Fibonacci

Probably everyone knows the Fibonacci numbers. Start with

$$1,\ 1,\ 2$$

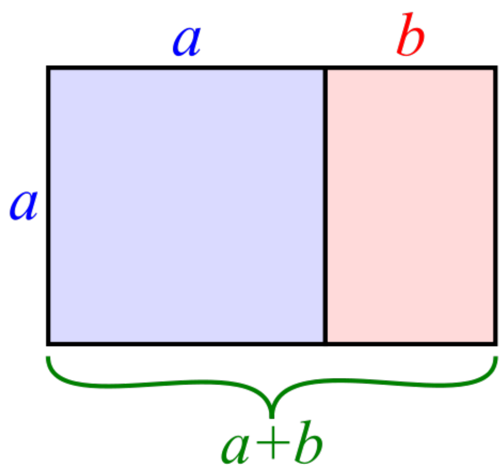
The third term is the sum of the previous two. And the next term is the sum of the previous two

$$1,\ 1,\ 2,\ 3$$

And so on..

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233\cdots$$

One interesting thing about the series is that the ratio of successive elements approaches the "golden ratio."



In the figure above, if the whole rectangle and the left-hand rectangle have the same proportions, that proportion is the golden mean

$$\frac{a+b}{a} = \frac{a}{b} = \phi$$

$$\phi = \frac{a}{b} = \frac{a+b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\phi}$$

so

$$\phi^2 - \phi - 1 = 0$$

$$\phi = \frac{1 \pm \sqrt{5}}{2} = 0.5 \pm 1.118034$$

These are often called ϕ_1 and ϕ_2

$$\phi_1 = 1.618034$$

$$\phi_2 = 0.618034$$

For right now, I prefer to stick with ϕ and $1/\phi$.

Going back to the Fibonacci numbers

$$89/55 = 1.6181818 \quad (> \phi)$$

$$144/89 = 1.617976 \quad (< \phi)$$

$$233/144 = 1.618055 \quad (> \phi)$$

Each successive number is first above and then below ϕ, and the absolute value of the difference gets smaller with each successive term.

One way of computing successive Fibonacci numbers would be to multiply by ϕ and then round to the nearest whole number. The next one after 233 is 377:

$$233 \times \phi = 233 \times 1.618034$$

$$= 377.001922 \cong 377 = 144 + 233$$

But what would be really nice would be to compute the nth Fibonacci number without finding all the ones that come before it.

Binet's formula will do that

$$F_n = \frac{1}{\sqrt{5}}(\phi^n + (1/\phi)^n)$$

For any $n > 4$ you can ignore the second term $((1/\phi)^4 = 0.014$, $(1/\phi)^n \to 0$.

$$F_n = \frac{\phi^n}{\sqrt{5}}$$

$$F_{13} = \frac{(1.618034)^{13}}{\sqrt{5}} = 232.999163$$

Here the formula is exact, the error is in the rounding of the value of ϕ. There is some really nice linear algebra related to Fibonacci, but I'll put that in a separate write-up.