REGRESSION LINES IN STATA

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1. Introduction to Regression

Regression analysis is about exploring linear relationships between a dependent variable and one or more independent variables. Regression models can be represented by graphing a line on a cartesian plane. Think back on your high school geometry to get you through this next part.

Suppose we have the following points on a line:

x	y
-1	-5
0	-3
1	-1
2	1
3	3

What is the equation of the line?

$$y = \alpha + \beta x$$

$$\beta = \frac{\Delta y}{\Delta x} = \frac{3-1}{3-2} = 2$$

$$\alpha = y - \beta x = 3 - 2(3) = -3$$

$$y = -3 + 2x$$

If we input the data into STATA, we can generate the coefficients automatically. The command for finding a regression line is `regress`. The STATA output looks like:

Date: February 25, 2014.

```
. regress y x

      Source |       SS       df       MS              Number of obs =       5
-------------+------------------------------           F(  1,     3) =       .
       Model |          40     1          40           Prob > F      =       .
    Residual |           0     3           0           R-squared     =  1.0000
-------------+------------------------------           Adj R-squared =  1.0000
       Total |          40     4          10           Root MSE      =       0

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |          2          .        .       .            .           .
       _cons |         -3          .        .       .            .           .
------------------------------------------------------------------------------
```

The first table shows the various sum of squares, degrees of freedom, and such used to calculate the other statistics. In the top table on the right lists some summary statistics of the model including number of observations, R^2 and such. However, the table we will focus most of our attention on is the bottom table. Here we find the coefficients for the variables in the model, as well as standard errors, p-values, and confidence intervals.

In this particular regression model, we find the x coefficient (β) is equal to 2 and the constant (α) is -3. This matches the equation we calculated earlier. Notice that no standard errors are reported. This is because the data fall exactly on the line so there is zero error. Also notice that the R^2 term is exactly equal to 1.0, indicating a perfect fit.

Now, let's work with some data that are not quite so neat. We'll use the `hire771.dta` data.

```
use hire771

. regress salary age

      Source |       SS       df       MS              Number of obs =    3131
-------------+------------------------------           F(  1,  3129) =  298.30
       Model |  1305182.04     1  1305182.04           Prob > F      =  0.0000
    Residual |  13690681.7  3129  4375.41762           R-squared     =  0.0870
-------------+------------------------------           Adj R-squared =  0.0867
       Total |  14995863.8  3130  4791.01079           Root MSE      =  66.147

------------------------------------------------------------------------------
      salary |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   2.335512   .1352248    17.27   0.000     2.070374    2.600651
       _cons |   93.82819   3.832623    24.48   0.000     86.31348    101.3429
------------------------------------------------------------------------------
```

The table here is much more interesting. We've regressed age on salary. The coefficient on age is 2.34 and the constant is 93.8 giving us an equation of:

$$salary = 93.8 + 2.34age$$

How do we interpret this? For every year older someone is, they are expected to receive another \$2.34 a week. A person with age zero is expected to make \$93.8 a week. We can find the salary of someone given their age by just plugging in the numbers into the above equation. So a 25 year old is expected to make:

$$salary = 93.8 + 2.34(25) = 152.3$$

Looking back at the results tables, we find more interesting things. We have standard errors for the coefficient and constant because the data are messy, they do not fall exactly on the line, generating some error. If we look at the R^2 term, 0.087, we find that this line is not a very good fit for the data.

