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Calculus

Calculation of Volumes Using Triple Integrals

The *volume of a solid* U in Cartesian coordinates xyz is given by

$$V = \iiint\limits_U dxdydz.$$

In cylindrical coordinates, the volume of a solid is defined by the formula

$$V = \iiint\limits_U \rho\, d\rho\, d\varphi\, dz.$$

In spherical coordinates, the volume of a solid is expressed as

$$V = \iiint\limits_U \rho^2 \sin\theta\, d\rho\, d\varphi\, d\theta.$$

Example 1

Find the volume of the cone of height H and base radius R (Figure 2).

Solution.

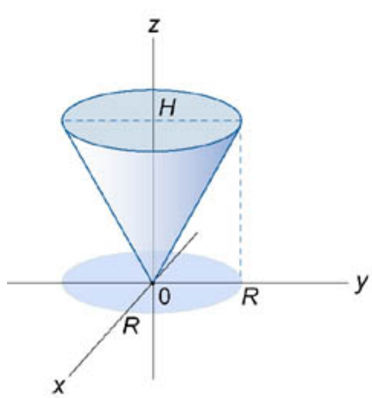
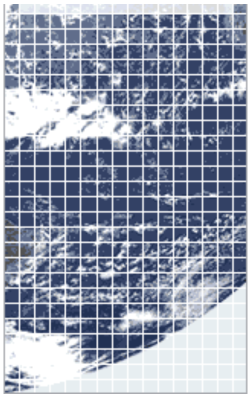


Fig.1



The cone is bounded by the surface $z = \frac{H}{R}\sqrt{x^2 + y^2}$ and the plane $z = H$ (Figure 1). Its volume in Cartesian coordinates is expressed by the formula

$$V = \iiint\limits_U dxdydz = \int\limits_{-R}^{R} dx \int\limits_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int\limits_{\frac{H}{R}\sqrt{x^2+y^2}}^{H} dz.$$

Calculate this integral in cylindrical coordinates that range within the limits:

$$0 \le \varphi \le 2\pi, \quad 0 \le \rho \le R, \quad \rho \le z \le H.$$

As a result, we obtain (do not forget to include the Jacobian ρ):

$$V = \int\limits_0^R \rho d\rho \int\limits_0^{2\pi} d\varphi \int\limits_{\frac{H}{R}\rho}^{H} dz.$$

Then the volume of the cone is

$$V = \int\limits_0^R \rho d\rho \int\limits_0^{2\pi} d\varphi \int\limits_{\frac{H}{R}\rho}^{H} dz = 2\pi \int\limits_0^R \rho d\rho \int\limits_{\frac{H}{R}\rho}^{H} dz = 2\pi \int\limits_0^R \rho d\rho \cdot \left[\left(z \right)\Big|_{z=\frac{H}{R}\rho}^{z=H} \right] = 2\pi \int\limits_0^R \rho \left(H - \frac{H}{R}\rho \right) d\rho$$

$$= 2\pi \int\limits_0^R \left(H\rho - \frac{H}{R}\rho^2 \right) d\rho = 2\pi \left[\left(\frac{\rho^2 H}{2} - \frac{\rho^3 H}{3R} \right)\Bigg|_{\rho=0}^{\rho=R} \right] = 2\pi \left(\frac{R^2 H}{2} - \frac{R^3 H}{3R} \right) = 2\pi R^2 H \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$= \frac{2\pi R^2 H}{6} = \frac{\pi R^2 H}{3}.$$

Example 2

Find the volume of the ball $x^2 + y^2 + z^2 \le R^2$.

