Solutions to final exam in FYS4160

Problem 1

a) The *weak equivalence principle* is the universality of free fall, i.e. the statement that the mass m appearing in Newton's force law and in Newtons's gravitational law is identical. Possible test: various test masses should fall with identical speeds (in the absence of any non-gravitational forces).
The *Einstein equivalence principle* states that the outcome of *any local* non-gravitational experiment in a freely falling laboratory is independent of the velocity of the laboratory, and of its location in spacetime. Possible test: Measuring the value of $\alpha_{\rm em}$ (through atomic transition lines) in various regions of the universe.
The *strong equivalence principle* is the same as the Einstein EP, but includes gravitational experiments; gravitational binding energy, in particular, thus gravitates in exactly the same way as any other form of mass/energy. Possible test: Check that a body with significant gravitational binding energy falls in the same way as if the binding energy was different, e.g. nuclear, in nature (in practice, one of the best limits derive from looking at the earth-moon system).

b) The equivalence principle states that every test particle of mass m is affected by gravity in exactly the same way, independent of its composition or other properties. This universality leads to the idea that gravity should not be described as a conventional force, but rather as a fundamental feature of the *background* on which test particles propagate – i.e. space-time itself. The interpretation of space-time being curved is strongly supported by the fact that there exist locally inertial coordinates on every (smooth, Riemannian) manifold, corresponding to the *local* undetectability of gravity (which also follows from the equivalence principle).

c) The motion of a test particle in *both* special (SR) and general relativity (GR) is given by the geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}=f^\mu\,, \tag{1}$$

where τ is the eigentime (or an affine parameter) and $f^\mu=0$ if there is no external force. The *difference* is that in SR the metric is always that of flat spacetime (though it might be expressed in 'strange' coordinates, such that it does not take the form of the Minkowski metric). In SR, furthermore, gravity would appear as an external force with $f^\mu\neq 0$ (though it turns out that it is not possible to consistently construct such a 4-vector), while for GR the effect of gravity is fully included in the left-hand side of the geodesic equation (by allowing for non-flat spacetimes).

d) The minimal coupling principle states that any Lorentz-invariant physical law can be extended to curved spacetime, and hence fully include the effect of gravity, by making it coordinate-invariant. Operationally, this typically just means that the Minkowski metric $\eta_{\mu\nu}$ has to be replaced with the full metric tensor $g_{\mu\nu}$, and every partial derivative ∂_μ with a covariant derivative ∇_μ. For *example*, the Klein-Gordon equation – i.e. the equation of motion for a scalar field ϕ – becomes

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu-\frac{dV(\phi)}{d\phi}\right)\phi=0\quad\rightarrow\quad\left(g^{\mu\nu}\nabla_\mu\nabla_\nu-\frac{dV(\phi)}{d\phi}\right)\phi=0. \tag{2}$$

Problem 2

Let t, x be the coordinates of the inertial observer Alice. The proper time τ of Bob, moving with constant acceleration a, can then be expressed in these coordinates as

$$d\tau^2=-ds^2=dt^2-d\mathbf{x}^2, \tag{3}$$

where $|d\mathbf{x}/dt|\simeq at$ as long as Bob moves slowly, $|d\mathbf{x}/dt|\ll 1$ (and assuming he is at rest at $t=0$). This implies $d\tau=dt[1-a^2t^2]^{1/2}$. Thus we can see that the eigentime τ measured by Bob, as observed by Alice, is less than the eigentime t measured by Alice. Now, the Weak Equivalence principle states that the effects of a gravitational field are locally indistinguishable from those of a an accelerating frame. Thus time passes slower in the presence of a gravitational field, too. In fact, we can use the same expression, with $\mathbf{a}\to\nabla\Phi$, as long as the Newtonian field Φ is weak.

