More about e

In other write-ups I introduced e as

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

and

$$e^x = \lim_{n \to \infty} (1 + \frac{1}{n})^{nx} = \lim_{n \to \infty} (1 + \frac{x}{n})^n$$

From this, I derived the infinite series for e and e^x using the Binomial Theorem

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and I showed that

$$\frac{d}{dx} e^x = e^x$$

makes complete sense in terms of the series. Starting from this, one can prove that

$$\ln(x) = \int \frac{1}{x}\ dx$$

I saw a couple of videos on Khan academy that show how to go the other way. So first, we will prove that

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

We use our basic definition for the slope of the tangent to the curve $f(x)$

$$\frac{d}{dx} \ln(x) = \lim_{h \to 0}\ \frac{1}{h} (\ln(x+h) - \ln(x))$$

using

$$log(a) - log(b) = log(\frac{a}{b})$$

$$\lim_{h \to 0}\ \frac{1}{h} (\ln(x+h) - \ln(x)) = \lim_{h \to 0}\ \frac{1}{h} [\ \ln \frac{(x+h)}{x}\ ]$$

$$= \lim_{h \to 0} \frac{1}{h} \ln(1 + \frac{h}{x}) = \lim_{h \to 0} \ln(1 + \frac{h}{x})^{1/h}$$

Substitute

$$u = \frac{h}{x}, \quad xu = h, \quad \frac{1}{xu} = \frac{1}{h}$$

and the new limit is $\lim u \to 0$ instead of $\lim h \to 0$

$$= \lim_{u \to 0} \ln((1 + u)^{1/xu})$$

using

$$a^{bc} = (a^b)^c$$

$$= \lim_{u \to 0} \ln(((1 + u)^{1/u})^{1/x})$$

????

$$= \lim_{u \to 0} \frac{1}{x} \ln((1 + u)^{1/u})$$

and using

$$log\ ((a^b)^c) = c\ log(a^b)$$

and since x isn't involved in the limit ????

$$= \frac{1}{x} \lim_{u \to 0} \ln((1 + u)^{1/u}) = \frac{1}{x} \ln(e) = \frac{1}{x}$$

$$\boxed{\frac{d}{dx} \ln(x) = \frac{1}{x}} \tag{1}$$

The second thing we want to do is to go from this to show that the derivative of the function $f(x) = e^x$ is itself. Start with

$$\frac{d}{dx} \ln(e^x)$$

since $\ln(e^x) = x$

$$\frac{d}{dx} \ln(e^x) = \frac{d}{dx} x = 1$$

but using the property we just proved and the chain rule, this is also

$$\frac{d}{dx} \ln(e^x) = \frac{1}{e^x} \frac{d}{dx} e^x$$

but

$$\frac{d}{dx} \ln(e^x) = \frac{1}{e^x} \frac{d}{dx} e^x = 1$$

$$\boxed{\frac{d}{dx} e^x = e^x} \tag{2}$$