Fourier series, introduction

If we consider the two series $\cos kx$ and $\sin kx$:

$$\cos x, \cos 2x, \cos 3x \ldots \cos kx$$

$$\sin x, \sin 2x, \sin 3x \ldots \sin kx$$

where $k \in \mathbb{N} \quad (k = 1, 2, 3 \ldots)$

Any function in this set is *orthogonal* to each of the others, by which we mean that:

$$\int f(x)\ g(x)\ dx = 0$$

Thus, considering the cosine-cosine pair, for $m, n \in \mathbb{N}$:

$$\int \cos mx \cos nx\ dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \\ 2\pi, & m = n = 0 \end{cases}$$

The usual intervals over which we will integrate are $[0, 2\pi]$ and especially $[-\pi, \pi]$ — these are bounds that are each a multiple of π and the length of the interval is equal to 2π.

The sine-sine pair is almost the same, except that the only non-zero case is $m = n \neq 0$, where the result is π.

$$\int \sin mx \sin nx\ dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \\ 0, & m = n = 0 \end{cases}$$

And finally, for the sine-cosine pair all cases are zero.

If you want to be really fancy, you can use what's called the Kronecker Delta:

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

Then

$$\int \cos mx \cos nx \ dx = \pi \delta_{mn}$$

$$\int \sin mx \sin nx \ dx = \pi \delta_{mn}$$

$$\int \cos mx \sin nx \ dx = 0$$

`http://mathworld.wolfram.com/FourierSeries.html`

(This doesn't cover the special behavior of the first function for $m = n = 0$).

We will use these facts to develop Fourier series that approximate $f(x)$ as an infinite sum of sine and cosine functions.

cosine-cosine

Before we do that, let's look at all three integrals in more detail.

Recalling the cosine addition formulas:

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

Add them together and multiply through by 1/2:

$$\frac{1}{2} \ [\ \cos(s + t) + \cos(s - t) \ ] \ = \cos s \cos t$$

So then

$$\int \cos mx \cos nx \; dx$$

$$= \frac{1}{2} \int \cos(m+n)x + \cos(m-n)x \; dx$$

For the first case, $m = n = 0$, we have

$$\frac{1}{2} \int \cos 0 + \cos 0 \; dx = \int dx$$

and so we obtain just x evaluated between some limits.

We will choose as the bounds $[-\pi, \pi]$.

$$\int_{-\pi}^{\pi} dx = 2\pi$$

For this integral ($\int dx$) no matter what limits separated by 2π we might choose, whether $\int_0^{2\pi}$ or any $\int_{-\pi}^{\pi}$ or $\int_a^{a+2\pi}$, we would obtain 2π for the result.

On the other hand, if $m = n \neq 0$, then in exactly the same way, we obtain the value of π as the result from the second term (remember the factor of $\frac{1}{2}$). The first term is zero, as follows:

For any non-zero integer k, whether $k = m + n$ or $k = m - n$ (as we will have below):

$$\int_{-\pi}^{\pi} \cos kx \; dx = \frac{\sin kx}{k} \Bigg|_{-\pi}^{\pi}$$

whether we choose bounds $\int_0^{2\pi}$ or $\int_{-\pi}^{\pi}$ or some other $[a, a + 2\pi]$, we obtain 0 for the result.

Graph the sine function between any bounds separated by 2π and the area of the function above zero is equal to the area below zero, no matter what the value of a. This is also true for the cosine.

sine-sine

$$\int \sin mx \sin nx \ dx$$

Go back to the sum of cosines above and subtract the first equation from the second to obtain

$$\frac{1}{2}(\cos s - t - \cos s + t) = \sin s \sin t$$

Hence

$$\int \sin mx \sin nx \ dx$$

$$= \frac{1}{2} \int \cos(m-n)x - \cos(m+n)x \ dx$$

This is very similar to what we had before. The difference is the minus sign. Here, if $m = n = 0$ the two terms are identical ($\cos(0) = 1$) and they cancel.

On the other hand, if $m = n \neq 0$ we get a value of π from the first term as before.

