Flux in Space

By now you should have seen Green's Theorem in the plane.

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \iint_R curl(\mathbf{F})\ dA$$

Often $curl(\mathbf{F}$ is written as $\nabla\times\mathbf{F}$. If

$$\mathbf{F} = < M, N >$$

then we have the equivalent expression

$$\oint_C M\ dx + N\ dy = \iint_R (N_x - M_y)\ dA$$

remembering that the integral on the left is a single integral, so we must eventually get x in terms of y or both in terms of t to use the formula. There is a great trick where if $N_x = 1/2$ and $M_y = -1/2$ then the right-hand side is

$$\iint_R (\frac{1}{2} - -\frac{1}{2})\ dA = \iint_R dA = A$$

and this leads to a simple calculation of the area of an ellipse. More often, we can avoid a complicated line integral by converting it to the right-hand side. And of course if $\nabla\times\mathbf{F} = N_x - M_y = 0$, we are dealing with a conservative vector field and we just evaluate the potential at the end-points of the curve, obtaining zero for the example with a closed curve.

An alternative version of Green's Theorem involves the divergence of $\mathbf{F}$

$$Flux = \oint_C \mathbf{F}\cdot\mathbf{n}\ ds = \iint_R \nabla\cdot\mathbf{F}\ dx\ dy$$

where

$$div(\mathbf{F}) = \nabla\cdot\mathbf{F} = M_x + N_y$$

The term $\mathbf{F} \cdot \mathbf{n}\ ds$ is the orthogonal counterpart (flow across the curve) of $\mathbf{F} \cdot \mathbf{t}\ ds = \mathbf{F} \cdot \mathbf{dr}$, the work done along C. $\mathbf{n}$ can be a little tricky to work with, but the point of this seems to be that we can convert the divergence (say, of a flow field) into a simpler calculation over the area of R. For reference

$$\mathbf{n} = \frac{1}{|\mathbf{r}'(t)|}\ < \frac{dy}{dt}, -\frac{dx}{dt} >$$

Velocity is in the direction we're headed $\mathbf{v} = < dx/dt, dy/dt >$. This vector is orthogonal to it ($\mathbf{v} \cdot \mathbf{n} = 0$), and it's a unit vector.

In space

Moving on to the actual topic for this write-up, we start to think about space. The flux of $\mathbf{F}$ across a surface S is

$$Flux = \iint_S \mathbf{F} \cdot \mathbf{n}\ dS$$

where $\mathbf{n}$ is the unit normal to the surface. Let's look more carefully at $\mathbf{n}\ dS$. Back when we looked at parametrization of surfaces, we said that

$$dS = |\mathbf{r}_u \times \mathbf{r}_v|\ du\ dv$$

but the unit normal to the surface is just

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

so

$$\mathbf{n}\ dS = \mathbf{r}_u \times \mathbf{r}_v\ du\ dv$$

and

$$Flux = \iint_S \mathbf{F} \cdot \mathbf{n}\ dS = \iint_S \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ du\ dv$$

example

Suppose we have a unit sphere

$$x^2 + y^2 + z^2 = 1$$

and $\mathbf{F} = < x, y, z >$. The standard parametrization of the (unit) sphere is

$$\mathbf{r}(\phi, \theta) = < \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi >$$

$\mathbf{F}$ is actually just the same. The cross-product is

$$\mathbf{r}_\phi \times \mathbf{r}_\theta =$$

$$\mathbf{r}_\phi = < \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi >$$

$$\mathbf{r}_\theta = < \sin\phi\sin\theta, \sin\phi\cos\theta, 0 >$$

The cross-product is

$$< -\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi >$$

The dot product is

$$\sin\phi\cos\theta\sin^2\phi\cos\theta + \sin\phi\sin\theta\sin^2\phi\sin\theta + \cos\phi\sin\phi\cos\phi$$

$$= \sin^3 phi + \sin\phi\cos^2\phi = \sin\phi$$

So we have

$$\int_0^{2\pi}\int_0^{\pi} \sin\phi \; d\phi \; d\theta = \int_0^{2\pi} 2 \; d\theta = 4\pi$$

divergence theorem

There is an easier way to do this calculation! It uses the divergence theorem in space, which states the following identity

$$flux = \iint\limits_S \mathbf{F}\cdot\mathbf{n} \; dS = \iiint\limits_V div\mathbf{F} \; dV$$

Remember that

$$\mathbf{F} = < x, y, z >$$

$$div(\mathbf{F}) = \nabla\cdot\mathbf{F} = P_x + Q_y + R_z = 1 + 1 + 1 = 3$$

So we have

$$= \iiint\limits_V 3 \; dV = 3V = 4\pi$$