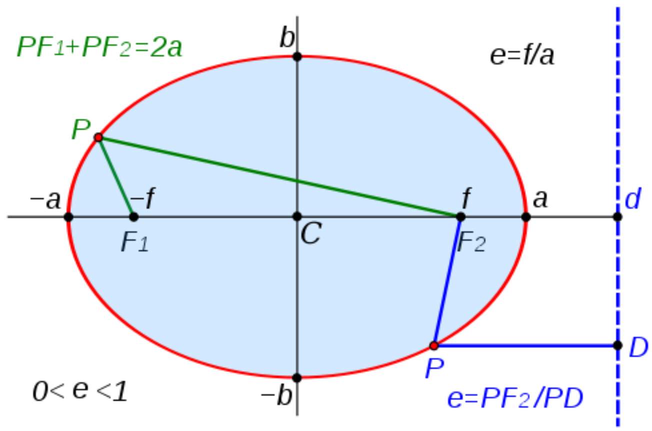
Ellipse-parametrization



I want to compute the volume of an ellipsoid. We imagine the solid formed by rotating the ellipse around the x-axis. For each value of x, this solid will have a cross-section whose radius is equal to y, so to get the volume of the ellipse we do

$$V = \int_{-a}^{a} \pi y^2 dx$$

Now,

$$x = a\cos t$$

$$dx = -a\sin t \; dt$$

And we will have to find new limits for the integral. Let's set it up first So

$$V = \pi \int (b^2 \sin^2 t)(-a \sin t)\ dt$$

Previously we had

$$x = -a \to a$$

The lower limit corresponds to $t = \pi$ and the upper limit to $t = 0$.

$$V = \pi a b^2 \int_{\pi}^{0} (\sin^2 t)(-\sin t)\ dt$$

$$= \pi a b^2 \int_{\pi}^{0} (1 - \cos^2 t)(-\sin t)\ dt$$

$$= \pi a b^2\ [\ \cos t - \frac{1}{3} cos^3 t\ ]\ \Bigg|_{\pi}^{0}$$

$$= \pi a b^2\ [\ (1 - \frac{1}{3}) - (-1 + \frac{1}{3})\ ]$$

$$= \frac{4}{3} \pi a b^2$$

This is quite beautiful. If we consider the three axes in space, for y and z the surface passes through at b, so b counts twice in the volume. If we rotated the other way (around the y axis), we would obtain $\frac{4}{3}\pi a^2 b$.