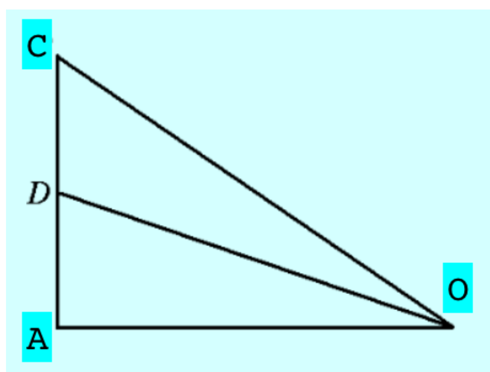
Archimedes and pi

We're going to follow a page that in turn follows Archimedes argument for the approximation of π.

`https://itech.fgcu.edu/faculty/clindsey/mhf4404/archimedes/archimedes.html`

overview

There are three steps which we will repeat (eventually, four times).



• We will have ratios for the cosecant ($OC : AC$) and cotangent ($OA : AC$), computed in the previous round.

• Add them together, obtaining $(OC + OA) : AC$, but observe that this ratio is also equal to $OA : AD$, by the corollary of the angle bisector theorem. We thereby obtain the cotangent of the half-angle.

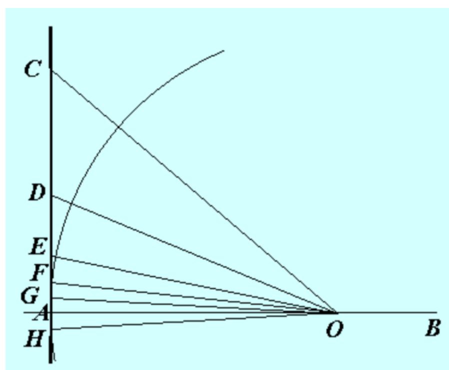
• Obtain the cosecant of the half-angle, $OD : AD$, by using the Pythagorean theorem. Namely:

$$OA^2 + AD^2 = OD^2$$

$$\sqrt{\frac{OA^2}{AD^2} + 1} = \frac{OD}{AD}$$

Part A, round 1

Draw a circle with radius OA and tangent AC, and let $\angle AOC$ be one-third of a right angle. (Note that these desginations differ from the previous figure and proofs).



The figure appears to have been compressed in width at some point. The original angle looks more like 45 than 30. It doesn't really matter.

Now, since the triangle is a 30-60-90 triangle, $OA : AC = \sqrt{3}$.

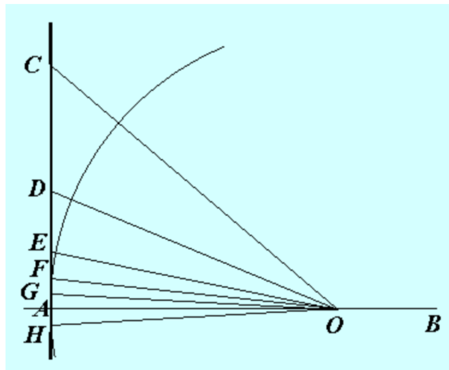
265/153 is a (very good) approximation, just slightly smaller than the true value.

• $OA : AC > 265 : 153$

The value of the cosecant is 2. The denominator has been chosen to match the previous ratio.

• $OC : AC = 306 : 153$

Now draw the angle bisector OD.



• $(CO + OA) : CA = OA : AD$

We just add numerators for the first two ratios above, leaving the result over the common denominator.

• $OA : AD > 571 : 153$

Finally, we want $OD : AD$. By the Pythagorean Theorem

$$(OD : AD)^2 = (OA : AD)^2 + 1$$

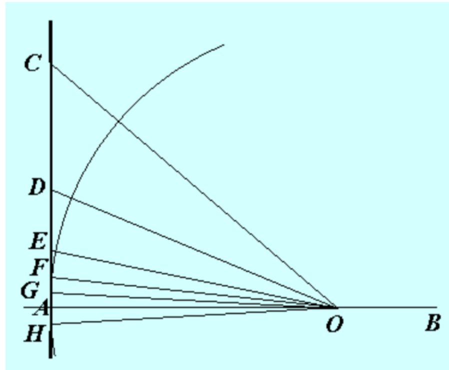
Math: $571^2 = 326041; 153^2 = 23409; \sqrt{+} = 591.143$

• $OD : AD > 591\ 1/8 : 153$

Archimedes approximates the result as 591 1/8 : 153.

