Binomial theorem: proof by induction

The binomial theorem gives the terms and coefficients of $(a+b)^n$ as follows. The terms are

$$(a+b)^n = \sum_{k=0}^{n} c_k \; a^{n-k} \; b^k$$

$$= c_0 \; a^n + c_1 \; a^{n-1} b + \dots c_{n-1} \; a^2 b^{n-1} + c_n \; b^n$$

where the coefficients are determined by the formula n "choose" k which you may have learned when studying probability. It is often written as

$$\binom{n}{k} = \frac{n!}{k! \; (n-k)!}$$

but can be typeset more simply as $C(n,k)$, so we can rewrite the equation as

$$(a+b)^n = C(n,0) \; a^n + C(n,1) \; a^{n-1} b + \cdots + C(n,n) \; b^n$$

or in summation form as

$$(a+b)^n = \sum_{k=0}^{n} C(n,k) \; a^{n-k} b^k$$

(Note that 0! is defined to be equal to 1, partly so that this formula works for the first term)

$C(n,k)$ gives the terms of Pascal's Triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

and so on.

We easily check that this theorem gives the correct values in the expansion of $(a+b)^n$, for small n. For example

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

$$(a+b)^3 = (a+b)(a^2 + 2ab + b^2) = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\begin{aligned}(a+b)^4 &= (a+b)(a^3 + 3a^2b + 3ab^2 + b^3)\\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

As an example of computing any particular coefficient, the second-to-last line is $n = 4$ and $k = 0 \to n$, so the third term ($k = 2$) is

$$C(4,2) = \frac{4!}{2!(4-2)!} = \frac{4 \cdot 3 \cdot 2}{2 \cdot 2} = 6$$

while for $n = 5$ and $k = 2$ we have:

$$C(5,2) = \frac{5!}{2!(5-2)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 2} = 10$$

simplification

You may have noticed that in the above calculations, there are terms that cancel. We expand $n!$ partially

$$n! = n \cdot (n-1) \dots (n-k+1) \cdot (n-k)!$$

The last term is also present in the denominator of our formula

$$C(n,k) = \frac{n!}{k!\ (n-k)!}$$

so we can simplify

$$C(n,k) = \frac{n \cdot (n-1) \dots (n-k+1)}{k!}$$

We will be interested in coefficients with $k+1$, so let's take a look at n "choose" $k+1$. The original definition would give:

$$C(n,k+1) = \frac{n!}{(k+1)!\ (n-(k+1))!}$$

Remove one set of parentheses

$$= \frac{n!}{(k+1)!\ (n-k-1)!}$$

By the same argument, expand $n!$

$$= \frac{n \cdot (n-1) \dots (n-k+1) \cdot (n-k) \cdot (n-k-1)!}{(k+1)!\ (n-k-1)!}$$

$$= \frac{n \cdot (n-1) \dots (n-k+1) \cdot (n-k)}{(k+1)!}$$

You should convince yourself that the last term in the numerator is correct. It is somewhat counterintuitive that for $k+1$ the last term is $(n-k)$ (rather than say, $(n-k+2)$, as I thought at first).

induction

Now we wish to use induction to show that if the formula works for n, it also works for $n+1$.

$$(a+b)^n = c_0\ a^n + c_1\ a^{n-1}b + \dots c_{n-1}\ ab^{n-1} + c_n\ b^n$$

$$(a+b)^{n+1} = (a+b)\ [c_0\ a^n + c_1\ a^{n-1}b + \ldots c_{n-1}\ ab^{n-1} + c_n\ b^n\ ]$$

When we do the multiplication we will have terms on the "ends" containing only powers of a^{n+1} or b^{n+1}, and these will retain the same coefficients, namely

$$c_0 = c_n = 1$$

For the rest of the terms each one will have two contributions, one from multiplying one term from previous line by b, and one from multiplying the next term by a.

