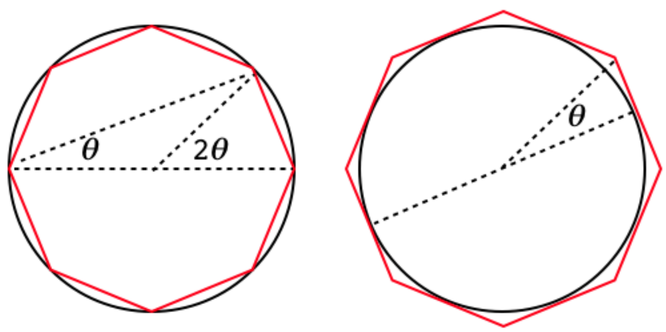
Value of pi using sine and cosine

We can approximate the value of π by squeezing it between the perimeter of an inscribed polygon, which is less than the circumference of the circle, and the perimeter of a circumscribed polygon, which is greater than the circumference of the circle.



We use a circle of *diameter* equal to 1 (rather than the radius, which is more usual). The circumference of the circle is then equal to π, the value which gets squeezed between the two perimeters.

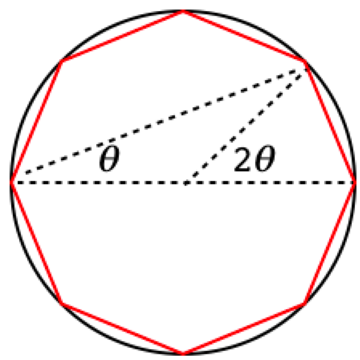
The figure shows a sketch of the polygons when $n = 8$. We will be increasing the number of sides by a factor of 2 at each step, so these are really 2^n-gons with $n = 3$ here.

Finding perimeters in terms of angle θ

For the left panel, we have 8 sides, so the central angle (marked 2θ) is equal to

$$\frac{2\pi}{8} = \frac{\pi}{4} = 45^\circ$$

and θ is one-half that.



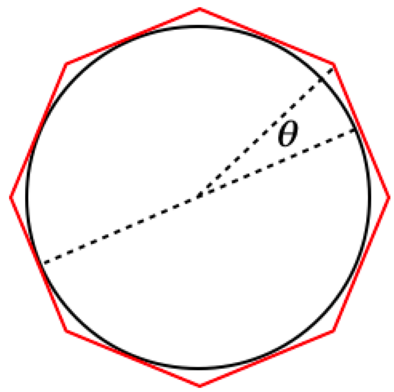
By a standard theorem (from Thales), the triangle above containing angle θ, with the diameter as one side, and two other vertices also on the circle, is a right triangle. The inscribed n-gon side of length S (shown in red) is equal to $\sin\theta$, since the hypotenuse of the triangle is the diameter of the circle, which is equal to 1.

The total perimeter is $8 \cdot S$.

[Alternatively, use half the angle at the center of the circle (i.e. θ). Then half the length of the red line $S/2$, divided by the radius ($r = 1/2$) gives $S = \sin\theta$, the same result.]

For the right panel, we have the same circle (now showing the outside polygon, circumscribing the circle), it is just rotated slightly.. One dashed line extends a bit further to the vertex of the n-gon outside. The angle marked θ is one-half the angle we marked as 2θ previously since now the diameter comes down to the middle of the side.

We compute the whole length of the side T as follows. The half-side is $T/2$ and the hypotenuse of the triangle is one-half the unit diameter, which is $1/2$, so $T = \tan\theta$. The total perimeter is $8 \cdot T$.



All of this gives us two simple equations for the two perimeters. At each stage there are 2^n sides, the length of each short side S on the inside equals $\sin\theta$ and the length of each short side on the outside T is equal to $\tan\theta$, where $\theta = 2\pi/2^n$.

The total length of the inside perimeter is $nS = n\sin\theta$ and that of the outside is $nT = n\tan\theta$. When we go from θ to $\theta/2$ and n to $2n$, we must compute the new values S' and T' from S and T using the half-angle formulas, and then also multiply by 2 to take account of the change from n to $2n$ for the total circumference.

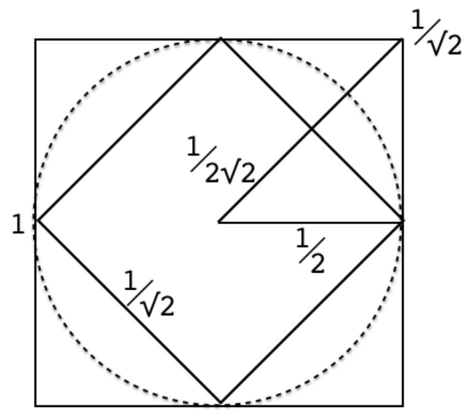
Base case: square

If we go back to the square ($n = 2, 2^n = 4$), then the angle θ is $\pi/4$.

The tangent is $T = \tan\pi/4 = 1$ and the sine is $S = \sin\pi/4 = 1/\sqrt{2}$.

Our formulas say that on the inside, the perimeter is $4S = 4/\sqrt{2} = 2\sqrt{2}$ and on the outside, the perimeter is $4T = 4$.

From simple geometry, we can calculate that the circumscribing square has a side length which is twice the radius of the circle, that is, 1 for our circle with unit diameter, so its perimeter is 4, which checks.



Similarly, an inscribed square can be decomposed into four isosceles right triangles with sides of length $1/2$ and hypotenuse $1/\sqrt{2}$, so the total perimeter is $4/\sqrt{2}$, which also checks.

Now, what we are going to do is to increase n in steps of 1, that increases 2^n by a factor of $2^1 = 2$ each time. Doubling n halves the angle. So all we need is a way to compute trigonometric functions of $\theta/2$, knowing the values for θ, so we can calculate what happens to the perimeter. We already know how to do that.

Half angle formulas

We have derived these elsewhere. Refer to this **chapter**.

The unprimed values refer to angle θ, while the primed ones have angle $\theta/2$.

$$C' = \sqrt{\frac{1}{2}(1 + C)}$$

This can be rearranged (e.g.) to give $2[C']^2 = 1 + C$, which we'll use in a second.

$$S' = \frac{S}{2C'}$$

$$T' = \frac{S'}{C'} = \frac{S}{2[C']^2} = \frac{S}{1+C}$$

So, given S, C and T, first calculate C' and T' and then S'.

To get the perimeters, remember that factor of two from doubling n, the number of sides. The inside perimeter is the sine S' and the outside perimeter is the tangent T'.

Calculation

Above we have for the square that the angle is 45 degrees or $\pi/4$ and the inside (p) and outside (P) perimeters are:

- $p = 4/\sqrt{2}$
- $P = 4$

Also, the $\sin\theta = \cos\theta = 1/\sqrt{2}$ while $\tan\theta = 1$.

one round

$$C' = \sqrt{\frac{1}{2}(1+C)} = \sqrt{\frac{1}{2}(1+\frac{1}{\sqrt{2}})} = 0.92387953$$

This checks out as equal to $\cos \pi/8$.

$$S' = \frac{S}{2C'} = 0.38268343$$

$$T' = \frac{S'}{C'} = 0.41421356$$

Then

$$p = 8S' = 3.06146744$$

$$P = 8T' = 3.31370848$$

I wrote a script to do this:

`https://gist.github.com/telliott99/433a73eb708e25a1d6f6ba5291f48ebf`

Output:

```
> python script.py
round 1
4
2.82842712475
4.0

round 2
8
3.06146745892
3.31370849898

round 3
16
3.12144515226
3.18259787807

...
```

```
round 15
65536
3.14159265239
3.141592656

3.14159265359
>
```