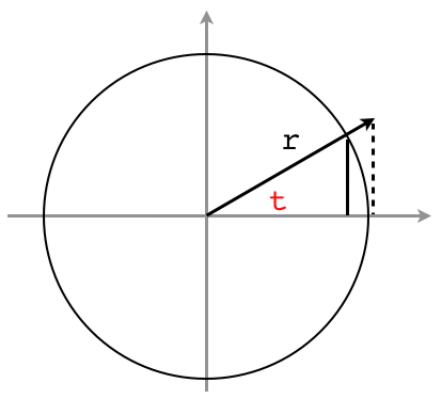
The limit of x/sin x

The fundamental result of calculus with respect to trigonometric functions depends on

$$\lim_{x \to 0} \frac{x}{\sin x}$$

The ratio of the angle to its sine as the angle gets very small. Here is a simple proof that the ratio is equal to 1.



Consider the triangle that lies entirely inside the circle. Its base is $r \cos t$ and its height is $r \sin t$, so its area is

$$A = \frac{1}{2} \cdot r \cos t \cdot r \sin t = \frac{1}{2} r^2 \sin t \cos t$$

Consider next the segment of the circle with angle t. Its area is that fraction of 2π times the total area

$$A = \frac{t}{2\pi} \pi r^2 = \frac{1}{2} r^2 t$$

Finally, consider the dotted line. By similar triangles its length is in the same ratio to r (which is the base of that triangle), as sine is to cosine of the angle. That is, its length is $r \tan t$ and so the area of the triangle is

$$A = \frac{1}{2} \cdot r \cdot r \tan t = \frac{1}{2} r^2 \frac{\sin t}{\cos t}$$

Since these areas get progressively larger

$$\frac{1}{2} r^2 \sin t \cos t < \frac{1}{2} r^2 t < \frac{1}{2} r^2 \frac{\sin t}{\cos t}$$

Now divide by $(1/2) r^2 \sin t$

$$\cos t < \frac{t}{\sin t} < \frac{1}{\cos t}$$

Now, as $t \to 0$, both $\cos t$ and $1/\cos t$ also go to 1. Therefore the ratio gets squeezed, and it goes to 1 as well. Thus

$$\lim_{x \to 0} \frac{x}{\sin x} = 1$$

There is another limit that also comes up in the basic derivations, but it turns out to be related to this one. That limit is: