Automated and Early Detection of Disease Outbreaks

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Section for Dynamical Systems

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  - Shiga- and verotoxin producing E. coli.
- State-of-the-art
  - Farrington
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- Hierarchical models
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- Comparison of methods
- References
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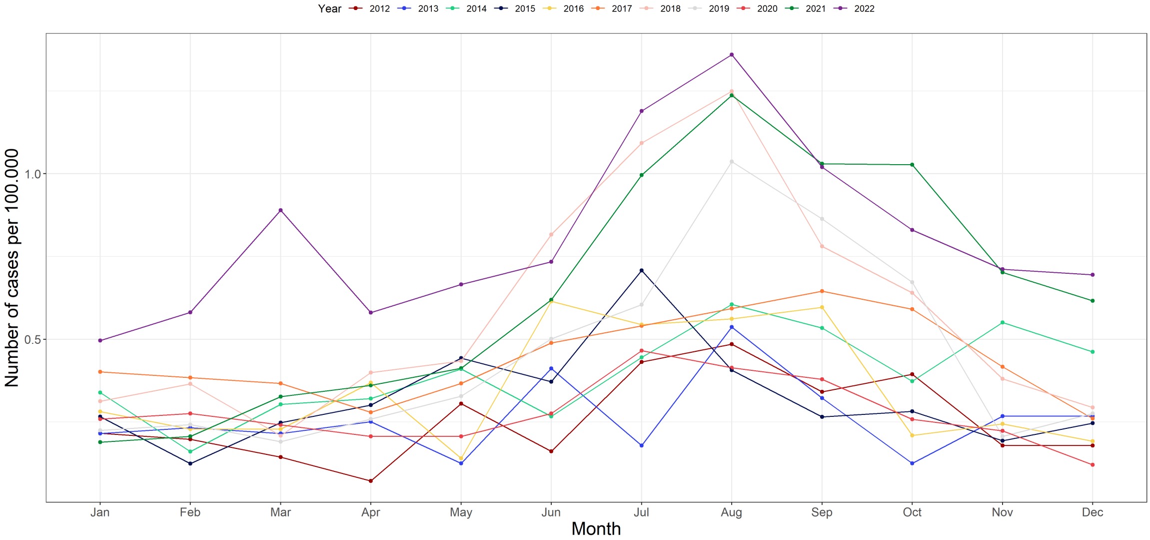
Shiga- and verotoxin producing E. coli.

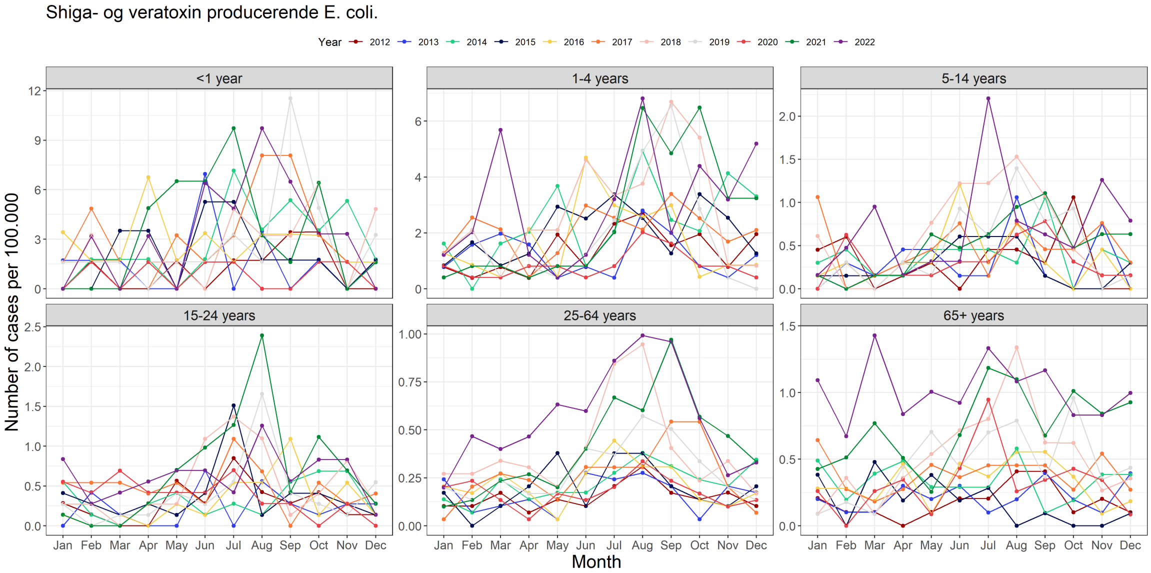
Date	ageGroup	y_{it}	x_{it}
2008-01-01	<1 year	2	64137
2008-01-01	1-4 years	2	259910
2008-01-01	5-14 years	2	680529
2008-01-01	15-24 years	1	635838
...	...	...	...
2022-12-01	5-14 years	5	634139
2022-12-01	15-24 years	1	721286
2022-12-01	25-64 years	10	3031374
2022-12-01	65+ years	12	1204892

Shiga- and verotoxin producing E. coli.

