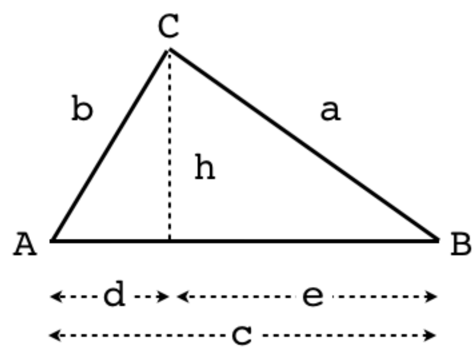
Law of Cosines

In this short write-up, we'll work through an algebraic proof of the law of cosines. In the triangle below, A, B, and C are the angles, with side lengths a, b, and c.



An altitude has been drawn from angle C to side c opposite. The altitude is perpendicular to the side c, which is thereby divided into lengths d and e.

We'll use these four facts

$$c = d + e$$
$$a^2 = e^2 + h^2$$
$$b^2 = d^2 + h^2$$
$$\frac{d}{b} = \cos A$$

Start with

$$a^2 = e^2 + h^2$$

Knowing that

$$b^2 = d^2 + h^2$$

$$h^2 = b^2 - d^2$$

substitute for h^2

$$a^2 = e^2 + b^2 - d^2$$

Since $e = c - d$, substitute for e^2

$$a^2 = (c - d)^2 + b^2 - d^2$$

$$= c^2 - 2cd + d^2 + b^2 - d^2$$
$$= b^2 + c^2 - 2cd$$

Finally, substitute for d knowing that $d = b \cos A$

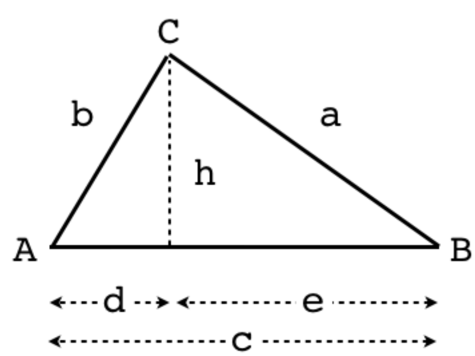
$$a^2 = b^2 + c^2 - 2bc \cos A$$

QED.

This is the Law of Cosines.

Notice that if $A = 90°$, $d = 0$ and $cosA = 0$, and this becomes the Pythagorean Theorem.

Another way, which is slightly shorter:



$$d = b\cos A$$

$$e = c - d = c - b\cos A$$

$$h = b\sin A$$

So, using Pythagoras

$$a^2 = h^2 + e^2$$

$$= b^2\sin^2 A + c^2 - 2bc\cos A + b^2\cos^2 A$$

$$= b^2 + c^2 - 2bc\cos A$$