Integrating cosine squared

We want to find the integral of

$$\int \cos^2 x \; dx$$

It's a very common integral in problems with trig substitution and otherwise. The first thing to note is that

$$\int \sin^2 x \; dx$$

is the same problem, because

$$\sin^2 x + \cos^2 x = 1$$

so

$$\int \sin^2 x \; dx + \int \cos^2 x \; dx = \int 1 \; dx = x$$

method 0

I call this method 0 because it's not really methodical. The first approach is to guess. If you play around differentiating products of functions (like e^x, $\ln x$, $\sin x$, $\cos x$ and x), you will discover that

$$\frac{d}{dx} \left[\; \sin x \cos x \; \right] \; = -\sin^2 x + \cos^2 x$$

$$
\begin{aligned}
&= \cos^2 x - 1 + \cos^2 x \\
&= 2\cos^2 x - 1
\end{aligned}
$$

Integrating both sides, we obtain

$$\sin x \cos x = 2 \int \cos^2 x \; dx - x$$

and rearranging:

$$\int \cos^2 x \; dx = \frac{1}{2}(x + \sin x \cos x)$$

method 1

There are two other systematic approaches that can be contrasted. The first, which is arguably the simpler one, is to remember the addition formula for cosine

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

The trick I use to remember these formulas is to work out the consequences for this one:

$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

This makes perfect sense since if $s = t$ then we get

$$\cos 0 = \cos^2 s + \sin^2 s = 1$$

which we know is correct. So

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

If $s = t$ then (changing to x)

$$\cos 2x = \cos^2 x - \sin^2 x$$

and using the standard identity $\cos^2 x + \sin^2 x = 1$ this becomes

$$\cos 2x = 2\cos^2 x - 1$$

The "double angle" formula.

$$2\cos^2 x = 1 + \cos 2x$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Integrating

$$\int \cos^2 x \; dx = \int \frac{1}{2}(1 + \cos 2x) \; dx$$
$$\frac{1}{2}(x + \frac{1}{2}\sin 2x)$$

We check by differentiating. Leaving the factor of 1/2 out, we obtain for d/dx:

$$1 + \cos 2x$$

which, as we saw above, is equal to $2\cos^2 x$. Remembering the factor of 1/2, we obtain the expected result.

Comparing our results so far, we have obtained different answers, namely

$$\int \cos^2 x \; dx = \frac{1}{2}(x + \sin x \cos x)$$

$$\int \cos^2 x \; dx = \frac{1}{2}(x + \frac{1}{2}\sin 2x)$$

which indicates (if there is no mistake), that

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

to see that this is correct, recall the addition formula for sine:

$$\sin(s + t) = \sin s \cos t + \sin t \cos s$$

then if $s = t$

$$\sin 2s = 2 \sin s \cos s$$

with a slight rearrangement, this is indeed what we had.

method 2

In the second "method", we do a substitution to take advantage of the integration by parts formula

$$\int u \; dv = uv - \int v \; du$$

Let $u = \cos x$, then $du = -\sin x \; dx$, and let $dv = \cos x \; dx$ then $v = \sin x$, so

$$\int \cos^2 x \; dx = \sin x \cos x + \int \sin^2 x \; dx$$

This still seems like not much progress since (as we saw) $\int \sin^2 x \; dx$ is really the same problem as $\int \cos^2 x \; dx$

$$\int \sin^2 x \; dx = \int (1 - \cos^2 x) dx = \int dx - \int \cos^2 x dx$$

but, forging ahead

$$\int \cos^2 x \ dx = \sin x \cos x + \int \sin^2 x \ dx$$

$$\int \cos^2 x \ dx = \sin x \cos x + x - \int \cos^2 x dx$$

Rearranging:

$$\int \cos^2 x \ dx = \frac{1}{2} \ [\ \sin x \cos x + x \ ]$$

which is what we had before.

Dealing with higher powers

Most higher powers of sine and cosine are fairly easy to work with, after using the basic trig identity. Here is the cube:

$$\int \cos^3 x \ dx = \int (1 - \sin^2 x) \cos x \ dx$$

We end up with two terms, $\int \cos x \ dx$, which is trivial, and $\int \sin^2 x \cos x \ dx$, which yields to substitution (let $u = \sin x$).

But, sometimes, we get an even power, for example:

$$\int \cos^4 x \ dx$$

What this problem needs is to forget about the integration for a moment and do two applications of the double angle formula:

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Rewrite

$$\cos^4 x = (\cos^2 x)^2$$
$$= \ [\ \frac{1}{2}(1+\cos 2x)\ ]^2$$
$$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$

now use the formula a second time to substitute for

$$\cos^2 2x = \frac{1}{2}(1+\cos 4x)$$

I get

$$\cos^4 x = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$
$$\int \cos^4 x \ dx = \frac{1}{8}\int \ [\ 3 + 4\cos 2x + \cos 4x\ ]\ dx$$
$$= \frac{1}{8}\ [\ 3x + 2\sin 2x + \frac{1}{4}\sin 4x\ ]$$

See Strang (pp 289-290).