## Why Do We Name the Integral for Someone Who Lived in the Mid-Nineteenth Century?

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When I ask my students to tell me what the Fundamental Theorem of Calculus (FTC) says, I usually get a response very similar to the definition given in the Wikipedia as of spring 2005 (http://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus): "The two central operations of calculus, differentiation and integration, are inverses of each other." In one sense, this is right on the mark. In another, it is very misleading.

The actual statement of the FTC comes in two parts that are true for "nice" functions:

**Evaluation part of the FTC:**
If *f* is continuous on $[a,b]$, and $F$ is any antiderivative of *f*, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

**Antiderivative part of the FTC:**
If f is continuous on an open interval containing a, then $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

The evaluation part says that if we know that f is the derivative of *F*, then integrating *f* gets us back to *F* (up to a constant). The antiderivative part says that if we integrate *f* and then differentiate the resulting function, we get back to the function *f* with which we started.

The problem with this understanding of the FTC is that most students think of integration as the inverse of differentiation: integration means reversing differentiation (up to the addition of a constant). Too often, the FTC is remembered as the definition of integration.

Of course, no modern calculus text actually defines integration this way. Most texts are careful to define the definite integral as the limit of Riemann sums:

> Given a function *f* and an interval $[a,b]$ on which *f* is bounded, the definite integral $\int_a^b f(x)\,dx$ is the limit of the Riemann sums, $\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$, where $a = x_0 < x_1 < \cdots < x_n = b$, each $c_k$ can be any

value in the interval $\left[x_{k-1}, x_k\right]$, and the limit is obtained by restricting the length of the longest subinterval. In other words, we can force $\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$ to be as close as we wish to the value designated as $\int_a^b f(x)\,dx$ just by restricting the size of $x_k - x_{k-1}$ for all $k$.

What the FTC really says is that if you define the definite integral as the limit of Riemann sums, then it has the very desirable property of being the inverse operation to differentiation.

The power of calculus comes from the fact that two operations that appear to be doing very different things, reversing differentiation and taking limits of Riemann sums, are, in fact, linked. The FTC is this link. This is what students need to know about the FTC.

It was not always the case that textbooks defined integration as a limit of Riemann sums. Until the 1820s, mathematicians defined integration as the inverse operation of differentiation. This continued in textbooks until the mid-twentieth century. The most widely used English language calculus textbook from 1900 until well into the 1950s was Granville's *Elements of the Differential and Integral Calculus*. It defines integration as "having given the differential of a function, to find the function itself." In other words, integration is the inverse of differentiation. The definite integral is then defined as "the difference of the values of $\int y\,dx$ for $x = a$ and $x = b$." Though he never labels it as such, Granville does make the connection that constitutes the FTC. He explains how the definite integral can be used to evaluate the limit of a summation of the form $\sum f(x_i)\Delta x$ as the length of the subintervals $(\Delta x)$ approaches zero. The link is still there. All that differs is the choice of which interpretation of the integral should be used as the definition.

The problem with defining the integral as the inverse of differentiation is that there are many functions for which we want to be able to evaluate a definite integral but for which there is no explicit formula for an antiderivative. If integration is antidifferentiation, then there are too many functions for which we do not have an easy answer to the question, What do we mean by an antiderivative?

# Special Focus: The Fundamental Theorem of Calculus

In the 1820s, Cauchy saw that he needed a definition of integration that was independent of the idea of differentiation. For example, even though there is no closed form for the antiderivative of $e^{-x^2}$, we should be able to assign a value to

$$\int_0^1 e^{-x^2}\,dx.$$

Cauchy became the first to insist that the definite integral be defined as a limit of what would later become known as Riemann sums. Specifically, Cauchy used left-hand sums:

$$\int_a^b f(x)\,dx = \lim_{|x_k - x_{k-1}| \to 0} \sum_k f(x_{k-1})(x_k - x_{k-1}).$$

Mathematicians recognized the usefulness of this definition, but it was slow to appear in the popular textbooks.