2. Testing Assumptions

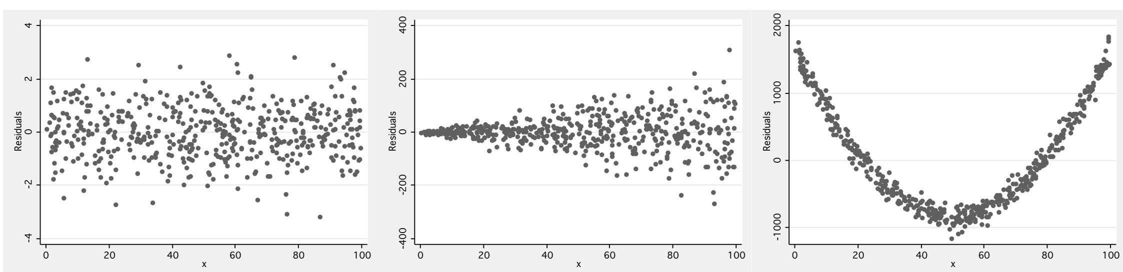
The OLS regression model requires a few assumptions to work. These are primarily concerned with the residuals of the model. The residuals are the same as the error - the vertical distance of each data point from the regression line. The assumptions are:

- **Homoscedasticity** - the probability distribution of the errors has constant variance
- **Independence of errors** - the error values are statistically independent of each other
- **Normality of error** - error values are normally distributed for any given value of x

The easiest way to test these assumptions are simply graphing the residuals on x and see what patterns emerge. You can have STATA create a new variable containing the residual for each case after running a regression using the `predict` command with the `residual` option. Again, you must first run a regression before running the `predict` command.

```
regress y x1 x2 x3
predict res1, r
```

You can then plot the residuals on x in a scatterplot. Below are three examples of scatterplots of the residuals.



(A) What you want to see (B) Not Homoscedastic (C) Not Independent

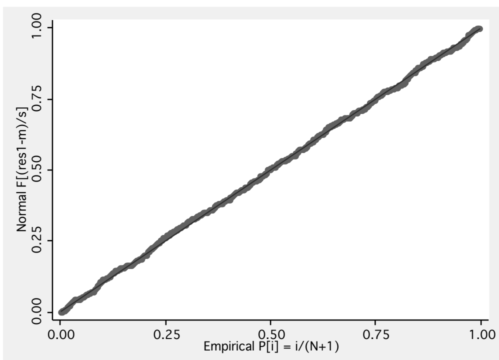
Figure (A) above shows what a good plot of the residuals should look like. The points are scattered along the x axis fairly evenly with a higher concentration at the axis. Figure (B) shows a scatter plot of residuals that are not homoscedastic. The variance of the residuals increases as x increases. Figure (C) shows a scatterplot in which the residuals are not independent - they are following a non-linear trend line along x. This can happen if you are not specifying your model correctly (this plot comes from trying to fit a linear regression model to data that follow a quadratic trend line).

If you think the residuals exhibit heteroscedasticity, you can test for this using the command `estat hettest` after running a regression. It will give you a chi^2 statistic and a p-value. A low p-value indicates the likelihood that the data is heteroscedastic. The consequences of heteroscedasticity in your model is mostly minimal. It will not bias your coefficients but it may bias your standard errors, which are used in calculating the test statistic and p-values for each coefficient. Biased standard errors may lead to finding significance for your coefficients when there isn't any (making a type I error). Most statisticians will tell you that you should only worry about heteroscedasticity if it is pretty severe in your data. There are a variaty of fixes (most of them complicated) but one of the easiest is specifying `vce(robust)` as an option in your regression command. This uses a more robust method to calculate standard errors that is less likely to be biased by a number of things, including heteroscedasticity.

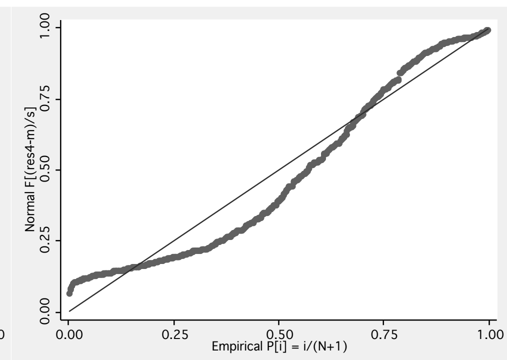
If you find a pattern in the residual plot, then you've probably misspecified your regression model. This can happen when you try to fit a linear model to non-linear data. Take another look at the scatterplots for your dependent and independent variables to see if any non-linear relationships emerge.

