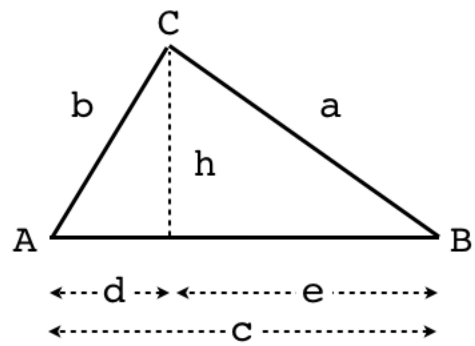
Heron's Formula

Once again, in the triangle below, A, B, and C are the angles, with side lengths a, b, and c.



One other formula we want to prove is called Heron's Formula for the area of a triangle. This formula does not explicitly include the altitude h or the parts of side c, which are d and e.

If s is the semi-perimeter

$$s = \frac{1}{2}(a + b + c)$$

$$A = \sqrt{s + (s - a) + (s - b) + (s - c)}$$

Start with the well-known formula for area

$$A = \frac{1}{2} \text{ base } \times \text{ height } = \frac{1}{2}\; c\; h = \frac{1}{2}\; cb\; \sin A$$

We will come back to this and substitute for the sine of A. But first, rearrange the equation for the law of cosines

$$a^2 = c^2 + b^2 - 2bc\cos A$$

$$\cos A = \frac{(c^2 + b^2 - a^2)}{2bc}$$

$$\sin A = \sqrt{1 - \cos^2 A} = \sqrt{1 - \frac{(c^2 + b^2 - a^2)^2}{(2bc)^2}}$$

So finally we have

$$A = \frac{1}{2}\ c\ b\ \sqrt{1 - \frac{(c^2 + b^2 - a^2)^2}{(2bc)^2}}$$

$$A = \frac{1}{4}\sqrt{4b^2c^2 - (c^2 + b^2 - a^2)^2}$$

Now we just need to work on what is under the square root. It looks like a mess but will simplify quite a bit.

For the next part, we won't write $A = \frac{1}{4}\sqrt{\ldots}$, but we'll recall that it's there near the end, when we will write it as $A = \sqrt{\frac{1}{16}\ldots}$

Look at what's inside

$$4b^2c^2 - (c^2 + b^2 - a^2)^2$$

This looks familiar, it is a difference of squares

$$(2bc + (c^2 + b^2 - a^2))(2bc - (c^2 + b^2 - a^2))$$

In the first term, we can rearrange

$$2bc + c^2 + b^2 - a^2$$

$$(c + b)^2 - a^2)$$

$$(c+b+a)(c+b-a)$$

Similarly in the second term

$$-(c^2-2bc+b^2-a^2)$$

$$-((c-b)^2-a^2)$$

$$-((c-b+a)(c-b-a))$$

$$(c-b+a)(a+b-c)$$

Putting it all together, we have

$$(c+b+a)(c+b-a)(c-b+a)(a+b-c)$$

Recall that the perimeter

$$p=a+b+c=2s$$

The first term above, $(a+b+c)$, is the perimeter, that is, twice the semi-perimeter or $2s$. The second term is $p-a-a=2s-2a=2(s-a)$. The third and fourth terms can be seen to be equal, by the same logic, to $2(s-b)$ and $2(s-c)$. Recalling the square root, etc. from above, we have finally:

$$A=\sqrt{\frac{1}{16}\ 2(s)2(s-a)\ 2(s-b)\ 2(s-c)}$$

Canceling

$$A=\sqrt{s(s-a)(s-b)(s-c)}$$

$\square$

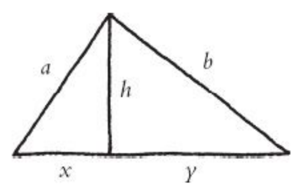
As a simple example, if we have a right triangle with sides 3,4,5, then the area is one-half of 3 times 4 = 6. The semi-perimeter is s

$$s = \frac{(3+4+5)}{2} = \frac{12}{2} = 6$$

We have

$$A = \sqrt{6(6-5)(6-4)(6-3)} = \sqrt{6(1)(2)(3)} = 6$$

Lockhart's version



The triangle is labeled slightly differently than the one above. The bottom side c is split into x and y. We can write three equations:

$$x^2 + h^2 = a^2$$
$$y^2 + h^2 = b^2$$
$$x + y = c$$

Lockhart gives us a target:

$$2xc = c^2 + a^2 - b^2$$

Let's just start manipulating equations to get there. Subtract the second from the first:

$$x^2 - y^2 = a^2 - b^2$$

Square the third

$$x^2 + 2xy + y^2 = c^2$$

Add the two new equations

$$2x^2 + 2xy = c^2 + a^2 - b^2$$

Substitute for y

$$2x^2 + 2x(c - x) = c^2 + a^2 - b^2$$

$$2xc = c^2 + a^2 - b^2$$

Finally a slight rearrangement:

$$x = \frac{c}{2} + \frac{a^2 - b^2}{2c} = \frac{c^2 + a^2 - b^2}{2c}$$

This says that to find the point where x meets y we move from the center $c/2$ a distance of $(a^2 - b^2)/2c$.

The corresponding equation for y is

$$y = \frac{c}{2} - \frac{a^2 - b^2}{2c}$$

which is easily checked by adding together the final two equations, obtaining $x + y = c$.

For the area, we will need h somehow. It is easier to use h^2.

$$h^2 = a^2 - x^2$$

$$= a^2 - \frac{(c^2 + a^2 - b^2)^2}{(2c)^2}$$

The area squared is

$$A^2 = \frac{1}{4}c^2h^2$$

$$= \frac{1}{4}c^2a^2 - \frac{1}{4}c^2\frac{(c^2 + a^2 - b^2)^2}{(2c)^2}$$

> the algebraic form of this measurement is aesthetically unacceptable. First of all, it is not symmetrical; second, it's hideous. I simply refuse to believe that something as natural as the area of a triangle should depend on the sides in such an absurd way. It must be possible to rewrite this ridiculous expression...

Here's a start:

$$16A^2 = (2ac)^2 - (c^2 + a^2 - b^2)^2$$

This is better, and actually, quite like what we had before. We will now go through two difference of squares manipulations. First

$$\begin{aligned} 16A^2 &= \left[\ 2ac + (c^2 + a^2 - b^2)\ \right] \left[\ 2ac - (c^2 + a^2 - b^2)\ \right] \\ &= \left[\ (a+c)^2 - b^2)\ \right] \left[\ b^2 - (a-c)^2\ \right] \\ &= (a+c+b)(a+c-b)(b+a-c)(b-a+c) \end{aligned}$$

So

$$A = \sqrt{\frac{a+b+c}{2} \cdot \frac{a+c-b}{2} \cdot \frac{a+b-c}{2} \cdot \frac{-a+b+c}{2}}$$

Of course, we recognize the semi-perimeter $s = (a+b+c)/2$ and then we see that each of the other terms is $(s-a)$, and so on

$$A = \sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)}$$