Curvature

We seek to expressions for curvature. Consider two circles of differing radius. We choose our measure of curvature κ to be larger for the circle of smaller curvature. That is, $\kappa \approx 1/R$. For a path in space the "radius" of the curve followed can vary continuously.

The first definition says: suppose that a particle moves with unit speed. Then

$$\kappa = \frac{d}{dt}\,\hat{\mathbf{T}}$$

The first example is uniform circular motion on a circle of radius R. We have

$$\begin{aligned}\mathbf{r}(t) &= \langle x(t), y(t)\rangle \\ &= \langle R\cos\omega t, R\sin\omega t\rangle\end{aligned}$$

The velocity is

$$\frac{d}{dt}\,\mathbf{r} = \mathbf{r}' = \omega R\langle -\sin\omega t, \cos\omega t\rangle$$

The speed squared is

$$|\mathbf{v}|^2 = \omega^2 R^2((-\sin\omega t)^2 + \cos^2\omega t) = \omega^2 R^2$$

so the speed is

$$v = |\mathbf{v}| = \omega R$$

and it's a unit speed if we set $\omega = 1/R$. So rewrite

$$\mathbf{r} = \langle R\cos\frac{t}{R}, R\sin\frac{t}{R}\rangle$$

$$\mathbf{v} = \langle -\sin\frac{t}{R}, \cos\frac{t}{R}\rangle$$

The tangent vector $\hat{\mathbf{T}}$ is a unit vector in the same direction as the velocity. Since we have set the speed to be unit speed

$$\hat{\mathbf{T}} = \mathbf{v} = \langle -\sin\frac{t}{R}, \cos\frac{t}{R}\rangle$$

Recalling the definition

$$\kappa = \frac{d}{dt}\,\hat{\mathbf{T}}$$

we see that κ is just the acceleration, namely

$$\kappa = |\mathbf{a}| = |\mathbf{v}'|$$

$$= |-\frac{1}{R}\langle\cos\frac{t}{R}, \sin\frac{t}{R}\rangle|$$

$$= \frac{1}{R}$$

For an arbitrary curve (not necessarily unit speed)

$$\kappa = \frac{|\hat{\mathbf{T}}'|}{|\mathbf{r}'|}$$

Problem: find the curvature of the parabola $y = x^2$ at the point $(1, 1)$.

First, we can parametrize the curve as

$$\mathbf{r}(t) = \langle t, t^2\rangle$$

with the corresponding t for $(1, 1)$ equal to 1. Then

$$\mathbf{r}' = \langle 1, 2t\rangle$$

$$|\mathbf{r}'| = \sqrt{1 + 4t^2}$$

And

$$\hat{\mathbf{T}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$$

We have

$$\frac{\mathbf{r}'}{|\mathbf{r}'|} = \frac{1}{\sqrt{1+4t^2}}\ \langle 1, 2t\rangle$$

$$= \langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \rangle$$

So

$$\hat{\mathbf{T}}' = \frac{d}{dt}\ \frac{\mathbf{r}'}{|\mathbf{r}'|}$$

$$= \langle -4t(1+4t^2)^{-3/2}, ??\rangle$$

We need the derivative:

$$\frac{d}{dt}\ \frac{2t}{\sqrt{1+4t^2}} = \frac{2\sqrt{1+4t^2} + 8t^2(1+4t^2)^{-3/2}}{1+4t^2}$$

$$= \frac{2}{\sqrt{1+4t^2}} + 8t^2(1+4t^2)^{-5/2}$$

What a mess! We need to find the length of this thing and then divide by $|\mathbf{r}|$ but life is too short for that.