To test for normality in the residuals, you can generate a normal probability plot of the residuals:

```
pnorm varname
```



(D) Normally Distributed



(E) Not Normal

What this does is plot the cumulative distribution of the data against a cumulative distribution of normally distributed data with similar means and standard deviation. If the data are normally distributed, then the plot should create a straight line. The resulting graph will produce a scatter plot and a reference line. Data that are normally distributed will not deviate far from the reference line. Data that are not normally distributed will deviate. In the figures above, the graph on the left depicts normally distributed data (the residuals in (A) above). The graph on the right depicts non-normally distributed data (the residuals in (C) above). Depending on how far the plot deviates from the reference line, you may need to use a different regression model (such as poisson or negative binomial).

3. Interpreting Coefficients

Using the `hire771` dataset, the average salary for men and women is:

Table 1. Average Salary

	Avg. Salary
Male	218.39
Female	145.56
Total	156.80

We can run a regression of salary on sex with the following equation:

$$\text{Salary} = \alpha + \beta\text{Sex}$$
$$\text{Salary} = 218.39 - 72.83\text{Sex}$$

Remember that regression is a method of averages, predicting the average salary given values of x. So from this equation, we can calculate what the predicted average salary for men and women would be from this equation:

Table 2. Predicted Salary

		equation	predicted salary
Male	$x = 0$	α	218.39
Female	$x = 1$	$\alpha + \beta$	146.56

How might we interpret these coefficients? We can see that α is equal to the average salary for men. α is always the predicted average salary when all x values are equal to zero. β is the effect that x has on y. In the above equation, x is only ever zero or one so we can interpret the β as the effect on predicted average salary when x is one. So the predicted average salary when x is zero, or for men, is \$218.39 a week. When x is one, or for women, the average predicted salary decreases by \$72.83 a week (remember that β is negative). So women are, on average, making \$72.83 less per week than men.

Remember: this only works for single regression with a dummy variable. Using a continuous variable or including other independent variables will not yield cell averages. Quickly, let's see what happens when we include a second dummy variable:

$$\text{Salary} = \alpha + \beta_{\text{sex}}x_{\text{sex}} + \beta_{\text{HO}}x_{\text{HO}}$$
$$\text{Salary} = 199.51 - 59.75x_{\text{sex}} + 47.25x_{\text{HO}}$$

TABLE 3. Average Salaries

	Male	Female
Field Office	211.46	138.27
Home Office	228.80	197.68

TABLE 4. Predicted Salaries

	Male	Female
Field Office	α 199.51	$\alpha + \beta_{\text{sex}}$ 139.76
Home Office	$\alpha + \beta_{\text{HO}}$ 246.76	$\alpha + \beta_{\text{sex}} + \beta_{\text{HO}}$ 187.01

We can see that the average salaries are not given using the regression method. As we learned in lecture, this is because we only have three coefficients to find four average salaries. More intuitively, the regression is assuming equal slopes for the four different groups. In other words, the effect of sex on salary is the same for people in the field office and in the home office. Additionally, the effect of office location is the same for both men and women. If we want the regression to accurately reflect the cell averages, we should allow the slope of one variable to vary for the categories of the other variables by including an interaction term (see interaction handout). Including an interacting term between sex and home office will reproduce the cell averages accurately.

