Auroux 22: Green's Theorem

$$curl(F) = N_x - M_y$$

for $F = < M, N >$. It measures how far the field is from being conservative. Green's Theorem is

$$\oint_C \mathbf{F} \cdot \mathbf{dr} = \int\int_R curl(\mathbf{F})\ dA$$

$$\oint_C M\ dx + N\ dy = \int\int_R (N_x - M_y)\ dA$$

where $\oint$ is an integral over a *closed path*, traveling in a ccw direction. The left-hand side "lives on the curve," whereas the right-hand side "lives over the whole region."

He gives an example where this is very useful. Suppose

$$M = ye^{-x}, \quad N = \frac{1}{2}x^2 - e^{-x}$$

and our curve is a shifted unit circle, centered at $(2, 0)$, so $x = 2 + cos\theta$ and $y = sin\theta$. This is trouble to parametrize, because of e^{-x}. Instead do

$$\int\int_R (N_x - M_y) dA$$

$$M = ye^{-x}, \quad M_y = e^{-x}$$

$$N = \frac{1}{2}x^2 - e^{-x}, \quad N_x = x + e^{-x}$$

$$\int\int_R x dA$$

From lecture 21, we had $\bar{x} = (1/Area) \int\int x dA$

$$\int\int_R x\ dA = Area(R)\ \bar{x} = \pi\ 2$$

Alternatively, we could calculate this directly

$$x = 2 + cos\theta, \quad dx = -2sin\theta \ d\theta$$

$$y = sin\theta, \quad dy = cos\theta d\theta$$

$$\int\int_R x \ dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{r=1}(2 + cos\theta) \ r \ dr \ d\theta$$

inner

$$(2 + cos\theta)\frac{r^2}{2}\Big|_0^1 = 1 + \frac{1}{2}cos\theta$$

outer

$$\int_{\theta=0}^{2\pi}(1 + \frac{1}{2}cos\theta) \ d\theta = (\theta + \frac{1}{2}sin\theta)\Big|_0^{2\pi} = 2\pi$$

Derivation

Part 1. We will show that

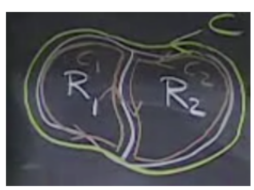
$$\oint_C Mdx = \int\int_R -M_y \ dA$$

we will do the special case where N = 0, there is only an x-component in the vector field. but by symmetry

$$\oint_C Ndy = \int\int_R N_x \ dA$$

and the sum is equivalent to the theorem.

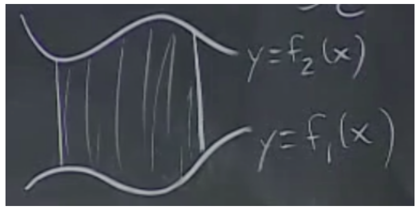
Part 2. Any complex curve (with some exceptions) can be decomposed into a set of regions, we do the integrals for each one, and the boundary curves between regions cancel.



Part 3. To prove:

$$\oint_C Mdx = \int\int_R -M_y \ dA$$

For a vertically simple region, we have a total of four curves going around



but two of them are vertical and have $dx = 0$, the other two are

$$\oint_C M\ dx = \oint_{C1} M\ dx + \oint_{C2} M\ dx = \int_a^b M(x, f_1(x))\ dx - \int_a^b M(x, f_2(x))\ dx$$

where f_1 is the lower curve and f_2 the upper one. At each point along the curve, we have $y = f(x)$, so we can evaluate what $M(x, y)$ is at that point and then integrate with respect to x. Notice that we have switched the bounds on the second integral, and added a minus sign.

Leaving that aside, now look at the right-hand side in the theorem, the integral over the region

$$\int\int_R -M_y\ dA = \int\int_R -M_y\ dy\ dx = -\int\int_R M_y\ dy\ dx$$

$$M_y = \frac{\partial M}{\partial y}$$

$$-\int_{x=a}^{x=b} \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\ dy\ dx$$

but

$$\frac{\partial M}{\partial y}\ dy = M$$

so (remembering the minus sign) the inner integral is just

$$\int_{f_1(x)}^{f_2(x)} M dy = M(x, f_2(x)) - M(x, f_1(x))$$

and the outer integral is

$$-\int_a^b [\ M(x, f_2(x)) - M(x, f_1(x))\ ]\ dx$$

but that is the same as what we had above (taking account of the signs).