For any non-zero k in the argument to the cosine, we have

$$\int_{-\pi}^{\pi} \cos kx \ dx = \frac{\sin kx}{k} \Big|_{-\pi}^{\pi}$$

which, again as we saw before is zero for any $\int_a^{a+2\pi}$.

sine-cosine

One way to do this is to remember the addition formula for sine:

$$\sin(s+t) = \sin s \cos t + \sin t \cos s$$

$$\sin(s-t) = \sin s \cos t - \sin t \cos s$$

$$\frac{1}{2}\ [\ \sin(s+t) + \sin(s-t)\ ]\ = \sin s \cos t$$

So we have

$$\int \sin mx \cos nx\ dx$$

$$= \frac{1}{2} \int \sin(m+n)x + \sin(m-n)x\ dx$$

Here, the case with $m = n = 0$ is $\int \sin(0)\ dx$ which is just 0.

For $m = n \neq 0$, the second term is zero and the first is

$$\int_a^b \sin kx\ dx = -\frac{\cos kx}{k}\ \Big|_{-\pi}^{\pi}$$

which is zero for any $\int_a^{a+2\pi}$.

Lastly, for $m \neq n$ we obtain

$$\int_{-\pi}^{\pi} \sin(m+n)x + \sin(m-n)x\ dx$$

$$= -\frac{\cos(m+n)x}{m+n} - \frac{\cos(m-n)x}{m-n}\ \Big|_{-\pi}^{\pi}$$

which is zero for any $\int_a^{a+2\pi}$ for the same reason. Hence all the cases for sine-cosine are zero.

writing a series

Now, suppose we try to represent a function $f(x)$ as a series using sine and cosine

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \ldots$$

We need to determine the cofactors (and we'll see the reason for the factor of $1/2$ on a_0 shortly). If we multiply both sides by $\cos mx$ and integrate over the interval $[-\pi, \pi]$, all of the infinite number of terms on the right-hand side vanish except for the one with cofactor a_m:

$$\int_{-\pi}^{\pi} f(x)\ \cos mx\ dx = a_m \int_{-\pi}^{\pi} \cos mx \cos mx\ dx$$

Remember from the previous section that for $m = n \neq 0$ the right-hand integral is equal to π so

$$\int_{-\pi}^{\pi} f(x)\ \cos mx\ dx = \pi a_m$$

Thus

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ \cos mx\ dx$$

Similarly, we can determine the coefficients b_m by multiplying by $\sin mx$ and integrating. By the same reasoning as before, we obtain:

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ \sin mx\ dx$$

Last, we have $m = 0$,

$$\int_{-\pi}^{\pi} f(x) \cos 0\ dx = \frac{1}{2}\ \int_{-\pi}^{\pi} a_0 \cos 0\ dx$$

$$\int_{-\pi}^{\pi} f(x)\ dx = \frac{a_0}{2}\ 2\pi = a_0 \pi$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ dx$$

We can use any interval of length 2π, here it was $[-\pi, \pi]$ as given here.

For reference then, the cofactors are

$$\begin{cases} a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ dx \\ a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos mx f(x)\ dx \\ b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin mx f(x)\ dx \end{cases}$$

application: odd step function

Consider the step function:

$$\begin{cases} f(x) = -1 & x < 0 \\ f(x) = 1 & x > 0 \end{cases}$$

This function is an *odd* function: $f(x) = -f(-x)$, while the cosine is an even function ($\cos x = \cos -x$). An even function times an odd function is an odd function. The integral of an odd function that is symmetric across zero vanishes.

Therefore, on the interval $[-\pi, \pi]$, all the terms with cosine vanish:

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ \cos mx\ dx = 0$$

To drive this point home:

$$= \frac{1}{\pi}\ [\ \int_{-\pi}^{0} f(x)\ \cos mx\ dx + \int_{0}^{\pi} f(x)\ \cos mx\ dx\ ]$$

For every value x from $0 < x < \pi$ from the second term and its $f(x)$, there is a value $f(-x)$ from the first term with opposite sign, multiplied by the same value $\cos(-mx) = \cos mx$, and they all cancel.

Thus the coefficients that do remain are those for sine:

$$b_m = \frac{1}{\pi}\ [\ \int_{-\pi}^{0} f(x)\ \sin mx\ dx + \int_{0}^{\pi} f(x)\ \sin mx\ dx\ ]$$

$$= \frac{1}{\pi}\ [\ -\int_{-\pi}^{0}\ \sin mx\ dx + \int_{0}^{\pi}\ \sin mx\ dx\ ]$$

For the sine function, the integral over $[-\pi, 0]$ is equal and opposite in sign to the integral over $[0, \pi]$

$$= \frac{1}{\pi}\ [\ \int_{0}^{\pi}\ \sin mx\ dx + \int_{0}^{\pi}\ \sin mx\ dx\ ]$$

$$\frac{2}{\pi}\int_{0}^{\pi}\ \sin mx\ dx$$

$$= -\frac{2}{m\pi}\ [\ \cos mx\ \Big|_{0}^{\pi}\ ]$$

For even m, the terms $(\cos 2\pi, \cos 4\pi \dots)$ are all the same as at the lower bound, leaving us with zero, while for odd m we get a factor of -2 from the integral so the whole thing is

$$b_m = \frac{4}{m\pi}$$

We also have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$$

$$= \frac{1}{\pi}\ [\ \int_{-\pi}^{0} f(x)\ dx + \int_{0}^{\pi} f(x)\ dx\ ]$$

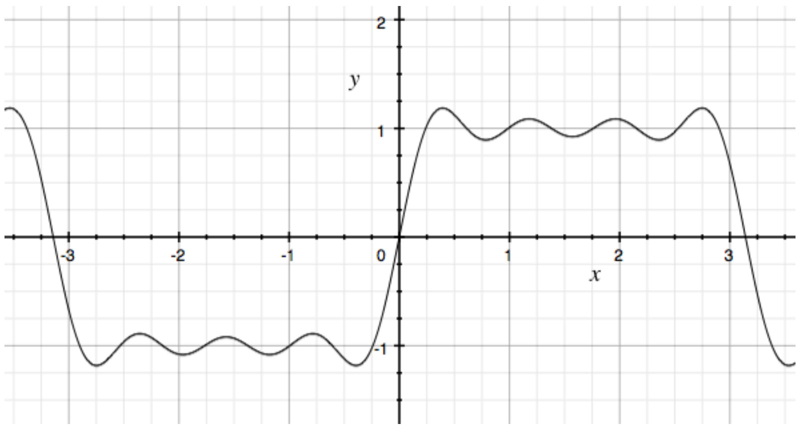
$$= \frac{1}{\pi}(-\pi + \pi) = 0$$

If that was confusing, just remember that $f(x)$ is odd, and the interval of integration is symmetric around 0.

So the series is $4/\pi$ times the values $\sin kx/k$ for odd k:

$$f(x) = \frac{4}{\pi}\ [\ \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \dots\ ]$$

which we can approximate with four terms



Notice there is one little hump in the step for each term we include.

application: even step function

Consider the step function centered on zero:

$$\begin{cases} f(x) = 1 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ f(x) = 0 & \text{otherwise} \end{cases}$$

Since this is an even function, all the b_m will be zero:

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin mx\ dx = 0$$

The cosine terms are:

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \cos mx\ dx = 0$$

$f(x) = 0$ outside $[-\pi/2, \pi/2]$:

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \ \cos mx \ dx$$

$f(x) = 1$ inside $[-\pi/2, \pi/2]$:

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \ \cos mx \ dx$$

Cosine is an even function so:

$$= \frac{2}{\pi} \int_{0}^{\pi/2} \cos mx \ dx$$

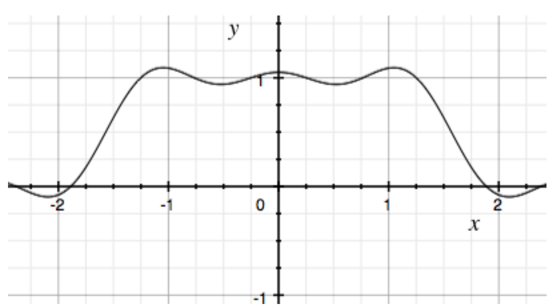
$$= \frac{2}{\pi m} \sin mx \ \Big|_{0}^{\pi/2}$$

Only the odd terms survive, and these terms alternate in sign. Check a_0

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ dx$$
$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \ dx$$
$$= 1$$

Recall that the a_0 term has a coefficient of $\frac{1}{2}$

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \ldots \right]$$



application: f(x) = x

This function is an odd function: $f(x) = -f(-x)$, while the cosine is an even function ($\cos x = \cos -x$). An even function times an odd function is an odd function, and so all the cosine terms vanish (between $[-\pi, \pi]$, as before.

