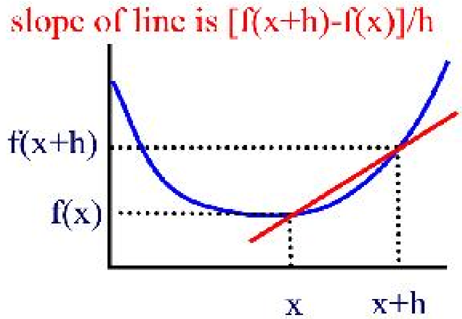
Tangent to the curve $f(x)$

In this short write-up we'll look at the geometric interpretation of the derivative—the beginning of an introduction to calculus. Suppose we have a curve like in the figure, corresponding to some function. Let's think for a minute about the general case, $f(x)$.



We pick a point P on the curve. The value of x at P is x (of course), and the value of y is $f(x)$. That is, the point P has coordinates $P = (x, f(x))$. Now consider moving to a point Q near P but also on the curve, by adding a small amount to x. We could call that small amount Δx, but many books use h, so we'll try that. The value of the function at $x + h$ is $f(x + h)$ and $Q = (x + h, f(x + h))$.

The slope of the line (the secant) connecting Q and P is

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

This is a famous quantity, it's called the *difference quotient*.

The goal of this part of calculus is to find the slope of the tangent to the curve at the point P. What we have is an expression for the slope of the line PQ, which is close but not quite the same. To go from the secant to the tangent, we ask "what happens to this expression as h gets smaller and smaller and approaches zero." In mathematical language, we say the slope of the tangent is equal to

$$\boxed{\lim_{h \to 0} \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}} \tag{1}$$

Let's try a couple of examples and look for a pattern.

Example 1. $f(x) = x^2$

Let's go without the limit sign to start with. For this function, we write that the difference quotient is

$$\frac{(x+h)^2 - x^2}{h}$$
$$\frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$\frac{2xh + h^2}{h}$$

Now we *divide by* h

$$2x + h$$

Finally, to get the slope of the tangent, we evaluate the limit part

$$\lim_{h \to 0} 2x + h = 2x$$

At every point on the curve $y = x^2$, the slope of the tangent line to the curve is $2x$. So the slope at $x = 0$ is 0, and the slope at $x = 2$ is 4, and so on. We call this process of computing the difference quotient and then taking the limit as $h \to 0$, "taking the derivative." It produces an expression which is the derivative of y with respect to x, in this case

$$\frac{dy}{dx} = 2x$$

Another useful shorthand uses the f from $f(x)$, we adopt the convention that the derivative of $f(x)$ is $f'(x)$.

If we repeat this exercise with a leading constant a (that is, for $f(x) = ax^2$), we find that every term in the numerator of the difference quotient will contain a, and the final result will be $2ax$.

Example 2. $f(x) = \sqrt{x}, \quad (x \geq 0)$

The difference quotient for this function is

$$\frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Clean up the numerator by multiplying by the conjugate

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} \; \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$\frac{x+h-x}{h\;(\sqrt{x})(\sqrt{x+h})}$$

$$\frac{h}{h\;\sqrt{x}\;\sqrt{x+h}}$$

$$\frac{1}{\sqrt{x}\;\sqrt{x+h}}$$

We evaluate the limit

$$m = \lim_{h \to 0} \; \frac{1}{\sqrt{x}\;\sqrt{x+h}} = \frac{1}{2\sqrt{x}}$$

Example 3. $f(x) = 1/x, \quad (x \neq 0)$

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Clean up the numerator

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h} \; \frac{(x)(x+h)}{(x)(x+h)}$$

$$\frac{x-(x+h)}{h\;(x)\;(x+h)}$$

$$\frac{-h}{h\;(x)\;(x+h)}$$

$$-\frac{1}{(x)\;(x+h)}$$

We put the limit part in

$$\lim_{h \to 0} \; -\frac{1}{(x)\;(x+h)} = -\frac{1}{x^2}$$

Summary

So there's a pattern here. We will use the notation $f'(x)$ to indicate the slope of the curve $f(x)$ at x, obtained as

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x$$

$$f(x) = \sqrt{x} = x^{1/2} \quad \Rightarrow \quad f'(x) = \frac{1}{2}x^{-1/2}$$

$$f(x) = \frac{1}{x} = x^{-1} \quad \Rightarrow \quad f'(x) = -\frac{1}{x^2} = -x^{-2}$$

The general formula is

$$f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1}$$

We will prove this next time.