Part A, round 2

Now draw the angle bisector OE.



Just carry out the arithmetic.

$$571 + 591\ 1/8 = 1162\ 1/8$$

$$1162^2 = 1350244; 153^2 = 23409; \sqrt{+} = 1172.15$$

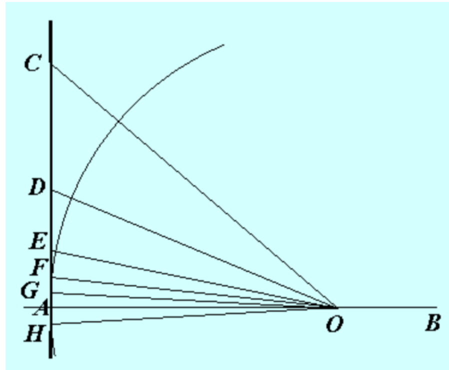
A reasonable first approximation for $\sqrt{a^2 + b^2}$ when $a has a small fractional part$ is to simply carry over the fraction. Above we start with 0.125 and end up with 0.153.

We use 1/8 as the resulting fraction. We are trying to fit an upper bound as tight as possible. However, we should not fit it tighter than justified. Here, the decimal result is a bit higher than 1/8, that is, we are fitting it tighter than justified, but it's a really small effect.

- $\cot\theta/2 > 1162\ 1/8 : 153$
- $\csc\theta/2 = 1172\ 1/8 : 153$

Part A, round 3

Now draw the angle bisector OF.



Just carry out our arithmetic, again.

$$1162\ 1/8 + 1172\ 1/8 = 2334\ 1/4$$

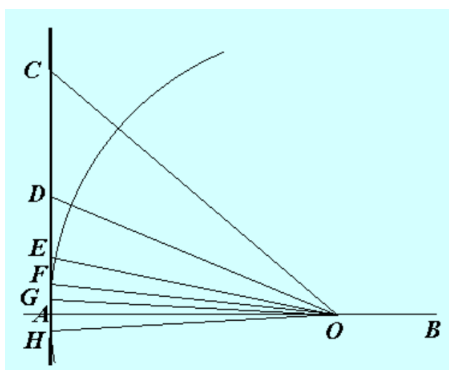
$$2334^2 = 5447556; +23409; \sqrt{+} = 2339$$

Remember the 1/4.

- $\cot\theta/2 > 2334\ 1/4 : 153$
- $\csc\theta/2 = 2339\ 1/4 : 153$

Part A, round 4

Now draw the angle bisector OG.



Arithmetic:

$$2334\ 1/4 + 2339\ 1/4 = 4673\ 1/2$$

We do not need to calculate the cosecant.

- $\cot \theta/2 > 4673\ 1/2 : 153$

We're almost done. The original distance AC was 1/12 the perimeter of a circumscribed hexagon, so we would multiply by 12 to get the ratio to the radius, but we want the ratio to the diameter so that gives a factor of 2 on the bottom for a total factor of 6.

There is an additional factor of 16 for the four "halvings" of 2^4. Hence we obtain

$$153 \times 96 = 14688$$

and then invert to get the ratio of the circumference to the diameter:

$$\frac{14688}{4673\ 1/2} = 3 + \frac{668\ 1/2}{4673\ 1/2}$$

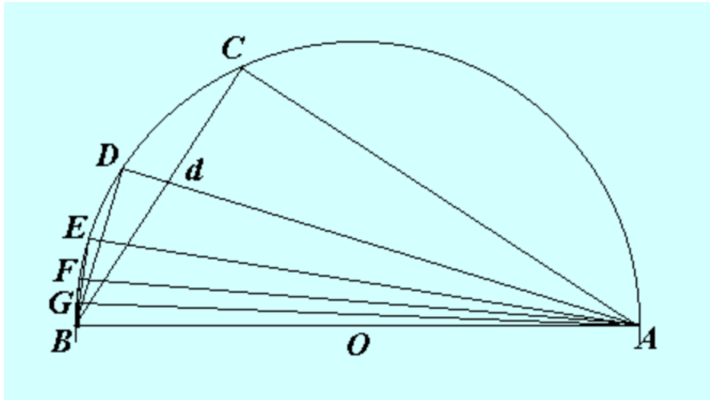
The fraction is just less than 1/7.

