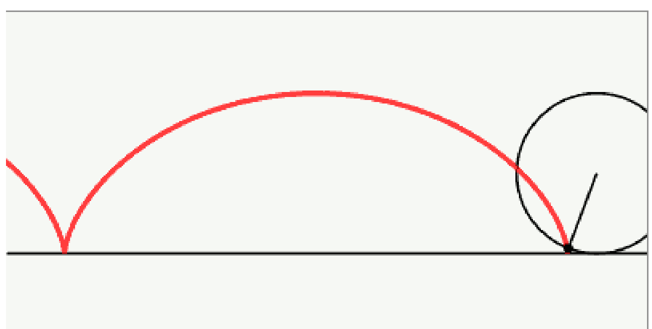
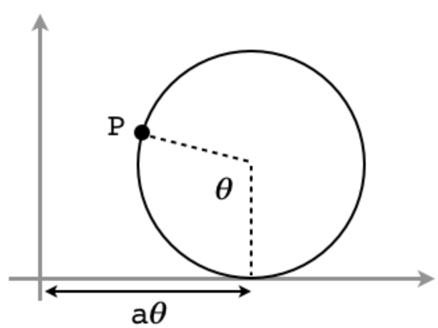
# The cycloid

We imagine a bicycle with one tire marked at a particular point on the rim, say with fluorescent paint or a small light. We start at time $t = 0$ with that point $P$ in contact with the $x$ axis at $(0, 0)$. Then we start rolling the bike. As the tire rotates our fixed point $P$ on the rim traces a curve






We want to find parametric equations $x(t)$, $y(t)$ that give the position of the point $P$ as a function of time. The second diagram above shows the angle through which the wheel has turned as $\theta$, but we will use

$t$ for $\theta$ here. The $x$ displacement of the vertical straight down from the center of the tire is just $at$, where $a$ is the radius of the wheel, it is equal to the arc on the circumference of the wheel from the point which is currently in contact with the ground, up to $P$.

It is fairly easy to derive the desired parametric equations, using vectors. For $x$, we have the vector that goes from $(0, 0)$ to the contact point with the ground. As indicated in the figure, that is $at$. We need to subtract the distance $a\ sin\ t$ from that. It's easier to see for $t < \pi/2$, but it is true always. This is the usual circular motion, just with the circle flipped so the motion is clockwise, and we started at the bottom.

For $y$, we have a constant factor of $a$ above the $x$ axis, then the additional displacement is $-a\ cos\ t$. So for $t = 0$ we have the additional displacement is -a (we were on the ground), for $t = \pi/2$ it is zero, and for $t = \pi$ it is plus $a$ for a total of $2a$.

The parametric equations are then

$$\begin{aligned} x(t) &= at - a\ sin\ t \\ y(t) &= a - a\ cos\ t \\ x'(t) &= a - a\ cos\ t \\ y'(t) &= a\ sin\ t \end{aligned}$$

The derivation above did a little mental gymnastics with the circle, flipping it and setting $t = 0$ when the point is at the bottom. As an alternative, leave the circle in its usual orientation, with an angle $s$ to the positive $x$ axis.

It can be seen easily that $s$ and $t$ are related by the equation

$$s = 3\pi/2 - t$$

The vector from the center of the circle to the point on the edge is just the standard one for a point on a circle of radius $a$:

$$a \langle \cos s, \sin s \rangle$$

For the $x$ component:

$$\cos s = \cos 3\pi/2 - t$$

$$= \cos 3\pi/2 \cos t + \sin 3\pi/2 \sin t$$

Recall that $\cos 3\pi/2 = 0$ and $\sin 3\pi/2 = -1$ so

$$\cos s = -\sin t$$

And for the $y$ component

$$\sin s = \sin 3\pi/2 - t$$

$$= \sin 3\pi/2 \cos t - \sin t \cos 3\pi/2$$

$$= -\cos t$$

The vector is then

$$a \langle \cos s, \sin s \rangle = a \langle -\sin t, -\cos t \rangle$$

In addition, we have to add another vector, one extending from the origin to the center of the wheel. The $y$ component is constant, it is just $a$. The $x$-component is the distance the wheel has traveled from its initial position (the distance between the origin and the point of contact with the $x$-axis, which is $at$, shown as $a\theta$ in the figure).