4. Multiple Regression

So far, we've been talking about single regression with only one independent variable. For example, we've regressed salary on age:

```
      Source |       SS       df       MS              Number of obs =    3131
-------------+------------------------------           F(  1,  3129) =  298.30
       Model |  1305182.04     1  1305182.04           Prob > F      =  0.0000
    Residual |  13690681.7  3129  4375.41762           R-squared     =  0.0870
-------------+------------------------------           Adj R-squared =  0.0867
       Total |  14995863.8  3130  4791.01079           Root MSE      =  66.147

------------------------------------------------------------------------------
      salary |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   2.335512   .1352248    17.27   0.000     2.070374    2.600651
       _cons |   93.82819   3.832623    24.48   0.000     86.31348    101.3429
------------------------------------------------------------------------------
```

There are a couple problems with this model, though. First, it is not a very good fit ($R^2 = 0.087$). Second, there are probably confounding variables. For example, education may be confounded with age (the older you are, the more education you are likely to have). So it makes sense to try to control for one while regressing salary on the other. This is multiple regression. In STATA, we simply include all the independent variables after the dependent variable:

```
. regress salary age educ

      Source |       SS       df       MS              Number of obs =    3131
-------------+------------------------------           F(  2,  3128) =  520.72
       Model |  3745633.26     2  1872816.63           Prob > F      =  0.0000
    Residual |  11250230.5  3128  3596.62101           R-squared     =  0.2498
-------------+------------------------------           Adj R-squared =  0.2493
       Total |  14995863.8  3130  4791.01079           Root MSE      =  59.972

------------------------------------------------------------------------------
      salary |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   2.129136   .1228567    17.33   0.000     1.888248    2.370024
        educ |   13.02054   .4998516    26.05   0.000     12.04046    14.00061
       _cons |   61.53663   3.689336    16.68   0.000     54.30286    68.77039
------------------------------------------------------------------------------
```

In the model above, we are controlling for education when analyzing age. We are also controlling for age when analyzing education. So, we can say that age being equal, for every advance in education, someone can expect to make \$13 more a week. We can also say that education being equal, for every year older someone is they can expect to make \$2 more a week. The effect of one variable controlled for when analyzing the other variable. One analogy would be that the regression model is dividing everyone up into similar age groups and then analyzing the effect of education within each group. At the same time, the model is dividing everyone into groups of similar education and then analyzing the effect of age within each group. This isn't a perfect analogy, but it helps to visualize what it means when we say the regression model is finding the effect of age on salary controlling for education and vice versa.

This is true of all the variables you include in your model. For example, maybe education doesn't have anything to do with salary, its just that men are more educated than women and so the more salient variable determining your salary is sex. We can find out the effect of education controlling for sex by including the sex variable:

```
. regress salary age educ sex

      Source |       SS       df       MS              Number of obs =    3131
-------------+------------------------------           F(  3,  3127) =  488.00
       Model |  4781981.18     3  1593993.73           Prob > F      =  0.0000
    Residual |  10213882.6  3127  3266.35196           R-squared     =  0.3189
-------------+------------------------------           Adj R-squared =  0.3182
       Total |  14995863.8  3130  4791.01079           Root MSE      =  57.152

------------------------------------------------------------------------------
      salary |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   2.137271    .117081    18.25   0.000     1.907708    2.366834
        educ |   10.24355   .5012123    20.44   0.000     9.260816    11.22629
         sex |  -53.00496   2.975742   -17.81   0.000    -58.83957   -47.17036
       _cons |   114.2192   4.594451    24.86   0.000     105.2108    123.2277
------------------------------------------------------------------------------
```

Here, we can describe the effects of the variables in the following way:

