Hyperbolic trig functions: inverse

We have looked at the hyperbolic sine and cosine elsewhere:

$$y = \sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$y = \cosh x = \frac{1}{2}(e^x + e^{-x})$$

In many ways these are similar to sine and cosine with a sign difference. For example

$$\cosh^2 x - \sinh^2 x = 1$$

$$\frac{d}{dx} \sinh x = \cosh x$$

$$\frac{d}{dx} \cosh x = \sinh x$$

Here, our first job is to derive the inverse functions. To do that we must solve the above equations for x. Take the first one

inverse of sinh

$$y = \sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$x = \sinh^{-1} y$$

Substitute $z = e^x$, then:

$$2y = z - \frac{1}{z}$$

$$z^2 - 2yz - 1 = 0$$

Solve using the quadratic equation

$$z = \frac{2y \pm \sqrt{4y^2 + 4}}{2}$$

$$= y \pm \sqrt{y^2 + 1}$$

Since $z = e^x$, $z > 0$ so we take the positive root. Substitute back to x

$$e^x = y + \sqrt{y^2 + 1}$$

$$x = \ln|y + \sqrt{y^2 + 1}|$$

Change back to the usual notation with y as the dependent variable

$$y = \sinh^{-1} x = \ln|x + \sqrt{x^2 + 1}|$$

For the derivative

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \; (1 + \frac{x}{\sqrt{x^2 + 1}})$$

$$= \frac{1}{x + \sqrt{x^2 + 1}} \; (\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}})$$

$$= \frac{1}{\sqrt{x^2 + 1}}$$

$$\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}}$$

Recall that

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

Just a change of sign on one term.

inverse of cosh

$$y = \cosh x = \frac{1}{2}(e^x + e^{-x})$$
$$x = \cosh^{-1} y$$

As before, substitute $z = e^x$

$$2y = z + \frac{1}{z}$$
$$z^2 - 2yz + 1 = 0$$
$$z = \frac{2y \pm \sqrt{4y^2 - 4}}{2}$$
$$= y \pm \sqrt{y^2 - 1}$$

Take the positive root and back substitute

$$e^x = y + \sqrt{y^2 - 1}$$
$$x = \ln |y + \sqrt{y^2 - 1}|$$

Change notation:

$$y = \cosh^{-1} x = \ln |x + \sqrt{x^2 - 1}|$$

Differentiate:

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 - 1}}(1 + \frac{x}{\sqrt{x^2 - 1}})$$
$$= \frac{1}{\sqrt{x^2 - 1}}$$
$$\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}$$

Compare with

$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}$$

inverse of tanh

Start with

$$y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$= \frac{e^x - 1/e^x}{e^x + 1/e^x}$$

Substitute $z = e^x$, then:

$$y = \frac{z - 1/z}{z + 1/z}$$
$$= \frac{z^2 - 1}{z^2 + 1}$$
$$(y-1)z^2 + (0)z + (y+1) = 0$$

The quadratic equation gives:

$$\frac{\pm\sqrt{-4(y-1)(y+1)}}{2(y-1)}$$

Factor out the $\sqrt{4}$

$$= \pm\frac{\sqrt{-(y-1)(y+1)}}{(y-1)}$$
$$= \pm\frac{\sqrt{(1-y)(y+1)}}{(y-1)}$$

Choose the negative root but multiply on the bottom by -1

$$= \frac{\sqrt{(1-y)(y+1)}}{(1-y)}$$

$$= \frac{\sqrt{y+1}}{\sqrt{1-y}}$$

Substitute back

$$e^x = \frac{\sqrt{y+1}}{\sqrt{1-y}}$$

$$x = \ln(\frac{\sqrt{y+1}}{\sqrt{1-y}})$$

$$x = \frac{1}{2}\ln(\frac{y+1}{1-y})$$

Change notation

$$y = \tanh^{-1} x = \frac{1}{2}\ln(\frac{x+1}{1-x})$$

Differentiate

$$\frac{d}{dx}\tanh^{-1} x = (\frac{1}{2})\frac{1-x}{x+1}\frac{(1-x+x+1)}{(1-x)^2}$$

$$= \frac{1-x}{(x+1)(1-x)^2}$$

$$= \frac{1}{(x+1)(1-x)}$$

$$= \frac{1}{1-x^2}$$