Don't be irrational, e

I found a nice proof of the irrationality of e in the calculus text by Courant and Robbins. It is a proof by contradiction. We start by assuming that e is rational.

$$e = \frac{p}{q}, \quad p, q \in \mathbb{N}$$

We make use of the infinite series representation of e

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

From this, it is obvious that $e > 2$. If you're interested, there is a proof that $e < 3$ in the book.

Equating the series representation to the rational fraction p/q:

$$\frac{p}{q} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Multiply both sides by $q!$. For the left-hand side, we have

$$e\ q! = \frac{p}{q}\ q! = p(q-1)!$$

We won't need to do anything more with this, but note that since $e\ q!$ is equal to $p(q-1)!$, we can see that the left-hand side, $e\ q!$, is clearly an integer. Therefore, the right-hand side must also be an integer. This is the series

$$q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{(q-1)!} + \frac{q!}{q!} + \frac{q!}{(q+1)!} + \cdots$$

Now, $q!$ is obviously an integer. And for every integer $k < q$, $k!$ divides $q!$ evenly

$$\frac{q!}{k!} = q \times (q-1) \times (q-2) \cdots \times (q-k+1)$$

In our series

$$q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{(q-1)!} + \frac{q!}{q!} + \frac{q!}{(q+1)!} + \cdots$$

all the terms to the left of $q!/(q-1)!$ are integers, as is $q!/(q-1)! = q$ and $q!/q! = 1$.

So now our concern is with the fractions that follow. We will show that these sum up to something less than 1. We have

$$\frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots$$

Since $q >= 2$

$$\frac{1}{(q+1)} <= \frac{1}{3}$$

$$\frac{1}{(q+1)(q+2)} <= (\frac{1}{3})^2$$

and so on, and the entire remaining series of fractions is less than or equal to

$$\frac{1}{3} + (\frac{1}{3})^2 + (\frac{1}{3})^3 + \cdots$$

This is the geometric series with $r = 1/3$ and first term equal to r, and the sum is known to be

$$\frac{1}{3}(1/(1-\frac{1}{3})) = \frac{1}{2}$$

Since the right-hand side is equal to an integer plus something "less than or equal to $\frac{1}{2}$", it is not an integer, and cannot be equal to the left-hand side, which is equal to an integer. We have reached a contradiction. Therefore e cannot be equal to p/q, for $p, q \in \mathbb{N}$.