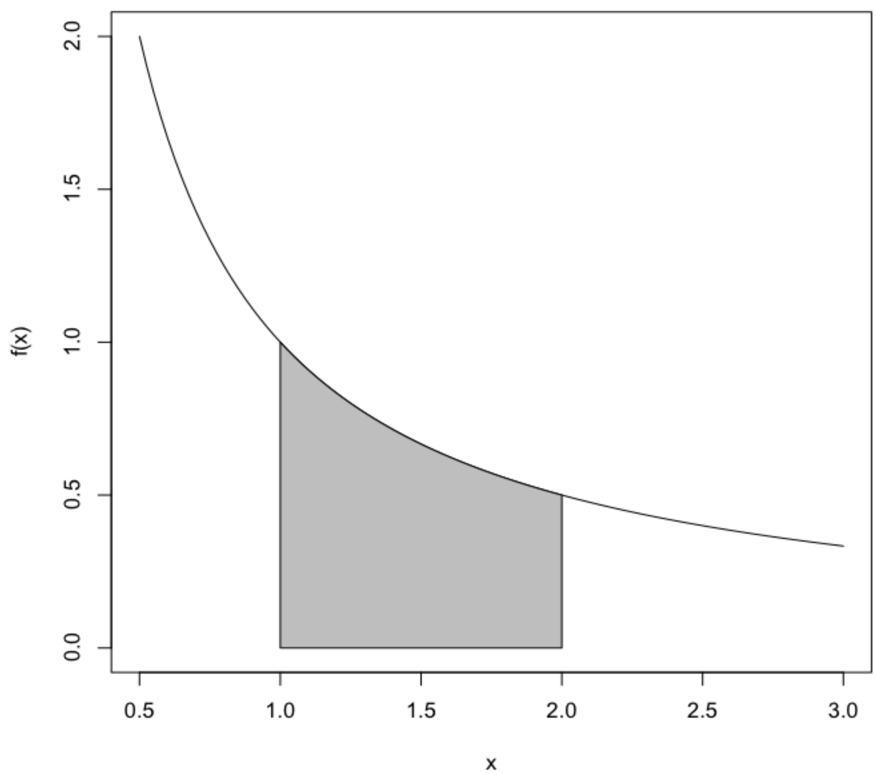
Properties of the logarithm

This comes straight from David Jerrison's lecture in Calculus 1. We define the logarithm function as

