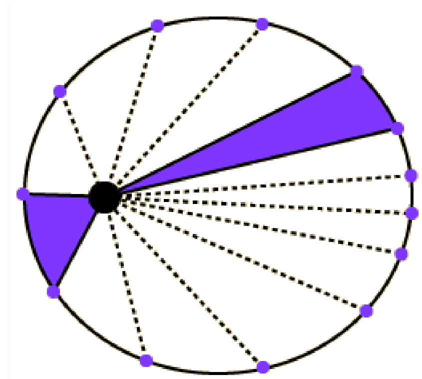
Newton to Kepler



I want to understand in detail how Kepler's Laws for the orbits of the planets can be derived from Newton's Laws, namely

$$\mathbf{F} = m\mathbf{a}$$

as well as the inverse square law of gravitation.

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{u}}$$

I've worked through derivations from several sources, and I think now that I have it figured out. My goal is to clearly document things here so that I will understand in a year (or five) when I re-read this.

Kepler's Laws are: first (K1), the orbits of the planets are not circles but ellipses (non-recurrent orbits may be other conic sections); second

(K2), the area or arc "swept out" per unit time is the same no matter where in the orbit the planet is; and third (K3) the period of the orbit is independent of the mass of the planet and its square is proportional to the cube of the length of the semi-major axis of the ellipse.

I also spent some time working on Newton's version of the proof as presented in the *Principia* (see Bressoud's vector calculus book), but he leaves out too many steps. There is also a version "cooked up" by Richard Feynman and discussed in a book called *Feynman's Lost Lecture*.

I never got either of these figured out, but if you want to go this route I recommend starting with Feynman.

For myself, I found that once I cleared up a couple of subtleties, and verified the application of the product rule for differentiation to vector cross-products, it was pretty easy.

circular approximation

The equation of an ellipse in xy-coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a is one-half the "diameter" in the long dimension and b is one-half the length perpendicular to that.

A second way is to give the focal length, the distance of each of the two foci from the center of the ellipse

$$f = \sqrt{a^2 - b^2}$$

Yet another way is to give the *eccentricity*, e, where

$$ea = f$$

Here is a table of planetary eccentricities I found on the web.

Planets: Orbital Properties

Planet	distance	revolution	eccentricity	inclination
	(A.U.)			(deg)
Mercury	0.387	87.969 d	0.2056	7.005
Venus	0.723	224.701 d	0.0068	3.3947
Earth	1.000	365.256 d	0.0167	0.0000
Mars	1.524	686.98 d	0.0934	1.851
Jupiter	5.203	11.862 y	0.0484	1.305
Saturn	9.537	29.457 y	0.0542	2.484
Uranus	19.191	84.011 y	0.0472	0.770
Neptune	30.069	164.79 y	0.0086	1.769
Pluto	39.482	247.68 y	0.2488	17.142

Mars is the planet that showed Kepler most clearly that the orbits are not circles, but its eccentricity is only 0.09. For Earth this value is only 0.017 which gives a focal length of roughly

$$0.0167 \times 149.6 \times 10^6 \text{ km} \approx 2.5 \times 10^6 \text{ km}$$

which is about four times the radius of the Sun.

However, there is a conflict with a simpler calculation, which I believe gives the correct answer, and I haven't figured that out yet. The center of mass for the Sun-Earth system is (picking the center of the Sun as the origin)

$$CM = \frac{M_E}{M_S} d_{ES} = \frac{6 \times 10^{24}}{2 \times 10^{30}}\, 1.5 \times 10^8 \text{ km} = 450 \text{ km}$$

compared to the radius of the Sun which is about 7×10^5 km.