Trig functions with $\pi/5$

I'm trying to reproduce here the argument I saw in a video by Math Dr. Bob about this subject.

We know already know how to calculate values for sine and cosine of π times $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$, and various increments multiplied by $n = 2, 3 \cdots$. The next obvious target is $\frac{1}{5}\pi$.

It turns out this can also be done pretty easily, and it leads to an interesting connection, which can be reached from a different perspective by looking at the properties of pentagons (remember that the interior angle of a pentagon is $\pi - \frac{2}{5}\pi = \frac{3}{5}\pi$.

Begin by writing down the double angle formulas for sine and cosine:

$$\sin 2s = 2 \sin s \cos s$$

$$\cos 2s = \cos^2 s - \sin^2 s = 2\cos^2 s - 1$$

If you visualize (or sketch) going around the unit circle in increments of $2\pi/5$, you will see that $\theta = \frac{2}{5}\pi$ and $\phi = 4\theta = \frac{8}{5}\pi$ form the same angle with the x-axis (convention adds a factor of -1). Therefore, these two angles have the same cosine, and the same sine as well, except for that factor of -1. For this particular angle $\theta = \frac{2}{5}\pi$

$$-\sin\theta = \sin 4\theta$$

Next, we use the double angle formulas to write out an expression for $\sin 4\theta$

$$\sin 4\theta = 2\ (\sin 2\theta)(\cos 2\theta)$$
$$= 2\ (2\sin\theta\cos\theta)(2\cos^2\theta - 1)$$

Combining with what we had before

$$-\sin\theta = 2(2\sin\theta\cos\theta)(2\cos^2\theta - 1)$$

Factor out $\sin\theta$

$$-1 = 2(2\cos\theta)(2\cos^2\theta - 1)$$

Substituting $x = \cos\theta$

$$-1 = 2(2x)(2x^2 - 1)$$
$$-1 = 8x^3 - 4x$$
$$8x^3 - 4x + 1 = 0$$

Now, it turns out that $x = \frac{1}{2}$ is a solution of this polynomial (easily checked), so we should be able to factor it into something like

$$8x^3 - 4x + 1 = (x - \frac{1}{2})(\cdots)$$

We deduce the rest of the factorization:

$$8x^3 - 4x + 1 = (x - \frac{1}{2})(8x^2 \cdots)$$
$$8x^3 - 4x + 1 = (x - \frac{1}{2})(8x^2 + 4x \cdots)$$
$$8x^3 - 4x + 1 = (x - \frac{1}{2})(8x^2 + 4x - 2)$$

We can get the solutions for the second factor

$$8x^2 + 4x - 2$$

from the quadratic equation:

$$\frac{-4 \pm \sqrt{16 + 4(16)}}{16}$$

$$-\frac{1}{4}(1 \pm \frac{1}{4}\sqrt{16+4(16)})$$
$$-\frac{1}{4}(1 \pm \sqrt{\frac{16+4(16)}{16}})$$
$$-\frac{1}{4}(1 \pm \sqrt{5})$$
$$-\frac{1}{2}\ (\frac{1 \pm \sqrt{5}}{2})$$
$$-\frac{1}{2}\phi_1, -\frac{1}{2}\phi_2$$

where

$$\phi_1 = \frac{1+\sqrt{5}}{2} \approx 1.618034$$
$$\phi_2 = \frac{1-\sqrt{5}}{2} \approx -0.618034$$

What does this mean? It means that if

$$-\sin\theta = \sin 4\theta$$

then solutions include θ for which

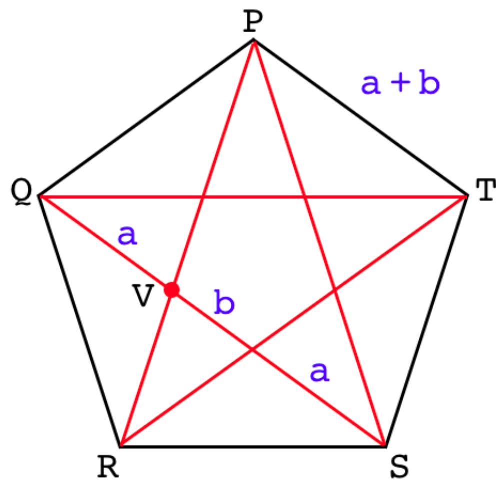
$$\cos\theta = \begin{cases} \frac{1}{2} \\ -\frac{1}{2}\phi_1 \\ -\frac{1}{2}\phi_2 \end{cases}$$

The third solution is numerically:

$$-\frac{1}{2}\ (-0.618034) = 0.309017$$

And the cosine of $2\pi/5$ is equal to 0.309017, which checks!

Another way to obtain this result is to start from our work with the pentagon.



We draw the chords of the pentagon, then consider the smallest triangle obtained. It is isosceles with two sides a and the last side b. The two larger angles measure $2\pi/5$ and the other is $\pi/5$. The law of cosines gives:

$$a^2 = a^2 + b^2 - 2ab\cos\,(\frac{2}{5}\pi)$$

$$-b^2 = -2ab\;\cos\,(\frac{2}{5}\pi)$$

$$\frac{b}{a} = 2\cos\,(\frac{2}{5}\pi)$$

and in the writeup about pentagons, what we found was that

$$\frac{a}{b} = \phi_1$$

so

$$\cos\,(\frac{2}{5}\pi) = -\frac{1}{2}\;\frac{1}{\phi_1} = -\frac{1}{2}\;\phi_2$$

The last step follows since $\phi_1\;\phi_2 = -1$.