Surface area of a solid of revolution

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Suppose we revolve a function y = f(x) around the x-axis. To compute the surface area of the solid, we imagine slicing it into disks in the usual way, moving along the x-axis in increments dx. Then we need to find the surface area of the disk. The elements of surface area ds are given by:

$$ds = \sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

After setting up ds, we will integrate

$$\int 2\pi y\ ds$$

As an example, consider the circle with unit radius centered at the origin.

$$x^2 + y^2 = 1; \quad y = f(x) = \sqrt{1 - x^2}$$

Using implicit differentiation, it is easy to show that

$$2x\ dx + 2y\ dy = 0$$

$$\frac{dy}{dx} = -x/y$$

Then

$$ds = \sqrt{1 + \frac{x^2}{y^2}}\ dx = \sqrt{1 + \frac{x^2}{1 - x^2}}\ dx$$

$$S = 2\pi \int \sqrt{1 - x^2}\ \sqrt{1 + \frac{x^2}{1 - x^2}}\ dx = 2\pi \int \sqrt{1 - x^2 + x^2}\ dx = 2\pi \int dx = 2\pi x$$

evaluate from x = -1 to x = 1, giving:

$$S = 4\pi$$

If we want the more general solution for radius R, then everything is the same except the limits of integration are -R and R, and the first term inside the integral is

$$\sqrt{R^2 - x^2}$$

So the second term in the integral simplifies to:

$$S = 2\pi \int \sqrt{R^2 - x^2}\,\sqrt{1 + \frac{x^2}{R^2 - x^2}}\,dx$$

$$= 2\pi \int \sqrt{R^2 - x^2 + x^2}\,dx = 2\pi \int R dx = 2\pi R x$$

evaluate from x = -R to x = R, giving $2R$ times $2\pi R$

$$S = 4\pi R^2$$