Perimeter formula

This web page originally got me started with this derivation

`http://personal.bgsu.edu/~carother/pi/Pi3d.html`

(Unfortunately, the link is dead now, probably because the University took Dr. Carother's pages down when he died, idiots). It has been preserved by the wayback machine:

`https://web.archive.org/web/20171024182015/http://personal.bgsu.edu/~carother/pi/Pi3d.html`

On that page, there was given a simple pair of formulas listed, namely, for an inside perimeter p and an outside perimeter P

$$P' = \frac{2pP}{p+P}$$

$$p' = \sqrt{pP'}$$

The first equation can be rearranged to give

$$\frac{1}{P'} = \frac{1}{2}\,[\frac{1}{P} + \frac{1}{p}\,]$$

which is the definition of the harmonic mean of p and P, while the second equation is the geometric mean.

Since in our derivation p and P are the same multiple of S and T, it seems like the same relationships should hold for the sine and tangent, but we must remember the extra factor of 2.

From the half-angle formulas, we said that

$$T' = \frac{S}{1+C}$$

Multiply top and bottom on the right by T:

$$T' = \frac{ST}{T+S}$$

Recall that S is the same as p, within a factor of n, and that T is the same as P, within the same factor.

$$p = nS$$

$$P = nT$$

while

$$P' = 2nT'$$

Going back to

$$T' = \frac{ST}{T+S}$$

$$2nT' = \frac{2 \cdot nS \cdot nT}{nT + nS}$$

$$P' = \frac{2pP}{p+P}$$

This is what was given.

For the second one

$$S' = \frac{S}{2C'}$$

$$= \frac{S}{2}\ \frac{T'}{S'}$$

Then

$$4[S']^2 = S \cdot 2T'$$

$$[2nS']^2 = nS \cdot 2nT'$$

Changing variables, $p' = 2nS'$

$$[p']^2 = pP'$$

Finally

$$p' = \sqrt{pP'}$$

which matches what was given.