Solution.
We calculate the volume of the part of the ball lying in the first octant ($x \ge 0, y \ge 0, z \ge 0$), and then multiply the result by 8. This yields:

$$V = 8 \iiint\limits_U dxdydz = 8 \iiint\limits_{U'} \rho^2 \sin\theta d\rho d\varphi d\theta = 8 \int\limits_0^{\pi/2} d\varphi \int\limits_0^R \rho^2 d\rho \int\limits_0^{\pi/2} \sin\theta d\theta$$

$$= 8 \int\limits_0^{\pi/2} d\varphi \int\limits_0^R \rho^2 d\rho \cdot \left[\left(-\cos\theta \right)\Big|_0^{\pi/2} \right] = 8 \int\limits_0^{\pi/2} d\varphi \int\limits_0^R \rho^2 d\rho \cdot \left(-\cos\frac{\pi}{2} + \cos 0 \right)$$

$$= 8 \int\limits_0^{\pi/2} d\varphi \int\limits_0^R \rho^2 d\rho \cdot 1 = 8 \int\limits_0^{\pi/2} d\varphi \cdot \left[\left(\frac{\rho^3}{3} \right)\Bigg|_0^R \right] = 8 \int\limits_0^{\pi/2} d\varphi \cdot \frac{R^3}{3} = \frac{8R^3}{3} \int\limits_0^{\pi/2} d\varphi$$

$$= \frac{8R^3}{3} \cdot \left[\left(\varphi \right)\Big|_0^{\pi/2} \right] = \frac{8R^3}{3} \cdot \frac{\pi}{2} = \frac{4\pi R^3}{3}.$$

As a result, we get the well-known expression for the volume of the ball of radius R.

Example 3

Find the volume of the tetrahedron bounded by the planes passing through the points A (1;0;0), B (0;2;0), C (0;0;3) and the coordinate planes Oxy, Oxz, Oyz (Figure 2).

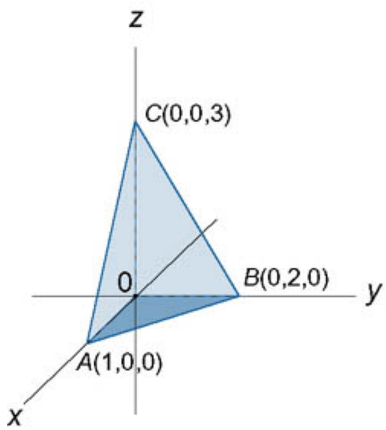


Fig.2

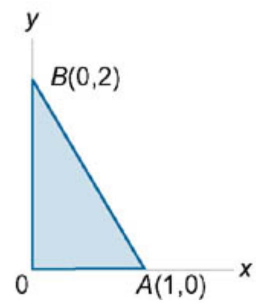


Fig.3

Solution.
The equation of the straight line AB in the xy-plane (Figure 3) is written as $y = 2 - 2x$. The variable x ranges here in the interval $0 \le x \le 1$, and the variable y ranges in the interval $0 \le y \le 2 - 2x$.

Write the equation of the plane ABC in segment form. Since the plane ABC cuts the line segments 1, 2, and 3, respectively, on the x-, y-, and z-axis, then its equation can be written as

$$\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1.$$

In general case the equation of the plane ABC is written as

$$6x + 3y + 2z = 6,$$
$$\text{or} \quad z = 3 - 3x - \frac{3}{2}y.$$

Hence, the limits of integration over the variable z range in the interval from $z = 0$ to $z = 3 - 3x - \frac{3}{2}y$. Now we can calculate the volume of the tetrahedron:

$$V = \iiint\limits_U dxdydz = \int_0^1 dx \int_0^{2-2x} dy \int_0^{3-3x-\frac{3}{2}y} dz = \int_0^1 dx \int_0^{2-2x} dy \cdot \left[\left(z \right)\Big|_0^{3-3x-\frac{3}{2}y} \right] = \int_0^1 dx \int_0^{2-2x} \left(3 - 3x - \frac{3}{2}y \right) dy$$

$$= \int_0^1 dx \cdot \left[\left(3y - 3xy - \frac{3}{4}y^2 \right)\Bigg|_{y=0}^{y=2-2x} \right] = \int_0^1 \left(3(2-2x) - 3x(2-2x) - \frac{3}{4}(2-2x)^2 \right) dx$$

