FYS 3120 Classical Mechanics and Electrodynamics
Spring semester 2014

Problem Set 2

Lagrange's equation

As discussed in the lecture notes, when the Lagrangian $L = T - V$ is expressed as a function of the generalized coordinates q_i and their velocities $\dot{q}_i$ (and possibly time t), $L = L(q_i, \dot{q}_i, t)$, then the dynamics of the system is expressed as a collection of Lagrange's equations, one for each coordinate,

$$\frac{d}{dt}\left(\frac{dL}{d\dot{q}_i}\right) - \frac{dL}{dq_i} = 0\,, \quad i = 1, 2, ..., d \tag{1}$$

Problem 2.1

A pendulum consists of a rigid rod, which we consider as massless, and a pendulum bob of mass m. The point of suspension of the pendulum has horizontal coordinate $x = s$ and vertical coordinate $y = 0$.

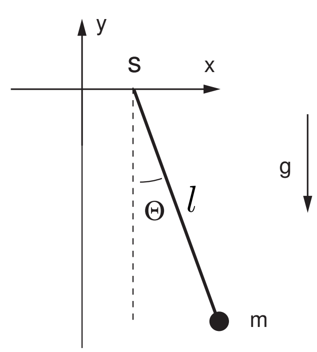

Figure 1: Problem 2.1

a) Assume first that the point of suspension is kept fixed, with $s = 0$. Use the angle θ as generalized coordinate, find the Lagrangian of the system and determine the form of Lagrange's equation for the system. Check that it has the standard form of a pendulum equation.

b) The point of suspension is now released so it can move freely in the horizontal direction (x-direction). Use s and θ as generalized coordinates for the system and determine the corresponding set of Lagrange's equations.

c) Show that s can be eliminated to give an equation which only depends on θ and its derivatives. Further show that this equation implies that the vertical motion of the pendulum bob is identical to free fall in the gravitational field (in reality restricted by the length l of the rod).

Problem 2.2

Two identical rods of mass m and length l are connected to each other with a frictionless joint (Fig. 3). The first rod is connected to a joint in the ceiling and to a joint at the center of the second rod. Assume that the motion takes place in the vertical plane.

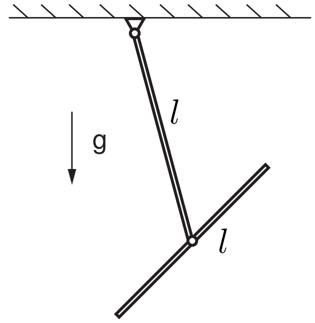

Figure 2: Two-rod problem

Choose suitable generalized coordinates for the system, and find the corresponding Lagrangian. Formulate Lagrange's equations for the system. As a reminder the moment of inertia of a rod (with even mass distribution) about an endpoint is $I_1 = ml^2/3$ and about the midpoint is $I_2 = ml^2/12$.

Problem 2.3

A rigid circular metal hoop rotates with constant angular velocity around an axis through the center. A particle slides without friction along the circle and there is no gravity. Use the angular variable θ as generalized coordinate (see figure).

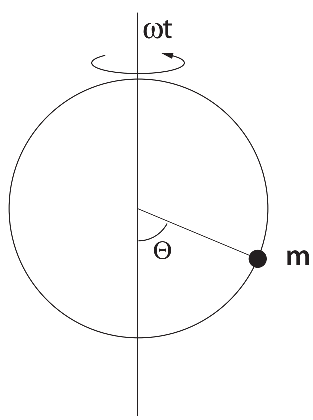

Figure 3: Rotating hoop

a) Find the Lagrangian expressed in terms of θ and $\dot{\theta}$, end derive Lagrange's equation for this variable.

b) Show that the Lagrangian is the same as the Lagrangian of a particle moving in a time-independent, periodic potential $V(\phi)$, with two stable and two stable equilibrium points (within a 2π interval).

c) With θ_0 as one of the stable equilibrium points, introduce a small deviation, $\phi = \theta - \theta_0$, and determine the small-angle form of the equation of motion (keeping only the lowest order in ϕ and its derivative). What is the angular frequency of oscillations about the stationary point?

Problem 2.4

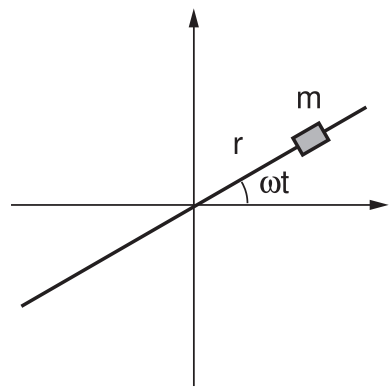

Figure 4: Problem 2.3

A small body with mass m moves without friction on a rod (see Fig. 2). The rod rotates in the horisontal plane, with constant angular velocity ω about a fixed point, which we assign the radial coordinate $r = 0$.

a) Find Lagrange's equation for the radial coordinate r, and show that, with the initial condition $\dot{r} = 0$ and $r = r_0$ at $t = 0$, the equation has the solution $r = r_0 \cosh \omega t$.

b) Make a plot of the orbit in the x, y-plane, with dimensionless parameters $r_0 = 1$, $\omega = 1$, and with t restricted by $\omega t \lessapprox \pi$.