A bit about series

One of the several equivalent definitions of the number e is

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

It's relatively easy to use the binomial theorem to derive an infinite series based on the above definition

$$(a+b)^n = c_0 \; a^n b^0 + c_1 \; a^{n-1} b^1 + c_2 \; a^{n-2} b^2 + c_3 \; a^{n-3} b^3 + \dots$$

The a terms drop out ($a = 1$) and $b = \frac{1}{n}$ so

$$(1 + \frac{1}{n})^n = c_0 \; \frac{1}{n^0} + c_1 \; \frac{1}{n^1} + c_2 \; \frac{1}{n^2} + c_3 \; \frac{1}{n^3} + \dots$$

The coefficients are from the combinations formula

$$c_k = \frac{n!}{(n-k)!k!}, \quad k = 0, 1, 2 \dots$$

if we expand this slightly we obtain

$$c_k = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}$$

Thus, the kth term is in the binomial expansion for e as defined above is:

$$\frac{n(n-1)(n-2)\dots(n-k+1)}{n^k \; k!}$$

There are k terms like $(n-1)$, $(n-2)$ and so on in the numerator, matched by k n terms in the denominator, so that as n gets very large these ratios all become 1, so we are left with simply

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} + \dots$$

And since k is just a "dummy variable" we will substitute it by n in the formulas below.

Similarly, by the same approach one can show that

$$e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n$$

$$e^x = \frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\ldots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

There are series expansions for sine and cosine as well. Proving these is not so easy as stated above for e. The method requires Taylor series approximations, which is moderately advanced calculus. Let's just assume the results

$$sin\ x = \frac{x^1}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

$$cos\ x = \frac{x^0}{0!}-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$$

However, it is easy to see that these must be correct, because differentiating term by term, we obtain

$$\frac{d}{dx}sin\ x = cos\ x, \quad \frac{d}{dx}cos\ x = -sin\ x$$

Now, what we would like to do is to show that Euler's formula follows from these definitions for the three series. Recall that the formula is

$$e^{ix} = cos\ x + i\ sin\ x$$

So let's try substituting ix for x in the series for e first, then that for sine. We have:

$$e^{ix} = \frac{(ix)^0}{0!}+\frac{(ix)^1}{1!}+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\frac{(ix)^4}{4!}+\frac{(ix)^5}{5!}\ldots$$

Now

$$i^0=1, \quad i^1=i, \quad i^2=-2, \quad i^3=-i, \quad i^4=1, \quad i^5=i$$

So

$$e^{ix} = \frac{(ix)^0}{0!}+\frac{(ix)^1}{1!}+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\frac{(ix)^4}{4!}+\frac{(ix)^5}{5!}\ldots$$

$$e^{ix} = \frac{x^0}{0!}+i\frac{x^1}{1!}-\frac{x^2}{2!}-i\frac{x^3}{3!}+\frac{x^4}{4!}+i\frac{x^5}{5!}\ldots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} \dots$$

The real terms (without i), when grouped together equal cosine x

$$cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

The terms containing i, when grouped together equal sine of ix

$$sin\ x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$i\ sin\ x = ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} + \dots$$

So

$$e^{ix} = cos\ x + i\ sin\ x$$

QED.