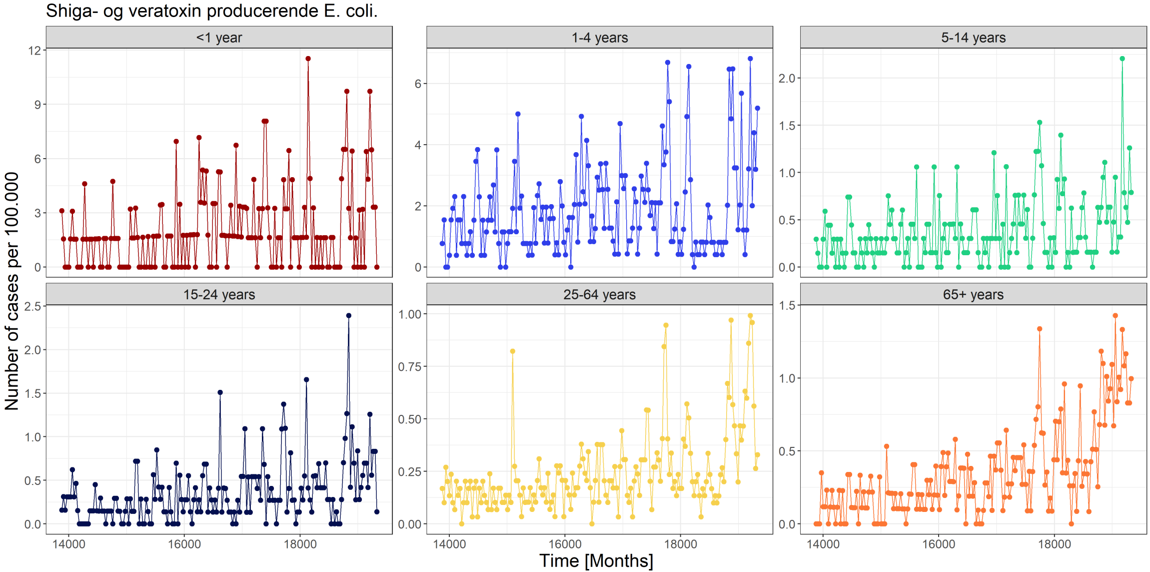
Shiga- og veratoxin producerende E. coli.



Shiga- and verotoxin producing E. coli.



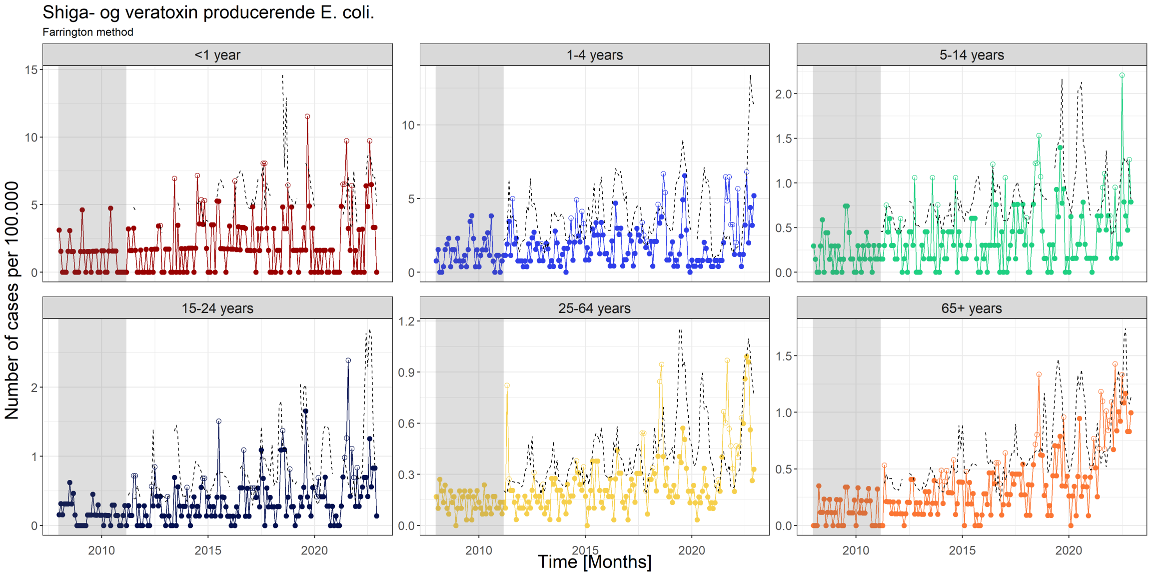
Shiga- and verotoxin producing E. coli.



