Elastic collisions

For an elastic collision, energy is conserved. Call v the velocity before collision and w the velocity after

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1w_1^2 + \frac{1}{2}m_2w_2^2$$

Rearrange (and multiply by 2):

$$m_1v_1^2 - m_1w_1^2 = m_2w_2^2 - m_2v_2^2$$

Factor

$$m_1(v_1 + w_1)(v_1 - w_1) = m_2(w_2 + v_2)(w_2 - v_2)$$

Conservation of momentum gives

$$m_1v_1 + m_2v_2 = m_1w_1 + m_2w_2$$

Rearrange the momentum equation and factor out the masses

$$m_1(v_1 - w_1) = m_2(w_2 - v_2)$$

Divide to obtain

$$v_1 + w_1 = v_2 + w_2$$

solve this linear system for w_1 by eliminating w_2

Isolate w_2

$$w_2 = v_1 + w_1 - v_2$$

multiply everything by m_2

$$m_2 w_2 = m_2 v_1 + m_2 w_1 - m_2 v_2$$

Go back to the momentum equation, isolate $m_2 w_2$

$$m_2 w_2 = m_1 v_1 + m_2 v_2 - m_1 w_1$$

Equate:

$$m_2 v_1 + m_2 w_1 - m_2 v_2 = m_1 v_1 + m_2 v_2 - m_1 w_1$$

Gather terms containing w_1

$$m_2 w_1 + m_1 w_1 = m_1 v_1 + m_2 v_2 - m_2 v_1 + m_2 v_2$$

$$(m_1 + m_2) w_1 = (m_1 - m_2) v_1 + 2 m_2 v_2$$

Divide by $M = m_1 + m_2$ to reach

$$w_1 = \frac{m_1 - m_2}{M} v_1 + \frac{2m_2}{M} v_2$$

We have the final velocity for m_1 in terms of the masses and the velocities of the objects before the collision.

Write the result with v' rather than w for those who like it that way:

$$v_1' = \frac{m_1 - m_2}{M} v_1 + \frac{2m_2}{M} v_2$$

The equation for v_2' is obtained by symmetry (switch 1 for 2 as needed):

$$v_2' = \frac{m_2 - m_1}{M} v_2 + \frac{2m_1}{M} v_1$$

Note that in the case where $m_1 = m_2$

$$v_1' = v_2$$

and similarly, $v_2' = v_1$.

Atwood machine

I ran into very similar equations in what seems to be a different context (in Fitzpatrick, Chapter 4). There is a device called an "Atwood machine".

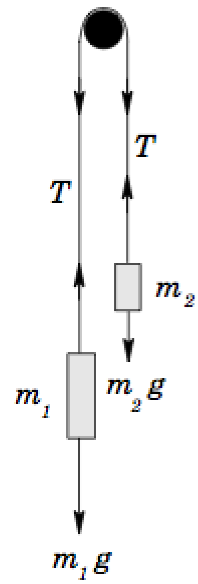

Figure 31: *An Atwood machine*

(massless, friction-less pulley, etc.) If mass m_1 is bigger, then it will move downward.

As a sign convention, assume that a is positive downward for m_1 *and positive upward for* m_2. Then, the forces on m_1 and m_2 are

$$m_1 a = m_1 g - T$$

$$m_2 a = T - m_2 g$$

By addition

$$m_1 a + m_2 a = m_1 g - m_2 g$$

$$a = \frac{m_1 - m_2}{m_1 + m_2} g$$

Alternatively, by solving for a and equating we obtain

$$g - \frac{T}{m_1} = \frac{T}{m2} - g$$

$$2g = T(\frac{1}{m_1} + \frac{1}{m_2})$$

$$2g = T(\frac{m_1 + m_2}{m_1 m_2})$$

$$T = g\frac{2m_1 m_2}{m_1 + m_2}$$

Not sure if there is a connection.