Classical Mechanics and Electrodynamics
Lecture notes – FYS 3120

Jon Magne Leinaas
Department of Physics, University of Oslo

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Preface

FYS 3120 is a course in classical theoretical physics, which covers the basics of three central parts of physics, Classical Mechanics, Relativity and Electrodynamics. The approach is analytical in form, where theoretical methods are used to derive central elements of these subjects from first principles.

Part I gives an introduction to Analytical Mechanics in the form of Lagrange and Hamilton theory, and it discusses the use of variational methods. In Part II, on Special Relativity, the relativistic four-vector notation and covariant formulations are introduced and applied to relativistic kinematics and dynamics. Part III is based on the relativistic formulation of Maxwell's equations, and on the methods used to find solutions of these equations with stationary and time dependent sources. A special focus is on the radiation phenomenon.

The lecture notes have been developed in parallel with my teaching of the course, and are still open for further modifications and improvements.

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Jon Magne Leinaas

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Part I
Analytical Mechanics

The form of classical mechanics we shall discuss here is often called *analytical mechanics*. It is essentially the same as the mechanics of Newton, but brought into a more abstract form. The analytical formulation of mechanics was developed in the 18th and 19th century by several physicists, but two of these had a particularly strong influence on the development of the field, namely Joseph Louis Lagrange (1736-1813) and William Rowan Hamilton (1805-1865). The mathematical formulation given to mechanics by these two, and developed further by others, is generally admired for its formal beauty. Although the formalism was developed a long time ago, it is still a basic element of modern theoretical physics and has influenced much the later theories of relativity and quantum mechanics.

Lagrange and Hamilton formulated mechanics in two different ways, which we refer to as the *Lagrangian* and *Hamiltonian* formulations. They are equivalent, and in principle we may make a choice between the two, but instead it is common to study both these formulations as two sides of the analytic approach to mechanics. This is because they have different useful properties and it is advantageous to able to choose between these methods the one that is best suited to solve the problem at hand. One should note, however, a certain limitation in both these formulations of mechanics, since they in the standard form assume the forces to be conservative. Thus mechanical systems that involve friction and dissipation are generally not handled by this formulation of mechanics. We refer to systems that can be handled by the Lagrangian and Hamiltonian formalism to be *Hamiltonian systems*.

In Newtonian mechanics *force* and *acceleration* are central concepts, and in modern terminology we often refer to this as a *vector formulation* of mechanics. Lagrangian and Hamiltonian mechanics are different since force is not a central concept, and *potential* and *kinetic energy* instead are functions that determine the dynamics. In some sense they are like extensions of the usual formulation of *statics*, where a typical problem is to find the minimum of a potential. As a curious difference the *Lagrangian*, which is the function that regulates the dynamics in Lagrange's formulation, is the *difference* between kinetic and potential energy, while the *Hamiltonian* which is the basic dynamical function in Hamilton's formulation, is usually the *sum* of kinetic and potential energy.

The Hamiltonian and Lagrangian formulations are generally more easy to apply to composite systems than the Newtonian formulation is. The main problem is to identify the physical degrees of freedom of the mechanical system, to choose a corresponding set of independent variables and to express the kinetic and potential energies in terms of these. The dynamical equations, or equations of motion, are then derived in a straight forward way as differential

equations determined either by the Lagrangian or the Hamiltonian. Newtonian mechanics on the other hand expresses the dynamics as motion in three-dimensional space, and all students who have struggled with the use of the vector equations of linear and angular momentum know that for a composite system such a vector analysis is not always simple. However, as is generally common when a higher level of abstraction is used, there is something to gain and something to lose. A well formulated abstract theory may introduce sharper tools for analyzing a physical system, but often at the expense of more intuitive physical interpretation. That is the case also for analytical mechanics, and the vector formulation of Newtonian mechanics is often indispensable for the physical interpretation of the theory.

In the following we shall derive the basic equations of the Lagrangian and Hamiltonian mechanics from Newtonian mechanics. In this derivation there are certain complications, like the distinction between *virtual* and *physical* displacements, but application of the derived formalism does not depend on these intermediate steps. The typical problem of using the Lagrangian or Hamiltonian formalism is based on a simple standardized algorithm with the following steps: First determine the degrees of freedom of the mechanical system and choose a set of independent coordinates, one for each degree of freedom. Next find the Lagrangian or Hamiltonian expressed in terms of the coordinates and their time derivatives and formulate the dynamics either as Lagrange's or Hamilton's equations. The final problem, which is the purely mathematical part, is then to solve the corresponding differential equations with the given initial conditions.

Chapter 1

Generalized coordinates

1.1 Physical constraints and independent variables

In the description of mechanical systems we often meet constraints, which means that the motion of one part of the system strictly follows the motion of another part. In the vector analysis of such a system there will be unknown forces associated with the constraints, and a part of the analysis of the system consists in eliminating the unknown forces by applying the constraint relations. One of the main simplifications of the Lagrangian and Hamiltonian formulations is that the dynamics is expressed in variables that from the beginning take these constraints into account. These independent variables are known as *generalized coordinates* and they are generally different from the Cartesian coordinates of the system. The number of generalized coordinates correspond to the number of degrees of freedom of the system, which is equal to the remaining number of variables after all constraint relations have been imposed.
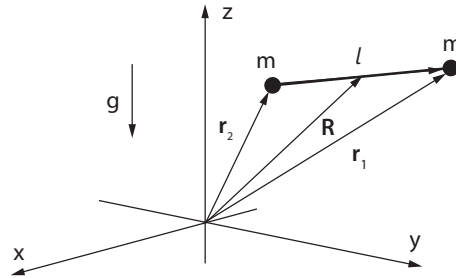
Figure 1.1: Two small bodies connected by a rigid rod

As a simple example let us consider two small bodies of equal mass m attached to the end points of a thin rigid, massless rod of length l moving in the earth's gravitational field, as shown in Fig. 1. In a vector analysis of the system we write the following equations

$$
\begin{aligned}
m\ddot{\mathbf{r}}_1 &= m\mathbf{g} + \mathbf{f} \\
m\ddot{\mathbf{r}}_2 &= m\mathbf{g} - \mathbf{f} \\
|\mathbf{r}_1 - \mathbf{r}_2| &= l
\end{aligned}
\tag{1.1}
$$

where we here, and in the following, use the standard notations for time derivatives, $\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}, \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2}$. The two first equations are Newton's equation applied to particle 1 and particle 2 with $\mathbf{f}$ as the force from the rod on particle 1, and $\mathbf{g}$ as the gravitational acceleration. The third equation is the constraint equation which expresses that the length of the rod is fixed. The number of degrees of freedom d is easy to find,

$$d = 3 + 3 - 1 = 5 \tag{1.2}$$

where each of the two vector equations gives the contribution 3, corresponding to the three components of each of the two vectors. The constraint equation, however removes one degree of freedom, since this equation can be used to express one of the variables in terms of the others, thus reducing the number of *independent* coordinates. As a set of generalized coordinates corresponding to these 5 degrees of freedom we may chose the center of mass vector $\mathbf{R} = (X, Y, Z)$ and the two angles (ϕ, θ) that determine the direction of the rod in space. Expressed in terms of these independent coordinates the position vectors of the end points of the rod are

$$\mathbf{r}_1 = (X + \frac{l}{2}\sin\theta\cos\phi)\mathbf{i} + (Y + \frac{l}{2}\sin\theta\cos\phi)\mathbf{j} + (Z + \frac{l}{2}\cos\theta)\mathbf{k}$$
$$\mathbf{r}_2 = (X - \frac{l}{2}\sin\theta\cos\phi)\mathbf{i} + (Y - \frac{l}{2}\sin\theta\cos\phi)\mathbf{j} + (Z - \frac{l}{2}\cos\theta)\mathbf{k} \tag{1.3}$$

The corresponding expressions for the kinetic energy T and the potential energy V are

$$\begin{aligned} T &= \frac{1}{2}m(\dot{\mathbf{r}}_1^2 + \dot{\mathbf{r}}_2^2) = m(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + \frac{1}{4}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \\ V &= mg(z_1 + z_2) = 2mgZ \end{aligned} \tag{1.4}$$

with the z-axis chosen in the vertical direction. These functions, which depend only on the 5 generalized coordinates are the input functions in Lagrange's and Hamilton's equations, and the elimination of constraint relations means that the unknown constraint force does not appear in the equations.

Let us now make a more general formulation of the transition from Cartesian to generalized coordinates. Following the above example we assume that a general mechanical system can be viewed as composed of a number of small bodies with masses $m_i, i = 1, 2, ..., N$ and position vectors $\mathbf{r}_i, i = 1, 2, ..., N$. We assume that these cannot all move independently, due to a set of constraints that can be expressed as a functional dependence between the coordinates

$$f_j(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N; t) = 0 \quad j = 1, 2, ..., M \tag{1.5}$$

One should note that such a dependence between the coordinates is not the most general possible. The constraints may, for example, also depend on velocities. However, the possibility of *time dependent* constraints are included in the expression. Constraints that can be written in the form (1.5) are called *holonomic* (or rigid constraints in simpler terms), and in the following we will restrict the discussion to constraints of this type.

The number of variables of the system is $3N$, since for each particle there are three variables corresponding to the components of the position vector $\mathbf{r}_i$, but the number of *independent* variables is smaller, since each constraint equation reduces the number of independent variables by 1. The number of degrees of freedom of the system is therefore

$$d = 3N - M \tag{1.6}$$

with M as the number of (independent) constraint equations. d then equals the number of generalized coordinates that is needed to give a complete description of the system, and we denote in the following such a set of coordinates by $\{q_k, k = 1, 2, ..., d\}$. Without specifying the constraints we cannot give explicit expressions for the generalized coordinates, but that is not needed for the general discussion. What is needed is to realize that when the constraints are imposed, the $3N$ Cartesian coordinates can in principle be written as functions of the smaller number of generalized coordinates,

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, ..., q_d; t) , \quad i = 1, 2, ..., N \tag{1.7}$$

The time dependence in the relation between the Cartesian and generalized coordinates reflects the possibility that the constraints may be time dependent. For convenience we will use the notation $q = \{q_1, q_2, ..., q_d\}$ for the whole set so that (1.7) gets the more compact form

$$\mathbf{r}_i = \mathbf{r}_i(q, t) , \quad i = 1, 2, ..., N \tag{1.8}$$

Note that the set of generalized coordinates can be chosen in many different ways, and often the coordinates will not all have the same physical dimension. For example some of them may have dimension of length, like the center of mass coordinates of the example above, and others may be dimensionless, like the angles in the same example. Below are some further examples of constraints and generalized coordinates.

1.1.1   Examples

A planar pendulum

We consider a small body with mass m attached to a thin, massless rigid rod of length l that can oscillate freely about one endpoint, as shown in Fig. 1.2 a). We assume the motion to be limited to a two-dimensional plane. There are two Cartesian coordinates in this case, corresponding to the components of the position vector for the small massive body, $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. The coordinates are restricted by one constraint equation, $f(\mathbf{r}) = |\mathbf{r}| - l = 0$. The number of degrees of freedom is $d = 2 - 1 = 1$, and therefore one generalized coordinate is needed to describe the motion of the system. A natural choice for the generalized coordinate is the angle θ indicated in the figure. Expressed in terms of generalized coordinate the position vector of the small body is

$$\mathbf{r}(\theta) = l(\sin\theta\mathbf{i} - \cos\theta\mathbf{j}) \tag{1.9}$$

and it is straight forward to check that the constraint is satisfied when $\mathbf{r}$ is written in this form. We may further use this expression to find the kinetic and potential energies expressed in terms of the generalized coordinate

$$\begin{aligned} T(\dot{\theta}) &= \frac{1}{2}ml^2\dot{\theta}^2 \\ V(\theta) &= -mgl\cos\theta \end{aligned} \tag{1.10}$$
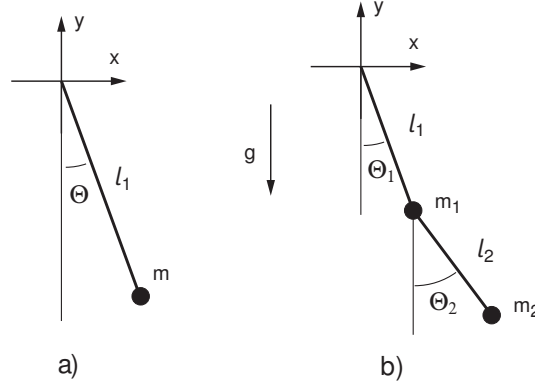
Figure 1.2: A planar pendulum a) and a double pendulum b)

A double pendulum

A slightly more complicated case is given by the double pendulum shown in Fig. 1.2 b). If we use the same step by step analysis of this system, we start by specifying the Cartesian coordinates of the two massive bodies, $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j}$. There are 4 such coordinates, x_1, y_1, x_2 and y_1. However they are not all independent due to the two constraints, $f_1(\mathbf{r}) = |\mathbf{r}| - l_1 = 0$ and $f_2(\mathbf{r}_1, \mathbf{r}_2) = |\mathbf{r}_1 - \mathbf{r}_2| - l_2 = 0$. The number of degrees of freedom is therefore $d = 4 - 2 = 2$, and a natural choice for the two generalized coordinates is the angles θ_1 and θ_2. The Cartesian coordinates are now expressed in terms of the generalized coordinates as

$$
\begin{aligned}
\mathbf{r}_1(\theta_1) &= l_1(\sin\theta_1\mathbf{i} - \cos\theta_1\mathbf{j}) \\
\mathbf{r}_2(\theta_1, \theta_2) &= l_1(\sin\theta_1\mathbf{i} - \cos\theta_1\mathbf{j}) + l_2(\sin\theta_2\mathbf{i} - \cos\theta_2\mathbf{j})
\end{aligned}
\tag{1.11}
$$

This gives for the kinetic energy the expression

$$
\begin{aligned}
T(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_1) &= \frac{1}{2}m(\dot{\mathbf{r}_1}^2 + \dot{\mathbf{r}_2}^2) \\
&= \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2
\end{aligned}
\tag{1.12}
$$

and for the potential energy

$$
\begin{aligned}
V(\theta_1, \theta_2) &= m_1gy_1 + m_2gy_2 \\
&= -(m_1 + m_2)gl_1\cos\theta_1 - m_2gl_2\cos\theta_2
\end{aligned}
\tag{1.13}
$$

Rigid body

As a third example we consider a three-dimensional rigid body. We may think of this as composed of a large number N of small parts, each associated with a position vector $\mathbf{r}_k, k = 1, 2, ..., N$. These vectors are not independent since the distance between any pair of the small

parts is fixed. This corresponds to a set of constraints, $|\mathbf{r}_k - \mathbf{r}_l| = d_{kl}$ with d_{kl} fixed. However, to count the number of independent constraints for the N parts is not so straight forward, and in this case it is therefore easier find the number of degrees of freedom by a direct argument.
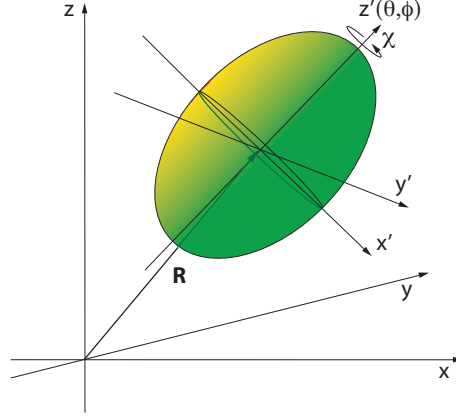
Figure 1.3: The six degrees of freedom of a rigid body. Three of these are described by the components of the center-of-mass vector $\mathbf{R}$, and three of them by the angular coordinates (θ, ϕ, χ) which determine the orientation of the body in three-dimensional space. The orientation is here illustrated by the rotation of a fixed frame (x, y, z) into a body-fixed frame (x', y', z'). The two tangles (θ, ϕ) define the direction of the z'-axis relative to the fixed frame, and the angle χ defines the rotation of the body-fixed frame around the z' axis.

The Cartesian components of the center-of-mass vector obviously is a set of independent variables

$$\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} \qquad (1.14)$$

When these coordinates are fixed, there is a further freedom to rotate the body about the center of mass. To specify the orientation of the body after performing this rotation, three coordinates are needed. This is easily seen by specifying the orientation of the body in terms of the directions of the axes of an imagined body-fixed orthogonal frame (see Fig. 1.3). For these axes, denoted by (x', y', z'), we see that two angles (ϕ, θ) are needed to fix the orientation of the z' axis, while the orientation of the remaining two axes may be fixed by a rotation angle χ in the x', y' plane. A complete set of generalized coordinates may thus be chosen as

$$q = \{X, Y, Z, \phi, \theta, \chi\} \qquad (1.15)$$

The number of degrees of freedom of a three-dimensional rigid body is consequently 6.

Time dependent constraint

We consider a small body sliding on an inclined plane, Fig. 1.4, and assume the motion to be restricted to the two-dimensional x, y-plane shown in the figure. The angle of inclination is α,
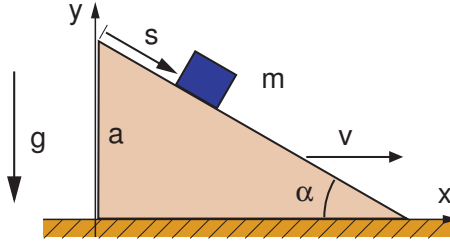
Figure 1.4: A body is sliding on an inclined plane while the plane moves with constant velocity in the horizontal direction.

and we first consider the case when the inclined plane is at rest ($v = 0$). With x and y as the two Cartesian coordinates of the body, there is one constraint equation

$$y = a - x \tan \alpha \qquad (1.16)$$

and therefore one degree of freedom for the moving body. As generalized coordinate we may conveniently choose the distance s along the plane. The position vector, expressed as a function of this generalized coordinate, is simply

$$\mathbf{r}(s) = s \cos \alpha \, \mathbf{i} + (a - \sin \alpha) \, \mathbf{j} \qquad (1.17)$$

Let us next assume that the inclined plane to be moving with constant velocity v in the x-direction. The number of degrees of freedom is still one, but the position vector is now a function of both the generalized coordinate s and of time t,

$$\mathbf{r}(s, t) = (s \cos \alpha - vt) \, \mathbf{i} + (a - \sin \alpha) \, \mathbf{j} \qquad (1.18)$$

This corresponds to a time-dependent constraint equation

$$y = a - (x - vt) \tan \alpha \qquad (1.19)$$

In the general discussion to follow we will assume that the constraints may be time dependent, since this possibility can readily be taken care of by the formalism.

Non-holonomic constraint

Even if, in the analysis to follow, we shall restrict the constraints to be holonomic, it may be of interest to to consider a simple example of a non-holonomic constraint. Let us study the motion of one of the skates of a person who is skating on ice. As coordinates for the skate we may choose the two Cartesian components of the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ together with the angle θ that determines the orientation of the skate. There is no functional relation between these three coordinates, since for an arbitrary position $\mathbf{r}$ the skate can have any angle θ. However, under normal skating there is a constraint on the motion, since the direction of the velocity will be the same as the direction of the skate. This we may write as

$$\dot{\mathbf{r}} = v(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) \qquad (1.20)$$
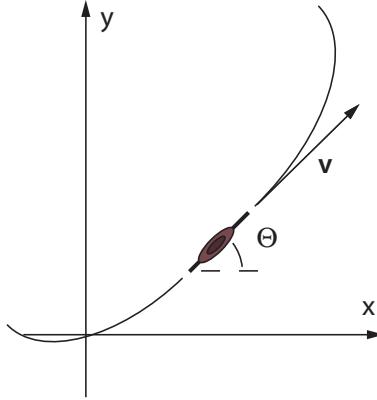
Figure 1.5: Example of a non-holonomic constraint. The velocity **v** of a skate moving on ice is related to the direction of the skate, here indicated by the angle θ. However there is no direct relation between the position coordinates (x, y) and the angle θ.

which gives the following relation

$$\dot{y} = \dot{x} \tan \theta \tag{1.21}$$

This is a non-holonomic constraint, since it is not a functional relation between coordinates alone, but between velocities and coordinates. Such a relation cannot simply be used to reduce the number of variables, but should be treated in a different way.

1.2 The configuration space

To sum up what we have already discussed: A three-dimensional mechanical system that is composed of N small parts and which is subject to M rigid (holonomic) constraints has a number of $d = 3N - M$ degrees of freedom. For each degree of freedom an independent generalized coordinate q_i can be chosen, so that the time evolution is fully described by the time dependence of the set of generalized coordinates

$$q = \{q_1, q_2, ..., q_d\} \tag{1.22}$$

The set q can be interpreted as the set of coordinates of a d-dimensional space (a *manifold*) that is referred to as the *configuration space* of the system. Each point q corresponds to a possible configuration of the composite system, which specifies the positions of all the parts of the system in accordance with the constraints imposed on the system.

In the Lagrangian formulation the time evolution in the configuration space is governed by the Lagrangian, which is a function of the generalized coordinates q, of their velocities $\dot{q}$ and possibly of time t (when the constraints are time dependent). The normal form of the Lagrangian is given as the difference between the kinetic and potential energy

$$L(q, \dot{q}, t) = T(q, \dot{q}, t) - V(q, t) \tag{1.23}$$

In the following we shall derive the dynamical equation expressed in terms of the Lagrangian. In this derivation we begin with the vector formulation of Newton's second law applied to the parts of the system and show how this can be reformulated in terms of the generalized coordinates.

For the discussion to follow it may be of interest to give a *geometrical* representation of the constraints and generalized coordinates. Again we assume the system to be composed of N parts, the position of each part being specified by a three-dimensional vector, $\mathbf{r}_k, k = 1, 2, ..., N$. Together these vectors can be thought of as a vector in a $3N$ dimensional space,

$$\mathbf{R} = \{\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N\} = \{x_1, y_1, z_1, ..., x_N, y_N, z_N\} \tag{1.24}$$

which is a Cartesian product of N copies of 3-dimensional, physical space, where each copy corresponds to one of the parts of the composite system. When the vector $\mathbf{R}$ is specified that means that the positions of all parts of the system are specified.

The constraints impose a restriction on the position of the parts, which can be expressed through the functional dependence of $\mathbf{R}$ on the generalized coordinates

$$\mathbf{R} = \mathbf{R}(q_1, q_2, ..., q_d; t) \tag{1.25}$$

When the generalized coordinates q are varied the vector $\mathbf{R}$ will trace out a surface (or submanifold) of dimension d in the $3N$ dimensional vector space. This surface[1], where the constraints are satisfied, represents the configuration space of the system, and the set of generalized coordinates are coordinates on this surface, as schematically shown in the figure. Note that the configuration space will in general *not* be a vector space like the $3N$ dimensional space.
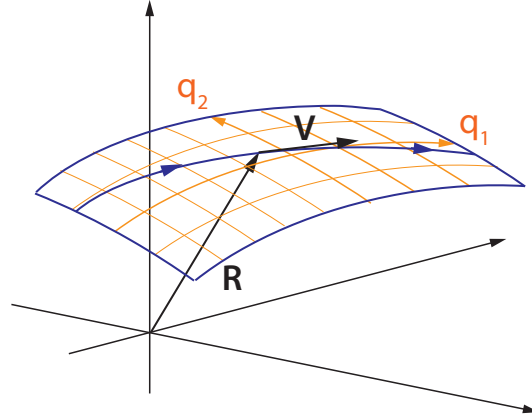
Figure 1.6: Geometrical representation of the configuration space as a hypersurface in the $3N$ dimensional vector space defined by the Cartesian coordinates of the N small parts of the physical system. The points $\mathbf{R}$ that are confined to the hypersurface are those that satisfy the constraints and the generalized coordinates define a coordinate system that covers the surface (orange lines). The time evolution of the system describes a curve $\mathbf{R}(t)$ on the surface (blue line). If the surface is time independent, the velocity $\mathbf{V} = \dot{\mathbf{R}}$ defines at all times a tangent vector to the surface.

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[1]Such a higher-dimensional surface is often referred to as a *hypersurface*.

Consider first the constraints to be *time independent*, with the d-dimensional surface as a fixed surface in the $3N$ dimensional vector space, and assume that we turn on the time evolution, so that the coordinates become time dependent, $q = q(t)$. The composite position vector $\mathbf{R}$ then describes the time evolution of the system in the form of a curve in $\mathbb{R}^{3N}$ that is constrained to the d-dimensional surface,

$$\mathbf{R}(t) = \mathbf{R}(q(t)) \tag{1.26}$$

and the velocity vector $\mathbf{V} = \dot{\mathbf{R}}$ is a tangent vector to the surface, as shown in the figure.

If the constraints instead are *time dependent*, and the surface therefore changes with time, the velocity vector $\mathbf{V}$ will in general no longer be a tangent vector to the surface at any given time, due to the motion of the surface itself. However, for the discussion to follow it is convenient to introduce a type of displacement which corresponds to a situation where we "freeze" the surface at a given time and then move the coordinates $q \to q + \delta q$. The corresponding displacement vector $\delta \mathbf{R}$ is a tangent vector to the surface. When the constraints are time dependent such a displacement can obviously not correspond to a physical motion of the system, since the displacement takes place at a fixed time. For that reason one refers to this type of change of position as a *virtual* displacement.

1.3 Virtual displacements

We again express the position vectors of each part of the system as functions of the generalized coordinates,

$$\mathbf{r}_k = \mathbf{r}(q_1, q_2, ..., q_d; t) \tag{1.27}$$

where we have included the possibility of time dependent constraints. We refer to this as an *explicit* time dependence, since it does not come from the change of the coordinates q during motion of the system. A general displacement of the positions, which satisfies the constraints, can then be decomposed in a contribution from the change of general coordinates q at fixed t and a contribution from change of t with fixed q,

$$d\mathbf{r}_k = \sum_{j=1}^{d} \frac{\partial \mathbf{r}_k}{\partial q_j} dq_j + \frac{\partial \mathbf{r}_k}{\partial t} dt \tag{1.28}$$

In particular, if we consider the dynamical evolution of the system, the velocities can be expressed as

$$\dot{\mathbf{r}}_k = \sum_{j=1}^{d} \frac{\partial \mathbf{r}_k}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_k}{\partial t} \tag{1.29}$$

The motion in part comes from the time evolution of the generalized coordinates, $q = q(t)$, and in part from the motion of the surface defined by the constraint equations.

Note that in the above expression for the velocity we distinguish between the two types of time derivatives, referred to as *explicit* time derivative,

$$\frac{\partial}{\partial t}$$

and *total* time derivative

$$\frac{d}{dt} = \sum_{j=1} \dot{q}_j \frac{\partial}{\partial q_j} + \frac{\partial}{\partial t}$$

The first one is simply the partial time derivative, which is well defined when acting on any function that depends on coordinates q and time t. The total time derivative, on the other hand, is meaningful only when we consider a particular time evolution, or path, expressed by time dependent coordinates $q = q(t)$. It acts on variables that are defined on such a path in configuration space.

A virtual displacement corresponds to a displacement δq_i at *fixed time* t. This means that it does not correspond in general to a real, physical displacement, which will always take a finite time, but rather to an imagined displacement, consistent with the constraints for a given instant. Thus a change caused by virtual displacements measures the functional dependence of a variable on the generalized coordinates q. For the position vectors $\mathbf{r}_k$, the change under a virtual displacement can be written as

$$\delta \mathbf{r}_k = \sum_{j=1}^{d} \frac{\partial \mathbf{r}_k}{\partial q_j} \delta q_j \tag{1.30}$$

There is no contribution from the explicit time dependence, as it is for the general displacement (1.28).

1.4   Applied forces and constraint forces

The total force acting on part k of the system can be thought of as consisting of two parts,

$$\mathbf{F}_k = \mathbf{F}_k^a + \mathbf{f}_k \tag{1.31}$$

where $\mathbf{f}_k$ is the generally unknown *constraint force*, and $\mathbf{F}_k^a$ is the so-called *applied force*. The constraint forces can be regarded as a response to the applied forces caused by the presence of constraints.

As a simple example, consider a body sliding on an inclined plane under the action of the gravitational force. The forces acting on the body are the gravitational force, the normal force from the plane on the body and finally the friction force acting parallel to the plane. The normal force is counteracting the normal component of the gravitational force and thus preventing any motion in the direction perpendicular to the plane. This is the force we identify as the constraint force, and the other forces we refer to as applied forces.

We assume now that a general constraint force is similar to the normal force, in the sense of being orthogonal to any *virtual* displacement of the system. We write this condition as

$$\mathbf{f} \cdot \delta \mathbf{R} = 0 \tag{1.32}$$

where we have introduced a $3N$ dimensional vector $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_N)$ for he constraint forces, similar to the $3N$ dimensional position vector $\mathbf{R} = (\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N)$. Thus $\mathbf{f}$ specifies the forces
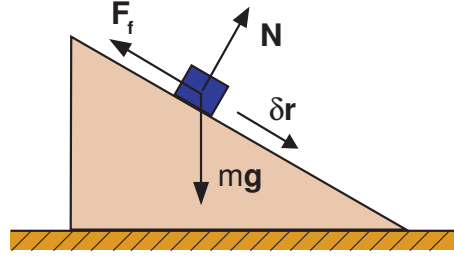
Figure 1.7: A body on an inclined plane. The applied forces are the force of gravity and the friction. The normal force is a constraint force. It can be viewed as a reaction to other forces that act perpendicular to the plane and neutralizes the component of the forces that would otherwise create motion in conflict with the constraints. The direction of virtual displacements $\delta\mathbf{r}$ is along the inclined plane. This is so even if the plane itself is moving since a *virtual* displacement is an imagined displacement at fixed time.

acting on all the N parts of the system. The vector $\mathbf{f}$ acts perpendicular to the surface defined by the constraints and can be viewed as a reaction to other (applied) forces that have components perpendicular to the surface. Since $\mathbf{f}$ has vanishing components along the surface, it will not affect the motion of the system in these directions and it can therefore be eliminated by changing from Cartesian to generalized coordinates. This we shall exploit in the following.

The orthogonality condition (1.32) can be re-written in terms of three-dimensional vectors as

$$\sum_k \mathbf{f}_k \cdot \delta\mathbf{r}_k = 0 \tag{1.33}$$

and we note that the expression can be interpreted as the *work* performed by the constraint forces under the displacement $\delta\mathbf{r}_k$. Thus, *the work performed by the constraint forces under any virtual displacement vanishes*. One should note that this does *not* mean that the work done by a constraint force under the time evolution will always vanish, since the *real* displacement $\mathbf{dr}_k$ may have a component along the constraint force if the constraint is time dependent.

1.5 Static equilibrium and the principle of virtual work

Let us first assume the mechanical system to be in static equilibrium. This means that there is a balance between the forces acting on each part of the system so that there is no motion,

$$\mathbf{F}_k^a + \mathbf{f}_k = 0 , \quad k = 1, 2, ..., N \tag{1.34}$$

Since the virtual work performed by the constraint forces always vanishes (see (1.33)), the virtual work done by the applied forces will in a situation of equilibrium also vanish,

$$\sum_k \mathbf{F}_k^a \cdot \delta\mathbf{r}_k = 0 \tag{1.35}$$
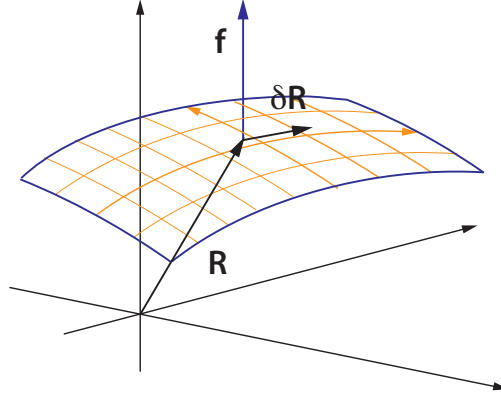
Figure 1.8: The constraint force $\mathbf{f}$ is a force that is perpendicular to the virtual displacements, and therefore to the hypersurface that defines the configuration space.

This form of the condition for static equilibrium is often referred to as the *principle of virtual work*.

This condition can be re-expressed in terms of the 3N-dimensional vectors as

$$\mathbf{F}^a \cdot \delta\mathbf{R} = 0 \tag{1.36}$$

Geometrically this means that in a point of equilibrium on the d dimensional surface in $\mathbb{R}^{3N}$, the applied force has to be orthogonal to the surface. This seems easy to understand: If the applied force has a non-vanishing component along the surface this will induce a motion of the system in that direction. That cannot happen in a point of static equilibrium.

Let us reconsider the virtual work and re-express it in terms of the generalized coordinates. We have

$$\begin{aligned}
\delta W &= \sum_k \mathbf{F}_k \cdot \delta\mathbf{r}_k \\
&= \sum_k \mathbf{F}_k^a \cdot \delta\mathbf{r}_k \\
&= \sum_k \sum_j \mathbf{F}_k^a \cdot \frac{\partial\mathbf{r}_k}{\partial q_j}\delta q_j \\
&= \sum_j \mathcal{F}_j \delta q_j
\end{aligned} \tag{1.37}$$

where, at the last step we have introduced the *generalized force*, defined by

$$\mathcal{F}_j = \sum_k \mathbf{F}_k^a \cdot \frac{\partial\mathbf{r}_k}{\partial q_j} \tag{1.38}$$

We note that the generalized force depends only on the applied forces, not on the constraint forces. At equilibrium the virtual work δW should vanish for *any* virtual displacement δq, and

since all the coordinates q_i are independent, that means that the coefficients of δq_i have all to vanish

$$\mathcal{F}_j = 0 , \quad j = 1, 2, ..., d \quad \text{(equilibrium condition)} \tag{1.39}$$

Thus at equilibrium all the generalized forces have to vanish. Note that the same conclusion cannot be drawn about the applied forces, since the coefficients of $\mathbf{r}_k$ may not all be independent due to the constraints.
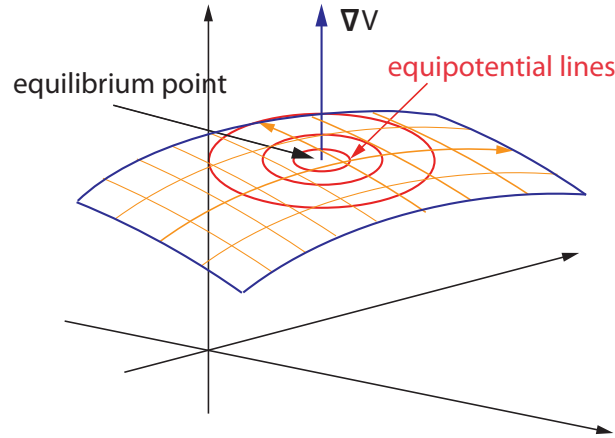
Figure 1.9: Equilibrium point. At this point the derivatives of the potential with respect to the generalized coordinates vanish and the gradient of the potential is perpendicular to the surface defined by the configuration space.

In the special cases where the applied forces can be derived from a potential $V(\mathbf{r}_1, \mathbf{r}_2, ...)$,

$$\mathbf{F}_k^a = -\boldsymbol{\nabla}_k V \tag{1.40}$$

with $\boldsymbol{\nabla}_k$ is the gradient with respect to the coordinates $\mathbf{r}_k$ of part k of the system, the generalized force can be expressed as a gradient in configuration space,

$$\mathcal{F}_j = -\sum_k \boldsymbol{\nabla}_k V \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} = -\frac{\partial V}{\partial q_j} \tag{1.41}$$

The equilibrium condition is then simply

$$\frac{\partial V}{\partial q_j} = 0 , \quad j = 1, 2, ..., d \quad \text{(equilibrium condition)} \tag{1.42}$$

which means that the potential has a local minimum (or more generally a stationary value) at a point of equilibrium in the d dimensional configuration space of the system.

Chapter 2

Lagrange's equations

2.1 D'Alembert's principle and Lagrange's equations

We now examine how the description of the equilibrium condition discussed in the previous section can be extended to non-equilibrium dynamics. The equilibrium condition is then replaced by Newton's second law, in the form

$$m_k \ddot{\mathbf{r}}_k = \mathbf{F}_k = \mathbf{F}_k^a + \mathbf{f}_k \tag{2.1}$$

and for a virtual displacement this implies

$$\sum_k (\mathbf{F}_k^a - m_k \ddot{\mathbf{r}}_k) \cdot \delta \mathbf{r}_k = 0 \tag{2.2}$$

which is referred to as *D'Alembert's principle*[1]. The important point is, like in the equilibrium case, that by introducing the virtual displacements in the equation one eliminates the (unknown) constraint forces. The expression is in fact similar to the equilibrium condition although the "force" which appears in this expression, $\mathbf{F}_k^a - m_k \ddot{\mathbf{r}}_k$, is not simply a function of the positions $\mathbf{r}_k$, but also of the accelerations. Nevertheless, the method used to express the equilibrium condition in terms of the generalized coordinates can be generalized to the dynamical case and that leads to Lagrange's equations. In order to show this we have to rewrite the expressions.

The first part of Eq. (2.2) is easy to handle and we write it as before as

$$\sum_k \mathbf{F}_k^a \cdot \delta \mathbf{r}_k = \sum_j \mathcal{F}_j \delta q_j \tag{2.3}$$

with $\mathcal{F}_j$ as the generalized force. Also the second part can be expressed in terms of variations in the generalized coordinates and we rewrite D'Alembert's principle as

$$\sum_j \left(\mathcal{F}_j - \sum_k m_k \ddot{\mathbf{r}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} \right) \delta q_j = 0 \tag{2.4}$$

---

[1] Jean le Rond d'Alembert (1717 – 1783) was a French mathematician, physicist and philosopher.

Since this should vanish for arbitrary virtual displacements, the coefficients of δq_j have to vanish, and this gives

$$\sum_k m_k \ddot{\mathbf{r}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} = \mathcal{F}_j, \quad j = 1, 2, ..., d \tag{2.5}$$

This can be seen as a generalized form of Newton's second law, and the objective is now to re-express the left hand side in terms of the generalized coordinates and their velocities.

To proceed we split the acceleration term in two parts,

$$\sum_k m_k \ddot{\mathbf{r}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} = \frac{d}{dt}\left(\sum_k m_k \dot{\mathbf{r}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j}\right) - \sum_k m_k \dot{\mathbf{r}}_k \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_k}{\partial q_j}\right) \tag{2.6}$$

and examine each of these separately. To rewrite the first term we first note how the velocity vector depends on the generalized coordinates and their velocities,

$$\dot{\mathbf{r}}_k \equiv \frac{d}{dt}\mathbf{r}_k = \sum_j \frac{\partial \mathbf{r}_k}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_k}{\partial t} \tag{2.7}$$

The expression shows that whereas the position vector depends only on the generalized coordinates, and possibly on time (when there is *explicit* time dependence),

$$\mathbf{r}_k = \mathbf{r}_k(q, t) \tag{2.8}$$

that is not the case for the velocity vector $\dot{\mathbf{r}}$, which depends also on the time derivative $\dot{q}$. At this point we make an extension of the number of independent variables in our description. We simply consider the generalized velocities $\dot{q}_j$ as variables that are independent of the generalized coordinates q_j. Is that meaningful? Yes, as long as we consider all possible motions of the system, we know that to specify the positions will not also determine the velocities. So, assuming all the positions to be specified, if we change the velocities that means that we change from one possible motion of the system to another.

In the following we shall therefore consider all coordinates $q = q_1, q_2, ..., q_d$, all velocities $\dot{q} = \dot{q}_1, \dot{q}_2, ..., \dot{q}_d$, and time t to be independent variables. Of course, when we consider a particular time evolution, $q = q(t)$ then both q_j and $\dot{q}_j$ become dependent on t. So the challenge is not to mix these two views, the first one when all 2d+1 variables are treated as independent, and the second one when all of them are considered as time dependent functions. However, the idea is not much more complicated than with the usual space and time coordinates (x, y, z, t). In general they can be considered as independent variables, but when applied to the motion of a particle, the space coordinates of the particle become dependent of time, $x = x(t)$ etc. As already discussed these two views are captured in the difference between the partial derivative with respect to time, $\frac{\partial}{\partial t}$ and the total derivative $\frac{d}{dt}$. The latter we may now write as

$$\frac{d}{dt} = \sum_j \left(\ddot{q}_j \frac{\partial}{\partial \dot{q}_j} + \dot{q}_j \frac{\partial}{\partial q_j}\right) + \frac{\partial}{\partial t} \tag{2.9}$$

since we have introduced $\dot{q}_j$ as independent variables.

From Eq.(2.7) we deduce the following relation between partial derivatives of velocities and positions

$$\frac{\partial \dot{\mathbf{r}}_k}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_k}{\partial q_j} \tag{2.10}$$

which gives

$$\dot{\mathbf{r}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} = \dot{\mathbf{r}}_k \cdot \frac{\partial \dot{\mathbf{r}}_k}{\partial \dot{q}_j} = \frac{1}{2}\frac{\partial}{\partial \dot{q}_j}\dot{\mathbf{r}}_k^2 \tag{2.11}$$

This further gives

$$\sum_k m_k \frac{d}{dt}(\dot{\mathbf{r}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j}) = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}(\sum_k \frac{1}{2}m_k \dot{\mathbf{r}}_k^2)\right] = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) \tag{2.12}$$

with T as the kinetic energy of the system. This expression simplifies the first term in the right-hand side of Eq.(2.6).

The second term we also re-write, and we use now the following identity

$$\begin{aligned}
\frac{d}{dt}(\frac{\partial \mathbf{r}_k}{\partial q_j}) &= \sum_l \frac{\partial^2 \mathbf{r}_k}{\partial q_j \partial q_l}\dot{q}_l + \frac{\partial}{\partial t}(\frac{\partial \mathbf{r}_k}{\partial q_j}) \\
&= \frac{\partial}{\partial q_j}(\sum_l \frac{\partial \mathbf{r}_k}{\partial q_l}\dot{q}_l + \frac{\partial \mathbf{r}_k}{\partial t}) \\
&= \frac{\partial \dot{\mathbf{r}}_k}{\partial q_j} \tag{2.13}
\end{aligned}$$

which shows that the order of differentiations $\frac{d}{dt}$ and $\frac{\partial}{\partial q_j}$ can be interchanged. This gives

$$\begin{aligned}
\sum_k m_k \dot{\mathbf{r}}_k \cdot \frac{d}{dt}(\frac{\partial \mathbf{r}_k}{\partial q_j}) &= \sum_k m_k \dot{\mathbf{r}}_k \cdot \frac{\partial \dot{\mathbf{r}}_k}{\partial q_j} \\
&= \frac{\partial}{\partial q_j}(\sum_k \frac{1}{2}m_k \dot{\mathbf{r}}_k^2) \\
&= \frac{\partial T}{\partial q_j} \tag{2.14}
\end{aligned}$$

We have then shown that both terms in Eq. (2.6) can be expressed in terms of partial derivatives of the kinetic energy.

By collecting terms, the equation of motion now can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \mathcal{F}_j, \quad j = 1, 2, ..., d \tag{2.15}$$

In this form the position vectors $\mathbf{r}_k$ have been eliminated from the equation, which only makes reference to the generalized coordinates and their velocities. The equation we have arrived at can be regarded as a reformulation of Newton's 2. law. It does not have the usual vector form. Instead there is one independent equation for each degree of freedom of the system.

We will make a further modification of the equations of motion based on the assumption that the applied forces can be derived from a potential. For the generalized forces $\mathcal{F}_j$ this means,

$$\mathcal{F}_j = -\frac{\partial V}{\partial q_j} \tag{2.16}$$

and the dynamical equation can then be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}, \quad j = 1, 2, ..., d \tag{2.17}$$

We further note that since the potential only depends on the coordinates q_j and not on the velocities $\dot{q}_j$ the equation can be written as

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0, \quad j = 1, 2, ..., d \tag{2.18}$$

This motivates the introduction of the *Lagrangian*, defined by $L = T - V$. In terms of this new function the dynamical equation can be written in a compact form, known as Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, 2, ..., d \tag{2.19}$$

Lagrange's equation gives a simple and elegant description of the time evolution of the system. The dynamics is specified by a single, scalar function - the Lagrangian -, and the dynamical equation has a form that shows similarities with the equation which determines the equilibrium in a static problem. In that case the coordinate dependent potential is the relevant scalar function. In the present case it is the Lagrangian, which will in general depend on velocities as well as coordinates. It may in addition depend explicitly on time, in the following way

$$L(q, \dot{q}, t) = T(q, \dot{q}, t) - V(q, t) \tag{2.20}$$

where explicit time dependence appears if the Cartesian coordinates are expressed as time dependent functions of the generalized coordinates (in most cases due to time dependent constraints).

Note that the potential is assumed to only depend on coordinates, but not on velocities, but the formalism has a natural extension to velocity dependent potentials. Such an extension is particularly relevant to the description of charged particles in electromagnetic fields, where the magnetic force depends on the velocity of the particles. We will later show in detail how a Lagrangian can be designed for such a system.

Lagrange's equation motivates a general, systematic way to analyze a mechanical system which satisfies the general condition (2.16). It consist of the following steps

1. Determine a set of generalized coordinates $q = (q_1, q_2, ..., q_d)$ that fits the system to be analyzed, one coordinate for each degree of freedom.

2. Find the potential energy V and the kinetic energy T expressed as functions of coordinates q, velocities $\dot{q}$ and possibly time t.

3. Write down the set of Lagrange's equations, one equation for each generalized coordinate.

4. Solve the set of Lagrange's equation with the relevant initial conditions.

Other ways to analyze the system, in particular the vector approach of Newtonian mechanics, would usually also, when the unknown forces are eliminated, end up with a set of equations, like in point. 4. However, the method outlined above is in many cases more convenient, since it is less dependent on a visual understanding of the action of forces on different parts of the mechanical system.

In the following we illustrate the Lagrangian method by some simple examples.

2.1.1 Examples

Particle in a central potential, planar motion

We consider a point particle of mass m which moves in a rotationally invariant potential $V(r)$. For simplicity we assume the particle motion to be constrained to a plane (the x, y-plane). We follow the schematic approach outlined above.

1. Since the particle can move freely in the plane, the system has two degrees of freedom and a convenient set of (generalized) coordinates is, due to the rotational invariance, the polar coordinates (r, ϕ), with $r = 0$ as the center of the potential.

2. The potential energy, expressed in these coordinates, is simply the function $V(r)$, while the kinetic energy is $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$, and the Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r) \tag{2.21}$$

3. There are two Lagrange's equations, corresponding to the two coordinates r and ϕ. The r equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad \Rightarrow \quad m\ddot{r} - mr\dot{\phi}^2 + \frac{\partial V}{\partial r} = 0 \tag{2.22}$$

and the ϕ equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \quad \Rightarrow \quad \frac{d}{dt}(mr^2\dot{\phi}) = 0 \tag{2.23}$$

From the last one follows

$$mr^2\dot{\phi} = \ell \tag{2.24}$$

with ℓ as a constant. The physical interpretation of this constant is the angular momentum of the particle.

4. Eq.(2.24) can be used to solve for $\dot{\phi}$, and inserted in (2.22) this gives the following differential equation for $r(t)$

$$m\ddot{r} - \frac{\ell^2}{mr^3} + \frac{\partial V}{\partial r} = 0 \tag{2.25}$$

To proceed one should solve this equation with given initial conditions, but since we are less focussed on solving the equation of motion than on illustrating the use of the Lagrangian formalism, we stop the analysis of the system at this point.

For the case discussed here Newton's second law, in vector form, would soon lead to the same equation of motion, with a change from Cartesian to polar coordinates. The main difference between the two approaches is then that with the vector formulation this change of coordinates is made *after* the (vector) equation of motion has been established, whereas in Lagrange's formulation the choice of coordinates is done *before* Lagrange's equations are established.

Atwood's machine

We consider the composite system illustrated in the figure. Two bodies of mass m_1 and m_2 are interconnected by a cord of fixed length that is suspended over a pulley. We assume that the two bodies move only vertically, and that the cord rolls over the pulley without sliding. The pulley has a moment of inertia I. We will establish the Lagrange equation for the composite system.
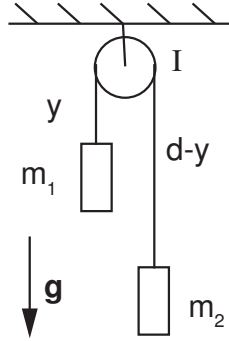
Figure 2.1: Atwood's machine with two weights.

1. The system has only one degree of freedom, and we may use the length of the cord on the left-hand side of the pulley, denoted y, as the corresponding (generalized) coordinate. This coordinate measures the (negative) height of the mass m_1 relative to the position of the pulley. The corresponding position of the mass m_2 is $d - y$, with d as the sum of the two parts of the cord on both sides of the pulley. With R as the radius of the pulley, the angular coordinate ϕ of the pulley is related to the coordinate y by $y = R\phi$.

2. The potential energy, expressed as a function of y is

$$V = -m_1 g y - m_2 (d - y) g = (m_2 - m_1) g y - m_2 d \tag{2.26}$$

where the last term is an unimportant constant. For the kinetic energy we find the expression

$$T = \frac{1}{2}m_1\dot{y}^2 + \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}I\dot{\phi}^2 = \frac{1}{2}(m_1 + m_2 + \frac{I}{R^2})\dot{y}^2 \tag{2.27}$$

This gives the following Lagrangian

$$L = \frac{1}{2}(m_1 + m_2 + \frac{I}{R^2})\dot{y}^2 + (m_1 - m_2)gy + m_2 d \tag{2.28}$$

It is the functional dependence of L on y and $\dot{y}$ that is interesting, since it is the partial derivative of L with respect to these two variables that enter into Lagrange's equations.

3. The partial derivatives of the Lagrangian, with respect to coordinate and velocity, are

$$\frac{\partial L}{\partial y} = (m_1 - m_2)g, \quad \frac{\partial L}{\partial \dot{y}} = ((m_1 + m_2 + \frac{I}{R^2})\dot{y} \tag{2.29}$$

and for the Lagrange equation this gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\Rightarrow \quad (m_1 + m_2 + \frac{I}{R^2})\ddot{y} + (m_2 - m_1)g = 0$$

$$\Rightarrow \quad \ddot{y} = \frac{m_1 - m_2}{m_1 + m_2 + \frac{I}{R^2}}g \tag{2.30}$$

This equation shows that the weights move with constant acceleration, and with specified initial data the solution is easy to find.

Pendulum with accelerated point of suspension

As discussed in the text, the Lagrangian formulation may include situations with explicit time dependence. We consider a particular example of this kind. Consider a pendulum that performs oscillations in the x, y-plane, with y as the vertical direction. The pendulum bob has mass m and the pendulum rod is rigid with fixed length l and is considered as massless. It is suspended in a point A which moves with constant acceleration in the x-direction, so that the coordinates of this point are

$$x_A = \frac{1}{2}at^2, \quad y_A = 0 \tag{2.31}$$

with a as the (constant) acceleration. We will establish the equation of motion of the pendulum.

1. The pendulum bob moves in a plane with a fixed distance to the point of suspension. This means that the system has one degree of freedom, and we choose the angle θ between the pendulum rod and the vertical direction as generalized coordinate. Expressed in terms of θ the Cartesian coordinates of the pendulum bob are

$$\begin{aligned} x &= x_A + l\sin\theta = l\sin\theta + \frac{1}{2}at^2 \\ y &= -l\cos\theta \end{aligned} \tag{2.32}$$
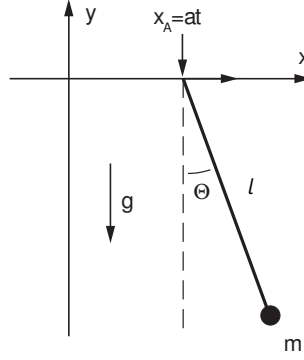
Figure 2.2: Pendulum with accelerated point of suspension

with the corresponding velocities

$$
\begin{aligned}
\dot{x} &= l\dot{\theta}\cos\theta + at \\
\dot{y} &= l\dot{\theta}\sin\theta
\end{aligned}
\tag{2.33}
$$

2. The potential energy is

$$
V = -mgl\cos\theta
\tag{2.34}
$$

and the kinetic energy is

$$
\begin{aligned}
T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\
&= \frac{1}{2}m(l^2\dot{\theta}^2 + 2atl\dot{\theta}\cos\theta + a^2t^2)
\end{aligned}
\tag{2.35}
$$

This gives the following expression for the Lagrangian

$$
L = \frac{1}{2}m(l^2\dot{\theta}^2 + 2atl\dot{\theta}\cos\theta + a^2t^2) + mgl\cos\theta
\tag{2.36}
$$

As expected it depends on the generalized coordinate θ, its velocity $\dot{\theta}$ and also explicitly on time t. The time dependence follows from the (externally determined) motion of the point of suspension.

3. Lagrange's equation has the standard form

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0
\tag{2.37}
$$

and can be expressed as a differential equation for θ by evaluating the partial derivatives of L with respect to θ and $\dot{\theta}$,

$$
\begin{aligned}
\frac{\partial L}{\partial \theta} &= -ma\,t\,l\dot{\theta}\sin\theta - mg\,l\sin\theta \\
\frac{\partial L}{\partial \dot{\theta}} &= ml^2\dot{\theta} + ma\,t\,l\cos\theta
\end{aligned}
\tag{2.38}
$$

This gives

$$ml^2\ddot{\theta} + ml(g\sin\theta + a\cos\theta) = 0 \tag{2.39}$$

We note that the term which is linear in $\dot{\theta}$ disappears from the equation. It is convenient to re-write the equation by introducing a fixed angle θ_0, defined by

$$g = \sqrt{g^2 + a^2}\cos\theta_0, \quad a = -\sqrt{g^2 + a^2}\sin\theta_0 \tag{2.40}$$

The equation of motion is then

$$ml^2\ddot{\theta} + ml\sqrt{g^2 + a^2}\sin(\theta - \theta_0) = 0 \tag{2.41}$$

and we recognize this as a standard pendulum equation, but with equilibrium position for the rotated direction $\theta = \theta_0$ rather than for the vertical direction $\theta = 0$, and with a stronger effective acceleration of gravity $\sqrt{g^2 + a^2}$.

Again we leave out the last point which is to solve this equation with given boundary conditions. We only note that the form of the equation of motion is in fact what we should expect from general reasoning. If we consider the motion in an accelerated reference frame, which follows the motion of the point of suspension A, we eliminate the explicit time dependence caused by the motion of the point A. However, in such an accelerated frame there will be be a fictitious gravitational force caused by the acceleration. The corresponding acceleration of gravity is a and the direction is opposite of the direction of acceleration, which means in the negative x-direction. In this frame the effective gravitational force therefore has two components, the true gravitational force in the negative y-direction and the fictitious gravitational force in the negative x-direction. The effective acceleration of gravity is therefore $\sqrt{g^2 + a^2}$ and the direction is given by the angle θ_0. The pendulum will perform oscillations about the direction of the effective gravitational force.

2.2 Symmetries and constants of motion

There is in physics a general and interesting connection between *symmetries* of a physical system and constants of motion. Well known examples of this kind are the relations between rotational symmetry and spin conservation and between translational symmetry and conservation of linear momentum. The Lagrangian formulation of classical mechanics gives a convenient way to *derive* constants of motion from symmetries in a direct way. A general form of this connection was shown in field theory by Emmy Noether (*Noether's theorem*) in 1918. In a simpler form it is valid also for systems with a discrete set of variables, as we discuss here. One of the important consequences of finding constants of motion is that they can be used to reduce the number of variables in the problem. And even if the equations of motion cannot be fully solved, the conserved quantities may give important partial information about the motion of the system.

Before discussing this connection between symmetries and constants of motion, it may be of interest with some general comments about symmetries in physics. *Symmetry* may have slightly different meanings depending on whether we consider a static or a dynamical situation. A body is symmetric under rotations if it looks identical when viewed from rotated positions.

Similarly a crystal is symmetric under a discrete group of rotations, translations and reflections, when the lattice structure is invariant under these transformations. These are static situations, where the symmetry transformations leave unchanged the body or structure that we consider.[2]

In a dynamical situation we refer to certain transformations as symmetries *when they leave the equations of motion invariant* rather than physical bodies or structures. In general the equations of motion take different forms depending on the coordinates we use, but in some cases a change of coordinates will introduce no change in the form of the equations. A well-known example is the case of inertial frames, where Newton's 2. law has the same form whether we use the coordinates of one inertial reference frame or another. It is this form of dynamical symmetry that is of interest for the further discussion.

Let us describe the time evolution of system by the set of coordinates $q = \{q_i, i = 1, 2, ..., d\}$, where d is the number of degrees of freedom of a system. A particular solution of the equations of motion we denote by $q = q(t)$. A coordinate transformation is a mapping

$$q \to q' = q'(q, t) \ , \tag{2.42}$$

where we may regard the new set of coordinates q' as a function of the old set q (and possibly of time t). This transformation is a symmetry transformation of the system if any solution $q(t)$ of the equation of motions *is mapped into a new solution* $q'(t)$ of the same equations of motion.

We shall here focus on symmetries that follows from *invariance of the Lagrangian* under coordinate transformations, in the sense

$$L(q', \dot{q}', t) = L(q, \dot{q}, t) \tag{2.43}$$

This equation means that under a change of variables $q \to q'$ the Lagrangian will have the same *functional dependence* of the new and old variables. Since the Lagrangian determines the form of the equations of motion, this implies that the time evolution of the system, described by coordinates $q(t)$ and by coordinates $q'(t)$ will satisfy the same equations of motion, and the mapping $q \to q'$ is thus a symmetry transformation of the system.

If the Lagrangian is invariant under a set of *continuous* coordinate transformations, then it follows from Lagrange's equations that there is one or more constants of motion associated with these transformations. This we will show quite generally in the following, but first we will discuss the special case where the Lagrangian has a *cyclic coordinate*.

2.2.1 Cyclic coordinates

We consider a Lagrangian of the general form

$$L = L(q, \dot{q}, t) \tag{2.44}$$

with $q = (q_1, q_2, ..., q_d)$ as the set of generalized coordinates. We further assume that the Lagrangian is independent of one of the coordinates, say q_1. This means

$$\frac{\partial L}{\partial q_1} = 0 \tag{2.45}$$

---

[2]The symmetries we consider are often restricted to space (or space-time) transformations, but more general types of symmetry transformations may be considered, which may involve the change of particles properties like charge and intrinsic spin.

and we refer to q_1 as a *cyclic coordinate*. From Lagrange's equation then follows

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = 0 \tag{2.46}$$

This means that the physical variable

$$p_1 \equiv \frac{\partial L}{\partial \dot{q}_1} \tag{2.47}$$

which we refer to as the *conjugate momentum*[3] to the coordinate q_1, is a constant of motion. Thus, for every cyclic coordinate there is a constant of motion.

