Matroid Theory

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Contents

Note for the readers:

The goal on writing this in english instead of writing it in my first language (spanish) is because this way I can start building a portfolio that could not be as important as a research project, but it is work after all. So it would be better to have something to show that nothing to show at all, and much better if the audience capable of read it is as wide as possible.

As my first language is not english, this text could have a lot of mispellings and bad use of english. I apologize in advance and I'll try to be as careful as possible.

Here is a link to this project's version controller repository:
https://github.com/tellezhector/matroids
There you can see what are the changes that have ocurred along all this project's lifetime.

If you want to leave comments you can do it there, where anyone else is able to notice it. Or you could write directly to my personal e-mail address, which is: `tellez.hector@gmail.com` where it will be read only by me.

1

Union Of Closed Sets That Is Not Closed

Definition 1 (Closure)**.** *Given a matroid $M = (E, \mathcal{J})$ with rank function f. The* ***closure*** *of a set $S \subset E$ is*

$$Cl(S) = \{x \in E | r(S \cup \{x\}) = r(S)\}.$$

Definition 2 (Closed set)**.** *Given a matroid $M = (E, \mathcal{J})$ with rank function f. We say that a set $S \subset E$ is* ***closed*** *if*

$$Cl(S) = S.$$

Problem 3. *Give an example of two closed sets whose union is not closed.*

— . —

Proof. We are going to use the linear matroid of $\mathbb{Z}^2$, where the independent sets are

$$\emptyset, \{(1,0)\}, \{(0,1)\}, \{(1,1)\}, \{(1,0),(0,1)\}$$

and the rank function is the size of the biggest independent set contained.

It is clear that $A = \{(1,0),(0,0)\}$ and $B = \{(0,1),(0,0)\}$ are both closed. But $C = A \cup B = \{(1,0),(0,1),(0,0)\}$ is not, because $r(C) = 2 = r(C \cup \{(1,1)\})$ and $(1,1) \notin C$. $\square$

2

First Test

February 17, 2014

Test for Oxley's book first chapter (Basic definitions and examples).

2.1 Oxley, Section 1.1, Problem 2

Problem 4. *Let A be the matrix*

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

For q in $\{2, 3\}$, let $M_q[A]$ be the vector matroid of A when A is viewed over $GF(q)$, the field of q elements. Show that:

(i) The sets of circuits of $M_2[A]$ and $M_3[A]$ are different.

(ii) $M_2[A]$ is graphic but $M_3[A]$ is not.

(iii) $M_2[A]$ is represntable over $GF(3)$, but $M_3[A]$ is not representable over $GF(2)$.

— . —

Proof. (i) Lets look for the circuits over $GF(2)$. As 0 and 1 are the only scalars over $GF(2)$, we can only pay attention to sums of the vector columns and forget about full linear combinations.

There are no loops, the only way to get a loop is having a zero column and it is not the case.

There are no circuits of length two (or parallel columns), the only way to get them is having two exactly equal columns and it is not the case.

Lets call the columns 1, 2, and 3 the "simple columns", and the columns 4, 5 and 6 the "double columns".

For any pair of the columns from the simple columns there is a column from the double columns such that the sum is zero. Such column is the one you can get from adding the chosen pair. That is, $\{1, 2, 4\}$, $\{1, 3, 5\}$, $\{2, 3, 6\}$ are three dependent sets.

If you try to add all the simple columns, you can't add up to zero.

If you add two of the the double columns, you get the third one, so $\{4, 5, 6\}$ is dependent.

These four sets are circuits given that there are no smaller dependent sets.

These four are all the circuits of size three. Lets think about the circuits of size four.

Again, if you add two of the double columns, you get the third one, and there are two simple columns that can add up this third one. So, $\{1, 2, 5, 6\}$, $\{1, 3, 4, 6\}$, $\{2, 3, 4, 5\}$ are dependent sets. And no one of the four already given circuits are contained in them, so these are circuits themselves.

At any time you choose the three simple columns, you will get no way to add up to zero just adding double columns. So, any set containing all of the simple columns will not be a circuit.

If you choose all the double columns, you have already chosen a circuit, so if you add any other vector, you will get a dependent set that is not a circuit.

So these seven are all the circuits of $M_2[A]$.

Now lets look for the circuits over $GF(3)$. Again and for the same reason, there are not loops or parallel columns.

If you choose two simple colums and you add them, you get one of the double columns, so if you add to them the negative of this double column, you get the zero vector. So $\{1, 2, 4\}$, $\{1, 3, 5\}$, $\{2, 3, 6\}$ are circuits.

$\{4, 5, 6\}$ is not dependent, you can check that the determinant of their submatrix is 2 (or -2 if you switched something).

If you choose two doble columns and add to them a simple one, again the determinant is not zero (it is always 1 or -1). So $\{1, 2, 4\}$, $\{1, 3, 5\}$, $\{2, 3, 6\}$ are all the circuits of size three.

Now lets look for circuits of size four.

If you choose two double columns and add one to the negative of the second one, you can find two simple columns that can make them add up the zero vector (using the correct signs), these two vectors are exactly the ones that add up to the double column that you didn't choose. So none of these selections contains the circuits of size three and therefore they are circuits. So this give us the circuits $\{1, 2, 5, 6\}, \{1, 3, 4, 6\}, \{2, 3, 4, 5\}$.

If choose all of the double columns, you can choose any of the simple one so the four of them add up the zero vector using an appropiate selection for the signs. So this give us the dependent sets $\{1, 4, 5, 6\}, \{2, 4, 5, 6\}, \{3, 4, 5, 6\}$ and they are circuits because any of the smaller circuits contain at least two simple columns.

The only other chance is choosing all of the simple columns, but wathever you add, you will get any of the three sized circuits as a subset. So these six are all of the circuits of size four.

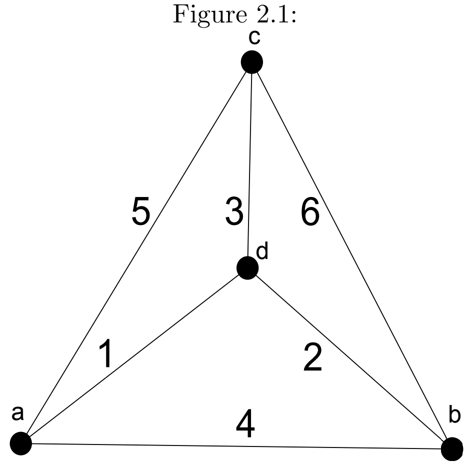
There are no circuits of size five, because there are only two posibilities, having three simple columns and two double columns, or having three double columns and two simple columns. The first one, again contains as a subset a circuit of size three. The second one always contains as a subset a circuit of size four.

