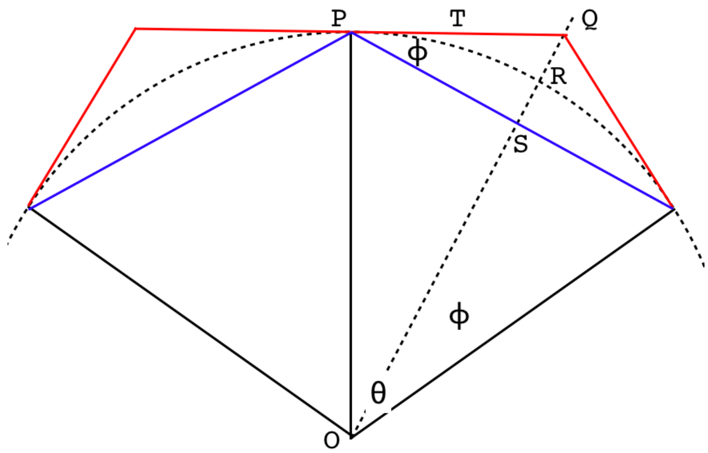
Value of pi and Gregory

In this write-up, we use basic geometry to derive the perimeter and area formulas.



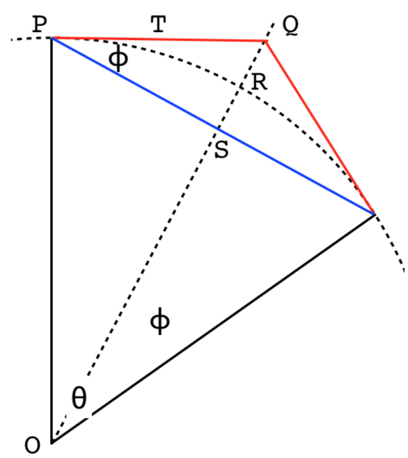
Draw a circle centered at O (only an arc of the circle is shown).

Points on the circle are chosen such that the arc length is an integral fraction of the whole. Equivalently, set $n\theta = 2\pi$.

Two adjacent sectors are shown in the figure above. The two polygons might be drawn so that the vertices of the internal and external figures are on the same ray, with parallel sides. However, the construction shown is more convenient.

The precise scale does not matter to the argument (nor the value of n). If it should turn out that the arc length as drawn is not exactly right, increase or decrease the radius of the circle and then fit it to the figure, keeping two points on the perimeter, and adjust O to be at the center of the adjusted circle.

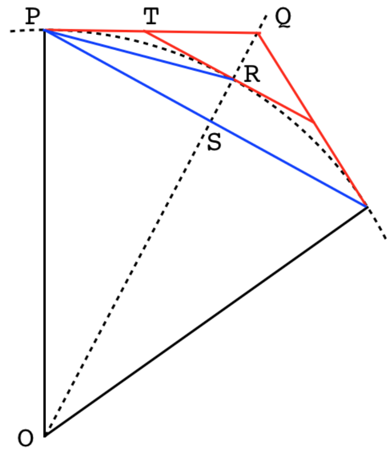
Two red lines comprise this sector's external perimeter P, while a single blue line is the inscribed perimeter p. The lines of the external perimeter are both tangent to the circle, and the whole figure is symmetric in each sector, with one blue and two red lines.



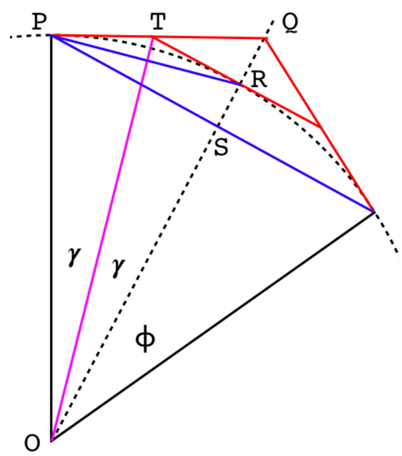
$\angle PSR$ is a right angle. Proof: we simply appeal to symmetry, or point out the congruent triangles. Since $\phi = \theta/2$, we have SAS.

Next, draw the perimeters p' and P' for the polygon with $2n$ sides and sector angle $\phi = \theta/2$.

It is convenient to rotate the internal perimeter by $\theta/2$ with respect to the external one, a bit to the left when we draw p' and a bit to the right for P'. Both p' and P' touch the circle at R.



A central relationship we use below is that $\triangle PRT$ is isosceles. For a proof, draw OT and appeal to symmetry.