$1/7 = 0.142857$, while $668\ 1/2/4673\ 1/2 = 0.14304$.

We conclude that $\pi < 3\ 1/7$.

Part B

For Part B we use this diagram for an inscribed polygon.



As before $\triangle ABC$ is a $30-60-90$ right triangle.

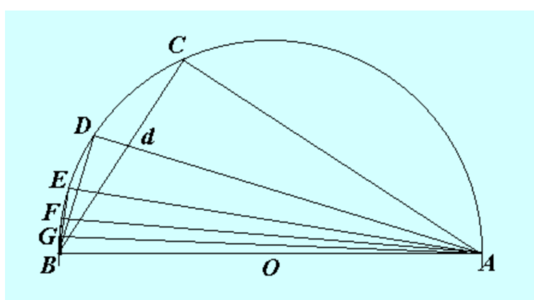
- $AC : BC < 1351 : 780$.

This ratio is an approximation for $\sqrt{3}$. It is an even better approximation than the previous one $(265 : 153)$, and also, crucially, it is just slightly *more* than the true value, whereas $265/153$ was slightly less. Since now we are trying to fit a lower bound, this is a good thing.

Part B, round 1

Let AD bisect the angle, and then join BD.

- $\angle BAD = \angle dAC = \angle dBD$.



The first statement just restates the construction as an angle bisector. The second follows from the fact that the two angles have vertices on the circle and cut off the same arc.

As a consequence, $\triangle dBD \sim \triangle dAC$.

Start with the similar triangles above and write three ratios of long side (not hypotenuse) to short side

$$AD : BD = BD : Dd = AC : Cd$$

Note: the source has $AB : Bd$ but this seems to be an error. That is a ratio of two hypotenuses and so is not equal to the others.

As a result, I hesitate to follow this part of the proof:

$$\begin{aligned} \boldsymbol{AD : BD = BD : Dd} &\boldsymbol{= AB : Bd} \\ &\boldsymbol{= (AB + AC) : (Bd + Cd)} \\ &\boldsymbol{= (AB + AC) : BC} \end{aligned}$$

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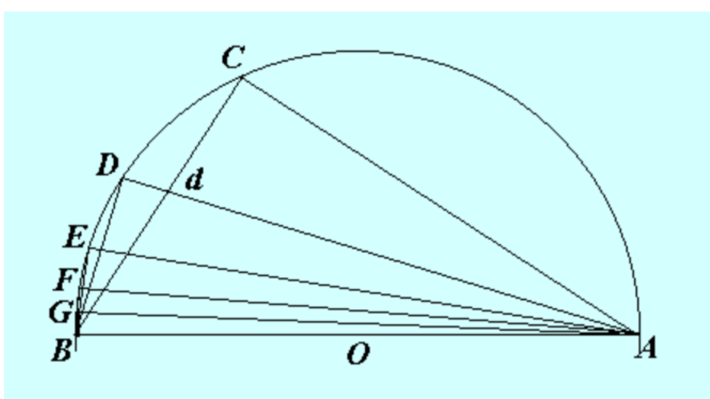
or $\boldsymbol{(BA + AC) : BC = AD : DB.}$

but actually, the next part is correct.

I worked out a proof of the last statement

$$(AB + AC) : BC = AD : DB$$



We have that $\triangle ABC$ is a right triangle and that AD and thus Ad is the angle bisector for $\angle BAC$. Therefore, we have by our favorite theorem that

$$(AB + AC) : BC = AC : Cd$$

We also have that $\triangle ABD$ is a right triangle and by virtue of the angle bisector construction, $\triangle ABD$ is similar to $\triangle ACd$. Therefore:

$$AC : Cd = AD : DB$$

These two lines combine to give the desired result.

But thinking about this later, it just follows from the corollary of the angle bisector theorem that we used before.

