Matroid Theory

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Contents

Note for the readers:

The goal on writing this in english instead of writing it in my first language (spanish) is because this way I can start building a portfolio that could not be as important as a research project, but it is work after all. So it would be better to have something to show that nothing to show at all, and much better if the audience capable of read it is as wide as possible.

As my first language is not english, this text could have a lot of mispellings and bad use of english. I apologize in advance and I'll try to be as careful as possible.

Here is a link to this project's version controller repository:
https://github.com/tellezhector/matroids
There you can see what are the changes that have ocurred along all this project's lifetime.

If you want to leave comments you can do it there, where anyone else is able to notice it. Or you could write directly to my personal e-mail address, which is: `tellez.hector@gmail.com` where it will be read only by me.

1

Homeworks

1.1 Union Of Closed Sets That Is Not Closed

Definition 1 (Closure)**.** *Given a matroid $M = (E, \mathcal{J})$ with rank function f. The **closure** of a set $S \subset E$ is*

$$Cl(S) = \{x \in E | r(S \cup \{x\}) = r(S)\}.$$

Definition 2 (Closed set)**.** *Given a matroid $M = (E, \mathcal{J})$ with rank function f. We say that a set $S \subset E$ is **closed** if*

$$Cl(S) = S.$$

Give an example of two closed sets whose union is not closed.

— **.** —

Proof. We are going to use the linear matroid of $\mathbb{Z}^2$, where the independent sets are

$$\emptyset, \{(1,0)\}, \{(0,1)\}, \{(1,1)\}, \{(1,0),(0,1)\}$$

and the rank function is the size of the biggest independent set contained.

It is clear that $A = \{(1,0),(0,0)\}$ and $B = \{(0,1),(0,0)\}$ are both closed. But $C = A \cup B = \{(1,0),(0,1),(0,0)\}$ is not, because $r(C) = 2 = r(C \cup \{(1,1)\})$ and $(1,1) \notin C$. $\square$

2

First Test

February 17, 2014

Test's exercices taken from Oxley's book, first chapter (Basic definitions and examples).

2.1 Oxley, Section 1.1, Problem 2

Let A be the matrix

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

For q in $\{2,3\}$, let $M_q[A]$ be the vector matroid of A when A is viewed over $GF(q)$, the field of q elements. Show that:

(i) The sets of circuits of $M_2[A]$ and $M_3[A]$ are different.

(ii) $M_2[A]$ is graphic but $M_3[A]$ is not.

(iii) $M_2[A]$ is represntable over $GF(3)$, but $M_3[A]$ is not representable over $GF(2)$.

— . —

Proof. (i) Lets look for the circuits over $GF(2)$. As 0 and 1 are the only scalars over $GF(2)$, we can only pay attention to sums of the vector columns and forget about full linear combinations.

There are no loops, the only way to get a loop is having a zero column and it is not the case.

There are no circuits of length two (or parallel columns), the only way to get them is having two exactly equal columns and it is not the case.

Lets call the columns 1, 2, and 3 the "simple columns", and the columns 4, 5 and 6 the "double columns".

For any pair of the columns from the simple columns there is a column from the double columns such that the sum is zero. Such column is the one you can get from adding the chosen pair. That is, $\{1,2,4\}$, $\{1,3,5\}$, $\{2,3,6\}$ are three dependent sets.

If you try to add all the simple columns, you can't add up to zero.

If you add two of the the double columns, you get the third one, so $\{4,5,6\}$ is dependent.

These four sets are circuits given that there are no smaller dependent sets.

These four are all the circuits of size three. Lets think about the circuits of size four.

Again, if you add two of the double columns, you get the third one, and there are two simple columns that can add up this third one. So, $\{1,2,5,6\}$, $\{1,3,4,6\}$, $\{2,3,4,5\}$ are dependent sets. And no one of the four already given circuits are contained in them, so these are circuits themselves.

At any time you choose the three simple columns, you will get no way to add up to zero just adding double columns. So, any set containing all of the simple columns will not be a circuit.

If you choose all the double columns, you have already chosen a circuit, so if you add any other vector, you will get a dependent set that is not a circuit.

So these seven are all the circuits of $M_2[A]$.

Now lets look for the circuits over $GF(3)$. Again and for the same reason, there are not loops or parallel columns.

If you choose two simple colums and you add them, you get one of the double columns, so if you add to them the negative of this double column, you get the zero vector. So $\{1,2,4\}$, $\{1,3,5\}$, $\{2,3,6\}$ are circuits.

$\{4,5,6\}$ is not dependent, you can check that the determinant of their submatrix is 2 (or -2 if you switched something).

If you choose two doble columns and add to them a simple one, again the determinant is not zero (it is always 1 or -1). So $\{1,2,4\}$, $\{1,3,5\}$, $\{2,3,6\}$ are all the circuits of size three.

Now lets look for circuits of size four.

If you choose two double columns and add one to the negative of the second one, you can find two simple columns that can make them add up the zero vector (using the correct signs), these two vectors are exactly the ones that add up to the double column that you didn't choose. So none of these selections contains the circuits of size three and therefore they are circuits. So this give us the circuits $\{1,2,5,6\}, \{1,3,4,6\}, \{2,3,4,5\}$.

If choose all of the double columns, you can choose any of the simple one so the four of them add up the zero vector using an appropiate selection for the signs. So this give us the dependent sets $\{1,4,5,6\}, \{2,4,5,6\}, \{3,4,5,6\}$ and they are circuits because any of the smaller circuits contain at least two simple columns.

The only other chance is choosing all of the simple columns, but wathever you add, you will get any of the three sized circuits as a subset. So these six are all of the circuits of size four.

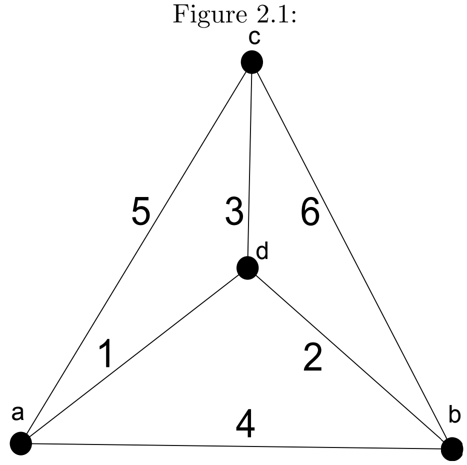
There are no circuits of size five, because there are only two posibilities, having three simple columns and two double columns, or having three double columns and two simple columns. The first one, again contains as a subset a circuit of size three. The second one always contains as a subset a circuit of size four.

So in total, there are nine circuits for $M_3[A]$.

Only by their cardinality, you can tell that the set of circuits of $M_2[A]$ and $M_3[A]$ are different.

(ii) We are going to give a graphic representation of a graph that has $M_2[A]$ as an isomorphic matroid. You can verify that it has all the seven circuits that we gave previously and only those.

Figure 2.1:



Now lets think what could happen if $M_3[A]$. For $M_3[A]$, we have the circuits $\{1,4,5,6\}, \{2,4,5,6\}, \{3,4,5,6\}$. Take any two of them, lets say $\{1,4,5,6\}, \{2,4,5,6\}$, they have three edges in common $\{4,5,6\}$.

If there is a graphic mathroid isomorphic to $M_3[A]$, $\{4,5,6\}$ defines three edges of a 4-cycle, that is, a 3-trajectory. In any 3-trajectory we will find 4 vertices, so the fourth edge is already determined by two of those vertices (the start vertex and the last vertex), but in this case we have that $\{1,4,5,6\}$, $\{2,4,5,6\}$ are both 4-cycles over the same set of vertices, and so 1 and 2 have exactly the same two vertices, that is, they are parallel. But in [2.1.(i)] we saw that $M_3[A]$ has no parallel columns. So this is a contradiction.

(iii) We are going to give a matrix B over $GF(3)$ such that $M_3[B]$ isomorfic to $M_2[A]$.

We are going to take the incidence matrix for the graphic representation [2.1], and choose one "1" from each column and change it by "-1". What we are doing is giving a orientation to each of the edges. So, if a cycle is such that each edge has exactly outdegree 1 and indegree 1, is easy to see that the sum of the respective columns will be the zero vector (each 1 will have exactly one -1 in its same row).

But lets remember that we are taking linear combination over $GF(3)$, that is, we can take a column, its negative, or take the zero vector instead. So, if we have a cycle, not necessarily "well-oriented", we can give it a "good-orientation" by multiplying its edges by -1 or 1 such way that each vertex has outdegree 1 and indegree 1. If we sum them, we will get the zero vector, and givin a "good-orientation" is no other thing than finding a linear combination that sums up 0.

So, let B be the matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

By the explanation just given, each circuit in $M_2[A]$ is a circuit in $M_3[B]$ (note that we used the same labels in both matrices so the subsets of labels can have the same representation).

Also, note that if you choose a subset of edges such that you have leaves, the row for any leaf cannot be zero under any linear combination unless you multiply it by 0. Having said these, is easy to see that the subsets of columns representing forests are independent.

Now, lets suppose there is a matrix C such that $M_2[C]$ is isomorphic to $M_3[A]$. Again, lets suppose that the labels are the same in both matrices. Then, you must have that the subsets of columns $\{1,4,5,6\}$ and

$\{2, 4, 5, 6\}$ are circuts. So, the sum of the columns 4, 5, 6 and 1 must be the zero vector, that is the column 1 is equal to the sum of the columns 4, 5 and 6. But the same occurs with the other circuit, the sum of columns 4, 5, 6 and 2 must be zero, so the colun 2 is equal to the sum of the columns 4, 5 and 6. Then we have that the columns 1 and 2 must be the equal. That is, the columns 1 and 2 are parallel. Which is a contradiction to what we proved in [2.1.(i)].

□

2.2 Oxley, Section 1.1, Problem 7

Let M_1 and M_2 be matroids on disjoint sets E_1 and E_2 and with independent sets $\mathcal{I}_1$ and $\mathcal{I}_2$ respectively. Let $E = E_1 \cup E_2$ and $\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}$. Prove that $(E, \mathcal{I})$ is a matroid.

— . —

Proof. Lets see that $(E, \mathcal{I})$ satisfies $I1, I2$ and $I3$.

(I1) $\emptyset \in \mathcal{I}$. This is clear, $\emptyset \in \mathcal{I}(M_1)$ and $\emptyset \in \mathcal{I}(M_1)$, so $\emptyset = \emptyset \cup \emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$, $J \subset I$ then $J \in \mathcal{I}$. $I \in \mathcal{I}$ means that there are $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$ such that $I = I_1 \cup I_2$. Given that $J \subset I$ it is clear that $J = J \cap I = J \cap (I_1 \cup I_2) = (J \cap I_1) \cup (J \cap I_2)$. We have that $(J \cap I_1) \subset I_1$ and $(J \cap I_2) \subset I_2$, given that M_1 and M_2 are matroids, then $(J \cap I_1) \in \mathcal{I}_1$ and $(J \cap I_2) \in \mathcal{I}_2$. Then $J = (J \cap I_1) \cup (J \cap I_2) \in \mathcal{I}$.

(I3) If $I, J \in \mathcal{I}$, $|J| < |I|$ then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$. By hypotesis, there are $I_1, J_1 \in \mathcal{I}_1$ and $I_2, J_2 \in \mathcal{I}_2$ such that $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$. Then $|J| = |J_1| + |J_2| < |I_1| + |I_2| = |I|$. Then there must be that $|J_1| < |I_1|$ or that $|J_2| < |I_2|$. Withtout loss of generality, lets say that $|J_1| < |I_1|$, as M_1 is a matroid, then, there exists $x \in I_1 \setminus J_1$ such that $J_1 \cup \{x\} \in \mathcal{I}_1$. Then, $x \in I$ and $J \cup \{x\} = (J_1 \cup \{x\}) \cup J_2 \in \mathcal{I}$. And we are done.

$\square$

2.3 Oxley, Section 1.1, Problem 9

Let M_1 and M_2 be matroids on a set E. Give an example to show that $(E, \mathcal{I}(M_1) \cap \mathcal{I}(M_2))$ need not be a matroid.

— . —

Proof. Let $E = \{a, b, c\}$.
Let $\mathcal{I}(M_1) = \{\{a, b\}, \{a, c\}, \{a\}, \{b\}, \{c\}, \emptyset\}$ Lets check that this is a matroid. It hass the empty set, given any set, any of its subests is independent. To see that the independence augmentation axiom (if $I, J \in \mathcal{I}$ and $|J| < |I|$ then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$), I will give you a table, where the first row is for the set I, the second for J and the third one for x.

I	J	x
$\{a, b\}$	$\{a\}$	b
$\{a, b\}$	$\{b\}$	a
$\{a, b\}$	$\{c\}$	a
$\{a, b\}$	$\emptyset$	a
$\{a, c\}$	$\{a\}$	c
$\{a, c\}$	$\{b\}$	a
$\{a, c\}$	$\{c\}$	a
$\{a, c\}$	$\emptyset$	a
$\{a\}$	$\emptyset$	a
$\{b\}$	$\emptyset$	b
$\{c\}$	$\emptyset$	c

So $(E, \mathcal{I}(M_1))$ is a matroid.

Now, let $\mathcal{I}(M_2) = \{\{a, b\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \emptyset\}$, to see that $(E, \mathcal{I}(M_2))$ the first two axioms can be justified just as above, and just below is the table to check the augmentation axiom.

