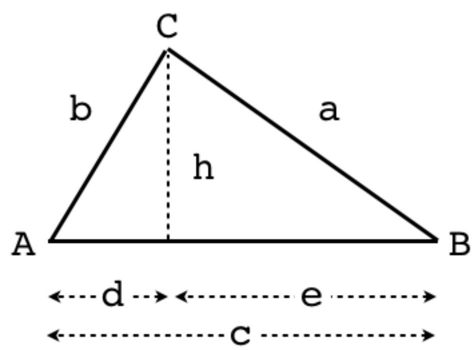
Assorted laws for trigonometry

Here, I will go through algebraic proofs of various laws used in trigonometry. Let's start with a look at the Pythagorean Theorem. In the triangle below, A, B, and C are the angles, with side lengths a, b, and c.



An altitude has been drawn from angle C to side c opposite. The altitude is perpendicular to the side c, which is thereby divided into lengths d and e.

Suppose angle C is a right angle. (I know it doesn't look exactly right, but allow me to just go with it). If C is a right angle, A and B are complementary:

$$A + B = 90^\circ$$

Referring to the small triangle on the left, it's clear that the angle (call it θ) between side b and the altitude h is equal to angle B, because

$$\theta + A + 90^\circ = 180^\circ = B + A + C$$

where $C = 90^\circ$. Therefore, the small triangle on the left and $\triangle ABC$ are similar. By similarity, the sides opposite equal angles are in the same ratio so

$$\frac{d}{b} = \frac{b}{c}$$

$$\frac{h}{b} = \frac{a}{c}$$

By exactly the same argument, the small triangle on the right is similar to both of these, so we can extend the ratios

$$\frac{d}{b} = \frac{b}{c} = \frac{h}{a}$$

$$\frac{h}{b} = \frac{a}{c} = \frac{e}{a}$$

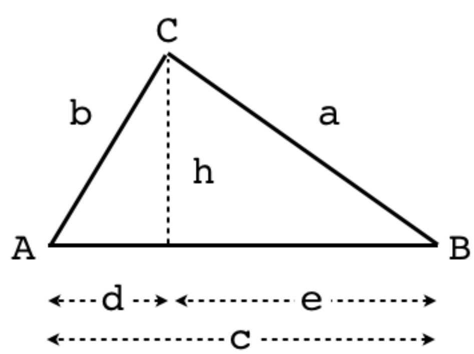
We have

$$a^2 = ce, \qquad b^2 = cd, \qquad a^2 + b^2 = ce + cd = c(e + d) = c^2$$

QED.

This is the Pythagorean Theorem.

Looking at the figure again (from now on angle C will not be equal to 90°), write a formula for the sine of A and sine of B



$$\frac{h}{b} = sin\ A, \qquad \frac{h}{a} = sin\ B$$

$$h = a\ sin\ B = b\ sin\ A$$

$$\frac{a}{sin\ A} = \frac{b}{sin\ B}$$

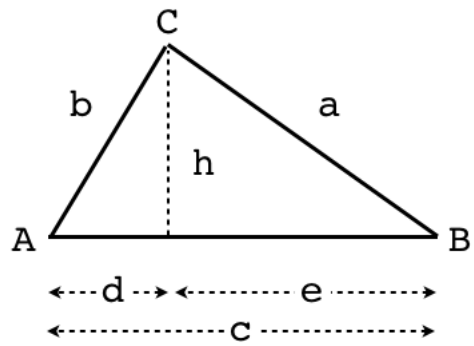
By symmetry, the ratio extends to side c and angle C

$$\frac{a}{sin\ A} = \frac{b}{sin\ B} = \frac{c}{sin\ C}$$

QED.

We have proved the Law of Sines.

With reference again to the figure, seeing h perpendicular to side c, and recalling the Pythagorean Theorem



We'll use these four facts

$$c = d + e$$

$$a^2 = e^2 + h^2$$

$$b^2 = d^2 + h^2$$

$$d = b\ cosA$$

Rewrite

$$h^2 = b^2 - d^2$$

Substitute into the expression for a^2

$$a^2 = e^2 + h^2 = (c - d)^2 + h^2 = c^2 - 2cd + d^2 + h^2$$

$$a^2 = c^2 - 2cd + d^2 + b^2 - d^2$$

$$a^2 = c^2 + b^2 - 2\ cd$$

$$a^2 = c^2 + b^2 - 2\ cb\ cosA$$

QED.

This is the Law of Cosines.

The last formula we want to prove is called Heron's Formula for the area of a triangle. If s is the semi-perimeter

$$s = \frac{1}{2}(a + b + c)$$

$$A = \sqrt{s + (s-a) + (s-b) + (s-c)}$$

Start with the well-known formula for area

$$A = \frac{1}{2}\ base\ times\ height = \frac{1}{2}\ c\ h = \frac{1}{2}\ cb\ sinA$$

We will come back to this and substitute for the sine of A. But first, rearrange the equation for the law of cosines

$$a^2 = c^2 + b^2 - 2bc\ cosA$$

$$cosA = \frac{(c^2 + b^2 - a^2)}{2bc}$$

$$sinA = \sqrt{1 - cos^2 A} = \sqrt{1 - \frac{(c^2 + b^2 - a^2)^2}{(2bc)^2}}$$

So finally we have

$$A = \frac{1}{2}\ c\ b\ \sqrt{1 - \frac{(c^2 + b^2 - a^2)^2}{(2bc)^2}}$$

$$A = \frac{1}{4}\sqrt{4b^2c^2 - (c^2 + b^2 - a^2)^2}$$

Now we just need to work on what is under the square root. It looks like a mess but will simplify quite a bit.

For the next part, we won't write $A = \frac{1}{4}\sqrt{\ldots}$, but we'll recall that it's there near the end, when we will write it as $A = \sqrt{\frac{1}{16}\ldots}$

Look at what's inside

$$4b^2c^2 - (c^2 + b^2 - a^2)^2$$

This looks familiar, it is a difference of squares

$$(2bc + (c^2 + b^2 - a^2))(2bc - (c^2 + b^2 - a^2))$$

In the first term, we can rearrange

$$2bc + c^2 + b^2 - a^2$$

$$(c + b)^2 - a^2)$$

$$(c + b + a)(c + b - a)$$

Similarly in the second term

$$-(c^2 - 2bc + b^2 - a^2)$$

$$-((c-b)^2 - a^2)$$

$$-((c-b+a)(c-b-a))$$

$$(c-b+a)(a+b-c)$$

Putting it all together, we have

$$(c+b+a)(c+b-a)(c-b+a)(a+b-c)$$

Recall that the perimeter

$$p = a + b + c = 2s$$

The first term above, $(a+b+c)$, is the perimeter, that is, twice the semi-perimeter or $2s$. The second term is $p - a - a = 2s - 2a = 2(s-a)$. The third and fourth terms can be seen to be equal, by the same logic, to $2(s-b)$ and $2(s-c)$. Recalling the square root, etc. from above, we have finally:

$$A = \sqrt{\frac{1}{16}\ 2(s)2(s-a)\ 2(s-b)\ 2(s-c)}$$

Canceling

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

QED.