So in total, there are nine circuits for $M_3[A]$.

Only by their cardinality, you can tell that the set of circuits of $M_2[A]$ and $M_3[A]$ are different.

(ii) We are going to give a graphic representation of a graph that has $M_2[A]$ as an isomorphic matroid. You can verify that it has all the seven circuits that we gave previously and only those.

Figure 2.1:



Now lets think what could happen if $M_3[A]$. For $M_3[A]$, we have the circuits $\{1,4,5,6\}, \{2,4,5,6\}, \{3,4,5,6\}$. Take any two of them, lets say $\{1,4,5,6\}, \{2,4,5,6\}$, they have three edges in common $\{4,5,6\}$.

If there is a graphic mathroid isomorphic to $M_3[A]$, $\{4,5,6\}$ defines three edges of a 4-cycle, that is, a 3-trajectory. In any 3-trajectory we will find 4 vertices, so the fourth edge is already determined by two of those vertices (the start vertex and the last vertex), but in this case we have that $\{1,4,5,6\}$, $\{2,4,5,6\}$ are both 4-cycles over the same set of vertices, and so 1 and 2 have exactly the same two vertices, that is, they are parallel. But in [4.(i)] we saw that $M_3[A]$ has no parallel columns. So this is a contradiction.

(iii) We are going to give a matrix B over $GF(3)$ such that $M_3[B]$ isomorfic to $M_2[A]$.

We are going to take the incidence matrix for the graphic representation [2.1], and choose one "1" from each column and change it by "-1". What we are doing is giving a orientation to each of the edges. So, if a cycle is such that each edge has exactly outdegree 1 and indegree 1, is easy to see that the sum of the respective columns will be the zero vector (each 1 will have exactly one -1 in its same row).

But lets remember that we are taking linear combination over $GF(3)$, that is, we can take a column, its negative, or take the zero vector instead. So, if we have a cycle, not necessarily "well-oriented", we can give it a "good-orientation" by multiplying its edges by -1 or -2 such way that each vertex has outdegree 1 and indegree 1. If we sum them, we will get the zero vector, and givin a "good-orientation" is no other thing than finding a linear combination that sums up 0.

So, let B be the matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

By the explanation just given, each circuit in $M_2[A]$ is a circuit in $M_3[B]$ (note that we used the same labels in both matrices so the subsets of labels can have the same representation).

Also, note that if you choose a subset of edges such that you have leaves, the row for any leaf cannot be zero under any linear combination unless you multiply it by 0. Having said these, is easy to see that the subsets of columns representing forests are independent.

Now, lets suppose there is a matrix C such that $M_2[C]$ is isomorphic to $M_3[A]$. Again, lets suppose that the labels are the same in both matrices. Then, you must have that the subsets of columns $\{1,4,5,6\}$ and

$\{2, 4, 5, 6\}$ are circuts. So, the sum of the columns 4, 5, 6 and 1 must be the zero vector, that is the column 1 is equal to the sum of the columns 4, 5 and 6. But the same occurs with the other circuit, the sum of columns 4, 5, 6 and 2 must be zero, so the colun 2 is equal to the sum of the columns 4, 5 and 6. Then we have that the columns 1 and 2 must be the equal. That is, the columns 1 and 2 are parallel. Which is a contradiction to what we proved in [4.(i)].

□

2.2 Oxley, Section 1.1, Problem 7

Problem 5. *Let M_1 and M_2 be matroids on disjoint sets E_1 and E_2 and with independent sets $\mathcal{I}_1$ and $\mathcal{I}_2$ respectively. Let $E = E_1 \cup E_2$ and $\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}$. Prove that $(E, \mathcal{I})$ is a matroid.*

— . —

Proof. Lets see that $(E, \mathcal{I})$ satisfies $I1, I2$ and $I3$.

(I1) $\emptyset \in \mathcal{I}$. This is clear, $\emptyset \in \mathcal{I}(M_1)$ and $\emptyset \in \mathcal{I}(M_1)$, so $\emptyset = \emptyset \cup \emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$, $J \subset I$ then $J \in \mathcal{I}$. $I \in \mathcal{I}$ means that there are $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$ such that $I = I_1 \cup I_2$. Given that $J \subset I$ it is clear that $J = J \cap I = J \cap (I_1 \cup I_2) = (J \cap I_1) \cup (J \cap I_2)$. We have that $(J \cap I_1) \subset I_1$ and $(J \cap I_2) \subset I_2$, given that M_1 and M_2 are matroids, then $(J \cap I_1) \in \mathcal{I}_1$ and $(J \cap I_2) \in \mathcal{I}_2$. Then $J = (J \cap I_1) \cup (J \cap I_2) \in \mathcal{I}$.

(I3) If $I, J \in \mathcal{I}$, $|J| < |I|$ then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$. By hypotesis, there are $I_1, J_1 \in \mathcal{I}_1$ and $I_2, J_2 \in \mathcal{I}_2$ such that $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$. Then $|J| = |J_1| + |J_2| < |I_1| + |I_2| = |I|$. Then there must be that $|J_1| < |I_1|$ or that $|J_2| < |I_2|$. Withtout loss of generality, lets say that $|J_1| < |I_1|$, as M_1 is a matroid, then, there exists $x \in I_1 \setminus J_1$ such that $J_1 \cup \{x\} \in \mathcal{I}_1$. Then, $x \in I$ and $J \cup \{x\} = (J_1 \cup \{x\}) \cup J_2 \in \mathcal{I}$. And we are done.

□

2.3 Oxley, Section 1.1, Problem 9

Problem 6. *Let M_1 and M_2 be matroids on a set E. Give an example to show that $(E, \mathcal{I}(M_1) \cap \mathcal{I}(M_2))$ need not be a matroid.*

— . —

Proof. Let $E = \{a, b, c\}$.
Let $\mathcal{I}(M_1) = \{\{a, b\}, \{a, c\}, \{a\}, \{b\}, \{c\}, \emptyset\}$ Lets check that this is a matroid. It hass the empty set, given any set, any of its subests is independent. To see that the independence augmentation axiom (if $I, J \in \mathcal{I}$ and $|J| < |I|$ then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$), I will give you a table, where the first row is for the set I, the second for J and the third one for x.