The presence of a cyclic coordinate can be used to reduce the number of independent variables from d to $d - 1$. The coordinate q_1 is already eliminated, since it does not appear in the Lagrangian, but $\dot{q}_1$ is generally present. However also this can be eliminated by using the fact that p_1 is a constant of motion. Let us write this condition in the following way,

$$\frac{\partial L}{\partial \dot{q}_1} = p_1(q_2, ..., q_d; \dot{q}_1, \dot{q}_2, ..., \dot{q}_d; t) = k \tag{2.48}$$

with k as a constant. In this equation we have written explicitly the functional dependence of p_1 on all coordinates and velocities. Since q_1 is cyclic it does not appear in the expression. The equation can be solved for $\dot{q}_1$,

$$\dot{q}_1 = f(q_2, ...q_d; \dot{q}_2, ...\dot{q}_d; k; t) \tag{2.49}$$

with the function f as the unspecified solution. In this way both q_1 and $\dot{q}_1$ are eliminated as basic variables, and the number of independent equations of motion are reduced from d to $d-1$. Note, however, that the $d - 1$ Lagrange equations will not only depend on the $d - 1$ remaining coordinates and their velocities, but also on the constant of motion k. The value of this constant is determined by the initial conditions.

2.2.2 Example: Point particle moving on the surface of a sphere

We consider a point particle of mass m that moves without friction on the surface of a sphere, under the influence of gravity. The gravitational field is assumed to point in the negative z-direction. This system has two degrees of freedom, since the three Cartesian coordinates (x, y, z) of the particle are subject to one constraint equation $r = \sqrt{x^2 + y^2 + z^2} = const.$ As generalized coordinates we chose the polar angles (ϕ, θ), so that the Cartesian coordinates are

$$\begin{aligned} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\theta \end{aligned} \tag{2.50}$$

---

[3]The conjugate momentum is also referred to as *generalized momentum* or *canonical momentum*.

with r as a constant. The corresponding velocities are

$$
\begin{aligned}
\dot{x} &= r(\cos\phi\cos\theta\,\dot{\theta} - \sin\phi\sin\theta\,\dot{\phi}) \\
\dot{y} &= r(\sin\phi\cos\theta\,\dot{\theta} + \cos\phi\sin\theta\,\dot{\phi}) \\
\dot{z} &= -r\sin\theta\,\dot{\theta}
\end{aligned}
\tag{2.51}
$$

The potential energy is

$$
V = mgz = -mgr\cos\theta
\tag{2.52}
$$

with g as the acceleration of gravitation, and the kinetic energy is

$$
T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}mr^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)
\tag{2.53}
$$

This gives the following expression for the Lagrangian

$$
L = \frac{1}{2}mr^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + mgr\cos\theta
\tag{2.54}
$$

Clearly ϕ is a cyclic coordinate,

$$
\frac{\partial L}{\partial\phi} = 0
\tag{2.55}
$$

and therefore Lagrange's equation for this variable reduces to

$$
\frac{\partial L}{\partial\dot{\phi}} = mr^2\dot{\phi}\sin^2\theta = l
\tag{2.56}
$$

with l as a constant. Lagrange's equation for the variable θ is

$$
mr^2\ddot{\theta} - mr^2\dot{\phi}^2\sin\theta\cos\theta + mgr\sin\theta = 0
\tag{2.57}
$$

To eliminate the variable ϕ from the equation, we express, by use of (2.56), $\dot{\phi}$ in terms of the constant of motion l,

$$
\dot{\phi} = \frac{l}{mr^2\sin^2\theta}
\tag{2.58}
$$

Inserted in (2.57) this gives

$$
mr^2\ddot{\theta} - \frac{l^2\cos\theta}{mr^2\sin^3\theta} + mgr\sin\theta = 0
\tag{2.59}
$$

This illustrates the general discussion of cyclic coordinates. In the present case the elimination of the coordinate ϕ has reduced the equations of motion to one, and the only remaining trace of the coordinate ϕ is the appearance of the conserved quantity l in the equation.

In this example the relation between symmetries and conserved quantities is clear. Thus, the independence of ϕ means that the Lagrangian is invariant under rotations about the z axis and at the same time the cyclic coordinate gives rise to a conserved quantity ℓ. It is straight forward to show that this constant has the physical interpretation as the z-component of the orbital angular momentum of the particle.

2.2.3 Symmetries of the Lagrangian

The existence of a cyclic coordinate will always imply the presence of a continuous symmetry in the form given by Eq.(2.43). Thus, the fact that q_1 is cyclic means that the Lagrangian is invariant under transformations

$$
\begin{aligned}
q_1 \to q_1' &= q_1 + a \\
q_i \to q_i' &= q_i , \quad i \neq 1
\end{aligned}
$$
(2.60)

where a is a parameter that can be continuously be varied. This continuous transformation has the form of *translation* in the cyclic coordinate, and in the previous example that corresponds to rotation about the z axis. This shows that there is a continuous symmetry transformation associated with each cyclic coordinate.

We shall now discuss more generally how invariance of the Lagrangian under a continuous coordinate transformation is related to the presence of a constant of motion. In the general case there may be no cyclic coordinate corresponding to this symmetry. We consider then a continuous *time independent* coordinate transformation

$$
q \to q' = q'(q) ,
$$
(2.61)

where we may assume the change in the coordinates to be arbitrarily small. We write this as $q_i' = q_i + \delta q_i$, and assume in the following that terms that are higher order in δq_i can be neglected. By expansion to first order in δq, we have the identity

$$
L(q', \dot{q}') = L(\dot{q}, q) + \sum_k \left(\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right) ,
$$
(2.62)

and invariance of the Lagrangian thus implies

$$
\sum_k \left(\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right) = 0 .
$$
(2.63)

This we may rewrite as

$$
\sum_k \left[\frac{\partial L}{\partial q_k} \delta q_k - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right) \right] = 0 .
$$
(2.64)

We will now apply this identity to a solution $q(t)$ of Lagrange's equations. The two first terms then will cancel, and this gives

$$
\frac{d}{dt} \left(\sum_k \frac{\partial L}{\partial \dot{q}_k} \delta q_k \right) = 0 .
$$
(2.65)

Therefore the following quantity is a constant of motion

$$
\delta K = \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta q_k .
$$
(2.66)

With δq_k as an infinitesimal change of the coordinates, it can be written as

$$\delta q_k = \epsilon J_k \tag{2.67}$$

where J_k is a *finite* parameter characteristic for the transformation, while ϵ is time independent and infinitesimal. The parameter ϵ can be omitted and that gives the following expression for the finite (non-infinitesimal) constant of motion associated with the symmetry

$$K = \sum_k \frac{\partial L}{\partial \dot{q}_k} J_k \ . \tag{2.68}$$

To summarize, if we can identify a symmetry of the system, expressed as invariance of the Lagrangian under a coordinate transformation, we can use the above expression to derive a conserved quantity corresponding to this symmetry. Clearly there is one constant of motion for each independent continuous symmetry of the Lagrangian.

2.2.4   Example: Particle in rotationally invariant potential

In order to illustrate the general discussion we examine a rotationally invariant system with kinetic and potential energies

$$T = \frac{1}{2} m \dot{\vec{r}}^2 \ , \quad V = V(r) \ , \tag{2.69}$$

which give the following Lagrangian in Cartesian coordinates

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(\sqrt{x^2 + y^2 + z^2}) \ , \tag{2.70}$$

and in polar coordinates

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - V(r) \ , \tag{2.71}$$

The system is obviously symmetric under all rotations about the origin (the center of the potential), but we note that expressed in Cartesian coordinates there is no cyclic coordinate corresponding to these symmetries. In polar coordinates there is one cyclic coordinate, ϕ. The corresponding conserved quantity is the conjugate momentum

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2 \theta \dot{\phi} \ , \tag{2.72}$$

and the physical interpretation of p_ϕ is the z-component of the angular momentum

$$(m \vec{r} \times \dot{\vec{r}})_z = m(x\dot{y} - y\dot{x}) = m r^2 \sin^2 \theta \dot{\phi} \ . \tag{2.73}$$

Clearly also the other components of the angular momentum are conserved, but there are no cyclic coordinates corresponding to these components.

We use the expression derived in the last section to find the conserved quantities associated with the rotational symmetry. First we note that an infinitesimal rotation can be expressed in the form

$$\vec{r} \to \vec{r}' = \vec{r} + \delta\vec{\alpha} \times \vec{r} \tag{2.74}$$

or

$$\delta\vec{r} = \delta\vec{\alpha} \times \vec{r} \,, \tag{2.75}$$

where the direction of the vector $\delta\vec{\alpha}$ specifies the direction of the axis of rotation and the absolute value $\delta\alpha$ specifies the angle of rotation.

We can explicitly verify that to first order in $\delta\vec{\alpha}$ the transformation (2.74) leaves $\vec{r}^{2}$ unchanged, and since the velocity $\dot{\vec{r}}$ transforms in the same way (by time derivative of (2.74)) also $\dot{\vec{r}}^{2}$ is invariant under the transformation. Consequently, the Lagrangian is invariant under the infinitesimal rotations (2.74), which are therefore symmetry transformations of the system.

By use of the expression (2.65) we find the following expression for the conserved quantity associated with the symmetry transformation,

$$K = \sum_{k=1}^{3} \frac{\partial L}{\partial \dot{x}_k} \delta x_k = m\dot{\vec{r}} \cdot \delta\vec{r} = m(\vec{r} \times \dot{\vec{r}}) \cdot \delta\vec{\alpha} \,. \tag{2.76}$$

Since this quantity is conserved for arbitrary values of the constant vector $\delta\vec{\alpha}$, we conclude that the vector quantity

$$\vec{l} = m\vec{r} \times \dot{\vec{r}} \tag{2.77}$$

is conserved. This demonstrates that the general expression we have found for a constant of motion reproduces, as expected, the angular momentum as a constant of motion when the particle moves in a rotationally invariant potential.

2.2.5 Time invariance and energy conservation

We consider a Lagrangian $L = L(q, \dot{q})$ that has no explicit time dependence, so that

$$\frac{\partial L}{\partial t} = 0 \tag{2.78}$$

This functional independence of t we note to be similar to the functional independence of q_1, when this is a cyclic coordinate. Time is certainly *not* a coordinate in the same sense as q_i, and in particular there is no conjugate momentum to t. Nevertheless, there is a conserved quantity that can be derived from the time independence of L. To show this we consider the total time derivative of L when evaluated for a path $q(t)$ that satisfies the equations of motion. The total time derivative picks up contributions both from the explicit dependence of L on time t and from the dynamical time dependence of L on the generalized coordinates $q_i(t)$,

$$\frac{dL}{dt} = \sum_{i} \left(\frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial L}{\partial q_i} \dot{q}_i \right) + \frac{\partial L}{\partial t} \tag{2.79}$$

We rewrite this equation and make use of the fact that the time dependence of q_i is determined by Lagrange's equation,

$$
\begin{aligned}
\frac{dL}{dt} &= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) - \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right]\dot{q}_i + \frac{\partial L}{\partial t} \\
&= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial L}{\partial t}
\end{aligned}
\tag{2.80}
$$

This shows that the following quantity

$$
H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L
\tag{2.81}
$$

which is called the *Hamiltonian* of the system, satisfies the equation

$$
\frac{dH}{dt} = -\frac{\partial L}{\partial t}
\tag{2.82}
$$

This means that if L has no explicit time dependence, $\frac{\partial L}{\partial t} = 0$, then H is time independent under the full time evolution of the system and is therefore a constant of motion.

When the Lagrangian has the standard form $L = T - V$, and when the constraints are *time independent*, the Hamiltonian corresponds to the sum of kinetic and potential energy, $H = T + V$. In that case conservation of H means that the total energy is conserved. To show this we note that with time independent constraints we have the following relation

$$
\dot{\mathbf{r}}_k = \sum_i \frac{\partial \mathbf{r}_k}{\partial q_i}\dot{q}_i
\tag{2.83}
$$

which implies that the kinetic energy is quadratic in the generalized velocity $\dot{q}$, and the Lagrangian therefore has the form

$$
L = \frac{1}{2}\sum_{ij} g_{ij}(q)\,\dot{q}_i\dot{q}_j - V(q)
\tag{2.84}
$$

with

$$
g_{ij}(q) = \sum_k \frac{\partial \mathbf{r}_k}{\partial q_i} \cdot \frac{\partial \mathbf{r}_k}{\partial q_j}
\tag{2.85}
$$

This gives

$$
\frac{\partial L}{\partial \dot{q}_i} = \sum_j g_{ij}(q)\,\dot{q}_j
\tag{2.86}
$$

and therefore

$$
\begin{aligned}
H &= \sum_j g_{ij}(q)\,\dot{q}_i\dot{q}_j - \left(\frac{1}{2}\sum_{ij} g_{ij}(q)\,\dot{q}_i\dot{q}_j - V(q)\right) \\
&= \frac{1}{2}\sum_{ij} g_{ij}(q)\,\dot{q}_i\dot{q}_j + V(q) \\
&= T + V
\end{aligned}
\tag{2.87}
$$

Note that with *time dependent* constraints we have

$$\dot{\mathbf{r}}_k = \sum_i \frac{\partial \mathbf{r}_k}{\partial q_i}\dot{q}_i + \frac{\partial \mathbf{r}}{\partial t} \tag{2.88}$$

where the last term gives rise to new terms in the kinetic energy, which now takes the form

$$T = \frac{1}{2}\sum_{ij} g_{ij}(q,t)\,\dot{q}_i\dot{q}_j + \sum_i h_i(q,t)\dot{q}_i + f(q,t) \tag{2.89}$$

The additional terms lead to an expression for the Hamiltonian that is in general different from $T + V$. One should note that even if the constraints are time *dependent*, the Lagrangian may in some cases be time *independent*, provided the functions g_{ij}, h_i, f and V are all independent of time. In that case the Hamiltonian H is a constant of motion, but it is *not* identical to the total energy of the system. (A particular example is included in Problem set 2.)

In a similar way as the Lagrangian L is the fundamental quantity in Lagrange's description of the dynamics, the Hamiltonian H is the fundamental quantity in Hamilton's description. We shall study that in some detail in Chapt. 3.

2.3 Generalizing the formalism

2.3.1 Adding a total time derivative

A change of the Lagrangian

$$L(q,\dot{q},t) \to L'(q,\dot{q},t) \tag{2.90}$$

will usually lead to a change in the corresponding equations of motions, but not always. Let us consider a change given by

$$L'(q,\dot{q},t) = L(q,\dot{q},t) + \frac{d}{dt}f(q,t) \tag{2.91}$$

where $f(q,t)$ is a differentiable function of the coordinates q_i, but is independent of the velocities $\dot{q}_i$. The additional term, which has the form of a total time derivative, does not change the (Lagrange) equations of motion, as we can easily demonstrate. We define the additional term as

$$g(q,\dot{q},t) \equiv \frac{d}{dt}f = \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial t} \tag{2.92}$$

and consider the contribution to the Lagrange equation from this additional term,

$$\frac{d}{dt}\Big(\frac{\partial L'}{\partial \dot{q}_i}\Big) - \frac{\partial L'}{\partial q_i} = \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} + \frac{d}{dt}\Big(\frac{\partial g}{\partial \dot{q}_i}\Big) - \frac{\partial g}{\partial q_i} \tag{2.93}$$

We have

$$\frac{\partial g}{\partial q_i} = \sum_m \frac{\partial^2 f}{\partial q_i \partial q_m}\dot{q}_m + \frac{\partial^2 f}{\partial q_i \partial t} \tag{2.94}$$

and

$$\frac{d}{dt}(\frac{\partial g}{\partial \dot{q}_i}) = \frac{d}{dt}(\frac{\partial f}{\partial q_i}) = \sum_m \frac{\partial^2 f}{\partial q_m \partial q_i}\dot{q}_m + \frac{\partial^2 f}{\partial t \partial q_i} \tag{2.95}$$

and provided the function f to be well behaved, so the order of differentiation can be inter-changed, these two expressions are equal. This means that the contribution to Lagrange's equation vanishes,

$$\frac{d}{dt}\left(\frac{\partial g}{\partial \dot{q}_i}\right) - \frac{\partial g}{\partial q_i} = 0 \tag{2.96}$$

Therefore, two Lagrangians that differ by a total time derivative, like in (2.91), are equivalent in the sense that they give rise to the same equations of motion. In particular, if the Lagrangian is given by the standard expression $L = T - V$, this implies that an equally valid Lagrangian for the same system, is obtained by adding (or subtracting) a total time derivative to the expression $T - V$. This observation is sometimes useful in order to simplify the expression for the Lagrangian.

One should also note, that even if a symmetry of a physical system will often correspond to *invariance* of the Lagrangian under a given transformation, *invariance up to a total time derivative* would more generally give rise to a symmetry of the equations of motion. Also in this case, when the Lagrangian is invariant up to the addition of a total time derivative, there is a constant of motion corresponding to the symmetry. This can be shown in essentially the same way as we have done for the case of an invariant Lagrangian.

2.3.2   Velocity dependent potentials

Let return to the equation of motion in the early form Eq.(2.15), which was established before the potential was introduced,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \mathcal{F}_j, \quad j = 1, 2, ..., d \tag{2.97}$$

We derived Lagrange's equations from this by writing the generalized force as $\mathcal{F}_j = -\frac{\partial V}{\partial q_j}$ with V as a velocity independent function. However, there is an obvious possibility of extending the formalism by assuming the potential to be velocity dependent, written as $U = U(q, \dot{q}, t)$, with the generalized force depending on U as

$$\mathcal{F}_i = \frac{d}{dt}(\frac{\partial U}{\partial \dot{q}_i}) - \frac{\partial U}{\partial q_i} = \sum_j \frac{\partial^2 U}{\partial q_j \partial \dot{q}_i}\dot{q}_j + \sum_j \frac{\partial^2 U}{\partial \dot{q}_j \partial \dot{q}_i}\ddot{q}_j + \frac{\partial^2 U}{\partial t \, \partial q_i} - \frac{\partial U}{\partial q_i} \tag{2.98}$$

We note the presence of an acceleration dependent term (depending on $\ddot{q}$) which may not seem natural to include in the generalized force, but if U depends linearly on the velocity this term is absent. The equation of motion (2.5) can now be written in the standard Lagrangian form, with the Lagrangian defined as

$$L(q, \dot{q}, t) = T(q, \dot{q}, t) - U(q, \dot{q}, t) \tag{2.99}$$

This generalized form of Lagrange's equation has an important application in the description of charged particles in electromagnetic fields, as we shall see. In that case the potential U depends linearly on the velocity and this dependence on the velocity gives rise to the magnetic force that acts on the particles.

2.4 Particle in an electromagnetic field

2.4.1 Lagrangian for a charged particle

We consider the motion of a charged particle in an electromagnetic field, and since there are no constraints the Cartesian coordinates of the particle are used as the generalized coordinates. The equation of motion is

$$m\mathbf{a} = e(\mathbf{E}(\mathbf{r},t) + \mathbf{v} \times \mathbf{B}(\mathbf{r},t)) \equiv \mathbf{F}(\mathbf{r},\mathbf{v},t) \tag{2.100}$$

with e as the charge of the particle, $\mathbf{E}$ as the electric field and $\mathbf{B}$ as the magnetic field. Only in the electrostatic case, with $\mathbf{B} = 0$, this equation of motion can be derived from a Lagrangian of the standard form $L = T - V$, with $V = e\phi$ as the electrostatic potential. However, as we shall see, in the general case the force can be expressed in terms of a velocity dependent potential as

$$F_i = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}_i}\right) - \frac{\partial U}{\partial x_i} \tag{2.101}$$

and therefore the equation of motion can be derived from the Lagrangian $L = T - U$.

In order to show this, we introduce the electromagnetic potentials ϕ and $\mathbf{A}$, with

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{2.102}$$

It follows from Maxwell's equations, to be discussed in Chapter 9, that such a representation of $\mathbf{E}$ and $\mathbf{B}$ in terms of potentials can always be made. The corresponding expression for the electromagnetic force is

$$\begin{aligned} \mathbf{F} &= e[-\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A})] \\ &= e[-\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}) - \mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{A}] \end{aligned} \tag{2.103}$$

which in component form is

$$\begin{aligned} F_i &= e(-\frac{\partial \phi}{\partial x_i} - \frac{\partial A_i}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{A}}{\partial x_i} - \mathbf{v} \cdot \boldsymbol{\nabla}A_i) \\ &= \frac{d}{dt}(-eA_i) - \frac{\partial}{\partial x_i}[e\phi - e\mathbf{v} \cdot \mathbf{A}] \end{aligned} \tag{2.104}$$

If the velocity dependent potential U is defined as

$$U = e\phi - e\mathbf{v} \cdot \mathbf{A} \tag{2.105}$$

that gives

$$\frac{\partial U}{\partial \dot{x}_i} = -eA_i \tag{2.106}$$

and it is clear from (2.104) that the Lorentz force $\mathbf{F}$ then is related to U by Eq. (2.101). It is then follows that the Lagrangian

$$L = T - U = \frac{1}{2}m\dot{\mathbf{r}}^2 - e\phi(\mathbf{r}, t) + e\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t) \tag{2.107}$$

will correctly reproduce the equation of motion (2.100).

Let us further examine the form of the conjugate momentum and the Hamiltonian in this case. We have

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + eA_i \tag{2.108}$$

which gives

$$m\dot{\mathbf{r}} = \mathbf{p} - e\mathbf{A} \tag{2.109}$$

This shows that the *canonical momentum* $\mathbf{p}$ in this case is not identical to the *mechanical momentum* $m\mathbf{v}$ of the charged particle. The Hamiltonian is now

$$\begin{aligned} H &= \mathbf{p} \cdot \dot{\mathbf{r}} - L \\ &= \mathbf{v} \cdot (m\mathbf{v} + e\mathbf{A}) - \frac{1}{2}\mathbf{v}^2 + e\phi - e\mathbf{v} \cdot \mathbf{A} \\ &= \frac{1}{2}m\mathbf{v}^2 + e\phi \\ &= \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + e\phi \end{aligned} \tag{2.110}$$

We note that this is different from $T + U$, but is identical to the total energy $T + V$, with $V = e\phi$ as the potential energy of the charge in the electromagnetic field.

According to the previous discussion H should be a constant of motion if the Lagrangian has no explicit time dependence. In the present case this can be related to a more direct argument for conservation of energy in the following way. We first note that time independence of L means that the potentials and therefore the electric and magnetic fields are time independent. The electric part of the force in (2.100) is $\mathbf{F}_e = -e\boldsymbol{\nabla}\phi$. This is a conservative force. It does not change the total energy, but only shifts it from the kinetic to the electrostatic part. The magnetic part of the force, $\mathbf{F}_m = e\mathbf{v} \times \mathbf{B}$, acts in a direction perpendicular to the direction of motion, and therefore performs no work on the particle, so the total energy is left unchanged. If the potentials on the other hand are time dependent, the electric force is no longer conservative and the interaction of the particle with the electric field will change the total energy.

There is one point about the Lagrangian that is worthwhile noting. It is not gauge invariant, even if the equation of motion is gauge invariant. A gauge transformation is a modification of the potentials of the form

$$\phi \to \phi' = \phi - \frac{\partial \chi}{\partial t}, \quad \mathbf{A} \to \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi \tag{2.111}$$

with $\chi = \chi(\mathbf{r}, t))$ as an arbitrary differentiable function of space and time. The fields $\mathbf{E}$ and $\mathbf{B}$ are left unchanged by this transformation, and usually gauge transformations are therefore considered as not corresponding to any physical change. The question is whether the non-invariance of the Lagrangian is consistent with this view. The transformation induces the following change of the Lagrangian

$$L \to L' = L + e\left(\frac{\partial \chi}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}\chi\right) = L + e\frac{d\chi}{dt} \tag{2.112}$$

So we see that the gauge transformation adds a term to the Lagrangian that can be written as a total time derivative. As already discussed Lagrangians that differ by a total time derivative are equivalent, so in this sense no essential change is made under the gauge transformation.

2.4.2 Example: Charged particle in a constant magnetic field

We assume the electromagnetic potentials are

$$\phi = 0, \quad \mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B} \tag{2.113}$$

with $\mathbf{B}$ constant. It is straight forward to check that $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, so that $\mathbf{B}$ represents a constant magnetic field. We use the established expression for the Lagrangian of a charged particle,

$$L = \frac{1}{2}m\mathbf{v}^2 + e\mathbf{v} \cdot \mathbf{A} = \frac{1}{2}m\mathbf{v}^2 - \frac{1}{2}\mathbf{v} \cdot (\mathbf{r} \times \mathbf{B}) \tag{2.114}$$

and will check that the corresponding Lagrange equation is consistent with the known expression for the equation of motion of a charged particle in a magnetic field.

The partial derivatives with respect to coordinates and velocities are

$$\frac{\partial L}{\partial x_i} = \frac{e}{2}(\mathbf{v} \times \mathbf{B})_i \tag{2.115}$$

and

$$\frac{\partial L}{\partial v_i} = mv_i - \frac{e}{2}(\mathbf{r} \times \mathbf{B})_i \tag{2.116}$$

The latter gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = ma_i - \frac{e}{2}(\mathbf{v} \times \mathbf{B})_i \tag{2.117}$$

Lagrange's equation, in the standard form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial L}{\partial x_i} = 0 \tag{2.118}$$

then gives

$$ma_i - e(\mathbf{v} \times \mathbf{B})_i = 0 \tag{2.119}$$

or in vector form

$$m\mathbf{a} = e\mathbf{v} \times \mathbf{B} \tag{2.120}$$

The left hand side is the well known Lorentz force which acts on a charged particle in a magnetic field.

We find the Hamiltonian

$$
\begin{aligned}
H &= \mathbf{v} \cdot \mathbf{p} - L \\
&= \mathbf{v} \cdot (m\mathbf{v} + e\mathbf{A}) - \frac{1}{2}m\mathbf{v}^2 + \frac{1}{2}\mathbf{v} \cdot (\mathbf{r} \times \mathbf{B}) \\
&= \frac{1}{2}m\mathbf{v}^2 \\
&= \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2
\end{aligned}
\tag{2.121}
$$

and note that this is identical to the kinetic energy. This is conserved, as follows from the fact that the Lagrangian has no explicit time dependence, and the energy conservation is consistent with the fact that the magnetic force can only change the direction of the velocity but not its absolute value.

Chapter 3

Hamiltonian dynamics

3.1 Hamilton's equations

In Lagrange's formulation the Lagrangian $L(q, \dot{q}, t)$ regulates the dynamical evolution of the physical system. It determines the motion of the system through its partial derivatives with respect to the variables q_i and $\dot{q}_i$. Hamilton's formulation of the dynamics of a physical system can be viewed as derived from Lagrange's formulation by a change of the dynamical steering function from the Lagrangian to the Hamiltonian,

$$L(q, \dot{q}, t) \rightarrow H(q, p, t) \tag{3.1}$$

The transformation form L to H is combined with a change of fundamental variables, from the set of generalized coordinates and velocities $(q, \dot{q})$, to the set of coordinates and conjugate momenta (q, p). This type of transformation is referred to as a *Legendre transformation*. The reason for combining the change of fundamental variables with the change in the dynamical function is that the equations of motion are expressed through the partial derivatives of this function with respect to the fundamental variables. Similar types of transformations are known from thermodynamics, where the thermodynamical variables $p, T, V, S, ...$ are related through partial derivatives of the relevant thermodynamical potential. There is a certain freedom in the choice of fundamental and derived variables, and a change in this choice is accompanied by a change of thermodynamic potential so that the derived variables can also after the transformation be expressed through partial derivatives of the potential.

To be more specific we return to the definition of the Hamiltonian

$$H = \sum_i p_i \dot{q}_i - L, \quad p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{3.2}$$

As already discussed, we may invert the relation between the conjugate momentum and velocity in the expression for p_i, to give the velocity as a function of momentum and coordinates (and possibly time),

$$\dot{q}_i = \dot{q}_i(p, q, t) \tag{3.3}$$

Thereby we may express the Hamiltonian as a function of q, p and t. To see how Lagrange's equation can be reformulated in terms of partial derivatives of H, we consider first the variation

in H under an infinitesimal change in the variables of the system. From the definition of H
follows

$$
\begin{aligned}
dH &= \sum_i (dp_i \dot{q}_i + p_i d\dot{q}_i) - dL \\
&= \sum_i \left[(dp_i \dot{q}_i + p_i d\dot{q}_i) - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right] - \frac{\partial L}{\partial t} dt \\
&= \sum_i \left[(p_i - \frac{\partial L}{\partial \dot{q}_i}) d\dot{q}_i + \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i \right] - \frac{\partial L}{\partial t} dt \\
&= \sum_i \dot{q}_i dp_i - \sum_i \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \qquad (3.4)
\end{aligned}
$$

and the important point to notice is that the differential $d\dot{q}_i$ has disappeared in the final expres-
sion due to the definition of the canonical momentum p_i. This means that only the differentials
for a set of *independent* variables (q, p) appear on the right-hand-side of the equation. The
coefficients in front of these can be interpreted as partial derivatives of H with respect to the
corresponding variables.

With H as a function of q, p and t, the general expression for the change in H due to a
change in the fundamental variables is

$$
dH = \sum_i \frac{\partial H}{\partial p_i} dp_i + \sum_i \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt \qquad (3.5)
$$

and by comparing with (3.4), we find the following relations

$$
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \qquad (3.6)
$$

One should note that, at this point, no dynamics is involved in these equations. They are simply
consequences of the definitions of the canonical momenta and the Hamiltonian. However, at
the next step we make use of Lagrange's equation, which may be written as

$$
\dot{p}_i = \frac{\partial L}{\partial q_i} \qquad (3.7)
$$

By use of this the two first of the above equations (3.6) can be written as

$$
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (3.8)
$$

These equations, which are known as *Hamilton's equations*, can be viewed as *equivalent* to
Lagrange's equations, in the sense that they constitute a complete set of equations of motion
for the physical system. As already shown, Hamilton's equations follow from Lagrange's equa-
tions, and in a similar way one can from Hamilton's equations re-derive Lagrange's equation.

Hamilton's equations (3.8) can be supplemented by a third equation

$$
\frac{dH}{dt} = \frac{\partial H}{\partial t} \qquad (3.9)
$$

This identity follows from (3.4) by use of Hamilton's equations for $\dot{q}$ and $\dot{p}$. This shows directly that if there is no explicit time dependence, which means $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = 0$, then the total time derivative of H vanishes and therefore the Hamiltonian is a constant of motion.

In the derivation of Hamilton's equation it was noticed that only the equations for $\dot{p}_i$ were dynamical, in the sense that only these equations depended on Lagrange's equation to be satisfied. However, after Hamilton's equation have been established, there is no reason for treating the equations for q and p differently. The standard way to view the equations is that both equations are parts of the full set of equations of motion for the system, with the coordinates and momenta being represented in a symmetric way.

Compared to Lagrange's formulation, it seems that we have doubled the set of equations, since now there are two equations for each degree of freedom, whereas in Lagrange's formulation there is only one. However, the two Hamilton's equations are *first order* in time derivatives, whereas Lagrange's equation is *second order*. The two first order differential equations can be replaced by a single second order differential equation, and we shall demonstrate this in a simple example.

3.1.1 Example: The one-dimensional harmonic oscillator

In this case there is no constraint (except for the reduction to one dimension) and we use the linear coordinate x as generalized coordinate. For kinetic and potential energy we have the expressions

$$T = \frac{1}{2}m\dot{x}^2, \quad V = \frac{1}{2}kx^2 \tag{3.10}$$

The Lagrangian is therefore

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{3.11}$$

and the canonical momentum conjugate to x is

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{3.12}$$

The Hamiltonian is defined by

$$H = p\dot{x} - L = \frac{1}{2m}p^2 + \frac{1}{2}kx^2 = T + V \tag{3.13}$$

and from this follows Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$
$$\dot{p} = -\frac{\partial H}{\partial x} = -kx \tag{3.14}$$

Position and momentum are therefore coupled through the two equations

$$\dot{x} = \frac{p}{m}, \quad \dot{p} = -kx \tag{3.15}$$

From these equations p can be eliminated to give

$$\ddot{x} + \frac{k}{m}x = 0 \tag{3.16}$$

which is the standard harmonic oscillator equation, with $\omega = \sqrt{k/m}$ as the circular frequency of the oscillator. This is the equation we would have derived directly from the Lagrangian through Lagrange's equation, and the reduction from the two Hamilton's equations to the single Lagrange's equation has been obtained by eliminating the momentum p. Although this is a very simple example, it illustrates the way in which Hamilton's equations are used, and how these equations relate to Lagrange's equations.

3.2   Hamilton's equations for a charged particle in an electromagnetic field

We have in an earlier section established the form of the Lagrangian for a charged particle in an electromagnetic field

$$L = \frac{1}{2}m\mathbf{v}^2 - e\phi + e\mathbf{v} \cdot \mathbf{A} \tag{3.17}$$

with ϕ and $\mathbf{A}$ as the electromagnetic potentials, m as the mass and e as the charge of the particle. The corresponding canonical momentum is

$$\mathbf{p} = m\mathbf{v} + e\mathbf{A} \tag{3.18}$$

and the Hamiltonian is

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + e\phi \tag{3.19}$$

This classical Hamiltonian has the same form as its quantum counterpart, and it represents the total energy of the system. If the potentials are time independent, the Hamiltonian H is also time independent and the energy is conserved.

We take the Cartesian coordinates of the particle as generalized coordinates, and write these as $x_i, i = 1, 2, 3$, with $x_1 = x$, $x_2 = y$ and $x_3 = z$ in the usual way. Hamilton's equations in this case give

$$\begin{aligned}
\dot{x}_i &= \frac{\partial H}{\partial p_i} = \frac{1}{m}(p_i - eA_i) \\
\dot{p}_i &= -\frac{\partial H}{\partial x_i} = \frac{e}{m}(\mathbf{p} - e\mathbf{A}) \cdot \frac{\partial \mathbf{A}}{\partial x_i} - e\frac{\partial \phi}{\partial x_i}
\end{aligned} \tag{3.20}$$

We will check that these two equations reproduce the well known form of Newton's second law applied to the charged particle in the electromagnetic field. We do this by eliminating $\mathbf{p}$

from the equations,

$$
\begin{aligned}
m\ddot{x}_i &= \dot{p}_i - e\frac{dA_i}{dt} \\
&= \dot{p}_i - e(\sum_j \frac{\partial A_i}{\partial x_j}\dot{x}_j + \frac{\partial A_i}{\partial t}) \\
&= \frac{e}{m}\sum_j (p_j - eA_j)\frac{\partial A_j}{\partial x_i} - e\frac{\partial \phi}{\partial x_i} - e(\sum_j \frac{\partial A_i}{\partial x_j}\dot{x}_j + \frac{\partial A_i}{\partial t}) \\
&= -e(\frac{\partial \phi}{\partial x_i} + \frac{\partial A_i}{\partial t}) + e\sum_j (\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j})\dot{x}_j]
\end{aligned}
\tag{3.21}
$$

This we can write in a more familiar form by use of the expressions for the electric and magnetic fields

$$
E_i = -(\frac{\partial \phi}{\partial x_i} + \frac{\partial A_i}{\partial t}), \quad B_k = \sum_{ij} \epsilon_{kij}\frac{\partial A_j}{\partial x_i}
\tag{3.22}
$$

with ϵ_{ijk} as the antisymmetric Levi-Civita symbol. The last equation can be inverted to give

$$
\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} = \sum_k \epsilon_{kij}B_k
\tag{3.23}
$$

and therefore the equation of motion (3.21) can be written as

$$
m\ddot{x}_i = eE_i + e\sum_{jk} \epsilon_{ijk}B_k\dot{x}_j
\tag{3.24}
$$

In vector form it gives the standard (non-relativistic) equation of motion for a charge particle in the electromagnetic field,

$$
m\mathbf{a} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})
\tag{3.25}
$$

This again demonstrates that Hamilton's (as well as Langrange's) equations have different form, but are equivalent to Newton's second law when applied to the same system. We shall next see how Hamilton's equations can be used in a direct way to find the motion of a charged particle in a constant magnetic field.

3.2.1   Example: Charged particle in a constant magnetic field

We assume the particle to be moving in a constant magnetic field with direction along the z-axis, $\mathbf{B} = B\mathbf{k}$. The vector potential we write as

$$
\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}
\tag{3.26}
$$

with components

$$
A_x = -\frac{1}{2}eBy, \quad A_y = \frac{1}{2}eBx, \quad A_z = 0
\tag{3.27}
$$

It is straight forward to check that the curl of this vector potential reproduces the correct magnetic field. The scalar potential vanishes, $\phi = 0$. The Hamiltonian (3.19) then gets the form

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 = \frac{1}{2m}[(p_x + \frac{1}{2}eBy)^2 + (p_y - \frac{1}{2}eBx)^2 + p_z^2] \tag{3.28}$$

We note that z is a cyclic coordinate[1], and it follows directly from Hamilton's equations that

$$\dot{p}_z = -\frac{\partial H}{\partial z} = 0 \tag{3.29}$$

so that $p_z = m\dot{z}$ is a constant of motion. Thus, the motion in the z-direction has the simple form of motion with constant velocity

$$z = z_0 + v_{z0}t \tag{3.30}$$

with $v_{z0} = p_z/m$ and z_0 as constants determined by the initial conditions.

From this follows that the motion in the x, y-plane (the plane orthogonal to the magnetic field) is decoupled from the motion in the z-direction. We write Hamilton's equations for this motion,

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p_x} = \frac{1}{m}(p_x + \frac{1}{2}eBy) \\
\dot{p}_x &= -\frac{\partial H}{\partial x} = \frac{eB}{2m}(p_y - \frac{1}{2}eBx) \\
\dot{y} &= \frac{\partial H}{\partial p_y} = \frac{1}{m}(p_y - \frac{1}{2}eBx) \\
\dot{p}_y &= -\frac{\partial H}{\partial y} = -\frac{eB}{2m}(p_x + \frac{1}{2}eBy)
\end{aligned} \tag{3.31}$$

By inspecting the right-hand-side of the equations we see that they can be grouped in pairs that are essentially identical. By combining these the following equations are established,

$$\begin{aligned}
\dot{p}_x - \frac{1}{2}eB\dot{y} &= 0 \\
\dot{p}_y + \frac{1}{2}eB\dot{x} &= 0
\end{aligned} \tag{3.32}$$

which means that there are two constants of motion

$$\begin{aligned}
K_x &= p_x - \frac{1}{2}eBy \\
K_y &= p_y + \frac{1}{2}eBx
\end{aligned} \tag{3.33}$$

Combined into a vector, this vector is

$$\begin{aligned}
\mathbf{K} &= \mathbf{p} - \frac{1}{2}e\mathbf{r} \times \mathbf{B} \\
&= m\mathbf{v} + e\mathbf{A} - \frac{1}{2}e\mathbf{r} \times \mathbf{B} \\
&= m\mathbf{v} - e\mathbf{r} \times \mathbf{B}
\end{aligned} \tag{3.34}$$

---

[1] If H does not depend on z, it is clear from the definition of the Hamiltonian that also the Lagrangian is independent of z.

and it is easy to verify directly from the equation of motion (3.25) that this vector is conserved.

We consider next the linear combinations of the equations (3.31) with opposite signs of those in (3.32),

$$
\begin{aligned}
\dot{p}_x + \frac{1}{2}eB\dot{y} &= \frac{eB}{m}(p_y - \frac{1}{2}eBx) \\
\dot{p}_y - \frac{1}{2}eB\dot{x} &= -\frac{eB}{m}(p_x + \frac{1}{2}eBy)
\end{aligned}
\tag{3.35}
$$

These equations can be expressed in terms of components of the *mechanical* momentum vector

$$
\boldsymbol{\pi} \equiv m\mathbf{v} = \mathbf{p} - e\mathbf{A} = \mathbf{p} + \frac{1}{2}e\mathbf{r} \times \mathbf{B}
\tag{3.36}
$$

They get the form

$$
\begin{aligned}
\dot{\pi}_x &= \frac{eB}{m}\,\pi_y \\
\dot{\pi}_y &= -\frac{eB}{m}\,\pi_x
\end{aligned}
\tag{3.37}
$$

which implies that each component satisfies a harmonic oscillator equation

$$
\ddot{\pi}_x + \omega^2 \pi_x = 0\,, \quad \ddot{\pi}_y + \omega^2 \pi_y = 0
\tag{3.38}
$$

with $\omega = eB/m$ as the circular frequency. This is known as the *cyclotron frequency*.

The solutions to the equations have the form

$$
\pi_x = A\cos\omega t\,, \quad \pi_y = -A\sin\omega t
\tag{3.39}
$$

where A is a constant to be determined by the initial conditions, and where a convenient choice of time $t = 0$ has been chosen. These expressions may be combined with the expressions for the components of the conserved vector $\mathbf{K}$, and we focus first on the x-component,

$$
p_x + \frac{1}{2}eBy = A\cos\omega t\,, \quad p_x - \frac{1}{2}eBy = K_x
\tag{3.40}
$$

By combining these we find

$$
\begin{aligned}
y &= \frac{1}{eB}(A\cos\omega t - K_x) \\
&\equiv y_0 + R\cos\omega t
\end{aligned}
\tag{3.41}
$$

where, in the last expression, we have introduced the constants

$$
y_0 = -\frac{K_x}{eB}\,, \quad R = \frac{A}{eB}
\tag{3.42}
$$

Similarly we have

$$
p_y - \frac{1}{2}eBx = A\cos\omega t\,, \quad p_y + \frac{1}{2}eBx = K_y
\tag{3.43}
$$

which gives

$$
\begin{aligned}
x &= \frac{1}{eB}(A\sin\omega t + K_y) \\
&\equiv x_0 + R\sin\omega t
\end{aligned}
\tag{3.44}
$$

The solutions for the components of the position vector show that the particle moves with constant speed on a circle of radius R about a point in the x,y-plane with coordinates (x_0, y_0). These coordinates, as well as the radius R are determine by the initial conditions. The circular frequency $\omega = eB/m$ is fixed by the strength of the magnetic field and the charge, and is independent of the energy of the particle. The direction of circulation in the circular orbit is determined by the sign of eB, so that negative sign corresponds to positive orientation of the motion in the x,y-plane.

When the circular motion in the x,y-plane is combined with the linear motion along the z-axis, this gives a spiral formed orbit of the particle around a magnetic flux line. The radius of the circular part is determined by the contribution to the kinetic energy of the particle from the motion in the x,y plane,

$$
T = \frac{1}{2}m\omega^2 R^2
\tag{3.45}
$$

A well known realization of this type of motion is for electrons and protons in the magnetic field of the earth. For these particles there is an additional effect, which is due to the convergence of the magnetic field lines towards the magnetic poles. This convergence induces a slow down of the component of the motion along the lines and, eventually a reversal of the motion. In this way the electrons may be trapped in a spiral like motion between the two poles with points of reflection above the atmosphere. The *van Allen* radiation belts are formed by charged particles from the sun, which are captured in this type of orbits.

3.3 Phase space

At an earlier stage we introduced the *configuration space* of the physical system as the d dimensional space described by the generalized coordinates $q = \{q_1, q_2, ..., q_d\}$. These d coordinates, one for each degree of freedom of the system, are all independent variables. Later, in the discussion of the Lagrangian formulation, we extended this to a larger set of $2d$ variables, by treating the velocities $\dot{q} = \{\dot{q}_1, \dot{q}_2, ..., \dot{q}_d\}$ as independent of the coordinates. The $2d$ dimensional space described by both coordinates and velocities $\{q, \dot{q}\}$ we refer to as the *phase space* of the system.

In Hamilton's formulation coordinates and momenta are treated on equal footing. Therefore the d dimensional configuration space seems less important than the $2d$ dimensional phase space. However, more commonly than using coordinates and velocities, one takes coordinates and momenta as the independent variables in phase space, since these are the standard variables in Hamilton's equations.

One of the interesting features of the phase space description becomes apparent when one considers the time evolution with given initial conditions. We know that $2d$ initial data are needed to give a unique trajectory. In the Lagrangian formulation this is because there are d
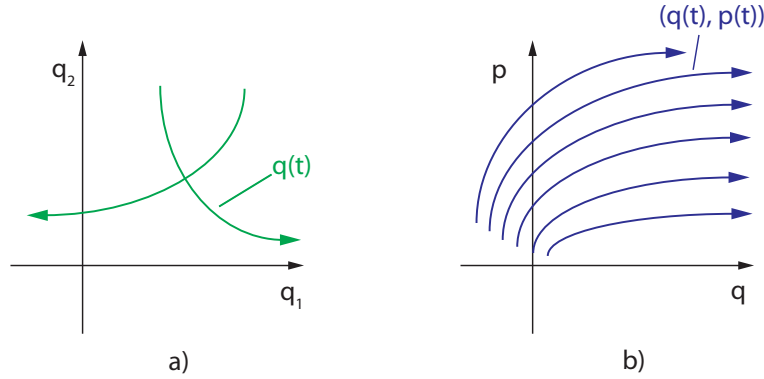
Figure 3.1: Motion in configuration space a) and phase space b). In configuration space the trajectory is determined by the time dependent coordinates $q(t)$, and many different trajectories, with different initial conditions, may pass through the same point. In phase space the trajectory is specified by the time dependent coordinates and momenta $(q(t), p(t))$. In this case only one trajectory will pass through a given point, and all dynamical trajectories (those that satisfy the equations of motion) will together form a flow pattern through phase space.

second order differential equations to determine the motion, and in the Hamiltonian formulation since there are $2d$ *first order* equations. In configuration space this means that through a given point (determined by the d coordinates) there are many possible trajectories, as we have already discussed. However, in phase space the number of coordinates needed to determine a point is $2d$ and that is also the number of data needed to determine uniquely a trajectory. This means that through a point in phase space there will pass only one dynamical trajectory (*i.e.*, a trajectory that satisfies the equations of motion).

This situation is illustrated in Fig. 3.1b for the case of a two-dimensional phase space. Through each point passes one and only one trajectory, specified by the initial conditions. If we continuously change these conditions, the trajectory will be deformed in such a way that, when we consider all possible motions of the system at the same time, the trajectories will form a flow pattern through phase space. These paths will be distinct, so that two paths will never meet (except at some singular, isolated points, which we shall discuss in an example to follow). This description of the dynamics, as a flow pattern in phase space, is particularly important in statistical mechanics, where one does not consider sharply defined initial conditions but rather a time evolution of the system with a statistical distribution over many initial data. As we shall see in examples, the phase space description is also sometimes useful to obtain a qualitative understanding of the motion of the system, without actually solving the equations of motion. Thus, if we find the special points of the flow, corresponding to points of equilibrium, and use the general properties of the phase space flow, we can derive a good qualitative picture of the full flow pattern, and thereby the motion of the system.

3.3.1   Examples

Phase space for the harmonic oscillator

We write the Hamiltonian of a one-dimensional harmonic oscillator in the following form

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 x^2) \tag{3.46}$$

with ω as the circular frequency of the oscillator. Since the Hamiltonian has no explicit time dependence, the total time dependence of H vanishes

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0 \tag{3.47}$$

The energy $H = E$ is therefore a constant of motion. This implies

$$p^2 + m^2\omega^2 x^2 = 2mE \tag{3.48}$$

and we recognize this as the equation for an ellipse in the two-dimensional plane with x and p as coordinates, which is the phase space of the harmonic oscillator. Since x and p have different physical dimensions, the eccentricity of the ellipse has no physical significance, and we can rescale one of the coordinates, for example by redefining the x coordinate, $\tilde{x} = m\omega x$ (which gives $\tilde{x}$ it the same physical dimension as p), so that the ellipse becomes a circle,

$$p^2 + \tilde{x}^2 = 2mE \tag{3.49}$$

The radius of the circle is determined by the energy and increases as $\sqrt{E}$ with energy. Since the energy is a constant of motion these circles of constant energy are the trajectories of the harmonic oscillator in phase space.
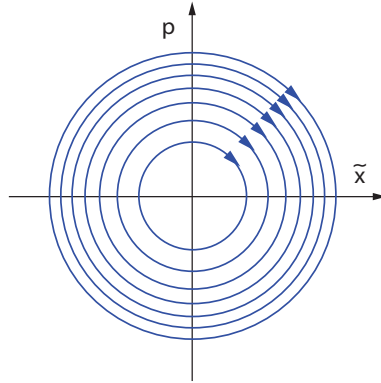
Figure 3.2: Phase space flow for the one dimensional harmonic oscillator. The time evolution define circles of constant energy with motion in the clockwise direction. The curves of constant energy are here plotted with constant energy differences.

We further have Hamilton's equations

$$
\begin{aligned}
\dot{x} &= m\omega\frac{\partial H}{\partial p} = \omega p \\
\dot{p} &= -\frac{\partial H}{\partial x} = -\omega\tilde{x}
\end{aligned}
\tag{3.50}
$$

which show that the system moves in the clockwise direction along a circle of constant energy. The initial conditions determine the energy and thereby the circle which the oscillator follows.

We may consider the Hamiltonian $H(x, p)$ as defining a *phase space potential*. Hamilton's equations show that the system moves in the direction orthogonal to the gradient of the potential, which means along one of the equipotential curves. As illustrated in Fig. 3.2 these (directed) curves of constant energy determine the phase space flow of the harmonic oscillator.

The pendulum

Let us next consider the phase space motion of a planar pendulum. With l as the length of the pendulum rod, m as the mass of the pendulum bob, and the angle of displacement θ chosen as the generalized coordinate, we find the following expression for the Lagrangian

$$
L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)
\tag{3.51}
$$

The canonical momentum conjugate to θ is

$$
p = \frac{\partial L}{\partial \dot{q}} = ml^2\dot{\theta}
\tag{3.52}
$$

and we find the following expression for the Hamiltonian

$$
\begin{aligned}
H &= p\dot{\theta} - L \\
&= \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos\theta) \\
&= \frac{p^2}{2ml^2} + 2mgl\sin^2\frac{\theta}{2}
\end{aligned}
\tag{3.53}
$$

Again there is no explicit time dependence, which means that the energy $H = E$ is a constant of motion. From this follows that a trajectory of the pendulum in phase space is given by

$$
p^2 + 4m^2gl^3\sin^2\frac{\theta}{2} = 2ml^2E
\tag{3.54}
$$

For small oscillations it simplifies to

$$
p^2 + m^2gl^3\theta^2 = 2ml^2E
\tag{3.55}
$$

It has the same form as the phase space equation of the harmonic oscillator, which we have already discussed, although the coordinates are different. In the present case p has the dimension of *angular* momentum rather than *linear* momentum, and θ is is a dimensionless variable. But
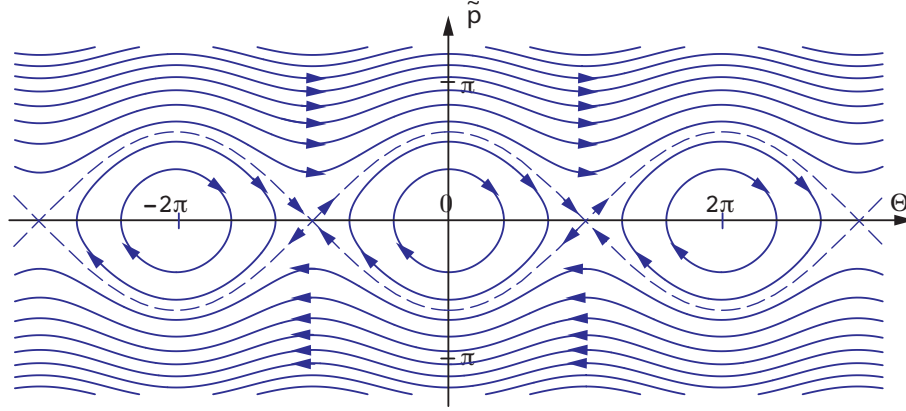
Figure 3.3: Phase space flow for the pendulum. There are two types of motion, where the closed curves represent oscillations of the pendulum about the stable equilibrium and the open curves represent full rotations. The dashed curves are limit curves that separate the two types of motion, and are referred to as *separatrices*. The singular crossing points of these curves are the points of unstable equilibrium. They are not real crossing points of the particle trajectories, since the pendulum velocity at these points vanishes.

that is not important for the phase space motion, and by a proper scaling of the variables it can also here be given the form of equation of a circle, with radius determined by the energy,

$$\tilde{p}^2 + \theta^2 = \frac{2E}{mgl}\ , \quad \tilde{p} = \frac{p}{m\sqrt{gl^3}} \tag{3.56}$$

When we include motion also for larger angles, we first note that that the Hamiltonian $H(p, \theta)$ is a periodic function of θ, and the equipotential curves in the θ, p-plane therefore will be periodic under shifts $\theta \to \theta + 2\pi$. Therefore the point of stable equilibrium will be periodically repeated at angles $(\theta, p) = (2n\pi, 0)$ with n as an integer. As the energy, and therefore the amplitude of oscillations, is increased, the motion will be represented by circles of increasing radii around each point of stable equilibrium. Due to the periodic structure these closed curves will necessarily get deformed for sufficiently large amplitudes, and at some point a singular situation is reached, where the closed curves belonging to neighboring equilibrium points will touch. This we interpret as corresponding to the situation where the pendulum reaches the upper point of *unstable* equilibrium. If the energy is increased even further, the motion is no longer oscillatory but rotational, corresponding to an unbounded increase or decrease of the angular variable θ.

This qualitative picture is in full agreement with the plot of phase space trajectories shown in the figure. There are solutions of bounded motion, corresponding to oscillations of the pendulum around the point of stable equilibrium, but there are also solutions of unbounded motion. The transition between these two different types of motion is represented by equipotential curves that intersect in singular points. These represent the point of unstable equilibrium, with the pendulum rod at rest in an upright vertical position. We see from this discussion that we can reach a rather complete, qualitative understanding of the phase space motion by using the knowledge of what happens for small oscillations together with implications of periodicity of

the motion.

3.4 The phase space fluid

The phase space trajectories of a Hamiltonian system can be viewed as the streamlines of a (fictitious) fluid that fills phase space. Each particle of the fluid represents a specific set of initial conditions for the phase space variables, and the collection of all particles in the fluid therefore represents a (large) collection of initial conditions. These are treated simultaneously in the fluid description. If a sufficiently dense set of such initial conditions has been chosen, the time evolution of the system can be described in terms of a time dependent particle density $\rho(q, p, t)$. Hamilton's equations, which determine the trajectories of the fluid particles, give rise to a hydrodynamical equation for the fluid density. As already mentioned, such a picture of the phase space motion is particularly useful in statistical physics, where the system is generally described by a statistical distribution over phase space rather than by a precisely determined phase space position. The statistical distribution then corresponds to the particle density $\rho(q, p, t)$ of the fluid.

Let us introduce a unified description of the phase space coordinates as x_i, $i = 1, 2, ..., 2d$, with $2d$ as the phase space dimension. We chose odd i to correspond to the generalized coordinates and even i to correspond to the canonical momenta, so that $x_{2k-1} = q_k$ and $x_{2k} = p_k$, with (q_k, p_k) as a conjugate pair of generalized coordinates and momenta. Hamilton's equations can then be written as

$$\dot{x}_i = \sum_{j=1}^{2d} J_{ij} \frac{\partial H}{\partial x_j} \tag{3.57}$$

with J_{ij} as the antisymmetric $2d \times 2d$ matrix

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.58}$$

here shown with $d = 3$. We introduce next a vector notation with $\mathbf{x}$ as the $2d$-dimensional position vector with components x_i, and $\mathbf{v}$ as the corresponding velocity vector, with components $\dot{x}_i$. Hamilton's equation can be seen as defining the velocity as a vector field $\mathbf{v}(\mathbf{x})$ in phase space, which is the velocity field of the fluid particles. We note that due to the antisymmetry of the matrix J the velocity field is divergence free,

$$\nabla \cdot \mathbf{v} = \sum_i \partial_i v_i = \sum_{ij} J_{ij} \frac{\partial^2 H}{\partial x_i \partial x_j} = 0 \tag{3.59}$$

The number of fluid particles are conserved during the time evolution. This implies that the particle density satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0 \tag{3.60}$$

with the meaning that the change in density at a particular point is caused by the flow of particles to and from this point. When this equation is combined with the condition of a divergence free velocity field, we find the time derivative of ρ as measured in a reference frame co-moving with the fluid, vanishes,

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}\rho = \mathbf{0} \tag{3.61}$$

This means that the phase fluid is incompressible. Note that this is not due to any interaction between the fluid particles, but follows from the general properties of flow patterns allowed by Hamilton's equation. There is, however, no constraint on the density of the fluid, which is determined by the chosen density function $\rho(\mathbf{x}, t)$ at some initial time t_0. The phase space fluid is rather different from normal fluids, where forces between the fluid particles are important.

   The property of the phase space motion, that the fluid density ρ is constant when measured in a co-moving frame, is referred to as *Liouville's theorem*.

3.5   Phase space description of non-Hamiltonian systems

So far we have linked the phase space description to the use of Hamilton's equations. However, the description of the dynamics as time evolution in phase space is not restricted to Hamiltonian systems. We may consider more generally systems, where the phase space variables satisfy a first order differential equation of the form

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t) \tag{3.62}$$

Here $\mathbf{v}(\mathbf{x}, t)$ is a fixed vector function that determines the motion of the system, but which can generally not be derived from a Hamiltonian. Such a system is quite generally referred to as a *dynamical system*. In this case the velocity field is not necessarily divergence free, and the continuity equation (3.60) will therefore not imply the fluid density to be conserved along the particle trajectory.

