Introduction to surface integrals 2

The theorem for surface integrals is that the area element is given by

$$dS = \sqrt{(f_x)^2 + (f_y)^2 + 1} \;\; dx \; dy$$

$$A(S) = \iint_D 1 \; dS = \iint_R \sqrt{(f_x)^2 + (f_y)^2 + 1} \;\; dA$$

In this short write-up, we want to see where this comes from. Consider a surface formed by

$$z = f(x, y)$$

at a point $P = (x_0, y_0)$. Slice by a plane parallel to the xz-axis ($y = y_0 = const$). Consider what happens to z as we move to $x = x_0 + \Delta x$. Using the linear approximation, the vector along our path is

$$\mathbf{u} = < \Delta x, 0, f_x \Delta x > = < 1, 0, f_x > \;\; \Delta x$$

The vector above is parallel to the surface in the direction where $\Delta y = 0$. Similarly $\mathbf{v}$ is parallel to the surface in the direction where $\Delta x = 0$

$$\mathbf{v} = < 0, \Delta y, f_y \Delta y > = < 0, 1, f_y > \;\; \Delta y$$

and

$$\mathbf{n} = \mathbf{u} \times \mathbf{v}$$

$$= < -f_x, -f_y, 1 > \;\; \Delta x \; \Delta y$$

dS

If we consider the relationship between the surface area element dS and the shadow that it casts in the xy-plane, dR, the "exchange rate" is

$$\cos\theta \; dS = dR$$

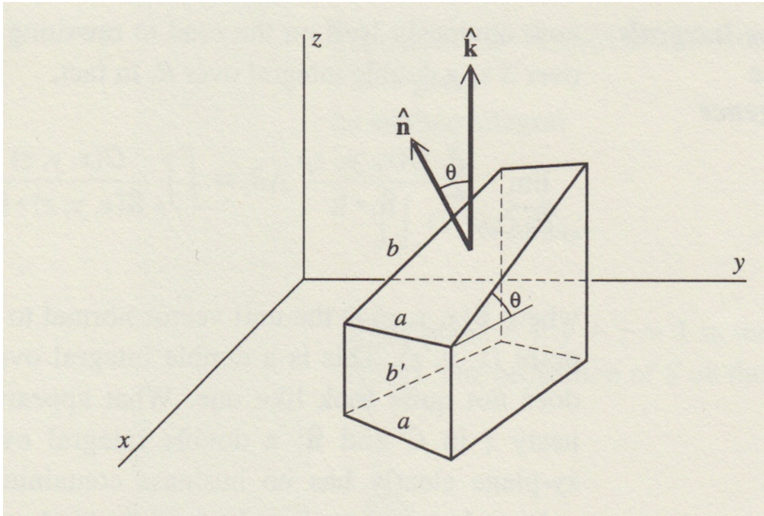

Figure II–14

dR is smaller than dS by a factor $\cos\theta$ which is just

$$\cos\theta = \hat{\mathbf{n}} \cdot \hat{\mathbf{k}}$$

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}\ dS = dR$$

$$dS = \frac{1}{\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}}\ dR$$

What is $\hat{\mathbf{n}}$? We need to divide $\mathbf{n}$ by $|\mathbf{n}|$. Compute compute $|\mathbf{n}|$ and set it equal to k

$$\mathbf{n} = < -f_x, -f_y, 1 > \ \ \Delta x\ \Delta y$$

$$|\mathbf{n}| = k = \sqrt{f_x^2 + f_y^2 + 1}\ \Delta x\ \Delta y$$

$$\hat{\mathbf{n}} = \frac{1}{k}\mathbf{n} = \frac{1}{k}\ \ < -f_x, -f_y, 1 >$$

So

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \frac{1}{k}$$

Finally

$$dS = k\ dR = \sqrt{f_x^2 + f_y^2 + 1}\ dR$$

Note then we finally do $\int \mathbf{F} \cdot \hat{\mathbf{n}}\ dS$ this will become

$$\int \mathbf{F} \cdot \frac{1}{k}\ \ < -f_x, -f_y, 1 > \ \ k\ dR = \int \mathbf{F} \cdot\ \ < -f_x, -f_y, 1 > \ \ dR$$