I	J	x
$\{a, b\}$	$\{a\}$	b
$\{a, b\}$	$\{b\}$	a
$\{a, b\}$	$\{c\}$	b
$\{a, b\}$	$\emptyset$	a
$\{b, c\}$	$\{a\}$	b
$\{b, c\}$	$\{b\}$	c
$\{b, c\}$	$\{c\}$	b
$\{b, c\}$	$\emptyset$	b
$\{a\}$	$\emptyset$	a
$\{b\}$	$\emptyset$	b
$\{c\}$	$\emptyset$	c

So $(E, \mathcal{I}(M_2))$ is also a matroid.

Then $\mathcal{I}' = \mathcal{I}(M_1) \cap \mathcal{I}(M_2) = \{\{a, b\}, \{a\}, \{b\}, \{c\}, \emptyset\}$ cannot be an independet set for E. It satisfies (I1) and (I2), but not I3 since $\{a, b\}, \{c\} \in \mathcal{I}'$, and $|\{c\}| < |\{a, b\}|$ but $\{c\} \cup \{a\} \notin \mathcal{I}'$ and $\{c\} \cup \{b\} \notin \mathcal{I}'$. $\square$

2.4 Oxley, Section 1.2, Problem 1

Prove that $\mathfrak{B}$ is the collection of bases of a matroid on E if and only if $\mathfrak{B}$ satisfies (B1) and the following two conditions:

(B2)' If $B_1, B_2 \in \mathfrak{B}$ and $e \in B_1$, then there is an element f of B_2 such that $(B_1 \setminus \{e\}) \cup \{f\} \in \mathfrak{B}$.

(B3) If $B_1, B_2 \in \mathfrak{B}$ and $B_1 \subset B_2$, then $B_1 = B_2$.

— . —

Proof.

Sufficiency

Suppose that $\mathfrak{B}$ is the collection of bases of a matroid.

Then, it satisfies (B1) and (B2). As $\mathfrak{B}$ is a clutter, if $B_1, B_2 \in \mathfrak{B}$, then $B_1 = B_2$. Proving that it satisfies $(B3)$.

Now, lets suppose that $B_1, B_2 \in \mathfrak{B}$ and $e \in B_1$.

If $e \in B_1 \cap B_2$ then we could choose $f = e \in B_2$ and then $(B_1 \setminus \{e\}) \cup \{f\} = B_1 \in \mathfrak{B}$.

If $e \in B_1 \setminus B_2$ then there exists an element $f \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e\}) \cup \{f\} \in \mathfrak{B}$.

Necessity

Now lets suppose that $\mathfrak{B}$ satisfies (B1), (B2)' and (B3).

Let be B_1, B_2 in $\mathfrak{B}$ and $x \in B_1 \setminus B_2$. By (B2)' we have that there is an element y in B_2 such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathfrak{B}$. If $y \in B_1$, then $(B_1 \setminus \{x\}) \cup \{y\} = B_1 \setminus \{x\} \in \mathfrak{B}$, but as $B_1 \setminus \{x\} \subset B_1$, (B3) implies that $B_1 \setminus \{x\} = B_1$, which is a contradition. So $y \in B_1 \setminus B_2$, proving that $\mathfrak{B}$ satisfies (B2) and therefore, is the collection of bases of a matroid. $\square$

2.5 Oxley, Section 1.2, Problem 6

Suppose B is a basis of a matroid M, $f \in E(M)$ and $e \in E(M) \setminus B$. Prove that $(B \cup \{e\}) \setminus \{f\}$ is a basis of M if and only if $f \in C(e, B)$.

— . —

Proof.

Sufficiency

Suppose that $f \in C(e, B)$. $C(e, B)$ is the only circuit contained in $B \cup \{e\}$ by definition. Then $(B \cup \{e\}) \setminus \{f\}$ does not contain any circuit, that means, it is independent. On the other hand, $e \notin B$ and $f \in B$, then $|(B \cup \{e\}) \setminus \{f\}| = (|B| + 1) - 1 = |B|$, therefore, $(B \cup \{e\}) \setminus \{f\}$ is a basis, as it is an independent set with the size of a basis.

Necessity

We know that $(B \cup \{e\})$ contains only one circuit (and it is $C(e, B)$). If $(B \cup \{e\}) \setminus \{f\}$ is a basis of M, then it can not contain $C(e, B)$. That is, $C(e, B) \subseteq (B \cup \{e\})$, but $C(e, B) \nsubseteq (B \cup \{e\}) \setminus \{f\}$, then f must be in $C(e, B)$.

$\square$

2.6 Oxley, Section 1.3, Problem 4

Prove that a matroid M is uniform if and only if it has no circuits of size less than $r(M) + 1$.

— . —

Proof.

Sufficiency

Suppose that M is a uniform matroid $U_{m,n}$, then $r(M) = m$ by definition. Moreover, any $A \subset E(M)$ such that $|A| < m$ will be subset of some basis and therefore independent. Then if there is any circuit C, it must have size greater than m, that is $r(C) \geq m + 1 = r(M) + 1$.

Necessity

Suppose now that M is such that it has no circuits of size less than $r(M) + 1$. Then, for any $A \subset E(M)$ with $|A| \leq r(M)$ it must be independent, if not, then A should contain a circuit, but any circuit has size at least $r(M) + 1$.

Then M must be isomorphic to an uniform matroid $U_{r(M),|E(M)|}$. □

2.7 Oxley, Section 1.3, Problem 5

(i) Characterize paving matroids in terms of their collection of independent sets and in terms of their collections of bases.

(ii) Characterize uniform matroids in terms of their collections of circuits.

— . —

Proof.

(i) Let M be a matroid. $\mathfrak{B}$ its collection of bases and $\mathcal{I}$ its collection of independent sets.

M is a paving matroid if and only if for every $A \subset E(M)$ with $|A| < r(M)$ there exists $B \in \mathfrak{B}$ such that $A \subset B$.

As M is matroid, by $I2$, this is exactly the same as saying:

M is a paving matroid if and only if for every $A \subset E(M)$ with $|A| < r(M)$, $A \in \mathcal{I}$.

Proof. If M is paving, by definition of paving matroid, any $A \in E(M)$ such that $|A| < r(M)$ then, $A \in \mathcal{I}$ (and therefore, there is some $B \in \mathfrak{B}$ such that $A \subset B$).

If M is a matroid such that for every $A \subset E(M)$ with $|A| \leq r(M)$ there exists $B \in \mathfrak{B}$ such that $A \subset B$ (and therefore, $A \in \mathcal{I}$). Then, suppose that C is a circuit of M. If $|C| < r(M)$, then there would be some $B \in \mathfrak{B}$ such that $C \subset B$, but then $C \in \mathcal{I}$, which is a contradiction and therefore $|C| \geq r(M)$. That is, M is paving.

□

(ii) A matroid M is uniform if and only if it has no circuits of size less than $r(M) + 1$.

Proof. See (2.6) □

□

2.8 Oxley, Section 1.4, Problem 2

Show that a subset X of a matroid is a basis if and only if X is both independent and spanning.

— . —

Proof.

Sufficiency

If X is a basis of a matroid M, then by definition of basis, it is independent. Also, as X is basis, then $r(X) = r(X \cup x)$ for any $x \in E(M)$ and, therefore, $cl(X) = E(M)$, that is, X is spanning.

Necessity

Suppose that X is both independent and spanning. As X is spanning, then $r(X) = r(M)$, that is, for any $B \in \mathfrak{B}(M)$, $|X| \geq r(X) = r(B) = |B|$. On the other hand, $X \in \mathcal{I}(M)$ so, for any $B \in \mathfrak{B}(M)$, $|X| \leq |B|$. We have then that $|B| \leq |X| \leq |B|$, that is $|B| = |X|$ for any $\mathfrak{B}(M)$ and therefore, X is a basis. $\square$

2.9 Oxley, Section 1.4, Problem 6

Prove that statments (a)-(g) below are equivalent for an element e of a matroid M:

(a) *e is in every basis.*

(b) *e is in no circuits.*

(c) *If $X \subseteq E(M)$ and $e \in cl(X)$, then $e \in X$.*

(d) $r(E(M) \setminus \{e\}) = r(E(M)) - 1$.

(e) *$E(M) \setminus \{e\}$ is a flat.*

(f) *$E(M) \setminus \{e\}$ is a hyperplane.*

(g) *If I is an independent set, then so is $I \cup \{e\}$.*

— . —

Proof.

[(a) ⇒ (b)]

Suppose there is a circuit C such that $e \in C$, then $C \setminus \{e\} \in \mathcal{I}(M)$, then there must be $B \in \mathfrak{B}$ such that $C \setminus \{e\} \subseteq B$, but $e \notin B$, otherwise $C \subset B$. But this contradicts that e is in every basis and therefor there is not such circuit C.

[(b) ⇒ (c)]

Let $X \subseteq E(M)$ such that $e \in cl(X)$. Suppose that $e \notin X$ and let B_X be a basis of X. we have that $x \in cl(X) = cl(B_X)$, then $cl(B_X) = cl(B_X \cup \{e\})$, which means that $r(B_X \cup \{e\}) = r(B_X) = |B_X| < |B_X \cup \{e\}|$, and therefore $B_X \cup \{e\}$ is a dependent set, and it must contain a circuit C, as C can not be subset of B_X (because B_X is independent) then $e \in C$, which is a contradiction to (b).

[(c) ⇒ (d)]

As $E(M) \setminus \{e\}$ is missing only one element of $E(M)$, then $r(E(M) \setminus \{e\}) \geq r(E(M)) - 1$. If $r(E(M) \setminus \{e\}) = r(E(M))$ then $cl(E(M) \setminus \{e\}) = E(M)$ and $e \in E(M)$ but $e \notin E(M) \setminus \{e\}$, which contradicts (c).

[(d) ⇒ (e)]

By definition of closure, $E(M) \setminus \{e\} \subseteq cl(E(M) \setminus \{e\})$. And $E(M) \setminus \{e\}$ is contained in only two subsets of $E(M)$, there are $E(M)$ itself and $E(M) \setminus \{e\}$. So $cl(E(M) \setminus \{e\}) = E(M) \setminus \{e\}$ or $cl(E(M) \setminus \{e\}) = E(M)$. If the second occurs, then $r(E(M) \setminus \{e\}) = r(E(M))$, which contradicts (d), and therefore $cl(E(M) \setminus \{e\}) = E(M) \setminus \{e\}$, which means that $E(M) \setminus \{e\}$ is flat.

[(e) ⇒ (f)]

As $E(M) \setminus \{e\}$ is flat, then $r(E(M) \setminus \{e\}) < r(M)$, but $E(M) \setminus \{e\}$ is missing only e, so $r(E(M) \setminus \{e\}) = r(M) - 1$ and then $E(M) \setminus \{e\}$ is an hyperplane.

[(f) ⇒ (g)]

Let $I \in \mathcal{I}(M)$ such that $e \notin I$ and let $B \in \mathcal{I}$ be a basis for $E(M) \setminus \{e\}$ containing I. As $|B| = r(B) = r(E(M) \setminus \{e\}) = r(M) - 1$, then $r(B \cup \{e\}) = r(M) = |B| + |\{e\}| = |B \cup \{e\}|$ and therefore $B \cup \{e\}$ is a basis, and $I \cup \{e\} \subseteq B \cup \{e\}$ so $I \cup \{e\}$ is independent.

[(g) ⇒ (a)]

Let $B \in \mathfrak{B}(M)$, as $B \cup \{e\} \in \mathcal{I}(M)$ and B is maximal by contention over independent sets, so there must be the case that $B \cup \{e\} = B$. $\square$

2.10 Oxley, Section 1.6, Problem 1

Show the following:

(i) All uniform matroids are transversal.

(ii) A transversal matroid need not be graphic.

(iii) A paving matroid need not be transversal.

— . —

Proof. (i) Let $U_{m,n}$ be a uniform matroid of rank m over n elements, define $\mathcal{A} = (A_j | j \in [1, m])$ and $A_j = E(U_{m,n})$ and $S = E(U_{m,n})$. Then, for every subset $J \subset [1, m]$ you can take $T = J$ and the bijection $\psi : J \to T$ such that $\psi(j) = j$, proving that every J is an indepentend set in the transversal matroid just defined. Just as every J is independent in $U_{m,n}$. By the definition of $\mathcal{A}$, there cannot be independent sets of size larger than m. Just as in $U_{m,n}$. These two observation determine $U_{m,n}$ proving that $U_{m,n}$ can be seen as a transversal matroid.

(ii) Let $U_{2,4}$ be a uniform matroid, as we proved above, $U_{m,n}$ is transversal. $\{1, 2, 3\}$ and $\{1, 2, 4\}$ are circuits (see 2.6), if they were triangles of a graph, then the edges 3 and 4 must be parallel. But {3, 4} is independent in $U_{\{}2, 4\}$, so $U_{2,4}$ is not graphic.

(iii) Take the uniform matroid $U_{2,3}$ again, and remove the set $\{1, 3\}$ and $\{2, 3\}$ from the collection of independent sets, what you get is a paving matroid that is not uniform. Now, we have only the independent sets $\{1, 2\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\emptyset$. Lets suppose that this is a transversal matroid, as the largest independent set has two elements, then we can assume that $\{A\}$ has only two elements, lets call them a and b. As $\{1, 2\}$ is independent without loss of generality, there would be a relation $(a, 1), (b, 2)$ and there is someone linked to 3, lets think it is $(a, 3)$, then we can have the matching $(b, 2), (a, 3)$, which lead us to a independent set that was not supposed to be in our matroid.

The same will happen no matter how you try to pair a, b with $1, 2, 3$ in a way that let you have the matching given by our collection of independent sets.