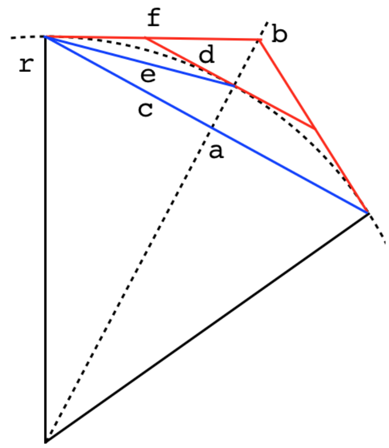


Or note that OT bisects $\angle POQ$ so $\triangle POT \cong \triangle ROT$ by SAS.

A consequence is that PR bisects $\angle QPS$. This can also be proved by an argument based on the sum of internal angles for an n-gon,

It looks as if the segment of the vertical that extends beyond the radius might be equal to that part below down to what looks like the "strut" of a kite. However, this is not true. We will show what this ratio is equal to in just a bit.

Rather than use the vertices as points of reference, we will label the line segments.

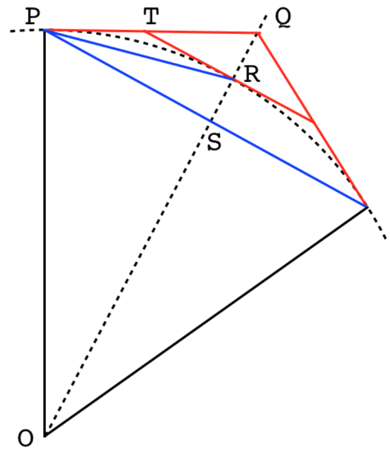


Just to be clear: a is the part of the radius extended to point S above, while b extends to Q. c and d are the lengths of the indicated lines *in the half-sector*, not all the way across, and f is the entire length of PQ.

We're ready to proceed.

basic geometry: perimeters

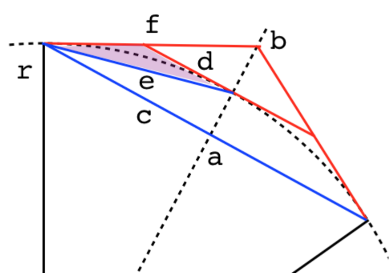
As we said, the key observation is that $\triangle PRT$ is isosceles.



Because of that, and since $\angle SPR = \angle PRT$ by the alternate interior angles theorem, $\angle SPR = \angle TPR$.

Therefore the cosines are also equal, namely:

$$\frac{c}{e} = \frac{e/2}{d}$$



(To see the midpoint of e, drop an altitude in the isosceles triangle, shown in purple).

Therefore:

$$2dc = e^2$$

Now, c is the entirety of p in this half-sector. But d is only one-half of P'.

Hence $2d \cdot c$ is equal to pP', and since $e = p'$, we have that

$$pP' = [p']^2$$

which was our second rule for the perimeters.

The first rule was

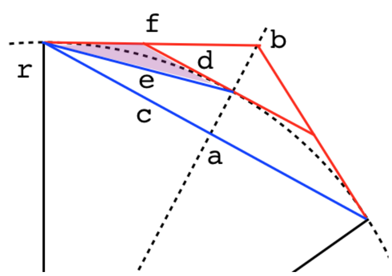
$$P' = 2\frac{pP}{p+P}$$

In geometric terms, we must show that

$$2d = 2\frac{cf}{c+f}$$

$$cd + df = cf$$

Taking another look at the diagram:



The small triangle with base d ($\triangle QRT$ above) has slanted side $f - d$ (subtracting d because, again, $\triangle PRT$ is isosceles). By similar triangles, we have

$$\frac{d}{f-d} = \frac{c}{f}$$

$$df = cf - cd$$

$$cd + df = cf$$

But this is what we needed to prove.

$\square$

basic geometry: areas

The area formulas for inside (a) and outside (A) polygons are those for a circle of unit radius (so that π is the area):

$$A' = 2\frac{a'A}{a' + A}$$

$$a' = \sqrt{aA}$$

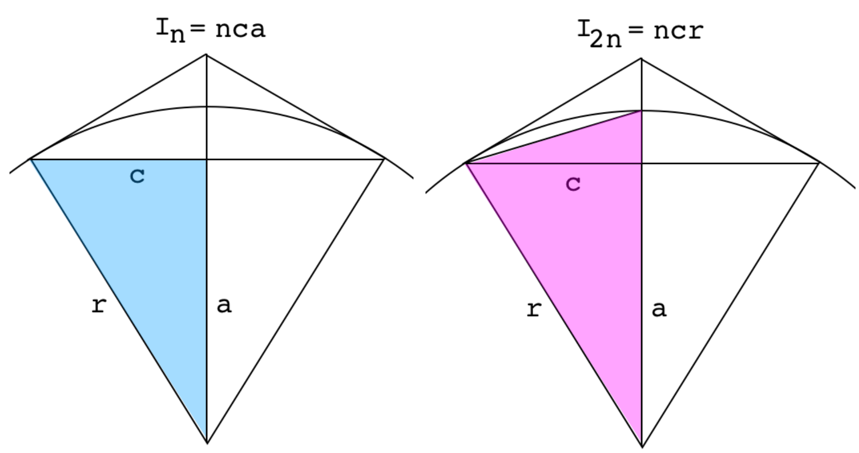
However, having reached this point, we need another symbol for area, because a is currently the line segment corresponding to p/n. Let's use I and C for the inside and outside areas, to match the source.

