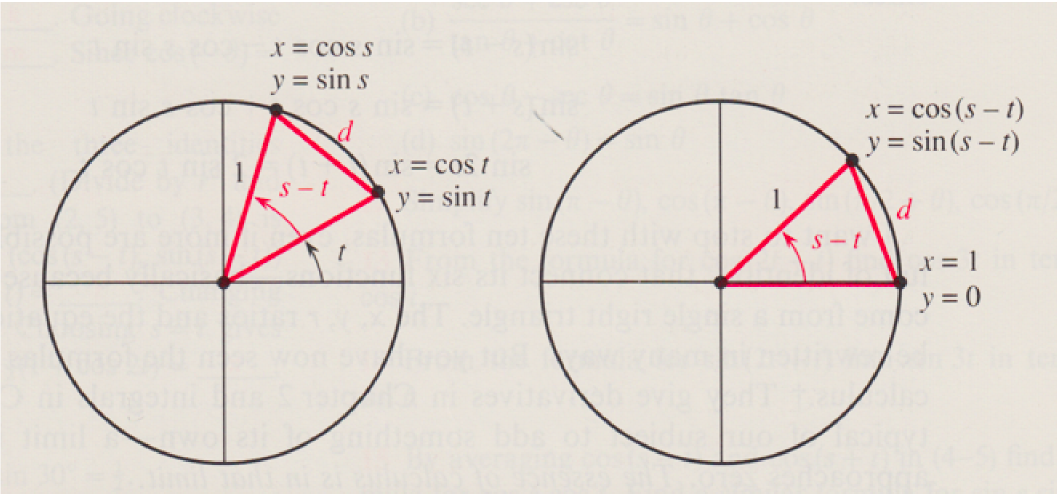
Sum of angles

Strang

For a geometric derivation of the sum of angles formula with minimal setup, I really like this figure from Strang



Distance between points and equal distances in two circles.

We have the same triangle in the two panels, just rotated clockwise on the right.

To compute the distance between two points in the plane we do

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(This is just the Pythagorean theorem in disguise).

We don't actually need to take the square root, let's stick with

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

In the first figure, t is the angle between the lower radius and the x-axis, s is the angle between the upper radius and the x-axis, and as labeled, $s - t$ is the angle between the two radii.

The distance d squared for the left panel is

$$d^2 = (\cos s - \cos t)^2 + (\sin s - \sin t)^2$$

Multiply out:

$$d^2 = \cos^2 s - 2\cos s \cos t + \cos^2 t + \sin^2 s - 2\sin s \sin t + \sin^2 t$$

We have two copies of $\sin^2 + \cos^2$, one for angle s and one for t

$$d^2 = 2 - 2\cos s \cos t - 2\sin s \sin t$$

In the right panel, the two radii have been rotated, preserving the same angle between them.

$$d^2 = (\cos(s-t) - 1)^2 + \sin(s-t)^2$$

(Don't forget the 1).

$$= \cos^2(s-t) - 2\cos(s-t) + 1 + \sin^2(s-t)$$

$$= 2 - 2\cos(s-t)$$

Because the included angle hasn't changed, neither has the distance, so we can equate the two expressions.

$$2 - 2\cos(s-t) = 2 - 2\cos s \cos t - 2\sin s \sin t$$

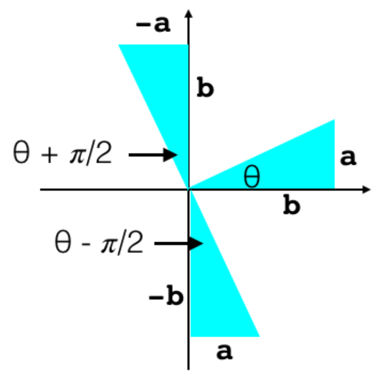
Subtract 2 from both sides, multiply by 1/2, and change sign to give

$$\cos(s-t) = \cos s \cos t + \sin s \sin t$$

This is our first formula, for the cosine of the difference of two angles.

getting to sine

Look at the relationships between sine and cosine for angles that are related by addition or subtraction of $\pi/2$.



