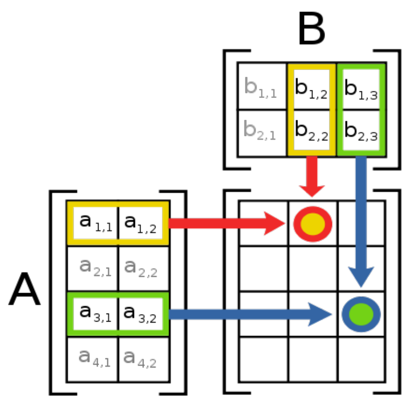
Multiplication of matrices

Standard picture

Here is the figure from wikipedia.



First of all, note that for two matrices A and B to be multiplied together $A \times B$, the number of columns in A must be equal to the number of rows in B. Here we have two columns in A (it is a 4 x 2 matrix) and two rows in B (a 2 x 3 matrix). Let's call the product $P = AB$. Then, P_{ij}, the entry in row i, column j of P, is obtained from the *dot product* of row i of A · column j of B. In this case, there are two terms in each sum. So

$$P_{12} = a_{11}\ b_{12} + a_{12}\ b_{22} = row\ a_1 \cdot col\ b_2$$

If A is an m x n matrix and B is an n x p matrix, the result will be an m x p matrix, and each cofactor will be formed as the sum of n terms.

And in general,

$$P_{ij} = \sum_{k=1}^{n} a_{ik}\ b_{kj}$$

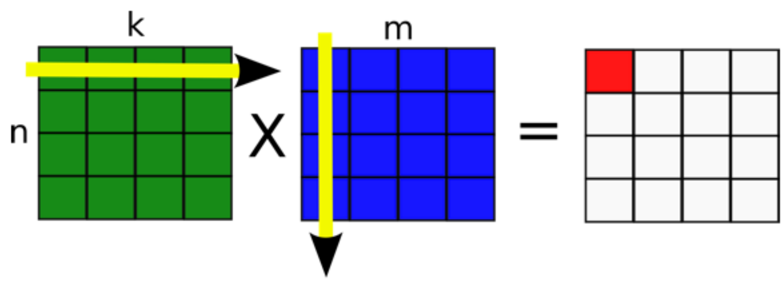
This is a little risky, but let's write out a 2×2 example where, as an experiment, I've labeled the subscripts going vertically in the second matrix

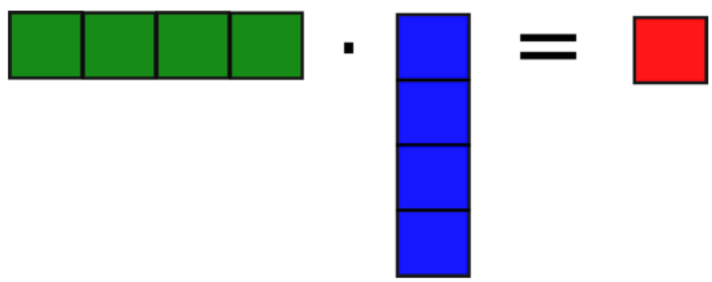
$$M1 \times M2 = M1\ M2$$

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \times \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1\ c_1 + a_2\ c_2 & a_1\ d_1 + a_2\ d_2 \\ b_1\ c_1 + b_2\ c_2 & b_1\ d_1 + b_2\ d_2 \end{bmatrix}$$

In this arrangement, it's clear that the upper left entry is $\mathbf{a} \cdot \mathbf{c}$, where $\mathbf{a}$ is the first row of $M1$ and $\mathbf{c}$ is the first column of $M2$.

Here are two great pictures that I found on the web, which shows the dot product clearly.





It can be confusing keeping track of which row and column you are doing, or even whether it should be a row or a column. That's why I really like the setup shown above in the wikipedia figure, where rather than write A and B on the same line, A is written to the left of the space where P will be filled in, and B is shown above it. Then it's clear which row and which column to choose when forming the sums.

