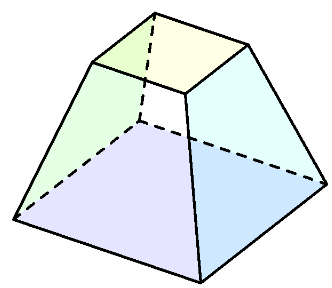
Frustum

A frustum is the (bottom) part of a larger pyramid or cone that remains after the original solid is cut by a horizontal plane and the upper, small pyramid or cone removed.



If we call the dimensions of the larger pyramid H (height) and B (base), then its volume is

$$V = \frac{1}{3}HB^2$$

Similarly, if we call the dimensions of the pyramid that has been removed h and b its volume is

$$V = \frac{1}{3}hb^2$$

The volume of the frustum is just the difference

$$V = \frac{1}{3}(HB^2 - hb^2)$$

If we're dealing with a cone rather than a pyramid, then replace B^2 with R^2 and b^2 with r^2 and multiply the whole thing by π.

alternative formula

However, there is another formula for the volume of the frustum, which is perhaps more interesting.

If we call the altitude or height of the frustum a, where $a = H - h$, this formula is

$$V = \frac{1}{3}a(B^2 + Bb + b^2)$$

$$V = \frac{1}{3}(H - h)(B^2 + Bb + b^2)$$

We'd like to derive this. The key insight here is that by similar triangles

$$\frac{b}{h} = \frac{B}{H}, \qquad h = \frac{b}{B}H$$

while

$$\begin{aligned} a &= H - h \\ &= H - \frac{b}{B}H \\ &= H(\frac{B-b}{B}) \end{aligned}$$

The proof proceeds in the reverse direction. Start with

$$V = \frac{1}{3}a(B^2 + Bb + b^2)$$

Substitute for a

$$V = \frac{1}{3}\ H(\frac{B-b}{B})(B^2 + Bb + b^2)$$

Part of this simplifies dramatically

$$(B - b)(B^2 + Bb + b^2)$$

$$= B^3 + B^2b + Bb^2 - bB^2 - Bb^2 - b^3$$

$$= B^3 - b^3$$

Hence we have that

$$V = \frac{1}{3}\ \frac{H}{B}(B^3 - b^3)$$

Multiplying out, the first term is $1/3\ HB^2$, as desired.

The second is

$$-\frac{1}{3}\ \frac{H}{B}b^3$$

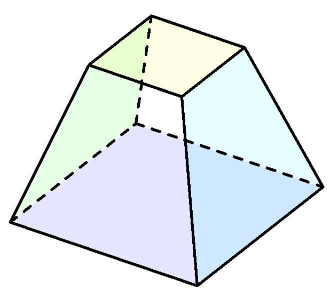
Recall that $h = bH/B$ so this is just $-1/3\ hb^2$, and we're done.

$\square$

slant height

The slant height is the length of a cone or frustum along its outside edge. In the case of a cone, we can obtain it from the height and 1/2 the length of the base using the Pythagorean theorem.

For a frustum



consider the triangle containing an altitude down from the outside edge on the top.

The height of the triangle is just a, and the base has length $(B - b)/2$.

If the slant height is c then Pythagoras says that

$$c^2 = a^2 + (\frac{B-b}{2})^2$$

$$a = \sqrt{c^2 - (\frac{B-b}{2})^2}$$