Derivatives of rational powers

An amazing fact about the power rule is that it applies not only to the positive integers $n = 1, 2, \ldots$ it also applies to the negative integers, and it even applies to rational exponents (powers) m/n.

$$\frac{d}{dx} x^{\frac{m}{n}} = \frac{m}{n}\ x^{\frac{m}{n}-1}$$

The original proof is by Newton's extension of the binomial. This short write-up gives two alternative proofs—the first uses the chain rule and the derivative of e^x.

For simpler notation, start by letting $r = m/n$ so that $x^{m/n} = x^r$. Then write the definition of the natural logarithm of x

$$x^r = e^{\ln(x^r)}$$

Take derivatives on both sides

$$\frac{d}{dx}\ x^r = \frac{d}{dx}\ e^{\ln(x^r)}$$

The left-hand side is what we seek. Just apply the *chain rule* on the right

$$\frac{d}{dx}\ e^{\ln(x^r)} = e^{\ln(x^r)}\ \frac{d}{dx}\ \ln(x^r)$$

Reversing the original substitution, the right-hand side becomes

$$x^r\ \frac{d}{dx}\ \ln(x^r)$$

But

$$\ln(x^r) = r\ln(x)$$

Now we've turned the rational exponent into a constant. We know how to deal with that.

$$\frac{d}{dx}\ r\ln(x) = r\frac{d}{dx}\ln(x) = \frac{r}{x}$$

Putting it all together

$$\frac{d}{dx}\ x^r = x^r\ \frac{d}{dx}\ \ln(x^r) = x^r\ \frac{r}{x} = r x^{r-1}$$

a third method

Yet another way is the following trick that is somewhat advanced but an especially nice piece of mathematics. It relies on implicit differentiation. I saw this in David Jerison's Calculus lectures.

$$y = x^{\frac{m}{n}}$$

$$y^n = x^m$$

$$\frac{d}{dx}y^n = \frac{d}{dy}y^n\frac{dy}{dx} = \frac{d}{dx}x^m$$

$$ny^{n-1}\frac{dy}{dx} = mx^{m-1}$$

$$\frac{dy}{dx} = \frac{m}{n}\ \frac{x^{m-1}}{y^{n-1}}$$

Now

$$y^{n-1} = [\ x^{\frac{m}{n}}\ ]^{(n-1)} = x^{\frac{m}{n}(n-1)}$$

If we write all the powers of x in the numerator we have

$$\frac{dy}{dx} = x^{m-1-\frac{m}{n}(n-1)} = x^{m-1-m+\frac{m}{n}} = x^{\frac{m}{n}-1}$$

which gives finally

$$\frac{dy}{dx} = \frac{m}{n}\ x^{\frac{m}{n}-1}$$

We've shown that powers of x with rational exponents obey the power rule. In fact irrational powers like $f(x) = x^{\pi}$ also obey the power rule, but we don't need to prove that here.