# Home exam FYS3140

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# 1 Problem 1: Fourier transforms and special functions

Some special functions are defined in terms of infinite sums. For example, the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, (\Re(s) > 1) \tag{1}$$

and similarly the Dirichlet eta function

$$\zeta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s}, (\Re(s) > 0). \tag{2}$$

One application of Fourier series is the possibility to compute certain values of this kind of functions.

## 1.a

Compute the Fourier series of $f(x) = x^2$ with basic interval $-\pi < x < \pi$.

**Solution:**
Since $f(x)$ is an even function I start by setting $b_n = 0$ for all $n$. The Fourier series can then be written

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(nx) \tag{3}$$

and the coefficients can be found with

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 cos(nx) dx. \tag{4}$$

First I find $a_0$:

$$\begin{aligned}
a_0 &= \frac{2}{\pi} \int_0^{\pi} x^2 dx \\
a_0 &= \frac{2}{\pi} \left[ \frac{x^3}{3} \right]_0^{\pi} \\
a_0 &= \frac{2}{\pi} \frac{\pi^3}{3} \\
a_0 &= \frac{2\pi^2}{3}
\end{aligned}$$

The rest of the coefficients can be found by using integration by parts on the integral in equation (4), and setting $u = x^2$, $u' = 2x$, $v' = cos(nx)$, $v = \frac{1}{n}sin(nx)$:

$$\int x^2 cos(nx)dx = \frac{x^2}{n}sin(nx) - \frac{2}{n}\int x sin(nx)dx$$

Where the integral on the right hand side can be solved with integration by parts. Setting $u = x$, $u' = 1$, $v' = sin(nx)$, $v = \frac{-1}{n}cos(nx)$ gives:

$$\begin{aligned}\int x sin(nx)dx &= -\frac{x}{n}cos(nx) + \frac{1}{n}\int cos(nx)dx \\ &= -\frac{x}{n}cos(nx) + \frac{1}{n^2}sin(nx)\end{aligned}$$

Inserting this back into (4) yields

$$\begin{aligned}a_n &= \frac{2}{\pi}\left[\frac{x^2}{n}sin(nx) - \frac{2}{n}\left(\frac{-x}{n}cos(nx) + \frac{1}{n^2}sin(nx)\right)\right]_0^\pi \\ &= \frac{2}{\pi}\left(\frac{2\pi}{n^2}cos(n\pi)\right) \\ &= \frac{4}{n^2}cos(n\pi) \\ &= \frac{4(-1)^n}{n^2}\end{aligned}$$

and thus we have found that

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}cos(nx) \tag{5}$$

## 1.b

Use your result to show that

$$\zeta(2) = \frac{\pi^2}{6} \tag{6}$$

and

$$\eta(2) = \frac{\pi^2}{12} \tag{7}$$

**Solution:**
We start with $\zeta(2)$. Using the definition of the function:

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

If one now takes equation (6) and sets $x = \pi$ one finds that

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} cos(n\pi)$$
$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n(-1)^n}{n^2}$$
$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$
$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{3\pi^2 - \pi^2}{12}$$
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$
$$\zeta(2) = \frac{\pi^2}{6}$$

Which was what I wanted to show.
From the definition of $\eta$ we have

$$\eta(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

Again I use equation (6) but this time I set $x = 0$ and get the following:

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}$$
$$-\frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$
$$-\frac{\pi^2}{3} = -4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$
$$\eta(2) = \frac{\pi^2}{12}$$

Which was what I wanted to show.

# 2 Problem 2: Contour integral

In this problem we will explore the complex integral

$$I = \oint_{\gamma} \frac{dz}{1+z^n}; n \geq 2 \tag{8}$$

along the closed contour $\Gamma = \gamma_1 + \gamma_2 + \gamma_3$ where
$\gamma_1$ : Straight line along the x-axis from the origin to $x = R$
$\gamma_2$ : Arc of a circle of radius R from $z = x = R$ to $z = Re^{2\pi i/n}$
$\gamma_3$ : Straight line from $z = Re^{2\pi i/n}$ to the origin,
where we will let $R \to \infty$ eventually.

## 2.a

Sketch the integration contour.

**Solution:**
From $\gamma_2$ it is clear that one starts with a half circle, and that this circle is made smaller by increasing $n$. Since it is clear that the integration contour is dependent on $n$ I've chosen to make a figure for $n = 4$.