$$= \int_0^1 \left(6 - 6x - 6x + 6x^2 - \frac{3}{4}\left(4 - 8x + 4x^2\right) \right) dx = \int_0^1 \left(6 - 12x + 6x^2 - 3 + 6x - 3x^2 \right) dx$$

$$= 3\int_0^1 \left(1 - 2x + x^2\right) dx = 3\left(x - x^2 + \frac{x^3}{3}\right)\Bigg|_0^1 = 3\cdot\left(1 - 1^2 + \frac{1^3}{3}\right) = 3\cdot\frac{1}{3} = 1.$$

Example 4

Find the volume of the tetrahedron bounded by the planes $x + y + z = 5$, $x = 0$, $y = 0$, $z = 0$ (Figure 4).

Solution.
The equation of the plane $x + y + z = 5$ can be rewritten in the form

$$z = 5 - x - y.$$

By setting $z = 0$, we get

$$5 - x - y = 0 \quad \text{or} \quad y = 5 - x.$$

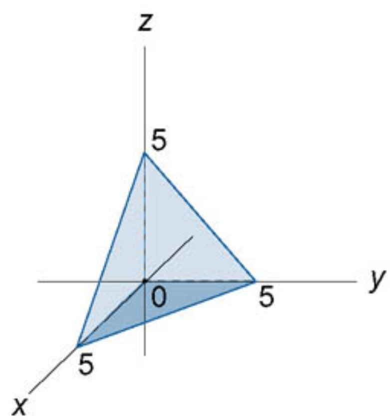


Fig.4

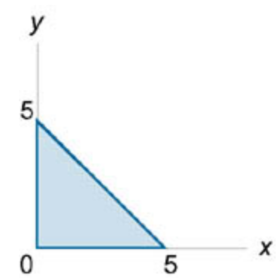


Fig.5

Hence, the region of integration D in the xy-plane is bounded by the straight line $y = 5 - x$ as shown in Figure 5.

Representing the triple integral as an iterated integral, we can find the volume of the tetrahedron:

$$V = \iiint_U dxdydz = \int_0^5 dx \int_0^{5-x} dy \int_0^{5-x-y} dz = \int_0^5 dx \int_0^{5-x} dy \cdot \left[\left(z\right)\Big|_0^{5-x-y}\right] = \int_0^5 dx \int_0^{5-x} \left(5 - x - y\right) dy$$

$$= \int_0^5 dx \cdot \left[\left(5y - xy - \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=5-x}\right] = \int_0^5 \left(5\left(5-x\right) - x\left(5-x\right) - \frac{\left(5-x\right)^2}{2}\right) dx$$

$$= \int_0^5 \left(25 - 5x - 5x + x^2 - \frac{25 - 10x + x^2}{2}\right) dx = \frac{1}{2}\int_0^5 \left(25 - 10x + x^2\right) dx$$

$$= \frac{1}{2}\left[\left(25x - \frac{10x^2}{2} + \frac{x^3}{3}\right)\Bigg|_0^5\right] = \frac{1}{2}\left(125 - 5\cdot 25 + \frac{125}{3}\right) = \frac{125}{6}.$$

Example 5

Find the volume of the solid formed by two paraboloids:

$$z_1 = x^2 + y^2 \quad \text{and} \quad z_2 = 1 - x^2 - y^2.$$

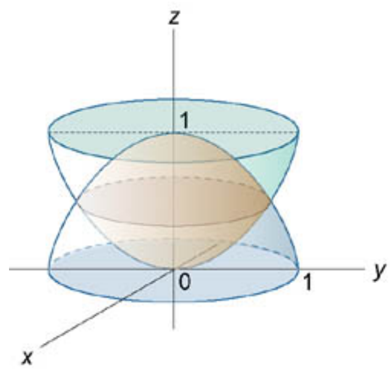


Fig.6

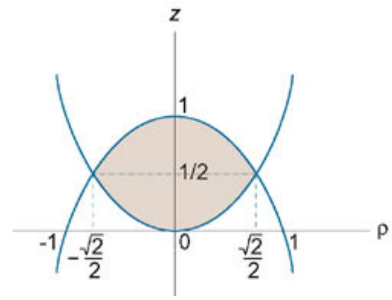


Fig.7

Solution.

