Kepler's Laws

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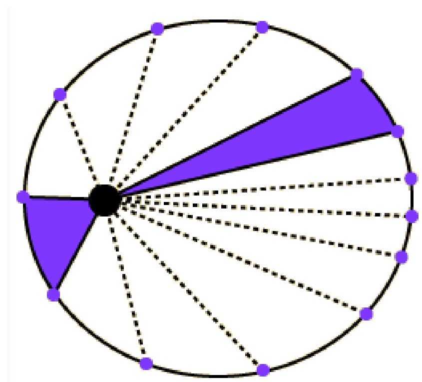
Chapter 1

Introduction



I want to understand in detail how Kepler's Laws for the orbits of the planets can be derived from Newton's Laws, namely

$$\mathbf{F} = m\mathbf{a}$$

as well as the inverse square law of gravitation.

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{u}}$$

I've worked through derivations from several sources, and I think now that I have it figured out. My goal is to clearly document things here so that I will understand in a year (or five) when I re-read this.

Kepler's Laws are:

◦ **K1**: the orbits of the planets are not circles but ellipses (non-recurrent orbits may be other conic sections);

◦ **K2**: the area or arc ”swept out” per unit time is the same no matter where in the orbit the planet is; and

◦ **K3**: the period of the orbit is independent of the mass of the planet and its square is proportional to the cube of the length of the semi-major axis of the ellipse.

I also spent some time working on Newton's version of the proof as presented in the *Principia* (see Bressoud's vector calculus book), but he leaves out too many steps. There is also a version ”cooked up” by Richard Feynman and discussed in a book called *Feynman's Lost Lecture*.

I never got either of these figured out, but if you want to go this route I recommend starting with Feynman.

For myself, I found that once I cleared up a couple of subtleties, and verified the application of the product rule for differentiation to vector cross-products, it was pretty easy.

eccentricity

The equation of an ellipse in xy-coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a is one-half the "diameter" in the long dimension and b is one-half the length perpendicular to that.

A second way to describe the shape of an ellipse is to give the focal length, the distance of each of the two foci from the center of the ellipse.

$$f = \sqrt{a^2 - b^2}$$

If we scale things so that $b = 1$ then a is determined.

Yet another way to give the shape is called its *eccentricity*, e, where

$$ea = f$$

Then

$$e^2 a^2 = a^2 - b^2$$

$$(1 - e)\ a = b$$

Here is a table of planetary eccentricities I found on the web.

Planets: Orbital Properties

Planet	distance	revolution	eccentricity	inclination
	(A.U.)			(deg)
Mercury	0.387	87.969 d	0.2056	7.005
Venus	0.723	224.701 d	0.0068	3.3947
Earth	1.000	365.256 d	0.0167	0.0000
Mars	1.524	686.98 d	0.0934	1.851
Jupiter	5.203	11.862 y	0.0484	1.305
Saturn	9.537	29.457 y	0.0542	2.484
Uranus	19.191	84.011 y	0.0472	0.770
Neptune	30.069	164.79 y	0.0086	1.769
Pluto	39.482	247.68 y	0.2488	17.142

Mars is the planet that showed Kepler most clearly that the orbits are not circles, but its eccentricity is only 0.09. For Earth this value is only 0.017 which gives a focal length of roughly

$$0.0167 \times 149.6 \times 10^6 \text{ km} \approx 2.5 \times 10^6 \text{ km}$$

which is about four times the radius of the Sun.

Chapter 2

Newton

Here, we look at the geometric proof of K2 used by Newton.

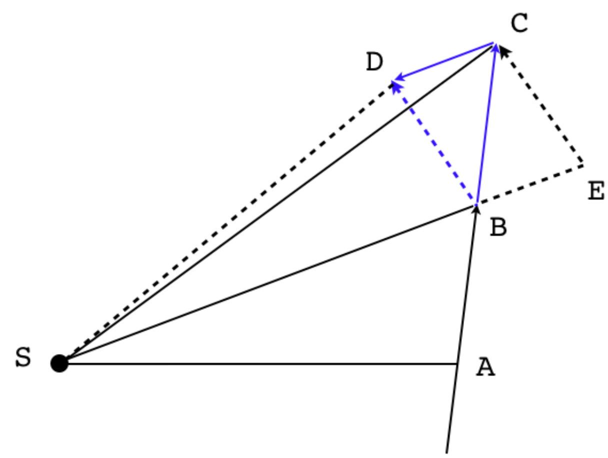
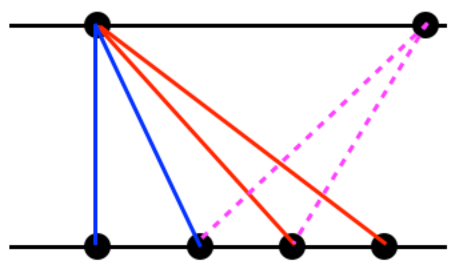


Diagram the sun S and a planet at A.

Imagine that the force toward the sun is applied discretely. That is, during a small interval Δt, the planet travels from A to B at constant velocity and if undisturbed, would travel to C in the next unit of time.

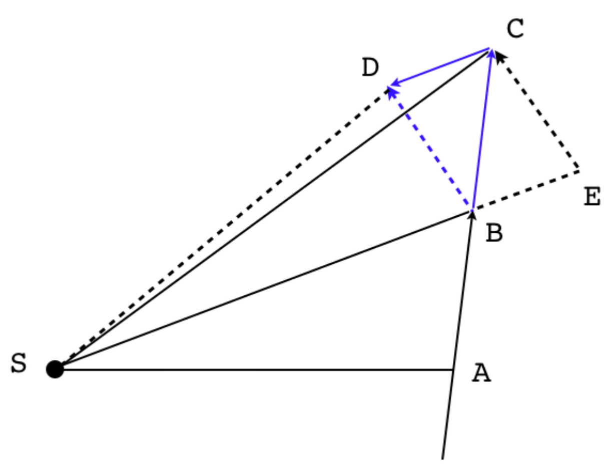
In the absence of a force, the velocity would be constant and the length of AB the same as that of BC, and then since AB is on the same line as BC, the area of $\triangle ABS$ is equal to the area of $\triangle BCS$.

Proof: draw the vertical line from S to the line containing ABC. The area of either triangle is one-half the length of that altitude times the distance, either AB or BC. The principle is illustrated in the next figure.



Given two parallel lines separated by a distance h, pick two points on one line separated by a distance d and *any* point on the other line. The triangles drawn using those points will all have equal area, namely $(1/2)dh$.

Now, suppose the force is applied at B *toward the sun* along EBS.



As a result, the trajectory BC is modified by the change in velocity resulting from application of the force toward the sun. The new path is the additional velocity times Δt. Call the length CD and add it to BC to give the actual trajectory, BD.

CD is parallel to SBE. Therefore, every point on CD has an altitude with respect to SBE of the same length. So any point on CD can be used to draw a triangle with the same base SB and the result will have the equal area no matter which point is chosen.

In particular, the area of $\triangle BDS$ is equal to the area of $\triangle BCS$, which was found earlier to be equal to the area of $\triangle ABS$. Since the two triangles from the actual motion have the same area, the area is constant.

Chapter 3

Feynman

Richard Feynman gave a famous series of talks at Cornell in 1964 that were videotaped and transcribed into a book. Bill Gates later purchased them and put them on the web, unfortunately with some Microsoft DRM.

Still, I have the book, called *The Character of Physical Law*. This argument is from Chapter 2, *The Relation of Mathematics to Physics*.

It depends on a tiny bit of calculus. Specifically, it uses the product rule for differentiation, plus the fact that the product rule is valid for vector cross products. Here is a short write-up on the subject, from my book *Best of Calculus*.

time-derivative of products

To take the derivative with respect to the parameter (such as time), we just go through each component of a vector one at a time:

$$\mathbf{r} = \langle x, y, z \rangle$$

$$\frac{d\mathbf{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$$

The question arises, what about products? Is there a product rule for vectors? Here's an example

$$\frac{d}{dt}\, \mathbf{r} \cdot \mathbf{v}$$

It turns out that there is.

$$\frac{d}{dt}\, \mathbf{r} \cdot \mathbf{v} = \frac{d\mathbf{r}}{dt} \cdot \mathbf{v} + \mathbf{r} \cdot \frac{d\mathbf{v}}{dt}$$

The reason is that the individual components of the dot product are simple functions of t, and our rule is to differentiate one component at a time.

Let's use Newton's dot notation for d/dt:

$$\mathbf{r} = \langle x, y, z \rangle$$

$$\dot{\mathbf{r}} = \mathbf{v} = \langle \dot{x}, \dot{y}, \dot{z} \rangle$$

$$\mathbf{r} \cdot \dot{\mathbf{r}} = x\dot{x} + y\dot{y} + z\dot{z}$$

The derivative is

$$\begin{aligned} \frac{d}{dt}\mathbf{r} \cdot \dot{\mathbf{r}} &= \frac{d}{dt}(x\dot{x} + y\dot{y} + z\dot{z}) \\ &= \dot{x}\dot{x} + x\ddot{x} + \dot{y}\dot{y} + y\ddot{y} + \dot{z}\dot{z} + z\ddot{z} \\ &= \dot{x}\dot{x} + \dot{y}\dot{y} + \dot{z}\dot{z} + x\ddot{x} + y\ddot{y} + z\ddot{z} \\ &= \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \mathbf{r} \cdot \ddot{\mathbf{r}} \end{aligned}$$

The same is true of the cross-product. The torque is $\mathbf{F} \times \mathbf{r}$ Let's take the derivative:

$$\frac{d}{dt}\, [\ \mathbf{F} \times \mathbf{r}\ ]\ =$$

Let's write $\mathbf{F} = \langle M, N, P \rangle$, then the cross-product gives a vector with components

$$[\ Nz - Py\ ]\ \hat{\mathbf{i}} + \ [\ Px - Mz\ ]\ \hat{\mathbf{j}} + \ [\ My - Nx\ ]\ \hat{\mathbf{k}}$$

where all of M, N, P and x, y, z are *functions* of time.

The time derivative is obtained by the product rule. Again, I will use dots, and here we separate the components onto different lines:

$$\begin{aligned} &[\ \dot{N}z + N\dot{z} - \dot{P}y - P\dot{y}\ ]\ \hat{\mathbf{i}} + \\ &+ [\ \dot{P}x + P\dot{x} - \dot{M}z - M\dot{z}\ ]\ \hat{\mathbf{j}} + \\ &+ [\ \dot{M}y + M\dot{y} - \dot{N}x - N\dot{x}\ ]\ \hat{\mathbf{k}} \end{aligned}$$

but this is just two different cross-products added together. The first one is

$$[\ \dot{N}z - \dot{P}y\ ]\ \hat{\mathbf{i}} + \ [\ \dot{P}x - \dot{M}z\ ]\ \hat{\mathbf{j}} + \ [\ \dot{M}y - \dot{N}x\ ]\ \hat{\mathbf{k}}$$
$$= \dot{\mathbf{F}} \times \mathbf{r}$$

and the second is:

$$[\ N\dot{z} - P\dot{y}\ ]\ \hat{\mathbf{i}} + \ [\ P\dot{x} - M\dot{z}\ ]\ \hat{\mathbf{j}} + \ [\ M\dot{y} - N\dot{x}\ ]\ \hat{\mathbf{k}}$$
$$= \mathbf{F} \times \dot{\mathbf{r}}$$

Putting it all together

$$\frac{d}{dt}\ [\ \mathbf{F} \times \mathbf{r}\ ]\ = \dot{\mathbf{F}} \times \mathbf{r} + \mathbf{F} \times \dot{\mathbf{r}}$$

The product rule for differentiation holds for both the dot product and the cross-product.

Do the dots

The rule is that if we have two vectors $\mathbf{a}$ and $\mathbf{b}$ which are changing (i.e. they are functions of time), then

$$\frac{d}{dt}\ (\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}$$

In our application the two vectors are the position vector of the planet with respect to the sun, $\mathbf{r}$, and the time-derivative of that vector.

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$

Or, using Newton's dot notation for the time-derivative:

$$\mathbf{v} = \dot{\mathbf{r}}$$

We are interested in the area of the triangle formed by the vectors $\mathbf{r}$ and $\dot{\mathbf{r}}$ over a small interval of time. The area swept out is constant, as Newton showed, and we will prove again here.