In the figure, I have simply rotated the same triangle.

From the figure we can easily read off these four identities

$$\sin(\theta + \pi/2) = b = \cos\theta$$

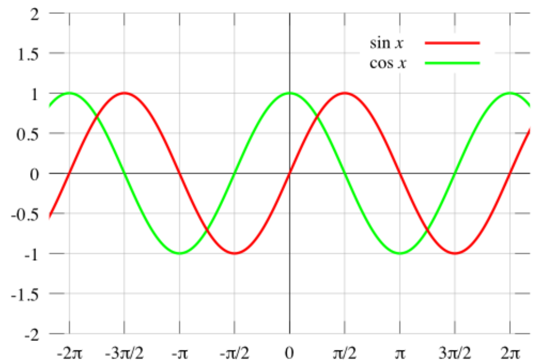
$$\cos(\theta + \pi/2) = -a = -\sin\theta$$

And

$$\sin(\theta - \pi/2) = -b = -\cos\theta$$

$$\cos(\theta - \pi/2) = a = \sin\theta$$

Here is an alternative derivation which proceeds from the graph of sine and cosine versus the angle.



Pick some angle (say $\theta = 0$), then $\cos\theta = 1$. What is the angle for which the sine gives the same result? The sine curve is exactly like the cosine, it is just shifted to the right by a *phase change* of $\pi/2$.

That angle is $\theta + \pi/2$. The phase change is added to the angle:

$$\cos\theta = \sin(\theta + \frac{\pi}{2})$$

Try the same reasoning in reverse.

The cosine curve is exactly like the sine, it is just shifted by a phase change of $-\pi/2$, i.e. to the left. Pick some angle (say $\theta = \pi/2$), then $\sin\theta = 1$. What is the value of the angle for which the cosine gives the same result?

It is $\theta - \pi/2$. The phase change is subtracted from the angle θ:

$$\sin\theta = \cos(\theta - \frac{\pi}{2})$$

In summary, switching sine for cosine gives a valid expression, but there is a difference of *sign* for the phase.

back to our task

So our sum of angles formula (well, really, the difference of angles) was

$$\cos(s-t) = \cos s \cos t + \sin s \sin t$$

Let

$$u = t - \frac{\pi}{2}$$
$$t = u + \frac{\pi}{2}$$

Substitute for t

$$\cos\ [\ s - (u + \frac{\pi}{2})\ ]\ = \cos s \cos(u + \frac{\pi}{2}) + \sin s \sin(u + \frac{\pi}{2})$$

Regroup the left-hand side

$$\cos\ [\ (s-u) - \frac{\pi}{2})\ ]\ = \cos s \cos(u + \frac{\pi}{2}) + \sin s \sin(u + \frac{\pi}{2})$$

Referring to the results we obtained above, cosine something minus $\pi/2$ is the sine of that something:

$$\sin(s-u) = \cos s \cos(u + \frac{\pi}{2}) + \sin s \sin(u + \frac{\pi}{2})$$

Cosine something plus $\pi/2$ is minus the sine:

$$\sin(s-u) = -\cos s \sin u + \sin s \sin(u + \frac{\pi}{2})$$

Sine something plus $\pi/2$ is cosine:

$$\sin(s-u) = -\cos s \sin u + \sin s \cos u$$

Rearrange:

$$\sin(s-u) = \sin s \cos u - \sin u \cos s$$

This is correct, but the path is fraught with error!

For now, memorize. Soon we will see a very simple and effective **aid to memory** due to Euler.

sum of tangents

It is also easy to derive the sum of tangents from the sum of sines and cosines.

$$\tan s + t = \frac{\sin s + t}{\cos s + t}$$

$$= \frac{\sin s \cos t + \cos s \sin t}{\cos s \cos t - \sin s \sin t}$$

Divide by $\cos s \cos t$

$$\tan s + t = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$

$$\tan s - t = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

We will use these for a few problems later in the book.