Convergence tests

The most important series is the geometric series:

$$s = \sum_{k=0}^{\infty} x^k$$

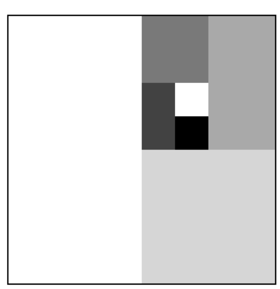
A famous example is $x = 1/2$

$$s = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

If we ignore the first term, and keep track of the sum after a number of terms, we see that at every step, the next step is to add precisely one-half of the difference between the current sum and the number 1. As a series, the cumulative sum is

$$\frac{1}{2},\ \frac{3}{4},\ \frac{7}{8},\ \frac{15}{16},\ \dots$$

The limit is clearly equal to 1. Here is a visual proof



If we assume that the series has a finite sum we can do the following:

$$s = 1 + x + x^2 + x^3 + \dots$$

$$sx = x + x^2 + x^3 + \dots$$

$$s - sx = 1$$

$$s = \frac{1}{1-x}$$

In the case $x = 1/2$, where we include the first term of 1, we find that $s = 2$ and this is consistent with what we had before.

Another way to check this is to multiply out

$$1 = (1-x)(1 + x + x^2 + x^3 + \dots)$$

On the right-hand side, for each term in the second factor, we multiply by $1-x$. Provided $x < 1$ the terms x^n as n gets larger become smaller and smaller, and finally negligible.

Or use distributivity with the first factor. We have

$$1 = (1 + x + x^2 + x^3 + \dots) - x(1 + x + x^2 + x^3 + \dots)$$

Each term in the first series (after the 1), has a matching identical term with the factor of -1 in the series on the right, which will cancel. However, this depends on x^n getting smaller and smaller as $n \to \infty$, which only happens for $x < 1$. In the limit, everything cancels except the 1 in front.

The more formal way to do this is to compute the sum to n places, for finite n:

$$s_n = 1 + x + x^2 + x^3 + \dots + x^n$$

This sum is definitely finite. Then we have

$$s_n x = x + x^2 + x^3 + \dots + x^{n+1}$$

$$s_n - s_n x = 1 - x^{n+1}$$
$$s_n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

Now, in the limit as $n \to \infty$, we see that the second term vanishes $\iff |x| < 1$. For $x = 1$ the sum is not defined. For $x > 1$ the sum goes to infinity. We say it diverges. For $x < -1$, the sum alternates sign, but each term gets larger in absolute value. This case is also divergent. We say that the geometric series has a radius of convergence $|x| < 1$.

series related to the geometric series

$$s = 1 + x + x^2 + x^3 + \dots$$

Differentiate

$$1 + 2x + 3x^2 + 4x^3 + \dots$$
$$\frac{d}{dx}\left[\ \frac{1}{1-x}\ \right] = \frac{1}{(1-x)^2}$$

Check by multiplying out

$$(1-x)(1 + 2x + 3x^2 + 4x^3 + \dots)$$
$$= 1 + 2x - x + 3x^2 - 2x^2 + 4x^3 - 3x^3 + \dots)$$
$$= 1 + x + x^2 + x^3 + \dots$$

And since

$$(\frac{1}{1-x})(\frac{1}{1-x}) = \frac{1}{(1-x)^2}$$
$$(1 + x + x^2 + \dots)(1 + x + x^2 + \dots)$$
$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$$

Integrate

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$-\ln|1-x| = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

Substitute $-x$ for x

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

Integrate

$$\ln|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

This series converges pretty slowly.

Tests for convergence

Series like the Taylor series can be very helpful in approximating a function. Here we review some common tests to see whether the sum of an infinite series converges to a finite limit, or instead diverges.

