Title

Rotation of coordinates

The simplest way to get the formulas is to think about how to rotate the unit vectors by an angle ϕ counter-clockwise. If we analyze this problem the question is, what values for a and c give a matrix will multiply $\mathbf{i}$ to give this result?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\ \phi \\ \sin\ \phi \end{bmatrix}$$

Similarly, what values for b and d give a matrix which will multiply $\mathbf{j}$ to give the vector $\mathbf{j}' = \langle -\sin\phi, \cos\phi \rangle$. The answer is clearly

$$\begin{bmatrix} \cos\ \phi & -\sin\ \phi \\ \sin\ \phi & \cos\ \phi \end{bmatrix}$$

The trick is that if we rotate every vector CCW by an angle ϕ, that is equivalent to rotating the coordinates CW by the same angle. For that reason, I will label the matrix above $R+$

$$R+ = \begin{bmatrix} \cos\ \phi & -\sin\ \phi \\ \sin\ \phi & \cos\ \phi \end{bmatrix}$$

with the understanding that it rotates *vectors* in the CCW direction, with ϕ increasing. Its inverse $R-$ rotates vectors in the direction of decreasing ϕ, and $(R-)(R+) = (R+)(R-) = I$. Thus

$$R- = \begin{bmatrix} \cos\ \phi & \sin\ \phi \\ -\sin\ \phi & \cos\ \phi \end{bmatrix}$$

It is easily verified that the products on the diagonal are equal to 1 and those off the diagonal are equal to 0. The relation we are asked to obtain is

$$(R-)\ \mathbf{A}' = \begin{bmatrix} \cos\ \phi & \sin\ \phi \\ -\sin\ \phi & \cos\ \phi \end{bmatrix} \begin{bmatrix} A'_x \\ A'_y \end{bmatrix} = \begin{bmatrix} A_x \\ A_y \end{bmatrix} = \mathbf{A}$$

The corresponding equations are

$$A_x = A'_x \cos\phi + A'_y \sin\phi$$

$$A_y = -A'_x \sin\phi + A'_y \cos\phi$$

and again, $R-$ rotates vectors CW which corresponds to a rotation of the coordinate system CCW. Before manipulating the equations, let's substitute remove the labels A and also write c for $\cos\phi$ and s for $\sin\phi$. We have

$$x = cx' + sy'$$

$$y = -sx' + cy'$$

Get rid of the coefficients for x'

$$\frac{x}{c} = x' + \frac{s}{c}y'$$

$$\frac{y}{s} = -x' + \frac{c}{s}y'$$

Add

$$\frac{x}{c} + \frac{y}{s} = \frac{s}{c}y' + \frac{c}{s}y'$$

$$xs + yc = (s^2 + c^2)y' = y'$$

Original notation

$$A'_y = A_x \sin\phi + A_y \cos\phi$$

Alternatively

$$\frac{x}{s} = \frac{c}{s}x' + y'$$

$$\frac{y}{c} = -\frac{s}{c}x' + y'$$

Subtract

$$\frac{x}{s} - \frac{y}{c} = \frac{c}{s}x' + \frac{s}{c}x'$$

$$xc - ys = (c^2 + s^2)x' = x'$$

$$A'_x = A_x \cos\phi - A_y \sin\phi$$

$$A'_y = A_x \sin\phi + A_y \cos\phi$$

This is equivalent to

$$(R+)\ \mathbf{A} = \begin{bmatrix} \cos\ \phi & -\sin\ \phi \\ \sin\ \phi & \cos\ \phi \end{bmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix} = \begin{bmatrix} A'_x \\ A'_y \end{bmatrix} = \mathbf{A}'$$