   A particular class of such systems are mechanical systems with friction. Due to dissipation of energy such a system is not conservative, and therefore in general not Hamiltonian. With $\mathbf{v}(\mathbf{x})$ as a time independent function, the system will typically end up, during the time evolution, at a stable equilibrium point. Such a point is referred to as an *attractive fixed points* of the phase space flow. More generally a fixed point $\mathbf{x}_0$ is defined as a point where the velocity vanishes,

$$\mathbf{v}(\mathbf{x}_0) = 0 \tag{3.63}$$

In Fig. 3.4 the flow diagram of a damped pendulum is shown. The equation of motion is

$$\ddot{\theta} + \lambda\sqrt{\frac{g}{l}}\,\dot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{3.64}$$

with λ as a dimensionless damping parameter. Expressed in terms of the same phase space variables as used for the undamped pendulum (se Eq. (3.56)), the corresponding phase space
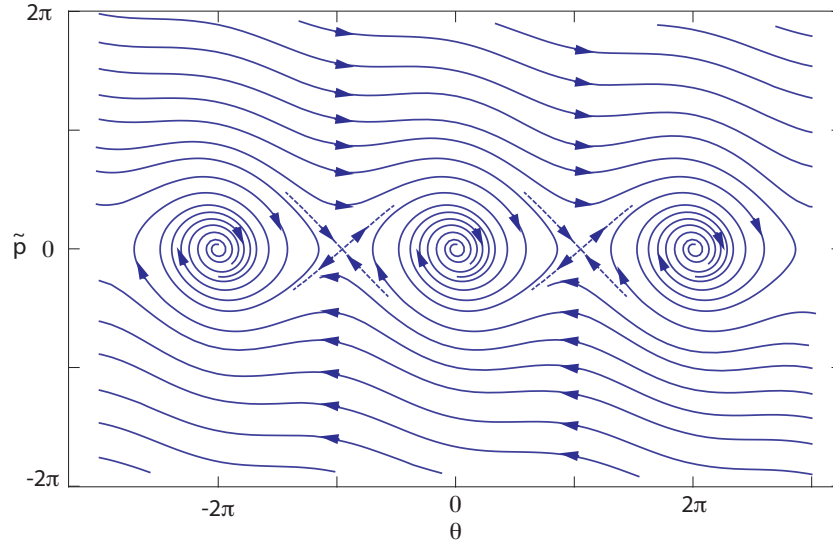
Figure 3.4: The phase space flow of a damped pendulum. With sufficiently large initial energy the pendulum will will complete a series of full rotations, corresponding to the left- and right-moving, oscillating flow lines. Eventually the damping makes the system end up at the equilibrium point. This is represented by the spiraling curves which approach one of the points with coordinates $\theta = 2\pi n$. The unstable equilibrium points are indicated by the crosses of dashed lines. They separate trajectories with different number of pendulum rotations.

equations are

$$\dot\theta = \sqrt{\frac{g}{l}}\tilde{p}, \quad \dot{\tilde{p}} = -\sqrt{\frac{g}{l}}(\lambda\tilde{p} + \sin\theta) \tag{3.65}$$

The right-hand side of the equations determine the velocity field in phase space, which is shown in the form of stream lines in 3.4 , for the parameter value $\lambda = 0.2$.

The stable equilibrium in the undamped case now is an attractive fixed points of the flow, while the unstable equilibrium separates trajectories which differ by the number of rotations performed by the pendulum before it settles in the attractive fixed point. (Remember that due to the periodicity of the θ variable there is actually just one fixed point of each kind, even if in the diagram they are repeated with distance 2π.) The velocity fields close to the two types of fixed points are shown in Fig.3.5.

3.6 Calculus of variation and Hamilton's principle

The motion in the configuration space of a physical system is described by the time dependent generalized coordinates $q(t)$. A specific time evolution may be determined by solving the equations of motion with initial conditions specified by the coordinates $q(t_0)$ and velocities $\dot{q}(t_0)$ for a given initial time $t = t_0$. For a d dimensional configuration space, these $2d$ initial data uniquely specifies the evolution of the system. However, the solution may be specified
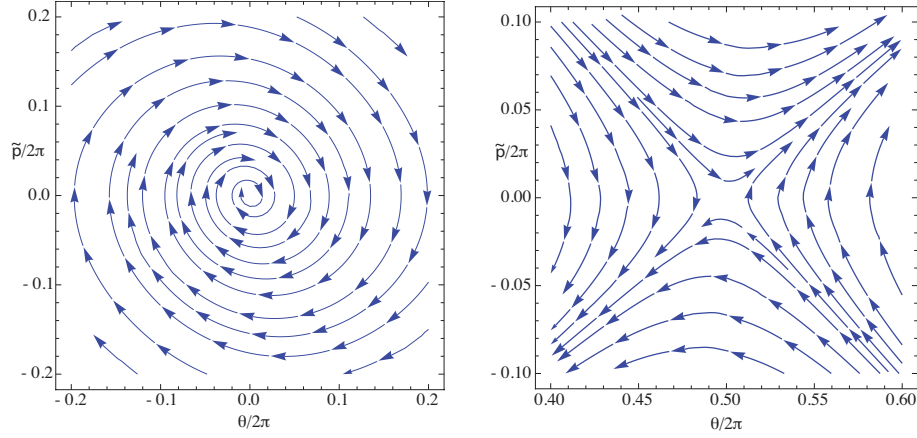
Figure 3.5: The flow patterns around two different types of fixed points. To the left is an attractive fixed point. For the damped pendulum this corresponds to the situation with small oscillations, with decreasing amplitude, about the stable equilibrium point. The diagram to the right shows a different type type of fixed points, which for the damped pendulum corresponds to motion close to the unstable equilibrium point. In more general two-dimensional phase space flows also other types of fixed points are possible.

also in other ways, in particular by fixing the coordinates at two different times, $q(t_1)$ and $q(t_2)$. Again such a set of $2d$ data will specify a unique solution[2].

Even if the two ways to specify a solution, either by initial data at a single time t_0 or by endpoint data at two different times t_1 and t_2, are equivalent, they may give rise to different points of view concerning the dynamics of the system. We consider the following problem motivated by choosing the latter type of boundary conditions:

When considering all possible paths $q = q(t)$ that satisfy the boundary conditions $q(t_1) = q_1$ and $q(t_2) = q_2$, with q_1 and q_2 as two given sets of coordinates, what characterizes the dynamical path (the one that satisfies the equations of motion), in comparison to other continuous paths between the given end points?

Hamilton formulated an answer to this question in the form of a variational problem, called *Hamilton's principle*. The principle is formulated by use of the *action integral* of paths between the end points. The definition of the action is

$$S[q(t)] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)dt \tag{3.66}$$

It is well defined for any continuous, differentiable path $q(t)$ between the end points, not only the one that satisfies the equation of motion. The action is a *functional* of the path, which means that it is a function of the function $q(t)$. Hamilton's principle refers to variations in the value of the action $S[q(t)]$ under small variations in in the path $q(t)$:

The path $q(t)$ between the fixed end points $q(t_1) = q_1$ and $q(t_2) = q_2$, which describes the dynamical evolution of the physical system, is characterized by the action being stationary

---

[2]In exceptional cases there may be more than one solution.

under small variations in the path, $q(t) \rightarrow q(t) + \delta q(t)$, with $\delta q(t_1) = \delta q(t_2) = 0$. We write the condition as

$$\delta S = 0 \qquad (3.67)$$

where the meaning of this equation is that the change in S vanishes *to first order* in the variation $\delta q(t)$, for the dynamical path $q(t)$ between the specified initial and final points.

We may say that Hamilton's principle expresses a *global* view on the evolution of the system in configuration space, with the correct, dynamical path being specified as the solution of a variational problem. Lagrange's equations, on the other hand, gives a *local* condition for the dynamical evolution, in the form of a differential equation that should be satisfied at any time t during the evolution. These two ways of describing the motion of the system are not in conflict, but are instead equivalent, as we shall demonstrate.

In order to show that Lagrange's equation and Hamilton's principle are two equivalent ways to describe the dynamics of the system, we examine how the change in the action S for a small variation in the coordinates around a given path can be expressed in terms of the Lagrangian. To first order in the variations in the coordinates we have

$$
\begin{aligned}
\delta S &= \int_{t_1}^{t_2} \delta L(q(t), \dot{q}(t), t) dt \\
&= \int_{t_1}^{t_2} \sum_k \left(\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right) dt \qquad (3.68)
\end{aligned}
$$

The integral can be manipulated in the following way

$$
\begin{aligned}
\delta S &= \int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} \delta q_k + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \right] dt \\
&= \int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} \delta q_k - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \right] dt + \left[\sum_k \frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \right] \delta q_k \, dt \qquad (3.69)
\end{aligned}
$$

where in the last step we have used the condition that the end points should be fixed during the variations in the coordinates, so that $\delta q(t_1) = \delta q(t_2) = 0$.

The expression we have derived for the change in the action shows that δS indeed vanishes to first order in variations of the coordinates for a path that satisfies Lagrange's equations. We note that the implication also works the other way, in the sense that if δS vanishes for *arbitrary* variations in the generalized coordinates, this implies that Lagrange's equation has to be satisfied for the path $q(t)$.

As pointed out, Hamilton's principle gives an interesting, different view on the evolution of the system. It is a global view on the dynamical path in configuration space, and this view may add something interesting to the understanding of the evolution of the system. However, in most cases, the equations of motion, expressed in Lagrange's or Hamilton's form, will give the most convenient way to actually determining the time evolution of the system.

Variational problems are met in many fields of physics. It is interesting to note that the relation we have discussed between the integral form of Hamilton's principle and the differential form of Lagranges's equations may be useful for such such problems more generally. Consider a problem where an integral of a physical variable (similar to $S = \int L dt$) should be stationary under variation in the physical variable (typically a minimum or maximum problem). In that case there should be a set of differential equations (similar to Lagrange's equations) corresponding to the variational problem. Such a reformulation in terms of differential equations may be useful for solving the problem, and we shall next illustrate this by an example.

3.6.1   Example: Rotational surface with a minimal area

We consider the following problem:

Two points (x_1, y_1) and (x_2, y_2) in a the x, y plane are selected. We want to determine the curve $y(x)$ in the plane which links the two points and which gives rise to a surface of minimal area when the curve is rotated in 3-dimensional space around the x axis.

This is a typical variational problem where we want to determine a curve $y(x)$ with fixed endpoints

$$y(x_1) = x_1 , \quad y(x_2) = x_2 \tag{3.70}$$

The area to be minimized can be written as

$$A[y(x)] = \int_{x_1}^{x_2} 2\pi y \sqrt{1 + y'^2} dx \tag{3.71}$$

where we have used the notation $y' = dy/dx$. This expression for the area is found by considering the contribution from an infinitesimal section of width dx in the x direction,

$$dA = 2\pi y \sqrt{dx^2 + dy^2} = 2\pi y \sqrt{1 + y'^2} dx \tag{3.72}$$

and then integrate this along the x axis.

The variational problem can be written as

$$\delta A = 0 \tag{3.73}$$

for variations $\delta y(x)$ with $\delta y(x_1) = \delta y(x_2) = 0$. The problem is seen to be of precisely the same form as in Hamilton's principle although the variables are different and the interpretation of the problem also. To exploit the formal correspondence we write the *area functional* as

$$A = 2\pi \int_{x_1}^{x_2} L(y, y') dx \tag{3.74}$$

with $L(y, y')$ as the function corresponding to the Lagrangian. We note that here x has taken the place of t in Hamilton's principle, and y' has taken the place of $\dot{q}$ with y as the equivalent of a generalized coordinate. (For convenience we have pulled out the constant factor 2π.) The correspondence makes it easy to write the differential equation that is equivalent to the variational problem. It has the form of Lagrange's equation

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0 \tag{3.75}$$

We calculate the partial derivatives,

$$\frac{\partial L}{\partial y} = \sqrt{1 + y'^2}, \quad \frac{\partial L}{\partial y'} = \frac{yy'}{\sqrt{1 + y'^2}} \tag{3.76}$$

and get the differential equation

$$\frac{d}{dx}\left(\frac{yy'}{\sqrt{1 + y'^2}}\right) - \sqrt{1 + y'^2} = 0 \tag{3.77}$$

By doing the differentiation with respect to x and simplifying the equation we get

$$yy'' - y'^2 = 1 \tag{3.78}$$

which is a *non-linear* differential equation that is second order in derivatives.

Usually a non-linear differential cannot be solved by analytic methods, but in the present case it can. We will in this case make a complete discussion of the problem by showing how to solve the differential equation. In order to do so we change to a new variable u in the following way

$$u = \frac{y'}{y} \tag{3.79}$$

This gives

$$u' = \frac{1}{y^2}(yy'' - y'^2) \tag{3.80}$$

By applying the equation (3.78) which y should satisfy, we find

$$u' = \frac{1}{y^2} \tag{3.81}$$

which gives

$$u'' = -2\frac{y'}{y^3} = -2uu' \tag{3.82}$$

This means that u should satisfy the differential equation

$$u'' + 2uu' = 0 \tag{3.83}$$

Since the expression on the left-hand side can be written as a derivative with respect to x, the equation can immediately be integrated once to give

$$u' + u^2 = k^2 \tag{3.84}$$

where k is a constant. (Note that we can write the integration constant in (3.83) as a *positive* constant k^2, since Eq.(3.81) shows that u' is positive.)

We have now a first order differential equation to solve, and we do this by integrating the equation in the following way,

$$\frac{u'}{k^2 - u^2} = 1 \quad \Rightarrow \quad \int \frac{du}{k^2 - u^2} = x + C \tag{3.85}$$

with C as an unspecified integration constant.

The integral, which determines u as a function of x can be solved, and we do this by the following substitution (the result is also listed in standard integration tables)

$$u = k \tanh w \tag{3.86}$$

By differentiating the expression we find

$$du = \frac{k}{\cosh^2 w} dw \tag{3.87}$$

and by combining this with

$$k^2 - u^2 = k^2(1 - \tanh^2 w) = \frac{k^2}{\cosh^2 w} \tag{3.88}$$

we find

$$\frac{du}{k^2 - u^2} = \frac{\cosh^2 w}{k^2} \frac{k}{\cosh^2 w} dw = \frac{1}{k} dw \tag{3.89}$$

This means that the integral in (3.85) is reduced to the simple form

$$\int dw = k(x + C) \tag{3.90}$$

with solution

$$w = kx + w_0 \tag{3.91}$$

where w_0 is an integration constant.

The expression for u is then found to be

$$u = k \tanh(kx + w_0) \tag{3.92}$$

with derivative

$$u' = \frac{k^2}{\cosh^2(kx + w_0)} = \frac{1}{y^2} \tag{3.93}$$

For y this finally gives the solution

$$y = \frac{1}{k} \cosh(kx - w_0) \tag{3.94}$$
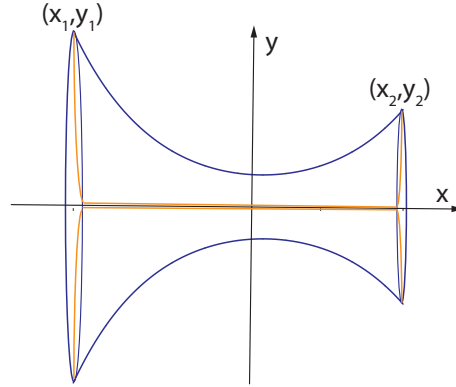
Figure 3.6: Minimal rotational surface derived from the variational problem, here shown in blue. The yellow curves indicate the collapsed surface, which may have an even smaller area .

where the two integration constants k and w_0 are (implicitly) determined from the boundary conditions,

$$y(x_1) = y_1 , \quad y(x_2) = x_2 , \quad \Rightarrow$$
$$\cosh(kx_1 - w_0) = ky_1 , \quad \cosh(kx_2 - w_0) = ky_2 \qquad (3.95)$$

The above expressions solve the variational problem. However, some further comments may be appropriate. In the case of Hamilton's principle we note that any solution to the variational problem gives a solution of the equations of motion. It is not important whether the solution corresponds to a minimum, a maximum or a saddle point of the action. In the present case, on the other hand, we are specifically interested in finding the minimum. By finding the variation in the area for infinitesimal variations in the function, $\delta y(x)$, calculated to *second order*, we can decide whether the solution we have found is a local minimum. This is similar to deciding whether a function has a minimum in a point where the derivative vanishes, by calculating and checking the sign of the *second* derivative of the function. It is straight forward to check in this way that the solution we have found is in fact a local minimum.

Another question is if we have found the *global* minimum. In fact, it is almost obvious that it is so only when the two points (x_1, y_1) and (x_2, y_2) are not too far apart, in the sense that $(x_2 - x_1)$ is not too large compared to y_1 and y_2. Therefore, if we separate x_1 and x_2 with y_1 and y_2 fixed it is clear that the area of the surface which is generated by the curve we have found will increase with the separation between the two points. At some point it will become preferable to collapse the surface in the following way: Close to each boundary point the curve $y(x)$ falls abruptly to 0, and between the two points differs only infinitesimally from 0, to form a narrow cylinder of vanishing area. Such a surface will have the area $A = \pi(y_1^2 + y_2^2)$ independent of the distance between x_1 and x_2.

The reason we do not see this surface in our analysis is that it corresponds to a curve in the x, y plane that is not differentiable. It can in this sense be excluded, but the point is that close to this curve there are differentiable curves with almost the same area. (The situation is similar to the one when we search a minimum of a function in a bounded region. In the interior of the region a (local) minimum is characterized by the derivative of the function being zero, but for

a minimum on the boundary that does not need to be the case.) From this we conclude that the minimum we have found is a global minimum only when the area satisfies

$$A \leq \pi(y_1^2 + y_2^2) \tag{3.96}$$

We do not go further in examining this point which is specific for the present example.

It is interesting to note that the minimization problem we have discussed in this example has a simple physical application. It is well known that due to the surface tension a soap film will make a surface with minimum area, adjusted to the given boundary of the film. If we therefore attach the soap film to two circular hoops that are positioned symmetrically about an axis, we will have created a situation like the one discussed in the example. According to the analysis we have made the film should make a surface similar to the one shown in the figure. For physical reasons it seems also clear that if the distance between the two hoops increases, at some point the surface will collapse to two independent surfaces that cover each of the two hoops.

Summary

We have in this part of the lectures discussed some of the basic elements of analytical mechanics. The focus has been on how to define a set of *independent*, generalized coordinates q that describe the physical degrees of freedom of the system, and to use these in a reformulation of the equations of motion. A main motivation for introducing the generalized coordinates is to eliminate from the description the explicit reference to constraints, and thereby to the corresponding (unknown) constraint forces.

The types of motion that are consistent with the constraints at a fixed time t, are referred to as virtual displacements. They correspond to changes δq in the generalized coordinates with t fixed. Application of Newton's second law, combined with virtual displacements of the system, allows a reformulation of the dynamics in a form which only refers to time evolution of the generalized coordinates. Two equivalent form of this dynamics are defined by Lagrange's and Hamilton's equations. Lagrange's equation defines a set of differential equations that primarily determines the motion in configuration space, $q(t)$. Hamilton's equations on the other hand treat the generalized coordinates q and their conjugate momenta p on equal footing and therefore determine primarily the motion in phase space, $(q(t), p(t))$.

One of the advantages of Lagrange's and Hamilton's formulations is that they specify the dynamics in a compact form through a scalar function, either the Lagrangian or the Hamiltonian. They further give an explicit scheme to follow when analyzing the physical system, where only the physical degrees of freedom participate. In addition, symmetries of the system that are represented as symmetries of the Lagrangian (and Hamiltonian) can be directly exploited to derive constants of motion and thereby effectively to reduce the number of independent variables of the system.

Hamilton's principle gives a description of the dynamics that is in a sense complementary to that of Lagrange's and Hamilton's equation. It is a variational principle which selects the path defined by the dynamical evolution between two fixed end points $q(t_1)$ and $q(t_2)$ in configuration space. This gives a global view on the time evolution which is however equivalent to the local view given by the differential equations of Lagrange and Hamilton.

The theory discussed here gives the basis for other related formulations of the dynamics of physical systems. Let me mention some of those that are not discussed in these notes. One important generalization of the theory is to the Lagrangian description of classical (and quantum) fields. In that case the continuous field variables replace the discrete generalized coordinates of mechanical systems, and in modern field theory this formulation is almost indispensable. Also for systems with discrete variables there are important generalizations. There is an underlying mathematical structure of Hamiltonian systems that is referred to as a *symplectic* structure. This can be refined and further developed by use of algebraic relations known as Poisson brack-

ets. Furthermore, the phase space flow discussed in the lectures can be extended to a the fluid like description of physical systems known as the *Hamilton-Jacobi* theory. The phase space description also has extensions to the description of *non-linear* systems, where a richer set of physical phenomena can be found than in the linear differential equations of Lagrange and Hamilton.

The theoretical reformulations of classical mechanics mentioned above also give physics a form that lies close to the formulations of quantum mechanics. That is seen clearly in the fact that many of the central objects of the classical theory, like the Hamiltonian and the conjugate coordinates and momenta, are also central objects in the quantum description, although with a reinterpretation of these as Hilbert space operators. Other correspondences are also close, for example between the Hamilton-Jacobi theory and Schrödinger's wave mechanics and between Hamilton's principle and Feynman's path integral description of quantum mechanics. This underscores the point mentioned in the introduction, that the formulations of Lagrange and Hamilton continues to hold a central position in modern theoretical physics.

Part II
Relativity

At the beginning of last century Maxwell's equations, which unifies the electromagnetism phenomena, seemed to pose a challenge to the old symmetry principle of physics, which we now refer to as *Galilean relativity*. This principle was first formulated through Galilei's observation that the laws of nature were the same in all *inertial frames*. The problem was that Maxwell's equations contains a constant with physical dimension of velocity, and the only way to make this compatible with the Galilean principle, was to assume that Maxwell's equations were valid, not generally, but only in a special inertial frame. This was thought to be the rest frame of the *luminiferous aether*, which was the name of the physical medium where the electromagnetic waves could propagate.

However, problems remained concerning the somewhat mysterious aether. It should fill the whole universe and it should have rather peculiar mechanical properties, but the most important problem was that there should be measurable corrections to Maxwell's equation in reference frames that moved relative to the aether. Michelson and Morley unsuccessfully tried to find such effects experimentally. The idea was that the earth could not at all times be at rest with respect to the aether, because of its orbital motion about the sun. If it at a particular time of the year was at rest with respect to the aether, it should half a year later have its maximal relative velocity. Measurements of the velocity of light at different times of the year did not show even tiny variations in the velocity.

In 1905 Albert Einstein offered a solution to this problem that made the discussion about the aether completely irrelevant. He insisted on the fundamental character of Maxwell's equations and at the same time he upheld the idea of all inertial systems to be equivalent with respect to the fundamental laws of nature. His way of making this possible was to change the relations between coordinates and velocities as measured in different inertial frames. In what we know as the *special theory of relativity* he introduced a new description of space and time by assuming the *Lorentz transformations* to give the correct transformations between inertial frames. These transformations were not new, at the mathematical level they had been identified and discussed as symmetry transformations of Maxwell's equations by Larmor, Lorentz and Poincare. But the fundamental character of the transformations had not been realized.

Einstein's idea was indeed revolutionary. It changed the perspective on space and time since the transformation formula showed that space and time were not independent concepts. The idea about the larger *space-time* emerged, where a distinction between space and time is not universal, but will change from one inertial frame to another. This idea had important implications, as Einstein showed. The length contraction and time dilatation of moving bodies

are well known consequences, and also the relativistic relation between mass and energy. But
the impact was deeper, since the *principle of relativity* should apply to all physical laws, and all
physical laws therefore should in some way reflect the new relation between space and time.
Later, in 1915, Einstein extended his ideas further in the *general theory of relativity*, where
gravitation was included in the fundamental description of space and time. In this theory the
geometrical properties of space-time itself has dynamical properties, and this gives rise to the
gravitational effects.

In these lectures we study some of the basic elements of Einstein's special theory of rela-
tivity. Our starting point is the Lorentz transformations, which define the fundamental relations
between coordinates and velocities in different inertial frames. We derive from these impor-
tant kinematical relations such as length contraction and time dilatation and also the relation
between relativistic mass and energy. We further discuss relativistic dynamics, where the prin-
ciple of relativity is used to guide us in how to bring Newton's equations into relativistic form.
Our approach will be to introduce and to make use of the natural formalism for theories where
space and time are treated on the same footing. This is the *four-vector formalism* where vec-
tors in three-dimensional space are replaced with vectors in four-dimensional space-time. With
the use of four-vectors (and their relatives - the *relativistic tensors*) the physical laws can be
expressed in *covariant* form, a form which is explicitly invariant under transitions between
inertial frames. This formalism may initially appear somewhat cumbersome, but applications
show that it is useful, and if one goes deeper into relativistic theory than we do in this course it
becomes indispensable. In addition to working with equations we will make extensive use of
Minkowski diagrams to illustrate the space time physics.

Chapter 4

The four-dimensional space-time

Space and time set the scene for the physical phenomena. To describe the phenomena we apply space and time coordinates, and these coordinates depend on our choice of *reference frame*. Such a reference frame we may view as a physical object which position and velocity are measured relative to, but in theoretical considerations we usually replace this object by an imagined frame with axes that define the origin and orientation of our reference system. A specific set of reference frames are the *inertial frames* which we may characterize as being non-accelerated.[1] We begin the description of the relativistic view of four-dimensional space-time by considering the coordinate transformation formulas between inertial frames, both in Galilean physics and in the special theory of relativity.

4.1 Lorentz transformations

Let us for simplicity assume that all motion is restricted to one direction, which we take as the x-direction in a Cartesian coordinate system. The *Galilean transformation* between two inertial frames with relative velocity v is then given by

$$x' = x - vt, \quad y' = y, \quad z' = z \quad (t' = t) \tag{4.1}$$

with (x, y, z, t) as the position and time coordinates in the first inertial reference frame (S) and (x', y', z', t) as the corresponding coordinates in the second frame (S'). These are the coordinate transformations used in elementary physics (and implicitly also in every day life), and to specify that the time coordinate is the same in the two reference frames seems almost unnecessary. Assume now a small body moves with velocity $u = \frac{dx}{dt}$ relative to reference frame S, and velocity $u' = \frac{dx'}{dt}$ relative to reference frame S'. The coordinate transformation (4.1) then gives us the standard velocity transformation formula

$$u' = u - v \tag{4.2}$$

---

[1]Note that even if position and velocity have no absolute meaning acceleration is different. An object far from the influence of any other object will have zero acceleration, and that defines a reference value for acceleration, both in Galilean physics and in Einstein's special relativity. However, in Einstein's general relativity this is changed, and even acceleration is no longer absolute, since gravitational effects and effects of acceleration are intermixed.

The transition from one inertial system to another means simply to correct the velocities by adding or subtracting the relative velocity of the two reference systems. This situation is illustrated in Fig.4.1 with the two sets of orthogonal coordinate axes representing the inertial frames. The transformation formula clearly shows that a theory which contains a velocity as constant parameter cannot be invariant under Galilean transformations.
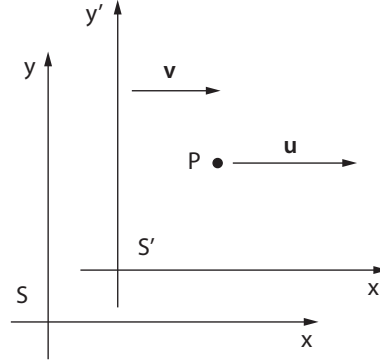
Figure 4.1: Transition from one inertial frame S to another S', here illustrated by two coordinate systems in relative motion along the x axis. The velocity u of a particle P and and the velocity v of the reference frame S' are given relative to reference frame S. The Galilean transformations give the velocities in S' by subtraction of the velocity of the reference frame itself, so that the velocity of the particle in this frame is $u - v$ and the velocity of the frame S is $-v$. In special relativity this rule for transforming velocities is no longer valid.

The Lorentz transformation, which give the correct *relativistic* formula for the transition between two inertial frames S and S' is

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma(t - \frac{v}{c^2}x) \tag{4.3}$$

where γ is defined as

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{4.4}$$

with v as the relative velocity of the two inertial frames. This transformation is not dramatically different in form from the Galilean transformation, but it is dramatically different in interpretation and in consequences.

The most prominent change in the transformation formula is that the time coordinate is no longer universal, but depends on the chosen inertial frame. It is *observer dependent.* Another important change is that the formula contains a constant c with the dimension of velocity. It has the physical interpretation as the speed of light. However, it is clear that when the relative velocity v is small compared to the speed of light c will there be essential no difference between the Galilean and the relativistic formulas. This is seen by making an expansion in v/c

$$\gamma = 1 - \frac{v^2}{2c^2} + ... \tag{4.5}$$

When only the leading terms in the v/c expansion are kept, the transformation equations (4.3) reduce to the Galilean equations, as one can readily check.

Let us show that an object which moves with the velocity of light will have the same velocity in all reference systems related by the Lorentz transformations (4.3). The important point is that the transformation formula for velocity is now changed. The definition of velocity is the same as before, but the Lorentz transformation between the coordinates of the two inertial frames will change the relation between u and u'. For an infinitesimal change in the position coordinates we have

$$
\begin{aligned}
dx' &= \gamma(dx - vdt) = \gamma(u - v)dt \\
dt' &= \gamma(dt - \frac{v}{c^2}dt) = \gamma(1 - \frac{uv}{c^2})dt
\end{aligned}
\tag{4.6}
$$

and from this follows

$$
u' = \frac{dx'}{dt'} = \frac{u - v}{1 - \frac{uv}{c^2}}
\tag{4.7}
$$

This is the new transformation formula, which is valid when the velocity u of the object is colinear with the relative velocity v of the two inertial frames. If we now set $u = c$ in the formula it follows directly that $u' = c$. So there is no addition of the relative velocity of the two frames in this case, and the speed of light is indeed the same in all reference frames[2].

The Lorentz transformations thus imply that the speed of light is universal, the same in all inertial reference frames. This explains the classic observational results of Michelson and Morley, where no change of the speed of light could be detected over the year. However, the way the problem is solved is radical, since the time coordinate now gets mixed with the space coordinates in transformations between inertial reference frames.

The relativistic velocity formula implies that if a particle moves with subliminal velocity in one inertial frame, it will move with subliminal velocity in any other inertial frame. The transformation formula does not, in a strict sense, exclude the possibility of superluminal velocity, which then, again as a consequence of the velocity formula, will have to be superluminal in all inertial frames. However, as we shall later see, this will create a problem for a causal understanding of processes where energy and momentum is exchanged over distance. The usual understanding of relativity is therefore that any signal that transports energy, or any form of information, has to propagate with a speed slower than or equal to the speed of light.

4.2 Rotations, boosts and the invariant distance

The Lorentz transformations (4.3) are often referred to as *boosts* or *special* Lorentz transformations. Such a transformation can be viewed as taking the first reference frame S and changing its velocity in some direction (here the x-direction) without rotating its coordinate axes, and thereby creating the new reference frame S'. The *general* Lorentz transformations are considered as transformations that include booth boosts and rotations.

---

[2]The special form of the Lorentz transformations (4.3) implies the speed of light is invariant. However, it is of interest to see that this works also the other way. Thus, if we only assume the space-time transformations to be *linear*, the assumption that the speed of light is unchanged will imply the special form (4.3) of the transformations.

There is in fact a formal resemblance between rotations and the boosts. To see this we first consider a rotation in the x, y-plane, which in Cartesian coordinates takes the form,

$$
\begin{aligned}
x' &= \cos\phi \, x + \sin\phi \, y \\
y' &= -\sin\phi \, x + \cos\phi \, y
\end{aligned}
\tag{4.8}
$$

where ϕ is the rotation angle. The typical feature of the rotations is that the *distance* between two points is left invariant by the transformations. For the transformation (4.8) this invariance is expressed by

$$
\Delta s'^2 = \Delta x'^2 + \Delta y'^2 = \Delta x^2 + \Delta y^2 = \Delta s^2
\tag{4.9}
$$

with Δx and Δy representing the coordinate difference between two points and Δs the relative distance between the points.

Let us next consider the Lorentz transformations (4.3) and introduce a new parameter χ in the following way[3]

$$
\cosh\chi = \gamma \,, \quad \sinh\chi = \gamma\beta
\tag{4.10}
$$

with β as the standard abbreviation for the dimensionless velocity $\beta = v/c$. This is a consistent parametrization, since the two expressions satisfy the requirement of hyperbolic functions,

$$
\cosh^2\chi - \sinh^2\chi = \gamma^2(1 - \beta^2) = 1
\tag{4.11}
$$

The parameter χ, which is related to the relative velocity v of the two reference frames by the equation

$$
v = c \tanh\chi
\tag{4.12}
$$

is referred to as *rapidity* and is sometimes a more convenient parameter to use than the velocity. It is here introduced in order to give the Lorentz transformations a form similar to that of rotations. For the special transformation (4.3) it takes the form

$$
\begin{aligned}
x' &= \cosh\chi \, x - \sinh\chi \, ct \\
ct' &= -\sinh\chi \, x + \cosh\chi \, ct
\end{aligned}
\tag{4.13}
$$

We note the formal similarity with the rotations (4.8), where the time coordinate ct has taken the place of the space coordinate y and the rapidity χ has taken the place of the angle ϕ. But χ is no angle, which is shown by the fact that the trigonometric functions are replaced by hyperbolic functions. The geometric difference between the two types of transformations are demonstrated in Fig.4.2.

For the Lorentz transformations the distance (in three-dimensional space) between two points is no longer invariant, but another quantity, which includes also the difference in time coordinate, takes its place. For the transformation (4.13) we find that with a new definition of

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[3]As a reminder the hyperbolic functions are defined by $\cosh\chi = \frac{1}{2}(e^\chi + e^{-\chi})$ and $\sinh\chi = \frac{1}{2}(e^\chi - e^{-\chi})$.
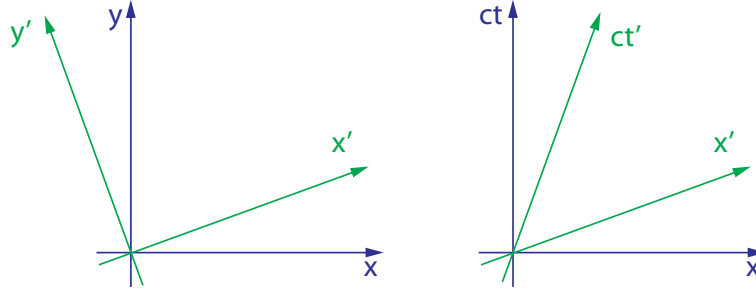
Figure 4.2: Comparison between a rotation in the (x, y) plane and a boost in the (ct, x) plane. In the first case the rotation will transform an orthogonal coordinate frame (blue) into a rotated orthogonal frame (green). In the second case the orthogonal frame will not appear as orthogonal after the boost transformation. However, the meaning of *orthogonality* is in fact changed when time is introduced as a new *space-time* coordinate.

Δs^2 the following combination of relative coordinates between two space-time points is the same in the two frames,

$$\Delta s'^2 = \Delta x'^2 - c^2 \Delta t'^2 = \Delta x^2 - c^2 \Delta t^2 = \Delta s^2 \tag{4.14}$$

This follows from the properties of the hyperbolic functions. We note the important change in relative sign of the two terms, compared to that of the distance in three-dimensional space.

Distance in three-dimensional space has an immediate physical meaning as a measurable quantity that is independent of our choice of coordinate system. From a mathematical point of view it is natural to consider distance, $\Delta s^2 = \Delta \mathbf{r}^2$, as a property of space itself. It defines the *geometry* of three-dimensional space, which we then consider as equipped with a property referred to as a *metric*. The metric of three-dimensional physical space is *Euclidean*, which means that it is geometrically a flat space with a positive measure of distance. The rotations we may regard as *symmetry transformations* of the space, which are transformations which leave the metric invariant. The Galilean transformations are time dependent extensions of these, which also leave all distances between points in three-dimensional space unchanged.

To change the fundamental transformations between inertial frames from the Galilean to the Lorentz transformations, implies a change in our view of space itself. The invariant metric is no longer defined by the (Euclidean) distance between points in three dimensional space, but rather by a generalized distance which involves in also the time coordinate. The expression for the generalized distance between two points in space and time (often referred to as two space-time *events*) is given by

$$\Delta s^2 = \Delta \mathbf{r}^2 - c^2 \Delta t^2 \tag{4.15}$$

This new metric, unlike the metric in three-dimensional space, does not have an immediate, physical interpretation. It can be expressed in terms of the three-dimensional distance $|\Delta r|$ (and the time difference Δt), and under certain conditions a special reference frame can be chosen where Δt vanishes and the four dimensional distance is identical to the three-dimensional one.

But it is important to note that the metric of four-dimensional space- time, defined by the invariant (4.15) is *not* a Euclidean metric. We refer to this as a *Minkowski metric*.

The important difference between the Euclidean and Minkowski metrics is that in three-dimensional space the invariant Δs^2 is always positive, while in the four-dimensional case that is not necessary the case. Even so it is conventional to write the invariant as a square, Δs^2. Depending on the relative position of the two space-time points the generalized invariant (4.15) may be positive, zero or negative. If it is positive we refer to the separation of the two space-time points as being *spacelike*, if it is zero the separation is called *lightlike* and if it is negative the separation is *timelike*. Since distance in three-dimensional space is the square root of Δs^2, this lack of positivity in four dimensions shows that the change in metric strictly speaking is not simply a change in the definition of distance.

The Lorentz invariance of the *line element* $\Delta s^2 = \Delta \mathbf{r}^2 - c^2 \Delta t^2$ is directly related to the fact that the speed of light is the same in all inertial frames. To see this we note that if c denotes the speed of light in a given reference frame, two space-time points on the path of a light signal through space and time will have lightlike separation,

$$\Delta s^2 = \Delta \mathbf{r}^2 - c^2 \Delta t^2 = 0 \tag{4.16}$$

Furthermore, since Δs^2 is invariant under Lorentz transformations, if this equation is satisfied in *one* inertial frame it will be satisfied in *all* inertial frames. This means that a signal that connects the two space-time points will travel with the same speed c in all inertial reference frames.

4.3   Relativistic four-vectors

A point in three-dimensional space can be specified by a position vector, often written as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{4.17}$$

with x, y and z as the Cartesian coordinates of the vector in a particularly chosen coordinate system, and with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as the unit vectors along the orthogonal coordinate axes. These vectors define the physical three-dimensional space as a *vector space*. The vector space is defined with respect to an arbitrarily chosen reference point, corresponding to the origin $\mathbf{r} = 0$, but the position vectors $\mathbf{r}$ may otherwise be considered as being independent of any choice of coordinate system in this space. The coordinates x, y and z, on the other hand, do depend on such a choice. This is consistent with our physical picture of a vector $\mathbf{r}$ in physical, three-dimensional space; it has a well-defined length and direction and can be viewed as a *geometrical object* that exists independent of any choice of coordinate system. The coordinates are however a convenient way to characterize the vector by a set of numbers (with a physical unit), and these will then vary from one reference frame to another.

Let us write the coordinate expansion in the following way,

$$\mathbf{r} = \sum_{k=1}^{3} x_k \mathbf{e}_k \tag{4.18}$$

with $\{\mathbf{e}_k, k = 1, 2, 3\}$ as a set of three orthogonal unit vectors,

$$\mathbf{e}_k \cdot \mathbf{e}_l = \delta_{kl} \tag{4.19}$$

A change from one set of orthogonal vectors to another, we write as a transformation

$$\mathbf{e}_k \rightarrow \mathbf{e}'_k = \sum_{l=1}^{3} R_{kl} \mathbf{e}_l \tag{4.20}$$

where orthogonality of the vectors means that the coefficients R_{kl} satisfy the condition

$$\sum_{i=1}^{3} R_{ki} R_{li} = \delta_{kl} \tag{4.21}$$

This equation gives the condition for the transformation (4.20) to be a rotation. With the vector $\mathbf{r}$ being independent of the transformation, the change of the unit vectors $\mathbf{e}_k$ has to be compensated by a rotation of the coordinates x_k,

$$x_k \rightarrow x'_k = \sum_{l=1}^{3} R_{kl} x_l \tag{4.22}$$

Due to the property (4.21) of the coefficients R_{kl} it is straight forward to check that the combined transformation of the coordinates and unit vectors leaves the vector $\mathbf{r}$ unchanged.

In a similar way as three-dimensional space is viewed as a three-dimensional vector space, space-time may be described as a four-dimensional vector space. The extension from three-dimensional space to four-dimensional space-time then leads to the extension of vectors $\mathbf{r}$ with Cartesian coordinates (x, y, z) to four-dimensional vectors with coordinates (x, y, z, t), where t is the time coordinate. In order to have the same physical dimension for all four directions in space-time, we introduce, in the standard way, a time coordinate with dimension of length, $x^0 = ct$, where c is the speed of light[4]. Note the convention that the coordinates of space-time are written with lifted indices, so that,

$$x^0 = ct \ , \quad x^1 = x \ , \quad x^2 = y \ , \quad x^3 = z \ , \tag{4.23}$$

We shall later explain the reason for this convention.

To distinguish the 4-vectors of space-time from the 3-vectors of space, we shall in these notes underline the 4-vectors. In particular, the position vector of a space time point, when decomposed in Cartesian components, we may write as

$$\underline{\mathbf{x}} = ct\underline{\boldsymbol{\tau}} + x\underline{\mathbf{i}} + y\underline{\mathbf{j}} + z\underline{\mathbf{k}} \tag{4.24}$$

where we have expanded the set of three unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ with a fourth vector $\boldsymbol{\tau}$, which points in the direction of the time axis, and by underlining the unit vectors we have indicated

---

[4]For historical reasons the time component, in the form discussed here, is taken to be the 0 component rather than the 4 component. Originally a 4 component that was *imaginary* was introduced for time, so that $x_4 = ict$. The reason for that was to formally give boost transformations the same form as rotations. However, this convention is not so often used anymore.

that they are now vectors in the extended four dimensional space-time. More often we will write the expansion in the general form

$$\underline{\mathbf{x}} = \sum_{\mu=0}^{3} x^{\mu} \underline{\mathbf{e}}_{\mu} \tag{4.25}$$

with $\{\underline{\mathbf{e}}_{\mu}\}$ as a orthogonal set of unit vectors in four-dimensional space-time. Note that these basis vectors are written with the indices as subscript, as opposed to the coordinates where the indices are written as superscript. This is a standard convention, which means that the coordinate independent sum (4.25) appears as a sum over pairs of equal indices, where one is an upper index and the other a lower index. We shall later discuss this convention in some detail.

One important point to note is that such a set of four-dimensional unit vectors will identify uniquely an inertial reference frame. This is different from the situation with three-dimensional vectors, where a set of three orthogonal unit vectors will define the orientation of a reference frame, but not its velocity.

A four-dimensional vector can be decomposed in its time component and its three-vector part. We often write it simply as

$$\underline{\mathbf{x}} = (x^0, \mathbf{r}) \tag{4.26}$$

Note however that this formulation is somewhat sloppy since the four-vector $\underline{\mathbf{x}}$ is to be considered as independent of any choice of reference frame, while the decomposition (4.25) refers to a specific class of inertial reference frames (with the same velocity), since the separation in time and space depends on the velocity of the inertial frame[5]. In any case, such a decomposition is often useful. Even if the four-vector formulation is attractive since it gives a compact relativistic form of physical equations, the decomposition is often needed in order to make a physical interpretation of the results.

The space-time vector $\underline{\mathbf{x}}$, which does not refer to any specific reference frame, we often refer to as an *abstract* vector. A concrete representation of the vector is given by its matrix representation, which is composed by its coordinates as

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{4.27}$$

As opposed to $\underline{\mathbf{x}}$, this matrix does depend on the choice of reference frame, and the Lorentz transformations specify how the matrix elements change under a change of the inertial frame. In the following we shall refer to this matrix, or more generally the collection of coordinates $\{x^{\mu}, \mu = 0, ..., 3\}$, simply by the symbol x. It represents the set of *coordinates* of a space time point in a particular inertial frame.

A transition between two inertial reference frames can now be viewed as a linear transformation of unit vectors and of coordinates in much the same as way as transformation of

---

[5]The position *vector* of a space-time point does however depend on the (arbitrary) choice of a reference point in space and time, corresponding to the vector $\underline{\mathbf{x}} = 0$.

three-dimensional unit vectors and coordinates given by (4.20) and (4.22). We write the relativistic transformations as

$$\begin{aligned}
\underline{\mathbf{e}}_\mu \quad &\rightarrow \quad \underline{\mathbf{e}}'_\mu = \sum_{\nu=0}^{3} L_\mu{}^\nu \, \underline{\mathbf{e}}_\nu \\
x^\mu \quad &\rightarrow \quad x'^\mu = \sum_{\nu=0}^{3} L^\mu{}_\nu \, x^\nu
\end{aligned}$$
(4.28)

Again we notice the different positions of space-time indices, with the coefficient of the basis vector transformations written as $L_\mu{}^\nu$ and the coefficients of the coordinate transformation written as $L^\mu{}_\nu$. These are not identical, but closely related, as we shall later see. Here we notice that since the space-time vector $\underline{\mathbf{x}}$ is coordinate independent, the expansion of this vector in the given basis, implies that the transformation coefficients have to satisfy the equation

$$\sum_{\mu=0}^{3} L^\mu{}_\rho \, L_\mu{}^\sigma = \delta^\sigma_\rho$$
(4.29)

where δ^σ_ρ is the Kronecker delta written in four-vector notation. This equation is a direct generalization of the condition (4.21) satisfied by the transformation coefficients of rotations in three dimensions.

Since the transitions between inertial frames is described by Lorentz transformations, such a transformation is now identified by the set of coefficients $L^\mu{}_\nu$. It is straight forward to check that the coordinate transformation (4.3) is a special case, with coefficients given by, $L^0{}_0 = L^1{}_1 = \gamma, L^0{}_1 = L^1{}_0 = \gamma\beta, L^2{}_2 = L^3{}_3 = 1$, while other coefficients vanish.

4.4 Minkowski diagrams

The vector space of four-dimensional space-time, with the relativistic metric (4.15), is referred to as *Minkowski space*. When discussing motion in this space it is often useful to make a graphical representation of the space, but since we cannot make a good representation of *all* four dimensions we usually make a restriction to the two-dimensional subspace spanned by the coordinates (x^0, x^1) or the three-dimensional subspace spanned by (x^0, x^1, x^2). Such a restricted representation may be sufficient when we consider motion in one or two (space) dimensions. The graphical representations of the subspaces are referred to as *Minkowski diagrams*. Such diagrams are especially useful in order to show the causal relations between space-time points.

In Figure 4.3a a two-dimensional Minkowski diagram is shown, which is similar to the space time diagram already used in Fig.4.2, with ct and x as coordinate axes of a chosen inertial system. The coordinate axes of another inertial frame, which moves in the x-direction relative to the first one are also shown, together with the basis vectors of the two coordinate systems. In the diagram also the lines $x = \pm ct$ are shown, which indicate space-time paths for light signals that pass through the reference point O.

Let us first consider the information given by the direction of the coordinate axes in the diagram. The ct coordinate we may view as the space-time trajectory, often called the *world*
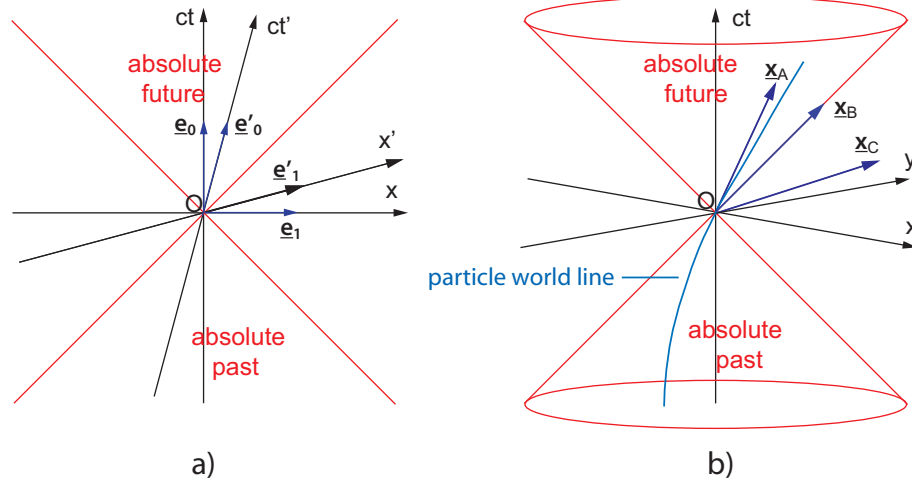
Figure 4.3: Two-dimensional and three-dimensional Minkowski diagrams. In both diagrams the location of the light cones relative to the point O are shown. They indicate which space time points are causally connected with or causally disconnected from O. In figure a) the coordinate axes of two inertial frames are drawn, as well as the corresponding basis vectors. In figure b) three different types of four-vectors are shown, with $\underline{\mathbf{x}}_A$ as a timelike vector, $\underline{\mathbf{x}}_B$ as a light like vector, and $\underline{\mathbf{x}}_C$ as a space-like vector. In figure b) the world line of a massive particle is also shown. It moves with subluminal velocity, which means that the four-vector velocity is timelike.

line, of (an imagined) observer at rest at the origin of inertial frame S, and in the same way the ct' axis describes the world line of an observer at rest with respect to the (moving) reference frame S'. The tilted direction of the ct' axis simply means that the observer at rest in S' moves relative to reference frame S. However, the x' axis is also tilted relative to the x axis, and that is an effect that one does not see in a similar Galilean diagram. Since the x axis describe points that are simultaneous in reference frame S, this means that the two reference frames disagree on what are simultaneous space-time events. This is one of the important predictions of relativity, that *simultaneity is not universally defined, but is reference-frame dependent*.

Let us next consider the implications of the fact that the location of the (red) light paths in the diagram are independent of the choice of inertial frames. The points on these lines have *lightlike* separation from the origin O. All space time point that lie between any of these lines and the time axis, either in the upward or the downward direction have *timelike* separation from O. Space-time points that have timelike separation from O and appear *later* we refer to as lying in the *absolute future* of the point O, while points with timelike separation that appear *earlier* than O we refer to as lying in the *absolute past*. "Absolute" here means that this ordering of events is independent of the choice of inertial frame.

However, for events that lie outside of the light paths, either to the right or to the left, the situation is different. These are points at *spacelike* separation from the origin O. For a specific reference frame like S also these points can be characterized as being either in the past ($t < 0$) or in the future ($t > 0$), but such a characterization is now reference frame dependent. In fact for any point at spacelike separation from O there exist some inertial frames that will place this

point in the past relative to the origin O and other inertial frames that will place the point in the future.

This relativity in the characterization of space-time points as being in the past or in the future may seem somewhat confusing, but is in reality not in conflict with causality, which orders events with respect to cause and effect. This is so since two points with spacelike separation are *causally disconnected* in the sense that no physical influence can propagate from one of the space-time points to the other. The speed of light sets in relativity theory an upper limit to the propagation speed of any physical signal and such a signal therefore cannot propagate between points with spacelike separation. This is a point to stress. In relativity theory there are space-time points (events) that are causally disconnected, *i.e.*, no physical signal can even in principle connect such a pair of events. In non-relativistic theory this feature of space and time is absent, since there is no definite upper limit to the speed with which a signal can propagate.

In Fig4.3b we show a three-dimensional representation of Minkowski space. The light paths to and from the origin O now form a double cone, consisting of a *future light cone* and a *past light cone*. Space-time points inside the light cone are causally connected to O, in the sense that points inside the future light cone can be reached by a physical signal sent from O and a point inside the past light cone can reach O with a physical signal. In the diagram three four-vectors are drawn, where $\underline{\mathbf{x}}_A$ is a *timelike* vector, $\underline{\mathbf{x}}_B$ is a *lightlike* vector and $\underline{\mathbf{x}}_C$ is a *spacelike* vector. In the diagram the world line of a (massive) particle that passes through the origin is also drawn. Since its velocity at all times is lower than the speed of light this space-time curve is restricted to lie within the light cone.

In these diagrams the light cones associated with the origin O have been drawn. In reality *any* space-time point E can be associated with a past and a future light cone. These cones order the points of space time in those that are causally connected to E and those that are causally disconnected.

As mentioned above, the Minkowski diagrams are particularly well suited for showing the causal relations between space-time points. However, one should be aware of the fact that there are in other respects certain shortcomings. This has to do with the point that the Minkowski geometry of space-time is not well represented in diagrams with Euclidean geometry. This is seen in Fig.4.3a and Fig.4.4, where the coordinate axes of reference frame S seems to have a special status, since the time and space axis have orthogonal directions. That is not the case for the coordinate axes of reference frame S', even if we know that these two inertial frames in reality are equivalent. The length scale along two different sets of coordinate axes are also not represented as equal in the diagram, even if they have the same length when measured in the corresponding reference frames. So one has to be aware of this, that angles and lengths are not correctly represented in the Minkowski diagrams.

4.5 General Lorentz transformations

So far we have focussed on the *special* Lorentz transformations. These are the transformations that change the velocity of the inertial frame without rotating its axes. A special case of these is the boosts in the x direction, but the velocity of a general boost can have an arbitrary direction. The special Lorentz transformations (or boosts) are therefore characterized by *three* parameters, namely the three components of the velocity vector $\mathbf{v}$ that relates the two inertial frames
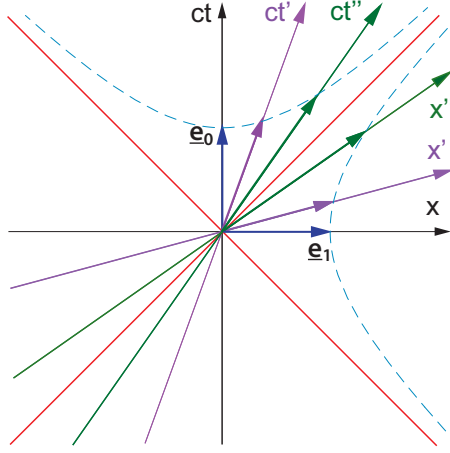
Figure 4.4: The geometry of the two-dimensional Minkowski diagram. The coordinate axes are shown (with different colors) for three different inertial frames. The length of the unit vectors of the three frames appear as different, and also the angle between the vectors, in spite of the equivalence between the frames. This is due to the difference between the Euclidean geometry of the plane and the Minkowski metric of space-time. Note that the set of unit vectors in the time, as well as in the space direction fall on hyperbolas.

of the transformation. Let us denote a general transformation of this type by B (with reference to this as a *boost*).

A *general* Lorentz transformation is a transformation between inertial frames that may also include a rotation of the axes of the second reference frame with respect to the first one. Such a transformation can therefore be seen as a composite operation, first a boost and then a rotation[6]

$$L = RB \qquad (4.30)$$

The Lorentz transformation L defines a linear map of the vector coordinates of the first reference frame (S) into the vector coordinates of the second reference frame (S'). We may write it as

$$x' = Lx \qquad (4.31)$$

where x, x' and L are matrices. Written out explicitly the matrix equation is

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} L^0{}_0 & L^0{}_1 & L^0{}_2 & L^0{}_3 \\ L^1{}_0 & L^1{}_1 & L^1{}_2 & L^1{}_3 \\ L^2{}_0 & L^2{}_1 & L^2{}_2 & L^2{}_3 \\ L^3{}_0 & L^3{}_1 & L^3{}_2 & L^3{}_3 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad (4.32)$$

The decomposition of the Lorentz transformation L in (4.30) can similarly be read as a matrix product of the boost matrix B and the rotation matrix R. Both these are 4x4 matrices, but the

---

[6]It can also be defined with the operations in opposite order, $L = B'R'$, but in general B will then be different from B' and R will be different from R' since these operations do not commute.

rotation matrix only mix the space coordinates x^1, x^2 and x^3, and leaves the time coordinate x^0 unchanged.

The general Lorentz transformations, as defined above, are *homogeneous* linear transformations, which imply that the origin of the two coordinate systems are mapped into each other by the transformation. However, a transformation between inertial frames can also involve a shift of the origin. This leads to the *inhomogeneous* Lorentz transformations, which we may write as

$$x' = Lx + a \qquad (4.33)$$

where a represents the displacement of the origin. In matrix form a is

$$a = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} \qquad (4.34)$$

where the four parameters describe the shift of the origin along the four space-time axes.

The inhomogeneous Lorentz transformations depend all together on 10 parameters, 3 of these are rotation parameters, another 3 are boost parameters and finally 4 are translation parameters. In mathematical terms this set define a 10 parameter *transformation group* referred to as the *inhomogeneous Lorentz group* or the *Poincaré group*. The group property of the set implies that the successive application of two transformations will create a new transformation from the same set.[7] The homogeneous transformation define a smaller *subgroup*, which is the 6 parameter *homogeneous Lorentz group* or simply the *Lorentz group*. The rotations form an even smaller, 3 parameter subgroup of the Lorentz group.

However, one should note that the set of boosts do not form a group, since the composition of two boosts with different directions will not be a pure boost, but will also include a rotation. This is purely relativistic effect with interesting physical consequences. A particular consequence is the *Thomas precession* effect, where a spinning particle which follows a bended path will show precession of the spin even if no force acts on the spin.

The full set of inhomogeneous Lorentz transformations define the fundamental *symmetry group* of special relativity. These symmetry transformations can in fact be interpreted in two different ways. They can be interpreted as *passive* transformations, which is the picture we use here. This means that the transformation of coordinates follows from a change of reference frame while the physical systems that are described are not changed in position or motion. When a symmetry transformation is instead interpreted as an *active* transformation this means that the change of coordinates corresponds to a physical change in the location of the processes described by the coordinates, while the reference frame is left unchanged. Such an active transformation could be to change the motion of a physical body by shifting its position, by rotating it and by changing its velocity. It is of interest to note that when working with coordinates, the formalism makes no distinction between these two types of transformation. This is a consequence of the fact that the transformations describe symmetries of the theory.

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[7]The group property of the Lorentz transformations means that the composition of any two Lorentz transformation will define a new Lorentz transformation and the inverse of a Lorentz transformation is also a Lorentz transformation. This group property is almost obvious, with the Lorentz transformations being defined as mappings between inertial frames.

