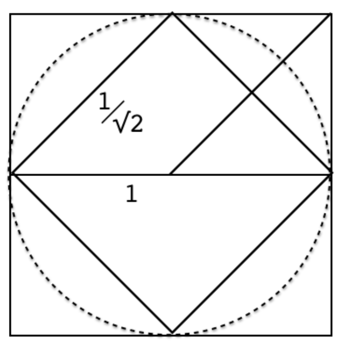
Estimating pi

We can get an estimate for the value of π by looking at the perimeters for the series square, hexagon, octagon. The circle has diameter of 1, so the value of π lies between the internal and external perimeters.

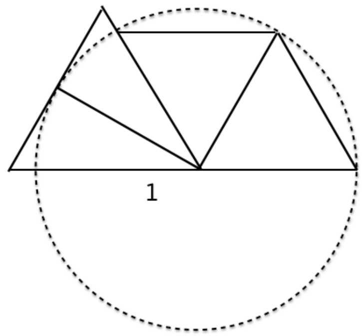
square



Dividing the internal square along the diagonal, the right triangle is isosceles with hypotenuse equal to 1 and sides equal to $1/\sqrt{2}$. The internal perimeter is $4/\sqrt{2} = 2\sqrt{2} = 2.828$.

The external perimeter is easy, it is just 4 times the diameter, or 4.

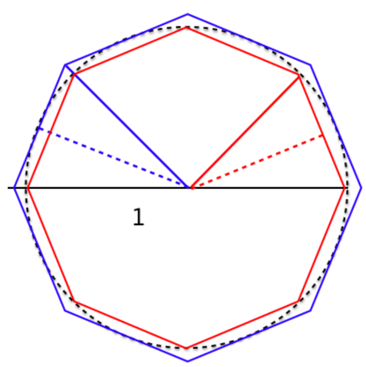
hexagon



Each of the six sectors of a hexagon consists of an equilateral triangle, with side equal to $1/2$, so the internal perimeter is $6 \cdot 1/2 = 3$.

The six sectors of the external perimeter each have an altitude equal to $1/2$. The half-sectors are $30-60-90$ triangles with sides $1-2-\sqrt{3}$. Hence the ratio of one-half an external side to the radius is $1/\sqrt{3}$ and since the radius is $1/2$, the actual value is $1/2\sqrt{3}$. So each whole side is $1/\sqrt{3}$, giving $6/\sqrt{3} = 3.464$.

octagon



The octagon suddenly becomes quite challenging. I really don't know any way to do this except by using trigonometry, and the corollary of the angle bisector rule that says:

$$\cot\theta + \csc\theta = \cot\theta/2$$

For a 45 degree angle the cotangent is 1 and the cosecant is $\sqrt{2}$ so

$$\cot\theta/2 = 1 + \sqrt{2}$$

Hence for the external perimeter, let the half-side be h, then

$$\frac{r}{h} = \cot\theta/2 = 1 + \sqrt{2}$$

$$h = \frac{1/2}{1+\sqrt{2}}$$

and there are sixteen of them so

$$P = \frac{8}{1+\sqrt{2}} = 3.3137$$

For the internal perimeter, we have the hypotenuse, hence we will need the cosecant. Label the sides of a right triangle a, b, c. The Pythagorean theorem:

$$a^2 + b^2 = c^2$$

$$\frac{a^2}{b^2} + 1 = \frac{c^2}{b^2}$$

If we choose a as the adjacent side, then a/b is the cotangent and c/b is the cosecant. Hence

$$\csc\theta = \sqrt{\cot^2\theta + 1}$$

and in our case

$$\csc\pi/8 = \sqrt{(1+\sqrt{2})^2 + 1} = 2.613$$

Let h be the half-side then

$$\frac{1/2}{h} = \csc\theta = 2.613$$