Double and half angles

We will find it useful in several problems to be able to compute the sine, cosine and tangent of angle 2θ, knowing the values for θ. These formulas can be rearranged to give the values of $\theta/2$ in terms of θ.

I can't remember these formulas, but derive them from the sum of angles when needed.

cosine

Start with our old friend:

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

Let $s = t$:

$$\cos 2s = \cos^2 s - \sin^2 s$$

Since $\sin^2 s + \cos^2 s = 1$, $-\sin^2 = \cos^2 - 1$ so

$$\cos 2s = 2\cos^2 s - 1$$

We can use this formula to compute the value for $2s$ given that for s. To go from 2θ to θ:

$$\cos^2 s = \frac{1}{2}(1 + \cos 2s)$$

$$\cos s = \sqrt{\frac{1}{2}(1 + \cos 2s)}$$

sine

$$\sin s + t = \sin s \cos t + \cos t \sin s$$

Let $s = t$:

$$\sin 2s = 2 \sin s \cos s$$

Put the other way

$$\sin s = \frac{\sin 2s}{2 \cos s}$$

tangent

The formulas for the tangent are easily obtained by substitution. Let us simplify the notation a bit by setting $S = \sin 2t$ and $S' = \sin t$ and similarly for cosine and tangent. From above we have the basic relationships

$$S' = \frac{S}{2C'}$$

and

$$C' = \sqrt{\frac{1}{2}(1 + C)}$$

$$2[C']^2 = 1 + C$$

So the tangent ($T' = \tan s$) is:

$$T' = \frac{S'}{C'} = \frac{S}{2C'} \frac{1}{C'} = \frac{S}{2[C']^2}$$

$$= \frac{S}{1 + C}$$

That's a fairly remarkable simplification!

Another way to say the same thing:

$$\frac{1}{T'} = \frac{1}{T} + \frac{1}{S}$$

This result can be massaged in various ways. Multiply on the top and bottom of the right-hand side by T

$$T' = \frac{ST}{S+T}$$

Also, since

$$T' = \frac{S}{1+C} = \frac{S'}{C'}$$
$$C' = \frac{S'(1+C)}{S}$$

In going from unprimed (2θ) to prime (θ), it seems that the most straightforward way is to compute

$$C' = \sqrt{\frac{1}{2}(1+C)}$$
$$T' = \frac{S}{1+C}$$

and then the sine last

$$S' = \frac{S}{2C'}$$