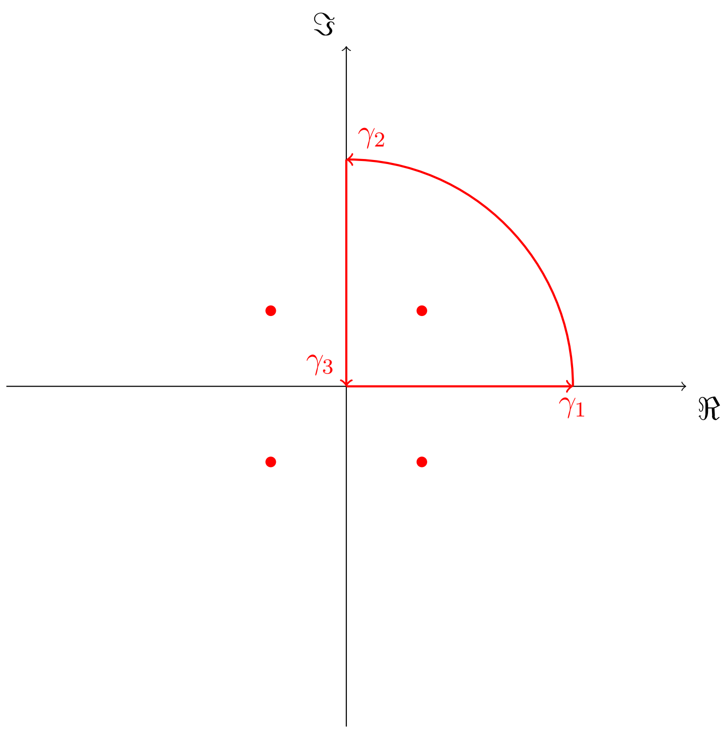


Figure 1: The integration path when $n = 4$.

## 2.b

Find all singularities of the integrand and indicate them in your figure from a).

**Solution:**

Since the integrand is $\frac{dz}{1+z^n}$, one gets a singularity when

$$z^n = -1 = e^{i\pi(1+2k)}$$
$$z = e^{\frac{i\pi(1+2k)}{n}}; k = 0, 1, ..., n-1$$

From this it is clear that the number of, and the position of, the singularities is dependent on $n$. As an example I get for $n = 2$:

$$z_1 = e^{\frac{i\pi}{2}} = i$$
$$z_2 = e^{\frac{3i\pi}{2}} = -i$$

while for $n = 3$ I get:

$$z_1 = e^{\frac{i\pi}{3}}$$
$$z_2 = e^{\frac{3i\pi}{3}} = -1$$
$$z_3 = e^{\frac{5i\pi}{3}}$$

In fact, there are $n$ singularities, and if one makes a straight line from one to the next they will form an n-sided polygon. Since it would be a complete mess to mark of singularities for all $n$, I've chosen to mark the ones for $n = 4$, since that is the $n$ I chose for the figure earlier.

## 2.c

Compute the integral I in the limit $R \to \infty$.

**Solution:**

From the residue theorem we have that

$$\int_\Gamma f(z)dz = 2\pi i \sum_N Res[f(z), z_n]. \tag{9}$$

So instead of calculating the integral I can just find all the residues inside $\Gamma$. Now I need to find how many are inside $\Gamma$ depending on what $n$ is. For a singularity to be inside $\Gamma$ the following must be true:

$$\begin{aligned}
\frac{2\pi}{n} &\geq \frac{\pi(1+2k)}{n} \\
2\pi &\geq \pi(1+2k) \\
2 &\geq 1+2k \\
\frac{1}{2} &\geq k \\
k &\leq \frac{1}{2}
\end{aligned}$$

and since k increases in integer steps, this only happens when $k = 0$. This means that

$$\sum_N Res[f(z), z_n] = Res[f(z), z_0].$$

And this means that I can conclude that no matter what $n$ is, there will always be a singularity inside the contour $\Gamma$ at $z_0 = e^{i\pi/n}$. To find this residue I use that for rational functions $f(z) = \frac{P(z)}{Q(z)}$, we have

$$Res[f(z), z_0] = \frac{P(z_0)}{Q'(z_0)}. \tag{10}$$

In our case $P(z) = 1$, and $Q(z) = 1 + z^n$. Thus $Q'(z) = nz^{n-1}$, and

$$\begin{aligned}
I &= 2\pi i Res[f(z), z_0] = 2\pi i \frac{1}{nz_0^{n-1}} = \frac{2\pi i}{n} e^{-\frac{i\pi(n-1)}{n}} \\
I &= -\frac{2\pi i}{n} e^{\frac{i\pi}{n}}
\end{aligned}$$

