Kepler (from Hartig Math 304)

This is a derivation of Kepler's laws from the a class handout (Math 304) by Hartig.

1

We start by defining M as the mass of the sun and m as the mass of a planet and $\mathbf{r}$ as the position vector from the sun to the planet. Combining Newton's second law and the inverse square law of gravitation we have that

$$\mathbf{F} = m\mathbf{a} = -\frac{GMm}{r^2}\,\frac{\mathbf{r}}{r}$$

$$\mathbf{a} = -\frac{GM}{r^2}\,\frac{\mathbf{r}}{r}$$

We take $\hat{\mathbf{u}}$ as a unit vector in the same direction as $\mathbf{r}$. I will write $\hat{\mathbf{u}}$ without its hat as $\mathbf{u}$ so as not to confuse it with the derivative $\dot{\mathbf{u}}$.

$$\mathbf{r} = r\mathbf{u}$$

$$\mathbf{a} = -\frac{GM}{r^2}\,\mathbf{u}$$

2

Now, the velocity is the time-derivative of the position vector $\mathbf{r}$.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

and the acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{r}}$$

because the force is toward the sun, the acceleration vector is parallel to the position vector, but with a change of sign.

3 Feynman's dots

We set up the angular momentum as

$$\mathbf{L} = \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}}$$

We compute the time-derivative

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}})$$

by a standard vector application of the product rule which we've looked at elsewhere this is equal to

$$= \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}}$$

and this is equal to zero, since any vector's cross product with itself is zero, including a reversed version of itself, as is present in the second term. This is just conservation of angular momentum

$$\frac{d\mathbf{L}}{dt} = 0$$

we will define a constant vector $\mathbf{c}$ rather than use $\mathbf{L}$ and say that

$$\mathbf{c} = \mathbf{r} \times \dot{\mathbf{r}}$$

Since $\mathbf{c}$ is a constant, unchanging in both direction and magnitude, it defines a normal vector to the plane containing $\mathbf{r}$ and $\dot{\mathbf{r}}$. We align $\mathbf{c}$ with the z-axis. All the motion occurs in the xy-plane. Note that

$$c = |\mathbf{c}|$$

4 Equal area

We consider the triangle formed by the position vector before and after a short period of time Δt, and the vector $\Delta\mathbf{r}$ connecting these two positions

$$\Delta\mathbf{r} \approx \dot{\mathbf{r}}\Delta t$$

The little bit of area that is swept out during this time is

$$\Delta A \approx \frac{1}{2}\;|\mathbf{r} \times \dot{\mathbf{r}}\Delta t| = \frac{1}{2}\;|\mathbf{c}|\Delta t$$

So we have that

$$\frac{\Delta A}{\Delta t} \approx \frac{1}{2}\;c$$

and in the limit as $\Delta t \to 0$

$$\frac{dA}{dt} = \frac{1}{2}\;c$$

5 Manipulating $\mathbf{a} \times \mathbf{c}$

The next step is to prove that

$$\mathbf{a} \times \mathbf{c} = GM\dot{\mathbf{u}}$$

This takes a bit of work, so I'd like to put it off until the end. We'll just assume it for now. Take this equality and integrate it with respect to time, obtaining

$$\int \mathbf{a} \times \mathbf{c} = \int GM\dot{\mathbf{u}}$$

$$\dot{\mathbf{r}} \times \mathbf{c} = GM\mathbf{u} + \mathbf{d}$$

where $\mathbf{d}$ is a constant *vector* of integration.

6 Dot product

We're almost there now. Take the left-hand side from above and form the dot product