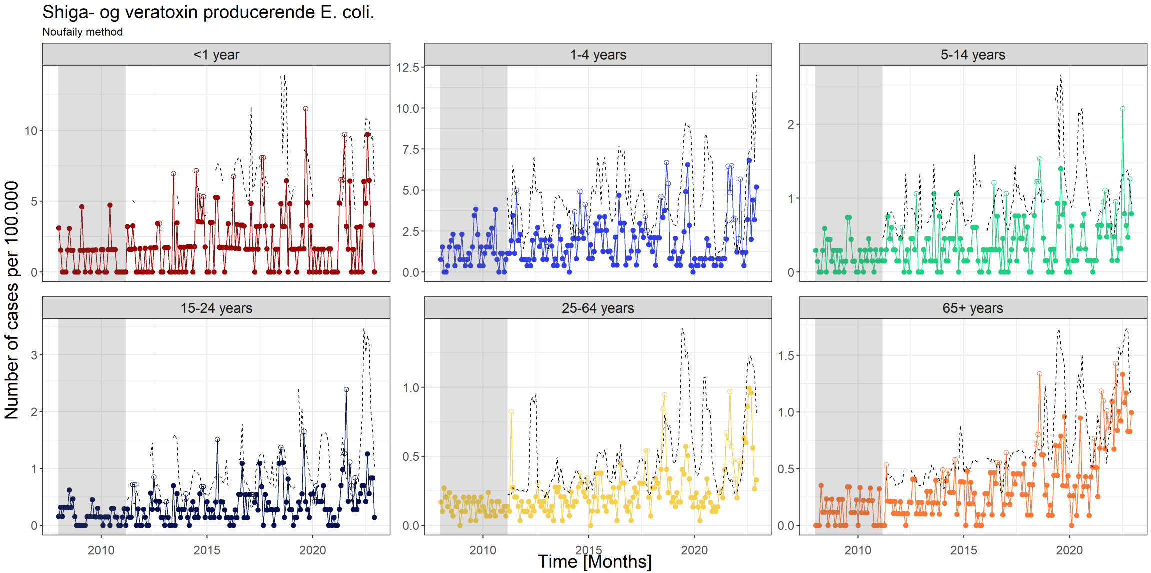
State-of-the-art

State-of-the-art methods for aberration detection is presented in Salmon, Schumacher, and Höhle 2016 and implemented in the R package **surveillance**. The R package includes methods such as the Farrington method introduced by Farrington et al. 1996 together with the improvements proposed by Noufaily et al. 2013.

Farrington


Shiga- og veratoxin producerende E. coli.
Farrington method
<1 year
1-4 years
5-14 years
15-24 years
25-64 years
65+ years
Number of cases per 100.000
Time [Months]
2010
2015
2020

Noufaily


Shiga- og veratoxin producerende E. coli.
Noufaily method
<1 year
1-4 years
5-14 years
15-24 years
25-64 years
65+ years
Number of cases per 100.000
Time [Months]
2010
2015
2020
0
5
10
0.0
2.5
5.0
7.5
10.0
12.5
1
2
3
0.5
1.0
1.5

Hierachical Poisson-Normal model

The conditional distribution, $Y|u$, of the count observations are assumed to be a Poisson distribution with intensities λ_{it}. Also, we shall assume that the count is proportional to the population size, x_{it}, within each age group, i, at a given time point, t. Hence, in terms of the canonical link for the Poisson distribution the model for the fixed effect is

$$\log(\lambda_{it}) = \mathbf{X}_i^T \beta_{it} + \log(x_{it}) \tag{1}$$

Here $\mathbf{X}_i$ is $T \times 6$-dimensional, and β_{it} contains the corresponding fixed effect parameter. The random effects u_{it} are assumed to be Gaussian.

$$u_{it} = \epsilon_{it} \tag{2}$$

where $\epsilon_{it} \sim \mathrm{N}(0, \sigma^2)$ is a white noise process, and σ is a model parameter.

