Integrating functions of a complex variable

Complex functions are differentiated and integrated in a way that is similar to real functions, with these differences:

- we typically restrict attention to analytic functions
- the integrals that we compute are line integrals along some path
- we pay attention to points in the complex plane with poles (or singularities)

In general terms, if we have

$$z = x + iy$$

$$dz = dx + idy$$

and the function

$$w = f(z) = u(x, y) + iv(x, y)$$

we can write

$$\int f(z)\ dz = \int (u + iv)(dx + idy)$$

$$= \int u\ dx - \int v\ dy + i\ [\ \int v\ dx + \int u\ dy\ ]$$

What was an integral of a complex function has been transformed into two integrals of real variables, where the real variables are related by the curve over which we will integrate.

Just as with line integrals for real functions of x and y, this is *not* some kind of double integral in both variables. We can view y as a function of x or perhaps, we can parametrize both x and y as functions of t.

Recall that for the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = M\ dx + N\ dy$$

we parametrize the curve to get the integral over a single variable.

Suppose our function is simply $z = x + iy$. The integral is

$$\int z\ dz = \int (x + iy)(dx + idy)$$

$$= \int x\ dx - y\ dy + ix\ dy + iy\ dx$$

We use the curve to get $y = f(x)$ or both x and y as functions of some parameter t.