Convergence

bounded, monotone sequences

We combine the previous concepts of a bounded monotone sequence and convergence:

Axiom (Monotone Sequence Property). *Any bounded monotone sequence converges.*

Beck:

> This axiom (or one of its many equivalent statements) gives arguably the most important property of the real number system; namely, that we can, in many cases, determine that a given sequence converges without knowing the value of the limit. In this sense we can use the sequence to define a real number.

theorem

• A bounded monotonic sequence converges. A bounded, monotonic increasing sequence converges to its least upper bound.

proof

Let α be the least upper bound of the sequence.

Given $\epsilon > 0$, we will show that all the terms of the sequence after some finite number of initial terms are in the interval $(\alpha - \epsilon, \alpha + \epsilon)$.

$\circ$ Since $\alpha + \epsilon$ is an upper bound of the sequence, all the terms certainly satisfy $a_n < \alpha + \epsilon$.

$\circ$ And since $\alpha - \epsilon$ is *not* an upper bound of the sequence, we must have $a_N > \alpha - \epsilon$ for some N.

$\circ$ But then all the later terms a_n (for $n > N$) will satisfy $a_n > \alpha - \epsilon$ (by the monotonic property), and so we have our condition for convergence.

restated theorem

If (a_n) is a monotone sequence of real numbers, then (a_n) is convergent if and only if it is bounded.

expanded proof

Let (a_n) be a monotone sequence.

$\Rightarrow$

Suppose that (a_n) is convergent. We showed previously that any convergent sequence is bounded.

$\Leftarrow$

There are two symmetric cases, increasing and decreasing. We consider only the first.

Suppose that (a_n) is an increasing sequence that is bounded. We look at the set $\mathbf{a} = \{a_n : n \in \mathbb{N}\}$. The set $\mathbf{a}$ is also bounded. By the completeness property, it has a least upper bound or supremum $L \in \mathbb{R}$.

Let $\epsilon > 0$ be given. Since L is the supremum of $\mathbf{a}$, $L - \epsilon$ cannot be an upper bound for the set so $\exists\ a_N$ such that $L - \epsilon < a_N$.

Since a_n is increasing, we have that for all $n \geq N$, $a_N \leq a_n$ so:

$$L - \epsilon < a_N \leq a_n \leq L < L + \epsilon$$

$$L - \epsilon < a_n < L + \epsilon$$

$$|a_n - L| < \epsilon$$

This proves that (a_n) converges to L.

$\square$