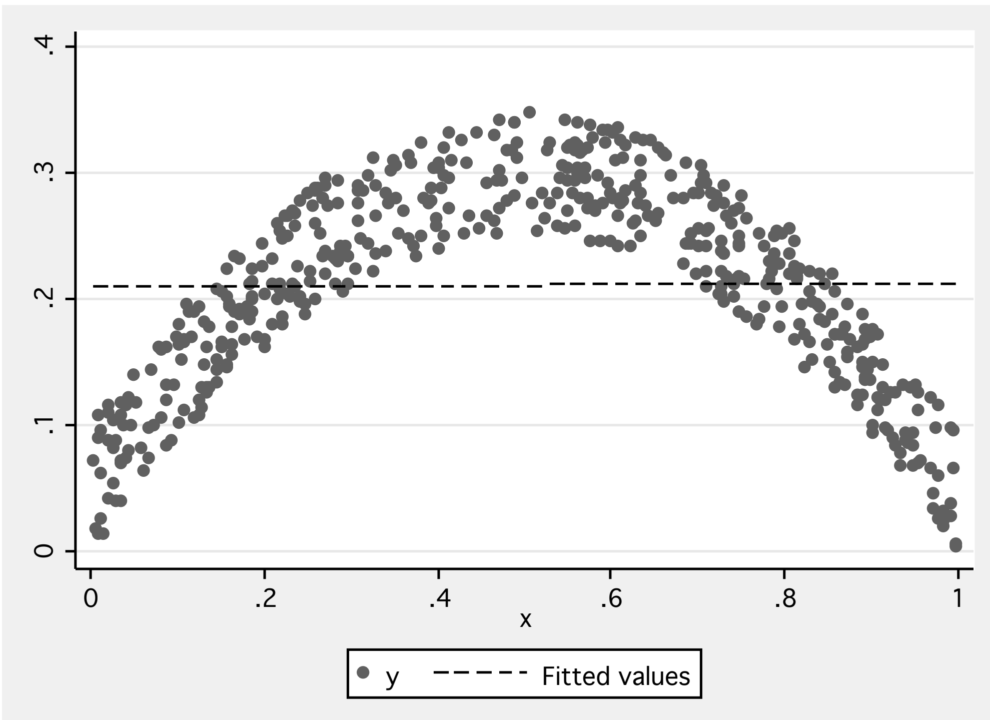
- When comparing men and women with similar ages and educations, women are expected to make \$53 less a week than men.
- Comparing people who are the same sex and have the same education, each year older confers an additional \$2.14 per week on average.
- Comparing people who are the same sex and age, each additional advancement in education confers, on average, an additional \$10.24 more per week.

Comparing the two models, we see that the effect of education decreases slightly when we include sex, but the education variable is still significant ($p < 0.000$). Additionally, if we ran a regression model with just sex, the sex coefficient would be -72.83 (the difference in average salary for men and women) but here the coefficient is only -53.00 indicating that age and education explain a fair amount of the difference in the salaries of men and women.

5. Quadratic Terms

Normal OLS regression assumes a linear relationship between the dependent variable and the independent variables. When you run a regression of y on x, the model assumes that y increases at the same rate for each value of x. But what happens when this isn't true. Assume a dataset in which the relationship between x and y is quadratic rather than linear. The scatterplot for such data may look like Figure 1.

Figure 1. Scatterplot



Briefly, let's review the algebra behind quadratic equations. Quadratic equations describe parabolas, which are part of the family of conic shapes (along with circles, ellipses, and hyperbolas). The quadratic equation typically looks like:

$$y = ax^2 + bx + c$$

c is our y-intercept (or our α). If a is positive, the parabola opens up (forms a bowl). If a is negative, the parabola opens down (forms a hill). The inflection point is the point at which the sign of the slope changes (from negative to positive or vice versa). So in the scatterplot above, we should expect the coefficient on the x^2 term to be negative.

As I said before, normal OLS regression assumes a linear relationship so if we were to run a regression of y on x alone, we get the results in Figure 2. The linear equation is included in the scatter plot in Figure 1. Notice that in the regression output, the coefficient on x is very small and not statistically significantly different from zero. Essentially, the linear model says that x has no effect on y. However, we can clearly see that x and y do have a relationship, it simply is not linear. How do we specify a proper model for this data?

