Archimedes' Approximation of Pi

One of the major contributions Archimedes made to mathematics was his method for approximating the value of pi. It had long been recognized that the ratio of the circumference of a circle to its diameter was constant, and a number of approximations had been given up to that point in time by the Babylonians, Egyptians, and even the Chinese. There are some authors who claim that a biblical passage[1] also implies an approximate value of 3 (and in fact there is an interesting story[2] associated with that).

At any rate, the method used by Archimedes differs from earlier approximations in a fundamental way. Earlier schemes for approximating pi simply gave an approximate value, usually based on comparing the area or perimeter of a certain polygon with that of a circle. Archimedes' method is new in that it is an iterative process, whereby one can get as accurate an approximation as desired by repeating the process, using the previous estimate of pi to obtain a new one. This is a new feature of Greek mathematics, although it has an ancient tradition among the Chinese in their methods for approximating square roots.

Archimedes' method, as he did it originally, skips over a lot of computational steps, and is not fully explained, so authors of history of math books have often presented slight variations on his method to make it easier to follow. Here we will try to stick to the original as much as possible, following essentially Heath's translation[3].

The Approximation of Pi

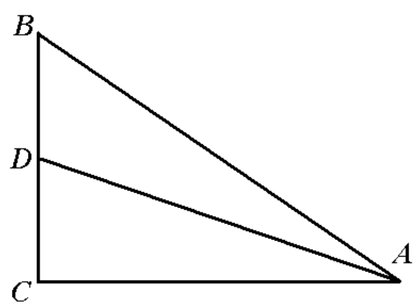
The method of Archimedes involves approximating pi by the perimeters of polygons inscribed and circumscribed about a given circle. Rather than trying to measure the polygons one at a time, Archimedes uses a theorem of Euclid to develop a numerical procedure for calculating the perimeter of a circumscribing polygon of $2n$ sides, once the perimeter of the polygon of n sides is known. Then, beginning with a circumscribing hexagon, he uses his formula to calculate the perimeters of circumscribing polygons of 12, 24, 48, and finally 96 sides. He then repeats the process using inscribing polygons (after developing the corresponding formula). The truly unique aspect of Archimedes' procedure is that he has eliminated the geometry and reduced it to a completely arithmetical procedure, something that probably would have horrified Plato but was actually common practice in Eastern cultures, particularly among the Chinese scholars.

The Key Theorem

The key result used by Archimedes is Proposition 3 of Book VI of Euclid's *Elements*. The full statement of the theorem is as follows:

> *If an angle of a triangle be bisected and the straight line cutting the angle cut the base also, the segments of the base will have the same ratio as the remaining sides of the triangle; and, if the segments of the base have the same ratio as the remaining sides of the triangle, the straight line joined from the vertex to the point of section will bisect the angle of the triangle.*[4]

We will just prove one direction of this theorem here, namely that the angle bisector cuts the opposite side in the ratio claimed. More precisely, in the diagram shown, if AD bisects angle BAC, then $\boldsymbol{BD : CD = BA : AC}$.



Animated GIF Proof of Theorem (99K)	QuickTime Video Proof of Theorem (243K)

Our proof differs from the original somewhat: the proof (and diagram) given here makes it more clear how Archimedes will use the theorem in his approximation scheme. For Euclid's original, complete proof, along with a *very* neat interactive diagram, see David Joyce's *Elements* Web site.

