Fourier series, introduction

Suppose we try to represent a function $f(x)$ as a series using sine and cosine

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \ldots$$

We need to determine the cofactors. If we multiply both sides by $\cos mx$ and integrate over the interval $[0, 2\pi]$, all of the terms on the right-hand side vanish except for the one with a_m:

$$\int_{-\pi}^{\pi} f(x)\ \cos mx\ dx = a_m \int_{-\pi}^{\pi} \cos mx \cos mx\ dx$$

Remember from the previous section that for $m = n \neq 0$ the right-hand side is equal to π so

$$\int_{-\pi}^{\pi} f(x)\ \cos mx\ dx = \pi a_m$$

Thus

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ \cos mx\ dx$$

Similarly, we can determine the coefficients b_m by multiplying by $\sin mx$ and integrating. We obtain:

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ \sin mx\ dx$$

Last, we have $m = 0$,

$$\int_{-\pi}^{\pi} f(x) \cos 0 \; dx = \frac{1}{2} \; \int_{-\pi}^{\pi} a_0 \cos 0 \; dx$$

$$\int_{-\pi}^{\pi} f(x) \; dx = \frac{a_0}{2} \; 2\pi = a_0 \pi$$

$$a_0 = \frac{1}{\pi} \; \int_{-\pi}^{\pi} f(x) \; dx$$

We can use any interval of length 2π, commonly it is $[-\pi, \pi]$ as given here.

For reference then, the cofactors are

$$\begin{cases} a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \; dx \\ a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos mx f(x) \; dx \\ b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin mx f(x) \; dx \end{cases}$$

application: odd step function

Consider the step function:

$$\begin{cases} f(x) = -1 & x < 0 \\ f(x) = 1 & x > 0 \end{cases}$$

This function is an *odd* function: $f(x) = -f(-x)$, while the cosine is an even function ($\cos x = \cos -x$). An even function times an odd function is an odd function.

Therefore, on the interval $[-\pi, \pi]$, all the terms with cosine vanish:

$$\int_{-\pi}^{\pi} f(x) \; \cos mx \; dx = 0 = a_m$$

$$= \frac{1}{\pi} \; [\int_{-\pi}^{0} f(x) \;\; \cos mx \;\, dx + \int_{0}^{\pi} f(x) \;\; \cos mx \;\, dx \;]$$

For every value x from $0 < x < \pi$ from the second term and its $f(x)$, there is a value $f(-x)$ from the first term with opposite sign, multiplied by the same $\cos -mx = \cos mx$, and they all cancel.

Thus the coefficients that do remain are those for sine:

$$b_m = \frac{1}{\pi} \; [\int_{-\pi}^{0} f(x) \;\; \sin mx \;\, dx + \int_{0}^{\pi} f(x) \;\; \sin mx \;\, dx \;]$$

$$= \frac{1}{\pi} \; [- \int_{-\pi}^{0} \;\; \sin mx \;\, dx + \int_{0}^{\pi} \;\; \sin mx \;\, dx \;]$$

$$= \frac{1}{\pi} \; [\int_{0}^{\pi} \;\; \sin mx \;\, dx + \int_{0}^{\pi} \;\; \sin mx \;\, dx \;]$$

$$\frac{2}{\pi} \int_{0}^{\pi} \;\; \sin mx \;\, dx$$

$$= -\frac{2}{m\pi} \; [\; \cos mx \; \Big|_{0}^{\pi} \;]$$

For odd m, these $(\cos \pi, \cos 3\pi \dots)$ are all zero, while for even m we get

$$b_m = \frac{4}{m\pi}$$

We also have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \;\, dx$$

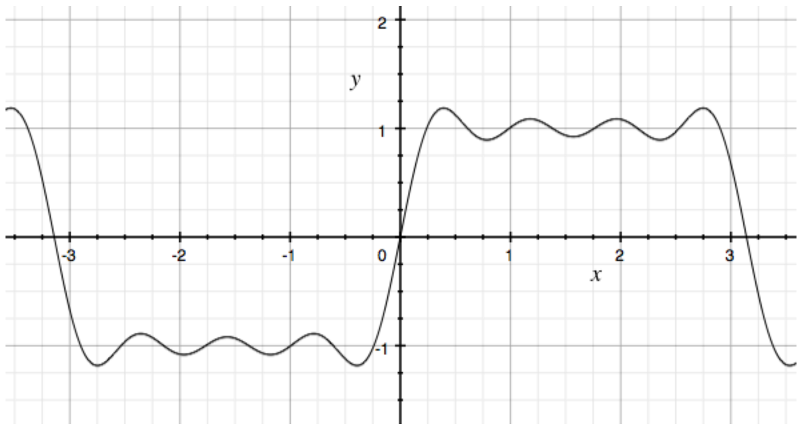
$$= \frac{1}{\pi} \; [\int_{-\pi}^{0} f(x) \;\, dx + \int_{0}^{\pi} f(x) \;\, dx \;]$$

$$= \frac{1}{\pi}(-\pi + \pi) = 0$$

So series is

$$f(x) = \frac{4}{\pi}\ [\ \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \ldots\ ]$$

which we can approximate with four terms



Notice there is one little hump in the step for each term we include.

application: even step function

Consider the step function centered on zero:

$$\begin{cases} f(x) = 1 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ f(x) = 0 & \text{otherwise} \end{cases}$$

Since this is an even function, all the b_m will be zero:

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin mx\ dx = 0$$

The cosine terms are:

