Python for Bioinformatics

adventures in bioinformatics

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Euler's gem

Here is a sketch of the derivation of Euler's famous formula:

$$e^{i\theta} = \cos\theta + i \sin\theta$$

as presented by William Dunham in his book *Euler, The Master of Us All*.

The first part of the proof is similar to when we used Euler's formula to derive other formulas for trig functions of sums and differences of angles (post), only backward. Start from the definition of i:

$$i = \sqrt{-1}$$

To begin with, having i allows us to factor new expressions:

$$\begin{aligned} 1 &= \cos^2 s + \sin^2 s \\ &= (\cos s + i \sin s)(\cos s - i \sin s) \end{aligned}$$

(I'm going to use s and t, as before, rather than θ and φ).

This shows where the original idea of cos + i sin comes from. (Of course, we could just as well do sin + i cos, that would result in a different convention for the orientation of the complex plane).

Suppose we have two angles s and t, we can multiply and then use the formulas from before (obtained by the geometric proof):

$$\begin{aligned} &(\cos s + i \sin s)(\cos t + i \sin t) = \\ &= (\cos s \cos t - \sin s \sin t) + i(\sin s \cos t + \cos s \sin t) \\ &= \cos(s + t) + i \sin(s + t) \end{aligned}$$

Set s = t:

$$(\cos s + i \sin s)^2 = \cos(2s) + i \sin(2s)$$

In fact Euler showed it works for fractional n but I'll assume that part:

$$[1]\ (\cos s + i \sin s)^n = \cos(ns) + i \sin(ns)$$
$$n >= 1$$

If we multiply the difference rather than the sum:

Jackson's Mill WV

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$$(\cos s - i \sin s)(\cos t - i \sin t) =$$
$$= (\cos s \cos t - \sin s \sin t) - i (\sin s \cos t + \cos s \sin t)$$
$$= \cos(s + t) - i \sin(s + t)$$

Again, with s = t we have:

$$(\cos s - i \sin s)^2 = \cos(2s) - i \sin(2s)$$
$$[2]\ (\cos s - i \sin s)^n = \cos(ns) - i \sin(ns)$$

Adding [1] and [2] we have:

$$2 \cos(ns) = (\cos s + i \sin s)^n + (\cos s - i \sin s)^n$$

The middle part of the proof is where the magic happens. Let:

$$s = x/n$$

As

$$n \to \infty$$
$$s \to 0$$
$$\cos s \to 1$$
$$\sin s \to s$$

So..

$$\cos x = \cos ns = 1/2\ [(\cos s + i \sin s)^n + (\cos s - i \sin s)^n]$$
$$\cos x = 1/2\ [(1 + is)^n + (1 - is)^n]$$
$$\cos x = 1/2\ [(1 + ix/n)^n + (1 - ix/n)^n]$$

But..

$$e^{ix} = (1 + ix/n)^n$$
$$\text{as } n \to \infty$$

So

$$\cos x = 1/2\ [e^{ix} + e^{-ix}]$$

By very similar manipulation to what's in the first part we can also handle the sine:

$$2i \sin(ns) = (\cos s + i \sin s)^n - (\cos s - i \sin s)^n$$

We will obtain:

$$\sin x = 1/(2i)\ [e^{ix} - e^{-ix}]$$

Now it's just a matter of addition:

$$\cos x + i \sin x = 1/2\,[e^{ix} + e^{-ix} + e^{ix} - e^{-ix}]$$
$$= 1/2\,[e^{ix} + e^{ix}]$$
$$= e^{ix}$$

Wow!

Posted by telliott99 at 7/13/2011 05:42:00 PM

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telliott99

I teach and do research in Microbiology. This blog started as a record of my adventures learning bioinformatics and using Python. It has expanded to include Cocoa, R, simple math and assorted topics. As bbum says, it's so "google can organize my head." The programs here are developed on OS X using R and Python plus other software as noted. YMMV

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