A common property of all the space-time transformations discussed above is that they leave invariant the line element between space time points,

$$\Delta s^2 = \mathbf{r}^2 - c^2 \Delta t^2 \tag{4.35}$$

and this was in fact, for a long time, regarded as the basic condition that defined the relativistic symmetry transformations. However, there exist some *discrete* space-time transformations that leave the line element (4.35) unchanged, but which have been shown, by experiments, not to be fundamental symmetries in the same sense. These are the space inversion and time reversal transformations defined by the transformation matrices,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.36}$$

Since they only change the sign of either $\Delta \mathbf{r}$ or Δt obviously Δs^2 is left unchanged. Most physical processes are in fact invariant under these transformations, but in elementary particle physics small effects which break P and T symmetry have been detected. These are associated with the *weak nuclear forces*.

Chapter 5

Consequences of the Lorentz transformations

The relativistic form of the fundamental space-time symmetries, expressed by the Lorentz transformations, has consequences for all physical theories at the fundamental level. Some of these may refer to as kinematical consequences, since they are directly linked to the relativistic transformations of space and time. One of these is the *length contraction* effect, which is the effect that a body in motion appears as shorter in the reference frame where the body moves than in the reference frame where it is at rest. Another kinematical effect is the *time dilatation* effect, which is the effect that time seems to run slower for a body in motion than for a body at rest. We shall discuss these effects and some further consequences of them, in particular the famous *twin paradox*, which has to do with the effect that two persons that follow different space-time paths between a common point where they depart and another point where they meet again, will perceive a difference in the time spent on the journey.

5.1 Length contraction

We consider a situation where the length of a moving body is measured. For simplicity, let the body be a rod with length L_0 when measured in its inertial *rest frame*. It is oriented along the x-axis in this reference frame, which we refer to as S'. Another inertial frame S, called the *laboratory frame*, is oriented with the axes parallel to those of S', and relative to this frame the rod is moving in the x-direction with the velocity v, as illustrated in Fig. 5.1.

We shall refer to the front end of the rod as A and the rear end as B. The space-time coordinates of these points in the two reference frames are related by the Lorentz transformation

$$
\begin{aligned}
x'_A &= \gamma(x_A - vt_A), \quad t'_A = \gamma\left(t_A - \frac{v}{c^2}x_A\right) \\
x'_B &= \gamma(x_B - vt_B), \quad t'_B = \gamma\left(t_B - \frac{v}{c^2}x_B\right)
\end{aligned}
\tag{5.1}
$$

where the time coordinates of the two end points are independently chosen.

We note that for the measurement of length in the rest frame S' the time coordinates of the end points are unimportant, since the space coordinates do not change with time. The length
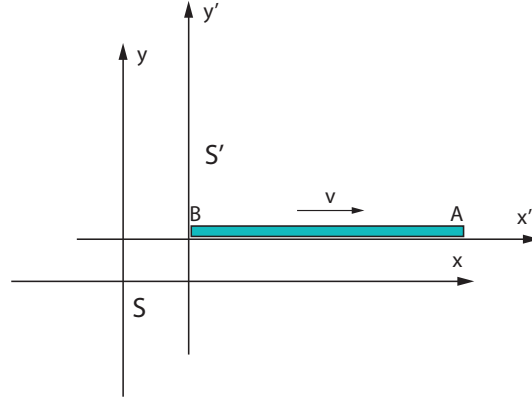
Figure 5.1: Measurement of the length of a moving body. S' is the rest frame of the moving body which has velocity v relative to the laboratory frame S. In the rest frame the measured length has its maximum value L_0, while in the lab frame it seems contracted to a shorter length $L < L_0$.

of the rod is simply the difference between the (time independent) x coordinates of the ends of the rod,

$$L_0 = x'_A - x'_B \qquad (5.2)$$

However, in S the positions of the endpoints change with time, and therefore it is meaningless to define the length as the difference in x coordinates unless we specify for what time the positions should be determined. The natural definition is that length should be defined as the distance *measured between simultaneous events* on the space-time paths of the two end points. Note that this is how length is measured also in non-relativistic physics. If distance is measured between the positions at different times, any value could be found for the length. The important point is that in non-relativistic physics simultaneity is universally defined, whereas in relativity it is reference frame dependent. Therefore we state that

The length of a moving body measured in an inertial frame S is the space distance between the end points of the body measured at equal times in the same reference frame S.

This means that we for the moving rod, to find the correct expression for the length in reference frame S, should fix the time coordinates of the end points so that $t_A = t_B$ (rather than $t'_A = t'_B$). From the Lorentz transformation formula we then derive

$$
\begin{aligned}
L_0 &= x'_A - x'_B \\
&= \gamma\left[(x_A - x_B) - v(t_A - t_B)\right] \\
&= \gamma\,(x_A - x_B) \\
&= \gamma\,L \qquad (5.3)
\end{aligned}
$$

This is in fact the length contraction formula, which we may also write as

$$L = \frac{1}{\gamma}L_0 \leq L_0 \qquad (5.4)$$

where the last inequality follows from the fact that $\gamma = 1/\sqrt{1 - \frac{v^2}{c^2}} \geq 1$. The formula tells us that the length has it maximum when measured in the rest frame of the body. When measured in an inertial frame where the body is moving it seems length contracted in the direction of motion.
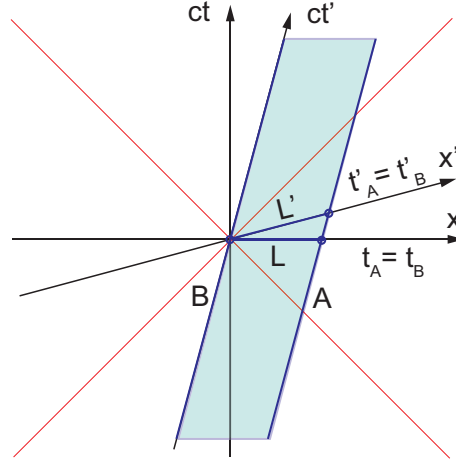
Figure 5.2: Minkowski diagram for the length measurement. The shaded area shows the space-time trajectory of the moving rod, with A and B as the trajectories of the end points. Length measurement in reference frame S (with unmarked coordinates) should be performed for space-time points with $t_A = t_B$ as indicated in the figure. This is different from measurements for points with $t'_A = t'_B$, which is the natural choice in the rest frame S'.

In Fig.5.2 the measurement of the length between the end points of the body in reference frame S is illustrated in a Minkowski diagram.

5.2 Time dilatation

Next we consider the relativistic effect that the clock in motion seems to be slower than a clock at rest. We have to specify precisely how the comparison is done, and also here the reference dependence of *simultaneity* is important. Let us consider a situation similar to that of the previous section. An inertial frame S' is the rest frame of a clock that is localized at the space origin ($x' = y' = z' = 0$) of S'. It measures the time coordinate t', and is therefore often called a *coordinate clock*. The clock and the reference frame is moving with velocity v along the x axis relative to a second inertial frame S (the *laboratory frame*). The time coordinate t of this frame we may consider as being measured by a second clock which is at rest at the space origin of S'. The coordinate transformation between the two reference frames is given by the same Lorentz transformation formula (5.1) as in the discussion of the length contraction effect.

The situation is illustrated in the Minkowski diagram of Fig.5.3. The time axis of reference frame S' is the world line of the moving clock, and the ticks of the clock as regular intervals τ
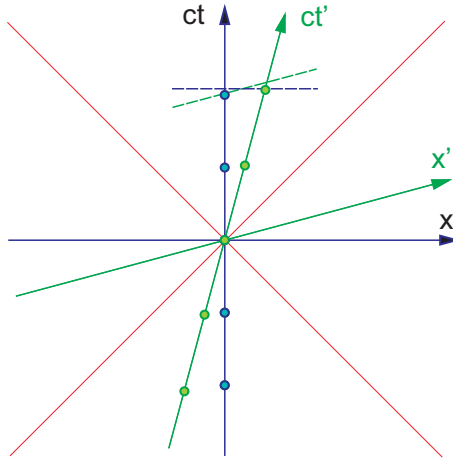
Figure 5.3: The time dilatation effect illustrated in a Minkowski diagram. The orthogonal (blue) coordinate axes define the laboratory frame S while the tilted (green) coordinate axes define a reference frame S' that moves relative to S. The green dots on the time axis denote the events where a co-moving coordinate clock in S' makes clicks at equal time intervals. Similarly the blue dots on the time axis of S denote the clicks of a coordinate clock in the lab frame. An observer in S finds that the clicks of the moving clock come with larger time separation than shown by his own clock. This is illustrated by the dashed blue line which shows that the click of the moving clock has larger t coordinate than the corresponding click on his own. This is the time dilatation effect, usually stated as *a moving clock runs more slowly than a clock at rest*. The diagram shows how this effect is symmetric with respect to the two clocks. An observer in S' will note that the clicks of the clock in S will have larger t' coordinate than the corresponding clicks on her own clock. The comparison is now performed with equal times in S', as shown by the dashed green line. The comparison between the two clocks is thus performed for different pairs of events on the world lines of the clocks by the two observers.

are indicated in the diagram (for example with $\tau = 1s$). In the same way the ticks of the clock of the laboratory frame is indicated on the time axis of S. Now we want to examine how the time scale of the moving clock is perceived in the laboratory frame, when compared with the clock at rest in this frame. Since the two clocks are not located at the same space-time points we have to make clear how this comparison should be done.

The important point is that we should choose one of the frames for the comparison, and in the present case we would like to compare the two clocks in the laboratory system S. Let us then focus on two events that correspond to two subsequent clicks of the moving clock. The first event we may take as the coincidence of the origins of the two reference frames: $t = t' = 0, \quad x = x' = 0$. We assume this event to correspond to the first click of both clocks. The next event corresponds to the second click of the moving clock. It has coordinates in S', $(x', t') = (0, \tau)$, while the corresponding coordinates in S we refer simply to as (x, t). The time t is then the time of the second click as registered in S, and this is the time we would like to compare with the time τ shown on the moving clock.

The time t is readily found by using the inverse of the Lorentz transformation formula

applied in (5.1),

$$ct = \gamma \left(ct' + \frac{v}{c^2}x'\right) = \gamma \, c\tau \qquad (5.5)$$

This gives the time dilatation formula

$$t = \gamma\tau \geq \tau \qquad (5.6)$$

where τ is referred as the *proper time* of the moving clock, which is identical to the time (of the second click) as measured in the rest frame S', while t is the *coordinate time* in the frame S of the same event. The time dilatation formula shows that the proper time is *less* than the coordinate time of any inertial frame which is not identical to the rest frame of the clock. This is to be compared with the length contraction formula which says that the length of a body measured in the rest frame is *larger* than the length measured in any other inertial frame.

It is interesting to note that even if the situation may seem asymmetric between the two reference systems S and S', that is not really the case. The coordinate clock of system S' seems to be slow when viewed from reference frame S, but at the same time the coordinate clock of S seems slow when viewed from S'. The explanation for this apparently paradoxical situation is again the difference in perception of simultaneity in the two reference systems. When comparing the time difference of the two clocks in the two reference systems, this is done by comparing two different sets of space time points of the clocks. In both cases the comparison is made for simultaneous events, but that means for the two reference frames to use different sets of points for the world lines of the two clocks. The situation is illustrated in the Fig. 5.3.

Let us now illustrate the length contraction and time dilatation effects in a slightly different way. We introduce a set of coordinate clocks for each of the reference frames in the following way. With equal spacing L_0 along the x-axis of reference system S there are placed clocks that are stationary in this system. They are *synchronized*, so they all show the coordinate time of S. This synchronization can be done by sending radio signals between the clocks. In the same way we introduce a set of coordinate clocks with the same spacing L_0 in reference frame S'. Since the two reference frames have a common origin for their coordinate systems, the two sets of coordinate clocks will also be synchronized. Thus, the clocks at position $x = 0$ in S and at $x' = 0$ in S' will show the same time when $t = t' = 0$.

In Fig. 5.4 the situation is illustrated by viewing the two sets of clocks at time $t = 0$ in reference frame S. All the coordinate clocks in S show the same time $t = 0$ and are located with separation L_0. However the moving clocks (coordinate clocks of S') have a different separation L_0/γ due to the length contraction effect and they seems to go slower due to the time dilatation effect. In addition they seems not to be synchronized when viewed from S. This is demonstrated by the Lorentz transformation formula. For space time points with $t = 0$, which are *simultaneous in S* the coordinates in S' are

$$x' = \gamma x \,, \quad t' = -\gamma\frac{v}{c^2}x = -\frac{v}{c^2}x' \qquad (5.7)$$

The first equation is simply the length contraction formula. The second equation shows that the time shown by the moving clocks depend on their positions. This is again a consequence of the reference system dependence of simultaneity. In the present case the pictured events are simultaneous in S ($t = 0$) but not in S'.
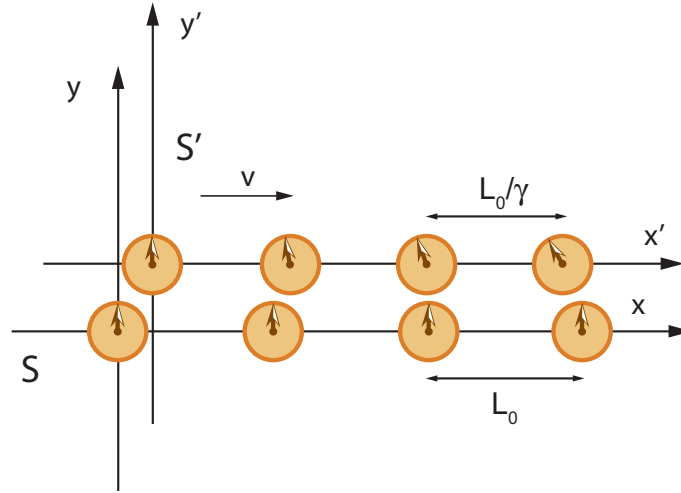
Figure 5.4: Moving coordinate clocks. Two sets of coordinate clocks are attached to two inertial reference frames S and S' in relative motion. The situation is here registered in reference frame S. The coordinate clocks in S show all equal time $t = 0$, but the coordinate clocks of S' seem not to be synchronized. Due to the length contraction effect they seem more densely spaced than the clocks in S and due to the time dilatation effect they seem to be running more slowly.

5.3   Proper time

Let us assume that a body is moving with constant velocity and that S' is the *rest frame* of the body. By the *proper time* of the body we mean simply the coordinate time in the rest frame. The time dilatation effect shows that this time will be different from the coordinate time of any other inertial frame that is moving relative to the body. The definition of proper time can be generalized to moving bodies in the case where the velocity is no longer constant, as we shall now discuss.

Let us us then consider a small body (a particle) with a velocity that is changing with time. As a consequence of this change in the velocity, there is no inertial reference frame which at all times is the rest frame of the particle. Even if there is no single inertial rest frame, valid for all points on the body's world line, there will be such a rest frame for any given point. This is an inertial frame that moves with the same velocity as the body *at that particular instant*. We refer to this as the *instantaneous rest frame* of the body. As soon as the particle changes its velocity this inertial frame ceases to be the rest frame of the particle. The important point is that the instantaneous rest frames at different points of the world line will in general be different inertial frames.

The world line of the particle we shall consider as being divided into a sequence of small line elements. For each of these the change in velocity is negligible and the instantaneous inertial rest frame can therefore be treated as the rest frame not only at a single space-time point, but for the line element. Strictly speaking this is true only for an element of infinitesimal length, and that is what we shall consider. For such an infinitesimal element of the particle path

the time dilatation formula is valid, and we write it as

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}\, dt \tag{5.8}$$

where $d\tau$ is the time measured in the instantaneous rest frame and dt is the time measured in an inertial frame S which we use as a fixed reference frame for the full journey of the particle.

Since the expression (5.8) is valid for any part of the particle trajectory, we can now define the proper time of this trajectory between two space-time points A and B as being identical to the integrated time

$$\tau_{AB} = \int_A^B \sqrt{1 - \frac{v(t)^2}{c^2}}\, dt \tag{5.9}$$

The proper time is then defined as the sum (integral) of the time intervals measured in the instantaneous rest frames along the path. These do not define a single reference frame, but rather a continuous sequence of inertial frames. The variation in velocity means that the time dilatation factor becomes a time dependent function.

The proper time we may consider as the time measured on an imagined clock that is fixed to the small body during its space-time journey. It should then be clear that the proper time will not depend on the choice of the reference frame S in the description of the motion. However, that is not obvious from the expression (5.9) which does seem to depend on the choice of reference frame. So, it is of interest to demonstrate more directly that proper time, as defined above, is independent of such a choice, or stated differently, that the proper time τ_{AB} is a Lorentz invariant.

We then focus again on an infinitesimal element of the space time curve, and consider the corresponding Lorentz invariant line element, which we have earlier introduced. In the present case it takes the form

$$\begin{aligned}
ds^2 &= d\mathbf{r}^2 - c^2 dt^2 \\
&= -c^2(1 - \frac{v^2}{c^2}) dt^2 \\
&= -c^2 d\tau^2 \tag{5.10}
\end{aligned}$$

This shows that $d\tau^2$ is proportional to the invariant ds^2 and is therefore also a Lorentz invariant. The minus sign in the relation is explained by the fact that the world line of the particle has a timelike orientation.

If we now compare the proper time for different world lines between *the same end points* A and B, the expression (5.9) indicates that the proper time may be *path dependent* so that the path which at average has the *largest* values of v^2 will have the shortest proper time. This is indeed a real physical effect, and it is the basis for the *twin paradox* which we shall discuss next.

5.4 The twin paradox

We consider a situation where a pair of twins, who we refer to as A and B, separate for several years, with twin B leaving the earth on a space ship, while twin A is staying behind on earth.

B travels at high speed far out in the universe to visit a distant space station. After a short stay he returns to the earth, where he arrives several years after his departure. When he meets his twin sister A he realizes that she has aged more than himself. Since he is well acquainted with Einstein's theory of relativity, this does not come as a surprise. It may seem paradoxical, but he knows that twin A has been at rest with respect to the inertial reference frame S of the earth, while he has performed a journey with large velocities. Her proper time should therefore be longer than his own, as shown by the proper time formula (5.9). (Only in in an approximate sense the earth defines an inertial frame, but since the variations in the orbital velocity of the earth is so small relative to the speed of light he knows that is ok to neglect this effect.)

However, there is something else that makes this situation look like a paradox. Let us assume that the velocity of the space ship of twin B is constant and the same on the way out to the space station and on the way back, except for its direction. The time dilatation factor is then the same for the two parts of the full journey. Of course, this cannot be fully correct, since there must be a period of acceleration at the beginning and at end of the journey as well as when B is close to the space station. However, we may assume these periods to be very short compared to the time spent on the rest of the journey, and therefore it seems very reasonable to assume that these short periods should only contribute with negligible corrections.

On the way out to the space station the relation between the rest frames of the two twins is symmetric, so the clocks of B seems to be slow measured with the clocks of A and *vice versa*. The time dilatation factor γ is constant and it is the same whether viewed from twin A or twin B. The situation is the same on the way back from the space station, with the same value for the time dilatation factor as on the way out. Based on this twin A will find that the proper time of B is reduced with the factor γ relative to her proper time, and that is consistent with the time dilatation formula (5.9). But based on the symmetry between the two twins on each of the halves of the journey and the fact that the time dilatation factor is the same for the two parts, it seems that twin B could also claim that the proper time of twin A should be shorter than his. That would clearly create an inconsistency. So what is the explanation of this apparent contradiction?

In order to resolve the paradox we have to analyze the situation more carefully. First let us consider the situation from the point of view of twin A. Her own proper time is identical to the inertial reference frame S of the earth. Let us denote her proper time for the whole journey by τ_A. The space-time path of B is assumed to be symmetric with respects to its two halves, to and from the space station, and therefore when A applies the time dilatation formula to each part of the journey she obtains for the total proper time of twin B

$$\tau_B = \frac{1}{\gamma} \, \tau_A/2 + \frac{1}{\gamma} \, \tau_A/2 = \frac{1}{\gamma} \, \tau_A \tag{5.11}$$

This is consistent with the time dilatation formula (5.9).

Next we consider the situation from twin B's point if view. He can also apply the time dilatation formula - if he does it with some care. An important point to observe is that even if the speed of his space ship is the same on the way out and on the way back, the inertial rest frames on the two parts of the trip *are not the same*. Let us refer to these two part of the journey as I and II and the corresponding inertial frames as S_I and S_{II}. The main point is now to observe that when using the time dilatation formula he should refer to events that are
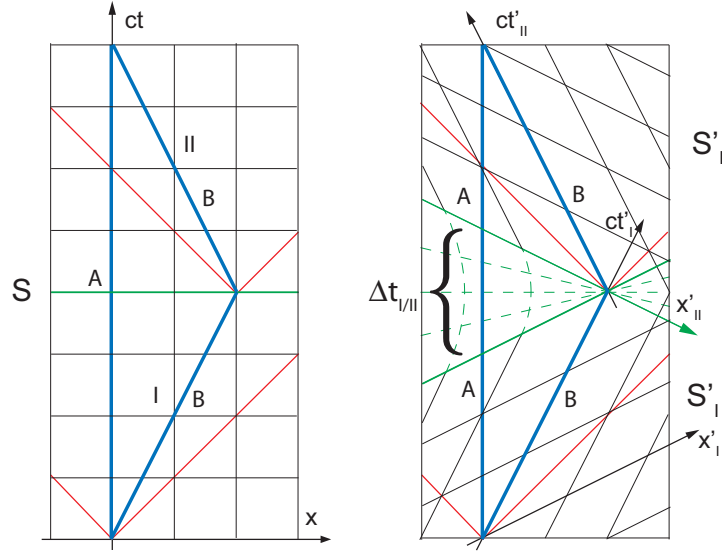
Figure 5.5: Illustration of the Twin Paradox. The Minkowski diagrams show the asymmetry between the two twins when they use the time dilatation formula. The space-time journeys of the two twins A and B are shown by the blue lines in two Minkowski diagrams, with the world line of twin A (who remains at earth) represented as a single straight line, while the world line of B consists of two straight lines, denoted I for the outgoing part and II for the return part of the journey. In the first diagram the coordinate lines of the rest frame S of twin A are shown. The coordinate time of the mid-journey event of B is indicated by the green line. In the second diagram coordinate lines of the rest frames of twin B are shown. There is a discontinuity since the rest frame of the journey out (S'_I) is different from the rest frame of the journey back (S'_{II}). The point on earth which is simultaneous with the mid-journey event now splits in two, with time difference $\Delta t_{I/II}$, since the simultaneous events of frame S_I and S'_{II} are different, now shown by the two unbroken green lines. The dotted green lines indicate the rapid transition between the two frames, due to the rapid change of velocity at the space station. The red lines are included in the diagrams to show the world lines of light signals emitted at the beginning of the journey and at the mid-journey event.

simultaneous in his own reference frame. Let us apply this to the first part of his journey, when his rest frame is S_I. The time dilatation formula can be written as

$$\Delta t_A = \frac{1}{\gamma}\tau_B/2 \tag{5.12}$$

with Δt_A as the time registered on the clock on earth during the time twin B is on the way to the space station. This has the same form as the time dilatation formula used by A. But note that $\Delta t_A \neq \tau_A/2$ since the time on earth that is simultaneous in S_I with the arrival of B at the space station is *not* the half time of the full journey. It is in fact an earlier time. A similar reasoning is applicable also for the second part for the journey. The time dilatation formula (5.12) is also here valid, now with τ_B as the time measured in S_{II}. In this frame the beginning of the return journey is simultaneous with an event on earth that is *later* than the midjourney event. The situation is illustrated in Fig. 5.5.

This means that the time dialtation formula (5.12) is valid both for the travel out and the travel back, but the contributions on the left hand side do not sum up to the total time measured on earth. The formula that relates the proper time of B, for the full journey, to the time registered on earth should therefore be written as

$$\tau_A = 2\Delta t_A + \Delta t_{I/II} = \frac{1}{\gamma}\tau_B + \Delta t_{I/II} \tag{5.13}$$

where $\Delta t_{I/II}$ is the correction which accounts for the jump in the definition of simultaneous events when B changes fro reference frame S_I to S_{II}. Consistency between (5.12) and (5.13) determines this to be

$$\Delta t_{I/II} = (1 - \frac{1}{\gamma^2})\tau_A = \frac{v^2}{c^2}\tau_A \tag{5.14}$$

and a more direct calculation of the time jump based on the use of the conditions for simultaneous events in the two inertial reference frames gives the same result. The conclusion is that the situation is not symmetric with respect to describing the journey for twin A and B. Both twins may use the time dilatation formula to compare the proper times of the two of them, but twin B has to be careful to add the time jump associated with the change of inertial frames.

Let us finally note that if we take into account that the change between the two rest frames S_I and S_{II} of twin B in reality is not infinitely rapid, then the time jump $\Delta t_{I/II}$ will be caused by a rapid but smooth change in reference frames, beginning with S_I and ending with S_{II}. This will affect the registering performed by B of simultaneous events on earth, so that during the first and second part of the journey the clocks on earth are registered as being slower than the ones on the space ship, but this is compensated by a very rapid speed up of the clocks on earth during the period of acceleration. The total effect is that, when correctly calculated, twin B should like twin A find the proper time τ_B to be shorter than the proper time τ_A between the start and end point of the space-time journey.

At the end, the easiest way to compare the times registered by the twins is to use the proper time formula (5.9) for the two space-time paths, since this formula does not depend on transforming between different inertial frames along the space-time trajectory.

Chapter 6

The four-vector formalism and covariant equations

In this chapter we discuss in a more systematic way the use of four-vectors, and in particular how to give physical equations a *covariant* form. In the covariant formulation all physical variables are expressed in terms of four-vectors and related objects, called (relativistic) tensors, and this formulation secures that the equations are valid in any inertial reference frame. We discuss how tensors are defined and what are their transformation properties under Lorentz transformations.

6.1 Notations and conventions

6.1.1 Einstein's summation convention

When using the four-vector notation some conventions are commonly used, and we shall make use of them also here. For example when a vector index is running over all the four values taken by the *space-time* coordinates, we label the index by a greek letter, while the use of a latin letter instead will normally indicate a restriction to the three values taken by the *space* components. For example when we write x^μ, μ is allowed to take values from 0 to 3. If however we write x_i the index runs instead from 1 to 3.

Another convention we shall apply is Einstein's summation convention. Thus a repeated space-time index in an expression normally means that we should sum over the index. As an example we write in the following for the decomposition of a four-vector $\underline{\mathbf{x}}$ on an orthogonal set of basis vectors,

$$\underline{\mathbf{x}} = x^\mu \underline{\mathbf{e}}_\mu \tag{6.1}$$

where the summation symbol is simply omitted. The repeated index tells us that we should sum over μ, and since it is a greek letter we know that the summation is from 0 to 3. If we at some stage should meet a case where a repeated index should *not* be taken as a summation index, we simply state that explicitly.

In the four-vector notation it is also important to correctly place the index up or down, while a similar distinction is not important for vectors in three-dimensional space. We shall

soon have a closer look at this distinction. The consistent use of four-vectors (and tensors) we refer to as *covariant notation*, and we note as a particular rule that in the covariant notation we only sum over pairs of indices, where one is an upper index and the other a lower index. This summation is commonly referred to as a *contraction.*

6.1.2   Metric tensor

Physical three-dimensional space is in non-relativistic physics considered to be equipped with a Euclidean metric, defined by the invariant *distance* between two neighboring points. The distance squared, which in Cartesian coordinates is

$$ds^2 = dx^2 + dy^2 + dy^2 = d\mathbf{r}^2 \tag{6.2}$$

is invariant under rotations of the coordinate axes.

As already discussed, the four-dimensional space-time of special relativity has a different metric, called Minkowski metric. It is defined by the Lorentz invariant line element

$$ds^2 = dx^2 + dy^2 + dy^2 - c^2 dt^2 \equiv d\underline{\mathbf{x}}^2 \tag{6.3}$$

When written in this way we have to remember that $d\underline{\mathbf{x}}^2$ does not have to be positive. It is positive for spacelike vectors, zero for lightlike vectors and negative for timelike vectors.

We may write the invariant line element in the following form,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{6.4}$$

where $g_{\mu\nu}$ is referred to as the *metric tensor*. (Note that in (6.4) Einstein's summation convention has been used.) The metric teonsor can be thought of as defining a 4×4 symmetric matrix, which in Cartesian coordinates is a diagonal matrix of theform

$$g = (g_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.5}$$

From the decomposition of the vector $d\underline{\mathbf{x}} = dx^\mu \underline{\mathbf{e}}_\mu$ and from the writing of the invariant line element as a generalized scalar product, it follows that the basis vectors satisfy a generalized orthogonality condition

$$\underline{\mathbf{e}}_\mu \cdot \underline{\mathbf{e}}_\nu = g_{\mu\nu} \tag{6.6}$$

This means that the vectors are orthogonal and space vectors have a standard normalization $\underline{\mathbf{e}}_k^2 = 1$, $k = 1, 2, 3$, while that time vector has the normalization $\underline{\mathbf{e}}_0^2 = -1$. The last one is negative since the basis vector $\underline{\mathbf{e}}_0$ is timelike.

6.1.3   Upper and lower indices

We have already stressed the convention that the coordinates of a four-vector $\underline{\mathbf{x}}$ are written with upper indices, as x^μ. However also coordinates with lower indices may be defined. The precise definition is,

$$x_\mu = g_{\mu\nu} x^\nu \tag{6.7}$$

Thus a four-vector can be associated with two sets of coordinates, those with upper indices which are the standard ones (referred to as *contravariant* components) and those with lower indices (referred to as *covariant* components). The metric tensor acts as a lowering operator on the indices. This gives a simple relation

$$x_0 = -x^0, x_1 = x^1, x_2 = x^2, x_3 = x^3 \tag{6.8}$$

Note that the only change introduced by lowering the indices is that the sign of the 0'th component is reversed.

Initially it may seem cumbersome to operate with two sets of coordinates for a four-vector, which are even so closely related. However, if one is careful to place the indices correctly the relativistic equations can be simplified, and if the positions of the indices are consistently used in a relativistic equation one will gain a guarantee that it keeps the form unchanged when transforming from one reference frame to another.

We note, as a special case, that the invariant line element can now be written without the metric tensor as

$$ds^2 = dx_\mu dx^\mu \tag{6.9}$$

More generally, summation over a pair of four-vector indices, one lower and one upper will produce a Lorentz invariant quantity.

The metric tensor acts as a lowering operator on the vector indices. Clearly there must be an inverse to this which acts as a raising operator. We write it as

$$x^\mu = g^{\mu\nu} x_\nu \tag{6.10}$$

Since it is the inverse to $g_{\mu\nu}$ we have the relation

$$g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu \tag{6.11}$$

Note that the relativistic form of the Kronecker delta is written with one upper and one lower index. This is to have the indices of the two sides of the equation consistently placed.

We note from the matrix form of $g_{\mu\nu}$ that the square of the matrix is identical to the identity matrix. This means that the matrix is its own inverse and therefore $g_{\mu\nu}$ and $g^{\mu\nu}$ represent the same 4×4 matrix. Nevertheless, we insist on writing this matrix with lower indices when it is used as a lowering operator of vector indices in an equation and with upper indices when it is used as a raising operator. This is to be able to place consistently all vector indices in the relativistic equations. [1]

6.2 Lorentz transformations in covariant form

A Lorentz transformation, which relates the coordinates x of an inertial frame S to the coordinates x' of another inertial frame S' can be written in component form as

$$x'^\mu = L^\mu{}_\nu \, x^\nu \tag{6.12}$$

---

[1]The notation with covariant and contravariant components is even more important in the *general theory of relativity* where more general coordinate systems are applied. In that case the metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ will usually no longer correspond to the same 4x4 matrix.

where we again stress the convention for placing the indices of L. In matrix form it is

$$x' = L\,x \tag{6.13}$$

For a *boost* in the x-direction the Lorentz transformation matrix is

$$L = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.14}$$

with $\beta = v/c$ and $\gamma = 1/\sqrt{1 - \beta^2}$ and v as the relative velocity of the two reference frames.

If a 4×4 matrix L should represent a more general Lorentz transformation, it has to satisfy a certain restriction, which follows from the requirement that the velocity of light is left unchanged by the transformation. As already noted this is related to the Lorentz invariance of the line element, which implies

$$\begin{aligned} g_{\mu\nu}dx'^{\mu}dx'^{\nu} &= g_{\mu\nu}L^{\mu}{}_{\rho}\,L^{\nu}{}_{\sigma}\,dx^{\rho}dx^{\sigma} \\ &= g_{\rho\sigma}dx^{\rho}dx^{\sigma} \end{aligned} \tag{6.15}$$

Since this should be valid for any displacement dx^{μ}, the L matrix has to satisfy the restriction

$$g_{\mu\nu}L^{\mu}{}_{\rho}L^{\nu}{}_{\sigma} = g_{\rho\sigma} \tag{6.16}$$

In matrix form this can be written as

$$L^{T}gL = g \tag{6.17}$$

where L^{T} represents the transposed matrix. This equation, which is the condition for the 4×4 matrix L to represent a Lorentz transformation, corresponds to the following condition satisfied by the 3×3 rotation matrices R in three-dimensional space,

$$R^{T}R = \mathbb{1} \tag{6.18}$$

where $\mathbb{1}$ represents the identity matrix.

6.3 General four-vectors

So far we have considered four-vectors $\underline{x}$ associated with points in four-dimensional space time. These are in a sense the fundamental vectors of the relativistic theory. However, exactly as in three dimensions vectors can represent more general objects, such as velocity, momentum, acceleration, etc. All these have the properties which characterizes a general four-vector $\underline{\mathbf{A}}$,

- it has four components A^{μ}, with $\mu = 0, 1, 2, 3$,

- the components transform like the coordinates x^{μ} under Lorentz transformations, $A^{\mu} \rightarrow A'^{\mu} = L^{\mu}{}_{\nu}A^{\nu}$.

The Minkowski diagram is convenient to give a geometric representation of general four-vectors, in the same way as with the use of a Minkowski diagram for space-time itself. A reference frame corresponds also here to a choice of basis vectors $\{\underline{e}_\mu, \mu = 0, 1, 2, 3\}$ and a vector $\underline{\mathbf{A}}$ can be decomposed on any set of basis vectors, corresponding to different inertial reference frames,

$$\underline{\mathbf{A}} = A^\mu \underline{e}_\mu = A'^\mu \underline{e}'_\mu \tag{6.19}$$

Thus a Lorentz transformation of the four components of $\underline{\mathbf{A}}$ simply corresponds to a change of basis in the same way as for the components of the space-time coordinates x^μ.
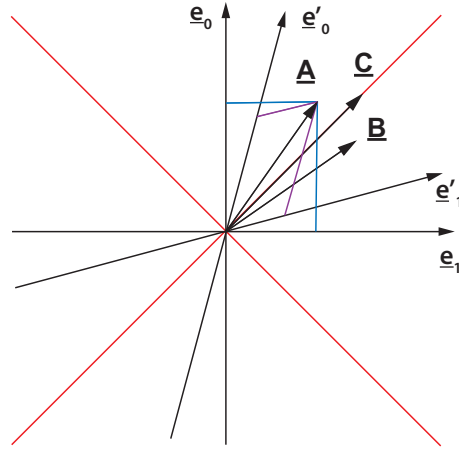
Figure 6.1: A two-dimensional Minkowski diagram with general four-vectors. The diagram does not represent space-time itself, but the metric is the same as the space-time metric, and the vectors can be separated in the same three classes. $\underline{\mathbf{A}}$ represents a timelike vector, $\underline{\mathbf{B}}$ a spacelike vector and $\underline{\mathbf{C}}$ a lightlike vector, also referred to as a *null vector*. The null vectors define the light cone which separates the timelike and spacelike vectors. The different sets of basis vectors $\underline{e}_\mu$ correspond to different inertial reference frames. The decomposition of the four-vector $\underline{\mathbf{A}}$ on two different basis sets are illustrated in the figure.

The Lorentz invariant *scalar product*, is defined by

$$\underline{\mathbf{A}} \cdot \underline{\mathbf{B}} = g_{\mu\nu} A^\mu B^\nu = A^\mu B_\mu \tag{6.20}$$

The scalar product is indefinite (not always positive) and separates the general four-vectors, like the space time vectors $d\underline{\mathbf{x}}$, in three classes: space-like ($\underline{\mathbf{A}}^2 > 0$), light-like ($\underline{\mathbf{A}}^2 = 0$) and time-like ($\underline{\mathbf{A}}^2 < 0$). In the Minkowski diagram these three classes are represented, like space-time vectors, as vectors lying outside the light cone, on the light cone or inside the light cone, respectively (see Fig. 6.1).

As already noticed, orthogonality in the sense that the scalar product of two four-vectors vanishes does not mean that they *appear* as orthogonal in the Minkowski diagram. In the Fig. 6.1 the two vectors $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ are orthogonal in the sense $\underline{\mathbf{A}} \cdot \underline{\mathbf{B}} = 0$. This means that the two vectors have directions symmetrically about the light cone. In particular a lightlike vector, with this definition, is orthogonal to itself, $\underline{\mathbf{C}}^2 = 0$.

6.4 Lorentz transformation of vector components with lower index

The index of a general four-vector can be lowered by applying the metric tensor, in the same way as for the position vector x^μ,

$$A_\mu = g_{\mu\nu} A^\nu \tag{6.21}$$

This relation leads to different transformation properties for vector components with upper indices (contravariant components) and lower indices (covariant components). We find the following expression for the transformed covariant components

$$
\begin{aligned}
A'_\mu &= g_{\mu\nu} A'^\nu \\
&= g_{\mu\nu} L^\nu{}_\rho A^\rho \\
&= g_{\mu\nu} L^\nu{}_\rho g^{\rho\sigma} A_\sigma \\
&\equiv L_\mu{}^\sigma A_\sigma
\end{aligned}
\tag{6.22}
$$

Note in the last line we have introduced a modified symbol for the transformation matrix

$$L_\mu{}^\sigma = g_{\mu\nu} L^\nu{}_\rho g^{\rho\sigma} \tag{6.23}$$

where we have followed the general rule that $g_{\mu\nu}$ acts as a lowering operator and $g^{\mu\nu}$ as a raising operator for the vector indices. With $L^\mu{}_\nu$ as the matrix elements of the 4×4 matrix L, $L_\mu{}^\nu$ then are the matrix elements of the matrix

$$
\begin{aligned}
\tilde{L} &= gLg^{-1} \\
&= (L^T)^{-1}
\end{aligned}
\tag{6.24}
$$

The last expression is derived from the identity (6.17), which is satisfied by all Lorentz transformation matrices L.

Note that the covariant and contravariant components transform in inverse ways. This is in accordance with the fact that the scalar product of two vectors, which can be written as a product of the covariant components of one of the vectors and the contravariant components of the other, is invariant under Lorentz transformations. Also note that the transformation coefficients $L_\mu{}^\sigma$ of the covariant components A_μ are the same as the transformation coefficients of the basis vectors $\underline{e}_\mu$, which have earlier been introduced in (4.28) and (4.29). This is consistent with a general property of the covariant formalism, namely that the position of the space-time index of an object, as an upper or lower index, indicates uniquely the transformation property of this object under Lorentz transformations.

6.5 Tensors

Four-vectors are in a sense the simplest geometrical objects that transform in a non-trivial way under space-time transformations. Many of the basic physical variables are represented as four-vectors, but there are also variables that cannot be represented in this way. These variables typically have more than four components that are mixed by the transformations. They are

generally represented as *tensors*, which are geometrical objects related to vectors, but labelled with more than one space-time index.

Tensors are not restricted to relativity theory. They appear also in non-relativistic theory, for example in the form of the stress tensor which measures the response of an elastic medium to external forces. However, several of the basic physical variables which in non-relativistic theory are represented as vectors, are in relativistic theory described as tensors. For this reason one tends to meet tensors at an earlier stage in relativity theory than in non-relativistic theory.

As an example, take the angular momentum of a particle, which in non-relativistic theory is usually represented by

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p} \tag{6.25}$$

which is a vector with components $\ell_i, \ i = 1, 2, 3$. We note that it can also be represented as the following two-index variable

$$\ell_{ij} = x_i p_j - x_j p_i \tag{6.26}$$

which is an example of an antisymmetric *rank two* tensor, *i.e.*, a tensor with two space indices. The relation between the vector and tensor form of the angular momentum is

$$\ell_i = \frac{1}{2} \sum_{jk} \epsilon_{ijk} \ell_{jk} \tag{6.27}$$

with ϵ_{ijk} as the Levi-Civita symbol. However, this mapping between vectors and antisymmetric tensors has no direct extension to four-dimensional space-time, since the Levi-Civita symbol in four dimensions has four indices rather than three. For this reason the relativistic angular momentum has a natural representation as an antisymmetric tensor rather than a vector. The form of this relativistic tensor is

$$\ell^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \tag{6.28}$$

where x^μ are the components of the position four-vector, and p^μ of the momentum four-vector.

In mathematics tensors appear in a natural way as generalizations of vectors, in the form of *tensor products*. Thus, starting with two vectors with components A^μ and B^ν, a tensor with two indices can be defined as the product

$$C^{\mu\nu} = A^\mu B^\nu \tag{6.29}$$

much like what we did for the angular momentum. This is referred to as the tensor product of the two original vectors. If we further consider all sums of such products they define a new vector space which we refer to as the tensor product of the two vector spaces (or here, rather of two copies of the original vector space). This vector space consists of all rank two tensors.

The transformation properties of a tensor, defined in this way, are obviously determined by the transformation properties of the original vectors. For relativistic tensors, the rank two tensors thus transform under Lorentz transformations as

$$C^{\mu\nu} \to C'^{\mu\nu} = L^\mu{}_\rho L^\nu{}_\sigma C^{\rho\sigma} \tag{6.30}$$

This property can in fact be used as a definition of tensors. An object characterized by two space-time indices, which transforms as (6.30) *is* a rank two tensor.

We have already considered the angular momentum of a particle as one example of such a tensor. Since it is antisymmetric in the two indices it has six independent components. Another example of an antisymmetric rank two tensor is the electromagnetic field tensor, conventionally written as $F^{\mu\nu}$. Also this has six independent components, which can be identified with the vector components of the electric and magnetic fields in the non-relativistic description. The energy and momentum densities of the fields define the components of yet another relativistic tensor, the symmetric energy-momentum tensor $T^{\mu\nu}$ of the electromagnetic field. A special symmetric rank two tensor that we have already encountered is the metric tensor $g_{\mu\nu}$. This has the unusually property that it is a constant, and nevertheless satisfies the transformation equation (6.30) of a rank two tensor. This peculiarity follows as a consequence of the identity (6.16) satisfied by the Lorentz transformation matrices.

Tensors may, like vectors, be written with upper indices or lower indices. These are related by the action of the metric tensor. For rank 2, we then have four related tensors

$$C^{\mu\nu}, \quad C^{\mu}{}_{\nu} = g_{\nu\rho}C^{\mu\rho}, \quad C_{\mu}{}^{\nu} = g_{\mu\rho}C^{\rho\nu}, \quad C_{\mu\nu} = g_{\mu\rho}g_{\nu\sigma}C^{\rho\sigma} \tag{6.31}$$

In the same way as we can view the set of covariant and the set of contravariant components of a four-vector as two different representation of the same vector, we can view all the different sets of tensor components in (6.31) as being different representations of the same geometrical object.

So far we have focussed on rank two tensors, that is on variables with two space-time indices. However, there is an obvious generalization of tensors to arbitrary rank, that is to tensors with any number of space-time indices. Vectors is then a special type of tensors, of rank 1, and scalars, which are invariants that carry no space-time index, are tensors of rank 0. We have the following list of tensors of increasing rank,

A	rank 0 (scalar)	no vector index	(1 component)
B^{μ}	rank 1 (vector)	one vector index	(4 components)
$C^{\mu\nu}$	rank 2	two vector indices	(16 components)
$D^{\mu\nu\rho}$	rank 3	three vector indices	(64 components)
etc.			

What define these as tensors is their transformation properties under Lorentz transformations. Thus, the transformed tensors are multiplied with one Lorentz matrix for each space-time index, as an obvious generalization of (6.30).

Like vectors, tensors of any rank can be considered as representing *geometrical objects*, which are well defined without specifying any particular reference frame. The components of the tensor, on the other hand refer to a specific set of basis vectors, and thus to a choice of reference frame. Equations describing physical laws in relativistic form can, in principle, be written in coordinate independent form, like vector equations in non-relativistic physics. However, usually this is not done, since it is cumbersome to introduce symbols for different types of tensors and for different types of multiplications between tensors. Relativistic equations are usually instead written in *covariant* form, which means that they are expressed in terms of tensor components, in such a way that that they keep their form unchanged under Lorentz transformations.

When the equations are written in covariant form they are expressed in terms of variables with simple, standardized transformation properties. One can therefore easily check that the equation is valid in any reference system. To check that an equation has the correct covariant form we note that

- all terms that appear additively in the equation should be tensors of the same rank

- free indices, which are indices that are not summed over, should have the same positions, either up or down, in all terms of the equation,

- repeated indices that are summed over should appear with one in the upper position and one in the lower position.

We note in particular that a contraction, *i.e.*, summation over a pair of repeated pair of indices, will reduce the rank of a tensor by 2. For example $A = A^{\mu}{}_{\mu}$ is a scalar, $B^{\mu} = B^{\mu\nu}{}_{\nu}$ is a vector etc.

As an example of a covariant equation, the equation of motion of a charged particle in an electromagnetic field can be written in the following compact form

$$m\ddot{x}^{\mu} = eF^{\mu\nu}\dot{x}_{\nu} \tag{6.32}$$

where the time derivative is here with respect to the Lorentz invariant *proper time* τ of the particle. The equation is a relativistic vector equation, *i.e.*, with one free space-time index, μ, which is written in the upper position on both sides of the equation. On the right-hand side there is a contraction of index ν, with a correct repetition, where one index is in the upper and the other in the lower position. The equation clearly has a correct covariant form, which shows that it satisfies the requirement of being invariant under Lorentz transformations. The same equation can be written in the following non-covariant form

$$\dot{\mathbf{p}} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{6.33}$$

where $\mathbf{E}$ is the electric field strength and $\mathbf{B}$ is the magnetic field. This equation, which is expressed in terms of three-vectors, is obviously invariant under rotations in three-dimensional space. However, the invariance under Lorentz transformations is not explicit in this equation. It depends on the transformations of $\mathbf{E}$ and $\mathbf{B}$, which generally will mix these two vector fields. A further discussion of covariance of electromagnetic equation will follow in Part III of this course.

6.6 Vector and tensor fields

In the same way as vectors in three-dimensional space often appear in the form of *vector fields*, vectors and tensors in four-dimensional space-time also often appear in the form of vector and tensor fields. As a particular example the electromagnetic field, in covariant relativistic form, is described by the rank two tensor field $F^{\mu\nu}(x)$. Let us list some of the tensor fields we may meet in relativistic theories:

$$
\begin{aligned}
\phi &= \phi(x) && \text{scalar field} \\
A^\mu &= A^\mu(x) && \text{vector field} \\
F^{\mu\nu} &= F^{\mu\nu}(x) && \text{rank two tensor field field}
\end{aligned}
$$
etc.

The fields are here written in component form and the space time variable x here means the full set of coordinates $x = (x^0, x^1, x^2, x^3)$.

Under a change of inertial reference frames, defined by a Lorentz transformation L, the fields transform in the following way

$$
\begin{aligned}
\text{Scalar field} && \phi(x) &\rightarrow \phi'(x') &=& \phi(x) \\
\text{Vector field} && A^\mu(x) &\rightarrow A'^\mu(x') &=& L^\mu{}_\nu A^\mu(x) \\
\text{Tensor field} && F^{\mu\nu}(x) &\rightarrow F'^{\mu\nu}(x') &=& L^\mu{}_\rho L^\nu{}_\sigma \, redF^{\rho\sigma}(x)
\end{aligned}
$$
etc.

One should note that there are two changes under the transformation. The field components transform according to the rank of the tensors, with the number of Lorentz matrices determined by their rank. But also the space-time argument changes, with $x'^\mu = L^\mu{}_\nu x^\nu$. This change simply means that the untransformed as well as the transformed fields refer to the same space-time point, but this point is represented by different sets of coordinates in the two inertial reference frames connected by the Lorentz transformation.

Physical fields, like the electromagnetic field, will usually satisfy a set of *field equations*, and when formulated as relativistic equations, these will often be expressed in covariant form. They are typically differential equations, and therefore we will discuss in general terms how differentiation with respect to space-time coordinates are treated in the covariant formalism.

We first examine the four gradient of a scalar field $\phi(x)$, written as

$$
A_\mu(x) = \frac{\partial \phi}{\partial x^\mu}(x) \equiv \partial_\mu \phi(x) \tag{6.34}
$$

The symbol ∂_μ has here been introduced to represent the derivative with respect to x^μ, and in the following we will use this as a convenient notation. We have also by writing the partial derivative of ϕ as A_μ indicated that the components of the derivative transform as covariant four-vector components, but that needs to be proven. In order to do so we note that the change of space time coordinates $x \rightarrow x'$ can be viewed as a change of variables for the fields. Derivatives with respect to x' can then be related to derivatives with respect to x by the chain rule. For the differentiation operators we write this as

$$
\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} \tag{6.35}
$$

or simply as

$$
\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \tag{6.36}
$$

Since the Lorentz transformation can be written as

$$
x'^\mu = L^\mu{}_\nu x^\nu \tag{6.37}
$$

we find

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = L^{\mu}{}_{\nu} \tag{6.38}$$

but it is actually the derivative for the inverse transformation that we need.

To invert the transformation we make use of the property of the Lorentz transformation matrix

$$g_{\mu\nu} L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} = g_{\rho\sigma} \tag{6.39}$$

and rewrite the transformation equation (6.37) as

$$g_{\sigma\rho} L^{\rho}{}_{\mu} x'^{\sigma} = g_{\sigma\rho} L^{\rho}{}_{\mu} L^{\sigma}{}_{\nu} x^{\nu} = g_{\mu\nu} x^{\nu} \tag{6.40}$$

By further applying the raising operator on the μ index and changing the name of some of the summation indices we find the inverse transformation formula

$$x^{\nu} = g_{\mu\rho} L^{\rho}{}_{\sigma} g^{\sigma\nu} x'^{\nu} = L_{\mu}{}^{\nu} x'^{\mu} \tag{6.41}$$

where we have made use of the definition $L_{\mu}{}^{\nu} = g_{\mu\rho} L^{\rho}{}_{\sigma} g^{\sigma\nu}$. As a result we find

$$\frac{\partial x^{\nu}}{\partial x'^{\mu}} = L_{\mu}{}^{\nu} \tag{6.42}$$

to be compared with the transformation matrix (6.38).

The relation between derivatives with respect to the original and the transformed space-time coordinates can then be written as

$$\partial'_{\mu} = L_{\mu}{}^{\nu} \partial_{\nu} \tag{6.43}$$

This shows that the partial derivatives transform in the same way as the covariant components of a vector. In particular this gives for the four gradient

$$\partial'_{\mu}\phi(x') = L_{\mu}{}^{\nu} \partial_{\nu}\phi(x) \tag{6.44}$$

which is identical to the transformation equation for a covariant vector field (see(6.22)).

The rule for writing an equation in covariant form when it involves derivatives is therefore simple. The equation should be written in tensor form (including scalars and vectors) with each space-time derivative adding a covariant four-vector index to the expression. The equation (6.34) for the four-gradient therefore has a correct covariant form. In the same way the four-divergence of a vector field $A^{\mu}(x)$ can be written in the covariant form as

$$\chi(x) = \partial_{\mu} A^{\mu}(x) \tag{6.45}$$

with $\chi(x)$ as a scalar field. We finally note that the transformation properties of the partial derivatives means that we can form a Lorentz invariant quadratic derivative operator

$$\partial_{\mu}\partial^{\mu} = g^{\mu\nu}\partial_{\mu}\partial_{\nu} = \boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{6.46}$$

This is called the *d'Alembertian* and is an extension from the *Laplacian* in three space dimensions to an operator in four space-time dimensions. As indicated by the contraction between an upper and a lower space-time index this operator transform as a scalar under Lorentz transformation.

We finish here by writing two fundamental field equations in covariant form. The first one is the *Klein Gordon* equation,

$$(\partial_\mu \partial^\mu + \mu^2)\phi(x) = 0 \qquad (6.47)$$

which is a relativistic wave equation for scalar particles of mass $m = \mu \hbar c$, and the other is one of Maxwell's equations

$$\partial_\nu F^{\mu\nu}(x) = \mu_0 j^\mu(x) \qquad (6.48)$$

where μ_0 is the permeability of vacuum and $j^\mu(x)$ is the four-vector current density. The latter we will meet again in Part III of the lecture notes. Here these field equations are included only to show their attractive, compact form when written in terms of relativistic tensors.

Chapter 7

Relativistic kinematics

We discuss in this chapter how to describe velocity and acceleration as relativistic four-vectors. The concept of *proper acceleration* is introduced and an example of motion with constant proper acceleration is investigated.

7.1 Four-velocity and four-acceleration

We consider the motion of a point particle through space and time. It can be described by a time dependent position vector, which we decompose in its time and space parts, defined with respect to some unspecified inertial frame,

$$\underline{\mathbf{x}}(t) = (ct, \mathbf{r}(t)) \tag{7.1}$$

Let us introduce the particle velocity by the time derivative of the four-vector, this also decomposed in its time and space parts,

$$\frac{d\underline{\mathbf{x}}}{dt} = (c, \frac{d\mathbf{r}}{dt}) \tag{7.2}$$

However, the derivative of the four-vector $\underline{\mathbf{x}}(t)$, when differentiated with respect to time t of the chosen inertial frame, is itself not a four-vector. As a direct demonstration of this we consider the special case where a particle is moving along the x-axis with velocity u relative to a coordinate system S. Assume another inertial frame S' is moving relative to this frame with velocity v, also in the direction of the x axis, so that the coordinates of the two frames are related by a special Lorentz transformation (boost) in the x direction. The time derivative of the position vector, when decomposed in the coordinates of the two frames, will have the form

$$S: \quad \frac{d\underline{\mathbf{x}}}{dt} = (c, u, 0, 0), \qquad S': \quad \frac{d\underline{\mathbf{x}}}{dt'} = (c, u', 0, 0) \tag{7.3}$$

when all the four space-time components are shown. The velocities are given by $u = \frac{dx}{dt}$ and $u' = \frac{dx'}{dt'}$ and the relation between these is given by the relativistic transformation formula for velocities, (4.7),

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} \tag{7.4}$$

Clearly the transformation of the components of $\frac{d\mathbf{x}}{dt}$ between the two inertial frames does not have the form of a four-vector transformation.

The reason for this is easy to understand. The position vector is differentiated with respect to the time coordinate of a specific reference frame, and the resulting vector, $\frac{d\mathbf{x}}{dt}$, will therefore not be coordinate independent. This result suggests that we need to use a Lorentz invariant time parameter in order to define velocity as a four-vector. Such a parameter is in fact available in the form of the *proper time* of the moving particle. As already discussed the proper time of a particle is directly related to the invariant line element of the particle path and is therefore also a Lorentz invariant. We therefore define the four-velocity of the particle as

$$\underline{\mathbf{U}} = \frac{d\mathbf{x}}{d\tau} \tag{7.5}$$

or in the component form

$$U^\mu = \frac{dx^\mu}{d\tau} \tag{7.6}$$

with τ as the proper time coordinate. The Lorentz invariance of the proper time is shown explicitly by the definition of the time difference $d\tau$ for an infinitesimal section of the particles world line,

$$d\tau^2 = -\frac{1}{c^2} dx^\mu dx_\mu \tag{7.7}$$

With τ as a Lorentz invariant it is clear that the vector components U^μ and x^μ transform in the same way, which secures that $\underline{\mathbf{U}}$ as defined above is a four-vector.

The definition of the proper time (7.7) furthermore shows that all the four components of $\underline{\mathbf{U}}$ cannot be independent. This is shown explicitly by evaluating the Lorentz invariant

$$\underline{\mathbf{U}}^2 = U^\mu U_\mu = -c^2 \tag{7.8}$$

For any motion of the particle the four-velocity is thus a timelike vector with a *fixed* (negative) norm squared. This can be seen in another way by expressing the four-velocity in terms of the (reference-frame dependent) velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt}$. We have

$$
\begin{aligned}
\underline{\mathbf{U}} &= \frac{d\mathbf{x}}{d\tau} \\
&= \frac{d}{d\tau}(ct, \mathbf{r}) \\
&= \frac{dt}{d\tau}\frac{d}{dt}(ct, \mathbf{r}) \\
&= \gamma(c, \mathbf{v})
\end{aligned} \tag{7.9}
$$

where we have decomposed the four-vector $\underline{\mathbf{x}}$ into its time and space-parts (with respect to the unspecified inertial frame), and where we have made use of the time dilatation formula $\frac{dt}{d\tau} = \gamma$. In this formulation the constant value for the Lorentz invariant $\underline{\mathbf{U}}^2$ follows form the identity

$$\underline{\mathbf{U}}^2 = \gamma^2(v^2 - c^2) = -c^2 \tag{7.10}$$

We note that the presence of the factor γ in the expression (7.9) for $\underline{\mathbf{U}}$ is important in order to define it as a four-vector.

It is now fairly obvious how to define the corresponding four-acceleration,

$$\underline{\mathbf{A}} = \frac{d\underline{\mathbf{U}}}{d\tau} = \frac{d^2\mathbf{x}}{d\tau^2} \tag{7.11}$$

Again, since τ is a Lorentz invariant parameter, the transformation properties of the components of $\underline{\mathbf{U}}$ and $\underline{\mathbf{A}}$ will be the same.

