Ellipse: area

The area of an ellipse can be computed in several different ways, all interesting. The simplest way is rescaling. In xy-coordinates, the formula is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$(\frac{bx}{a})^2 + y^2 = b^2$$

What this says is that if the x value of each point on the ellipse is re-scaled by a factor of b/a, the result is

$$u = \frac{b}{a}x$$

$$u^2 + y^2 = b^2$$

a circle of radius b and area $A = \pi b^2$. Because the scaling factor is only in the x-direction

$$x = \frac{a}{b}u$$

the area of the original ellipse is bigger by a factor of a/b

$$A = \pi b^2 \ \frac{a}{b} = \pi ab$$

We might also argue as follows. The area of the ellipse clearly depends on both a and b, so we write

$$A = kab$$

where k is an unknown constant. Now, if $a = b$, we obtain

$$A = ka^2$$

but this is just a circle, with known area

$$A = \pi a^2 = ka^2$$

Hence $k = \pi$ and $A = \pi ab$.

single variable calculus

Solve the equation of the ellipse for y

$$y = b\sqrt{1 - \frac{x^2}{a^2}}\ dx$$

We take the positive square root, and integrate from $x = 0 \to a$, and should obtain 1/4 the area of the ellipse.

$$A = 4b \int \sqrt{1 - \frac{x^2}{a^2}}$$

The first thing to do is to get rid of the a by substitution. Let $u = x/a$, so $au = x$ and $a\ du = dx$, then

$$A = 4ab \int \sqrt{1 - u^2}\ du$$

The next step is to recognize that $f(x) = \sqrt{1 - u^2}$ is the equation of a circle. Since we are integrating over the first quadrant, the value of the area is just $\pi/4$. The whole thing is π and we pick up the factor ab from outside to give $A = \pi ab$.

If you failed to see this, you can do a trig substitution. If u is the side opposite angle θ, and 1 is the hypotenuse, then

$$\sqrt{1 - u^2} = \cos\theta$$

$$u = \sin\theta$$

$$du = \cos\theta \quad d\theta$$

and the integral becomes

$$4ab \int \cos^2\theta \ d\theta$$

Before we do the integration, consider the changing bounds. We originally had $x = 0 \to a$, in changing to u by remembering that

$$au = x$$

we obtain $u = 0 \to 1$. Then, in changing to θ we have

$$u = \sin\theta$$

$$\theta = \sin^{-1} u$$

and we have $\theta = 0 \to 2\pi$. I'm not going to do the integral here, but just give the result

$$\int \cos^2\theta \ d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta)$$

(and there are other ways to write it). But we will take a moment to check that by differentiating

$$\frac{d}{d\theta}\ \frac{1}{2}(\theta + \sin\theta\cos\theta)$$

$$= \frac{1}{2}(1 + \cos^2\theta - \sin^2\theta)$$

$$= \frac{1}{2}(1 + \cos^2\theta + \cos^2\theta - 1) = \cos^2\theta$$

So we need to evaluate

$$4ab\ [\ \frac{1}{2}(\theta + \sin\theta\cos\theta)\ ]\ \Big|_0^{\pi/2}$$

Only one term is non-zero and that is $\theta = \pi/2$ at the upper limit. We obtain

$$A = 4ab\ (\frac{1}{2}\ \frac{\pi}{2}) = \pi ab$$

Green's Theorem

State the theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{r} = \iint_R \nabla \times \mathbf{F}\ dA$$

$$\int_C M\ dx + N\ dy = \iint_R (N_x - M_y)\ dx\ dy$$

The theorem equates the line integral around a closed path with an area over a region.

To start with, if $\mathbf{F}$ is the gradient of some function, we call such a function the potential, and the integral of the work over a closed path is just zero.

Of course, my favorite example is the area of the ellipse.

Suppose $N_x - M_y = 1$. Then the curl integral is the area of the region. An example would be if $\mathbf{F} = \langle M, N \rangle = \langle -y/2, x/2 \rangle$. Parametrize the ellipse.

$$x = a \cos\theta$$

$$y = b \sin\theta$$

So, for the left hand side we have

$$\int_C M\ dx + N\ dy = \int_C -\frac{1}{2} y\ dx + \frac{1}{2} x\ dy$$

$$= \int_0^{2\pi} (-\frac{1}{2})(b \sin\theta)\ (-a \sin\theta)\ d\theta\ + (\frac{1}{2})(a \cos\theta)\ (b \cos\theta)\ d\theta$$

$$= \int_0^{2\pi} (\frac{ab}{2} \sin^2 \theta + \frac{ab}{2} \cos^2 \theta)\ d\theta = \frac{ab}{2} \int_0^{2\pi}\ d\theta = \pi ab$$