Hence the vector to the point is:

$$a \langle -\sin t, -\cos t \rangle + \langle at, a \rangle$$

$$a \langle t - \sin t, 1 - \cos t \rangle$$

which matches what we had before.

**Arc length**

We wish to determine the arc length and area under the curve for a complete revolution of the wheel.

We want to use a slightly different version of the usual formula for arc length

$$ds^2 = dx^2 + dy^2$$

$$(\frac{ds}{dt})^2 = (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2$$

$$ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt = \sqrt{(a - a\cos\ t)^2 + (a\sin\ t)^2}\ dt$$

This expands to

$$a\sqrt{1 - 2\cos\ t + \cos^2 t + \sin^2 t}\ dt = a\sqrt{2 - 2\cos\ t}\ dt$$

The length is

$$L = \int_0^{2\pi} a\sqrt{2 - 2\cos\ t}\ dt$$

$$= a\sqrt{2}\ \int_0^{2\pi} \sqrt{1 - \cos\ t}\ dt$$

**double angle**

$$\cos(s - t) = \cos s \cos t + \sin s \sin\ t$$

(check: if $s = t$ then $\cos 0 = 1$, which is correct).

So

$$\cos(s + t) = \cos s \cos t - \sin s \sin\ t$$

Let $s = t$ and $u = 2s$, then

$$\cos 2s = \cos u = \cos^2\ (\frac{u}{2}) - \sin^2\ (\frac{u}{2})$$

$$\cos u = 1 - \sin^2\ (\frac{u}{2}) - \sin^2\ (\frac{u}{2})$$

$$2\sin^2\ (\frac{u}{2}) = 1 - \cos u$$

$u$ is just a dummy variable, so we can switch back to $t$

$$2\sin^2\ (\frac{t}{2}) = 1 - \cos t$$

**finishing up**

We have that

$$L = a\sqrt{2}\ \int_0^{2\pi} \sqrt{1 - \cos\ t}\ dt$$

$$1 - \cos t = 2\sin^2(\frac{t}{2})$$

$$\sqrt{1 - \cos t} = \sqrt{2}\sin(\frac{t}{2})$$

So

$$L = a\sqrt{2}\ \int_0^{2\pi} \sqrt{2}\sin\ (\frac{t}{2})\ dt$$

$$2a\ \int_0^{2\pi} \sin\ (\frac{t}{2})\ dt$$

$$= 2a\ (-2)\cos\ (\frac{t}{2})\ \Big|_0^{2\pi}$$

$$= -4a\ (\cos\ \pi - \cos\ 0)$$

$$= -4a\ (-1 - 1) = 8a$$

A simple answer to the problem.

**Area under the arc**

We want

$$A = \int_{t=0}^{t=2\pi} y\ dx$$

$$= \int_{t=0}^{t=2\pi} (a - a\cos\ t)(a - a\cos\ t)\ dt$$

$$a^2 \int_{t=0}^{t=2\pi} (1 - \cos\ t)(1 - \cos\ t)\ dt$$

$$a^2 \int_{t=0}^{t=2\pi} (1 - 2\cos\ t + \cos^2\ t)\ dt$$

If you don't remember the result for $\int \cos^2 t\ dt$, you can go back to the double angle formula above and convert from $\sin^2$ to $\cos^2$. Otherwise recall it and write:

$$A = a^2(t - 2\sin\ t + \frac{1}{2}t + \frac{1}{4}\sin 2t)\ \Big|_0^{2\pi}$$

$$a^2(2\pi - 0 + \pi + 0 - 0 + 0 - 0 - 0) = 3\pi a^2$$

Also a very simple answer.