Thus the coefficients that do remain are those for sine:

$$b_m = \frac{1}{\pi} \; [\int_{-\pi}^{\pi} f(x) \;\; \sin mx \; dx \;]$$

$$= \frac{1}{\pi} \; [\int_{-\pi}^{\pi} x \;\; \sin mx \; dx \;]$$

We can integrate $x \sin x$ using integration by parts, or we can just guess:

$$\int x \sin x \; dx = -x \cos x + \sin x$$

Check

$$\frac{d}{dx} \; - x \cos x + \sin x = x \sin x - \cos x + \cos x = x \sin x$$

We do have that factor of m

$$\frac{d}{dx} \; \frac{1}{m}(-x \cos mx + \sin mx)$$

$$= x \sin mx - \frac{\cos mx}{m} + \frac{\cos mx}{m} = x \sin mx$$

The limits are $[-\pi, \pi]$. We do not forget the factor of $1/\pi$ in front.

$$b_m = \frac{1}{\pi m}(-x \cos mx + \sin mx) \; \Big|_{-\pi}^{\pi}$$

Now the second term $\sin k\pi$ is zero for any integer k. That leaves

$$b_m = \frac{1}{\pi m}(-x\cos mx)\ \Big|_{-\pi}^{\pi}$$

$$= -\frac{1}{\pi m}(x\cos mx)\ \Big|_{-\pi}^{\pi}$$

$$= -\frac{1}{\pi m}\ [\ \pi\cos m\pi - (-\pi)\cos m(-\pi)\ ]$$

$$= -\frac{1}{\pi m}\ [\ \pi\cos m\pi + \pi\cos m(-\pi)\ ]$$

$$= -\frac{1}{\pi m}\ [\ \pi\cos m\pi + \pi\cos m\pi\ ]$$

$$= \frac{-2}{\pi m}\ [\ \pi\cos m\pi\ ]$$

$$= -\frac{2}{m}\cos m\pi$$

The terms for odd m are

$$a_m = \frac{2}{m}$$

while the even terms are

$$a_m = -\frac{2}{m}$$

For a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\ dx$$

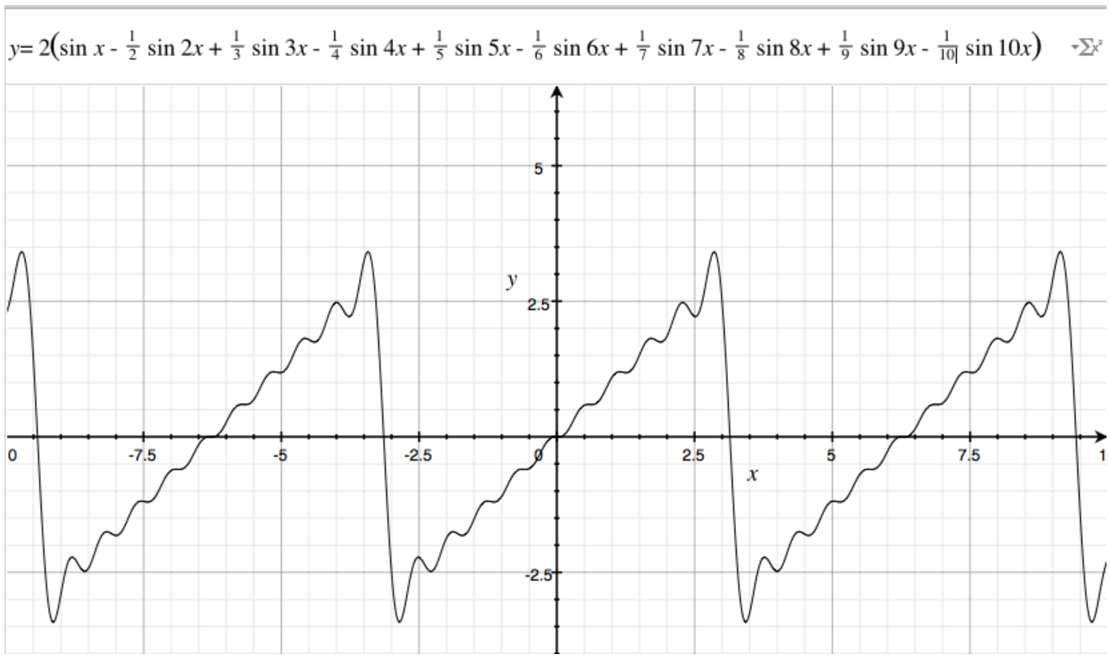
x is an odd function so the integral is zero

$$= \frac{1}{2\pi}x^2\ \Big|_{-\pi}^{\pi} = 0$$

So finally we have that

$$f(x) = 2\ [\ \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\ ]$$

Here are 10 terms



complex exponential

The derivation that we did previously can be accomplished in a simpler way using the complex exponential (Euler's formula):

$$e^{i\theta} = \cos\theta + i\sin\theta$$

switching to x

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

Adding or subtracting gives:

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

So why don't we investigate the orthogonality of the function e^{ikx} where $k, m \in \mathbb{N}$:

$$\int_{-\pi}^{\pi} e^{ikx} e^{-imx}\ dx$$

What we find is that, as before, the result is zero when $k \neq m$:

$$\int_{-\pi}^{\pi} e^{ikx} e^{-imx}\ dx = \int_{-\pi}^{\pi} e^{i(k-m)x}\ dx$$

$$= \frac{e^{i(k-m)x}}{k-m}\ \Big|_{-\pi}^{\pi}$$

Convert back to sin and cos to see the result more easily:

$$\int_{-\pi}^{\pi} e^{i(k-m)x} = \frac{1}{i(k-m)}\ [\ \cos(k-m)x + i\sin(k-m)x\ ]\ \Big|_{-\pi}^{\pi}$$

Consider $n = k - m$. Since n is an integer, the sine term is zero ($\sin n\pi = 0$). If n is odd, the cosine term is equal to 1 at both the upper and lower bounds, while if n is even the cosine term is equal to -1 at both the upper and lower bounds, so the difference is zero in both cases.

Actually, the total integral is zero for any bounds $\int_a^{a+2\pi}$ since

$$\cos a = \cos(a + 2\pi), \quad \sin a = \sin(a + 2\pi)$$

On the other hand, if $k = m$ then

$$\int_0^{2\pi} e^{ikx} e^{-imx}\ dx = \int_0^{2\pi}\ dx = 2\pi$$

So again, we will try to approximate $f(x)$ by a series with terms $c_k e^{ikx}$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

Notice that in this series we have negative integer k as well.

To determine the coefficient c_k, multiply both sides by e^{-ikx} and integrate over the interval $[-\pi, \pi]$. As before, all the terms of the series except those with $c_k e^{ikx} e^{-ikx}$ drop out and the result for the one remaining is just c_k times 2π so:

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \ dx$$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ dx$$

application: odd step function

Consider the step function:

$$\begin{cases} f(x) = -1 & x < 0 \\ f(x) = 1 & x > 0 \end{cases}$$

For $k \in \{\cdots -3, -2, -1, 0, 1, 2, 3 \dots\}$:

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ e^{-ikx} \ dx$$

Hamming solves this quickly but I couldn't follow a key step.

Let us go more slowly and carefully and write it as the cosine plus sine:

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ [\ \cos -kx + i \sin -kx \ ] \ dx$$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \; [\; \cos kx - i \sin kx \;] \; dx$$

$f(x)$ is an odd function, while cosine is an even function, hence the cosine term vanishes when integrated from $\int_{-\pi}^{\pi}$, so we have

$$c_k = -\frac{i}{2\pi} \int_{-\pi}^{\pi} f(x) \sin kx \; dx$$

Substitute for $f(x)$

$$= -\frac{i}{2\pi} \; [\; \int_{-\pi}^{0} - \sin kx \; dx + \int_{0}^{\pi} \sin kx \; dx \;]$$

$$= -\frac{i}{2\pi k} \; [\; \cos kx \; \Big|_{-\pi}^{0} \; - \cos kx \; \Big|_{0}^{\pi} \;]$$

Consider the two evaluations separately:

$$\cos kx \; \Big|_{-\pi}^{0}$$

$$\cos 0 - \cos(-k\pi) = 1 - \cos k\pi$$

For even k this is $1 - 1 = 0$. Alternatively just recall that the integral for sine or cosine over any interval of length 2π is equal to 0. So all the even k terms drop out.

