Chapter 41

Value of pi

Archimedes and π

Since Archimedes is a strong presence in this book, we will discuss his method for approximating the value of π, the ratio of the circumference of a circle to its diameter. The commonly cited result is

> The ratio of the circumference of any circle to its diameter is less than 3 1/7 but greater than 3 10/71.

In decimal that is $3.140845.. < \pi < 3.1428571$.

However, to some extent this misses the main idea, that Archimedes described an iterative procedure which can be used to calculate the value of π *to any desired accuracy.*

Although the idea is beautiful, his argument is somewhat unwieldy in detail, so instead we will use modern trigonometry to achieve the same result more economically.

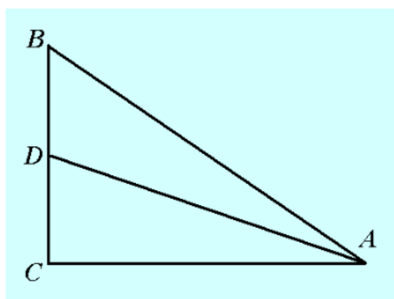
For a discussion of Archimedes actual method (based on a translation by Heath), see this web page https://itech.fgcu.edu/faculty/clindsey/mhf4404/archimedes/archimedes.html

and I have worked out the same proof in detail in this **chapter**.

In addition, we will connect the trigonometry to easy formulas for the perimeter and area of inscribed and circumscribed polygons. The first part is in this chapter, and the second part has been split out into another **chapter**, which is in the Addendum.

If this material is too esoteric, it can be skipped without loss of continuity in the rest of the book.

I should also point out that although we don't follow Archimedes exactly, a key element which he relies upon is the proof that, for an angle bisector in a right triangle, the adjacent sides are in the same proportion as the two segments formed where the bisector meets the other side.



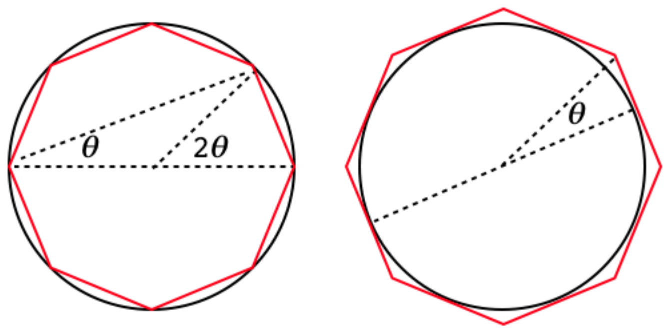
Here:

$$\frac{AB}{AC} = \frac{BD}{DC}$$

We showed a proof of this earlier (**here**).

the method

We will approximate the value of π by squeezing it between the perimeter of an inscribed polygon, which is less than the circumference of the circle, and the perimeter of a circumscribed polygon, which is greater than the circumference of the circle.



We use a circle of *diameter* equal to 1 (rather than the radius, which is more usual). The circumference of the circle is then equal to π, the value which gets squeezed between the two perimeters.

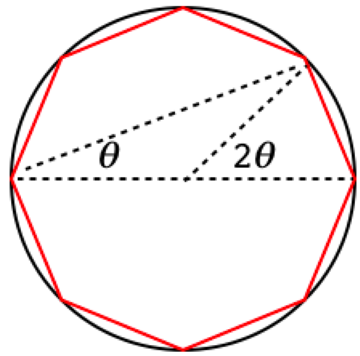
The figure shows a sketch of the polygons when $n = 8$. We will be increasing the number of sides by a factor of 2 at each step, so these are really 2^n-gons with $n = 3$ here.

Finding perimeters in terms of angle θ

For the left panel, we have 8 sides, so the central angle (marked 2θ) is equal to

$$\frac{2\pi}{8} = \frac{\pi}{4} = 45^\circ$$

and θ is one-half that.



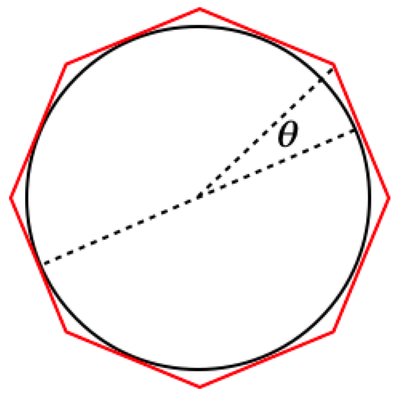
By a standard theorem (from Thales), the triangle above containing angle θ, with the diameter as one side, and two other vertices also on the circle, is a right triangle. The inscribed n-gon side of length S (shown in red) is equal to $\sin\theta$, since the hypotenuse of the triangle is the diameter of the circle, which is equal to 1.

The total perimeter is $8 \cdot S$.

[Alternatively, use half the angle at the center of the circle (i.e. θ). Then half the length of the red line $S/2$, divided by the radius ($r = 1/2$) gives $S = \sin\theta$, the same result.]

For the right panel, we have the same circle (now showing the outside polygon, circumscribing the circle), it is just rotated slightly.. One dashed line extends a bit further to the vertex of the n-gon outside. The angle marked θ is one-half the angle we marked as 2θ previously since now the diameter comes down to the middle of the side.

We compute the whole length of the side T as follows. The half-side is $T/2$ and the hypotenuse of the triangle is one-half the unit diameter, which is $1/2$, so $T = \tan\theta$. The total perimeter is $8 \cdot T$.



All of this gives us two simple equations for the two perimeters. At each stage there are 2^n sides, the length of each short side S on the inside equals $\sin\theta$ and the length of each short side on the outside T is equal to $\tan\theta$, where $\theta = 2\pi/2^n$.

The total length of the inside perimeter is $nS = n\sin\theta$ and that of the outside is $nT = n\tan\theta$. When we go from θ to $\theta/2$ and n to $2n$, we must compute the new values S' and T' from S and T using the half-angle formulas, and then also multiply by 2 to take account of the change from n to $2n$ for the total circumference.

The base case

If we go back to the square ($n = 2, 2^n = 4$), then the angle θ is $\pi/4$.

The tangent is $T = \tan\pi/4 = 1$ and the sine is $S = \sin\pi/4 = 1/\sqrt{2}$.

Our formulas say that on the inside, the perimeter is $4S = 4/\sqrt{2} = 2\sqrt{2}$ and on the outside, the perimeter is $4T = 4$.

From simple geometry, we can calculate that the circumscribing square has a side length which is twice the radius of the circle, that is, 1 for our circle with unit diameter, so its perimeter is 4, which checks.

Similarly, an inscribed square can be decomposed into four isosceles right triangles with sides of length $1/2$ and hypotenuse $1/\sqrt{2}$, so the total perimeter is $4/\sqrt{2}$, which also checks.

Now, what we are going to do is to increase n in steps of 1, that increases 2^n by a factor of $2^1 = 2$ each time. Doubling n halves the angle. So all we need is a way to compute trigonometric functions of $\theta/2$, knowing the values for θ, so we can calculate what happens to the perimeter. We already know how to do that.

Half angle formulas

We have derived these elsewhere. Refer to this **chapter**.

The unprimed values refer to angle θ, while the primed ones have angle $\theta/2$.

$$C' = \sqrt{\frac{1}{2}(1 + C)}$$

This can be rearranged (e.g.) to give $2[C']^2 = 1 + C$, which we'll use in a second.

$$S' = \frac{S}{2C'}$$

$$T' = \frac{S'}{C'} = \frac{S}{2[C']^2}$$
$$= \frac{S}{1 + C}$$

So, given S, C and T, first calculate C' and T' and then S'. To get the perimeters, remember that factor of two from doubling n, the number of sides.

another approach

This web page originally got me started with this derivation

http://personal.bgsu.edu/~carother/pi/Pi3d.html

(Unfortunately, the link is dead now, probably because the University took Dr. Carother's pages down when he died, idiots). It has been preserved by the wayback machine:

https://web.archive.org/web/20171024182015/http://personal.bgsu.edu/~carother/pi/Pi3d.html

On that page, there was given an arguably simpler pair of formulas listed, namely, for an inside perimeter p and an outside perimeter P

$$P' = \frac{2pP}{p+P}$$

$$p' = \sqrt{pP'}$$

The first equation can be rearranged to give

$$\frac{1}{P'} = \frac{1}{2}\,[\frac{1}{P} + \frac{1}{p}\,]$$

which is the definition of the harmonic mean of p and P, while the second equation is the geometric mean.

Since in our derivation p and P are the same multiple of S and T, it seems like the same relationships should hold for the sine and tangent, but we must remember the extra factor of 2.

From the half-angle formulas, we said that

$$T' = \frac{S}{1 + C}$$

Multiply top and bottom on the right by T:

$$T' = \frac{ST}{T + S}$$

Recall that S is the same as p, within a factor of n, and that T is the same as P, within the same factor.

$$p = nS$$

$$P = nT$$

while

$$P' = 2nT'$$

Going back to

$$\begin{aligned} T' &= \frac{ST}{T + S} \\ 2nT' &= \frac{2 \cdot nS \cdot nT}{nT + nS} \\ P' &= \frac{2pP}{p + P} \end{aligned}$$

This is what was given.

For the second one

$$\begin{aligned} S' &= \frac{S}{2C'} \\ &= \frac{S}{2} \, \frac{T'}{S'} \end{aligned}$$

Then

$$4[S']^2 = S \cdot 2T'$$

$$[2nS']^2 = nS \cdot 2nT'$$

Changing variables, $p' = 2nS'$

$$[p']^2 = pP'$$

Finally

$$p' = \sqrt{pP'}$$

which matches what was given.

Calculation

Let's run a simulation to see what kind of numbers we get. Start with the square ($n = 2$, $2^n = 4$) Previously we found that $S = 1/\sqrt{2}$ and $T = 1$ so

$$p = 2^n S = \frac{4}{\sqrt{2}} = 2.8284$$

$$P = 2^n T = 4$$

Let's try a script to calculate this to larger n.

https://gist.github.com/telliott99/19f521c807210171a4847b319104b3df

Output:

```
> python pi.py
 2 2.8284271247  4.0000000000
 3 3.0614674589  3.3137084990
 4 3.1214451523  3.1825978781
 5 3.1365484905  3.1517249074
 6 3.1403311570  3.1441183852
 7 3.1412772509  3.1422236299
 8 3.1415138011  3.1417503692
```

```
 9 3.1415729404  3.1416320807
10 3.1415877253  3.1416025103
11 3.1415914215  3.1415951177
12 3.1415923456  3.1415932696
13 3.1415925766  3.1415928076
14 3.1415926343  3.1415926921
15 3.1415926488  3.1415926632
16 3.1415926524  3.1415926560
17 3.1415926533  3.1415926542
18 3.1415926535  3.1415926537
19 3.1415926536  3.1415926536
>
```

That looks pretty good to me, although it's a bit slow to converge.

This is really quite amazing. Archimedes has not only calculated π to 3 significant figures. More important, he has provided us with an iterative procedure that can be used to calculate the value to *any precision we desire*. As an engineer, Archimedes knew that 3.1416 is precise enough, so he stopped.

After all, no one wants to be William Shanks, or one of these guys:

https://en.wikipedia.org/wiki/Chronology_of_computation_of_

Quote:

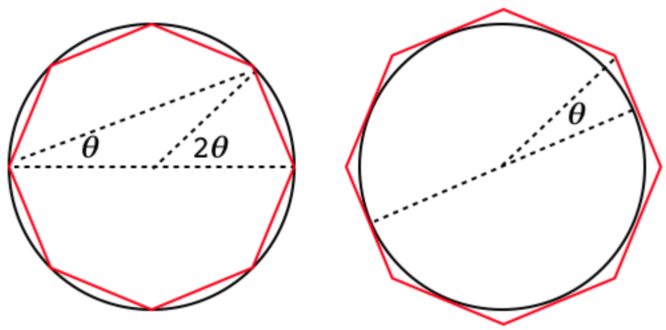
> [He] calculated pi to [n] digits, but *not all were correct.*

There is an additional **chapter** which substantially extends the above discussion, showing a geometric derivation of the basic relationships and developing new formulas involving the areas as well as the perimeters of the sectors of inscribed and circumscribed polygons.

area

I became aware later that there is yet another way to apply the method, and that is to calculate the *areas* of inscribed and circumscribed polygons. We'll go through this briefly.

For this approach we use a unit circle (radius 1) rather than a diameter of 1, as we did above. As before, we define θ to be the central angle of the half-sector (i.e. $\theta = 2\pi/2n$).



Rather than draw an entirely new figure, just imagine in the left panel that we draw the angle bisector of angle 2θ. The area of each new triangle is then $\sin\theta\cos\theta/2$ and the total area of the inner polygon is

$$a = n \sin\theta \cos\theta = nSC$$

in the notation we adopted previously in this chapter. And, as before, to progress to a' we have a factor of 2 as well as the new values S' and C':

$$a' = 2nS'C'$$

For the circumscribed or outer polygon, we just have what we had before, that the side length of the triangle in the right panel is $\tan\theta$ so the total area is

$$A = nT$$

Bring in the half-angle formulas as follows:

$$a' = 2nS'C' = 2n \cdot \frac{S}{2C'} \cdot C' = nS$$

That is slick, but we need an expression for nS:

$$aA = nSC \cdot n\frac{S}{C} = [nS]^2$$

$$aA = [a']^2$$

$$a' = \sqrt{aA}$$

This is like, and yet subtly different than what we had when calculating the perimeter.

Since

$$A = nT$$

and

$$\begin{aligned} A' &= 2nT' \\ &= 2n\frac{ST}{S+T} = 2\frac{nS \cdot nT}{nS+nT} \\ A' &= 2\frac{a'A}{a'+A} \end{aligned}$$

Compare

$$a' = \sqrt{aA} \qquad A' = 2\frac{a'A}{a'+A}$$

$$p' = \sqrt{pP'} \qquad P' = 2\frac{pP}{p+P}$$

Chapter 158

Value of pi revisited

As discussed in a previous **chapter**, Archimedes used paired inscribed and circumscribed polygons to develop an iterative procedure that can be used to calculate the value of π *to any desired accuracy.* Although the method is beautiful, his argument is unwieldy in detail, so we used modern trigonometry to achieve the same result more economically.

There are, in addition, two other sets of formulas that also reach this end, one based on perimeters, and the other on areas. These formulas are intriguing because they are simple, and it is not surprising that they are connected.

