Integrate the square root of z

Consider

$$\int \sqrt{z}\ dz$$

along the half-circle of radius 3 starting from the point $z = R$ on the x-axis and proceeding counter-clockwise. We can do this integral even if the "branch" of the square root function that we're using is only defined for $\theta > 0$. We have that

$$z = Re^{i\theta}, \quad \theta = 0 \to \pi$$

$$dz = iz = iRe^{i\theta}\ d\theta$$

$$\sqrt{z} = \sqrt{R}e^{i\theta/2}$$

so

$$I = \int_0^\pi iR\sqrt{R}e^{i3\theta/2}\ d\theta$$

We need

$$\int e^{i3\theta/2}\ d\theta = \frac{2}{3i}e^{i3\theta/2}\ \Big|_0^\pi$$

easiest to write it out as

$$e^{i3\theta/2}\ \Big|_0^\pi = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} - \cos 0 - i\sin 0$$

$$= 0 + i(-1) - 1 - 0 = -(1+i)$$

Going back to pick up all the factors we left behind:

$$I = -iR\sqrt{R}\ \frac{2}{3i}\ (1+i) = -R\sqrt{R}\ \frac{2}{3}\ (1+i)$$

In the problem, R was actually specified as 3, leading to the cancellation:

$$I = -2\sqrt{3}\ (1+i)$$

We can also do this problem by antiderivatives:

$$\int_{R}^{-R} \sqrt{z}\ dz = \frac{2}{3}\ z^{3/2}\ \Big|_{R}^{-R}$$

$$= \frac{2}{3}(R^{3/2}e^{i3\pi/2} - R^{3/2}e^{0})$$

$$= \frac{2}{3}R^{3/2}(e^{i3\pi/2} - 1)$$

and, as we showed above:

$$e^{i3\pi/2} = -i$$

If $R = 3$ we get the same answer as before.