Introduction to surface integrals

The theorem for surface integrals is that the area element is given by

$$dS = \sqrt{(f_x)^2 + (f_y)^2 + 1} \;\; dx \; dy$$

$$A(S) = \iint_D 1 \; dS = \iint_R \sqrt{(f_x)^2 + (f_y)^2 + 1} \;\; dA$$

If you want to see where this comes from, you should look at the write-up for Schey Ch 2-1. But basically what is happening is that the exchange rate between area on the surface and area in the plane is

$$dR = \hat{\mathbf{n}} \cdot \hat{\mathbf{k}} \; dS$$

where

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}} \;\; < \; -f_x, -f_y, 1 \; >$$

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}} \; (1)$$

dS is larger than dA by a factor of the part of the unit normal vector that points up, this is $\cos\theta$, where θ is the angle between $\hat{\mathbf{n}}$ and $\hat{\mathbf{k}}$.

Plane

A plane is almost too easy. Suppose we take the plane which goes through the points $(2,0,0)$, $(0,2,0)$ and $(0,0,2)$ (forming an equilateral triangle with side lengths $2\sqrt{2}$, altitude $\sqrt{3}\sqrt{2}$ and area $2\sqrt{3}$.

The equation of the surface is $x + y + z = 2$ or $z = f(x,y) = 2 - x - y$. The multiplicative constant is

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{3}$$

The region R in the plane has bounds $y = 0 \to 1$ and $x = 0 \to 2 - y$ so we have

$$A = \int_0^2 \int_0^{2-y} \sqrt{3}\ dx\ dy$$

$$= \sqrt{3} \int_0^2 2 - y\ dy$$

$$= \sqrt{3}\ (2y - \frac{1}{2}y^2)\Big|_0^2 = 2\sqrt{3}$$

Of course, we could put a density function or some other scalar function $G(x, y, z)$ under the integral as well. For example $xy + z$

$$A = \int_0^2 \int_0^{2-y} \sqrt{3}\ (xy + z)\ dx\ dy$$

$$A = \sqrt{3} \int_0^2 \int_0^{2-y} xy + 2 - x - y\ dx\ dy$$

$$A = \sqrt{3} \int_0^2 (\frac{1}{2}x^2 y + 2x - \frac{1}{2}x^2 - yx)\Big|_0^{2-y}\ dy$$

$(2 - y)^2 = 4 - 2y + y^2$, so the inner integral is

$$\frac{1}{2}(4 - 2y + y^2)y + 2(2 - y) - \frac{1}{2}(4 - 2y + y^2) - y(2 - y)$$

$$= 2y - y^2 + \frac{1}{2}y^3 + 4 - 2y - 2 + y - \frac{1}{2}y^2 - 2y + y^2$$

$$= \frac{1}{2}y^3 - \frac{1}{2}y^2 - y + 2$$

Integrate and evaluate

$$(\frac{1}{8}y^4 - \frac{1}{6}y^3 - \frac{1}{2}y^2 + 2y)\Big|_0^2 = 2 - \frac{4}{3} - 2 + 4 = \frac{11}{3}$$

Remember the $\sqrt{3}$ for the final answer.

Paraboloid

Our purpose here is to develop two simple examples, the surface areas of the paraboloid and the hemisphere. For the paraboloid, consider one which has its vertex at $z = 1$ and opens down

$$z = 1 - x^2 - y^2$$

When $z = 0$ this is just $x^2 + y^2 = 1$. We find that

$$f_x = -2x$$

$$f_y = -2y$$

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$$

This would be a good time to switch to polar coordinates.

$$x^2 + y^2 = r^2$$

$$\sqrt{(f_x)^2 + (f_y)^2 + 1} = \sqrt{4r^2 + 1}$$

So we have the integral

$$\int \int \sqrt{4r^2 + 1} \ \ r \ dr \ d\theta$$

with the r term coming from the usual source. The inner integral is

$$\frac{1}{12} \ (4r^2 + 1)^{3/2}$$

For the unit circle ($r = 0 \to 1$), and multiplying by 2π for the outer integral, this is

$$2\pi \ \frac{1}{12} \ [\ (5)^{3/2} - 1^{3/2}] = \frac{\pi}{6} \ [\ 5\sqrt{5} - 1 \ ]$$