Problem 3

a) For a star composed of perfect fluid with density ρ and pressure p, the energy momentum tensor is given by

$$T_{\mu\nu}=(\rho+p)U_\mu U_\nu+pg_{\mu\nu}\,. \tag{4}$$

The four velocity $U_\mu=(U_0,0,0,0)$ is normalized as $g_{\mu\nu}U^\mu U^\nu=-1$, hence $U^0=(-g_{00})^{-1/2}=e^{-\alpha(r)}=(U_0)^{-1}$. Thus the relevant components of $T_{\mu\nu}$ are given by

$$\begin{aligned} T_{tt} &= (\rho+p)e^{2\alpha(r)}+p\,g_{tt}=\rho e^{2\alpha(r)} && (5)\\ T_{rr} &= p\,g_{rr}=pe^{2\beta(r)} && (6)\end{aligned}$$

The tt and rr components of Einstein's equations, $G_{\mu\nu}=8\pi GT_{\mu\nu}$, then become

$$\begin{aligned}\rho &= \frac{1}{8\pi Gr^2}e^{-2\beta}\left(2r\partial_r\beta-1+e^{2\beta}\right), && (7)\\ p &= \frac{1}{8\pi Gr^2}e^{-2\beta}\left(2r\partial_r\alpha+1-e^{2\beta}\right). && (8)\end{aligned}$$

b) In the Schwarzschild case, we have $g_{rr}=\left(1-\frac{2GM}{r}\right)^{-1}$, so the asked-for replacement is defined by $e^{2\beta(r)}\equiv\left(1-\frac{2Gm(r)}{r}\right)^{-1}$ or $m(r\leq R)\equiv\frac{r}{2G}(1-e^{-2\beta(r)})$. For $r\geq R$ we are in vacuum and hence must recover the Schwarzschild solution: $m(r\geq R)=M$.

c) Using the above replacement we can calculate

$$\begin{aligned}\partial_r e^{2\beta(r)} &= \partial_r\left(1-\frac{2Gm(r)}{r}\right)^{-1} && (9)\\ \rightsquigarrow 2e^{2\beta}\partial_r\beta &= -\left(1-\frac{2Gm}{r}\right)^{-2}\left(\frac{2Gm}{r^2}-\frac{2G\partial_r m}{r}\right) && (10)\\ \rightsquigarrow e^{-2\beta}\partial_r\beta &= \frac{G}{r}\left(\partial_r m-\frac{m}{r}\right). && (11)\end{aligned}$$

Thus, the tt component of Einstein's field equations, Eq. (7), becomes

$$\begin{aligned}\rho &= \tfrac{1}{8\pi Gr^2}\left(2re^{-2\beta}\partial_r\beta+1-e^{-2\beta}\right)\\ &= \tfrac{1}{8\pi Gr^2}\left(2G\left(\partial_r m-\tfrac{m}{r}\right)+1-\left(1-\tfrac{2Gm}{r}\right)\right)=\tfrac{1}{4\pi r^2}\partial_r m,\end{aligned} \tag{12}$$

or $m(r)=4\pi\int_0^r\rho(r')r'^2dr'$.

d) For a proper 3-D volume integral, we first need to determine the determinant γ of the spatial metric γ_{ij}, where

$$\gamma_{ij}dx^idx^j=e^{2\beta}dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2\,. \tag{13}$$

We thus have $\gamma=r^4e^{2\beta}\sin^2\theta$, leading to an integrated energy density of

$$\begin{aligned}\tilde m(r) &= \int\rho(r')\,\sqrt{\gamma}\;d^3x\\ &= \int\rho(r')\;r'^2\;e^\beta\sin\theta\;dr'd\theta d\phi\\ &= 4\pi\int_0^r\rho(r')\;r'^2\;e^\beta dr'.\end{aligned} \tag{14}$$

As $e^\beta>1$, we can see that $\tilde m(r)>m(r)$. If we interpret $m(r)$ as the mass within a radius r – which makes sense because of the matching condition to the Schwarzschild metric – the difference $\tilde m(r)-m(r)$ should be interpreted as the gravitational binding energy, i.e. the energy required to disperse the content of the star to infinity.