FIGURE 2. Linear Regression Model

```
. regress y x

      Source |       SS       df       MS              Number of obs =     500
-------------+------------------------------           F(  1,   498) =    0.04
       Model |  .000245043     1  .000245043           Prob > F      =  0.8455
    Residual |  3.21217977   498   .00645016           R-squared     =  0.0001
-------------+------------------------------           Adj R-squared = -0.0019
       Total |  3.21242481   499  .006437725           Root MSE      =  .08031

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .0023507   .0120604     0.19   0.846    -.0213449    .0260463
       _cons |   .2104006   .0070504    29.84   0.000     .1965484    .2242528
------------------------------------------------------------------------------
```

We can write the quadratic equation in regression parlance:

$$y = \alpha + \beta_1 x + \beta_2 (x * x)$$

So we include the x variable twice, once by itself and a second time multiplied by itself once. In STATA, we can generate that quadratic term easily:

```
gen x2 = x*x
```

And then include it in the regression (see Figure 3)

FIGURE 3. Quadratic Regression Model

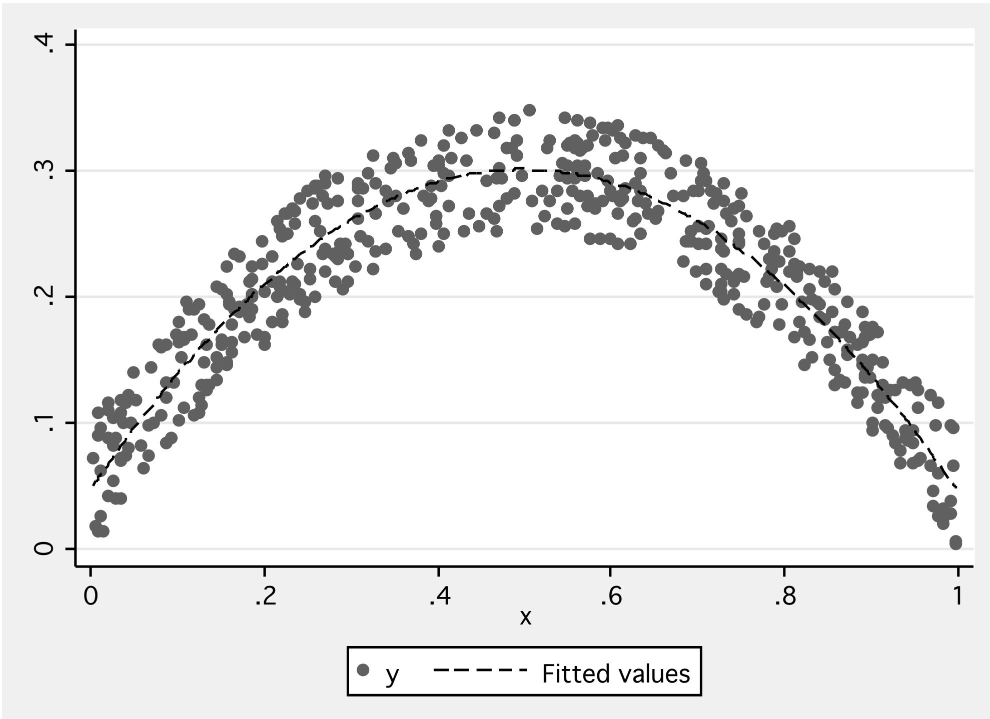
```
. regress y x x2

      Source |       SS       df       MS              Number of obs =     500
-------------+------------------------------           F(  2,   497) = 1665.43
       Model |  2.79533235     2  1.39766617           Prob > F      =  0.0000
    Residual |  .417092466   497   .00083922           R-squared     =  0.8702
-------------+------------------------------           Adj R-squared =  0.8696
       Total |  3.21242481   499  .006437725           Root MSE      =  .02897

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   1.014641    .018072    56.14   0.000     .9791338    1.050148
          x2 |  -1.016923   .0176209   -57.71   0.000    -1.051543   -.9823022
       _cons |   .0487006   .0037839    12.87   0.000     .0412662    .0561351
------------------------------------------------------------------------------
```

And we see that this model does much better at explaining the data. Both the x and the x^2 terms are significant and our R^2 is fairly high at 0.87. The regression equation for this model would be:

$$y = 0.0487 + 1.0146x - 1.0169x^2$$

Graphing the fitted regression line to the scatter plot shows a much better description of the data (see Figure 4).

