Inverse

This short write-up discusses the matrix inverse. Consider a basic matrix equation such as

$$A\mathbf{x} = \mathbf{b}$$

where A is a 2×2 matrix, and $\mathbf{b}$ is a *known* vector of size 2, while $\mathbf{x}$ is a vector of *unknowns.*

In other words, the above is shorthand for

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Which can in turn be written as two simultaneous linear equations

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

There are several ways to solve a system like this one. One method is Gaussian elimination. A second way is to find the inverse of the matrix A. The inverse has the notation A^{-1} and it is defined by this property

$$AA^{-1} = I = A^{-1}A$$

where I is the identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since

$$I\mathbf{x} = \mathbf{x}$$

in other words

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

it should be clear that if we multiply our original equation on both sides by A^{-1}

$$A^{-1}A\mathbf{x} = \mathbf{x} = A^{-1}\mathbf{b}$$

the left-hand side has become just $\mathbf{x}$, which is our unknown, while we know how to compute $A^{-1}\mathbf{b}$, given A^{-1}.

finding the inverse

For this part I want to switch symbols:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

We know $abcd$ and we need to find $efgh$. The result we want is $A \times A^{-1} = I$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We have four equations, but if you write them out you'll see it looks like a bit of a mess! Luckily, I know an easy formula for the 2×2 case:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We switch a and d, negate b and c, and multiply by 1 over the determinant of A. I claim that

$$A^{-1}A = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

I think it is clear by inspection that this is correct. For example, the first term is $1/(ad - bc) \times (ad - bc) = 1$.

There are many other formulas for finding the inverse (see wikipedia). A standard approach is to go through the steps of Gaussian elimination to convert A to I. These can be represented as a series of matrix multiplications. As an example, suppose we have

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$$

The steps of Gaussian elimination (all the way to I) generate in turn

$$\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix multiplications that do this are:

$$E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$E_1 A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix}$$

$$E_2(E_1 A) = \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$$

$$E_3(E_2(E_1 A))) = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Putting it all together

$$E_3 \times E_2 \times E_1 \times A = I$$

Therefore

$$E_3 \times E_2 \times E_1 = A^{-1}$$

$$E_3 \times E_2 \times E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/4 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 \\ -1/2 & 1/4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}$$

If you remember the rule for 2×2 from above, you'll see that we did indeed switch a with d, negate b and c, and multiply by $1/det(A)$.