Column picture

The second way of looking at this same operation is introduced by thinking of the multiplication of a matrix A times a vector $\mathbf{x}$ to give a second vector $A \times \mathbf{x} = \mathbf{b}$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

This is the same thing as

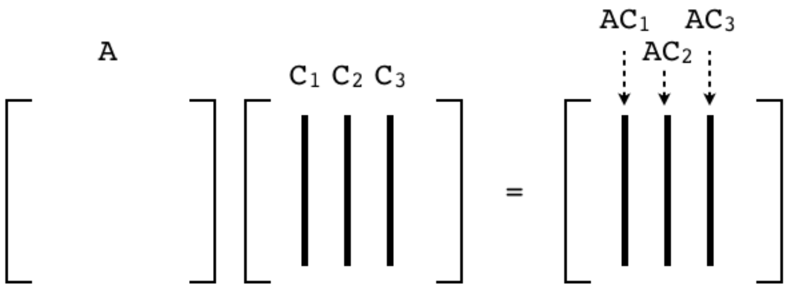
$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

The result **b** is a *combination* of the columns of A. Here is a numerical example

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

This is the same thing as

$$1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$



This extends to a full matrix B instead of the single vector $\mathbf{x}$.

Each of its columns C_1, C_2, etc. picks out a combination of the columns of A, and those combinations are what end up as the columns of the product.

Row picture

In the same way, if we multiply (on the left) by a row vector—the transpose of a column vector—we see that the product is a combination of the rows of A. $A \times \mathbf{x} = \mathbf{b}$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$$

This is the same thing as

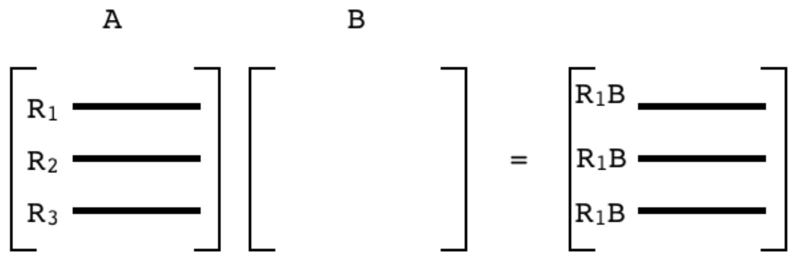
$$x_1 \begin{bmatrix} a_{11} & a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$$

Here is a numerical example

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 8 \end{bmatrix}$$

This is the same thing as

$$1 \begin{bmatrix} 1 & 2 \end{bmatrix} + 2 \begin{bmatrix} 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 8 \end{bmatrix}$$



One row and one column

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1\, b_1 & a_1\, b_2 & a_1\, b_3 \\ a_2\, b_1 & a_2\, b_2 & a_2\, b_3 \\ a_3\, b_1 & a_3\, b_2 & a_3\, b_3 \end{bmatrix}$$

For each row of A we find the correct column of B and do this multiplication, generating a whole series of matrices of full size. Then we add together all the matrices.

Blocks

The fifth and last view of matrix multiplication involves thinking about regions derived from the original matrix but containing a number of elements.

$$\begin{array}{cc} \mathbf{A} & \mathbf{B} \\ \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} & \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \end{array}$$

The upper-left hand corner (2 x 2) would be formed by $A_11 \times B_11 + A_21 \times B_12$. For example, with these two 4 x 4 matrices

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \times \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{bmatrix} =$$

The entry in row 1, column 1 is computed in the standard "dot product" way as $aA + bE + cI + dM$, but if you broke it up into blocks the upper left-hand corner (a 2 x 2 matrix) would be computed as follows

$$\begin{bmatrix} a & b \\ e & f \end{bmatrix} \times \begin{bmatrix} A & B \\ E & F \end{bmatrix} + \begin{bmatrix} c & d \\ g & h \end{bmatrix} \times \begin{bmatrix} I & J \\ M & N \end{bmatrix}$$

$$= \begin{bmatrix} aA + bE & aB + bF \\ eA + fE & eB + fF \end{bmatrix} + \begin{bmatrix} cI + dM & cJ + dN \\ gI + hM & gJ + hN \end{bmatrix}$$

I won't fill in the whole thing, but you can see that the entry in row 1, column 1 is indeed $aA + bE + cI + dM$.