Karkhar Theorems

Power Series Theorems

16A

Suppose there is a $z_1 \neq z_0$ such that

$$\sum_{n=0}^{\infty} a_n \ (z_1 - z_0)^n \text{ converges}$$

Then

$$\forall\ z : |z - z_0| < |z_1 - z_0|$$

$$\sum_{n=0}^{\infty} a_n \ (z - z_0)^n \text{ converges absolutely}$$

If we know a point z_1 where the series is convergent, then for *every* point that is closer to z_0 than is z_1, the series converges absolutely.

16B

If

$$f(z) = \sum_{n=0}^{\infty} a_n \ (z - z_0)^n$$

has a positive (or infinite) radius of convergence, then inside the disk $|z - z_0| < R$, then inside that disk $f(z)$ is infinitely differentiable. Each derivative is given by a power series:

$$f^{(k)}(z) = \sum_{n=0}^{\infty} n \cdot (n-1) \ldots (n-k-1) \; a_n \; (z - z_0)^n$$

17A

Suppose f is analytic on domain D. Let γ be a piecewise smooth simple closed curve in D whose inside Ω is also in D. Then

$$\int_\gamma f(z) \; dz = 0$$

(One proof uses the CRE).

17B

Let D be a simply connected domain and γ a closed curve in D. If f is analytic in D then

$$\int_\gamma f(z) \; dz = 0$$

(no requirement for piecewise smooth or closed).

17C

If f is analytic in a simply connected domain D, then there is an analytic function F on D with $F' = f$ throughout D.

18A

Cauchy's formula:

Suppose f is analytic on a domain D and γ is a piecewise smooth, positively oriented, simple, closed curve in D whose inside Ω is in D. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw = f(z) \qquad \forall z \in \Omega$$

Let's change this to use z for the complex variable and z_0 for the point, that I'm used to.

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz = f(z_0) \qquad \forall z_0 \in \Omega$$

So z_0 is any fixed point in Ω, and depending on its value, there is a singularity at that point.

19A

Suppose f is analytic in a domain D and $z_0 \in D$. If the disk $\{z : |z-z_0| < R\}$ lies in D, then f has a power series:

$$f(z) = \sum_0^\infty a_k\ (z-z_0)^k$$

valid in the disk. Also

$$a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}\,dw$$

19B

A corollary of $19A$ is: if $f(z)$ is analytic on D, then so is $f'(z)$. So, f has derivatives *of all orders*, and all of them are analytic on D.

19C

Another corollary: suppose f is analytic on D, and at $z_0 \in D$

$$f^{(k)}(z_0) = 0 \quad k = 0, 1, 2 \ldots$$

then

$$f(z) = 0 \ \ \forall \ z \in D$$

That is, if there is any point z_0 in a domain D where an analytic function f is zero (and so its derivatives are all zero), then the value of the function is zero for all z in the domain.