I	J	x
$\{a, b\}$	$\{a\}$	b
$\{a, b\}$	$\{b\}$	a
$\{a, b\}$	$\{c\}$	a
$\{a, b\}$	$\emptyset$	a
$\{a, c\}$	$\{a\}$	c
$\{a, c\}$	$\{b\}$	a
$\{a, c\}$	$\{c\}$	a
$\{a, c\}$	$\emptyset$	a
$\{a\}$	$\emptyset$	a
$\{b\}$	$\emptyset$	b
$\{c\}$	$\emptyset$	c

So $(E, \mathcal{I}(M_1))$ is a matroid.

Now, let $\mathcal{I}(M_2) = \{\{a, b\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \emptyset\}$, to see that $(E, \mathcal{I}(M_2))$ the first two axioms can be justified just as above, and just below is the table to check the augmentation axiom.

I	J	x
$\{a, b\}$	$\{a\}$	b
$\{a, b\}$	$\{b\}$	a
$\{a, b\}$	$\{c\}$	b
$\{a, b\}$	$\emptyset$	a
$\{b, c\}$	$\{a\}$	b
$\{b, c\}$	$\{b\}$	c
$\{b, c\}$	$\{c\}$	b
$\{b, c\}$	$\emptyset$	b
$\{a\}$	$\emptyset$	a
$\{b\}$	$\emptyset$	b
$\{c\}$	$\emptyset$	c

So $(E, \mathcal{I}(M_2))$ is also a matroid.

Then $\mathcal{I}' = \mathcal{I}(M_1) \cap \mathcal{I}(M_2) = \{\{a, b\}, \{a\}, \{b\}, \{c\}, \emptyset\}$ cannot be an independet set for E. It satisfies (I1) and (I2), but not I3 since $\{a, b\}, \{c\} \in \mathcal{I}'$, and $|\{c\}| < |\{a, b\}|$ but $\{c\} \cup \{a\} \notin \mathcal{I}'$ and $\{c\} \cup \{b\} \notin \mathcal{I}'$. $\square$

2.4 Oxley, Section 1.2, Problem 1

Problem 7. *Prove that $\mathfrak{B}$ is the collection of bases of a matroid on E if and only if $\mathfrak{B}$ satisfies (B1) and the following two conditions:*

(B2)' If $B_1, B_2 \in \mathfrak{B}$ and $e \in B_1$, then there is an element f of B_2 such that $(B_1 \setminus \{e\}) \cup \{f\} \in \mathfrak{B}$.

(B3) If $B_1, B_2 \in \mathfrak{B}$ and $B_1 \subset B_2$, then $B_1 = B_2$.

— . —

Proof.

Sufficiency

Suppose that $\mathfrak{B}$ is the collection of bases of a matroid.

Then, it satisfies (B1) and (B2). As $\mathfrak{B}$ is a clutter, if $B_1, B_2 \in \mathfrak{B}$, then $B_1 = B_2$. Proving that it satisfies $(B3)$.

Now, lets suppose that $B_1, B_2 \in \mathfrak{B}$ and $e \in B_1$.

If $e \in B_1 \cap B_2$ then we could choose $f = e \in B_2$ and then $(B_1 \setminus \{e\}) \cup \{f\} = B_1 \in \mathfrak{B}$.

If $e \in B_1 \setminus B_2$ then there exists an element $f \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e\}) \cup \{f\} \in \mathfrak{B}$.

Necessity

Now lets suppose that $\mathfrak{B}$ satisfies (B1), (B2)' and (B3).

Let be B_1, B_2 in $\mathfrak{B}$ and $x \in B_1 \setminus B_2$. By (B2)' we have that there is an element y in B_2 such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathfrak{B}$. If $y \in B_1$, then $(B_1 \setminus \{x\}) \cup \{y\} = B_1 \setminus \{x\} \in \mathfrak{B}$, but as $B_1 \setminus \{x\} \subset B_1$, (B3) implies that $B_1 \setminus \{x\} = B_1$, which is a contradition. So $y \in B_1 \setminus B_2$, proving that $\mathfrak{B}$ satisfies (B2) and therefore, is the collection of bases of a matroid. □

2.5 Oxley, Section 1.2, Problem 6

Problem 8. *Suppose B is a basis of a matroid M, $f \in E(M)$ and $e \in E(M) \setminus B$. Prove that $(B \cup \{e\}) \setminus \{f\}$ is a basis of M if and only if $f \in C(e, B)$.*

— . —

Proof.

Sufficiency

Suppose that $f \in C(e, B)$. $C(e, B)$ is the only circuit contained in $B \cup \{e\}$ by definition. Then $(B \cup \{e\}) \setminus \{f\}$ does not contain any circuit, that means, it is independent. On the other hand, $e \notin B$ and $f \in B$, then $|(B \cup \{e\}) \setminus \{f\}| = (|B| + 1) - 1 = |B|$, therefore, $(B \cup \{e\}) \setminus \{f\}$ is a basis, as it is an independent set with the size of a basis.

Necessity

We know that $(B \cup \{e\})$ contains only one circuit (and it is $C(e, B)$). If $(B \cup \{e\}) \setminus \{f\}$ is a basis of M, then it can not contain $C(e, B)$. That is, $C(e, B) \subseteq (B \cup \{e\})$, but $C(e, B) \nsubseteq (B \cup \{e\}) \setminus \{f\}$, then f must be in $C(e, B)$.

$\square$

2.6 Oxley, Section 1.3, Problem 4

Problem 9. *Prove that a matroid M is uniform if and only if it has no circuits of size less than $r(M) + 1$.*

— . —

Proof.

Sufficiency

Suppose that M is a uniform matroid $U_{m,n}$, then $r(M) = m$ by definition. Moreover, any $A \subset E(M)$ such that $|A| < m$ will be subset of some basis and therefore independent. Then if there is any circuit C, it must have size greater than m, that is $r(C) \geq m + 1 = r(M) + 1$.

Necessity

Suppose now that M is such that it has no circuits of size less than $r(M) + 1$. Then, for any $A \subset E(M)$ with $|A| \leq r(M)$ it must be independent, if not, then A should contain a circuit, but any circuit has size at least $r(M) + 1$.

Then M must be isomorphic to an uniform matroid $U_{r(M),|E(M)|}$. $\square$

2.7 Oxley, Section 1.3, Problem 5

Problem 10.

(i) Characterize paving matroids in terms of their collection of independent sets and in terms of their collections of bases.

(ii) Characterize uniform matroids in terms of their collections of circuits.

— . —

Proof.

(i) Let M be a matroid. $\mathfrak{B}$ its collection of bases and $\mathcal{I}$ its collection of independent sets.