e) Energy-momentum conservation in the r-direction, $\nabla_\mu T^{\mu\nu}=0$ for $\nu=r$, gives:

$$\begin{aligned}0 &= \nabla_\mu T^{\mu r}=\partial_\mu T^{\mu r}+\Gamma^\mu_{\mu\lambda}T^{\lambda r}+\Gamma^r_{\mu\lambda}T^{\mu\lambda}\\ &= \partial_r T^{rr}+(\Gamma^t_{tr}+\Gamma^r_{rr}+\Gamma^\theta_{\theta r}+\Gamma^\phi_{\phi r})T^{rr}+\Gamma^r_{tt}T^{tt}+\Gamma^r_{rr}T^{rr}+\Gamma^r_{\theta\theta}T^{\theta\theta}+\Gamma^r_{\phi\phi}T^{\phi\phi},\end{aligned} \tag{15}$$

because of the diagonal nature of the metric and $T^{\mu\nu}$. The eight relevant Christoffel symbols are easy to calculate (as above, the repeated indices are **not** summed over),

$$\begin{aligned}\Gamma^\mu_{\mu r} &= \tfrac{1}{2}g^{\mu\mu}\partial_r g_{\mu\mu}\\ \Gamma^r_{\mu\mu} &= \tfrac{1}{2}g^{rr}(2\partial_\mu g_{\mu r}-\partial_r g_{\mu\mu})\ \rightsquigarrow\Gamma^r_{\mu\mu}=\tfrac{1}{2}\left\{\begin{array}{ll} g^{rr}\partial_r g_{rr} & \text{for }\mu=r\\ -g^{rr}\partial_r g_{\mu\mu} & \text{for }\mu\neq r\end{array}\right. .\end{aligned} \tag{16}$$

Eq. (15) then becomes

$$\begin{aligned}0 &= \partial_r T^{rr}+(\partial_r\alpha+\partial_r\beta+\tfrac{1}{r}+\tfrac{1}{r})T^{rr}+e^{2(\alpha-\beta)}\partial_r\alpha\;T^{tt}+\partial_r\beta\;T^{rr}-re^{-2\beta}T^{\theta\theta}-re^{-2\beta}\sin^2\theta\;T^{\phi\phi}\\ &= e^{-2\beta}(-2p\partial_r\beta+\partial_r p)+pe^{-2\beta}(\partial_r\alpha+\partial_r\beta+\tfrac{2}{r})+\rho e^{-2\beta}\partial_r\alpha+pe^{-2\beta}\partial_r\beta-pe^{-2\beta}\tfrac{1}{r}-pe^{-2\beta}\tfrac{1}{r}\\ &= \partial_r p+(p+\rho)\partial_r\alpha.\end{aligned} \tag{17}$$

Using Eq. (8), we can get rid of $\partial_r\alpha$:

$$\partial_r\alpha=\left(8\pi Gpr^2e^{2\beta}-1+e^{2\beta}\right)/(2r)=\partial_r\alpha=\frac{G(4\pi pr^3+m)}{r^2-2Gmr} \tag{18}$$

Inserting this into Eq. (17) give the TOV equation as stated in the problem.

Problem 4

The four-velocity $U^\mu=dx^\mu/d\tau$ satisfies $U_\mu U^\mu=-1$. For a stationary observer ($U^i=0$) in Schwarzschild coordinates, this implies

$$-1=U_\mu U^\mu=g^{00}(U^0)^2=-\left(1-\frac{2GM}{r}\right)^{-1}\quad\rightsquigarrow\quad U^0=\left(1-\frac{2GM}{r}\right)^{-1/2}. \tag{19}$$

The acceleration A^μ is given by the covariant directional derivative of the 4-velocity:

$$\begin{aligned}A^\mu &= \frac{d}{d\tau}U^\mu=U^\nu\nabla_\nu U^\mu=U^\nu\partial_\nu U^\mu+\Gamma^\mu_{\nu\rho}U^\nu U^\rho && (20)\\ &\overset{(19)}{=} \Gamma^\mu_{00}\left(1-\frac{2GM}{r}\right)^{-1} && (21)\\ &= \frac{1}{2}g^{\mu\rho}\left(\partial_t g_{0\rho}+\partial_t g_{\rho 0}-\partial_\rho g_{00}\right)\left(1-\frac{2GM}{r}\right)^{-1} && (22)\\ &= -\frac{1}{2}\left(1-\frac{2GM}{r}\right)^{-1}g^{\mu\rho}\delta^r_\rho\partial_r g_{00}\,, && (23)\end{aligned}$$

where the last step follows because the Schwarzschild metric is static (and g_{00} only depends on r). With $g^{rr}=-g_{00}$ this simplifies to

$$A^\mu=\left(0,-\frac{1}{2}\partial_r\left(1-\frac{2GM}{r}\right)=\frac{GM}{r^2},0,0\right) \tag{24}$$

because the Schwarzschild metric is diagonal. The magnitude of the actual acceleration felt by the observer is given by the magnitude of the spatial acceleration,

$$a=\sqrt{-a^ia_i}=\left(1-\frac{2GM}{r}\right)^{-\frac{1}{2}}\frac{GM}{r^2}\,. \tag{25}$$