## 2.d

Examine the contribution to the integral from the circle arc $\gamma_2$ as $R \to \infty$. (Hint: perform an upper bound estimate)

**Solution:**

I want to estimate an upper bound for $\int_{\gamma_2} \frac{1}{1+z^n} dz$. This can be calculated by

$$\left| \int_{\gamma_2} f(z) dz \right| \leq ML \tag{11}$$

Where $L$ is the length of the contour, and $M$ is the maximum value of $|f(z)|$ on the contour $\Gamma$.

$L$ is straightforward to find:

$$dL = Rd\theta$$
$$L = \frac{R2\pi}{n}$$

Finding M:

$$\left|\frac{1}{1+z^n}\right| = \frac{|1|}{|1+z^n|} \leq \frac{1}{|z^n|-1} = \frac{1}{R^n-1} = M$$
$$M = \frac{1}{R^n-1}$$

And thus

$$\begin{aligned}\left|\int_{\gamma_2} f(z)dz\right| &\leq \lim_{R\to\infty} \frac{R2\pi}{n}\frac{1}{R^n-1} \\ &\leq \lim_{R\to\infty} \frac{2\pi}{n^2R^{n-1}} \\ &\leq 0 \\ \left|\int_{\gamma_2} f(z)dz\right| &\leq 0 \\ \int_{\gamma_2} f(z)dz &= 0.\end{aligned}$$

Where I've used L'Hopital's rule to calculate the limit. So the conclusion is that $\gamma_2$ contributes nothing to the integral as long as $R \to \infty$.

## 2.e

Using your results from c) and d), show that the real integral

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi/n}{sin(\pi/n)} \tag{12}$$

(Hint: Find a suitable parametrization along $\gamma_3$).

**Solution:**

I found earlier that

$$I = \frac{-2\pi i}{n}e^{i\pi/n}$$

but one can also write the integral for I as

$$\begin{aligned}
I =& \int_{\gamma_1} \frac{dx}{1+x^n} + \int_{\gamma_2} \frac{dz}{1+z^n} + \int_{\gamma_3} \frac{dz}{1+z^n} \\
=& \lim_{R\to\infty} \int_0^R \frac{dx}{1+x^n} + 0 + \lim_{R\to\infty} e^{2i\pi/n} \int_R^0 \frac{dx}{1+x^n} \\
=& \int_0^\infty \frac{dx}{1+x^n} - e^{2i\pi/n} \int_0^\infty \frac{dx}{1+x^n} \\
=& (1-e^{2i\pi/n}) \int_0^\infty \frac{dx}{1+x^n}.
\end{aligned}$$

Since I actually want to find the integral in this last expression, I set this expression equal to the one I found earlier for $I$:

$$\begin{aligned}
\frac{-2\pi i}{n} e^{i\pi/n} =& (1-e^{2i\pi/n}) \int_0^\infty \frac{dx}{1+x^n} \\
\int_0^\infty \frac{dx}{1+x^n} =& \frac{\frac{-2\pi i}{n} e^{i\pi/n}}{1-e^{2i\pi/n}} \\
=& \frac{\pi}{n} \frac{2i}{e^{\frac{i\pi}{n}} - e^{\frac{-i\pi}{n}}} \\
\int_0^\infty \frac{dx}{1+x^n} =& \frac{\pi/n}{sin(\pi/n)}.
\end{aligned}$$

Which was what I wanted to show.