Archimedes' Method

Here we outline the method used by Archimedes to approximate pi. The specific statement of Archimedes is Proposition 3 of his treatise *Measurement of a Circle:*

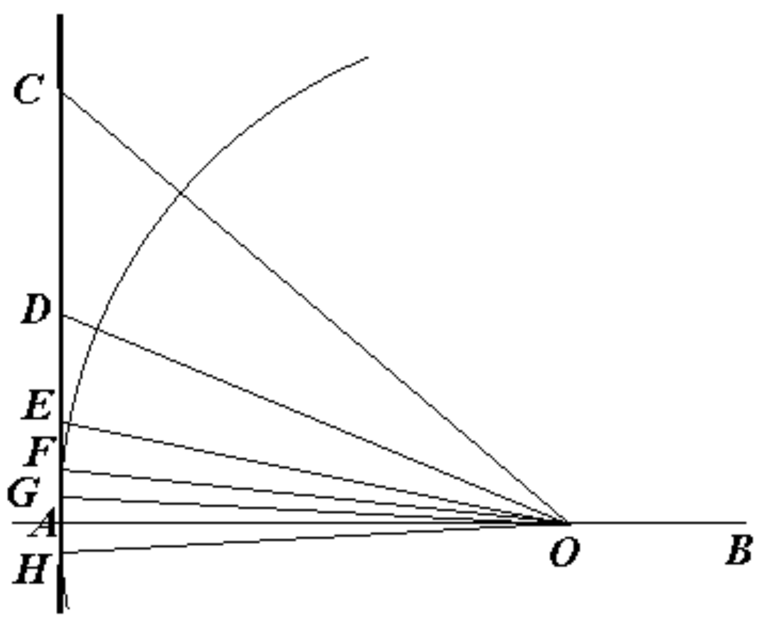
> *The ratio of the circumference of any circle to its diameter is less than $3^{1}/_{7}$ but greater than $3^{10}/_{71}$.*

The proof we give below essentially follows that of Archimedes, as set out in Heath's translation[5]. Much of the text skips over steps in the proof; rather than adding intermediate steps as Heath does[6], we are putting those in pop-up windows. Look for buttons like this: 

. Clicking on these will bring up pop-up windows showing intermediate steps that Archimedes has left out of this text (HTGT stands for How'd They Get That?).

Proof:

[Note: throughout this proof, Archimedes uses several rational approximations to various square roots. Nowhere does he say how he got those approximations--they are simply stated without any explanation--so how he came up with some of these is anybody's guess.]



I. Let *AB* be the diameter of any circle, *O* its center, *AC* the tangent at *A*; and let the angle *AOC* be one-third of a right angle.

Then

(1) $$OA : AC > 265 : 153$$

and

(2) $$OC : AC = 306 : 153.$$

First, draw *OD* bisecting the angle *AOC* and meeting *AC* in *D*. Now

$$CO : OA = CD : DA$$

so that

$$(CO + OA) : CA = OA : AD$$

Therefore

(3) $$OA : AD > 571 : 153.$$

Hence

$$OD^2 : AD^2 > 349450 : 23409$$

so that

(4) $$OD : DA > 591\,{}^1/_8 : 153.$$

Secondly, let *OE* bisect the angle *AOD*, meeting *AD* in *E*. Therefore

(5) $$OA : AE > 1162\,{}^1/_8 : 153$$

Thus

(6) $$OE : EA > 1172\,{}^1/_8 : 153.$$

Thirdly, let OF bisect the angle AOE and meet AE in F.
We thus obtain the result that

$$OA : AF > 2334\tfrac{1}{4} : 153 \tag{7}$$

Thus

$$OF : FA > 2339\tfrac{1}{4} : 153. \tag{8}$$

Fourthly, let OG bisect the angle AOF, meeting AF in G.
We have then

$$OA : AG > 4673\tfrac{1}{2} : 153.$$

Now the angle AOC, which is one-third of a right angle, has been bisected four times, and it follows that angle AOG = 1/48 (a right angle).
Make the angle AOH on the other side of OA equal to the angle AOG, and let GA produced meet OH in H.
Then angle GOH = 1/24 (a right angle).
Thus GH is one side of a regular polygon of 96 sides circumscribed to the given circle.
And, since