□

2.11 Oxley, Section 1.6, Problem 3

Let $S = \{1, 2, \ldots, 6\}$ and $\mathcal{A} = \{A_1, A_2, A_3\}$ where $A_1 = \{1, 2, 3\}$ $A_2 = \{2, 3, 4\}$, and $A_3 = \{4, 5, 6\}$.

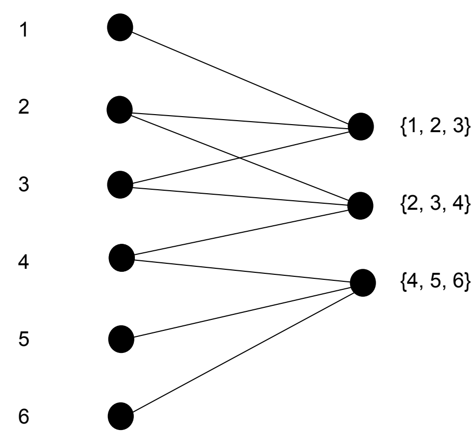
(i) Find $\Delta[\mathcal{A}]$.

(ii) Give a geometric representation for $M[\mathcal{A}]$.

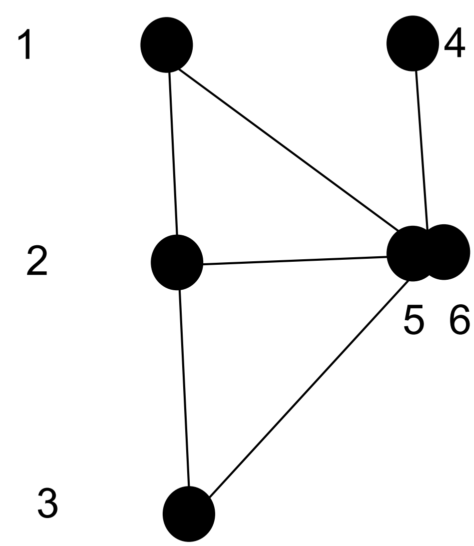
— . —

Proof.

1. Graphic representation for for $\Delta[\mathcal{A}]$



2. Geometric representation for $M[\mathcal{A}]$:



The dependent sets are $\{1, 2, 3\}$, $\{1, 5, 6\}$, $\{2, 5, 6\}$, $\{3, 5, 6\}$, $\{4, 5, 6\}$ and $\{5, 6\}$

□

2.12 Oxley, Section 1.6, Problem 4

Characterize the circuits of $M[\mathcal{A}]$ in terms of the bipartite graph $G = \Delta[\mathcal{A}]$.

— . —

Proof. Lets remember that $G = \Delta[\mathcal{A}]$ is a bipartite graph with partition $(S, \mathcal{A})$ and that in terms of transversal matroids, a subset $T \subset S$ is independent if there is an injection $\phi : T \to \mathcal{A}$ such that $(t, \phi(t))$ is and edge for every $t \in T$.

Halls theorem states that a susbset $T \subset S$ is independent if and only if $|U| \leq |N_G(U)|$. For every $U \subset T$.

That is, a subset $T \subset S$ is dependent if and only if there is $U \subset T$ such that $|U| > |N_G(U)|$. We are looking for subsets T that satisfies this condition and that are minimal.

If there is $U \subsetneq T$ such that $|U| > |N_G(U)|$, then T can not be minimal.
Then T must has the property $|T| > |N_G(T)|$ (we will refer to this property as the $(*)$-property) and its minimal.

Then, T is a circuit if and only if it is has the $(*)$-property and its minimal with such condition. (first approach). We can learn a little more about these T's. Lets check what happen if the graph induced by T has at least 2 connected components. Lets call T_1 and T_2 such that T_1 induces one connected component and $T_2 = T \backslash T_1$. Notice that $N_G(T_1) \cap N_G(T_2) = \emptyset$, and then $|N_G(T)| = |N_G(T_1)| + |N_G(T_2)|$. As $|T| = |T_1| + |T_2| > |N_G(T_1)| + |N_G(T_2)| = |N_G(T)|$ it must be that either $|T_1| > |N_G(T_1)|$ or $|T_2| > |N_G(T_2)|$, without loss of generality, if the first one is true, then at least T_1 is smaller than T and then T is not minimal. Then the graph induced by T must be connected.

T is circuit if and only if the graph induced by T (lets call it $G[T]$) is isomorphic to $K_{|T|,|T|-1}$.

Necessity

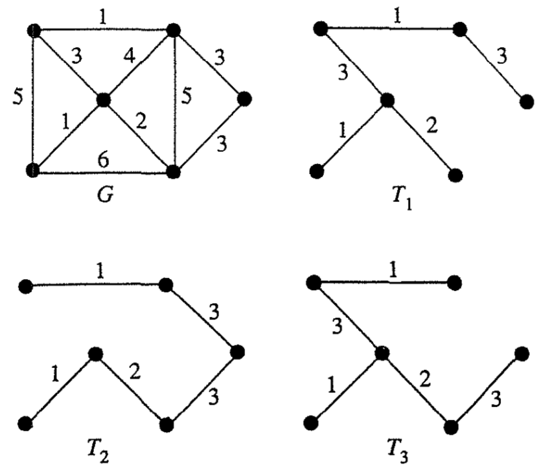
Suppose the $G[T]$ is isomorphic to $K_{|T|,|T|-1}$, T is minimal with $(*)$-property, if you remove any vertex from T, its induced graph will be $K_{|T|-1,|T|-1}$ wich has perfect matches, if you remove more you will have a $K_{m,n}$ with $m < n$ which has $K_{m,m}$ as subgraph.

Sufficiency

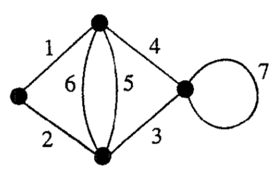
Suppose you have a circuit T, as T is bipartite and connected, then it must contain a $K_{m,n}$ as subgraph induced by vertices in T. If $n < m$, for one of them the necessity proves that T must be as desired, or then it has a subset that is dependent. If it is not the case and $m \leq n$ for every such graph, then T doesn't have the $(*)$-property. $\square$

2.13 Oxley, Section 1.8, Problem 1

(i) Find a maximum-weight spanning tree of the graph in the next figure. Is this the unique such tree?

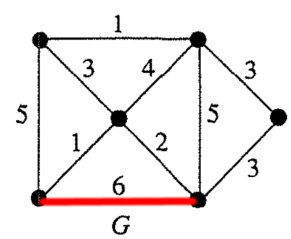


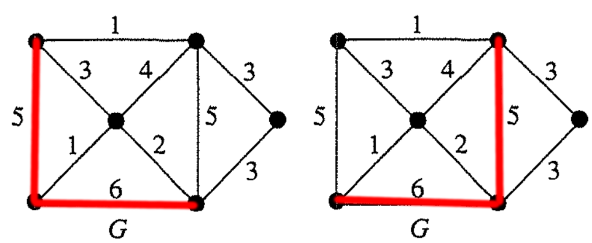
(ii) Find all maximum-weight spanning trees and all minimum-weight spanning trees of the graph in the next figure, where the edge labels are interpreted as weights.

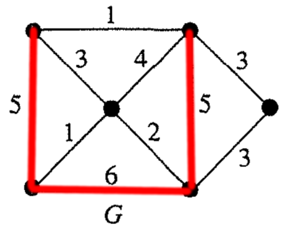


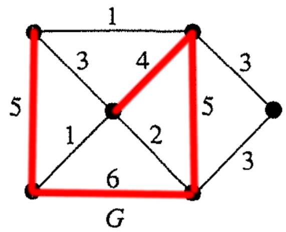
— . —

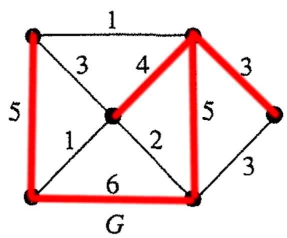
Proof. 1. There are only 2 maximum-weight spanning trees of the graph. I have ilustrated what will greedy do step by step, and with all possible desicions being made.

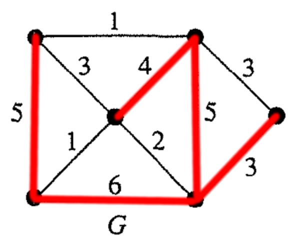








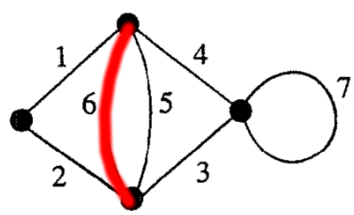


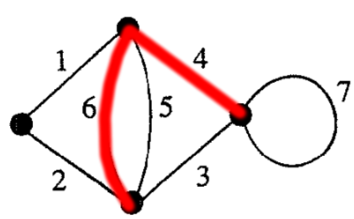


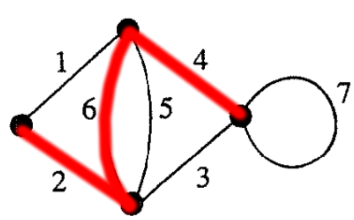
2. As the weight function is injective, the maximum-weight and minimum-weight spanning trees are unique.

   Here is again an ilustration of what greedy would do in both scenarios.

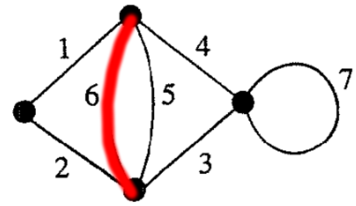
   Maximum-weight:

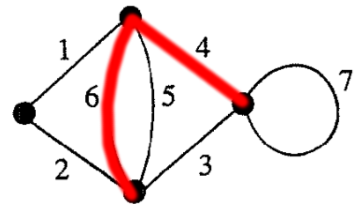


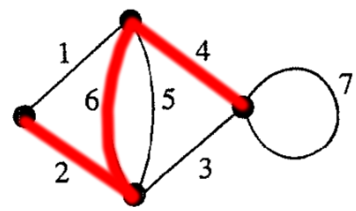




Minimum-weight:







□

2.14 Oxley, Section 1.8, Problem 4

Let M be a matroid and $\omega : E(M) \to \mathbb{R}$ be a one-to-one function. Prove that M has a unique basis of maximum weight.

— . —

Proof. Let B be a maximum-weight basis found using greedy. Suppose that there is another basis B' with maximum-weight also and suppose that they are different.

Let $e \in B' \setminus B$ of minimum-weight (it exists because ω is one-to-one), and then, there exists $f \in B \setminus B'$ such that $(B' \setminus \{e\}) \cup \{f\}$ is basis. We can choose such f of maximum weight. If $\omega(e) < \omega(f)$ then we have increased the weight of B', which is a contradiction. If $\omega(f) < \omega(e)$ we can change the roles of B and B' and get the same contradiction.

So, there is no other basis B' with maximum-weight and different from B. $\square$

3

Second Test

April 17, 2014

Test's exercices taken from Oxley's book, second and third chapters (Duality and Minors).

3.1 Oxley, Section 2.1, Problem 1

Find each of the following:

(i) all self-dual uniform matroids;

(ii) all identically self-duall uniform matroids;

(iii) all self-dual graphic matroids on six or fewer elements;

(iv) all identically self-dual graphic matroids on six or fewer elements;

(v) an infinite family of simple graphic self-dual matroids.

— . —

Proof.

(i) Let $U_{n,m}$ be an uniform matroid.

Lets asume that it is self-dual. As any basis B for $U_{n,m}$ has size n, if it is going to be self-dual, then its complement must have size n as well. This means that $m = 2n$.

Now lets suppose that $m = 2n$. So, any subset of size n is a basis for $U_{n,m}$, but it is also true that any subset of size n is the complement of another subset of size n and then, any basis is the complement of another basis. Then $U_{n,m}$ is not only self-dual, but also identically self-dual.

(ii) Just as we saw above, if an uniform matroid is self-dual, then it is identically self-dual, and an uniform matroid $U_{n,m}$ is self-dual if and only if $m = 2n$.

(iii) It is clear that if a matroid is to be self-dual, then its size must be two times the size of any of its basis (any forest that is not contained in any other forest). This means that there are only self-dual graphic matroids with an even number of edges.

Lemma 2.3.7 from [Oxl92] stays that if G^* is a geometric dual of the planar graph G, then $M(G^*) \cong M^*(G)$. In particular for self-dual matroids we will have that $M(G^*) \cong M^*(G) \cong M(G)$.

From now on we will use this fact to look for the remaining self-dual matroids. As the maximum size of these matroids is 6 and K_4 is a complete planar graph with 6 edges, then every one of the matroids will have a planar representation. And the previously mentioned lemma will let us work with graph representations instead of a complicated structure of subsets and its intersections.

Also, notice that if G is a planar graph. And $H = G \cup G^*$, then $M(H)$ is self-dual. The only trick is to send $M(G)$ to $M(G)$ in the dual and $M(G^*)$ to $M(G^*)$ in the dual.

Considering this. We will try to find self-duality through blocks (2-connected maximal subgraphs) when it is possible. Also, the drawings will be done thinking of these, we are going to separate in as many blocks as possible, given that it will not affect the resultant matroid.

We will try to order our analysis by the size of the maximum cycle contained in a graph or its dual matroid. And for this remember that the dual of a n-cycle is a set of n edges all of them parallel to each other.

2 edges:

- **No cycles.** There is no such graph. If a graph has no cycles, then its dual contains only loops (which are cycles). This will happen in any of the other graphs so we are not going to list this case anymore.
- **Cycles of size at most 1.** If there has to be a loop, then it has to be an edge that is not a loop. The loop will be maped to the dual of the edge that is not a loop and viceversa. This one is not identically self-dual.