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \cos mx\ dx = 0$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\ \cos mx\ dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \cos mx \; dx$$

$$= \frac{2}{\pi m} \sin mx \; \Big|_0^{\pi/2}$$

Only the odd terms survive, and these terms alternate in sign. Check a_0

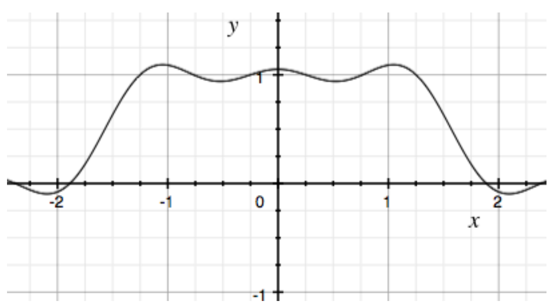
$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \; dx$$

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \; dx$$

$$= 1$$

Recall that the a_0 term has a coefficient of $\frac{1}{2}$

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \; [\; \cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \dots]$$



application: f(x) = x

This function is an odd function: $f(x) = -f(-x)$, while the cosine is an even function ($\cos x = \cos -x$). An even function times an odd function is an odd function, and so all the cosine terms vanish (between $[-\pi, \pi]$, as before.

Thus the coefficients that do remain are those for sine:

$$b_m = \frac{1}{\pi} \; [\int_{-\pi}^{\pi} f(x) \; \sin mx \; dx \;]$$

$$= \frac{1}{\pi} \; [\int_{-\pi}^{\pi} x \; \sin mx \; dx \;]$$

We can integrate $x \sin x$ using integration by parts:

$$u = x$$

$$du = dx$$

$$dv = \sin mx \; dx$$

$$v = -\frac{1}{m} \cos mx$$

So IBP gives

$$= -\frac{1}{m} x \cos mx - \int -\frac{1}{m} \cos mx \; dx$$

The limits are $[-\pi, \pi]$.

Integrate the second term and find that the result is just 0

$$\frac{1}{m^2} \sin mx \; \Big|_{-\pi}^{\pi} = 0$$

We go back for the factor of $1/\pi$:

$$b_1 = \frac{1}{\pi} \frac{(-x \cos mx)}{m} \; \Big|_{-\pi}^{\pi}$$

Now we use symmetry

$$= \frac{2}{\pi} \frac{(-x \cos mx)}{m} \; \Big|_{0}^{\pi}$$

At the lower bound of 0 the result is zero so now we have just:

$$= \frac{2}{\pi}\frac{(-\pi \cos m\pi)}{m}$$

$$= -\frac{2}{m}\cos m$$

The terms for odd m are

$$a_m = \frac{2}{m}$$

while the even terms are

$$a_m = -\frac{2}{m}$$

For a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \ dx$$

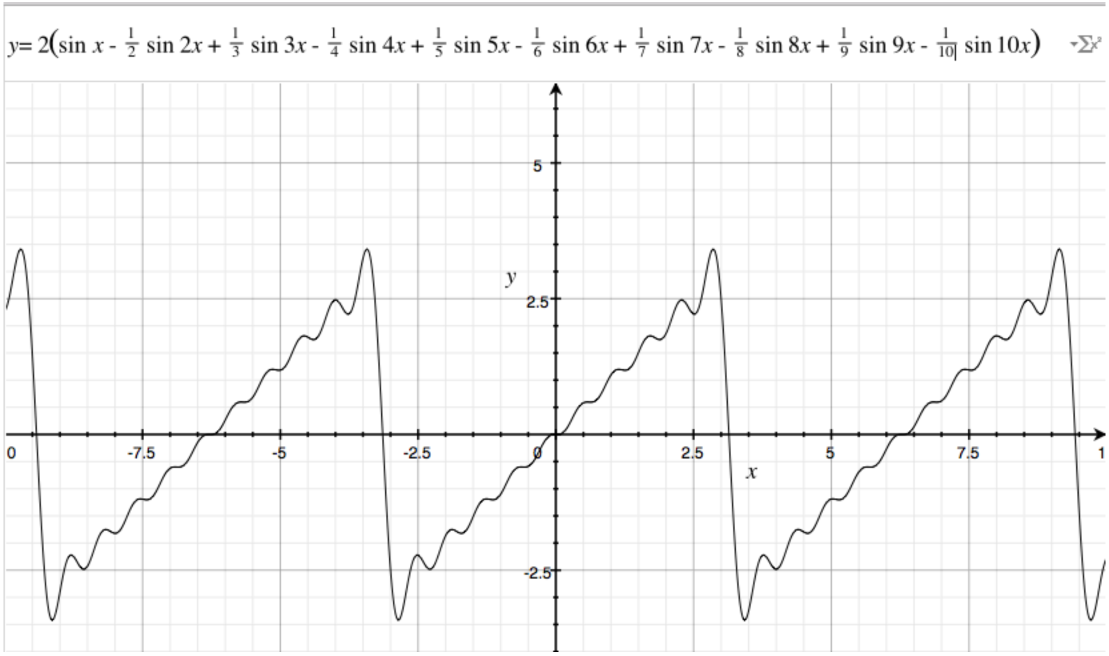
x is an odd function so the integral is zero

$$= \frac{1}{2\pi}x^2 \Big|_{-\pi}^{\pi} = 0$$

So finally we have that

$$f(x) = 2 \ [\ \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots]$$

Here are 10 terms


$y = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x - \frac{1}{6}\sin 6x + \frac{1}{7}\sin 7x - \frac{1}{8}\sin 8x + \frac{1}{9}\sin 9x - \frac{1}{10}\sin 10x\right)$
y
5
2.5
-2.5
-7.5
-5
-2.5
0
2.5
5
7.5
x