Stokes Theorem and the divergence

Stokes

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, dS$$

Stokes' theorem applies to a curve in space (it does not have to lie in a plane). The theorem says that the work going around the curve is equal to the integral over *any surface* with that curve as its boundary, of the component of the curl of $\mathbf{F}$ normal to the surface.

An alternative form (using the fact that $\mathbf{dr} = \langle dx, dy, dz \rangle$, and computing the curl $\nabla \times \mathbf{F}$) is

$$\int_C P \, dx + Q \, dy + R \, dz = \iint_R \; < R_y - Q_z, P_z - R_x, Q_x - P_y > \; \cdot \hat{\mathbf{n}} \, dS$$

When Stokes' theorem is applied to a region in the xy-plane, $\hat{\mathbf{n}} = \hat{\mathbf{k}}$, so only the third term of the curl is non-zero, and $dS = dA$. On the left-hand side $dz = 0$, so this just reduces to

$$\int_C P \, dx + Q \, dy = \iint_R \, (Q_x - P_y) \, dA$$

which is Green's theorem for work.

Divergence in space

The last theorem is called the divergence theorem, or Gauss's theorem (and there are other names).

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_V \nabla \cdot \mathbf{F} \, dV$$
$$= \iiint_V (P_x + Q_y + R_z) \, dV$$

As a simple example, consider the unit sphere and a radial field $\mathbf{F} = \langle x, y, z \rangle$. We can certainly do the surface integral, but notice that $\nabla \cdot \mathbf{F} = 3$. Using the theorem, the result (the flux out of the sphere) is simply 3 times the volume of the unit sphere, or 4π.

Other coordinates

Above, in $R3$, the divergence was given as

$$\iiint_V \nabla \cdot \mathbf{F} \, dV$$

The "del" operator is

$$\nabla = < \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >$$

The divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F}$$

if $\mathbf{F} = < P, Q, R >$

$$\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$$

The divergence of a vector field is a scalar quantity.

In cylindrical and spherical coordinates, the divergence is more complicated. Although the expression above is often given as the *definition* of divergence, Schey makes a big deal out of saying that he prefers this definition

$$div \; \mathbf{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS$$

As further illustration, the divergence has a more complicated form in cylindrical and spherical coordinates. In the former it is

$$div\ \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(F_\theta) + \frac{\partial}{\partial z}(F_z)$$

where F_z is not a partial derivative but just the component of $\mathbf{F}$ in the z-direction, and so on.

Similarly, in spherical coordinates the divergence is

$$div\ \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\ \sin\phi}\frac{\partial}{\partial \phi}(\sin\phi F_\phi) + \frac{1}{r\ \sin\phi}\frac{\partial}{\partial \theta}(F_\theta)$$

example

Here is an example to explore divergence in cylindrical coordinates. Suppose we have a cylinder oriented along the z-axis with its length equal to 1 and its radius $r = 2$.

Further, we have a field with some divergence, like $\mathbf{F} = \langle x, y, 0 \rangle$.

This field is written in x, y, z coordinates, when we switch to cylindrical coordinates (of course) $x = r\cos\theta$ and $y = r\sin\theta$.

We want to check the divergence theorem

$$\iint_S \mathbf{F} \cdot \mathbf{n}\ dS = \iiint_V \nabla \cdot \mathbf{F}\ dV$$

(The integral above is an integral over a closed surface, in this case, the cylinder with top and bottom included).

When we rewrite $\mathbf{F}$ in cylindrical coordinates, we will have

$$\mathbf{F} = \langle r, \theta, z \rangle$$

The given field is independent of θ and z and and since

$$r = \sqrt{x^2 + y^2}$$

$$\mathbf{F} = \langle r, 0, 0 \rangle$$

Using the definition of divergence from above, we have

$$div\ \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(F_\theta) + \frac{\partial}{\partial z}(F_z)$$

since $\mathbf{F}$ has no z or θ component and the r component is $F_r = r$ (this is *not* a derivative), we have

$$div\ \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(r^2) = 2$$

So the triple integral uses the cylindrical volume element and is just

$$\int_0^{2\pi} \int_0^1 \int_0^2 2\ r\ dr\ dz\ d\theta$$

$$= \int_0^{2\pi} \int_0^1 [\ r^2 \Big|_0^2\ ]\ dz\ d\theta = 8\pi$$

Notice that the value of the integral scales linearly with z and like r^2. Now for the surface integral. In standard form, the cylinder has

$$\hat{\mathbf{n}}\ dS = \langle x, y, 0 \rangle\ d\theta\ dz$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\ dS = \langle x, y, 0 \rangle \cdot \langle x, y, 0 \rangle\ d\theta\ dz$$

$$\iint_S x^2 + y^2\ d\theta\ dz$$

$$\int_0^1 \int_0^{2\pi} r^2\ d\theta\ dz = 2\pi r^2 = 8\pi$$

I almost forgot the top and bottom of the cylinder. However the flux $\mathbf{F} \cdot \hat{\mathbf{n}} = 0$ everywhere on these two surfaces, so the total is still just 8π.