Figure 3.1: self-dual graphic matroid representation

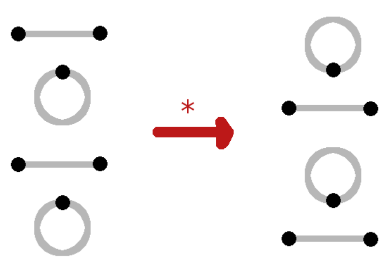
- **Cycles of size at most 2.** There is only one way to get a cycle of size two with two edges, and it is easy to see that it has not only a self-dual matroid but an identically auto-dual matroid.

Figure 3.2: identically self-dual graphic matroid representation

There cannot be larger cycles with only 3 edges.

4 edges:

- **Cycles of size at most 1.** Again. For every loop ther must be an edge that is not a loop. And for every edge that is not a loop there must be a loop. So the only posibility left is to have two loops and two edges that are not loops.



- **Cycles of size at most 2.** If we have a block as in 3.2, then, under the matroid isomorphisim are two posibilities, that such block is mapped into itself, or that it is mapped into another block.

  If it is mapped into another block, then the other block must be essentialy the same graph and then whe have something like

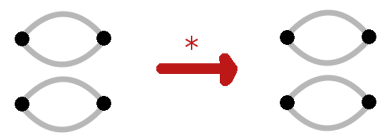


Figure 3.3: identically self-dual graphic matroid representation

(Notice that this will result in an identically self-dual matroid also. Every forest is a pair of non-parallel edges, and its complement is also a pair of non-parallel edges.)

If the first block is mapped into itself, then the other block must be mapped into itself also. Then the other block must be a representation of a self-dual matroid on 2 edges. And then, there are two posibilities. If the other block is like in 3.2, then we will obtain again 3.3. If the other block is like 3.1 then we will obtain something like

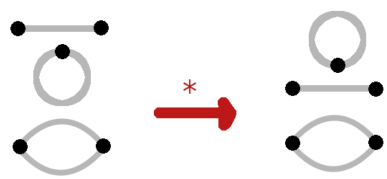


Figure 3.4: self-dual graphic matroid representation

(Notice that this could not be identically self-dual. As it has loops, any loop will be in a cobasis, but no basis contains loops.)

- **Cycles of size at most 3.** A graph that will represent a self-dual matroid and that has a 3-cycle and at most 4 edges, must have at most one block. Otherwise, on block must be the triangle, but a triangle is not self-dual and could not be mapped in the other block that only could have 1 edge.

  This gives us essentially only one opportunity. A triangle adding a parallel edge to one of its edges. An isomorphism would be sending the simple edges to the parallel edges and the parallel edges to the simple edges. The two simple edges form a set that intersects any of the basis of the matroid, so it is a cocircuit. The proposed isomorphism send that cocircuit to a circuit in the original matroid. The two parallel edges along with any of the simple edges form another set that intersects any of the basis of the original matroid, so it is another cocircuit in the dual matroid. Those edges are sent to a triangle in the original matroid. There are two of these cocircuits and each one is sent two one of the two possible triangles in the original matroid. Any basis formed for a simple edge and one of the two parallel edges has as complement the other simple edge and the other parallel edge, and under the proposed isomorphism any of such basis are sent to other of the same type. The only other basis that is left is the one that consits of the two simple edges, and they are sent to the pair of parallel edges, which is independent in the dual matroid. We have proved so far that this isomorphism sends any of the basis in the original matroid to a basis in the dual matroid and any cocircuit to a circuit in the original matroid. Then such isomorphism is a matroid isomorphism between the graphic matroid and its dual.

Figure 3.5: self-dual graphic matroid representation

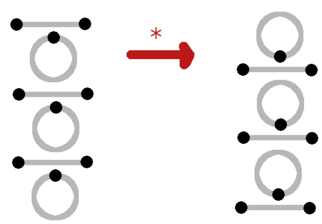
This could not be identically self-dual as it has a cobasis that contains parallel edges.

- **Cycles of size at most 4.** There is none, as the only cycle of size 4 has 4 edges, and as the 4-cycle is not self-dual, there cannot be self-dual matroids with 4-elements and 4-cycles.

There cannot be larger cycles with only 4 edges.

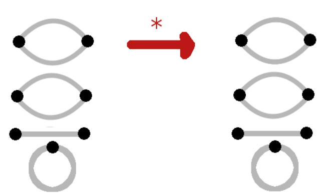
6 edges:

- **Cycles of size at most 1.** Again, there is only one posibility, the one shown below.

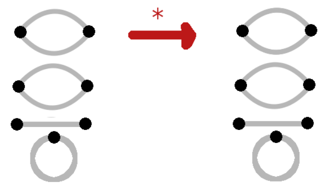


This case cannot be identically self-dual because it has loops.

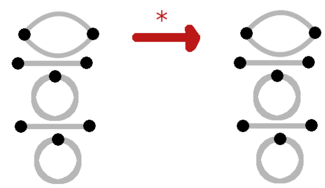
- **Cycles of size at most 2.** All the blocks will have at most two edges. Then all the posibilities are combinations of self-dual graphic matroids with 2 edges and at least two parallel edges. Cases shown below:



This case is identically self-dual by the same arguments as in [3.3] and [3.2].



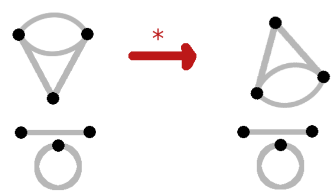
This case cannot be identically self-dual because it has loops.



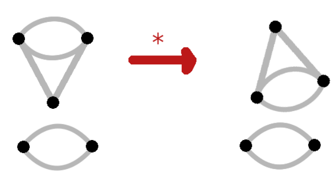
This case cannot be identically self-dual because it has loops.

- **Cycles of size at most 3.** If there is to be a triangle, and there are two blocks, there are two posibilities. First notice that there is not possible to have two triangles. Then the isomorphism will map the block with the triangle two itself, or it will map the block with the triangle to the other block.

  If it is mapped to itself, then the block with the triangle must have more than the trhee edges of the triangle, this is because a triangle is not self-dual. Then, such block will have at least 4 edges. It cannot have 5 edges, or else the block with the other edge must be mapped to itself, but there are not self-dual graphic matroids with one edge. Then we have again a triangle with a parallel edge as in [3.5]. Then these leads us to two new posibliities shown below.

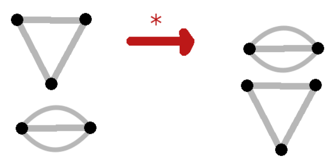


This case cannot be identically self-dual because it has loops.



This case cannot be identically self-dual because it has a cobasis with a circuit.

If the triangle is mapped to the other block then there is nothing else to do. The other block is obligated to be the dual of a triange.



This case cannot be identically self-dual because it has a cobasis with a circuit.

- **Cycles of size at most 4.** This case is oblicated to consist of only one block, and that is because a 4-cycle is not self-dual.

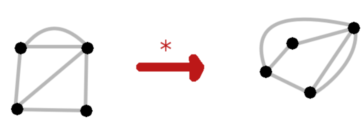
  Then, there are two main posibilities. The 4-cycle has adjacent antipodes or not.

  If it has, there are three posibilities.

  The other pair of antipodes are adjacent also, in this case what we have is K_4, which is W_3, which in other excercises is proved to be self-dual (but not identically self-dual as it has a cobasis with a cycle).

  The edge that is adjacent to two antipodes has a parallel edge. (This case is not self-dual, the triangles share two edges with the exterior face, this will casue two pairs of parallel edges and the original graph has only one).

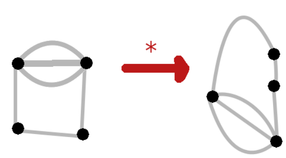
  The any edge of the cycle has a parallel edge. This case is self-dual.



  This case is not identically self-dual as it has a cobasis that contains a cycle.

  If the 4-cycle doesn't have adjacent antipodes. Then there are essencially two cases. One triple edge, or two pair of parallel edges.

  In the case of a triple edge we have a self-dual graph.



  This case is not identically self-dual as it has a cobasis that contains a cycle.

  And, finally, the case with two pairs of parallel edges is not self-dual. Its dual doesn't have two paris of parallel edges.

(iv) Any graph on six or fewer elements obtained from a forest adding a parallel edge to each edge in the forest. We have proved this in the last part, indicating which of the self-dual matroids are identically self-dual or non-identically self-dual.

(v) Any wheel. Proof in [3.11]

□

3.2 Oxley, Section 2.1, Problem 2

Let M be a matroid. Show that M^* has two disjoint circuits if and only if M has two hyperplanes whose union is $E(M)$.

— . —

Proof.

Sufficiency

Let C^* and C'^* be two disjoint circuits in M^*. From Proposition 2.1.6 in [Oxl92] we know that $C = E(M) \setminus C^*$ and $C' = E(M) \setminus C'^*$ are hyperplanes in M. Moreover, as C^* and C'^* are disjoint disjoint, then $C^* \subset C'$ and $C'^* \subset C$. Then

$$\begin{aligned} C \cup C' &= (C \cup C'^*) \cup (C' \cup C^*) \\ &\quad \text{(we have added nothing new to each subset)} \\ &= (E(M) \setminus C^*) \cup C'^* \cup (E(M) \setminus C'^*) \cup C^* \\ &\quad \text{(reordering terms)} \\ &= (E(M) \setminus C^*) \cup C^* \cup (E(M) \setminus C'^*) \cup C'^* \\ &= E(M) \cup E(M) \\ &= E(M). \end{aligned}$$

Necessity

Let H and H' be two hyperplanes such that $H \cup H' = E(M)$. Againg, from Proposition 2.1.6 in [Oxl92] we know that $H^* = E(M) \setminus H$ and $H'^* = E(M) \setminus H'$ are cocircuits. Suppose there exists $x \in H^* \cap H'^*$, that is $x \in H^*$ and $x \in H'^*$. But $x \in H^*$ means $x \notin H$ and $x \in H'^*$ means $x \notin H'$, then $x \notin H \cup H' = E(M)$ which is a contradiction. Then there is not such x and $H^* \cap H'^*$ are cocircuits such that $H^* \cap H'^* = \emptyset$. $\square$

3.3 Oxley, Section 2.1, Problem 6

Let e and f be distinct elements of a matroid. Prove that every circuit containing e also contains f if and only if $\{e\}$ or $\{e, f\}$ is a cocircuit.

— . —

Proof.

Sufficiency

Suppose every circuit containing e also contains f.

If every basis contains e, then, no cobasis contains e, which means $\{e\}$ is a cocircuit.

Suppose then that there is a basis B such that $e \notin B$. As $C(e, B)$ is a circuit that contains e, then $f \in B$. That is, any basis that doesn't contain e, contains f. That is, any cobasis that contains e, doesn't contain f.

Note that $B \setminus \{f\} \cup \{e\}$ is a basis given that it has the right size to be a basis and it is obtained from $B \cup \{e\}$ getting rid of the only circuit that it contains (that is, it is independent). Then, there is a basis that doesn't contains e and then $\{e\}$ is coindependent, and there is also a basis that doesn't contain f and then $\{f\}$ is also coindependent.

Now suppose there is a basis B' such that $f \notin B'$. If $e \notin B'$, then $C(e, B')$ is a circuit that contains e but not f whitch contradicts our hyphothesis. Then any basis that doesn't contain f, contains e. That is, any cobasis containg f, doesn't contain e.

Then $\{e, f\}$ is a subset that is not contained in any cobasis, then it is codependent but any of its subsets is coindependent. Then it is a cocircuit as we wished to show.

Necessity

Suppose $\{e\}$ is a cocircuit. Then, again by Proposition 2.1.6 in [Oxl92], $E(M) \setminus \{e\}$ is an hyperplane, and then e is contained in every basis, then e is in no circuits (by [2.9]). Then it is true that every circuit containing e also contains f, since there are not circuits containing e.

Now suppose that $\{e, f\}$ is a cocircuit. Then, again by Proposition 2.1.6 in [Oxl92], $H = E(M) \setminus \{e, f\}$ is an hyperplane, so any basis must contain e or f.

Let C be a circuit such that $e \in C$. Then $C \setminus \{e\}$ is independent. Let B be a basis such that $C \setminus \{e\} \subset B$. As B is basis and doesn't contain e, then $f \in B$. Notice that $C(e, B) = C$

Now, $B^* = E(M) \setminus B$ is a cobasis that contains e but doesn't contain f, then $C * (f, B^*) = \{e, f\}$ is the only cocircuit in $B^* \cup \{f\}$, that means that $B^* \cup \{f\} \setminus \{e\}$ is a coindependent of maximum size, that is a cobasis. Taking its complement we will get a basis, and it is

$$E(M) \setminus (B^* \cup \{f\} \setminus \{e\}) = B \cup \{e\} \setminus \{f\}.$$

Which means that $f \in C$. □

3.4 Oxley, Section 2.1, Problem 10

Let B be a basis of a matroid M and B^* be $E(M) - B$. If $e \in B$, let $C^*(e, B^*)$ denote the fundamental cocircuit of e with respect to the cobasis B^* of M, that is, $C^*(e, B^*) = C_{M^*}(e, B^*)$.

(i) Show that $C^*(e, B^*)$ is the unique cocircuit that is disjoint from $B - e$.

(ii) If $f \in B^*$, prove that $f \in C^*(e, B^*)$ if and only if $e \in C(f, B)$.

— . —

Proof.