M is a paving matroid if and only if for every $A \subset E(M)$ with $|A| < r(M)$ there exists $B \in \mathfrak{B}$ such that $A \subset B$.

As M is matroid, by $I2$, this is exactly the same as saying:

M is a paving matroid if and only if for every $A \subset E(M)$ with $|A| < r(M)$, $A \in \mathcal{I}$.

Proof. If M is paving, by definition of paving matroid, any $A \in E(M)$ such that $|A| < r(M)$ then, $A \in \mathcal{I}$ (and therefore, there is some $B \in \mathfrak{B}$ such that $A \subset B$).

If M is a matroid such that for every $A \subset E(M)$ with $|A| \leq r(M)$ there exists $B \in \mathfrak{B}$ such that $A \subset B$ (and therefore, $A \in \mathcal{I}$). Then, suppose that C is a circuit of M. If $|C| < r(M)$, then there would be some $B \in \mathfrak{B}$ such that $C \subset B$, but then $C \in \mathcal{I}$, which is a contradiction and therefore $|C| \geq r(M)$. That is, M is paving.

□

(ii) A matroid M is uniform if and only if it has no circuits of size less than $r(M) + 1$.

Proof. See (9) □

□

2.8 Oxley, Section 1.4, Problem 2

Problem 11. *Show that a subset X of a matroid is a basis if and only if X is both independent and spanning.*

— . —

Proof.

Sufficiency

If X is a basis of a matroid M, then by definition of basis, it is independent. Also, as X is basis, then $r(X) = r(X \cup x)$ for any $x \in E(M)$ and, therefore, $cl(X) = E(M)$, that is, X is spanning.

Necessity

Suppose that X is both independent and spanning. As X is spanning, then $r(X) = r(M)$, that is, for any $B \in \mathfrak{B}(M)$, $|X| \geq r(X) = r(B) = |B|$. On the other hand, $X \in \mathcal{I}(M)$ so, for any $B \in \mathfrak{B}(M)$, $|X| \leq |B|$. We have then that $|B| \leq |X| \leq |B|$, that is $|B| = |X|$ for any $\mathfrak{B}(M)$ and therefore, X is a basis. $\square$

2.9 Oxley, Section 1.4, Problem 6

Problem 12. *Prove that statments (a)-(g) below are equivalent for an element e of a matroid M:*

(a) *e is in every basis.*

(b) *e is in no circuits.*

(c) *If $X \subseteq E(M)$ and $e \in cl(X)$, then $e \in X$.*

(d) $r(E(M) \setminus \{e\}) = r(E(M)) - 1$.

(e) *$E(M) \setminus \{e\}$ is a flat.*

(f) *$E(M) \setminus \{e\}$ is a hyperplane.*

(g) *If I is an independent set, then so is $I \cup \{e\}$.*

— . —

Proof.

[(a) ⇒ (b)]

Suppose there is a circuit C such that $e \in C$, then $C \setminus \{e\} \in \mathcal{I}(M)$, then there must be $B \in \mathfrak{B}$ such that $C \setminus \{e\} \subseteq B$, but $e \notin B$, otherwise $C \subset B$. But this contradicts that e is in every basis and therefor there is not such circuit C.

[(b) ⇒ (c)]

Let $X \subseteq E(M)$ such that $e \in cl(X)$. Suppose that $e \notin X$ and let B_X be a basis of X. we have that $x \in cl(X) = cl(B_X)$, then $cl(B_X) = cl(B_X \cup \{e\})$, which means that $r(B_X \cup \{e\}) = r(B_X) = |B_X| < |B_X \cup \{e\}|$, and therefore $B_X \cup \{e\}$ is a dependent set, and it must contain a circuit C, as C can not be subset of B_X (because B_X is independent) then $e \in C$, which is a contradiction to (b).

[(c) ⇒ (d)]

As $E(M) \setminus \{e\}$ is missing only one element of $E(M)$, then $r(E(M) \setminus \{e\}) \geq r(E(M)) - 1$. If $r(E(M) \setminus \{e\}) = r(E(M))$ then $cl(E(M) \setminus \{e\}) = E(M)$ and $e \in E(M)$ but $e \notin E(M) \setminus \{e\}$, which contradicts (c).

[(d) ⇒ (e)]

By definition of closure, $E(M) \setminus \{e\} \subseteq cl(E(M) \setminus \{e\})$. And $E(M) \setminus \{e\}$ is contained in only two subsets of $E(M)$, there are $E(M)$ itself and $E(M) \setminus \{e\}$. So $cl(E(M) \setminus \{e\}) = E(M) \setminus \{e\}$ or $cl(E(M) \setminus \{e\}) = E(M)$. If the second occurs, then $r(E(M) \setminus \{e\}) = r(E(M))$, which contradicts (d), and therefore $cl(E(M) \setminus \{e\}) = E(M) \setminus \{e\}$, which means that $E(M) \setminus \{e\}$ is flat.

[(e) ⇒ (f)]

As $E(M) \setminus \{e\}$ is flat, then $r(E(M) \setminus \{e\}) < r(M)$, but $E(M) \setminus \{e\}$ is missing only e, so $r(E(M) \setminus \{e\}) = r(M) - 1$ and then $E(M) \setminus \{e\}$ is an hyperplane.

[(f) ⇒ (g)]

Let $I \in \mathcal{I}(M)$ such that $e \notin I$ and let $B \in \mathcal{I}$ be a basis for $E(M) \setminus \{e\}$ containing I. As $|B| = r(B) = r(E(M) \setminus \{e\}) = r(M) - 1$, then $r(B \cup \{e\}) = r(M) = |B| + |\{e\}| = |B \cup \{e\}|$ and therefore $B \cup \{e\}$ is a basis, and $I \cup \{e\} \subseteq B \cup \{e\}$ so $I \cup \{e\}$ is independent.

[(g) ⇒ (a)]

Let $B \in \mathfrak{B}(M)$, as $B \cup \{e\} \in \mathcal{I}(M)$ and B is maximal by contention over independent sets, so there must be the case that $B \cup \{e\} = B$. $\square$

2.10 Oxley, Section 1.6, Problem 1

Problem 13. *Show the following:*

(i) All uniform matroids are transversal.

(ii) A transversal matroid need not be graphic.

(iii) A paving matroid need not be transversal.

