we can use our formula from above to conclude that

$$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} f(g(x)) = f\left(\lim_{x\to\infty} g(x)\right) = f(0) = \sin(0) = 0.$$

A more intuitive way of expressing this is that $1/x \to 0$ as $x \to \infty$, so $\sin(1/x) \to \sin(0) = 0$ as $x \to \infty$.

A.4.2 Proof of the Intermediate Value Theorem

In Section 5.1.4, we looked at the Intermediate Value Theorem, which says that if f is continuous on $[a, b]$, and also $f(a) < 0$ and $f(b) > 0$, then there is some number c such that $f(c) = 0$. Now we're going to look at the idea of the proof of this theorem.

Consider the set of values x in the interval $[a, b]$ such that $f(x) < 0$. We know that a is in this set, since $f(a) < 0$, and that b isn't in the set. We'd like to find the largest number c which is in the set, but that might not be possible. For example, what's the largest number less than 0 itself? There isn't one—for any negative number, you can always find a negative number closer to zero, for example, by dividing your number by 2. On the other hand, we can find a number c that is a sort of right-hand bookend of the set. In particular, we can insist that no member of the set is to the right of c, and also that any open interval with right-hand endpoint c includes at least one member of the set. (This is due to a nice property of the real line called *completeness*.) So here's what we know, written in symbols:

1. for any $x > c$, we have $f(x) \geq 0$; and
2. for any interval $(c - \delta, c)$ where $\delta > 0$, there is at least one point x in the interval such that $f(x) < 0$.

Now let's get busy. Here's the big question: what is $f(c)$? Suppose that it's negative. In that case, $c \neq b$ since $f(b) > 0$. Because f is continuous, the values of $f(x)$ should be near $f(c)$ when x is near c; this will be a problem when x is a little to the right of c, because $f(x)$ is supposed to be positive but $f(c)$ is negative. More formally, you can choose $\varepsilon = -f(c)/2$ (which is positive); then your tolerance interval is $(3f(c)/2, f(c)/2)$, which consists only of negative numbers. I can't pick any interval of the form $(c - \delta, c + \delta)$ lying inside $[a, b]$ that works, since any such interval includes an x which is bigger than c. By condition #1 above, we know that $f(x)$ would have to be positive, which means that it doesn't lie in your tolerance region. So it can't be true that $f(c) < 0$. Intuitively, if it is, then your bookend still has books to the right of it!

Perhaps $f(c) > 0$. In this case, we can't have $c = a$ since $f(a) < 0$. Now, the values of $f(x)$ should be near $f(c)$ when x is near c; so in particular they should be positive. This is a problem because of condition #2 above. Specifically, this time you can choose $\varepsilon = f(c)/2$, so that your tolerance interval is $(f(c)/2, 3f(c)/2)$. I need to try to find an interval $(c-\delta, c+\delta)$ within $[a, b]$ such that for any x in my interval, $f(x)$ always lies in your tolerance interval. In particular, $f(x) > 0$. This means that $f(x) > 0$ for all x in the interval $(c - \delta, c)$, which violates condition #2. So $f(c) > 0$ isn't true either; if it were true, then the bookend could be pushed to the left some more, so it wouldn't be at c.

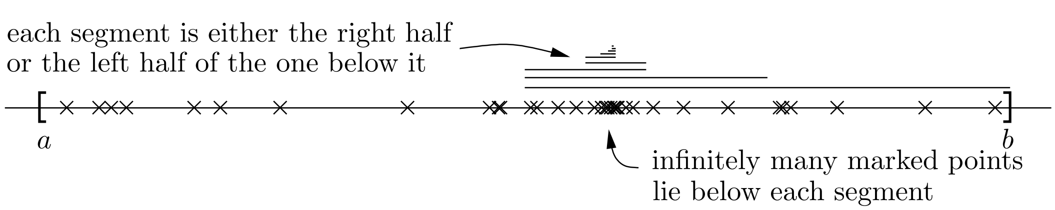
What's left? The only possibility is that $f(c) = 0$, so we have proved our theorem. By the way, it's easy to change the situation to the case when $f(a) > 0$ and $f(b) < 0$ instead; you can either rewrite the proof slightly differently, or you can just set $g(x) = -f(x)$ and apply the theorem to g instead of f.

A.4.3 Proof of the Max-Min Theorem

Now let's prove the Max-Min Theorem, which we looked at in Section 5.1.6. The idea is that we once again have a function f which is continuous on the closed interval $[a, b]$; the claim is that there is some number c in the interval which is a maximum for f. As we saw, this means that $f(c)$ is greater than or equal to every other value of $f(x)$ where x wanders over the whole interval $[a, b]$.

