SURFACE AREA.

Today we learn how to evaluate the surface area. We assume that the surface is given as a graph of function $z = f(x,y)$ and the domain of this function is a region D. We have

Theorem. *The area of the surface given as a graph of the function $z = f(x,y)$ over the region $(x,y) \in D$ is*

$$A(S) = \int\int_D \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2} dA \tag{2}$$

provided that the functions f_x and f_y are continuous over the region D.

Example 1. Find the area of the surface: The part of the plane $x+2y+z=4$ that lies inside of the cylinder $x^2+y^2=1$.

Solution. In our case the region D is the disk of radius one centered at $(0,0)$:

$$D = \{(x,y)|x^2+y^2 \leq 1\}.$$

We can rewrite the equation of the plane in the following form

$$z = 4 - x - 2y.$$

Hence $f(x,y) = 4 - x - 2y$ and $f_x = -1$ $f_y = -2$. By formula (2) we have

$$A(S) = \int\int_D \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2} dA = \int\int_D \sqrt{1+1+4} dA = \sqrt{6} \int\int_D 1 dA$$
$$= \sqrt{6} \cdot \text{Area of the disk} = \sqrt{6}\pi.$$

Example 2. Find the area of the surface: The part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies inside of the cylinders $x^2+y^2=1$ and $x^2+y^2=9$.

Solution. We have $f(x,y) = y^2 - x^2$ and $f_x(x,y) = -2x,\ f_y(x,y) = 2y$. By formula (2) we have

$$A(S) = \int\int_D \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2} dA = \int\int_D \sqrt{1 + 4x^2 + 4y^2} dA.$$

Starting from this point it is better to use the polar coordinate system. The region D is the polar rectangle

$$D = \{(x,y) | 1 \leq r \leq 3, 0 \leq \theta \leq 2\pi.\}$$

$$A(S) = \int_0^{2\pi} \int_1^3 \sqrt{1 + 4r^2} r dr d\theta = \int_0^{2\pi} \frac{1}{12} (1+4r^2)^{\frac{3}{2}} |_1^3 d\theta = \frac{1}{12} ((37)^{\frac{3}{2}} - (5)^{\frac{3}{2}}) \int_0^{2\pi} 1 d\theta = \frac{\pi}{6} ((37)^{\frac{3}{2}} - (5)^{\frac{3}{2}}).$$

Example 3. Find the area of the surface: The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$.

Solution. The intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = 1$ is the circle $x^2 + y^2 = 3$ on the plane $z = 1$. Therefore the projection of the part of the sphere $x^2 + y^2 + z^2 = 4$ which lies above the plane $z = 1$ is the disk

$$D = \{(x,y) | x^2 + y^2 \leq 3\}.$$

We can solve the equation for sphere respect to the variable z

$$z = \sqrt{4 - x^2 - y^2}.$$

Hence $f(x,y) = \sqrt{4 - x^2 - y^2}$ and $f_x(x,y) = \frac{-x}{\sqrt{4 - x^2 - y^2}}$, $f_y(x,y) = \frac{-y}{\sqrt{4 - x^2 - y^2}}$.
By formula (2) we have

$$A(S) = \int\int_D \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2} dA = \int\int_D \sqrt{1 + \frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2}} dA = \int\int_D \frac{2}{\sqrt{4 - x^2 - y^2}} dA.$$

Starting from this point it is better to use the polar coordinate system. The region D is the polar rectangle

$$D = \{(x,y) | 0 \leq r \leq \sqrt{3}, 0 \leq \theta \leq 2\pi.\}$$

$$A(S) = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r}{\sqrt{4 - r^2}} dr d\theta = 2 \int_0^{2\pi} -\sqrt{4 - r^2} |_0^{\sqrt{3}} d\theta = 2 \int_0^{2\pi} 1 d\theta = 4\pi.$$