Funny series

In Strogatz book (*The Joy of* x), he gives the following series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

and he says that the sum of the series is equal to the natural logarithm of 2:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

with the provision that you have to calculate the sum in the order given.

For example, the second, third and fourth partial sums are:

$$S_2 = \frac{1}{2}; \quad S_3 = \frac{5}{6}; \quad S_4 = \frac{14}{24}; \quad S_5 = \frac{94}{120}$$

with $S_4 = 0.583$ and $S_5 = 0.783$. For any partial sum S_n and the previous sum S_{n-1} the value of the series will be bounded by the two sums.

I thought I would try to show that $\ln 2$ is the correct value for series, by using a Taylor series for the logarithm. Taylor says we can write a function $f(x)$ (near the value $x = a$) as an infinite sum

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x - a)^n$$

where f^n means the nth derivative of f and f^0 is just f, and these derivatives are to be evaluated at $x = a$. Near $a = 0$ this simplifies to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n$$

Let's calculate the derivatives of the logarithm:

$$f^0 = \ln x; \quad f^1 = \frac{1}{x} = x^{-1}; \quad f^2 = -x^{-2}; \quad f^3 = 2x^{-3}; \quad f^4 = -3!\ x^{-4}$$

The first thing I notice is that we can't use $a = 0$, since $f^1 = 1/x$ is undefined there. So, let's try $a = 1$. Then (evaluated at $a = 1$)

$$f^0 = \ln x = 0; \quad f^1 = \frac{1}{x} = 1; \quad f^2 = -x^{-2} = -1; \quad f^3 = 2; \quad f^4 = -3!$$

Going back to the definition

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

I get the following series near $a = 1$:

$$\ln x = \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \cdots$$

For the special value $x = 2$, all the terms $(x-1)^n$ go away (which confirms that $a = 1$ is an excellent choice!). We have then

$$\begin{aligned}\ln x &= \frac{0}{0!} + \frac{1}{1!} - \frac{1}{2!} + \frac{2}{3!} - \frac{3!}{4!} + \cdots \\ &= 0 + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\end{aligned}$$

which is what was to be proved.