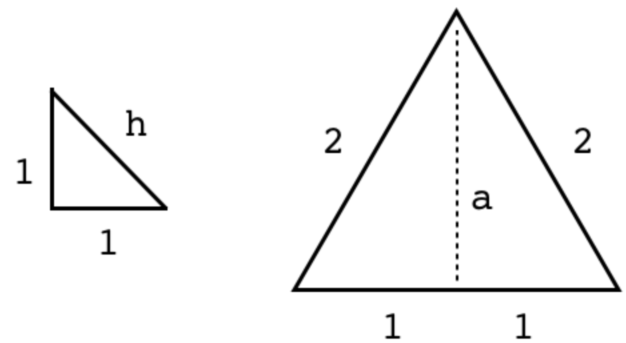
Sine and cosine of simple angles

We'll consider the two simple triangles shown below.



The one on the left is a right triangle with both side lengths equal to 1. Since it is an isosceles triangle, the two angles that flank the hypotenuse are equal. Since the sum of those angles is 90°, each one is 45°.

Furthermore, by the Pythagorean theorem $h^2 = 2$ so $h = \sqrt{2}$. Therefore

$$sin(45^\circ) = cos(45^\circ) = \frac{1}{\sqrt{2}}$$

Or, using radians for the angle measure:

$$sin\frac{\pi}{4} = cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

The triangle on the right is an equilateral triangle. If we drop an altitude a, the two smaller triangles are congruent. Since both are right triangles and the angles at the base (left and right) are 60°, these are 30-60-90 triangles. The angles at the top are both equal to 30°, so we see that

$$sin(30^\circ) = cos(60^\circ) = \frac{1}{2}$$

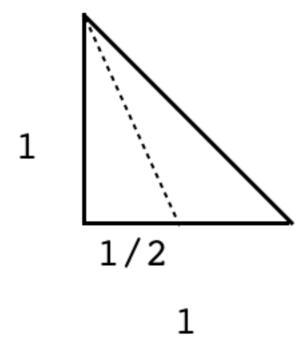
Using radians for the angle measure

$$sin(\frac{\pi}{6}) = cos(\frac{\pi}{3}) = \frac{1}{2}$$

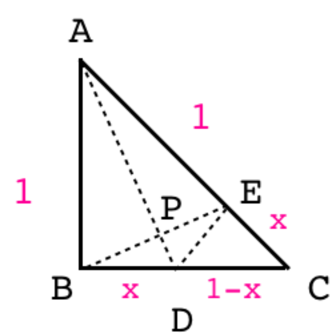
We can find the length of a by the Pythagorean Theorem, it is equal to $\sqrt{3}$. So the cosine of 30°

$$cos(30°) = cos(\frac{\pi}{6}) = sin(60°) = sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$$

Finally, let's do 22.5°



While the figure above looks tempting, it is not correct! $tan^{-1}(1/2) = 26.565°$.



We draw AP, the bisector of 45° $\angle BAC$ and also draw BE, a perpendicular to the bisector, as shown. BE divides the hypotenuse into two pieces. Now we have an isosceles $\triangle ABE$ with sides of 1 and an apical angle of 45°, so the angle $\angle ABP$ at its base is 67.5°.

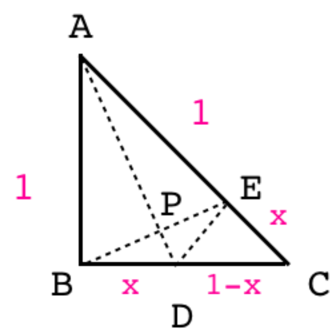
We also connect the two new points DE.

Since $\angle APE$ is a right angle, and AP is the angle bisector, $\triangle APB \cong \triangle APE$, by AAA plus one side (all four angles at P are right angles). Therefore, $BP = PE$.

By SAS, $\triangle BPD \cong \triangle DPE$. Therefore, $DE = BD = x$.

Also, $\angle BAP$ and ABP add to $90°$, but so do $\angle ABP$ and $\angle PBD$, so $\angle PBD = \angle BAP$, and the two pairs of triangles are all similar.

Finally, since $\triangle APE \sim \triangle DPE$, $\angle AEP$ and $\angle PED$ are complementary and so $\angle AED = \angle DEC$ is a right angle. Since $\angle DCE = 45°$, $\angle EDC = 45° = \angle DCE$ and thus, $\triangle DEC$ is isosceles, and EC is also equal to x.



We look at the figure again and focus on the side lengths. From $\triangle DEC$ we see that

$$\frac{x}{1-x} = cos(45°) = \frac{1}{\sqrt{2}}$$

Inverting we have

$$\frac{1-x}{x} = \sqrt{2}$$

$$\frac{1}{x} = 1 + \sqrt{2}$$

$$x = \frac{1}{1+\sqrt{2}}$$

From $\triangle ABD$ we see that

$$x = \frac{1}{1+\sqrt{2}} = \frac{1}{2.414} = 0.4142 = tan(22.5°)$$

And this can be easily checked on a calculator. To get the sine and cosine requires computing the hypotenuse.

a better way

However, after all that, I must admit there is another way. It involves the formula for sum of cosine

$$cos(s + t) = cos\ s\ cos\ t - sin\ s\ sin\ t$$

If $s = t$

$$cos(2s) = cos^2 s - sin^2 s = 2cos^2 s - 1$$

$$cos^2 s = \frac{1}{2}(1 + cos(2s))$$

$$cos\ s = \sqrt{\frac{1}{2}(1 + cos(2s))}$$

since

$$cos(45^\circ) = cos(2s) = 1/\sqrt{2}$$

$$cos\ s = cos(22.5^\circ) = \sqrt{\frac{1}{2}(1 + \frac{1}{\sqrt{2}})} = 0.9239$$

$$sin(22.5^\circ) = \sqrt{1 - cos^2(22.5^\circ)} = \sqrt{1 - (0.9239)^2} = 0.3826$$

And in general, the full complement of sum and difference equations

$$sin(s + t) = sin\ s\ cos\ t + sin\ t\ cos\ s$$

$$sin(s - t) = sin\ s\ cos\ t - sin\ t\ cos\ s$$

$$cos(s - t) = cos\ s\ cos\ t = sin\ s\ sin\ t$$

can be used to obtain sine and cosine for other angles, starting with these four (plus 90°). We can get 15° as half of 30° and 7.5° as half of 15°, or as the difference between 30° and 22.5°.