More sums

We have shown in other write-ups, and you can easily verify by searching that the sum of the integers between 1 and n is

$$1 + 2 + \cdots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

The next one, the sum of the squares of the first n integers, is useful for certain derivations in calculus (e.g. the Riemann sum to integrate $y = x^2$)

$$1^2 + 2^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

I was quite surprised to find that the sum of cubes is also simple and frankly, amazing

$$1^3 + 2^3 + \cdots + n^3 = \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{2^2} = [\ \frac{n(n+1)}{2}\ ]^2 = [\ \sum_{k=1}^{n} k\ ]^2$$

$$\boxed{\sum_{k=1}^{n} k^3 = [\ \sum_{k=1}^{n} k\ ]^2} \tag{1}$$

Let's just try to prove the last formula using induction.

The "base case" is pretty simple. For $n = 2$

$$1^3 + 2^3 = 1 + 8 = 9$$

and

$$\frac{n^2(n+1)^2}{2^2} = \frac{2^2(3^2)}{2^2} = 3^2 = 9$$

Check. Now for the induction step what we need to show is that what we get assuming the formula for n is correct and then adding the term $(n+1)^3$

$$\boxed{\frac{n^2(n+1)^2}{2^2} + (n+1)^3} \tag{2}$$

is equal to what we get by plugging $n+1$ into the formula.

$$\boxed{\frac{(n+1)^2(n+2)^2}{2^2}} \tag{3}$$

We need to show that eqn 2 is equal to eqn 3.

$$\frac{n^2(n+1)^2}{2^2} + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{2^2}$$

First, we can factor out and cancel $(n+1)^2$ from both sides. So then we have

$$\frac{n^2}{2^2} + (n+1) \stackrel{?}{=} \frac{(n+2)^2}{2^2}$$

$$n^2 + 4(n+1) \stackrel{?}{=} (n+2)^2$$

Sure, that looks correct! And we're done with the proof by induction, so we can put a little box.
$\blacksquare$

Looking deeper

$$\sum_{k=1}^{n} k^3 = [\ \sum_{k=1}^{n} k\ ]^2$$

I wanted to try to understand something more about why this is true. A simple web search revealed the answer. Here's an interesting pattern for the cubes of integers that I'd never seen before

$$1^3 = 1$$
$$2^3 = 8 = 3 + 5$$
$$3^3 = 27 = 7 + 9 + 11$$
$$4^3 = 64 = 13 + 15 + 17 + 19$$
$$5^3 = 125 = 21 + 23 + 25 + 27 + 29$$

If you want a formula for n^3, notice that the first term is $n^2 - n + 1$ and the last term is $n^2 - n + 2n - 1$, and the number of terms for each sum equals n. (There are n odd numbers between 1 and $2n - 1$).

In other words, the sum of all the cubes of integers from 1^3 to n^3 is equal to the sum of all the odd numbers up to $n^2 - n + 2n - 1 = n^2 + n - 1$.

How many of these numbers are there? A little thought should convince you that the correct answer is $(n^2 + n)/2$. For example, with $n = 5$, our last odd number is $5^2 + 5 - 1 = 29$, and we have $(25 + 5)/2 = 15$ terms.

We want the sum of the first $(n^2 + n)/2$ odd numbers.

Let's look at another pattern

$$1 = 1$$

$$2^2 = 4 = 1 + 3$$

$$3^2 = 9 = 1 + 3 + 5$$

$$4^2 = 16 = 1 + 3 + 5 + 7$$

$$5^2 = 25 = 1 + 3 + 5 + 7 + 9$$

The *odd number theorem* says that the sum of the first n odd numbers is equal to n^2. We want the sum of the first $(n^2 + n)/2$ odd numbers, so that's $((n^2 + n)/2)^2$. And that's how we get our formula.