Orthonormal basis

We know that if we start with a matrix with independent columns, we have a "basis" for the space. That is, we can express any vector as a combination of the columns of A.

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Here, we calculate $det(A) = -(2)(6-8) - (2)(6-8) = 8 \neq 0$. So we know the columns of A are independent.

However, we are asked to find an orthonormal basis, one with vectors that are orthogonal ("ortho") and unit length ("normal").

One thing to notice before we start: the sum of the diagonal (called the trace) is 6, so when we get the eigenvalues they should add up to 6, while they multiply to give the determinant.

$$det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

I will put the three terms of the determinant on separate lines

$$(3-\lambda)[(-\lambda)(3-\lambda) - 4)] = (3-\lambda)(\lambda^2 - 3\lambda - 4) = -\lambda^3 + 6\lambda^2 - 5\lambda - 12$$

$$(-2)[(2)(3-\lambda) - 8] = (-2)(6 - 2\lambda - 8) = 4\lambda + 4$$

$$(4)[4 - (4)(-\lambda)] = (4)(4 + 4\lambda) = 16 + 16\lambda$$

Adding them all up

$$-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$$

I'm not too good at factoring cubics, but I notice that $\lambda = -1$ is a solution, so that means that $(\lambda + 1)$ is a factor leading to

$$(\lambda + 1)(-\lambda^2 + 7\lambda + 8) = (-1)(\lambda + 1)(\lambda - 8)(\lambda + 1) = 0$$

So finally we have eigenvalues $\lambda = 8, -1, -1$. Notice that these do indeed add up to give the trace, and multiply to give the determinant.

Sidestep finding eigenvectors for now

$$\mathbf{u} = < 0, -2, 1 >$$

$$\mathbf{v} = < 1, -2, 0 >$$

$$\mathbf{w} = < 2, 1, 2 >$$

$\mathbf{w}$ is the eigenvector for eigenvalue $\lambda = 8$. Notice that although $\mathbf{u} \perp \mathbf{w}$ and $\mathbf{v} \perp \mathbf{w}$, $\mathbf{u} \cdot \mathbf{v} \neq 0$.

I didn't mention this before, but notice now that A is a symmetric matrix. It turns out that the eigenvectors of a symmetric matrix are guaranteed to be orthogonal, *unless there are repeated eigenvalues.* In that case the combinations of the eigenvectors form what is called an eigenspace, and any vector in that eigenspace will be orthogonal to $\mathbf{w}$.

This is easily checked. For constants c and d, we have the combinations $c\mathbf{u}$ and $d\mathbf{v}$ are

$$< 0, -2c, c > + < d, -2d, 0 > = < d, -2c - 2d, c >$$

$$< d, -2c - 2d, c > \cdot < 2, 1, 2 > = 2d - 2c - 2d + 2c = 0$$

Let's take $\mathbf{w}$ and $\mathbf{v}$ as already orthogonal, and then produce from $\mathbf{u}$ a new vector orthogonal to both. The formula we want (see the Projections write-up) is

$$\mathbf{u}' = \mathbf{u} - \frac{w^T u}{w^T w} w - \frac{v^T u}{v^T v} v$$

$$\mathbf{w} \cdot \mathbf{u} = 0$$

So this means that the whole second term is zero.

$$\mathbf{v} \cdot \mathbf{u} = 4$$

$$\mathbf{v} \cdot \mathbf{v} = 5$$

So we have

$$\mathbf{u}' = \mathbf{u} - (4/5)\mathbf{v}$$

$$\mathbf{u} = < 0, -2, 1 > = (1/5) < 0, -10, 5 >$$

$$(4/5)\mathbf{v} = (1/5) \ < 4, -8, 0 >$$

$$\mathbf{u}' = \mathbf{u} - (4/5)\mathbf{v} = (1/5)(< 0, -10, 5 > \ - \ < 4, -8, 0 >) = (1/5) < -4, -2, 5 >$$

Check that $\mathbf{u}' \perp \mathbf{v}$ and $\mathbf{u}' \perp \mathbf{w}$.

The last step is to scale these to be unit vectors. We have

$$\mathbf{u}' = (1/5) < -4, -2, 5 >$$

As a unit vector, this is

$$\mathbf{u}' = (1/3\sqrt{5}) < -4, -2, 5 >$$

Then

$$\mathbf{v} = < 1, -2, 0 >$$

becomes

$$\mathbf{v}' = \ (1/\sqrt{5}) < 1, -2, 0 >$$

The easiest one is

$$\mathbf{w} = < 2, 1, 2 >$$

$$\mathbf{w}' = < 2/3, 1/3, 2/3 >$$

I want to check that the diagonalization works properly.

$$\mathbf{u}' = (1/3\sqrt{5}) < -4, -2, 5 >$$

$$\mathbf{v}' = \ (1/\sqrt{5}) < 1, -2, 0 >$$

$$\mathbf{w}' = \ (1/3) < 2, 1, 2 >$$

We should have that

$$Q\Lambda Q^{-1} = A$$

Luckily, for an orthonormal matrix

$$Q^{-1} = Q^T$$

But even so, I don't want to do this by hand. In R

[1]

```
> A = c(3,2,4,2,0,2,4,2,3)
> dim(A) = c(3,3)
> A
     [,1] [,2] [,3]
[1,]    3    2    4
[2,]    2    0    2
[3,]    4    2    3
> result = eigen(A)
> result
```

```
$values
[1]  8 −1 −1

$vectors
          [,1]       [,2]       [,3]
[1,] 0.6666667  0.7453560  0.0000000
[2,] 0.3333333 −0.2981424 −0.8944272
[3,] 0.6666667 −0.5962848  0.4472136
```

Our first vector is $\mathbf{w}'$ (they have listed the largest eigenvalue first). The second vector is $\mathbf{u}'$. They have listed the vectors with their components in opposite order (bottom to top). Remember that $\mathbf{u}' =< -4, -2, 5 >$ with a factor of $(1/3\sqrt{5}) = 0.1491$, which gives the result shown.

[1]
```
>>> f = 5**0.5 * 3
>>> 1/f
0.14907119849998599
>>> 2/f
0.29814239699997197
>>> 4/f
0.59628479399994394
>>> 5/f
0.7453559924999299
```

$\mathbf{v}' =< 1, -2, 0 >$ with a leading factor of $(1/\sqrt{5}) = 0.4472$.

[1]
```
>>> f = 5**0.5
>>> 1/f
0.4472135954999579
>>> 2/f
0.8944271909999159
```

So this also matches. Now we just do

[1]
```
> L = diag(result$values)
> L
     [,1] [,2] [,3]
[1,]    8    0    0
[2,]    0   −1    0
[3,]    0    0   −1
```

```
> S = result$vectors
> S
          [,1]       [,2]       [,3]
[1,] 0.6666667  0.7453560  0.0000000
[2,] 0.3333333 -0.2981424 -0.8944272
[3,] 0.6666667 -0.5962848  0.4472136
> Si = solve(S)
> Si
          [,1]       [,2]       [,3]
[1,] 0.6666667  0.3333333  0.6666667
[2,] 0.7453560 -0.2981424 -0.5962848
[3,] 0.0000000 -0.8944272  0.4472136
```

"solve" is R's way of finding the inverse. Notice that $Q^{-1} = Q^T$, as we said.

```
[1]
> S %*% L %*% Si
     [,1]         [,2] [,3]
[1,]    3 2.000000e+00    4
[2,]    2 1.776357e-15    2
[3,]    4 2.000000e+00    3
```

If you round the second column to $< 2, 0, 1 >$, you will see that we have re-generated A.