# Integrate z squared

Consider $f(z) = z^2$. For the path, take the unit circle over the first quadrant from $(1,0)$ to $(0,1)$. There is an easy way to do this, and a hard way.

Let's start by checking that this function is analytic, and then doing the hard way first.

Write $z$ in terms of $x$ and $y$:

$$z = x + iy$$

$$z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$$

$$u_x = 2x = v_y$$

$$u_y = -2y = -v_x$$

The CRE hold.

Also

$$dz = dx + i\ dy$$

So

$$\int z^2\ dz = \int (x^2 - y^2 + 2ixy)(dx + i\ dy)$$

$$= \int (x^2 - y^2)\ dx - \int 2xy\ dy + i \int 2xy\ dx + i \int (x^2 - y^2)\ dy$$

As before, we must parametrize this using the relationship between $x$ and $y$ along the curve.

$$x = \cos t$$

$$y = \sin t$$
$$dx = -\sin t \ dt$$
$$dy = \cos t \ dt$$

and then

$$x^2 - y^2 = \cos^2 t - \sin^2 t = \cos 2t$$
$$2xy = 2 \cos t \sin t = \sin 2t$$

so the integral is

$$= \int -\cos 2t \ \sin t \ dt - \int \sin 2t \ \cos t \ dt + \dots$$
$$+ \ i \ [ \int -\sin 2t \ \sin t \ dt + \int \cos 2t \ \cos t \ dt$$

Looks pretty wild! In the book they use some trig identities I hadn't seen before, namely starting with the standard

$$\sin s + t = \sin s \cos t + \sin t \cos s$$

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

then, if $s = 2t$ then

$$\sin 3t = \sin 2t \cos t + \sin t \cos 2t$$

$$\cos 3t = \cos 2t \cos t - \sin 2t \sin t$$

Looking at the real part of the integral we had (combining terms)

$$\int -\cos 2t \ \sin t - \sin 2t \ \cos t \ dt = \int -\sin 3t \ dt = \frac{\cos 3t}{3}$$

and for the imaginary part of the integral

$$i \ [ \int -\sin 2t \ \sin t + \cos 2t \ \cos t \ dt = i \int \cos 3t \ dt = i \frac{\sin 3t}{3}$$

That looks a lot better.

$$\frac{\cos 3t}{3} + i\frac{\sin 3t}{3}\Big|_0^{\pi/2} = -\frac{1}{3} - i\frac{1}{3} = -\frac{1}{3}(1+i)$$

For one version of the easy way, since $z^2$ is analytic, we can just treat $z$ as if it were a real variable

$$\int z^2 \ dz = \frac{z^3}{3}\Big|_1^i = -\frac{1}{3}i - \frac{1}{3}$$

Note that if we go all the way around the unit circle the integral is just zero.

Alternatively, parametrize the unit circle as $z = e^{i\theta}$, then $dz = ie^{i\theta} \ d\theta$ and

$$\begin{aligned}\int z^2 \ dz &= \int e^{i2\theta} \ i \ e^{i\theta} \ d\theta \\ &= i\int \ e^{i3\theta} \ d\theta \\ &= \frac{1}{3} \ e^{i3\theta} \ \Big|_{\theta_1}^{\theta_2}\end{aligned}$$

From Euler's identity:

$$e^{i3\theta} = \cos 3\theta + i\sin 3\theta$$

If

$$\theta_2 = \theta_1 + 2\pi$$

(going all the way around, the integral is zero). Over the first quadrant only, we have

$$\begin{aligned}e^{i3\theta} \ \Big|_0^{\pi/2} &= \cos 3\pi/2 + i\sin 3\pi/2 - \cos 0 - i\sin 0 \\ &= 0 + i(-1) - 1 - i(0) = -(1+i)\end{aligned}$$

Multiply by the factor of 1/3, and we match the previous result.