Matrix proofs

associativity of matrix multiplication

This is the first important property to prove. If A,B and C are three matrices with the correct shape to be multiplied together as ABC, then

$$ABC = (AB)C = A(BC)$$

Example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} =$$

The entry at the upper left will be

$$(ae+bg, af+bh) \cdot (i,k) = aei + bgi + afk + bhk$$

Alternatively

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} ei+fk & ej+fl \\ gi+hk & gj+hl \end{bmatrix} =$$

The entry at the upper left will be

$$(a,b) \cdot (ei+fk, gi+hk) = aei + afk + bgi + bhk$$

Thus we see that associativity of matrix multiplication depends on associativity of the dot product. Using parentheses for the first dot product and angle brackets for the second, multiplying AB first gives

$$\langle (a,b)\cdot(e,g),(a,b)\cdot(f,h)\rangle \cdot \langle i,k\rangle$$
$$= \langle ae+bg, af+bh\rangle \cdot \langle i,k\rangle$$
$$= aei+bgi+afk+bhk$$

while multiplying BC first gives

$$\langle a,b\rangle \cdot \langle (e,f)\cdot(i,k),(g,h)\cdot(i,k)\rangle$$
$$\langle a,b\rangle \cdot \langle ei+fk, gi+hk\rangle$$
$$= aei+afk+bgi+bhk$$

inverse of AB

Suppose we have two square, invertible matrices, A and B, that are of the same size and can be multiplied together to give the product AB. Then

$$(AB)^{-1} = B^{-1}A^{-1}$$

Let $C = B^{-1}A^{-1}$. We need to show that

$$C(AB) = (AB)C = I$$

Substitute

$$B^{-1}A^{-1}(AB) = (AB)B^{-1}A^{-1} = I$$

Having proved associativity (above), this is trivial. For multiplication with the inverse on the left side.

$$B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

Multiplication on the right side is similar.

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

transpose of A

The transpose exchanges columns and rows. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \Rightarrow \quad M^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

Clearly, $(A^T)^T = A$ and $(M^T)^T = M$. Because of the exchange, if we reverse the order of multiplication, we get the same terms, but in transposed positions:

$$(AB)^T = B^T A^T$$

Consider the 2×2 case

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} e & g \\ f & h \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} ae + bg & ce + dg \\ af + bh & cf + dh \end{bmatrix}$$

We saw a formal proof of this property in the short write-up on properties of the determinant.

transpose and inverse

Finally,

$$(A^T)^{-1} = (A^{-1})^T$$

Example:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Recalling the quick formula for inverse in the 2×2 case, the cofactor is the determinant, switch positions along the diagonal, and change signs off the diagonal:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$(A^{-1})^T = \frac{1}{ad - bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$(A^T)^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

Here is a quick proof:

$$AA^{-1} = I$$

Take the transpose of both sides

$$(AA^{-1})^T = I^T = I$$

by the property discussed in the previous section, the left-hand side is

$$= (A^{-1})^T A^T$$

That is

$$(A^{-1})^T A^T = I$$

Thus, by the definition of the inverse, $(A^{-1})^T$ is the inverse of A^T for left-multiplication

$$(A^T)^{-1} = (A^{-1})^T$$

Reverse the order of A and A^{-1} to prove the same for right-multiplication.