Investigate intersection of the two paraboloids (Figure 6). Since $\rho^2 = x^2 + y^2$, the equations of the paraboloids can be written as

$$z_1 = \rho^2 \quad \text{and} \quad z_2 = 1 - \rho^2.$$

By setting $z_1 = z_2$ for the intersection curve, we obtain

$$\rho^2 = 1 - \rho^2, \quad \Rightarrow \quad 2\rho^2 = 1,$$
$$\Rightarrow \quad \rho^2 = \frac{1}{2} \quad \text{or} \quad z = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

For this value of ρ (Figure 7), the coordinate z is

$$z = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}.$$

The volume of the solid is expressed through the triple integral as

$$V = \iiint\limits_U dxdydz.$$

This integral in cylindrical coordinates becomes

$$V = \iiint\limits_U dxdydz = \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}/2} \rho d\rho \int_{\rho^2}^{1-\rho^2} dz = \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}/2} \rho d\rho \cdot \left[z\Big|_{\rho^2}^{1-\rho^2} \right] = \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}/2} \rho\left(1 - \rho^2 - \rho^2\right) d\rho$$
$$= 2\pi \int_0^{\sqrt{2}/2} \rho\left(1 - 2\rho^2\right) d\rho = 2\pi \int_0^{\sqrt{2}/2} \left(\rho - 2\rho^3\right) d\rho = 2\pi \left[\left(\frac{\rho^2}{2} - \frac{2\rho^4}{4} \right)\Bigg|_0^{\sqrt{2}/2} \right]$$
$$= 2\pi \left(\frac{\left(\sqrt{2}/2\right)^2}{2} - \frac{\left(\sqrt{2}/2\right)^4}{2} \right) = \pi\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{4}.$$

Example 6

Calculate the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Solution.
It is easier to calculate the volume of the ellipsoid using generalized spherical coordinates. Let

$$x=a\rho\cos\varphi\sin\theta,\quad y=b\rho\sin\varphi\sin\theta,\quad z=c\rho\cos\theta.$$

Since the absolute value of the Jacobian for transformation of Cartesian coordinates into generalized spherical coordinates is

$$|I|=abc\rho^2\sin\theta,$$

hence,

$$dxdydz=abc\rho^2\sin\theta d\rho d\varphi d\theta.$$

The volume of the ellipsoid is expressed through the triple integral:

$$V=\iiint\limits_U dxdydz=\iiint\limits_{U'} abc\rho^2\sin\theta d\rho d\varphi d\theta.$$

By symmetry, we can find the volume of 1/8 part of the ellipsoid lying in the first octant ($x\ge 0, y\ge 0, z\ge 0$), and then multiply the result by 8. The generalized spherical coordinates will range within the limits:

$$0\le\rho\le 1,\quad 0\le\varphi\le\frac{\pi}{2},\quad 0\le\theta\le\frac{\pi}{2}.$$

Then the volume of the ellipsoid is

$$V=\iiint\limits_{U'} abc\rho^2\sin\theta d\rho d\varphi d\theta=8abc\int_0^{\pi/2}d\varphi\int_0^1\rho^2d\rho\int_0^{\pi/2}\sin\theta d\theta=8abc\int_0^{\pi/2}d\varphi\int_0^1\rho^2d\rho\cdot\left[\left(-\cos\theta\right)\Big|_0^{\pi/2}\right]$$

$$=8abc\int_0^{\pi/2}d\varphi\int_0^1\rho^2d\rho\cdot\left(-\cos\frac{\pi}{2}+\cos 0\right)=8abc\int_0^{\pi/2}d\varphi\int_0^1\rho^2d\rho=8abc\int_0^{\pi/2}d\varphi\cdot\left[\left(\frac{\rho^3}{3}\right)\Bigg|_0^1\right]=\frac{8abc}{3}\int_0^{\pi/2}d\varphi$$

$$=\frac{8abc}{3}\cdot\left[\left(\varphi\right)\Big|_0^{\pi/2}\right]=\frac{8abc}{3}\cdot\frac{\pi}{2}=\frac{4}{3}\pi abc.$$

