Distance

inequalities

For any two numbers $a < b$ it is also true that $-b < -a$. Multiplication by -1 switches the direction of the inequality.

• Proof: add $-a - b$ to both sides above.

We define $x < 0$ below as follows: first define $m > 0$ and then let $x = -m$.

Absolute value function

If $x \geq 0$ then $|x| = x$.

For $x < 0$: $|x| = |-m| = m$.

The following theorem involves a given constant real number $a > 0$ and a real number x.

Theorem

$-a < x < a \Rightarrow |x| < a$.

• Proof:

∘ $x \geq 0$, then $|x| = x$. We are given $x < a \Rightarrow |x| < a$.

$\circ$ $x < 0$, then $|x| = m$. We are given $x > -a$, $-m > -a$, $m < a \Rightarrow |x| < a$.

Theorem

$|x| < a \Rightarrow -a < x < a$.

• Proof:

$\circ$ $x \geq 0$, and we are given $|x| < a$. Since $|x| = x$, we have $x < a$. Also, since $0 \leq x$ and $-a < 0$, we have $-a < x$. Hence $-a < x < a$.

$\circ$ $x < 0$, then certainly $x < a$. Since $|x| = m$, that implies $m < a$, so $-x < a$ and therefore $-a < x$. Hence $-a < x < a$.

We have shown that $-a < x < a \iff |x| < a$. This completes the proof of the theorem.

Theorem

Consider a point p in the middle of an interval of length 2δ so $I = [p-\delta, p+\delta]$.

In other words, $p - \delta < x < p + \delta$ implies that $-\delta < x - p < \delta$ but by the theorem above, this is true $\iff |x - p| < \delta$.