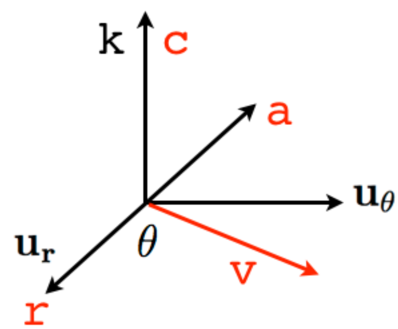
Kepler (part 3): Axes



Here is a sketch of the situation. $\mathbf{r}$ is the position vector, extending radially out from the sun to the planet. $\mathbf{u_r}$ is a unit vector in the $\mathbf{r}$ direction, so that

$$\mathbf{r} = r\mathbf{u_r}$$

By the central force hypothesis, the acceleration $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$ is in the $-\mathbf{u_r}$ direction. The source of all our complexity is that the velocity $\mathbf{v} = \dot{\mathbf{r}}$ is not perpendicular to $\mathbf{u_r}$ but makes an angle θ with it.

Earlier we proved that

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{c}$$

is a constant. Here we give that vector the label $\mathbf{c}$ and a direction. We align $\mathbf{c}$ with $\hat{\mathbf{k}}$. All the motion takes place in the xy-plane. Finally, we define $\mathbf{u}_\theta$ as orthogonal to $\mathbf{u_r}$ (and to $\hat{\mathbf{k}}$). $\mathbf{u}_\theta$ is aligned with $\hat{\mathbf{j}}$. As a result of these definitions:

$$\mathbf{u_r} \times \mathbf{u}_\theta = \hat{\mathbf{k}}$$

$$\hat{\mathbf{k}} \times \mathbf{u_r} = \mathbf{u}_\theta$$

$$\mathbf{u}_\theta \times \hat{\mathbf{k}} = \mathbf{u_r}$$

At any given time, $\mathbf{r}$ makes an angle θ with the x-axis, and is at a distance r from the origin, so we write:

$$\mathbf{r} = \ \langle r\cos\theta, r\sin\theta\rangle$$

$$\mathbf{u_r} = \ \langle \cos\theta, \sin\theta\rangle$$

$$\mathbf{u}_\theta \perp \mathbf{u_r}$$

$$\mathbf{u}_\theta = \ \langle -\sin\theta, \cos\theta\rangle$$

Verify that the cross-product is zero and that both vectors are unit length. Now, differentiate $\mathbf{u_r}$ and $\mathbf{u}_\theta$ (and recall that θ is also a function of time):

$$\frac{d}{dt}\ \mathbf{u_r} = \dot{\mathbf{u}}_\mathbf{r} = \frac{d\theta}{dt}\ \langle -\sin\theta, \cos\theta\rangle = \frac{d\theta}{dt}\ \mathbf{u}_\theta$$

$$\frac{d}{dt}\ \mathbf{u}_\theta = \dot{\mathbf{u}}_\theta = \frac{d\theta}{dt}\ \langle -\cos\theta, -\sin\theta\rangle = \frac{d\theta}{dt}\ \mathbf{u_r}$$

We can also get parametric expressions for the velocity

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}\ (r\mathbf{u_r}) = \frac{dr}{dt}\mathbf{u_r} + r\frac{d\theta}{dt}\mathbf{u}_\theta$$

and (with a little work) we can get the acceleration

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d}{dt}\ (\frac{dr}{dt}\mathbf{u_r} + r\frac{d\theta}{dt}\mathbf{u}_\theta)$$

$$= \frac{d^2r}{dt^2}\mathbf{u_r} + \frac{dr}{dt}\dot{\mathbf{u}}_\mathbf{r} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + r\frac{d\theta}{dt}\dot{\mathbf{u}}_\theta$$

We get three terms from differentiating the triple product $r\ d\theta/dt\ \mathbf{u}_\theta$, by a variation on the product rule. Substitute for the dotted terms from above

$$= \frac{d^2r}{dt^2}\mathbf{u_r} + \frac{dr}{dt}\frac{d\theta}{dt}\ \mathbf{u}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + r\frac{d\theta}{dt}\frac{d\theta}{dt}\ \mathbf{u_r}$$

Group common terms together

$$= (\frac{d^2r}{dt^2} + r(\frac{d\theta}{dt})^2)\mathbf{u_r} + (2\ \frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2})\mathbf{u}_\theta$$

Now for a trick, look at the factors multiplying $\mathbf{u}_\theta$ and recognize that

$$\frac{d}{dt}(r^2\frac{d\theta}{dt}) = 2r\frac{dr}{dt}\frac{d\theta}{dt} + r^2\frac{d^2\theta}{dt^2}$$

Therefore, the cofactors for $\mathbf{u}_\theta$ can be re-written as

$$\frac{1}{r}(\frac{d}{dt}(r^2\frac{d\theta}{dt})$$

and since if the acceleration is to be only radial (pointed toward the sun), this term must be equal to zero.

$$\frac{1}{r}(\frac{d}{dt}(r^2\frac{d\theta}{dt}) = 0$$

$$\frac{d}{dt}(r^2\frac{d\theta}{dt}) = 0$$

$$r^2\frac{d\theta}{dt} = h = \text{constant}$$

If we write $d\theta/dt = \omega$, the angular velocity, then $r\omega$ is the speed of the planet, and r times that, times the mass, is the angular momentum. This result is the conservation of angular momentum.