Limits-Problems

The *Calculus Lifesaver* has a very nice treatment of limits, both theoretically and practically. The first thing is to note that the techniques differ depending on whether the limit is at a number a which x approaches, $x \to a$, or as x approaches infinity, either plus or minus. Start with the first type, $x \to a$.

Try plugging in $x = a$ and see what happens. (Maybe there is no problem).

Next, for rational functions such as

$$\lim_{x \to a} \frac{p(x)}{q(x)}$$

try factoring. (Since we are interested in the limit, and not directly in what happens when x is equal to a, if we can get a term $(x - a)$ in both the numerator and denominator, it will lead to simplification).

Factoring

Factoring is OK when evaluating a limit. For example

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x}$$

Plugging in $x = 2$ gives $0/0$. We notice that we can factor the numerator and the denominator

$$= \lim_{x \to 2} \frac{(x + 6)(x - 2)}{x(x - 2)}$$

Now, since we are *not interested* in what happens at $x = 2$, we can cancel here, giving

$$= \lim_{x \to 2} \frac{(x+6)}{x} = \frac{8}{2} = 4$$

A harder example might be cubic or higher.

$$\lim_{x \to 3} \frac{x^3 - 27}{x^4 - 5x^3 + 6x^2}$$

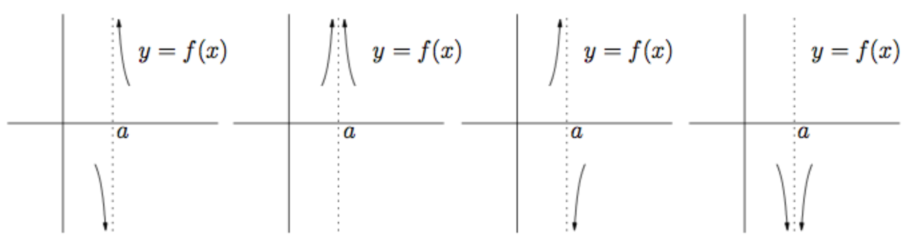
Recall that

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Hence we have

$$\begin{aligned}
&\lim_{x \to 3} \frac{(x-3)(x^2+3x+9)}{x^2(x^2-5x+6)} \\
&= \lim_{x \to 3} \frac{(x-3)(x^2+3x+9)}{x^2(x-3)(x-2)} \\
&= \lim_{x \to 3} \frac{x^2+3x+9}{x^2(x-2)} = \frac{9+9+9}{9} = 3
\end{aligned}$$

If, when we plug in $x = a$, only the denominator but not the numerator is zero, then we have one of the following situations:



$\lim_{x \to a^+} f(x) = \infty$	$\lim_{x \to a^+} f(x) = \infty$	$\lim_{x \to a^+} f(x) = -\infty$	$\lim_{x \to a^+} f(x) = -\infty$
$\lim_{x \to a^-} f(x) = -\infty$	$\lim_{x \to a^-} f(x) = \infty$	$\lim_{x \to a^-} f(x) = \infty$	$\lim_{x \to a^-} f(x) = -\infty$
$\lim_{x \to a} f(x)$ DNE	$\lim_{x \to a} f(x) = \infty$	$\lim_{x \to a} f(x)$ DNE	$\lim_{x \to a} f(x) = -\infty$

An example:

$$\lim_{x \to 1} \frac{2x^2 - x - 6}{x(x-1)^3}$$

At $x = 1$, the numerator is equal to -5, and will not change sign if x is slightly smaller or larger than 1. Similarly for x in the denominator. However, $x - 1$ does change sign, and so does $(x-1)^3$.

We have the situation shown in the left-hand panel above, and since the two limits $x \to a+$ and $x \to a-$ are not equal, the limit $x \to a$ does not exist (DNE).

On the other hand, if we have the similar but slightly changed

$$\lim_{x \to 1} \frac{2x^2 - x - 6}{x(x-1)^2}$$

Now the denominator does not change sign and is positive, hence both the one-sided limits are equal to $-\infty$, and so the limit is also equal to $-\infty$.

For limits involving square roots, we use the conjugate.

Conjugate

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$$

Substitution gives 0/0.

