Completeness

This property of completeness distinguishes the set of real numbers $\mathbb{R}$ from the rationals $\mathbb{Q}$.

One informal statement is that "the real number line has no holes in it." In contrast, the set of rational numbers is missing the irrationals such as $\sqrt{2}$ and e and so on.

Elsewhere we defined the *supremum* or least upper bound of a set $\mathbf{A}$ by saying that

• u is an upper bound for $\mathbf{A}$ if $\forall x \in \mathbf{A}, x \leq u$,

• U is the supremum or least upper bound for $\mathbf{A}$ if for all upper bounds u, $U \leq u$.

Note that for $U = sup(\mathbf{A})$ it is not necessary that $U \in \mathbf{A}$. 0 is the supremum of both $\{x \mid x < 0\}$ and $\{x \mid x \leq 0\}$, but only in the second case is 0 a member of the set.

The completeness axiom then says that

• Every non-empty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

This is not a true statement for $\mathbb{Q}$. Suppose we define the subset of $\mathbb{Q}$ that contains all numbers $r \mid r^2 < 2$. Then, $\sqrt{2}$ is the supremum of this set, but $\sqrt{2} \notin \mathbb{Q}$.

How does this definition of completeness lead to the properties that we really care about, like: $\forall\ \epsilon > 0, \exists\ \delta \mid a - \epsilon < a - \delta < a$. No matter how small ϵ is and so no matter how close we are to a real number a, we can always find another real number that is even closer.

approximation property

Apostol gives this corollary to the completeness axiom:

Let $\mathbf{S}$ be a non-empty set of real numbers with a supremum, say $b = sup(\mathbf{S})$. Then for every $a < b$ there is some x in $\mathbf{S}$ such that

$$a < x \leq b$$

Proof: First, $x \leq b$ for all x in $\mathbf{S}$. If we had $x \leq a$ for all x in $\mathbf{S}$, then a would be an upper bound for $\mathbf{S}$ smaller than the least upper bound. Therefore, $x > a$ for at least one x in $\mathbf{S}$.