ANALYZING AC CIRCUITS WITH COMPLEX NUMBERS

This analysis note shows how AC circuits can be described by treating voltage and current as complex numbers. Generally speaking, the reason that complex numbers are so useful here is that, in contrast to its DC counterpart, an AC voltage or current needs two numbers to describe it completely - both an amplitude and a phase. The "complex number" is a pair of real numbers, which are to be operated on together in a well-defined way in addition, multiplication, and other mathematical operations. The rules for operating on complex numbers turn out to be exactly the ones we want that will treat the amplitudes and phases correctly as AC currents and voltages are combined.

Once this method is developed, we end up with the AC analogy of Ohm's law, in which the expression v = iR becomes V = iZ, with all three quantities to be considered as complex numbers. The quantity Z is called the impedance, and plays much the same role in the general circuit that the resistance R plays in the DC case. In fact, you will see that if we allow the frequency of the AC to go to zero, the impedance will turn into a real number (the imaginary part will go to zero) that is equal to the DC resistance.

The particular circuit components that introduce phase changes are capacitors and inductors. This note is only concerned with mathematical descriptions, however. We state the circuit behavior of capacitors and inductors in order to explain the way complex numbers serve to describe it. The physical basis for the behavior is not treated here.

The following matters of notation are important:

(1) Because i is the common symbol for current, it is customary to denote the square root of -1 by j:

$$j^2 = -1$$

(2) Because the arguments of the exponential function will often be algebraic expressions, we will denote the exponential by the term exp x or exp(x):

$$\exp x = \exp (x) = e^x$$

Describing Sinusoidal Functions of Time.

Consider a sinusoidal function of time such as

Eq. (1)

$$y = A\sin(\omega t)$$

as shown in Fig. 1(b). The figure illustrates graphically how the sine wave is generated as the projection of a vector that is uniformly rotating counter-clockwise with an angular velocity w (radians per second). The vector, which we denote by V, has a length that is equal to the constant amplitude, A. The phase of the function is equal to the angular position of the rotating vector at any instant, measured relative to the initial (horizontal) position. When the angular velocity is ω, the frequency of the function is

Eq. (2)

$$F = \omega/2\pi$$

and is measured in cycles per second, or Hertz (Hz).