Formulation

Henceforth, the model can be formulated as a two-level hierarchical model

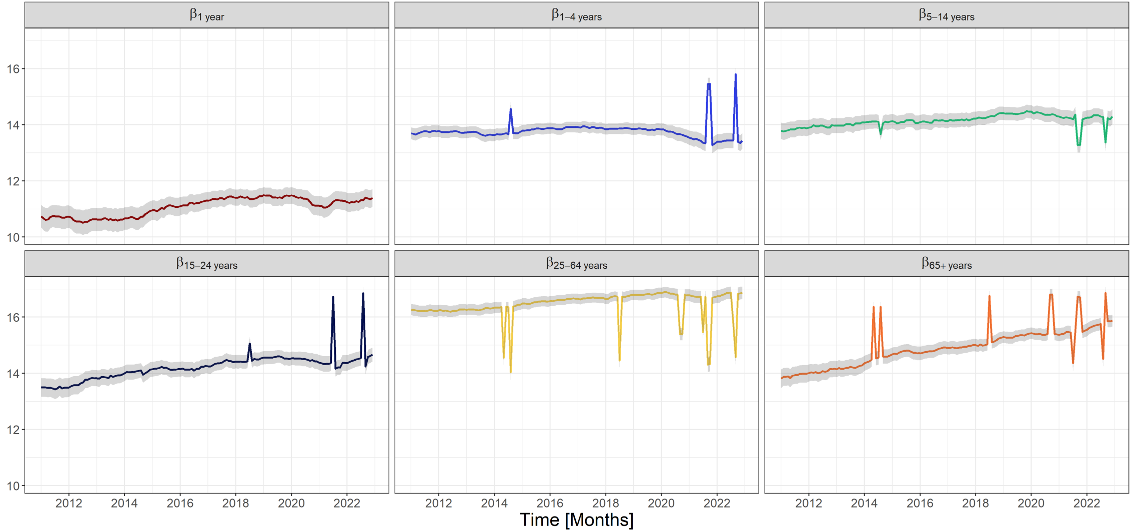
$$Y_{it}|u_{it} \sim \mathrm{Pois}(\lambda_{it} e^{u_{it}}) \quad (3a)$$

$$u_{it} \sim \mathrm{N}(0, \sigma^2) \quad (3b)$$

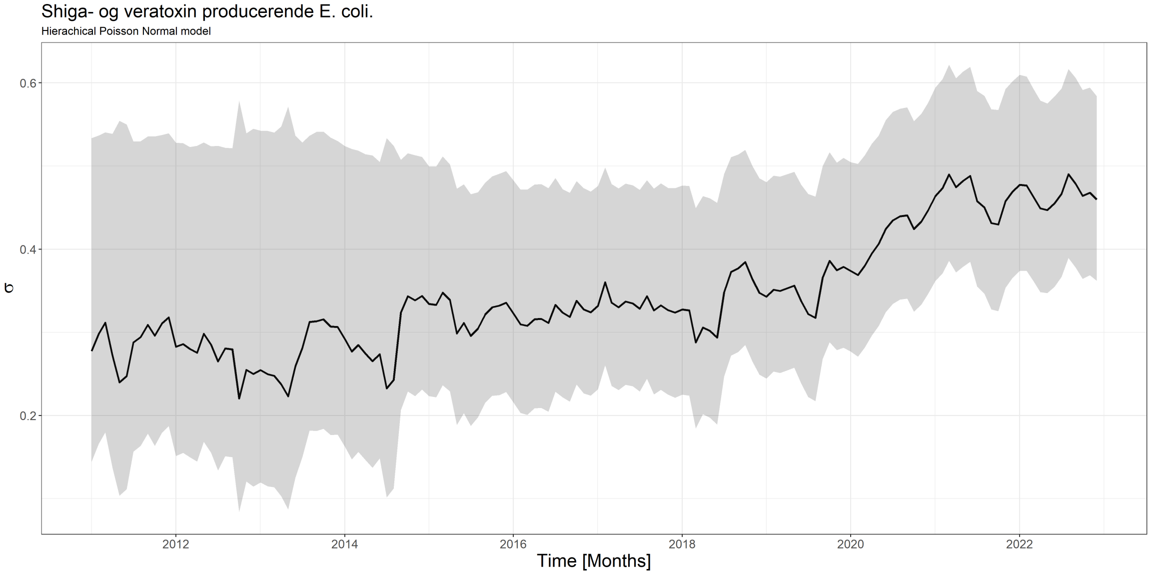
Results

Shiga- og veratoxin producerende E. coli.