The conjugate of $\sqrt{x} - 2$ is $\sqrt{x} + 2$. The point is that the product does not contain a square root:

$$(\sqrt{x} - 2) \times (\sqrt{x} + 2) = x - 4$$

We try multiplication by the conjugate

$$\frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{x} - 2}{x - 4} \, \frac{\sqrt{x} + 2}{\sqrt{x} + 2}$$

The numerator gives $x - 4$, and before you rush to do the denominator, notice the cancellation:

$$= \frac{x-4}{(x-4)(\sqrt{x}+2)}$$

So this is just

$$\lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$$

A slightly more complicated example:

$$\lim_{x\to 5} \frac{\sqrt{x^2-9}-4}{x-5}$$
$$= \lim_{x\to 5} \frac{\sqrt{x^2-9}-4}{x-5} \times \frac{\sqrt{x^2-9}+4}{\sqrt{x^2-9}+4}$$
$$= \lim_{x\to 5} \frac{x^2-9-16}{(x-5)(\sqrt{x^2-9}+4)}$$

The numerator becomes $x^2 - 25 = (x-5)(x+5)$ leading to a cancellation. The result is

$$= \lim_{x\to 5} \frac{x+5}{\sqrt{x^2-9}+4} = \frac{10}{4+4}$$

Now we deal with ∞.

Polynomial as $x \to \infty$

$$\lim_{x\to\infty} 3x^2 + 2x + 1$$
$$= \lim_{x\to\infty} x^2(3 + \frac{2}{x} + \frac{1}{x^2})$$
$$= \lim_{x\to\infty} x^2(3+0+0)$$
$$= \lim_{x\to\infty} 3x^2 = \infty$$

This method can be adapted to more complex examples.

$$= \lim_{x\to\infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7}$$

$$= \lim_{x\to\infty} \frac{(2 - \frac{1}{x^2} + \frac{8}{x^3})(x^4)}{(-5 + \frac{7}{x^4})(x^4)}$$

$$= \lim_{x\to\infty} \frac{(2 - 0 + 0)(x^4)}{(-5 + 0)(x^4)} = -\frac{2}{5}$$

And one with a square root

$$\lim_{x\to\infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{3 + \frac{6}{x^2}}}{x(\frac{5}{x} - 2)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{3 + 0}}{x(0 - 2)}$$

But we have to be careful here because

$$\sqrt{x^2} = |x|$$

So this is

$$= -\frac{\sqrt{3}}{2} \lim_{x\to\infty} \frac{|x|}{x}$$

which we deal with now.

absolute value

Consider

$$\lim_{x\to 0} \frac{|x|}{x}$$

At $x = 0$, this is equal to $0/0$. Also, for any $x \neq 0$

$$|\frac{|x|}{x}| = 1$$

Now, what happens as we approach $x = 0$ from either side? Approaching from the right, for $x > 0$, the sign of the expression is positive, while approaching from the left, for $x < 0$, the sign is negative. Thus, the limit as $x \to 0-$ is not equal to the limit as $x \to 0+$ and so the two-sided limit as $x \to 0$ does not exist.

For the problem we had above

$$\lim_{x \to \infty} \frac{|x|}{x}$$

Since as $x \to \infty$, $x > 0$, since both terms of the fraction are positive, the limit is $+1$.

This problem is only slightly different:

$$\lim_{x \to -2} \frac{|x+2|}{x+2}$$

Again, for every $x \neq -2$, the absolute value of the expression is equal to 1, but the sign changes depending from which side we are approaching to the limit. From the left, the value is -1 because the sign of the denominator is less than zero. Hence, the two-sided limit does not exist.

Sine

As you know, $\sin x$ does not approach any limit because it is periodic. So what about

$$\lim_{x \to 0} x \sin x$$

We may guess that since the right-hand term is always a number between -1 and 1, when multiplied by 0 we'll get 0. The way to do this is to use the "squeeze" theorem.

$$-1 \leq \sin x \leq 1$$

When we multiply by x, we have to take account of sign. So let's do the two cases separately. For $x > 0$, we obtain

$$-x \leq x \sin x \leq x$$

Since both $-x$ and x go to 0, so does $x \sin x$. Suppose $x < 0$. Then, when we multiply we have to flip the inequality

$$-x \geq x \sin x \geq x$$

but it doesn't matter because both left and right-hand terms still tend to 0, and since $x \sin x$ is squeezed between them, it does too.