Limit Problems

Factoring

Factoring is OK when evaluating a limit. For example

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x}$$

Plugging in $x = 2$ gives $0/0$. We notice we can factor the numerator and the denominator

$$= \lim_{x \to 2} \frac{(x+6)(x-2)}{x(x-2)}$$

Now, since we are *not interested* in what happens at $x = 2$, we can cancel here, giving

$$= \lim_{x \to 2} \frac{(x+6)}{x} = \frac{8}{2} = 4$$

Conjugate

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$$

Substitution gives $0/0$. We try multiplication by the conjugate

$$\frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{x} - 2}{x - 4} \; \frac{\sqrt{x} + 2}{\sqrt{x} + 2}$$

The numerator gives $x - 4$, and before you rush to do the denominator, notice the cancellation:

$$= \frac{x - 4}{(x - 4)(\sqrt{x} + 2)}$$

So this is just

$$\lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4}$$

Polynomial as $x \to \infty$

$$\lim_{x\to\infty} 3x^2 + 2x + 1$$
$$= \lim_{x\to\infty} x^2(3 + \frac{2}{x} + \frac{1}{x^2})$$
$$= \lim_{x\to\infty} 3x^2 = \infty$$

This method can be adapted to more complex examples.

$$= \lim_{x\to\infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7}$$
$$= \lim_{x\to\infty} \frac{(2 - \frac{1}{x^2} + \frac{8}{x^3})(x^4)}{(-5 + \frac{7}{x^4})(x^4)}$$
$$= \lim_{x\to\infty} \frac{(2 - 0 + 0)(x^4)}{(-5 + 0)(x^4)} = -\frac{2}{5}$$

And one with a square root

$$\lim_{x\to\infty} \frac{\sqrt{3x^2 + 6}}{5 - 2x}$$
$$= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{3 + \frac{6}{x^2}}}{x(\frac{5}{x} - 2)}$$
$$= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{3 + 0}}{x(0 - 2)}$$

But we have to be careful here because

$$\sqrt{x^2} = |x|$$

So this is

$$= -\frac{\sqrt{3}}{2} \lim_{x\to\infty} \frac{|x|}{x}$$

Now, if $x > 0$, then the fraction under the limit sign is 1, but if $x < 0$, then it's -1. So there are two limits for these two cases.

Sine

As you know, $\sin x$ does not approach any limit because it is periodic. So what about

$$\lim_{x \to 0} \ x \sin x$$

We may guess that since the right-hand term is always a number between -1 and 1, when multiplied by 0 we'll get 0. The way to do this is to use the "squeeze" theorem.

$$-1 \leq \sin x \leq 1$$

When we multiply by x, we have to take account of sign. So let's do the two cases separately. For $x > 0$, we obtain

$$-x \leq x \sin x \leq x$$

Since both $-x$ and x go to 0, so does $x \sin x$. Suppose $x < 0$. Then, when we multiply we have to flip the inequality

$$-x \geq x \sin x \geq x$$

but it doesn't matter because both left and right-hand terms still tend to 0, and since $x \sin x$ is squeezed between them, it does too.