Derivatives

The basic method for finding the slope of a tangent line to a function $f(x)$ at x is to compute the difference quotient

$$\frac{f(x+h)-f(x)}{h}$$

for a small change h. Since $h = \Delta x$ and $\Delta y = f(x+h) - f(x)$, this is $\Delta y/\Delta x$, the slope of a secant line between the points $(x, f(x))$ and $(x+h, f(x+h))$ on the curve $y = f(x)$. To find the derivative we find the limit

$$lim_{h\to 0}\ \frac{f(x+h)-f(x)}{h}$$

Last time we went through some examples

$$f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x$$

$$f(x) = \sqrt{x} = x^{1/2} \quad \Rightarrow \quad f'(x) = \frac{1}{2}x^{-1/2}$$

$$f(x) = \frac{1}{x} = x^{-1} \quad \Rightarrow \quad f'(x) = -\frac{1}{x^2} = -x^{-2}$$

We deduce that the general formula is

$$f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1}$$

We would like to find a general expression for $f'(x)$, when $f(x) = x^n$ with integer n $(n = 1, 2, 3\cdots)$. Recall that the Binomial Theorem gives the expansion of $(x+h)^n$. We only need the first three terms.

$$(x+h)^n = x^n + nx^{n-1}h + n(n-1)x^{n-2}h^2 + \cdots$$

Now we compute the difference quotient and find the limit

$$\frac{f(x+h)-f(x)}{h}$$

$$\frac{x^n + nx^{n-1}h + n(n-1)x^{n-2}h^2 + \cdots - x^n}{h}$$
$$\frac{nx^{n-1}h + n(n-1)x^{n-2}h^2 + \cdots}{h}$$
$$nx^{n-1} + n(n-1)x^{n-2}h + \cdots$$

and find the limit

$$lim_{h\to 0}\ nx^{n-1} + n(n-1)x^{n-2}h = nx^{n-1}$$

This is called the power rule

$$f(x) = x^n, \qquad f'(x) = nx^{n-1}$$

Another question is what to do with a sum or difference of polynomials, such as

$$f(x) + g(x)$$

If you write out the difference quotient in the second case

$$\frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

everything can be exactly as before, just grouping all the terms from $f(x)$ and those from $g(x)$ separately.

$$[f(x) + g(x)]' = f'(x) + g'(x)$$