Let's start by considering a concrete example: the second ($k = 1$) and third ($k = 2$) terms in the expansion for $(a+b)^n$:

$$C(n,1)\ a^{n-1}b + C(n,2)\ a^{n-2}b^2$$

We multiply the first term by b and the second by a, obtaining

$$C(n,1)\ a^{n-1}b^2 + C(n,2)\ a^{n-1}b^2$$

$$=\ [\ C(n,1) + C(n,2)\ ]\ a^{n-1}b^2$$

It isn't so easy to see from looking at the power of a because we are still numbering according to the initial expansion for $(a+b)^n$, but clearly this is the third term in the expansion for $(a+b)^{n+1}$ because it contains b^2.

The coefficient for this term according to the binomial theorem or formula should be $C(n+1,2)$. So, for the proof by induction, we must show that

$$C(n,1) + C(n,2) = C(n+1,2)$$

Generalizing from this, we need

$$C(n,k) + C(n,k+1) = C(n+1,k+1)$$

That is what we will prove.

coefficient

Let us examine the general statement

$$C(n,k) + C(n,k+1)$$

Rewriting it as the factorial using the simplification we found above

$$\frac{n \cdot (n-1) \ldots (n-k+1)}{k!} + \frac{n \cdot (n-1) \ldots (n-k+1) \cdot (n-k)}{(k+1)!}$$

We can factor out $(n-k)/(k+1)$ from the second term:

$$= (\frac{n-k}{k+1}) \ \cdot \frac{n \cdot (n-1) \ldots \ (n-k+1)}{k!}$$

and now this is just that factor multiplied by the first term.

So the complete sum becomes

$$[\ 1 + (\frac{n-k}{k+1})\ ] \ \cdot \frac{n \cdot (n-1) \ldots \ (n-k+1)}{k!}$$

Take the leading factor and put it over a common denominator

$$\frac{(k+1) + (n-k)}{k+1} = \frac{n+1}{k+1}$$

so the sum now becomes

$$= (\frac{n+1}{k+1}) \ \cdot \frac{n \cdot (n-1) \ldots \ (n-k+1)}{k!}$$

$$= \frac{(n+1) \cdot n \cdot (n-1) \ldots \ (n-k+1)}{(k+1)!}$$

rearranging the last term in the numerator slightly

$$= \frac{(n+1) \cdot n \cdot (n-1) \ldots \ ((n+1)-k)}{(k+1)!}$$

$$= C(n+1, k+1)$$

This is the correct expression for $n+1$ (because it has $n+1$ in the right places), and it is the correct expression for $k+1$ because it ends with $n+1$ minus k (rather than $k+1$). Go back to the simplification section to see this.

recap

We assume that $C(n, k)$ is the correct coefficient for $a^{n-k}b^k$ in the expansion of $(a+b)^n$, and that $C(n, k+1)$ is the correct coefficient for the succeeding term $a^{n-(k+1)}b^{k+1}$ in the same expansion. Multiplication of the first by b and the second by a leads to:

$$[\ C(n, k) + C(n, k+1)\ ]\ a^{n-k}b^{k+1}$$

We need to tweak one exponent slightly when considering this as part of the expansion for $(a+b)^{n+1}$. The n in the exponent for a should be expressed in terms of n referring to the incremented value $n+1$ so it needs to step down one unit, becoming a^{n-k-1} which is equal to $a^{n-(k+1)}$. Thus we have

$$[\ C(n, k) + C(n, k+1)\ ]\ a^{n-(k+1)}b^{k+1}$$

We showed that

$$C(n, k) + C(n, k+1) = (1 + \frac{n-k}{k+1}) \cdot C(n, k)$$

$$= \frac{n+1}{k+1}) \cdot C(n, k)$$

$$= C(n+1, k+1)$$

which is what the binomial theorem gives. This completes the proof by induction. $\square$

It is worth looking back at Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

and recognizing that all we have really done is to show that, indeed, when we take two successive terms in the k and $k+1$ positions, multiplying the first by b and the second by a, we generate the coefficient of the $k+1$ term in the row below by adding the first two coefficients.