# 3 Problem 3: Fröbenius method

Here you will use the Fröbenius method to address the differential equation

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0 \tag{13}$$

## 3.a

Set up and solve the indicial equation

**Solution:**

Start by assuming a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$$

which has first derivative

$$y'(x) = \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1}$$

and second derivative

$$y''(x) = \sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2}.$$

Inserting these into the DE (14):

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s} + (x^2 - \frac{1}{4})\sum_{n=0}^{\infty} a_n x^{n+s} = 0.$$

Since this has to hold for every power of $x$ we start with the first one, $x^s$ :

$$\begin{aligned} s(s-1)a_0 + sa_0 + 0 - \frac{1}{4}a_0 &= 0; a_0 \neq 0 \\ s^2 - s + s - \frac{1}{4} &= 0 \\ s^2 &= \frac{1}{4} \\ s &= \pm\frac{1}{2} \end{aligned}$$

## 3.b

Find the general solution for $y(x)$.

**Solution:**
Since $s1 - s2 = integer$, there is a possibility that the smallest of them will generate a complete solution. Because of this I want to try using $s = -1/2$, and see if I can find the coefficients $a_n$. To do this I start with the lowest power of x, e.g $x^{-1/2}$:

$$\begin{aligned} \Big(-\frac{1}{2}\Big)\Big(-\frac{1}{2}-1\Big)a_0 - \frac{1}{2}a_0 - \frac{1}{4}a_0 &= 0 \\ \Big(\frac{3}{4} - \frac{3}{4}\Big)a_0 &= 0 \end{aligned}$$

Which means $a_0$ is undetermined.

Next is $x^{1/2}$:

$$\begin{aligned}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{2}-1\right)a_1+\left(1-\frac{1}{2}\right)a_1-\frac{1}{4}a_1&=0\\ -\frac{1}{4}a_1+\frac{1}{2}a_1-\frac{1}{4}a_1&=0\\ \left(\frac{1}{2}-\frac{1}{2}\right)a_1&=0\end{aligned}$$

Which means $a_1$ is also undetermined. And since I now have two undetermined coefficients, this has produced the complete solution.

Next thing then is to find an equation for $a_n$ with $n \geq 2$:

$$\begin{aligned}\left(n-\frac{1}{2}\right)\left(n-\frac{1}{2}-1\right)a_n+\left(n-\frac{1}{2}\right)a_n+a_{n-2}-\frac{1}{4}a_n&=0\\ \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)a_n+\left(n-\frac{1}{2}\right)a_n-\frac{1}{4}a_n&=-a_{n-2}\\ \left(n^2-\frac{3n}{2}-\frac{n}{2}+\frac{3}{4}+n-\frac{1}{2}-\frac{1}{4}\right)a_n&=-a_{n-2}\\ \left(n^2-n\right)a_n&=-a_{n-2}\\ a_n&=-\frac{a_{n-2}}{n(n-1)}\end{aligned}$$

This means that the complete solution is

$$y(x)=a_0x^{-1/2}+a_1x^{1/2}-\sum_{n=2}^{\infty}\frac{a_{n-2}}{n(n-1)}x^{n-1/2} \tag{14}$$

or if one doesn't like infinite sums one can write out the start of the sum

$$y(x)=a_0x^{-1/2}+a_1x^{1/2}-\frac{a_0}{2!}x^{3/2}-\frac{a_1}{3!}x^{5/2}+\frac{a_0}{4!}x^{7/2}+\frac{a_1}{5!}x^{9/2}+...$$

and notice that this can be written

$$y(x)=x^{-1/2}[a_0cos(x)+a_1sin(x)] \tag{15}$$

## 3.c

Show that the solution can be written as

$$y(x)=AJ_{1/2}(x)+BJ_{-1/2}(x) \tag{16}$$

where A and B are constants, and $J_{\pm 1/2}(x)$ are Bessel functions of the first kind, of order $\pm\frac{1}{2}$. Bessel functions are important in a number of physics applications, and information about them can be found in mathematical tables.

**Solution:**

To start with I need to know what these Bessel functions look like. It turns out that Bessel functions of the first kind of order p are defined as

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n+p} \tag{17}$$

which means that

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+3/2)} \left(\frac{x}{2}\right)^{2n+1/2} \tag{18}$$

and

$$J_{-1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1/2)} \left(\frac{x}{2}\right)^{2n-1/2.} \tag{19}$$

I will also need that

$$cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{20}$$

and

$$sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \tag{21}$$

First I write $y(x)$ with these other forms for $sin(x)$ and $cos(x)$:

$$y(x) = x^{-1/2}\left[a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right]$$

$$y(x) = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n-1/2}}{(2n)!} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1/2}}{(2n+1)!}$$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{2^{2n-1/2}(-1)^n}{(2n)!} \left(\frac{x}{2}\right)^{2n-1/2} + a_1 \sum_{n=0}^{\infty} \frac{2^{2n+1/2}(-1)^n}{(2n+1)!} \left(\frac{x}{2}\right)^{2n+1/2}$$