For $r\gg 2GM$, this agrees as expected with the Newtonian acceleration (GM/r^2), but for smaller distances the actual acceleration is larger. Approaching the Schwarzschild radius, it would require an infinite amount of acceleration to escape.

Problem 5

a) Let us first look at the Christoffel symbols, $\Gamma^\rho_{\mu\nu}=\frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\mu\sigma}-\partial_\sigma g_{\mu\nu}\right)$. For $\mu=\nu=0$, this becomes

$$\Gamma^\rho_{00}=\frac{1}{2}g^{\rho\sigma}\left(\partial_t g_{0\sigma}+\partial_t g_{0\sigma}-\partial_\sigma g_{00}\right)=0 \tag{26}$$

because $g_{\mu\nu}$ is diagonal and g_{00} constant. Similarly, we have

$$\begin{aligned}\Gamma^\rho_{i0} &= \frac{1}{2}g^{\rho\sigma}\left(\partial_i g_{0\sigma}+\partial_t g_{i\sigma}-\partial_\sigma g_{i0}\right) && (27)\\ &= \frac{1}{2}g^{\rho j}\partial_t\left[a^2(t)\gamma_{ij}(\mathbf{x})\right]=g^{\rho j}g_{ji}H=H\delta^\rho_i && (28)\\ & && (29)\end{aligned}$$

$$
\begin{aligned}
\Gamma^0_{ij} &= \frac{1}{2}g^{00}\left(\partial_i g_{j0} + \partial_j g_{i0} - \partial_t g_{ij}\right) && (30)\\
&= \frac{1}{2}\partial_t\left[a^2(t)\gamma_{ij}(\mathbf{x})\right] = H g_{ij} && (31)\\
\Gamma^i_{jk} &= \frac{1}{2}g^{il}\left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}\right) && (32)\\
&= \frac{1}{2}\gamma^{il}\left(\partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk}\right) = {}^{(3)}\tilde{\Gamma}^i_{jk}\,. && (33)
\end{aligned}
$$

In order to solve Einstein's equation, we need to calculate

$$
\begin{aligned}
\mathcal{R}_{00} &= \mathcal{R}^{\mu}{}_{0\mu 0} = \mathcal{R}^{i}{}_{0i0} && (34)\\
\mathcal{R}_{ij} &= \mathcal{R}^{\mu}{}_{i\mu j} = \mathcal{R}^{0}{}_{i0j} + \mathcal{R}^{k}{}_{ikj} && (35)
\end{aligned}
$$

Now let us use the basic definition of the curvature tensor,

$$
\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu)\,, \qquad (36)
$$

and compute the relevant components:

$$
\begin{aligned}
\mathcal{R}^{i}{}_{0i0} &= \partial_i \Gamma^i_{00} + \Gamma^i_{i\lambda}\Gamma^\lambda_{00} - \partial_t \Gamma^i_{i0} - \Gamma^i_{0\lambda}\Gamma^\lambda_{i0} && (37)\\
&= -\partial_t(H\delta^i_i) - (H\delta^i_j)(H\delta^j_i) = -3\frac{\ddot{a}}{a} && (38)\\
\mathcal{R}^{0}{}_{i0j} &= \partial_t \Gamma^0_{ji} + \Gamma^0_{0\lambda}\Gamma^\lambda_{ji} - \partial_j \Gamma^0_{0i} - \Gamma^0_{j\lambda}\Gamma^\lambda_{0i} && (39)\\
&= \partial_t(H g_{ji}) - (H g_{jl})(H\delta^l_i) = \partial_t(a\dot{a}\gamma_{ji}) - H^2 g_{ji} = \frac{\ddot{a}}{a}g_{ji} && (40)\\
\mathcal{R}^{k}{}_{ikj} &= {}^{(3)}\tilde{\mathcal{R}}^{k}{}_{ikj} + \Gamma^k_{k0}\Gamma^0_{ji} - \Gamma^k_{j0}\Gamma^0_{ki} && (41)\\
&= 2\kappa a^{-2} g_{ij} + H^2(\delta^k_k g_{ij} - \delta^k_j g_{ki}) = 2(H^2 + \kappa a^{-2})g_{ij} && (42)
\end{aligned}
$$

