Elementary complex functions

First of all, write

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \end{aligned}$$

We can thus visualize the complex exponential as having modulus e^x and argument y.

Reversing Euler:

$$\begin{aligned} &= e^x(\cos y + i \sin y) \\ &= e^x \cos y + ie^x \sin y \end{aligned}$$

So the real part of e^z is

$$u(x, y) = e^x \cos y$$

$$\begin{aligned} u_x &= e^x \cos y \\ u_y &= -e^x \sin y \end{aligned}$$

and the imaginary part is

$$v(x, y) = e^x \sin y$$

$$\begin{aligned} v_x &= e^x \sin y \\ v_y &= e^x \cos y \end{aligned}$$

Hence

$$u_x = v_y$$

$$u_y = -v_x$$

The CRE conditions are satisfied and the complex exponential e^z is analytic. (Which, according to Shankar, we could have predicted, since it depends only on z and not on $z*$).

Notice also that (evaluating the derivative along $\Delta y = 0$:

$$f'(z) = u_x + iv_x$$

$$= e^x \cos y + ie^x \sin y = z$$

The exponential is its own derivative.

Which is good because we want our definitions for the complex functions to give the standard results when z has only a real part, i.e. when $y = 0$.

Now, once more we recall Euler's formula (for a real variable θ or x):

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{ix} = \cos x + i\sin x$$

Substitute $-x$ for x:

$$e^{-ix} = \cos -x + i\sin -x$$

$$= \cos x - i\sin x$$

By addition and subtraction we obtain:

$$2\cos x = e^{ix} + e^{-ix}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

and

$$2i\sin x = e^{ix} - e^{-ix}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

We will also need the hyperbolic sine and cosine later so let's just remind ourselves:

$$2\cosh x = e^x + e^{-x}$$

$$2\sinh x = e^x - e^{-x}$$

The derivative of the complex exponential is as we would hope and expect:

$$\frac{d}{dz}e^z = e^z$$

This can be proved by using a Taylor series. Shankar says to define e^z in the same way as e^x. That is:

$$e^x = \sum_0^\infty \frac{x^n}{n!}$$

which we know converges because

$$|x| < R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

where $a_n = 1/n!$. So

$$e^z = \sum_0^\infty \frac{z^n}{n!}$$

and again we see that

$$\frac{d}{dz}\; e^z = e^z$$

differentiating the series term by term.

The complex exponential

$$e^z = e^x e^{iy}$$

has some properties that are not shared with the real exponential. As we saw before, the angle $\theta + 2\pi = \theta$ (and $2\pi = 0$), so any number is really a family of numbers with different $\theta + 2\pi k$ for integer k.

In particular, e^z is periodic with a period of $2\pi i$. Additionally, it is possible for e^z to be negative. Consider that it is possible that

$$e^z = -1$$

as follows. Let $z = 0 + i\pi$. Then

$$e^x = e^0 = 1$$

and

$$e^{iy} = e^{i\pi} = -1$$

So

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y) = 1(-1) = -1$$

On the other hand, e^z *cannot be zero.*

$$e^z = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y)$$

For $x \in \mathbb{R}$, $e^x > 0$ so the only way this could be zero would be if we can find a y such that $\sin y$ and $\cos y$ were both zero. Since there is no such y, we conclude that e^z cannot be equal to zero.