FIGURE 4. Scatterplot with Fitted Line



Note: You *must* include both orders of your independent variable in a quadratic model. Do not include just the quadratic term - this is wrong and misspecifies the model.

To learn how to interpret the coefficients, let's try this with some real data, using the `hire771` dataset.

First, regress salary and age (See Figure 5).

We've seen this before but let's review interpreting the coefficients. First, our regression equation is:

$$\text{Salary} = 93.828 + 2.336 \text{ age}$$

So for every year older someone is, the average starting salary increases by \$2.34 per week.

20 year olds' average starting salary: $93.828 + 2.336(20) = 140.55$ or \$140.55 a week.

50 year olds' average starting salary: $93.828 + 2.336(50) = 210.628$ or \$210.63 a week.

Now age often has diminishing returns, indicating a potential quadratic relationship. Let's generate a quadratic age variable and run a regression (See Figure 6).

FIGURE 5. Salary on Age; Linear

```
. regress salary age

      Source |       SS       df       MS              Number of obs =    3131
-------------+------------------------------           F(  1,  3129) =  298.30
       Model |  1305182.04     1  1305182.04           Prob > F      =  0.0000
    Residual |  13690681.7  3129  4375.41762           R-squared     =  0.0870
-------------+------------------------------           Adj R-squared =  0.0867
       Total |  14995863.8  3130  4791.01079           Root MSE      =  66.147

------------------------------------------------------------------------------
      salary |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   2.335512   .1352248    17.27   0.000     2.070374    2.600651
       _cons |   93.82819   3.832623    24.48   0.000     86.31348    101.3429
------------------------------------------------------------------------------
```

FIGURE 6. Salary on Age; Quadratic

```
. gen age2 = age*age

. regress salary age age2

      Source |       SS       df       MS              Number of obs =    3131
-------------+------------------------------           F(  2,  3128) =  235.77
       Model |     1964445     2  982222.498           Prob > F      =  0.0000
    Residual |  13031418.8  3128   4166.0546           R-squared     =  0.1310
-------------+------------------------------           Adj R-squared =  0.1304
       Total |  14995863.8  3130  4791.01079           Root MSE      =  64.545

------------------------------------------------------------------------------
      salary |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   12.63113    .829006    15.24   0.000     11.00568    14.25658
        age2 |  -.1524362   .0121177   -12.58   0.000    -.1761957   -.1286767
       _cons |  -61.29655   12.88607    -4.76   0.000    -86.56256   -36.03054
------------------------------------------------------------------------------
```

We interpret the `age` coefficient the same as before, but the `age2` coefficient requires some explaining. First, the regression equation is:

