Theorems

We say that a function f approaches the **limit** L near a if, for every $\epsilon > 0$ there exists $\delta > 0$ such that for all

$$0 < |x - a| < \delta$$

implies

$$|f(x) - L| < \epsilon$$

And we say that a function is **continuous** at a if

$$\lim_{x \to a} f(x) = f(a)$$

preliminary theorem

Suppose f is continuous at a and that $f(a) > 0$. Then $f(x) > 0$ for all x in some interval containing a. More precisely, there is a number $\delta > 0$ such that $f(x) > 0$ for all x satisfying $|x - a| < \delta$, that is, all x in $[a - \delta, a + \delta]$.

proof

Since f is continuous, for every $\epsilon > 0$ there is a $\delta > 0$ such that for all x satisfying $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

$$-\epsilon < f(x) - f(a) < \epsilon$$

This must be true for $\epsilon = f(a)$ (since $f(a) > 0$). Hence

$$-f(a) < f(x) - f(a) < f(a)$$

so

$$f(x) > 0$$

1 Bolzano's Theorem

theorem

If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$ then there is some x in $[a, b]$ such that $f(x) = 0$.

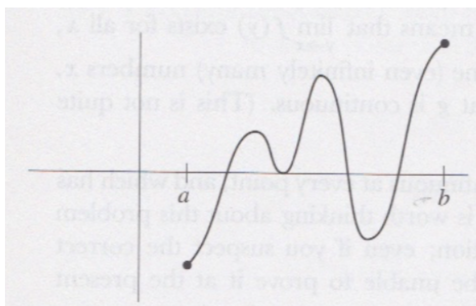

FIGURE 1

Spivak

Recall the completeness axiom in this formulation: if A is a set of real numbers, $A \neq \varnothing$, and A is bounded above, then A has a *least upper bound*.

Define $A = \{\ x : a \leq x \leq b$ and f is negative on $[a, x]\ \}$.

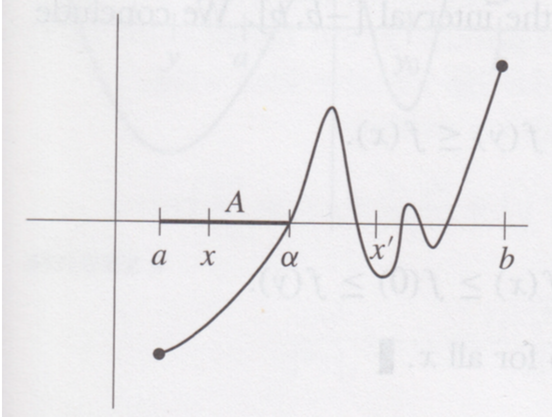

FIGURE 11

(This definition of A will exclude points like x' where $f(x') < 0$, but not every point in $[a, x']$ is < 0. In effect, we are focusing on the first time $f(x)$ crosses zero. We prove there is at least one such x).

Clearly, $a \in A$ since $f(a) < 0$, so $A \neq \varnothing$.

Since f is continuous, points x close to a also have the property that $f(x) < 0$.

Similarly $b \notin A$ and points near b have the property that $f(x) > 0$ and so these points are also not in A. Hence b is certainly an upper bound for A.

Therefore, by the completeness axiom, A has a least upper bound. Let us call it α.

We claim that $f(\alpha) = 0$. The proof is by contradiction.

proof

∘ Suppose $f(\alpha) < 0$

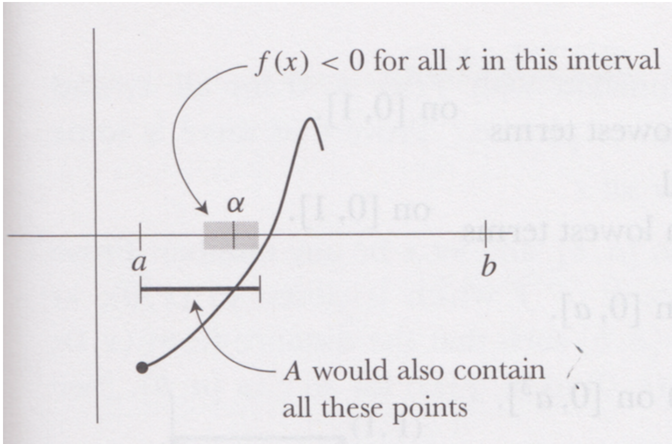

FIGURE 12

If $f(\alpha) < 0$ then nearby points also have this property. In particular we can find δ such that if x is in $[\alpha, \alpha + \delta]$, $f(x) < 0$. But this contradicts the fact that α is an upper bound for A, since there would be points $x > \alpha$ that are in A.

∘ Suppose $f(\alpha) > 0$

If $f(\alpha) > 0$ then points in the neighborhood of x also have this property.

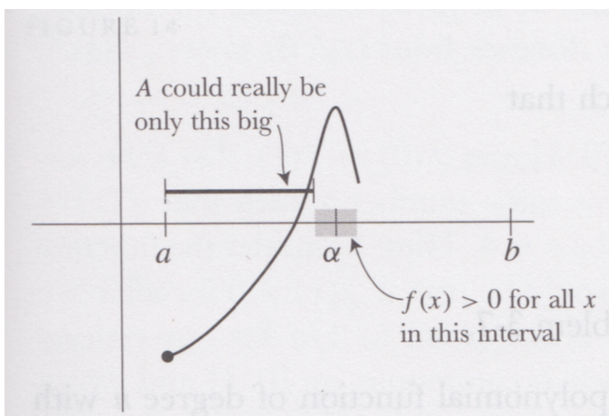

FIGURE 13

In particular we can find δ such that if x is in $[\alpha - \delta, \alpha]$, $f(x) > 0$.

But this contradicts the fact that α is the least upper bound for A, since there would be points $x < \alpha$ that are not in A and so also upper bounds.

□

I found another version of this proof with the same logic but with more δ and ϵ stuff.

proof

We look for the largest x on this interval such that $f(x) \leq 0$.

∘ Let $\mathbf{S}$ be the set of all $x \in [a, b]$ such that $f(x) \leq 0$.

∘ $\mathbf{S}$ is non-empty ($\mathbf{S} \neq \varnothing$) since $a \in \mathbf{S}$.

∘ Since $f(b) > 0$, $b \notin \mathbf{S}$, and since every x in the relevant interval is $\leq b$, b is larger than all members of $\mathbf{S}$, and so b is an upper bound of $\mathbf{S}$.

The completeness axiom says that every non-empty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

∘ Therefore, by the property of **completeness**, there exists a least upper bound or supremum of $\mathbf{S}$.

∘ Define c to be that supremum, the largest x in this interval with the property that $f(x) \leq 0$.

∘ Since $f(x)$ is continuous, $\lim_{x \to c} f(x) = f(c)$.

Exactly one of three things is true: $f(c) > 0, f(c) < 0$ or $f(c) = 0$.

We claim that $f(c) = 0$. The proof is by contradiction.

Suppose $f(c) > 0$.

∘ We define $\epsilon_1 = f(c)/2$. Then ϵ_1 is positive and $2\epsilon_1 = f(c) > 0$.

∘ By the definition of continuity, there exists δ_1, such that for all $0 < |x - c| < \delta_1$ it is true that

$$|f(x) - f(c)| < \epsilon_1$$

∘ Then, by a preliminary theorem from the section on the triangle inequality:

$$-\epsilon_1 < f(x) - f(c) < \epsilon_1$$

$$-\epsilon_1 < f(x) - 2\epsilon_1 < \epsilon_1$$

$$\epsilon_1 < f(x) < 3\epsilon_1$$

since $\epsilon_1 > 0$ this implies that $f(x) > 0$ everywhere in the interval $c - \delta_1 < x < c + \delta_1$.

∘ It would appear that we have found a smaller upper bound for the set $\mathbf{S}$ in the interval $[c - \delta_1, c)$. But by assumption, c is a supremum or least upper bound, so this is a contradiction.

To summarize: by focusing on the least upper bound c of the set of all numbers x in $[a, b]$ for which $f(x) \leq 0$, we conclude that $f(c) > 0$ is impossible.