Example 7

Find the volume of the solid bounded by the sphere $x^2+y^2+z^2=6$ and the paraboloid $x^2+y^2=z$.

Solution.
We first determine the curve of intersection of these surfaces. Substituting the equation of the paraboloid into the equation of the sphere, we find:

$$z+z^2=6\quad\text{or}\quad z^2+z-6=0,$$

$$\Rightarrow\ z_{1,2}=\frac{-1\pm 5}{2}=2,-3.$$

The second root $z_2=-3$ corresponds to intersection of the sphere with the lower shell of the paraboloid. So we do not consider this case. Thus, intersection of the solids happens at $z=2$. Obviously, the projection of the region of integration on the xy-plane is the circle (Figure 8) defined by the equation $x^2+y^2=2$.

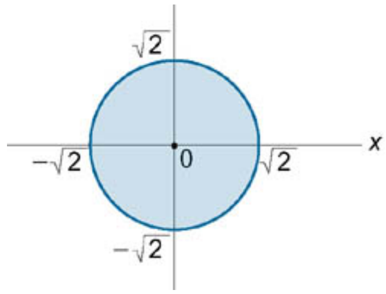


Fig.8

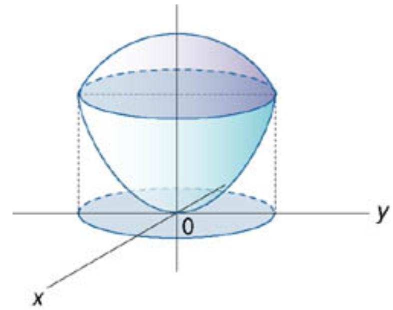


Fig.9

The region of integration is bounded from above by the spherical surface, and from below by the paraboloid (Figure 9).
The volume of the solid region is expressed by the integral

$$V = \iiint\limits_U dxdydz = \int\limits_{-\sqrt{2}}^{\sqrt{2}} dx \int\limits_0^{\sqrt{2-x^2}} dy \int\limits_{x^2+y^2}^{\sqrt{6-x^2-y^2}} dz.$$

It is convenient to convert the integral to cylindrical coordinates:

$$V = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\sqrt{2}} \rho d\rho \int\limits_{\rho^2}^{\sqrt{6-\rho^2}} dz,$$

where $\rho^2 = x^2 + x^2$ and the integral includes the Jacobian ρ. As a result, we have:

$$V = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\sqrt{2}} \rho d\rho \int\limits_{\rho^2}^{\sqrt{6-\rho^2}} dz = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\sqrt{2}} \rho d\rho \cdot \left[\left. (z) \right|_{\rho^2}^{\sqrt{6-\rho^2}} \right] = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\sqrt{2}} \rho \left(\sqrt{6-\rho^2} - \rho^2 \right) d\rho$$

$$= 2\pi \int\limits_0^{\sqrt{2}} \rho \left(\sqrt{6-\rho^2} - \rho^2 \right) d\rho = \pi \int\limits_0^{\sqrt{2}} \left(\sqrt{6-\rho^2} - \rho^2 \right) d\rho^2.$$

We change the variable: $\rho^2 = t$. Here $t = 0$ when $\rho = 0$, and, respectively, $t = 2$ when $\rho = \sqrt{2}$.

