Kepler (part 10): Quick Summary

unit vectors and velocity

The position vector (from the sun to the planet) is $\mathbf{r}$. Starting from our definition of the unit vector in the $\mathbf{r}$ direction as

$$\mathbf{u_r} = \ \langle \cos\theta, \sin\theta \rangle$$

where θ is the angle with the positive x-axis, we find $\mathbf{u}_\theta \perp \mathbf{u_r}$

$$\mathbf{u}_\theta = \ \langle -\sin\theta, \cos\theta \rangle$$

and confirm orthognality

$$\mathbf{u_r} \cdot \mathbf{u}_\theta = 0$$

Remembering that $\theta = \theta(t)$, we easily obtain by the chain rule

$$\dot{\mathbf{u}}_\mathbf{r} = \dot{\theta}\mathbf{u}_\theta$$

$$\dot{\mathbf{u}}_\theta = -\dot{\theta}\mathbf{u_r}$$

r is the magnitude of $\mathbf{r}$

$$\mathbf{r} = r\mathbf{u_r}$$

The velocity $\mathbf{v}$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{u_r} + r\dot{\mathbf{u}}_\mathbf{r} = \dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta$$

We use a vector identity that is easy to prove

$$\frac{d}{dt}\,(\mathbf{a}\times\mathbf{b}) = \dot{\mathbf{a}}\times\mathbf{b} + \mathbf{a}\times\dot{\mathbf{b}}$$

to calculate with Feynman's "dots"

$$\frac{d}{dt}\,(\mathbf{r}\times\mathbf{v})$$

$$= \frac{d}{dt}\,(\mathbf{r}\times\dot{\mathbf{r}})$$

$$= \dot{\mathbf{r}}\times\dot{\mathbf{r}} + \mathbf{r}\times\ddot{\mathbf{r}} = 0$$

because any vector's cross-product with itself is zero (including minus itself), which is true for the second term involving the acceleration.

acceleration

An actual expression for the acceleration is just a matter of working through the dots

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d}{dt}\,(\dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta)$$

$$= \ddot{r}\mathbf{u_r} + \dot{r}\dot{\mathbf{u}}_\mathbf{r} + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta$$

substituting for $\dot{\mathbf{u}}_\mathbf{r}$ and $\dot{\mathbf{u}}_\theta$ from above

$$= \ddot{r}\mathbf{u_r} + \dot{r}\dot{\theta}\mathbf{u}_\theta + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta - r\dot{\theta}^2\mathbf{u_r}$$

$$= (\ddot{r} - r\dot{\theta}^2)\mathbf{u_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{u}_\theta$$

Rewrite the coefficient for $\mathbf{u}_\theta$ as

$$\frac{1}{r}(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

angular momentum

We find that the acceleration $\mathbf{a} = \dot{\mathbf{v}}$ has two parts of which the second (in $\mathbf{u}_\theta$)

$$\frac{1}{r}\ \frac{d}{dt}\ r^2\dot{\theta} = 0$$

is zero because $\mathbf{a}$ is all radial. Hence $r^2\dot{\theta} = h$ where h is a constant. Multiplied by the mass m, mh becomes the conserved quantity, angular momentum. It is also twice the area "swept out" and this is the statement of K2.

We get the vector $\mathbf{h}$ by defining the plane of motion as the xy-plane ($\mathbf{u_r} \times \mathbf{u}_\theta = \hat{\mathbf{k}}$) and

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = r\mathbf{u_r} \times (\dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta)$$

the first term is zero so

$$= r^2\dot{\theta}(\mathbf{u_r} \times \mathbf{u}_\theta) = r^2\dot{\theta}\ \hat{\mathbf{k}}$$

key step

With these preliminary steps we come to the key part of the derivation. I like Varberg's version best. The radial acceleration is

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{u_r}$$

Compute $\mathbf{a} \times \hat{\mathbf{k}}$ (recall that $\mathbf{a}$ is in the $-\mathbf{u_r}$ direction) by recognizing that $-\mathbf{u_r} \times \hat{\mathbf{k}} = \mathbf{u}_\theta$ so

$$\mathbf{a} \times \hat{\mathbf{k}} = \frac{GM}{r^2}\mathbf{u}_\theta$$

but from above $\dot{\mathbf{u}}_\mathbf{r} = \dot{\theta}\mathbf{u}_\theta$ so we have the crucial substitution:

$$\mathbf{a} \times \hat{\mathbf{k}} = \frac{GM}{r^2\dot{\theta}}\dot{\mathbf{u}}_\mathbf{r}$$

$$\mathbf{a} \times \hat{\mathbf{k}} = \frac{GM}{h}\dot{\mathbf{u}}_{\mathbf{r}}$$

Now we just integrate with respect to time and get

$$\int \mathbf{a} \times \hat{\mathbf{k}} = \int \frac{GM}{h}\dot{\mathbf{u}}_{\mathbf{r}}$$

$$\mathbf{v} \times \hat{\mathbf{k}} = \frac{GM}{h}\mathbf{u}_{\mathbf{r}} + \mathbf{d}$$

where $\mathbf{d}$ is a constant *vector* of integration. One last trick, we dot with $\mathbf{r}$ and simplify the left-hand side dramatically

$$\mathbf{r} \cdot (\mathbf{v} \times \hat{\mathbf{k}}) = (\mathbf{r} \times \mathbf{v}) \cdot \hat{\mathbf{k}} = \mathbf{h} \cdot \hat{\mathbf{k}} = h$$

So

$$h = \mathbf{r} \cdot (\frac{GM}{h}\mathbf{u}_{\mathbf{r}} + \mathbf{d})$$

$$\frac{h^2}{GM} = \mathbf{r} \cdot (\mathbf{u}_{\mathbf{r}} + \frac{h}{GM}\mathbf{d})$$

Define $k = h^2/GM$ and $e = hd/GM$ and θ as the angle between the constant vector $\mathbf{d}$ and $\mathbf{u}_{\mathbf{r}}$, so finally

$$k = r(1 + e\cos\theta)$$

which for $e < 1$ is an ellipse.