1. $C^*(e, B^*)$ is the only cocircuit contained in $B^* \cup e$, then any other circvuit must intersect the complement of $B^* \cup e$, but the complement of $B^* \cup e$ is $B - e$. Then any other cocircuit must intersect $B - e$

2. Suppose $e \in C(f, B)$. Now, $B \setminus e \cup f$ must be a basis, it has the same size as B and $B \cup f$ has an unique circuit which is $C(f, B)$ and which contains f, so $B \setminus e \cup f$ cannot contain a circuit and then it is independent with maximum size, that is, $B \setminus e \cup f$ is a basis.

   Then, its complement must be a cobasis, but its complement is $B^* \setminus f \cup e$, we know that $B^* \setminus f \cup e \cup f = B^* \cup e$ has an unique cocircuit and then it must contain f.

   We have already proved that $e \in C(f, B)$ implies $f \in C^*(e, B)$, the proof for the sufficiency is completely analog.

□

3.5 Oxley, Section 2.2, Problem 2

Let A be an $m \times n$ matrix over a field F.

(i) For each of the row operations, specify an $m \times m$ matrix L such that multiplying A on the left by L has the same effect as the row operation.

(ii) Fore each of the column operations, specify an $n \times n$ matrix R such that multiplying A on the right by R has the same effect as the column operation.

— . —

Proof. ,

1. Take the $m \times m$ identity matrix. Perform any of the row operations to it and let L be the result.

   Such matrix is called elemental matrix, and if you multiply A on the left by L it will have the same effect as the row operation.

2. The proof is exaclty the same, but working with columns. Perform any of the column operations to the $n \times n$ identity matrix, and let R be the result.

$\square$

3.6 Oxley, Section 2.2, Problem 4

Show that, in a binary matroid, a circuit and cocircuit cannot have an odd number of common elements.

— . —

Proof. Let M be a binary matroid with rank n and such that $|E(M)| = n + r$.

Theorem 2.2.8 from [Oxl92] sates that if a linear matroid is expresed as $[I_n|D]$, then its dual matroid can be represented with the matrix $[-D^T|I_r]$.

In the case of binary matroids is the same $[-D^T|I_r]$ than $[D^T|I_r]$ (In general it is always the same, multiplying columns by -1 doesn't change dependency/independency conditions).

Lets say that C is a circuit of M and C^* is a cocircuit of M. As C is a circuit in a binary matroid, the sum of their elements must be the zero vector.

The representation of M can always be arranged in such way that C consists of $|C| - 1$ columns from I_n (lets call them I) and one column of D (lets call it d_j assuming that d_j is the j-th column of D).

With such representation is easy to see that if $e_i \in I$ is the i-th colmun of I_n, then $(d_j[i] = 1$ (that is, the i-th entry of d_j). Conversely, if $d_j[i] = 1$ then $e_i \in I$.

In the j-th row of $[D^T|I_r]$ the first n entries are exactly the entries of d_j, in the next r entries are exactly one one and it is in the j-th entry of I_r. This is, the j-th row of $[D^T|I_r]$ has a one in its k-entry if and only if the k column of $[I_n|D]$ was in C.

Given that C^* is a circuit in M^*, the sum of its columns must be the zero vector. In particular, in $[D^T|I_r]$ the sum of their j-th entry, must be 0. That is, the number of ones in their j-entry must be even. As we saw that the j-th row of $[D^T|I_r]$ have ones exactly in the entries corresponding of columns of C, then the intersection of C and C^* must be even. □

3.7 Oxley, Section 2.2, Problem 8

Let T_8 and R_8 be the vector matroids of the following matrices over $GF(3)$:

$$\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \qquad \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

$$\left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 0 \end{array}\right], \quad \left[\begin{array}{c|cccc} & -1 & 1 & 1 & 1 \\ I_4 & 1 & -1 & 1 & 1 \\ & 1 & 1 & -1 & 1 \\ & 1 & 1 & 1 & -1 \end{array}\right]$$

(i) *Show that T_8 and R_8 ar both self-dual.*

(ii) *Show that R_8 is identically self-dual but T_8 is not.*

(iii) *Give geometric representation for T_8 and R_8.*

(iv) *Show that if $M \in \{T_8, R_8\}$ and $X = E(M) \setminus \{8\}$, then $(M|X)^* \cong F_7^-$.*

(v) *Consider the following matrices over $GF(3)$:*

$$A_1 = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & -1 \\ & 1 & 1 & 1 & 0 \end{array}\right], \quad A_2 = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & -1 \\ I_4 & 1 & 0 & 1 & -1 \\ & 1 & 1 & 0 & -1 \\ & 1 & 1 & 1 & 1 \end{array}\right]$$

Show, by applying a sequence of the row and clolumn opeartions to A_1 and A_2 that $M[A_1] \cong T_8$ and $M[A_2] \cong R_8$.

(vi) *Show that R_8 can be obtained from $AG(3,2)$ by relaxing two disjoint circuit-hyperplanes.*

— . —

Proof.

(i) Theorem 2.2.8 from [Oxl92] sates that if a linear matroid is expresed as $[I_n|D]$, then its dual matroid can be represented with the matrix $[-D^T|I_r]$.

As multiplying columns by -1 doesn't change dependency/independency conditions. The dual matroid can be represented as $[D^T|I_r]$.

For both, T_8 and R_8, we have that $n = 4$, $r = 4$, and $D = D^T$. Then, in both cases, $[D^T|I_4] = [D^T|I_4] = [D|I_4]$. And with this representation is clear how we would build an isomorphism from T_8 to T_8^* and from R_8 to R_8^*.

(ii) Lets write $T_8 = [I_4|D_1]$, it is easy to prove that T_8 is not identically self-dual by simply pointing out a cobasis that is not a basis.

All the columns of I_4 form a basis. Then all the columns of D form a cobasis, but such cobasis is dependent, if you sum up all of their columns you will get the zero vector.

Now lets show that R_8 is identically self-dual by showing that every cobasis is a basis. For this, lets clasify the subsets of columns with cardinality 4 by how they intersect I_4.

There are three main types that intersect the columns in I_4

(a) **four columns from I_4.** There is only one of these, and its complement (the columns from D) is also independent, its determinant is -16 and $-16 \equiv 2 \mod 3$.

So, in this case, every cobasis is basis.

(b) **trhee columns from I_4 and one from D.** Without loss of generality, lets say that the 3 columns from I_4 are the first three. If it is not the case we can always exchange rows to make them look like the first three. Here are two subcases. The column from D is 8 or it is 5. If it is among $5, 6, 7$ we can always exchange rows to make them look as if the column chosen from D is 5.

If the column from D is 8, the determinant of the columns $\{1, 2, 3, 8\}$ is -1, and the determinant of the complement $\{4, 5, 6, 7\}$ is -4 and $-4 \equiv -1 \mod 3$.

If the column from D is 5, the determinant of the columns $\{1, 2, 3, 5\}$ is 1, and the determinant of the complement $\{4, 6, 7, 8\}$ is -4 and $-4 \equiv -1 \mod 3$.

Then, in this case, every cobasis is basis.

(c) **two columns from I_4 and two from D.** Here are essentialy 3 subcases. And those are $\{1, 2, 5, 6\}$, $\{1, 2, 6, 7\}$ and $\{1, 2, 7, 8\}$. Any other can be reduced to one of those by exchanging rows.

In the subcase $\{1, 2, 5, 6\}$ the determinant is 0. And the determinant of the complement $\{3, 4, 7, 8\}$ is 0 also.

In the subcase $\{1, 2, 6, 7\}$ the determinant is 2 which is $-1 \mod 3$. And the determinant of the complement $\{3, 4, 5, 8\}$ is -2, which is $1 \mod 3$.

In the subcase $\{1, 2, 7, 8\}$ the determinant is 0. And the determinant of the complmement $\{3, 4, 5, 6\}$ is 0 also.

Then, in this case also, every cobasis is basis.

The case **one colum from I_4 and three from D** is alredy covered in the complements of the case **trhee columns from I_4 and one from D.**, Similarly, the case **four columns from D** is covered in the complement of the case **four columns from I_4**.

Then, in any case, every cobasis is basis and then R_8 is identically self-dual.

(iii)

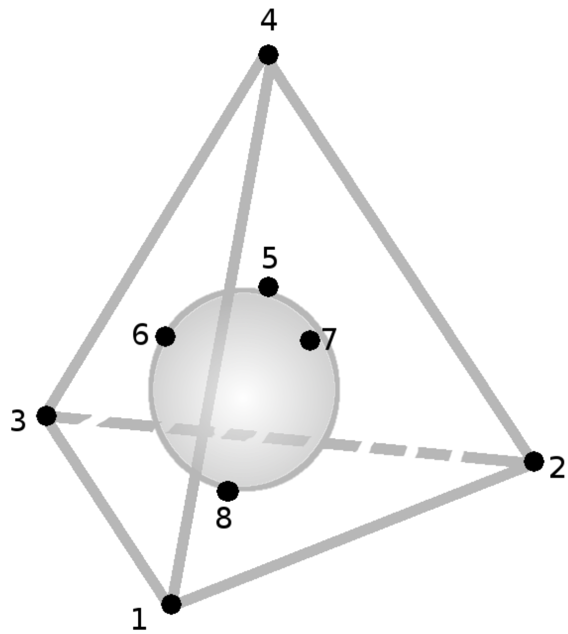


Figure 3.6: Geometric representation of T8

In this representation, there are points in the center of each face of the thetrahedron and the sphere that is inside the thetrahedron tangent to all the faces, is a plane that contains all the centers of the faces (just as in Fano's plane the circle inside the triangle is a line).

Assuming that the thetrahedron is regular, all the other planes that have 4 vertices and that are not suggested in the figure are those that can be found in the euclidean space. For example $\{1, 2, 5, 6\}$.

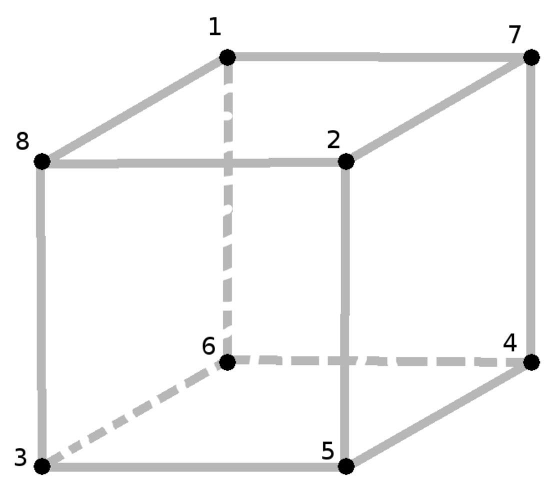


Figure 3.7: Geometric representation of R_8

This representation is simply a cube. All the other planes that contain 4 vertices and that are not suggested in the figure are those that can be found in the euclidean space. For Example $\{7, 8, 3, 4\}$.

(iv) Given the first parenthesis of this excercise, we know that an isomorphism from T_8 to T_8^* is given by simply sending 1 to 5, 5 to 1, 2 to 6, 6 to 2, 3 to 7, 7 to 3, 4 to 8, 8 to 4. Then we can relabel our geometric representation of T_8 and obtain one of T_8^*.

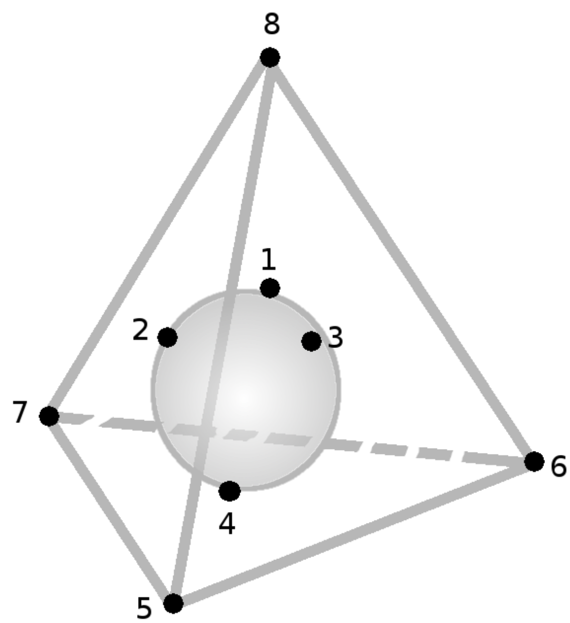


Figure 3.8: Geometric representation of T_8^*

We are being asked to contract $\{8\}$ in this representation. To contract is simply to create new circuits from those who contained the contracted elements, the new circuits will be those subsets that when we join $\{8\}$ to them, what we obtain is one of the old circuits.

In a geometric representation this is equivalent to "getting rid of one dimension", or projecting.

What we obtain from projecting with respect to 8, is the following representation.

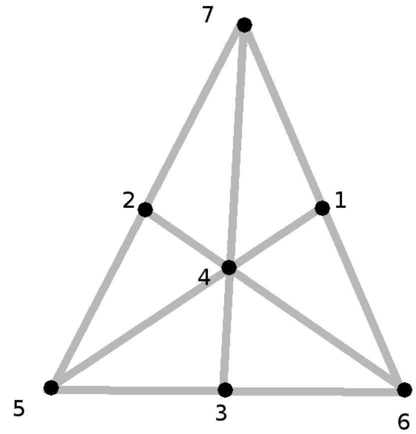


Figure 3.9: Geometric representation of $T_8^*/\{8\}$

This is exactly the geometric representation of F_7^-.
As R_8 is identically self-dual, the same representation from 3.7 will work.

Then we must project from 8 and se what is the result.