Here's how it's done. The first thing we want to show is that you can plonk down some horizontal line at $y = N$, say, such that the function values $f(x)$ all lie below that line. If you couldn't do that, then the function would somehow grow bigger and bigger somewhere inside $[a, b]$, and it wouldn't have a maximum. So, let's suppose you **can't** draw such a line. Then for every positive integer N, there's some point x_N in $[a, b]$ such that $f(x_N)$ is above the line $y = N$. That is, we have found some points x_N such that $f(x_N) > N$ for every N. Let's mark them on the x-axis with an X.

Now, where are these marked points? There are infinitely many. So if we chop the interval $[a, b]$ in half to get two new intervals, one of them must still have infinitely many marked points. Perhaps they both do, but they can't both have finitely many marked points or else the total would be finite. Let's focus on the half of the original interval that has infinitely many marked points; if they both do, choose your favorite one (it doesn't matter). Now repeat the exercise with the new, smaller interval: chop it in half. One of the halves must have infinitely many marked points. Continue doing this for as long as you like, and you will get a collection of intervals which get smaller and smaller, all nested inside each other, and each of which has infinitely many marked points. Stacking the intervals on top of each other, this is what the situation might look like:



Intuitively, there has to be some real number which is inside every single one of these intervals.* Let's call the number q. What is $f(q)$? We can use the

*Again, one needs to use the completeness property of the real line to show this. Actually, there has to be exactly one such number—can you see why?

continuity of f to get some idea of what it should be. Indeed, we know that

$$\lim_{x \to q} f(x) = f(q).$$

So if you pick your ε to be 1, for example, then I should be able to find an interval $(q - \delta, q + \delta)$ so that $|f(x) - f(q)| < 1$ for all x in the interval. The problem is that the interval $(q - \delta, q + \delta)$ contains infinitely many marked points! This is because eventually one of the little nested intervals that we chose will lie within $(q - \delta, q + \delta)$, no matter how small δ is. This is a real problem: we are supposed to have all these marked points inside our interval $(q - \delta, q + \delta)$, but when you take f of any of them, you get a number between $f(q) - 1$ and $f(q) + 1$. So, no matter what $f(q)$ is, we're going to get in trouble: some of the marked points are going to have function values which are much bigger than $f(q) + 1$. The whole thing is out of control. So we were wrong about not being able to draw in a line like $y = N$ which had the whole function beneath it!

We're still not done. We have this line $y = N$ which lies above the graph of $y = f(x)$ on $[a, b]$, but now we need to move it down until it hits the graph in order to find the maximum. So, let's pick N as small as possible so that $f(x) \leq N$ for all x in $[a, b]$. (We have used completeness once again.) Now we need to show that $N = f(c)$ for some c. To do this, we're going to repeat the same trick as we did above with marked points, except this time they'll be circled. Pick a positive integer n; we must be able to find some number c_n in $[a, b]$ such that $f(c_n) > N - 1/n$. If not, then we should have drawn our line at $y = N - 1/n$ (or even lower) instead of $y = N$. So there is such a c_n, and there's one for every positive integer n. Circle all of these points. There are infinitely many of them, and when you apply f to them, the resulting values get closer and closer—arbitrarily close, in fact—to N. (None of the values can be bigger than N because $f(x) \leq N$ for all x!) Now all we have to do is keep bisecting the interval $[a, b]$ over and over again, such that each little interval has infinitely many circled points in it. As before, there is a number c in all the intervals. This number is really surrounded by a fog of circled points.

What is $f(c)$? It can't be more than N, but maybe it can be less than N. Let's suppose that $f(c) = M$, where $M < N$, and let's set $\varepsilon = (N - M)/2$. Since f is continuous, we really need

$$\lim_{x \to c} f(x) = f(c) = M.$$

You have your ε, and so I need to find an interval $(c - \delta, c + \delta)$ so that $f(x)$ lies in $(M - \varepsilon, M + \varepsilon)$ for x in my interval. The problem is that $M + \varepsilon = N - \varepsilon$, and also that there are infinitely many circled points lying in $(c - \delta, c + \delta)$, no matter how I choose $\delta > 0$. Some of them might have function values lying in $(M - \varepsilon, M + \varepsilon)$, but since the function values get closer to N, most of them won't. So I can't make my move. The only way out is that $f(c) = N$ after all. This means that c is a maximum, and we're done!

To get the minimum version of the theorem, just reapply the theorem to $g(x) = -f(x)$. After all, if c is a maximum for g, then it is a minimum for f.