Orthogonality of sine and cosine: using exp

I want to take another look at the basic identities that are used in building up Fourier series. We're interested in products of sine and cosine like

$$\cos nx \ \ \cos mx$$

$$\sin nx \ \ \sin mx$$

$$\sin nx \ \ \cos mx$$

There may be other factors as well

$$\cos \frac{n\pi}{L} x \ \ \cos \frac{n\pi}{L} mx$$

but I'm going to treat the simple case here. The result we had before was that (over an appropriate interval which is a multiple of π), such products are always 0 if $m \neq n$, whereas if $m = n$ then the integrals are very simple, but do depend on whether $m = n = 0$ or $m = n \neq 0$.

$\cos nx \ \ \cos mx$

This one is more easily treated with the trig approach. Let's see

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

$$\cos s - t = \cos s \cos t + \sin s \sin t$$

Adding them together

$$\cos s + t + \cos s - t = 2 \cos s \cos t$$

So

$$\cos nx \ \cos mx = \frac{1}{2} \ [\ \cos(n+m)x + \cos(n-m)x \ ]$$

When integrated over an interval like

$$\int_{-\pi}^{\pi} \cos nx \ \cos mx \ dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(n+m)x + \cos(n-m)x \ dx$$

$$= \frac{1}{2} \ [\ \frac{\sin(n+m)x}{n+m} + \frac{\sin(n-m)x}{n-m} \ ] \ \Big|_{-\pi}^{\pi}$$

if $n \neq m$ then since n and m are integers both terms on the right side are equal to zero, since $\sin k\pi = 0$ for integer k, including $k = 0$.

However, if $n = m$ then there are two cases. If $n = m \neq 0$ the left-hand term is zero as we just saw. Looking at the integrated form, it seems we have a problem, since we're attempting to divide by zero, luckily the right-hand term in the integral above it is $\int \cos(0) \ dx = \int dx$ which is just x, which gives us 2π times one-half, which is π.

If $m = n = 0$, then we have double this value. Let's apply this logic to the more complicated argument and limits used by Paul

$$\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \ dx$$

$$= \frac{1}{2} \ [\ \int_{-L}^{L} \cos \frac{(n+m)\pi x}{L} + \cos \frac{(n-m)\pi x}{L} \ dx \ ]$$

For integer n and m, the left-hand term is

$$\cos(n+m)\pi - \cos -(n+m)\pi$$

$$= \cos(n+m)\pi - \cos(n+m)\pi = 0$$

and the right is the same. Otherwise, we have either one or two terms of L (remembering the factor of one-half from outside the integral).

If we were to do this with exponentials, it is arguably more complicated

$$\cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$$

$$\cos nx \cos mx = \frac{1}{4}(e^{inx} + e^{-inx})(e^{imx} + e^{-imx})$$

$$= \frac{1}{4}(e^{i(m+n)x} + e^{i(n-m)x)} + (e^{-i(n-m)x} + e^{-i(n+m)x})$$

$$= \frac{1}{2}(\frac{e^{i(m+n)x} + e^{-i(n+m)x}}{2} + \frac{e^{i(n-m)x} + e^{-i(n-m)x}}{2})$$

$$= \frac{1}{2}\ [\ \cos(n+m)x + \cos(n-m)x\ ]$$

which is just what we had.

I'm not going to do the sine times sine. But let's look at

$$\sin nx\ \ \cos mx$$

$$= \frac{1}{4i}(e^{inx} - e^{-inx})(e^{imx} + e^{-imx})$$

$$= \frac{1}{4i}(e^{i(m+n)x} + e^{i(n-m)x} - e^{i(m-n)x} - e^{-i(m+n)x})$$

$$= \frac{1}{4i}(e^{i(m+n)x} + e^{i(n-m)x} - e^{-i(n-m)x} - e^{-i(m+n)x})$$

$$= \frac{1}{4i}(e^{i(m+n)x} - e^{-i(m+n)x})$$

$$= \frac{1}{2}(\sin(m+n)x)$$

so

$$\int_{-\pi}^{\pi} \sin nx\ \ \cos mx\ dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(m+n)x\ dx = 0$$

since $\sin k\pi = 0$ for integer k, including $k = 0$.