The shift in textbooks to defining integration in terms of Riemann sums began with Richard Courant's *Differential and Integral Calculus*, first published in English in 1934. This is also the first time that the term Fundamental Theorem of Calculus was used in its modern sense. Courant realized that if the integral is defined as a limit of Riemann sums, then the fact that integration is the inverse operation to differentiation is something that requires proof and elevation to the level of a fundamental theorem. When George Thomas wrote the first edition of his influential calculus text in the late 1940s, he chose Courant's approach to the integral. Calculus texts since then have followed his example.

Mathematically, the definition chosen by Cauchy, Courant, and Thomas makes sense. Pedagogically, it carries a potential trap. Students at this stage in their mathematical development usually ignore formal definitions in favor of a working understanding in terms of the examples they have seen. Traditional texts move quickly from the definition of the integral to extensive practice in how to use antidifferentiation to accomplish integration. Students internalize integration as the inverse of differentiation, and so the FTC lacks meaning.

There are two practices that can be used to help counter this. The first is to spend time on the implications of the integral as a limit of Riemann sums before exhibiting the power of the FTC. This means familiarizing students with a variety of problems that can be formulated as limits of Riemann sums: areas and volumes, population counts derived from density functions, distances from velocities, and velocities from accelerations. Twenty years ago, the complexity of calculating these limits without the aid of the FTC made this approach very difficult, but today we have calculators. A calculator that is not using a computer algebra system is using the Riemann sum definition to approximate the definite integral. Students can learn the power of the definite integral as a model for a wide class of problems, problems for which the simplest case requires multiplying two quantities (distance = velocity × change in time, mass = density × volume), but for which the first quantity varies as a function of a variable whose change is measured by the second quantity (velocity depends on time, density depends on location).

The second practice is to emphasize what is truly important about the FTC. It informs us that two very different ways of understanding the integral are, in fact, equivalent (or almost so). This is the true power of calculus. We can take a problem modeled by a definite integral that reflects the nature of the problem as a limit of a Riemann sum, and we can find an elegant solution to this problem using antidifferentiation. Students need to be fully aware that the power runs both ways. Given a problem that seems to call for an antiderivative:

$$\text{Solve for } f: \quad \frac{df}{dx} = e^{-x^2}, \quad f(0) = 0,$$

the Riemann sum definition can provide a solution:

$$f(x) = \int_0^x e^{-t^2}\,dt.$$

Students must realize that this is a meaningful solution.

## So Where Does Riemann Come into This?

Riemann's definition of the integral was published in 1867 in a paper on Fourier series. Riemann wanted to determine when a function could be integrated. Cauchy had shown in the 1820s that every continuous function could be integrated. It is not too hard to see that there are also discontinuous functions that can be integrated, and the study of Fourier series had created situations where scientists needed to integrate discontinuous functions.

A simple example of an integrable discontinuous function is the signum function, $\operatorname{sgn}(x)$ which is 1 when $x$ is positive, −1 when $x$ is negative, and 0 when $x = 0$. We can integrate this function from −1 to 1 and get 0 or from −2 to 5 and get 3. Even though this function is discontinuous at $x = 0$, its definite integral is well defined over any interval. In fact, with just a little thought about what it means to integrate this function from 0 to $x$ when $x$ is negative, we see that

$$\int_0^x \operatorname{sgn}(t)\,dt = |x|.$$

Riemann wanted to know how discontinuous a function could be and still be integrable. To even begin to ask this question, he needed a very clear definition of what it means to be integrable. As he showed, his definition of integration allows for very strange functions. One of his examples is a function that makes a discontinuous jump at every rational number with an even denominator (every interval contains an infinite number of these), that is continuous at all other numbers, and that is integrable.[1]

The ability to integrate discontinuous functions creates problems for the antiderivative part of the FTC: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$ Derivatives cannot have jumps. This statement is deeper than the Intermediate Value Theorem. If a function has a derivative, then it must be continuous, but if a function is a derivative, it does not have to be continuous. Nevertheless, even discontinuous derivatives are almost continuous. If the limit from the left and the limit from the right at *a* exist, then they must be equal. If we let *f* be Riemann's very discontinuous but still integrable function, then the derivative of $\int_a^x f(t)\,dt$ cannot exist at rational numbers with even denominators.

If *f* is continuous at every point in $\left[a,b\right]$, then it is true that $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ at every $x$ in $(a,b)$. The remainder of the nineteenth century would be spent figuring out what could be said about this antiderivative part of the FTC when *f* is not continuous at every point in $\left[a,b\right]$.