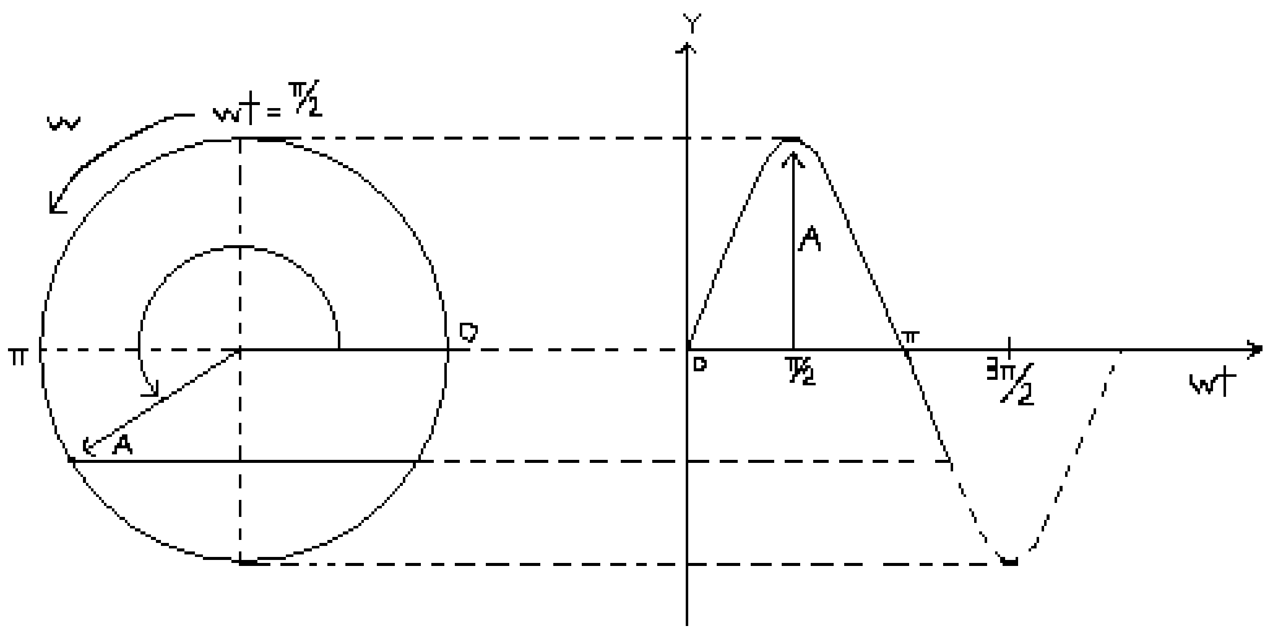


Fig. 1

Next consider two such sinusoidal functions which are not "in phase" — that is, they each reach their maximum values at different times. Using the graphical method, Fig. 2 illustrates that if we consider two vectors rotating at the same frequency, but with different lengths and with a phase difference between them, the vector sum of the pair also traces out a sinusoidal function of time. The graph shows that the sum of two sine waves of the same frequency is again a sine wave of that frequency, and gives the amplitude and phase of the summed wave. The algebraic proof of the relation between the summed wave and the two components is not simple if you do not have a table of trigonometric identities at

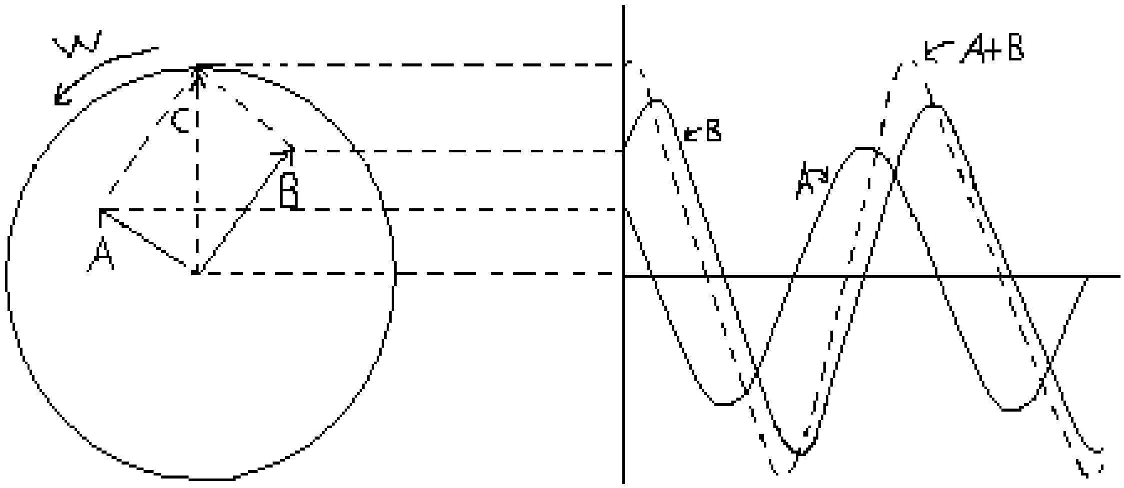
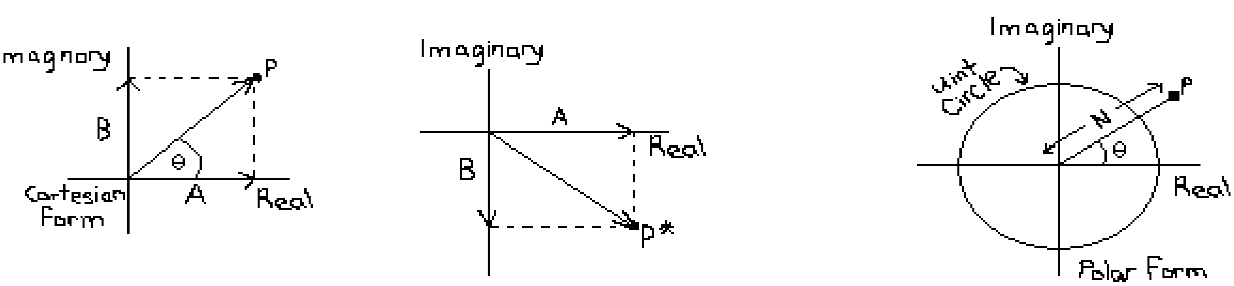


Fig. 2

The result is important because it illustrates why, in dealing with sinusoidal functions of time, each function can be treated as a vector. But it also illustrates how a formal analysis can get rather complicated in the combining of circular functions. The purpose of this note is to present a more concise way of handling these rotating vectors as complex quantities. The technique is particularly helpful in treating AC because the derivatives and integrals that relate capacitor and inductor action to current and voltage produce simple multiples of the complex function when it is written in its exponential form.

Complex Numbers.

