Irrational numbers

continued fractions

Square roots can be represented as continued fractions. Some smart person figured out that we can write this:

$$(\sqrt{2}-1)(\sqrt{2}+1) = 2-1 = 1$$

Now, rearrange to get a substitution we will use repeatedly

$$\sqrt{2}-1 = \frac{1}{\sqrt{2}+1}$$

Add one and subtract one on the bottom right:

$$\sqrt{2}-1 = \frac{1}{2+\sqrt{2}-1}$$

And substitute for $\sqrt{2}-1$:

$$= \frac{1}{2+\frac{1}{\sqrt{2}+1}}$$

Lather, rinse, and repeat:

$$= \frac{1}{2+\frac{1}{2+\sqrt{2}-1}} = \frac{1}{2+\frac{1}{2+\frac{1}{\sqrt{2}+1}}}$$

Clearly, this goes on forever.

$$\sqrt{2} - 1 = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$$

Add 1 to the value of the *continued fraction* to get an expression for the square root of 2.

The numerators are all 1, so this is called a simple continued fraction. The continued fraction representation of $\sqrt{2}$ is usually written as $[1 : 2]$, meaning that there is an initial 1 followed by repeated 2's.

This fraction goes on forever (since $\sqrt{2}$ is irrational). One can view the existence of the infinite continued fraction as a proof of irrationality.

We can turn the above into an approximate decimal representation of $\sqrt{2}$, by truncating the infinite expansion at the .... Then the last fraction is 5/2. Invert and add, repeatedly:

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2 + 1/2 = 5/2
2 + 2/5 = 12/5
2 + 5/12 = 29/12
2 + 12/29 = 70/29
2 + 29/70 = 169/70
2 + 70/169 = 408/169
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To terminate we need to use that initial 1:

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1 + 169/408 = 577/408 = 1.414216
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To six places, $\sqrt{2} = 1.414213$. We have five places, and can easily get more.

approximations

In all cases we write particular real numbers as *approximations*. For example, the square root of 2 lies between 1 and 2 because

$$1^2 = 1 < 2$$

$$2^2 = 4 > 2$$

Implying that $\sqrt{2} < 2$. At the second place:

$$1.4^2 = 1.96 < 2$$

$$1.5^2 = 2.25 > 2$$

Implying that $\sqrt{2} < 1.5$. At the third:

$$1.41^2 = 1.9881 < 2$$

$$1.42^2 = 2.0164 > 2$$

Implying that $\sqrt{2} < 1.42$.

This process may be continued for as long as desired.

We can never write down the decimal value of $\sqrt{2}$ exactly, but only approximate it to greater and greater precision. It goes on forever.

In carrying out this recursive process, suppose we know 1.41 and we seek the next digit. Rather than try all the digits in order starting with 1, there is a better way.

Try to estimate the error from the previous round.

For example $1.41^2 = 1.9881$ so we are short of 2.0000 by 0.0119.

$1.42^2 = 2.0164$ so the difference is 0.0283 and the fraction of the difference that we're under is $119/283 = 0.4205$. In fact, the next two digits of the approximation to $\sqrt{2}$ are 42.

However, we will see a much better method for obtaining this value later, called Newton's method.

At the seventh place

$$1.414213^2 = 1.9999984093689998.. < 2$$

$$1.414214^2 = 2.0000012377960004 > 2$$

Because any repeating decimal can be written as a fraction, we know that the sequence cannot repeat (any apparent repeat will be illusory).

It is a curious fact that all the digits of π, *to whatever accuracy you desire*, can be found in the correct order, somewhere within the digital expansion of e or ϕ or indeed, any irrational number. The converse is also true.

Another way to say the same thing is that *any* finite sequence can be found within *any* infinite sequence, and in as many copies as you have the patience to discover. The sequence 271828 is found starting around digit 33,790 of π, but 2718281 (adding the next digit of e) is not found within the first million digits of π. You just need more.

limit of a sequence

The real number $\sqrt{2}$ is defined to be the limit of the sequence

$1.4, 1.41, 1.414, \dots 1.414214 \dots$

as the number of terms $n \to \infty$.

In a similar way, the number e can be viewed as

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n$$

And the number π can be viewed as the limit of the method of exhaustion applied to the area of a unit circle.