Integrate 1/z

$$\int_0^{2\pi} \frac{1}{z}\ dz$$

Examining the inverse function, let's first confirm that it is analytic by calculating the partial derivatives. We have

$$\frac{1}{z} = \frac{1}{x+iy}$$

One way to simplify is to multiply on top and bottom by $z*$:

$$= \frac{1}{x+iy}\ \frac{x-iy}{x-iy}$$

$$= \frac{x-iy}{x^2+y^2}$$

Thus

$$u = \frac{x}{x^2+y^2}$$

$$u_x = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = \frac{-2xy}{(x^2+y^2)^2}$$

And

$$v = \frac{-y}{x^2+y^2}$$

$$v_y = -\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$v_x = \frac{2xy}{(x^2+y^2)^2}$$

CRE are satisfied and the inverse of z is indeed analytic.

If we are on the unit circle, then

$$z = e^{i\theta}$$

$$dz = ie^{i\theta} d\theta$$
$$\int \frac{dz}{z} = \int e^{-i\theta}\ ie^{i\theta}\ d\theta = 2\pi i$$

If we're centered on the origin but we don't have a unit circle, there will be an R in both the numerator and the denominator, which cancel. The result is thus independent of the radius of the circle.

In general
$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 0, & n \neq 1 \\ 2\pi i, & n = 1 \end{cases}$$

We can also integrate the inverse function in terms of x and y:
$$\oint \frac{1}{z}\ dz = \oint \frac{dx + idy}{x+iy}$$
$$= \oint \frac{1}{x^2+y^2}\ [\ x\ dx - y\ dy + ix\ dy + iy\ dx\ ]$$

Suppose we go on a circle of radius R centered on the origin and parametrize in terms of θ. We obtain:

$$x = R\cos\theta$$

$$y = R\sin\theta$$

$$x^2 + y^2 = R^2$$

$$dx = -R \sin\theta \; d\theta$$

$$dy = R \cos\theta \; d\theta$$

We have for the integral

$$\oint \frac{1}{x^2 + y^2} \; [\; x \; dx - y \; dy + ix \; dy + iy \; dx \;]$$

$$= \int \frac{1}{R^2} \, [\, -R^2 \cos\theta \sin\theta \; d\theta + R^2 \sin\theta \cos\theta \; d\theta + iR^2 \cos^2\theta \; d\theta + iR^2 \sin^2\theta \; d\theta \,]$$

$$= \int \frac{1}{R^2} \; [\; iR^2 \cos^2\theta \; d\theta + iR^2 \sin^2\theta \; d\theta \;]$$

$$= \int i \cos^2\theta \; d\theta + i \sin^2\theta \; d\theta$$

$$= \int i \; d\theta = 2\pi i$$

Note that if we integrate the same function around a unit square, we run into problems. First let's do $[0, 0 \times 1, 1]$. We have

$$\int u \; dx - \int v \; dy + i \; [\; \int v \; dx + \int u \; dy \;]$$

Along $C1$, $y = 0$ and $dy = 0$ so:

$$\int \frac{x}{x^2 + y^2} \; dx + i \; [\; \int \frac{-y}{x^2 + y^2} \; dx$$

$$= \int_0^1 \frac{1}{x} \; dx = \ln x \; \Big|_0^1$$

Since $\ln 0$ is not defined, we can't do this.

Logarithms are tricky, no doubt. If the complex logarithm $Log z$ is defined and differentiable along the curve (say the semicircle from $-i$ to i), we can do this:

$$I = \int_{-i}^{i} \frac{1}{z}\ dz = Log z \ \Big|_{-i}^{i}$$

Recall that $z = re^{i\theta}$ with $r = 1$ so this is

$$= (\ln 1 + i\ \frac{\pi}{2}) - (\ln 1 + i\frac{-\pi}{2}) = 2i\ \frac{\pi}{2} = \pi i$$

For any value of r (except $r = 0$), we get the same answer, since $\ln r - \ln r = 0$.

example

We can extend this to

$$\oint \frac{1}{z^2}\ dz$$

As before, on the unit circle

$$z = e^{i\theta}$$

$$dz = iz\ d\theta$$

so the integral is

$$\int_0^{2\pi}\ \frac{i}{z}\ d\theta = \int_0^{2\pi}\ ie^{-i\theta}\ d\theta$$

Now

$$\int e^{-i\theta}\ d\theta = -ie^{-i\theta}$$

so cancel $i \cdot -i$ and we have just

$$= e^{-i\theta}\ \Big|_0^{2\pi}$$

Evaluate the first term using Euler's formula:

$$e^{-2\pi i} = \cos -2\pi + i \sin -2\pi$$

$$= \cos 2\pi - i \sin 2\pi = 1$$

So the whole thing is zero.

In fact, for any negative integer power of z

$$\int z^{-n} \ dz$$

around the unit circle $z = e^{i\theta}$ we have

$$i \int e^{-i(n-1)\theta} \ d\theta$$

$$= \frac{1}{n-1} \ e^{-i(n-1)\theta} \ \Big|_0^{2\pi}$$

$$= \frac{1}{n-1} \ [\ (\cos 2(n-1)\pi - i \sin 2(n-1)\pi) \ \ -1]$$

$$= \frac{1}{n-1} \ [\ 1 - 1] \ \ = 0$$