Cauchy integral theorem

Cauchy' First Integral Theorem

Cauchy's first theorem says that the integral of an analytic function over a closed path is equal to zero (when the enclosed region is without a singularity).

$$\oint_C f(z)\ dz = 0$$

This will turn out to be a consequence of Green's Theorem, which I've written about a lot before. Let

$$z = x + iy$$

$$dz = dx + idy$$

$$z = f(x,y) = u(x,y) + iv(x,y)$$

Our integral is

$$\oint_C z\ dz = \int (u(x,y) + iv(x,y))\ (dx + idy)$$

$$= \oint_C u(x,y)\ dx - \int v(x,y)\ dy + i \int v(x,y)\ dx + i \int u(x,y)\ dy$$

As before, because we are moving along a curve there is a relationship between x and y, so we can either express that relationship or parametrize the curve. In any case, these become integrals in a single variable. We remove extra (x,y) notation:

$$= \oint_C u\ dx - \int v\ dy + i \int v\ dx + i \int u\ dy$$

proof of Cauchy 1

Back in vector calculus we proved Green's theorem, which says that for two real functions of x and y: $M(x, y)$ and $N(x, y)$:

$$\oint_C M dx + N dy = \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\ dx\ dy$$

Back then, M and N were components of a vector field $\mathbf{F}$ and we wrote the shorthand for curl:

$$= \iint_R \nabla \times \mathbf{F}\ dA$$

but the important thing is that they are real functions of two variables $f : \mathbb{R}^2 \to \mathbb{R}^1$.

In terms of u and v we have for the real part of Cauchy's Theorem that $M = u$ and $N = -v$ (notice the minus sign!).

So:

$$\oint u\ dx - \oint v\ dy = \iint_R (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\ dx\ dy$$

$$= - \iint_R (v_x + u_y)\ dx\ dy$$

But, according to the CRE

$$u_y = -v_x$$

Hence, this integral is zero.

For the imaginary part:

$$\oint v\ dx + \oint u\ dy = \iint_R (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})\ dx\ dy$$

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$$= \iint_R (u_x - v_y)\ dx\ dy$$

But, again, according to the CRE

$$u_x = v_y$$

So the integral for the imaginary part is also zero, and thus the whole thing is zero as well:

$$\oint u\ dx - \oint v\ dy + i \oint v\ dx + i \oint u\ dy = 0$$

Remember how important it was (for Green's theorem) that the function being integrated be defined everywhere in the region. For example, it is *not* true that

$$\oint_C \frac{1}{z}\ dz = 0$$

if the curve C includes the origin, but it *is* true otherwise. A simple demonstration for the former case is the unit circle centered at the origin. We write

$$z = re^{i\theta}$$
$$\frac{dz}{d\theta} = rie^{i\theta} = iz$$

Hence

$$\oint_C \frac{1}{z}\ dz = \oint_C \frac{1}{z}\ iz\ d\theta$$
$$= i \oint_C d\theta = 2\pi i$$