Integration problems

Definite integrals

$$\int_0^{\pi/2} \cos x \; dx$$

$$\int_0^{\pi/2} \cos\theta \; d\theta$$

$$\int_{-\pi/2}^{\pi/2} \sin x \; dx$$

$$\int_0^1 x^2 \; dx$$

$$\int_0^1 \sqrt{x} \; dx$$

$$\int_1^3 \frac{1}{x} \; dx$$

$$\int_1^2 3x^2 + 2x + 1 \; dx$$

$$\int_0^8 x^{2/3} \; dx$$

$$\int_0^{\pi/4} \cos 2t \ dt$$

$$\int_1^3 \frac{1}{x^2} \ dx$$

$$\int_0^3 \frac{2t}{1+t^2} \ dt \ ; \quad \text{hint: watch the bounds}$$

$$= \int_{u=1}^{u=10} \frac{1}{u} \ du$$

$$= \ln u \ \Big|_{u=1}^{u=10} = \ln 10 - \ln 1 = \ln 10$$

$$\int_0^1 (3x-2)^3 \ dx$$

$$\int_0^1 x e^{-x^2} \ dx$$

$$\int_{-\pi/4}^{\pi/4} \cos 2x \ dx$$

$$\int_{-1}^1 x e^x \ dx$$

$$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}} \ dx$$

$$\int_0^{e-1} \ln(x+1) \ dx; \quad \text{hint: what is } \frac{d}{dx} x \ln x \ ?$$

$$\int_{\pi/3}^{\pi/2} \tan\frac{\theta}{2}\ \sec^2\frac{\theta}{2}\ d\theta$$

$$\int_{\pi/2}^{x} \cos t\ dt$$

$$\int_{0}^{\ln 2} e^{2x}\ dx$$

$$\int_{0}^{2} (x^3 + k)\ dx = 10\ ; \quad \text{find } k$$

$$\int_{0}^{\ln 2} e^{2x}\ dx$$

$$\int_{1}^{e} \frac{\ln t}{t}\ dt$$

$$\int_{0}^{1} x e^{x^2+1}\ dx$$

Indefinite integrals

$$\int \tan x\ dx$$

$$\int \ln x\ dx$$

$$\int \sec^2 x\ dx$$

$$\int \csc^2\theta\ d\theta$$

$$\int \tan\theta \sec\theta\ d\theta$$

$$\int (\sqrt{x} + \frac{1}{x^3})\ dx$$

$$\int \frac{3x^2 + x - 1}{x^2}\ dx$$

$$\int \frac{1}{u-3}\ du$$

$$\int \cos^2(2x)\ \ \sin 2x\ dx$$

$$\int \frac{x}{\sqrt{3-4x^2}}\ dx$$

$$\int \frac{1}{\sqrt{9-x^2}}\ dy$$

$$\int \frac{x}{(2-x^2)^3}\ dx$$

$$\int \frac{e^x}{1-2e^x}\ dx$$

$$\int e^{x+e^x}\ dx$$

$$\int (x^3 - \sin 2x)\ dx$$

$$\int \frac{e^{3x}}{e^x}\ dx$$

$$\int \frac{z}{1-4z^2}\ dz$$

$$\int \frac{5}{1+x^2}\ dx$$

$$\int \frac{\cos x}{\sin^2 x}\ dx$$

$$\int \tan^4 t\ \ \sec^2 t\ dt$$

$$\int e^x \cos(e^x)\ dx$$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\ dx$$

$$\int \frac{x+1}{x^2+1}\ dx$$

$$\int \frac{x}{x+a}\ dx\ ;\quad \text{hint:}\ = \int \frac{x+a-a}{x+a}\ dx$$

$$\int a^u\ du\ ;\quad a = const$$

$$\int e^{4-\ln x}\ dx$$

$$\int x\ \sqrt{x+2}\ dx\ ;\quad \text{hint:}\ u = x+2$$

$$\int \frac{x}{\sqrt{x+3}}\ dx\ ;\quad \text{hint:}\ u = x+3$$

$$\int \frac{1+x}{\sqrt{x}}\ dx$$

$$\int \frac{1}{x^2+2x+5}\ dx$$

$$\int \frac{x^2+3}{x-1}\ dx \ : \quad \text{hint: make top divisible by } x-1$$

$$\int \frac{\ln x}{3x}\ dx$$

$$\int \frac{e^x}{e^x+1}\ dx$$

$$\int \frac{1+x}{\sqrt{x}}\ dx$$

$$\int \sin\theta\cos\theta\ d\theta$$

for the last, give both versions of the answer and show they are equal

$$\int (\sin x + \cos x)^2\ dx$$

$$\int (1+\tan x)^2\ dx$$

$$\int \frac{\cos^2 x}{1+\sin x}\ dx$$

$$\int \frac{\sin x}{1+\sin x}\ dx$$

$$\int \sin^3\ dx$$

$$\int \sec^2 x\ \sqrt{5+\tan x}\ dx$$

$$\int \cos x\ e^{1+\sin x}\ dx$$

$$\int e^x \cos(e^x)\ dx$$

$$\int x \sin x\ dx$$

$$\int e^x \sin x\ dx$$

$$\int \frac{\sin x + \cos x}{e^{-x} + \sin x}\ dx\ ; \quad \text{hint: multiply by } e^x/e^x$$

$$\int \frac{2^{\ln x}}{x}\ dx$$

$$\int \frac{1}{x \ln x}\ dx$$

$$\int \frac{\ln \sqrt{x}}{x}\ dx$$

For absolute value problems, recall that

$$|x| = \begin{cases} x & : x \geq 0 \\ -x & : x < 0 \end{cases}$$

The method is to find the place where the expression inside the absolute value symbols is equal to zero, then integrate piecewise, substituting as shown above.

$$\int_0^2 |t-1|\ dt$$

Since $t - 1 = 0$ when $t = 1$ this is

$$\int_0^1 -(t-1)\ dt + \int_1^2 (t-1)\ dt$$

$$= (-\frac{1}{2}t^2 + t)\ \Big|_0^1 + (\frac{1}{2}t^2 - t)\ \Big|_1^2$$

$$= (-\frac{1}{2} + 1 - 0 + 0) + (2 - 2 - \frac{1}{2} + 1) = 1$$

Tricky to evaluate.

