i=0; i < nobs; i++){
        f -= dnorm(u[i],mean_ran,sigma_u,true);
        j = ageGroup[i];
        f -= dpois(y[i],exp(log(lambda[j])-log(w[j]))*exp(u[i]),true);
    }

    return f;
}

```

Hierachical Poisson-Normal model

Implementation - Call from R

```
# Import libraries
library(readr)
library(dplyr)
library(TMB)

# Import the data
dat <- read_rds(file = "../data/processed/dat.rds")

# Only consider some of the data
y <- dat %>%
  filter(caseDef == "Shiga- og veratotoxin producerende E. coli.") %>%
  group_by(Date, ageGroup) %>%
  mutate(y = sum(cases)) %>%
  select(Date, ageGroup, y, n)

compile(file = "PoissonLognormal.cpp") # Compile the C++ file
dyn.load(dynlib("PoissonLognormal"))  # Dynamically link the C++ code

# Function and derivative
PoisLN <- MakeADFun(
  data = list(y = y$, ageGroup = y$ageGroup, w = y$n),
  parameters = list(u = rep(1, length(y$y)),
                    lambda = rep(1, nlevels(y$ageGroup)),
                    log_sigma_u = log(1)),
  random = "u",
  DLL = "PoissonLognormal"
)

opt <- nlminb(start = PoisLN$par, PoisLN$fn, PoisLN$gr, lower = c(0.01, 0.01))
```

Hierarchical Poisson-Normal model

Results

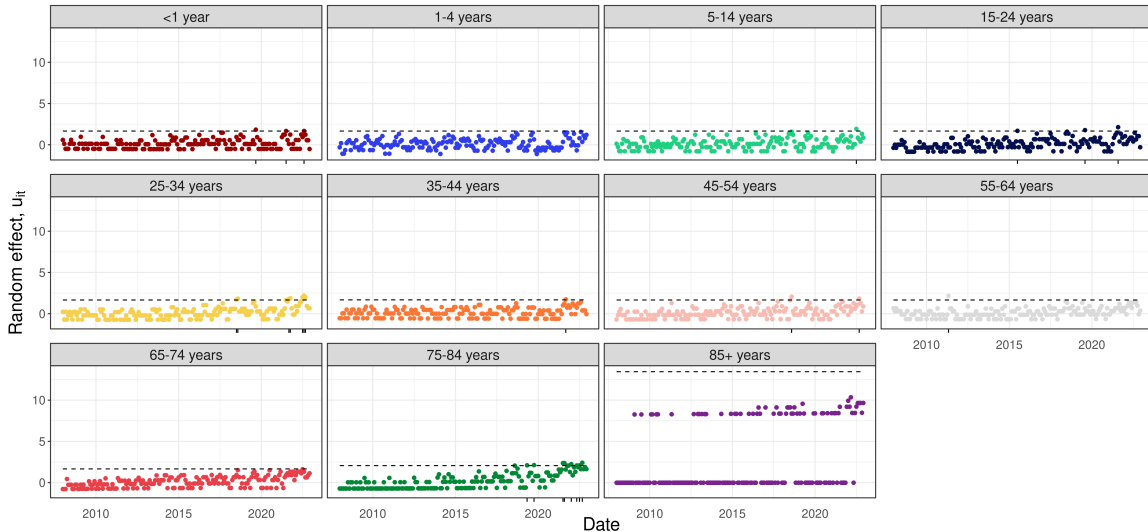
ageGroup	λ	σ
<1 year	2.38	1.01
1-4 years	2.61	1.01
5-14 years	2.63	1.01
15-24 years	2.64	1.01
25-34 years	2.62	1.01
35-44 years	2.60	1.02
45-54 years	2.63	1.01
55-64 years	2.62	1.01
65-74 years	2.61	1.01
75-84 years	2.52	1.25
85+ years	1.16	8.18

Hierarchical Poisson-Normal model

Results

Shiga- og veratoxin producerende E. coli.

Hierarchical Poisson Normal model



$$Y_{it}|u_{it} \sim \text{Pois}(\lambda_i u_{it}) \quad (2a)$$

$$u_{it} \sim G(1/\beta_i, \beta_i) \quad (2b)$$

$$\begin{aligned} P[Y = y] &= g_Y(y; \lambda, \beta) \\ &= \frac{\lambda^y}{y! \Gamma(1/\beta) \beta^{1/\beta}} \frac{\beta^{y+1/\beta} \Gamma(y + 1/\beta)}{(\lambda\beta + 1)^{y+1/\beta}} \\ &= \frac{\Gamma(y + 1/\beta)}{\Gamma(1/\beta) y!} \frac{1}{(\lambda\beta + 1)^{1/\beta}} \left(\frac{\lambda\beta}{\lambda\beta + 1} \right)^y \\ &= \binom{y + 1/\beta - 1}{y} \frac{1}{(\lambda\beta + 1)^{1/\beta}} \left(\frac{\lambda\beta}{\lambda\beta + 1} \right)^y, \text{ for } y = 0, 1, 2, \dots \end{aligned} \quad (3)$$

where we have used the convention

$$\binom{z}{y} = \frac{\Gamma(z + 1)}{\Gamma(z + 1 - y) y!} \quad (4)$$

The marginal distribution of Y is a negative binomial distribution, $Y \sim \text{NB}(1/\beta, 1/(\lambda\beta + 1))$

The probability function for the conditional distribution of Y for given u

$$f_{Y|u}(y; \lambda, u) = \frac{(\lambda u)^y}{y!} \exp(-\lambda u) \quad (5)$$

and the probability density function for the distribution of u is

$$f_u(u; \beta) = \frac{1}{\beta \Gamma(1/\beta)} \left(\frac{u}{\beta} \right)^{1/\beta-1} \exp(-u/\beta) \quad (6)$$

Given (5) and (6), the probability function for the marginal distribution of Y is determined from

$$\begin{aligned} g_Y(y; \lambda, \beta) &= \int_{u=0}^{\infty} f_{Y|u}(y; \lambda, u) f_u(u; \beta) du \\ &= \int_{u=0}^{\infty} \frac{(\lambda u)^y}{y!} \exp(-\lambda u) \frac{1}{\beta \Gamma(1/\beta)} \left(\frac{u}{\beta}\right)^{1/\beta-1} \exp(-u/\beta) du \\ &= \frac{\lambda^y}{y! \Gamma(1/\beta) \beta^{1/\beta}} \int_{u=0}^{\infty} u^{y+1/\beta} \exp(-u(\lambda\beta + 1)/\beta) du \end{aligned} \quad (7)$$

In (7) it is noted that the integrand is the *kernel* in the probability density function for a Gamma distribution, $G(y + 1/\beta, \beta/(\lambda\beta + 1))$. As the integral of the density shall equal one, we find by adjusting the norming constant that

$$\int_{u=0}^{\infty} u^{y+1/\beta} \exp\left(-u/(\beta/(\lambda\beta + 1))\right) du = \frac{\beta^{y+1/\beta} \Gamma(y + 1/\beta)}{(\lambda\beta + 1)^{y+1/\beta}} \quad (8)$$

and then (3) follows

Consider the hierarchical Poisson-Gamma model in (2), and assume that a value $Y = y$ has been observed. Then the conditional distribution of u for given $Y = y$ is a Gamma distribution,

$$u|Y = y \sim G(y + 1/\beta, \beta/(\lambda\beta + 1)) \quad (9)$$

with mean

$$E[u|Y = y] = \frac{y\beta + 1}{\lambda\beta + 1} \quad (10)$$

and variance

$$V[u|Y = y] = \frac{(y\beta^2 + \beta)}{(\lambda\beta + 1)^2} \quad (11)$$

The conditional distribution is found using Bayes Theorem

$$\begin{aligned}
 g_u(u|Y = y) &= \frac{f_{y,u}(y, u)}{g_Y(y; \lambda, \phi)} \\
 &= \frac{f_{y|u}(y; u)g_u(u)}{g_Y(y; \lambda, \beta)} \\
 &= \frac{1}{g_Y(y; \lambda, \beta)} \left(\frac{(\lambda u)^y}{y!} \exp(-\lambda u) \frac{1}{\beta \Gamma(1/\beta)} \left(\frac{u}{\beta} \right)^{1/\beta-1} \exp(-u/\beta) \right) \\
 &\propto u^{y+1/\beta-1} \exp(-u(\lambda\beta + 1)/\beta)
 \end{aligned} \tag{12}$$

We identify the *kernel* of the probability density function

$$u^{y+1/\beta-1} \exp(-u(\lambda\beta + 1)/\beta) \tag{13}$$

as the kernel of a Gamma distribution, $G(y + 1/\beta, \beta/(\lambda\beta + 1))$

Implementation - Define negative likelihood function in R