For example, consider a circle of unit *diameter*, so that π is equal to the perimeter. If p and P are the inside and outside perimeters for polygons whose sectors have central angle θ, and the same symbols are used with primes for angle $\theta/2$, then:

$$P' = 2\frac{pP}{p+P}$$

$$p' = \sqrt{pP'}$$

The corresponding formulas for inside (a) and outside (A) areas are (for a circle of unit radius)

$$A' = 2\frac{a'A}{a' + A}$$

$$a' = \sqrt{aA}$$

Notice that these two similar sets of formulas are subtly different. For example, to go from p and P to the primed version, we start with the first formula, while for area we must start with the square root. Part of our purpose in this chapter is to show that this works. (I must confess, I still do not have a simple explanation for *why* it is true).

inspiration

Originally, I was thinking about trying to implement Archimedes actual method for calculating π. However, the details of the approach are pretty painful. Instead, I worked through the problem using trigonometry.

It's striking that the formulas for the inside and outside perimeters are so simple, namely $n \sin \theta$ and $n \tan \theta$. The rest just follows from the half-angle formulas.

The web page which originally got me started with the harmonic and geometric mean formulas has been preserved by the wayback machine:

https://web.archive.org/web/20171024182015/http://personal.bgsu.edu/~carother/pi/Pi3d.html

On the very same day that I was revising the previous chapter to better integrate these two approaches, I came across another page which gives a "proof without words" of Gregory's Theorem (that is our subject).

https://divisbyzero.com/2018/09/28/proof-without-word-gregorys-theorem/

It gives these two formulas:

$$I_{2n} = \sqrt{I_n C_n}$$

$$C_{2n} = \frac{2}{1/I_{2n} + 1/C_n}$$

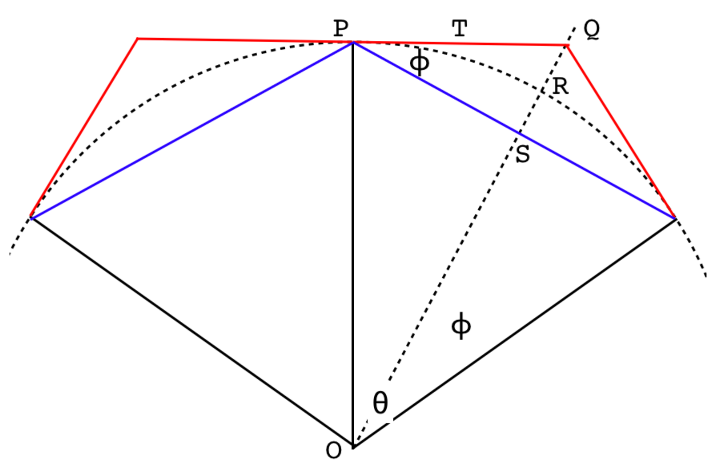
I found this notation a bit awkward, so I substituted the versions given above:

$$a' = \sqrt{aA}$$

$$A' = 2\frac{a'A}{a' + A}$$

Here, we mainly follow the development from that page and its "proof without words". One difference is that we will start with the geometry and work backward to the formulas. Let's deal with the perimeter first and then do the area.

basic setup



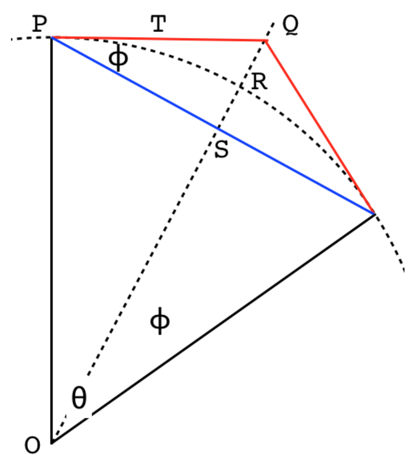
Draw a circle centered at O (only an arc of the circle is shown).

Points on the circle are chosen such that the arc length is an integral fraction of the whole. Equivalently, set $n\theta = 2\pi$.

Two adjacent sectors are shown in the figure above. The two polygons might be drawn so that the vertices of the internal and external figures are on the same ray, with parallel sides. However, the construction shown is more convenient.

The precise scale does not matter to the argument (nor the value of n). If it should turn out that the arc length as drawn is not exactly right, increase or decrease the radius of the circle and then fit it to the figure, keeping two points on the perimeter, and adjust O to be at the center of the adjusted circle.

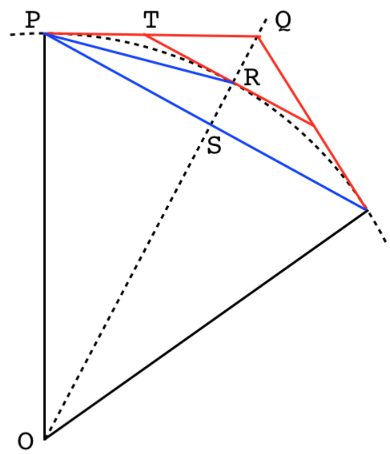
Two red lines comprise this sector's external perimeter P, while a single blue line is the inscribed perimeter p. The lines of the external perimeter are both tangent to the circle, and the whole figure is symmetric in each sector, with one blue and two red lines.



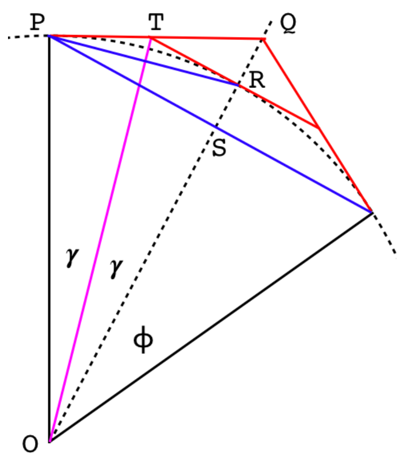
$\angle PSR$ is a right angle. Proof: we simply appeal to symmetry, or point out the congruent triangles. Since $\phi = \theta/2$, we have SAS.

Next, draw the perimeters p' and P' for the polygon with $2n$ sides and sector angle $\phi = \theta/2$.

It is convenient to rotate the internal perimeter by $\theta/2$ with respect to the external one, a bit to the left when we draw p' and a bit to the right for P'. Both p' and P' touch the circle at R.



A central relationship we use below is that $\triangle PRT$ is isosceles. For a proof, draw OT and appeal to symmetry.

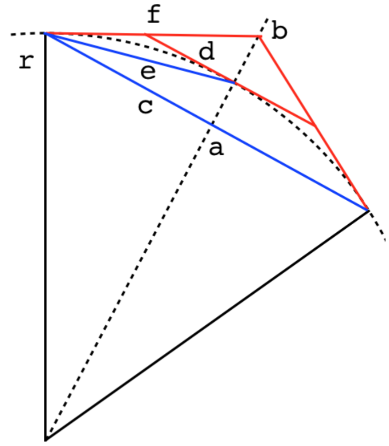


Or note that OT bisects $\angle POQ$ so $\triangle POT \cong \triangle ROT$ by SAS.

A consequence is that PR bisects $\angle QPS$. This can also be proved by an argument based on the sum of internal angles for an n-gon,

It looks as if the segment of the vertical that extends beyond the radius might be equal to that part below down to what looks like the "strut" of a kite. However, this is not true. We will show what this ratio is equal to in just a bit.

