Integrating functions of a complex variable

Complex functions are differentiated and integrated in a way that is similar to real functions, with these differences: normally, we restrict our attention to functions that are analytic, and we pay attention to points in the complex plane where they have poles (or singularities).

Also, the integrals that we compute are line integrals. Let's do one.

example

In general terms, if we have

$$z = x + iy$$

$$dz = dx + idy$$

and the function

$$\begin{aligned} w &= f(z) \\ &= u(x,y) + iv(x,y) \end{aligned}$$

we can write either

$$\int f(z)\ dz = \int (x+iy)(dx+idy)$$

or what may be less confusing:

$$\int f(z)\ dz = \int (u+iv)(dx+idy)$$

$$= \int u\ dx - \int v\ dy + i\ [\ \int v\ dx + \int u\ dy\ ]$$

What was an integral of a complex function has been transformed into two integrals of real variables, where the real variables are related by the curve over which we will integrate.

Just as with line integrals for real functions of x and y, this is *not* some kind of double integral in both variables. We can view y as a function of x or perhaps, we can parametrize both x and y as functions of t.

Recall that for the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = M\ dx + N\ dy$$

we parametrize the curve to get the integral over a single variable.

Suppose our function is simply $z = x + iy$. The integral is

$$\int z\ dz = \int (x + iy)(dx + idy)$$

$$= \int x\ dx - y\ dy + ix\ dy + iy\ dx$$

Now we must get y in terms of x from the curve. Suppose the curve goes from $(1, i)$ to $(3, i)$, then to $(3, 3i)$ and finally back to where we started.

We have three segments. Along the first part, we are moving in the positive x direction, with no change in y, so $dy = 0$ and $y = 1$, a constant, and the integral is

$$\int x\ dx - y\ dy + ix\ dy + iy\ dx$$

$$= \int x\ dx + iy\ dx$$

$$= \int_{x=1}^{x=3} x + i\ dx$$

$$= \frac{x^2}{2} + ix\ \Big|_1^3$$

$$= 4 + 2i$$

Along the second part, we are moving in the positive y direction with $dx = 0$ and $x = 3$ so

$$\int x\ dx - y\ dy + ix\ dy + iy\ dx$$

$$= \int y_{y=1}^{y=3} - y\ dy + 3i\ dy$$

$$= -\frac{y^2}{2} + 3iy\ \Big|_1^3$$

$$= -4 + 6i$$

And for the third, we have that both dx and dy are non-zero, so we must actually do the parametrization. The curve is $y = x$. Hence $dy = dx$.

$$\int x\ dx - y\ dy + ix\ dy + iy\ dx$$

$$\int x\ dx - x\ dx + ix\ dx + ix\ dx$$

Suppose we take the path as going from $(1, 1)$ to $(3, 3)$.

$$= 2i \int_{x=1}^{x=3} x\ dx$$

$$= 2i\ \frac{x^2}{2}\ \Big|_1^3$$

$$= 2i\,\frac{8}{2} = 8i$$

Notice that

$$\int_{C1} + \int_{C2} = \int_{C3} = 8i$$

Alternatively, if we follow the curve $C3$ from $(3, 3)$ to $(1, 1)$, the whole thing is just zero. We will see later that this is not a coincidence.

example

Suppose that the function is

$$f(z) = y - x - i3x^2$$

and we proceed from the origin to the point $z = 1 + i$ either directly (C_2) or by first going up vertically and then across (C_1).

For the vertical part of C_1 we have that $x = 0$ and $dx = 0$.