A nice feature of the vector cross-product is that it provides (twice) this area. Namely

$$A = \mathbf{r} \times \dot{\mathbf{r}} = |\mathbf{r}||\dot{\mathbf{r}}| \sin\theta$$

where θ is the angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$, and A is the little bit of additional area.

Our hypothesis is that A is the same no matter where the planet is in its orbit. Another way to say the same thing is that A doesn't change with time

$$\frac{d}{dt}\ A = \dot{A} = 0$$

Now

$$A = \mathbf{r} \times \dot{\mathbf{r}}$$

and we want to compute $\dot{A}$. Using the product rule it's easy.

$$\dot{A} = \frac{d}{dt}\ (\mathbf{r} \times \dot{\mathbf{r}})$$

$$\dot{A} = \dot{\mathbf{r}} \times \dot{\mathbf{r}}\ +\ \mathbf{r} \times \ddot{\mathbf{r}}$$

As Feynman says: it's just playing with dots. So let's look at those two terms. Another nice fact about the cross-product is that if the two vectors point in the same direction, then the cross-product is zero.

Any vector points in the same direction as itself, so the first term is certainly zero.

$$\dot{\mathbf{r}} \times \dot{\mathbf{r}} \ = 0$$

Next, recall that the second derivative with respect to time of the position is the acceleration vector. According to Newton's second law, the force of gravity points toward the sun, radially.

But of course the position vector also points out radially from the sun. $\mathbf{r}$ and $\ddot{\mathbf{r}}$ are in the same direction (the opposite direction *is* the same direction, multiplied by -1), so the cross-product is again zero.

$$\mathbf{r} \times \ddot{\mathbf{r}} \ = 0$$

So that means the whole thing is zero.

$$\dot{A} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} \ + \ \mathbf{r} \times \ddot{\mathbf{r}} = 0 + 0 = 0$$

We have shown that the area is constant, which is Kepler's second law. By the way, the invariant quantity

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}}$$

(times the mass) is the angular momentum, and the lack of change is the principle of the conservation of angular momentum.

Note: in this section we have followed Feynman (who was trying to make the argument as simple as possible). He called the area swept out in a small interval of time A and showed that $\dot{A} = 0$.

It would be more appropriate it to call it dA/dt, the instantaneous rate of change of the area. Then the quantity that is equal to zero is the time-derivative of *that*, namely: d^2A/dt^2.

Chapter 4

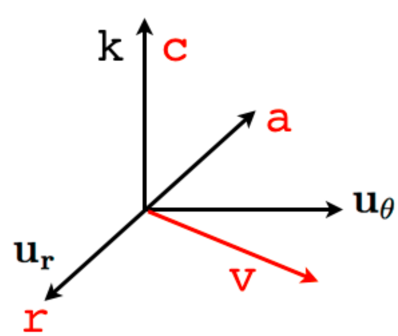
Axes

In this chapter, we introduce some notation and obtain the value of the thing that is constant.

The invariant quantity

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}}$$

(times the mass) is the angular momentum, and the lack of change is the principle of the conservation of angular momentum.



Here is a sketch of the situation. $\mathbf{r}$ is the position vector, extending radially out from the sun to the planet. $\mathbf{u_r}$ is a unit vector in the $\mathbf{r}$

direction, so that

$$\mathbf{r} = r\mathbf{u_r}$$

As the planet moves, it makes an angle θ with the x-axis, which changes with time.

By the central force hypothesis, the acceleration $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$ is in the $-\mathbf{u_r}$ direction. The source of complexity is that the velocity $\mathbf{v} = \dot{\mathbf{r}}$ is not perpendicular to $\mathbf{u_r}$.

Earlier we proved that

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{c}$$

is a constant. Here we give that constant vector the label $\mathbf{c}$.

The initial motion is in the xy-plane but so is the acceleration, thus all the motion takes place in the xy-plane.

By the properties of the cross product, $\mathbf{c}$ is aligned with $\hat{\mathbf{k}}$.

Define $\mathbf{u}_\theta$ as orthogonal to $\mathbf{u_r}$ (and to $\hat{\mathbf{k}}$).

As a result of these definitions:

$$\mathbf{u_r} \times \mathbf{u}_\theta = \hat{\mathbf{k}}$$

$$\hat{\mathbf{k}} \times \mathbf{u_r} = \mathbf{u}_\theta$$

$$\mathbf{u}_\theta \times \hat{\mathbf{k}} = \mathbf{u_r}$$

If you reverse the order of any of the pairs, the result switches sign.

At any given time, $\mathbf{r}$ makes an angle θ with the x-axis, and the position is at a distance r from the origin, and we write:

$$\mathbf{r} = \ \langle r\cos\theta, r\sin\theta, 0\rangle = r\ \mathbf{u_r}$$

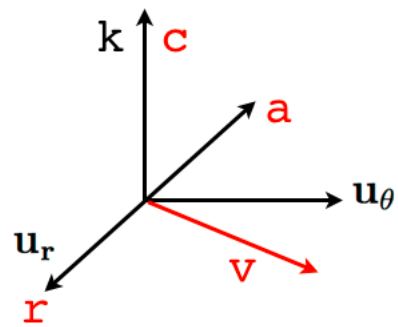
so

$$\mathbf{u_r} = \ \langle \cos\theta, \sin\theta, 0\rangle$$

$$\mathbf{u}_\theta \perp \mathbf{u_r}$$

$$\mathbf{u}_\theta = \ \langle -\sin\theta, \cos\theta, 0\rangle$$

Verify that the dot-product is zero and that both vectors are unit length.



Now, differentiate $\mathbf{u_r}$ and $\mathbf{u}_\theta$ (realizing that θ is also a function of time):

$$\dot{\mathbf{u}}_\mathbf{r} = \frac{d}{dt}\,[\mathbf{u_r}\,]$$

$$= \frac{d}{dt}\,\langle\cos\theta, \sin\theta, 0\rangle$$

$$= \langle -\sin\theta, \cos\theta, 0\rangle \cdot \frac{d\theta}{dt}$$

$$= \mathbf{u}_\theta \cdot \frac{d\theta}{dt}$$

and

$$\frac{d}{dt}\,\mathbf{u}_\theta = \dot{\mathbf{u}}_\theta$$

$$= \langle -\cos\theta, -\sin\theta, 0\rangle \cdot \frac{d\theta}{dt}$$

$$= -\mathbf{u_r} \cdot \frac{d\theta}{dt}$$

With appropriate choice of units for time t, we can have $\theta = t$, so all of these factors of $d\theta/dt = 1$ so $\dot{\mathbf{u}}_\mathbf{r} = \mathbf{u}_\theta$ and $\dot{\mathbf{u}}_\theta = -\mathbf{u}_\mathbf{r}$.

calculation

We can get a parametric expression for the velocity

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}\ (r\mathbf{u}_\mathbf{r}) = \frac{dr}{dt}\mathbf{u}_\mathbf{r} + r\frac{d\theta}{dt}\mathbf{u}_\theta$$

and (with a little more work) we can get the acceleration

$$\begin{aligned}\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} \\ = \frac{d}{dt}\ (\frac{dr}{dt}\mathbf{u}_\mathbf{r} + r\frac{d\theta}{dt}\mathbf{u}_\theta) \\ = \frac{d^2r}{dt^2}\mathbf{u}_\mathbf{r} + \frac{dr}{dt}\dot{\mathbf{u}}_\mathbf{r} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + r\frac{d\theta}{dt}\dot{\mathbf{u}}_\theta\end{aligned}$$

We get three terms from differentiating the triple product $r\ d\theta/dt\ \mathbf{u}_\theta$, by a variation on the product rule.

Substitute for the dotted terms from above

$$= \frac{d^2r}{dt^2}\mathbf{u}_\mathbf{r} + \frac{dr}{dt}\frac{d\theta}{dt}\ \mathbf{u}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta - r\frac{d\theta}{dt}\frac{d\theta}{dt}\ \mathbf{u}_\mathbf{r}$$

Group common terms together

$$= (\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2)\mathbf{u}_\mathbf{r} + (2\ \frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2})\mathbf{u}_\theta$$

Now for a nice simplification, look at the factors multiplying $\mathbf{u}_\theta$ and recognize that

$$r\ [\ 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\ ]\ = \frac{d}{dt}(r^2\frac{d\theta}{dt})$$

Therefore, the cofactors for the acceleration in the $\mathbf{u}_\theta$ direction can be re-written as

$$\frac{1}{r}\ [\ \frac{d}{dt}(r^2\frac{d\theta}{dt})\ ]$$

and since if the acceleration is to be only radial (pointed toward the sun), there is no torque (no θ component) and this term must be equal to zero.

$$\frac{1}{r}\ [\ \frac{d}{dt}(r^2\frac{d\theta}{dt})\ ] = 0$$

$$\frac{d}{dt}(r^2\frac{d\theta}{dt}) = 0$$

$$r^2\frac{d\theta}{dt} = h = \text{constant}$$

If we write $d\theta/dt = \omega$, the angular velocity, then $r\omega$ is the speed of the planet, and r times that, times the mass, is the angular momentum. This result is the conservation of angular momentum.

advantage of dot notation

It's interesting to see how much more compact and thus readable dot notation for all the above is. By the definition of $\mathbf{u}_\theta$ above we had that

$$\dot{\mathbf{u}}_r = \mathbf{u}_\theta\dot{\theta}$$

$$\dot{\mathbf{u}}_\theta = -\mathbf{u}_r\dot{\theta}$$

so velocity is:

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}\ (r\mathbf{u_r})$$

$$= \dot{r}\mathbf{u_r} + r\dot{\mathbf{u}}_r = \dot{r}\mathbf{u_r} + r\mathbf{u}_\theta\dot{\theta}$$

and acceleration:

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{d}{dt}\ [\ \dot{r}\mathbf{u_r} + r\mathbf{u}_\theta\dot{\theta}\ ]$$

$$= \ddot{r}\mathbf{u_r} + \dot{r}\dot{\mathbf{u}}_\mathbf{r} + \dot{r}\mathbf{u}_\theta\dot{\theta} + r\dot{\mathbf{u}}_\theta\dot{\theta} \ + r\mathbf{u}_\theta\ddot{\theta}$$

$$= \ddot{r}\mathbf{u_r} + \dot{r}\mathbf{u}_\theta\dot{\theta} + \dot{r}\mathbf{u}_\theta\dot{\theta} - r\mathbf{u_r}\dot{\theta}^2 \ + r\mathbf{u}_\theta\ddot{\theta}$$
$$= \ddot{r}\mathbf{u_r} + 2\dot{r}\mathbf{u}_\theta\dot{\theta} - r\mathbf{u_r}\dot{\theta}^2 \ + r\mathbf{u}_\theta\ddot{\theta}$$

Easy to read, but a nightmare to typeset.

Chapter 5

Area again

At this point, we have almost all the tools we need to follow the derivation of Kepler's laws. We just need a bit more discussion of area "swept out" by a planet in a short time. Our approach is based on Varberg *Calculus* (online version only, Chapter 14).

We revisit the triangle formed by the motion of the planet, and confirm that twice the area of the triangle is equal to

$$h = r^2 \frac{d\theta}{dt}$$

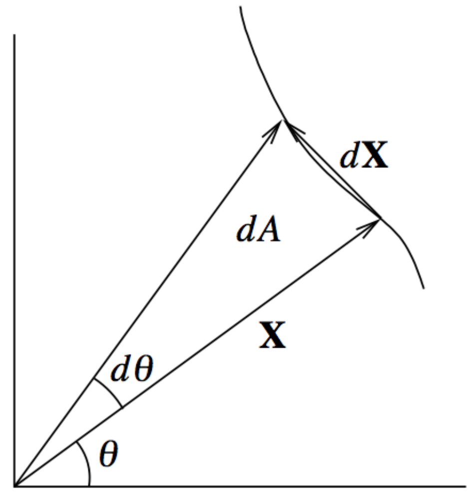
Whereas Feynman used the velocity for one of the sides of the triangle and showed that

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$$

is a constant, Varberg use $d\mathbf{r}$ for their triangle. It leads to some confusing aspects in the presentation, which I want to work through since I like everything else about their derivation except some of the notation.

