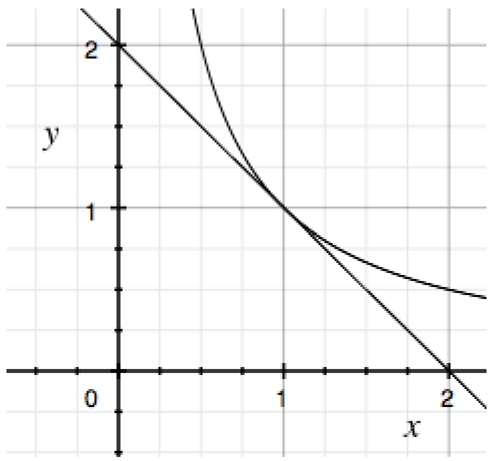
Linear Approximation



Consider the branch of the hyperbola $xy = 1$ for $x > 0$, in the first quadrant. Pick a point x_0, y_0—in the figure it is at $P = (1, 1)$ but it could be anywhere on the curve. The linear approximation to the curve at P is the line that goes through P and which has the same slope as the curve does at P.

$$f'(x) = \frac{d}{dx}\,\frac{1}{x} = -\frac{1}{x^2} = -\frac{1}{x_0^2} \quad \text{(at } x_0\text{)}$$

The equation of the line with this slope and going through P is

$$y - y_0 = -\frac{1}{x_0^2}(x - x_0)$$

Let's find the x-intercept ($y = 0$).

$$-y_0 = -\frac{1}{x_0^2}(x - x_0)$$

But remember that $y_0 = 1/x_0$!

$$-\frac{1}{x_0} = -\frac{1}{x_0^2}(x - x_0)$$

$$1 = \frac{1}{x_0}(x - x_0)$$
$$x_0 = x - x_0$$
$$x = 2x_0$$

We could do a similar calculation to find the y-intercept, but life is too short. By symmetry, x and y can be interchanged, so

$$y = 2y_0$$

And now a neat result is that the area of the triangle determined by this line and the two axes is

$$A = \frac{1}{2}\ 2x_0\ 2y_0 = 2x_0 y_0 = 2$$

The area is the same no matter which point P we pick. Maybe you could sketch a couple of these lines to check the result.