# Trig Subs 2

There are at least a couple more trig substitutions that are very useful for finding integrals. One that we saw in looking at Euler's equation was:

$$\int \frac{1}{\sqrt{1+z^2}}\ i\ dz = i\ \ln\ (\sqrt{1+z^2}+z)$$

or generally

$$\int \frac{1}{\sqrt{1+x^2}}\ dx = \ln\ (\sqrt{1+x^2}+x)$$

Of course, one way to see that this is correct is to differentiate the right-hand side:

$$\frac{d}{dx}\ \ln\ (\sqrt{1+x^2}+x)$$

by the chain rule

$$= \frac{1}{\sqrt{1+x^2}+x}\ (\frac{x}{\sqrt{1+x^2}}+1)$$

We put the second term over a common denominator

$$= \frac{1}{\sqrt{1+x^2}+x}\ (\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}})$$

And now it's clear!

$$= \frac{1}{\sqrt{1+x^2}}$$

The integral is normally written in a completely general way as

$$\int \frac{1}{x^2 + a^2}\ dx$$

with $a$ a constant. Our trig substitution is to draw a right-triangle with angle $y$ and side opposite $x$, and side adjacent $a$. Then the hypotenuse is $\sqrt{x^2 + a^2}$. We have:

$$\frac{x}{a} = \tan y$$

$$\frac{1}{a}\ dx = \sec^2 y\ dy$$

$$\frac{a}{\sqrt{x^2 + a^2}} = \cos y$$

So with the substitution the integral becomes

$$\int \frac{1}{a}\ \cos y\ a\ \ \sec^2 y\ dy$$

which is just

$$\int \sec y\ dy$$

We did this in the other section. The trick is to multiply by

$$\int \sec y\ \frac{\sec y + \tan y}{\sec y + \tan y}\ dy$$

$$\int \frac{\sec^2 y + \sec y \tan y}{\sec y + \tan y}\ dy$$

This is just

$$\int \frac{1}{u}\ du$$

so we have

$$= \ln\,(\sec y + \tan y)$$

If we undo the substitution we obtain

$$= \ln\,(\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a})$$

and with $a = 1$ this is just

$$= \ln\,(\sqrt{x^2+1} + x)$$

How about

$$\int x^3\sqrt{1-x^2}\ dx$$

This looks difficult, with that extra factor of $x^2$. Try substituting

$$x = \sin\theta$$

$$dx = \cos\theta\ d\theta$$

We have

$$= \int \sin^2\theta\sqrt{1-\sin^2\theta}\sin\theta\ d\theta$$

$$= \int (1-\cos^2\theta)\cos^2\theta\sin\theta\ d\theta$$

$$= \int (-\cos^2\theta + \cos^4\theta)(-\sin\theta\ d\theta)$$

$$= -\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C$$

$$= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C$$

We can check this by differentiating, and then we'll understand what happened to the extra factor of $x^2$

$$\frac{d}{dx}\ [\ -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C\ ]$$

$$= x(1-x^2)^{1/2} - x(1-x^2)^{3/2}$$

$$= x(1-x^2)^{1/2}\ [\ 1 - 1 + x^2\ ]$$

and there it is!

$$= x^3(1-x^2)^{1/2}$$

$$= x^3\sqrt{1-x^2}$$

Here's another one that looks weird at first:

$$\int \sqrt{x^2+6x}\ dx$$

Try completing the square

$$= \int \sqrt{x^2+6x+9-9}\ dx$$

$$= \int \sqrt{(x+3)^2-3^2}\ dx$$

Now substitute

$$\frac{x+3}{3} = \sec t$$

$x+3$ is the hypotenuse, 3 the side adjacent to angle $t$, and $\sqrt{(x+3)^2-3^2}$ is the side opposite. So

$$\sqrt{(x+3)^2 - 3^2} = 3\tan t$$

and for $dx$:

$$\frac{x+3}{3} = \sec t$$
$$\frac{1}{3}\ dx = \sec t \tan t\ dt$$

So our integral is

$$= \int 3\tan t\ 3\sec t \tan t\ dt$$

Recall that $\tan^2 t + 1 = \sec^2 t$

$$= 9\int \sec^3 t - \sec t\ dt$$

We had a trick for $\sec t$ which gives

$$\int \sec t\ dt = \ln|\sec t + \tan t| + C$$

(easily checked by differentiating). The other term is solved by integration by parts. I'll just give a sketch here:

$$\int sec^3 t\ dt = \sec t \tan t - \int \sec^3 t\ dt + \int \sec t\ dt$$
$$\int sec^3 t\ dt = \frac{1}{2}\ [\ \sec t \tan t + \int \sec t\ dt\ ]$$

combined with what was above, we end up subtracting (one-half) $\int \sec t \; dt$

$$= (9)\ \frac{1}{2}\ [\ \sec t \tan t - \ln |\sec t + \tan t|\ ] + C$$

I'll leave it to you to substitute back for $x$.