Here is their diagram

Figure 14.3



They use $\mathbf{X}$ for the position vector, but I will label it as $\mathbf{r}$, following Feynman. This vector "stays always in the xy-plane." The reason is that the force is radial, so also confined to this plane.

I will use $\mathbf{u_r}$ for the vector they call $\mathbf{L}$ and similarly $\mathbf{u}_\theta$ for the vector they call $\mathbf{L}^\perp$.

We are asked to show that

$$2\,\frac{dA}{dt} = \left|\frac{d\mathbf{r}}{dt} \times \mathbf{r}\right|$$

This presentation is a bit different than Feynman's, and it confused me for a while, because, just for starters, we proved before that $dA/dt = 0$ but that will not be the case here, because now dA/dt means something different.

What I'm going to do is to change the notation and say that in a short time Δt, the area that is swept out is ΔA, corresponding to a length $d\mathbf{r} = \mathbf{v}\Delta t$, and that by the geometry we have

$$2\,\Delta A = |\mathbf{r} \times \mathbf{v}\Delta t|$$

I assert that it is OK to bring Δt out of the cross product (by the rule of scalar multiplication), since it is a scalar quantity, and is a constant at any stage of its future journey to the limit when $\Delta t \to 0$, so I write

$$2\ \Delta A = |\mathbf{r} \times \mathbf{v}|\Delta t$$

Now we have

$$2\ \frac{\Delta A}{\Delta t} = |\mathbf{r} \times \mathbf{v}|$$

and in the limit

$$2\ \frac{dA}{dt} = |\mathbf{r} \times \mathbf{v}|$$

Now this is not quite what we were asked to prove, the order is reversed, but recall that

$$\frac{d\mathbf{r}}{dt} \times \mathbf{r} = \dot{\mathbf{r}} \times \mathbf{r} = -\mathbf{r} \times \dot{\mathbf{r}}$$

so the absolute values are the same. Again, the result is that

$$2\ \frac{dA}{dt} = |\dot{\mathbf{r}} \times \mathbf{r}| = |\mathbf{r} \times \dot{\mathbf{r}}| = |\mathbf{h}| = h$$

If you're not completely happy with the argument allowing this step:

$$2\ \Delta A = |\mathbf{r} \times \mathbf{v}\Delta t| = |\mathbf{r} \times \mathbf{v}|\Delta t$$

recall that

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$$

so

$$|\mathbf{r} \times \mathbf{v}\Delta t| = |\mathbf{c}\Delta t| = h\Delta t$$

Varberg *et al.* also give a second argument which which we will go through because it gives us the term

$$r^2\ \frac{d\theta}{dt} = h$$

By the geometry of the triangle, the area is

$$2\ dA = r\ r d\theta = r^2 d\theta$$

where r is $|\mathbf{r}|$. And then they say

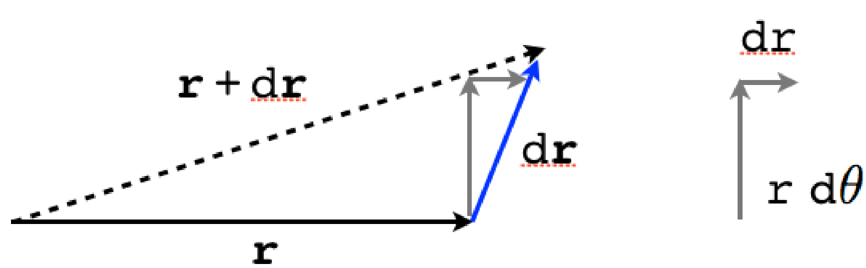
$$2\ \frac{dA}{dt} = r^2\ \frac{d\theta}{dt}$$

They do this without comment, but this result assumes that r does not vary with time, although clearly it does (that's the whole point of everything we are doing here). The product rule would give us:

$$\frac{d}{dt}\ r^2 d\theta = 2r\ d\theta\ \frac{dr}{dt} + r^2\ \frac{d\theta}{dt}$$

This looks a little weird because of the single differential $d\theta$, but I think what it means is that in the limit, the first term goes to zero.

Another way of explaining this is to break the area into two parts.



The first part is $(1/2)r$ times $r\ d\theta$, the length of (almost straight) arc perpendicular to $\mathbf{r}$. This is the part we get by assuming that r is constant. And in the limit as $t \to 0$, the resulting $d\theta/dt$ has some value, namely, the angular velocity.

The second part is $(1/2)r\ d\theta$ times dr. This is the part that accounts for the change in r, but it contains two differentials, only one of which gets rescued into some quantity by dt. The other just goes to zero, so the whole thing is zero.

Anyway, let's continue with the argument.

Go back to the right-hand side of what we were asked to prove above

$$2\ \frac{dA}{dt} = |\frac{d\mathbf{r}}{dt} \times \mathbf{r}|$$

and show that it turns into $r^2 d\theta/dt$. Using $\mathbf{u_r}$ for the unit vector in the $\mathbf{r}$ direction, we have

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\mathbf{u_r}) = \frac{dr}{dt}\mathbf{u_r} + r\dot{\mathbf{u}}_\mathbf{r}$$

where the first part is just separating the scalar and unit vector part of $\mathbf{r}$ and the rest is from the product rule. At this point we recall the result that $\dot{\mathbf{u}}_\mathbf{r} = d\theta/dt\ \mathbf{u}_\theta$, so we have

$$= \frac{dr}{dt}\mathbf{u_r} + r\frac{d\theta}{dt}\ \mathbf{u}_\theta$$

So now this is what we need to cross with $\mathbf{r}$, also known as $r\mathbf{u_r}$. We write

$$\begin{gathered}(\frac{dr}{dt}\mathbf{u_r} + r\frac{d\theta}{dt}\ \mathbf{u}_\theta) \times r\mathbf{u_r} \\ = (\frac{dr}{dt}\mathbf{u_r} \times r\mathbf{u_r}) + (r\frac{d\theta}{dt}\ \mathbf{u}_\theta \times r\mathbf{u_r})\end{gathered}$$

The first term is zero (the cross-product of $\mathbf{u_r}$ with itself), and because these are unit vectors the absolute value of the second term's vector cross-product is 1

$$|r\frac{d\theta}{dt}\ \mathbf{u}_\theta \times r\mathbf{u_r}| = r^2\frac{d\theta}{dt}\ |\mathbf{u}_\theta \times \mathbf{u_r}| = r^2\frac{d\theta}{dt}$$

So what we've shown is that

$$2\ \frac{dA}{dt} = |\mathbf{r} \times \dot{\mathbf{r}}|$$

and

$$2\ \frac{dA}{dt} = r^2\ \frac{d\theta}{dt}$$

This term ($r^2\ d\theta/dt$) is what Hartig calls c and the other guys call h. As the vector $\mathbf{h}$, it points in the $\hat{\mathbf{k}}$ direction and is the angular momentum but without the mass component.

Chapter 6

Kepler1

K2 says that orbits ”sweep out equal areas in equal times” or equivalently, that $\ddot{A} = d^2A/dt^2 = 0$. So we start from Newton's force directed toward the sun, the centripetal force, and we will show that the motion stays in a plane, and also implies K2.

Again, this is Feynman's argument (and notation)

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = 0$$

This is zero because you get two terms from the derivative of the cross-product: one is $\dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$, and the second one is $\mathbf{r} \times \ddot{\mathbf{r}}$, which is zero because these two vectors point in opposite directions by the centripetal force postulate. Therefore, $\mathbf{r} \times \dot{\mathbf{r}}$ is constant. We will say that

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$$

$$|\mathbf{h}| = h = 2\,\frac{dA}{dt}$$

If $\mathbf{h} = 0$, there is no force, and just straight-line motion. But for $\mathbf{h} \neq 0$, then $\mathbf{r}$ and $\dot{\mathbf{r}}$ are in a plane that doesn't change with time, and $\mathbf{h}$ is the normal vector of that plane.

$$h = |\mathbf{r} \times \dot{\mathbf{r}}| = |\mathbf{r} \times \frac{d\mathbf{r}}{dt}|$$

by the discussion about area previously (Varberg's Example 14.5)

$$h = r^2 \frac{d\theta}{dt} = 2 \frac{dA}{dt}$$

This is the statement of K2. dA/dt is constant, equal to $h/2$.

At this point, Varberg reverse the argument and show that planar motion and K2 imply a centripetal force. But this is just Feynman's dots, which we already went through.

Kepler's First Law K1

Now, we make an additional hypothesis due to Newton, which is that the acceleration is proportional to the inverse square of the distance from the sun (origin), and pointed toward it.

$$\mathbf{a} = \ddot{\mathbf{r}} = -\frac{GM}{r^2}\mathbf{u_r}$$

where (as before) $\mathbf{u_r}$ is the unit vector in the $\mathbf{r}$ direction (i.e. equal to $\mathbf{r}/|\mathbf{r}|$), and GM is a constant.

The first of three main steps in the proof is to take the cross-product with $\hat{\mathbf{k}}$ (as the text says, "this allows us to introduce the area information in vectorial form")

$$\ddot{\mathbf{r}} \times \hat{\mathbf{k}} = -\frac{GM}{r^2} \mathbf{u_r} \times \hat{\mathbf{k}} = \frac{GM}{r^2} \mathbf{u}_\theta$$

(recall that we "go to the left" for $\mathbf{u}_\theta$).

It should not be surprising that the cross-product $\mathbf{u_r} \times \hat{\mathbf{k}}$ brings us back to $-\mathbf{u}_\theta$, since we defined

$$\mathbf{u_r} \times \mathbf{u}_\theta = \hat{\mathbf{k}}$$

$$\hat{\mathbf{k}} \times \mathbf{u_r} = \mathbf{u}_\theta$$

and thus

$$\mathbf{u_r} \times \hat{\mathbf{k}} = -\mathbf{u}_\theta$$

From our discussion of the unit vectors and parametrization,

$$\frac{d}{dt}\ \mathbf{u_r} = \dot{\mathbf{u}}_\mathbf{r} = \frac{d\theta}{dt}\ \mathbf{u}_\theta = \omega \mathbf{u}_\theta$$

$$\mathbf{u}_\theta = \frac{\dot{\mathbf{u}}_\mathbf{r}}{\omega}$$

and from K2

$$\frac{d\theta}{dt} = \omega = \frac{h}{r^2}$$

Hence

$$\mathbf{u}_\theta = \frac{\dot{\mathbf{u}}_\mathbf{r}}{\omega} = \frac{\dot{\mathbf{u}}_\mathbf{r}}{h/r^2}$$

So the cross-product which we had as

$$\ddot{\mathbf{r}} \times \hat{\mathbf{k}} = \frac{GM}{r^2}\ \mathbf{u}_\theta$$

is equal to

$$= \frac{GM}{r^2}\ \frac{\dot{\mathbf{u}}_\mathbf{r}}{h/r^2}$$

$$= \frac{GM}{h}\ \dot{\mathbf{u}}_\mathbf{r}$$

This is really the key step in the whole adventure.

The second clever thing is to integrate with respect to time

$$\int \ddot{\mathbf{r}} \times \hat{\mathbf{k}} = \int \frac{GM}{h}\ \dot{\mathbf{u}}_{\mathbf{r}}$$

(remember that GM, h and $\hat{\mathbf{k}}$ are all constant)

$$\dot{\mathbf{r}} \times \hat{\mathbf{k}} = \frac{GM}{h}\ (\mathbf{u}_{\mathbf{r}} + \mathbf{E})$$

where $\mathbf{E}$ is a constant (vector) of integration.

