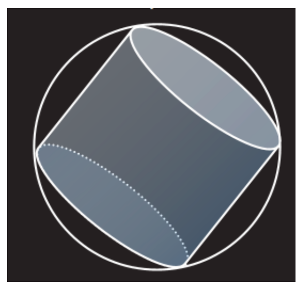
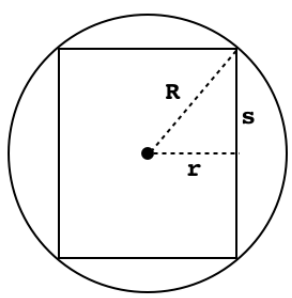
Napkin ring problem

Given a sphere of radius R, construct a cylinder of radius r and height $2s$ inside the sphere.



Both the upper and lower edges of the cylinder lie completely on the surface of the sphere.

To see this, visualize the cross-section through the center of the sphere (and along the axis of the cylinder): it is just a rectangle inside a circle.



Rotate this cross section around its vertical axis and obtain our object. Now we want to treat r as a variable and impose the constraint that the volume of the cylinder is a maximum. The crucial relationships are that

$$0 < r < R$$

because at the extremes the volume of the cylinder will be zero. And that

$$r^2 + s^2 = R^2$$

Compute the two volumes, take their ratio, and maximize that.

$$V_c = \pi r^2 \cdot 2s$$
$$V_s = \frac{4}{3}\pi R^3$$

$$F = \frac{V_c}{V_s} = \frac{\pi r^2 \cdot 2s}{4/3\ \pi R^3} = \frac{3}{2}\ \frac{r^2 s}{R^3}$$

From above, we have that

$$s = \sqrt{R^2 - r^2}$$

So

$$F = \frac{3}{2}\ \frac{r^2\sqrt{R^2 - r^2}}{R^3}$$

Set the slope of the derivative equal to zero

$$\frac{dF}{dr} = 0 = \frac{3}{2R^3}\ \frac{d}{dr} r^2\sqrt{R^2 - r^2}$$

Since $R \neq 0$, we must have

$$\frac{d}{dr} r^2 \sqrt{R^2 - r^2} = 0$$
$$-2r^3 \ \frac{1}{2} \frac{1}{\sqrt{R^2 - r^2}} + 2r \ \sqrt{R^2 - r^2} = 0$$
$$r^2 \ \frac{1}{2} \frac{1}{\sqrt{R^2 - r^2}} = \sqrt{R^2 - r^2}$$
$$r^2 = 2(R^2 - r^2)$$
$$3r^2 = 2R^2$$
$$r = \sqrt{\frac{2}{3}} R$$

The answer given here:

urlwww.datagenetics.com/blog/july22014/index.html

is that the ratio of the volumes at the maximum is $1/\sqrt{3}$.

Our expression for the ratio was

$$F = \frac{3}{2} \ \frac{r^2 \sqrt{R^2 - r^2}}{R^3}$$

Our answer above:

$$r = \sqrt{\frac{2}{3}} R$$
$$r^2 = \frac{2}{3} R^2$$

So, plugging in:

$$F = \frac{3}{2}\ \frac{2}{3}R^2\ \frac{1}{R^3}\ \sqrt{R^2 - \frac{2}{3}\ R^2}$$

$$= \frac{1}{R}\ \sqrt{R^2 - \frac{2}{3}\ R^2}$$

$$= \sqrt{\frac{1}{3}}$$