Project 3 - FYS3150*

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An abstract

I. INTRODUCTION

An introduction.

II. THEORY AND METHODS

A. Newton's law of gravitation

Newton's law of gravitation states for two objects of mass m_1 and m_2, the force on object 1 from object 2 is given by [1],

$$\boldsymbol{F_{1,2}} = \frac{Gm_1m_2}{r^2}\boldsymbol{u_r} = \frac{Gm_1m_2}{r^3}\boldsymbol{r} \tag{1}$$

where G is the gravitational constant and $\boldsymbol{u_r} = \boldsymbol{r}/r$ is a radial unit vector. $\boldsymbol{r}$ is a radial vector pointing at object 2 and $r = |\boldsymbol{r}|$ is the distance. Newton's third law gives us that the force on object 2 from object 1 is $\boldsymbol{F_{2,1}} = -\boldsymbol{F_{1,2}}$. Newton's third law gives us the differential equation governing the motion of object 1

$$\ddot{\boldsymbol{r(t)}} = \boldsymbol{a(t)} = \boldsymbol{F_{1,2}}(t, \boldsymbol{r(t)})/m_1, \tag{2}$$

where $\boldsymbol{a}$ is the acceleration, and we can solve this equation to find the motion $\boldsymbol{r(t)}$. For a two-body system this equation will produce closed elliptical orbits around a common center of mass.
If we assume that the orbit of object 2 around object 1 is circular we know that the force obeys the following equation

$$F_{2,1} = \frac{Gm_1m_2}{r^2} = \frac{m_1v_1^2}{r}, \tag{3}$$

which implies that

$$v_1^2 r = Gm_2. \tag{4}$$

Introducing 1 AU $= 1.5 \cdot 10^{11}$m; the distance from the Earth to the sun, we let object 1 be the Sun and object 2 the Earth, and we obtain

$$v_{earth}^2 r = GM_\odot = 4\pi^2 AU^3/yr^2, \tag{5}$$

where I have used that the circular velocity of the Earth is $v_{earth} = 2\pi r/T = 2\pi AU/yr$. This lets us rewrite our differential equation in a more natural length scale for the solar system. For a solar system with 1 sun and 9 planets (including Pluto) we may write the differential equation governing the motion of each celestial object i as

$$\begin{aligned}\ddot{\boldsymbol{r}}_{\boldsymbol{i}} = \frac{\boldsymbol{F_{tot}}}{M_i} &= \sum_{j\neq i}\frac{\boldsymbol{F_{i,j}}}{M_i} \\ &= -\sum_{j\neq i}\frac{4\pi^2 M_j}{r^3}\boldsymbol{r_{i,j}} \quad ,\end{aligned} \tag{6}$$

where the acceleration $\boldsymbol{a} = \ddot{\boldsymbol{r}}$ is measured in units of $4\pi^2 AU/yr^2$. The minus sign stems from the fact that the position vectors point in the opposite direction of the force vectors. We may also measure the mass of each celestial object by solar mass, by setting $M_\odot = 1$.

B. Ordinary differential equations

We discretize the differential equation by setting the number of time steps n, the number of years we want to evolve the system y and the length of each time step $\Delta t = y/n$. The simplest way of solving equation (6) numerically is Euler's method [2]:

$$\begin{aligned}&\boldsymbol{v_{i+1}} = \boldsymbol{v_i} + \Delta t \cdot \boldsymbol{F_{tot}/M_i} \qquad i = 0, 1, 2, .., n-1 \\ &\boldsymbol{x_{i+1}} = \boldsymbol{x_i} + \Delta t \cdot \boldsymbol{v_i}\end{aligned} \tag{7}$$

A significant improvement would be the Velocity Verlet algorithm. For a second order differential equation of the form

$$\frac{d^2}{dt^2}\boldsymbol{r} = \boldsymbol{F}(\boldsymbol{r}, t), \tag{8}$$

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(such as ours!) we may rewrite our differential equations as:

$$\begin{aligned} \boldsymbol{r_{i+1}} &= \boldsymbol{r_i} + \Delta t \cdot \boldsymbol{v_i} + \frac{(\Delta t)^2}{2}\boldsymbol{a_i} \\ \boldsymbol{v_{i+1}} &= \boldsymbol{v_i} + \frac{\Delta t}{2}(\boldsymbol{a_{i+1}} + \boldsymbol{a_i}). \end{aligned} \tag{9}$$

Note that the velocity depends on the acceleration at time t_{i+1}. This is a function of $\boldsymbol{r_{i+1}}$ which needs to be calculated first. The error in this method goes like $\mathcal{O}(\Delta t^3)$ locally, which means we get a global error $\mathcal{O}(\Delta t^2)$, compared to Euler's method which has a total error $\mathcal{O}(\Delta t)$.