## Beyond Riemann

A derivative cannot have jumps, but it does not have to be continuous. One example is the derivative of

$$F(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

As long as $x$ is not zero, we can use the standard techniques of differentiation to find its derivative:

$$F'(x) = 2x\sin\left(\frac{1}{x}\right) + x^2\left(\frac{-1}{x^2}\right)\cos\left(\frac{1}{x}\right) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), \quad x \neq 0.$$

When $x$ is 0, we have to rely on the limit definition:

$$F'(0) = \lim_{h\to 0}\frac{F(h)-F(0)}{h} = \lim_{h\to 0}\frac{h^2\sin\left(\frac{1}{h}\right)-0}{h} = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0.$$

This function is differentiable at all values of $x$, but the limits from the left and right of $F'(x)$ as $x$ approaches 0 do not exist. The derivative $F'(x)$ oscillates forever between 1 and –1 as $x$ approaches 0, and so $F'$ is not continuous at $x = 0$.

If we modify this function slightly:

$$G(x) = \begin{cases} x^2\sin\left(\dfrac{1}{x^2}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

then we get a function that is still differentiable at every value of $x$, including $x = 0$, but the derivative is

$$G'(x) = \begin{cases} 2x\sin\left(\dfrac{1}{x^2}\right) - 2x^{-1}\cos\left(\dfrac{1}{x^2}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

This derivative is not bounded near $x = 0$. For this function, the conclusion of the evaluation part of the FTC fails. Since $G'$ is not bounded, the Riemann integral $\int_a^b G'(x)\,dx$ does not exist if 0 is in the interval $[a,b]$, so it does not equal $G(b)-G(a)$.

You can get around this difficulty for the function $G$ by using improper integrals, only integrating up to a value ε near 0 and then taking the limit as ε approaches 0. But in 1881, Vito Volterra showed how to build a function (call it $V$) that is differentiable at every $x$ and whose derivative stays bounded, but for which the derivative cannot be integrated.[2] Although the derivative stays bounded, the limit of the Riemann sums of the derivative does not exist, and we cannot get around the problem by using improper integrals. Volterra's function is an even stranger example of a function for which the

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[1] You can find this function in my book, *A Radical Approach to Real Analysis* (Mathematical Association of America, 1994), pages 254–256.

evaluation part of the FTC is not true: $\frac{d}{dx}V(x) = v(x)$, but $\int_a^b v(x)\,dx$ does not exist, even if we allow improper integrals, and so it cannot equal $V(b) - V(a)$.

These problems would eventually lead to an entirely different definition of integration, the Lebesgue integral, first proposed by Henri Lebesgue in his doctoral thesis of 1902. It takes a very different approach from Riemann integration. Instead of dividing up the $x$-axis (the domain of $f$) and choosing a value of the function at some point in each subinterval, it divides up the $y$-axis (the range of $f$) and looks at the length of the set of $x$'s that yield values of $f$ in that subinterval of the range. For nice functions, the limits of Riemann sums and of Lebesgue sums are the same, but Lebesgue's approach turns out to be much more useful. In particular, the first statement of the FTC always holds: If $f$ is the derivative of $F$, then the Lebesgue integral of $f$ from $a$ to $b$ will always exist and will equal $F(b) - F(a)$.

I find it deliciously ironic that the standard definition of integration found in all calculus texts today, Riemann's definition, was created almost two centuries after the beginnings of calculus, was developed specifically to identify strange functions that would create problems for integration, and lasted barely over 30 years before it was superceded by a more inclusive and useful definition of integration, the Lebesgue integral. Despite the irony of this situation, I am not arguing to replace Riemann's definition. Students in first-year calculus are not ready for the subtleties of Lebesgue's approach. More importantly, Riemann's definition is useful for driving home the critical observation that integrals enable us to evaluate limits of sums of products.

But it is also worth dropping a few hints to your students that the FTC is, in fact, more complicated than it may look at first glance.

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[2] Volterra's function is explained in Chae's *Lebesgue Integration*, 2nd ed. (Springer-Verlag, 1995), pages 46–47.