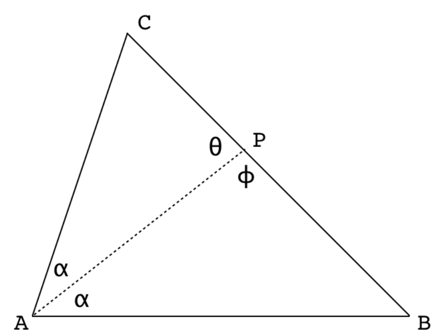
Angle bisector

In this write-up we will prove a theorem about the angle bisector. We will prove it only for the case of an acute triangle, but it is true for any triangle.



Angle A in $\triangle ABC$ is bisected by AP. In general, the lengths BP and CP are not equal. However, the ratios to their respective adjacent sides are equal:

$$\frac{CP}{AC} = \frac{BP}{AB}$$

To prove this theorem, we use the law of sines, which says that for any triangle with vertices A, B and C and opposing sides a, b and c:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Thus

$$\frac{\sin\alpha}{CP} = \frac{\sin\theta}{AC}$$

$$\frac{\sin \alpha}{BP} = \frac{\sin \phi}{AB}$$

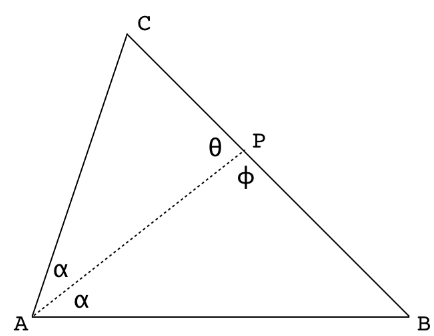
Combining these two results and eliminating α:

$$\sin \phi \frac{BP}{AB} = \sin \theta \frac{CP}{AC}$$

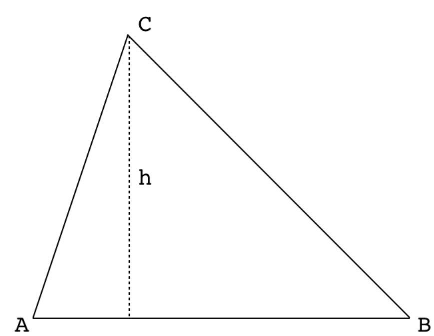
But θ and ϕ are supplementary angles, so $\sin \theta = \sin \phi$ and we obtain

$$\frac{BP}{AB} = \frac{CP}{AC}$$

which is what we wanted to prove.



The law of sines is easily demonstrated. Drop an altitude in the triangle:



$$h = AC \sin A = BC \sin B$$
$$\frac{\sin A}{BC} = \frac{\sin B}{AC}$$

By symmetry, the same can be done for angle-opposing side pair.

The other theorem we used is that the sines of supplementary angles are equal. Suppose

$$\theta + \phi = 180$$

$$\theta = 180 - \phi$$

Then write the double-angle formula for sine:

$$\sin 180 - \phi = \sin 180 \cos \phi - \sin \phi \cos 180$$

$$= 0 - \sin \phi \ (-1) = \sin \phi$$

Hence

$$\sin \theta = \sin \phi$$

A simple construction of the angle $180 - \phi$ makes this obvious.