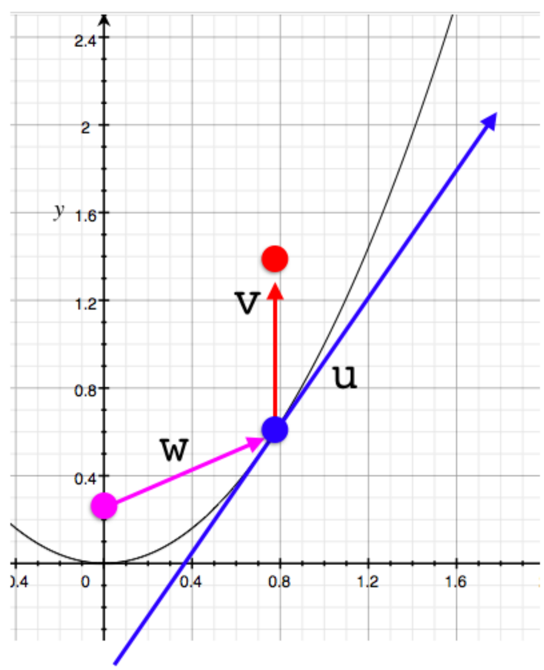
Vectors to solve the headlight problem




The parabola has the property that every ray of light parallel to the $y$-axis will bounce off the curve and go to the same point, which is called the *focus* of the parabola. We will prove this here using vectors, making use of the property from optics that a reflected ray makes equal angles of incidence and reflection.

We pick an arbitrary point on the parabola $P = (x, ax^2)$, shown in blue, and draw the tangent to the curve at that point. The slope of the tangent is $2ax$. The point $Q = (0, p)$, in magenta, is the focus.

Consider three vectors, one of each color. The one in blue is

$$\mathbf{u} = \langle 1, 2ax \rangle$$

The one in red is

$$\mathbf{v} = \langle 0, 1 \rangle$$

And the one in magenta is

$$\mathbf{w} = \langle x, ax^2 - p \rangle$$

Hopefully, you can derive these yourself.

**setup**

The key fact we use is the property from optics, which says that the angle between the blue and red vectors (let's call it $\theta$) is equal to that between the magenta and blue vectors ($\phi$). If that is true, then their cosines are also equal. Thus we can write:

$$\cos\theta = \cos\phi$$

But by the definition of the dot product:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\cos\phi = \frac{\mathbf{w} \cdot \mathbf{u}}{|\mathbf{w}||\mathbf{u}|}$$

Substituting into first equality, we write

$$\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{\mathbf{w} \cdot \mathbf{u}}{|\mathbf{w}||\mathbf{u}|}$$

$$\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{w} \cdot \mathbf{u}}{|\mathbf{w}|}$$

**calculation**

What remains is just arithmetic.

$$\mathbf{u} \cdot \mathbf{v} = \langle 1, 2ax \rangle \cdot \langle 0, 1 \rangle = 2ax$$

$$\mathbf{w} \cdot \mathbf{u} = \langle x, ax^2 - p \rangle \cdot \langle 1, 2ax \rangle = x + 2ax(ax^2 - p)$$

To make things a bit simpler, define $h = ax^2 - p$. Then

$$\mathbf{w} \cdot \mathbf{u} = \langle x, h \rangle \cdot \langle 1, 2ax \rangle = x + 2axh$$

Substitute for the dot products

$$\frac{2ax}{|\mathbf{v}|} = \frac{x + 2axh}{|\mathbf{w}|}$$

We chose $|\mathbf{v}| = 1$ and $|\mathbf{w}| = \sqrt{x^2 + (ax^2 - p)^2} = \sqrt{x^2 + h^2}$. Substituting for $|\mathbf{w}|$ we obtain:

$$2ax = \frac{x + 2axh}{\sqrt{x^2 + h^2}}$$

$$2ax\sqrt{x^2 + h^2} = x + 2axh$$

Divide by $x$

$$2a\sqrt{x^2 + h^2} = 1 + 2ah$$

Square both sides:

$$4a^2 \; [\; x^2 + h^2 \;] = 1 + 4ah + 4a^2h^2$$

$$4a^2x^2 + 4a^2h^2 = 1 + 4ah + 4a^2h^2$$

$$4a^2x^2 = 1 + 4ah$$

Reverse the substitution

$$4a^2x^2 = 1 + 4a(ax^2 - p) = 1 + 4a^2x^2 - 4ap$$

We obtain:

$$0 = 1 - 4ap$$

$$4ap = 1$$

$$p = \frac{1}{4a}$$

This is indeed the definition of the focal point. Furthermore, $p$ is not dependent on $x$, so we obtain the desired result: every perpendicular ray ends up at the focus. Conversely, in the case of a headlight, every ray emitted at the focus ends up exiting vertically upwards.