From the form of $T_{\mu\nu}$ given in the problem, the only non-vanishing components of the stress-energy tensor are given by

$$
\begin{aligned}
T_{00} &= g_{0\mu}T^\mu_{\ 0} = \sum_f \rho_f && (43)\\
T_{ij} &= g_{i\mu}T^\mu_{\ j} = g_{ij}\sum_f p_f && (44)
\end{aligned}
$$

This implies that

$$
T = g^{\mu\nu}T_{\mu\nu} = -T_{00} + g^{ij}T_{ij} = \sum_f(-\rho_f + 3p_f) \qquad (45)
$$

The asked-for components of Einstein's equations are thus

$$
-3\frac{\ddot{a}}{a} = 8\pi G\left(\sum_f \rho_f - \frac{1}{2}\sum_f(-\rho_f + 3p_f)(-1)\right) - \Lambda = 4\pi G\sum_f(\rho_f + 3p_f) - \Lambda \quad (46)
$$

and

$$
\begin{aligned}
\frac{\ddot{a}}{a}g_{11} + 2(H^2 + \kappa a^{-2})g_{11} &= 8\pi G\left(g_{11}\sum_f p_f - \frac{1}{2}\sum_f(-\rho_f + 3p_f)g_{11}\right) + \Lambda g_{11} && (47)\\
\rightsquigarrow \quad \frac{\ddot{a}}{a} + 2H^2 + 2\kappa a^{-2} &= 4\pi G\sum_f(\rho_f - p_f) + \Lambda && (48)
\end{aligned}
$$

b) Taking $\frac{1}{2}\times$ Eq. (48) + $\frac{1}{6}\times$ Eq. (46) gives

$$
H^2 = 4\pi G\sum_f\left[\rho_f\left(\frac{1}{2}+\frac{1}{6}\right) + p_f\left(-\frac{1}{2}+\frac{3}{6}\right)\right] + \Lambda\left(\frac{1}{2}-\frac{1}{6}\right) - \frac{\kappa}{a^2}\,. \qquad (49)
$$

With

$$
\rho_{\rm tot} \equiv \sum_f \rho_f + \frac{\Lambda}{8\pi G}\,, \qquad p_{\rm tot} \equiv \sum_f p_f - \frac{\Lambda}{8\pi G}\,, \qquad (50)
$$

this takes the form of the first Friedman equation as stated in the question – while Eq. (48) takes the form of the second Friedman equation.

c) Covariant conservation of the stress-energy tensor, in the time-component, gives

$$
\begin{aligned}
0 = \nabla_\mu T^{\mu 0} &= \partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\nu}T^{\nu 0} + \Gamma^0_{\mu\nu}T^{\mu\nu} && (51)\\
&= \partial_t T^{00} + \Gamma^\mu_{\mu 0}T^{00} + \Gamma^0_{ij}T^{ij} && (52)\\
&= (\partial_t + 3H)T^{00} + H g_{ij}T^{ij}\,, && (53)
\end{aligned}
$$

where we used the explicit forms of the Christoffel symbols from a). With $T^{00} = g^{0\mu}T^0_{\ \mu} = +\rho$ and $T^{ij} = g^{j\mu}T^i_{\ \mu} = g^{ij}p$, this becomes (for each component, the subscript on ρ and p denoting the different species is left out for ease of notation)

$$
0 = \partial_t\rho + 3\frac{\dot{a}}{a}(\rho + p) = \left(\partial_t + 3(1+w)\frac{\dot{a}}{a}\right)\rho\,. \qquad (54)
$$

Assuming a constant w, this can be integrated to give

$$
\rho(t) \propto a^{-3(1+w)}\,. \qquad (55)
$$

For non-relativistic matter ($w = 0$), the energy density thus dilutes as expected with the volume of the expanding space, $\rho \propto a^{-3}$. For radiation ($w = 1/3$), we have $\rho \propto a^{-4}$, where the physical origin of the additional factor of a is the redshift. For vacuum energy Λ ($w = -1$), the energy density stays constant even if space is expanding. At early times, the universe was thus necessarily radiation-dominated.