$$I = \int (y - x - i3x^2)\ (dx + idy)$$

$$= \int yi\ dy$$

It's important to recognize that although we are proceeding from the point $z = 0$ to the point $z = i$, the upper bound on this integral is not i but 1! Hence

$$I = i\frac{y^2}{2}\ \Big|_0^1 = \frac{i}{2}$$

For the horizontal part of C_1 we have that $y = 1$ and $dy = 0$ so

$$I = \int (y - x - i3x^2)\ (dx + idy)$$

$$= \int (1 - x - i3x^2)\ dx$$

$$= x - \frac{x^2}{2} - ix^3 \Big|_0^1 = \frac{1}{2} - i$$

Therefore the total

$$I = \frac{i}{2} + \frac{1}{2} - i = \frac{1}{2}\ (1 - i)$$

When going directly from the origin to $1 + i$ we related x to y by the equation of the line $y = x$ so $dy = dx$ and

$$I = \int (y - x - i3x^2)\ (dx + idy)$$

$$= \int -i3x^2\ (dx + idx)$$

$$= -i\ (x^3 \Big|_0^1 + ix^3 \Big|_0^1) = -i(1 + i) = 1 - i$$

And around the closed curve going backward along C_2:

$$\oint f(z)\ dz = \frac{1}{2}\ (1 - i) - (1 - i) = -\frac{1}{2}(1 + i)$$

example

If the contour (curve) of integration C is parametrized in terms of t, then

$$\int_C f(z)\ dz = \int_a^b f[z(t)]\ z'(t)\ dt$$

For this example, take $f(z) = z*$ (note that this function is *not* analytic). Suppose our curve is the circle of radius 2 centered at the origin, and we proceed between the endpoints $z = -2i \to 2i$. On this curve

$$z = 2e^{i\theta}$$

we have

$$dz = 2ie^{i\theta}\ d\theta$$

In radial coordinates

$$z* = 2e^{-i\theta}$$

so we have

$$\int z*\ dz = \int 2e^{-i\theta} 2ie^{i\theta}\ d\theta$$

$$= 4i \int_{-\pi/2}^{\pi/2} d\theta = 4\pi i$$

Alternatively,

$$zz* = |z|^2 = 2^2 = 4$$

$$\int z*\ dz = 4 \int \frac{1}{z}\ dz$$

Again

$$z = 2e^{i\theta}$$

$$dz = 2ie^{i\theta}\ d\theta = iz\ d\theta$$

So the integral is just

$$\int z*\ dz = 4 \int \frac{1}{z}\ dz = 4 \int \frac{1}{z}\ iz\ d\theta = 4i \int d\theta = 4\pi i$$

example

Consider $f(z) = z^2$. For the path, take the unit circle over the first quadrant from $(1, 0)$ to $(0, 1)$. There is an easy way to do this, and a hard way. Let's start by checking that this function is analytic, and then doing the hard way first.

Write z in terms of x and y:

$$z = x + iy$$

$$z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$$

$$u_x = 2x = v_y$$

$$u_y = -2y = -v_x$$

The CRE hold.

Also

$$dz = dx + i\ dy$$

So

$$\int z^2\ dz = \int (x^2 - y^2 + 2ixy)(dx + i\ dy)$$

$$= \int (x^2 - y^2)\ dx - \int 2xy\ dy + i \int 2xy\ dx + i \int (x^2 - y^2)\ dy$$

As before, we must parametrize this using the relationship between x and y along the curve.

$$x = \cos t$$

$$y = \sin t$$

$$dx = -\sin t\ dt$$

$$dy = \cos t\ dt$$

and then

$$x^2 - y^2 = \cos^2 t - \sin^2 t = \cos 2t$$

$$2xy = 2\cos t \sin t = \sin 2t$$

so the integral is

$$= \int -\cos 2t\ \ \sin t\ dt - \int \sin 2t\ \ \cos t\ dt + \ldots$$

$$+\ i\ [\int -\sin 2t\ \ \sin t\ dt + \int \cos 2t\ \ \cos t\ dt$$

Looks pretty wild! In the book they use some trig identities I hadn't seen before, namely starting with the standard

$$\sin s + t = \sin s \cos t + \sin t \cos s$$

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

then, if $s = 2t$ then

$$\sin 3t = \sin 2t \cos t + \sin t \cos 2t$$

$$\cos 3t = \cos 2t \cos t - \sin 2t \sin t$$

Looking at the real part of the integral we had (combining terms)

$$\int -\cos 2t \ \ \sin t - \sin 2t \ \ \cos t \ dt = \int -\sin 3t \ dt = \frac{\cos 3t}{3}$$

and for the imaginary part of the integral

$$i \ [\int -\sin 2t \ \ \sin t + \cos 2t \ \ \cos t \ dt = i \int \cos 3t \ dt = i \frac{\sin 3t}{3}$$

