Fibonacci - Linear Algebra

If we want to find the nth Fibonacci number

$$F = 1, 1, 2, 3, 5, 8, 13, 21 \cdots$$

Binet's formula will do it

$$F_n = \frac{1}{\sqrt{5}}(\phi_1^n + \phi_2^n)$$

where ϕ_1 and ϕ_2 are the roots of $x^2 - x - 1 = 0$, as we saw in the first write-up on these numbers. Here is a matrix that also generates the Fibonacci numbers

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad M * M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad M * M * M = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

If we do M^{10} we get

$$\begin{bmatrix} 55 & 89 \\ 89 & 144 \end{bmatrix}$$

The formal statement of the relationship is that

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = M \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix} = M^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

By investigating the factorization of M, we can see where Binet's formula comes from.

Eigenvalues

In linear algebra, if we want to do M^n or as a specific example, M^{100}, we first find the eigenvalues and eigenvectors of the matrix. The characteristic equation is

$$(-\lambda)(1 - \lambda) - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

which is exactly the same as we had before. The solutions of this equation are the golden mean ϕ_1 and ϕ_2. These are

$$\phi_1 = \frac{1}{2}(1+\sqrt{5})$$

$$\phi_2 = \frac{1}{2}(1-\sqrt{5})$$

Useful properties

These are perhaps not so easy to see right away, but they check out

$$\phi_1\phi_2 = -1$$

$$1-\phi_1 = \phi_2$$

$$1-\phi_2 = \phi_1$$

Also, since $\lambda^2 = 1+\lambda$

$$\phi_1^2 = 1+\phi_1$$

$$\phi_2^2 = 1+\phi_2$$

Eigenvectors

So we have ϕ_1 and ϕ_2 as the eigenvalues of M, now we have to find the eigenvectors. One way is to solve

$$(M-\lambda I)\mathbf{v} = 0$$

For both ϕ_1 and ϕ_2 the process is the same. The matrix is

$$\begin{bmatrix} -\phi & 1 \\ 1 & 1-\phi \end{bmatrix}$$

So we have two equations

$$-\phi x + y = 0$$

$$x + (1-\phi)y = 0$$

so

$$y = \phi x$$

$$x + (1-\phi)\phi x = 0$$

$$x + \phi x - \phi^2 x = 0$$

$$\phi^2 = 1 + \phi$$

$$x + \phi x - (1+\phi)x = 0$$

$$x(1+\phi) - x(1+\phi) = 0$$

x can be anything, so why not $x = 1$

$$y = \phi x = \phi$$

These solutions are the same for both eigenvalues. And we can see that this works, that the eigenvectors are $< 1, \phi_1 >$ and $< 1, \phi_2 >$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ 1+\phi_1 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_1^2 \end{bmatrix} = \phi_1 \begin{bmatrix} 1 \\ \phi_1 \end{bmatrix}$$

and similarly

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \phi_2 \\ 1+\phi_2 \end{bmatrix} = \begin{bmatrix} \phi_2 \\ \phi_2^2 \end{bmatrix} = \phi_2 \begin{bmatrix} 1 \\ \phi_2 \end{bmatrix}$$

The matrix of eigenvectors is

$$S = \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{bmatrix}$$

The inverse of S is obtained by getting the determinant $(-\sqrt{5})$, multiplying by its inverse, switching a and d, and negating b and c

$$S^{-1} = -\frac{1}{\sqrt{5}} \begin{bmatrix} \phi_2 & -1 \\ -\phi_1 & 1 \end{bmatrix}$$

So the complete factorization of M is

$$S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix} (-\frac{1}{\sqrt{5}}) \begin{bmatrix} \phi_2 & -1 \\ -\phi_1 & 1 \end{bmatrix}$$

We should just check this

$$S\Lambda = \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1 & \phi_2^2 \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1+\phi_1 & 1+\phi_2 \end{bmatrix}$$

$$S\Lambda S^{-1} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1+\phi_1 & 1+\phi_2 \end{bmatrix} (-\frac{1}{\sqrt{5}}) \begin{bmatrix} \phi_2 & -1 \\ -\phi_1 & 1 \end{bmatrix}$$

$$S\Lambda S^{-1} = (-\frac{1}{\sqrt{5}}) \begin{bmatrix} 0 & \phi_2 - \phi_1 \\ \phi_2 - \phi_1 & \phi_2 - \phi_1 \end{bmatrix} = (-\frac{1}{\sqrt{5}}) \begin{bmatrix} 0 & -\sqrt{5} \\ -\sqrt{5} & -\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Exponentiation

To raise the matrix M to the power M^n, we first compute

$$D^n = \begin{bmatrix} \phi_1^n & 0 \\ 0 & \phi_2^n \end{bmatrix}$$

Actually, here we will do D^{n-1} because we'll get another factor of ϕ_1 and ϕ_2, as you will see.

$$D^{n-1} = \begin{bmatrix} \phi_1^{n-1} & 0 \\ 0 & \phi_2^{n-1} \end{bmatrix}$$

Then bring back the eigenvectors by first doing

$$\begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} \phi_1^{n-1} & 0 \\ 0 & \phi_2^{n-1} \end{bmatrix} = \begin{bmatrix} \phi_1^{n-1} & \phi_2^{n-1} \\ \phi_1^n & \phi_2^n \end{bmatrix}$$

Now multiply this times the inverse

$$\begin{bmatrix} \phi_1^{n-1} & \phi_2^{n-1} \\ \phi_1^n & \phi_2^n \end{bmatrix} (-\frac{1}{\sqrt{5}}) \begin{bmatrix} \phi_2 & -1 \\ -\phi_1 & 1 \end{bmatrix} =$$

This is too complicated! So instead, we look forward to the last step, where we will multiply

$$M^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

to get the numbers we seek (F_n and F_{n+1}).

If we do

$$P^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

now, it will simplify things.

$$(-\frac{1}{\sqrt{5}}) \begin{bmatrix} \phi_2 & -1 \\ -\phi_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-\frac{1}{\sqrt{5}}) \begin{bmatrix} \phi_2 - 1 \\ -\phi_1 + 1 \end{bmatrix} = (-\frac{1}{\sqrt{5}}) \begin{bmatrix} -\phi_1 \\ \phi_2 \end{bmatrix}$$

The last step follows from the fact (see useful properties, above) that

$$\phi_2 - 1 = -\phi_1$$

$$1 - \phi_1 = \phi_2$$

Now, finally

$$P D^{n-1} D^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-\frac{1}{\sqrt{5}}) \begin{bmatrix} \phi_1^{n-1} & \phi_2^{n-1} \\ \phi_1^n & \phi_2^n \end{bmatrix} \begin{bmatrix} -\phi_1 \\ \phi_2 \end{bmatrix} = (\frac{1}{\sqrt{5}}) \begin{bmatrix} \phi_1^n + \phi_2^n \\ \phi_1^{n+1} + \phi_2^{n+1} \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$$

That is

$$F_n = \frac{1}{\sqrt{5}}(\phi_1^n + \phi_2^n)$$

which is Binet's formula!