Karkhar Video 27

Karkhar gives this problem

$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\ d\theta$$

Before we start I'd just point out that the equation for an ellipse in polar coordinates (with one focus at the origin) is

$$r = \frac{b^2}{a - c\cos\theta}$$

If we neglect the minus sign (which just flips the orientation along the x-axis), let $a = 2$ and $c = 1$ and

$$b^2 = a^2 - c^2 = 3$$

rewrite

$$\int_0^{2\pi} \frac{3}{2+\cos\theta}\ d\theta$$

What this looks like to me is the integral of $r\ d\theta$ around an ellipse with $a = 2$ and $b = \sqrt{3}$. This would be $rd\theta$ added up over the perimeter of that ellipse, i.e. the area.

Go back to the given problem. Let

$$z = e^{i\theta}$$

$$dz = iz\ d\theta$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

Use this result, but go back to z:

$$\int_{|z|=1} \frac{1}{2 + (1/2)(z + 1/z)} \, \frac{1}{iz} \, dz$$

$$= \frac{1}{i} \int_{|z|=1} \frac{1}{2z + (1/2)(z^2 + 1)} \, dz$$

$$= \frac{2}{i} \int_{|z|=1} \frac{1}{z^2 + 4z + 1} \, dz$$

The roots of the denominator are $-2 \pm \sqrt{3}$. One of these roots $(-2 + \sqrt{3})$ lies within our contour, which is just the unit circle.

Carry out partial fractions:

$$\frac{1}{z^2 + 4z + 1} = \frac{A}{z - (-2 + \sqrt{3})} + \frac{B}{z - (-2 - \sqrt{3})}$$

$$Az + A2 + A\sqrt{3} + Bz + B2 - B\sqrt{3} = 1$$

Hence $A = -B$ and

$$A2 + A\sqrt{3} - A2 + A\sqrt{3} = 1$$

$$2A\sqrt{3} = 1$$

$$A = \frac{1}{2\sqrt{3}}$$

The term we want is the one with $z_0 = (-2 + \sqrt{3})$ and that has coefficient A. Hence the value is

$$2\pi i \, (\frac{1}{2\sqrt{3}}) = \frac{\pi i}{\sqrt{3}}$$

Pick up the leading factor of $2/i$ and obtain $2\pi/\sqrt{3}$.

Going back to the argument about the ellipse at the beginning, multiplied by 3 this gives $2\sqrt{3}\pi$. This is exactly the area of an ellipse with $a = 2$ and $b = \sqrt{3}$.