Examples for Stokes theorem

State the theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, dS$$

By the usual reasoning, since $d\mathbf{r} = < dx, dy, dz >$, the left-hand side is

$$P \ dx + Q \ dy + R \ dz$$

Now, suppose we have

$$\mathbf{F} = < z, x, y >$$

and C is the unit circle in the xy-plane, then

$$P \ dx + Q \ dy + R \ dz = \oint_C z \ dx + x \ dy + y \ dz = \oint_C x \ dy$$

Parameterize

$$C = \begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

we have

$$\oint_C x \ dy = \int_0^{2\pi} \cos t \ \cos t \ dt$$

$$= \frac{1}{2}(t + \frac{1}{2} \sin t) \Big|_0^{2\pi} = \pi$$

For the surface, we can use anything that passes through C, let's use the paraboloid for fun.

$$z = 1 - x^2 - y^2$$

We need the curl of $\mathbf{F} = < z, x, y >$

$$\nabla \times \mathbf{F} = < 1, 1, 1 >$$

We need

$$\hat{\mathbf{n}} \ dS = < -f_x, -f_y, 1 > \ dx \ dy = < 2x, 2y, 1 > \ dx \ dy$$

so

$$\iint_R (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\ dS = \iint_R\ 2x + 2y + 1\ dx\ dy$$

Again, C is the unit circle in the xy-plane. To save effort, we can notice that

$$\int x\ dx = \overline{x}$$

What is the *average* value of x over the unit circle? It is just equal to 0. The same thing is true for the second integrand (reverse the order of integration). So we have just

$$\iint_R\ 1\ dx\ dy = \pi$$

which matches what we had above.

Suppose we hadn't seen this. We could just do

$$\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\ x\ dy\ dx$$

$$= \int_{x=-1}^{1} 2\sqrt{1-x^2}\ x\ dy\ dx$$

$$= -\frac{2}{3}\ (1-x^2)^{3/2}\ \Big|_{-1}^{1}$$

At both bounds, $1 - x^2 = 0$, so the whole thing is 0.