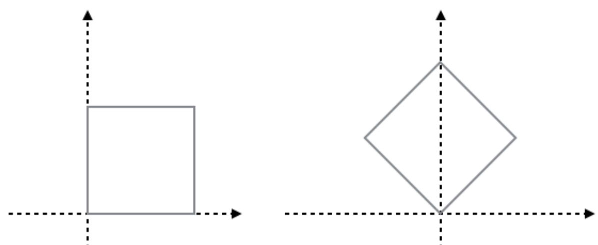
Tilted square

I want to compute the area of the "tilted square" — in this example, rotated by 45 degrees with vertices at $(0,0)$, $(1,1)$, $(2,0)$, and $(-1,1)$.



It is also stretched, note that this is a square with sides of length $\sqrt{2}$ and thus area equal to 2.

Suppose we integrate first over y and then over x. to the right of the origin, the lower bound is the line $y = x$ and the upper bound is $y = 2 - x$.

To the left of the origin, the lower bound is $y = -x$ and the upper bound is $y = 2 + x$. But we can use symmetry and just compute the part for $x \geq 0$, then double it to get the final answer.

As we said, we integrate first over y:

$$A = \int_0^1 \int_{y=x}^{y=2-x} \; dy \; dx$$

The inner integral is

$$\int_x^{2-x} dy = y \Big|_x^{2-x}$$

$$= 2 - x - x = 2 - 2x = 2(1 - x)$$

and the outer integral is then

$$= 2 \int_0^1 1 - x \ dx$$

$$= 2 \ [\ x - \frac{x^2}{2} \ ] \ \Big|_0^1 = 2 \ \frac{1}{2} = 1$$

For the whole area we have another factor of 2 and the total is thus 2. This matches what we get from basic geometry.

Now, let's compute the average value of y over the region. We need to get what is called the *moment* of y:

$$\int \int y \ dy \ dx$$

We will then divide by the area to get the average value. This is equivalent to the single variable case:

$$\bar{x} = \frac{\int_a^b x \ dx}{b - a}$$

The integral is:

$$\int_0^1 \int_{y=x}^{y=2-x} y \ dy \ dx$$

The inner integral is

$$\frac{y^2}{2} \Big|_x^{2-x} = \frac{1}{2}(4 - 4x + x^2 - x^2) = 2(1 - x)$$

which is the same as before. The outer integral is also the same.

$$2\int_0^1 1 - x \; dx = 2 \; [\; (x - \frac{x^2}{2}) \Big|_0^1 \;] = 1$$

And then $\bar{y}$ is equal to this value divided by the area of the region, which is also equal to 1, leaving the answer as just 1. We can see from the symmetry of the problem that this must be correct. Moving along the x-axis (for each value of x), we're looking at a line where for each value of $y = 1+\delta$ above the value $y = 1$ there is another value $y = 1-\delta$ below. So the average value of y is indeed just 1.

For practice, suppose we reverse the order and compute the x integral first.

Because the equation relating the upper bound of x as a function of y changes at $y = 1$ we split the integral into two parts:

$$\int_0^1 \int_{x=0}^{x=y} \; dx \; dy + \int_1^2 \int_{x=0}^{x=2-y} \; dx \; dy$$

The inner integral is just y for the first term and $2 - y$ for the second (but remember the bounds are different). So we obtain

$$= \int_0^1 y \; dy + \int_1^2 (2 - y) \; dy$$

$$= \frac{y^2}{2} \Big|_0^1 + 2y \Big|_1^2 - \frac{y^2}{2} \Big|_1^2$$

$$= \frac{1}{2} + 2 - \frac{3}{2} = 1$$

This matches what we got by computing the y-integral first, as it must.

Now we compute the average value of the function $f(x, y) = y$ over the same region.

$$\int_0^1 \int_{x=0}^{x=y} y \; dx \; dy + \int_1^2 \int_{x=0}^{x=2-y} y \; dx \; dy$$

The inner integrals are

$$xy \Big|_0^y = y^2$$

and

$$xy \Big|_0^{2-y} = y(2-y)$$

The outer integral is

$$\int_0^1 y^2 \; dy + \int_1^2 2y \; dy - \int_1^2 y^2 \; dy$$

$$= \frac{y^3}{3} \Big|_0^1 + y^2 \Big|_1^2 - \frac{y^3}{3} \Big|_1^2$$

$$= \frac{1}{3} + 3 - \; [\; \frac{8}{3} - \frac{1}{3} \;] = 1$$

A different way to do the area problem to use a change of variable, which simplifies that problem quite a bit, at the expense of changing the area element.

We would like to tilt the square back to horizontal so that

$$(0,0) \to (0,0)$$

$$(1,1) \to (1,0)$$

$$(0,2) \to (1,1)$$

$$(-1,1) \to (0,1)$$

By guessing I find that the transformation that does this is:

$$u = \frac{1}{2}(x+y)$$

$$v = \frac{1}{2}(y-x)$$

Since this is a linear transformation and it gives the correct answers for the vertices, it works for all interior points as well.

For the area problem, the area of the re-tilted square is equal to 1 in the u, v-coordinate system.

That's a good reason to do this transformation. We get to the correct value for the area in x, y-coordinates by remembering that the area elements are not the same. That is

$$dx\ dy \neq du\ dv$$

The scaling factor is obtained computing the absolute value of the determinant of the Jacobian (a mouthful for sure), but it is just

$$J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}$$

So the area elements are related by

$$du\ dx = J\ dx\ dy$$

The area as computed under the transformation is one-half the area in the standard coordinate system. We obtained 1 as the answer in u, v-coordinates so we multiply by 2 and obtain the answer for x, y-coordinates, which matches what we had before.

We can also compute the average value of y, but first we need an expression for y in terms of u and v. I add the above equations to obtain:

$$y = u + v$$

So the integral we must compute for the average value is

$$\frac{1}{2} \iint y\ dy\ dx = \iint (u + v)\ du\ dv$$

The limits are easy

$$= \int_0^1 \int_0^1 u + v \; du \; dv$$

The inner integral is

$$\int u + v \; du = \frac{u^2}{2} + uv \,\Big|_0^1 = \frac{1}{2} + v$$

The outer integral is then

$$\int_0^1 \frac{1}{2} + v \; dv = \frac{1}{2} + \frac{1}{2} = 1$$

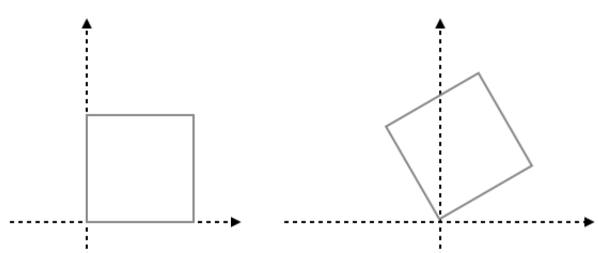
remembering the factor of 1/2, the final answer is 1/2. This, divided by the area is the average value of y in the u, v-coordinate system.

So how does this stack up against the original answer for $\bar{y}$ and what the geometry tells us?

Still working on this, but my feeling about it is that we are looking at y as we slide from $(-1, 1) \to (1, 1)$ in x, y coordinates which is the same as sliding from $(1, 0)$ to $(0, 1)$ in u, v-coordinates.

Since we have $y = 1 \to 0$ linearly with x, the answer is obviously correct.

The real advantage of the transformation approach is that we can tilt to any angle.



Recall that if we turn a vector by an angle ϕ counter-clockwise, the equations are

$$u = x\cos\phi - y\sin\phi$$

$$v = x\sin\phi + y\cos\phi$$

It is easy to get mixed up and do the calculations with the wrong sign. When we *start* with the tilted square and then transform it to the standard orientation we can think of it as either a CW orientation of the points, or a CCW rotation of the coordinates. In any event, the sign of sine in the equations is the opposite of what we have above.

$$u = x\cos\phi + y\sin\phi$$

$$v = -x\sin\phi + y\cos\phi$$

We check this by asking about $\phi = \pi/2$. Then we have

$$u = y$$

$$v = -x$$

The vectors or points:

$$(0,1) \to (1,0)$$

$$(-1,0) \to (0,1)$$

which is correct for a clockwise turn of 90 degrees. Since the equations work for unit vectors along the x and y-axes, they will work for any vector or any point.

In contrast to what we did above, this is not a stretching transformation.

$$J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} -\sin\phi & \cos\phi \\ -\cos\phi & -\sin\phi \end{vmatrix} = \sin^2\phi - (-\cos^2\phi) = 1$$

The limits are great: $0 \to 1$. The area of the square is

$$\frac{1}{J} \int_0^1 \int_0^1 \, du \; dv = 1$$

To find the moment of y we can go back to our equations

$$u = x \cos \phi + y \sin \phi$$

$$v = -x \sin \phi + y \cos \phi$$

To get, say, y as a function of u and v we can isolate y as follows

$$u \sin \phi = x \sin \phi \cos \phi + y \sin^2 \phi$$

$$v \cos \phi = x \sin \phi \cos \phi + y \cos^2 \phi$$

Adding:

$$y = u \sin \phi + v \cos \phi$$

or we can remember to just switch the sign of the sines when we switch letters in the original equation:

$$x = u \cos \phi - v \sin \phi$$

$$y = u \sin \phi + v \cos \phi$$

Hence

$$\iint y \; dA = \int \int u \sin \phi + v \cos \phi \; du \; dv$$

where ϕ is a constant.

The inner integral is:

$$\int_0^1 (u \sin \phi + v \cos \phi) \; du = \frac{1}{2} \sin \phi + \cos \phi \; v$$

and the outer integral is

$$\int_0^1 (\frac{1}{2} \sin \phi + \cos \phi \; v) \; dv$$

$$= \frac{1}{2}(\cos\phi + \sin\phi)$$

If we do the integrals for x as well then we will have:

$$\bar{x} = \frac{1}{2}(\cos\phi - \sin\phi)$$

$$\bar{y} = \frac{1}{2}(\cos\phi + \sin\phi)$$

The average value of x and y are these values divided by the area (which was equal to 1).

We can see that we have our signs correct here. If we tilt counter-clockwise, then the average value of x passes through zero and goes negative, which the average value of y will stay positive, reaching a maximum at $\phi = \pi/4$.

Notice that if there is no turning ($\phi = 0$) or if $\phi = \pi/2$, then the average value $\bar{x} = \bar{y} = 1/2$, as expected.

If $\phi = \pi/4$ then $\bar{x} = 0$, as expected, and $\bar{y} = 1/\sqrt{2} \approx 0.7$.

And if $\phi = \pi/2$ then $\bar{x} = -1/2$ and $\bar{y} = 1/2$.

If we differentiate and set the derivative equal to zero, the maximum $\bar{y}$ is:

$$\frac{d}{d\phi}\bar{y} = \frac{1}{2}(-\sin\phi + \cos\phi) = 0$$

$$\sin\phi = \cos\phi$$

$$\phi = \frac{\pi}{4}$$