Paraboloid Volume

A paraboloid is a solid whose vertical cross-section is a parabola (often, it is oriented along the z-axis). It may open up, or down. The cross-sections parallel to the xy-plane are typically circles, though the shape factors for the parabolas in the xz- and yz-planes could be different, leading to an ellipse for the cross-sections.

Consider

$$z = 2 - x^2 - y^2$$

This is a paraboloid that opens down (z gets large and negative when either x or y get large). The vertex is at $z = 2$. When $z = 0$, the cross-section is a circle

$$x^2 + y^2 = 2$$

of radius $r = \sqrt{2}$.

Usually, cylindrical coordinates are good for dealing with this solid. For example, in those coordinates, the surface area element is $dS = r\ dz\ d\theta$.

But let's start with a method from 1-D calculus. Suppose we turn the paraboloid so that its symmetry axis is the x-axis, and its vertex is at the origin. We can model this as being generated by revolution of the graph of $f(x) = \sqrt{x}$ around the x-axis.

The general formulation is that at each point x, we have a circle of radius $f(x)$ and area $\pi f(x)^2$.

Here, $f(x) = \sqrt{x}$, and area $A = \pi r^2 = \pi x$. If we consider the width of each slice to be dx, then for the volume just add all these up

$$V(x) = \int A(x)\ dx = \int \pi x\ dx$$
$$= \frac{\pi}{2}\ x^2\ \Big|_a^b$$

For the unit paraboloid at $b = 1$, we get $\pi/2$. And our first paraboloid ($b = 2$) has a volume of

$$V = 2\pi$$

Now, consider the shape factor for the parabola, a. In the standard equation $y = ax^2$, and the larger a is, the faster the parabola grows in the y-direction. But here, we have the parabola "opening" in the $+x$-direction. That is, we have $x = ay^2$ and so $y = \sqrt{x/a}$. Thus

$$V(x) = \int A(x)\ dx = \pi \int \frac{x}{a}\ dx = \frac{\pi}{a} \int x\ dx$$

and we have a factor of $1/a$ for the final volume.

We can ask in another way if these formulas make sense. Consider the parabola $y = x^2$ in the unit square. The area "under" the curve is $1/3$, which is another way of saying that the area "over" the curve and inside the parabola is $2/3$. Compare with the circle, whose area over the unit square is $\pi/4 \approx 3/4$. The circle is a bit "fatter" than the parabola and we expect its volume, when we do the rotation, to be larger in proportion. So this looks reasonable.

another method

I want to find the volume using cylindrical coordinates. I'd also like to generalize the problem. In 1D we orient the vertex at the origin and integrate (usually) from $0 \to b$. When we turn the volume so that it aligns with the z-axis, we usually place it with the bottom of the desired region in the xy-plane. So I'm going to re-write the equation as

$$z = f(x, y) = c - x^2 - y^2$$

where c is the height of the parabola we are measuring.

We are going to use r, θ, z. So we need to find the limits on r. The "shadow" of the paraboloid is a circle in the x, y-plane

$$\begin{aligned} z = 0 &= c - x^2 - y^2 \\ x^2 + y^2 &= r^2 = c \\ r &= \sqrt{c} \end{aligned}$$

The volume in cylindrical coordinates is

$$V = \iiint dV = \iiint r \ dz \ dr \ d\theta$$

What are the bounds on z? Remember, if we integrate first with respect to z then r is *fixed.* For a given r

$$z = c - r^2$$

So, the bounds on z are $z = 0 \to c - r^2$, and the inner integral is just

$$\int_0^{c-r^2} r \ dz = r(c - r^2)$$

This gives us what we would have if we just started by thinking about the double integral of $f(x, y)$ or $f(r, \theta)$ over the region R in the plane.

$$V = \iint f(x, y) \ dx \ dy = \iint f(r, \theta) \ r \ dr \ d\theta$$

The middle integral is

$$\int_0^{\sqrt{c}} cr - r^3 \ dr$$

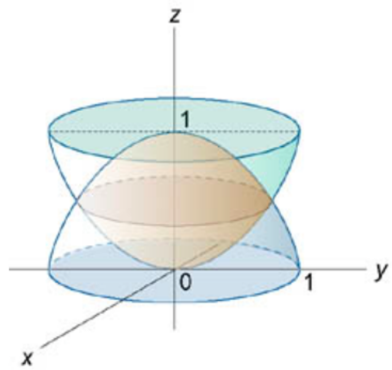
$$= \frac{1}{2}cr^2 - \frac{1}{4}r^4 \ \Big|_0^{\sqrt{c}}$$

$$= \frac{1}{2}c^2 - \frac{1}{4}c^2 = \frac{1}{4}c^2$$

times 2π from the outer integral. Hence $V = \pi/2 \cdot c^2$.

The way we've set up this problem, the region that we want starts at the xy-plane. So there's not much point in preserving the option of starting evaluation of the middle integral at $a \neq 0$. But if you did decide to do this you could. Just remember to go back and fix the lower bound on z in the inner integral. That would also have to change.

double paraboloid



Just for fun, let's try to do a double paraboloid. In the figure, the paraboloid that opens up is $z = x^2 + y^2$ while the one that opens down is $z = 1 - x^2 - y^2$. To match what we had in the earlier section, I'm going to change the upper one to be $z = 2 - x^2 - y^2$.

We will use cylindrical coordinates to integrate. The key to the problem, as usual, is to find the limits for r and z. First, solve for the intersection of the two surfaces:

$$z = 2 - x^2 - y^2 = x^2 + y^2$$
$$x^2 + y^2 = 1 = r^2$$

So $r = 0 \to 1$. Easy enough. And z ranges from the lower surface to the upper one. Our integral is

$$V = \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} \ dz \ r \ dr \ d\theta$$

The middle integral is

$$\int_0^1 2 - 2r^2 \ r \ dr$$

$$= 2 \ (\frac{r^2}{2} - \frac{r^4}{4}) \ \Big|_0^1$$

$$= 2 \cdot \frac{1}{4} = \frac{1}{2}$$

Multiply by 2π from the outer integral and that gives simply, π. Notice that we have a duplicated version of the first volume, for which we found the answer $\pi/2$. It checks.