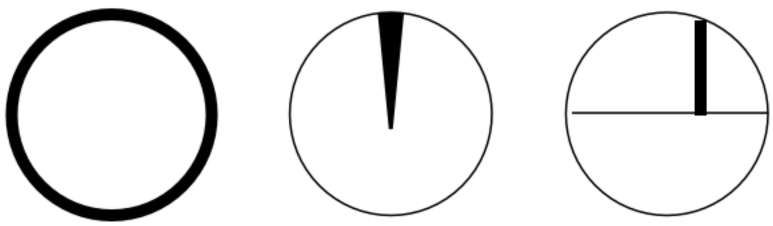
Area of a circle by integration



Integration is used to compute areas and volumes, and other sums, by adding up many little pieces. To calculate the area of a circle, we find the pieces we will use with one of three basic strategies: rings, slices of pie, and rectangles of area underneath the function obtained by solving $x^2 + y^2 = R^2$, and using the positive square root. These three approaches are illustrated in the figure above.

Rings

In the first approach (left panel), we imagine the area being computed by adding up the individual areas of a series of very thin, concentric rings.

The total area to be computed is that of a circle of a definite, fixed size, and we denote the radius of this circle by capital R, a constant. On the other hand, the series of rings ranges from the origin of the circle to the circumference of the outmost ring. Each one of this progression of rings has a radius, so we use the lowercase r to describe them, with r being a variable—r varies from 0 at the origin to R at the outside of the circle.

Think about an individual ring, for example the outermost ring, which is similar to the peel or rind surrounding a slice of lemon. We are working with areas here, in two dimensions, so the slice we imagine to be infinitely thin, and we are working with it as a cross-section or ring. The area of the ring is the length times the width. The length is the circumference, $2\pi R$ for the outermost ring, but in general, for any of the inner rings it is $2\pi r$. The length is multiplied by the width of the slice, which is a small element of radius, dr. The small element of area contributed by an individual ring is dA:

$$dA = 2\pi r \ dr$$

Another way to explain this equation is to ask the question: *how does area change with increasing radius*? If we take a circle and increase its radius by a bit, how does the area change? The answer is, it changes in proportion to the circumference, $2\pi r$.

Proceeding from the equation, the total area is the sum of the areas for the series of rings.

$$A = \int dA = \int_0^R 2\pi r \ dr$$

It's worth emphasizing how this view is much different than our first uses of integration: the pieces of area are no longer rectangles but circles. But it poses most clearly the question we are trying to answer, "how does area vary with r"?

The solution is

$$\int_0^R 2\pi r \ dr = 2\pi \ \frac{1}{2} r^2 \ \Big|_{r=0}^{r=R} = \pi R^2$$

Wedges

In the second method, we need to first find the area of a wedge. For a thin enough slice, this is a triangle, with a similar formula: one-half the base times the height. The height is R, the radius of the circle.

For the base we need the length of a piece of arc of a circle. Recall that by definition, if we have a unit circle, then the angle of a wedge is equal to the arc it cuts out, and vice-versa, the arc is equal to the angle. (Thus, the total length if we go all the way around the unit circle is 2π). For a circle with radius R, the length going all the way around is $2\pi R$, and the length of arc for any angle θ is θ times R.

Our area is built up of a series of wedges that are almost infinitely slender, with angle $d\theta$, so these wedges have bases measuring $R \ d\theta$. The area of each wedge is

$$dA = \frac{1}{2} R \ R \ d\theta$$

And we have for the total area

$$
\begin{aligned}
A = \int dA &= \int \frac{1}{2} R \ R \ d\theta \\
&= \frac{1}{2} R^2 \int_{\theta=0}^{\theta=2\pi} \ d\theta \\
&= \frac{1}{2} R^2 \theta \ \Big|_{\theta=0}^{\theta=2\pi} \\
&= \pi R^2
\end{aligned}
$$

Area under the curve

The third view is the most familiar, but has a substantially harder calculation. We need to find the area under the positive square root in the equation for a circle, between some limits, which could be $x = 0 \Rightarrow x = R$ or $x = -R \Rightarrow x = R$.

$$x^2 + y^2 = R^2$$

$$y = \sqrt{R^2 - x^2}$$

We use a trigonometric substitution

$$x = R \sin \theta$$

$$y = R \cos \theta$$

$$dx = R \cos \theta \ d\theta$$

The integral we want to calculate is:

$$\int \sqrt{R^2 - x^2} \ dx$$

We *could* substitute $x = R \sin \theta$:

$$
\begin{aligned}
&\sqrt{R^2 - x^2} \\
&= \sqrt{R^2 - R^2 \sin^2 \theta} \\
&= R \sqrt{1 - \sin^2 \theta} \\
&= R \sqrt{\cos^2 \theta}
\end{aligned}
$$

$$= R\cos\theta$$

plugging in for dx we obtain:

$$= \int R\cos\theta \ R\cos\theta \ d\theta$$

$$= \int R^2\cos^2\theta \ d\theta$$

Alternatively, just recognize that we want to integrate value of y

$$\int y \ dx$$

$$= \int R\cos\theta \ R \ cos \ \theta \ d\theta$$

$$= R^2 \int \cos^2\theta \ d\theta$$

We have worked this integral out elsewhere (or you can solve it by substituting from the double angle formula). We obtain

$$R^2 \int \cos^2\theta \ d\theta = \frac{1}{2}R^2 \int (1 + \cos \ 2\theta) \ d\theta$$

$$= \frac{1}{2}R^2 \ [\ \theta + \frac{1}{2}\sin \ 2\theta \ ]$$

Area under the curve: limits

To actually compute the are, we need limits. Suppose we take the limits for the original integral as

$$x = 0, \quad x = R$$

and agree that we need to multiply the final result by 4, because we're calculating only the upper-right quarter of the circle.

After the substitution of θ for x, these limits become

$$x = 0 = R\sin\theta, \quad \theta = \frac{\pi}{2}$$

$$x = R = R\sin\theta, \quad \theta = 0$$

But there's a subtlety lurking here. If we integrate from $\theta > 0$ to $\theta = 0$, the area will be negative. To fix this, we must reverse the order of the limits.

With an upper limit equal to $\pi/2$, and 0 as the lower limit, the term in brackets is

$$(\frac{\pi}{2} + \frac{1}{2}\sin\ \pi) - (0 + \frac{1}{2}\sin 0) = \frac{\pi}{2}$$

$$A = \frac{1}{4}\pi R^2$$

This is one-fourth of the total, hence $A_T = \pi R^2$.