Fermat and Euler's Theorems

theorem

Fermat's Theorem (often called Fermat's "little" Theorem to distinguish it from his "last" theorem) says that for any prime number p and any integer $1 < a < n$

$$a^p \bmod p = a$$

Two equivalent statements are

$$a^{p-1} \bmod p = 1$$

$$(a^p - a) \bmod p = 0$$

The second statement is what we use in analyzing the RSA method, and the third is used in the combinatorial proof of the theorem.

Examples:

```
2^3 mod 3 =   2
2^5 mod 5 =  32 mod 5 = 2
3^5 mod 5 = 243 mod 5 = 3
```

Consider $p = 7$

```
1^6       1 mod 7 = 1
2^6 =    64 mod 7 = 1
3^6 =   729 mod 7 = 1
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4^6 =  4096 mod 7 = 1
5^6 = 15625 mod 7 = 1
6^6 = 46656 mod 7 = 1
```

Here is a table from Laws of Cryptography for $p = 13$, which computes such powers more efficiently (computing mod p at each step).

p	a	a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}
13	2	***2***	***4***	***8***	***3***	***6***	***12***	***11***	***9***	***5***	***10***	***7***	***1***
13	3	***3***	***9***	***1***	3	9	1	3	9	1	3	9	1
13	4	***4***	***3***	***12***	***9***	***10***	***1***	4	3	12	9	10	1
13	5	***5***	***12***	***8***	***1***	5	12	8	1	5	12	8	1
13	6	***6***	***10***	***8***	***9***	***2***	***12***	***7***	***3***	***5***	***4***	***11***	***1***
13	7	***7***	***10***	***5***	***9***	***11***	***12***	***6***	***3***	***8***	***4***	***2***	***1***
13	8	***8***	***12***	***5***	***1***	8	12	5	1	8	12	5	1
13	9	***9***	***3***	***1***	9	3	1	9	3	1	9	3	1
13	10	***10***	***9***	***12***	***3***	***4***	***1***	10	9	12	3	4	1
13	11	***11***	***4***	***5***	***3***	***7***	***12***	***2***	***9***	***8***	***10***	***6***	***1***
13	12	***12***	***1***	12	1	12	1	12	1	12	1	12	1

Table 1.4 Fermat's Theorem for $p = 13$.

`www.cs.utsa.edu/~wagner/lawsbookcolor/laws.pdf`

Let's do the calculation for $a = 7, p = 13$:

```
7**1  =                7
7**2  = 49 - 3(13) = 10
7**3  = 70 - 5(13) =  5
7**4  = 35 - 2(13) =  9
7**5  = 63 - 4(13) = 11
7**6  = 77 - 5(13) = 12
7**7  = 84 - 6(13) =  6
```

```
7**8  = 42 - 3(13) =  3
7**9  = 21 - 1(13) =  8
7**10 = 56 - 4(13) =  4
7**11 = 28 - 2(13) =  2
7**12 = 14 - 1(13) =  1
```

As the theorem says, we cycle around to $7^{12} \bmod 13 = 1$. The number 7 is called a *generator* because its powers generate all the values in the field Z_{13}.

$2, 6$ and 11 are also generators for Z_{13}.

Other values for a have shorter repeats (2, 3, 4 or 6 long). The lengths of such runs are divisors of 12.

proofs of Fermat's Theorem

Note that Fermat didn't actually prove his theorem. Euler proved something more general, the theorem we introduce below, and Fermat is a specific case of that.

There is a beautiful combinatorial proof called the "necklace proof". Here is a write-up:

`http://scienceblogs.com/evolutionblog/2010/04/15/a-combinatorial-proof-of-ferma/`

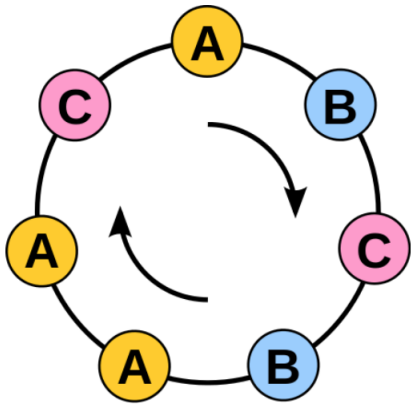
combinatorial proof

We construct a necklace of p beads, choosing from a different colors. We start by threading beads onto a linear piece of string.