The third step is to realize that we can simplify a lot by forming the dot product of both sides with $\mathbf{r}$

$$\mathbf{r} \cdot (\dot{\mathbf{r}} \times \hat{\mathbf{k}}) = \frac{GM}{h}\ \mathbf{r} \cdot (\mathbf{u}_{\mathbf{r}} + \mathbf{E})$$

using a vector identity, the left-hand side is

$$\mathbf{r} \cdot (\dot{\mathbf{r}} \times \hat{\mathbf{k}}) = (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \hat{\mathbf{k}}$$

but

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h} = h\ \hat{\mathbf{k}}$$

so we have

$$h\ \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = h$$

Bringing back the right-hand side:

$$\mathbf{r} \cdot (\ddot{\mathbf{r}} \times \hat{\mathbf{k}}) = h = \frac{GM}{h}\ \mathbf{r} \cdot (\mathbf{u}_{\mathbf{r}} + \mathbf{E})$$

$$\frac{h^2}{GM} = \mathbf{r} \cdot (\mathbf{u}_{\mathbf{r}} + \mathbf{E})$$

Recall that $\mathbf{u}_{\mathbf{r}}$ is the unit vector in the same direction as $\mathbf{r}$ so that $\mathbf{r} \cdot \mathbf{u}_{\mathbf{r}} = r$.

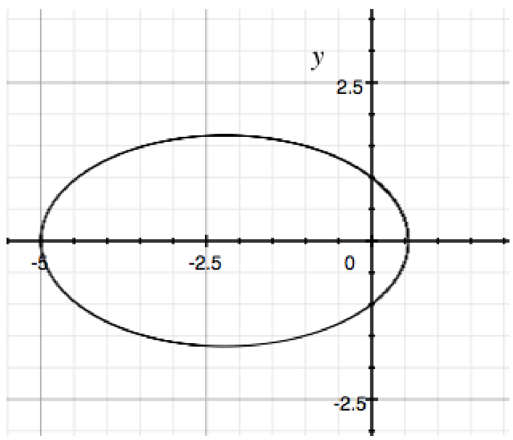
We can take $\mathbf{E}$ to be in the direction of $\mathbf{r}$ at time-zero so $\mathbf{r} \cdot \mathbf{E}$ is equal to r times e times the cosine of the angle between them at some later time. (Since $\mathbf{E}$ is a constant vector of integration, its magnitude e can be anything).

This becomes

$$r(1 + e\cos\theta) = \frac{h^2}{GM}$$

which is an equation in polar coordinates.

This is a family of curves which are conic sections. If $e < 1$ it's an ellipse.



The curve in the figure is an ellipse with the formula

$$r(1 + 0.8\cos\theta) = 1$$

e is the eccentricity of the ellipse

$$e^2 + \frac{b^2}{a^2} = 1$$

In the figure

$$e^2 = 0.8^2 = 0.64$$
$$\frac{b^2}{a^2} = 1 - 0.64 = 0.36$$

$$\frac{b}{a} = \sqrt{0.36} = 0.6$$

Chapter 7

Kepler3

The last part of Varberg's derivation is about K3. To prove:

$$T^2 = \frac{(2\pi)^2}{GM}\, a^3$$

where T is the period, GM is our constant from before, and a is the length of the half-major axis of the ellipse. In other words, the period of an orbit is the 3/2 power of the "radius", technically the semi-major axis of the ellipse.

Start with K2

$$2\, \frac{dA}{dt} = h$$

Integrate with respect to time over one revolution obtaining an ellipse with area πab and the period T for the time

$$2\pi ab = hT$$

$$T^2 = (\frac{2\pi ab}{h})^2$$

Now, go back to the equation for the orbit

$$r(1 + e\cos\theta) = \frac{h^2}{GM}$$

Consider one-half an orbit between $\theta = 0 \to \theta = \pi$. The length of the axis is $2a$, equal to $2r$ for this orbit, so

$$
\begin{aligned}
2a &= \frac{h^2}{GM(1 + e\cos\pi)} + \frac{h^2}{GM(1 + e\cos 0)} \\
&= \frac{h^2}{GM}\left(\frac{1}{1-e} + \frac{1}{1+e}\right) \\
&= \frac{h^2}{GM}\,\frac{2}{1-e^2}
\end{aligned}
$$

So

$$a = \frac{h^2}{GM}\,\frac{1}{1-e^2}$$

For an ellipse

$$\frac{b^2}{a^2} = 1 - e^2$$

so

$$a = \frac{h^2}{GM}\,\frac{a^2}{b^2}$$

$$b^2 = \frac{ah^2}{GM}$$

We had

$$
\begin{aligned}
T^2 &= \left(\frac{2\pi ab}{h}\right)^2 \\
&= \left(\frac{2\pi a}{h}\right)^2 \frac{ah^2}{GM} \\
&= \frac{(2\pi)^2}{GM} a^3
\end{aligned}
$$

which is K3. GM is the gravitational constant times the mass of the sun. The term due to the angular momentum h has dropped out.

Note that we can get an estimate for GM from observation of the orbits of the planets, and that G can be determined very simply, allowing us to find M, and "weigh the sun".

Chapter 8

Hartig

This is a derivation of Kepler's laws from a class handout I found on the web for Math 304 by Hartig.

We start by defining M as the mass of the sun and m as the mass of the planet and $\mathbf{r}$ as the position vector from the sun to the planet. Combining Newton's second law and the inverse square law of gravitation we have that

$$\mathbf{F} = m\mathbf{a} = -\frac{GMm}{r^2}\;\frac{\mathbf{r}}{r}$$

$$\mathbf{a} = -\frac{GM}{r^2}\;\frac{\mathbf{r}}{r}$$

We take $\mathbf{u_r}$ as a unit vector in the same direction as $\mathbf{r}$. I write $\hat{\mathbf{u}}_\mathbf{r}$ without its hat ($\hat{}$) as $\mathbf{u_r}$ so as not to confuse it with the derivative $\dot{\mathbf{u}}_\mathbf{r}$.

$$\mathbf{r} = r\mathbf{u_r}$$

$$\mathbf{a} = -\frac{GM}{r^2}\;\mathbf{u_r}$$

The acceleration is along the line of the radial vector, pointing toward the sun.

The velocity is the time-derivative of the position vector $\mathbf{r}$.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

and the acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{r}}$$

Feynman's dots, again

We set up the angular momentum as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$$

For a unit mass this is

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}}$$

We compute the time-derivative

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}})$$

by the standard vector application of the product rule which we've looked at above, this is equal to

$$= \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}}$$

and this is equal to zero, since any vector's cross product with itself is zero, including a reversed version of itself, as in the second term. We define a constant vector $\mathbf{h}$ such that

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$$

Since $\mathbf{h}$ is a constant, unchanging in both direction and magnitude, it defines a normal vector to the plane containing $\mathbf{r}$ and $\dot{\mathbf{r}}$. Align $\mathbf{h}$ with the z-axis so all the motion occurs in the xy-plane. Note that

$$h = |\mathbf{h}| = |\mathbf{r} \times \dot{\mathbf{r}}| = rv \sin\theta$$

where these are all scalar quantities and θ is the angle between $\mathbf{r}$ and $\dot{\mathbf{r}} = \mathbf{v}$.

Equal area

We consider the triangle formed by the position vector before and after a short period of time Δt, and the vector $\Delta\mathbf{r}$ connecting these two positions, where

$$\Delta\mathbf{r} \approx \dot{\mathbf{r}}\Delta t$$

The little bit of area ΔA that is swept out during this time is

$$\Delta A \approx \frac{1}{2}\ |\mathbf{r} \times \dot{\mathbf{r}}\Delta t|$$

$$\Delta A = \frac{h}{2}\ \Delta t$$

So we have that

$$\frac{\Delta A}{\Delta t} \approx \frac{h}{2}$$

and in the limit as $\Delta t \to 0$

$$\frac{dA}{dt} = \frac{h}{2}$$

(Note a difference with Feynman. He uses A for the area, but never actually computes its value $|\mathbf{r} \times \dot{\mathbf{r}}|$. Here, dA/dt is the area and it's the second derivative d^2A/dt^2 that is equal to zero. Which is another way of saying that $\mathbf{h}$ is constant).

Manipulating $\mathbf{a} \times \mathbf{h}$

The crucial step is to prove that

$$\mathbf{a} \times \mathbf{h} = GM\dot{\mathbf{u}}_{\mathbf{r}}$$

This takes a bit of work, so I'd like to defer it until the end. We'll just assume it for now. Take the equality and integrate with respect to time, obtaining

$$\int \mathbf{a} \times \mathbf{h} = \int GM\dot{\mathbf{u}}_{\mathbf{r}}$$

$$\dot{\mathbf{r}} \times \mathbf{h} = GM\mathbf{u}_{\mathbf{r}} + \mathbf{d}$$

where $\mathbf{d}$ is a constant *vector* of integration.

Dot product

We're almost there now. Take the left-hand side from above and form the dot product

$$\mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{h})$$

Use another vector identity to switch it around

$$= (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \mathbf{h}$$

But $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$ so

$$= \mathbf{h} \cdot \mathbf{h} = h^2$$

conic sections

What we've shown is that

$$h^2 = \mathbf{r} \cdot (GM\mathbf{u}_{\mathbf{r}} + \mathbf{d})$$

$$= r(GM + d\cos\theta)$$

$$= rGM(1 + \frac{d}{GM}\cos\theta)$$

Define $k = h^2/GM$ and $e = d/GM$. Then

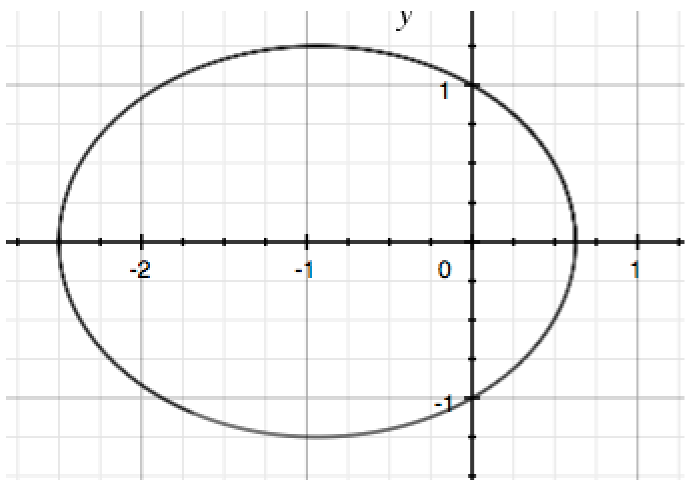
$$k = r(1 + e\cos\theta)$$

This is the equation of a conic section. In particular, if $e < 1$, then

$$r = \frac{k}{1 + e\cos\theta}$$

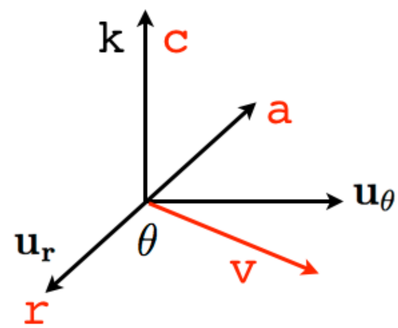
is the equation of an ellipse. Here is an example with $k = 1$ and $e = 0.6$

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} \frac{1}{1+0.6\cos t} \\ t \end{bmatrix}, t=0\ldots10$$



Cleaning up

Here is a sketch of the situation



As we've said all along, $\mathbf{u_r}$ is a unit vector in the $\mathbf{r}$ direction, so that $\mathbf{r} = r\mathbf{u_r}$. By the central force hypothesis, the acceleration $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$ is in the $-\mathbf{u_r}$ direction. The source of all our complexity is that $\mathbf{v} = \dot{\mathbf{r}}$ is not perpendicular to $\mathbf{u_r}$ but forms an angle θ with it.

Also, we defined

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}$$

and aligned $\mathbf{h}$ with the $\hat{\mathbf{k}}$ direction. We analyzed $\mathbf{r} \times \mathbf{v}$ to show that $\mathbf{h}$ is a constant vector. $\mathbf{u}_\theta$ is the unit vector orthogonal to $\mathbf{u_r}$.

