Approximation

Some functions are easy to compute, while some are harder. Any polynomial in x is easy

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

On the other hand, how would we compute $f(x) = e^x$? It's not difficult for $x = 0, 1, 2 \cdots$, but what about for $x = 0.1$? To begin with, let's try finding a polynomial function of x that gives a value *approximately equal* to that for the function e^x in the neighborhood of the value $x = 0$.

The very least we should require is that the value of $g(0) = f(0)$.

$$f(x) = e^x \approx f(0) = e^0 = 1$$

For $g(x)$ to give the best answer

$$g(0) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 + \cdots = a_0$$

Evidently, if $g(0) = f(0)$ then a_0 must be equal to 1.

The next step is to look for a linear approximation. How much will $f(x)$ change from $f(0)$ for x near 0? It will change by approximately $f'(0)$ times x. So a better approximation is that

$$f(x) = e^x \approx f(0) + f'(0)\ x$$

If $f(x) = e^x$ then $f'(x) = e^x$ and $f'(0) = 1$ so

$$f(x) \approx 1 + x$$

For $g(x)$ to give the best answer

$$g(x) = a_0 + a_1(x) = 1 + a_1(x)$$

Evidently, $a_1 = 1$ as well. We want the slope of $g(x)$ to be equal to the slope of $f(x)$ at $x = 0$.

For a quadratic approximation, the secret is that the second derivatives must match. $f''(x) = e^x$ at $x = 0$ is still equal to 1, but $g''(x)$ at $x = 0$ is equal to $2a_2$. So we have

$$f''(0) = 1 = g''(0) = 2a_2$$

$$a_2 = \frac{1}{2}$$

$$g(x) = 1 + x + \frac{1}{2}x^2$$

Let's stop and see how accurate this is. To ten places:

$$e^{0.1} = 1.105170918$$

Compare

$$g(0.1) = 1 + 0.1 + \frac{1}{2}(0.1)^2 = 1 + 0.1 + 0.005000000 = 1.105$$

To be even more accurate, we need the third derivatives to match. Notice that when we differentiate x^3 twice we get first a factor of 3 and then a factor of 2, i.e. 3!

$$f'''(0) = 1 = g'''(0) = 3!\ a_3$$

$$a_3 = \frac{1}{3!}$$

$$g(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$$

Notice the pattern. We have

$$g(x) = \sum_{k=0}^{k=\infty} \frac{1}{k!}\ x^k$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

When we use the whole (infinite) series, :), we have an *exact* approximation

$$e^0 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

In general, we have Taylor's Series as an approximation for $f(x)$ near $x = a$

$$f(x) \approx f(a) + f'(a)\ (x-a) + \frac{f''(x)}{2!}(x-a)^2 + \frac{f'''(x)}{3!}(x-a)^3 + \cdots$$