Integrating complex functions—Karkar

This one is from Ritvik Karkar on Youtube: Computing Line Integrals

Suppose $f(z) = |z|^2$ and the curve is from $2 + 0i \to 3 + 1i$. We should *always* draw a picture, but I'm going to skip it for this one. This is a straight line which goes across one unit and up one unit.

My way to do this is to say:

$$|z|^2 = zz* = (x + iy)(x - iy) = x^2 + y^2$$

That is

$$u(x, y) = x^2 + y^2$$

$$v(x, y) = 0$$

$$I = \int u\ dx - \int v\ dy + i\ [\ \int v\ dx + \int u\ dy\ ]$$

$$= \int u\ dx + i \int u\ dy$$

Furthermore, along the curve, $y = x - 2$ so $dx = dy$ and

$$x^2 + y^2 = x^2 + (x - 2)^2 = 2x^2 - 4x + 4$$

Since $x = 2 \to 3$ our integral is

$$\int_2^3 (2x^2 - 4x + 4)\ dx + i \int_2^3 (2x^2 - 4x + 4)\ dx$$

$$= (1+i) \int_2^3 (2x^2 - 4x + 4)\ dx$$

$$= (1+i)\ [\ \frac{2}{3}x^3 - 2x^2 + 4x\ ]\ \Big|_2^3$$

The term in brackets is

$$\frac{2}{3}(3^3 - 2^3) - 2(3^2 - 2^2) + 4$$

$$= \frac{38}{3} - 10 + 4 = \frac{20}{3}$$

So the final answer is $(1+i) \cdot 20/3$.

alternatively

His method is to first start with a parametrization with $t = [0,1]$. So the curve is "start point plus (end point - start point) times t". The start point is $2 + 0i$, while the end minus the start is $(1+i)$. Hence

$$\gamma(t) = (2 + 0i) + (1+i)t = (2+t) + it$$

So everywhere along the curve $z = \gamma(t)$. Now evaluate the integral as

$$\int_\gamma f(\gamma(t))\ \gamma'(t)\ dt$$

The function is

$$|z|^2 = |\gamma(t)|^2$$

$$= (2+t)^2 + t^2 = 4 + 4t + 2t^2$$

(just using the complex conjugate).

The derivative is

$$\gamma'(t) = (1+i)$$

Our integral is

$$\int_0^1 (4 + 4t + 2t^2)(1 + i)\ dt$$

Notice that $(1 + i)$ is just a number

$$= (1 + i) \int_0^1 4 + 4t + 2t^2\ dt$$

The integral is

$$4t + 2t^2 + \frac{2}{3}t^3 \Bigg|_0^1$$

Evaluation is easier because the limits are $0 \to 1$:

$$= 6 + \frac{2}{3} = \frac{20}{3}$$

Putting it all together we have

$$(1 + i) \cdot \frac{20}{3}$$

another example

Another typical parametrization is a circle or part of a circle. Suppose we are on the unit circle starting at 1 and curving counter-clockwise through i and ending at -1.

Use t as our parameter, so

$$\gamma(t) = e^{it}$$

$$\gamma'(t) = ie^{it}$$

with $t = [0, \pi]$. Looking at the function $f(z) = z^2$ (subtly different than last time) we have

$$\int_\gamma z^2\ dz = \int_0^\pi e^{i2t}\ i\ e^{it}\ dt$$

$$= i \int_0^{\pi} e^{i3t} \ dt$$

$$= i \ \frac{1}{3i} \ e^{i3t} \ \Big|_0^{\pi} = \frac{1}{3} \ [\ e^{i3\pi} - 1 \ ]$$

where

$$e^{i3\pi} = \cos 3\pi + i \sin 3\pi = -1$$

Hence the final answer is $-2/3$.