Parametrizing the sphere

A sphere centered at the origin is defined as the set of points x, y, z at a distance ρ away from $(0, 0, 0)$, leading to the equation $x^2 + y^2 + z^2 = \rho^2$. We are looking for a "parametrization" or relationship between x, y, z coordinates and spherical coordinates in terms of one radial and two angular variables. These are usually called ρ, θ, and ϕ.

If we think of the vector $< x, y, z >$ to a point on the sphere, then θ is the angle it makes going ccw from the positive x-axis and ranges from $0 \leq \theta \leq 2\pi$. ϕ is the "polar" angle that the same vector makes with the positive z-axis and ranges from $0 \leq \phi \leq \pi$.

The projection of ρ in the xy-plane is r.

$$r = \rho \cos(\frac{\pi}{2} - \phi) = \rho \sin \phi$$

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

$$z = \rho \sin(\frac{\pi}{2} - \phi) = \rho \cos \phi$$

Now, above we said that $x^2 + y^2 + z^2 = \rho^2$, as if it were obvious. For a circle, we know that $x^2 + y^2 = r^2$ by using the Pythagorean theorem. To get the same thing for a sphere, we use it in 3-dimensions, i.e. $x^2 + y^2 = r^2$ and then $r^2 + z^2 = \rho^2$.

Let's just check that

$$\begin{gathered} x^2 + y^2 + z^2 \\ = \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \cos^2 \phi \\ = \rho^2 (\sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta + \cos^2 \phi) \\ = \rho^2 (\sin^2 \phi + \cos^2 \phi) \\ = \rho^2 \end{gathered}$$

For what comes below we will need all 9 partial derivatives.

$$x_\rho = \sin\phi\cos\theta$$

$$x_\phi = \rho\cos\phi\cos\theta$$

$$x_\theta = -\rho\sin\phi\sin\theta$$

$$y_\rho = \sin\phi\sin\theta$$

$$y_\phi = \rho\cos\phi\sin\theta$$

$$y_\theta = \rho\sin\phi\cos\theta$$

$$z_\rho = \cos\phi$$

$$z_\phi = -\rho\sin\phi$$

$$z_\theta = 0$$

When we change variables from x, y, z to ρ, θ, ϕ, the scaling factor for the volume element dV is the Jacobian:

$$dx\ dy\ dz = J\ d\rho\ d\phi\ d\theta$$

where J is the absolute value of the determinant of this matrix:

$$J = \begin{vmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{vmatrix}$$

If you notice, $z_\theta = 0$, which suggests we compute using either the third row or the third column.

$$J = x_\theta(y_\rho z_\phi - y_\phi z_\rho) - y_\theta(x_\rho z_\phi - x_\phi z_\rho)$$

Now we just plug in from our list above. The first term is

$$-\rho\sin\phi\sin\theta\ (\sin\phi\sin\theta\ (-\rho\sin\phi) - \rho\cos\phi\ \ \sin\theta\cos\phi)$$

$$= -\rho\sin\phi\sin\theta\ (-\rho\ \ \sin\theta)$$

$$= \rho^2\sin\phi\sin^2\theta$$

while the second term is

$$-\rho\sin\phi\cos\theta\ (\sin\phi\cos\theta\ (-\rho\sin\phi) - \rho\cos\phi\cos\theta\ \ \cos\phi)$$

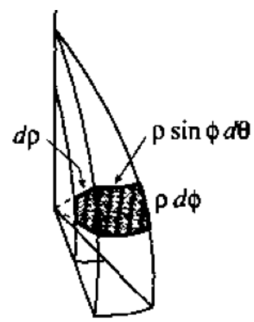
$$= -\rho \sin\phi \cos\theta \; (-\rho\cos\theta)$$
$$= \rho^2 \sin\phi \cos^2\theta$$

Putting them together

$$J = \rho^2 \;\sin\phi(\sin^2\theta + \cos^2\theta) = \rho^2 \;\sin\phi$$

So our volume element is

$$dV = dx\; dy\; dz = \rho^2 \sin\phi \; d\rho\; d\phi\; d\theta$$



Notice that the top of the box is $\rho \sin\phi \; d\theta = r d\theta$, varying with ϕ, while the sides do not depend on the polar angle but are just $\rho \; d\phi$.

We might as well check this

$$V = \iiint dV = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{a} \rho^2 \sin\phi \; d\rho\; d\phi\; d\theta$$
$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \frac{1}{3} a^3 \sin\phi \; d\phi\; d\theta$$
$$= \int_{\theta=0}^{2\pi} \frac{1}{3} a^3 (-\cos\phi)\Big|_0^{\pi} \; d\theta$$
$$= \int_{\theta=0}^{2\pi} \frac{1}{3} a^3 (2)\; d\theta$$
$$= \frac{1}{3} a^3 (2)(2\pi)$$
$$= \frac{4}{3}\pi a^3$$

which seems to be correct.

surface

How about parametrizing the surface of the sphere? In this case ρ is a constant, and we will have only two variables, similar to longitude and latitude. The standard parametrization of the (unit) sphere is

$$\mathbf{r}(\phi, \theta) = < \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi >$$

$$\mathbf{r}_\phi = < \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi >$$

$$\mathbf{r}_\theta = < -\sin\phi\sin\theta, \sin\phi\cos\theta, 0 >$$

The cross-product is

$$\mathbf{r}_\phi \times \mathbf{r}_\theta =$$

$$< -\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi >$$

If we want

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi}$$

$$= \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi}$$

$$= \sqrt{\sin^2\phi}$$

$$= \sin\phi$$

In my writeup of the first part of Schey's book (chapter 2), we saw that the normal vector to a surface is

$$\hat{\mathbf{n}} = \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|}$$

Dividing the cross-product above by its absolute value we get

$$\frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|}$$

$$= \frac{1}{\sin\phi} \ < -\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi >$$

$$= < -\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi >$$

$$= < -x, -y, -z >$$

???