Auroux Ch. 23: Flux

To review, we compute work as

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

As a shorthand we say that

$$W = \int_C M dx + N dy$$

We get from one to the other by deconstructing $d\mathbf{r}$

$$d\mathbf{r} = \hat{\mathbf{T}}\ ds = \frac{d\mathbf{r}}{dt}\ dt$$

where

$$\frac{d\mathbf{r}}{dt} = < \frac{dx}{dt}, \frac{dy}{dt} >$$

Hence, if we have $\mathbf{F} =< M, N >$ then

$$W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\ dt = \int_C < M, N > \cdot < \frac{dx}{dt}, \frac{dy}{dt} >\ dt = \int_C M dx + N dy$$

example

If

$$\mathbf{F} =< -y, x >$$

$$x = t, \quad y = t^2$$

we obtain

$$W = \int_C < M, N > \cdot < \frac{dx}{dt}, \frac{dy}{dt} >\ dt$$

$$= \int_C < -t^2, t > \cdot < 1, 2t >\ dt$$

$$= \int_C t^2$$

$$= \frac{1}{3}t^3 \Big|_{P1}^{P2}$$

Flux

In this write-up, we're concerned with Flux, which also has a shorthand

$$Flux = \int_C -Ndx + Mdy$$

We switched M for N and changed signs.

Work is done by the component of the force in the direction of $\hat{\mathbf{T}}$. It's the "tail wind", if you will. Flux is the "cross-wind", it is the component $\perp \hat{\mathbf{T}}$.

$$Flux = \int_C \mathbf{F} \cdot \hat{\mathbf{n}} \; ds$$

We'll show a derivation below.

Example

Sometimes we don't need to calculate! Suppose

$$\mathbf{F} = < x, y >$$

a radial field. Let C be a circle of radius a centered at the origin. In this situation the formula with $\hat{\mathbf{n}}$ is useful.

$$\mathbf{F} \cdot \hat{\mathbf{n}} = |F||n|cos\theta = |F| = \sqrt{x^2 + y^2} = a$$

So

$$Flux = \int_C \mathbf{F} \cdot \hat{\mathbf{n}} \; ds$$

$$= \int_C a \; ds = a \int_C \; ds$$

$$= a \; 2\pi a$$

$$= 2\pi a^2$$

Note that $\hat{\mathbf{n}}$ is just $\hat{\mathbf{T}}$ rotated by 90° cw. (The convention is that the integral should be positive if we move along the curve with the region on our left). Now

$$d\mathbf{r} = < dx, dy >$$
$$= \hat{\mathbf{T}}\ ds$$

so

$$\hat{\mathbf{n}}\ ds =< dy, -dx >$$

and

$$\int_C \mathbf{F} \cdot \hat{\mathbf{n}}\ ds$$
$$= \int_C < M, N > \cdot < \frac{dy}{dt}, -\frac{dx}{dt} >\ dt$$

which we can try to remember as

$$= \int_C -Ndx + Mdy$$

as shown above.

Green's Theorem

Our statement of the theorem was that

$$\boxed{\int_C Mdx + Ndy = \int\int_R (N_x - M_y)\ dA} \qquad (1)$$

We can use the "del" notation to make this shorter

$$\int_C Mdx + Ndy = \int\int_R (N_x - M_y)\ dA$$
$$= \int\int_R \nabla \times \mathbf{F}\ dA$$

I will come back to that cross-product in a minute. But $N_x - M_y$ is the curl of $\mathbf{F}$. Now we just switch letters! Put $-N$ for M and M for N

$$\boxed{\int_C -Ndx + Mdy = \int\int_R (M_x + N_y)\ dA = \int\int_R \nabla \cdot \mathbf{F}\ dA} \qquad (2)$$

This is Green's Theorem for Flux. The left-hand side is the Flux, the right-hand side is a way to calculate the same quantity using $\nabla \cdot \mathbf{F}$

example

Suppose $\mathbf{F} = < x, y >$ and the curve is a circle of radius a, but centered at $(0, 2)$. So now parametrization gets a little trickier.. But notice that

$$\nabla \cdot \mathbf{F} = 2$$

So we can calculate the Flux by using Green's Thm (for Flux). It is just

$$\int \int_R \nabla \cdot \mathbf{F} \; dA = 2 \int \int_R dA = 2\pi a^2$$

and this is true regardless of where the circle is.

Notation

So we have the symbol ∇ which we use as an operator

$$\nabla = \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$$

We have already used this in defining the *gradient* of f

$$\nabla f = \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} = < f_x, f_y, f_z >$$

Now we introduce a dot product and cross product employing ∇. The first is the divergence of $\mathbf{F}$

$$\mathbf{F} = < P, Q, R >$$

$$\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$$

And the second is the curl of $\mathbf{F}$

$$\nabla \times \mathbf{F}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

This determinant has three components

$$|(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}) - (\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})|$$

What we have in Green's Theorem (with different letters, substitute N for Q and M for P), is a vector with only x and y components and hence only one of the terms.