Suppose $f(c) < 0$.

∘ We can define $\epsilon_2 = -f(c)/2$. Then $\epsilon_2 > 0$ and $-f(c) = 2\epsilon_2$.

∘ By the definition of continuity, there exists δ_2, such that for all $0 < |x - c| < \delta_2$ it is true that

$$|f(x) - f(c)| < \epsilon_2$$

∘ Then

$$-\epsilon_2 < f(x) - f(c) < \epsilon_2$$

We have $-f(c) = 2\epsilon_2$

$$-\epsilon_2 < f(x) + 2\epsilon 2 < \epsilon_2$$

$$-3\epsilon_2 < f(x) < -\epsilon_2$$

which implies that $f(x) < 0$ everywhere in the interval $c - \delta_2 < x < c + \delta_2$.

$\circ$ It would appear that we have found a value for $x < 0$ in the interval $[c, c+\delta_2)$. But c is a least upper bound for $\mathbf{S}$, there are not supposed to be any negative values of $f(x)$ larger than c, so this is a contradiction.

We conclude that $f(c) < 0$ is impossible.

The last remaining possibility is that $f(c) = 0$.

There is one more issue. We assumed above that $f(b) > 0 > f(a)$.

Suppose that $f(b) < 0$ and $f(a) > 0$. Define $g(x) = -f(x)$. Note that $g(x)$ is continuous on the same interval, and repeat the argument. The conclusion does not depend on this assumption.

This completes the proof of Bolzano's Theorem.

2 Existence of the square root of 2

The above proof is basically the same as a proof that $\sqrt{2}$ exists (for example).

We find $\sqrt{2}$ as the least upper bound of the set

$$\mathbf{A} = \{a \in \mathbb{R} \mid a^2 < 2\}$$

We know that $\mathbf{A}$ is bounded above (certainly, by 2), and so it has a least upper bound b by the completeness axiom.

We claim that $b^2 = 2$

We will prove this by showing that assuming that $b^2 < 2$ or $b^2 > 2$ both lead to contradictions.

• Suppose that $b^2 > 2$.

Consider a number just a bit smaller than b, namely $b - 1/n$. Then we can always find n so that $(b - 1/n)^2 > 2$. Multiplying out:

$$(b - \frac{1}{n})^2 = b^2 - \frac{2b}{n} + \frac{1}{n^2} > b^2 - 2\frac{b}{n}$$

We show that we can find n such that

$$b^2 - 2\frac{b}{n} > 2$$

$$b^2 - 2 > 2\frac{b}{n}$$

$$\frac{b^2 - 2}{2b} > \frac{1}{n}$$

We can always find such an n, by the Archimedean property.

Since $(b - 1/n)^2 > 2$ it is an upper bound on $\mathbf{A}$ (numbers whose square is less than 2.

This contradicts the assumption that b is the *least* upper bound.

• Similarly, assume $b^2 < 2$.

In this case we will prove that b is not an upper bound at all for $\mathbf{A}$.

We do this by showing that $(b+1/n)^2 < 2$ and so is $\in \mathbf{A}$ but $(b+1/n)^2 > b^2$. Thus it is an element in the set which is larger than the supposed upper bound.

We have

$$(b + \frac{1}{n})^2 = b^2 + \frac{2b}{n} + \frac{1}{n^2}$$

and we need

$$b^2 + \frac{2b}{n} + \frac{1}{n^2} < 2$$
$$b^2 - 2 < -\frac{2b}{n} - \frac{1}{n^2}$$
$$b^2 - 2 > \frac{2b}{n} + \frac{1}{n^2}$$

We can always find such an n. As n gets large, the second term will get small much faster than the first. We need to find n large enough that

$$\frac{b^2 - 2}{2b} > \frac{1}{n}$$

and we already did, above.

Now we have that b is not an upper bound on the set **A** (numbers whose square is less than 2) since $(b + 1/n)^2 < 2$.

Since neither of $b^2 > 2$ and $b^2 < 2$ is true, we conclude that $b^2 = 2$.