According to Hartig, what we have to prove is that

$$\mathbf{a} \times \mathbf{h} = GM\dot{\mathbf{u}}_\mathbf{r}$$

Go back to basic definitions.

$$\mathbf{r} = r\mathbf{u_r}$$

$$\mathbf{v} = \dot{r}\mathbf{u_r} + r\dot{\mathbf{u}}_\mathbf{r}$$

Recall that $\dot{\mathbf{u}}_\mathbf{r} = \dot{\theta}\mathbf{u}_\theta$ so

$$\mathbf{v} = \dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta$$

$$\begin{aligned}\mathbf{h} = \mathbf{r} \times \mathbf{v} &= r\mathbf{u_r} \times (\dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta) \\ &= r^2\dot{\theta}\hat{\mathbf{k}}\end{aligned}$$

The acceleration is

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{u_r}$$

So

$$\begin{aligned}\mathbf{a} \times \mathbf{h} &= -\frac{GM}{r^2}\mathbf{u_r} \times r^2\dot{\theta}\hat{\mathbf{k}} \\ &= -GM\dot{\theta}(-\mathbf{u}_\theta) \\ &= GM\dot{\theta}\mathbf{u}_\theta\end{aligned}$$

Again, recall that $\dot{\mathbf{u}}_\mathbf{r} = \dot{\theta}\mathbf{u}_\theta$ so

$$\mathbf{a} \times \mathbf{h} = GM\dot{\mathbf{u}}_\mathbf{r}$$

Now, integrate

$$\int \mathbf{a} \times \mathbf{h} = \int GM\dot{\mathbf{u}}_\mathbf{r}$$

$$\mathbf{v} \times \mathbf{h} = \dot{\mathbf{r}} \times \mathbf{h} = GM\mathbf{u_r}$$

Chapter 9

Fitzpatrick

This is a derivation of Kepler's laws from a book I found on the web for Fitzpatrick's course on Mechanics.

part one

He starts by establishing unit vectors in polar coordinates as $\mathbf{e_r}$ and $\mathbf{e}_\theta$ and then parametrically

$$\mathbf{e_r} = \langle \cos\theta, \sin\theta \rangle$$

$$\mathbf{e}_\theta = \langle -\sin\theta, \cos\theta \rangle$$

$$\mathbf{e}_\theta \perp \mathbf{e_r}$$

So

$$\dot{\mathbf{e}}_\mathbf{r} = \dot{\theta}\mathbf{e}_\theta$$

$$\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e_r}$$

Writing the position vector as

$$\mathbf{r} = r\ \mathbf{e_r}$$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e_r} + r\dot{\mathbf{e}}_\mathbf{r} = \dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e}_\theta$$

For the acceleration

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d}{dt}\ (\dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e}_\theta)$$

$$= \ddot{r}\mathbf{e_r} + \dot{r}\dot{\mathbf{e}}_\mathbf{r} + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta$$

$$= \ddot{r}\mathbf{e_r} + \dot{r}\dot{\theta}\mathbf{e}_\theta + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e_r}$$

$$= (\ddot{r} - r\dot{\theta}^2)\mathbf{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta$$

As before we recognize the coefficient for $\mathbf{e}_\theta$ as

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})$$

and this term is also equal to zero because the acceleration is all radial and so the term in parentheses must be zero and so

$$2r\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

if we integrate

$$\int 2r\dot{r}\dot{\theta} + r\ddot{\theta} = r^2\dot{\theta} = h$$

where h is a constant.

The physical interpretation comes from angular momentum, which is defined as

$$\mathbf{l} = m\mathbf{r} \times \dot{\mathbf{r}}$$

$$= m(r\mathbf{e_r} \times (\dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e}_\theta)$$

$$= mr^2\dot{\theta}\ \hat{\mathbf{k}}$$

That is,

$$mh\ = |\mathbf{l}|$$

At this point he goes through the standard analysis to obtain that the area swept out in a small time $\delta A/\delta t = h/2$. I think we can skip this part.

part two

This derivation has an unusual approach to using the information from the inverse square law. Define a new radial variable, the inverse of r

$$r = \frac{1}{u}$$

Differentiate with respect to time

$$\dot{r} = -\frac{\dot{u}}{u^2}$$

obviously. But what is $\dot{u}$?

$$\dot{u} = \frac{du}{dt} = \frac{du}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\ \frac{du}{d\theta}$$

So

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{\dot{\theta}}{u^2}\frac{du}{d\theta}$$

Recall $r^2\dot{\theta} = \dot{\theta}/u^2 = h$ so

$$= -h\ \frac{du}{d\theta}$$

Differentiate again with respect to time

$$\ddot{r} = -h\frac{d}{dt}\ (\frac{du}{d\theta}) = -h\dot{\theta}\ \frac{d^2u}{d\theta^2}$$

but $\dot{\theta} = hu^2$ so

$$= -h^2u^2\ \frac{d^2u}{d\theta^2}$$

Now, go back to our previous expression for the acceleration, it is

$$-\frac{GM}{r^2} = \ddot{r} - r\dot{\theta}^2$$

Plug in for $\ddot{r}$ and multiply everything by -1:

$$\frac{GM}{r^2} = h^2u^2\ \frac{d^2u}{d\theta^2} + r\dot{\theta}^2$$

Rearrange ($ru = 1$):

$$\frac{GM}{h^2} = \frac{d^2u}{d\theta^2} + \frac{r^3}{h^2}\dot{\theta}^2$$

but $h = r^2\dot{\theta}$ and $h^2 = r^4\dot{\theta}^2$ so

$$\frac{GM}{h^2} = \frac{d^2u}{d\theta^2} + \frac{1}{r}$$

$$\frac{GM}{h^2} = \frac{d^2u}{d\theta^2} + u$$

How about that? Now we have a basic differential equation in u

We guess the solution has, say $\cos\theta$ and constants A and C.

$$u = A\cos\theta + C$$

because

$$\frac{d^2u}{d\theta^2} = -A\cos\theta$$

So

$$C = \frac{GM}{h^2}$$

$$u = A\cos\theta + \frac{GM}{h^2}$$

Technically, we should have θ_0 in the solution, but we can just set that equal to zero, since we don't care about where we start. Go back to r

$$1 = r(A\cos\theta + \frac{GM}{h^2})$$

$$\frac{h^2}{GM} = r(A\frac{h^2}{GM} + A\cos\theta)$$

Define
$$e = A = \frac{GM}{h^2}$$
so now we have
$$\frac{h^2}{GM} = r(1 + e\cos\theta)$$
which is exactly what we had with Varberg.

Chapter 10

Brilliant

I found another derivation here:

https://brilliant.org/wiki/deriving-keplers-laws/

We start with Newton's second law:

$$\mathbf{F} = m\mathbf{a}$$

where $\mathbf{a}$ describes the acceleration, which in radial coordinates is $\ddot{\mathbf{r}}$. By hypothesis, the force is

$$\mathbf{F} = -\frac{GmM}{r^2}\,\hat{\mathbf{r}}$$

To solve this, convert to Cartesian xy-coordinates, breaking the acceleration into two components

$$m\ddot{\mathbf{x}} = -\frac{GmM}{r^2}\cos\theta$$

$$m\ddot{\mathbf{y}} = -\frac{GmM}{r^2}\sin\theta$$

We will use a trick to solve the system. Getting ahead of ourselves a little bit, cancel the m, and multiply the first equation by $\cos\theta$ and the second by $\sin\theta$:

$$\ddot{\mathbf{x}}\cos\theta = -\frac{GM}{r^2}\cos^2\theta$$

$$\ddot{\mathbf{y}}\sin\theta = -\frac{GM}{r^2}\sin^2\theta$$

then add

$$\ddot{\mathbf{x}}\ \cos\theta + \ddot{\mathbf{y}}\ \sin\theta = -\frac{GM}{r^2}$$

We will provide another equality involving the left-hand side.

derivatives

According to the source: "we need to find the second derivative of the x and y coordinates in terms of the polar coordinates." Repeat what we've done elsewhere:

$$x = r\cos\theta$$

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

By the chain rule. Notice there are three components in the second term.

$$\ddot{x} = \ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \ [\ \dot{r}\dot{\theta}\sin\theta + r\ddot{\theta}\sin\theta + r\dot{\theta}^2\cos\theta\ ]$$

$$= \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta$$

It looks like a mess, but it will all work out. Next:

$$y = r\sin\theta$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$\ddot{y} = \ddot{r}\sin\theta + \dot{r}\dot{\theta}\cos\theta + \ [\ \dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta\ ]$$

$$= \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta$$

Now, when we multiply the equation for $\ddot{x}$ by $\cos\theta$, there are two terms with $\sin\theta\cos\theta$:

$$\cdots - 2\dot{r}\dot{\theta}\sin\theta\cos\theta - r\ddot{\theta}\sin\theta\cos\theta\ldots$$

which exactly cancel the second and third terms from the expression for $\ddot{y}$, multiplied by $\sin\theta$, when the two parts are added together.

We obtain:

$$\begin{aligned}\ddot{\mathbf{x}}\ \cos\theta + \ddot{\mathbf{y}}\ \sin\theta = \\ = \ddot{r}\cos^2\theta - r\dot{\theta}^2\cos^2\theta + \ddot{r}\sin^2\theta - r\dot{\theta}^2\sin^2\theta \\ = \ddot{r} - r\dot{\theta}^2\end{aligned}$$

which you must admit is an impressive simplification.

Combining with the previous result we have:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}$$

Suppose r is constant, then $\ddot{r} = 0$ and

$$r\dot{\theta}^2 = \frac{GM}{r^2}$$

We found previously that for a circular orbit

$$v = \sqrt{\frac{GM}{R}}$$

$$\frac{v^2}{R} = \frac{GM}{R^2}$$

But $v = R\omega = R\dot{\theta}$. This is the same equation.

Kepler 2

Rather than multiply as we did, switch and multiply $\ddot{x}$ by $\sin\theta$ and $\ddot{y}$ by $\cos\theta$:

$$\ddot{x}\sin\theta = \ddot{r}\sin\theta\cos\theta - 2\dot{r}\dot{\theta}\sin^2\theta - r\ddot{\theta}\sin^2\theta - r\dot{\theta}^2\sin\theta\cos\theta$$

$$\ddot{y}\cos\theta = \ddot{r}\sin\theta\cos\theta + 2\dot{r}\dot{\theta}\cos^2\theta + r\ddot{\theta}\cos^2\theta - r\dot{\theta}^2\sin\theta\cos\theta$$

And subtract:

$$\ddot{x}\sin\theta - \ddot{y}\cos\theta = -2\dot{r}\dot{\theta} - r\ddot{\theta}$$

But the left-hand side is zero (go back to the original equation)

$$0 = -2\dot{r}\dot{\theta} - r\ddot{\theta}$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

Multiply by r and recognize

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0 = \frac{d}{dt}\ r^2\dot{\theta}$$

m times this quantity whose derivative is zero is the angular momentum. Hence the angular momentum is conserved. Multiply by m and integrate with respect to time:

$$L = mr^2\dot{\theta}$$

Integrate again with respect to time:

$$\frac{L}{m}t = \int r^2\theta$$

The quantity on the right-hand side is twice the area swept out from θ_1 to θ_2, which is now seen to be independent of r (since it is equal to a constant times the time).

Kepler 1

In the first section, we had

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}$$

We must now solve this equation. Substitute $u = 1/r$.

$$\dot{r} = \frac{dr}{dt} = -\frac{1}{u^2}\ \frac{du}{dt} = -\frac{1}{u^2}\ \frac{du}{d\theta}\ \dot{\theta}$$

And recalling the previous result:

$$\frac{L}{mr^2} = \dot{\theta} = \frac{L}{m}u^2$$

So substitute for $\dot{\theta}$ in the equation with u:

$$= -\frac{1}{u^2}\ \frac{du}{d\theta}\ \frac{L}{m}u^2$$

$$= -\frac{L}{m}\ \frac{du}{d\theta}$$

Differentiating again:

$$\ddot{r} = -\frac{L}{m}\ \frac{d}{dt}\ (\frac{du}{d\theta})$$

$$\ddot{r} = -\frac{L}{m}\ \frac{d\theta}{dt}\ \frac{d}{d\theta}\ (\frac{du}{d\theta})$$

(By the chain rule).