Rather than use the vertices as points of reference, we will label the line segments.

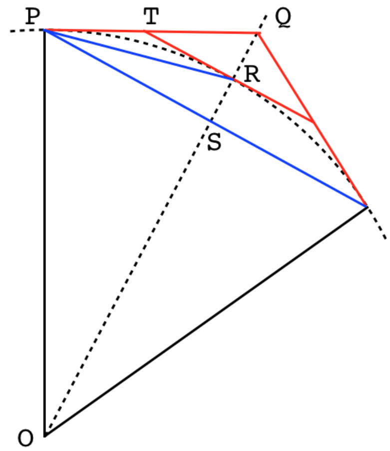


Just to be clear: a is the part of the radius extended to point S above, while b extends to Q. c and d are the lengths of the indicated lines *in the half-sector*, not all the way across, and f is the entire length of PQ.

We're ready to proceed.

basic geometry: perimeters

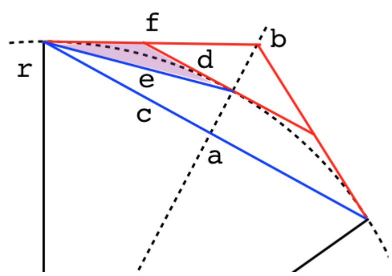
As we said, the key observation is that $\triangle PRT$ is isosceles.



Because of that, and since $\angle SPR = \angle PRT$ by the alternate interior angles theorem, $\angle SPR = \angle TPR$.

Therefore the cosines are also equal, namely:

$$\frac{c}{e} = \frac{e/2}{d}$$



(To see the midpoint of e, drop an altitude in the isosceles triangle, shown in purple).

Therefore:

$$2dc = e^2$$

Now, c is the entirety of p in this half-sector. But d is only one-half of P'.

Hence $2d \cdot c$ is equal to pP', and since $e = p'$, we have that

$$pP' = [p']^2$$

which was our second rule for the perimeters.

The first rule was

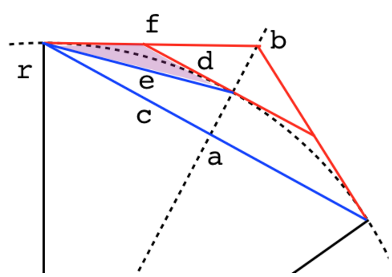
$$P' = 2\frac{pP}{p+P}$$

In geometric terms, we must show that

$$2d = 2\frac{cf}{c+f}$$

$$cd + df = cf$$

Taking another look at the diagram:



The small triangle with base d ($\triangle QRT$ above) has slanted side $f - d$ (subtracting d because, again, $\triangle PRT$ is isosceles). By similar triangles, we have

$$\frac{d}{f-d} = \frac{c}{f}$$

$$df = cf - cd$$

$$cd + df = cf$$

But this is what we needed to prove.

$\square$

basic geometry: areas

The area formulas for inside (a) and outside (A) polygons are those for a circle of unit radius (so that π is the area):

$$A' = 2\frac{a'A}{a' + A}$$

$$a' = \sqrt{aA}$$

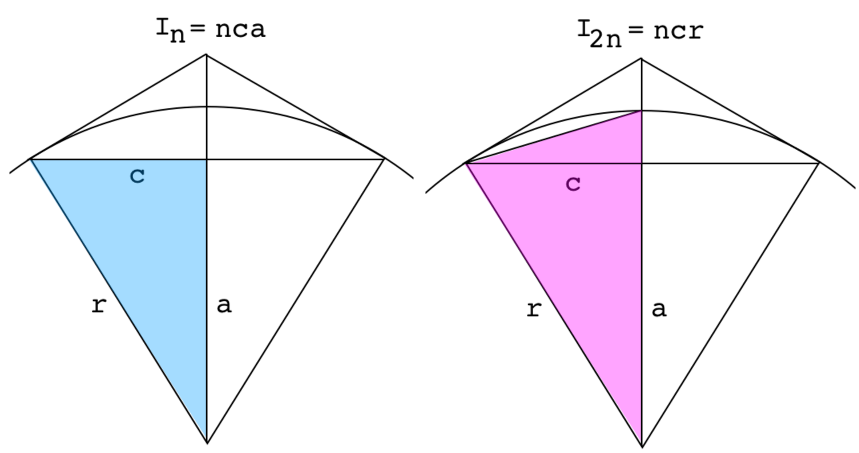
However, having reached this point, we need another symbol for area, because a is currently the line segment corresponding to p/n. Let's use I and C for the inside and outside areas, to match the source.

We will also adopt their n and $2n$ notation, It's a bit clumsy but that will make it easier to match things up.

$$C_{2n} = 2 \cdot \frac{I_{2n} C_n}{I_{2n} + C_n}$$

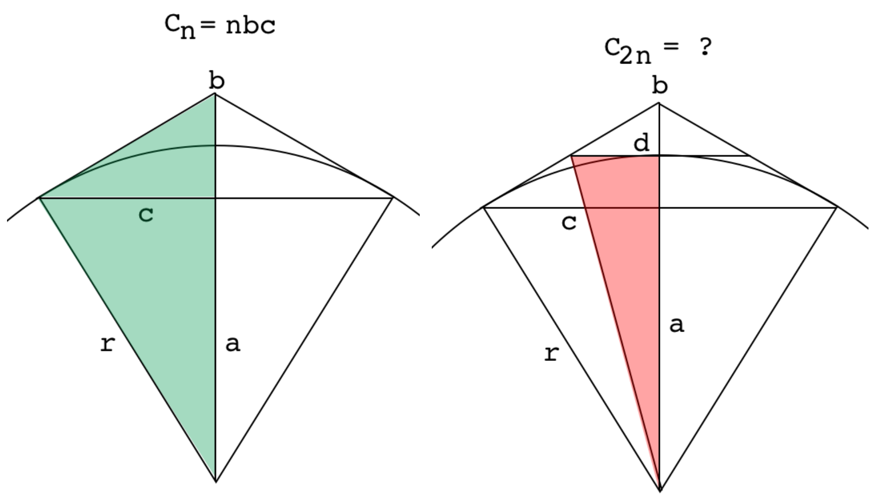
$$I_{2n} = \sqrt{I_n C_n}$$

The first two areas are I_n and I_{2n}

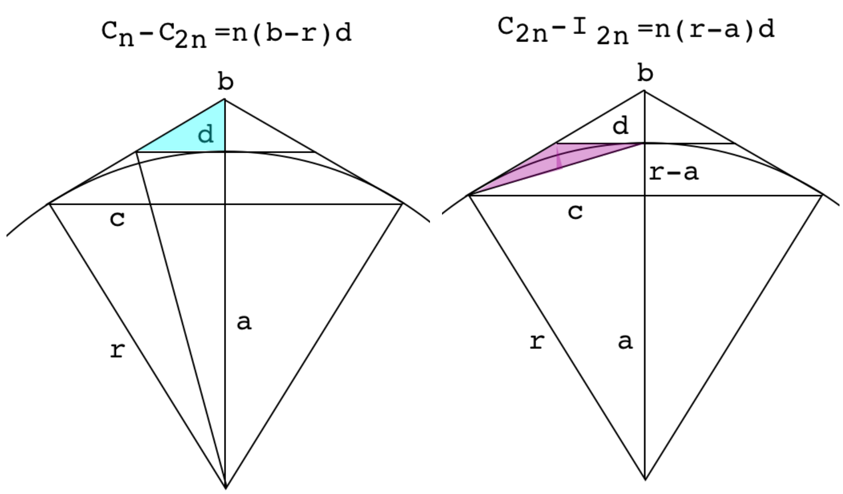


We compute these areas for the whole sector of angle θ, so there are two congruent triangles with base a (or base r) and height c. Multiply by n if you like to get the entire polygon, but every expression will have a factor of n, and we'll be looking at ratios, so we can just not worry about it.

