Fractional powers

In this short write-up we'll look at the difference quotient for simple roots of x

$$\lim_{h \to 0} \frac{(x+h)^{1/n} - x^{1/n}}{h}$$

In the case of $\sqrt{x}$, we were able to clean up the numerator by multiplying by the conjugate

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} \quad \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

Giving

$$\frac{x+h-x}{h \sqrt{x+h} + \sqrt{x}}$$

$$\frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$\frac{1}{2\sqrt{x}}$$

in the limit as $h \to 0$.

4th root

That approach does not work (at least not directly) for the cube root, but it does work for the fourth root

$$[\ (x+h)^{1/4} - x^{1/4}\ ]\ [\ (x+h)^{1/4} + x^{1/4}\ ] = [\ (x+h)^{1/2} - x^{1/2}\ ]$$

We get back to the difference of square roots (in the numerator) and then just repeat, doing what we did above. The denominator is finally

$$\frac{1}{[(x+h)^{1/4} + x^{1/4}][(x+h)^{1/2} + x^{1/2}]}$$

and in the limit $\lim \to 0$

$$\frac{1}{[\ 2x^{1/4}\ ]\ [\ 2x^{1/2}\ ]} = \frac{1}{4}x^{-3/4}$$

which matches the power rule. And it will work for any root that is a power of 1/2.

Cube root

A slight modification will do the same thing for the cube root. The necessary factor is

$$(a^{1/3} - b^{1/3})\ (a^{2/3} + a^{1/3}b^{1/3} + b^{2/3})$$

The inner terms cancel and we get

$$a - b$$

So we have

$$[\ (x+h)^{1/3} - x^{1/3}\ ]\ [\ (x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}\ ] = \ x + h - x$$

and after multiplying out the difference quotient (and subtracting x and dividing by h) we will have

$$\frac{1}{(x+h)^{2/3} - (x+h)^{1/3}x^{1/3} + x^{2/3}}$$

in the limit as $h \to 0$.

$$\frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}$$

General method

There are ways to solve this kind of equation for more complicated powers, and I will show one below. However, there are better solutions to the general problem of $x^{m/n}$. First, we can use Newton's expansion for the binomial, which has been proved for rational powers (though I don't think Newton actually proved it). And there is another very elegant proof that uses implicit differentiation. I have written about those elsewhere.

Cleaning up $x^{m/n}$

The general method is essentially the same as for the cube root. If we had

$$(a^{m/n} - b^{m/n})(a^{n/m} + a^{m/n}b^{m/n} + b^{n/m})$$
$$= a + a^{2m/n}b^{m/n} + a^{m/n}b^{n/m} - a^{n/m}b^{m/n} - a^{m/n}b^{2m/n} - b$$

which really looks like a mess! But substitute $x + h$ for a and x for b and what do we get? Let's do the denominator first. We will have

$$h\ (a^{n/m} + a^{m/n}b^{m/n} + b^{n/m})$$

$$h\ [\ (x+h)^{n/m} + (x+h)^{m/n}x^{m/n} + x^{n/m}\ ]$$

We'll be canceling the leading h, as you'll see in a second. The numerator is really a bit of a mess, but we have

$$= (x+h) + (x+h)^{2m/n}x^{m/n} + (x+h)^{m/n}x^{n/m} - (x+h)^{n/m}x^{m/n} - (x+h)^{m/n}x^{2m/n} - x$$

Let's continue (boldly)

$$= h + (x+h)^{2m/n}x^{m/n} + (x+h)^{m/n}x^{n/m} - (x+h)^{n/m}x^{m/n} - (x+h)^{m/n}x^{2m/n}$$

Now, separate this into two parts. Take that first h over the denominator from above, which cancels to give

$$\frac{1}{(x+h)^{n/m} + (x+h)^{m/n}x^{m/n} + x^{n/m}}$$

The second term, which is added to this one, has the same denominator (but without canceling h) and its numerator is what we had just before

$$(x+h)^{2m/n}x^{m/n} + (x+h)^{m/n}x^{n/m} - (x+h)^{n/m}x^{m/n} - (x+h)^{m/n}x^{2m/n}$$

Now what will happen to this in the limit as $h \to 0$? This becomes

$$x^{2m/n}x^{m/n} + x^{m/n}x^{n/m} - x^{n/m}x^{m/n} - x^{m/n}x^{2m/n}$$

$$x^{3m/n} + x - x - x^{3m/n}$$

This is just 0, so the whole second term is 0, and we are left with

$$\frac{1}{x^{n/m} + x^{m/n}x^{m/n} + x^{n/m}}$$

$$\frac{1}{x + x^{2n/m}}$$