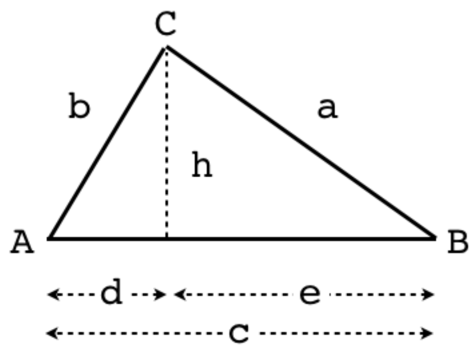
Law of Sines

Here, I will go through algebraic proofs of the Pythagorean Theorem and the Law of Sines. In the triangle below, A, B, and C are the angles, with side lengths a, b, and c.



An altitude has been drawn from angle C to side c opposite. The altitude is perpendicular to the side c, which is thereby divided into lengths d and e.

Suppose angle C is a right angle. (I know it doesn't look exactly right, but allow me to just go with it). If C is a right angle, A and B are complementary:

$$A + B = 90°$$

Referring to the small triangle on the left, it's clear that the angle (call it θ) between side b and the altitude h is equal to angle B, because

$$\theta + A + 90° = 180° = B + A + C$$

where $C = 90°$. Therefore, the small triangle on the left and $\triangle ABC$ are similar. By similarity, the sides opposite equal angles are in the same ratio so

$$\frac{d}{b} = \frac{b}{c}$$

$$\frac{h}{b} = \frac{a}{c}$$

By exactly the same argument, the small triangle on the right is similar to both of these, so we can extend the ratios

$$\frac{d}{b} = \frac{b}{c} = \frac{h}{a}$$

$$\frac{h}{b} = \frac{a}{c} = \frac{e}{a}$$

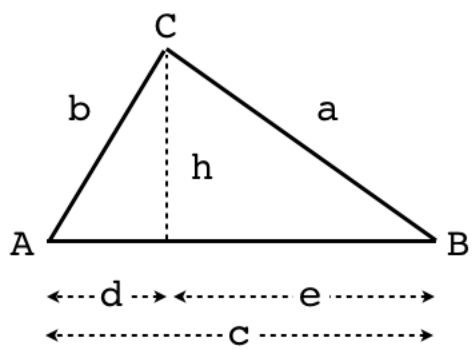
We have

$$a^2 = ce, \qquad b^2 = cd, \qquad a^2 + b^2 = ce + cd = c(e + d) = c^2$$

QED.

This is the Pythagorean Theorem.

Looking at the figure again (from now on angle C will not be equal to 90°), write a formula for the sine of A and sine of B



$$\frac{h}{b} = sin\ A, \qquad \frac{h}{a} = sin\ B$$

$$h = a\ sin\ B = b\ sin\ A$$

$$\frac{a}{sin\ A} = \frac{b}{sin\ B}$$

By symmetry, the ratio extends to side c and angle C

$$\frac{a}{sin\ A} = \frac{b}{sin\ B} = \frac{c}{sin\ C}$$

QED.

We have proved the Law of Sines.