Our use of complex numbers is such that the most applicable treatment employs the concept of the <u>complex plane</u>, a two-dimensional space illustrated in Fig. 3(a). The space is characterized by an origin and two axes at right angles to each other, a <u>real</u> horizontal axis and an <u>imaginary</u> vertical axis. Each complex number can then be represented as a point[1] in the plane.



(a) (b) (c)

Fig. 3

The point P represents a complex number, and is described by a vector from the origin. It is the vector sum of the real number represented by the length A parallel to real axis and the imaginary number represented by the length B parallel to the imaginary axis. The imaginary number will be written as the

[1] A complex number will be represented by an underscored symbol. The components of the number are real numbers, and not underscored. For simplicity, the complex number j itself will not be underscored.

real length B multiplied by the quantity j. Thus we have

$$P = A + jB \tag{3}$$

We recall that the complex conjugate of any number is obtained by reversing the signs of any of its terms that contain j. Fig. 3(b) shows the complex conjugate of P, which is written as

$$P^* = A - jB$$

The modulus of P, its magnitude N, is a real number that can be found by taking the square root of the product of P and its complex conjugate, which always yields a real positive number

$$PP^* = A^2 + B^2 = N^2$$

The expression of a complex number specifically as the sum a real and an imaginary part, such as in Eq. (3) is similar to the expression of a vector in terms of x and y components. The number can also be expressed in polar form. This is done through the use of the use of Euler's formula, which relates a complex exponential to the circular functions:

$$\exp(j\theta) = \cos\theta + j\sin\theta \tag{4}$$

The formula can be proved by expanding each of the three functions in a power series. In Fig. 3(c),the example complex number, P, is shown also to have the polar representation (since $PP^* = N^2$)

$$P = N\exp(j\theta) \tag{5}$$

with the phase angle θ related to the Cartesian components by

$$\tan\theta = B/A \tag{6}$$

The Relation of Complex Numbers to AC Circuit Theory.

We began by associating sinusoidal functions of time graphically with rotating two-dimensional vectors. Then we showed how complex numbers can be considered as two-dimensional vectors. It follows that complex numbers also provide a framework to describing sinusoidal functions of time - in particular, the voltage and current functions encountered in AC circuits.

The complex exponential form is especially useful because the fundamental relation between voltage and current is a multiplicative one. In DC circuits it is called Ohm's law

$$V = IR \tag{7}$$

with R a constant that expresses the potential difference needed to cause a current to flow through a circuit element.

When the voltage is sinusoidal, other kinds of impedance to the flow of current exist. The particular impedances we deal with here are capacitors and inductors — this excludes more "non-linear" devices among which the diode is a familiar example. The capacitive impedance is proportional to the time integral of the current, since this measures the charge on the capacitor. The inductive impedance is proportional to the time derivative of current, because this expresses the rate at which a magnetic field around a conductor is changed. Both devices, when present in a circuit, introduce a phase difference between the sinusoidal impressed voltages and the resulting currents.

The "generalized Ohm's law" that can describe AC circuit behavior is written as

$$v(t) = i(t)Z \qquad \text{Eq. (8)}$$

where all three functions are complex[2]. The impedance Z, is not a function of time in the circuits we will consider. This is because we will only deal with the steady state of a circuit, not with the transient states that occur when elements are switched in and out of a circuit, or the frequency is changed, for example. We now write down the form of the impedance for the resistor, the capacitor, and the inductor.

Impedance of a resistor. When a sinusoidal voltage v(t) is impressed on a resistor, where v(t) has the form

$$\underline{v}(t) = V \exp (j\omega t) \qquad \text{Eq. (9)}$$

the circuit equation gives

$$\underline{i}(t)R = V \exp (j\omega t) \qquad \text{Eq. (10)}$$

so the current is

$$\underline{i}(t) = (V/R) \exp (j\omega t) \qquad \text{Eq. (11)}$$

Therefore the current through a resistor is in phase with the voltage dropped across it. The AC resistive impedance of a resistor is the real number R.

Impedance of a capacitor. Starting with the same voltage given in Eq. (9) across a capacitor, the charge on it is

$$\underline{q}(t) = CV \exp (j\omega t) \qquad \text{Eq. (12)}$$

so that differentiating with respect to time gives the current

$$\underline{i}(t) = dq/dt = j\omega CV \exp (j\omega t) \qquad \text{Eq. (13)}$$

Using Eq. (8), we see that the complex form of the capacitive reactance, which we will denote as X_c,

---

[2] While it is true in general that both the voltage and the current are complex, it will often be possible in a particular case to choose some particular voltage or current to be real. This merely reflects the fact that the entire equation can be multiplied by a complex number of magnitude unity, shifting all phases by a fixed amount.