For this to be the same as equation (16), the following must be true:

$$\frac{2^{2n-1/2}a_0}{(2n)!} = \frac{B}{\Gamma(n+1)\Gamma(n+1/2)} \tag{22}$$

and

$$\frac{2^{2n+1/2}a_1}{(2n+1)!} = \frac{A}{\Gamma(n+1)\Gamma(n+3/2)} \tag{23}$$

Start with (22):

$$\begin{aligned}
&\frac{2^{2n-1/2}a_0}{(2n)!} = 2^{-1/2}a_0\frac{2^{2n}}{(2n)!}\\
&= 2^{-1/2}a_0\frac{2^{2n}}{(2n)(2n-1)(2n-2)(2n-3)...3\times 2\times 1}\\
&= 2^{-1/2}a_0\frac{1}{(n)(n-1/2)(n-1)(n-3/2)...3/2\times 1\times 1/2}\\
&= 2^{-1/2}a_0\frac{\Gamma(n)\Gamma(n-\frac{1}{2})\Gamma(n-1)...\Gamma(1)\Gamma(\frac{1}{2})}{(n)\Gamma(n)(n-\frac{1}{2})\Gamma(n-\frac{1}{2})(n-1)\Gamma(n-1)...\Gamma(1)\frac{1}{2}\Gamma(\frac{1}{2})}\\
&= 2^{-1/2}a_0\frac{\Gamma(n)\Gamma(n-\frac{1}{2})\Gamma(n-1)...\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(n+1)\Gamma(n+\frac{1}{2})\Gamma(n)...\Gamma(1)1/2\Gamma(\frac{1}{2})}\\
&= 2^{-1/2}a_0\frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(n+1)\Gamma(n+\frac{1}{2})}\\
&= 2^{-1/2}a_0\frac{\pi^{1/2}}{\Gamma(n+1)\Gamma(n+\frac{1}{2})}
\end{aligned}$$

which works if I define

$$B = a_0\left(\frac{\pi}{2}\right)^{1/2}. \tag{24}$$

This is okay, since all I've actually done to get $B$ is multiply $a_0$ by some constant. Similarly for (23):

$$\begin{aligned}
&\frac{2^{2n+1/2}a_1}{(2n+1)!} = 2^{1/2}a_1\frac{2^{2n}}{(2n)!}\\
&= 2^{1/2}a_1\frac{2^{2n}}{(2n+1)(2n)(2n-1)(2n-2)(2n-3)...3\times 2\times 1}\\
&= 2^{1/2}a_1\frac{1}{(n+1/2)(n)(n-1/2)(n-1)(n-3/2)...3/2\times 1\times 1}\\
&= 2^{1/2}a_1\frac{\Gamma(n+\frac{1}{2})\Gamma(n)\Gamma(n-\frac{1}{2})...\Gamma(1)}{(n+1/2)\Gamma(n+\frac{1}{2})(n)\Gamma(n)(n-\frac{1}{2})\Gamma(n-\frac{1}{2})...\Gamma(1)}\\
&= 2^{1/2}a_1\frac{\Gamma(n+\frac{1}{2})\Gamma(n)\Gamma(n-\frac{1}{2})\Gamma(n-1)...\Gamma(1)}{\Gamma(n+\frac{3}{2})\Gamma(n+1)\Gamma(n+\frac{1}{2})\Gamma(n)...\Gamma(1)}\\
&= 2^{1/2}a_1\frac{1}{\Gamma(n+1)\Gamma(n+\frac{3}{2})}
\end{aligned}$$

which works if I define

$$A = a_1 2^{1/2}. \tag{25}$$

Notice that $A$ is only $a_1$ multiplied by some constant, just like $B$. Then finally

$$y(x) = x^{-1/2}[a_0 cos(x) + a_1 sin(x)] = AJ_{1/2}(x) + BJ_{-1/2}(x) \tag{26}$$

where

$$A = a_1 2^{1/2}, B = a_0 \left(\frac{\pi}{2}\right)^{1/2}.$$

Which is what I wanted to show.