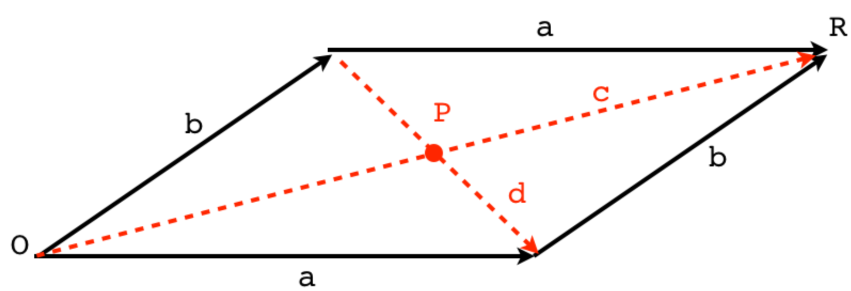
Proof for Ceva with vectors

Ceva's Theorem says that if we start with a triangle and draw the line segments connecting each vertex with the midpoint of the opposite side, the three line segments cross at a single, unique point. Furthermore, it is possible to show that for any of these line segments, the *centroid* lies one-third of the length from the side, and two-thirds of the length from the vertex.

I'd like to show a proof of this using vectors, which makes everything particularly simple. As a warmup, let's start by looking at the mid-point of the diagonals for a parallelogram including the triangle of interest. We will prove that the two diagonals cross at their mid-points (at P).



by construction:

$$\mathbf{c} = \mathbf{a} + \mathbf{b}$$

$$\mathbf{b} + \mathbf{d} = \mathbf{a} \;\Rightarrow\; \mathbf{d} = \mathbf{a} - \mathbf{b}$$

Let's define P as the point we reach by going halfway along $\mathbf{c}$

$$\mathbf{c}/2 = (\mathbf{a} + \mathbf{b})/2$$

What we need to show is that if we do

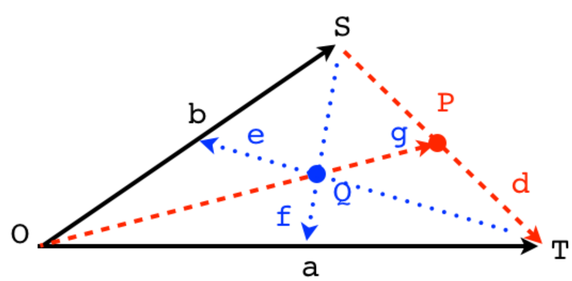
$$\mathbf{b} + \mathbf{d}/2$$

we arrive at P. Since $\mathbf{d} = \mathbf{a} - \mathbf{b}$

$$\mathbf{b} + \mathbf{d}/2 = \mathbf{b} + (\mathbf{a} - \mathbf{b})/2 = (\mathbf{a} + \mathbf{b})/2 \qquad \blacksquare$$

Vectors make that pretty easy.

Now, here is the triangle.



By construction

$$\mathbf{b} + \mathbf{f} = \mathbf{a}/2 \;\Rightarrow\; \mathbf{f} = \mathbf{a}/2 - \mathbf{b}$$
$$\mathbf{a} + \mathbf{e} = \mathbf{b}/2 \;\Rightarrow\; \mathbf{e} = \mathbf{b}/2 - \mathbf{a}$$

Define $\mathbf{c}/2 = \mathbf{g}$

$$\mathbf{b} + \mathbf{d}/2 = \mathbf{g}$$

(For the last one, refer back to the first diagram).

It makes it a little easier when we know that Q is two-thirds of the way along the line segment. We have three paths to move to Q.

Starting from O:

$$\frac{2}{3}\mathbf{g} = \frac{2}{3}\,\frac{1}{2}\mathbf{c} = (\mathbf{a}+\mathbf{b})/3$$

from S:

$$\mathbf{b} + \frac{2}{3}\mathbf{f} = \mathbf{b} + \frac{2}{3}(\mathbf{a}/2 - \mathbf{b}) = (\mathbf{a}+\mathbf{b})/3$$

or from T:

$$\mathbf{a} + \frac{2}{3}\ \mathbf{e} = \mathbf{a} + \frac{2}{3}\ (\mathbf{b}/2 - \mathbf{a}) = (\mathbf{a}+\mathbf{b})/3$$

How would we find the factor of 2/3 if we didn't already know? Call that unknown factor r

$$r\ (\mathbf{a}+\mathbf{b})/2 + (1-r)\ \mathbf{e} = \mathbf{b}/2$$
$$r\ (\mathbf{a}+\mathbf{b})/2 + (1-r)\ (\mathbf{b}/2 - \mathbf{a}) = \mathbf{b}/2$$

Expand:

$$r(\mathbf{a}/2) + r(\mathbf{b}/2) + \mathbf{b}/2 - r(\mathbf{b}/2) - \mathbf{a} + r\mathbf{a} = \mathbf{b}/2$$
$$r(\mathbf{a}/2) + \mathbf{b}/2 - \mathbf{a} + r\mathbf{a} = \mathbf{b}/2$$
$$r(\mathbf{a}/2) - \mathbf{a} + r\mathbf{a} = 0$$
$$(3/2)r\mathbf{a} - \mathbf{a} = 0$$
$$(3/2)r = 1$$
$$r = 2/3$$