We will also adopt their n and $2n$ notation, It's a bit clumsy but that will make it easier to match things up.

$$C_{2n} = 2 \cdot \frac{I_{2n}C_n}{I_{2n} + C_n}$$

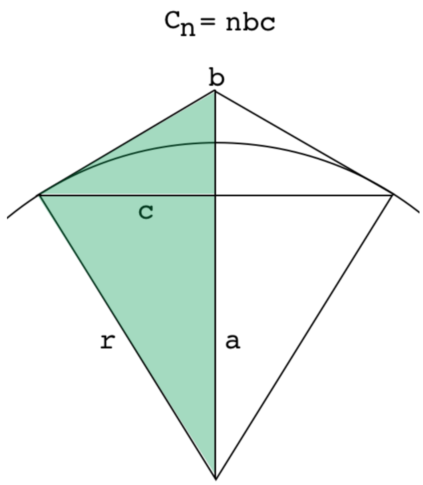
$$I_{2n} = \sqrt{I_n C_n}$$

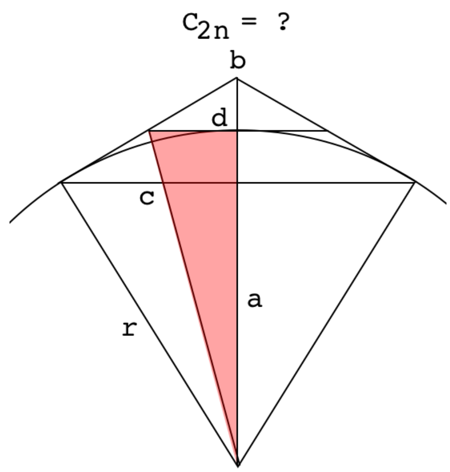
The first two areas are I_n and I_{2n}



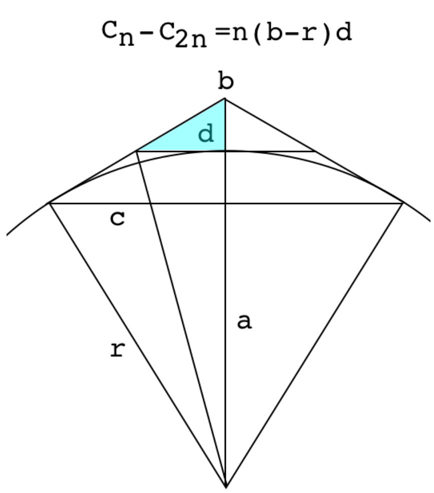
We compute these areas for the whole sector of angle θ, so there are two congruent triangles with base a (or base r) and height c. Multiply by n if you like to get the entire polygon, but every expression will have a factor of n, and we'll be looking at ratios, so we can just not worry about it.

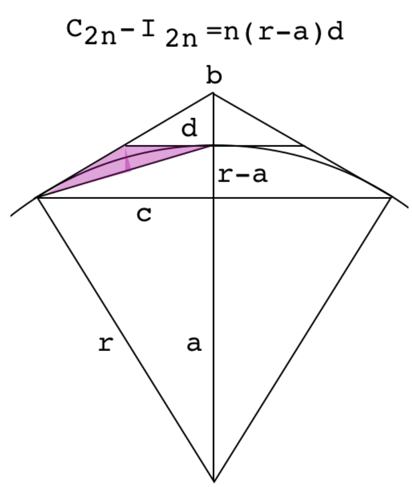
The third easy one is C_n:





We write the last one (C_{2n}) as two different differences.





Let's gather all these expressions in one place, forming ratios:

$$\frac{I_{2n}}{I_n} = \frac{ncr}{nca} = \frac{r}{a}$$

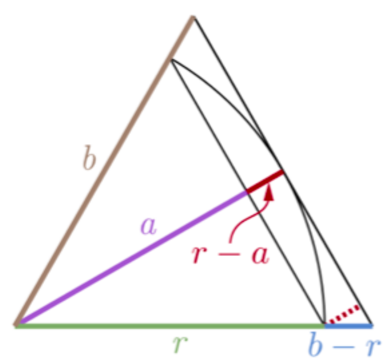
$$\frac{C_n}{I_{2n}} = \frac{ncb}{ncr} = \frac{b}{r}$$

$$\frac{C_n - C_{2n}}{C_{2n} - I_{2n}} = \frac{n(b-r)d}{n(r-a)d} = \frac{b-r}{r-a}$$

We will prove that these three ratios are all equal to each other.