Hierachical Poisson Normal model



Results


Shiga- og veratoxin producerende E. coli.
Hierachical Poisson Normal model
0.6
0.4
0.2
σ
2012
2014
2016
2018
2020
2022
Time [Months]

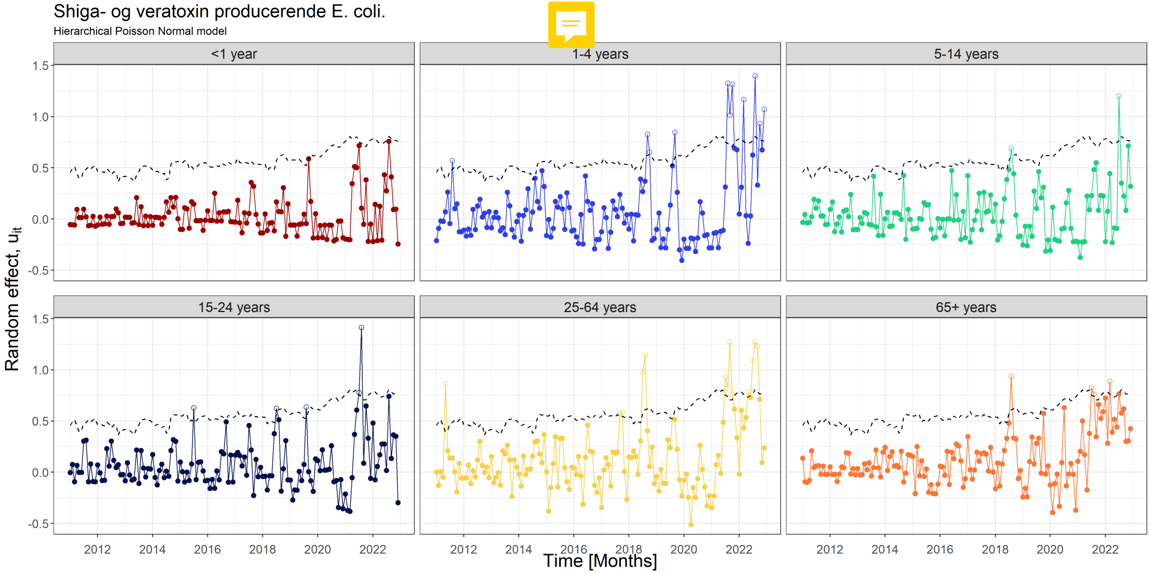
Threshold calculation

The critical value, C_α, is computed from the $1-\alpha$-quantile of the Normal distribution with the maximum-likelihood estimate for the variance, $\hat{\sigma}$.

$$C_\alpha = \mathrm{N}(0, \hat{\sigma}^2)_{1-\alpha} \tag{4}$$

NOTE: For this presentation a default of $\alpha = 0.05$ is used.

Results


Shiga- og veratoxin producerende E. coli.
Hierarchical Poisson Normal model
<1 year
1-4 years
5-14 years
15-24 years
25-64 years
65+ years
Random effect, u_{it}
Time [Months]
1.5
1.0
0.5
0.0
-0.5
2012
2014
2016
2018
2020
2022

Hierachical Poisson-Gamma model

Likewise, in the compound Poisson-Gamma model the conditional distribution, $Y|u$, of the count observations are assumed to be a Poisson distribution, but this time the intensities, λ_{it}, are defined as

$$\log(\lambda_{it}) = \mathbf{X}_i^T \beta_{it} + \log(x_{it}) \tag{5}$$

Here $\mathbf{X}_i$ is $T \times 6$-dimensional, and β_{it} contains the corresponding fixed effect parameter. Additionally, the random effects u_{it} are assumed to follow a reparametrized Gamma distribution with a mean 1.

Formulation

Subsequently, the model can be formulated as a two-level hierarchical model

$$Y_{it}|u_{it} \sim \mathrm{Pois}(\lambda_{it}u_{it}) \tag{6a}$$

$$u_{it} \sim \mathrm{G}(1/\phi, \phi) \tag{6b}$$

Probability function for Y

$$
\begin{aligned}
P[Y = y_i] &= g_Y(y; \lambda, \phi) \\
&= \frac{\lambda^y}{y!\Gamma(1/\phi)\phi^{1/\phi}} \frac{\phi^{y+1/\phi}\Gamma(y+1/\phi)}{(\lambda\phi+1)^{y+1/\phi}} \\
&= \frac{\Gamma(y+1/\phi)}{\Gamma(1/\phi)y!} \frac{1}{(\lambda\phi+1)^{1/\phi}} \left(\frac{\lambda\phi}{\lambda\phi+1}\right)^y \\
&= \binom{y+1/\phi-1}{y} \frac{1}{(\lambda\phi+1)^{1/\phi}} \left(\frac{\lambda\phi}{\lambda\phi+1}\right)^y \;, \; for\; y = 0, 1, 2, \ldots
\end{aligned} \tag{7}
$$

where we have used the convention

$$
\binom{z}{y} = \frac{\Gamma(z+1)}{\Gamma(z+1-y)y!} \tag{8}
$$

The marginal distribution of Y is a negative binomial distribution, $Y \sim \mathrm{NB}\left(1/\phi, 1/(\lambda\phi+1)\right)$

