Double and half angles

We will find it useful in several problems to be able to compute the sine, cosine and tangent of angle 2θ, knowing the values of the functions for angle θ. These formulas can be rearranged to give the values of $\theta/2$ in terms of θ.

I usually can't remember these formulas, but easily derive them from the sum of angles when needed.

cosine

Start with our old friend:

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

Let $s = t$:

$$\cos 2s = \cos^2 s - \sin^2 s$$

Since $\sin^2 s + \cos^2 s = 1$, $- \sin^2 = \cos^2 -1$ so

$$\cos 2s = 2\cos^2 s - 1$$

We can use this formula to compute the value for $2s$ given that for s. To go from 2θ to θ:

$$\cos^2 s = \frac{1}{2}(1 + \cos 2s)$$

$$\cos s = \sqrt{\frac{1}{2}(1 + \cos 2s)}$$

sine

$$\sin s + t = \sin s \cos t + \cos t \sin s$$

Let $s = t$:

$$\sin 2s = 2 \sin s \cos s$$

Put the other way

$$\sin s = \frac{\sin 2s}{2 \cos s}$$

tangent

The formulas for the tangent are easily obtained by substitution. Let us simplify the notation a bit by setting $S = \sin 2t$ and $S' = \sin t$ and similarly for cosine and tangent. From above we have the basic relationships

$$S' = \frac{S}{2C'}$$

and

$$C' = \sqrt{\frac{1}{2}(1 + C)}$$

$$2[C']^2 = 1 + C$$

So the tangent $(T' = \tan s)$ is:

$$T' = \frac{S'}{C'} = \frac{S}{2C'}\frac{1}{C'} = \frac{S}{2[C']^2}$$

$$= \frac{S}{1+C}$$

That's a fairly remarkable simplification!

Another way to say the same thing:

$$\frac{1}{T'} = \frac{1}{T} + \frac{1}{S}$$

The last version is used extensively by Archimedes, although it isn't labeled as such. Another way to say it in modern language is

$$\cot \theta/2 = \cot \theta + \csc \theta$$

This comes from our half-angle formula and is also proven easily using the angle bisection theorem.

This result can be massaged in various ways. Multiply on the top and bottom of the right-hand side by T

$$T' = \frac{ST}{S+T}$$

Also, since

$$T' = \frac{S}{1+C} = \frac{S'}{C'}$$

$$C' = \frac{S'(1+C)}{S}$$

In going from unprimed (2θ) to prime (θ), it seems that the most straightforward way is to compute one or both of these first

$$C' = \sqrt{\frac{1}{2}(1+C)}$$

$$T' = \frac{S}{1+C}$$

and then the sine requires C' so it comes later.

$$S' = \frac{S}{2C'}$$