Vectors and planes

This is a continuation of the first write-up about vectors and the dot product. Let's talk now about another type of vector multiplication, called the cross-product. Unlike the dot product, which gives a number, the cross-product of two vectors is another vector. The resulting vector is perpendicular to each of the original ones. Among other things, this provides a way to find such vectors. The symbol for the cross product of u and v is

$$u \times v$$

There is a trick or device to remember how to compute the cross product. We form the matrix

$$\begin{bmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

and then pretend we're taking its determinant. Since i, j, and k are vectors, this is not really a valid operation, but we just do it anyway. I get

$$(u_2v_3 - u_3v_2)\ i - (u_1v_3 - u_3v_1)\ j + (u_1v_2 - u_2v_1)\ k$$

$$w = <(u_2v_3 - u_3v_2), -(u_1v_3 - u_3v_1), (u_1v_2 - u_2v_1)>$$

This looks kind of ugly, but it's usually not too bad.

Notice that if we do the dot product with u or v, we get 0. For example, with u

$$u_1(u_2v_3 - u_3v_2) - u_2(u_1v_3 - u_3v_1) + u_3(u_1v_2 - u_2v_1)$$

$$= u_1u_2v_3 - u_1u_3v_2 - u_2u_1v_3 - u_2u_3v_1 + u_3u_1v_2 - u_3u_2v_1$$

$$= (u_1u_2v_3 - u_2u_1v_3) + (u_3u_1v_2 - u_1u_3v_2) + (u_2u_3v_1 - u_3u_2v_1)$$

$$= 0$$

This is also try with v and therefore it is true for $cu + dv$, i.e. for every vector in the plane formed by u and v.

I also want to show where we can get the equation of a plane. The standard format is

$$ax + by + cz = d$$

where a, b, c, d are some constants.

Suppose we are trying to find the equation of a plane and all we know is one single point that lies in it. (We will need something more than this, the vector perpendicular to the plane, but that's coming in a bit).

So we are given $P = (x_0, y_0, z_0)$ and we know it satisfies the equation of the plane, which is what we want to find.

We also know that for each point $Q = (x, y, z)$ in the plane we can construct the vector that starts at P and ends at Q, by just subtracting each component of P from the corresponding component of Q

$$v = < x - x_0, y - y_0, z - z_0 >$$

(remembering that x_0, y_0, z_0 will be known values—given in the problem statement, while x, y, z are variables). Since v runs between two points in the plane, the vector v itself lies in the plane.

The definition of a normal vector to a plane is that it is perpendicular to every vector in the plane. That is

$$N = < a, b, c >$$

and

$$N \perp v$$

so

$$N \cdot v = 0 = < a, b, c > \cdot < x - x_0, y - y_0, z - z_0 >$$

$$0 = ax - ax_0 + by - by_0 + cz - cz_0$$

Rearranging

$$ax + by + cz = ax_0 + by_0 + cz_0$$

Define

$$d = ax_0 + by_0 + cz_0$$

(Remember, we know a, b, c, x_0, y_0, and z_0). Now we have the usual equation of a plane.

$$ax + by + cz = d$$

And we see that, given such an equation, we can find the normal vector very easily. It is just $N = < a, b, c >$!