For odd k this is $1 - (-1) = 2$.

Similarly

$$-\cos kx \; \Big|_{0}^{\pi}$$

for even k this is $-(1-1) = 0$, and again there are no even terms. For odd k we have $-(-1-1) = 2$. Adding them together, the total is 4 and we have then:

$$c_k = -\frac{2i}{\pi k}$$

so our series for $f(x)$ is

$$f(x) = \sum c_k e^{ikx}$$

$$= -\frac{2i}{\pi k}\ (\cos kx + i \sin kx)$$

At this point, we recall that the sum is over both negative and positive k. So if we consider the terms with the same absolute value $|k|$ as a pair, replace k with $|k|$ and put in minus signs appropriately.

For the first term in each pair we have $-k$, which gives $\cos -kx = \cos kx$ and $\sin -kx = -\sin kx$ so we obtain:

$$= -\frac{2i}{\pi(-k)}(\cos kx - i \sin kx) - \frac{2i}{\pi k}(\cos kx + i \sin kx)$$

$$= \frac{2i}{\pi k}(\cos kx - i \sin kx) - \frac{2i}{\pi k}(\cos kx + i \sin kx)$$

We see that the imaginary (cosine) terms cancel. The result is purely real has the form

$$= \frac{2}{\pi k} \sin kx + \frac{2}{\pi k} \sin kx$$

$$= \frac{4}{\pi k} \sin kx$$

for odd k only. That gives

$$f(x) = \sum \frac{4}{\pi}\ \cdot \frac{\sin kx}{k}$$

$$= \frac{4}{\pi}\ [\ \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots$$

Compare with what we had before:

$$f(x) = \frac{4}{\pi}\ [\ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \frac{1}{7} \sin 7x + \dots\ ]$$

Now the only issue is that we have ignored c_0 so far.

Recall

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ dx$$

Since $f(x)$ switches value from -1 to $+1$ at 0, we split the integral and substitute for $f(x)$

$$= -\frac{1}{2\pi} \int_{-\pi}^{0}\ dx + \frac{1}{2\pi} \int_{0}^{\pi}\ dx$$

The second term is clearly $1/2$, while the first integral is

$$x \Big|_{-\pi}^{0} = 0 - (-\pi) = \pi$$

times $-1/2\pi$, which equals $-1/2$ and cancels the second term. So we have that $c_0 = 0$.

This is a *lot* harder than Hamming's sketch of the proof, but he had a step I couldn't justify. We reach the same result.

f(x) = x

Once again, the Fourier series with the complex exponential is

$$f(x) = \sum_{k=-\infty}^{\infty} c_k\ e^{ikx}$$

The individual c_k can be found by multiplying both sides by e^{-ikx} and integrating:

$$\int f(x)\ e^{-ikx}\ dx = \int c_k\ e^{ikx}\ e^{-ikx}\ dx$$

$$= c_k \int dx$$

On the right-hand side all the other terms drop out due to orthogonality, leaving the one shown. We use the bounds $[-\pi, \pi]$, obtaining

$$\int_{-\pi}^{\pi} f(x)\ e^{-ikx}\ dx = 2\pi c_k$$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ e^{-ikx}\ dx$$

For this problem, we have $f(x) = x$ so

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\ e^{-ikx}\ dx$$

For c_0 $(k = 0)$:

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\ dx = \frac{1}{2\pi} \frac{x^2}{2}\ \Big|_{-\pi}^{\pi} = 0$$

or just say: x is odd, $-(x) = (-x)$, and so the integral is zero.