$$\mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{c})$$

Use another vector identity to switch it around

$$= (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \mathbf{c}$$

But this is just the definition of $\mathbf{c}$

$$= \mathbf{c} \cdot \mathbf{c} = c^2$$

7 Conic section

So what we've shown is that

$$c^2 = \mathbf{r} \cdot (GM\mathbf{u} + \mathbf{d})$$

$$= r(GM + d\cos\theta) = rGM(1 + \frac{d}{GM}\cos\theta)$$

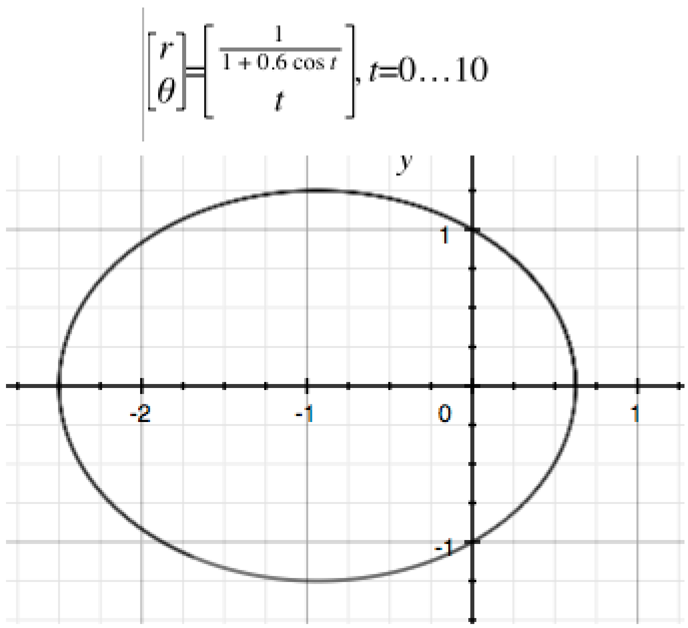
Define $k = c^2/GM$ and $e = d/GM$. Then

$$k = r(1 + e\cos\theta)$$

This is the equation of a conic section. In particular, if $e < 1$, then

$$r = \frac{k}{1 + e\cos\theta}$$

is the equation of an ellipse. Here is an example with $k = 1$ and $e = 0.6$



8 Cleaning up

To prove:

$$\mathbf{a} \times \mathbf{c} = GM\dot{\mathbf{u}}$$

where $\hat{\mathbf{u}}$ is a unit vector and

$$\mathbf{r} = r\hat{\mathbf{u}}$$

$$\mathbf{r} = r\mathbf{u}$$

We recall that

$$\mathbf{c} = \mathbf{r} \times \mathbf{v}$$

so substitute into the left-hand side

$$\mathbf{a} \times (\mathbf{r} \times \mathbf{v})$$

A standard result is that

$$\mathbf{a} \times (\mathbf{r} \times \mathbf{v}) = (\mathbf{a} \cdot \mathbf{v})\mathbf{r} - (\mathbf{a} \cdot \mathbf{r})\mathbf{v}$$

Recall that

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{u}$$

$$\mathbf{r} = r\mathbf{u}$$

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}\ r\mathbf{u} = \frac{dr}{dt}\mathbf{u} + r\dot{\mathbf{u}}$$

So let's work through it. The first term is

$$\mathbf{a} \cdot \mathbf{v} = -\frac{GM}{r^2}\mathbf{u} \cdot (\frac{dr}{dt}\mathbf{u} + r\dot{\mathbf{u}})$$

$$= -\frac{GM}{r^2}\ \frac{dr}{dt} - \frac{GM}{r}\ \mathbf{u} \cdot \dot{\mathbf{u}}$$

and that all multiplies $\mathbf{r} = r\mathbf{u}$

$$= -\frac{GM}{r}\ \frac{dr}{dt}\mathbf{u} - GM\ (\mathbf{u} \cdot \dot{\mathbf{u}})\mathbf{u}$$

The second term is

$$\mathbf{a} \cdot \mathbf{r} = -\frac{GM}{r^2}\mathbf{u} \cdot r\mathbf{u}$$

$$= -\frac{GM}{r}$$

which multiplies $\mathbf{v}$

$$-\frac{GM}{r}(\frac{dr}{dt}\mathbf{u} + r\dot{\mathbf{u}})$$

$$= -\frac{GM}{r}\frac{dr}{dt}\mathbf{u} - GM\dot{\mathbf{u}})$$

Now we subtract the second term from the first, so we have a cancellation and what is left is

$$GM\dot{\mathbf{u}} - GM\ (\mathbf{u}\cdot\dot{\mathbf{u}})\mathbf{u}$$

So somehow we need to show that

$$\mathbf{u}\cdot\dot{\mathbf{u}} = 0$$