Karplan on Residues (8.15)

Kaplan gives some rules for computing residues:

RULE I At a simple pole z_0 (that is, a pole of first order),

$$\text{Res } [f(z), z_0] = \lim_{z \to z_0} (z - z_0) f(z)$$

example

$$f(z) = \frac{1}{z^2 + 1} = \frac{1}{(z-i)(z+i)}$$

$$\text{Res } [f(z), z = i] = \lim_{z \to i} (z - i) \frac{1}{(z-i)(z+i)}$$

$$= \lim_{z \to i} \frac{1}{z + i} = \frac{1}{2i}$$

RULE II At a pole of order N ($N = 2, 3, \dots$),

$$\text{Res } [f(z), z_0] = \lim_{z \to z_0} (z - z_0) \frac{g^{(N-1)}(z)}{(N-1)!}$$

where

$$g(z) = (z - z_0)^N f(z)$$

example

$$f(z) = \frac{e^z}{z(z-1)^2}$$

We have a pole of first order at $z = 0$ and one of second order at $z = 1$. At the first

$$\text{Res }[f(z), z = 0] = \lim_{z \to 0} \frac{e^z}{(z-1)^2} = 1$$

For the other one, remove the factor of $(z-1)^2$ and compute the $N-1$ (first) derivative of what's left

$$\text{Res }[f(z), z = 1] = \lim_{z \to 1} [\ \frac{e^z}{z}\ ]'$$

$$= \left.\frac{e^z z - e^z}{z^2}\right|_1 = 0$$

Hence

$$\oint f(z)\ dz = 2\pi i\ [\ \sum \text{Res}\ \ ]\ = 2\pi i$$

RULE III If $A(z)$ and $B(z)$ are analytic in a neighborhood of z_0, $A(z_0) \neq 0$, and $B(z)$ has a zero at z_0 of order 1, then

$$f(z) = \frac{A(z)}{B(z)}$$

has a pole of first order at z_0 and

$$\text{Res }[f(z), z_0] = \frac{A(z_0)}{B'(z_0)}$$

example

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$$

$$B = z^2 + 1, \quad B' = 2z$$

$$\text{Res } [f(z), z = i] = \frac{1}{2i}$$

$$\text{Res } [f(z), z = -i] = -\frac{1}{2i}$$

example

$$f(z) = \frac{ze^z}{z^2 - 1}$$

Both top and bottom are analytic. The poles of $B(z)$ are at $\pm$ 1. $A(z) \neq 0$ at those points. We have

$$\frac{A(z)}{B'(z)} = \frac{ze^z}{2z}$$

$$\left.\frac{ze^z}{2z}\right|_{z=1} = \frac{e}{2}$$

$$\left.\frac{ze^z}{2z}\right|_{z=-1} = e^{-1}$$

RULE IV If $A(z)$ and $B(z)$ are analytic in a neighborhood of z_0, $A(z_0) \neq 0$, and $B(z)$ has a zero at z_0 of order 2, then

$$\text{Res } [f(z), z_0] = \frac{6A'B'' - 2AB'''}{3B''^2}$$

example

$$f(z) = \frac{e^z}{z(z-1)^2}$$

$$B = z(z^2 - 2z + 1) = z^3 - 2z^2 + z$$

$$B' = 3z^2 - 4z + 1$$

$$B'' = 6z - 4$$

$$B''' = 6$$

So

$$\frac{6A'B'' - 2AB'''}{3B''^2} = \frac{6(e^z)(6z-4) - 2e^z(6)}{3(6z-4)^2}$$

Evaluate at $z = 1$:

$$\frac{6e(2) - 2e(6)}{12} = 0$$

So only the pole at $z = 0$ contributes.