The third easy one is C_n:



We write the last one (C_{2n}) as two different differences.



Let's gather all these expressions in one place, forming ratios:

$$\frac{I_{2n}}{I_n} = \frac{ncr}{nca} = \frac{r}{a}$$

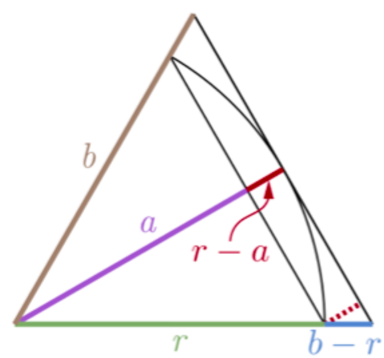
$$\frac{C_n}{I_{2n}} = \frac{ncb}{ncr} = \frac{b}{r}$$

$$\frac{C_n - C_{2n}}{C_{2n} - I_{2n}} = \frac{n(b-r)d}{n(r-a)d} = \frac{b-r}{r-a}$$

We will prove that these three ratios are all equal to each other.

We will have used the geometry to prove what the source calls their Lemmas, and those can be used in turn to prove the original Gregory formulas.

But the proof is easy:



It's just a matter of similar triangles:

$$\frac{r}{a} = \frac{b}{r} = \frac{b-r}{r-a}$$

That's the "without words" part.

For that very last part, you can work out the dimensions of the tiny similar triangle, or you can say:

$$\frac{r}{a} = \frac{b}{r}$$

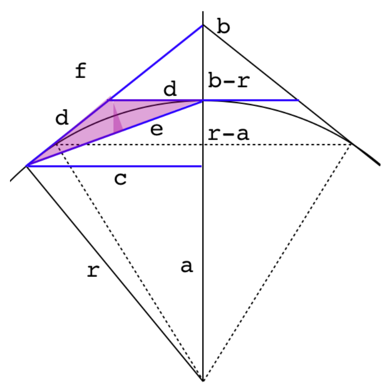
$$\frac{r}{a} - \frac{a}{a} = \frac{b}{r} - \frac{r}{r}$$

$$\frac{r-a}{a} = \frac{b-r}{r}$$

which is easily rearranged to give the desired result.

□

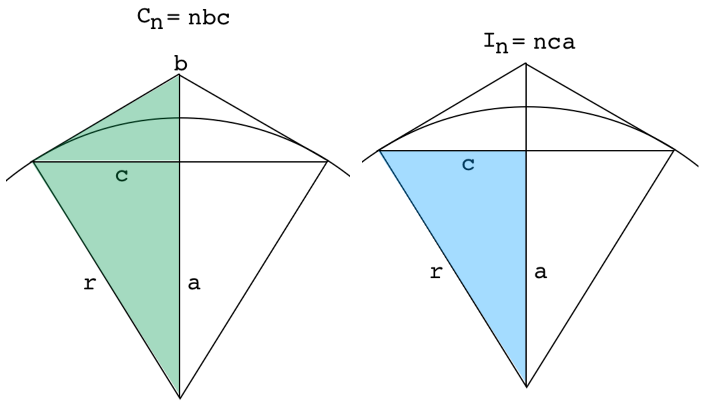
This can also be proved using the **angle bisector theorem**.



The side labeled e bisects the angle formed by the two sides labeled c and f. Therefore

$$\frac{b-r}{f} = \frac{r-a}{c} \quad \Rightarrow \quad \frac{b-r}{r-a} = \frac{f}{c}$$

But f and c are two sides of a triangle which is similar to the colored portions below:



Therefore

$$\frac{b}{r} = \frac{r}{a} = \frac{f}{c} = \frac{b-r}{r-a}$$

As we said.

algebra

Moving on to the geometric mean formula is not hard. From above we have that

$$\frac{I_{2n}}{I_n} = \frac{C_n}{I_{2n}}$$

$$[I_{2n}]^2 = I_n C_n$$

Translated back into the A, a area notation

$$a' = \sqrt{aA}$$

This is just what we wanted to show.

For the other formula, what we have is:

$$\frac{C_n - C_{2n}}{C_{2n} - I_{2n}} = \frac{C_n}{I_{2n}}$$

$$I_{2n}(C_n - C_{2n}) = C_n(C_{2n} - I_{2n})$$

$$
\begin{aligned}
2I_{2n}C_n &= C_nC_{2n} + I_{2n}C_{2n} \\
&= C_{2n}(C_n + I_{2n})
\end{aligned}
$$

So

$$
C_{2n} = 2 \cdot \frac{I_{2n}C_n}{C_n + I_{2n}}
$$

$$
C_{2n} = 2 \cdot \frac{1}{1/I_{2n} + 1/C_n}
$$

And we're done. In our preferred notation

$$
A' = 2 \cdot \frac{1}{1/a' + 1/A}
$$

historical note

The area-based formulas given above are due to James Gregory.

https://divisbyzero.com/2018/09/28/proof-without-word-gregorys-theorem/

As an aside, the Fundamental Theorem of Calculus (FTC) is usually thought about (taught and learned) using the language of functions, and ascribed mainly to Leibnitz, with some credit to the two Isaacs, Newton and his university lecturer, Barrow.

https://arxiv.org/abs/1111.6145

Amazingly enough, Gregory published a geometric (Euclidean) proof of the FTC in 1668! That predates Liebnitz (1693) by more than 25 years. This is motivation to give considerable credit to individuals other than Newton and Liebnitz (e.g. Fermat, Pascal, Wallis, Gregory, etc.) in the invention of the calculus.

test

I wrote a simple test of the area formulas using Python.

The script is here:

https://gist.github.com/telliott99/5269b48672cdaeca95c9c9c9d163321d

It gives this output:

```
> python script.py
    4 2.0000000000 4.0000000000
    8 2.8284271247 3.3137084990
   16 3.0614674589 3.1825978781
   32 3.1214451523 3.1517249074
   64 3.1365484905 3.1441183852
  128 3.1403311570 3.1422236299
  256 3.1412772509 3.1417503692
  512 3.1415138011 3.1416320807
 1024 3.1415729404 3.1416025103
 2048 3.1415877253 3.1415951177
 4096 3.1415914215 3.1415932696
 8192 3.1415923456 3.1415928076
16384 3.1415925766 3.1415926921
32768 3.1415926343 3.1415926632
65536 3.1415926488 3.1415926560
>
```

The digits of the output appear to be identical or nearly so. The only difference is that in this script I computed 2^n to give the number of sides. In the previous chapter, we just print n.

details

That's very curious. The first four lines of output from the perimeter version:

```
  2 2.8284271247  4.0000000000
 3 3.0614674589  3.3137084990
 4 3.1214451523  3.1825978781
 5 3.1365484905  3.1517249074
```

and the first five from the area version:

```
    4 2.0000000000 4.0000000000
    8 2.8284271247 3.3137084990
   16 3.0614674589 3.1825978781
   32 3.1214451523 3.1517249074
   64 3.1365484905 3.1441183852
```

It's pretty clear that we are doing the same calculation. It's just that the first column is shifted up by one row.

