Cubics and complex numbers

Among the applications of complex numbers that are often cited is their use in solving cubic equations. I find the procedures for solving cubics difficult to grasp, but I discovered another approach that makes sense to me, building up to cubics from quadratics. So let's start with

$$f(x) = ax^2 + bx + c$$

As you know, the graph of this function is a parabola, pointing up or down depending on the sign of a. It's symmetric about the vertex, which can be found in various ways, but using calculus, it is the place where the slope is equal to zero

$$2ax + b = 0$$

$$x = -\frac{b}{2a}$$

$$y = a(-\frac{b}{2a})^2 + b(-\frac{b}{2a}) + c$$

$$y = \frac{b^2}{4a} - \frac{b^2}{2a} + c = -\frac{b^2}{4a} + c$$

Now, if the sign of a is positive and $y < 0$ at the vertex, then the parabola opens up and the vertex is below the x-axis and there will be two real roots. There are two places where $y = 0$ and the graph crosses the x-axis. If $y = 0$ then there is a single real root, that is repeated. And if $a < 0$ and y is above the x-axis, there are no real roots.

We obtain the quadratic equation by completing the square.

$$y = ax^2 + bx + c$$

When is $y = 0$?

$$0 = ax^2 + bx + c$$

$$-\frac{c}{a} = x^2 + \frac{b}{a}x$$

$$-\frac{c}{a} + \frac{b^2}{4a^2} = (x^2 + \frac{b}{a}x + \frac{b^2}{4a^2})$$

$$-\frac{c}{a} + \frac{b^2}{4a^2} = (x + \frac{b}{2a})^2$$

$$-4ac + b^2 = 4a^2(x + \frac{b}{2a})^2$$

$$\sqrt{b^2 - 4ac} = 2ax + b$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

As an aside, if $a = 1$, then this is the same as

$$x = \frac{b}{2} \pm \sqrt{(\frac{b}{2})^2 - c}$$

and of course, we can always divide the above equation by a to achieve this, changing the coefficients b, c to b', c'.

The term $b^2 - 4ac$ in the first version is called the discriminant, D. If $D < 0$ there are no real solutions, but we can get two complex solutions which have the form

$$x = p \pm iq$$

To see why this is so, suppose $b^2 < 4ac$. Factor out -1 to obtain $(-1)(b^2 - 4ac)$. Then the factor of -1 can come out as i and for the square root part we have $\pm i\sqrt{4ac - b^2}$.

These two solutions $x = p \pm iq$ are complex conjugates, so that

$$(p + iq)(p - iq) = p^2 + q^2$$

On the other hand, if $b^2 = 4ac$, there is only a single solution yielding $y = 0$ and notice that from above, the y-value of the vertex of the parabola is

$$\begin{aligned} y &= -\frac{b^2}{4a} + c \\ 0 &= -\frac{b^2}{4a} + c \\ b^2 &= 4ac \end{aligned}$$

and then D must be equal to zero.

Finally, if $D > 0$, there are two real solutions.

cubics

What about cubics? Well, any cubic has an x^3 in it. That means that as x gets large and positive $f(x)$ is also large and positive, while if x is large and negative, $f(x)$ is large and negative. Consequently, the graph must cross the x-axis at least once, and so there must be at least one real root.

When we are building a cubic from a quadratic, no matter what the quadratic, the last step must be to multiply by $(x - r)$, where r is real.

So in other words, these are the roots of any cubic

$$(x - r)(x + \sqrt{D})(x - \sqrt{D})$$

if $D < 0$ then the latter two factors are complex conjugates. They must be, so that their product gives a completely real quadratic. If $D = 0$ then we just have

$$(x - r)x^2$$

This has three roots, at $x = 0$ (repeated) and at $x = r$. It turns out that the graph does not cross the x-axis at $x = 0$, because the first derivative

$$2x$$

is equal to zero at $x = 0$, it is a local maximum or local minimum.