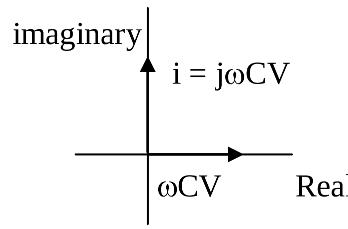
(note that we will not underscore this symbol, which is totally imaginary) is

Eq. (14)

$$X_c = 1/j\omega C = -j/\omega C$$

We have used the fact that $1/j = -j$, which follows immediately from the definition $j^2 = -1$. since the capacitive reactance is a pure imaginary number, the current through the capacitor will always be 90° out of phase with the voltage across the capacitor.

Let us sketch the voltage across the capacitor at the instant when it is represented by the real number V, that is at time t = 0, for example. The current through the capacitor at that instant is obtained by dividing the capacitive reactance given by Eq. (14). But this is equivalent to multiplying by jwC, so the current is represented at that instant by a vector along the positive complex axis.



This is the convenience of the complex expression for the reactance. It will insure that the correct phase is maintained between the voltage and the current. The current will always be 90° ahead of the voltage if Eq. (14) is used to describe the impedance of the capacitor.

<u>Impedance of an inductor</u>. When the voltage across an inductor is given by Eq. (9), the time rate of change of the current through an inductor L is given by

Eq. (15)

$$di/dt = (V/L) \exp(j\omega t)$$

which can immediately be put in an integrable form that gives

Eq. (16)

$$\underline{i}(t) = (V/j\omega L) \exp(j\omega t)$$

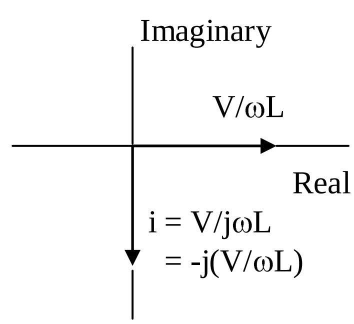
Using Eq. (8), we see that the complex form of the <u>inductive reactance</u> X_L is given by

Eq. (17)

$$X_L = j\omega L$$

Thus the inductive reactance is a pure imaginary number, like j itself, we will not underline this symbol

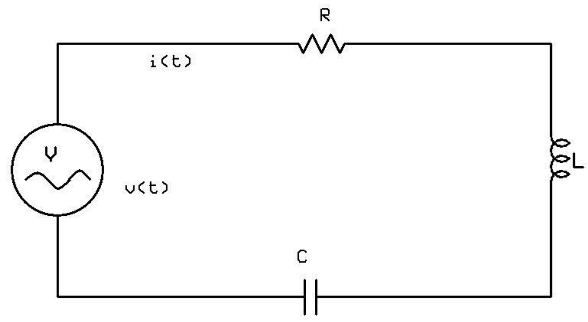
To interpret what this impedance implies for the relation between voltage and current, let us again choose an instant when the voltage across an inductor lies along the real axis. The current at that instant is $V/X_L = V/j\omega L$. Thus the current is represented by a vector directed along the negative imaginary axis. The complex inductive reactance of Eq. (17) carries the information that the current through the inductor lags the voltage across it by 90°.



These three elementary impedance forms will now be used to describe a series circuit containing all three elements. We will simply use the extended form of Ohm's law given by Eq. (8), and the complex impedances that automatically satisfy the phase requirements:

$$\underline{z}_R = R, \qquad \underline{z}_c = X_C = 1/k\omega C, \qquad \underline{z}_L = X_L = j\omega L$$

The Series R-L-C Circuit



When all three elements described above are connected in series to an AC source, the importance of the phase relations that were described for the individual elements will become clearer. The fact that the elements are connected in series means that there is only one current in the circuit – it passes out of the source and through all three impedances, returning again to the source.

Let us choose the form of the source

voltage to be that of Eq. (9). At any instant, the total voltage dropped across the resistor, the capacitor, and the inductor must add to equal the source voltage. But we know what those individual voltage drops are in terms of products of current and impedance.

Calling the single current of the circuit I(t), we have

Eq. (18)

$$\begin{aligned}\underline{v}(t) = V \exp(j\omega t) &= \underline{I}(t)\,[\underline{Z}_R + \underline{Z}_L + \underline{Z}_C] \\ &= \underline{i}(t)\,[R + j\omega L + 1/j\omega C] \\ &= \underline{i}(t)\,[R + j(\omega l - 1/\omega C)]\end{aligned}$$

It is clear from Eq. (18) that the complex impedance equals

Eq. (19)

$$\underline{Z} = R + j(\omega L - 1/\omega C)$$

This impedance equation contains all of the interesting characteristics of the series R-L-C circuit. Consider what it says both as to the magnitude and the phase of the current.