Inference on individual group means

Consider the hierarchical Poisson-Gamma model in (6), and assume that a value $Y = y$ has been observed. Then the conditional distribution of u for given $Y = y$ is a Gamma distribution,

$$u|Y = y \sim \mathrm{G}\left(y + 1/\phi, \phi/(\lambda\phi + 1)\right) \tag{9}$$

with mean

$$\mathrm{E}[u|Y = y] = \frac{y\phi + 1}{\lambda\phi + 1} \tag{10}$$

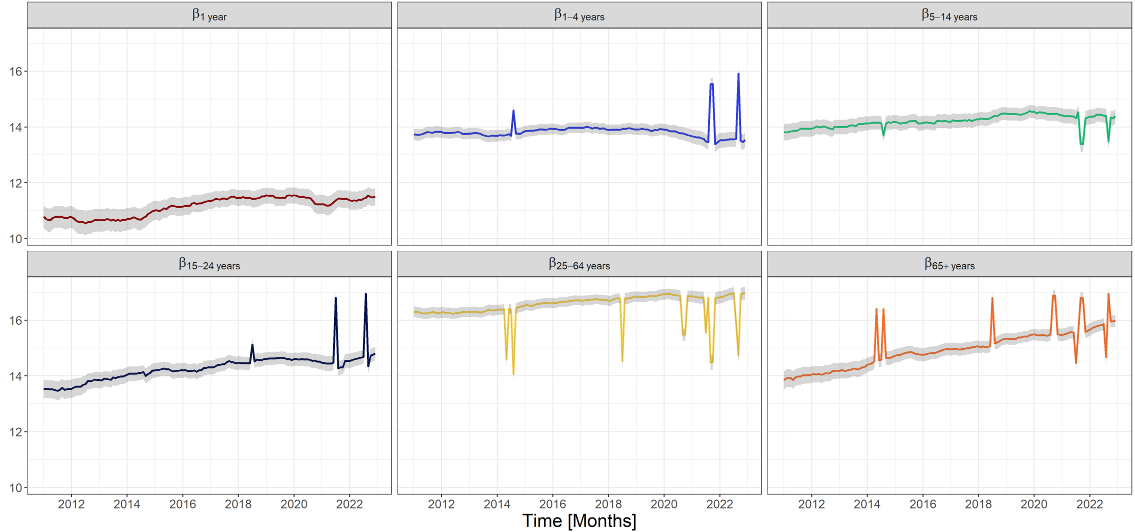
and variance

$$\mathrm{V}[u|Y = y] = \frac{(y\phi^2 + \phi)}{(\lambda\phi + 1)^2} \tag{11}$$

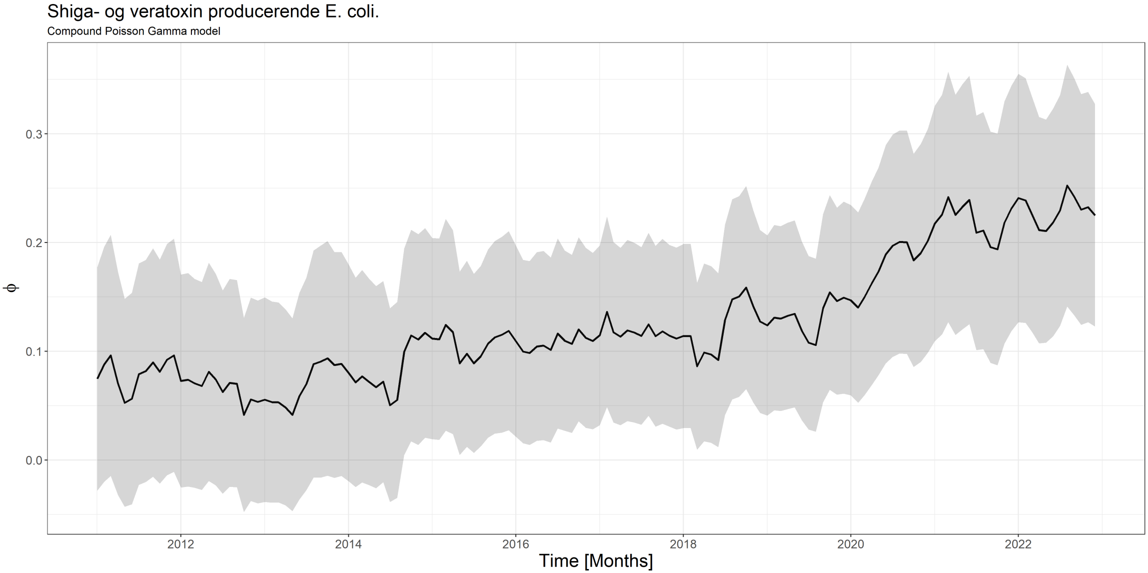
Results

Shiga- og veratoxin producerende E. coli.