Since we can choose any one of a colors, p times, there are a^p possible sequences. However, we wish to exclude arrangements where all the

beads have the same color There are a of these and so the number of arrangements is $a^p - a$.

Now, join the ends of the string. Some sequences become indistinguishable. There are p such cyclic shifts for each arrangement. For example, any cyclic permutation of $ABCBAAC$ looks the same:



Therefore, the total number of arrangements is reduced by a factor of p.

$$n = \frac{a^p - a}{p}$$

The key fact is that n, the number of different arrangements, is clearly an integer. So we have that

$$a^p - a = np$$

If we take the modulus of both sides (mod p) we have the result we seek.

$$(a^p - a) \bmod p = 0$$

proof by induction

This proof is just as simple. We claim that

$$a^p \bmod p = a$$

The base case is $1^p \bmod p = 1$, which is obviously true.

The binomial theorem gives

$$(a+1)^p = a^p + \binom{p}{1}a^{p-1} + +\binom{p}{2}a^{p-2} + \cdots + \binom{p}{p-1}a + 1$$

All of the middle terms have p as a factor, so mod p each is just zero. We have

$$(a+1)^p \bmod p = (a^p + 1) \bmod p$$

So then supposing that $a^p \bmod p = a$, this is just

$$(a+1)^p \bmod p = (a+1) \bmod p$$

This completes the proof.

□

consequence

A consequence of this theorem is that the sequence $a^1,\ a^2,\ a^3 \ldots a^{p-1}$ repeats, so the sequence $a^p, a^{p+1} \ldots a^{2p-1}$ gives exactly the same values.

As the source says:

Because a to a power x mod p always starts repeating after the power reaches $p-1$, one can reduce the power mod $p-1$ and still get the same answer."

Thus no matter how big the power x, let

$$y = x \bmod (p-1)$$

and then

$$a^x \bmod p = a^y \bmod p$$

For example, mod 13:

$$a^{29} \bmod 13 = a^{29 \bmod 12} \bmod 13$$
$$= a^5 \bmod 13$$

Euler

Euler's totient function $\phi(n)$ is defined like so:

$$\phi(n) = n \prod (1 - \frac{1}{p_i})$$

(where the p_i are the prime factors of n). In number theory, ϕ gives the count of the positive integers up to a given integer n that are relatively prime to n (that are not divisors of n).

theorem

Relevant to our study of cryptography, Euler's Theorem says that for an integer $a < p$ (not equal to 1):

$$a^{\phi(n)} \bmod n = 1$$

If n is prime this reduces to

$$\phi(n) = n \prod (1 - \frac{1}{n}) = n(1 - \frac{1}{n}) = n - 1$$

and then

$$a^{n-1} \bmod n = 1$$

showing that Fermat's Theorem is a special case of Euler's Theorem.

Here we are most interested in the situation where n has two large prime factors p and q and then:

$$\phi(n) = pq \ (1 - \frac{1}{p})(1 - \frac{1}{q}) = (p - 1)(q - 1)$$

Furthermore, we have that encryption followed by decryption is $m^{ed} \bmod n = m$. We would like to show that this follows as a consequence of our definitions.

From now on, I will write ϕ for $\phi(n)$. Recall that we set d to be the multiplicative inverse of e mod ϕ.

Write out what's given:

$$\phi = (p-1)(q-1)$$

$$m^{\phi} \bmod n = 1$$

$$ed \bmod \phi = 1$$

Our chained encryption/decryption is $(m^e)^d \bmod n$ or:

$$m^{ed} \bmod n$$

My source says: "similar to Fermat's Theorem, arithmetic in the exponent is taken mod ϕ."

The idea is that since

$$m^{\phi} \bmod n = 1$$

if we are computing any *other* power of m, say ed, we need only compute to the power of ed mod ϕ, because beyond that, the sequence just repeats. (Note: we saw the repetition for Fermat's Theorem, but didn't prove it. We are accepting the source that says it happens here as well, even though n is not prime).

Here we have m to the power ed and then mod pq, and we reduce the power ed mod ϕ since $\phi = (p-1)(q-1)$.

(Assuming m has no common divisors with n):

$$m^{ed} \bmod n = m^{ed} \bmod pq$$

$$= m^{ed \bmod \phi} \bmod pq$$

But of course ed mod $\phi = 1$ so this is just m.

Furthermore, $m^{ed} = m^{de}$, hence the ability to encrypt first with the private key and then decrypt with the public one.