Start with the geometric series

$$\sum_{k=0}^{\infty} x^k$$

Suppose we compute the sum of a number of terms n

$$s_n = x^0 + x^1 + x^2 + \dots + x^n$$

Since this is a finite series, the sum exists so

$$x s_n = x^1 + x^2 + \dots + x^{n+1}$$

$$(1-x)s_n = 1 + x^{n+1}$$

$$s_n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}, \quad x \neq 1$$

Clearly, if $x > 1$ then the second term is positive and $x^n \to \infty$ as n gets large. If $x < -1$ then the second term alternates in sign and its absolute value gets very large as n gets large. For $|n| < 1$, as $n \to \infty$, the second term vanishes and we have

$$s_n = \frac{1}{1-x}$$

For example,

$$\sum_{k=0}^{\infty} (\frac{1}{2})^k = \frac{1}{1-1/2} = 2$$

$$\sum_{k=0}^{\infty} (\frac{1}{3})^k = \frac{1}{1-1/3} = \frac{3}{2}$$

and so on.

A second famous series is the harmonic series

$$\sum_{k=0}^{\infty} \frac{1}{k^p}$$

especially with $p = 1$

$$\sum_{k=0}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

This series diverges, as we'll see below. Now let's look at some tests of convergence.

Divergence test

The first test requires that the limit of the individual terms a_k must tend to zero

$$\lim_{k \to \infty} a_k = 0$$

if not, then the sum diverges. For example,

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots$$

This clearly diverges, since

$$\lim_{k \to \infty} a_k = 1$$

Or

$$\begin{aligned} &\lim_{k \to \infty} \frac{k}{2k+1}, \quad k \in \{1, 2, \dots\} \\ &= \lim_{k \to \infty} \frac{1}{2 + 1/k} = \frac{1}{2} \neq 0 \end{aligned}$$

Another example is the harmonic series

$$\lim_{k \to \infty} \frac{1}{k} = 0, \quad k \in \{1, 2, \dots\}$$

Despite passing this test, the harmonic series diverges. Thus, a pass is necessary but not sufficient.

Integral test

The integral test says that a (well-behaved) function $f(x)$

$$\int_1^\infty f(x)\ dx$$

converges $\iff$

$$\sum_{k=1}^\infty f(k)$$

also converges. The function must be continuous and ..

Let's apply this test to the harmonic series

$$\sum_{k=0}^\infty \frac{1}{k}$$

We have

$$\int_1^\infty x \ dx = \ln|x| \ \Big|_1^\infty$$

but the upper bound has the limit

$$\lim_{k\to\infty} \ln|k| = \infty$$

In general, for

$$\sum_{k=0}^\infty \frac{1}{k^p}$$

if $p > 1$, the sum converges, but not otherwise:

$$\int_1^\infty x \ dx = \frac{1}{1-p} x^{1-p} \ \Big|_1^\infty$$

For $p > 1$

$$\lim_{x\to\infty} x^{1-p} = 0$$

On the other hand

$$\int_1^\infty \frac{1}{n^2} \ dn = -\frac{1}{n} \ \Big|_1^\infty = 0 - -1 = 1$$

so the $\sum 1/n^2$ converges.

Comparison test

If we compare a series and a convergent series and the test series is smaller term-by-term, then it also converges. Similarly, if a series is larger than a divergent series when compared term-by-term, it also diverges. Any finite number of terms from the beginning of a series may be disregarded before starting the comparison.

Since

$$\sum_{k=0}^{\infty} \frac{1}{k^2}$$

converges, so does

$$\sum_{k=0}^{\infty} \frac{1}{k^2+10}$$

And since

$$\sum_{k=0}^{\infty} \frac{1}{k}$$

diverges, so does

$$\sum_{k=0}^{\infty} \frac{1}{\ln|k+1|}$$

since for $k > 2$

$$\ln|k+1| < k$$

so

$$\frac{1}{\ln|k+1|} > \frac{1}{k}$$

Ratio test

Consider

$$\sum_{k=0}^{\infty} a_k, \quad a_k > 0$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L$$

$$f(x) = \begin{cases} L < 1 & : \text{converges} \\ L > 1 & : \text{diverges} \\ L = 1 & : \text{inconclusive} \end{cases}$$

As an example

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

Check

$$\lim_{k\to\infty} \frac{1/(k+1)!}{1/k!} = \frac{1}{k+1} < 0$$

This one is also easily checked by the comparison test since

$$k! > k^2, \quad k > 3$$

Since $1/k^2$ converges, so does $1/k!$.

Or

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

$$\lim_{k\to\infty} \frac{1/n+1}{1/n} = \frac{n}{n+1} = 1$$

so the test is inconclusive.