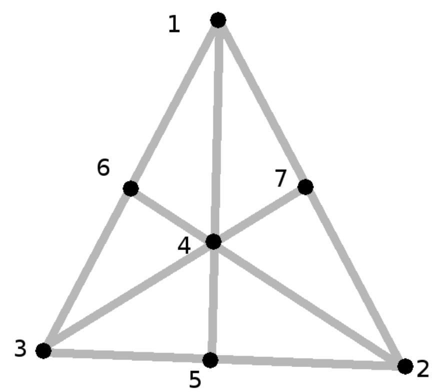


Figure 3.10: Geometric representation of $R_8^*/\{8\}$

Which is againg the geometric representation of F_7^-.

(v) I have constructed a set of software tools to solve this parenthesis by brute force.

From the first parenthesis we know that all the basis and all the circuits in T_8 and R_8 have all exactly 4 elements.

I constructed a tool that looks in every of the 70 subsets of 4 columns in a 4×8 matrix and determines if it is a basis and another to determine if it is a circuit, both by taking the determinant modulo 3.

Then, to find an equivalence between A_1 and T_8, I made a another tool that will crawl among all the column permutations of A_1 and for each one determine the set of labels that are basis (or those that are circuits).

If such sets of labels are the same than in T_8, then we have an equivalency.

After such analysis, I found 24 permutations of the columns of A_1 that determines exactly the same matroid as T_8.

Exactly the same procedure will work for A_2 and R_8 and in that case, I found 96 permutations of the columns of A_2 that give a matroid that is equivalent to R_8.

My first approach was to demonstrate that A_1 and T_8 where equivalent matrices and the same for A_2 and R_8.

In both cases, making use of gaussian-jordan elimination, I couldn't get the original matrices of T_8 and R_8. It would be not an elegant method, but I could determine that A_1 and T_8 are not equivalent and the same for R_8 and A_2 and it would take me a very long time to find any permutation that would work.

I will include my code here:

This is the code that gives all the combinations of m elements in a set of n elements

```
function C = combinations(n, m)
        C = [];
        if(m > n)
                return;
        endif;

        if(m == 1)
                C = [1:n]';
                return;
        endif;

        Y = combinations(n-1, m-1) + 1;
        Y = [(ones(size(Y, 1), 1)), Y];

        X = combinations(n-1, m) + 1;

        C = [Y; X];
endfunction
```

This is the code that determines which are all the basis of the matroid, it works only if you give it the rank of the matroid, but it could be changed to determine it by itself.

```
function Ind = column_basis(M, r)
        columns = size(M, 2);
        C = combinations(columns, r);
        Ind = [];
        for (i=1:size(C, 1))
                subset = C(i, :);
                N = M(:, subset);
                c = det(N);
                if(mod(c, 3) != 0)
                        Ind = [Ind; subset];
                endif
        endfor
endfunction
```

This is the code that determines which are all the circuits of the matroid, it works only if all the circuits have the same size (as in this case) and you give such size to the algorithm.

```
function Circ = column_circuits(M, r)
        columns = size(M, 2);
        C = combinations(columns, r);
        Circ = [];
        for (i=1:size(C, 1))
                subset = C(i, :);
                N = M(:, subset);
                c = det(N);
                if(mod(c, 3) == 0)
                        Circ = [Circ; subset];
                endif
        endfor
endfunction
```

This is the code that crawls among all the permutations of one of the matrices and compares the set of basis with the set of basis of the other matrix.

```
function void = busca_bases_iguales(A, T, r)
        n = size(A, 2);
        P = perms([1:n]);

        B = column_basis(T, r);
        for i = 1:factorial(n);
                p = P(i, :);
                pA = A(:, p);
                pa = rref(pA);
                Bpa = column_basis(pa, r);
                if(isequal(B, Bpa))
                        p
                        pA
                        pa
                        "____________________________________________"
                endif
        endfor
endfunction
```

This is the code that crawls among all the permutations of one of the matrices and compares the set of circuits with the set of circuits of the other matrix.

```
function void = look_for_common_circuits(A, T, r)
        n = size(A, 2);
        P = perms([1:n]);

        B = column_circuits(T, r);
        for i = 1:factorial(n);
                p = P(i, :);
                pA = A(:, p);
                pa = rref(pA);
                Bpa = column_circuits(pa, r);
                if(isequal(B, Bpa))
                        p
                        pA
                        pa
                        "____________________________________________"
                endif
        endfor
endfunction
```

This last two algoritmhs are redundant, but I made both just to double-check the results from any of them. With both, I obtained exactly the same set of permutations for both A_1 and A_2.

The result thrown by these algorithms is too long to be displayed here. But I'll include one case of each one.

T_8 is isomorphic to the matroid determined by:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Which is obtained from A_1 by permuting its columns.

R_8 is isomorphic to the matroid determined by:

$$\begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Which is obtained from A_2 by permuting its columns.

(vi) $AG(3,2)$ accepts the following representation, where every point is a vector in $\mathbb{Z}_2^3$

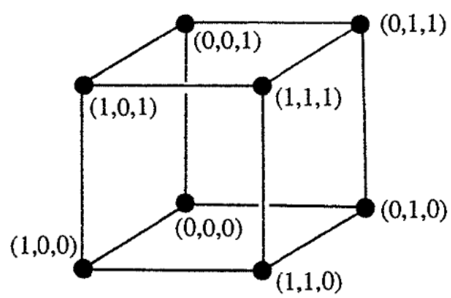


Figure 3.11: Representation of $AG(3,2)$

In this representation, every 4 vertices that lay on the same plane are dependient. Any set of 3 vertices is independent. This can be seen by simply affinity in $\mathbb{Z}_2^3$.

There are also a pair of circuits that are not laying in a plane. And those are the two thetahedrons obtained by taking any vertex and all the neighbors of its antipode.
By example let $T_1 = \{(0,0,0),(1,1,0),(1,0,1),(1,1,0)\}$.

It is complement is the other thetahedron. Both are circuits (If you sum them up, you will obtain the zero vector). Now, if you add the vector $(0,0,1)$ to T_1, you can generate $(1,0,0) = (1,0,1) + (0,01)$, and you can generate $(0,1,0) = (1,1,0) + (1,0,0)$, and $(1,0,0) = (1,1,0) + (0,1,0)$, and $(1,1,1) = (1,0,0) + (0,1,0) + (0,0,1)$. Then T_1 is an hyperplane. It doesn't matter what you add to it, you are able to generate any other vector in $\mathbb{Z}_2^3$. The same will occur the other thetahedron.

Then, you can relax these two circuit-hyperplanes, and you will get exactly the representation of R_8 given in [3.7].

□

3.8 Oxley, Section 2.3, Problem 1

(i) Show that the geometric dual of a plane graph is connected.

(ii) Give an example of a plane graph G for which $(G^*)^* \neq G$.

— . —

Proof.

(i) Take any embeding of a plane graph, take its dual, and take any pair of vertices of its dual. such vertices are faces in the original embeding. It is always possible to find a curve that joins that pair of vertices and that avoids any other vertice in the original embeding. Such curve defines a path bewteen the pair of vertices.

(ii) Any disconnected graph will work, $(G^*)^*$ will be connected but G not. Look at the following example.

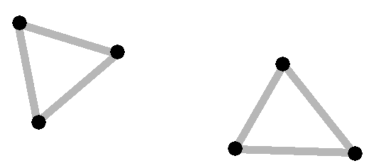


Figure 3.12: disconnected graph

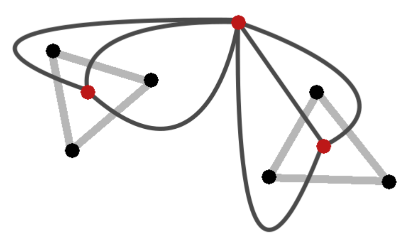


Figure 3.13: disconnected graph and its dual

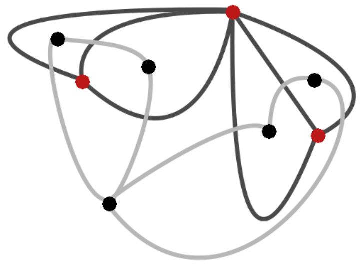


Figure 3.14: disconnected graph dual and its dual

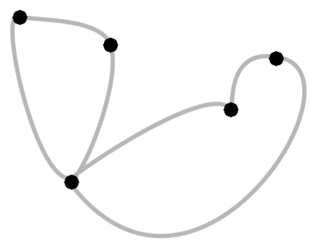


Figure 3.15: disconnected graph double dual

□

3.9 Oxley, Section 2.3, Problem 2

Find geometric duals of the graphs obtained from K_5 and $K_{3,3}$ by deleting a single edge of each.

— . —

Proof.

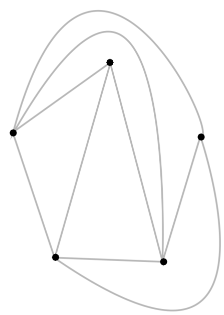


Figure 3.16: K_5 without one edge

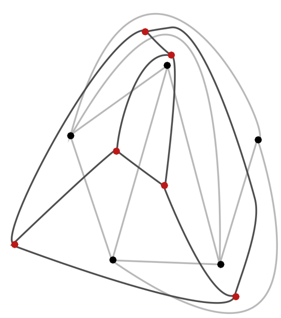


Figure 3.17: K_5 without one edge and its dual

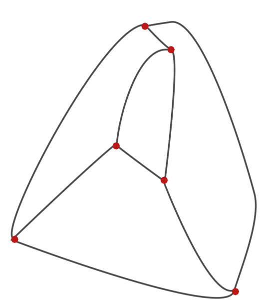


Figure 3.18: Dual of K_5 without one edge

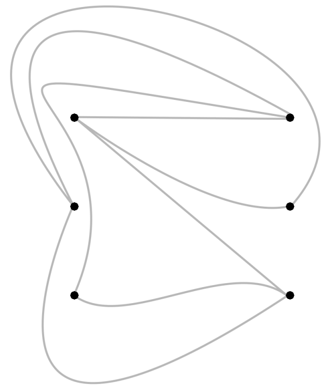


Figure 3.19: $K_{3,3}$ without one edge

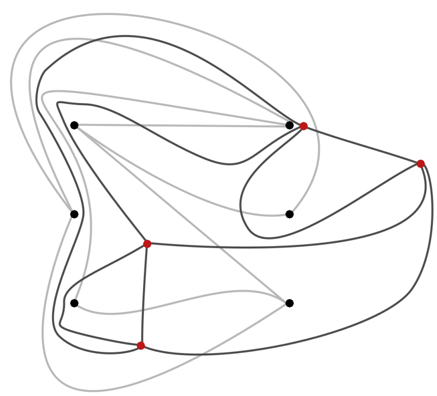


Figure 3.20: $K_{3,3}$ without one edge and its dual

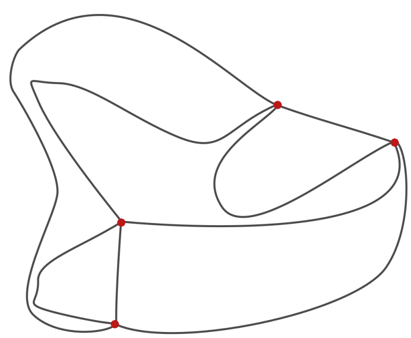
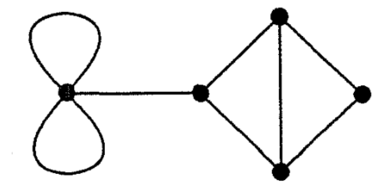


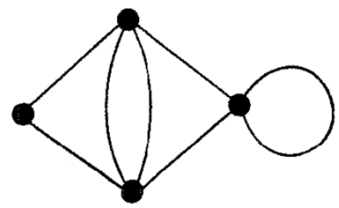
Figure 3.21: Dual of $K_{3,3}$ without one edge (K_4)

$\square$

3.10 Oxley, Section 2.3, Problem 6

Construct the geometric duals of the plane graphs in the next figures.





— . —

Proof.

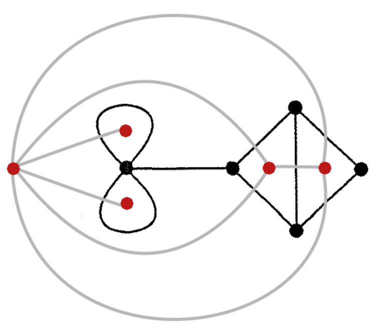


Figure 3.22: First graph and its dual

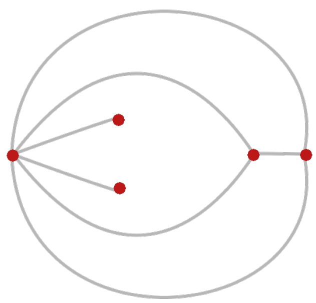


Figure 3.23: First graph's dual

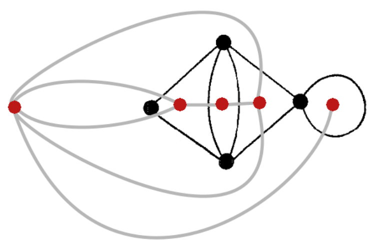


Figure 3.24: Second graph with its dual

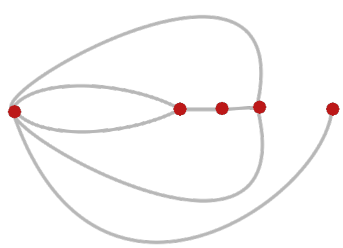


Figure 3.25: Second graph's dual

□

3.11 Oxley, Section 2.3, Problem 10

Let r be an integer exceeding one and W_r be the r-wheel. Show that:

(i) $W_3 \cong K_4$.

(ii) $W_r^* \cong W_r$.

(iii) $M(W_4)$ *is isomorphic to a restriction of* $M^*(K_{3,3})$.

— . —

Proof.

