Integrating zz + 1

This problem is from wikipedia. Consider

$$g(z) = \frac{z^2}{z^2 + 2z + 2}$$

We want to evaluate the integral:

$$I = \oint g(z)\, dz$$

The zeroes of the denominator are

$$\frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm \frac{\sqrt{-4}}{2} = -1 \pm i$$

Confirm

$$(z + (1-i))\ (z + (1+i))$$
$$= z^2 + z + iz + z + 1 + i - iz - i + 1$$
$$= z^2 + 2z + 2$$

The denominator can then be rewritten as

$$= \frac{A}{z + (1-i)} + \frac{B}{z + (1+i)}$$

From the numerator we have

$$A(z + (1+i)) + B(z + (1-i)) = 1$$

So $A = -B$ and

$$A(1+i) + B(1-i) = 1$$

$$A(1+i) - A(1-i) = 1$$
$$A2i = 1$$
$$A = -\frac{1}{2i}, \quad B = \frac{1}{2i}$$

The integral is

$$\oint z^2 \left[\frac{-1}{2i} \frac{1}{z+(1-i)} + \frac{1}{2i} \frac{1}{z+(1+i)} \right] dz$$

If the contour is $|z| = 2$ (the circle of radius 2, then both of the points lie within the contour.

We have two points

$$z = -1 - i, \quad z = -1 + i$$

We evaluate $2\pi i f(z_0)$ for each and add them, where

$$f(z) = -\frac{z^2}{2i}$$

$$f(z_0) = -\frac{1}{2i} (-1-i)^2 = -\frac{1}{2i} (1 + 2i - 1) = -1$$

for the first and for the second

$$f(z) = \frac{z^2}{2i}$$

$$= \frac{1}{2i} (-1+i)^2 = \frac{1}{2i}(1 - 2i - 1) = -1$$

Adding them together, the answer is just $I = -2 \times 2\pi i = -4\pi i$.