Substituting again for $\dot{\theta}$

$$\ddot{r} = -(\frac{L}{m})^2\ u^2\ \frac{d^2u}{d\theta^2}$$

Go back to

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}$$

and substitute:

$$-(\frac{L}{m})^2\ u^2\ \frac{d^2u}{d\theta^2} - \frac{1}{u}\ [(\frac{L}{m})u^2]^2 = -GMu^2$$

$$-(\frac{L}{m})^2\ \frac{d^2u}{d\theta^2} - (\frac{L}{m})^2 u = -GM$$

$$-\frac{d^2u}{d\theta^2} + GM(\frac{m}{L})^2 = u$$

u is just a periodic function of θ.

$$u = A\cos(\theta + \delta) + C$$

where $C = GMm^2/L^2$. We can always define coordinates such that $\delta = 0$ so

$$u = A\cos\theta + C$$

$$r = \frac{1}{A\cos\theta + C}$$

$$= \frac{1}{C(A/C\cos\theta + 1)}$$

With an appropriate definition ($e = A/C$) we can write

$$r = \frac{e/A}{e\cos\theta + 1}$$

This is the equation of an ellipse in radial coordinates. The closest and furthest approaches occur when $\theta = 0, \pi$ and then

$$r = \frac{e/A}{1 \pm e}$$

The semi-major axis is

$$a = \frac{1}{2}(r_{min} + r_{max})$$

$$= \frac{e/a}{1 - e^2}$$

Chapter 11

Ellipse review

Now, a bit about ellipses. We will not assume this for the derivations, since we are going the other way, from Newton to Kepler. But it might help to clarify things.

This whole problem of the orbits would be trivial if they were circular. Then, the velocity vector $\mathbf{v} = \dot{\mathbf{r}}$ would be perpendicular to the position vector $\mathbf{r}$. On an ellipse, that isn't true. Newton's diagram is drawn above with AB not perpendicular to AS.

But in an elliptical orbit, the planet's velocity is in the same direction as the tangent vector to the ellipse at that position. And the tangent vector has a nice relationship to the radial vector. We can get to the point more quickly by writing the parametrization

$$x = \cos\theta = \cos\omega t$$

$$y = \sin\theta = \sin\omega t$$

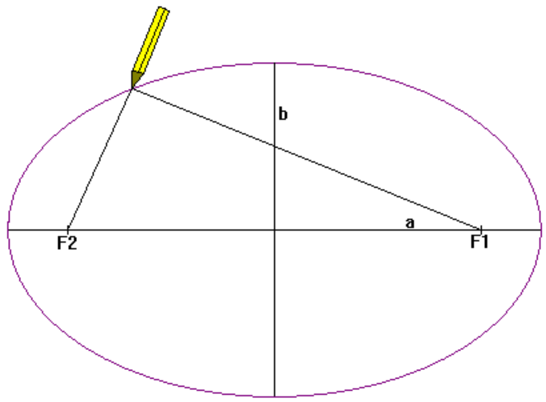
$$\mathbf{r}(t) = \ \langle x(t), y(t)\rangle$$

$$= \ \langle \cos\omega t, \sin\omega t\rangle$$

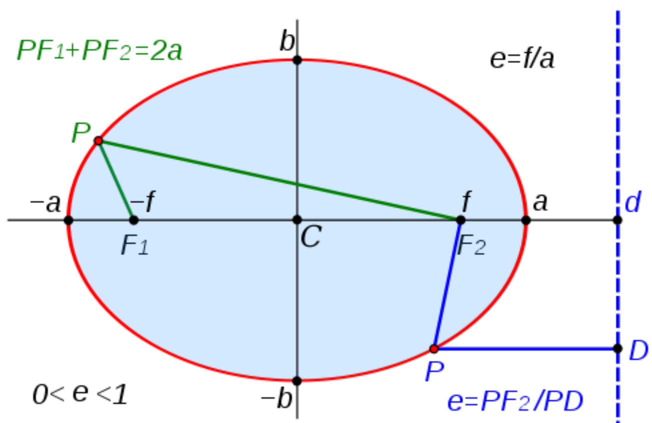
$$\mathbf{v} = \dot{\mathbf{r}}$$

$$= \langle -\omega \sin \omega t, \omega \cos \omega t \rangle$$

What follows is a longer presentation borrowed from my short write-up on parametrization of the ellipse.



Learning how to draw an ellipse using two pins and a circular string holding a pencil is an early adventure in mathematics. The ellipse is the set of all points whose combined distance to the two pins (foci) is the same.



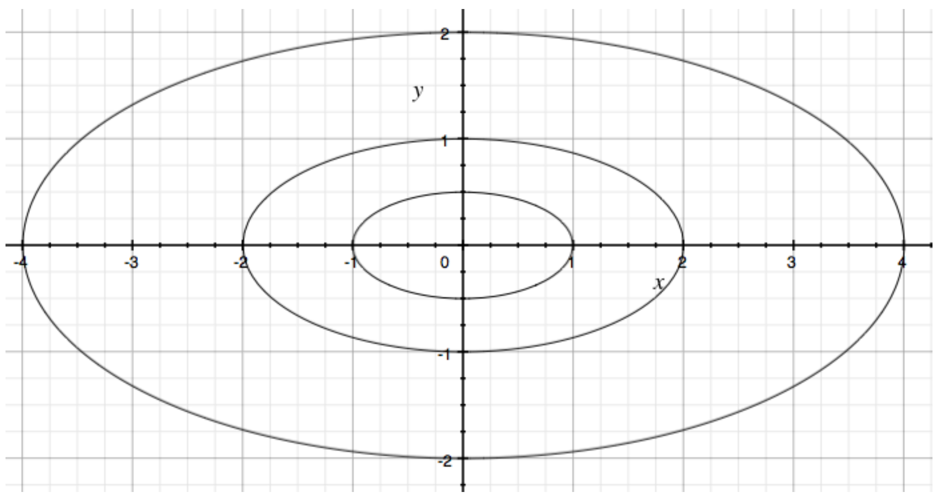
The pin positions with respect to the origin or center are called the foci, lying at the points $(\pm f, 0)$. The lengths of the axes (called semi-major and semi-minor) are usually labeled a and b. Consider the situation when the pencil is at the point $(0, a)$. The length L of the string is equal to twice the distance to the left focus, $L = 2(f + a)$, so

$$a = \frac{L}{2} - f$$

We learn in algebra that the equation for an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Here are three ellipses drawn with the same center.



They were drawn by adjusting the value on the right-hand side of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$$

where $r = \{1/2, 1, 2\}$. This is equivalent to scaling both a and b by the same factor of r

$$\frac{x^2}{(ra)^2} + \frac{y^2}{(rb)^2} = 1$$

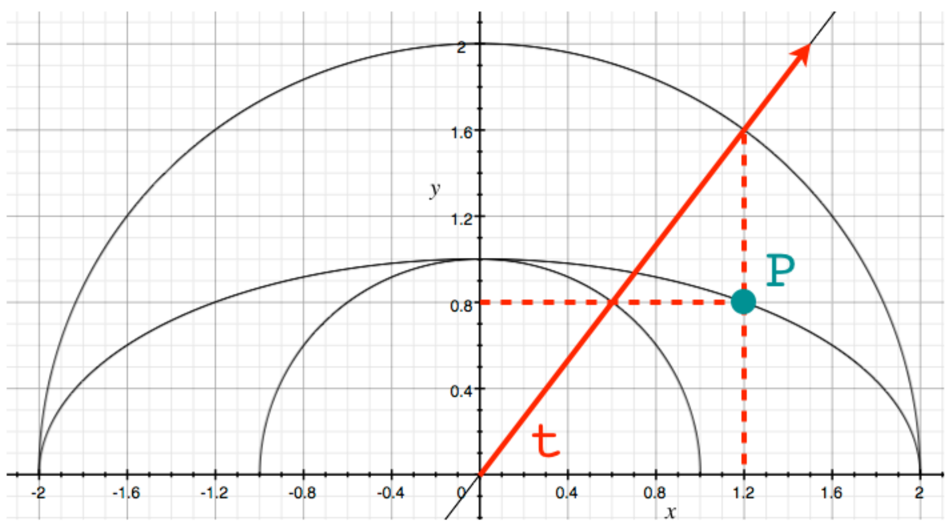
When $r = 2$ we need to make the string a bit less than twice as long, because the length f is also involved:

$$ra = r(\frac{L}{2} - f)$$

parametrization

An alternative view is the one below, which shows (black curves) the upper half of two circles of radius $r = 1$ and $r = 2$ and an ellipse whose equation is

$$\frac{x^2}{2^2} + \frac{y^2}{1} = 1$$



Here $a = 2$ and $b = 1$.

The standard parametrization of the ellipse is

$$x = a \cos t$$

$$y = b \sin t$$

which I had trouble visualizing, until I drew the picture. The point is that the parameter t is *not* the angle that a ray to P makes with the x-axis, as it is for the circle. Instead, to find the x value of P corresponding to t, we extend the ray with angle t to the larger circle, with radius a, where we read off the x-value as

$$x = a \cos t$$

We go back to find the intersection of the same ray with the small circle to get

$$y = b \sin t$$

The algebraic way to do this is to show that the parametrization is equivalent to the original formulation

$$x^2 = a^2 \cos^2 t$$

$$y^2 = b^2 \sin^2 t$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$$

as expected.

rotation

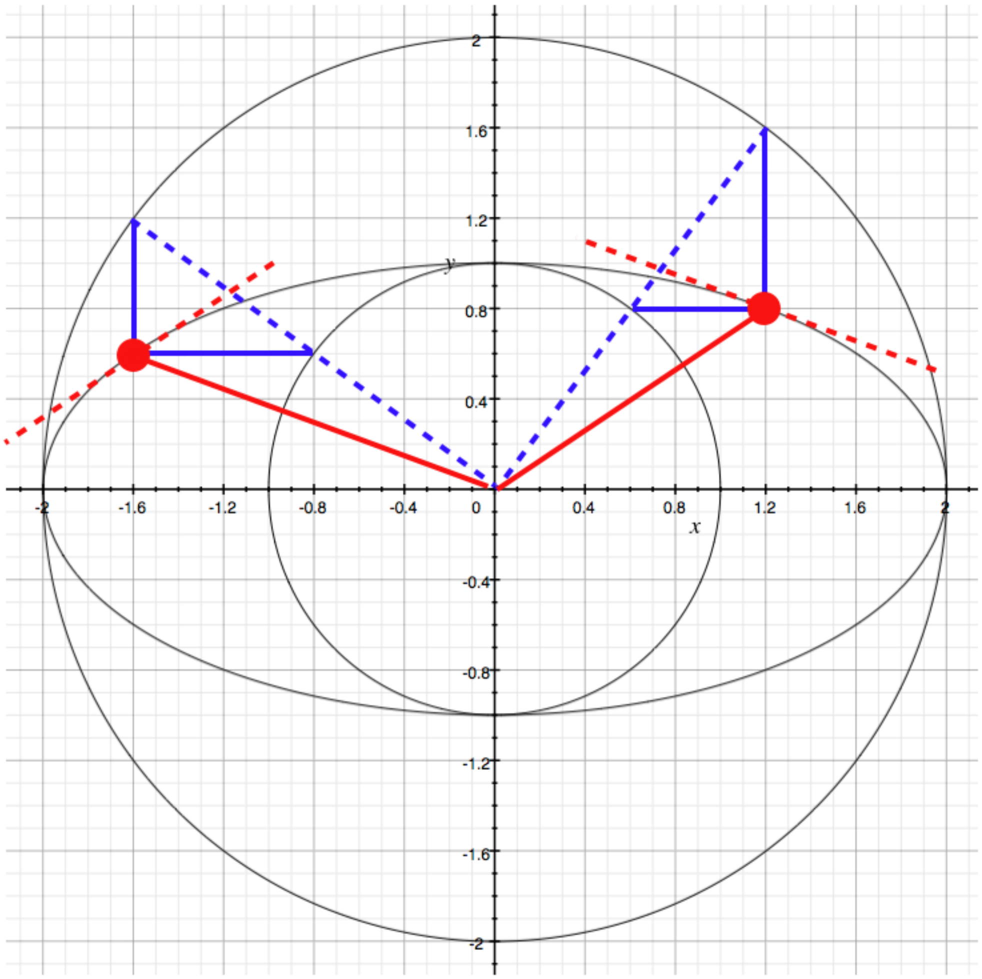
Let's return to the diagram of the ellipse with two bounding circles of radius a and radius b. I have a new diagram on the next page.

Consider the coordinates of the point $P = (x, y)$ (the red dot in the first quadrant) as functions of the angle t. As we said, t is *not* the angle of a ray from the origin to P.