That looks a lot better.

$$\frac{\cos 3t}{3} + i \frac{\sin 3t}{3} \ \Big|_0^{\pi/2} = -\frac{1}{3} - i\frac{1}{3} = -\frac{1}{3}(1+i)$$

For the easy way, just treat z as if it were a real variable

$$\int z^2 \ dz = \frac{z^3}{3} \ \Big|_1^i = -\frac{1}{3}i - \frac{1}{3}$$

Note that if we go all the way around the unit circle the integral is just zero.

Going back to the first example we had

$$\int z \ dz = \frac{z^2}{2} \ \Big|_{1+i}^{3+3i}$$

$$= \frac{9 - 9 + 18i - \ [\ 1 - 1 + 2i \ ]}{2}$$

$$= 8i$$

example

$$\int_0^{2\pi} \frac{1}{z}\ dz$$

Examining the inverse function, let's first confirm that it is analytic by calculating the partial derivatives. We have

$$\frac{1}{z} = \frac{1}{x+iy}$$

One way to simplify is to multiply on top and bottom by $z*$:

$$= \frac{1}{x+iy}\ \frac{x-iy}{x-iy}$$

$$= \frac{x-iy}{x^2+y^2}$$

Thus

$$u = \frac{x}{x^2+y^2}$$

$$u_x = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = \frac{-2xy}{(x^2+y^2)^2}$$

And

$$v = \frac{-y}{x^2+y^2}$$

$$v_y = -\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$v_x = \frac{2xy}{(x^2+y^2)^2}$$

CRE are satisfied and the inverse of z is indeed analytic.

If we are on the unit circle, then

$$z = e^{i\theta}$$

$$dz = ie^{i\theta} d\theta$$

$$\int \frac{dz}{z} = \int e^{-i\theta} \; ie^{i\theta} \; d\theta = 2\pi i$$

If we're centered on the origin but we don't have a unit circle, there will be an R in both the numerator and the denominator, which cancel. The result is thus independent of the radius of the circle.

In general

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 0, & n \neq 1 \\ 2\pi i, & n = 1 \end{cases}$$

We can also integrate the inverse function in terms of x and y:

$$\oint \frac{1}{z} \; dz = \oint \frac{dx + idy}{x + iy}$$

$$= \oint \frac{1}{x^2 + y^2} \; [\; x \; dx - y \; dy + ix \; dy + iy \; dx \;]$$

Suppose we go on a circle of radius R centered on the origin and parametrize in terms of θ. We obtain:

$$x = R\cos\theta$$

$$y = R\sin\theta$$

$$x^2 + y^2 = R^2$$

$$dx = -R\sin\theta \; d\theta$$

$$dy = R\cos\theta \; d\theta$$

We have for the integral

$$\oint \frac{1}{x^2+y^2} \; [\; x\;dx - y\;dy + ix\;dy + iy\;dx \;]$$

$$= \int \frac{1}{R^2} [\, -R^2 \cos\theta \sin\theta \; d\theta + R^2 \sin\theta \cos\theta \; d\theta + iR^2 \cos^2\theta \; d\theta + iR^2 \sin^2\theta \; d\theta \,]$$

$$= \int \frac{1}{R^2} \; [\; iR^2 \cos^2\theta \; d\theta + iR^2 \sin^2\theta \; d\theta \;]$$

$$= \int i \cos^2\theta \; d\theta + i \sin^2\theta \; d\theta$$

$$= \int i \; d\theta = 2\pi i$$

Note that if we integrate the same function around a unit square, we run into problems. First let's do $[0, 0 \times 1, 1]$. We have

$$\int u \; dx - \int v \; dy + i \; [\; \int v \; dx + \int u \; dy \;]$$

Along $C1$, $y = 0$ and $dy = 0$ so:

$$\int \frac{x}{x^2+y^2} \; dx + i \; [\; \int \frac{-y}{x^2+y^2} \; dx$$

$$= \int_0^1 \frac{1}{x} \; dx = \ln x \; \Big|_0^1$$

Since $\ln 0$ is not defined, we can't do this.