— . —

Proof. (i) Let $U_{m,n}$ be a uniform matroid of rank m over n elements, define $\mathcal{A} = (A_j | j \in [1, m])$ and $A_j = E(U_{m,n})$ and $S = E(U_{m,n})$. Then, for every subset $J \subset [1, m]$ you can take $T = J$ and the bijection $\psi : J \to T$ such that $\psi(j) = j$, proving that every J is an indepentend set in the transversal matroid just defined. Just as every J is independent in $U_{m,n}$. By the definition of $\mathcal{A}$, there cannot be independent sets of size larger than m. Just as in $U_{m,n}$. These two observation determine $U_{m,n}$ proving that $U_{m,n}$ can be seen as a transversal matroid.

(ii) Let $U_{2,4}$ be a uniform matroid, as we proved above, $U_{m,n}$ is transversal. $\{1, 2, 3\}$ and $\{1, 2, 4\}$ are circuits (see 9), if they were triangles of a graph, then the edges 3 and 4 must be parallel. But $\{3, 4\}$ is independent in $U_{\{}2, 4\}$, so $U_{2,4}$ is not graphic.

(iii) Take the uniform matroid $U_{2,3}$ again, and remove the set $\{1, 3\}$ and $\{2, 3\}$ from the collection of independent sets, what you get is a paving matroid that is not uniform. Now, we have only the independent sets $\{1, 2\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\emptyset$. Lets suppose that this is a transversal matroid, as the largest independent set has two elements, then we can assume that $\{A\}$ has only two elements, lets call them a and b. As $\{1, 2\}$ is independent without loss of generality, there would be a relation $(a, 1), (b, 2)$ and there is someone linked to 3, lets think it is $(a, 3)$, then we can have the matching $(b, 2), (a, 3)$, which lead us to a independent set that was not supposed to be in our matroid.

The same will happen no matter how you try to pair a, b with $1, 2, 3$ in a way that let you have the matching given by our collection of independent sets.

□

2.11 Oxley, Section 1.6, Problem 3

Problem 14. *Let* $S = \{1, 2, \ldots, 6\}$ *and* $\mathcal{A} = \{A_1, A_2, A_3\}$ *where* $A_1 = \{1, 2, 3\}$ $A_2 = \{2, 3, 4\}$, *and* $A_3 = \{4, 5, 6\}$.

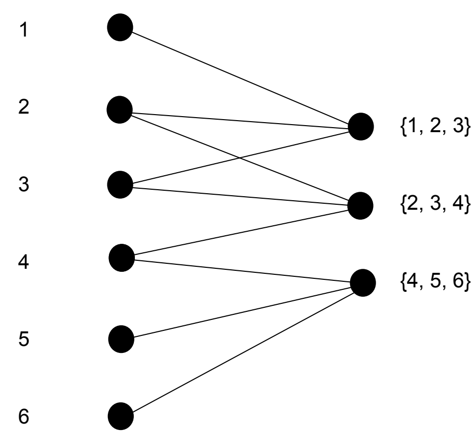
(i) Find $\Delta[\mathcal{A}]$.

(ii) Give a geometric representation for $M[\mathcal{A}]$.

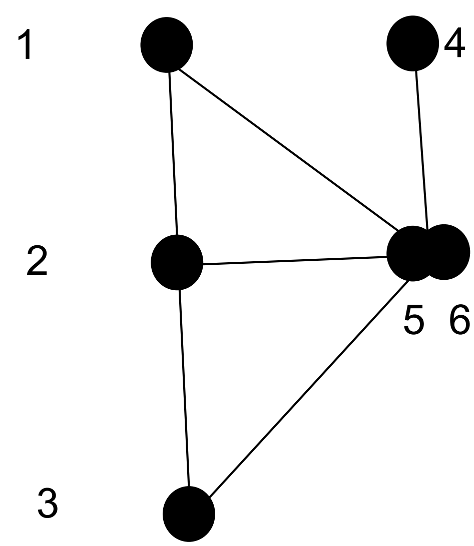
— . —

Proof.

1. Graphic representation for for $\Delta[\mathcal{A}]$



2. Geometric representation for $M[\mathcal{A}]$:



The dependent sets are $\{1, 2, 3\}$, $\{1, 5, 6\}$, $\{2, 5, 6\}$, $\{3, 5, 6\}$, $\{4, 5, 6\}$ and $\{5, 6\}$

□

2.12 Oxley, Section 1.6, Problem 4

Problem 15. *Characterize the circuits of $M[\mathcal{A}]$ in terms of the bipartite graph $G = \Delta[\mathcal{A}]$.*

— . —

Proof. Lets remember that $G = \Delta[\mathcal{A}]$ is a bipartite graph with partition $(S, \mathcal{A})$ and that in terms of transversal matroids, a subset $T \subset S$ is independent if there is an injection $\phi : T \to \mathcal{A}$ such that $(t, \phi(t))$ is and edge for every $t \in T$.

Halls theorem states that a susbset $T \subset S$ is independent if and only if $|U| \leq |N_G(U)|$. For every $U \subset T$.

That is, a subset $T \subset S$ is dependent if and only if there is $U \subset T$ such that $|U| > |N_G(U)|$. We are looking for subsets T that satisfies this condition and that are minimal.

If there is $U \subsetneq T$ such that $|U| > |N_G(U)|$, then T can not be minimal.
Then T must has the property $|T| > |N_G(T)|$ (we will refer to this property as the $(*)$-property) and its minimal.

Then, T is a circuit if and only if it is has the $(*)$-property and its minimal with such condition. (first approach).
We can learn a little more about these T's. Lets check what happen if the graph induced by T has at least 2 connected components. Lets call T_1 and T_2 such that T_1 induces one connected component and $T_2 = T \backslash T_1$. Notice that $N_G(T_1) \cap N_G(T_2) = \emptyset$, and then $|N_G(T)| = |N_G(T_1)| + |N_G(T_2)|$. As $|T| = |T_1| + |T_2| > |N_G(T_1)| + |N_G(T_2)| = |N_G(T)|$ it must be that either $|T_1| > |N_G(T_1)|$ or $|T_2| > |N_G(T_2)|$, without loss of generality, if the first one is true, then at least T_1 is smaller than T and then T is not minimal. Then the graph induced by T must be connected.

T is circuit if and only if the graph induced by T (lets call it $G[T]$) is isomorphic to $K_{|T|,|T|-1}$.

Necessity

Suppose the $G[T]$ is isomorphic to $K_{|T|,|T|-1}$, T is minimal with $(*)$-property, if you remove any vertex from T, its induced graph will be $K_{|T|-1,|T|-1}$ wich has perfect matches, if you remove more you will have a $K_{m,n}$ with $m < n$ which has $K_{m,m}$ as subgraph.