FTC

There is a perverse desire to make sure you understand the FTC (part 1).

If $F(x)$ is "nice" and

$$F(x) = \int_a^x f(t)\ dt$$

then..

$$F'(x) = \frac{d}{dx}\ F(x) = \frac{d}{dx}\ \int_a^x f(t)\ dt = f(x)$$

Problems: for each $G(x)$ below, find $G'(x)$

$$G(x) = \int_1^x 2t\ dt$$

$$G(x) = \int_0^x (2t^2 + \sqrt{t})\ dt$$

$$G(x) = \int_0^x \tan t\ dt$$

$$G(x) = \int_{x^2}^x \frac{t^2}{1+t^2}\ dt$$

The last problem needs first to be manipulated into a (sum of) integrals between a constant (0) on the lower bound and x above, and then the one with x^2 must take account of the fact that if $t = -x^2$ then $dt = -2x\ dx$.

$$G(x) = -\int_0^{-x^2} \frac{t^2}{1+t^2}\ dt + \int_0^x \frac{t^2}{1+t^2}\ dt$$

$$G'(x) = -\frac{x^4}{1+x^4}\ (-2x) + \frac{x^2}{1+x^2}$$

hard one

Here is a problem involving the actual integral we had above. I didn't know how to solve it completely, but I found the answer on the web and can work backward and see that it's correct. Call it a challenge. It looks simple enough:

$$\int \frac{\sqrt{x}}{1-x}\ dx$$

substitute

$$u = \sqrt{x}, \quad u^2 = x, \quad 2u\ du = dx$$

we obtain

$$\int \frac{u}{1-u^2}\ 2u\ du$$

$$2\int \frac{u^2}{1-u^2}\ du$$

If this were x in the numerator rather than x^2, it would be simple. Still, it looks like it ought to be easy, somehow. The answer is here:

(http://integrals.wolfram.com/index.jsp)

Let's change to x:

$$\int \frac{x^2}{1-x^2}\ dx$$

The first part of the answer is a useful trick for many problems. If the numerator is the same as the denominator, within a constant, then:

$$= -\int \frac{1-x^2-1}{1-x^2}\ dx$$
$$= -\int 1\ dx - \int \frac{1}{1-x^2}\ dx$$

Now the real trick is that the second part can be re-worked because it is a difference of squares

$$(1-x)(1+x) = 1-x^2$$
$$\int \frac{1}{1-x^2}\ dx = \frac{1}{2}\int (\frac{1}{1+x} + \frac{1}{1-x})\ dx$$

If we put the two terms on the right over the common denominator $1-x^2$, then for the numerator we have $1-x+1+x=2$. !! So the whole integral is

$$\int \frac{x^2}{1-x^2}\ dx$$
$$= -x - \frac{1}{2}\ [\ (\ln(1+x) - \ln(1-x))\ ]\ + C$$
$$= -x - \frac{1}{2}\ \ln\frac{(1+x)}{(1-x)} + C$$

I'll leave it to you to work out the answer to the original problem with $\sqrt{x}$.

another hard one

$$\int \frac{\sqrt{x^2+1}}{x}\ dx$$

Substitution: let $x = \tan t$. So opp $= x$, adj $= 1$, hyp $= \sqrt{1+x^2}$.

$$x = \tan t$$
$$dx = \sec^2 t\ dt$$
$$\sqrt{1+x^2} = \sec t$$

So the integral is

$$\int \frac{\sec t}{\tan t}\ \sec^2 t\ dt$$
$$= \int \frac{\sec t}{\tan t}(1+\tan^2 t)\ dt$$

The first term is

$$\int \frac{1}{\sin t}\ dt$$

and the second is

$$\int \sec t \tan t\ dt$$
$$= \int \frac{\sin t}{\cos^2 t}\ dt$$

The second part is easy $(1/\cos t)$. But the first requires more work. Let

$$u = \cos t$$
$$du = -\sin t \ dt$$

We rewrite the integral as

$$\int \frac{\sin t}{\sin^2 t} \ dt$$

$$= -\int \frac{1}{1-u^2} \ du$$
$$= -\frac{1}{2} \int \frac{1}{1+u} + \frac{1}{1-u} \ du$$
$$= -\frac{1}{2}(\ln(1+u) - \ln(1-u))$$

So, in terms of t we have (combining)

$$\frac{1}{\cos t} - \frac{1}{2}(\ln(1+\cos t) - \ln(1-\cos t))$$

In order to substitute back to x, we recall that

$$\frac{1}{\sqrt{1+x^2}} = \cos t$$

and I think we'll just leave it right there. Well, in the original problem we had a definite integral with limits $\sqrt{15}$ and $\sqrt{3}$, so that $\cos t = 1/4$ at the high end and $\cos t = 1/2$ at the low end which makes it considerably easier to evaluate.

$$= 4 - \frac{1}{2}(\ln 5/4 - \ln 1/2) - 2 + \frac{1}{2}(\ln 3/2 - \ln 1/2)$$

$$= 2 - \frac{1}{2}(\ln 5/2 + \ln 3)$$