Now we can calculate the volume of the solid:

$$V = \pi \int\limits_0^{\sqrt{2}} \left(\sqrt{6-\rho^2} - \rho^2 \right) d\rho^2 = \pi \int\limits_0^{2} \left(\sqrt{6-t} - t \right) dt = \pi \left[\left. \left(-\frac{2(6-t)^{3/2}}{3} - \frac{t^2}{2} \right) \right|_0^2 \right] = \pi \left[-\frac{2}{3} \left(4^{3/2} - 6^{3/2} \right) - 2 \right]$$

$$= \pi \left[\frac{2}{3} \left(6\sqrt{6} - 8 \right) - 2 \right] = \pi \left(4\sqrt{6} - \frac{16}{3} - 2 \right) = \pi \left(4\sqrt{6} - \frac{22}{3} \right) = 2\pi \left(\frac{6\sqrt{6} - 11}{3} \right).$$

Example 8

Calculate the volume of the solid bounded by the paraboloid $z = 2 - x^2 - y^2$ and the conic surface $z = \sqrt{x^2 + y^2}$.

Solution.
First we investigate intersection of the two surfaces. By equating the coordinates z, we get the following equation:

$$2 - x^2 - y^2 = \sqrt{x^2 + y^2}.$$

Let $x^2 + y^2 = t^2$. Then

$$2 - t^2 = t \quad \text{or} \quad t^2 + t - 2 = 0,$$

$$\Rightarrow t_{1,2} = \frac{-1 \pm 3}{2} = -2, 1.$$

Only the root $t = 1$ has the sense in the context of the given problem, i.e.

$$z = \sqrt{x^2 + y^2} = 1 \quad \text{or} \quad x^2 + y^2 = 1.$$

Thus, both the surfaces intersect at $z = 1$, and the intersection is a circle (Figure 10).

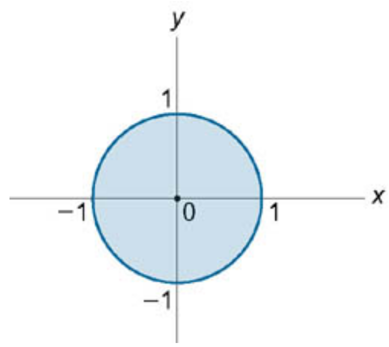


Fig.10

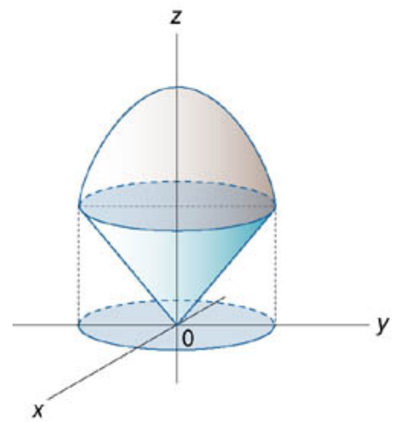


Fig.11

The region of integration is bounded from above by the paraboloid, and from below by the cone (Figure 11). To calculate the volume of the solid we use cylindrical coordinates:

$$x^2 + y^2 = \rho^2, \quad \sqrt{x^2 + y^2} = \rho, \quad dxdydz = \rho d\rho d\varphi dz.$$

As a result, we find:

$$V = \iiint\limits_U \rho d\rho d\varphi dz = \int_0^{2\pi} d\varphi \int_0^1 \rho d\rho \int_{\rho}^{2-\rho^2} dz = \int_0^{2\pi} d\varphi \int_0^1 \rho d\rho \cdot \left(2 - \rho^2 - \rho\right) = \int_0^{2\pi} d\varphi \int_0^1 \left(2\rho - \rho^3 - \rho^2\right) d\rho$$

$$= \int_0^{2\pi} d\varphi \cdot \left[\left.\left(\rho^2 - \frac{\rho^4}{4} - \frac{\rho^3}{3}\right)\right|_0^1\right] = \int_0^{2\pi} \left(1 - \frac{1}{4} - \frac{1}{3}\right) d\varphi = \frac{5}{12} \int_0^{2\pi} d\varphi = \frac{5\pi}{6}.$$