We would like to relate the four-vector $\underline{\mathbf{A}}$ to the usual (reference-frame dependent) acceleration $\mathbf{a} = \frac{d\mathbf{v}}{dt}$. This we do by decomposing the four-vector into its time and space parts in a similar way as we have done for the four-velocity $\underline{\mathbf{U}}$,

$$\underline{\mathbf{A}} = \gamma(\frac{dU^0}{dt}, \frac{d\mathbf{U}}{dt}) \tag{7.12}$$

and we examine the two parts separately. For the 0 component we have

$$\frac{dU^0}{dt} = c\frac{d\gamma}{dt} \tag{7.13}$$

and for the three-vector part

$$\frac{d\mathbf{U}}{dt} = \frac{d}{dt}(\gamma\mathbf{v}) = \gamma\frac{d\mathbf{v}}{dt} + \frac{d\gamma}{dt}\mathbf{v} = \gamma\mathbf{a} + \frac{d\gamma}{dt}\mathbf{v} \tag{7.14}$$

The time derivative of the γ-factor is

$$
\begin{aligned}
\frac{d\gamma}{dt} &= \frac{d}{dt}(1 - \frac{v^2}{c^2})^{-\frac{1}{2}} \\
&= (1 - \frac{v^2}{c^2})^{-\frac{3}{2}}\frac{v}{c^2}\frac{dv}{dt} \\
&= \frac{1}{2}\gamma^3\frac{1}{c^2}\frac{d(\mathbf{v}^2)}{dt} \\
&= \gamma^3\frac{1}{c^2}\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} \\
&= \gamma^3\frac{1}{c^2}\mathbf{v}\cdot\mathbf{a}
\end{aligned}
\tag{7.15}
$$

This gives for the time and space components of the four-acceleration

$$A^0 = \gamma c\frac{d\gamma}{dt} = \gamma^4\frac{\mathbf{v}\cdot\mathbf{a}}{c}$$

$$\mathbf{A} = \gamma^2\mathbf{a} + \gamma\frac{d\gamma}{dt}\mathbf{v} = \gamma^2\mathbf{a} + \gamma^4\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\mathbf{v} \tag{7.16}$$

These expressions are valid in any inertial frame, with $\mathbf{v}$ as the (time dependent) velocity of the particle in this frame and $\mathbf{a}$ as the time derivative of the velocity in the same frame. If we focus on the space part of the four-vector $\underline{\mathbf{A}}$, we note that it has one part which is proportional to the acceleration $\mathbf{a}$ (in the chosen inertial frame), with a proportionality factor that can by interpreted as a time dilatation factor of the proper time relative to the coordinate time. However

there is another term, which in direction is proportional to $\mathbf{v}$ rather than $\mathbf{a}$. This comes from the time derivative of the time dilatation factor.

There are now two new Lorentz invariants that we can construct with the help of $\underline{\mathbf{A}}$ (and $\underline{\mathbf{U}}$). The first one is

$$\underline{\mathbf{U}} \cdot \underline{\mathbf{A}} = \underline{\mathbf{U}} \cdot \frac{d\underline{\mathbf{U}}}{dt} = \frac{1}{2}\frac{d\underline{\mathbf{U}}^2}{dt} = 0 \tag{7.17}$$

This result follows from the fact that $\underline{\mathbf{U}}^2$ is a constant. The other Lorentz invariant is

$$\underline{\mathbf{A}}^2 = \mathbf{A}^2 - A^{02} = \gamma^4 \mathbf{a}^2 + \gamma^6 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2} \tag{7.18}$$

where the last expression is valid in any inertial reference system.

We have already noticed that the four-velocity $\underline{\mathbf{U}}$ is a timelike vector. Since $\underline{\mathbf{A}}$ is orthogonal (in the relativistic sense) to a timelike vector, it has itself to be a spacelike vector, as one can also show by a direct calculation. Since $\underline{\mathbf{A}}$ is spacelike it means that one can by properly choosing the inertial frame transform the time component A^0 to zero. As shown by the expressions (7.16) this happens when $\mathbf{v}$ and $\mathbf{a}$ are orthogonal. In particular this is the case in the *instantaneous inertial rest frame* of the particle, where $\mathbf{v} = 0$. The acceleration measured in instantaneous inertial rest frame is referred to as the *proper acceleration* of the particle.

Let us denote the proper acceleration as $\mathbf{a}_0$. We should stress that this acceleration is for any point on the particle's world line measured in the *inertial* reference frame where the particle is instantaneously at rest. This means that when we follow the motion of the particle, the proper acceleration refers to a (continuous) sequence of inertial frames, each of them associated with a particular point on the particle path. The proper acceleration will in general vary along the path, so that it can be regarded as a function of the proper time of the particle's world line, $\mathbf{a}_0 = \mathbf{a}_0(\tau)$. When decomposed in the instantaneous rest frame at proper time τ, the four-acceleration then gets a particularly simple form,

$$\underline{\mathbf{A}}(\tau) = (0, \mathbf{a}_0(\tau)) \qquad \text{(instantaneous rest frame)} \tag{7.19}$$

This means that we can identify the Lorentz invariant (7.18) with $\mathbf{a}_0^2$, and therefore we have the following relation between the proper acceleration and the acceleration measured in another inertial frame

$$\mathbf{a}_0^2 = \gamma^4 \mathbf{a}^2 + \gamma^6 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2} \tag{7.20}$$

This shows that the proper acceleration is larger than the acceleration measured in any other inertial frame, $a_0 \geq a$.

Let us consider two special cases. For motion in a circular orbit we have $\mathbf{v} \cdot \mathbf{a} = 0$ and therefore

$$a_0 = \gamma^2 a \tag{7.21}$$

In the rest frame the acceleration is enhanced by the factor γ^2, and this we may see as a time dilatation effect, due to the double differentiation with respect to proper time rather than

coordinate time. The other special case is linear acceleration where $\mathbf{v} \cdot \mathbf{a} = v\,a$. In this case we find

$$
\begin{aligned}
a_0^2 &= \gamma^4 (a^2 + \gamma^2 \frac{v^2}{c^2} a^2) \\
&= \gamma^4 (1 + \gamma^2 \beta^2) a^2 \\
&= \gamma^6 (1 - \beta^2 + \beta^2) a^2 \\
&= \gamma^6 a^2
\end{aligned}
\tag{7.22}
$$

so in this case the enhancement factor is even larger,

$$
a_0 = \gamma^3 a
\tag{7.23}
$$

7.1.1 Hyperbolic motion through space and time

We will illustrate the discussion of the previous section by considering a space travel with constant proper acceleration. Let us therefore assume that a space ship is leaving earth for a travel far out in the universe. The ship is maintaining a constant direction of velocity and the engines are providing a thrust so that the effective gravitational field on board is kept constant and equal in strength to the gravitational field on the surface of the earth. This means that the acceleration relative to an inertial rest frame, that is the proper acceleration of the ship, is the same at all times of the travel, with $a_0 = g = 9.8 m/s^2$.

The problem to be discussed is how this travel appears in an earth-fixed frame, which we can assume to be (to a good approximation) an inertial reference frame.

Since the motion of the space ship is assumed to be linear we have for the velocity and acceleration, as seen in the earth-fixed reference frame, $\mathbf{v} \cdot \mathbf{a} = va$, and the relation between the (constant) proper acceleration and the acceleration measured in the earth-fixed frame is

$$
a = \frac{a_0}{\gamma^3} = \frac{g}{\gamma^3}
\tag{7.24}
$$

The acceleration therefore seems to decrease with time, when measured at earth, and this we can view as a consequence of the time dilatation effect. By integrating the above equation we can find the position of the space ship as a function of its proper time. We choose the x axis of the inertial frame in the direction of the motion.

First we rewrite the equation as a differential equation for $\beta = v/c$,

$$
\frac{d\beta}{d\tau} = \frac{dt}{d\tau}\frac{d\beta}{dt} = \gamma \frac{1}{c} a = \frac{g}{c}\frac{1}{\gamma^2} = \frac{g}{c}(1 - \beta^2)
\tag{7.25}
$$

It is now convenient to substitute β with the *rapidity* χ, which we have earlier introduced. It is related to β by $\beta = \tanh \chi$, which gives

$$
\frac{d\beta}{d\tau} = (\frac{d}{d\chi} \tanh \chi)\frac{d\chi}{d\tau} = \frac{1}{\cosh^2 \chi}\frac{d\chi}{d\tau}
\tag{7.26}
$$

and

$$
1 - \beta^2 = 1 - \tanh^2 \chi = \frac{1}{\cosh^2 \chi}
\tag{7.27}
$$

The differential equation for β therefore gets the following simple form when expressed in term of the rapidity

$$\frac{d\chi}{d\tau} = \frac{g}{c} \tag{7.28}$$

with solution

$$\chi = \frac{g}{c}\tau \tag{7.29}$$

We have then assumed that the time coordinates are $t = \tau = 0$ at the beginning of the journey, when the velocity vanishes, and therefore $\beta = \chi = 0$. The solution for the velocity is then

$$\beta = \tanh(\frac{g}{c}\tau) \tag{7.30}$$

which for the γ factor gives

$$\gamma = \cosh(\frac{g}{c}\tau) \tag{7.31}$$

The relation between the coordinate time t and the proper time τ can be determined from the time dilatation formula

$$dt = \gamma d\tau = \cosh(\frac{g}{c}\tau)d\tau \tag{7.32}$$

which by integration gives

$$t = \frac{c}{g}\sinh(\frac{g}{c}\tau) \tag{7.33}$$

In a similar way the x coordinate can be found by integrating the expression for the velocity,

$$\frac{dx}{d\tau} = \gamma\frac{dx}{dt} = \gamma\beta c = c\sinh(\frac{g}{c}\tau) \tag{7.34}$$

which gives

$$x = \frac{c^2}{g}\cosh(\frac{g}{c}\tau) \tag{7.35}$$

In the last expression we have for simplicity chosen the integration constant to be zero. Note that this means that the x coordinate is not zero at the beginning of the journey, but rather $x(0) = c^2/g$.

To sum up, the coordinates of the space ship in the inertial frame of the earth are given by

$$ct = \frac{c^2}{g}\sinh(\frac{g}{c}\tau), \quad x = \frac{c^2}{g}\cosh(\frac{g}{c}\tau), \quad y = z = 0 \tag{7.36}$$

when the proper time τ is used as the time parameter of the space ship's world line. From this follows that the coordinates satisfy the equation

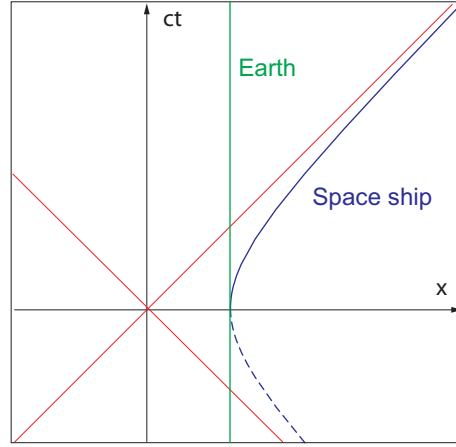$$x^2 - (ct)^2 = \frac{c^4}{g^2} \tag{7.37}$$

Figure 7.1: The hyperbolic space-time path of the accelerated space ship. In this Minkowski diagram of the earth fixed frame the path of the earth is the solid green line parallel to the time axis. The path of the space ship, which has constant proper acceleration, defines the part of a hyperbola, shown by the solid blue line in the diagram. The dashed blue line shows the remaining part of the hyperbola. The asymptotes of the hyperbola (red lines) correspond to motion with the speed of light.

In a two-dimensional Minkowski diagram, with ct and x as coordinate axes, the world line of the space ship will therefore define a hyperbel. This is illustrated in the Fig. 7.1.

To get some feeling for what this means let us consider how time and position of the space ship, as registered on earth, develops as a function of the time coordinate τ registered on the space ship. We first note that the proper acceleration $a_0 = g$ defines a time constant

$$\tau_0 = \frac{c}{g} = 0.97 \text{ year} \tag{7.38}$$

when $g = 9.81 m/s^2$. So this time constant is very close to 1 year. This also means that the start value of the space ship's x coordinate, which is also the x coordinate of the earth is

$$x_0 = \frac{c^2}{g} = 0.97 \text{ lightyear} \tag{7.39}$$

In the table the change in position and coordinate time is shown for a sequence of increasing proper times of the space ship.

τ	1 y	2 y	3 y	5 y	7 y	11 y	15 y
t	1.2 y	3.6 y	10.0 y	74 y	548 y	30 000 y	$1.6 \cdot 10^6$ y
$x - x_0$	0.5 ly	2.8 ly	9.1 ly	73 ly	547 ly	30 000 ly	$1.6 \cdot 10^6$ ly

Table 1: Space-time positions of a space ship with hyperbolic motion. The table shows a list of distances and coordinate times for increasing proper times τ. For large τ the distance and coordinate time increase exponentially with the proper time of the space ship.

The numbers shown in the table are quite remarkable. Even if the acceleration as felt in the space ship is quite modest, it is no more than the acceleration of gravity experienced at

the surface at the earth, the speed and the distance to the earth increases rapidly. Already after one year, as measured onboard the space ship, the distance to the space ship is half a light year. After a little more than 2 years the space ship will have a distance equal to the distance to our nearest star. Then the velocity really becomes large. After 11 years onboard the space ship it will pass the distance to the center of the galaxy and after 15 years the distance to the Andromeda galaxy. All this is due to the time dilatation effect, or alternatively due to the length contraction effect, since distances between heavenly objects seem to shrink when observed from the space ship. As shown by the Minkowski diagram the speed of the space ship seems to approach the speed of light, so that already after 3 years it has reached a velocity $v = 0.995c$.

The numbers of the table also seem to indicate that space travels even into distant parts of the universe may be possible with a travel time of a few years and under conditions that seem quite agreeable. However, as shown by the corresponding coordinate time on earth, and by comparison with conclusions made in the discussion of the twin paradox, it is clear that if the ship return to the earth it will experience a major jump forward in earth time as compared to the time experienced on board the space ship.

There is another major obstacle to carrying out such a travel. If the time dilatation effect should cut down the time of the journey in a substantial way, the space ship has to reach velocities close to the speed of light. This creates a serious energy problem. How should it be possible to feed the engines with the large amount of energy needed? It seems impossible to bring along all this fuel along, even with the most efficient conversion of fuel into energy. So the only possibility seems for the space ship to be recharged with energy during the travel. But the safest conclusion may seem to be that for a space ship to maintain a constant proper acceleration on the time scale of years, even at the modest value of $a = g$, is outside the reach of any practical setting. However, we shall include a further discussion of this energy problem in the next chapter.

7.2   Relativistic energy and momentum

The relativistic space-time symmetries introduce important changes in the description of energy and momentum as compared to that of non-relativistic physics. Also a new understanding of the energy contained in matter is introduced, as captured by the famous Einstein formula $E = mc^2$. In this section we examine first the relation between energy and momentum for a single particle. We next consider consequences of conservation of these physical quantities for systems of particles.

Consider a point particle of mass m. When moving, this particle will, in the non-relativistic description, carry momentum $\mathbf{p} = m\mathbf{v}$ and kinetic energy $E = \frac{1}{2}m\mathbf{v}^2$. For a free particle these are both constants of motion and for a collection of particles they are conserved, even if they are interacting, when we sum over the contributions from all particles. Also in special relativity energy and momentum are conserved, provided we modify the definitions of these quantities. The changes in the definitions of energy and momentum are important only when the velocities approach the speed of light. For small velocities they reduce to their non-relativistic form.

To find the correct relativistic form of energy and momentum we apply the formalism of four-vectors, with the idea to rewrite the non-relativistic three-vector momentum as a rela-

tivistic four-vector. The four-vector form makes the expression independent of any particular inertial frame, and if it reproduces correctly the non-relativistic three-momentum in reference frames where v/c is small, this gives a strong indication that the correct relativistic expression has been found. That this is indeed the case has been demonstrated in many ways experimentally in relativistic processes where energy and momentum are conserved. A similar formal approach will later be used when non-relativistic equations are updated to their *covariant* relativistic form.

The natural assumption is to replace the three-vector velocity $\mathbf{v}$ by the four-velocity $\underline{\mathbf{U}}$ in the definition of the momentum. The expression for the four-momentum of a particle will then be

$$\underline{\mathbf{P}} = m\underline{\mathbf{U}} \tag{7.40}$$

Written as a four-vector the expression for the momentum should be independent of any choice of reference frame.

We next consider the non-relativistic limit of this four-vector. It is then convenient to separate the time component from the space component (in an arbitrarily chosen inertial frame) in the same way as we have earlier done with the four-velocity (see (7.9)),

$$\underline{\mathbf{P}} = (\gamma mc, \gamma m\mathbf{v}) \tag{7.41}$$

Since γ approach the value 1 for low velocities, the three-vector part has the correct non-relativistic limit

$$\mathbf{p} = \gamma m\mathbf{v} \quad \underset{v \ll c}{\longrightarrow} \quad m\mathbf{v} \tag{7.42}$$

We therefore conclude that the correct three-vector part of the relativistic momentum is

$$\mathbf{p} = \gamma m\mathbf{v} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{7.43}$$

At this point we make a comment on the notations that we apply. When decomposing the four-velocity and four-acceleration, we write these with capital letters,

$$\begin{aligned} \underline{\mathbf{U}} &= (U^0, \mathbf{U}) \\ \underline{\mathbf{A}} &= (A^0, \mathbf{A}) \end{aligned} \tag{7.44}$$

This is because the space components of these four-vectors *are not identical* to the three-vectors $\mathbf{v}$ and $\mathbf{a}$. Even in the relativistic context the original definitions of velocity and acceleration are valid as the quantities measured in a specific inertial frame, and we therefore make a distinction between these and the three-vector parts of $\underline{\mathbf{U}}$ and $\underline{\mathbf{A}}$. As far as the momentum is concerned the situation is different. The measured three-vector part is identical to $\mathbf{p} = \gamma m\mathbf{v}$, and the expression $m\mathbf{v}$ is only to be considered as the non-relativistic approximation. For this reason we do not make any distinction between $\mathbf{P}$ and $\mathbf{p}$, and use in the following the relativistic definition for $\mathbf{p}$ with the old expression valid only for velocities $v \ll c$.

When we make the transition from non-relativistic to relativistic theory by replacing three-vectors with four-vectors, an additional component is introduced, the time component of the vector. It is of interest to understand the meaning of this additional component. For the four-momentum this is

$$P^0 = \gamma mc = \frac{mc}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{7.45}$$

To see the physical interpretation we consider its non-relativistic form by making an expansion to first order in v^2/c^2,

$$P^0 = mc + \frac{1}{2}m\frac{v^2}{c} + ... \tag{7.46}$$

When multiplied with c this gives

$$cP^0 = mc^2 + \frac{1}{2}mv^2 + ... \tag{7.47}$$

The second term is identical to the (non-relativistic) kinetic energy of the particle, while the first term is a constant with physical dimension of energy. It is called the *rest energy* of the particle and is here simply a constant. We refer to the full expression as the *relativistic energy* of the particle,

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{7.48}$$

and the expression for the rest energy is

$$E_0 = mc^2 \tag{7.49}$$

Since E_0 is a constant we may simply subtract it to get the correct relativistic form for the kinetic energy,

$$T = E - E_0 = (\gamma - 1)mc^2 \tag{7.50}$$

When T is expanded in powers of $\frac{v^2}{c^2}$, the first terms are

$$T = \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + ... \tag{7.51}$$

So for small velocities the expression for the kinetic energy reduces to the non-relativistic expression, but there are higher order relativistic corrections.

However, even if the rest energy here only appears as an innocent looking constant, the formula indicates the presence of a relation between mass and energy, and we know that this relation has far-reaching consequences. Mass can be converted to energy, and as we shall discuss that can be seen already in a study of inelastic collisions. But the true significance is, as we all know, in the field of nuclear physics, where large amounts of free energy are created by converting small amounts of mass, either in nuclear reactors or in nuclear bombs. The basis

for this is the large conversion factor c^2 which is present in the rest energy formula. This shows that a small mass of $m = 1g$ is equivalent to a large rest energy $E_0 = 0.9 \cdot 10^{14} J$.

To sum up, the relativistic four-momentum can be separated in a time component which is the energy of the particle divided by c, and a space component which is the relativistic momentum three-vector. The expressions are

$$\underline{\mathbf{P}} = (\frac{E}{c}, \mathbf{p}) = (\frac{mc}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{m\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}}) \tag{7.52}$$

7.3 The relativistic energy-momentum relation

From the four-moment $\underline{\mathbf{P}}$ we can form the following Lorentz invariant

$$\underline{\mathbf{P}}^2 = P_\mu P^\mu = \mathbf{p}^2 - \frac{E^2}{c^2} \tag{7.53}$$

A direct calculation gives

$$\begin{aligned} \underline{\mathbf{P}}^2 &= \gamma^2 m^2 v^2 - \gamma^2 m^2 c^2 \\ &= -m^2 c^2 \gamma (1 - \frac{v^2}{c^2}) \\ &= -m^2 c^2 \end{aligned} \tag{7.54}$$

From this follows the relativistic relation between energy and momentum for a freely moving particle

$$E^2 - c^2 \mathbf{p}^2 = m^2 c^4 \tag{7.55}$$

or

$$E = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \tag{7.56}$$

This replaces the non-relativistic relation

$$E = \frac{1}{2m} \mathbf{p}^2 \tag{7.57}$$

The connection between the two expressions is found by making an expansion in p^2/mc^2,

$$E = mc^2 + \frac{1}{2m} \mathbf{p}^2 + ... \tag{7.58}$$

which is essentially the same as the expansion (7.47). The first term is the rest energy and the second term the non-relativistic kinetic energy.

The presence of the rest energy in the energy-momentum relation has one important consequence. This is seen by considering the limit $m \to 0$. In this limit the expansion in powers of p/mc makes no sense, and that is reflected in the difference the limit $m \to 0$ makes for the relativistic and non-relativistic energy. In the relativistic case we get in this limit

$$E = \sqrt{p^2 c^2 + m^2 c^4} \to cp, \quad p = |\mathbf{p}| \tag{7.59}$$

The limit is well defined and gives an energy which is proportional to the absolute value of the momentum. In the non-relativistic case the limit gives instead

$$E = \frac{1}{2m}\mathbf{p}^2 = \frac{1}{2}m\mathbf{v}^2 \to 0 \tag{7.60}$$

where we have assumed that the velocity is finite also in this limit. Similarly the momentum tends to zero, $\mathbf{p} = m\mathbf{v} \to 0$. Since both momentum and energy vanish when $m \to 0$, the reasonable conclusion is that the non-relativistic formalism has no place for massless particles. The conclusion is different in the theory of relativity, where the formalism is open for the presence of massless particles. That is fortunate, since nature seems to provide such particles, with the photons being the most well-known example.

Let us derive some further consequences for massless particles. We first note that if $m \neq 0$, the relativistic expressions for E and $\mathbf{p}$ give the following expression for the velocity

$$\mathbf{v} = c^2\frac{\mathbf{p}}{E} \tag{7.61}$$

which should be compared with the non-relativistic expression $\mathbf{v} = \mathbf{p}/m$. In the limit $m \to 0$ the relativistic expression gives

$$\mathbf{v} = \frac{\mathbf{p}}{p}c \tag{7.62}$$

which means that in absolute value the speed of the particle is identical to the speed of light. Thus *a massless particle always moves with the speed of light*, and this is independent of what the energy carried be the particle is. Therefore, we cannot think of a massless particle as being accelerated to the speed of light, it has simply to be born with the speed of light. This is contrasted by the property of massive particles: A particle with mass $m \neq 0$ can never reach the speed of light. This is demonstrated by the form of the relativistic energy

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \underset{v \to c}{\longrightarrow} \quad \infty \tag{7.63}$$

7.3.1   Space ship with constant proper acceleration

We return to the situation discussed in Sect. 7.1.1 where a space ship was assumed to perform a space-time journey with constant proper acceleration far out in the universe. The acceleration would give a monotonic increase in the velocity of the space ship, which then would asymptotically approach the speed of light. In the discussion of the space-time motion we only briefly commented on the point that such a journey cannot go on indefinitely, since the limitation of available energy will end the journey after a finite time. Let us now consider this limitation in some detail.

We assume that the total mass of the space ship at the beginning of the journey is m_0 with m_1 as the mass of the ship without fuel. Since we do not know what kind of engine the space ship has, we only seek an upper limit to its efficiency. Let us for simplicity assume that *all* the mass of the fuel is converted to energy according to the Einstein formula $E = mc^2$. This energy is used to increase the velocity and therefore the momentum of the space ship. This

is done by sending the exhaust gas with *maximum momentum* in the opposite direction of the velocity of the space ship. The energy momentum relation (7.56) tells us that this happens if massless particles are emitted from the space ship. So we assume that photons are emitted in one direction, and the space ship due to this emission is accelerated in the opposite direction.

Let us consider what happens in a short time interval $d\tau$ on the space ship. In this time interval an amount of mass dm is converted to energy, and the photons that carry the energy away also carry an amount of momentum $dp = dm\, c$. This gives the same amount of momentum to the ship, but in the opposite direction. In the inertial frame which is the instantaneous rest frame of the space ship at time τ, the space ship at a little time later, at $\tau + d\tau$, will have a velocity slightly different from zero. The velocity is $dv = -dm\, c/m$ and this gives for the proper acceleration

$$a_0 = g = \frac{dv}{d\tau} = -\frac{c}{m}\frac{dm}{d\tau} \tag{7.64}$$

where we have assumed that the proper acceleration is kept fixed at the level of the gravitational acceleration on the surface of the earth. We note that this gives a differential equation for the change with time of the mass of the space ship

$$\frac{dm}{d\tau} = -\frac{mg}{c} \tag{7.65}$$

with an exponential function as solution

$$m(\tau) = m_0 \exp(-\frac{g}{c}\tau) \tag{7.66}$$

We denote by T the time onboard when all fuel has been consumed, so that $m(T) = m_1$. This gives

$$m_1 = m_0 \exp(-\frac{g}{c}T) \tag{7.67}$$

If we make the assumption that 90% of the space ship's weight at the beginning of the journey is fuel this gives the following (proper) time onboard the ship when it runs out of fuel

$$T = \frac{c}{g}\ln 10 \approx 2.3 \text{ years} \tag{7.68}$$

The speed of the space ship is then

$$v = \tanh(\frac{g}{c}T)c = \frac{m_0^2 - m_1^2}{m_0^2 + m_1^2} \approx 0.98c \tag{7.69}$$

and the time dilatation factor is

$$\gamma = \cosh(\frac{g}{c}T)c = \frac{1}{2}(\frac{m_0}{m_1} + \frac{m_1}{m_0}) \approx 5 \tag{7.70}$$

This is indeed a large velocity and gamma factor. The coordinate time at earth and the distance to the ship is at this point

$$t = \frac{c}{g}\sinh(\frac{g}{c}T) \approx 5 \text{ years}\,, \quad x - x_0 = \frac{c^2}{g}\cosh(\frac{g}{c}T) \approx 4 \text{ lightyears} \tag{7.71}$$

Even if this does not bring the space ship out to distant galaxies, the distance is still very impressive, comparable to the distance to the closest star. One should however note that the assumptions we have made are rather unrealistic. In particular this is so for the assumption that all mass of the fuel is converted to energy, which should be compared to the efficiency of mass conversion of about 1% for the nuclear fusion process where hydrogen is transformed into helium. A more realistic estimate would definitely limit the space travel much more than shown by the numbers above. However, the idea that a rocket engine based on emission of photons could give a constant acceleration over a long time, and thereby bring a space ship in an efficient way far outside the solar system, may not be such a bad idea.

7.4   Doppler effect with photons

Even if the speed of a light signal does not change when changing from one inertial reference frame to another, the frequency of the light signal will appear different in the two frames. This is the Doppler effect, which is well known for wave propagation also in non-relativistic physics. The correct relativistic Doppler shift formula can be found by considering light as a propagating wave, but another way to derive it, which is in fact simpler, is to make use of the transformation formula for relativistic four-momenta. This is the approach we take here, when we consider the transformation of four-momentum for a massless photon between two inertial frames and use the de Broglie relations to translate this to a transformation of frequencies.

Let us then consider the situation where a photon is emitted from a space-time point O which is the origin of an inertial reference frame S. In this frame the photon momentum is directed with angle θ relative to the x axis in the x, y plane,

$$\mathbf{p} = p(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \tag{7.72}$$

Since the photon is massless the components of the four-momentum in this frame are

$$\underline{\mathbf{P}} = p\,(1, \cos\theta, \sin\theta, 0) \qquad (S \ \text{frame}) \tag{7.73}$$

Let us assume that the photon is absorbed by a detector in another inertial reference frame S' that moves with velocity v in the x direction relative to S. The four momentum in this frame has components

$$\underline{\mathbf{P}}' = p'(1, \cos\theta', \sin\theta', 0) \qquad (S' \ \text{frame}) \tag{7.74}$$

The components of the two reference frames are related by the Lorentz transformation

$$\begin{aligned}
p'^0 &= \gamma(p^0 - \beta p^1) \\
p'^1 &= \gamma(p^1 - \beta p^0) \\
p'^2 &= p^2 \\
p'^3 &= p^3
\end{aligned} \tag{7.75}$$

which gives

$$\begin{aligned}
p' &= \gamma p(1 - \beta\cos\theta) \\
p'\cos\theta' &= \gamma p(\cos\theta - \beta) \\
p'\sin\theta' &= p\sin\theta
\end{aligned} \tag{7.76}$$

Only two of these are independent equations, as one can readily check, and these two equations can be used to solve for p' and $\cos\theta'$,

$$p' = \gamma(1 - \beta\cos\theta)p \tag{7.77}$$

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta} \tag{7.78}$$

The first one of these gives the Doppler shift formula. To show this we make use of the de Broglie formula which gives the link between the particle and wave nature of the photon, $p = E/c = h\nu/c$, with ν as the photon frequency. Eq. (7.77) then gives the frequency transformation formula

$$\nu' = \gamma(1 - \beta\cos\theta)\nu \tag{7.79}$$

This equation shows how the frequency of a light signal changes between two inertial frames in relative motion. The frame S' moves with velocity βc relative to S and θ is the angle between the photon and the relative velocity of the two frames, as measured in S. Clearly the same formula should be applicable if we interchange the two frames. This gives

$$\nu = \gamma(1 + \beta\cos\theta')\nu' \tag{7.80}$$

where we have introduced a sign change for the relative velocity. The formula can be rewritten as

$$\nu' = \frac{1}{\gamma(1 + \beta\cos\theta')}\nu \tag{7.81}$$

Consistency between (7.79) and (7.81) then gives a relation between the angle measured in the two frames, and this is the same as the equation (7.78).

We conclude that the Doppler shift can be expressed either as in (7.79) or in (7.81), depending on whether the angle of the light signal refers to the inertial frame S where it is emitted or the inertial frame S' where it is absorbed. We consider now some special cases.

a) $\theta = 0$: The light signal is emitted in the direction of motion of reference frame S'. Seen from S' the emitter of the signal is moving away from the receiver. The formula is

$$\nu' = \gamma(1 - \beta)\nu = \sqrt{\frac{1 - \beta}{1 + \beta}}\,\nu \tag{7.82}$$

The light is now redshifted since the frequency in S' is lower than in S.

b) $\theta = \pi$: The light signal is emitted against the direction of motion of reference frame S', so that the emitter is moving towards the receiver. The formula is

$$\nu' = \gamma(1 + \beta)\nu = \sqrt{\frac{1 + \beta}{1 - \beta}}\,\nu \tag{7.83}$$

and the light is blue shifted in reference frame S'.
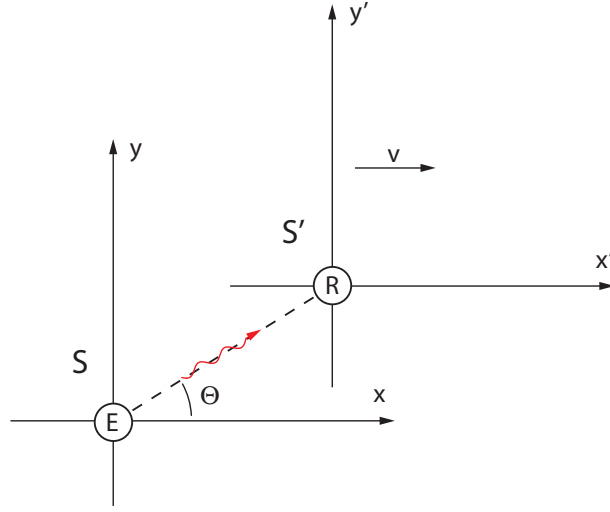
Figure 7.2: The Doppler effect. A photon is emitted from a sender E in an inertial frame S at an angle θ with respect to the x axis. The photon is received by a receiver R in another inertial frame S' which moves with velocity v relative to the first reference frame. The photon is received at a different frequency in S' than the frequency of the emitted photon in the reference frame S. Also the direction of the photon appears different in the two frames, as discussed in the text.

c) $\theta' = \pi/2$: The light signal is now *received* with direction orthogonal to the velocity of reference frame S'. This gives

$$\nu' = \frac{1}{\gamma}\nu \tag{7.84}$$

Even in this case there is a Doppler shift. We may view this as a time dilatation effect, where time in S is seen as slow when viewed from reference frame S'. The light signal is redshifted.

d) $\theta = \pi/2$: The light signal is now *emitted* at $90°$ degrees in S, and formula is now

$$\nu' = \gamma\nu \tag{7.85}$$

The time dilatation effect works the other way, and the light signal is blue shifted. In reference frame S' the angle θ' is larger than $90°$, as follows from Eq.(7.78). This means that the signal is received with a velocity component against the motion of the frame, which is consistent with the blue shift.

7.5   Conservation of relativistic energy and momentum

An important property of energy and momentum is that these physical quantities are conserved, when we consider the total sum of contributions from all parts of a physical system. This is true both in non-relativistic and in relativistic physics. However, the relativistic form of the

Figure 7.3: A collision process. Two particles are moving freely until they reach a collision region (yellow circular area). As a result of the collision a set of new particles emerge. Relativistic four-momentum is conserved in the process.
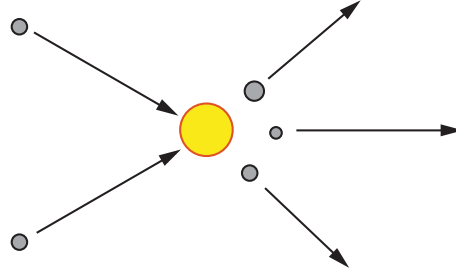
conservation laws is different from the non-relativistic one, and there are differences in physical consequences. We will examine these differences.

Let us consider this for a general collision process, as schematically shown in Fig. 7.3. In this process a set of particles are initially freely moving, but then enter a region of interaction. From this region another set of particles are emerging and these, in the final state, are again freely moving. The collision process may be elastic, in which case the initial and final sets of particles are identical, but it may also be inelastic, with the outgoing particles being different from the incoming. For simplicity we assume that radiation can be neglected during the collision process, which means in the relativistic description that we assume that *massless* particles are not emitted.

In the non-relativistic case we may formulate three conservation laws which apply to the collision process. They are

$$
\begin{aligned}
\sum_i \mathbf{p}_i &= \sum_f \mathbf{p}_f \\
\sum_i \frac{\mathbf{p}_i^2}{2m_i} &= \sum_f \frac{\mathbf{p}_f^2}{2m_f} + Q \\
\sum_i m_i &= \sum_f m_f
\end{aligned}
\tag{7.86}
$$

where the index i refers to the incoming particles and f to the outgoing ones. The first equations states that total momentum is conserved. The second one states that total energy is conserved. This does not mean that total *kinetic* energy needs to be conserved. In inelastic collisions that is not the case, and Q then measures how much energy that is transformed from kinetic to other forms of energy (internal energy) in such an inelastic collision. The third equation expresses the conservation of total mass.

In the relativistic setting these conservation laws are replaced by a single four-vector equation,

$$
\sum_i \underline{\mathbf{P}}_i = \sum_i \underline{\mathbf{P}}_f
\tag{7.87}
$$

which states that the total four-momentum is preserved in the process. If the non-relativistic limit should be reached in the correct way from this equation, then Eqs.(7.86) should follow from (7.87) when the particle velocities are small compared to the speed of light. We shall check that this is the case.

The space component of the four-vector equation has the same form as the non-relativistic equation for conservation of momentum. But the meaning is different because of the relativistic form of momentum. Expressed in terms of velocities it is

$$\sum_i \gamma_i \, m_i \, \mathbf{v}_i \;\; = \;\; \sum_f \gamma_f \, m_f \, \mathbf{v}_f \tag{7.88}$$

where the gamma factors of the particles are missing in the non-relativistic equations. However, in the non-relativistic limit, $v/c \to 0$, we have $\gamma \to 1$ and the relativistic equation reproduces the non-relativistic equation (as it should).

We next consider the time component of the four-vector equation (7.87), which we may write as

$$\sum_i \gamma_i \, m_i \, c^2 \;\; = \;\; \sum_f \gamma_f \, m_f \, c^2 \tag{7.89}$$

Obviously, if the non-relativistic limit also here is taken as $\gamma \to 1$ the equation will reproduce the non-relativistic mass conservation equation. However, this raises the question how the non-relativistic equation for conservation of energy should be reproduced. The answer is that this is also contained in the time component of (7.87), but we have to keep the first order contributions in v^2/c^2 when we make an expansion in this small quantity. This gives the following equation

$$\sum_i m_i \, c^2 + \frac{1}{2} m_i v_i^2 \;\; = \;\; \sum_f m_f \, c^2 + \frac{1}{2} m_f \, v_f^2 \tag{7.90}$$

By use of the non-relativistic form of the momentum $\mathbf{p}$ it can be rewritten as

$$\sum_i \frac{\mathbf{p}_i^2}{2m_i} \;\; = \;\; \sum_f \frac{\mathbf{p}_f^2}{2m_f} + \left(\sum_f m_f \, c^2 - \sum_i m_i \, c^2 \right) \tag{7.91}$$

This is seen to have the same form as the non-relativistic energy conservation equation, but with an explicit expression for the Q term,

$$Q = \sum_i m_i \, c^2 - \sum_f m_f \, c^2 \tag{7.92}$$

This result is interesting and important. It shows that mass is not conserved in a strict sense in special relativity. Instead, in an inelastic collision, where $Q \neq 0$, there will be a mass difference between the initial and final states, that is determined by the ratio Q/c^2. So mass is in such a process converted to kinetic energy or kinetic energy is converted to mass. (The kinetic energy may in part take the form of radiation, *i.e.*, massles particles.) This gives a concrete, physical interpretation of the rest energy $E = mc^2$ of a massive body. A dramatic application of this relation between mass and energy is in nuclear fission reactions, where a fraction of the mass of an unstable nucleus is converted to kinetic energy and radiation energy.

We consider two examples of such inelastic collisions. The first is a completely inelastic collision where two bodies collide and create a single larger body. In the process heat is created and we assume that the heat energy is stored in the body as internal energy.

Let us for simplicity assume the two bodies before the collision to have equal masses m_0. We consider the collision process in the *center-of-mass system* where the two bodies before the collision have momenta of equal size but with opposite directions. The larger body that is produced in the collision has mass M and sits at rest in this reference frame. The three-vector part of the relativistic momentum conservation equation simply states that the total momentum vanishes in this frame. The conservation equation for energy is

$$2\gamma mc^2 = Mc^2 \tag{7.93}$$

with γ as the (common) gamma factor of each of the two colliding particles. The equation shows that the mass of the body that is formed in the collision is larger than the sum of the masses of the two particles before the collisions,

$$\Delta m = M - 2m = 2m(\gamma - 1) > 0 \tag{7.94}$$

When expanded in powers of v^2/c^2, we have $\gamma = 1 + \frac{1}{2}\frac{v^2}{c^2} + \dots.$ This gives for $v << c$

$$2(\frac{1}{2}mv^2) = \Delta m\, c^2 \tag{7.95}$$

which shows that the kinetic energy of the colliding particles is present, after the collision, as an increase in the mass of the larger body that is formed by the collision. This result is independent of how energy is stored in the body, but in the present case it seems natural to identify $Q = \Delta mc^2$ with the heat created by the collision. The mass energy formula in fact suggests quite generally that if a body is heated, the increase in internal energy will lead to an increase in its mass. However, the mass increase obtained by heating the body is under normal conditions extremely small.

The inverse of the process considered above is a fission process where a body is split in two parts by an explosion of some sort. In that case the total mass after the explosion is smaller than the total mass before the explosion, and the missing mass is converted to kinetic energy (and radiation) according to the mass conversion formula $\Delta E = \Delta mc^2$.

To conclude, in special relativity the total four-momentum of an isolated system is always conserved. This conservation law reduces to the standard expressions for conservation of energy and momentum in the non-relativistic limit. However, a consequence of the relativistic formula is that the total mass is not strictly conserved. The change of mass in a physical process relates to the difference in total kinetic energy and radiation energy in the initial and final states.

7.6 The center of mass system

Consider a composite system which is isolated from the surroundings so that no external forces act on the system. In non-relativistic physics the center of mass of the system, $\mathbf{R}$, is defined by

$$M\mathbf{R} = \sum_k m_k\mathbf{r}_k \tag{7.96}$$

with $\mathbf{P}$ as the total three-momentum of the system and M as the total mass, and where $\mathbf{r}_k, k = 1, 2, ...$ denote the position vectors of the small parts of the system, with masses m_k. Without external forces the center of mass is non-accelerated and therefore we can find an inertial reference frame where it is at rest. This is the *center-of-mass system*, which is then characterized by

$$\mathbf{P} \equiv M\dot{\mathbf{R}} = \sum_k m_k \dot{\mathbf{r}}_k = 0 \qquad (7.97)$$

In the center-of-mass system (the *CM system* for short) the total momentum of the physical system therefore vanishes.

In relativistic physics the center of mass is not a well defined concept. This can be seen in the following way. In the definition of the CM-position vector $\mathbf{R}$ it is essential that the sum (7.96) is performed at *equal times* for all parts for the system. This is a definition which is independent of choice of inertial frame in non-relativistic physics, since there *equal time* is a universal concept. However in relativistic physics that is no longer true. If we therefore define the sum over a three-dimensional space with time coordinate $t = constant$ in one inertial frame, that is different from defining the sum over three-dimensional space with $t' = constant$ in another inertial frame. There will in general be no simple relation between the result of to such different summations. As a result we simply give up the idea of defining, in general, the center of mass of an extended system.

Even so, the *center-of-mass system* is both a well defined and useful concept in relativistic physics. This follows from the fact that the total four-momentum $\underline{\mathbf{P}}$ of the system is well defined, and the condition that the space part vanishes, as in (7.97), specifies an inertial frame which we identify as the center-of mass-system. The condition that identifies the center-of mass system therefore is

$$\mathbf{P} = \sum_k \mathbf{p}_k = 0 \qquad (7.98)$$

which is now written in a form that is correct both in non-relativistic and relativistic physics, but in the latter case one has to remember that for massive particles the right definition of relativistic momentum is $\mathbf{p} = \gamma m \mathbf{v}$.

The total momentum $\underline{\mathbf{P}}$ is a reference-frame independent four-vector, in spite of the fact the sum over contributions from all part of the physical system seems to depend on the choice of frame. This is in fact a consequence of conservation of the total four-momentum for an isolated physical system. For a system of *non-interacting* particles this is quite clear, since the momentum is conserved for each particle individually. This means that the sum of the particles four momenta will be independent of the points on the particle world lines that are chosen when performing the sum. In particular the result is the same whether they are summed at equal times in one inertial frame or another. For the same reason the sum of the four-vectors will result in a new four-vector. In the general case the same conclusion can be reached by use of momentum conservation expressed as a *local* conservation law. However, we will not here give a detailed derivation of this result.

We conclude that the components of the total four-momentum of an isolated system transform as components of a four-vector, and since it is a time-like vector an inertial frame can

always be found where the space part of the vector vanishes. This is the center-of-mass system, and it is a unique reference frame up its orientation in space.

Chapter 8

Relativistic dynamics

In non-relativistic physics Newton's second law is the basic dynamical equation. It is not valid in relativistic physics, unless some changes are introduced. We will here examine the question of how to correctly update the law to relativistic form. Our approach is based on the general idea of rewriting the non-relativistic equation in a *covariant* relativistic form. This means that we express the equation in terms of four-vectors and tensors, in such a way that it has the correct non-relativistic limit for low velocities $v << c$. The covariant form secures that the equation is valid in all inertial frames. Whether the equation is really correct is at the end a question to check experimentally, but at least the formal properties demanded by relativistic invariance will be satisfied by this approach.

8.1 Newton's second law in relativistic form

Our starting point is the (non-relativistic) Newton's second law, which we write as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{8.1}$$

with $\mathbf{p} = m\mathbf{v}$. It is is here assumed to apply to a small body (point particle) which carry momentum $\mathbf{p}$ and is subject to a force $\mathbf{F}$. This is a three-vector equation, which in relativistic form should be generalized to a four-vector equation. As an obvious attempt to do so we write the it in the following relativistic form

$$\underline{\mathbf{K}} = \frac{d\underline{\mathbf{P}}}{d\tau} \tag{8.2}$$

The non-relativistic momentum then is replaced by the relativistic four-momentum and coordinate time is replaced by proper time of the particle. The time derivative of the four-vector then is also a four-vector. On the left-hand side we have simply replaced the three-vector force $\mathbf{F}$ with a four-vector $\underline{\mathbf{K}}$, which we refer to as the four-force. We shall examine what constraints that physics puts on this vector, but for the moment we just note that the new equation has a correct covariant form.

The equation can also be written as

$$\underline{\mathbf{K}} = m\underline{\mathbf{A}} \tag{8.3}$$

with m as the mass of the particle and $\underline{\mathbf{A}}$ as the proper acceleration. This follows since $\underline{\mathbf{P}} = m\underline{\mathbf{U}}$ and therefore $\frac{d\underline{\mathbf{P}}}{d\tau} = m\frac{d\underline{\mathbf{U}}}{d\tau}$ with $\underline{\mathbf{U}}$ as the four-velocity of the particle. Note however, since the three-vector part of $\underline{\mathbf{A}}$ is generally not identical to the acceleration $\mathbf{a}$ in a chosen inertial frame, the three-vector part of the right-hand side of (8.3) is *not* simply $m\mathbf{a}$.

The next step is to relate the four-force $\underline{\mathbf{K}}$ to the (non-relativistic) three-vector force $\mathbf{F}$. To this end we decompose the four-vector in its time and space components, with reference to some unspecified inertial reference frame,

$$\underline{\mathbf{K}} = (K^0, \mathbf{K}) \tag{8.4}$$

The three-vector part of the equation (8.2) is then

$$\mathbf{K} = \frac{d\mathbf{p}}{d\tau} = \gamma\frac{d\mathbf{p}}{dt} \tag{8.5}$$

with $\mathbf{p} = \gamma m\mathbf{v}$ as the relativistic momentum. The factor γ appears in the equation as a time dilatation effect.

Let us now return to the original form (8.1) of Newton's second law, and assume that also in the context of relativistic physics the three-vector force $\mathbf{F}$ is defined so that (8.1) is correct. However, since $\mathbf{p}$ should then be the relativistic momentum, then generally $\mathbf{F} \neq m\mathbf{a}$. This means that the non-relativistic form of Newton's second law is valid also in relativistic theory, but only when expressed in terms of $\frac{d\mathbf{p}}{dt}$ and *not* when expressed in terms of $\mathbf{a}$. It is then clear that the correct non-relativistic equation is obtained when $\gamma \to 1$ and $\mathbf{p}$ therefore is changed from its relativistic to its non-relativistic form.

We have now established the relation

$$\mathbf{K} = \gamma\mathbf{F} \tag{8.6}$$

and we examine further the time component of $\underline{\mathbf{K}}$,

$$K^0 = \frac{dP^0}{d\tau} = \gamma\frac{1}{c}\frac{dE}{dt} \tag{8.7}$$

The relativistic energy-momentum relation is

$$E^2 = p^2c^2 + m^2c^4 \tag{8.8}$$

and the time derivative of this equation gives

$$E\frac{dE}{dt} = c^2\,\mathbf{p}\cdot\frac{d\mathbf{p}}{dt} \tag{8.9}$$

This further gives

$$\frac{dE}{dt} = c^2\frac{\mathbf{p}}{E}\cdot\frac{d\mathbf{p}}{dt} = \mathbf{v}\cdot\mathbf{F} \tag{8.10}$$

where we have made use of the relativistic relation $\mathbf{v} = c^2\mathbf{p}/E$. It is interesting to note that with the relativistic generalization introduced for the three-vector force $\mathbf{F}$, the expression for

the power is $\mathbf{v} \cdot \mathbf{F}$, precisely as in non-relativistic physics. The four-force, when decomposed in time and space components can then be written as

$$\underline{\mathbf{K}} = \gamma(\frac{1}{c}\mathbf{v} \cdot \mathbf{F}, \mathbf{F}) \tag{8.11}$$

While the space part is proportional to the three-vector force $\mathbf{F}$, the time component is proportional to the power of the force, $\mathbf{v} \cdot \mathbf{F}$.

One should note from the above expressions that even when the three-vector force $\mathbf{F}$ is a velocity *independent* force, the four-force $\underline{\mathbf{K}}$ will quite generally depend on the velocity of the particle. This point is also demonstrated by the fact that the four-force, which is proportional to the four-acceleration, is always orthogonal, in the relativistic sense, to the four-velocity,

$$\underline{\mathbf{K}} \cdot \underline{\mathbf{U}} = \underline{\mathbf{A}} \cdot \underline{\mathbf{U}} = 0 \tag{8.12}$$

We further note that since $\underline{\mathbf{U}}$ is a time-like vector this implies that $\underline{\mathbf{K}}$ is a spacelike vector. From this follows that we can always find an inertial frame where the time component of the force vanishes. The expression (8.11) shows that this happens in the instantaneous rest frame of the particle, where the four-force reduces to the form

$$\underline{\mathbf{K}} = (0, \mathbf{F}) \qquad \text{(rest frame)} \tag{8.13}$$

8.1.1 The Lorentz force

We shall consider, as a special case, the force that acts on a charged particle in an electromagnetic field, and derive the corresponding covariant form of the equation of motion. The non-relativistic form of the three-vector force is

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{8.14}$$

with e as the charge and $\mathbf{v}$ as the velocity of the particle, and this is in fact a valid expressions also for relativistic velocities. The corresponding expressions for the time and space components of the four-force are

$$\underline{\mathbf{K}} = \gamma e(\frac{1}{c}\mathbf{v} \cdot \mathbf{E}, \, \mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{8.15}$$

We would now like to write this in covariant form, and that requires that we introduce the electromagnetic field tensor. This is an antisymmetric tensor where the electric field appears as the time components and the magnetic field as the space components in the following way,

$$\begin{aligned} F^{0k} &= \frac{1}{c}E_k \quad k = 1, 2, 3 \\ F^{kl} &= \sum_m \epsilon_{klm}B_m \quad k, l = 1, 2, 3 \end{aligned} \tag{8.16}$$

where ϵ_{klm} is the three-dimensional Levi-Civita symbol and the Cartesian components are here numbered 1,2,3. In matrix form the field tensor is

$$F = (F^{\mu\nu}) = \begin{pmatrix} 0 & \frac{1}{c}E_1 & \frac{1}{c}E_2 & \frac{1}{c}E_3 \\ -\frac{1}{c}E_1 & 0 & B_3 & -B_2 \\ -\frac{1}{c}E_2 & -B_3 & 0 & B_1 \\ -\frac{1}{c}E_2 & B_2 & -B_1 & 0 \end{pmatrix} \tag{8.17}$$

Constructed in this way $F^{\mu\nu}$ transforms indeed as a relativistic tensor under Lorentz transformations.

We consider first how the time component of the four-force can be expressed in terms of the electromagnetic field tensor. The expression is

$$K^0 = e\gamma\frac{\mathbf{v}}{c}\cdot\mathbf{E} = eF^{0\nu}U_\nu \tag{8.18}$$

with U_ν as the four-velocity of the particle. The space part we rewrite in a similar way,

$$
\begin{aligned}
K^k &= e\gamma(E_k + \sum_{lm} v_l B_m) \\
&= e\gamma(cF^{0k} + \sum_l F^{kl} v_l)
\end{aligned}
\tag{8.19}
$$

We next make use of the following identities,

$$U^0 = -U_0 = \gamma c\,, \quad U^l = U_l = \gamma v_l\,, \quad F^{0k} = -F^{k0} \tag{8.20}$$

This gives

$$K^k = e(F^{k0}U_0 + \sum_l F^{kl}U_l) = eF^{k\nu}U_\nu \tag{8.21}$$

We can now combine the expressions for K^0 and K^k into a single equation for the components of the four-force,

$$K^\mu = eF^{\mu\nu}\,U_\nu \tag{8.22}$$

With the above expression for the four-force the equation of motion of the charged particle can be rewritten in covariant form. The general relativistic form of Newton's second law,

$$\underline{\mathbf{K}} = m\underline{\mathbf{A}} = m\frac{d^2\underline{\mathbf{x}}}{d\tau^2} \tag{8.23}$$

here gives the equation

$$m\ddot{x}^\mu = eF^{\mu\nu}\dot{x}_\nu \tag{8.24}$$

where the time derivative marked by the *dot* here means differentiation with respect to the proper time of the particle.

We finally note that this covariant equation is equivalent to the non-covariant equation of motion,

$$\frac{d\mathbf{p}}{dt} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{8.25}$$

which has the same form in the non-relativistic limit. However, in the relativistic case the left-hand side of the equation cannot simply be replaced by $m\mathbf{a}$, due to the presence of the gamma factor in the expression for the momentum, $\mathbf{p} = \gamma m\mathbf{v}$.

The two versions, (8.24) and (8.25), of the relativistic equation of motion for a charged particle in the electromagnetic field, we have earlier referred to at the end of Sect.6.5.

8.1.2 Example: Relativistic motion of a charged particle in a constant magnetic field

We first briefly examine the non-relativistic motion of the charged particle in the constant magnetic. This has in Sect.3.2.1 been done by use of Hamilton's equations. Here we derive the motion in a more direct way from Newton's second law.

The equation of motion is

$$m\mathbf{a} = e\mathbf{v} \times \mathbf{B} \tag{8.26}$$

with e as the charge of the particle. There is no force along the direction of the field, and the motion in this direction is therefore a constant drift. For simplicity we assume initial conditions with no velocity in this direction, and the motion is therefore restricted to a plane orthogonal to $\mathbf{B}$. The velocity is also restricted to this plane, and the equation of motion shows that the force is orthogonal to the velocity, so that $\mathbf{a} \cdot \mathbf{v} = 0$. Consequently the kinetic energy is conserved, $T = \frac{1}{2}mv^2$ is constant.

There is another constant of motion which can be derived from the equation of motion. The equation can be written as

$$\frac{d}{dt}(m\mathbf{v} - e\mathbf{r} \times \mathbf{B}) = 0 \tag{8.27}$$

which gives the following constant of motion,

$$m\mathbf{v} - e\mathbf{r} \times \mathbf{B} \equiv -e\mathbf{r}_0 \times \mathbf{B} \tag{8.28}$$

with $\mathbf{r}_0$ as a constant vector. The form we have chosen for the constant vector, on the right-hand side of the equation, is consistent with the fact that this vector is restricted to the plane orthogonal to $\mathbf{B}$ when $\mathbf{v}$ has no component along the magnetic field. We rewrite the equation as

$$m\mathbf{v} - e(\mathbf{r} - \mathbf{r}_0) \times \mathbf{B} = 0 \tag{8.29}$$

and this shows that $\mathbf{r}_0$ can be absorbed in a shift of the origin in the plane of motion. We assume in the following that to have been done, so the motion satisfies the equation

$$m\mathbf{v} = e\mathbf{r} \times \mathbf{B} \tag{8.30}$$

This equation shows that the velocity is orthogonal to the position vector, $\mathbf{r} \cdot \mathbf{v} = 0$, so that $\mathbf{r}^2$ is a constant of motion.

From the arguments given above we conclude that the particle moves in a circle with constant velocity. With $\mathbf{r}_0 = 0$ the center of the orbit is at the origin of the coordinate system, but without this restriction the center can be placed anywhere in the plane. The circular frequency is

$$\omega = \frac{v}{r} = \frac{eB}{m} \tag{8.31}$$

which is the cyclotron frequency. The kinetic energy is $T = \frac{1}{2}m\omega^2 r^2$ which shows that the radius increases with energy so that T is proportional with r^2.

The expressions given above are valid for non-relativistic velocities, $v << c$, which means that $T << \frac{1}{2}mc^2$. That restricts the kinetic energy to be much smaller than the rest energy of the particle. In the following we will lift this restriction and study how the motion changes when we use the correct relativistic equation of motion.

The relativistic equation of motion can be written as

$$\frac{d\mathbf{p}}{dt} = e\mathbf{v} \times \mathbf{B} \tag{8.32}$$

with $\mathbf{p} = \gamma m\mathbf{v}$. The power of the force vanishes, $\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v} = 0$, which means that γmc^2 is constant. As a consequence the velocity is constant in the relativistic description as well as in the non-relativistic approximation. There is also a constant of motion corresponding to (8.28), but now written as

$$\gamma m\mathbf{v} - e\mathbf{r} \times \mathbf{B} \equiv -e\mathbf{r}_0 \times \mathbf{B} \tag{8.33}$$

We may also in this case chose to shift the origin in order to have $\mathbf{r}_0 = 0$. The equation is then

$$\gamma m\mathbf{v} = e\mathbf{r} \times \mathbf{B} \tag{8.34}$$

which is a relativistic generalization of (8.30). This shows that $\mathbf{v} \cdot \mathbf{r} = 0$ and therefore r is a constant, just as in the non-relativistic case.

We conclude that in the relativistic case, just as in the non-relativistic case the charged particle moves with constant speed in a circle. There is however a difference between the two cases as far as the circular frequency is concerned. Eq.(8.34) gives

$$\gamma mv = erB \tag{8.35}$$

from which follows

$$\omega = \frac{eB}{\gamma m} \equiv \frac{\omega_0}{\gamma} \tag{8.36}$$

where we have introduced the symbol $\omega_0 = eB/m$ for the non-relativistic cyclotron frequency. This shows that in the relativistic case the circular frequency decreases with the speed, and therefore with the energy of the particle. We can also find the frequency as a function of the radius of the orbit if we write the equation as

$$\omega = \omega_0 \sqrt{1 - \frac{\omega^2 r^2}{c^2}} \tag{8.37}$$

Solving this for ω we find

$$\omega = \frac{\omega_0}{\sqrt{1 + \frac{\omega_0^2 r^2}{c^2}}} \tag{8.38}$$

The corresponding expression for the relativistic energy of the particle is

$$E = \sqrt{1 + \frac{\omega_0^2 r^2}{c^2}}\, mc^2 \tag{8.39}$$

As shown by this expression the energy is a quadratic function of r for non-relativistic velocities but this changes to a linear dependence in the relativistic regime.

