# Taylor Series—Shankar derivation

Suppose we have a function $f(x)$, but

> "imagine that you don't have access to the whole function. You cannot see the whole thing. You can only zero-in on a tiny region."

around $f(0)$, where you know the value. So the question is, what do we guess the function will do near $f(0)$? The first approximation is that

$$f(x) \approx f(0)$$

We really can't say anything more. But now suppose we know the slope of the function at 0, $f'(0)$. Then, since $\Delta y = f'(0)\Delta x = f'(0)x$, we can get a better approximation as

$$f(x) \approx f(0) + f'(0)\ x + \dots$$

For most functions, there will be more terms. If $f$ is not a linear function, then the slope won't be constant. So

> "the rate of change itself has a rate of change .. the second derivative."

The term we are going to add is

$$f''(0)\ \frac{x^2}{2}$$

so

$$f(x) \approx f(0) + f'(0)\ x + f''(0)\ \frac{x^2}{2} + \dots$$

A simple way to see why we have $x^2/2$ is to take derivatives on both sides. The terms like $f'(0)$ and $f''(0)$ are constants, they have been evaluated at $x = 0$. So the first derivative is

$$f'(x) \approx f'(0) + f''(0)\ x + \dots$$

We evaluate at $x = 0$ and the term $f''(0)\ x$ goes away because of the 0 multiplying the constant $f''(0)$. So we have just

$$f'(x) \approx f'(0)$$

and that matches. Now take the second derivative

$$f''(x) \approx f''(0)$$

and that matches too. We can see a pattern here. The next term is

$$f(x) \approx f(0) + f'(0)\ x + f''(0)\ \frac{x^2}{2!} + f'''(0)\ \frac{x^3}{3!} + \dots$$

You might not be expecting the factorial which I snuck in there. But if you go back to the exercise above, where we evaluated derivatives, you can see why it works. When we take the first derivative

$$\frac{d}{dx}(f'''(0)\ \frac{x^3}{3!}) = f'''(0)\ \frac{x^2}{2!}$$

the 3 comes down from the power and then turns 3! in the denominator into 2!. The next derivative will bring down the 2. So everything cancels properly. If you like $\Sigma$ notation, we can write

$$f(x) = \sum_{n=0}^{\infty} f^n(0) \frac{x^n}{n!}$$

with the understanding that $0! = 1$. The approximation is better the closer $x$ is to 0, and the more terms the better as well. There is one final wrinkle to this derivation. The series can be modified deal with $x$ near any value $a$, not just near 0. The modification is

$$f(x) = \sum_{n=0}^{\infty} f^n(a) \frac{(x-a)^n}{n!}$$

The series near $a = 0$ is known as the Maclaurin series.

**example 1**

The first example is

$$f(x) = \frac{1}{1-x}$$

We know the answer to this.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3$$

Proof:

$$1 = (1-x)(1 + x + x^2 + x^3)$$

Multiplying by 1, the second term $x$ is matched by $-x$ from the first term in the multiplication by $-x$, and so on. The whole thing vanishes, leaving just 1.

We want to evaluate $f(x)$ near 0, let's say, at $x = 0.1$. The correct value of the function is

$$f(x) = \frac{1}{0.9} = 1.11111\ldots$$

Let's try to approximate using the series. We need derivatives

$$f(x) = \frac{1}{1-x}$$

$$f'(x) = \frac{1}{(1-x)^2} = (1-x)^{-2}$$

$$f'(0) = 1$$

so the linear approximation is

$$f(x) \approx 1 + 1x = 1.1$$

For the next term we obtain

$$f''(x) = 2(1-x)^{-3}$$

$$f''(0)\ \frac{x^2}{2} = x^2 = 0.01$$

And I think we can see where this one is going.

Another very useful series is the binomial.

$$f(x) = (1+x)^n$$

$$f(0) = 1$$

$$f'(0) = n(1+x)^{n-1} = n$$

$$f''(0) = n(n-1)(1+x)^{n-2} = n(n-1)$$

So the series is

$$(1+x)^n \approx 1 + nx + n(n-1)\frac{x^2}{2}$$

A nice application is relativistic energy

$$E = mc^2 f$$

$$f = 1/\sqrt{1 - \frac{v^2}{c^2}}$$

This is, in disguise, a binomial with $n = -1/2$ and $x = -v^2/c^2$ so the expansion is

$$f \approx 1 + nx = 1 + \frac{v^2}{2c^2}$$

so the energy is

$$E \approx mc^2(1 + \frac{v^2}{2c^2})$$

And we see that the second term is just the kinetic energy, $mv^2/2$.

**more examples**

Let's take a look at $e^x$. The nice thing about $e^x$ is the first derivative, in fact all the derivatives, are just $e^x$, and since we're evaluating them at 0, all those factors become 1. So the series is just

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Let's try sine and cosine. Cosine first

$$f'(0) = -\sin(x)\Big|_{x=0} = 0$$

$$f''(0) = -\cos(x)\Big|_{x=0} = -1$$

$$f'''(0) = \sin(x)\Big|_{x=0} = 0$$

$$f''''(0) = \cos(x)\Big|_{x=0} = 1$$

So the pattern is, every other term, with alternating signs.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

The sine function loses the first term (because $\sin 0 = 0$), then we have the same pattern of every other term and alternating sign.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

Finally, we'll sneak in an oddball. Suppose we consider

$$f(x) = e^{ix}$$

where $i = \sqrt{-1}$. Let's just follow the rules and see where we get to. First of all, the derivatives look like this

$$f'(x) = ie^{ix}$$

$$f''(x) = i^2 e^{ix} = -e^{ix}$$
$$f'''(x) = i^3 e^{ix} = -ie^{ix}$$
$$f''''(x) = i^4 e^{ix} = e^{ix}$$

Evaluated at $x = 0$, the exponentials become 1 and we are left with the pattern $1, i, -1, -i \ldots$. So our series is

$$e^x = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} \ldots$$
$$= \cos x + i \sin x$$

These series then lead to definitions of the sine and cosine in terms of the exponential:

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos(-x) + i \sin(-x) = \cos x - i \sin x$$

Add

$$e^{ix} + e^{-ix} = 2 \cos x$$

or subtract

$$e^{ix} - e^{-ix} = 2i \sin x$$

So

$$\frac{d}{dx}(2 \cos x) = \frac{d}{dx}(e^{ix} + e^{-ix}) = i(e^{ix} - e^{-ix}) = i(2i \sin x) = -2 \sin x$$

$$\frac{d}{dx}(2i \sin x) = \frac{d}{dx}(e^{ix} - e^{-ix}) = i(e^{ix} + e^{-ix}) = i(2 \cos x)$$