Sufficiency

Suppose you have a circuit T, as T is bipartite and connected, then it must contain a $K_{m,n}$ as subgraph induced by vertices in T. If $n < m$, for one of them the necessity proves that T must be as desired, or then it has a subset that is dependent. If it is not the case and $m \leq n$ for every such graph, then T doesn't have the $(*)$-property. $\square$

2.13 Oxley, Section 1.7, Problem 5

MISSING

Problem 16. *Prove that a finite lattice $\mathcal{L}$ is semimodular if, for all x, y in $\mathcal{L}$, the following condition holds:*

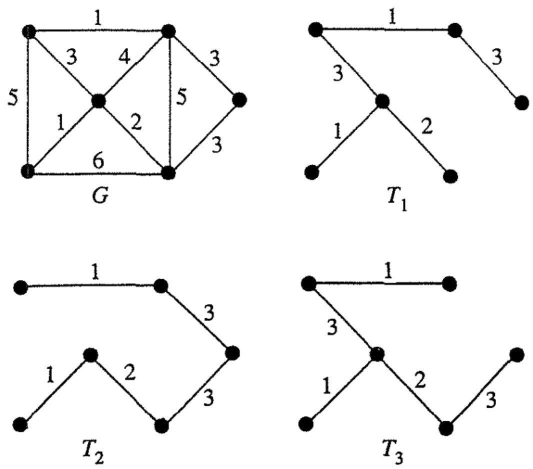
IF BOTH x AND y COVER $x \wedge y$, THEN $x \vee y$ COVERS BOTH x AND y.

— . —

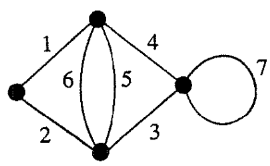
Proof. □

2.14 Oxley, Section 1.8, Problem 1

Problem 17. *(i) Find a maximum-weight spanning tree of the graph in the next figure. Is this the unique such tree?*

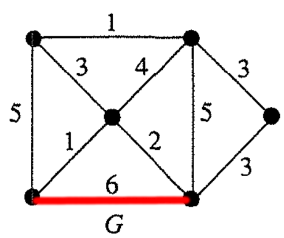


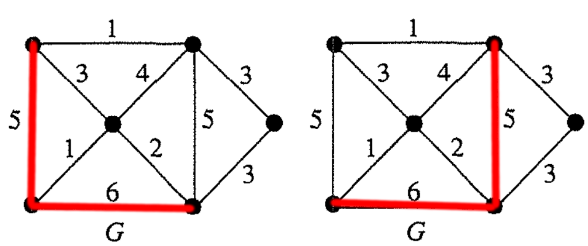
(ii) Find all maximum-weight spanning trees and all minimum-weight spanning trees of the graph in the next figure, where the edge labels are interpreted as weights.

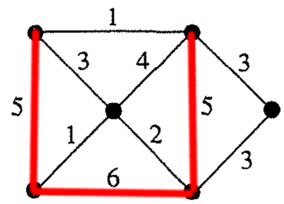


— . —

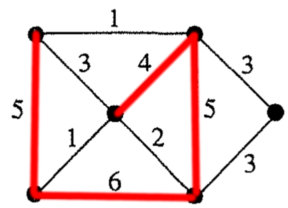
Proof. 1. There are only 2 maximum-weight spanning trees of the graph. I have ilustrated what will greedy do step by step, and with all possible desicions being made.







G



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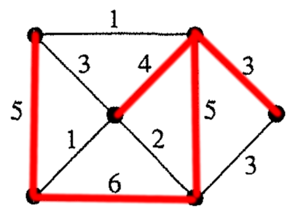

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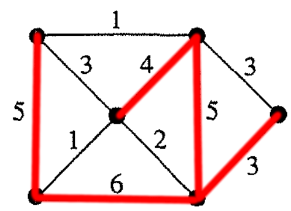
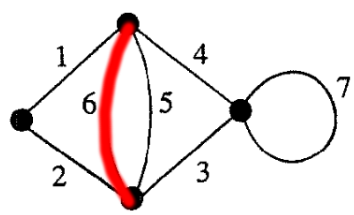


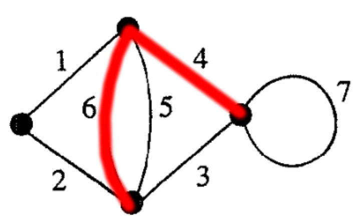
G

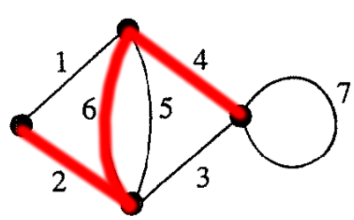
2. As the weight function is injective, the maximum-weight and minimum-weight spanning trees are unique.

   Here is again an ilustration of what greedy would do in both scenarios.

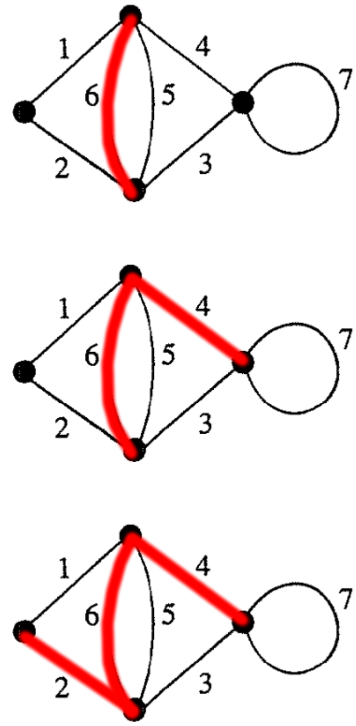
   Maximum-weight:







Minimum-weight:



□

2.15 Oxley, Section 1.8, Problem 4

Problem 18. *Let M be a matroid and $\omega : E(M) \to \mathbb{R}$ be a one-to-one function. Prove that M has a unique basis of maximum weight.*

— . —

Proof. Let B be a maximum-weight basis found using greedy. Suppose that there is another basis B' with maximum-weight also and suppose that they are different.

Let $e \in B' \setminus B$ of minimum-weight (it exists because ω is one-to-one), and then, there exists $f \in B \setminus B'$ such that $(B' \setminus \{e\}) \cup \{f\}$ is basis. We can choose such f of maximum weight. If $\omega(e) < \omega(f)$ then we have increased the weight of B', which is a contradiction. If $\omega(f) < \omega(e)$ we can change the roles of B and B' and get the same contradiction.