$$L(x) = \int_1^x \frac{dt}{t}$$

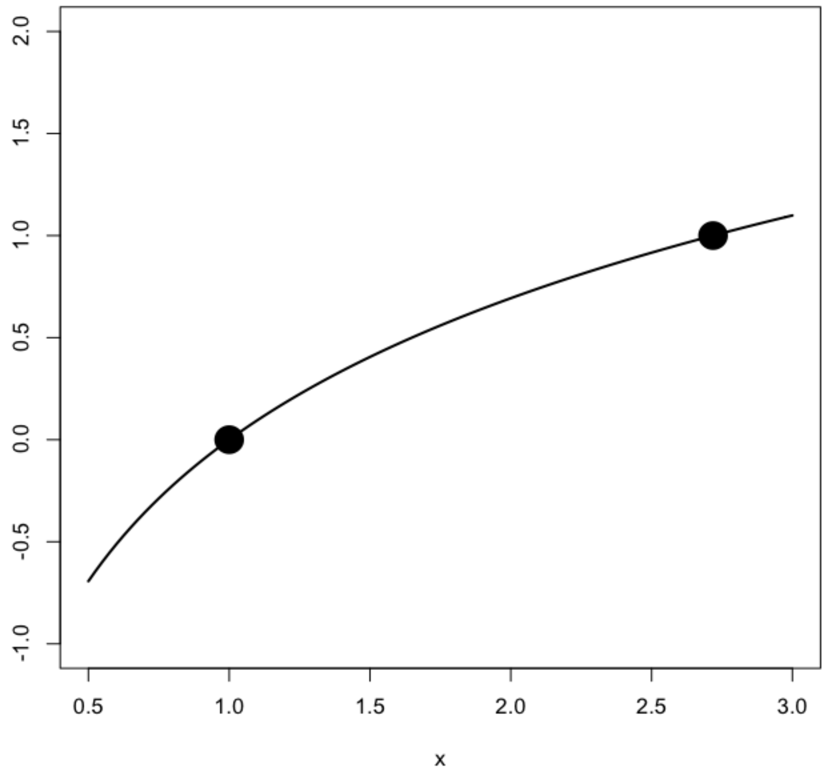


The logarithm of 2 is the area under the curve above, $f(x) = 1/x$, between $1 < x < 2$. By the Fundamental Theorem of Calculus (part II) we have

Property 1

$$L'(x) = \frac{1}{x}$$

The slope of the logarithm function is always positive ($x > 0$), but is undefined for $x = 0$



Property 2

$$L(1) = \int_1^1 \frac{dt}{t} = 0$$

This property is by definition. It fits with our use of exponents, where $b^0 = 1$.

Property 3

$$L''(x) = -\frac{1}{x^2}$$

Although the area under the curve $ln(x)$ is always increasing, so the slope is always positive, the rate of increase of the slope is always decreasing, so the shape is concave down.

Property 4

$$L(e) = 1$$

This is by definition as well. In extending to exponents it means we can write $y = ln(x) \iff e^y = x$.

Property 5

$$L(ab) = L(a) + L(b)$$

To show that this last statement is true involves showing that this is equivalent

$$\int_1^{ab} \frac{dt}{t} = \int_1^{a} \frac{dt}{t} + \int_a^{ab} \frac{dt}{t}$$

For the arguments a and ab we have

$$L(ab) = \int_1^{ab} \frac{dt}{t}$$

$$L(a) = \int_1^{a} \frac{dt}{t}$$

Both of these are true by definition. The one that takes a little work is

$$L(b) = \int_a^{ab} \frac{dt}{t}$$

Substitute $au = t$, then $a\ du = dt$ and

$$L(b) = \int \frac{a\ du}{au} = \int \frac{du}{u}$$

with a change in the limits

$$t = a \Rightarrow u = 1$$

$$t = ab \Rightarrow u = b$$

So it's just

$$L(b) = \int_1^{b} \frac{du}{u}$$

which is again, true by definition. So the function L has the property that $L(ab) = L(a) + L(b)$, which is one of the two major properties of logarithms.

To see that the second is also true, start with

$$L(a^r) = \int_1^{a^r} \frac{dt}{t}$$

Substitute $t = u^r$, so $dt = ru^{r-1}du$, and the limits become

$$t = 1 \Rightarrow u = 1$$

$$t = a^r \Rightarrow u = a$$

$$L(a^r) = \int_{t=1}^{t=a^r} \frac{dt}{t} = \int_{u=1}^{u=a} \frac{1}{u^r}(ru^{r-1})du = r\int_{u=1}^{u=a} \frac{du}{u} = rL(a)$$

As Dunham says (using A for L) "these properties of the hyperbolic area—namely $A(ab) = A(a) + A(b)$ and $A(a^r) = rA(a)$—exactly mirror the corresponding properties of logarithms. Clearly something interesting is afoot."

R code

```
f <- function(x) { return (1/x) }
plot(f, 0.5,3,cex=2,ylim=c(0,2))
xvals = seq(1,2,length=100)
yvals = f(xvals)
x = c(xvals,rev(xvals))
y = c(rep(0,100),rev(yvals))
polygon(x,y,col='gray')

f <- function(x) { return (log(x)) }
plot(f, 0.5,3,lwd=2,ylim=c(-1,2))
points(1,f(1),pch=16,cex=3)
points(2.71828,f(2.71828),pch=16,cex=3)
```