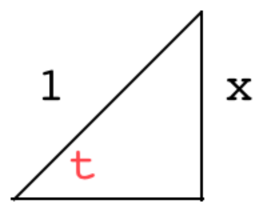
Injection, surjection and inverse sine

The question of whether a function has a reverse (is reversible) comes up naturally in the study of techniques of integration. Consider this triangle



We have constructed the triangle so that the sine of the angle t is equal to $x/1$:

$$\sin t = x$$

The reverse of sine is the function arcsin (or $\sin^{-1}$):

$$\sin^{-1} x = t$$

Read this as: arcsine (inverse sine) of x is equal to t. Alternatively, we can say that t is the angle whose sine is equal to x. It doesn't really matter that there is no technique for computation other than trying different values for x, calculating $\sin x$, and comparing that with the t we are given.

Now, by the Pythagorean Theorem, the third side of the triangle has length $\sqrt{1-x^2}$, and in terms of angle t, we have that

$$\cos t = \sqrt{1-x^2}$$

This becomes useful where we have an integral like:

$$\int \frac{1}{\sqrt{1-x^2}}\, dx$$

We cannot just substitute for $1-x^2$ because we don't have the derivative (we don't have an x on top). But doing a trigonometric substitution, we have that

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{\cos t}$$

To complete the substitution, we need to find dx in terms of dt:

$$x = \sin t$$

$$dx = \cos t \; dt$$

So that gives:

$$\int \frac{1}{\cos t} \; \cos t \; dt$$

which simplifies nicely to:

$$\int dt = t$$

To complete the solution, we need to switch back to the original variable x.

$$t = \sin^{-1} x$$

Thus:

$$\int \frac{1}{\sqrt{1-x^2}} \; dx = \sin^{-1} x + C$$

It may seem strange at first that this integral in Cartesian coordinates (xy-"land") gives a result in terms of the angle t, but consider that the equation of the unit circle is

$$x^2 + y^2 = 1$$

so

$$y = +\sqrt{1-x^2}$$

is the equation of the top half of the circle (above the x-axis). The integral that we have computed is the area under this curve, under the right limits it is the area of the unit circle, so naturally this result involves π.

Another way to approach this (which involves the same relationships) is to use differentials

$$x = \sin t$$

$$
\begin{aligned}
dx &= \cos t \; dt \\
\frac{dt}{dx} &= \frac{1}{\cos t} \\
\frac{d}{dx} t &= \frac{1}{\cos t} \\
\frac{d}{dx} \sin^{-1} x &= \frac{1}{\sqrt{1-x^2}} \\
\int \frac{d}{dx} \sin^{-1} x \; dx &= \int \frac{1}{\sqrt{1-x^2}} \; dx \\
\sin^{-1} x &= \int \frac{1}{\sqrt{1-x^2}}
\end{aligned}
$$

In working with reverse functions, we have to consider carefully the domain and range of each. (more)

We also have to consider whether a given function even has an inverse. Consider (following Koblitz)

$$f = x^3; \quad g = x^{1/3}$$

For these two functions to qualify as inverses we require that

$$x = f(g(x))$$

and

$$x = g(f(x))$$

However, this is not true of the square root function

$$f = x^2; \quad g = x^{1/2}$$

because there are two real numbers that satisfy $x^2 = 2$ for example, but when we take the square root, we define the positive root as the result of the function g.

injective

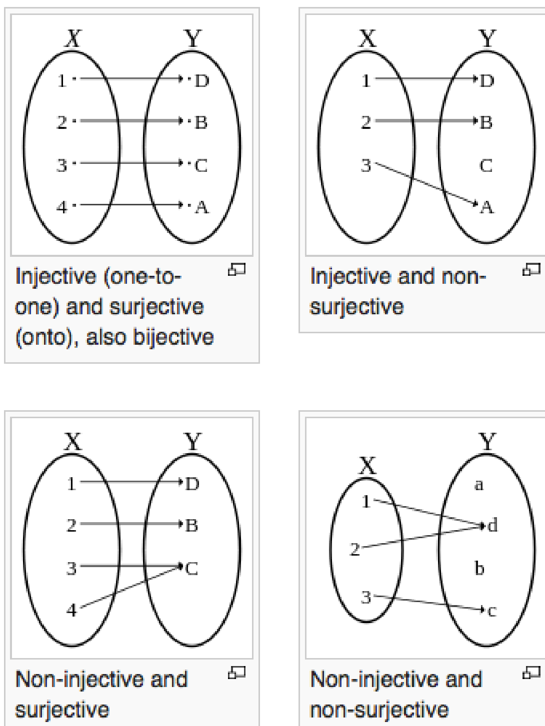
In the language of set theory, consider (the set of) all real numbers that are in the domain of f—call that set A, and then call the corresponding values of $f(x)$ the set B. B is range of f.

