The exponential function

In Algebra, you probably met this equation for principal and interest

$$A = P(1 + \frac{r}{n})^{nt}$$

where n is the number of periods in a year, t is the number of years, and r is the interest rate per year. This equation has been known for a long time, and people started playing around with this variant

$$(1 + \frac{1}{n})^n$$

and they wondered what happens as n gets very large. What is

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n \stackrel{?}{=}$$

method 1

I saw an interesting approach to this in the book *Mooculus*. We will use L'Hospital's Rule. With their notation, we're looking for

$$\lim_{x\to\infty} (1 + \frac{1}{x})^x$$

the first thing is to rewrite this as the exponential

$$= \lim_{x\to\infty} e^{x\ln(1+1/x)}$$

To evaluate the limit, we need to evaluate the limit of the exponent

$$\lim_{x\to\infty} x \ln(1+\frac{1}{x})$$

It doesn't look like we can use the rule (there is no quotient), but there is a standard trick for these situations. Just rearrange like so

$$= \lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}$$

Both the top and the bottom limits are easily evaluated to be equal to 0. So we will differentiate. The derivative of the numerator is (by the chain rule)

$$f'(x) = \frac{-x^{-2}}{1+1/x}$$

while the denominator is just

$$g'(x) = -x^{-2}$$

So we need to evaluate

$$= \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

The factor of $-x^{-2}$ cancels from both top and bottom, leaving us with

$$= \lim_{x\to\infty} \frac{1}{1+1/x} = 1$$

Substituting back, we see that

$$\lim_{x\to\infty} (1+\frac{1}{x})^x = e^1 = e$$

method 2

If we use the standard binomial expansion

$$\frac{1}{0!}a^n + \frac{n}{1!}a^{n-1}b^1 + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 \ + \cdots$$

with a = 1

$$\frac{1}{0!} + \frac{n}{1!}b^1 + \frac{n(n-1)}{2!}b^2 + \frac{n(n-1)(n-2)}{3!}b^3 \ + \cdots$$

plug in for b = 1/n

$$\frac{1}{0!}(\frac{1}{n})^0 + + \frac{n}{1!}(\frac{1}{n})^1 + \frac{n(n-1)}{2!}(\frac{1}{n})^2 + \frac{n(n-1)(n-2)}{3!}(\frac{1}{n})^3 + \cdots$$

Now, in the limit, n and $n-1$ are nearly equal, and so are n and $n-2$, and all the terms n, $(n-1)$, $(n-2)$ find just the right number of n's in the denominator to cancel and we get

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

which is one of several equivalent definitions for e. The first three terms are 1+1+1/2, which is reasonably close. After six terms, we have $e = 2.718$. The series converges rapidly because the inverse factorials get small very quickly.

So e is just a number. We can define the exponential function, which means we raise e to the power x, that is $f(x) = e^x$. We have

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

We want

$$e^x = \lim_{n \to \infty} (1 + \frac{1}{n})^{nx}$$

It turns out this is exactly the same as

$$e^x = \lim_{n \to \infty} (1 + \frac{x}{n})^n$$

That's because

$$(1+b)^{mn} = (1+bm)^n$$

To confirm this, look at the binomial expansion for these two expressions. Remember that the terms $n(n-1)(n-2)$ etc. come from this formula

$$\frac{n!}{(n-k)!k!}$$

so for example, with $k = 2$ we have

$$\frac{n(n-1)(n-2)(n-3)!}{(n-3)!2!} = \frac{n(n-1)(n-2)}{2!}$$

With nx as the exponent

$$\frac{(nx)!}{((nx)-k)!k!}$$

for example with $k = 2$

$$\frac{nx(nx-1)(nx-2)}{2!} = x^3\frac{n(n-1)(n-2)}{2!}$$

Thus, in the first one

$$e^x = \lim_{n\to\infty}(1+\frac{1}{n})^{nx}$$

we get a power of x coming in for each term in $n(n-1)(n-2)$, so that's

$$x^0\frac{1}{0!} + +x^1\frac{n}{1!}(\frac{1}{n}) + x^2\frac{n(n-1)}{2!})(\frac{1}{n})^2 + x^3\frac{n(n-1)(n-2}{3!}(\frac{1}{n})^3 + \cdots$$

while the second one is

$$\frac{1}{0!}(\frac{x}{n})^0 + +\frac{n}{1!}(\frac{x}{n})^1 + \frac{n(n-1)}{2!}(\frac{x}{n})^2 + \frac{n(n-1)(n-2)}{3!}(\frac{x}{n})^3 + \cdots$$

In both of these, the n's cancel as before and we have

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The exponential is its own derivative

A most important fact about the exponential function $f(x) = e^x$ is that this function is its own derivative. To see that, take $\frac{d}{dx}$ of the last series.

$$\frac{d}{dx}\ e^x = 0 + (1)\frac{x^{1-1}}{1!} + (2)\frac{x^{2-1}}{2!} + (3)\frac{x^{3-1}}{3!} + \cdots$$

$$\frac{d}{dx}\ e^x = 0 + \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots = e^x$$

Each exponent n that comes down through the power rule, finds an n in $n!$ to cancel, leaving $n-1$ in the exponent as well as $(n-1)!$.

We can find the derivative of a generic exponential using the standard method with the difference quotient. Let b be the base, then $f(x) = b^x$ and

$$\lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$$

But

$$b^{x+h} = b^x b^h$$

so this is just

$$\lim_{h \to 0} \frac{b^x b^h - b^h}{h} = \lim_{h \to 0} \frac{b^x(b^h - 1)}{h} = b^x \lim_{h \to 0} \frac{(b^h - 1)}{h} = cb^x$$

At this point we don't know what the value of c is, but it is just a number which depends on b, but *not on* x, so we know it's a constant. It turns out that if $b = e$, then $c = 1$. You can see that by looking at

$$\lim_{h \to 0} \frac{(e^h - 1)}{h}$$

For small h we can approximate $e^h = 1 + h$ (see the series above), then

$$\lim_{h \to 0} \frac{(1 + h - 1)}{h} = \frac{h}{h} = 1$$