So, there is no other basis B' with maximum-weight and different from B. □

3

Second Test

3.1 Oxley, Section 2.1, Problem 1

Problem 19. *Find each of the following:*

(i) all self-dual uniform matroids;

(ii) all identically self-duall uniform matroids;

(iii) all self-dual graphic matroids on six or fewer elements;

(iv) all identically self-dual graphic matroids on six or fewer elements;

(v) an infinite family of simple graphic self-dual matroids.

— . —

Proof.

(i) Let $U_{n,m}$ be an uniform matroid.

Lets asume that it is self-dual. As any basis B for $U_{n,m}$ has size n, if it is going to be self-dual, then its complement must have size n as well. This means that $m = 2n$.

Now lets suppose that $m = 2n$. So, any subset of size n is a basis for $U_{n,m}$, but it is also true that any subset of size n is the complement of another subset of size n and then, any basis is the complement of another basis. Then $U_{n,m}$ is not only self-dual, but also identically self-dual.

(ii) Just as we saw above, if an uniform matroid is self-dual, then it is identically self-dual, and an uniform matroid $U_{n,m}$ is self-dual if and only if $m = 2n$.

(iii) Lets asume that the graphs are connected. For not connected graphs the following conditions must hold for each of the connected compontents.

Any graph with no edges.

A graph with two vertices and two parallel edges.

Any graph with three vertices and four edges that such that any edge has at most a parallel one.

MISSING!

(iv) Any graph on six or fewer elements obtained from a forest adding a parallel edge to each edge in the forest. MISSING!

(v) Any wheel.

□

3.2 Oxley, Section 2.1, Problem 2

Problem 20. *Let M be a matroid. Show that M^* has two disjoint circuits if and only if M has two hyperplanes whose union is $E(M)$.*

— . —

Proof.

Sufficiency

Let C^* and C'^* be two disjoint circuits in M^*. From Proposition 2.1.6 in [Oxl92] we know that $C = E(M) \setminus C^*$ and $C' = E(M) \setminus C'^*$ are hyperplanes in M. Moreover, as C^* and C'^* are disjoint disjoint, then $C^* \subset C'$ and $C'^* \subset C$. Then

$$\begin{aligned}
C \cup C' &= (C \cup C'^*) \cup (C' \cup C^*) \\
&\quad \text{(we have added nothing new to each subset)} \\
&= (E(M) \setminus C^*) \cup C'^* \cup (E(M) \setminus C'^*) \cup C^* \\
&\quad \text{(reordering terms)} \\
&= (E(M) \setminus C^*) \cup C^* \cup (E(M) \setminus C'^*) \cup C'^* \\
&= E(M) \cup E(M) \\
&= E(M).
\end{aligned}$$

Necessity

Let H and H' be two hyperplanes such that $H \cup H' = E(M)$. Againg, from Proposition 2.1.6 in [Oxl92] we know that $H^* = E(M) \setminus H$ and $H'^* = E(M) \setminus H'$ are cocircuits. Suppose there exists $x \in H^* \cap H'^*$, that is $x \in H^*$ and $x \in H'^*$. But $x \in H^*$ means $x \notin H$ and $x \in H'^*$ means $x \notin H'$, then $x \notin H \cup H' = E(M)$ which is a contradiction. Then there is not such x and $H^* \cap H'^*$ are cocircuits such that $H^* \cap H'^* = \emptyset$. □

3.3 Oxley, Section 2.1, Problem 6

Problem 21. *Let e and f be distinct elements of a matroid. Prove that every circuit containing e also contains f if and only if $\{e\}$ or $\{e, f\}$ is a cocircuit.*

— . —

Proof.

Sufficiency

Suppose every circuit containing e also contains f.

If every basis contains e, then, no cobasis contains e, which means $\{e\}$ is a cocircuit.

Suppose then that there is a basis B such that $e \notin B$. As $C(e, B)$ is a circuit that contains e, then $f \in B$. That is, any basis that doesn't contain e, contains f. That is, any cobasis that contains e, doesn't contain f.

Note that $B \setminus \{f\} \cup \{e\}$ is a basis given that it has the right size to be a basis and it is obtained from $B \cup \{e\}$ getting rid of the only circuit that it contains (that is, it is independent). Then, there is a basis that doesn't contains e and then $\{e\}$ is coindependent, and there is also a basis that doesn't contain f and then $\{f\}$ is also coindependent.

Now suppose there is a basis B' such that $f \notin B'$. If $e \notin B'$, then $C(e, B')$ is a circuit that contains e but not f whitch contradicts our hyphothesis. Then any basis that doesn't contain f, contains e. That is, any cobasis containg f, doesn't contain e.

Then $\{e, f\}$ is a subset that is not contained in any cobasis, then it is codependent but any of its subsets is coindependent. Then it is a cocircuit as we wished to show.

Necessity

Suppose $\{e\}$ is a cocircuit. Then, again by Proposition 2.1.6 in [Oxl92], $E(M) \setminus \{e\}$ is an hyperplane, and then e is contained in every basis, then e is in no circuits (by [12]). Then it is true that every circuit containing e also contains f, since there are not circuits containing e.

Now suppose that $\{e, f\}$ is a cocircuit. Then, again by Proposition 2.1.6 in [Oxl92], $H = E(M) \setminus \{e, f\}$ is an hyperplane, so any basis must contain e or f.

Let C be a circuit such that $e \in C$. Then $C \setminus \{e\}$ is independent. Let B be a basis such that $C \setminus \{e\} \subset B$. As B is basis and doesn't contain e, then $f \in B$. Notice that $C(e, B) = C$

Now, $B^* = E(M) \setminus B$ is a cobasis that contains e but doesn't contain f, then $C * (f, B^*) = \{e, f\}$ is the only cocircuit in $B^* \cup \{f\}$, that means that $B^* \cup \{f\} \setminus \{e\}$ is a coindependent of maximum size, that is a cobasis. Taking its complement we will get a basis, and it is

$$E(M) \setminus (B^* \cup \{f\} \setminus \{e\}) = B \cup \{e\} \setminus \{f\}.$$

Which means that $f \in C$. □

3.4 Oxley, Section 2.1, Problem 10

Problem 22. *Let B be a basis of a matroid M and B^* be $E(M) - B$. If $e \in B$, let $C^*(e, B^*)$ denote the fundamental cocircuit of e with respect to the cobasis B^* of M, that is, $C^*(e, B^*) = C_{M^*}(e, B^*)$.*

(i) Show that $C^*(e, B^*)$ is the unique cocircuit that is disjoint from $B - e$.

(ii) If $f \in B^*$, prove that $f \in C^*(e, B^*)$ if and only if $e \in C(f, B)$.

