Green's Theorem for complex variables

We can state Green's Theorem for complex variables as

$$\oint f(z)\ dz = i \iint \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\ dx\ dy$$

As an example, consider the curve going around the square $[-1, -i] \times [1, i]$ and the function $f(z) = |z|^2$. So

$$f(z) = |z|^2 = zz*$$

$$= (x + iy)(x - iy)$$

$$= x^2 + y^2$$

Then

$$f_x = 2x, \quad f_y = 2y$$

and

$$I = \iint 2x + i2y\ dx\ dy$$

The inner integral is

$$\int_{-1}^{1} 2x + i2y\ dx = 2x^2 + i2xy\ \Big|_{-1}^{1} = 4iy$$

The outer integral is

$$I = \int_{-1}^{1} 4iy\ dy = 2iy^2\ \Big|_{-1}^{1} = 0$$

$C_1 + C_3$

$$I = \int u\ dx - \int v\ dy + i\ [\ \int u\ dy + \int v\ dx\ ]$$

Along C_1, $dy = 0$ and so the line integral is

$$I = \int u\ dx + i \int v\ dx$$

The function is $u(x, y) = x^2 + y^2$, $v(x, y) = 0$ so

$$I = \int x^2 + y^2\ dx$$

With $y = 1$ this is

$$I = \int_1^{-1} x^2 + 1\ dx = \frac{x^3}{3} + x\ \Big|_1^{-1}$$

$$= (-\frac{1}{3} - 1) - (\frac{1}{3} + 1) = -\frac{8}{3}$$

Along C_3, with $y = -1$ (and $y^2 = 1$) this is

$$I = \int_{-1}^{1} x^2 + 1\ dx = \frac{x^3}{3} + x\ \Big|_{-1}^{1}$$

$$= (\frac{1}{3} + 1) - (-\frac{1}{3} - 1) = \frac{8}{3}$$

so these two line integrals cancel.

$C_2 + C_4$

$$I = \int u\ dx - \int v\ dy + i\ [\ \int u\ dy + \int v\ dx$$

Along C_2 and C_4, $dx = 0$ and so the line integral is

$$I = -\int v\ dy + i\ \int u\ dy$$

The function is $u(x, y) = x^2 + y^2$, $v(x, y) = 0$ so

$$I = i\ \int x^2 + y^2\ dy$$

With $x = -1$ or on C_4, $x = 1$, in either case $x^2 = 1$ so we have the same integral along opposite paths, (multiplied by i), so we have the same answer for the two together: 0.