Logarithms are tricky, no doubt. If the complex logarithm $Logz$ is defined and differentiable along the curve (say the semicircle from $-i$ to i), we can do this:

$$I = \int_{-i}^{i} \frac{1}{z} \; dz = Logz \; \Big|_{-i}^{i}$$

Recall that $z = re^{i\theta}$ with $r = 1$ so this is

$$= (\ln 1 + i\ \frac{\pi}{2}) - (\ln 1 + i\frac{-\pi}{2}) = 2i\ \frac{\pi}{2} = \pi i$$

For any value of r (except $r = 0$), we get the same answer, since $\ln r - \ln r = 0$.

example

We can extend this to

$$\oint \frac{1}{z^2}\ dz$$

As before, on the unit circle

$$z = e^{i\theta}$$

$$dz = iz\ d\theta$$

so the integral is

$$\int_0^{2\pi} \frac{i}{z}\ d\theta = \int_0^{2\pi} ie^{-i\theta}\ d\theta$$

Now

$$\int e^{-i\theta}\ d\theta = -ie^{-i\theta}$$

so cancel $i \cdot -i$ and we have just

$$= e^{-i\theta}\ \Big|_0^{2\pi}$$

Evaluate the first term using Euler's formula:

$$e^{-2\pi i} = \cos -2\pi + i \sin -2\pi$$

$$= \cos 2\pi - i \sin 2\pi = 1$$

So the whole thing is zero.

In fact, for any negative integer power of z

$$\int z^{-n}\ dz$$

around the unit circle $z = e^{i\theta}$ we have

$$i \int e^{-i(n-1)\theta}\ d\theta$$

$$= \frac{1}{n-1}\ e^{-i(n-1)\theta}\ \Big|_0^{2\pi}$$

$$= \frac{1}{n-1}\ [\ (\cos 2(n-1)\pi - i \sin 2(n-1)\pi)\ \ - 1]$$

$$= \frac{1}{n-1}\ [\ 1 - 1]\ \ = 0$$

example

Consider

$$\int \sqrt{z}\ dz$$

along the half-circle of radius 3 starting from the point $z = R$ on the x-axis and proceeding counter-clockwise. We can do this integral even if the "branch" of the square root function that we're using is only defined for $\theta > 0$. We have that

$$z = Re^{i\theta}, \quad \theta = 0 \to \pi$$

$$dz = iz = iRe^{i\theta}\ d\theta$$

$$\sqrt{z} = \sqrt{R}e^{i\theta/2}$$

so

$$I = \int_0^{\pi} iR\sqrt{R}e^{i3\theta/2}\ d\theta$$

We need

$$\int e^{i3\theta/2}\ d\theta = \frac{2}{3i} e^{i3\theta/2}\ \Big|_0^\pi$$

easiest to write it out as

$$e^{i3\theta/2}\ \Big|_0^\pi = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} - \cos 0 - i\sin 0$$

$$= 0 + i(-1) - 1 - 0 = -(1+i)$$

Going back to pick up all the factors we left behind:

$$I = -iR\sqrt{R}\ \frac{2}{3i}\ (1+i) = -R\sqrt{R}\ \frac{2}{3}\ (1+i)$$

In the problem, R was actually specified as 3, leading to the cancellation:

$$I = -2\sqrt{3}\ (1+i)$$

We can also do this problem by antiderivatives:

$$\int_R^{-R} \sqrt{z}\ dz = \frac{2}{3}\ z^{3/2}\ \Big|_R^{-R}$$

$$= \frac{2}{3}(R^{3/2}e^{i3\pi/2} - R^{3/2}e^0)$$

$$= \frac{2}{3}R^{3/2}(e^{i3\pi/2} - 1)$$

and, as we showed above:

$$e^{i3\pi/2} = -i$$

If $R = 3$ we get the same answer as before.