This is conveniently examined by expressing the complex impedance in the polar form similar to Eq. (5). Let us denote the magnitude of the impedance $\underline{Z}$ by the small letter z, and its phase angle by Ψ. According to the discussion of the polar representation of complex numbers, the impedance can then be expressed as

Eq. (20)

$$\underline{Z} = x \exp(j\psi)$$

Where the amplitude of the impedance is

Eq. (21)

$$z = [R^2 + (\omega L - 1/\omega C)^2]^{1/2}$$

and its phase angle is related to the "Cartesian" form Eq. (19) by

Eq. (22)

$$\tan\ \Psi = (\omega L - 1/\omega C) / R$$

To understand the significance of expressing the impedance in polar form, use it in Eq. (18) in place of the Cartesian form and we obtain

Eq. (23)

$$V \exp (j\omega t) = \underline{i}\ z \exp (j\psi)$$

so that the current can be written

Eq. (24)

$$\underline{i}(t) = V/z \exp [j(\omega t - \psi)]$$

Therefore the polar form of the impedance gives a polar form for the current, in which we can separately discuss the current amplitude, and its phase relative to the driving voltage. Both exhibit interesting behavior as a function of frequency.

<u>Current Amplitude in the Series Circuit</u>.

The current amplitude is the fraction

Eq. (25)

$$I(\omega) = V/z$$

which we write as a function of the frequency. We consider its behavior, which varies inversely with the impedance z, as we keep the magnitude of the driving voltage (v) constant and sweep through the frequency range. When ω is small enough, the inductive reactance becomes negligible, but the capacitive reactance becomes very large. If we bring the frequency low enough, the impedance is essentially that of the capacitor alone, for any values of R and L, according to Eq. (21), and the current amplitude becomes approximately proportional to the frequency:

Eq. (26)

$$I(\omega) \cong \omega CV \qquad \omega\text{-very small}$$

This is reasonable, if we remember that the low frequency limit is DC, which cannot pass through a capacitor, and so the current vanishes.

In the high frequency limit, the inductive reactance becomes the dominant impedance term. The resistance and capacitance become negligible, and the impedance varies like the function ωL. The current becomes approximately an inverse function of frequency

Eq. (27)

$$I(\omega) \cong V/\omega L \qquad \omega\text{-very large}$$

and again vanishes in the limit.

So at both low and high frequency, the current amplitude goes to zero. Between these limits, the current rises with frequency, reaches a maximum, and then falls with increasing frequency. The point of maximum current is the point of minimum impedance, occurring when the reactive terms in Eq. (21) cancel each other:

Eq. (28)

$$\omega_r L - 1/\omega_r C = 0$$

where we use the subscript "r" to denote the "resonance" conditions. The corresponding linear frequency at which the R-L-C

Circuit exhibits resonance follows when the above equation is solved for ω_r:

Eq. (29)

$$f_r = \omega_r/2\pi = 1/2\pi\ (1/LC)^{1/2}$$

At that frequency, z = R, and the current maximum has the value V/R. Although we will not attempt to show it here, the "sharpness" of the current maximum varies inversely with the resistance of the circuit, for all other parameters constant, so that a low resistance gives a very sharp frequency selecting resonance and a large resistance gives a broad maximum that is not very selective in frequency.

<u>Phase of the Current in the Series Circuit</u>.

If we compare Eq. (24), we see that there is a phase difference, in general, between the current and the voltage. We might have expected this from the discussion of the previous section, without resorting further to equations. Remember that the impedance behavior can be summed up in the statements:

(1) The impedance is "capacitive" below the resonant frequency, down to the lowest frequencies.

(2) It is purely resistive at the resonant frequency.

(3) It is "inductive above the resonant frequency, up to the highest frequencies.

Now these are rather crude statements, except for the second, because the presence of the inductor is felt below resonance, and that of the capacitor is felt above resonance, but they are borne out by the expression for the phase angle given in Eq. (22). The current leads the voltage below resonance, is in phase with it at resonance, and lags the driving voltage above resonance. Again the amount of lead and lag depends on the total resistance of the circuit, consistent with the observations made above regarding the amplitude of the current and the sharpness of the resonance, and we leave it to you and the laboratories to explore the matter in detail.