In general, if the limits for the radius are $r = a \to r = b$ we will have

$$\frac{\pi}{6} \ [\ (4b^2 + 1)^{3/2} - (4a^2 + 1)^{3/2} \ ]$$

We can check this using the surface of a solid of revolution. Turn the function to have more familiar variables. It is $y = \sqrt{x}$. Rotated around the x-axis, this solid has a cross-section at each point with circumference $2\pi y = 2\pi\sqrt{x}$. The surface area is

$$\int 2\pi\sqrt{x} \ ds$$

The surface area element is

$$ds = \sqrt{1 + f'(x)^2}\ dx$$

(Looks familiar!) So $f'(x)^2 = \frac{1}{4x}$ and we have

$$\int 2\pi\sqrt{x}\ \sqrt{1 + \frac{1}{4}x}\ dx$$

$$2\pi \int \sqrt{x + \frac{1}{4}}\ dx$$

$$\frac{4}{3}\pi\ (x + \frac{1}{4})^{3/2}$$

$$\frac{\pi}{6}\ (4x + 1)^{3/2}$$

evaluated between $x = 0 \to 1$, we obtain

$$\frac{\pi}{6}\ [\ (5)^{3/2} - 1^{3/2}\ ]\ \ = \frac{\pi}{6}\ [\ 5\sqrt{5} - 1\ ]$$

which matches what we had by the new method.

Hemisphere

For the unit hemisphere we have that

$$x^2 + y^2 + z^2 = 1$$

$$z = \sqrt{1 - x^2 - y^2}$$

$$f_x = \frac{1}{z}\frac{1}{2}(-2x) = -\frac{x}{z}$$

$$f_y = \frac{1}{z}\frac{1}{2}(-2y) = -\frac{y}{z}$$

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{1 + (\frac{x}{z})^2 + (\frac{y}{z})^2}$$

Here, the trick is to multiply on the bottom by $1/z$ and on top by $\sqrt{z^2}$

$$= \frac{1}{z}\sqrt{z^2 + x^2 + y^2} = \frac{1}{z}$$

So our integral is

$$\iint_R \frac{1}{z} \ dA$$

$$= \iint_R \frac{1}{\sqrt{1 - x^2 - y^2}} \ dA$$

This would be a good time to switch to polar coordinates.

$$= \iint_R \frac{1}{\sqrt{1 - r^2}} \ r \ dr \ d\theta$$

The inner integral is

$$-\sqrt{1 - r^2}$$

If the limits for the radius are $r = 0 \to r = 1$ this is just 1, multiplied by 2π from the outer integral. If the sphere's radius is $\rho = a$, then there are two differences. First, we multiply by a/z above and our integral is

$$\iint_R \frac{a}{z} \ dA$$

and then second, the inner integral is evaluated as

$$-\sqrt{a^2 - r^2}$$

between $r = 0 \to r = a$, which is of course a, multiplied by by $2a\pi$ it becomes $2\pi a^2$ for the surface area of one-half of a sphere.

Note: my difficulty with these problems has come from wanting to switch to cylindrical or spherical coordinates. But this is a double integral over the region R in the plane, and the appropriate choice is to switch to polar coordinates in r and θ.

A modification of this simple type of problem is to suppose we want the area of the object above a plane slicing through at $z = const$. Suppose the radius of the sphere is $\rho = a$. Then, at the simplification step above, we get

$$= \frac{1}{z}\sqrt{z^2 + x^2 + y^2} = \frac{a}{z}$$

$$a \iint_R \frac{1}{z} \ dA$$

$$= a \iint_R \frac{1}{\sqrt{a^2 - x^2 - y^2}} \ dA$$

This would be a good time to switch to polar coordinates.

$$= a \iint_R \frac{1}{\sqrt{a^2 - r^2}}\ r\ dr\ d\theta$$

What are the limits on r? We have

$$z^2 = a^2 - x^2 - y^2$$

$$z = 1$$

$$x^2 + y^2 = r^2 = a^2 - 1$$

so $r = 0 \to r = \sqrt{a^2 - 1}$. The inner integral is then

$$-\sqrt{a^2 - r^2}\ \Big|_0^{\sqrt{a^2-1}}$$

$$= a - 1$$

times $2\pi a$

$$A(S) = 2\pi a(a - 1)$$