Ellipse equations

The standard equation for a simple ellipse (aligned with the x- and y-axes) is

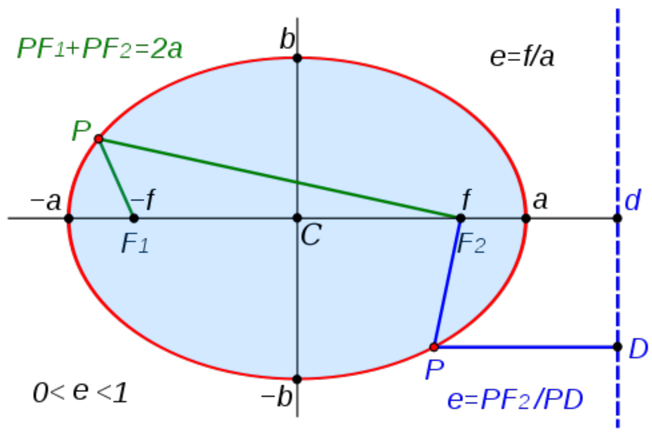
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

From the formula it is easy to see that when

$$x = 0; \quad y = \pm b$$

and similarly when

$$y = 0; \quad x = \pm a$$



The graphic, which is from wikipedia, shows a typical ellipse. The geometric definition of an ellipse is the set of points which has

$$PF_1 + PF_2 = const$$

The sum of the distances from any point to each of the two foci is a constant. The foci lie on the semi-major axis (the long axis of the ellipse), which here is the x-axis. The formula for the focal distance is

$$c = \sqrt{a^2 - b^2}$$

For example, if $a = 5$ and $b = 3$ then $c = 4$, so the foci are the two points $(0, -c), (0, c)$. (Some folks switch the labels a and b depending on the orientation of this axis, but I find it confusing).

The distance from F_1 to P ($P_x > 0$) is

$$\sqrt{(x + c)^2 + y^2}$$

and from F_2 it is

$$\sqrt{(x - c)^2 + y^2}$$

This sum is a constant, and equal to $2a$ as seen from the point $P = (a, 0)$

$$PF_1 + PF_2 = a - c + a + c = 2a$$

The relation to b can be seen from the point $Q = (0, b)$ where

$$PF_1 + PF_2 = 2\sqrt{b^2 + c^2} = 2a$$

$$b^2 + c^2 = a^2$$

Given a and b one can then find c easily.

Additional comments

The first comment is that just as for the other quadratic formulas, one can move the center of the ellipse by including offsets

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

When presented with an equation in which terms like $-2xh$ or $-2yk$ present, you need to "complete the square" to rearrange the terms as $(x - h)^2$ and $(y - k)^2$, respectively.

Recall that if

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

the "discriminant" is $B^2 - 4AC$ and if this is < 0, then the equation is an ellipse. The presence of an xy term indicates a rotated quadratic (though not necessarily an ellipse).

Parametric equations

These are simply

$$x = a\ cos\theta$$

$$y = b\ sin\theta$$

It can be verified easily that these solve the basic equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and note, for example that if $\theta = \pi/4$ then $x = a/\sqrt{2};\quad y = b/\sqrt{2}$.

Derivation of the equation of the ellipse

We have enough to derive the equation of the ellipse, with just a bit of algebra

$$2a = \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2}$$

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

$$x^2 - 2xc + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2xc + c^2 + y^2$$

$$-4xc = 4a^2 - 4a\sqrt{(x+c)^2 + y^2}$$

$$a^2 + xc = a\sqrt{(x+c)^2 + y^2}$$

$$a^4 + 2a^2xc + x^2c^2 = a^2(x^2 + 2xc + c^2 + y^2)$$

$$a^4 + 2a^2xc + x^2c^2 = a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2$$

$$a^4 + x^2c^2 = a^2x^2 + a^2c^2 + a^2y^2$$

$$(a^2 - c^2)x^2 = (a^2 - c^2)a^2 + a^2y^2$$

Recall that $c^2 = a^2 - b^2$

$$b^2x^2 = b^2a^2 + a^2y^2$$

$$\frac{b^2x^2}{a^2} = b^2 + y^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$