To confirm that, the perimeter calculation is:

initialization:

$$p = 2\sqrt{2} \qquad P = 4$$

recurrence:

$$P' = \frac{2pP}{p + P} \qquad p' = \sqrt{pP'}$$

The area version is:

initialization:

$$a = 2 \qquad A = 4$$

recurrence:

$$a' = \sqrt{aA} \qquad A' = \frac{2a'A}{a' + A}$$

They give identical results: $A = P$, at each round, but a matches p', or to put it the other way around, p' is retarded by one cycle compared to a'.

Let's try one round of calculation by hand:

$$p = 2\sqrt{2} \qquad P = 4$$

$$P' = \frac{2pP}{p+P} = \frac{2 \cdot 2\sqrt{2} \cdot 4}{2\sqrt{2}+4} = \frac{2 \cdot 2\sqrt{2} \cdot 4}{2\sqrt{2}(1+\sqrt{2})} = \frac{8}{1+\sqrt{2}} = 3.31371$$

$$p' = \sqrt{pP'} = \sqrt{2\sqrt{2} \cdot \frac{8}{1+\sqrt{2}}} = 4\sqrt{\frac{1}{1+1/\sqrt{2}}} = 3.06147$$

The area calculation:

$$a' = \sqrt{aA} = \sqrt{2 \cdot 4} = \sqrt{8} = 2.828427$$

$$A' = \frac{2a'A}{a'+A} = \frac{2 \cdot \sqrt{8} \cdot 4}{\sqrt{8}+4} = \frac{8}{1+\sqrt{2}}$$

A' is the same as P'.

The next round for a' is

$$a' = \sqrt{aA} = \sqrt{\sqrt{8} \cdot \frac{8}{1+\sqrt{2}}} = 4\sqrt{\frac{1}{1+1/\sqrt{2}}}$$

Perhaps someday I'll have a deeper understanding, Undoubtedly, there is a series here.

Chapter 159

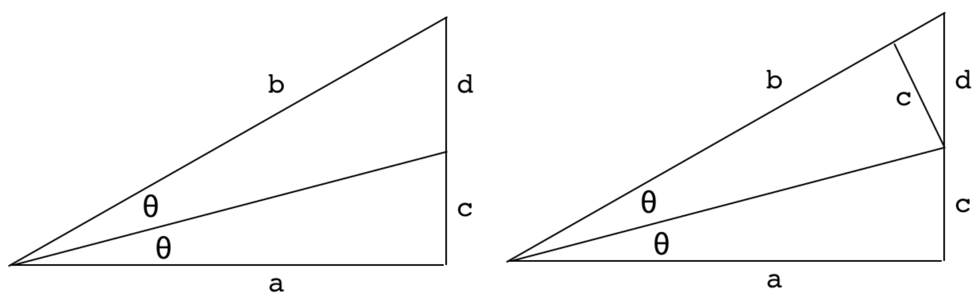
Archimedes and pi

We're going to follow a page that in turn follows Archimedes argument for the approximation of pi.

https://itech.fgcu.edu/faculty/clindsey/mhf4404/archimedes/archimedes.html

Before we start, let's review a theorem. Recall that if we have an angle bisector in a right triangle (left panel), the theorem says that

$$\frac{a}{c} = \frac{b}{d}$$



The proof of this involves drawing the altitude of the top triangle, forming two congruent triangles and a smaller one which is easily shown to be similar to the original triangle (right panel). By similar triangles, we have

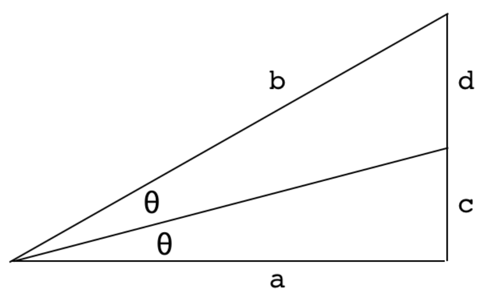
$$\frac{d}{c} = \frac{b}{a}$$

which can be rearranged to give the desired statement. A corollary follows:

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a+b}{b} = \frac{c+d}{d}$$

$$\frac{a+b}{c+d} = \frac{b}{d} = \frac{a}{c}$$

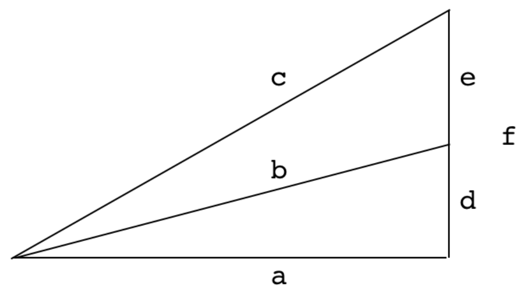


This doesn't seem obvious to me, in fact it seems counter-intuitive. Nonetheless, we will use it extensively for what follows.

overview

There are three steps which we will repeat (eventually, four times).

Let's relabel the figure now:



• Obtain a ratio for the cosecant and secant (c/f and a/f). In the first round, these are just $\sqrt{3}$ and 2.

• Add these together, obtaining $(c+a)/f$ and observe that this is also equal to a/d by the corollary of the angle bisector theorem above.

• Obtain b/d by the Pythagorean theorem. Namely:

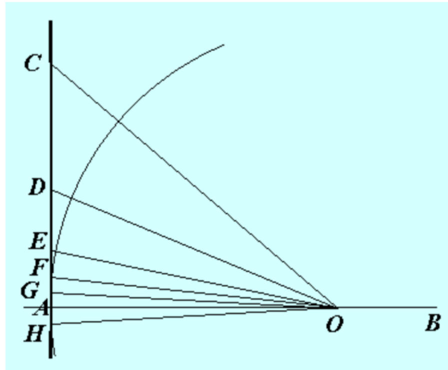
$$a^2 + d^2 = b^2$$

$$\sqrt{\frac{a^2}{d^2} + 1} = \frac{b}{d}$$

But b/d and a/d are the cosecant and secant of the half-angle.

Part A, round 1

Draw a circle with radius OA and tangent AC, and let $\angle AOC$ be one-third of a right angle.



Note: the figure appears to have been compressed in the width. The angle bisectors don't look right and the original angle looks more like 45 than 30. We'll use it anyway.

- $OA : AC > 265 : 153$

Since the triangle is a 30-60-90 triangle, $OA = \sqrt{3}$ and $AC = 1$, so the ratio is just $\sqrt{3}$. 265/153 is a (very good) approximation.

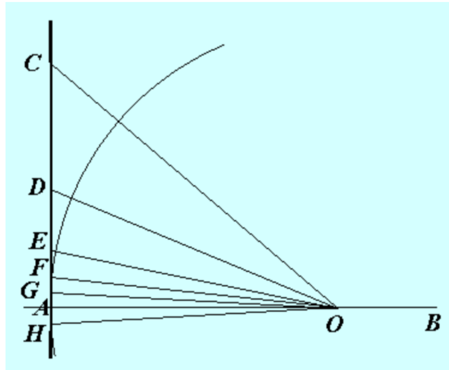
- $OC : AC = 306 : 153$

The cosecant = 2. The denominator has been chosen to match the previous ratio.

Now draw the angle bisector OD.

- $CO : OA = CD : DA$

This is just the angle bisector theorem.



- $(CO + OA) : CA = OA : AD$

This ratio is equal to that from the bisector theorem by the corollary. This crucial step gives us the secant of the half-angle formed by the angle bisector OD.

- $OA : AD > 571 : 153$

We just add numerators for the first two ratios above, leaving the result over their common denominator.

- $OD : AD > 591\ 1/8 : 153$

We have $OD : AD$. By the Pythagorean Theorem

$$OD^2 = OA^2 + AD^2$$

$$(OD : AD)^2 = (OA : AD)^2 + 1$$

The calculation is

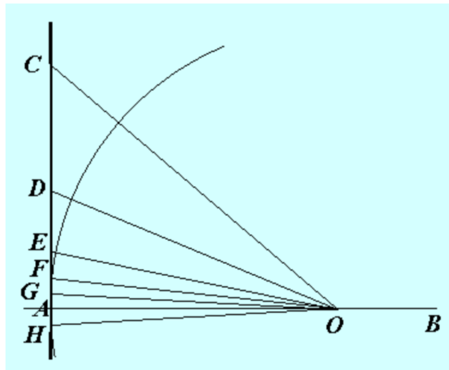
$$571^2/153^2 + 1 = 349450/23409$$

The decimal value of the ratio is 14.9280 and its square root is 3.8637.

Archimedes approximates the above value as 591 1/8 : 153. The decimal value is 3.8636, once again, a very good approximation.

Part A, round 2

Now draw the angle bisector OE.



- From above, we have that $OA : AD > 571 : 153$ and $OD : AD > 591\ 1/8 : 153$.
- $OA : AE > 1162\ 1/8 : 153$

This calculation invokes the angle bisector corollary again.

$$\frac{OD+OA}{DE+EA} = \frac{OA}{EA}$$
$$\frac{OD+OA}{OA} = \frac{DE+EA}{EA} = \frac{DA}{EA}$$
$$\frac{OA}{EA} = \frac{OD}{DA} + \frac{OA}{DA}$$
$$591\ 1/8 : 153 + 571 : 153$$

which adds to give the result above, $1162\ 1/8 : 153$.

- $OE : AE > 1172\ 1/8 : 153$

Use the Pythagorean theorem to write:

$$OE^2 = AE^2 + OA^2$$

$$\frac{OE^2}{AE^2} = \frac{OA^2}{AE^2} + 1$$

We have $(1162\ 1/8)^2$ and $153^2 = 23409$.

Write

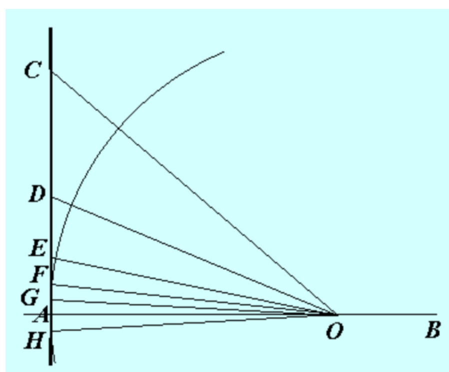
$$1162^2 + 1162/4 + 153^2 = 1350244 + 290\ 1/2 + 23409$$

$$= 1373943\ 1/2 + 1/64 = 1373943\ 33/64$$

The square root is 1172 1/8.

Part A, round 3

Now draw the angle bisector OF.



• From above, we have that $OA : AE > 1162\ 1/8 : 153$ and $OE : AE > 1172\ 1/8 : 153$.

• $OA : AF > 2334\ 1/4 : 153$

This calculation invokes the angle bisector corollary again.

$$\frac{OA}{FA} = \frac{OE}{EA} + \frac{OA}{EA}$$
$$1162\ 1/8 : 153 + 1172\ 1/8 : 153$$

which adds to give the result above.

• $OF : FA > 23391/4 : 153$

Use the Pythagorean theorem to write:

$$\frac{OF^2}{FA^2} = \frac{OA^2}{FA^2} + 1$$

We have $(2334\ 1/4)^2$ and $153^2 = 23409$.

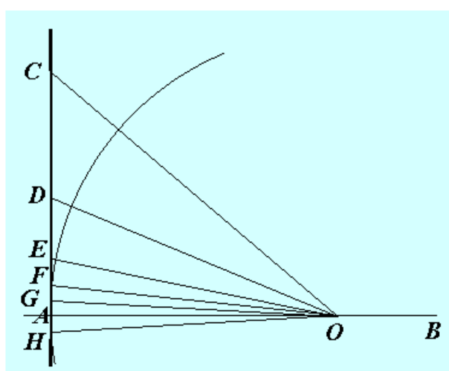
Write

$$\begin{aligned} 2334^2 + 2334/2 + 153^2 &= 5447556 + 1167 + 23409 \\ &= 5472132 + 1/16 \end{aligned}$$

The square root is 2339 1/4.

Part A, round 4

Now draw the angle bisector OG.



• From above, we have that $OA : FA > 2334\ 1/4 : 153$ and $OF : FA > 23391/4 : 153$

Add

• $OA : AG > 4673\ 1/2 : 153$

We're almost done. The original distance AC was 1/12 the perimeter of a circumscribed polygon, so we would multiply by 12 to get the ratio to the radius, but we want the ratio to the diameter so that gives a factor of 2 on the bottom for a total factor of 6.

There is an additional factor for the four "halvings" of $2^4 = 16$. Hence we obtain

$$153 \times 96 = 14688$$

and then

$$\frac{14688}{4673\ 1/2} = 3 + \frac{668\ 1/2}{4673\ 1/2}$$

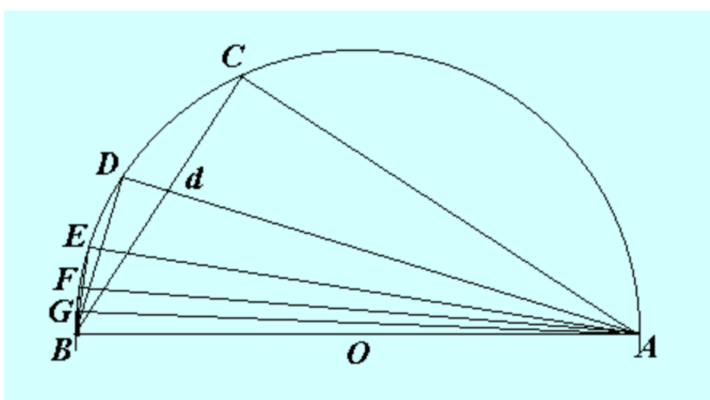
The fraction is just less than 1/7.

$1/7 = 0.142857$, while $\frac{668\ 1/2}{4673\ 1/2} = 0.14304$.

We conclude that $\pi < 3\ 1/7$.

Part B

For Part B we use this diagram:



As before $\triangle ABC$ is a $30 - 60 - 90$ right triangle.

