Solving the spring equation

The spring equation describes the motion of a mass attached to a spring on a frictionless table. The mass is pulled to extend the spring, and then released. The equation is

$$F = ma = m\ddot{x} = -kx$$

Rearranging

$$\ddot{x} + \frac{k}{m}x = 0$$

One solution is

$$x(t) = A\cos\omega t$$

$$\ddot{x} = -\omega^2 A\cos\omega t$$

$$(-\omega^2 + \frac{k}{m})A\cos\omega t = 0$$

since $A = 0$ is a trivial solution, and $\cos\omega t$ cannot be always zero, we have

$$\omega^2 = \frac{k}{m}$$

We can also try an exponential

$$x(t) = Ae^{\alpha t}$$

$$\ddot{x} = \alpha^2 Ae^{\alpha t} = \alpha^2 x$$

but this doesn't work, because $\alpha^2 \geq 0$, but k and m are both positive.

friction

We can add a term in $\dot{x}$.

$$\ddot{x} + b\dot{x} + cx = 0$$

Now try the exponential

$$x = Ae^{\omega t}$$
$$\dot{x} = \omega Ae^{\omega t}$$
$$\ddot{x} = \omega^2 Ae^{\omega t}$$

so

$$(\omega^2 + b\omega + c)Ae^{\omega t} = 0$$

becomes a quadratic

$$\omega^2 + b\omega + c = 0$$

with solution

$$\omega = -\frac{b}{2} \pm \sqrt{(\frac{b}{2})^2 - c}$$

Now there are various possibilities depending on the values of b and c. There are real solutions for $c \leq b/2$. For both $\pm$ the second term, ω is negative. Then

$$x(t) = Ae^{-|\omega|t}$$

dies out with time, which is what we expect to happen.

There are also solutions for $c > b/2$.

$$\omega = -\frac{b}{2} \pm i\sqrt{(c - \frac{b}{2})^2}$$

Define

$$\omega' = \sqrt{(c - \frac{b}{2})^2}$$
$$\omega = -\frac{b}{2} \pm i\omega'$$
$$x(t) = Ae^{-b/2}e^{\pm i\omega' t}$$

non-zero force

$$\ddot{x} + b\dot{x} + cx = d$$

The approach here is to say, rather than solve this equation, we will solve the equation for a complex number z where

$$\ddot{z} + b\dot{z} + cz = d$$

because the real part of z will also be a solution, and using z makes the manipulations easier.

We guess

$$z = Ae^{i\omega t}$$

$$\dot{z} = i\omega Ae^{i\omega t}$$

$$\ddot{z} = -\omega^2 Ae^{i\omega t}$$

We use the following trick. z containing both $\pm i$ are solutions, so add them together

$$z = Ae^{i\omega t} + Be^{-i\omega t}$$

Second, we need to take the real part of this. The way that works is to add the complex conjugate of z, which requires that $B = A^*$ so

$$x = \frac{1}{2}(Ae^{i\omega t} + A^* e^{-i\omega t})$$