Derivation of Taylor Series

first

We consider a function f that is analytic inside and on the circle $C_0 : z = r_0$, centered at 0. The point z is interior to C_0 and the Cauchy integral formula applies (w is on C_0):

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(w)\ dw}{w - z}$$

The factor $1/(w - z)$ can be rewritten as

$$\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-(z/w)}$$

aside on the geometric series

We recall that the nth partial sum of the geometric series can be written

$$S_N = \sum_{n=0}^{N-1} z^n = 1 + z + z^2 \cdots + z^{N-1}$$

(starting our numbering with the 0th term, the last term is $N - 1$).

So

$$z\ S_N = \sum_{n=0}^{N-1} z^n = z + z^2 \cdots + z^N$$

$$(1 - z)\ S_N = 1 - z^N$$

$$S_N = \frac{1 - z^N}{1 - z}$$

$$S_N = \frac{1}{1 - z} - \frac{z^N}{1 - z}$$

so finally

$$\frac{1}{1 - z} = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1 - z}$$

return to the problem

We had

$$\frac{1}{w - z} = \frac{1}{w} \cdot \frac{1}{1 - (z/w)}$$

$$= \frac{1}{w} \cdot \ [\ \sum_{n=0}^{N-1} (z/w)^n + \frac{(z/w)^N}{1 - z/w}\ ]$$

$$= \sum_{n=0}^{N-1} \frac{1}{w^{n+1}} z^n + \frac{z^N}{w^{N-1}(w - z)}$$

Multiply through by $f(w)$ and then integrate:

$$\int_C \frac{f(w)}{w - z}\ dw = \sum_{n=0}^{N-1} \int_C \frac{f(w)\ dw}{w^{n+1}} z^n + z^N \int_C \frac{f(w)\ dw}{w^{N-1}(w - z)}$$

The left-hand side is $2\pi i f(z)$.

Recall that (around $z = 0$) we had as an extension of the Cauchy Integral theorem:

$$f^n(0) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{w^{n+1}}\ dw$$

Hence the first term on the right-hand side in the previous expression is

$$\frac{2\pi i}{n!}\sum_{n=0}^{N-1} f^n(0)\ z^n$$

They show that the second term on the right-hand side is the remainder that will go to zero as $N \to \infty$, so (after dividing by $2\pi i$, we obtain

$$f(z) = \lim_{N\to\infty} \frac{1}{n!}\sum_{n=0}^{N} f^n(0)\ z^n +$$