Let's draw a ray (blue dotted line) from the origin that does have angle t with the x-axis. How to find x and y from the diagram. For x, extend the ray to the outer circle. The radius is a, the angle is t, and

$$a \cos t = x$$

This is the parametrization of the ellipse introduced above.



The ray drawn with angle t has the same x-intercept with the outer circle as our point P on the ellipse. Similarly, the intercept of the ray with the inner circle has the same y-value as the point P on the ellipse.

We estimate the point $P = (1.2, 0.8) = (6/5, 4/5)$. Using our algebraic equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Recall that $a = 2$ and $b = 1$ so

$$x^2 + 4y^2 = 4$$

Plugging in for x^2 and y^2 we get

$$\frac{36}{25} + 4\ (\frac{16}{25}) = \frac{100}{25} = 4$$

as expected. Reading off the intercepts for the ray with angle t (dotted blue line) with the outer circle, we have the point $(1.2, 1.6)$ at a distance 2 from the origin. Thus,

$$\frac{1.2}{2} = 0.6 = \cos t$$
$$t \approx 0.927 \text{ rad} \approx 53^\circ$$

Looking again at the figure, we want to consider what happens for the angle $u = t + \pi/2$. This is the dotted blue ray in the second quadrant.

We might calculate the values of sine and cosine for u, but notice that if we view u as a vector, its *dot product* with t must be equal to zero. The coordinates of the intercept of the rotated vector with the outer circle are $(-1.6, 1.2)$, so the cosine of the angle u is

$$\cos u = -0.8$$
$$u \approx 2.498 = t + \frac{\pi}{2} \text{ rad} \approx 143^\circ$$

We confirm that

$$2.498 - 0.927 = 1.57 = \frac{\pi}{2}$$

The coordinates of the point on the ellipse are $(-1.6, 0.6)$, which we check against the formula

$$x^2 + 4y^2 = 4$$
$$(1.6)^2 + 4(0.6)^2 = 2.56 + 4(0.36) = 4$$

(no clean fractions this for this one).

tangent

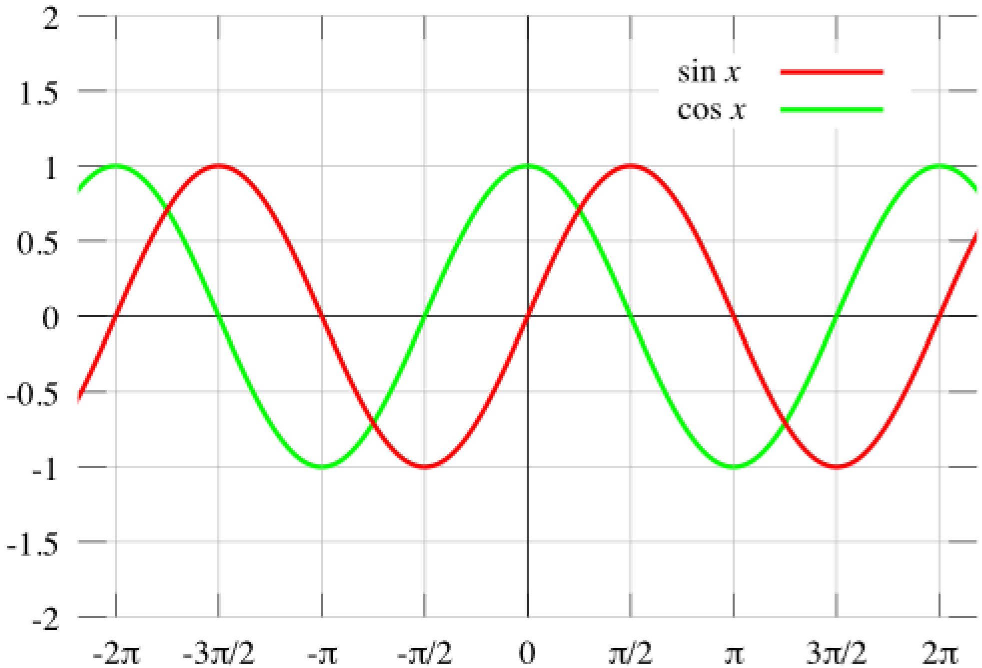
Finally, and this is really the point of the write-up, the vector to the point, call it Q, on the ellipse (red dot in the second quadrant) is the tangent to the ellipse for the point P in the first quadrant, and vice-versa.

How did this happen? Recall what we did. We had

$$x = a \cos t$$
$$y = b \sin t$$

The rotated point $Q = (x', y')$ is

$$x' = a \cos(t + \frac{\pi}{2})$$
$$y' = b \sin(t + \frac{\pi}{2})$$



Sine is like cosine, but shifted to the right by $\pi/2$

$$\cos\theta = \sin(\theta + \frac{\pi}{2})$$
$$\sin\theta = -\cos(\theta + \frac{\pi}{2})$$

So

$$x' = a\cos(t + \frac{\pi}{2}) = -a\sin t$$
$$y' = b\sin(t + \frac{\pi}{2}) = b\cos t$$

So what, you say. Well, let's look at the position vector, which can be written $\mathbf{r}(t)$, since it's a function of the angle t or the time, but we will just use $\mathbf{r}$. It has components x and y.

$$\mathbf{r} = \langle x, y \rangle = \langle a \cos t, b \sin t \rangle$$

Now, the tangent to the ellipse is precisely the direction in which a particle at (x, y) is currently moving on the ellipse. The tangent vector points in the same direction as the velocity vector, but $\mathbf{v}$ is just the time-derivative of the position vector.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle = \langle -a \sin t, b \cos t \rangle = \langle x', y' \rangle$$

And that's the point. :)

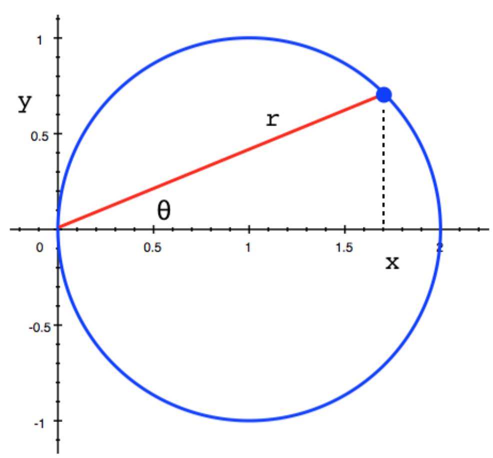
Chapter 12

Polar conics

circle

A very simple circle in polar coordinates is $r = a$. There is no θ-dependence when the circle has its center at the origin.

For a circle of radius a centered at $(a, 0)$ then



$$a^2 = (x - a)^2 + y^2$$

$$x^2 - 2ax + y^2 = 0$$

Always, $x = r\cos\theta$ and $y = r\sin\theta$ so

$$r^2(\sin^2\theta + \cos^2\theta) - 2ar\cos\theta = 0$$
$$r^2 - 2ar\cos\theta = 0$$
$$r = 2a\cos\theta$$

If the center of the circle is on the y-axis the equation is similar but with $\sin\theta$. A more general equation is

$$r = 2h\cos\theta + 2k\sin\theta$$

which is a circle that touches the origin, and has its center at (h, k).

The most general equation is with the circle anywhere in the plane. If we remember to specify the center at (s, ϕ) in *radial* coordinates, then the law of cosines easily yields

$$r^2 + s^2 - 2rs\cos(\theta - \phi) = a^2$$

reverse

Start from

$$r = 2h\cos\theta + 2k\sin\theta$$

Always, $x = r\cos\theta$ and $y = r\sin\theta$ so

$$r = 2h\frac{x}{r} + 2k\frac{y}{r}$$
$$r^2 = 2hx + 2ky$$
$$x^2 + y^2 = 2hx + 2ky$$

Easily rearrange and complete the square:

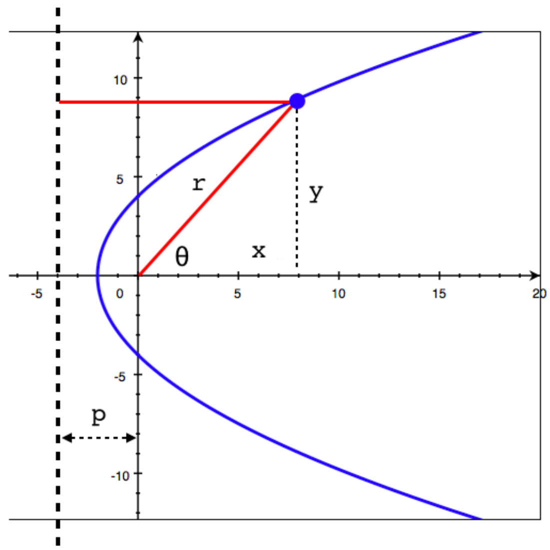
$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = h^2 + k^2$$
$$(x-h)^2 + (y-k)^2 = h^2 + k^2$$

For a circle touching the origin, $h^2 + k^2 = a^2$

$$(x-h)^2 + (y-k)^2 = a^2$$

parabola

To derive the equation for a parabola in polar coordinates it is convenient to rotate from the standard orientation by 90 degrees CW. In this way θ will have its usual relationship with the x-axis.



The origin of coordinates is placed at the focus, the distance to the vertex for this parabola is 2, and the distance from the origin to the directrix is $p = 4$.

Note that in Cartesian coordinates, this parabola will be of the form is $x = ay^2$ because of the rotation.

The distance from the focus to a general point (x, y) is just r. The distance from the directrix to the point is $p+x$. The geometric constraint gives simply:

$$r = p + x$$

We make the standard substitution $x = r\cos\theta$.

$$r = p + r\cos\theta$$

Some rearrangement gives the standard equation

$$r = \frac{p}{1 - \cos\theta}$$

For a vertically oriented parabola we would have $\sin\theta$ instead.

reverse

To go back to Cartesian coordinates, reverse the substitution for x:

$$r = \frac{p}{1 - x/r}$$

$$r - x = p$$

$$r^2 = (x + p)^2$$

Use $r^2 = x^2 + y^2$:

$$x^2 + y^2 = x^2 + 2px + p^2$$

$$y^2 = 2px + p^2$$

$$\frac{1}{2p}y^2 = x + \frac{p}{2}$$

This looks unusual. However, the equation that was actually plotted was $r = 4/(1 + \cos\theta)$ $(p = 4)$.

Note: here we have used p as the distance from the focus to the directrix, which is twice the distance to the vertex. If we call the latter distance c, the $p = 2c$. Previously we showed that $4ac = 1$, so $a = 1/4c$. Thus we obtain $a = 1/8$:

$$\frac{1}{8}y^2 = x + 2$$

This shape factor matches the plot (four units above the axis (at $x = 0$ is two units to the right of the vertex) and the vertex is at $(-2, 0)$. a is unusually small, the reason is so the parabola will open quickly, giving room to put all the labels in the diagram.

ellipse

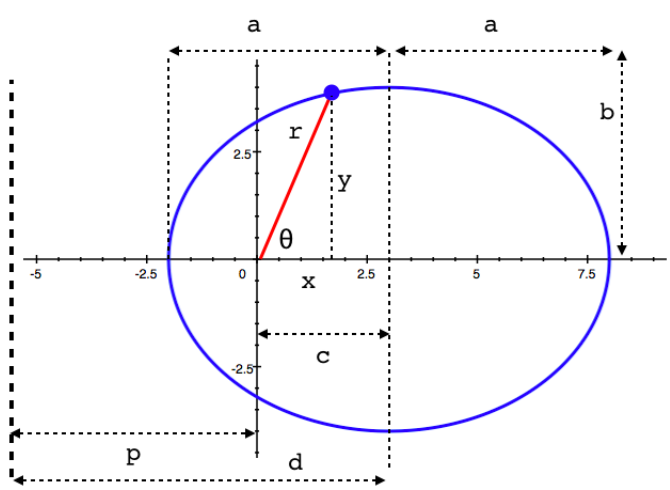
From the geometry of the ellipse, with the center at the origin, it is fairly easy to show that

$$a^2 = b^2 + c^2$$

and derive the equation of the ellipse in Cartesian coordinates:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

To derive the equation for an ellipse in polar coordinates it is convenient to shift the origin of coordinates to be the left focus of the ellipse at $(-c, 0)$.



