Exponential and logarithm

As you know, the basic idea of logarithms is that

$$b^p \cdot \ b^q = b^{p+q}$$

for any base b. So if we know

$$x = b^p$$

$$y = b^q$$

then

$$xy = b^{p+q}$$

and the logarithm to the base b of xy equals $p + q$. To use this, we need some way of finding the number that is equal to b^{p+q} (the anti-logarithm). In the old days, tables of logarithms were used for this purpose, but today we don't bother, we use a calculator.

There is a lot of good discussion on the web about how these values were calculated. Feynman in his *Lectures on Physics* (Chapter 22 of Vol 1) has a particularly fun one.

Another fact about logarithms is the change of base formula:

$$log_b(x) = \frac{log_a(x)}{log_a(b)}$$

We can try to remember this by noting that both terms on the right are logarithms to base a. We can also check it quickly by noting that if $a < b$ then $log_a(b) > 1$ and so the factor $1/log_a(b) < 1$ so that $log_b(x) < log_a(x)$, as we expect.

differential calculus

For the first part of calculus, the most important thing is that

$$\frac{d}{dx}e^x = e^x$$

the derivative of this function is just the function itself. If you accept that this is a valid formula for e^x

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

then it is easy to verify that the derivative is equal to the function, since each term $x^k/k!$ becomes $x^{k-1}/(k-1)!$ — except the first, which is a constant with derivative equal to 0.

We can also use the chain rule to differentiate more complicated functions, e.g.

$$\frac{d}{dx}e^{x^2} = 2x\ e^{x^2}$$

Also, if

$$x = e^y$$

then

$$y = log_e(x) = \ln(x)$$

We say that y is the *natural* logarithm of x.

It is worth remembering that e is just a number (although a very special one). One problem that seems tricky at first is

$$\frac{d}{dx}3^x = \ ?$$

But remember that

$$e^{ln(3)} = 3$$

so we have

$$3^x = (e^{ln(3)})^x = e^{\ln(3)\cdot x}$$

So

$$\frac{d}{dx}3^x = \frac{d}{dx}e^{\ln(3)\cdot x} = \ln(3)e^{x\ln(3)} = \ln(3)\cdot 3^x$$

How about

$$\frac{d}{dx}\ln(x) = ?$$

Using the identity $x = e^{\ln(x)}$

$$\frac{d}{dx}x = \frac{d}{dx}\ e^{\ln(x)}$$

But the left-hand side is just 1 so

$$1 = \frac{d}{dx}\ e^{\ln(x)} = e^{\ln(x)} \cdot \frac{d}{dx}\ln(x)$$

(again, by the chain rule). But we can substitute back for x

$$\begin{aligned} 1 &= e^{\ln(x)} \cdot \frac{d}{dx}\ln(x) \\ &= x \cdot \frac{d}{dx}\ln(x) \\ \frac{d}{dx}\ln(x) &= \frac{1}{x} \end{aligned}$$

This will become really useful when we start with integral calculus. There are a lot of problems where the rate of change of x is proportional to x (bacterial growth, radioactive decay).

$$\begin{aligned} \frac{dx}{dt} &= kx \\ \frac{1}{x}\ \frac{dx}{dt} &= k \end{aligned}$$

Solving such a problem involves finding a function whose derivative is equal to $1/x$

$$\frac{d}{dt}F(x) = \frac{1}{x} \cdot \frac{dx}{dt}$$

That function is $\ln(x)$. We will have

$$F(x) = \ln(x) = kt$$

$$x = e^{kt}$$

Without too much explanation, x depends also on its initial value at *time-zero* ($t = 0$)

$$x = x_0 \ e^{kt}$$

Normally, we solve for $x/x_0 = 2$. The corresponding value for t is usually called T, the half-life or doubling time, depending on the problem type.

$$2 = \ e^{kT}$$

$$\ln(2) = kT$$

$$\frac{\ln(2)}{T} = k$$

so we can substitute back into the original equation

$$x = x_0 \ e^{(\ln(2)/T) \ t}$$