Limit of (1 + 1/n) to the nth power

Our goal for this write-up is to find this limit:

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n$$

Massage it a bit

$$(1 + \frac{1}{n})^n = e^{\ln\ [\ (1+1/n)^n\ ]}$$
$$= e^{n \ln(1+1/n)}$$

so we want

$$= \lim_{n\to\infty} e^{n \ln(1+1/n)}$$

We will show that the limit of the exponent is equal to 1:

$$\lim_{n\to\infty} n \ln(1 + \frac{1}{n}) = 1$$

Rearrange slightly

$$= \lim_{n\to\infty} \frac{\ln(1 + \frac{1}{n})}{1/n}$$

Note that the limit of both numerator and denominator is zero. Therefore, we can apply L'Hopital's Rule. If $f(n)$ is the numerator and $g(n)$ is the denominator

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

then by the chain rule

$$f'(n) = \frac{1}{1 + 1/n} \cdot (-n^{-2})$$

Since $g'(n) = -n^{-2}$, we have just

$$= \lim_{n\to\infty} \frac{1}{1 + 1/n}$$

which is indeed, just equal to 1. We have proved that

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n = e^1 = e$$

alternative

Start by *defining* a function that turns out to have all the properties of the logarithm (we've explored this approach elsewhere):

$$L(x) = \int_1^x \frac{1}{t}\ dt$$

By standard properties of integrals:

$$L(1) = \int_1^1 \frac{1}{t}\ dt = 0$$

Define one more property, namely

$$L(e) = \int_1^e \frac{1}{t}\ dt = 1$$

Now, let t be any number in the interval $[1, 1 + 1/n]$. We're interested in what happens as n gets large. This means that

$$1 \le t \le 1 + \frac{1}{n}$$

We will invert each term and rearrange as well:

$$\frac{1}{1+1/n} \leq \frac{1}{t} \leq 1$$

For each of these terms, we integrate $f(t)dt$ between $t = 1 \to 1 + 1/n$. Remembering that n is just a number and so is $1 + 1/n$ we have:

$$\int_1^{1+1/n} \frac{1}{1+1/n}\, dt \leq \int_1^{1+1/n} \frac{1}{t}\, dt \leq \int_1^{1+1/n} 1\, dt$$

The first term is

$$\int_1^{1+1/n} \frac{1}{1+1/n}\, dt = \frac{1}{1+1/n} \cdot t\, \Big|_1^{1+1/n}$$

$$= \frac{1}{1+1/n} \cdot (1 + 1/n - 1) = \frac{1}{n+1}$$

The second term is

$$\int_1^{1+1/n} \frac{1}{t}\, dt = L(1+1/n)$$

and the third is just $1/n$ so going back to the inequality we have established that

$$\frac{1}{n+1} \leq L(1+1/n) \leq \frac{1}{n}$$

Exponentiate each term:

$$e^{1/n+1} \leq (1+1/n) \leq e^{1/n}$$

Now, take the two inequalities separately. We have

$$e^{1/n+1} \leq (1+1/n)$$

Raise to the power $n + 1$

$$e^1 \leq (1+1/n)^{n+1}$$

Divide by $1 + 1/n$

$$= \frac{e}{1 + 1/n} \leq (1 + 1/n)^n$$

Take the limit as $n \to \infty$ and we see that

$$e \leq \lim_{n \to \infty} (1 + 1/n)^n$$

Similarly for the right-hand inequality:

$$(1 + 1/n) \leq e^{1/n}$$

Raise to the power n

$$(1 + 1/n)^n \leq e$$

So in the limit as $n \to \infty$

$$\lim_{n \to \infty} (1 + 1/n)^n \leq e$$

And since

$$e \leq \lim_{n \to \infty} (1 + 1/n)^n \leq e$$

By the squeeze theorem, the middle term must be *equal* to e

$$\lim_{n \to \infty} (1 + 1/n)^n = e$$