1. It is sufficient to prove that W_3 is complete.

   The central vertex is always adjacent to the rest of the vertices.

   Any vertex in the exterior cycle is adjacent to extactly other two of its type, but as W_3 has only 3 vertices in its exterior cycle, then any vertex in the exterior cycle is adjacent to the other two, plus, it is adjacent to the center.

   We have proved that any vertex has degree 3 and there are 4 vertices. Then the graph is complete and we are done.

2. Any triangle that is induced by the central vertex and two of the external vertices under the dual operator will end up as a vertex of degree tree, adjacent to two triangles of its type and to the exterior face.

   There are exactly r triangles of this type in W_r, then the external face will be adjacent to every one of these triangles.

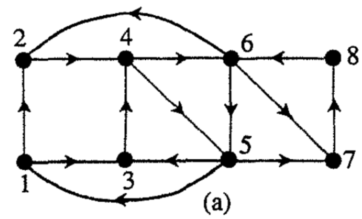
   Then the dual operator will make a graph such that it has a vertex adjacent to the rest of the vertices and the rest of the vertices are adjacent in addition to exactly two of its type. This is, the dual of W_r is W_r itself.

3. It is easy to see that $W_4 \cong K_{3,3}/\{e\}$ for any $e \in E(K_{3,3})$. Using [3.14], we have that $M^*(K_{3,3}/\{e\}) \cong M^*(W_4) \cong M(W_4^*) \cong M(W_4)$.

□

3.12 Oxley, Section 2.4, Problem 3

Let G be the directed graph shown in next figure.



(i) Construct the corresponding bipartite graph $\hat{G}$.

(ii) Let $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $X = \{5, 7\}$, $Y = \{4, 2\}$, and the paths linking X to Y in G be 5134 and 7862. Construct the corresponding matching of $V \setminus X$ to $\hat{V} \setminus \hat{Y}$ in $\hat{G}$.

(iii) Now take the matching $\{2\hat{1}, 3\hat{5}, 4\hat{3}, 5\hat{4}, 6\hat{8}, 8\hat{7}\}$ in $\hat{G}$ and construct the corresponding disjoint paths in G as in the proof of Lemma 2.4.3.

— . —

Proof.

(i)

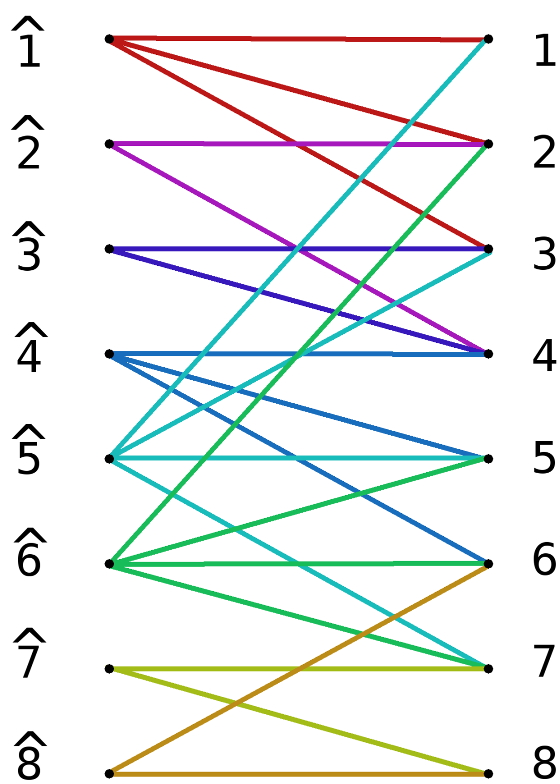


Figure 3.26: Bipartite graph

(ii)

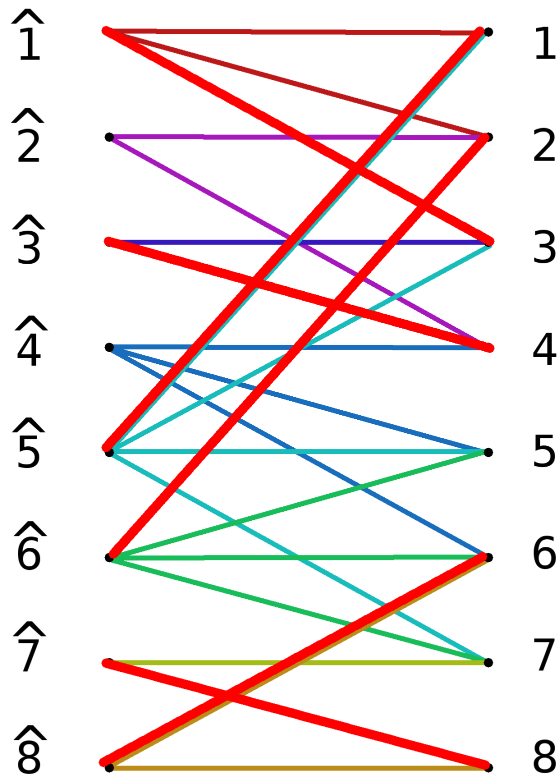


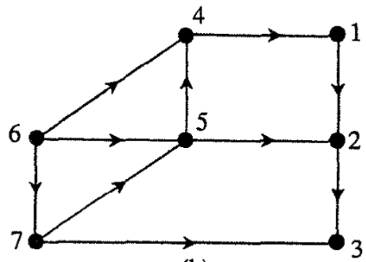
Figure 3.27: Bipartite graph and matching

(iii) The corresponding paths are $(1, 2)$ and $(7, 8, 6)$. It is generated also a cycle

$\square$

3.13 Oxley, Section 2.4, Problem 4

Let G be the directed graph shown in the next figure



(b)

and let $M = L(G, B_0)$ where $B_0 = \{1, 2, 3\}$.

(i) Find geometric representations for M and M^*.

(ii) Give a presentation for the transversal matroid M^*.

(iii) Reverse the directions on the arcs $(5, 4)$ and $(7, 5)$ of G and repeat (i) and (ii).

— . —

Proof. (i)

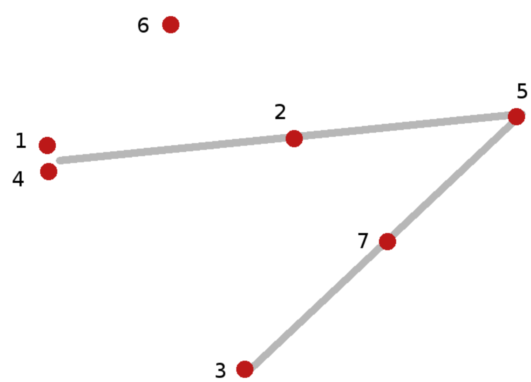


Figure 3.28: Geometric representation for M

(ii) Following the method exposed in the book, the family $\mathcal{A} = \{\{1, 4\}, \{2, 4, 5\}, \{4, 5, 6, 7\}, \{3, 5, 7\}\}$ is a presentation for M^*.

(iii) As the directions inside B_0 doesn't affect the gamoid, to invert $(5, 4)$ and $(7, 5)$ produces exactly the same matroid but exchanging the labels of 4 and 7.

□

3.14 Oxley, Section 3.1, Problem 1

Show that if $T \subset E(M)$, then

(i) $M \setminus T = (M^*/T)^*$;

(ii) $(M/T)^* = M^* \setminus T$; *and*

(iii) $M^*/T = (M \setminus T)^*$.

— . —

Proof. (i) Let B be a basis for $M \setminus T$. We would like to prove that B is also a basis for $(M^*/T)^*$, or, what is the same, that $E \setminus (B \cup T)$ is a base for M^*/T.

As B is a basis for $M \setminus T$, B is independent in M. And then there is a basis B' of M such that $B \subset B'$. Now $E \setminus B'$ is basis for M^*, and then $B_T^* = E \setminus B' \cap T$ is basis for T in M^*.

Now

$$\begin{aligned} E \setminus ((E \setminus (T \cup B)) \cup B_T^*) &= (T \cup U) \cap (E \setminus B_T^*) \\ &= (T \cup U) \cap (E \setminus ((E \setminus B') \cap T)) \\ &= (T \cup U) \cap (B' \cup (E \setminus T)) \\ &= ((T \cup B) \cap B) \cup ((T \cup B) \cap (E \setminus T)) \\ &= (T \cap B') \cup B \\ &= (T \cap B') \cup ((E \setminus T) \cap B') \\ &= B' \end{aligned}$$

Then, every basis for $M \setminus T$ is a basis for $(M^*/T)^*$.

Now, let B be a basis for $(M^*/T)^*$, then $(E \setminus T) \setminus B = E \setminus (B \cup T)$ is a basis for M^*/T, then, there is a basis B_T^* for T in M^* such that $(E \setminus (B \cup T)) \cup B_T^*$ is a basis for M^*.

Let $B' = (B \cup T) \cap (E \setminus B_T^*)$, B' is a basis for M. Now

$$\begin{aligned} (E \setminus T) \cap B' &= (E \setminus T) \cap (B \cup T) \cap (E \setminus B_T^*) \\ &\quad \text{(remember that } B \cap T = \emptyset) \\ &= B \cap (E \setminus B_T^*) \\ &= B. \end{aligned}$$

This menas that B is a basis for $M \setminus T$ also. And we have finished.

(ii) Applying the previous part,

$$\begin{aligned} (M/T)^* &= ((M^*)^*/T)^* \\ &= M^* \setminus T. \end{aligned}$$

(iii) Applying the previous part (the first one),

$$\begin{aligned} M^*/T &= ((M^*/T)^*)^* \\ &= (M \setminus T)^*. \end{aligned}$$

□

3.15 Oxley, Section 3.1, Problem 4

Suppose that, for all elements f of a loopless matroid M, $r(M \setminus \{f\}) = r(M)$, but that, for some elements e and g, $r(M \setminus \{e, g\}) = r(M \setminus \{e\}/g)$. Show that $\{e, g\}$ is a cocircuit of M.

— . —

Proof. Given that for every f, there is a basis of M that doesn't contain f, then as $\{f\}$ doesn't intersect every basis, $\{f\}$ is coindependent. That is M^* is loopless.

Now, let $e, g \in M$ such that $r(M \setminus \{e, g\}) = r(M \setminus \{e\}/g)$.

As M is loopless and M^* is loopless, $r(M \setminus \{e\}/g) < r(M \setminus \{e\})$.

Let B is a basis for M such that $e, g \notin B$, then $B \subset E(M) \setminus \{e\}$ and then B is a basis for $M \setminus \{e\}$ and also is a basis for $M \setminus \{e, g\}$. Then

$$\begin{aligned} |B| &= r(M \setminus \{e\}) \\ &> r(M \setminus \{e\}/g) \\ &= r(M \setminus \{e, g\}) \\ &= |B| \end{aligned}$$

Which is a contradiction. Then there is not such B that doesn't intersect $\{e, g\}$, and then $\{e, g\}$ is coindependent. As M^* is loopless then $\{e, g\}$ is a cocircuit of M, as we wished to prove. □

3.16 Oxley, Section 3.2, Problem 3

Consider the wheel graph W_r. Show that if $r \geq 3$, then $M(W_r)$ has $M(K_4)$ as a minor.

— . —

Proof. Thinking of graphs, we can obtain K_4 which is the same as W_3 from W_r by simply erasing $r-3$ contiguous edges of the central vertex and contracting all those edges that contain vertices that were disconected from the central vertex.

As in graph matroids it is the same to take the minor matroid or taking the matroid of the minor graph, what we have said is already sufficient. $\square$

3.17 Oxley, Section 3.2, Problem 5

Let T_8 and R_8 be matroids in [3.7]. Give geometric representations for each of $T_8/8$, $T_8/1$, $R_8/8$ and $R_8/1$.

— . —

Proof. The only interesting case is $T_8/8$. $T_8/1$ will be exactly the same case as in [3.9] (which is the representation of F_7^-) with a posible relabeling of the points. As every point in R_8 does behave exactly the same, it doesn't matter if you contract 8 or 1, the result will be completely analog to the one in [3.10] (which is again the representation of F_7^-).
The case of $T_8/8$ is only one that we have not seen yet.

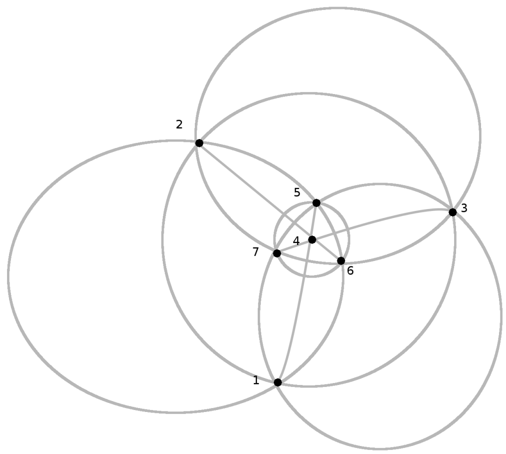


Figure 3.29: Geometric representation for $T_8/8$

□

4

Third Test

May 1, 2014

Some test's exercices were taken from Oxley's book, fifth and ninth chapters (Duality and Minors).

4.1 K_4 as a totally unimodular real matrix

Give a real representation for $M(K_4)$ that is totally unimodular.

— . —

Proof. Lets take a incidence matrix for K_4, and give it any orientation by replacing exaclty one 1 by a -1 in each column. It is a representation on $GF(3)$, we have already justified it in (iii), and for the same justification, it is a representation on $\mathbb{R}$ as well. Les call M such matrix.

