scratch

example

`https://www.youtube.com/watch?v=1m7OlAiEzOY`

Find the power series expansion for the following function about the given point c, valid for the given region R

$$f(z) = \frac{1}{(z+1)(z+3)}, \quad c = 1$$

$$R = \{z : 2 < |z-1| < 4\}$$

We're expanding around $c = 1$ (i.e. $(1+0i)$). We're working with an annulus of inner radius 2 and outer radius 4, which include the two singularities at $z = -1$ and $z = -3$.

We expand the function using partial fractions

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

We have for the numerator that

$$Az + 3A + Bz + B = 1$$

so $A = -B$ and then $-2B = 1$ so $B = -1/2$ and $A = 1/2$ and:

$$f(z) = \frac{1}{2}\ [\ \frac{1}{z+1} - \frac{1}{z+3}\ ]$$

and now, according to the video, we are looking for a Laurent series for the first one and a Taylor series for the second one (we're inside the singularity).

For the Taylor series (around the point 1):

$$\frac{1}{z+3} = \frac{1}{(z-1)+4} = \frac{1}{4}\,\frac{1}{1+\frac{(z-1)}{4}}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{4^n}, \quad \text{for } z-1<4$$

So where does this come from? Recall that the geometric series is the Taylor series for $1/1-x$ since

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

We can see that this is correct by multiplying out. Or, we can take the derivatives:

$$f'(x) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{2}{(1-x)^3}$$

$$f'''(x) = \frac{3!}{(1-x)^4}$$

The terms (evaluated at $a=0$) are

$$\frac{1}{n!}f^n(a)(x-a)^n = x^n$$

For our series we must substitute $-x = x$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

which accounts for the $(-1)^n$ term. The rest can be obtained by rescaling the variable.