Problem 6

The coordinate distance covered by a photon during the time interval from t_1 to t_2 is given by $\Delta r = \int_{t_1}^{t_2}\frac{dt}{a(t)}$. During radiation domination, the scale factor a grows

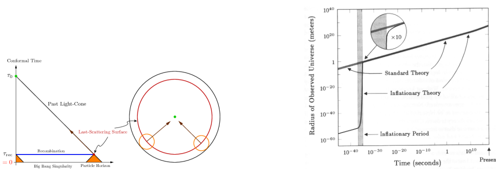

Figure 1: **Left:** Spacetime diagram showing how the standard big bang model results in causally disconnected regions [*Fig. from* arXiv:0907.5424]. **Right:** A comparison of the evolution of the size of the universe according to the standard big bang cosmology and Inflationary theory [*Fig. from* http://cosmology.berkeley.edu/∼yuki/CMBpol/].

as $a(t) = a_{\rm eq}(t/t_{\rm eq})^{1/2}$, while during matter domination we have $a = (t/t_0)^{2/3}$. The coordinate size of the event horizon at $t_{\rm CMB}$ is thus given by

$$
\Delta r_{\rm hor} \simeq \int_0^{t_{\rm CMB}}\frac{dt}{a_{\rm CMB}(t/t_{\rm CMB})^{1/2}} = \frac{2t_{\rm CMB}}{a_{\rm CMB}} \simeq 2t_0 a_{\rm CMB}^{1/2}\,. \qquad (56)
$$

The coordinate distance to the surface where all CMB photons originate from, on the other hand, is given by

$$
\Delta r_{\rm dist} = \int_{t_{\rm CMB}}^{t_0}\frac{dt}{(t/t_0)^{2/3}} \simeq 3t_0\,, \qquad (57)
$$

which means that this surface spans an area $A^{\rm CMB}_{\rm all\ sky} = 4\pi\Delta r^2_{\rm dist} \sim 36\pi t_0^2$. The (projected) area of a region that could have been in causal contact, in contrast, is given by $A^{\rm CMB}_{\rm causal} = \pi\Delta r^2_{\rm hor} \sim 4a_{\rm CMB}t_0^2$. This implies that only the fraction

$$
\frac{A^{\rm CMB}_{\rm causal}}{A^{\rm CMB}_{\rm all\ sky}} \sim \frac{1}{9\pi z_{\rm CMB}} \sim \frac{1}{10\cdot 3\cdot 1000} \sim 3\times 10^{-5} \qquad (58)
$$

of the total observable region was causally connected at the time of CMB emission. Expressed in steradians, this result has to be multiplied by $4\pi \sim 10$. The angular diameter of such a region is thus $\theta \sim 1^\circ$ since then $2\pi(1-\cos\frac{\theta}{2}) \simeq \pi\theta^2/4 \sim \pi^3(1/180)^2/4 \sim 3\times 10^{-4}$ This is illustrated in the left panel of Fig.1.

An early period of accelerated expansion ('*inflation*') can explain this discrepancy because it gives a large contribution to $\Delta r_{\rm hor}$. For exponential expansion $a = a_i \exp H_I(t - t_i)$ during t_i and t_f, e.g., we pick up a contribution of

$$
\Delta r^{\rm inflation}_{\rm hor} \simeq \int_{t_i}^{t_f} a_i^{-1}e^{H_I(t-t_i)}dt = (a_i H_I)^{-1}\left[e^{H_I(t-t_i)}\right]_{t_i}^{t_f} \simeq \frac{a_f}{a_i}(a_i H_I)^{-1}\,. \qquad (59)
$$

If inflation lasts long enough, one can thus easily arrange for $\Delta r_{\rm hor} \gtrsim \Delta r_{\rm dist}$ (for typical values of H_I, this requires $a_f/a_i \gtrsim e^{30}$). In that case, the CMB photons observed today can have been in thermal equilibrium at early times. A comparison of standard big bang cosmology and inflationary theory is shown in Fig.1 (right panel).