We will have used the geometry to prove what the source calls their Lemmas, and those can be used in turn to prove the original Gregory formulas.

But the proof is easy:



It's just a matter of similar triangles:

$$\frac{r}{a} = \frac{b}{r} = \frac{b-r}{r-a}$$

That's the "without words" part.

For that very last part, you can work out the dimensions of the tiny similar triangle, or you can say:

$$\frac{r}{a} = \frac{b}{r}$$

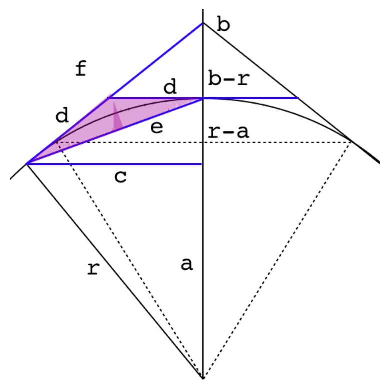
$$\frac{r}{a} - \frac{a}{a} = \frac{b}{r} - \frac{r}{r}$$

$$\frac{r-a}{a} = \frac{b-r}{r}$$

which is easily rearranged to give the desired result.

□

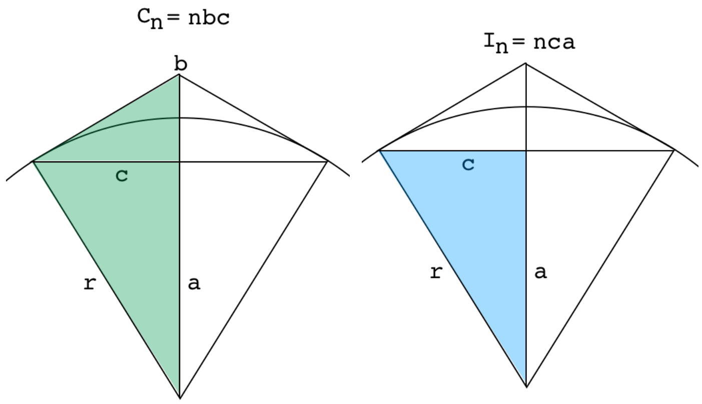
This can also be proved using the **angle bisector theorem**.



The side labeled e bisects the angle formed by the two sides labeled c and f. Therefore

$$\frac{b-r}{f} = \frac{r-a}{c} \quad \Rightarrow \quad \frac{b-r}{r-a} = \frac{f}{c}$$

But f and c are two sides of a triangle which is similar to the colored portions below:



Therefore

$$\frac{b}{r} = \frac{r}{a} = \frac{f}{c} = \frac{b-r}{r-a}$$

As we said.

algebra

Moving on to the geometric mean formula is not hard. From above we have that

$$\frac{I_{2n}}{I_n} = \frac{C_n}{I_{2n}}$$

$$[I_{2n}]^2 = I_n C_n$$

Translated back into the A, a area notation

$$a' = \sqrt{aA}$$

This is just what we wanted to show.

For the other formula, what we have is:

$$\frac{C_n - C_{2n}}{C_{2n} - I_{2n}} = \frac{C_n}{I_{2n}}$$

$$I_{2n}(C_n - C_{2n}) = C_n(C_{2n} - I_{2n})$$

$$2I_{2n}C_n = C_nC_{2n} + I_{2n}C_{2n}$$
$$= C_{2n}(C_n + I_{2n})$$

So

$$C_{2n} = 2 \cdot \frac{I_{2n}C_n}{C_n + I_{2n}}$$
$$C_{2n} = 2 \cdot \frac{1}{1/I_{2n} + 1/C_n}$$

And we're done. In our preferred notation

$$A' = 2 \cdot \frac{1}{1/a' + 1/A}$$

historical note

The area-based formulas given above are due to James Gregory.

`https://divisbyzero.com/2018/09/28/proof-without-word-gregorys-theorem/`

As an aside, the Fundamental Theorem of Calculus (FTC) is usually thought about (taught and learned) using the language of functions, and ascribed mainly to Leibnitz, with some credit to the two Isaacs, Newton and his university lecturer, Barrow.

`https://arxiv.org/abs/1111.6145`

Amazingly enough, Gregory published a geometric (Euclidean) proof of the FTC in 1668! That predates Liebnitz (1693) by more than 25 years. This is motivation to give considerable credit to individuals other than Newton and Liebnitz (e.g. Fermat, Pascal, Wallis, Gregory, etc.) in the invention of the calculus.