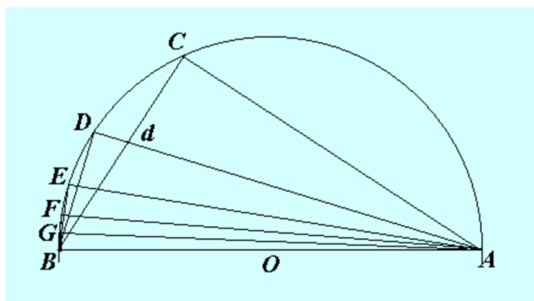
• $AC : BC < 1351 : 780$.

This ratio is an approximation for $\sqrt{3}$. It is an even better approximation and also, crucially, it is just slightly *more* than the true value, whereas 265/153 was slightly less.

Part B, round 1

Let AD bisect the angle, and then join BD.

• $\angle BAD = \angle dAC = \angle dBD$.



The first statement just restates the construction as an angle bisector. The second follows from the fact that the two angles have vertices on the circle and cut off the same arc. As a consequence, $\triangle dBD$ is similar to $\triangle dAC$.

• $AD : DB < 2911 : 780$

To get this, start with the similar triangles above and write three ratios of long sides (not hypotenuse) to short sides

$$AD : BD = BD : Dd = AC : Cd$$

Note: the source has $AB : Bd$ but this seems to be an error. It is a ratio of hypotenuses and is not equal to the others). I was unable to follow this part of the proof:

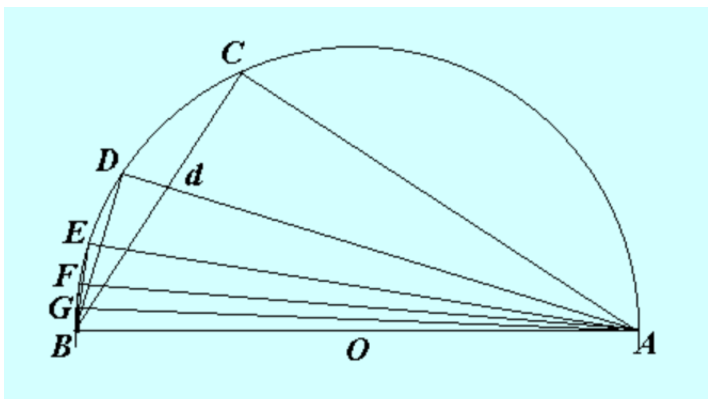
$$\begin{aligned} AD : BD = BD : Dd &= AB : Bd \\ &= (AB + AC) : (Bd + Cd) \\ &= (AB + AC) : BC \end{aligned}$$

or $(BA + AC) : BC = AD : DB.$

However, I was able to prove the last statement

$$(AB + AC) : BC = AD : DB$$

The proof is as follows.



We have that $\triangle ABC$ is a right triangle and that AD and thus Ad is the angle bisector for $\angle BAC$. Therefore, we have by our favorite theorem that

$$(AB + AC) : BC = AC : Cd$$

We also have that $\triangle ABD$ is a right triangle and by virtue of the angle bisector construction, $\triangle ABD \approx \triangle ACd$. Therefore:

$$AC : Cd = AD : DB$$

These two lines combine to give the desired result.

Now, $AD : DB$ is what we need going forward, and we get a value for that from the other part: $(AB + AC) : BC$. We have that AC :

$BC < 1351 : 780$, and $AB : BC$ is the cosecant whose value is 2 so we obtain $780 \times 2 = 1560$, add 1351 to get the numerator of the result listed above.

- $AB : BD < 3013\ 3/4 : 780$

From the Pythagorean theorem: $AD^2 + BD^2 = AB^2$ so

$$AB^2 : BD^2 = AD^2 : BD^2 + 1$$

$AD : DB < 2911 : 780$ So we obtain $2911^2 = 8473921$ and $BD^2 = 608400$ so we have that

$$AB^2 : BD^2 = 9082321 : 608400$$

$$AB : BD < 3013\ 3/4 : 780$$

Part B, round 1 summary

Let's summarize what we did in round 1. We started with the cosecant and the cotangent for $\triangle ABC$, namely $AB : BC$ and $: AC : BC$.

We used this relationship $(AB + AC) : BC = AD : DB$ to obtain the cotangent of the bisected angle, and then we used the Pythagorean theorem in this form

$$AB^2 : BD^2 = AD^2 : BD^2 + 1$$

to get the cosecant from the cotangent. Thus we have

- $AD : DB < 2911 : 780$ cotangent
- $AB : BD < 3013\ 3/4 : 780$ cosecant

Part B, round 2

Now, let AE bisect the angle, and then join BE.

Rather than go through the geometry again, let's just substitute letters. First the cotangent

$$(AB + AD) : BD = AE : EB$$

Then the cosecant.

$$AB^2 : BE^2 = AE^2 : BE^2 + 1$$

For the first part we have $2911 : 780 + 3013\ 3/4 : 780 = 5924\ 3/4 : 780$. We will reduce the denominator to 240. This amounts to dividing by 3 1/4. So 5924 3/4 divided by 3 1/4 is exactly equal to 1823.

- $AE : EB = 1823 : 240$ cotangent.

For the second part we have the previous number squared and added to 1 and then take the square root. $1823^2 = 3323329; 240^2 = 57600$; so we have 3380929 and the square root is 1838 3/4, but the source gives the fraction as a bit larger 1838 9/11.

- $AB : BE = 1838\ 9/11 : 240$ cosecant.

Two more rounds.

Part B, round 3

Now, let AF bisect the angle, and then join BF. Substitute letters (carefully, looking at the diagram). First the cotangent

$$(AB + AE) : BE = AF : FB$$

Then the cosecant.

$$AB^2 : BF^2 = AF^2 : BF^2 + 1$$

For the first part we have 1838 9/11 : 240 + AE : EB = 1823 : 240 = 3661 9/11 : 240. Again, we reduce the denominator, this time to 66. This amounts to multiplication by 11/40. So the numerator is multiplied by the same factor giving

- $AF : FB = 1007 : 66$ cotangent.

For the second part we have the previous number squared and added to 1 and then take the square root.$1007^2 = 1014049; 66^2 = 4356$, so we have 1018405 and the square root is 1009 1/6.

- $AB : FB = 1009\ 1/6 : 66$ cosecant.

Part B, round 4

Finally, let AG bisect the angle, and then join BG. Substitute letters (carefully, looking at the diagram). First the cotangent

$$(AB + AF) : BF = AG : GB$$

Then the cosecant.

$$AB^2 : BG^2 = AG^2 : GF^2 + 1$$

For the first part we have

- $AG : GB = 2016\ 1/6 : 66$ cotangent.

Now do $2016^2 = 4064256; 66^2 = 4356$ so that's 4068612 and the square root is 2017 1/12 but the source gives

- $AB : GB = 2017\ 1/4 : 66$ cosecant.

Almost done. The side BG is a side of an inscribed regular polygon of 96 sides. We multiply $66 \times 96 = 6336$ and compute the ratio of the inverse.

[Go through the logic here]

I am not sure how he came up with it, but it is easy to verify that the ratio which is less than π is greater than:

$$\frac{6336}{2017\ 1/4} > 3\ 10/71$$

We combine parts A and B to state that

$$3\ 10/71 < \pi < 3\ 1/7$$