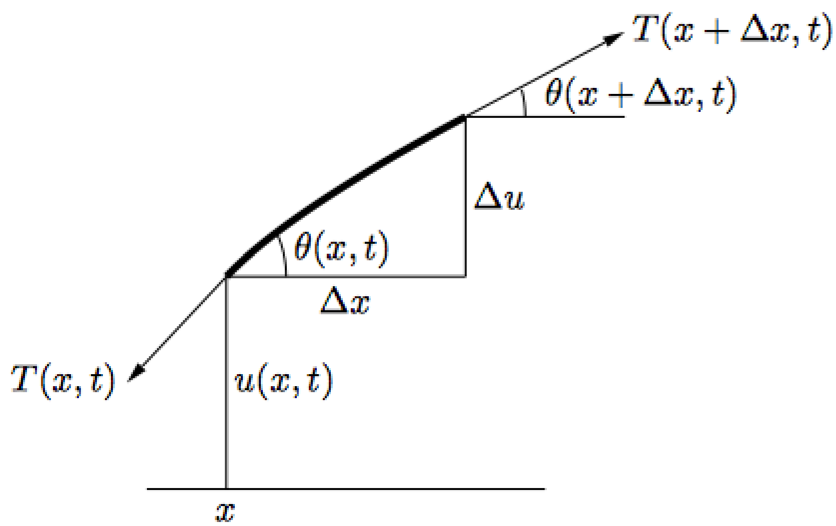
The Wave Equation

This short write-up contains a derivation of the wave equation. We consider a violin string pinned down at the ends and then plucked. Here is a short segment of the string (the notation doesn't match exactly what I'm going to use, but it's a place to start).



Here, x is not a variable but just a label for a position on the string. We start to solve this problem by an approximation, saying that the tension T (the force in the direction shown by the arrows), has the *same magnitude* at both ends of the short interval shown as Δx in the figure.

What differs between the two ends of the interval and provides a net force is the difference in the angle θ at the two positions x and $x+\Delta x$. That force is

$$T \sin \theta_{x+\Delta x} - T \sin \theta_x$$

which, by Newton's Law, is equal to ma. For this small segment of the string

$$T \sin \theta_{x+\Delta x} - T \sin \theta_x = dm \ a$$

where dm is the mass of this small segment. You might be tempted to write $\ddot{x}$ (d^2x/dt^2) for a here, but as we said, in this problem x is just a label for a position on the string.

The value which changes is the displacement, which we will call ψ. Furthermore, if you think about it, it is clear that the displacement ψ is a function of both time and the horizontal coordinate x, so we need the partial derivative

$$T \sin \theta_{x+\Delta x} - T \sin \theta_x = dm \ \frac{\partial^2 \psi}{\partial t^2}$$

Now, dm is the mass of this small segment, which is equal to the mass per unit length times dx.

$$T \sin \theta_{x+\Delta x} - T \sin \theta_x = \mu \ dx \ \frac{\partial^2 \psi}{\partial t^2}$$

On the left hand side we are going to apply the small angle approximation. Recall that

$$\sin \theta \approx \theta$$

(where the next term in the series for $\sin \theta$ is $-\theta^3/3!$). Since $\cos \theta \approx 1$ then

$$\theta \approx \sin \theta \approx \tan \theta$$

If you look back at the figure you will see that according to the labels there

$$\frac{\Delta u}{\Delta x} = \tan\theta$$

Now, u is what we are calling ψ and this is really a partial derivative

$$\frac{\partial \psi}{\partial x} = \tan\theta \approx \sin\theta$$

$$T(\frac{\partial \psi}{\partial x}\bigg|_{x+dx} - \frac{\partial \psi}{\partial x}\bigg|_{x}) = \mu\ dx\ \frac{\partial^2 \psi}{\partial t^2}$$

Now, divide both sides by T and by dx and let $dx \to 0$ and we get

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\mu}{T}\ \frac{\partial^2 \psi}{\partial t^2}$$

This is the wave equation, but we will re-write it as

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\ \frac{\partial^2 \psi}{\partial t^2}$$

$$v = \sqrt{T/\mu}$$

It will turn out that v is the velocity of the wave.

We just guess the solution

$$\psi(x,t) = A\cos kx + \omega t$$

where k is called the *wave number.*

$$\frac{\partial^2}{\partial x^2}\ \psi(x,t) = -k^2\ \psi(x,t)$$

$$\frac{\partial^2}{\partial t^2}\ \psi(x,t) = -\omega^2\ \psi(x,t)$$

So

$$
\begin{aligned}
-k^2 &= -\frac{\omega^2}{v^2} \\
k &= \pm\frac{\omega}{v} \\
\pm kv &= \omega \\
\psi(x,t) &= A\cos kx - \omega t
\end{aligned}
$$

At time zero, this function has a maximum at $x = 0$. Wait a time dt, then the maximum is when $k\ dx - \omega\ dt = 0$.

$$\frac{dx}{dt} = \frac{\omega}{k}$$

Substituting $\omega = \pm kv$

$$\frac{dx}{dt} = \pm v$$

and

$$\psi(x,t) = A\cos kx \pm kvt = A\cos k(x \pm vt)$$

Clearly, the crest of the wave is moving at the velocity v.

$$\psi(x,t) = A\cos k(x - vt)$$

describes a wave moving to the right, and the opposite choice of sign means a wave moving to the left.

Note that *any* function $f(x - vt)$ satisfies the wave equation, even

$$Ae^{-k^2(x-vt)^2}$$

If $kx = 2\pi$ the wave repeats and by definition

$$
\begin{aligned}
k\lambda &= 2\pi \\
k &= \frac{2\pi}{\lambda}
\end{aligned}
$$

$$v = \frac{\omega}{k} = \frac{\omega\lambda}{2\pi}$$

since $\omega = 2\pi f$

$$v = f\lambda$$

The wavelength times the frequency is equal to the velocity.