$$\text{Salary} = -61.297 + 12.631\ \text{Age} - 0.152\ \text{Age}^2$$

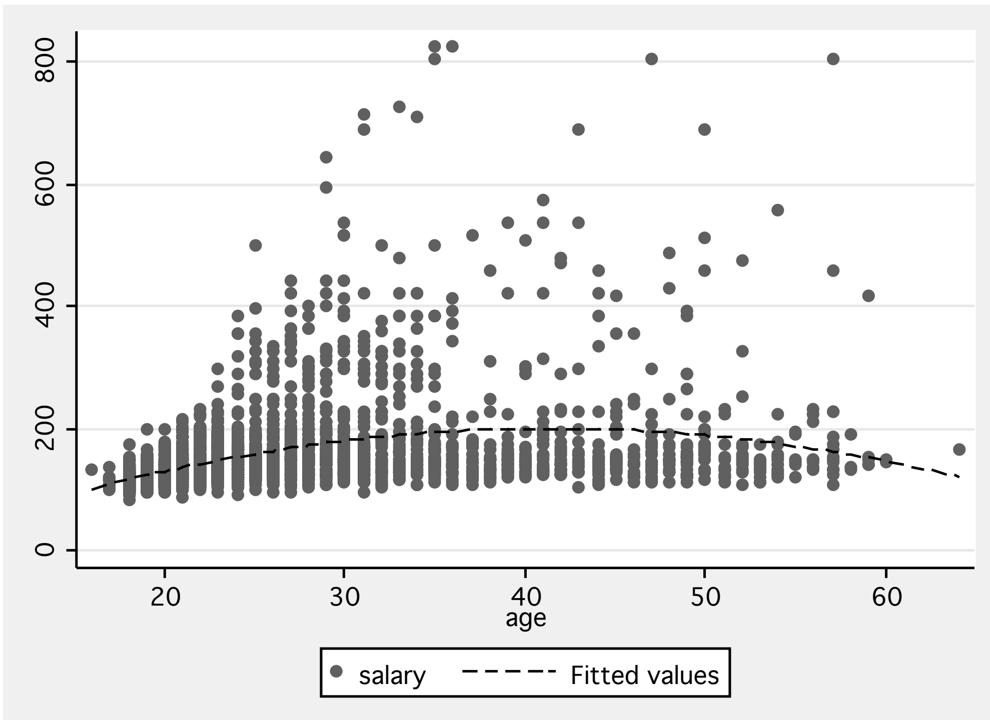
As before, we can say that for each age older someone is, their average starting salary will increase by \$12.63 a week. The `age2` coefficient we can think of as an interaction effect of age on itself. For each year older someone is, the effect of age decreases by 0.152 so the difference in average starting salary for 21 year olds over 20 year olds is bigger than the difference between 51 and 50.

Average starting salary for 20 year olds is: $-61.297+12.631(20)-0.152(20^2) = 130.523$

Average starting salary for 50 year olds is: $-61.297+12.631(50)-0.152(50^2) = 190.253$

Plotting the regression line along with the scatter plot shows a better description of the relationship than a linear model (See Figure 7).

Figure 7. Salary on Age Scatterplot



NOTE: You do not need to know how to do the following, but I include it for those who are curious.

To find the effect of age on salary at any specific age, you will need to derive the derivative of the regression equation. Remember that the derivative of an equation gives you the slope of the curve at any given point. So the derivative of our regression equation is:

$$y = -61.297 + 12.634x - 0.152x^2$$
$$\frac{dy}{dx} = 12.634 - 2(0.152)x$$
$$\frac{dy}{dx} = 12.634 - 0.304x$$

Now we can plug any age into the derivative equation above and it will give us the effect of age on salary at that age. So we plug in 20 and that will give us the effect of age on salary when one moves from 20 to 21 (more or less). Similarly, plugging in 50 will give us the effect of age on salary when one moves from 50 to 51:

$$\frac{dy}{dx} = 12.634 - 0.304(20) = 6.554$$

$$\frac{dy}{dx} = 12.634 - .304(50) = -2.566$$

So according to the results above, moving from 20 to 21 is expected to add \$6.55 per week to one's salary on average. However, moving from 50 to 51 is expected to subtract \$2.57 per week from one's salary on average. So age is beneficial to salary until some age between 20 and 50, after which increases in age will decrease salary. The point at which the effect of age switches from positive to negative is the inflection point (the very top of the curve). The effect of age at this point will be zero as it transitions from positive to negative. We can find the age at which this happens by substituting in zero for the slope:

$$0 = 12.634 - 0.304x$$

$$x = \frac{12.634}{0.304} = 41.55$$

So increases in age adds to one's salary until 41 years, after which age subtracts from salary. Remember: this data is starting salary for 3000 new employees at the firm so people are not getting pay cuts when they get older. Rather, the firm is offering lower starting salaries to people who are older than 41 years than people who are around 41 years old.