The ellipse plotted here has $a = 5, b = 4$ and so $c = \sqrt{a^2 - b^2} = 3$. It has been shifted so the focus at $(-3, 0)$ is the origin of coordinates.

The eccentricity e is defined by the geometric constraint (next), which can be shown to be equivalent to $c/a = 0.6$.

Let p be the distance from the focus to the directrix, and let d be the distance from the directrix to the center of the ellipse.

The ellipse can be defined by its **geometric constraint**.

This says that for any point on the ellipse, the ratio of the distance from the focus (and here, the origin) to the point (that is, r), when divided by the distance from the point to the directrix, $x + p$, is equal to a constant, which we will call the eccentricity e.

$$\frac{r}{x+p} = \frac{r}{p + r\cos\theta} = e$$

$$r = e(p + r\cos\theta)$$

We simply rearrange to isolate r

$$r(1 - e\cos\theta) = ep$$

$$r = \frac{ep}{1 - e\cos\theta}$$

reverse

Going back is more complicated for the ellipse. Reverse the substitution $x/r = \cos\theta$.

$$r(1 - ex/r) = ep$$

$$r - ex = ep$$

$$r = ex + ep$$

There's a *magic* substitution that we will justify below:

$$ep = a(1 - e^2)$$

Using that, we have

$$r = ex + a(1 - e^2)$$

Use $r^2 = x^2 + y^2$:

$$x^2 + y^2 = e^2x^2 + 2exa(1 - e^2) + a^2(1 - e^2)^2$$

Combine cofactors for x^2, obtaining $(1 - e^2)$ and then divide through by $(1 - e^2)$:

$$x^2 + \frac{y^2}{1 - e^2} = 2exa + a^2(1 - e^2)$$

Complete the square for x by adding $(ea)^2$ to both sides

$$x^2 - 2exa + (ea)^2 + \frac{y^2}{1 - e^2} = a^2(1 - e^2) + (ea)^2$$

$$(x - ea)^2 + \frac{y^2}{1 - e^2} = a^2(1 - e^2) + (ea)^2$$

We asserted that $ea = c$. Simplify the right-hand side at the same time:

$$(x - c)^2 + \frac{y^2}{1 - e^2} = a^2$$

This is great, because we need to shift the origin of coordinates back to the center of the ellipse by exactly this amount.

Unfortunately, I have not discovered any way to make that derivation simpler.

solve for $1 - e^2$

To deal with $1 - e^2$, recall that the basic geometry says

$$a^2 - c^2 = b^2$$

$$1 - (\frac{c}{a})^2 = \frac{b^2}{a^2}$$

Since $c = ea$

$$1 - e^2 = \frac{b^2}{a^2}$$

so what we had simplifies as the inverse of that times y^2

$$(x-c)^2 + \frac{a^2}{b^2}\, y^2 = a^2$$

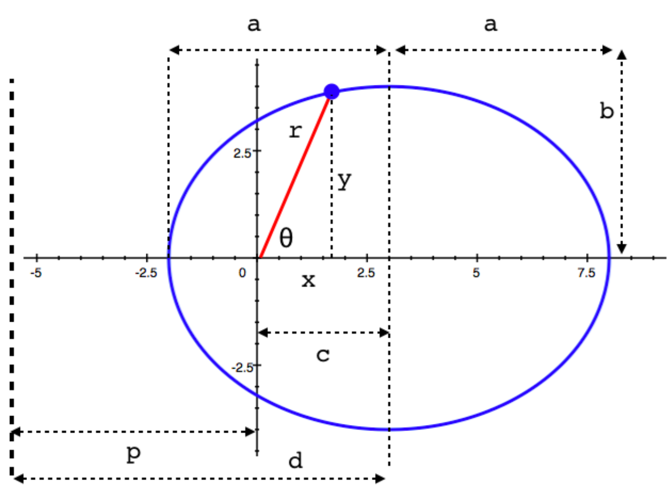
$$\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

Exactly what we want. Furthermore, we can view this derivation in reverse as a proof of $c = ea$ for an ellipse with this equation in Cartesian coordinates, shifted out to focus $(-c, 0)$.

Now let's explain the substitution:

$$ep = a(1-e^2)$$

It is easiest to start by finding an expression for d, then getting e and p.



solve for d and e

Applying the geometric constraint to the extreme left end:

$$\frac{a-c}{d-a} = e$$

At the very top of the ellipse the distance to the focus is $\sqrt{b^2+c^2}$ but this is also just a, which means

$$\frac{a}{d} = e = \frac{a-c}{d-a}$$

so

$$ad - a^2 = ad - cd$$

thus

$$d = \frac{a^2}{c}$$

One can also obtain this result by equating the ratios for the left and right ends of the ellipse.

Notice that the ratio a/d obeys the geometric constraint:

$$\frac{a}{d} = e = \frac{ac}{a^2} = \frac{c}{a}$$

We have proved $ae = c$, using only the geometry.

A longer, but pretty, proof is to start from the ratio for the extreme right end:

$$e = \frac{a+c}{d+a}$$

substitute $d = a^2/c$

$$= \frac{a+c}{a^2/c + a}$$

Multiply top and bottom by $1/a$

$$e = \frac{1 + c/a}{a/c + 1}$$

Put top and bottom over common denominators

$$e = \frac{(a+c)/a}{(a+c)/c} = \frac{c}{a}$$

solve for p

$$
\begin{aligned}
p = d - c &= \frac{a^2}{c} - c \\
&= \frac{a^2 - c^2}{c} = \frac{b^2}{c}
\end{aligned}
$$

finally

$$
\begin{aligned}
pe &= \frac{b^2}{c} \cdot \frac{c}{a} = \frac{b^2}{a} \\
&= \frac{a^2 - c^2}{a} \\
&= \frac{a^2 - e^2 a^2}{a} \\
&= a(1 - e^2)
\end{aligned}
$$

which is the special substitution we used.

summary of the summary

The circle (touching the origin), parabola (rotated to the right), and the ellipse are, in order:

$$
\begin{aligned}
r &= 2h\cos\theta + 2k\sin\theta \\
r &= \frac{p}{1 - \cos\theta} \\
r &= \frac{ep}{1 - e\cos\theta}
\end{aligned}
$$

In the last case, note that $0 < e < 1$, so the parabola is the same but with $e = 1$.

Chapter 13

Summary

unit vectors and velocity

The position vector (from the sun to the planet) is $\mathbf{r}$. Starting from our definition of the unit vector in the $\mathbf{r}$ direction as

$$\mathbf{u_r} = \langle \cos\theta, \sin\theta \rangle$$

where θ is the angle with the positive x-axis, we find $\mathbf{u}_\theta \perp \mathbf{u_r}$

$$\mathbf{u}_\theta = \langle -\sin\theta, \cos\theta \rangle$$

and confirm orthognality

$$\mathbf{u_r} \cdot \mathbf{u}_\theta = 0$$

Remembering that $\theta = \theta(t)$, we easily obtain by the chain rule

$$\dot{\mathbf{u}}_\mathbf{r} = \dot{\theta}\mathbf{u}_\theta$$

$$\dot{\mathbf{u}}_\theta = -\dot{\theta}\mathbf{u_r}$$

r is the magnitude of $\mathbf{r}$

$$\mathbf{r} = r\mathbf{u_r}$$

The velocity $\mathbf{v}$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{u_r} + r\dot{\mathbf{u}}_\mathbf{r} = \dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta$$

We use a vector identity that is easy to prove

$$\frac{d}{dt}\ (\mathbf{a} \times \mathbf{b}) = \dot{\mathbf{a}} \times \mathbf{b} + \mathbf{a} \times \dot{\mathbf{b}}$$

to calculate with Feynman's "dots"

$$\frac{d}{dt}\ (\mathbf{r} \times \mathbf{v})$$

$$= \frac{d}{dt}\ (\mathbf{r} \times \dot{\mathbf{r}})$$

$$= \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = 0$$

because any vector's cross-product with itself is zero (including minus itself), which is true for the second term involving the acceleration.

acceleration

An actual expression for the acceleration is just a matter of working through the dots

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d}{dt}\ (\dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta)$$

$$= \ddot{r}\mathbf{u_r} + \dot{r}\dot{\mathbf{u}}_\mathbf{r} + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta$$

substituting for $\dot{\mathbf{u}}_\mathbf{r}$ and $\dot{\mathbf{u}}_\theta$ from above

$$= \ddot{r}\mathbf{u_r} + \dot{r}\dot{\theta}\mathbf{u}_\theta + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta - r\dot{\theta}^2\mathbf{u_r}$$

$$= (\ddot{r} - r\dot{\theta}^2)\mathbf{u_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{u}_\theta$$

Rewrite the coefficient for $\mathbf{u}_\theta$ as

$$\frac{1}{r}(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

angular momentum

We find that the acceleration $\mathbf{a} = \dot{\mathbf{v}}$ has two parts of which the second (in $\mathbf{u}_\theta$)

$$\frac{1}{r}\,\frac{d}{dt}\,r^2\dot{\theta} = 0$$

is zero because $\mathbf{a}$ is all radial. Hence $r^2\dot{\theta} = h$ where h is a constant. Multiplied by the mass m, mh becomes the conserved quantity, angular momentum. It is also twice the area "swept out" and this is the statement of K2.

We get the vector $\mathbf{h}$ by defining the plane of motion as the xy-plane ($\mathbf{u_r} \times \mathbf{u}_\theta = \hat{\mathbf{k}}$) and

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = r\mathbf{u_r} \times (\dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta)$$

the first term is zero so

$$= r^2\dot{\theta}(\mathbf{u_r} \times \mathbf{u}_\theta) = r^2\dot{\theta}\ \hat{\mathbf{k}}$$

key step

With these preliminary steps we come to the key part of the derivation. I like Varberg's version best. The radial acceleration is

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{u_r}$$

Compute $\mathbf{a} \times \hat{\mathbf{k}}$ (recall that $\mathbf{a}$ is in the $-\mathbf{u_r}$ direction) by recognizing that $-\mathbf{u_r} \times \hat{\mathbf{k}} = \mathbf{u}_\theta$ so

$$\mathbf{a} \times \hat{\mathbf{k}} = \frac{GM}{r^2}\mathbf{u}_\theta$$

but from above $\dot{\mathbf{u}}_\mathbf{r} = \dot{\theta}\mathbf{u}_\theta$ so we have the crucial substitution:

$$\mathbf{a} \times \hat{\mathbf{k}} = \frac{GM}{r^2\dot{\theta}}\dot{\mathbf{u}}_\mathbf{r}$$

$$\mathbf{a} \times \hat{\mathbf{k}} = \frac{GM}{h} \dot{\mathbf{u}}_{\mathbf{r}}$$

Now we just integrate with respect to time and get

$$\int \mathbf{a} \times \hat{\mathbf{k}} = \int \frac{GM}{h} \dot{\mathbf{u}}_{\mathbf{r}}$$

$$\mathbf{v} \times \hat{\mathbf{k}} = \frac{GM}{h} \mathbf{u}_{\mathbf{r}} + \mathbf{d}$$

where $\mathbf{d}$ is a constant *vector* of integration. One last trick, we dot with $\mathbf{r}$ and simplify the left-hand side dramatically

$$\mathbf{r} \cdot (\mathbf{v} \times \hat{\mathbf{k}}) = (\mathbf{r} \times \mathbf{v}) \cdot \hat{\mathbf{k}} = \mathbf{h} \cdot \hat{\mathbf{k}} = h$$

So

$$h = \mathbf{r} \cdot (\frac{GM}{h} \mathbf{u}_{\mathbf{r}} + \mathbf{d})$$

$$\frac{h^2}{GM} = \mathbf{r} \cdot (\mathbf{u}_{\mathbf{r}} + \frac{h}{GM} \mathbf{d})$$

Define $k = h^2/GM$ and $e = hd/GM$ and θ as the angle between the constant vector $\mathbf{d}$ and $\mathbf{u}_{\mathbf{r}}$, so finally

$$k = r(1 + e \cos \theta)$$

which for $e < 1$ is an ellipse.