The decrease in circular frequency with velocity or energy of the circulating charge is important in a type of particle accelerator called *cyclotrons*. In these accelerators charged particles are circulating in a strong magnetic field and energy is fed to the particles by applying an electric field which oscillates with the circular frequency of the particles. In the early cyclotrons where the particles moved non-relativistically the frequency of the field was kept fixed. Later on accelerators were built to accelerate beams of particles to relativistic speeds. In these accelerators, called *synchrocyclotrons*, the frequency was synchronized with the decreasing circular frequency of the accelerated particles. In *isochronous cyclotrons* a different approach is taken to compensate for the relativistic effect. These accelerators work with constant electric field frequency, but the strength of the magnetic field is increased with time. As shown by Eq.(8.36) the circular frequency of the circulating particles can be kept fixed by compensating for the increase of γ by a similar increase in the value of B.

8.2 The Lagrangian for a relativistic particle

In the Lagrangian formulation of Newton's mechanics introduced earlier in the course, the time coordinate plays a different role than the generalized coordinates of the system. Time is there a parameter for the path of the system through configuration space. For a particle moving through three dimensional space this means that the space coordinates and the time coordinate appear in different ways in the formalism. This difference seems to create a problem when extending the formalism to relativistic theory, where time and space coordinates are mixed by the Lorentz transformations.

We consider the question of how to introduce a relativistic particle Lagrangian, and in order to do so we make an attempt to follow the same approach as in the previous section, where relativistic generalizations of physical formulas were introduced by re-writing the equations in covariant form. We apply this first to a freely moving particle of mass m.

Instead of considering the Lagrangian directly we start with the action, which is the integral of the Lagrangian for an arbitrarily chosen time dependent path in configuration space. For a free particle it has the form

$$S = \int T dt = \int \frac{1}{2} m \dot{\mathbf{r}}^2 dt \tag{8.40}$$

with T as the kinetic energy of the particle. As a first step in a relativistic modification of the integral we note the following correspondence

$$T dt \to E dt = c P^0 dt \tag{8.41}$$

where the expressions to the right have the one to the left as the non-relativistic limit, except for the presence of the rest energy $E_0 = mc^2$. However, a constant can always be added to the Lagrangian, since it does not affect Lagrange's equations. This immediately suggests that one could add the term $-\mathbf{P} \cdot d\mathbf{r}$ to make the expression Lorentz invariant

$$T dt \to -P^\mu dx_\mu \tag{8.42}$$

However, with this modification we have to check again the non-relativistic limit. We have

$$
\begin{aligned}
P^\mu dx_\mu &= \mathbf{P} \cdot d\mathbf{r} - cP^0 dt = (\mathbf{P} \cdot \mathbf{v} - cP^0)dt \\
&\rightarrow (mv^2 - mc^2 - \frac{1}{2}mv^2 - ...)dt = (\frac{1}{2}mv^2 - mc^2 - ...)dt \quad (8.43)
\end{aligned}
$$

where $-...$ denotes higher order terms in v^2/c^2 which are omitted in the non-relativistic approximation. We note that the term we have added has in fact changed the sign of the non-relativistic kinetic energy. To compensate for this we may simply change the sign in the correspondence (8.42).

As a result we have found the following expression for the action integral which is Lorentz invariant and has the correct non-relativistic limit,

$$
S = \int P^\mu dx_\mu = m \int U^\mu dx_\mu = m \int U^\mu U_\mu d\tau = -mc^2 \int d\tau \quad (8.44)
$$

We have here applied the identities $P^\mu = mU^\mu$, $dx^\mu = U^\mu d\tau$, $U^\mu U_\mu = -c^2$, with τ as the proper time of the particle.

It may initially seem very natural that in the Lorentz invariant form the proper time should be used as a time parameter rather than the coordinate time. However, this choice of time parameter creates a problem, since the Lagrangian then is simply a constant, which cannot be used to derive Lagrange's equations. The origin of the problem is following. When Lagrange's equation of motion is derived from Hamilton's principle, one considers changes in the action under variations in the space-time path where the end points are kept fixed. Also the time parameter along the paths has to take fixed values at the end points. However, proper time does not satisfy this requirement, since the proper time between two space-time points depends on the path between the points, as we have previously discussed.

The conclusion we therefore draw is that the expression we have found for the action may be fine, but proper time is not a good parameter to use if we want to derive the equation of motion from the action integral. To circumvent the problem we may simply introduce a new, unspecified parameter λ, which is assumed to take fixed values at the end points of any given space-time path when we perform variations in the path with the end points fixed. For this new parameter we find the following expression

$$
c d\tau = \sqrt{-dx^\mu dx_\mu} = \sqrt{-\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}} d\lambda \quad (8.45)
$$

where one should note that the minus sign under the square rote is simply to compensate for the fact that dx^μ is a timelike vector for the path of the particle. The expression given by the equation is clearly invariant under arbitrary changes in the parameter, $\lambda \rightarrow \lambda'$.

The action can then be written as

$$
S = -mc^2 \int_{\lambda_0}^{\lambda_1} \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\lambda \quad (8.46)
$$

where the parameter values at the end points are called λ_0 and λ_1 and where we here define $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$. The corresponding Lagrangian is then

$$
L = -mc\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \quad (8.47)
$$

and Lagrange's equations have the standard form

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^{\mu}}\right) - \left(\frac{\partial L}{\partial x^{\mu}}\right) = 0, \quad \mu = 0, 1, 2, 3 \tag{8.48}$$

It should be straight forward to check whether the equations found in this way are the correct equations of motion for a free particle. We first note that all the coordinates x^{μ} are cyclic, $\frac{\partial L}{\partial x^{\mu}} = 0$, so that we have the following set of constants of motion

$$\frac{\partial L}{\partial \dot{x}^{\mu}} = mc\frac{\dot{x}^{\mu}}{\sqrt{-\dot{x}^{\nu}\dot{x}_{\nu}}} \equiv k^{\mu} \tag{8.49}$$

where k^{μ} satisfies the condition $k^{\mu}k_{\mu} - m^2c^2$. This gives

$$k^{\mu} = mc\,n^{\mu} \tag{8.50}$$

with n^{μ} as a timelike unit vector, $n^{\mu}n_{\mu} = -1$. If we now introduce the four-velocity, $U^{\mu} = \dot{x}^{\mu}\frac{d\lambda}{d\tau}$, Eq.(8.49) implies,

$$U^{\mu} = c\,n^{\mu} \tag{8.51}$$

which shows that the four-velocity is a constant timelike vector with relativistic norm squared, $U^{\mu}U_{\mu} = -c^2$. This is clearly the correct expression for a free particle which moves with constant velocity.

Next we consider the Lagrangian of a charged particle in an electromagnetic field. This can be obtained from the free field Lagrangian by simply adding a contribution from the electromagnetic potentials. The Lagrangian has the following Lorentz invariant form

$$L = -mc\sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}} + eA_{\mu}\dot{x}^{\mu} \tag{8.52}$$

where the scalar potential ϕ and vector potential $\mathbf{A}$ are identified as the components of the electromagnetic *four-potential* $\underline{\mathbf{A}} = (\frac{1}{c}\phi, \mathbf{A})$. It is again a straight forward exercise to check that the corresponding Lagrange's equations give the correct relativistic equations of motion in the relativistic form (8.24).

Let us finally point out that even if covariant expressions have been used in order to construct the Lagrangians with correct relativistic form, there is no problem to re-express them with the coordinate time as parameter, for any chosen inertial frame. After all the parameter λ can be freely chosen and in particular chosen to coincide with such a coordinate time. The main point is that the action we have found does not depend on the choice of parameter. In particular this means that if the coordinate time is chosen, the action of a free particle should be written as

$$S = -mc\int d\tau = \int L dt \tag{8.53}$$

By use of the time dilatation formula $\frac{dt}{d\tau} = \gamma$ this gives the following expression for the Lagrangian, when the coordinate t is chosen as parameter

$$L = -\frac{mc^2}{\gamma} = -mc^2\sqrt{1 - \frac{v^2}{c^2}} \tag{8.54}$$

We note that this expression, even if there is some resemblance, is not identical to the energy $E = \gamma mc^2$. In this regard it is different from the non-relativistic Lagrangian.

For a charged particle in an electromagnetic field the corresponding Lagrangian is

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} - e\phi + e\mathbf{v} \cdot \mathbf{A} \qquad (8.55)$$

The expressions (8.54) and (8.55) are not Lorentz invariant, but nevertheless correct expressions for the relativistic Lagrangians, as our derivation has shown.

Summary

This part of the lectures has focussed on some of the basic elements of the special theory of relativity. The starting point has been the fundamental space-time symmetries expressed in the form of Lorentz transformations. They define the transition between Cartesian coordinates of different inertial frames, and the basic difference between these transformations and the Galilean symmetry transformations of non-relativistic physics is the mixing of space and time coordinates in the relativistic case. An important consequence of this is that *distance* in non-relativistic theory, in the form of the length of the three-vector $\Delta\mathbf{r}$, is replaced by another invariant which also includes the time difference, $\Delta s^2 = \Delta\mathbf{r}^2 - c^2\Delta t^2$. Invariance of this quantity is directly related to the basic property of Lorentz transformation, that the speed of light does not depend on the choice of reference frame.

The change from Galilean to Lorentz transformations as the fundamental symmetry transformations has many important consequences for relativistic kinematics and dynamics, as first demonstrated by Albert Einstein. We have here derived the kinematical effects of *length contraction* and *time dilatation* and have stressed the important point that measurements should be performed at *simultaneity* of the observer who makes the measurement. For the time dilatation effect this understanding is applied to the *twin paradox*, which is resolved by taking into account the change in definition of simultaneous events that is performed by one of the twins during his space-time journey.

An introduction to the formalism of four-vectors and tensors has been given, and we have discussed how to apply this formalism when defining *covariant* relativistic equations. The formalism has been used at several places to *derive* the relativistic expressions that correspond to known non-relativistic quantities. The idea is to seek covariant expressions, which secures that they are valid in any inertial frame, and to impose the condition that the expressions have the correct non-relativistic limit. This formal approach indeed produces correctly the relativistic extensions of the non-relativistic expressions for the physical quantities and equations. In particular we have introduced the four-vector description of velocity and acceleration, and we have discussed the meaning of *proper acceleration*. As an example we have studied the so-called *hyperbolic motion* of a space ship with constant proper acceleration and effects of the time dilatation for time registered on the space ship and on earth.

The definition of the *conserved* four-momentum has been shown to have important consequences. The energy and three-vector momentum are there combined into a four-component object, and Lorentz invariance imposes a particular form to the relation between energy and momentum for a moving object. This involves in particular a conversion formula between mass and energy, which is Einstein's famous equation $E = mc^2$. By considering the case of inelastic collisions between particles we demonstrate that this relation is not only a curious co-

incidence, but it shows that mass can be converted to energy in real physical processes, with a large conversion factor between mass and energy. This points to the well-known and dramatic effects that release huge amounts of energy in nuclear processes.

In the chapter on Relativistic Dynamics we have examined how to update Newton's second law to a relativistic equation and to give meaning to the four-vector force. As a particular application we have examined how to give the equation of motion for a charged particle in an electromagnetic field the correct covariant form. Finally we have discussed how to bring relativistic equations into Lagrangian form and have shown how to resolve the problem which appears when using proper time as the time parameter. The approach has been illustrated by deriving Lagrangians for a free particle and for a charged particle in an electromagnetic field, both in the covariant and the non-covariant forms.

Part III
Electrodynamics

Each of the three parts of this course is associated with one or two scientists that have played a particularly important role in developing the theory, and who have in this way put their finger prints in a lasting way on the development of the science of physics. In the first part on analytical mechanics the key figures were Lagrange and Hamilton and in the second part on relativity the central person was Einstein. In this third part of the course, on electrodynamics, the physicist that played the most decisive part in developing the theory was James Clark Maxwell (1831 - 1879). Maxwell collected and modified the equations that are now known as Maxwell's equations, and in this way built the foundation for the classical theory of electromagnetism. On the basis of these equations it was shown convincingly that light is an electromagnetic wave phenomena, and that the many other electric and magnetic phenomena can be understood as different realizations of the underlying fundamental theory of electromagnetism.

The intention of this part of the lectures is to analyze the fundamentals of electrodynamics on the basis of Maxwell's equations. We begin by discussing the non-relativistic form of the equations and then show how to bring them into relativistic, covariant form. The use of electromagnetic potentials is important in this discussion and the following applications. The idea is to examine solutions of Maxwell's equations under different types of conditions. This include solutions of the free wave equations, solutions with stationary sources and solutions to the equations with time dependent charge and current distributions. The expansion in terms of multipoles is important in this discussion, and we put emphasis on the study of the radiation phenomena.

Chapter 9

Maxwell's equations

In this chapter we establish the fundamental electromagnetic equations. Historically they were developed by studying the different forms of electric and magnetic phenomena and first formulated as independent laws. These phenomena included the creation of electric fields by charges (Gauss' law) and by time dependent magnetic fields (Faraday's law of induction), and the creation of magnetic fields by electric currents (Ampère's law). We will first recall the form of each of these individual laws and next follow the important step of Maxwell by collecting these in a set of coupled equations for the electromagnetic phenomena. Maxwell's equations, which gain their most attractive form when written in relativistic, covariant form, is the starting point for the further discussion, where we examine different types of solutions to these equations.

9.1 Charge conservation

The electric and magnetic fields are produced by charges, either at rest or in motion. These charges satisfy the important law of *charge conservation*. This is a law that seems to be strictly satisfied in nature. The carriers of electric charge, at the microscopic level these are the elementary particles, may be created and may disappear, but in these processes the total charge is always preserved.

With Q as the total electric charge within a given volume, charge conservation may simply be expressed as

$$\frac{dQ}{dt} = 0 \tag{9.1}$$

However, this equation is correct only when there is no charge passing through the boundary surface of the selected volume. A more general expression is therefore

$$\frac{dQ}{dt} = -I \tag{9.2}$$

where I is the current through the boundary. This equation for the integrated charge and current can be reformulated in terms of the local charge density ρ and current density $\mathbf{j}$, defined by

$$Q(t) = \int_V \rho(\mathbf{r}, t) dV \,, \quad I(t) = \int_S \mathbf{j}(\mathbf{r}, t) \cdot d\mathbf{S} \tag{9.3}$$
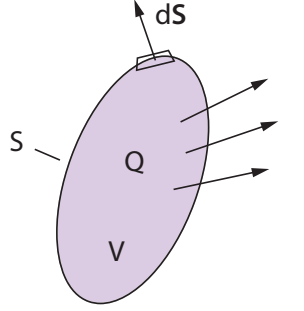
Figure 9.1: Charge conservation: The change in the charge Q within a volume V is caused by a current I through the boundary surface S.

 In these expressions V is the (arbitrarily) chosen volume with S as the corresponding boundary surface, dV is the three dimensional volume element and $d\mathbf{S}$ is the surface element, with direction orthogonal to the closed surface. Charge conservation then gets the form

$$\frac{d}{dt}\int_V \rho(\mathbf{r},t)\,dV + \int_S \mathbf{j}(\mathbf{r},t)\cdot d\mathbf{S} = 0 \tag{9.4}$$

The last term can be rewritten as a volume integral by use of Gauss' theorem, and this gives the following integral form of the equation

$$\int_V (\boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r},t) + \frac{\partial\rho}{\partial t}(\mathbf{r},t))\,dV = 0 \tag{9.5}$$

Since charge conservation in the form (9.5) is valid at any time t and for an arbitrarily small volume centered at any chosen point $\mathbf{r}$, it can be reformulated as the following local condition on the charge and current densities

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0 \tag{9.6}$$

This form for the condition of charge conservation, as a *continuity equation*, we will later apply repeatedly.

When expressed in terms of *densities* we have a view of charge as something continuously distributed in space. However, we know that at the microscopic level charge has a granular structure, since it is carried by small (pointlike) particles. We may take the view that the continuum description is based on a macroscopic approximation where the local charge is averaged over a volume that is small on a macroscopic scale but sufficiently large on the microscopic scale to smoothen the granular distribution of charge. In most cases this will be sufficient for our purpose. However, the description of charged particles can also be included by use of Dirac's delta function. For a system of pointlike particles the charge density and current density then take the form

$$\rho(\mathbf{r},t) \;=\; \sum_i e_i\,\delta(\mathbf{r}-\mathbf{r}_i(t))$$
$$\mathbf{j}(\mathbf{r},t) \;=\; \sum_i e_i\mathbf{v}_i(t)\,\delta(\mathbf{r}-\mathbf{r}_i(t)) \tag{9.7}$$

In these expressions the label i identifies a particle in the system, with charge e_i, time dependent position $\mathbf{r}_i(t)$ and velocity $\mathbf{v}_i(t)$.

In general, when the motion of the charged particles can be described by a (smooth) velocity field $\mathbf{v}(\mathbf{r}, t)$, we have the following relation between current and charge densities

$$\mathbf{j}(\mathbf{r}, t) = \mathbf{v}(\mathbf{r}, t)\rho(\mathbf{r}, t) \tag{9.8}$$

Note, however, for currents in a conductor there are two independent contributions, from the electrons and from the ions, and these move with different velocities, $\mathbf{v}_e$ and $\mathbf{v}_a$, so that the total current has the form

$$\mathbf{j}(\mathbf{r}, t) = \mathbf{v}_e(\mathbf{r}, t)\rho_e(\mathbf{r}, t) + \mathbf{v}_a(\mathbf{r}, t)\rho_a(\mathbf{r}, t) \tag{9.9}$$

For the usual situation, with total charge neutrality and with the ions sitting at rest, the expressions for total charge and current densities are

$$\rho(\mathbf{r}, t) = 0, \quad \mathbf{j}(\mathbf{r}, t) = \mathbf{v}_e(\mathbf{r}, t)\rho_e(\mathbf{r}, t) \tag{9.10}$$

9.2 Gauss' law

This law expresses how electric charge acts as a source for the electric field. Like all of the other electromagnetic equations it can be given both an integral and a differential form. In integral form it relates the flux of the electric field through any given closed surface S to the total charge Q within the surface,

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0} \tag{9.11}$$

In this equation ϵ_0 is the permittivity of vacuum, with the value

$$\epsilon_0 = 8.85 \cdot 10^{-12} \text{ C}^2/\text{Nm}^2 \tag{9.12}$$

Eq.(9.11) can be rewritten in terms of volume integrals as

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{E} \, dV = \int_V \frac{\rho}{\epsilon_0} dV \tag{9.13}$$

where on the left hand side Gauss' theorem has been used to rewrite the surface integral as a volume integral. Since this equality should be satisfied for any chosen volume V, the integrands should be equal, and that gives Gauss' law in differential form

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{9.14}$$

Gauss' law is the fundamental equation of *electrostatics*, where the basic problem is to determine the electric field from a given charge distribution, with specified boundary conditions satisfied by the field. In its simplest form the problem is to determine the field from an isolated

point charge, in which case Gauss' law in integral form can easily be solved under the assumption of rotational symmetry. Thus with the charge located at $\mathbf{r} = 0$ and with the electric field of the form $\mathbf{E} = E\mathbf{r}/r$, Gauss' law gives

$$4\pi r^2 E = \frac{q}{\epsilon_0} \tag{9.15}$$

with q as the charge. This gives the expression for the Coulomb field of a stationary point charge

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r^2} \frac{\mathbf{r}}{r} \tag{9.16}$$

Due to the fact that Gauss' law gives a linear differential equation for the electric field, the solution for a point charge can be extended to the full solution for a charge distribution. We shall return to the discussion of stationary solutions of Maxwell's equations later on.

9.3   Ampère's law

This law expresses how electric currents produce a magnetic field. The integral form is

$$\oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \frac{1}{c^2} \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{S} \tag{9.17}$$

and it shows that the line integral of the magnetic field around any closed curve C gets two contributions, one from the total electric current I passing through C and the other from the "displacement current", which is defined by the time derivative of the electric flux through a surface S with C as boundary. In this equation the vacuum permeability has been introduced. The value of this constant is given by

$$\frac{\mu_0}{4\pi} = 10^{-7} \text{N/A}^2 \tag{9.18}$$

To rephrase this in differential form, the left-hand-side is rewritten as a surface integral by use
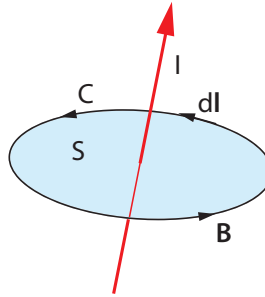

Figure 9.2: Ampère's law: The circulation of the magnetic field $\mathbf{B}$ around a closed curve C is determined by the current and the time derivative of the electric flux through the loop.

of Stoke's theorem and the current is expressed as a surface integral of the current density

$$\int_S (\boldsymbol{\nabla} \times \mathbf{B}) \cdot d\mathbf{S} = \int_S \mu_0 \mathbf{j} \cdot d\mathbf{S} + \frac{1}{c^2} \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{S} \tag{9.19}$$

Since this should be satisfied for an arbitrarily chosen surface S we conclude that there is a point wise equality which gives Ampere's law in differential form

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{E} = \mu_0 \mathbf{j} \tag{9.20}$$

Ampere's law shows that an electric current gives rise to a magnetic field that circulates the current, but it also shows that a changing electric field produces a magnetic field. The origin of the time derivative of the electric field in the equation may not be so obvious, but this term, which was introduced by Maxwell, is needed for the full set of equations to be consistent. An important consequence of this is that the equations have solutions in the form of propagating waves. Thus, the propagation of the waves may be seen as a consequence of the properties that time variations in $\mathbf{E}$ produce a magnetic field $\mathbf{B}$ (Ampere's law) and, at the next step, time variations in $\mathbf{B}$ recreate the electric field $\mathbf{E}$ (Faraday's law).

An interesting point to notice is that without the contribution from the time derivative of $\mathbf{E}$, the equation (9.20) would be in conflict with the conservation of electric charge. This is seen by taking the divergence of the equation, which would without the electric term give rise to the equation $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$ for the current. However, by comparison with the continuity equation for the charge current one sees that this is correct only if the charge density is not changing with time. The form of the electric term is in fact precisely what is needed to reproduce the continuity equation, provided there is a specific connection between the constants ϵ_0 and μ_0. To demonstrate this we take the divergence of Eq.(9.20) and apply Gauss' law,

$$\mu_0 \boldsymbol{\nabla} \cdot \mathbf{j} = -\frac{1}{c^2} \frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \mathbf{E} = -\frac{1}{c^2 \epsilon_0} \frac{\partial \rho}{\partial t} \tag{9.21}$$

This is identical to the continuity equation, provided

$$\epsilon_0 \mu_0 = \frac{1}{c^2} \tag{9.22}$$

which is indeed a correct relation. This shows that conservation of electric charge is not a condition that should be viewed as being independent of the electromagnetic equations. It can be *derived* from the laws of Gauss and Ampere, and can therefore be seen as a consistency requirement for these two electromagnetic equations.

9.4 Gauss' law for the magnetic field and Faraday's law of induction

An important property of the magnetic field is that there exists no isolated magnetic pole. This means that the total magnetic flux through any closed surface S vanishes,
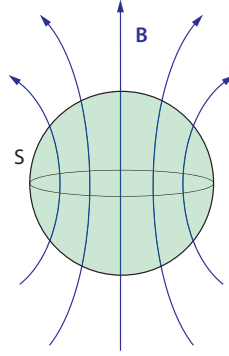
Figure 9.3: Gauss' law for magnetic fields: The total magnetic flux through any closed surface S is zero. Therefore magnetic flux lines have no end points.

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{9.23}$$

and in differential form this gives

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{9.24}$$

It has a similar form as Gauss' law for the electric field, but in this case there is no counterpart to the electric charge density. Expressed in terms of field lines, this means that magnetic field lines are always closed, whereas the electric field lines may be open, with end points on the electric charges.

Finally, Faraday's law of induction states that the integral of the electric field around a closed curve C is determined by the time derivative of the magnetic flux through a surface S with C as boundary

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{9.25}$$

There is an obvious similarity between this equation and Ampere's law, with the electric field interchanged with the magnetic field. By use of the same method as in the discussion of Ampere's law, we rewrite the equation in differential form,

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} = 0 \tag{9.26}$$

The main difference is that there is no counterpart to the electric current in this equation. We also note from this equation the electric field will in general not be a conservative field.

Faraday's law of induction describes the important phenomenon of induction of an electric field by a variable magnetic field. This effect is the basis for electromagnetic generators, where mechanical work is transformed into electric energy.

9.5 Maxwell's equations in vacuum

Maxwell's equations consists of the four coupled equations for the electromagnetic field that we have discussed separately,

$$a) \qquad \boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$b) \qquad \boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} = \mu_0 \mathbf{j}$$

$$c) \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

$$d) \qquad \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} = 0 \qquad\qquad (9.27)$$

These equation show how electromagnetic fields are produced by electric charges and currents. They should be supplemented with the continuity equation for charge

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0 \qquad\qquad (9.28)$$

which however, as we have seen, does not appear as an independent equation, but rather as a consistency condition for Maxwell's equations. They should also be supplemented with an equation which shows how the electromagnetic fields act back on the charges, here represented by the equation of motion for a charged point particle

$$\frac{d\mathbf{p}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad\qquad (9.29)$$

with $\mathbf{p}$ as the (mechanical) momentum of the particle. Together these equations form a closed set that describe the complete dynamics of a the physical system of electromagnetic fields and charged particles.

Maxwell's equations have several interesting symmetry properties. One of these is the symmetry under Lorentz transformations. This symmetry of the equations was found even before special relativity was formulated as a theory. Maxwell's equations define in fact a fully relativistic theory, developed before Einstein formulated the theory of relativity. This is seen most clearly when the equations are formulated in the language of four-vectors and tensors.

Another symmetry that is clearly seen in Maxwell's equations (9.27) is the symmetry under an interchange of electric and magnetic fields. In fact, without the source terms, *i.e.*, with vanishing charge and current densities, equations a) and b) are transformed into equations c) and d) (and *vice versa*) by the following change in the fields, $\mathbf{E} \to c\mathbf{B}$ and $c\mathbf{B} \to -\mathbf{E}$[1]. Even with sources there are symmetries between the electric and magnetic equations, and this can be exploited when solving problems in electrostatics and magnetostatics.

There have been speculations in the past whether the symmetry in Maxwell's equations between $\mathbf{E}$ and $\mathbf{B}$ should be extended to the general form of the equations, by including source terms also for the equations c) and d) The lack of sources for these equations in (9.27) can be

---

[1]This transformation, which is referred to as a *duality transformation*, is a special case of field rotations of the form $\mathbf{E} \to \cos\theta\ \mathbf{E} + \sin\theta\ c\mathbf{B}$ and $\mathbf{B} \to \cos\theta\ \mathbf{B} - \sin\theta\ \mathbf{E}/c$. Without the source terms ρ and $\mathbf{j}$, Maxwell's equations are invariant under general transformations of this form.

understood as reflecting the lack of *magnetic monopoles* in nature, with magnetic poles always seeming to appear in pairs of opposite sign. However, the existence of magnetic monopoles in the form of magnetic charges carried by new types of elementary particles can not be fully excluded. To take this possibility into account in Maxwell's equations would mean to include source terms also in equations c) and d), in the form of *magnetic* charge and current densities. In that case there would be two different types of sources for the electromagnetic field, electric charges and currents, and magnetic charges and currents. There have been performed several experimental searches for elementary particles with magnetic charge, but so far with negative results. We shall here proceed in the usual way, by assuming that no magnetic charges and currents exist, and therefore by keeping Maxwell's equations in the standard form (9.27).

9.5.1 Electromagnetic potentials

When the possibility of magnetic charges have been excluded and equations c) and d) therefore are homogeneous, the electric and magnetic fields can be expressed in terms of the electromagnetic potentials. These are referred to as the scalar potential ϕ and the vector potential $\mathbf{A}$, and they are defined by

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \qquad (9.30)$$

These expressions depend on the fact that Maxwell's equations c) and d) are source free and in fact make these two equations satisfied as identities, as one can readily check. The use of electromagnetic potentials therefore effectively reduce the set of field equations to a) and b) In addition to reducing the number of field equations, the use of potentials is helpful when solving the field equations.

Expressed in terms of the potentials Maxwell's equations get the form

$$a) \quad \boldsymbol{\nabla}^2\phi + \frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{A} = -\frac{\rho}{\epsilon_0}$$

$$b) \quad \boldsymbol{\nabla}^2\mathbf{A} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{\nabla}\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0\mathbf{j} \qquad (9.31)$$

These equations can be simplified further by imposing certain *gauge conditions* on the potentials.

Gauge transformations are transformations of the potentials that leave the electromagnetic fields unchanged. They have the form

$$\phi \;\rightarrow\; \phi' = \phi - \frac{\partial \chi}{\partial t}$$

$$\mathbf{A} \;\rightarrow\; \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi \qquad (9.32)$$

where $\chi = \chi(\mathbf{r}, t)$ is an arbitrary differentiable function of the space and time coordinates. It is straight forward to check that such a transformation does not change $\mathbf{E}$ or $\mathbf{B}$,

$$\mathbf{E} \;\rightarrow\; \mathbf{E}' = -\boldsymbol{\nabla}\phi' - \frac{\partial \mathbf{A}'}{\partial t} = \mathbf{E} + \boldsymbol{\nabla}\frac{\partial \chi}{\partial t} - \frac{\partial}{\partial t}\boldsymbol{\nabla}\chi = \mathbf{E}$$

$$\mathbf{B} \;\rightarrow\; \mathbf{B}' = \boldsymbol{\nabla}\times\mathbf{A}' = \mathbf{B} + \boldsymbol{\nabla}\times\boldsymbol{\nabla}\chi = \mathbf{B} \qquad (9.33)$$

The usual understanding is that gauge transformations do not correspond to any physical operation, since they leave $\mathbf{E}$ or $\mathbf{B}$ unchanged, but only reflects a certain freedom in the choice of electromagnetic potentials which represent a given electromagnetic field configuration. This freedom can be exploited by making specific gauge choices in the form of conditions that the potentials should satisfy. Two commonly used gauge conditions are the following

$$1)\quad \boldsymbol{\nabla} \cdot \mathbf{A} \;=\; 0 \qquad \text{Coulomb gauge} \tag{9.34}$$
$$2)\quad \partial_\mu A^\mu \;=\; 0 \qquad \text{Lorentz gauge} \tag{9.35}$$

where the abbreviation $\partial_\nu = \frac{\partial}{\partial x^\nu}$, introduced in Part II of the lecture notes, has been used. The Lorentz gauge condition has a covariant form when expressed in terms of the 4-vector potential with components A^μ. This potential is defined by $\underline{\mathbf{A}} = (\frac{1}{c}\phi, \mathbf{A})$ so that the scalar potential is (up to a factor $1/c$) the time component and the vector potential is the space component of the 4-potential. This gauge condition is often used when it is important to keep the relativistic form of the equations. The Coulomb gauge condition, on the other hand is often used when charged particles, such as electrons in atoms, move with non-relativistic velocities and the relativistic form of the equations is not so important. Also other types of gauge conditions can be imposed in order to simplify the electromagnetic equations, but it is important that the constraints they impose on the potentials should not correspond to any constraint on the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$.

9.5.2 Coulomb gauge

For the Coulomb and Lorentz gauge conditions one can show explicitly that these only affect the choice of potentials, but do not constrain the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ in any way. Let us consider how this can be demonstrated for the Coulomb gauge. Assume $\mathbf{A}$ is an arbitrary vector potential, which does *not* satisfy the Coulomb gauge condition. We will change this to a vector potential $\mathbf{A}'$ that does satisfy the condition $\boldsymbol{\nabla} \cdot \mathbf{A}' = 0$, and since the two potentials should be equivalent in the sense that they represent the same magnetic field $\mathbf{B}$, they should be related by a gauge transformation, $\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi$. The Coulomb gauge condition then implies that the function χ should satisfy the equation

$$\boldsymbol{\nabla}^2\chi = -\boldsymbol{\nabla} \cdot \mathbf{A} \tag{9.36}$$

We recognize this as having the same form as Gauss' law in the static case, where the electric field is determined by the electrostatic potential, $\mathbf{E} = -\boldsymbol{\nabla}\phi$. Expressed in terms of the potential the electrostatic equation is $\boldsymbol{\nabla}^2\phi = -\rho/\epsilon_0$, and this equation we know, for an arbitrary charge distribution, to have a well defined solution as the superposition of the Coulomb potentials from all the parts of the distribution. (We shall later discuss the electrostatic case explicitly.) The solution of (9.36) should then have the same form as the solution of the Coulomb problem for a charge distribution, with ρ/ϵ_0 replaced by $\boldsymbol{\nabla} \cdot \mathbf{A}$. This shows that, for any electromagnetic field configuration, one can make a gauge transformation of the vector potential to a form that satisfies the Coulomb gauge condition.

With the Coulomb gauge condition satisfied, $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, Maxwell's equations take the

form

$$a) \qquad \mathbf{\nabla}^2 \phi = -\frac{\rho}{\epsilon_0}$$

$$b) \qquad \mathbf{\nabla}^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0 \mathbf{j}_\perp \qquad (9.37)$$

where in the second equation we have introduced the *transverse* current density, defined by

$$\mathbf{j}_\perp = \mathbf{j} - \epsilon_0 \frac{\partial}{\partial t} \mathbf{\nabla} \phi \qquad (9.38)$$

It is called "transverse" since it is divergence free, $\mathbf{\nabla} \cdot \mathbf{j}_\perp = 0$. This follows by applying equation 1. and the continuity equation for charge,

$$
\begin{aligned}
\mathbf{\nabla} \cdot \mathbf{j}_\perp &= \mathbf{\nabla} \cdot \mathbf{j} - \epsilon_0 \frac{\partial}{\partial t} \mathbf{\nabla}^2 \phi \\
&= \mathbf{\nabla} \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} \\
&= 0
\end{aligned}
\qquad (9.39)
$$

Eq. (9.38) can therefore be re-interpreted as a standard (*Helmholtz*) decomposition of the vector field, in a divergence-free (transverse) and a curl-free (longitudinal) component,

$$\mathbf{j} = \mathbf{j}_\perp + \mathbf{j}_\parallel \qquad (9.40)$$

and Eq.(9.37b). then shows that only the divergence-free component contributes to the equation.

One should also note that Eq. 1. is non-dynamical in the sense that it involves no time derivative. It can thus be solved like the electrostatic equation, to give the potential ϕ expressed in terms of the charge distribution ρ.. That is the case even if ϕ is time dependent. This means that dynamical evolution of the electromagnetic field, in the Coulomb gauge, is described by the vector potential $\mathbf{A}$ alone, while the scalar potential ϕ is uniquely defined as the Coulomb potential of the charge distribution at any given time.

9.6   Maxwell's equations in covariant form

The covariant form of Maxwell's equations is based on the introduction of the *electromagnetic field tensor*. It is an antisymmetric, relativistic tensor $F^{\mu\nu}$, constructed by the $\mathbf{E}$ and $\mathbf{B}$ fields in the following way

$$
\begin{aligned}
F^{0k} &= \frac{1}{c}E_k, \qquad k = 1,2,3 \\
F^{kl} &= \epsilon_{klm}B_m, \quad k,l = 1,2,3
\end{aligned}
\qquad (9.41)
$$

with summation over the repeated index m, and with ϵ_{klm} representing the three dimensional Levi-Civita tensor. This tensor is antisymmetric in any pair of indices and is consequently different from 0 only when all indices klm are different. This set of indices then define a

permutation of the set 123, and with the definition $\epsilon_{123} = 1$ the value of ϵ_{klm} for a permutations of 123 is determined by the antisymmetry property of the tensor.

Written as a 4×4 matrix the field tensor takes the form

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & B_3 & -B_2 \\ -E_2/c & -B_3 & 0 & B_1 \\ -E_3/c & B_2 & -B_1 & 0 \end{pmatrix} \tag{9.42}$$

The reason for the electric and magnetic fields to be arranged into a common object $F^{\mu\nu}$ is that the two fields are mixed under Lorentz transformations. Such a mixing is implicit both in Maxwell's equations and in the equation of motion for a charged particle (9.29). In the latter case this is obvious since a reference frame can be chosen where the particle is instantaneously at rest. In such a reference frame there is no contribution to the force from the magnetic field, and therefore the effect of the electric field in this frame must be equivalent to both the electric and magnetic fields in another frame where the particle is moving. It is an interesting fact that the mixing of the **E** and **B** fields is correctly expressed by combining them in a linear way into the antisymmetric, rank two tensor $F^{\mu\nu}$.

Maxwell's equations, in the form (9.27), gets a simple compact form when expressed in terms of the electromagnetic field tensor. The two first equations can be rewritten as

$$a) \quad \frac{\partial}{\partial x_k} F^{0k} = \frac{1}{c\epsilon_0} \rho$$

$$b) \quad \frac{\partial}{\partial x^l} F^{kl} + \frac{\partial}{\partial x^0} F^{k0} = \mu_0 j^k \tag{9.43}$$

and these two equations can now be merged into a single covariant equation

$$\partial_\nu F^{\mu\nu} = \mu_0 j^\mu \tag{9.44}$$

In the equation we have also introduced the 4-vector current density j^μ, which is composed by the charge and current densities in the following way

$$(j^\mu) = (c\rho, \mathbf{j}) \tag{9.45}$$

so that the original 3-vector **j** is extended to a 4-vector $\underline{\mathbf{j}}$ by taking $c\rho$ as the time component of the current.

The continuity equation for charge can also be written in covariant form when the four-vector current is introduced. The covariant form is

$$\partial_\mu j^\mu = 0 \tag{9.46}$$

as we can readily verify by separating the time derivative from the space derivative and using the fact that the time component of the 4-current is the charge density (up to a factor c). We have already noticed that charge conservation is needed if Maxwell's equations should be consistent. This is seen very clearly from the covariant equation (9.44), where the continuity equation of the current follows from the antisymmetry of the electromagnetic tensor.

We continue with bringing the source free equations c) and d) into covariant form. We first note that the two equations can be expressed in terms of the electromagnetic field tensor as

$$c) \qquad \epsilon_{klm} \frac{\partial}{\partial x^k} F^{lm} = 0 \tag{9.47}$$

$$d) \qquad \epsilon_{klm} \frac{\partial}{\partial x^l} F^{0m} + \frac{1}{2}\epsilon_{klm}\frac{\partial}{\partial x^0}F^{lm} = 0 \tag{9.48}$$

where we in these equations have used $E_k = cF^{0k}$ and $B_k = \frac{1}{2}\epsilon_{klm}F^{lm}$. We introduce now the four dimensional Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$ which is fully antisymmetric in interchange of the indices, and which further satisfies $\epsilon_{0klm} = \epsilon_{klm}$. As one can readily check the two equations now can be merged into a single, compact equation

$$\epsilon_{\mu\nu\rho\sigma}\,\partial^\lambda F^{\rho\sigma} = 0 \tag{9.49}$$

To simplify the covariant form of Maxwell's equations even further we introduce the *dual field tensor*

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{9.50}$$

If we check what this definition means for the components of the dual tensor, we find that it is derived from the field tensor by a *duality transformation*

$$F^{\mu\nu} \to \tilde{F}^{\mu\nu}: \quad \frac{1}{c}\mathbf{E} \to -\mathbf{B}, \quad \mathbf{B} \to \frac{1}{c}\mathbf{E} \tag{9.51}$$

Written as a 4×4 matrix the dual field tensor is therefore

$$(\tilde{F}^{\mu\nu}) = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3/c & -E_2/c \\ B_2 & -E_3/c & 0 & E_1/c \\ B_3 & E_2/c & -E_1/c & 0 \end{pmatrix} \tag{9.52}$$

The original four Maxwell equation can now be written as two compact, covariant equations

$$\begin{aligned} \partial_\nu F^{\mu\nu} &= \mu_0 j^\mu \\ \partial_\nu \tilde{F}^{\mu\nu} &= 0 \end{aligned} \tag{9.53}$$

In this form the symmetry of the equations under duality transformation, which interchanges $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$, is seen very clearly and also the difference, that the "magnetic current" of the second equation is missing.

9.7   The electromagnetic 4-potential

The lack of a magnetic current in Maxwell's equations makes the symmetry between the electric and magnetic fields not fully complete. However, for the same reason the field tensor can be expressed in terms of the *electromagnetic 4-potential* A^μ in the following way

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{9.54}$$

As previously discussed, the 4-potential is composed of the non-relativistic potentials in such a way that the time component is $A^0 = \phi/c$ with ϕ as the original scalar potential and the space part of A^μ is identical to the vector potential **A**. When the field tensor is expressed in terms of the 4-potential the second of the two covariant Maxwell equations is satisfied as an identity, as one now can easily verify by expressing $\tilde{F}^{\mu\nu}$ in terms of A^μ. This means that Maxwell's equations are reduced to one 4-vector equation, which is

$$\partial_\nu \partial^\nu A^\mu - \partial^\mu \partial_\nu A^\nu = -\mu_0 j^\mu \tag{9.55}$$

As a last step to simplify the equation we again make use of the freedom to change the potential by a gauge transformation. In covariant form such a transformation is

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi \tag{9.56}$$

where χ is an unspecified differentiable function of the space time coordinates. In the covariant formulation it is straight forward to check that such a transformation of the 4-potential will not change the field tensor. The freedom to change the potential in this way can be used to bring it to the form where the covariant Lorentz gauge condition is satisfied,

$$\partial_\mu A^\mu = 0 \tag{9.57}$$

When this condition is satisfied we find Maxwell's equation reduced to its simplest form

$$\partial_\nu \partial^\nu A^\mu = -\mu_0 j^\mu \tag{9.58}$$

One sometimes use the symbol $\Box^2$ (or $\Box$) for the differential operator,

$$\Box^2 \equiv \partial_\nu \partial^\nu = \boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{9.59}$$

It is called the *d'Alembertian* operator and is an extension of the three dimensional Laplacian $\boldsymbol{\nabla}^2$ to four dimensions.

9.8   Lorentz transformations of the electromagnetic field

When the electric and magnetic fields are collected in the electromagnetic field *tensor*, this means that the correct transformation of **E** and **B** under Lorentz transformations have been implicitly assumed. This is of course not simply postulated, it is based on the assumption that Maxwell's equations (as well as the equation of motion of charged particles) have the same form in all inertial reference frames, and this is in turn a well established fact based on experimental tests. We will here take the tensor properties of $F^{\mu\nu}$ as the starting point, and show from this how Lorentz transformations mix the electric and magnetic components.

We consider first field transformations under a simple boost in the x direction. It is given by the Lorentz transformation matrix

$$L = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9.60}$$

which means that the only non-vanishing matrix elements are

$$
\begin{aligned}
L^0{}_0 &= L^1{}_1 = \gamma \\
L^0{}_1 &= L^1{}_0 = -\beta\gamma \\
L^2{}_2 &= L^3{}_3 = 1
\end{aligned}
\tag{9.61}
$$

The tensor properties of the electromagnetic field tensor implies that the transformed field is related to the original field by the equation

$$
F'^{\mu\nu} = L^\mu{}_\rho L^\mu{}_\sigma F^{\rho\sigma}
\tag{9.62}
$$

We extract from this formula the transformation equations for the components of the electric and magnetic fields, in the case where the matrix elements of the Lorentz transformation are given by (9.61). The x component of the electric field is,

$$
\begin{aligned}
E'_1 &= cF'^{01} \\
&= cL^0{}_0 L^1{}_1 F^{01} + cL^0{}_1 L^1{}_0 F^{10} \\
&= c(L^0{}_0 L^1{}_1 - L^0{}_1 L^1{}_0) F^{01} \\
&= \gamma^2(1 - \beta^2) E_1 \\
&= E_1
\end{aligned}
\tag{9.63}
$$

which shows that the component in the direction of the boost is unchanged. In the orthogonal directions the components of the transformed field are

$$
\begin{aligned}
E'_2 &= cF'^{02} \\
&= cL^0{}_0 L^2{}_2 F^{02} + cL^0{}_1 L^2{}_2 F^{12} \\
&= \gamma E_2 - \gamma\beta c B_3 \\
&= \gamma(E_2 - vB_3)
\end{aligned}
\tag{9.64}
$$

and

$$
\begin{aligned}
E'_3 &= cF'^{03} \\
&= cL^0{}_0 L^3{}_3 F^{03} + cL^0{}_1 L^3{}_3 F^{13} \\
&= \gamma E_3 + \gamma\beta c B_2 \\
&= \gamma(E_3 + vB_2)
\end{aligned}
\tag{9.65}
$$

These expressions can be written in a form which is independent of the choice of coordinate axes by introducing the *parallel* and *transverse* components of the electric field

$$
E_\parallel = E_1 , \quad \mathbf{E}_\perp = E_2\,\mathbf{j} + E_3\,\mathbf{k}
\tag{9.66}
$$

The transformation formulas then are

$$
E'_\parallel = E_\parallel , \quad \mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B})
\tag{9.67}
$$

which shows that the component of the field in the direction of the boost velocity $\mathbf{v}$ is unchanged, while the transverse components (orthogonal to $\mathbf{v}$) are mixtures of the of the original transverse components of the electric and magnetic fields.

The transformeation of the magnetic components of the field tensor can be found in the same way, and the result is

$$B'_\parallel = B_\parallel \,, \quad \mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \frac{\mathbf{v}}{c^2} \times \mathbf{E}) \tag{9.68}$$

The transformation formulas for $\mathbf{E}$ and $\mathbf{B}$ have almost the same form, and they are related by the duality transformation already discussed,

$$\frac{1}{c}\mathbf{E} \to -\mathbf{B}\,, \quad \mathbf{B} \to \frac{1}{c}\mathbf{E}$$

The symmetry under this transformation, which gives a mapping between the two field tensors $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$, can in fact be used to *derive* the transformation formula for the $\mathbf{B}$ field directly from the transformation formula for the $\mathbf{E}$ field.

9.8.1 Example

As a simple example we assume that in the reference frame S there is no electric field, and a constant magnetic field $\mathbf{B} = B_0 \mathbf{k}$ directed along the z axis. The moving frame S' has a velocity $\mathbf{v}$ in the x direction. We split the fields in parallel and transverse components,

$$E_\parallel = \mathbf{E}_\perp = 0\,, \quad B_\parallel = 0\,, \, \mathbf{B}_\perp = B_0 \mathbf{k} \tag{9.69}$$

For the parallel components of the transformed fields we find

$$E'_\parallel = E_\parallel = 0\,, \quad B'_\parallel = B_\parallel = 0 \tag{9.70}$$

and for the transverse components

$$\begin{aligned}
\mathbf{E}'_\perp &= \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}) = \gamma v B_0 \, \mathbf{i} \times \mathbf{k} = -\gamma v B_0 \, \mathbf{j} \\
\mathbf{B}'_\perp &= \gamma(\mathbf{B}_\perp - \frac{\mathbf{v}}{c^2} \times \mathbf{E}) = \gamma \mathbf{B}_\perp = -\gamma B_0 \mathbf{k}
\end{aligned} \tag{9.71}$$

Collecting these terms we find that the fields in the reference frame S are

$$\mathbf{E}' = -\gamma v B_0 \, \mathbf{j}\,, \quad \mathbf{B}' = \gamma B_0 \, \mathbf{k} \tag{9.72}$$

Also in this reference frame the magnetic field points in the z direction, but it is stronger than in S due to the factor γ which is larger than 1. In addition there is an electric field in the direction orthogonal to both the velocity of the transformation and to the magnetic field.

9.8.2 Lorentz invariants

From the electromagnetic tensor $F^{\mu\nu}$ we can construct two Lorentz invariant quantities. These are combinations of the electric and magnetic field strengths that take the same value in all inertial frames. For a general tensor $T^{\mu\nu}$ the trace $T^\mu{}_\mu$ is such an invariant, but in the present

case the trace vanishes since $F^{\mu\nu}$ is antisymmetric. This means that there is no invariant that is *linear* in the components of $\mathbf{E}$ and $\mathbf{B}$. However, there are two quadratic expressions that are Lorentz invariants. These are

$$
\begin{aligned}
I_1 &= \frac{1}{2} F^{\mu\nu} F_{\mu\nu} = \mathbf{B}^2 - \frac{1}{c^2} \mathbf{E}^2 \\
I_2 &= \frac{1}{4} F^{\mu\nu} \tilde{F}_{\mu\nu} = \frac{1}{c} \mathbf{E} \cdot \mathbf{B}
\end{aligned}
\tag{9.73}
$$

It is easy to check that for the example just discussed we get the same expression for the two invariants, whether we evaluate them in reference frame S or S',

$$
I_1 = B_0^2, \quad I_2 = 0
\tag{9.74}
$$

We note in particular that even if $\mathbf{E}$ and $\mathbf{B}$ get mixed by the Lorentz transformation, the fact that $\mathbf{E}$ dominates $\mathbf{B}$ ($\mathbf{E}^2 > c^2 \mathbf{B}^2$) or $\mathbf{B}$ dominates $\mathbf{E}$ ($\mathbf{E}^2 < c^2 \mathbf{B}^2$) can be stated without reference to any particular inertial frame.

9.9   Example: The field from a linear electric current

In this Example we consider the situation where a constant current is running in a straight conducting wire, as shown in the Fig. 9.4. In a reference frame S which is stationary with respect to the conductor, we assume the current to take the value I and the conductor to be electrically neutral. The magnetic field circulates the current and outside the conductor the field strength $\mathbf{B}$ is determined by Ampère's law as

$$
\oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 I
\tag{9.75}
$$

Assuming the conductor to be rotationally symmetric this determines the field to be

$$
\mathbf{B} = \frac{\mu_0 I}{2\pi r} \mathbf{e}_\phi
\tag{9.76}
$$

with r as the distance from the centre of the conductor and $\mathbf{e}_\phi$ as unit vectors circulating the current. With charge neutrality the electric field orthogonal to the current vanishes, but there is an electric field inside the conductor that drives the current. It is given by

$$
\mathbf{j}_e = \sigma \mathbf{E}_0
\tag{9.77}
$$

with σ as the *conductivity*. We assume $\mathbf{E}$ to have a constant value inside the conductor, with the same value also outside, close to the conductor.

The field strength $\mathbf{B}$ refers to the reference frame S where the ions of the conducting material are at rest. In this frame the electrons are moving with an average velocity v_e. We have

$$
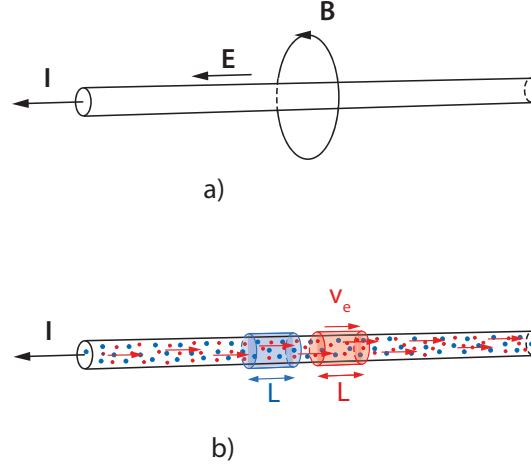I = A j_e = A v_e \rho_e
\tag{9.78}
$$

Figure 9.4: Electromagnetic fields of a linear current. In figure a) the directions of the electric and magnetic fields are indicated as seen in the rest frame S of the conductor. In figure b) are indicated two volumes of equal length in S, where one of them is stationary with respect to the ions (blue dots) and the other moves with the electrons (red dots). In S the charges neutralize each other, while in the rest frame S' of the electrons that is not so due to the length contraction effect. The non-vanishing charge density of the conductor in S' explains the presence of a radially directed $\mathbf{E}$ field in this frame, which follows from the Lorentz transformation of the fields from S to S'.

where ρ_e and j_e are the average charge and current densities of the electrons and A is the cross section area of the conductor. (For simplicity we assume the current density to be constant over the cross section.) We will now introduce a second inertial reference frame S' that moves with the average velocity of the electrons. In this frame the ions move with the velocity $-v_e$, while the electrons are (on average) at rest.

The fields in the reference frame S' are given by the field transformation formulas. To use these we need first to split the fields in a parallel component (along the conductor) and a normal component (orthogonal to the conductor). For the fields in S these components are

$$E_\parallel = E_0\,, \quad \mathbf{E}_\perp = 0\,, \quad B_\parallel = 0\,, \quad \mathbf{B}_\perp = \frac{\mu_0 I}{2\pi r}\mathbf{e}_\phi \tag{9.79}$$

The transformation formulas, with $-\mathbf{v}_e$ as the velocity of reference frame S' along the conducting wire, give

$$E'_\parallel = E_\parallel = E_0\,, \qquad \mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp - \mathbf{v}_e \times \mathbf{B}) = -\gamma v_e \frac{\mu_0 I}{2\pi r}\mathbf{e}_r$$

$$B'_\parallel = B_\parallel = 0\,, \qquad \mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp + \frac{\mathbf{v}_e}{c^2} \times \mathbf{E}) = \gamma\frac{\mu_0 I}{2\pi r}\mathbf{e}_\phi \tag{9.80}$$

The magnetic field is also in this reference frame circulating the current, but now it is stronger, enhanced by the factor γ. The electric field we note to have, in addition to the parallel component E_0, a normal component that is radially directed, out from the conductor. This normal component is somewhat unexpected, since it indicates that the conducting wire in reference frame S' is not charge neutral. Thus, a charge density is needed along the wire in order to

create a radially directed electric field. We will check whether this result is consistent with Maxwell's equations by evaluating the charge and current densities in the transformed reference frame.

In reference frame S the charge and current densities of the electrons are ρ_e and $j_e = v_e\rho_e$, while the charge and current densities of the ions are $\rho_i = -\rho_e$ (due to charge neutrality) and $j_i = 0$. To find the corresponding quantities in reference frame S' we use the fact that charge and current densities together form a 4-vector $\mathbf{j} = (c\rho, \mathbf{j})$. The standard transformation formula for 4-vectors to give the charge and current densities in S',

$$
\begin{aligned}
\rho'_e &= \gamma(\rho_e - \frac{v_e}{c^2}j_e) = \gamma(1 - \frac{v_e^2}{c^2})\rho_e = \frac{1}{\gamma}\rho_e \\
j'_e &= \gamma(j_e - v_e\rho_e) = 0 \\
\rho'_i &= \gamma(\rho_i - \frac{v_e}{c^2}j_i) = -\gamma\rho_e \\
j'_i &= \gamma(j_i - v_i\rho_i) = \gamma v_e\rho_e
\end{aligned}
\tag{9.81}
$$

This gives for the total charge and current densities in S',

$$
\begin{aligned}
\rho' &= \rho'_e + \rho'_i = \frac{1}{\gamma}\rho_e - \gamma\rho_e = \frac{1}{\gamma}\rho_e - \gamma\rho_e = -\gamma\beta^2\rho_e \\
j' &= j'_e + j'_i = \gamma v_e\rho_e
\end{aligned}
\tag{9.82}
$$

The charge density in S' is indeed different from zero and the current density is modified by a factor γ. We check now that the expressions given above for the transformed the charge and current densities are consistent with what we have found for the transformed fields. We note that the enhancement of current in S', by the factor γ, is consistent with the corresponding enhancement of the transformed magnetic field. To the check the consistency of the transformation of the charge density and the electric field we consider Gauss' law in S'. We denote by $\Delta Q'$ the charge in a piece of the conducting wire of length ΔL, so that $\Delta Q' = \rho'\,\Delta L A$. According to Gauss' law this charge should create a radially directed electric field given by

$$
2\pi r\,\Delta L\,E'_r = \frac{\Delta Q'}{\epsilon_0}
\tag{9.83}
$$

which gives

$$
E'_r = \frac{\rho' A}{2\pi r\epsilon_0} = -\gamma\beta^2\frac{\rho_e A}{2\pi r\epsilon_0} = -\gamma\beta\frac{j_e A}{2\pi r c\epsilon_0} = -\gamma v_e\frac{\mu_0 I}{2\pi r}
\tag{9.84}
$$

where in the last step we have used the relation $\epsilon_0\mu_0 = 1/c^2$. The expression for the radial component of the electric field found in this way is indeed consistent with the result found by applying the transformation formula for the electromagnetic field.

Although it may initially seem strange that the conducting wire is charge neutral in one reference frame, but not in the other, it is a clear consequence of the description of charge and current as components of the same 4-vector current. Lorentz transformations will mix the time and space components of of the 4-current.

As a final point, we shall examine how the results we have obtained can also be understood as a consequence of length contraction. Let us then consider an imaginary container, which

includes a part of the conducting wire and moves with the speed v_e of the electrons along the wire. The length measured in S is L and the number of electrons within the container is N. The total electron charge within the container is therefore

$$Q_e = Ne = AL\rho_e \tag{9.85}$$

with e as the electron charge. Let us next consider another container of the same length L in S, but which is at rest. At a given instant the two imagined containers will overlap, and due to charge neutrality the number of electrons within the first container is the same as the number of ions within the second container (if we assume each ion contributes with one conduction electron). For the total ion charge within the second container we therefore have

$$Q_i = -Q_e = -Ne = -AL\rho_e \tag{9.86}$$

We now regard the situation as it appears in reference frame S'. The lengths of the containers appear with a different lengths from those in S, due to the length contraction effect, but the number of particles in each container is unchanged. If we first consider the length of the electron container, we find that it is $L'_e = \gamma L$. The container is longer in S' than in S, since S' is the *rest frame* of the electrons. As far as the other container is concerned the situation is opposite, since S is the rest frame of the ions. Therefore the length of this in S' is $L'_i = L/\gamma$. Thus the two containers still contain the same amount of charge as in S, but since the length of the containers are different, the charge *densities* have changed. We have

$$\rho'_e = \frac{Q_e}{L'_e A} = \frac{1}{\gamma}\frac{Q_e}{LA} = \frac{1}{\gamma}\rho_e \tag{9.87}$$

and

$$\rho'_i = \frac{Q_i}{L'_e A} = -\gamma\frac{Q_e}{LA} = -\gamma\rho_e \tag{9.88}$$

The total charge density in S' is therefore

$$\rho' = \rho'_e + \rho'_i = \frac{1}{\gamma}\rho_e - \gamma\rho_e = -\gamma\beta^2\rho_e \tag{9.89}$$

This gives a result for the charge density in reference frame S' that agrees is with what we have already found by use of the Lorentz transformation formulas.