Compound Poisson Gamma model



Results

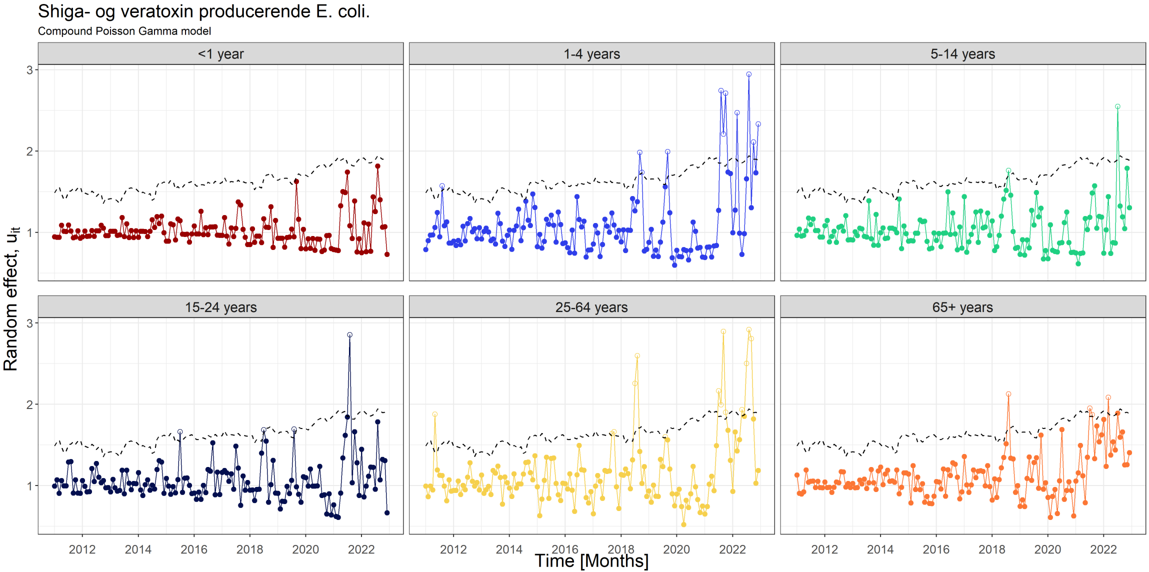

Shiga- og veratoxin producerende E. coli.
Compound Poisson Gamma model
0.3
0.2
0.1
0.0
φ
2012
2014
2016
2018
2020
2022
Time [Months]

Threshold calculation

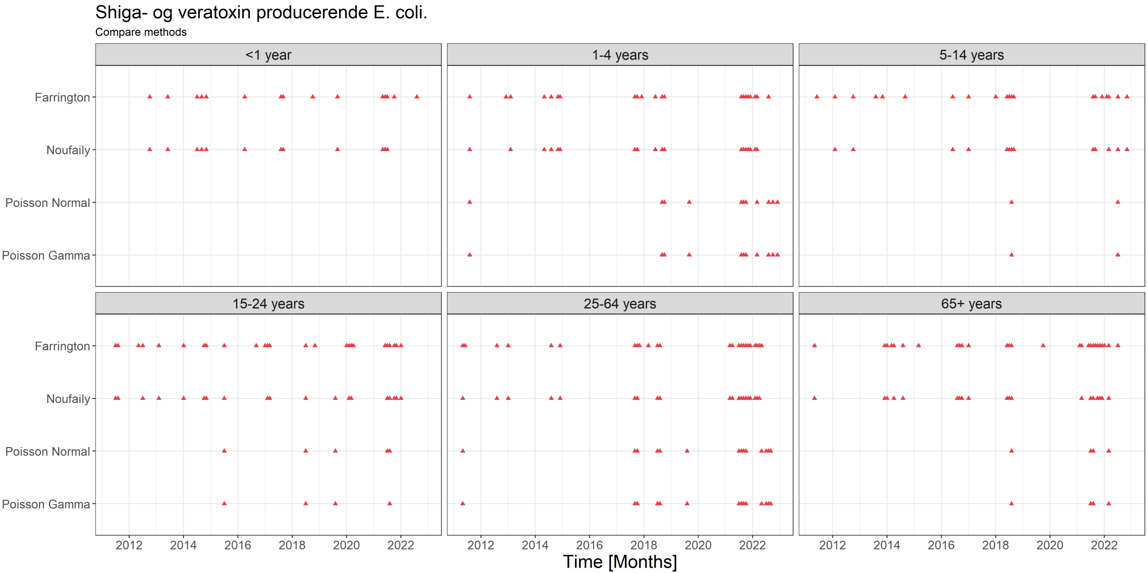
The critical value, C_α, is computed from the $1-\alpha$-quantile of the reparametrized Gamma distribution with the maximum-likelihood estimate for the variance, $\hat{\phi}$.

