$\hat{\mathbf{n}}\ dS$ six ways

plane

In the xy-plane things are really easy. The normal vector points straight up. It is $\hat{\mathbf{k}}$ or

$$\hat{\mathbf{n}} = <0, 0, 1>$$

Similarly, the surface element is just dA, so we have

$$\hat{\mathbf{n}}\ dS = <0, 0, 1>\ \ dx\ dy$$

cylinder

On the surface of a cylinder the normal vector points straight out (no z component). If we look down from the top at the circular cross-section, the normal vector points out with components

$$\mathbf{n} =< x, y, 0 >$$

The length of $\mathbf{n}$ is a, the radius of the cylinder, so

$$\hat{\mathbf{n}} = \frac{1}{a}\ < x, y, 0 >$$

The surface area element in the vertical direction is just Δz, while in the horizontal direction it is r or a times $\Delta\theta$, thus

$$\Delta S = a\ \Delta\theta\ \Delta z$$

$$dS = a\ d\theta\ dz$$

$$\hat{\mathbf{n}}\ dS = < x, y, 0 >\ \ d\theta\ dz$$

Although it seems weird to mix x, y with θ, the idea is to keep things like this until we do the dot product with the field as will usually happen.

sphere

On the surface of a sphere, the normal vector again points straight out. It is

$$\mathbf{n} =< x, y, z >$$

as we've seen before. If the sphere has radius a, then the length of $\mathbf{n}$ is a, and

$$\hat{\mathbf{n}} = \frac{1}{a}\ \ < x, y, z >$$

The surface of the sphere is parametrized by just ϕ and θ (no r since it is fixed $r = a$). Looking down at the horizontal circular cross-section for a given ϕ, the radius of that circle is $a \sin\phi$, so the horizontal component of the surface area element is $a \sin\phi\ \Delta\theta$. The vertical component is a great circle (radius $r = a$), so its length is just $a\ \Delta\phi$.

$$\Delta S = a^2\ \sin\phi\ \Delta\phi\ \Delta\theta$$

$$dS = a^2\ \sin\phi\ d\phi\ d\theta$$

Let's just check:

$$\int dS = \int_0^{2\pi} \int_0^{\pi} a^2\ \sin\phi\ d\phi\ d\theta$$

$$= 2\pi a^2 \int_0^{\pi}\ \sin\phi\ d\phi$$

$$= 2\pi a^2 \ [\ -\cos\phi \ \Big|_0^\pi \ ]$$

$$= 4\pi a^2$$

and

$$\hat{\mathbf{n}}\ dS = a \ \ < x, y, z > \ \ \sin\phi \ d\phi \ d\theta$$

graph of a function

If our surface is the graph of a function $g(x, y)$ then we can just remember the formula

$$\hat{\mathbf{n}}\ dS = < -f_x, -f_y, 1 > \ \ dx \ dy$$

If we forget and need to work it out, the deal is that we get a linear approximation to the plane surface using f_x and f_y so that

$$\mathbf{u} =< 1, 0, f_x > \ \ dx$$

$$\mathbf{v} =< 0, 1, f_y > \ \ dy$$

The cross-product $\mathbf{u} \times \mathbf{v}$ gives

$$\mathbf{N} = < -f_x, -f_y, 1 >$$

The length of $\mathbf{N}$ is

$$|\mathbf{N}| = \sqrt{f_x^2 + f_y^2 + 1}$$

so

$$\hat{\mathbf{n}} = \frac{\mathbf{N}}{|\mathbf{N}|} = \ \frac{< -f_x, -f_y, 1 >}{\sqrt{f_x^2 + f_y^2 + 1}}$$

However, dS is larger than its shadow in the xy-plane by exactly this same factor

$$dS \ \ \cos\theta = dA$$

where

$$\cos\theta = \frac{\mathbf{n}\cdot\hat{\mathbf{k}}}{|\mathbf{n}|\ |\mathbf{k}|} = \frac{1}{\sqrt{f_x^2+f_y^2+1}}$$

Hence

$$\hat{\mathbf{n}}\ dS = \hat{\mathbf{n}}\ \frac{1}{\cos\theta}\ dA = \frac{< -f_x, -f_y, 1 >}{\sqrt{f_x^2+f_y^2+1}}\ \sqrt{f_x^2+f_y^2+1}\ dA$$

$$= < -f_x, -f_y, 1 >\ \ dA$$

$$= < -f_x, -f_y, 1 >\ \ dx\ dy$$

Depending on whether **n** points up or down we may change the sign.

parametrization (parameterization)

More generally, we may have only a parametrization of the surface (it's not an explicit function $f(x, y)$).

$$S = \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

Then

$$\hat{\mathbf{n}}\ dS = |\mathbf{r_u} \times \mathbf{r_v}|\ du\ dv$$

normal vector

Auroux has a last example, in which we only know a normal vector **N** to the surface S. Examples include a plane

$$ax + by + cz = d$$

$$\mathbf{N} = < a, b, c >$$

or S is given by

$$g(x, y, z) = 0$$

$$\mathbf{N} = \nabla g$$

Then

$$dS = \frac{1}{\cos\theta}\ dA = \frac{|\mathbf{N}|}{\mathbf{N} \cdot \hat{\mathbf{k}}}\ dA$$

$$\hat{\mathbf{n}}\ dS = \frac{|\mathbf{N}|\ \hat{\mathbf{n}}}{\mathbf{N} \cdot \hat{\mathbf{k}}}\ dA$$

As Auroux says, what happens if I take the unit normal $\mathbf{n}$, and I multiply it by the length of my other normal $|\mathbf{N}|$? It's just $\mathbf{N}$.

$$\hat{\mathbf{n}}\ dS = \frac{\mathbf{N}}{\mathbf{N} \cdot \hat{\mathbf{k}}}\ dA$$

And again, this is "within sign", depending on how $\hat{\mathbf{n}}$ is oriented.