As a final comment we should point out that in a normal conducting wire the effect we have discussed is extremely small. It is a relativistic effect determined by the ratio of the drift velocity (average velocity) of the electrons to the speed of light. A typical value of the drift velocity may be $v_e = 3 \cdot 10^{-4} m/s$, corresponding to $\beta = 10^{-12}$ and $\gamma - 1 = 0.5 \cdot 10^{-24}$, indeed a very small number.

Chapter 10

Dynamics of the electromagnetic field

Maxwell's equations show that the electromagnetic field has it own dynamics. It can propagate as waves through empty space and it can carry energy and momentum. This was realized by Maxwell, and since the propagation velocity is identical to the speed of light, that convinced him that light is such an electromagnetic wave phenomenon. In this chapter we first discuss the wave solutions of Maxwell's equations with particular focus on *polarization* of electromagnetic waves and next examine how Maxwell's equation determine the energy and momentum densities of the electromagnetic field.

10.1 Electromagnetic waves

In the source free case, with $j^\mu = 0$, Maxwell's equations have solutions in the form of electromagnetic waves. We study this situation in the Lorentz gauge, with $\partial_\mu A^\mu = 0$. The field equation then is

$$\partial_\nu \partial^\nu A^\mu = 0 \tag{10.1}$$

where the differential operator

$$\partial_\nu \partial^\nu = \boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{10.2}$$

has the form of a *wave operator* in three dimensional space. The wave equation (10.1), which is a *linear* differential equation, has a complete set of *normal modes* as solutions. In the open space, without any physical boundaries, a natural choice for such a set of normal modes is the set of *monochromatic* plane waves, of the form

$$A^\mu(x) = A^\mu(0)\, e^{ik_\mu x^\mu} \tag{10.3}$$

with $A^\mu(0)$ as the amplitude of field component μ. The four-vector $\underline{\mathbf{k}}$, with components k_μ, has a time component that is proportional to the frequency of the wave and a space component that is identified as the wave vector,

$$\underline{\mathbf{k}} = (\frac{\omega}{c}, \mathbf{k}) \tag{10.4}$$

The plane wave solution (10.3) is a *complex* solution of Maxwell's equation. Such a complex form is often convenient to use, since it makes the expressions more compact. However, one should keep in mind that the *physical* field is real, and should be identified with the real (or imaginary) part of the solution.

To check that plane waves of the form (10.3) are solutions of the wave equation (10.1) is straight forward. We find

$$\partial_\nu \partial^\nu A^\mu = -k_\nu k^\nu A^\mu \tag{10.5}$$

which shows that the function (10.3) is a solution provided

$$k_\nu k^\nu = 0 \tag{10.6}$$

This means that the 4-vector $\underline{\mathbf{k}}$ is a *light like* vector (also called a *null vector*), and this gives rise to the well-known linear relation between frequency and wave number for electromagnetic waves,

$$\omega = c|\mathbf{k}| \tag{10.7}$$

The Lorentz gauge condition further demands the two 4-vectors k_μ and A^μ to be orthogonal in the relativistic sense

$$k_\mu A^\mu = 0 \tag{10.8}$$

One should note that the Lorentz gauge condition does not fix uniquely the 4-potential for a given electromagnetic field. This is readily seen by assuming A^μ to be a general potential which satisfies no particular gauge condition. By a gauge transformation

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi \tag{10.9}$$

it can be brought to a form which does satisfy the Lorentz gauge condition $\partial_\mu A'^\mu = 0$, provided χ satisfies the equation

$$\partial_\mu \partial^\mu \chi = -\partial_\mu A^\mu \tag{10.10}$$

χ is not uniquely determined by the equation, since to any particular solution of this differential equation one can add a general solution of the homogeneous (wave) equation $\partial_\mu \partial^\mu \chi = 0$. In the present case, with $j^\mu = 0$, one can use this freedom to set the time component of the potential to zero, $A^0 = 0$, and the remaining vector part will then satisfy the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$. When written in non-covariant form the wave equation for $\mathbf{A}$ is

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(\mathbf{r}, t) = 0 \tag{10.11}$$

This shows that the three components of the vector potential satisfy three identical, uncoupled wave equations, but the three components are coupled by the Coulomb gauge condition.

With $A^0 = 0$ the non-covariant form of the plane wave solution is

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{10.12}$$

where the amplitude $\mathbf{A}_0$ is a complex vector that should be a orthogonal to $\mathbf{k}$,

$$\mathbf{k} \cdot \mathbf{A}_0 = 0 \tag{10.13}$$

in order to satisfy the Coulomb gauge condition.

The general solution to the electromagnetic wave equation (10.11) can now be written as a superposition of plane waves

$$\mathbf{A}(\mathbf{r}, t) = \int d^3 k \, \mathbf{A}(\mathbf{k}) \, e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad \mathbf{k} \cdot \mathbf{A}(\mathbf{k}) = 0 \tag{10.14}$$

where each Fourier component $\mathbf{A}(\mathbf{k})$ has to satisfy the transversality condition.

We have in this discussion of electromagnetic waves assumed that they propagate in the open infinite space. The plane waves then define a complete set of normal modes of the field. If the situation instead correspond to wave propagation inside some given boundaries, for example inside a *wave guide* the normal modes are not the infinite plane waves but solutions that are adjusted to the given boundary conditions. To find the normal modes of the electromagnetic field is then more demanding, but the general solution is again a (general) superposition of these modes.

10.2 Polarization

The plane wave solution (10.12) for the electromagnetic potential gives related expressions for the electric and magnetic fields. For the electric field we find

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{A}}{\partial t} = i\omega \mathbf{A}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} = i\omega \mathbf{A}(\mathbf{r}, t) \tag{10.15}$$

and for the magnetic field

$$\mathbf{B}(\mathbf{r}, t) = \boldsymbol{\nabla} \times \mathbf{A} = i\mathbf{k} \times \mathbf{A}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} = i\mathbf{k} \times \mathbf{A}(\mathbf{r}, t) \tag{10.16}$$

We note that both these fields satisfy the transversality condition

$$\mathbf{k} \cdot \mathbf{E} = \mathbf{k} \cdot \mathbf{B} = 0 \tag{10.17}$$

and are related by

$$\mathbf{B} = \frac{1}{c}\mathbf{n} \times \mathbf{E}, \quad \mathbf{E} = -c\mathbf{n} \times \mathbf{B} \tag{10.18}$$

with $\mathbf{n} = \mathbf{k}/k$ as the unit vector in the direction of propagation of the plane wave. Thus the triplet $(\mathbf{k}, \mathbf{E}, \mathbf{B})$ form a right handed, orthogonal set of vectors. We further note that for a monochromatic plane wave the two electromagnetic Lorentz invariants previously discussed both vanish

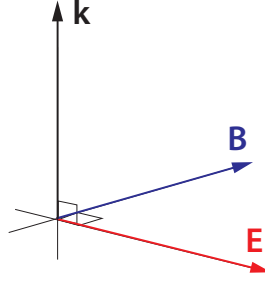$$\mathbf{E}^2 - c^2 \mathbf{B}^2 = 0, \quad \mathbf{E} \cdot \mathbf{B} = 0 \tag{10.19}$$

Figure 10.1: Field vectors of a plane wave. The wave number **k** together with the two field vectors **E** and **B** form a righthanded set of orthogonal vectors.

The monochromatic plane wave is, as we see, specified on one hand by the wave number **k**, which gives the direction of propagation and the frequency of the wave, and on the other hand by the orientation of the electric field vector **E** in the plane orthogonal to **k**. The degree of freedom specified by the *direction* of **E** we identify as the *polarization* of the electromagnetic wave. We shall take a closer look at the description of different types of polarization. As follows from Eq.(10.18) it is sufficient to focus on the electric field **E**, since the magnetic field **B** is uniquely determined by **E**.

Written in complex form the electric field strength of the plane wave has the form

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{10.20}$$

where the amplitude $\mathbf{E}_0$ is in general a complex vector. We consider the real part of the field (10.20) as the physical field. When decomposed on two arbitrarily chosen orthogonal real unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in the plane orthogonal to **k**, the real field gets the general form

$$\mathbf{E}(\mathbf{r}, t) = E_{10} \, \mathbf{e}_1 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi_1) + E_{20} \, \mathbf{e}_2 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi_2) \tag{10.21}$$

where the two amplitudes E_{10} and E_{20} as well as the two phases ϕ_1 and ϕ_2 may be different. When the two components are in phase, $\phi_1 = \phi_2$ the wave is *linearly polarized*, but in the general case the wave is *elliptically polarized*, with *circular polarization* as a special case. We will discuss these different types of polarization in some detail, but let us first reintroduce the complex notation in the following way.

We write the complex amplitude of the electric field as

$$\mathbf{E}_0 = E_0 \, \boldsymbol{\epsilon}_1 \tag{10.22}$$

where E_0 is now a real (positive) amplitude and $\boldsymbol{\epsilon}_1$ is a complex unit vector. The real part has the form (10.21) if we make the following identifications

$$
\begin{aligned}
E_0 &= \sqrt{E_{10}^2 + E_{20}^2} \\
\boldsymbol{\epsilon}_1 &= \cos\chi \, e^{i\phi_1} \, \mathbf{e}_1 + \sin\chi \, e^{i\phi_2} \, \mathbf{e}_2
\end{aligned}
\tag{10.23}
$$

with

$$\cos\chi = \frac{E_{10}}{E_0}, \quad \sin\chi = \frac{E_{20}}{E_0} \tag{10.24}$$

In complex notation the general monochromatic plane wave then has the form

$$\mathbf{E}(\mathbf{r}, t) = E_0 \, e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \, \boldsymbol{\epsilon}_1 \tag{10.25}$$

This expression is equivalent to (10.21) in the sense that the latter is the real part of the former, but the complex field (10.25) is usually more convenient to work with due to its more compact form. The corresponding magnetic field strength is

$$\mathbf{B}(\mathbf{r}, t) = B_0 \, e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \, \boldsymbol{\epsilon}_2 \tag{10.26}$$

with $B_0 = E_0/c$ and

$$\boldsymbol{\epsilon}_2 = \mathbf{n} \times \boldsymbol{\epsilon}_1 = -\sin \chi e^{i\phi_2} \mathbf{e}_1 + \cos \chi e^{i\phi_1} \mathbf{e}_2 \tag{10.27}$$

The two unit vectors $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ are referred to as *polarization vectors*. They satisfy the orthonormalization relations

$$\boldsymbol{\epsilon}_i^* \cdot \boldsymbol{\epsilon}_j = \delta_{ij} \,, \quad \boldsymbol{\epsilon}_i \cdot \mathbf{n} = 0 \quad (\mathbf{n} = \mathbf{k}/k) \tag{10.28}$$

Note that only the difference $\phi_1 - \phi_2$ is relevant for the polarization, since the sum of the angles can be adjusted by a shift in the time parameter, as demonstrated explicitly by the form of the real field in (10.21).

The different types of polarization can be analyzed by considering the orbit described by the real vector $\mathbf{E}(\mathbf{r}, t)$ in the two-dimensional plane when the time coordinate t changes for a fixed point $\mathbf{r}$ in physical space. We consider first some special cases.
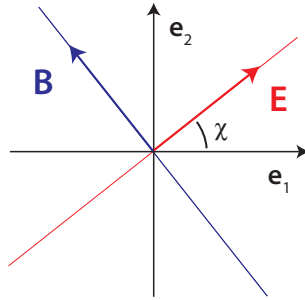
Linear polarization

Figure 10.2: Linear polarization. The electric field vector $\mathbf{E}$ oscillates in a fixed direction orthogonal to the wave number $\mathbf{k}$, while the magnetic field vector $\mathbf{B}$ oscillates in the direction orthogonal to $\mathbf{E}$. The vector $\mathbf{k}$ is in this figure directed out of the plane, towards the reader.

This corresponds to the case where the two orthogonal components of the real electric field oscillates *in phase*, with $\phi_2 = \phi_1$ (or $\phi_2 = \phi_1 + \pi$). The $\mathbf{E}$ field then oscillates along a fixed axis orthogonal to $\mathbf{k}$ and the $\mathbf{B}$ field oscillates in the direction orthogonal to both $\mathbf{k}$ and $\mathbf{E}$. The axis of oscillation of $\mathbf{E}$ together with the axis defined by $\mathbf{k}$ span a two dimensional plane,

which is the *polarization plane* of the electromagnetic field. The realvalued electric field then has the form

$$\mathbf{E}(\mathbf{r}, t) = E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi)[\cos \chi \mathbf{e}_1 + \sin \chi \mathbf{e}_2] \tag{10.29}$$

with $\phi_1 = \phi_2 = \phi$. The field oscillates along a fixed line which is rotated by an angle χ relative to the chosen unit vector $\mathbf{e}_1$ in the plane orthogonal to $\mathbf{k}$.

Circular polarization

In this case the two orthogonal components of the $\mathbf{E}$ field are $90°$ out of phase, so that $\phi_2 = \phi_1 + \pi/2$ or $\phi_2 = \phi_1 - \pi/2$, while the amplitudes of these components are equal, so that $\cos \chi = \sin \chi = 1/\sqrt{2}$. The complex polarization vector then has the form

$$\boldsymbol{\epsilon}_1 = \frac{1}{\sqrt{2}} e^{i\phi}(\mathbf{e}_1 \pm i\mathbf{e}_2) \tag{10.30}$$

For the electric field this gives

$$\mathbf{E}(\mathbf{r}, t) = \frac{E_0}{\sqrt{2}}[\cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi)\mathbf{e}_1 \pm \sin(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi)\mathbf{e}_2] \tag{10.31}$$

where the sign $\pm$ determines whether the field vector rotates in the positive or negative direction when t is increasing.
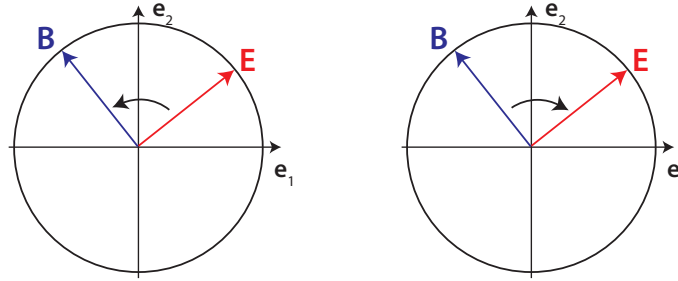

Figure 10.3: Circular polarization. The electric field vector now rotates in the plane orthogonal to $\mathbf{k}$, and the magnetic field $\mathbf{B}$ also rotates, but $90°$ out of phase with $\mathbf{E}$. The direction of $\mathbf{k}$ is also here out of the plane, towards the reader. Two cases are shown, corresponding to right handed and left-handed circular polarization.

Elliptic polarization

Next we consider the case where the two orthogonal components are still $90°$ out of phase, but where the absolute value of the two components now are different. The complex polarization vector $\boldsymbol{\epsilon}_1$ now has the form

$$\boldsymbol{\epsilon}_1 = e^{i\phi}(\cos \chi \, \mathbf{e}_1 + i \sin \chi \, \mathbf{e}_2) \tag{10.32}$$

and the (real) electric field is

$$
\begin{aligned}
\mathbf{E}(\mathbf{r}, t) &= E_0[\cos \chi \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi)\mathbf{e}_1 + \sin \chi \sin(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi)\mathbf{e}_2] \\
&\equiv E_1(\mathbf{r}, t)\, \mathbf{e}_1 + E_2(\mathbf{r}, t)\, \mathbf{e}_2
\end{aligned} \tag{10.33}
$$

The expression shows that the two components for the field satisfy the ellipse equation

$$
\frac{E_1^2}{a^2} + \frac{E_2^2}{b^2} = 1 \tag{10.34}
$$

when we define $a = E_0 \cos \chi = E_{10}$ and $b = E_0 \sin \chi = E_{20}$. This means that when we consider the field for a fixed point $\mathbf{r}$ in space, the time dependent vector $\mathbf{E}$ will trace out an ellipse in the plane orthogonal to the direction of propagation of the wave, $\mathbf{n}$. The symmetry axes of the ellipse are in this case along the directions of the real unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ and the half axes of the ellipse are given by a and b. This is a case of elliptic polarization.
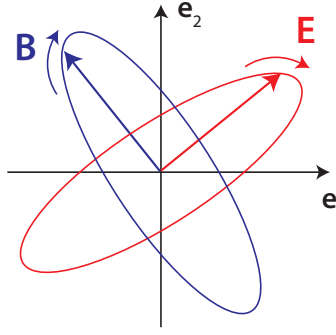

Figure 10.4: Elliptic polarization. The time dependent electric field vector $\mathbf{E}$, in the generic case, describes an ellipse in the plane orthogonal to $\mathbf{k}$, while the magnetic field vector $\mathbf{B}$ describes a similar ellipse, rotated by $90°$. Also here the direction of $\mathbf{k}$ is out of the plane, towards the reader.

The general case

The case of elliptic polarization discussed above seems not to be the most general one, since we have fixed the relative phase of the two orthogonal components of the polarization vector to be $\pi/2$. However, the most general case does in fact correspond to elliptic polarization, with the only modification that the ellipse is rotated relative to the axes defined by the two chosen real unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. To demonstrate this we start with the general expression for the polarization vector

$$
\boldsymbol{\epsilon}_1 = \cos \chi \, \mathbf{e}^{i\phi_1} \mathbf{e}_1 + \sin \chi e^{i\phi_2} \, \mathbf{e}_2 \tag{10.35}
$$

which for the real electric field corresponds to the expression (10.21). We now write the complex phases in the form

$$
\phi_1 = \phi + \theta_1 , \quad \phi_2 = \phi + \theta_2 \tag{10.36}
$$

with ϕ as a free variable, where a change in this variable can be compensated for by a change in θ_1 and θ_2. The expression for the polarization vector (10.35) can be rewritten in a form similar to (10.32)

$$\epsilon_1 = e^{i\phi}(\cos\chi'\,\mathbf{e}_1' + i\sin\chi'\,\mathbf{e}_2') \tag{10.37}$$

provided we make the following identifications

$$
\begin{aligned}
\cos\chi'\mathbf{e}_1' &= \cos\chi\cos\theta_1\,\mathbf{e}_1 + \sin\chi\cos\theta_2\,\mathbf{e}_2 \\
\sin\chi'\mathbf{e}_2' &= \cos\chi\sin\theta_1\,\mathbf{e}_1 + \sin\chi\sin\theta_2\,\mathbf{e}_2
\end{aligned} \tag{10.38}
$$

The two new vectors $\mathbf{e}_1'$ and $\mathbf{e}_2'$ should be orthogonal (and normalized), which gives the following additional condition,

$$\cos^2\chi\cos\theta_1\sin\theta_1 + \sin^2\chi\cos\theta_2\sin\theta_2 = 0 \tag{10.39}$$

The last equation can be regarded as an equation to determine ϕ (and therefore θ_1 and θ_2), when ϕ_1 and ϕ_2 are fixed, thereby exploiting the freedom in the choice of this variable. Eq. (10.38) can then be used to determine $\cos\chi'$, $\sin\chi'$ and the rotated unit vectors $\mathbf{e}_1'$ and $\mathbf{e}_2'$. This shows that the general polarization vector (10.35), when re-expressed in terms of the rotated unit vectors, has the form of the polarization vector for an elliptically polarized plane wave, with rotated symmetry axes of the ellipse. Fig. 10.4 illustrates a case of elliptic polarization where the electric field vector and the magnetic field vector trace out two orthogonal ellipses under the time evolution.

One should note that all the effects of polarization that we have discussed in this section can be viewed as being consequences of *superposition*. In all cases the monochromatic plane wave can be viewed as a superposition of two linearly polarized plane waves, with the same wave vector $\mathbf{k}$, and with polarization along two arbitrarily chosen orthogonal directions. The different types of polarization are then produced by varying the relative amplitudes and phases of these two partial waves.
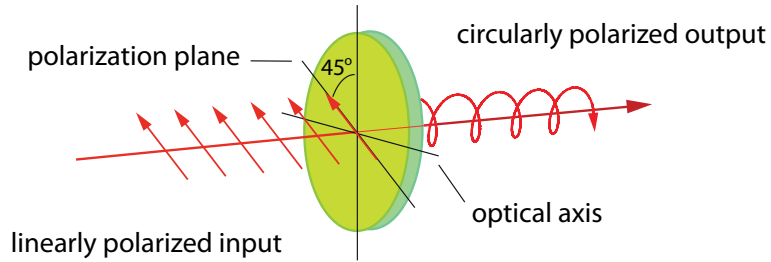

Figure 10.5: Transforming linear to circular polarization in a briefingent crystal. The vertical line refers to the *optical axis* of the crystal. Plane waves with polarization in this direction propagate with a different speed than waves with polarization in the orthogonal direction. In the figure the incoming light is linearly polarized with the polarization plane tilted $45°$ relative to the optical axis. The width of the crystal is adjusted to give a quarter-wavelength relative phase shift of the two polarization components during the propagation in the crystal. This gives rise to a circularly polarized output.

Electromagnetic waves can propagate not only in vacuum, but also in transparent media. The speed of propagation in such a medium is generally slower than in vacuum, as a consequence of the fact that the effective values of the permittivity ϵ and permeability μ is different from their values in vacuum. A particularly interesting medium is found in birefringent crystals where the speed is not unique, but depends on the state of polarization of the propagation wave. This is exploited in so-called *wave plates* which have the property of changing the polarization of a transmitted wave. There are for example wave plates that can change linear polarization of the incoming wave to circular polarization of the transmitted wave.

10.3   Electromagnetic energy and momentum

Maxwell's equations describe how moving charges give rise to electromagnetic fields and the *Lorentz force* describes how the fields act back on the charges. Since the field acts with forces on charged particles, this implies that energy and momentum is transferred between the field and the particles, and consequently the electromagnetic field has to be a carrier of energy and momentum. The precise form of the energy and momentum density of the field is determined by Maxwell's equation and the Lorentz force, under the assumption of *conservation of energy* and *conservation of momentum*.

To demonstrate this we examine the change in energy and momentum of a pointlike test particle, which is influenced by the field. The charge and momentum density in this case can be expressed as

$$\begin{aligned}
\rho(\mathbf{r}, t) &= q\,\delta(\mathbf{r} - \mathbf{r}(t)) \\
\mathbf{j}(\mathbf{r}, t) &= q\,\mathbf{v}(t)\,\delta(\mathbf{r} - \mathbf{r}(t))
\end{aligned} \tag{10.40}$$

with q as the charge of the particle, $\mathbf{r}(t)$ as the time dependent position vector, and $\mathbf{v}(t)$ as the velocity of the particle. The Lorentz force which acts on the particle is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{10.41}$$

This force will in general change the energy of the particle, and the time derivative of the energy is

$$\frac{d}{dt}\mathcal{E}_{part} = \mathbf{F} \cdot \mathbf{v} = q\mathbf{v} \cdot \mathbf{E} \tag{10.42}$$

Energy conservation means that the total energy of both field and particle is left unchanged. With $\mathcal{E}_{field}$ as the field energy within a finite (but large) volume V with boundary surface Σ, energy conservation takes the form

$$\frac{d}{dt}(\mathcal{E}_{field} + \mathcal{E}_{part}) = -\int_{\Sigma} \mathbf{S} \cdot d\mathbf{A} \tag{10.43}$$

where $\mathbf{S}$ is the energy current density of the field and $d\mathbf{A}$ is the area element of the surface. The right hand side of the equation is the energy loss in V due to the energy current through the boundary surface, for example due to radiation. The time derivative of the field energy can now be written

$$\frac{d}{dt}\mathcal{E}_{field} = -q\mathbf{v} \cdot \mathbf{E} - \int_{\Sigma} \mathbf{S} \cdot d\mathbf{A} = -\int_{V} \mathbf{j} \cdot \mathbf{E}\, dV - \int_{\Sigma} \mathbf{S} \cdot d\mathbf{A} \tag{10.44}$$

where the expression for the time derivative of the particle energy has been rewritten by use of expression (10.40) for the current density. In the last form the equation is in fact valid for arbitrary charge configurations within the volume V.

By use of Ampere's law the current density can be replaced by the electric and and magnetic fields in the following way

$$\mathbf{j} = \frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{B} - \epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{10.45}$$

and this gives for the volume integral in (10.44)

$$\int_V \mathbf{j}\cdot\mathbf{E}\,dV = \int_V \left[\frac{1}{\mu_0}\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) - \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}\right]dV \tag{10.46}$$

We further modify the integrand of the first term by using field identities and Faraday's law of induction,

$$\begin{aligned}
\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) &= \boldsymbol{\nabla}\cdot(\mathbf{B}\times\mathbf{E}) + \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{E}) \\[2mm]
&= -\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B}) - \mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}
\end{aligned} \tag{10.47}$$

This gives

$$\begin{aligned}
\int_V \mathbf{j}\cdot\mathbf{E}\,dV &= -\int_V\left[\epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} - \frac{1}{\mu_0}\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B})\right]dV \\[2mm]
&= -\frac{d}{dt}\int_V\frac{1}{2}\left[\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right]dV - \frac{1}{\mu_0}\int_\Sigma(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{A}
\end{aligned} \tag{10.48}$$

where in the last step a part of the volume integral has been rewritten as a surface integral by use of Gauss' theorem.

Writing the field energy as a volume integral, $\mathcal{E}_{field} = \int_V u\,dV$, with u as the energy density of the field, and separating the volume and surface integrals, we get the following form for Eq. (10.44)

$$\frac{d}{dt}\int_V (u - \frac{1}{2}(\epsilon_0\mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2))dV = -\int_\Sigma(\mathbf{S} - \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}))\cdot d\mathbf{A} \tag{10.49}$$

Since this equation should be satisfied for an arbitrarily chosen volume and for general field configurations, we conclude that the integrands of the volume and surface integrals should vanish separately. This determines the energy density as a function of the field strength,

$$u = \frac{1}{2}\left[\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right] \tag{10.50}$$

and the current density

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{10.51}$$

These are the standard expressions for the energy density and the energy current density of the electromagnetic field, and the derivation shows that the field equations combined with energy conservation leads to these expressions. The vector $\mathbf{S}$ is called *Poynting's vector*.

The expression for the momentum density of the electromagnetic field can be derived in the same way. We start with the expression for the time derivative of the particle momentum,

$$\frac{d}{dt}\mathbf{P}_{part} = \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{10.52}$$

and for energy conservation

$$\frac{d}{dt}(\mathbf{P}_{field} + \mathbf{P}_{part}) = 0 \tag{10.53}$$

In this case we assume for simplicity the momentum density to be integrated over the infinite space in order to avoid the surface contribution.

We follow the same approach as for the field energy, by applying Maxwell's equations to replace the charge and current densities with field variables. By further manipulating the expression, and in particular assuming surface integrals to vanish, we get

$$
\begin{aligned}
\frac{d}{dt}\mathbf{P}_{field} &= -\int \left[\rho\mathbf{E} + \mathbf{j} \times \mathbf{B}\right] dV \\
&= -\int \left[\epsilon_0\mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{E}\right) + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}) \times \mathbf{B}\right] dV \\
&= -\int \left[-\epsilon_0(\boldsymbol{\nabla}\times\mathbf{E})\times\mathbf{E} + (\frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{B} - \epsilon_0\frac{\partial\mathbf{E}}{\partial t}) \times \mathbf{B}\right] dV \\
&= \int \left[-\epsilon_0\frac{\partial\mathbf{B}}{\partial t}\times\mathbf{E} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}\right] dV \\
&= \frac{d}{dt}\int \epsilon_0\mathbf{E}\times\mathbf{B}\, dV
\end{aligned}
\tag{10.54}
$$

This gives the following expression for the field momentum density

$$\mathbf{g} = \epsilon_0\mathbf{E}\times\mathbf{B} \tag{10.55}$$

and we note that, up to a factor $1/c^2 = \epsilon_0\mu_0$, it is identical to the energy current density $\mathbf{S}$.

In the relativistic formulation the energy density and the momentum density are combined in the symmetric *energy-momentum tensor*,

$$T^{\mu\nu} = -(F^{\mu\rho}F^{\nu}{}_{\rho} + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}) \tag{10.56}$$

The energy density corresponds here to the component T^{00} and Poynting's vector to (c times) the components T^{0i}, $i = 1, 2, 3$.

10.3.1 Energy and momentum density of a monochromatic plane wave

We consider a plane wave with the electric and magnetic field vectors related by

$$\mathbf{B} = \frac{1}{c}\mathbf{n}\times\mathbf{E}, \quad \mathbf{E} = -c\mathbf{n}\times\mathbf{B} \tag{10.57}$$

where $\mathbf{n} = \mathbf{k}/k$ is a unit vector in the direction of propagation of the wave. This gives $\mathbf{B}^2 = \mathbf{E}^2/c^2$ and therefore the energy density of the field is

$$u = \frac{1}{2}(\epsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2) = \epsilon_0 \mathbf{E}^2 \tag{10.58}$$

with equal contributions from the electric and magnetic fields.

Poynting's vector, which determines the energy current and momentum densities of the plane wave, is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = \epsilon_0 c \mathbf{E}^2 \mathbf{n} = u\,c\mathbf{n} \tag{10.59}$$

It is directed along the direction of propagation of the wave, and the last expression in (10.59) is consistent with the interpretation that the field energy is transported in the direction of the propagating wave with the speed of light.

10.3.2   Field energy and potential energy

Let us consider two static charges q_1 and q_2 at relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. The Coulomb energy of the system is

$$U(r) = \frac{q_1 q_2}{4\pi\epsilon_0 r} \tag{10.60}$$

and the usually this energy is considered as a potential energy of the two *charges*. However, in the preceding discussion we have found an expression for the local energy density of the electromagnetic field, which should also apply to this static situation. This raises the question of how the potential energy of the charges is related to the electromagnetic *field energy*. An important point to notice is that we should *not* consider the two energies as something we should add in order to obtain the total energy of the system of charges and fields. Instead the integrated field energy is identical to the *total* electromagnetic energy of the charges and fields and the potential energy can be extracted as the part of this energy that depends on the position of the static charges. We demonstrate this by calculating the integrated field energy of the two charges.

The integrated field energy is

$$\mathcal{E} = \frac{1}{2}\epsilon_0 \int \mathbf{E}(\mathbf{r}')^2 d^3 r' \tag{10.61}$$

where the electrostatic field $\mathbf{E}$ is the superposition of the Coulomb field from the two charges,

$$\mathbf{E}(\mathbf{r}') = \mathbf{E}_1(\mathbf{r}') + \mathbf{E}_2(\mathbf{r}') = \frac{q_1}{4\pi\epsilon_0}\frac{\mathbf{r}' - \mathbf{r}_1}{|\mathbf{r}' - \mathbf{r}_1|} + \frac{q_2}{4\pi\epsilon_0}\frac{\mathbf{r}' - \mathbf{r}_2}{|\mathbf{r}' - \mathbf{r}_2|} \tag{10.62}$$

The field energy then has a natural separation into three parts

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_{12} \tag{10.63}$$

where the first two parts are the contributions from the Coulomb energies of each of the two charges disregarding the presence of the other,

$$\mathcal{E}_1 = \frac{1}{2}\epsilon_0 \int \mathbf{E}_1(\mathbf{r}')^2 d^3r' = \frac{q_1{}^2}{32\pi^2\epsilon_0} \int \frac{d^3r'}{r'^4} = \frac{q_1{}^2}{8\pi\epsilon_0} \int_0^\infty \frac{dr'}{r'^2}$$

$$\mathcal{E}_2 = \frac{1}{2}\epsilon_0 \int \mathbf{E}_2(\mathbf{r}')^2 d^3r' = \frac{q_2{}^2}{32\pi^2\epsilon_0} \int \frac{d^3r'}{r'^4} = \frac{q_1{}^2}{8\pi\epsilon_0} \int_0^\infty \frac{dr'}{r'^2} \tag{10.64}$$

We note that these two terms are independent of the positions of the particles. They are referred to as the *self energies* of the particles and these energies are in a sense always bound to the particles in the Coulomb field surrounding each of them. Except for the different charge factors the self energy of the two charges are the same, but we note that for *point particles* the integrated self energy diverges in the limit $r' \to 0$. This is a separate point to discuss and we shall return to this question briefly.

The third contribution to the field energy comes from the superposition of the Coulomb fields of the two particles,

$$\mathcal{E}_{12}(r) = \epsilon_0 \int \mathbf{E}_1(\mathbf{r}') \cdot \mathbf{E}_2(\mathbf{r}') d^3r' \tag{10.65}$$

As indicated in the equation it depends on the distance between the two charges. To calculate this term it is convenient to introduce the Coulomb potential of one of the particles $\mathbf{E}_1 = -\boldsymbol{\nabla}\phi_1$. We restrict the volume integral to a finite volume V and extract, by partial integration, a surface term as an integral over the boundary surface S of the volume V,

$$
\begin{aligned}
\mathcal{E}_{12} &= -\epsilon_0 \int_V \boldsymbol{\nabla}\phi_1 \cdot \mathbf{E}_2 \, d^3r' \\
&= -\epsilon_0 \int_V \boldsymbol{\nabla} \cdot (\phi_1 \mathbf{E}_2) \, d^3r' + \epsilon_0 \int_V \phi_1 \boldsymbol{\nabla} \cdot \mathbf{E}_2 \, d^3r' \\
&= -\epsilon_0 \int_S \phi_1 \mathbf{E}_2 \cdot d\mathbf{S} + \epsilon_0 \int_V \phi_1(\mathbf{r}') \, q_2 \, \delta(\mathbf{r}' - \mathbf{r}_2) \, d^3r'
\end{aligned}
\tag{10.66}
$$

Gauss' theorem has here been applied to rewrite the volume integral of the divergence in the first term as a surface integral, and in the second term Gauss' law for the electromagnetic field has been applied to rewrite the divergence of the electric field as a charge density. Since we consider point charges this density is proportional to a Dirac delta function.

Let us now assume the volume extends to infinity. We note that the surface integral tends to zero, since far from the charges the product $\phi_1 E_2$ falls off with distance as $1/r'^3$. We are then left with the volume integral, which is easy to evaluate due to the presence of the delta function,

$$\mathcal{E}_{12}(r) = q_2\phi_1(\mathbf{r}_2) = q_2\frac{q_1}{4\pi\epsilon_0|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{q_1 q_2}{4\pi\epsilon_0 r} = U(r) \tag{10.67}$$

This shows that the Coulomb potential can be identified as the part of the total field energy that depends on the distance between the charges and is due to the overlap of the electric fields of the two charges.

We return now to the question of how to understand the expression for the self energy terms. For an isolated point charge q located at the origin the energy of the Coulomb field is

$$\mathcal{E} = \frac{1}{2}\epsilon_0 \int \mathbf{E}^2 d^3r = \frac{q^2}{8\pi\epsilon_0} \int_0^\infty \frac{dr}{r^2} \tag{10.68}$$

and this energy is obviously infinite due to the divergence of the integral as $r \to 0$. A reasonable assumption is that there is nothing wrong with the expression for the field energy, but that the idealization of treating the charge as being located at a mathematical point is the origin of the problem. Thus, as soon as we assume that the charge has a finite size a, with this as an effective cutoff of the integral, the energy becomes finite,

$$\mathcal{E}_a = \frac{q^2}{8\pi\epsilon_0} \int_a^\infty \frac{dr}{r^2} = \frac{q^2}{8\pi\epsilon_0 a} \tag{10.69}$$

This, at least formally, solves the problem with the infinite energy. However, to make a consistent picture of physical particles like electrons as small charged bodies is not so simple. That is a problem that exists not only in the classical theory; also in the quantum description of particles and fields there are infinities associated with the electromagnetic self energies that have to be taken care of by the theory.

A standard way to treat the self energy problem is based on the fact that the self energy is bound to each individual charge and therefore is not important for the interactions between the particles. One may therefore avoid the problem of a precise theory of pointlike particles by simply assuming that the energy carried by the field is finite, and that the only physical effect of this energy is to change the mass of the charged particle. This change is given by Einstein's relation

$$\Delta m c^2 = \mathcal{E}_a = \frac{q^2}{8\pi\epsilon_0 a} \tag{10.70}$$

The physical mass of the particle can then be written as a sum

$$m = m_b + \Delta m \tag{10.71}$$

where m_b is the so called bare mass, which is the (imagined) mass of the particle without the Coulomb field. When the physical mass enters the equations of motion that means that the *mass renormalization* effect of the self energy has been included and all other effects of the self energy can be neglected.

Finally, let us use this interpretation of the self energy to give an estimate the value of the length parameter a for an electron. We know that $\Delta m \leq m_e$, with m_e as the physical electron mass and with equality meaning that *all* the electron mass is due to the electromagnetic energy of its Coulomb field. In this limit we get

$$\frac{e^2}{8\pi\epsilon_0 a} = m_e c^2 \quad \Rightarrow \quad a = \frac{e^2}{8\pi\epsilon_0 m_e c^2} \tag{10.72}$$

With a more explicit model of the electron as a charged spherical shell of radius r_e a similar calculation of the electromagnetic energy gives the same result as the one obtained by a simple cutoff in the integral, except for a factor 2,

$$r_e = \frac{e^2}{4\pi\epsilon_0 m_e c^2} \tag{10.73}$$

This value is called the *classical electron radius*. It numerical value is

$$r_e = 2.818 \times 10^{-15} \text{m} \tag{10.74}$$

which shows that it is indeed a very small radius, comparable to the radius of an atomic nucleus.

Chapter 11

Maxwell's equations with stationary sources

We return to the original form of Maxwell's equations in the Lorentz gauge,

$$\partial_\nu \partial^\nu A^\mu = -\mu_0 j^\mu \tag{11.1}$$

and assume the 4-current j^μ, and therefore both the charge and current densities to be independent of time,

$$\rho = \rho(\mathbf{r}), \quad \mathbf{j} = \mathbf{j}(\mathbf{r}) \tag{11.2}$$

Note that this is the case only in a preferred inertial frame (which refer to as the *laboratory frame*). With time independent sources we may also assume the electromagnetic potential A^μ, as a solution of (11.1), to be time independent. This means that the Lorentz gauge condition again reduces to the Coulomb gauge condition, $\nabla \cdot \mathbf{A} = 0$ and Maxwell's equation has a natural decomposition in two independent equations

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \tag{11.3}$$

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j} \tag{11.4}$$

where the scalar potential $\phi = A^0/c$ determines the electric field and vector potential $\mathbf{A}$ determines the magnetic field.

Since there is no coupling between the two equations (11.3) and (11.4), they can be studied separately. Equation (11.3) is the basic equation in *electrostatics*, where static charges give rise to a time independent electric field, while equation (11.4) is the basic equation in *magnetostatics* where stationary currents give rise to a time independent magnetic field. As differential equations they are of the same type, known as the *Poisson equation*, and even if there are some differences, the methods of finding the electrostatic and magnetostatic fields with given sources, are much the same. We examine now the two cases separately.

11.1 The electrostatic equation

Since the electrostatic equation (11.3) is a linear differential equation, the solution can be seen as a linear superposition of contributions from pointlike parts of the charge distributions. For

a single point charge located at the origin, the charge density is $\rho(\mathbf{r}) = q\delta(\mathbf{r})$, with q as the electric charge. In this case the electric field is most easily determined by use of the integral form of Gauss' law, and by exploiting the rotational invariance of the field, $\mathbf{E}(\mathbf{r}) = E(\mathbf{r}) \, \mathbf{r}/r$. For a spherical surface centered at the origin, Gauss' law then takes the form,

$$4\pi r^2 E(r) = \frac{q}{\epsilon_0} \tag{11.5}$$

which determines the electric field as

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r^2} \frac{\mathbf{r}}{r} \tag{11.6}$$

which is the standard form of the Coulomb field. The corresponding Coulomb potential is also easily found to be

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r} \tag{11.7}$$
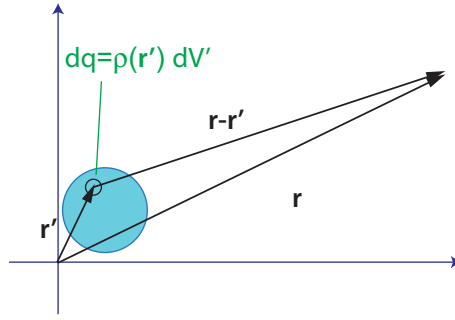

Figure 11.1: The electrostatic potential. The potential at a point $\mathbf{r}$ is determined as a linear superposition of contributions from small pieces dq of the charge, located at points $\mathbf{r}'$ in the charge distribution.

For a charge distribution $\rho(\mathbf{r})$ which is no longer pointlike, the potential can be written directly as a sum (or integral) over the Coulomb potential from all parts of the distribution,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \tag{11.8}$$

The corresponding electric field strength is

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}') d^3 r' \tag{11.9}$$

In reality the above solution is a *particular* solution of the differential equation. A general solution will therefore be of the form

$$\phi(\mathbf{r}) = \rho_c(\mathbf{r}) + \phi_0(\mathbf{r}) \tag{11.10}$$

where ϕ_c denotes the solution given above and ϕ_0 is a general solution of the source free *Laplace equation*

$$\boldsymbol{\nabla}^2 \phi_0 = 0 \qquad (11.11)$$

The solution (11.8) written above *implicitly* assumes certain boundary conditions that are natural in the open, infinite space, namely that the potential falls to zero at infinity.

When we consider the electric field in a finite region V of space, with given boundary conditions on the on the boundary surface S, the contribution from ϕ_0 will generally be important. This contribution to the potential will correct the contribution from the integrated Coulomb potential so that the total potential satisfies the boundary conditions. As a particular situation we may consider the electric field produced by a charge within a cavity of an electric conductor. Since the boundary surface of the conductor is an *equipotential* surface, the function ϕ_0 is determined as a solution of the Laplace equation (11.11) with the following boundary condition on the surface of the conductor

$$\phi_0(\mathbf{r}) = -\phi_c(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \quad \mathbf{r} \in S \qquad (11.12)$$

More generally we often make a distinction between two types of boundary conditions where either the potential ϕ is specified on the boundary (Dirichlet condition) or the electric field $\mathbf{E} = -\boldsymbol{\nabla}\phi$ is specified (Neuman condition). To determine the potential that satisfies the correct boundary conditions is generally a non-trivial problem, and several methods have been developed to solve the problem for different types of boundary condition. This we shall not discuss further, but instead we shall assume that simple form (11.8) of the potential in the open, infinite space to be valid. The integral expression for $\phi(\mathbf{r})$ in a sense solves the electrostatic problem, even if the integral has to be evaluated for a specified charge distribution in order to determine the electrostatic potential. However, when far from the charges the integral can be simplified by use of a *multipole expansion*, and we will next study how such an expansion can be performed to give useful approximations to the electrostatic potential.

11.1.1 Multipole expansion

This expansion is based on the assumption that the point $\mathbf{r}$ where we are interested in determining the potential and the electric field is at some distance from the charge distribution. To be more precise let us assume that the charge distribution has a finite extension of linear dimension a, as illustrated in Fig. 11.2. We assume the point where to find the potential lies at a distance form the charges which is much larger than a. With the origin chosen to lie close to the charges we may write this assumption as

$$r >> r' \approx a \qquad (11.13)$$

where $\mathbf{r}'$ is the variable in the integration over the charge distribution. In the integral formula for the potential we may introduce the small vector $\boldsymbol{\xi} \equiv \mathbf{r}'/r$, and make a Taylor expansion in powers of the vector.
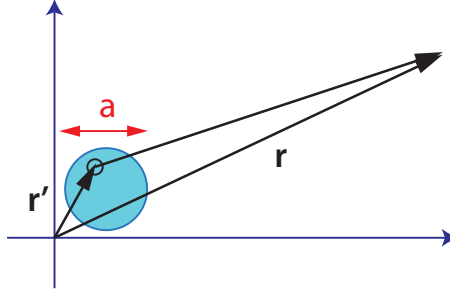
Figure 11.2: Multipole expansion. The expansion is based on the assumption that the distance to the point where the electric field is determined (measured) is large compared to the linear size a of the charge distribution. The ratio r'/r, between the distance r' to a part of the charge and the distance r to where the potential is evaluated is used as expansion parameter in the multipole expansion.

The inverse distance between the points $\mathbf{r}'$ and $\mathbf{r}$, when expressed in terms of $\boldsymbol{\xi}$ is

$$
\begin{aligned}
\frac{1}{|\mathbf{r} - \mathbf{r}'|} &= (r^2 + r'^2 - 2\mathbf{r} \cdot \mathbf{r}')^{-\frac{1}{2}} \\
&= \frac{1}{r} \left[1 + \left(\frac{r'}{r}\right)^2 - 2\frac{\mathbf{r}}{r} \cdot \frac{\mathbf{r}'}{r} \right]^{-\frac{1}{2}} \\
&= \frac{1}{r} \left[1 - 2\frac{\mathbf{r}}{r} \cdot \boldsymbol{\xi} + \xi^2 \right]^{-\frac{1}{2}} \\
&\equiv \frac{1}{r} f(\boldsymbol{\xi})
\end{aligned}
\tag{11.14}
$$

We make a Taylor expansion of the function $f(\boldsymbol{\xi})$ introduced at the last step,

$$
\begin{aligned}
f(\boldsymbol{\xi}) &= f(0) + \sum_i \xi_i \frac{\partial f}{\partial \xi}(0) + \frac{1}{2} \sum_{ij} \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} + \dots \\
&= 1 + \frac{\mathbf{r}}{r} \cdot \boldsymbol{\xi} + \sum_{ij} \frac{1}{2}(3\frac{x_i x_j}{r^2} - \delta_{ij})\xi_i \xi_j + \dots
\end{aligned}
\tag{11.15}
$$

and reintroduce the integration variable $\mathbf{r}'$, in the corresponding expansion of the inverse distance

$$
\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r}(1 + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} + \frac{1}{2}\left(3\frac{(\mathbf{r} \cdot \mathbf{r}')^2}{r^4} - \frac{r'^2}{r^2}\right) + \dots)
\tag{11.16}
$$

For the electrostatic potential this gives the following expansion

$$
\begin{aligned}
\phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0 r} \int \rho(\mathbf{r}') \left[1 + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} + \frac{1}{2}\left(3\frac{(\mathbf{r} \cdot \mathbf{r}')^2}{r^4} - \frac{r'^2}{r^2}\right) + \dots \right] d^3r' \\
&\equiv \phi_0(\mathbf{r}) + \phi_1(\mathbf{r}) + \phi_2(\mathbf{r}) + \dots
\end{aligned}
\tag{11.17}
$$

with ϕ_n as the n'th term of the expansion of the potential in powers of ξ.

We consider the first terms in the expansion, beginning with the *monopole* term,

$$\phi_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r} \int \rho(\mathbf{r}') \, d^3 r' = \frac{q}{4\pi\epsilon_0 r} \tag{11.18}$$

where $q = \int \rho(\mathbf{r}') \, d^3 r'$ as the total charge of the charge distribution. This shows that the lowest order term of the expansion gives a potential which is the same as if the total charge was collected in the origin of the coordinate system. This first term will give a good approximation to the true potential if the point $\mathbf{r}$ is sufficiently far away and the origin is chosen sufficiently close to the (center of the) charge distribution.

The second term of the expansion is the *dipole* term,

$$\phi_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3} \int \rho(\mathbf{r}') \mathbf{r} \cdot \mathbf{r}' \, d^3 r' = \frac{\mathbf{r} \cdot \mathbf{p}}{4\pi\epsilon_0 r^3} \tag{11.19}$$

where we have introduced the electric dipole moment,

$$\mathbf{p} = \int \rho(\mathbf{r}) \, \mathbf{r} \, d^3 r \tag{11.20}$$

This term gives a correction to the monopole term, and we note that for large r it falls off like $1/r^2$ while the monopole term falls off like $1/r$, so the monopole term will always dominate the dipole term for sufficiently large r (unless $q = 0$).

We include one more term of the expansion in our discussion. That is the electric quadrupole term,

$$\phi_2(\mathbf{r}) = \frac{1}{8\pi\epsilon_0 r^3} \int \rho(\mathbf{r}')(3(\mathbf{n} \cdot \mathbf{r}')^2 - r'^2) \, d^3 r' = \frac{Q_\mathbf{n}}{8\pi\epsilon_0 r^3} \tag{11.21}$$

with $\mathbf{n} = \mathbf{r}/r$ as the unit vector in direct of the point $\mathbf{r}$ and $Q_\mathbf{n}$ as the *quadrupole moment* about the axis $\mathbf{n}$. It can be written as $Q_\mathbf{n} = \sum_{ij} Q_{ij} n_i n_j$, with

$$Q_{ij} = \int \rho(\mathbf{r})(3x_i x_j - r^2 \delta_{ij}) \, d^3 r \tag{11.22}$$

as the *quadrupole moment tensor*.

The electric field can now be expanded in the same way,

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + \dots \tag{11.23}$$

with $\mathbf{E}_n = -\boldsymbol{\nabla}\phi_n$ for the $n'th$ term in the expansion. We give the explicit expressions for the first two terms. The monopole term is

$$\mathbf{E}_0 = -\boldsymbol{\nabla}\phi_0 = \frac{q}{4\pi\epsilon_0 r^3}\mathbf{r} \tag{11.24}$$

which is the Coulomb field of a point charge q located in the origin. The next term is

$$\mathbf{E}_1 = -\boldsymbol{\nabla}\phi_1 = -\boldsymbol{\nabla}\left(\frac{\mathbf{r} \cdot \mathbf{p}}{4\pi\epsilon_0 r^3}\right) = \frac{1}{4\pi\epsilon_0 r^3}(3\mathbf{n}(\mathbf{n} \cdot \mathbf{p}) - \mathbf{p}), \tag{11.25}$$

with **n=r**/r as before. This field is called the *electric dipole field.*

It should be clear from the above construction that the higher the multipole index n is, the faster the corresponding potentials and electric fields fall off with distance. Thus for large r the n'th multipole term of the potential falls off like $r^{-(n+1)}$, while the corresponding term of in the expansion of the electric field field falls off like $r^{-(n+2)}$. When considering the electric field far from the charges often it is sufficient to consider only the first terms of the multipole expansion. In particular that is the case when we are interested in electromagnetic radiation far from the radiation emitter, as we shall soon consider. In that case the field is determined by the time derivatives of the multipole momenta. Since the total charge is conserved there will be no contribution from the monopole term, but for large r the main contribution will be from the electric dipole term, unless this term is absent.

11.1.2   Elementary multipoles

Elementary multipole fields can be produced by point charges in the following way. An elementary monopole field is simply the Coulomb field of a point charge located at the origin. This Coulomb field has no higher monopole components. A dipole field is produced by two point charges of opposite sign,$\pm q$, located symmetrically about the origin, at positions $\pm \mathbf{d}/2$. The dipole moment of the charge configuration is $\mathbf{p} = q\mathbf{d}$. This field has no monopole component, and in the limit where $d \to 0$ with qd fixed all higher multipole components vanish and the electric field is a pure dipole field. Such an electric dipole field is illustrated in Fig. 11.3.
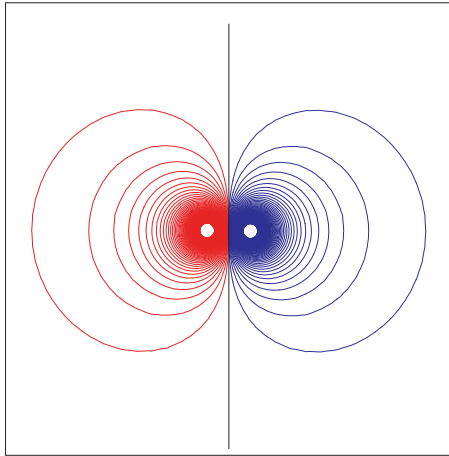

Figure 11.3: Electric dipole potential. Two electric point charges of opposite sign, but equal magnitude, are place at shifted positions. Equipotential lines are shown for a plane which includes the two charges. In the figure red corresponds to positive potential values and blue to negative values. The potential diverges towards the point charges.

In a similar way a pure quadrupole field can be produced by two dipoles of opposite signs that have positions with a relative shift l. For this charge configuration only the quadrupole component of the electric field survives in the limit $l \to 0$ with pl fixed. Such an elementary
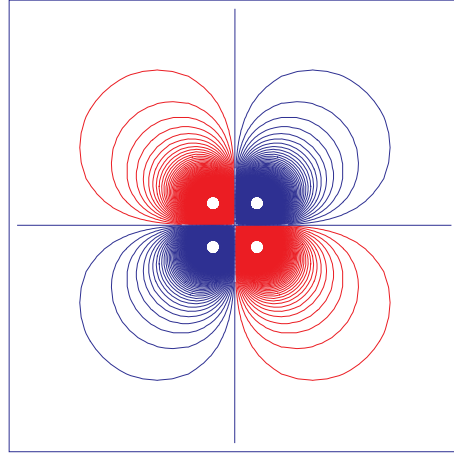
Figure 11.4: Electric quadrupole potential. In this case four point charges located in a plane, with pairwise opposite signs, produce the potential. The charges form two shifted dipoles of opposite orientation. Equipotential lines are shown for the plane which includes the four charges. In the figure red corresponds to positive potential values and blue to negative values. The potential diverges towards each of the four point charges.

quadrupole field is shown in Fig. 11.4.

11.2 Magnetostatics

When studying magnetic fields from stationary currents the basic equation is

$$\boldsymbol{\nabla}^2 \mathbf{A} = -\mu_0 \mathbf{j} \tag{11.26}$$

with $\mathbf{j} = \mathbf{j}(\mathbf{r})$ a s a time independent current density. We note that the equation has, for each component of $\mathbf{A}(\mathbf{r})$, the same form as the equation for a static electric potential, and the Coulomb field solution can immediately be translated to the following solution of the magnetic equation

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \tag{11.27}$$

The corresponding magnetic field is

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int (\boldsymbol{\nabla} \frac{1}{|\mathbf{r} - \mathbf{r}'|}) \times \mathbf{j}(\mathbf{r}') \, d^3 r' \tag{11.28}$$

The gradient in the integrand can easily be calculated by changing temporarily the position of the origin so that $\mathbf{r}' = 0$. We have

$$\boldsymbol{\nabla} \left(\frac{1}{r} \right) = \frac{d}{dr} \left(\frac{1}{r} \right) \boldsymbol{\nabla} r = -\frac{\mathbf{r}}{r^3} \tag{11.29}$$

Shifting the origin back to the correct position gives

$$\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \tag{11.30}$$

and therefore the following expression for the magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r' \tag{11.31}$$
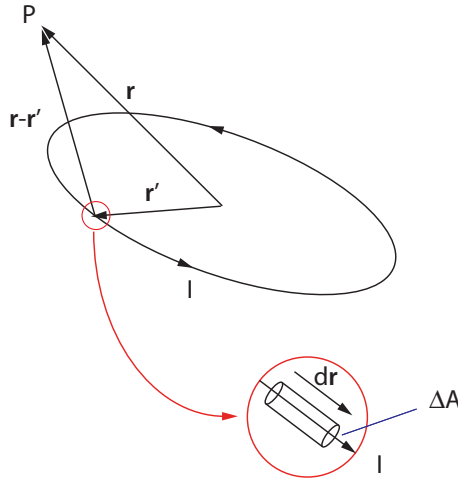

Figure 11.5: Magnetic field from a current loop. The magnetic field at a given point P is a superposition of contributions from each part of the current loop. The field expressed as a line integral around the loop is derived from the general volume integral of the current density by first integrating over the cross-section of the conductor, with area ΔA.

The above expression gives the magnetic field from a general stationary current distribution. However, another form is often more useful, and that corresponds to the situation where the magnetic field is produced by a current in a thin conducting cable. When the cross section can be regarded as vanishingly small the volume integral of the current density can be replaced by the line integral of the current along the curve defined by the thin cable. To find this expression we use the following replacement in the integral, as illustrated in Fig. 11.5,

$$\mathbf{j}\, d^3 r' \to \mathbf{j}\, \Delta A \, dr' = j\, \Delta A\, d\mathbf{r}' = I d\mathbf{r}' \tag{11.32}$$

Here ΔA is the cross section area of the cable and I is the current running in the cable. This gives the following line integral representation of the magnetic field

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi} \int_C \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \times d\mathbf{r}' \tag{11.33}$$

with C as the curve that the current follows. This expression for the magnetic field produced by a stationary current is known as *Biot-Savart's law*.

11.2.1 Multipole expansion for the magnetic field

For positions $\mathbf{r}$ far from the current a similar multipole expansion can be given for the magnetic as for the electric field. We then expand the integrand of (11.31) in powers of r'/r, and for the vector potential that gives

$$
\begin{aligned}
\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \\
&= \frac{\mu_0}{4\pi r} \int \mathbf{j}(\mathbf{r}') \left[1 + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} + \frac{1}{2} \left(3 \frac{(\mathbf{r} \cdot \mathbf{r}')^2}{r^4} - \frac{r'^2}{r^2} \right) + ... \right] d^3 r' \\
&\equiv \mathbf{A}_0(\mathbf{r}) + \mathbf{A}_1(\mathbf{r}) + \mathbf{A}_2(\mathbf{r}) + ...
\end{aligned}
\tag{11.34}
$$

We derive now the explicit expression for the first two terms of the expansion. The monopole term is

$$
\mathbf{A}_0(\mathbf{r}) = \frac{\mu_0}{4\pi r} \int \mathbf{j}(\mathbf{r}') \, d^3 r'
\tag{11.35}
$$

As we shall see this term in fact vanishes identically. We examine one of the vector components (the x component) of the integral

$$
\begin{aligned}
\int j_x(\mathbf{r}) \, d^3 r &= \int \boldsymbol{\nabla} x \cdot \mathbf{j}(\mathbf{r}) \, d^3 r' \\
&= \int \boldsymbol{\nabla} \cdot (x \, \mathbf{j}(\mathbf{r})) \, d^3 r - \int x \boldsymbol{\nabla} \cdot \mathbf{j}(\mathbf{r}) \, d^3 r
\end{aligned}
\tag{11.36}
$$

and first note that by use Gauss' theorem the first term can be rewritten as a surface integral

$$
\int_V \boldsymbol{\nabla} \cdot (x \, \mathbf{j}(\mathbf{r})) \, d^3 r = \int_S x \, \mathbf{j}(\mathbf{r}) \cdot d\mathbf{S}
\tag{11.37}
$$

when the integral is restricted to a finite volume V with boundary surface S. This shows that when V is expanded so that S is outside all the relevant currents then the integral vanishes.

