# Integration of $\sin x \cos x$

Integrals involving the double-angle formula can be tricky because there may be two different answers which *are both right* but can be seen to be equivalent only after some manipulation. Consider

$$\int \sin x \cos x \; dx$$

A simple answer is to substitute $u = \sin x$, then obviously we have $\int u \; du$ so the answer is

$$= \frac{1}{2} \sin^2 x + C$$

which is easily checked by differentiation. However, on an exam they may try some trickery like this with the double-angle formula:

$$\sin x \cos x = \frac{1}{2} \sin(2x)$$

So the integral is

$$= \int \frac{1}{2} \sin(2x) \; dx$$

$$= -\frac{1}{4} \cos(2x) + C$$

Is it really true that

$$\frac{1}{2}\sin^2 x \stackrel{?}{=} -\frac{1}{4}\cos(2x)$$

We can show that these are equal. The double-angle formula for cosine is:

$$\begin{aligned}\cos(2x) &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x\end{aligned}$$

substituting

$$\begin{aligned}-\frac{1}{4}\cos(2x) + C &= -\frac{1}{4}(1 - 2\sin^2 x) + C \\ &= -\frac{1}{4} + \frac{1}{2}\sin^2 x + C\end{aligned}$$

And now we see that they are the same, we just have to remember that the $C$ in the first answer *is not the same* as the $C$ in the second answer.

Try checking the second answer by differentiation:

$$\begin{aligned}&\frac{d}{dx}\ -\frac{1}{4}\cos(2x) \\ &= \frac{1}{2}\ \sin(2x) \\ &= \sin x \cos x\end{aligned}$$

**cosine squared**

We've solved this before, I thought I'd just repeat it here:

$$\int \cos^2 x \ dx$$

Start with

$$\begin{aligned}\cos(2x) &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1\end{aligned}$$

Thus

$$\cos^2 x = \frac{1}{2}(\cos(2x) + 1)$$

so the integral is

$$= \frac{1}{2}\int (\cos(2x) + 1)$$

$$= \frac{1}{2} \left( \frac{1}{2} \ \sin(2x) + x\right) + C$$

But this also has a second version. The simplest way is just to see what happens when we differentiate

$$\begin{aligned}\frac{d}{dx}\sin x\cos x &= -\sin^2 x + \cos^2 x \\ &= 2\cos^2 x - 1\end{aligned}$$

Hence

$$\sin x\cos x = -x + 2\int cos^2 x \ dx$$

$$\int cos^2 x \ dx = \frac{1}{2}(x + \sin x\cos x) + C$$

(writing the constant now).

Our two answers must be the same, somehow, within a constant:

$$\frac{1}{2}\ (\ \frac{1}{2}\ \sin(2x) + x) \stackrel{?}{=} \frac{1}{2}(x + \sin x \cos x)$$

$$\frac{1}{2}\ \sin(2x) + x \stackrel{?}{=} (x + \sin x \cos x)$$

$$\frac{1}{2}\ \sin(2x) = \sin x \cos x$$

The double-angle formula, again.