Calculus of variations

Consider two points in the xy-plane (x_1, y_1) and (x_2, y_2), and the paths that connect them. One path is the shortest path, namely a straight line. Our goal is to prove that.

There are many possible paths, any one of which we could try to write as $y = f(x)$ or to use fewer symbols we write $y = y(x)$.

As you know, any small element of the path ds is (by Pythagoras)

$$ds = \sqrt{dx^2 + dy^2}$$

Factoring out dx we get

$$ds = \sqrt{1 + (\frac{dy}{dx})^2}\ dx = \sqrt{1 + y'(x)^2}\ dx$$

The total path length is

$$L = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\ dx$$

The unknown is the function $y = y(x)$. The principle is that we will find $y(x)$ such that any infinitesimal change in $y(x)$ makes no difference in the length, to first order. We say that such a path makes the integral **stationary**.

Fermat's principle

The problem can be made slightly harder by including some function $f(x, y)$ under the integral. For example, consider the path taken by a light beam between two points. If the refractive index varies with position, then the path will not be a straight line. Instead, light takes the path that takes the *least time*.

$$\text{time} \ = \int_1^2 dt = \int_1^2 \frac{ds}{v}$$

where $v\ dt = ds$ and the velocity $v = c/n$ so

$$\text{time} \ = \int_1^2 dt = \int_1^2 \frac{ds}{v} = \frac{1}{c} \int_1^2 n\ ds$$

n is a function $n(x, y)$, and we substitute for ds:

$$\text{time} \ = \frac{1}{c} \int_{x_1}^{x_2} n(x, y)\ \sqrt{1 + y'(x)^2}\ dx$$

Since for any parametrized curve (which is all we can really deal with), $y = y(x)$, the function $n(x, y)$ is really only dependent on x and this is a single integral over x.

generally

We will have some integral over x where the integrand is a function that depends on x, on $y(x)$ and on $y'(x)$ so we write

$$S = \int_{x_1}^{x_2} f\ [\ y(x), y'(x), x\ ]\ dx$$

where $y(x)$ is an unknown curve and we seek the $y(x)$ which makes the integral stationary.

Let us denote the correct solution to the problem as $y(x)$ and all the other "wrong" curves as $Y(x)$ and then

$$Y(x) = y(x) + \eta(x)$$

η is a term that contains all the extra length of a wrong path Y compared to the shortest path y. We are only interested in paths $Y(x)$ that pass through the endpoints (x_1, y_1) and (x_2, y_2). At those points η must be zero because $Y = y$ there.

$$\eta(x_1) = \eta(x_2) = 0$$

As Taylor says, "there are infinitely many choices for the difference $\eta(x)$, for example we could choose $\eta = (x - x_1)(x - x_2)$ or $\sin\left[\pi(x - x_1)/(x_2 - x_1)\ \right]$."

The function η will have something that makes the difference in $Y(x) - y(x)$. Let us parametrize those things that make the difference and factor them out into a parameter α so that

$$Y(x) = y(x) + \alpha\eta(x)$$

The integral S now depends on α. The curve $y(x)$ is obtained by setting $\alpha = 0$. Our problem is now to make sure that $S(\alpha)$ is a minimum when $\alpha = 0$. We write

$$S(\alpha) = \int_{x_1}^{x_2} f\ [\ Y(x), Y'(x), x\ ]\ dx$$

$$= \int_{x_1}^{x_2} f\ [\ y + \alpha\eta(x), y' + \alpha\eta'(x), x\ ]\ dx$$

Notice that although η depends on x, α does not.

Next, we want to differentiate S with respect to α and set that derivative equal to zero. Differentiating the integrand:

$$\frac{\partial}{\partial\alpha} f\ [\ y + \alpha\eta(x), y' + \alpha\eta'(x), x\ ]\ = \eta\ \frac{\partial f}{\partial y} + \eta'\ \frac{\partial f}{\partial y'}$$

I don't understand the previous step.

But given this

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha}\ dx$$

$$= \int_{x_1}^{x_2} (\eta\ \frac{\partial f}{\partial y} + \eta'\ \frac{\partial f}{\partial y'})\ dx$$

This condition must be true for any choice of the path.

Now he says, we will rewrite the second term on the right using integration by parts.

$$\int_{x_1}^{x_2} \eta'\ \frac{\partial f}{\partial y'}\ dx = \eta(x) \frac{\partial f}{\partial y'}\ \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\ \frac{\partial f}{\partial y'}\ dx$$

Because $\eta(x_1) = \eta(x_2) = 0$, the first term is zero and we have then:

$$\int_{x_1}^{x_2} \eta'\ \frac{\partial f}{\partial y'}\ dx = -\int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\ \frac{\partial f}{\partial y'}\ dx$$

Substitute back into the derivative and set it equal to zero:

$$\int_{x_1}^{x_2} \eta(x) (\frac{\partial f}{\partial y} - \frac{d}{dx}\ \frac{\partial f}{\partial y'})\ dx = 0$$

Since this must be true for any choice of $\eta(x)$, the term in parentheses must be zero (at least if the functions involved are continuous functions, and our examples will be). So we have finally

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\ \frac{\partial f}{\partial y'}$$

which is the Euler-Lagrange equation.

examples