We are left with the contribution from the second term in the last line of (11.36). This is rewritten by use of the continuity equation for charge

$$
\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0
\tag{11.38}
$$

to give

$$
\int j_x(\mathbf{r}) \, d^3 r = \int x \frac{\partial \rho}{\partial t} \, d^3 r = \frac{d}{dt} \int x \rho \, d^3 r = \frac{d}{dt} p_x
\tag{11.39}
$$

where p_x is the x component of the electric dipole moment $\mathbf{p}$. Since similar expression are valid for the other vector components we conclude that the following identity is valid

$$
\int \mathbf{j}(\mathbf{r}, t) \, d^3 r = \dot{\mathbf{p}}
\tag{11.40}
$$

and it is valid not only for stationary, but also for time dependent currents. In the case we consider here, with stationary currents, the time derivative of the dipole moment vanishes and so does the integral over the current. This gives

$$\mathbf{A}_0 = \int \mathbf{j}\, d^3r = 0 \tag{11.41}$$

The vanishing of the monopole term seems reasonable from or previous discussion of the lack of magnetic monopoles in Maxwell's equations.

The next term to consider is the magnetic dipole term,

$$\mathbf{A}_1(\mathbf{r}) = \frac{\mu_0}{4\pi r^3} \int (\mathbf{r} \cdot \mathbf{r}')\, \mathbf{j}(\mathbf{r}')d^3r' \tag{11.42}$$

Also here we have to make use of some identities in order to rewrite the integral. We first consider the following vector identity,

$$(\mathbf{r}' \times \mathbf{j}) \times \mathbf{r} = \mathbf{j}(\mathbf{r} \cdot \mathbf{r}') - \mathbf{r}'(\mathbf{j} \cdot \mathbf{r}) \tag{11.43}$$

and examine further the volume integral of the last term,

$$\int \mathbf{r}'(\mathbf{j}(\mathbf{r}') \cdot \mathbf{r})\, d^3r' = \sum_k \mathbf{e}_k x_l \int x_k' j_l(\mathbf{r}')\, d^3r' \tag{11.44}$$

where $\mathbf{e}_k, k = 1, 2, 3$ are the Cartesian unit vectors. We manipulate the last integral, and leave for simplicity out the "prime" of the variables

$$
\begin{aligned}
\int x_k\, j_l(\mathbf{r})\, d^3r &= \int x_k\, (\boldsymbol{\nabla} x_l \cdot \mathbf{j}(\mathbf{r}))\, d^3r \\
&= \int \boldsymbol{\nabla} \cdot (x_k x_l \mathbf{j}(\mathbf{r}))\, d^3r - \int x_l\, (\boldsymbol{\nabla} x_k \cdot \mathbf{j}(\mathbf{r}))\, d^3r - \int x_k x_l\, \boldsymbol{\nabla} \cdot \mathbf{j}(\mathbf{r})\, d^3r
\end{aligned}
\tag{11.45}
$$

By the same argument as used before, the first term, which can be rewritten as a surface integral, vanishes when the boundary of the volume is outside the region with currents. For the second term we use $\boldsymbol{\nabla} x_k \cdot \mathbf{j}(\mathbf{r}) = \mathbf{e}_k \cdot \mathbf{j} = j_k$, with $\mathbf{e}_k$ as the unit vector in the direction of the x_k coordinate axis, and in the last term we apply the continuity equation, $\boldsymbol{\nabla} \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t}$. This gives

$$\int x_k\, j_l(\mathbf{r})\, d^3r = -\int x_l\, j_k(\mathbf{r})\, d^3r - \frac{d}{dt} \int x_k x_l\, \rho(\mathbf{r})\, d^3r \tag{11.46}$$

The last term is the time derivative of a part of the electric quadrupole moment. However, since we now consider a situation with time independent sources also the contribution from this term vanishes. We are therefore left with the identity

$$\int x_k\, j_l(\mathbf{r})\, d^3r = -\int x_l\, j_k(\mathbf{r})\, d^3r \tag{11.47}$$

which shows that the two indices k and l can be interchanged when combined with a change of sign.

When the symmetry under interchange of indices is introduced in the original integral expression, we get the following identity

$$\int \mathbf{r}' \left(\mathbf{j}(\mathbf{r}') \cdot \mathbf{r} \right) d^3 r' = - \int \mathbf{j}(\mathbf{r}') \left(\mathbf{r} \cdot \mathbf{r}' \right) d^3 r' \tag{11.48}$$

and together with Eq.(11.43) this implies

$$\int (\mathbf{r}' \times \mathbf{j}) \times \mathbf{r} \, d^3 r' = 2 \int (\mathbf{r} \cdot \mathbf{r}') \mathbf{j}(\mathbf{r}') \, d^3 r' \tag{11.49}$$

For the vector potential this finally gives the following expression

$$\begin{aligned} \mathbf{A}_1(\mathbf{r}) &= \frac{\mu_0}{8\pi r^3} \left[\int (\mathbf{r}' \times \mathbf{j}(\mathbf{r}')) \, d^3 r' \right] \times \mathbf{r} \\ &= \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3} \end{aligned} \tag{11.50}$$

where $\mathbf{m}$ is the magnetic dipole moment of the current distribution, defined as

$$\mathbf{m} = \frac{1}{2} \int (\mathbf{r} \times \mathbf{j}) \, d^3 r \tag{11.51}$$

The corresponding magnetic dipole field is

$$\begin{aligned} \mathbf{B}_1(\mathbf{r}) &= \boldsymbol{\nabla} \times \mathbf{A}_1(\mathbf{r}) \\ &= \frac{\mu_0}{4\pi} \boldsymbol{\nabla} \times \left(\frac{\mathbf{m} \times \mathbf{r}}{r^3} \right) \\ &= \frac{\mu_0}{4\pi r^3} \left(3\mathbf{n}(\mathbf{n} \cdot \mathbf{m}) - \mathbf{m} \right) \end{aligned} \tag{11.52}$$

with $\mathbf{n} = \mathbf{r}/r$ as before. We note that the form of the magnetic dipole field is precisely the same as that of the electric dipole field, with the electric dipole moment $\mathbf{p}$ replaced by the magnetic dipole moment $\mathbf{m}$.

11.2.2 Force on charge and current distributions

The electric and magnetic multipole moments appear in various ways in electromagnetic theory. One of these is when we consider electromagnetic radiation, and we shall discuss that in the next section, another one is when we consider the electromagnetic force on a body with a non-vanishing charge or current distribution. We consider the last situation here.

Let us first consider a body with a given charge density $\rho(\mathbf{r})$ that is subject to an electric field $\mathbf{E}(\mathbf{r})$ that varies slowly over the charge distribution. Assume we choose the origin at a central point of the body and make an expansion of the field around this point,

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}(0) + \mathbf{r} \cdot \boldsymbol{\nabla} \mathbf{E}(0) + \dots \tag{11.53}$$

The total force that acts on the body is then

$$
\begin{aligned}
\mathbf{F}_e &= \int \rho \mathbf{E} \, dV \\
&= \mathbf{E}(0) \int \rho \, dV + [\int \rho \mathbf{r} \, dV] \cdot \boldsymbol{\nabla} \mathbf{E}(0) + ... \\
&= q\mathbf{E} + (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E} + ...
\end{aligned}
\tag{11.54}
$$

with q as the total charge of the body and $\mathbf{p}$ as the electric dipole moment. We note in particular the expression for the dipole force acting on the body.

The multipole moments also appears in the torque acting on the body

$$
\begin{aligned}
\mathbf{M}_e &= \int \mathbf{r} \times \mathbf{E} \, \rho \, dV \\
&= \int \rho(\mathbf{r}) \, \mathbf{r} \times (\mathbf{E}(0) + \mathbf{r} \cdot \boldsymbol{\nabla} \mathbf{E}(0) + ...) \, dV \\
&= \mathbf{p} \times \mathbf{E} + ...
\end{aligned}
\tag{11.55}
$$

In the expressions above one should note that $\mathbf{E}$ is the *external* field acting on the charge distribution. The internal field from one part of the charge distribution to another part does not contribute, since internal forces do not contribute to the total force or torque.

We may describe in a similar way the magnetic force and torque acting on a current distribution. The force is

$$
\begin{aligned}
\mathbf{F}_m &= \int \mathbf{j} \times \mathbf{B}(\mathbf{r}) dV \\
&= \int \mathbf{j}(\mathbf{r}) \times (\mathbf{B}(0) + (\mathbf{r} \cdot \boldsymbol{\nabla})\mathbf{B}(0) + ...) dV
\end{aligned}
\tag{11.56}
$$

For a stationary current the first term gives no contribution since $\int \mathbf{j}(\mathbf{r}) dV = 0$ as previously discussed and the magnetic force is therefore

$$
\begin{aligned}
\mathbf{F}_m &= \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}) + ... \\
&= (\mathbf{m} \cdot \boldsymbol{\nabla})\mathbf{B} + ...
\end{aligned}
\tag{11.57}
$$

Similarly the torque is

$$
\mathbf{M}_m = \mathbf{m} \times \mathbf{B} + ...
\tag{11.58}
$$

In both these expressions we have only included the lowest non-vanishing multipole contributions which are the magnetic dipole terms. These terms have precisely the same form as the corresponding terms for the electric force and torque, with the electric moments and fields replaced by the magnetic moments and fields. Note however that we have assumed $\mathbf{E}$ and $\mathbf{B}$ to be time independent external fields. This means in particular that $\boldsymbol{\nabla} \times \mathbf{B} = 0$, which is necessary for the equivalence between the two expressions given for $\mathbf{F}_m$.

Chapter 12

Electromagnetic radiation

We consider now the full problem of solving Maxwell's equations with time dependent sources. The field equation, in the Lorentz gauge, is as before

$$\partial_\nu \partial^\nu A^\mu = -\mu_0 j^\mu, \quad \partial_\mu A^\mu = 0 \tag{12.1}$$

where the current may now depend both on space and time coordinates, $j^\mu = j^\mu(\mathbf{r}, t)$. The solution involves a *retardation effect*, since the field at some distance from the source will respond to changes in the source at a delayed time, in accordance with the fact that the speed of wave propagation is finite. We shall, as the next step, examine how this retardation effect gives rise to the phenomenon of electromagnetic radiation.

12.1 Solutions to the time dependent equation

We note that also in this general case, with time dependent sources, the equations for each vector component of A^μ can be solved separately, and the equations are all of the same form. There is a coupling between the components, as follows from the Lorentz gauge condition $\partial_\mu A^\mu = 0$, but this equation is automatically taken care of by the continuity equation $\partial_\mu j^\mu = 0$, as one can readily show. In non-covariant form the differential equation to be solved is

$$(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}) f(\mathbf{r}, t) = -s(\mathbf{r}, t) \tag{12.2}$$

where f represents one of the components of the potential and s represents the corresponding component of the current density. When discussing solutions of this equation we consider the source term $s(\mathbf{r}, t)$ as a known function while $f(\mathbf{r}, t)$ is the unknown function, to be determined as a solution of the differential equation.

To proceed we introduce the Fourier transformation of the equation with respect to time. For the function $f(\mathbf{r}, t)$ this transformation is

$$f(\mathbf{r}, t) = \int_{-\infty}^{\infty} \tilde{f}(\mathbf{r}, \omega) e^{-i\omega t} d\omega, \quad \tilde{f}(\mathbf{r}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\mathbf{r}, t) e^{i\omega t} dt \tag{12.3}$$

and the same type of transformation formulas are valid for the source function $s(\mathbf{r}, t)$. In the Fourier transformed version time t is then replaced by the frequency variable ω, while the space

coordinate $\mathbf{r}$ is left unchanged. Applied to the differential equation (12.2) the transformation gives the following equation for the Fourier transformed fields,

$$(\mathbf{\nabla}^2 + \frac{\omega^2}{c^2})\tilde{f}(\mathbf{r}, \omega) = -\tilde{s}(\mathbf{r}, \omega) \tag{12.4}$$

This differential equation, which only includes derivatives with respect to the space coordinates, shows a clear resemblance to the electrostatic equation. However, the presence of the constant $\frac{\omega^2}{c^2}$ makes it different. The differential equation (12.4) is known as *Helmholtz' equation*.

Even if there is a difference, we may take some inspiration from the Coulomb problem. As we have earlier discussed, the usual way to find the solution of the electrostatic problem is first to find the electrostatic potential of a point charge, and to use this to find a general solution by integrating over the actual charge distribution. For a point charge q the charge distribution is $\rho(\mathbf{r}) = q\delta(\mathbf{r})$, and the electrostatic equation is

$$\mathbf{\nabla}^2\phi = -\frac{\rho}{\epsilon_0} = -\frac{q}{\epsilon_0}\delta(\mathbf{r}) \tag{12.5}$$

with the Coulomb potential as solution

$$\phi = \frac{q}{4\pi\epsilon_0 r} \tag{12.6}$$

This shows that we (formally) have the following identity

$$\mathbf{\nabla}^2\left(\frac{1}{r}\right) = -4\pi\delta(\mathbf{r}) \tag{12.7}$$

We will show that a similar relation is valid when a constant is added to $\mathbf{\nabla}^2$, in the following way

$$(\mathbf{\nabla}^2 + \alpha^2)\left(\frac{e^{i\alpha r}}{r}\right) = -4\pi\delta(\mathbf{r}) \tag{12.8}$$

which gives the solution of the Helmholtz equation for a point source.

To this end we evaluate the action of the Laplacian on the function introduced above,

$$
\begin{aligned}
\mathbf{\nabla}^2\left(\frac{e^{i\alpha r}}{r}\right) &= \frac{1}{r}\mathbf{\nabla}^2 e^{i\alpha r} + 2\,\mathbf{\nabla}\frac{1}{r}\cdot\mathbf{\nabla}e^{i\alpha r} + e^{i\alpha r}\,\mathbf{\nabla}^2\left(\frac{1}{r}\right) \\
&= -\frac{\alpha^2}{r}e^{i\alpha r} + e^{i\alpha r}\mathbf{\nabla}^2\left(\frac{1}{r}\right)
\end{aligned}
\tag{12.9}
$$

which gives

$$(\mathbf{\nabla}^2 + \alpha^2)(\frac{e^{i\alpha r}}{r}) = -4\pi e^{i\alpha r}\delta(\mathbf{r}) = -4\pi\delta(\mathbf{r}) \tag{12.10}$$

In the last step we have used the fact that the delta function vanishes unless $r = 0$, and in this point the exponential function is equal to 1.

We immediately rewrite the above equation in a form directly related to the problem we would like to solve

$$(\nabla^2 + \frac{\omega^2}{c^2})(\frac{e^{\pm i\frac{\omega}{c}r}}{r}) = -4\pi\delta(\mathbf{r}) \tag{12.11}$$

Note that in this expression we have explicitly made use of the fact that α is only up to a sign determined by α^2. Our interpretation of this equation is now the following. Assume we modify the electrostatic equation by adding the term proportional to ω^2/c^2. This change in the field equation will modify the potential set up by a point charge so it is no longer a Coulomb potential. Actually the modification is not unique, the Coulomb potential can be modified either by the factor $\exp(+i\frac{\omega}{c}r)$ or $\exp(-i\frac{\omega}{c}r)$. However, as we shall soon see, there is a reason for choosing one these as the *physical* solution.

With the solution established for a point source, the potential for a source distribution is found by integrating over the distribution, in the same way as done for the Coulomb problem. This gives with $\tilde{s}(\mathbf{r},\omega)/4\pi$ as the source term the following solutions

$$\tilde{f}_{\pm}(\mathbf{r},\omega) = \frac{1}{4\pi}\int \frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{s}(\mathbf{r}',\omega)d^3r' \tag{12.12}$$

where the distance r from the point charge is now replaced by the distance $|\mathbf{r} - \mathbf{r}'|$ form the point of integration. The corresponding time dependent solution of the original equation is found as the Fourier integral

$$\begin{aligned}
f_{\pm}(\mathbf{r},t) &= \int_{-\infty}^{\infty} \tilde{f}_{\pm}(\mathbf{r},\omega)e^{-i\omega t}d\omega \\
&= \frac{1}{4\pi}\int \left(\int_{-\infty}^{\infty} e^{-i\omega(t\mp\frac{|\mathbf{r}-\mathbf{r}'|}{c})}\tilde{s}(\mathbf{r}',\omega)d\omega\right)\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r'
\end{aligned} \tag{12.13}$$

We recognize the integral in the brackets as the Fourier integral of the function $s(\mathbf{r}',t\mp\frac{|\mathbf{r}-\mathbf{r}'|}{c})$, and this gives for $f_{\pm}$ the following expression

$$f_{\pm}(\mathbf{r},t) = \frac{1}{4\pi}\int \frac{s(\mathbf{r}',t\mp\frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r}-\mathbf{r}'|}d^3r' \tag{12.14}$$

The solutions we have found are similar in form to the Coulomb potential, since the potential $f_{\pm}$ is determined as the integral of the source term divided by the distance between the source and the point of the potential. But there is one important difference which has to do with the time dependence. The potential at a given time t is determined by the source at another time $t_{\pm} = t \pm |\mathbf{r} - \mathbf{r}'|/c$. One of these is earlier than t and the other is later than t. The solution f_- is called the *retarded* solution, since $t_- < t$, and the effect that the source has on the field therefore is delayed in time. Similarly f_+ is called the *advanced* solution since $t_+ > t$ and the effect that the source has on the potential is advanced in time. For this reason we usually consider the retarded solution f_- as the physical one. Note however that Maxwell's equations accept both these solutions, since they are invariant under time reversal, $t \to -t$. The two types of solutions can be understood as corresponding to two different types of boundary conditions.

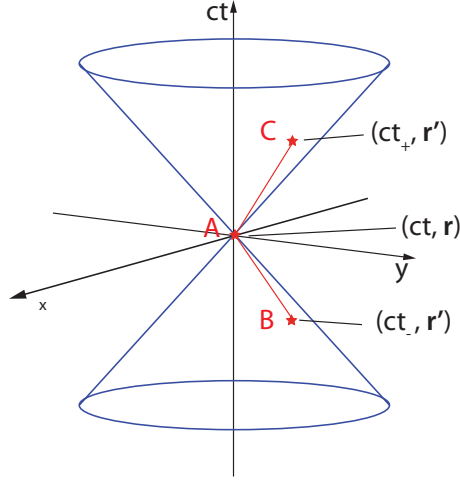Figure 12.1: Advanced and retarded space time points. Given a point A with coordinates $(ct, \mathbf{r})$ a point B with *retarded* time coordinate $t_- = t - |\mathbf{r} - \mathbf{r}'|/c$ is located on the *past light cone* of the point A. This means that a light signal emitted from B can reach the point A. Similarly a point C with *advanced* time coordinate $t_+ = t + |\mathbf{r} - \mathbf{r}'|/c$ is located on the *future light cone* of the point A. A light signal emitted from A is then able to reach the point C.

Usually we specify *initial conditions*, with the solution of Maxwell's equation given as the retarded potential, but it is also possible to specify *final conditions* with the solution given as the advanced potential.

It is of interest to note that the two space time points $(\mathbf{r}, t)$ and $(\mathbf{r}', t_\pm)$ can be connected by a light signal, since we have

$$(\mathbf{r} - \mathbf{r}')^2 - c^2(t - t_\pm)^2 = 0 \tag{12.15}$$

as we can readily check. Thus $(\mathbf{r}', t_-)$ lies on the past light cone relative to $(\mathbf{r}, t)$, while $(\mathbf{r}', t_+)$ lies on the future light cone.

12.1.1 The retarded potential

We now translate the results we have found to expressions for the electromagnetic potentials. In the following we shall consider only the retarded solutions, which we regard as the physical ones. For the scalar and vector potentials the expressions are

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_-)}{|\mathbf{r} - \mathbf{r}'|} d^3 r'$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}', t_-)}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \tag{12.16}$$

with $t_- = t - |\mathbf{r} - \mathbf{r}'|/c$ referred to as the *retarded time*. It is interesting to note that the potentials we have found have precisely the same form as the potentials previously found with static sources. The only effect of the time dependence sits in the retardation effect, the effect

that there is a time delay between the change in the charge and current distributions and the effects measured in the potentials. This means that the volume integrals in the expressions for ϕ and $\mathbf{A}$ are not integrals over space at constant t. Instead they are integrals over the three-dimensional past light cone of the point $\mathbf{r}$.

Even if the effect of time evolution of the source terms looks simple (and innocent) when we consider the potentials, that is not so when we consider the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$. This is because the retarded time t_- depends on $\mathbf{r}$ and $\mathbf{r}'$. When the fields are expressed through derivatives of the potentials, this dependence on $\mathbf{r}$ gives rise to new terms in the expressions for $\mathbf{E}$ and $\mathbf{B}$. These terms have an immediate physical interpretation. They describe radiation from the time dependent sources.

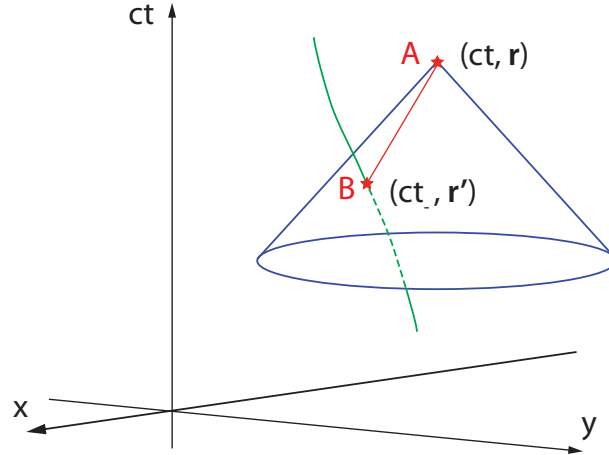## 12.2 Electromagnetic potential of a point charge

Figure 12.2: Retarded point on the world line of a point charge. Given a point A with coordinates $(ct, \mathbf{r})$ there is only one point B which is both located on the world line and on the past light cone of A. This means that there is only one point in the volume integral of the retarded electromagnetic potentials that gives a contribution at the point A.

In this case the charge and current densities are expressed as

$$
\begin{aligned}
\rho(\mathbf{r}, t) &= q\delta(\mathbf{r} - \mathbf{r}(t)) \\
\mathbf{j}(\mathbf{r}, t) &= q\mathbf{v}(t)\delta(\mathbf{r} - \mathbf{r}(t))
\end{aligned}
\tag{12.17}
$$

with q as the charge, $\mathbf{r}(t)$ as the time dependent position of the charge and $\mathbf{v}(t)$ as the velocity. The presence of the delta function means that in the expressions we have derived for the potentials produced by charges and currents, the integral over the densities will get contributions only from a single point. However, there is a complication due to the retardation effect. We consider first the scalar potential,

$$
\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \int \frac{\delta(\mathbf{r}' - \mathbf{r}(t_-))}{|\mathbf{r} - \mathbf{r}'|} d^3r'
\tag{12.18}
$$

One should note that the retarded time $t_- = t - |\mathbf{r} - \mathbf{r}'|/c$ is a function of the integration variable $\mathbf{r}'$ and this we have to take into account when integrating over the delta function. It is convenient to introduce the argument of the delta function as a new integration variable,

$$\mathbf{r}'' = \mathbf{r}' - \mathbf{r}(t_-) \tag{12.19}$$

where we note that the vector $\mathbf{r}$ in the definition of t_- is a constant under the integration. The change of variable introduce a change in the integration measure given by

$$d^3 r'' = J d^3 r' \tag{12.20}$$

where J is the Jacobian of the transformation, which is the determinant of the matrix with elements

$$J_{kl} = \frac{\partial x_k''}{\partial x_l'} \tag{12.21}$$

We find for this matrix element the following expression

$$
\begin{aligned}
J_{kl} &= \delta_{kl} - \frac{dx_k}{dt_-} \frac{\partial t_-}{\partial x_l'} \\
&= \delta_{kl} + \frac{1}{c} v_k(t_-) \frac{\partial}{\partial x_l'} \sqrt{r^2 + r'^2 - 2\mathbf{r} \cdot \mathbf{r}'} \\
&= \delta_{kl} - \frac{1}{c} v_k(t_-) \frac{x_l - x_l'}{|\mathbf{r} - \mathbf{r}'|}
\end{aligned}
\tag{12.22}
$$

To simplify expressions we introduce $\boldsymbol{\beta}(t) = \mathbf{v}(t)/c$ and $\mathbf{n} = (\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$. The matrix element of the Jacobian can then be written as

$$J_{kl} = \delta_{kl} - \beta_k n_l \tag{12.23}$$

When calculating the corresponding determinant it is useful temporarily to chose the x axis in the direction of $\mathbf{n}$, which gives $n_1 = 1,\ n_2 = n_3 = 0$. The result is simply $1 - \beta_1$ which we rewrite in a coordinate independent way as

$$J = 1 - \boldsymbol{\beta} \cdot \mathbf{n} \tag{12.24}$$

The integral in the expression for the potential can now be evaluated,

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \int \frac{\delta(\mathbf{r}'')}{|\mathbf{r} - \mathbf{r}'|} \frac{1}{1 - \boldsymbol{\beta} \cdot \mathbf{n}} d^3 r'' \tag{12.25}$$

In this integral the effect of the delta function is simply to put $\mathbf{r}'' = 0$, which is equivalent to $\mathbf{r}' = \mathbf{r}(t_-)$, and the potential can therefore be written as

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}(t_-)|(1 - \boldsymbol{\beta}(t_-) \cdot \mathbf{n}(t_-))} \tag{12.26}$$

To simplify this expression we introduce the relative vector $\mathbf{R}(t) = \mathbf{r} - \mathbf{r}(t)$ and use the label *ret* to indicate that expression should be evaluated at time $t = t_-$.

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 \left(R - \boldsymbol{\beta} \cdot \mathbf{R}\right)_{ret}} \tag{12.27}$$

The vector potential can be found in precisely the same way, and we simply give the result

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi} \left(\frac{\mathbf{v}}{R - \boldsymbol{\beta} \cdot \mathbf{R}} \right)_{ret} \tag{12.28}$$

The expressions we have found for the potentials of a moving point charge are called the *Lienard-Wiechert potentials*. We note that these expressions are valid with no restriction on the motion of the charge; it may be at rest, move with constant speed or be accelerated. Therefore the potentials implicitly contain all effects of charge in motion, in particular radiation from an accelerated charge.

There is a clear similarity between the expressions found here and that of the Coulomb potential of a stationary point charge. This we see most clearly if we choose as inertial frame the rest frame of the moving charge *at the retarded time* t_-. In this frame the potential are

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 R_{ret}}, \quad \mathbf{A}(\mathbf{r}, t) = 0 \tag{12.29}$$

This is simply the Coulomb potential with the distance to the charge determined by its position at the retarded time.

This gives us a simple picture of how the potential in the surrounding space-time is created by the moving charge. Each point along its trajectory determines the potential on the future light cone from the chosen point as a Coulomb potential in the rest frame of the charge. If the charge is accelerated this rest frame changes along the path and this means that the potential is not that of a Coulomb potential in a *fixed* inertial frame.

12.3 General charge and current distribution: The fields far away

We consider now the potentials of a general time-dependent charge and current distribution, but restrict the discussion to points $\mathbf{r}$ that are far away from the distribution. In that case the same approximation technique as used in the multipole expansions of static distributions can be used. With $\mathbf{r}$ denoting the position at which the potential is evaluated and $\mathbf{r}'$ as the integration variable over the charge distribution. We again assume the origin to be chosen close to the charges so that r'/r is a small quantity which we can use as an expansion parameter. The distance to the charge distribution is as before given by

$$|\mathbf{r} - \mathbf{r}'| = r - \frac{\mathbf{r} \cdot \mathbf{r}'}{r} + \dots \tag{12.30}$$

The expression for the retarded time can be expanded in a similar way,

$$t_- = t - \frac{r}{c} + \frac{\mathbf{r} \cdot \mathbf{r}'}{rc} + \dots \tag{12.31}$$

We include now only the terms to order r' in these expansions.

When considering the scalar potential we need to make an expansion of the charge density

$$
\begin{aligned}
\rho(\mathbf{r}', t_-) &= \rho(\mathbf{r}', t - \frac{r}{c}) + \frac{\mathbf{r} \cdot \mathbf{r}'}{rc} \frac{\partial \rho}{\partial t}(\mathbf{r}', t - \frac{r}{c}) + \dots \\
&= \rho(\mathbf{r}', t_r) + \frac{\mathbf{r} \cdot \mathbf{r}'}{rc} \frac{\partial \rho}{\partial t}(\mathbf{r}', t_r) + \dots
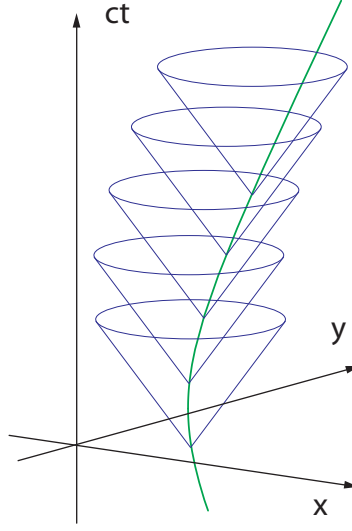\end{aligned}
\tag{12.32}
$$

Figure 12.3: Geometrical interpretation of the electromagnetic potentials produced by an accelerated charged point particle. The potential at any given space-time point is uniquely determined by the position and velocity of the charge at the *retarded* time. This is the time from which a light signal sent from the particle is able to reach the chosen space-time point. As a consequence all points on the future light cone of any chosen point on the world line of the particle will be determined by the position and velocity of the charge at this point. The figure illustrates how the full space-time dependent potential can be viewed as composed of a sequence of contributions defined on the future light cones (in blue) of the moving charge, here represented by the green curve. Note that even if the potential has no direct dependence the acceleration of the particle, the electromagnetic fields do, since they dependent of the space-time derivatives of the potential.

where we have introduced $t_r = t - r/c$, which is the retarded time, not for a general point $\mathbf{r}'$ of the charge distribution, but rather of the origin $\mathbf{r}' = 0$. This we assume to be a central point of the distribution. From the above expressions we find

$$\frac{\rho(\mathbf{r}', t_-)}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r}\rho(\mathbf{r}', t_r) + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2 c}\frac{\partial \rho}{\partial t}(\mathbf{r}', t_r) + ... \qquad (12.33)$$

where we in this expansion have kept only the terms that fall off with distance as $1/r$ or slower. This expression is now inserted in the integral expression for the potential, which gives

$$
\begin{aligned}
\phi(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_-)}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \\
&= \frac{1}{4\pi\epsilon_0 r} \int \rho(\mathbf{r}', t - \frac{r}{c}) d^3 r' + \frac{1}{4\pi\epsilon_0 r^2 c} \int (\mathbf{r} \cdot \mathbf{r}')\frac{\partial \rho}{\partial t}(\mathbf{r}', t - \frac{r}{c}) d^3 r' + ... \\
&= \frac{q}{4\pi\epsilon_0 r} + \frac{\mathbf{r} \cdot \dot{\mathbf{p}}_{ret}}{4\pi\epsilon_0 r^2 c} + ... \qquad (12.34)
\end{aligned}
$$

with q as the total charge and $\mathbf{p}$ as the electric dipole moment. In the expression for the potential it is the time derivative of the dipole moment at the retarded time $t_r = t - r/c$ that enters. The dipole term is only the first term of a multipole expansion of the potential, with the

quadrupole term as the next. One should note that when only terms that fall off like $1/r$ for large r are included, the static multipoles do not contribute, but the time derivative of these do. In fact there are contributions from all higher multipoles, but for the $1/r$ terms the number of time derivatives increases with the degree of the multipole, so that the second derivative of the quadrupole term contributes etc.

We continue now to analyze the vector potential in the same way. The general expression is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}', t_-)}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \tag{12.35}$$

where a Taylor expansion is introduced for the current density in the same way as done above for the charge density. We find

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r} \int \mathbf{j}(\mathbf{r}', t - \frac{r}{c}) d^3 r' + \frac{\mu_0}{4\pi c r^2} \int (\mathbf{r} \cdot \mathbf{r}') [\frac{\partial \mathbf{j}}{\partial t}(\mathbf{r}', t - \frac{r}{c})] d^3 r' + \dots \tag{12.36}$$

The first term can be expressed in terms of the electric dipole moment, since have

$$\int \mathbf{j}(\mathbf{r}, t) d^3 r' = \dot{\mathbf{p}}(t) \tag{12.37}$$

which is an identity that we have earlier demonstrated (see Eq.(10.47)).

To rewrite the second term another identity, given by Eq.(11.46), will be needed

$$\int x_k j_l(\mathbf{r}) \, d^3 r = - \int x_l j_k(\mathbf{r}) \, d^3 r - \frac{d}{dt} \int x_k x_l \rho(\mathbf{r}) \, d^3 r \tag{12.38}$$

This implies

$$\int (\mathbf{r} \cdot \mathbf{r}') \mathbf{j}(\mathbf{r}', t) d^3 r' = \frac{1}{2}[\int \mathbf{r}' \times \mathbf{j}(\mathbf{r}', t) d^3 r'] \times \mathbf{r} + \frac{1}{2} \frac{d}{dt} \int \mathbf{r}' (\mathbf{r} \cdot \mathbf{r}') \rho(\mathbf{r}') d^3 r'$$
$$= \mathbf{m} \times \mathbf{r} + \frac{1}{2} r \dot{\mathbf{D}}_\mathbf{n} \tag{12.39}$$

where we have introduced the magnetic dipole moment $\mathbf{m}$ and an electric quadrupole *vector* $\mathbf{D_n}$ defined by

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r}' \times \mathbf{j}(\mathbf{r}', t) d^3 r'$$
$$\mathbf{D_n} = \int \mathbf{r}'(\mathbf{r}' \cdot \mathbf{n}) \rho(\mathbf{r}', t) d^3 r' \tag{12.40}$$

with $\mathbf{n} = \mathbf{r}/r$ as the unit vector in direction of $\mathbf{r}$. By use of these expressions we are able to write vector potential as

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r} (\dot{\mathbf{p}} + \frac{1}{c} \dot{\mathbf{m}} \times \mathbf{n} + \frac{1}{2c} \ddot{\mathbf{D}}_\mathbf{n} + \dots)_{ret}, \quad \mathbf{n} = \frac{\mathbf{r}}{r} \tag{12.41}$$

where the subscript *ret* now means that the vectors should be taken at retarded time $t_r = t - r/c$. We note that both the electric and the magnetic dipole momenta, as well as the electric

quadrupole moment contributes to the potential. (In the case of the scalar potential we did not include all these terms.)

Let us next consider the magnetic field that corresponds to the vector potential that we have found,

$$
\begin{aligned}
\mathbf{B}(\mathbf{r}, t) &= \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}, t) \\
&= \frac{\mu_0}{4\pi r}(\boldsymbol{\nabla} t_r) \times \frac{d}{dt}(\dot{\mathbf{p}} + \frac{1}{c}\dot{\mathbf{m}} \times \mathbf{n} + \frac{1}{2c}\ddot{\mathbf{D}}_\mathbf{n} + ...)_{ret} + ...
\end{aligned} \tag{12.42}
$$

In this expression only the terms that fall off as $1/r$ are explicitly written out. The gradient of the retarded time is given by

$$
\boldsymbol{\nabla} t_r = \boldsymbol{\nabla}(t - \frac{r}{c}) = -\frac{1}{c}\mathbf{n} \tag{12.43}
$$

and this gives for the magnetic field the following expression

$$
\begin{aligned}
\mathbf{B}(\mathbf{r}, t) &= \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}, t) \\
&+ \frac{\mu_0}{4\pi rc}(\ddot{\mathbf{p}} + \frac{1}{c}\ddot{\mathbf{m}} \times \mathbf{n} + \frac{1}{2c}\dddot{\mathbf{D}}_\mathbf{n} + ...)_{ret} \times \mathbf{n} + ...
\end{aligned} \tag{12.44}
$$

One should note all these terms, which fall off as $1/r$, are obtained by differentiation through the retarded time variable t_r. These dominate, for large r, over the terms that fall off as higher powers of $1/r$. Among these are the static terms of the multipole expansion which we have examined earlier.

A similar expression as for the magnetic field (12.44) is found for the electric field. In the following we shall restrict the discussion to these fields, refereed to as the *radiation fields*, which are the parts of the electromagnetic field that dominate far from the sources.

12.4   Radiation fields

Sufficiently far away from the charge and current distributions only terms that fall off with distance as $1/r$ give substantial contributions. This region is called the *radiation zone*. There the first term in the expression (12.44) can be neglected and we get as expression for the magnetic component of the *radiation field*,

$$
\mathbf{B}_{rad}(\mathbf{r}, t) = \frac{\mu_0}{4\pi rc}(\ddot{\mathbf{p}} \times \mathbf{n} + \frac{1}{c}(\ddot{\mathbf{m}} \times \mathbf{n}) \times \mathbf{n} + \frac{1}{2c}\dddot{\mathbf{D}}_\mathbf{n} \times \mathbf{n} + ...)_{ret} \tag{12.45}
$$

To find the corresponding expression for the electric field we may write the $\mathbf{E}$ field in terms of the potentials and follow the same procedure as for $\mathbf{B}$. However, we may make a short cut in the following way. In the radiation zone the fields can in the neighborhood of a point $\mathbf{r}$ far from the sources be regarded as a plane wave which propagate in the direction $\mathbf{n}$. (It is not necessarily a *monochromatic* plane wave since the Fourier transform of the time dependent multipole momenta may contain more than one frequency.) But as previously shown, for electromagnetic plane waves we have a simple relation between $\mathbf{E}$ and $\mathbf{B}$ that is not dependent of the frequency of the wave, $\mathbf{E} = -c\mathbf{n} \times \mathbf{B}$. In the present case the electric field therefore takes the form

$$
\mathbf{E}_{rad}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r}((\ddot{\mathbf{p}} \times \mathbf{n}) \times \mathbf{n} - \frac{1}{c}\ddot{\mathbf{m}} \times \mathbf{n} + \frac{1}{2c}(\dddot{\mathbf{D}}_\mathbf{n} \times \mathbf{n}) \times \mathbf{n} + ...)_{ret} \tag{12.46}
$$

There are two conditions that should be satisfied in the radiation zone, where the radiation fields (12.45) and (12.46) give the dominating contributions to the full electromagnetic field. The first one is $r >> a$ with a as a typical linear size of the charge and current distribution. Our derivation so far has been based on this to be satisfied. If that is not the case there will be fields with a faster fall off with distance (which we have omitted in the expansions) that would compete with the radiation fields in strength. The other is $r >> \lambda$, with λ as a typical wave length of the radiation. If that is not satisfied there are contributions to the fields where a smaller number of time derivatives of the multipole momenta could compensate for a higher power in $1/r$. In particular this condition is necessary for the second term in (12.44) to dominate over the first term.

If furthermore we have the following condition satisfied , $\lambda >> a$, then the first terms of the multipole expansions of the radiation field, (12.45) and (12.46), will dominate over the later ones, so that the electric dipole contribution would be more important than the electric quadrupole contribution etc.

The electric and magnetic dipole contributions may seem to be giving comparable contributions to the radiation, but under normal conditions that is not the case. The reason for this is that the magnetic moment depends on the charge *currents* and therefore on the velocity of the charges (usually electrons). This implies that the magnetic dipole term is damped by a factor v/c relative to that of the electric dipole term, with v as the (average) velocity of the charges.

Electric dipole radiation will therefore usually be dominating, for example in the radiation from an antenna. This contribution to the radiation is described by the term which depends on $\ddot{\mathbf{p}}$, and it is for short referred to as the $E1$ radiation term. However, under certain conditions this type of radiation may be suppressed so that *magnetic dipole radiation* would be dominating. This is the term depending on $\ddot{\mathbf{m}}$, with the short hand notation $M1$. Similarly *electric quadrupole radiation*, referred to as $E2$, may also under certain conditions be important, etc.

As an interesting point to stress, the radiation fields described by (12.45) and (12.46) appear as a direct consequence of the retardation effects. This has been clearly demonstrated in our derivation of the magnetic field $\mathbf{B}$ in (12.44). It is the derivative of the retarded time t_r with respect to r that gives the field contributions that fall off like $1/r$, while the direct derivation of the potentials with respect to r gives field contributions that fall off like $1/r^2$.

12.4.1 Electric dipole radiation

When the electric dipole terms dominate the radiation, the expressions for the radiation fields simplify to

$$
\begin{aligned}
\mathbf{E}_{rad}(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0 r c^2} (\ddot{\mathbf{p}}_{ret} \times \mathbf{n}) \times \mathbf{n} \\
\mathbf{B}_{rad}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi rc} \ddot{\mathbf{p}}_{ret} \times \mathbf{n}
\end{aligned}
\tag{12.47}
$$

Poynting's vector for this field is

$$
\begin{aligned}
\mathbf{S}(\mathbf{r},t) &= \frac{1}{\mu_0}\mathbf{E}_{rad} \times \mathbf{B}_{rad} \\
&= \frac{c}{\mu_0}\mathbf{B}_{rad}^2\,\mathbf{n} \\
&= \frac{\mu_0}{16\pi^2 r^2 c}(\ddot{\mathbf{p}}_{ret} \times \mathbf{n})^2\,\mathbf{n} \\
&= \frac{\mu_0}{16\pi^2 r^2 c}(\ddot{\mathbf{p}}^2\sin^2\theta)_{ret}\,\mathbf{n} \tag{12.48}
\end{aligned}
$$

where the angle θ introduced in the last step is the angle between the vectors $\ddot{\mathbf{p}}$ and $\mathbf{n}$ and the subscript *ret* is a reminder that all variables at the source should be taken at the retarded time $t_r = t - r/c$.

Since $\mathbf{S}(\mathbf{r},t)$ gives the energy current density of the electromagnetic field, the above expression shows that the radiation is, as one should expect, directed in the radial direction $\mathbf{n}$ away from the source of the radiation. The total power radiated is given as the integral of $\mathbf{S}$ over all angles,

$$
\begin{aligned}
P &= \frac{\mu_0}{16\pi^2 c}\ddot{\mathbf{p}}_{ret}^2 \iint d\phi d\theta \sin^3\theta \\
&= \frac{\mu_0}{8\pi c}\ddot{\mathbf{p}}_{ret}^2 \int_{-1}^{1} du(1-u^2) \\
&= \frac{\mu_0}{6\pi c}\ddot{\mathbf{p}}_{ret}^2 \tag{12.49}
\end{aligned}
$$

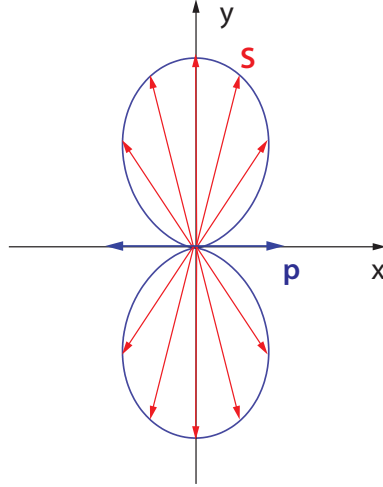Figure 12.4: Angular distribution of radiated energy in electric dipole radiation. The electric dipole moment $\mathbf{p}$ here oscillates along the x axis. The magnitude of the Poynting vector $\mathbf{S}$, which gives the angular distribution of the radiated power is indicated in the figure.

For radiation from a linear antenna the direction of the electric dipole moment is fixed by the direction of the antenna and only the amplitude oscillates in time. The angular distribution

of the radiation then has the simple form

$$\mathbf{S}(\mathbf{r}, t) = \frac{\mu_0 \, \ddot{p}_{ret}^2}{16\pi^2 r^2 c} \sin^2 \theta \, \mathbf{n} \tag{12.50}$$

where only the amplitude determined by $\ddot{p}_{ret}^2$ is time dependent, while the direction of the dipole, given by the angle θ is constant. The angular distribution of the radiated energy is illustrated in Fig. 12.4, and we note in particular that maximum of the radiation is in the direction perpendicular to the direction of the antenna.

Let us further assume the time variation of the electric dipole moment of the antenna to have a simple harmonic form,

$$p(t) = p_0 \cos \omega t \tag{12.51}$$

with oscillation period $T = 2\pi/\omega$. The expression for the time averaged radiated power from the antenna is then

$$\begin{aligned}
\bar{P} &= \frac{1}{T} \int_0^T P(t) dt \\
&= \frac{\mu_0 p_0^2 \omega^4}{6\pi c} \int_0^T \cos^2 \omega t \, dt \\
&= \frac{\mu_0 \, p_0^2 \, \omega^4}{12\pi c}
\end{aligned} \tag{12.52}$$

We note in particular that, for fixed p_0, the radiated power increases rapidly with the frequency of the oscillating dipole moment.

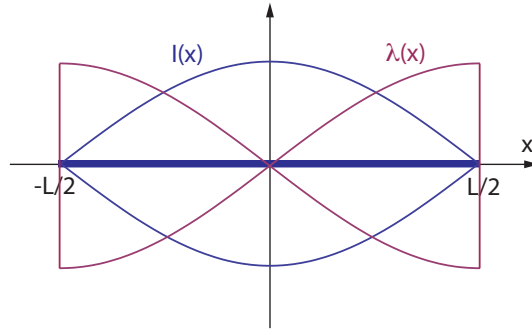### 12.4.2 Example: Electric dipole radiation from a linear antenna

Figure 12.5: Oscillating current and charge in a linear antenna. The figure shows the current $I(x)$ and charge density $\lambda(x)$ along the antenna, where the current here has a cosine form and the charge density a sinus form as functions of x. They both oscillate in time, with a phase shift of $\pi/2$, so that the charge density vanishes when the current has its maximum and *vice versa*.

Let us assume a linear antenna of length L is directed along the x axis as illustrated in the figure. Let us further assume an oscillating current is induced in the antenna, of the form

$$I(x,t) = I_0 \cos(\frac{x}{L}\pi) \cos \omega t \tag{12.53}$$

The x dependence of the current shows that it has its maximum at the midpoint of the antenna and that it vanishes, as it should, at the endpoints. Charge conservation now gives a connection between the space variation in current and the time variation in the charge density, which has the form

$$\frac{\partial I}{\partial x} + \frac{\partial \lambda}{\partial t} = 0 \tag{12.54}$$

where λ is the *linear* charge density, *i.e.*, the charge per unit length along the antenna. The equation is the one-dimensional form of the continuity equation for the charge, which we earlier have formulated as an equation in three space dimensions.

The electric dipole moment can in this case be expressed as a one dimensional integral along the antenna,

$$p(t) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \lambda(x,t)\, x\, dx \tag{12.55}$$

with direction $\mathbf{p} = p\,\mathbf{i}$ along the x axis. For the time derivative we find

$$
\begin{aligned}
\dot{p} &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\partial \lambda}{\partial t} x\, dx \\
&= -\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\partial I}{\partial x} x\, dx \\
&= -\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\partial}{\partial x}(xI)\, dx + \int_{-\frac{L}{2}}^{\frac{L}{2}} I\, dx \\
&= \int_{-\frac{L}{2}}^{\frac{L}{2}} I\, dx \\
&= \frac{2L}{\pi} I_0 \cos \omega t
\end{aligned}
\tag{12.56}
$$

The corresponding expression for the oscillating dipole moment is

$$p(t) = p_0 \sin \omega t \tag{12.57}$$

with $p_0 = 2L/\pi\omega$. The double time derivative of the dipole moment, which is needed for the radiation formula, is

$$\ddot{p}(t) = -p_0 \omega^2 \sin \omega t = \frac{2L}{\pi}\omega I_0 \sin \omega t \tag{12.58}$$

The formula for the radiated power now gives

$$P(t) = \frac{\mu_0}{6\pi c} \ddot{p}_{ret}^2$$

$$= \frac{\mu_0}{6\pi c} p_0^2 \omega^4 \sin^2 \omega t_r , \qquad t_r = t - r/c \tag{12.59}$$

which, when expressed in terms of the current amplitude, is

$$P(t) = \frac{2}{3\pi^3} \frac{\mu_0 L^2 \omega^2 I_0^2}{c} \sin^2 \omega t_r \tag{12.60}$$

For the time average of the power this gives

$$\bar{P} = \frac{\mu_0 L^2 \omega^2 I_0^2}{3\pi^3 c} \tag{12.61}$$

since the average value of $\sin^2 \omega t$ is 1/2. We note that the radiated power, for fixed current I_0, increases quadratically with the oscillation frequency of the dipole moment.

Let us finally consider the polarization of the radiation as it is measured by a receiver. As we already know both $\mathbf{E}$ and $\mathbf{B}$ are orthogonal to the direction of wave propagation which is given by $\mathbf{n}$, the unit vector pointing from the antenna to the receiver. Since the dipole moment oscillates in strength but not in direction, the general expressions for the electric and magnetic fields produced in electric dipole radiation, (12.47), shows that $\mathbf{B}$ will be oscillating along the fixed line $\mathbf{i} \times \mathbf{n}$ and $\mathbf{E}$ along the fixed line $(\mathbf{i} \times \mathbf{n}) \times \mathbf{n}$. This means that the radiation field will for any direction $\mathbf{n}$ of propagation be *linearly* polarized. The polarization plane, which is the plane defined by the direction of wave propagation and the direction of the oscillating $\mathbf{E}$ field is identical to the plane spanned by the direction of the antenna and the direction from the antenna to the receiver. This is so since $\mathbf{E}$ oscillates in this plane in the direction perpendicular to $\mathbf{n}$. The magnetic field will then oscillate in the direction orthogonal to the polarization plane.

12.5 Larmor's radiation formula

As a last point we shall consider radiation from an accelerated point charge. The fields produced by a moving point charge has previously been given in the form of the Lienard-Wiechert potentials (see (12.26) and (12.28)). We consider the non-relativistic form of these potentials, which correspond to $\beta = v/c \to 0$. The corresponding radiation fields are

$$\mathbf{E}(\mathbf{r}, t) = [\frac{q\mu_0}{4\pi R} (\mathbf{a} \times \mathbf{n}) \times \mathbf{n}]_{ret}$$

$$\mathbf{B}(\mathbf{r}, t) = [\frac{q\mu_0}{4\pi R c} \mathbf{a} \times \mathbf{n}]_{ret} \tag{12.62}$$

In these expressions we have

$$\mathbf{R}(t) = \mathbf{r} - \mathbf{r}(t), \quad \mathbf{n} = \frac{\mathbf{R}}{R}, \quad \mathbf{a}(t) = \ddot{\mathbf{r}}(t) \tag{12.63}$$

with $\mathbf{r}$ as the position where the fields are evaluated and $\mathbf{r}(t)$ is the time dependent position vector of the moving charge. Note that in Eq.(12.62) the retarded time is measured relative to the position of the moving charge,

$$t_- = t - \frac{1}{c}|\mathbf{r} - \mathbf{r}(t_-)| \tag{12.64}$$

We note that the fields given above have the same form as the electric dipole radiation fields previously found. Thus for a point charge the dipole moment is $\mathbf{p}(t) = q\mathbf{r}(t)$ and therefore $\ddot{\mathbf{p}} = q\mathbf{a}$. There is one difference, since R and $\mathbf{n}$ depend on the time dependent position of the charge. However, when sufficiently far from the charge this time dependence is less important. The lack of higher multipole contributions is clearly a consequence of the pointlike charge distribution of the moving charge.

Poynting's vector for the fields is of the same form as for electric dipole radiation

$$\mathbf{S}(\mathbf{r}, t) = \frac{q^2 \mu_0}{16\pi^2 r^2 c}[a^2 \sin^2 \theta \, \mathbf{n}]_{ret} \tag{12.65}$$

where θ is the angle between the direction of the acceleration $\mathbf{a}$ and the direction vector $\mathbf{n}$. The formula for the integrated radiated power is

$$P(t) = \frac{\mu_0 q^2}{6\pi c} a_{ret}^2 \tag{12.66}$$

This is called the Larmor radiation formula.

The expressions given above gives a simple picture of the radiation process. At a given time along the space-time trajectory of the charge, it will radiate energy at a rate proportionally to the square of the acceleration of the charge. The energy emitted in a time interval dt will then propagate as an expanding spherical shell radially outwards from the charge. The time delay when the shell moves outwards is the origin of the retardation effect. When the charge moves, the center of these shells of energy will continuously change, so that when viewed from a fixed point in space the radiation is at any time directed away from the position of the charge at the retarded time.

Summary

This part of the lectures, on electrodynamics, has been focussed on how Maxwell's equations form the basis for our understanding of the variety of electromagnetic phenomena. Beginning from the four equations that constitute the set of Maxwell's equations, we have first seen how these can be compactified into two covariant field equations which involve the electromagnetic field tensor rather than the electric and magnetic field separately. This covariant form is attractive, not only because of its compactness and elegance, but also because of the relativistic invariance of electromagnetic theory is made explicit in the covariant equations.

Relativistic invariance and symmetry under Lorentz transformations are important properties of the Maxwell theory. In fact this symmetry was realized as an interesting, although apparently somewhat formal, property of Maxwell's equations by people like Henri Poincaré even before the theory of relativity was introduced. But the true importance of these symmetries were understood only after Albert Einstein lifted the Lorentz transformation from being merely an interesting set of symmetries of Maxwell's equations to be the fundamental symmetry of all kinds of natural phenomena. When applied to the electromagnetic theory the relativistic invariance predicts the specific way in which the $\mathbf{E}$ and $\mathbf{B}$ fields are mixed when changing from one inertial reference frame to another. As we have seen, the covariant description of the field in terms of the electromagnetic field tensor gives a direct information of how this mixing takes place.

The problem addressed in these notes is how to solve Maxwell's equation under different conditions. As a first step it is then of interest to simplify the equations by introducing the electromagnetic potentials. These are not uniquely determined by the $\mathbf{E}$ and $\mathbf{B}$ fields, and we may therefore impose certain gauge conditions on the potentials to simplify the equations. Both the non-covariant Coulomb gauge and the covariant Lorentz gauge conditions are of interest to use, which one depends on under what conditions we will solve the equations. In these notes we have looked at three different situation. The first one when the sources of the fields, *i.e.*, the charge and current distributions, vanish. The second one is when the sources (in a given inertial frame) are time independent, and finally when we have the general situation with space and time dependent distributions of charge and current. In all these cases we assume that there are no non-trivial boundary conditions, so we look for solutions in the open infinite space, where all fields are finite or tend to zero at infinity.

In the first case, where the charge and current densities vanish, Maxwell' equations have solutions in the form of freely propagating waves. These are the electromagnetic waves that span a wide variety of phenomena, depending on the frequency of the waves, from the energetic γ radiation, through X-rays, light to microwaves and radio waves. In our somewhat brief discussion of electromagnetic waves we have focussed on the property of polarization which

characterizes all these different types of wave phenomena. The special cases of linear and circular polarization can be understood as depending on the phase and amplitude relations between two orthogonal components of the radiation, and that is also so for the general type of elliptic polarization.

When the charge and current distributions are time independent the equations for the electric and magnetic fields decouple completely and they can be examined separately in the form of electrostatic and magnetostatic equations. It is easy to find solutions of these equations by applying the linearity of the equations. Thus the general solutions of the electrostatic problem can be found as a linear superposition of the Coulomb potentials of all the small parts of the charge distributions. The magnetostatic equations are of the same form and solutions can be found by the same method. In both cases the general solutions for the (scalar or vector) potentials can be written as integrals over the charge or current distributions.

Even if explicit solutions can be found for the field equations with general stationary sources, it is often of interest to make simplifications for the resulting integral in the form of approximations that are valid for points not to close to the charges and currents. This have been done in the notes in the form of the multipole expansion. This expansion is based on the assumption that the distance from the source to the point where the potential should be determined is much larger than the extension of the source itself. For the electrostatic field the leading term is the Coulomb potential, the next is the electric dipole potential, then the electric quadrupole potential etc. For the magnetostatic potential there is a similar expansion, but here the leading term is the magnetic dipole potential. There is in fact a simple symmetry between the electrostatic and magnetostatic expansions, so that term by term the $\mathbf{E}$ and $\mathbf{B}$ fields, for dipole, quadrupole etc., are of precisely the same form.

The method used with stationary sources can with some modifications be used also to solve Maxwell's equations with general time and space dependent sources. As a first step in finding the general solution we have introduced the Fourier transform in time and thereby brought the equations into a form similar to the static cases. The type of differential equation we then meet is not identical to that of the electrostatic case, but the same general method can be used. This means that we first look for solution of the problem with a point source, which is now a modified Coulomb potential. We next extend this to the general case by making a linear superposition over contributions from all pointlike parts of the charge and current densities. Finally the inverse Fourier transformation gives the solution in the form of an integral over the time and space dependent charge and current distributions. As we have seen the solution is strikingly similar to the corresponding solutions for the electrostatic and magnetostatic potentials. The main difference with time dependent sources is the retardation effect. Thus the integral over the charge and current distributions is not taken at a fixed time, but the integral is instead over the past light cone relative to the point where the potential should be determined.

The retardation effect one should clearly expect from the theory of relativity, where the influence of a source on a the field at a distant point is delayed by the limit of propagation set by the speed of light. Even if it in this sense the effect may look innocent, it contains the important physical effect of radiation from a time dependent source. To see this explicitly we have made a multipole expansion similar to the one applied to the static cases. Far from the sources, in the radiation zone, the fields that fall off with distance as $1/r$ will dominate. These are the radiation fields, and in the derivation of the electromagnetic fields from the potentials they appear as a consequence of the position dependence of the retarded time. We have found

the expressions for the first few terms of the multipole expansion of the radiation fields, where normally the electric dipole contribution is the most important one, but where under certain conditions also magnetic dipole and electric quadrupole contributions may be significant.

Obviously there are a lot of interesting further developments of the theory that are not covered in these lecture notes. That is true for all the three parts of the notes, where the motivation has been to focus on some of the most important and simplest parts of the classical theory of mechanics and electrodynamics. One of the main objectives have been to show that the analytic approach applied in this part of physic gives the theory an attractive and elegant form, but also to show that these methods are important in solving the fundamental equations and revealing the underlying structure of the physical phenomena.