Continued fractions

Continued fractions are awkward, but they're powerful.

`http://www.mathpath.org/concepts/cont.frac.htm`

rational number

Consider 153/53

$$\frac{153}{53} = 2 + \frac{47}{53} = 2 + \frac{1}{53/47}$$

But

$$\frac{53}{47} = 1 + \frac{6}{47} = 1 + \frac{1}{47/6}$$

And

$$\frac{47}{6} = 7 + \frac{5}{6} = 7 + \frac{1}{6/5}$$

And

$$\frac{6}{5} = 1 + \frac{1}{5}$$

All together

$$\frac{153}{53} = 2 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{5}}}}$$

Hard to think about, and hard to write, let alone typeset.

If all the numerators are 1's like this, the continued fraction is called simple.

Really, this is just the Euclidean algorithm in disguise. And that algorithm must terminate, for a *rational* (fractional) number.

One alternative representation is

$$2 + \frac{1}{1+}\ \frac{1}{7+}\ \frac{1}{1+}\ \frac{1}{5}$$

The continued fraction representation may also be written as

```
153/53 = [2,1,7,1,5]
```

To go backwards:

```
1 + 1/5 = 6/5
7 + 5/6 = 47/6
1 + 6/47 = 53/47
2 + 47/53 = 153/53
```

square root of 2

One can also find continued fraction representations of irrational numbers like $\sqrt{2}$. However, since the difference is that these cannot terminate (because they are not rational, i.e. fractional).

$$(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$$

Rearrange to get a substitution we will use again

$$\sqrt{2} - 1 = \frac{1}{\sqrt{2} + 1}$$

At the same time, add one and subtract one on the bottom right:

$$\sqrt{2} - 1 = \frac{1}{2 + \sqrt{2} - 1}$$

substitute

$$= \frac{1}{2 + \frac{1}{\sqrt{2}+1}}$$

Add one and subtract one again and then substitute

$$= \frac{1}{2 + \frac{1}{2+\sqrt{2}-1}} = \frac{1}{2 + \frac{1}{2+\frac{1}{\sqrt{2}+1}}}$$

Clearly, this goes on forever.

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$$

The continued fraction representation of $\sqrt{2}$ is $[1 : 2]$, meaning that there is an initial 1 followed by repeated 2's.

To turn this into an approximate decimal representation of $\sqrt{2}$, ignore the .... Then the last fraction is 5/2. Invert and add, repeatedly:

```
2 + 1/2 = 5/2
2 + 2/5 = 12/5
2 + 5/12 = 29/12
2 + 12/29 = 71/29
2 + 29/71 = 171/71
2 + 71/171 = 413/171
```

To terminate:

```
1 + 171/413 = 584/413 = 1.414043
```

To six places, $\sqrt{2} = 1.414213$. We have three.

square root of 3

The continued fraction representation of $\sqrt{3}$ is $[1, 1, 2, 1, 2, \dots]$.

This can be shortened to $[1 : (1, 2)]$. Here is a derivation:

$$(\sqrt{3}-1)(\sqrt{3}+1) = 2$$
$$\sqrt{3}-1 = \frac{2}{\sqrt{3}+1}$$
$$\frac{\sqrt{3}-1}{2} = \frac{1}{\sqrt{3}+1}$$

both of which we will use again. However, going further, add and subtract on the bottom right

$$\sqrt{3}-1 = \frac{2}{\sqrt{3}+1} = \frac{2}{2+\sqrt{3}-1}$$

Divide top and bottom by 2

$$= \frac{1}{1+\frac{\sqrt{3}-1}{2}}$$

and substitute giving

$$= \frac{1}{1+\frac{1}{\sqrt{3}+1}}$$

That's the end of step 1.

Now, for the second step, we focus on that last fraction

$$\frac{1}{\sqrt{3}+1} = \frac{1}{2+\sqrt{3}-1} = \frac{1}{2+\frac{2}{\sqrt{3}+1}}$$

Then for step three, we focus again on the last fraction, which is what we worked with in the first part.

$$\frac{2}{\sqrt{3}+1} = \frac{1}{1+\frac{1}{\sqrt{3}+1}}$$

So now both terms repeat:

$$\sqrt{3} - 1 = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}}$$

which is $[1 : (1, 2)]$, as we said.

We can get approximations for $\sqrt{3}$ similar to what we did for $\sqrt{2}$. Unlike previously, here there are two possibilities. We start with either one of

$$1 + \frac{1}{2 + \dots}$$
$$2 + \frac{1}{1 + \dots}$$

and start by ignoring the dots.

The first gives

```
1 + 1/2 = 3/2
2 + 2/3 = 8/3
1 + 3/8 = 11/8
2 + 8/11 = 30/11
1 + 11/30 = 41/30
2 + 30/41 = 112/41
1 + 41/112 = 153/112

1 + 112/153 = 265/153 = 1.732026
```

The actual value is $\sqrt{3} = 1.732051$, to six places. We have four.

The second gives

```
2 + 1 = 3
1 + 1/3 = 4/3
2 + 3/4 = 11/4
1 + 4/11 = 15/11
2 + 11/15 = 41/15
1 + 15/41 = 56/41
2 + 41/56 = 153/56
1 + 56/153 = 209/153
2 + 153/209 = 571/209
1 + 209/571 = 780/571

1 + 571/780 = 1351/780 = 1.732051
```

The actual value is $\sqrt{3} = 1.732051$, to six places. We have all six.