If and only if each of these numbers yields a *different* value in B, then it may be possible to come up with an inverse function g which maps from the range back to the domain.

Such a function is described as *injective*. On the other hand, if two different values a_1, a_2 in the domain of f yield the same value $f(a_1) = f(a_2)$, then the function is non-injective.

Koblitz:

Let A and B be sets and let $f : A \to B$ be a function. We say that f is injective or one-to-one if $f(x) = f(y)$ implies that $x = y$. See the wikipedia figure, upper-left panel.



Injective (one-to-one) and surjective (onto), also bijective

Injective and non-surjective

Non-injective and surjective

Non-injective and non-surjective

surjective

Surjective is a statement about the range and co-domain of f. If there numbers in the co-domain that do not correspond to $f(x)$ for *any* x in the range of f, then f is not surjective. See the upper-right panel.

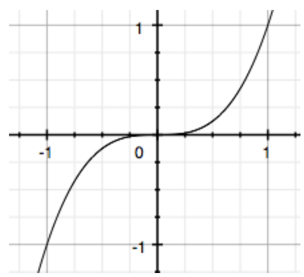
Koblitz:

We say that f is surjective or onto if for every $b \in B$ there is an $a \in A$ such that $f(a) = b$.

If f is both injective and surjective, then f is bijective or one-to-one and onto.

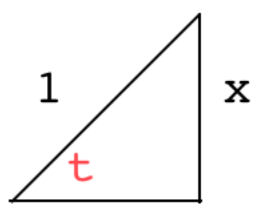
A simple test is the **horizontal line test**. If a horizontal line intersects the graph of $f(x)$ at more than one point, the function is not injective and not reversible. (Those two points are different values of x which share the same value $y = f(x)$).

The function $f(x) = x^2$ is a parabola and it fails the horizontal line test. On the other hand, the graph of $f(x) = x^3$ is:



and it passes the test. (However $x^3 - x^2$ fails).

Let's look at the graph of the arcsin and arccosine functions.



These graphs are simply plots of the more familiar sine and cosine that have been rotated counter-clockwise by 90 degrees, and also flipped left-to-right (to make x-values increase to the right, as usual.

By definition, a function assigns a unique value of y for each value of x. Since these functions repeat, we limit the domain appropriately. For arcsin, the domain is $y = -\pi/2 \to \pi/2$, while for arccose it is $y = 0 \to \pi$.

Note on constants

We looked at the function $1/\sqrt{1-x^2}$, but a more general form is $1/\sqrt{a^2-x^2}$, for a constant a.

There are two ways to deal with this. The first is to scale the triangle so that the hypotenuse is of length a and the side adjacent to angle t is $\sqrt{a^2-x^2}$. Then, we have

$$
\begin{aligned}
x &= a \sin t \\
\sqrt{a^2-x^2} &= a \cos t \\
dx &= a \cos t \; dt \\
\int \frac{1}{\sqrt{a^2-x^2}} \; dx &= \int \frac{1}{a \cos t} a \cos t \; dt = t \\
t &= \sin^{-1} \frac{x}{a}
\end{aligned}
$$

Another way to do this is to manipulate the original equation:

$$
\begin{aligned}
&\int \frac{1}{\sqrt{a^2-x^2}} \; dx \\
&= \int \frac{1}{a\sqrt{1-x^2/a^2}} \; dx \\
&= \frac{1}{a} \int \frac{1}{\sqrt{1-x^2/a^2}} \; dx
\end{aligned}
$$

Now let $u = x/a$ so that $a \; du = dx$ and then

$$
\begin{aligned}
&= a \; \frac{1}{a} \int \frac{1}{\sqrt{1-u^2}} \; du \\
&= \int \frac{1}{\sqrt{1-u^2}} \; du
\end{aligned}
$$

We obtain

$$
\begin{aligned}
t &= \sin^{-1} u \\
&= \sin^{-1} \frac{x}{a}
\end{aligned}
$$

as before.