C. Conservation laws

In Newtonian mechanics, for a system of objects without any external forces we have three fundamental conservations laws: The conservation of energy, the conservation of momentum and the conservation of angular momentum.
For the solar system, if we neglect any forces other than the pure Newtonian gravitational force (and the celestial objects are taken to be point masses) we may write the total energy as

$$E_{tot} = \sum_{i \neq j} \frac{1}{2} m_i v_i^2 - \frac{G m_i m_j}{r_{i,j}}, \tag{10}$$

This is a conserved, or time invariant quantity; that is to say it must be the same in our calculations for all times t_i.
The total momentum of the system (no external forces applied) is also a conserved quantity and can be given as

$$\boldsymbol{p_{tot}} = \sum_i m_i \boldsymbol{v_i}. \tag{11}$$

If we wish to study our system from the gravitational center of mass at the origin with $\boldsymbol{p_{tot} = 0}$ we require

$$\begin{aligned} m_1 \boldsymbol{r_1} + ... + m_N \boldsymbol{r_N} &= \boldsymbol{0} \\ m_1 \boldsymbol{v_1} + ... + m_N \boldsymbol{v_N} &= \boldsymbol{0}. \end{aligned} \tag{12}$$

This gives us $N-1$ degrees of freedom to choose the initial positions and velocitites. We can then fullfill these conditions by requiring the Sun to have an initial position and velocity

$$\begin{aligned} \boldsymbol{r_\odot} &= \frac{1}{M_\odot}(m_2 \boldsymbol{r_2} + ... + m_N \boldsymbol{r_N}) \\ \boldsymbol{v_\odot} &= \frac{1}{M_\odot}(m_2 \boldsymbol{v_2} + ... + m_N \boldsymbol{v_N}). \end{aligned} \tag{13}$$

As Newton's second law gives us conservation of momentum, a rotational analog of Newton's second law gives us conservation of angular momentum. For a system unaffected by external torque, the total angular momentum is conserved.

$$\boldsymbol{L_{tot}} = \sum_i \boldsymbol{r_i} \times \boldsymbol{v_i} \tag{14}$$

Note that momentum, energy and angular momentum will constantly be exchanged between the objects of the system via the gravitational forces, and these conservation laws are only valid when examining the system as a whole. If we didn't model the celestial objects as point particles there could also be several extra degrees of freedom in which energy and momentum could be distributed, but this is negligeble for our purposes.

D. Celestial mechanics

For a planet in orbit around a much more massive celestial object of mass M, we may find the escape velocity of the planet by finding the minimum kinetic energy required to move the planet infinitely far away. We thus acquire an analytical expression:

$$v_{esc} = \sqrt{\frac{2GM}{r}}. \tag{15}$$

When considering the Earth-Sun system this is simply $v_{esc} = \sqrt{8\pi^2}$, when expressed in AU/yr.

For a planet (mass m) in uniform circular orbit around a much more massive object of mass M we have that the distance between the objects r = constant. From equation (10) this implies that the potential energy is a conserved quantity. By studying for example the expression for the centripetal acceleration $a = F(r)/m = v^2/r$ it is easy to see that the speed must be constant and therefore the kinetic energy must be conserved.
From vector calculus we find that $\boldsymbol{r} \perp \boldsymbol{v}$, and from equation (14) we obtain conservation of angular momentum. As the velocity is constantly changing, the vector momentum $m\boldsymbol{v}$ is not conserved. The scalar quantity mv is however (as speed is constant).
The speed $v = |\boldsymbol{v}|$ is easily obtained for the Earth-Sun system, $v = 2\pi r/T = 2\pi AU/yr$.

E. Unit testing

Text

III. RESULTS AND DISCUSSION

A. First subpart

An equation reference

B. Second subpart

More text.

IV. CONCLUSION

Do stuff.

V. APPENDIX

All code used is available at: The programs used in this project are listed in this section:

main.cpp: Program1

plot.py: Program2

[1] All theory in this project adapted from FYS3150 Project 3 (Fall 2016) *link*.

[2] Adapted from FYS3150 Lecture Notes, Ordinary Differential Equation (Fall 2016) *link*.