Examples for limits

The limit of a function $f(x)$ at a point a is written

$$\lim_{x \to a} f(x) = L$$

The formal definition is:

$$\forall\ \epsilon > 0, \exists\ \delta > 0 \mid \forall\ x,$$

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

You tell me the ϵ you require with $|f(x) - L| < \epsilon$, and I will try to find the right δ.

For a typical function, it's a good guess that $L = f(a)$.

$$|f(x) - f(a)| < \epsilon$$

which we can write without the absolute value bars (see Triangle write-up):

$$-\epsilon < f(x) - f(a) < \epsilon$$

example 1

Suppose $f(x) = 3x$ and we're interested in the point $a = 5$. Then set $L = f(a) = 15$.

$$-\epsilon < f(x) - f(a) < \epsilon$$

$$-\epsilon < 3x - 15 < \epsilon$$
$$-\frac{\epsilon}{3} < x - 5 < \frac{\epsilon}{3}$$

If we set $\delta = \epsilon/3$ we'll be good. And in general for a function $f(x) = cx$ with c a constant, at the point a, we can use

$$|x| - a < \frac{\epsilon}{c}$$

example 2

Suppose $f(x) = x^2$ and we're interested in the point $a = 2$. Then set $L = f(a) = a^2 = 4$.

$$-\epsilon < f(x) - f(a) < \epsilon$$
$$-\epsilon < x^2 - a^2 < \epsilon$$

Now we argue as follows:

$$x^2 - a^2 = (x-a)(x+a) = |x-a|\ |x+a|$$

and to get started suppose we require that *at least*

$$|x-a| < 1$$

$$-1 < x - a < 1$$

$$a - 1 < x < a + 1$$

$$2a - 1 < x + a < 2a + 1$$

$$|x+a| < 2a + 1$$

Then going back to

$$|x-a|\ |x+a| < \epsilon$$

$$|x-a|\ |x+a| < |x-a|\ (2a+1) < \epsilon$$

and

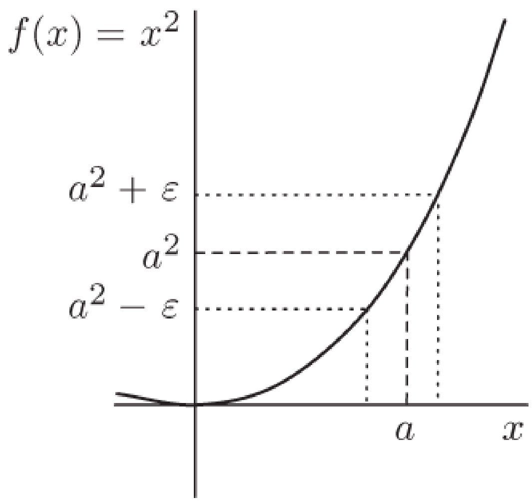
$$|x-a| < \frac{\epsilon}{(2a+1)}$$

Remembering the first condition we set:

$$|x-a| < \min\left(\frac{\epsilon}{(2a+1)}, 1\right) = \delta$$

And what we notice is that for $f(x) = x^2$, at least for some a (and depending on the value of ϵ that is chosen), the value of δ required depends on a.

That should not be too surprising.



The same ϵ will require a smaller δ the farther out we go on the curve.

example 3

Now consider the inverse function $f(x) = 1/x$. Suppose we're interested in the point $a = 3$ where we expect the limit to be $L = 1/3$. For this to be true we must guarantee that

$$\left|\frac{1}{x} - \frac{1}{3}\right| < \epsilon$$

for arbitrary ϵ.

Factor

$$|\frac{1}{x} - \frac{1}{3}| = |\frac{3-x}{3x}| = \frac{1}{3}\ \frac{1}{|x|}\ |3-x|$$

We showed in the write-up on the triangle inequality that $|a-x| = |x-a|$ so

$$= \frac{1}{3}\ \frac{1}{|x|}\ |x-3|$$

Here, we need to make sure that $|x|$ is not too *small*, so $1/|x|$ is not too large.

First require that $|x-3| < 1$. Then

$$-1 < x - 3 < 1$$

$$2 < x < 4$$

$$\frac{1}{4} < \frac{1}{x} < \frac{1}{2}$$

This means that $1/x > 0$ so

$$\frac{1}{|x|} = \frac{1}{x} < \frac{1}{2}$$

We now have

$$|\frac{1}{x} - \frac{1}{3}| = \frac{1}{3}\ \frac{1}{|x|}\ |3-x|$$

provided $|x-3| < 1$ and also with this condition $1/|x| < 1/2$ so

$$|\frac{1}{x} - \frac{1}{3}| < \frac{1}{6}\ |x-3|$$

Hence if $\delta = |x-3| < 6\epsilon$, the above expression is $< \epsilon$ and we're done. Officially we need:

$$|x-3| < \min\ (6\epsilon, 1)$$