Determinant Facts

Strang says that all the usual properties of the determinant can be deduced from three simple rules.

1. $|I| = 1$

The determinant of the identity matrix I equals 1.

$$|I| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

2. row exchange may change the sign

Each *single* row exchange changes the sign of the determinant. Some permutation matrices in $\mathbb{R}3$ are formed by two exchanges.

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1, \qquad \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1$$

The rotation matrices in $\mathbb{R}2$ have determinant equal to 1:

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

as they do in $\mathbb{R}3$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{vmatrix}$$

3. linearity

The determinant is a linear function of each row separately.

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

For example

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

and also

$$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

Thus

$$|A + B| = |A| + |B|$$

With those three rules, we can derive the rest. Important rules:

- row reduction operations do not change the determinant
- A^{-1} exists $\iff$ $|A| = 0$
- $|A| = |A^T|$
- $|AB| = |A||B|$

To continue systematically:

4. equal rows

If two rows of $\mathbf{A}$ are equal, $|\mathbf{A}| = 0$.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$$

Proof: exchange the two equal rows to give $\mathbf{A}'$. Now, $|\mathbf{A}| = -|\mathbf{A}'|$ because of the exchange, but $|\mathbf{A}| = |\mathbf{A}'|$ because $\mathbf{A} = \mathbf{A}'$. Therefore $|\mathbf{A}| = |\mathbf{A}'| = 0$.

5. no change with row reduction

Row reduction methods don't change the determinant.

$$\begin{vmatrix} a & b \\ c - ka & d - kb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - k \begin{vmatrix} a & b \\ a & b \end{vmatrix}$$

Proof: the above is true by linearity (sequential application of rule 3). But because of the equal rows in

$$-k \begin{vmatrix} a & b \\ a & b \end{vmatrix}$$

the second term is 0 by rule 4, and thus, row reduction has not changed the determinant.

6. row of zeros

If a matrix has a row of zeros, its determinant is equal to 0.

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} a & b \\ a & b \end{vmatrix}$$

Proof: use row reduction to add row 1 to row 2. Now there are equal rows, so the determinant is equal to 0 (by rule 4).

7. triangular matrix

A triangular matrix is one that looks like this

$$\begin{bmatrix} d_1 & a & b \\ 0 & d_2 & c \\ 0 & 0 & d_3 \end{bmatrix}$$

called upper triangular, U, or alternatively the lower triangular matrix L

$$\begin{bmatrix} d_1 & 0 & 0 \\ x & d_2 & 0 \\ y & z & d_3 \end{bmatrix}$$

In either case, we can use row reduction methods to zero out the off-diagonal entries without changing the entries along the diagonal. Thus

$$|U| = |L| = \begin{vmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{vmatrix}$$

And those entries can be factored out to give

$$d_1 d_2 d_3 |I|$$

Thus both U and L have determinant $d_1 \times d_2 \times d_3$.

8. singular matrix

$$|\mathbf{A}| = 0 \iff \mathbf{A} \text{ is } \textit{singular.}$$

The determinant is zero if and only if the matrix is singular. Proof: produce $\mathbf{U}$ by row reduction methods, not changing the determinant. If $\mathbf{U}$ is singular, it will have a zero row and $|\mathbf{U}| = 0$ by rule 6. Otherwise, use rule 7.