Hand calculating

analysis of the calculations: perimeter

∘ base case is a square

∘ perimeter of circle with $d = 1$

∘ rule: do P first then p

formulas

$$P' = 2pP/(p + P)$$
$$p' = \sqrt{pP'}$$

initialization

$$P = 4$$
$$p = 2\sqrt{2}$$

Let's switch variables to make the general case clear. Let $x = P = 4, y = p = 2\sqrt{2}$ and follow the rules above:

$$x' = 2xy/(x + y)$$

$$y' = \sqrt{x'y}$$

$$x = x'; y = y'$$
$$x' = 2xy/(x + y)$$

Area

- ∘ base case is still a square
- ∘ area of circle with r = 1
- ∘ rule: do a first then A

formulas

$$a' = \sqrt{aA}$$
$$A' = 2a'A(a' + A)$$

initialization

$$a = 2$$
$$A = 4$$

a special first step for area

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a’ = sqrt(aA)
   = 2 sqrt{2)
a = a’
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Now, let $x = A = 2, y = a = 2\sqrt{2}$. Note the switched order, lower case a second.

$$x' = 2xy/(x + y)$$

$$y' = \sqrt{x'y}$$

$$x = x'; y = y'$$
$$x' = 2xy/(x+y)$$

It's exactly the same calculation! The change in order of operations is seen to be due to the fact that we need a first step for the area, to convert $a = 2$ to $a = 2\sqrt{2}$.

Let's actually do a few steps of the calculation by hand.

perimeter

P first, then p. For a square on a circle of diameter 1, the outside perimeter is

$$P = 4$$

while the inside is

$$p = 2\sqrt{2}$$

Next: P (re-using the symbol):

$$P = \frac{2pP}{p+P} = \frac{16\sqrt{2}}{4+2\sqrt{2}} = \frac{8}{1+\sqrt{2}} = 3.3137$$

Then p:

$$p = \sqrt{pP} = \sqrt{2\sqrt{2} \cdot \frac{8}{1+\sqrt{2}}} = 4\sqrt{\frac{1}{1+1/\sqrt{2}}} = 3.0615$$

area

a first, then A. For a square on a circle of radius 1, the inside area is

$$a = \sqrt{2} \cdot \sqrt{2} = 2$$

while the outside is

$$A = 4$$

Next (re-using the symbol):

$$a = \sqrt{aA} = \sqrt{8} = 2\sqrt{2}$$

Then A

$$A = \frac{2aA}{a + A} = 16\frac{\sqrt{2}}{4 + 2\sqrt{2}} = \frac{8}{1 + \sqrt{2}} = 3.3137$$

Then a:

$$a = \sqrt{aA} = \sqrt{2\sqrt{2} \cdot \frac{8}{1 + \sqrt{2}}} = 4\sqrt{\frac{1}{1 + 1/\sqrt{2}}} = 3.0615$$

The calculations are identical, and will remain so from here on out.

sine and tangent

$n = 4$

$$n \sin \theta = 4 \cdot \frac{1}{\sqrt{2}} = 2\sqrt{2}$$

$$n \tan \theta = 4 \cdot 1 = 4$$

At this stage, the inner area by trigonometry is equal to the second value of a above, and the outer area is equal to the first value of A above.

Calculate S', etc:

$$C' = \sqrt{\frac{1}{2}(1+C)} = \sqrt{\frac{1}{2}(1+1/\sqrt{2})}$$

$$S' = S/2C' = \frac{1}{2\sqrt{2}} \cdot \sqrt{\frac{2}{1+1/\sqrt{2}}} = \frac{1}{2} \cdot \sqrt{\frac{1}{1+1/\sqrt{2}}}$$

$$T' = \frac{S}{1+C} = \frac{1/\sqrt{2}}{1+1/\sqrt{2}} = \frac{1}{1+\sqrt{2}}$$

so for $n = 8$

$$nS' = 4\sqrt{\frac{1}{1+1/\sqrt{2}}}$$

$$nT' = \frac{8}{1+\sqrt{2}}$$

It continues. At this stage, the inner area is equal to the third value of a above, while the outer area is equal to the second value of A above.

It's the same calculation, in disguise.