LU Decomposition

To find the "LU" decomposition of a matrix A is to find

$$A = LU$$

where L is a "lower triangular matrix" and U is an "upper triangular matrix." What is meant is that L has the form

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

Every entry *above* the diagonal is 0. Typically the entries of L on the diagonal are 1.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ d & e & 1 \end{bmatrix}$$

U is the reverse: every entry *below* the diagonal is 0. The 2x2 version is about as simple as linear algebra can be. Suppose we have

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \quad \rightarrow \quad U = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

produced by elimination. U is upper triangular. The matrix that can do this is

$$E = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = U$$

So we have

$$EA = U$$

But we want

$$A = LU$$

so clearly

$$E^{-1}EA = A = E^{-1}U = LU$$
$$E^{-1} = L$$

For a 2x2 with one E matrix it's easy

$$E = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$

We use the standard method for 2x2. Find $det(E)$, which is 1. If it were otherwise, we would multiply the result by $1/det$. Switch the values at the a and d positions, and negate the others.

$$E^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$$

Check that we have it right

$$EE^{-1} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$LU = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = A$$

Simple as that! An additional factorization could be done to bring the diagonal entries of U out, like this

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = A$$