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$$

Lets see that it is totally unimodular.

case 1×1 There is nothing to prove. The matrix consists of only 1×1 submatrices with 1, 0, or -1.

case 2×2 Cases with a column or row of zeros have determinat 0. So lets see the remaining cases.

There cannot be a case with no zero entries there are no two columns that contains non-zero entries in the same coordinates.

So, there should be at least one zero entry. Now, if you have a column with only one non-zero entry with value r, and the determinant of the resultant submatrix obtained by deleting the row and column of that non-zero entry is D, then the determinant of the original matrix will be $\pm rD$. So, in our case, $r \in \{-1, 1\}$ and $D \in \{-1, 0, 1\}$. So any 2×2 submatrix of M will have determinant $-1, 0$ or 1.

case 3×3 As we have already proved, if a 3×3 submatrix N of M contains a column with only zero entries or with only one non-zero entry, then its determinant will be $-1, 0$ or 1. Then the only case left is when any column contain two non-zero entries.

Which is equivalent to choose three edges of K_4 such that all of them are different and any of them is incident with the other two. Saying this, it is easy to see that N is the incidence matrix of an oriented triangle, and in (iii) we saw that such matrices have dependent columns set and then its determinant is always 0.

case 4×4 Again, the only case left to discuss will be the case in which every column has exactly two non-zero entries. That will be equivalent to choose four edges in the underlaying oriented K_4, as any tree in K_4 has exactly three edges, then any four edges will form a dependent set (because it will contain cycles and again thanks to 2.1) and then the determinant in such case will always be zero.

□

4.2 Totally unimodular matrix

Prove that the next matrix is totally unimodular.

$$\left[\begin{array}{c|ccccc} & -1 & 1 & 0 & 0 & 1 \\ & 1 & -1 & 1 & 0 & 0 \\ I_5 & 0 & 1 & -1 & 1 & 0 \\ & 0 & 0 & 1 & -1 & 1 \\ & 1 & 0 & 0 & 1 & -1 \end{array}\right]$$

This matrix represents R_{10} over all fields.
(In characteristic two, $-1 = 1$.)

— . —

Proof. As in the previous problem we are going to approach each case.

1×1 Any entry is -1, 0 or 1, so its determinat will be -1, 0 or 1.

2×2 As we saw before, any column containing only one non-zero entry will result in the determinant of a 1×1 with a possibly change of sign, and any column with only zeros will result in a determinant zero. Then, the only case left is the case with no zero entries.

There is essentially one case, which is

$$N = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

(given to ones in a column, there is no other column which contains non-zero values in the same rows). The determinant of N is 0 and then we have finished this part of the proof.

3×3 There is no case in cases that include columns from I_5. Every column of I_5 has at most one 1 and then, those cases can be reduced to 2×2 cases, which we have already proved to be totally unimodular.

We would do exhaustive search over all the $\binom{5}{3}\binom{5}{3} = 100$, 3×3 submatrices, but we won't. We will take advantage of the symmetries of the 5 last columns.

Because of the symmetries, we can fix column 6 and any other case will be similar up to permutations.

This has already reduced the space to $\binom{5}{3}\binom{4}{2} = 60$ cases. Which is still quite large.

Now, remember that any case in which we have a column with only one non-zero entry can be reduced to a case of a 2×2 submatrix (up to determinant sign), this gets rid of three possible row choices $(\{1,3,4\}, \{2,3,4\}, \{3,4,5\})$. Wich reduces the space to $\left(\binom{5}{3} - 3\right)\binom{4}{2} = 42$.

We will reduce cases applying this kind of arguments in even more specific subcases. Always think that column 6 is fixed.

rows 1, 2, 3. We cannot choose columns 9 or 10 because they will end up with only one non-zero entry. So the only case to check is choosing columns 7 and 8 (along with 6 which is fixed). This submatrix has determinant 1 and then we are done with this subcase.

rows 1, 2, 4. We cannot choose column 9 because of the same reason. We cannot choose column 7 or we will have two zeros in row 4. Then we must choose columns 8 and 10. And the resultant submatrix has determinant 0 and then we are done with this subcase.

rows 1, 2, 5. We cannot choose columns 8 or 9. We cannot choose columns 7 and 8 together or row 5 will end up with two zeros.

The two remaining case is choosing columns 7 and 10. And it is analog to the case when choosing rows 1, 2, 3 reordering rows as 5, 1, 2 and columns as 10, 6, 7.

rows 1, 3, 5. This subcase is analog to choosing rows 1, 2, 4 up to permutations.

rows 1, 4, 5. This subcase is analog to choosing rows 1, 2, 3 up to permutations.

rows 2, 3, 5. We cannot choose column 10 or it will have two zeros. We cannot choose columns 7 and 8 together or row 5 will have two zeros.

The two remaining cases are choosing 7 and 9 or 8 and 9. Whose determinats are both 0. And we are done with this case.

In the end we only needed to check the determinant seven cases.

4×4 Again, we will only check submatrices obtainded from the five last columns. Any other case can be reduced to a 3×3 case up to determinant sign.

Te symmetry in the five last columns lets us get rid of any row, it would be the same to delete any of them. So we will not take into account the row 5.

Now we only have five cases depending on which columns we choose.

columns 6, 7, 8, 9. The determinant is -1.

columns 6, 7, 8, 10. The determinant is 0.

columns 6, 7, 9, 10. The determinant is 1.

columns 6, 8, 9, 10. The determinant is -1.

columns 7, 8, 9, 10. The determinant is 0.

Any other case can be seen as one of these by permuting rows and columns.

5×5 Here we have only one interesting case. Choosing all the five last columns. And it has determinant 1.

□

4.3 Oxley, Section 5.2, Problem 5

(Whitney 1932a; Ore 1967, Section 3.3) The graph H is a Whitney dual of the graph G if there is a bijection $\psi : E(E) \to E(H)$ such that, for all subsets $Y \subset E(G)$,

$$r(M(H)) - r(M(H) \setminus \psi(Y)) = |Y| - r(M(G)|Y).$$

(i) Show that if H is a Whitney dual of G, then G is a Whitney dual of H.

(ii) Determine the relationship between Whitney duals and geometric and abstract duals.

(iii) Prove that a graph is planar if and only if it has a Whitney dual.

— . —

4.4 Oxley, Section 5.2, Problem 7

Let G be a connected plane graph.

(i) Use matroid duality to show that G satisfies Euler's polyhedron formula,

$$|V(G)| - |E(G)| + |F(G)| = 2,$$

where $F(G)$ is the set of faces of G.

(ii) Prove that if G is simple and $|V(G)| \geq 3$ then

$$|E(G)| \leq 3|V(G)| - 6.$$

(iii) Characterize the graphs for which equality is attained in (ii).

(iv) Show that if G is simple, then it has a vertex of degree at most five.

(v) Given an example of a simple plane graph in which every vertex has degree five.

— . —

Proof. (i) We know that $|F(G)| = |V(G*)|$ that $r(M(G)) = |V(G)| - 1$, that $r(M(G*)) = |V(G*)| - 1$, and that $|E(G)| = r(M(G)) + r(M(G*))$.

Then

$$\begin{aligned} |E(G)| &= r(M(G)) + r(M(G*)) \\ &= |V(G)| - 1 + |V(G*)| - 1 \\ &= |V(G)| + |V(G*)| - 2 \\ &= |V(G)| + |F(G)| - 2 \end{aligned}$$

And we are done.

(ii) Given that G is plane, then each face must have at least 3 edges. Then $3|F(G)| \leq 2|E(G)|$. From the equality from above

$$3|E(G)| + 6 = 3|V(G)| + 3|F(G)| \qquad \leq 3|V(G)| + 2|E(G)|$$

Which can be rewriten as $|E(G)| \leq 3|V(G)| - 6$.

(iii) If G is such that all the minimum cycles (cycles whose vertex are not contained in bigger cycles) are triangles. This means that any representation of G is a triangulation. That is, every face is a triangle.

If every face is a triangle then $\frac{3}{2}|F(G)| = |E(G)|$ and then

$$\begin{aligned} 2 &= |V(G)| - |E(G)| + |F(G)| \\ 2 &= |V(G)| - |E(G)| + \frac{2}{3}|E(G)| \\ 2 &= |V(G)| - \frac{1}{3}|E(G)| \\ \frac{1}{3}|E(G)| &= |V(G)| - 2 \\ |E(G)| &= 3|V(G)| - 6. \end{aligned}$$

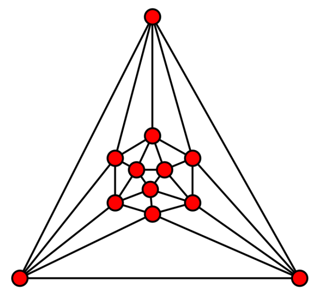
Now suppose that G is not a triangulation, in any representation there will be at least one face that is not a triangle and then such face has at least a pair of vertices that are not adjacent. We can draw a new edge between such pair of vertices such that the resulting graph is also plane. Given the inequality from above and this new graph, $E(G) + 1 \leq 3|V(G)| - 6$, which means that $E(G) < 3|V(G)| - 6$.

(iv) If every vertex has degree at least 6, given that $|E(G)| = \frac{\sum_{v\in V(G)} \delta(v)}{2}$, then

$$\begin{aligned}|E(G)| &= \frac{\sum_{v\in V(G)} \delta(v)}{2}\\ &\geq \frac{6|V(G)|}{2}\\ &\geq 3|V(G)|\end{aligned}$$

Which means that $|E(G)| \geq 3|V(G)| > 3|V(G)| - 6$, contradictint the inequality form above.

(v) The following graph meets the conditions.



□

4.5 Oxley, Section 9.1, Problem 7

(Seymour 1981b) If X is a 3-element subset of a matroid M, prove that M has a $U_{2,4}$-minor whose ground set contains X if and only if M has a circuit and a cocircuit whose intersection is X.

— . —

Proof. 9.1.5 □

4.6 Oxley, Section 9.2, Problem 2

For $r \geq 2$, consider an $r \times (2^r - 1)$ matrix A_r whose columns are all of the non-zero vectors in $V(r, 2)$. The vectors in the orthogonal subspace of $\mathcal{R}(A_r)$ form the binary Hamming code H_r is only determined up to a permutation of the coordinates. Show that:

(i) H_3 equals the row space of the following G matrix over $GF(2)$.

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

(ii) H_r has dimension $2^r - r - 1$.

(iii) There is a partition of $V(2^r - 1, 2)$ into m classes where m is the number of vectors in H_r and each class contains a unique member v of H_r together with all members of $V(2^r - 1, 2)$ that differ from v in exactly one coordinate.

— . —

Proof. (i) Lets write the following matrix.

$$A_3 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

From the definition. H_3 is the orthogonal subspace of $\mathcal{R}(A_3)$. Lets write A_3 as $A_3 = [M|I_3]$. It is easy to see that $G = [I_4|M^T]$.

Then $GA_3^T = [I_4|M^T][\frac{M^T}{I_3}] = [M^T + M^T] = 0_{4\times 3}$. Which means that $\mathcal{R}(G) \subset H_3$. As A_3 has rank 3 (In its right side it contains the I_3 identity matrix, which makes all of its rows linearly independent) and G has rank 4 (same reason) and we have proved that its genearted vector spaces are orthogonal, then they must be orthogonal complements (because $V(2^3 - 1, 2)$ has dimmension $2^3 - 1 = 7$).

(ii) $V(2^r - 1, 2)$ has dimmension $2^r - 1$, and A_r has dimmension r. As H_r is orthogonal to $\mathcal{R}(A_r)$ then the rank-nullity theorem says that

$$\begin{aligned} 2^r - 1 = \dim(V(2^r - 1, 2)) &= \dim(A_r) + \dim(H_r) \\ &= r + \dim(H_r) \end{aligned}$$

Which means $\dim(H_r) = 2^r - r - 1$

(iii) If $r = 1$ there is nothing to prove. Then Suppose that $r \geq 2$ and that there are distinct vectors $v, w \in H_r$ and vectors e_i, e_j (with zeros everywhere and ones in the entries i and j respectively) such that $v + e_i = w + e_j$. That is, there is a vector in $V(2^r - 1, 2)$ such that it belongs to two different classes. Then $v + w = e_i + e_j \in H_r$ which means that $e_i + e_j$ must be orthogonal to all the rows in A_r. Note that $e_i \neq e_j$, otherwise $v = w$ and we have assumed them to be distinct. The columns of A_r represent all the non-empty subsets of a set with r elements, the k-th row of A_r says if the element k is in a given subset or not. Given that the columns i and j are distinct, then the subsets that they represent are distinct, and then, ther must be at least one element that one has and the other one doesn't. Let u be the row that represent such element, then $(u)\cdot(e_i+e_j) = 0+1 = 1 \neq 0$ which means e_i+e_j cannot belong to H_r and then there are not elements that belong to two different classes.

He have proved that all the classes described are disjoint, now lets see that they cover $V(2^r-1,2)$. It is easy to see that each class has exactly 2^r elements and as H_r has dimension 2^r-r-1 it contains 2^{2^r-r-1} elements. Then the union of all the classes contain $2^{2^r-r-1}\cdot 2^r = 2^{2^r-1}$ elements, which is exactly de cardinality of $V(2^r-1,2)$.

$\square$

4.7 Oxley, Section 9.3, Problem 9

(Lehman 1964) Let e be an element of a connected binary matroid M and let $\mathcal{C}_e$ be the set of circuits containing e. Show that the circuits of M not containing e are precisely the minimal non-empty sets of the form $C_1 \triangle C_2$ where $C_1, C_2 \in \mathcal{C}_e$.

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Bibliography

[Oxl92] James G. Oxley. Matroid Theory. *Oxford University Press, 1992.*