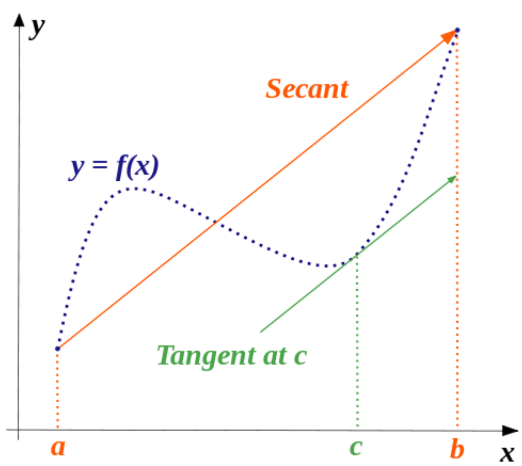
Mean Value Theorem

A man passes a police car at point A doing 60 mph (the speed limit) and 4 minutes later passes another police car at point B, also doing 60 mph, yet the second officer writes him a ticket for speeding, justified by the mean value theorem.

The reason: point A and point B are 5 miles apart, hence the average speed over this interval was 75 mph, and *must at least have been equaled at some point.*

If f is a "nice" function on the interval (a, b) then there exists at least one point c in that interval where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



What does it take to be "nice"? The function f must be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).

The proof of the MVT relies on Rolle's Theorem, which is similar. Rolle's Theorem says that for a "nice" f, if $f(a) = f(b)$, then there will exist at least one point c in the interval (a, b) such that $f'(c) = 0$.

The MVT proof basically turns Rolle's interval so that $f(a) = f(b)$.

Proof

We construct a new function utilizing the slope of the line connecting a and b. That slope is

$$m = \frac{f(b) - f(a)}{b - a}$$

The equation of the line connecting the two endpoints is

$$m(x - a) + f(a)$$

We subtract the above from $f(x)$ to construct our new function. This is what "turns" $f(x)$ so the values at the endpoints are equal. If we write it out in full:

$$g(x) = f(x) - \ [\ f(a) + \frac{f(b) - f(a)}{b - a}\ (x - a)\ ]$$

Notice that at $x = a$ the term with $(x - a)$ is just zero so $g(a) = f(a) - f(a) = 0$.

On the other hand, at $x = b$ we have $g(b) = f(b) - f(a) - f(b) + f(a) = 0$. Since $g(a) = g(b)$, Rolle's theorem applies.

What is the derivative of g? To make life easier remember that the slope

$$m = \frac{f(b) - f(a)}{b - a}$$

is *just a number* and so is $f(a)$. Hence

$$g(x) = f(x) - \ [\ f(a) + m(x - a)\ ]$$
$$g'(x) = f'(x) - m$$

Rolle's theorem says that there is at least one value $x = c$ where this expression is zero, which means at that point:

$$f'(c) = m = \frac{f(b) - f(a)}{b - a}$$

This completes the proof of the MVT.