— . —

Proof.

1. $C^*(e, B^*)$ is the only cocircuit contained in $B^* \cup e$, then any other circvuit must intersect the complement of $B^* \cup e$, but the complement of $B^* \cup e$ is $B - e$. Then any other cocircuit must intersect $B - e$

2. Suppose $e \in C(f, B)$. Now, $B \setminus e \cup f$ must be a basis, it has the same size as B and $B \cup f$ has an unique circuit which is $C(f, B)$ and which contains f, so $B \setminus e \cup f$ cannot contain a circuit and then it is independent with maximum size, that is, $B \setminus e \cup f$ is a basis.

   Then, its complement must be a cobasis, but its complement is $B^* \setminus f \cup e$, we know that $B^* \setminus f \cup e \cup f = B^* \cup e$ has an unique cocircuit and then it must contain f.

   We have already proved that $e \in C(f, B)$ implies $f \in C^*(e, B)$, the proof for the sufficiency is completely analog.

□

3.5 Oxley, Section 2.2, Problem 2

Problem 23. *Let A be an $m \times n$ matrix over a field F.*

(i) For each of the row operations, specify an $m \times m$ matrix L such that multiplying A on the left by L has the same effect as the row operation.

(ii) Fore each of the column operations, specify an $n \times n$ matrix R such that multiplying A on the right by R has the same effect as the column operation.

— . —

Proof. ,

1. Take the $m \times m$ identity matrix. Perform any of the row operations to it and let L be the result.

   Such matrix is called elemental matrix, and if you multiply A on the left by L it will have the same effect as the row operation.

2. The proof is exaclty the same, but working with columns. Perform any of the column operations to the $n \times n$ identity matrix, and let R be the result.

□

3.6 Oxley, Section 2.2, Problem 4

Problem 24. *Show that, in a binary matroid, a circuit and cocircuit cannot have an odd number of common elements.*

— . —

Proof. □

3.7 Oxley, Section 2.2, Problem 8

Problem 25. *Let T_8 and R_8 be the vector matroids of the following matrices over $GF(3)$:*

$$\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \qquad \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

$$\left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 0 \end{array}\right], \quad \left[\begin{array}{c|cccc} & -1 & 1 & 1 & 1 \\ I_4 & 1 & -1 & 1 & 1 \\ & 1 & 1 & -1 & 1 \\ & 1 & 1 & 1 & -1 \end{array}\right]$$

(i) *Show that T_8 and R_8 ar both self-dual.*

(ii) *Show that R_8 is identically self-dual but T_8 is not.*

(iii) *Give geometric representation for T_8 and R_8.*

(iv) *Show that if $M \in \{T_8, R_8\}$ and $X = E(M) \setminus \{8\}$, then $(M|X)^* \cong F_7^-$.*

(v) *Consider the following matrices over $GF(3)$:*

$$A_1 = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & -1 \\ & 1 & 1 & 1 & 0 \end{array}\right], \quad A_2 = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & -1 \\ I_4 & 1 & 0 & 1 & -1 \\ & 1 & 1 & 0 & -1 \\ & 1 & 1 & 1 & 1 \end{array}\right]$$

Show, by applying a sequence of the row and clolumn opeartions to A_1 and A_2 that $M[A_1] \cong T_8$ and $M[A_2] \cong R_8$.

(vi) *Show that R_8 can be obtained from $AG(3,2)$ by relaxing two disjoint circuit-hyperplanes.*

— . —

Proof. □

3.8 Oxley, Section 2.3, Problem 1

Problem 26. *(i) Show that the geometric dual of a plane graph is connected.*

(ii) Give an example of a plane graph G for which $(G^*)^* \neq G$.

— . —

Proof. $\square$

3.9 Oxley, Section 2.3, Problem 2

Problem 27. *Find geometric duals of the graphs obtained from K_5 and $K_{3,3}$ by deleting a single edge of each.*

— . —

Proof. □

3.10 Oxley, Section 2.3, Problem 6

Problem 28. *Construct the geometric duals of the plane graphs in Figures 1.4(a) and 1.28(a). MISSING!*

— . —

Proof. □

3.11 Oxley, Section 2.3, Problem 10

Problem 29. *Let r be an integer exceeding one and W_3 be the graph in Figure 2.10(b) MISSING!. Show that:*

(i) $W_3 \cong K_4$.

(ii) $W_3 r^* \cong W_r$.

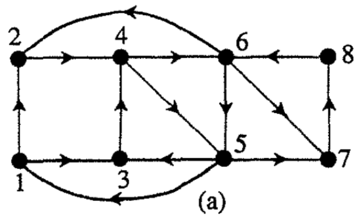
(iii) $M(W_4)$ *is isomorphic to a restriction of* $M^*(K_{3,3})$.

— . —

Proof. □

3.12 Oxley, Section 2.4, Problem 3

Problem 30. *Let G be the directed graph shown in Figure 2.17(a).*



(a)

(i) Construct the corresponding bipartite graph $\hat{G}$.

(ii) Let $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $X = \{5, 7\}$, $Y = \{4, 2\}$, and the paths linking X to Y in G be 5134 and 7862. Construct the corresponding matching of $V \setminus X$ to $\hat{V} \setminus \hat{Y}$ in $\hat{G}$.

(iii) Now take the matching $\{2\hat{1}, 3\hat{5}, 4\hat{3}, 5\hat{4}, 6\hat{8}, 8\hat{7}\}$ in $\hat{G}$ and construct the corresponding disjoint paths in G as in the proof of Lemma 2.4.3.

— . —

Proof. □

3.13 Oxley, Section 2.4, Problem 4

Problem 31. *Let G be the directed graph shown in Figure 2.17(b) MISSING! and let $L(G, B_0)$ where $B_0 = \{1, 2, 3\}$.*

(i) Find geometric representations for M and M^*.

(ii) Give a presentation for the transversal matroid M^*.

(iii) Reverse the directions on the arcs $(5, 4)$ and $(7, 5)$ of G and repeat (i) and (ii).

— . —

Proof. □

3.14 Oxley, Section 3.1, Problem 1

Problem 32. — . —

Proof. ☐

3.15 Oxley, Section 3.1, Problem 4

Problem 33. — . —

Proof. □

3.16 Oxley, Section 3.2, Problem 3

Problem 34. — . —

Proof. □

3.17 Oxley, Section 3.2, Problem 5

Problem 35. — . —

Proof. □

Bibliography

[Oxl92] James G. Oxley. Matroid Theory. *Oxford University Press, 1992.*