The ratio $AD : DB$ is what we need going forward, and we get that by evaluating $(AB + AC) : BC$. We have that the cotangent $AC : BC < 1351 : 780$, and $AB : BC$ is the cosecant whose value is 2 so we multiply $780 \times 2 = 1560$, and then add 1351 to get the numerator of the result.

- $AD : DB < 2911 : 780$

From the Pythagorean theorem: $AD^2 + BD^2 = AB^2$ so

$$AB^2 : BD^2 = AD^2 : BD^2 + 1$$

$AD : DB < 2911 : 780$

So we obtain $2911^2 = 8473921$ and $BD^2 = 608400$ so we have that

$$AB^2 : BD^2 = 9082321 : 608400$$

$$AB : BD < 3013\ 3/4 : 780$$

- $AB : BD < 3013\ 3/4 : 780$

Part B, round 1 summary

It is interesting that, although the construction is different, we are using the same relationships. We started with the cosecant and the cotangent for $\triangle ABC$, namely $AB : BC$ and $: AC : BC$.

We used the relationship $(AB + AC) : BC = AD : DB$ to obtain the cotangent of the bisected angle, and then we used the Pythagorean theorem in this form

$$AB^2 : BD^2 = AD^2 : BD^2 + 1$$

to get the cosecant from the cotangent. Thus we have

- $AD : DB < 2911 : 780$, the cotangent.
- $AB : BD < 3013\ 3/4 : 780$, the cosecant.

Part B, round 2

Now, let AE bisect the angle, and then join BE.

Rather than go through the geometry again, let's just substitute letters. First the cotangent

$$(AB + AD) : BD = AE : EB$$

Then the cosecant.

$$AB^2 : BE^2 = AE^2 : BE^2 + 1$$

For the first part we have $2911 : 780 + 3013\ 3/4 : 780 = 5924\ 3/4 : 780$.

At this point, we reduce the denominator to 240. This amounts to dividing by 3 1/4. 5924 3/4 divided by 3 1/4 is exactly equal to 1823.

- $AE : EB = 1823 : 240$, the cotangent.

For the second part we have the previous number squared and added to 1 and then take the square root. $1823^2 = 3323329; 240^2 = 57600$; so we have 3380929 and the square root is $< 1838\ 3/4$, but the source gives the fraction as a bit larger 1838 9/11.

- $AB : BE = 1838\ 9/11 : 240$, the cosecant.

Part B, round 3

Now, let AF bisect the angle, and then join BF. Substitute letters (carefully, looking at the diagram). First the cotangent

$$(AB + AE) : BE = AF : FB$$

Then the cosecant.

$$AB^2 : BF^2 = AF^2 : BF^2 + 1$$

For the first part we have $1838\ 9/11 : 240 + AE : EB = 1823 : 240 = 3661\ 9/11 : 240$.

We reduce the denominator, this time to 66. This amounts to multiplication by 11/40. So the numerator is multiplied by the same factor giving

- $AF : FB = 1007 : 66$, the cotangent.

For the second part we have the previous number squared and added to 1 and then take the square root.$1007^2 = 1014049; 66^2 = 4356$, so we have 1018405 and the square root is 1009 1/6.

- $AB : FB = 1009\ 1/6 : 66$, the cosecant.

Part B, round 4

Finally, let AG bisect the angle, and then join BG. Substitute letters. First the cotangent

$$(AB + AF) : BF = AG : GB$$

Then the cosecant.

$$AB^2 : BG^2 = AG^2 : BG^2 + 1$$

For the first part we have

- $AG : GB = 2016\ 1/6 : 66$, the cotangent.

Now do $2016^2 = 4064256; 66^2 = 4356$ so that's 4068612 and the square root is 2017 1/12 while the source gives

- $AB : GB < 2017\ 1/4 : 66$, the cosecant.

This time we do need the cosecant, because the hypotenuse of the triangle is lying on the diameter.

Almost done. The side BG is a side of an inscribed regular polygon of 96 sides. We multiply $66 \times 96 = 6336$ and compute the ratio of the inverse.

I am not sure how Archimedes came up with it, but it is easy to verify that the ratio which is less than π is greater than:

$$\frac{6336}{2017\ 1/4} > 3\ 10/71$$

We combine parts A and B to make our final statement that

$$3\ 10/71 < \pi < 3\ 1/7$$