$$C_\alpha = \mathrm{G}(1/\hat{\phi}, \hat{\phi})_{1-\alpha} \tag{12}$$

Results


Shiga- og veratoxin producerende E. coli.
Compound Poisson Gamma model
<1 year
1-4 years
5-14 years
15-24 years
25-64 years
65+ years
Random effect, u_{it}
Time [Months]
1
2
3
2012
2014
2016
2018
2020
2022

Comparison of methods



References

Farrington, C. P. et al. (1996). "A Statistical Algorithm for the Early Detection of Outbreaks of Infectious Disease". In: *Journal of the Royal Statistical Society. Series A (Statistics in Society)* 159.3, pp. 547–563. ISSN: 09641998, 1467985X. URL: http://www.jstor.org/stable/2983331 (visited on 01/27/2023).

Noufaily, Angela et al. (2013). "An Improved Algorithm for Outbreak Detection in Multiple Surveillance Systems". en. In: *Online Journal of Public Health Informatics* 32.7, pp. 1206–1222.

Salmon, Maëlle, Dirk Schumacher, and Michael Höhle (2016). "Monitoring Count Time Series in R: Aberration Detection in Public Health Surveillance". In: *Journal of Statistical Software* 70.10, pp. 1–35. DOI: 10.18637/jss.v070.i10. URL: https://www.jstatsoft.org/index.php/jss/article/view/v070i10.

Proof - Probability function for Y

The probability function for the conditional distribution of Y for given u

$$f_{Y|u}(y;\lambda,u) = \frac{(\lambda u)^y}{y!}\exp(-\lambda u) \tag{13}$$

and the probability density function for the distribution of u is

$$f_u(u;\phi) = \frac{1}{\phi\Gamma(1/\phi)}\left(\frac{u}{\phi}\right)^{1/\phi-1}\exp(-u/\phi) \tag{14}$$

Proof - Probability function for Y

Given (13) and (14), the probability function for the marginal distribution of Y is determined from

$$
\begin{aligned}
g_Y(y;\lambda,\phi) &= \int_{u=0}^{\infty} f_{Y|u}(y;\lambda,u) f_u(u;\phi)\,du \\
&= \int_{u=0}^{\infty} \frac{(\lambda u)^y}{y!} \exp(-\lambda u) \frac{1}{\phi\Gamma(1/\phi)} \left(\frac{u}{\phi}\right)^{1/\phi-1} \exp(-u/\phi)\,du \\
&= \frac{\lambda^y}{y!\Gamma(1/\phi)\phi^{1/\phi}} \int_{u=0}^{\infty} u^{y+1/\phi-1} \exp\big(-u(\lambda\phi+1)/\phi\big)\,du
\end{aligned}
\tag{15}
$$

Proof - Probability function for Y

In (15) it is noted that the integrand is the *kernel* in the probability density function for a Gamma distribution, $\mathrm{G}\left(y+1/\phi, \phi/(\lambda\phi+1)\right)$. As the integral of the density shall equal one, we find by adjusting the norming constant that

$$\int_{u=0}^{\infty} u^{y+1/\phi-1} \exp\left(-u\Big/\Big(\phi/(\lambda\phi+1)\Big)\right) du = \frac{\phi^{y+1/\phi}\Gamma(y+1/\phi)}{(\lambda\phi+1)^{y+1/\phi}} \tag{16}$$

and then (7) follows

Proof - Conditional distribution of Y

The conditional distribution is found using Bayes Theorem

$$
\begin{aligned}
g_u(u|Y=y) &= \frac{f_{y,u}(y,u)}{g_Y(y;\lambda,\phi)} \\
&= \frac{f_{y|u}(y;u)g_u(u)}{g_Y(y;\lambda,\phi)} \\
&= \frac{1}{g_Y(y;\lambda,\phi)}\left(\frac{(\lambda u)^y}{y!}\exp(-\lambda u)\frac{1}{\phi\Gamma(1/\phi)}\left(\frac{u}{\phi}\right)^{1/\phi-1}\exp(-u/\phi)\right) \\
&\propto u^{y+1/\phi-1}\exp\left(-u(\lambda\phi+1)/\phi\right)
\end{aligned}
\tag{17}
$$

We identify the *kernel* of the probability density function

$$
u^{y+1/\phi-1}\exp(-u(\lambda\phi+1)/\phi) \tag{18}
$$

as the kernel of a Gamma distribution, $\mathrm{G}(y+1/\phi, \phi/(\lambda\phi+1))$