```
# Define the negative likelihood function for the marginal distribution of Y
nll.age <- function(theta, data){
  # Extract counts
  y <- data$y
  # Extract agegroups
  ageGroup <- data$ageGroup
  # Extract number of agegroups
  n.ageGroup <- n_distinct(data$ageGroup)

  # Define parameters
  lambda <- theta[1:n.ageGroup]
  beta <- theta[(n.ageGroup+1):(n.ageGroup*2)]

  # Construct the size and probability for the negative binomial distribution
  r <- 1/beta
  p <- 1/(lambda*beta+1)

  # Initilize the log-likelihood
  ll <- 0
  for(i in 1:nrow(data)){
    ll = ll + dnbinom(x = y[i],
                     size = r[ageGroup[i]],
                     prob = p[ageGroup[i]],
                     log = TRUE)
  }

  # Return the negative log-likelihood
  -ll
}
```

Hierarchical Poisson-Gamma model

Results

ageGroup	λ	β
<1 year	1.24	0.33
1-4 years	4.57	0.36
5-14 years	2.47	0.42
15-24 years	2.57	0.48
25-34 years	2.08	0.57
35-44 years	1.47	0.20
45-54 years	1.88	0.39
55-64 years	1.77	0.25
65-74 years	1.89	0.40
75-84 years	1.45	1.08
85+ years	0.50	0.97

Hierarchical Poisson-Gamma model

Results

Shiga- og veratoxin producerende E. coli.

Compound Poisson Gamma model