$$OA : AG > 4673\tfrac{1}{2} : 153,$$

while

$$AB = 2\,OA, \quad GH = 2\,AG,$$

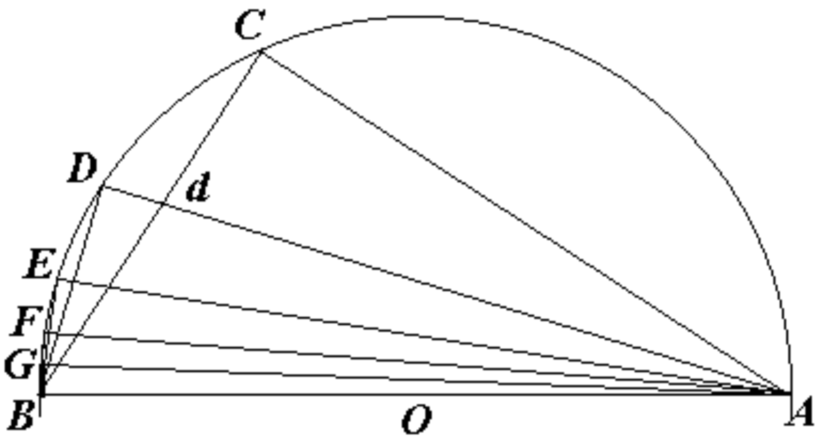
it follows that

$$AB : (\text{perimeter of a polygon of 96 sides}) > 4673\tfrac{1}{2} : 14688$$

But

$$\frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}} < 3\tfrac{1}{7}$$

Therefore the circumference of the circle (being less than the perimeter of the polygon) is *a fortiori* less than 3 1/7 times the diameter AB.



II. Next let *AB* be the diameter of a circle, and let *AC*, meeting the circle in *C*, make the angle *CAB* equal to one-third of a right angle. Join *BC*.

Then

$$AC : BC < 1351 : 780.$$

First, let *AD* bisect the angle *BAC* and meet *BC* in *d* and the circle in *D*. Join *BD*.
Then

$$\text{angle } BAD = \text{angle } dAC = \text{angle } dBD$$

and the angles at D, C are both right angles. It follows that the triangles ADB, BDd are similar.
Therefore

$$\begin{aligned} AD : BD = BD : Dd &= AB : Bd \\ &= (AB + AC) : (Bd + Cd) \\ &= (AB + AC) : BC \end{aligned}$$

or $(BA + AC) : BC = AD : DB$.

Therefore

(1) $$AD : DB < 2911 : 780.$$

Thus

(2) $$AB : BD < 3013\ {}^{3}/_{4} : 780.$$

Secondly, let *AE* bisect the angle *BAD,* meeting the circle in *E*; and let *BE* be joined. Then we prove, in the same way as before, that

(3) $$AE : EB < 5924\ {}^{3}/_{4} : 780 = 1823 : 240.$$

Therefore

(4) $$AB : BE < 1838\ {}^{9}/_{11} : 240.$$

Thirdly, let AF bisect the angle BAE, meeting the circle in F.
Thus,

(5) $$AF : FB < 3661\ {}^{9}/_{11} \text{ x } {}^{11}/_{40} : 240 \text{ x } {}^{11}/_{40}$$
$$= 1007 : 66.$$

Therefore,

(6) $$AB : BF < 1009\ {}^{1}/_{6} : 66.$$

Fourthly, let the angle *BAF* be bisected by *AG* meeting the circle in *G*.
Then

$$AG : GB < 2016\ {}^{1}/_{6} : 66,$$ by (5) and (6).

Therefore

(7) $$AB : BG < 2017\ {}^{1}/_{4} : 66.$$

Therefore *BG* is a side of a regular inscribed polygon of 96 sides.

It follows from (7) that

$$\text{(perimeter of polygon)} : AB > 6336 : 2017\ {}^{1}/_{4}.$$

And $\dfrac{6336}{2017\ {}^{1}/_{4}} > 3\ {}^{10}/_{71}$.

Much more then is the circumference to the diameter

$$< 3\ {}^{1}/_{7} \quad \text{but} \quad > 3\ {}^{10}/_{71}.$$

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