# Paraboloid Surface Area

A paraboloid is a solid whose vertical cross-section is a parabola (usually, it is centered along the $z$-axis). It may be oriented opening up, or down. The cross-sections parallel to the $xy$-plane are typically circles, though the shape factors for the parabolas in the $xz$- and $yz$-planes could be different, leading to an ellipse for the cross-sections.

Consider

$$z = 2 - x^2 - y^2$$

This is a paraboloid that opens down (it gets big when either $x$ or $y$ get large). The vertex is at $z = 2$. When $z = 0$, the cross-section is a circle of radius $r^2 = 2$.

Usually, cylindrical coordinates are good for dealing with this solid. For example, the volume element is $dV = \ dz\ r\ dr\ d\theta$.

As with the volume, there are two ways (at least) to do the surface area. The first is to lay the parabola down as $f(x)$.

$$f(x) = \sqrt{x}$$

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}}$$

$$f'(x)^2 = \frac{1}{4x}$$

For the surface area of a volume of revolution, we take the circumference of the solid at each value of $x$ times the path element $ds$ (*not* $dx$). This element is

$$\begin{aligned} ds &= \sqrt{1 + f'(x)^2}\ dx \\ &= \sqrt{1 + \frac{1}{4x}}\ dx \end{aligned}$$

So the surface area is

$$\begin{aligned} SA &= \int_a^b C(x)\ dx \\ &= 2\pi \int_a^b \sqrt{x}\ \sqrt{1 + \frac{1}{4x}}\ dx \\ &= 2\pi \int_a^b\ \sqrt{x + \frac{1}{4}}\ dx \\ &= \frac{4}{3}\pi\ (x + \frac{1}{4})^{3/2}\ \Big|_a^b \\ &= \frac{4}{3}\pi\ [\ (b + \frac{1}{4})^{3/2} - (a + \frac{1}{4})^{3/2}\ ] \end{aligned}$$

If $a = 0$

$$\begin{aligned} &= \frac{4}{3}\pi\ [\ (b + \frac{1}{4})^{3/2} - \frac{1}{4})^{3/2}\ ] \\ &= \frac{4}{3}\pi\ [\ (b + \frac{1}{4})^{3/2} - \frac{1}{8}\ ] \end{aligned}$$

For this problem, $b = 2$ and the answer simplifies a bit

$$= \frac{4}{3}\pi \; [ \; (2 + \frac{1}{4})^{3/2} - \frac{1}{8} \; ]$$

$$= \frac{4}{3}\pi \; [\frac{27}{8} - \frac{1}{8} \; ]$$

$$= \frac{26}{6}\pi$$

Let's try to do this by integrating over two variables. We have

$$f(x, y) = 2 - x^2 - y^2$$

$$f_x = -2x$$

$$f_y = -2y$$

The surface area element is

$$dS = \sqrt{1 + f_x^2 + f_y^2} \; dx \; dy$$

The surface area integral is

$$SA = \iint_R \sqrt{1 + f_x^2 + f_y^2} \; dx \; dy$$

$$= \iint_R \sqrt{1 + 4x^2 + 4y^2} \; dx \; dy$$

Now is a good time to switch to polar coordinates (remember the extra factor of $r$):

$$SA = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \sqrt{1 + 4r^2} \; r \; dr \; d\theta$$

$$= 2\pi \int_{r=0}^{R} \sqrt{1 + 4r^2} \; r \; dr$$

$$= 2\pi \ \frac{1}{12} \ [ \ (1 + 4r^2)^{3/2} \ ] \ \Big|_0^R$$

Here, recall that $R = \sqrt{2}$, so

$$\begin{aligned} &= \frac{\pi}{6}(27 - 1) \\ &= \frac{26}{6}\pi \end{aligned}$$