For the general term with integer k, a good approach is to replace the exponential using Euler's equation:

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\ e^{-ikx}\ dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} x\ (\cos -kx + i \sin -kx)\ dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} x\ (\cos kx - i \sin kx)\ dx$$

We notice that $x \cos kx$ is odd $\times$ even $=$ odd, so that term is zero. We're left with

$$= -\frac{i}{2\pi} \int_{-\pi}^{\pi} x \sin kx\ dx$$

To be safe, we use IBP

$$x = u, \quad dx = du$$

$$dv = \sin kx \ dx, \quad v = \frac{-\cos kx}{k}$$

So the integral is

$$-\frac{i}{2\pi} \ [\ x\frac{-\cos kx}{k} \ \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{-\cos kx}{k} \ dx \ ]$$

I've seen web pages say that the right-hand term is obviously zero (if it were $\cos x$ in the numerator, I would agree, but $\cos kx$ doesn't just cancel over an interval of length 2π like $\cos x$ does). Instead, we do the integration:

$$-\frac{i}{2\pi} \ [\ x\frac{-\cos kx}{k} \ \Big|_{-\pi}^{\pi} + \frac{\sin kx}{k^2} \ \Big|_{-\pi}^{\pi} \ ]$$

And now we see that the right-hand side *really is* zero, since $\sin k\pi$ for integer k is always 0. We go slowly so as not to mess up the minus signs:

$$= -\frac{i}{2\pi} \ [\ x\frac{-\cos kx}{k} \ \Big|_{-\pi}^{\pi} \ ]$$

$$= -\frac{i}{2\pi} \ [\ \pi\frac{-\cos k\pi}{k} - (-\pi)\frac{-\cos k(-\pi)}{k} \ ]$$

$$= -\frac{i}{2\pi} \ [\ \pi\frac{-\cos k\pi}{k} + \pi\frac{-\cos k(-\pi)}{k} \ ]$$

$$= -\frac{i}{2\pi} \ [\ \pi\frac{-\cos k\pi}{k} + \pi\frac{-\cos k\pi}{k} \ ]$$

$$= \frac{i}{\pi} \ [\ \pi\frac{\cos k\pi}{k} \ ]$$

$$= i \ \frac{\cos k\pi}{k}$$

$$= \frac{i}{k} \ (-1)^k$$

A dramatic simplification. Now, the series is

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \; e^{ikx}$$

For every $|k|$ there are two terms which are very similar, it is really $\pm k$.

And we especially need the negative k terms because we have to lose everything with i... (according to Hamming every term for $-k$ is the complex conjugate of the $+k$ term). We want a purely real answer.

$k = 1$

So, for example, suppose $k = 1$ then we have for the cofactor:

$$\frac{i}{k} \; (-1)^k = \frac{i}{1}(-1)^1 = -i$$

and that term is

$$-ie^{ikx}$$

while for $k = -1$ we have

$$\frac{i}{k} \; (-1)^k = \frac{i}{-1}(-1)^{-1} = i$$

and that term is

$$ie^{-ikx}$$

Adding them together we get

$$(-i)(e^{ikx} - e^{-ikx})$$

Recall that

$$2i \sin\theta = e^{i\theta} - e^{-i\theta}$$

So we have

$$= \frac{-i}{1}\ 2i \sin kx$$

$$= 2\ \frac{\sin x}{1}$$

Every odd k will behave the same way

$$= 2\ \frac{\sin x}{k}$$

$k = 2$

Suppose $k = 2$ then we have for the cofactor:

$$\frac{i}{k}\ (-1)^k = \frac{i}{2}(-1)^2 = \frac{i}{2}$$

and that term is

$$\frac{i}{2}e^{ikx}$$

while for $k = -2$ we have

$$\frac{i}{k}\ (-1)^{-2} = \frac{i}{-2}$$

and that term is

$$-\frac{i}{2}e^{-ikx}$$

Adding them together we get

$$(\frac{i}{2})(e^{ikx} - e^{-ikx})$$

Again,

$$2i \sin \theta = e^{i\theta} - e^{-i\theta}$$

So we have

$$= \frac{i}{2}\ 2i \sin kx$$

$$= -2 \ \frac{\sin 2x}{2}$$

Every even k will behave this way

$$= -2 \ \frac{\sin x}{k}$$

This gives us the series

$$f(x) = 2 \ [\ \frac{\sin x}{x} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \dots]$$

which matches what we had before.

After fooling around with this, I'm not convinced that complex is better. The original proof was easier, but the particular cases are very tricky. I worked much longer than I will admit on this last example. And as you can see, we go fairly quickly back to the trigonometric view of the the equations anyway.