Integrating zz + 1

This problem is Beck 4.26. Consider

$$f(z) = \frac{1}{z^2 + 1}$$

We see that the denominator is zero when

$$z^2 = -1, \quad z = \pm\, i$$

Suppose our curve is the unit circle centered at i, designated as $C[i, 1]$. Obviously, this curve contains the singularity $z = i$. Now

$$z^2 + 1 = (z + i)(z - i)$$

Rewrite the function as

$$f(z) = \frac{1/z + i}{z - i}$$

Then

$$\int \frac{1}{z^2 + 1}\ dz = \int \frac{1/z + i}{z - i}\ dz$$

Cauchy's Integral formula says that

$$f(w) = \frac{1}{2\pi i}\ \oint \frac{f(z)}{z - w}\ dz$$

The value of the function is

$$\frac{1}{z + i}(i) = \frac{1}{2i}$$

and the integral in question is then.

$$2\pi i f(w) = \pi$$

$$\int \frac{1}{z^2+1}\ dz = \pi$$

Similarly, if the unit circle is centered at $-i$, rewrite the function as

$$f(z) = \frac{1/z-i}{z+i}$$

The value of the function is

$$\frac{1}{z-i}(-i) = -\frac{1}{2i}$$

and that integral is then $-\pi$.

A contour that includes both singularities integrates to zero.

connection reals

Note that for the real function

$$\int_0^\infty \frac{1}{1+x^2}\ dx$$

We know the answer to this one, it is

$$\tan^{-1} x \ \Big|_0^\infty = \frac{\pi}{2}$$

Since $f(x)$ is an even function, the integral over $-\infty \to \infty = \pi$, and we feel there ought to be some connection between the two real and complex results.

Suppose we draw a part of the contour from $-\infty \to \infty$ on the real axis. That integral is $\int f(z)\ dz$ but y and dy are both zero so it is just $\int f(x)\ dx$ with the result which we just obtained.

How to complete the contour? Imagine the hemisphere in the upper half-plane with R at ∞. That is, parametrize

$$\gamma(\theta) = Re^{i\theta}, \quad \theta \in [0, \pi]$$

$$\gamma\,'(\theta) = iRe^{i\theta}$$

The integral is

$$\int_0^{\pi} \frac{1}{1 + R^2 e^{i2\theta}} \; iRe^{i\theta} \; d\theta$$

Now what? the It's clear that as $R \to \infty$, this integrand goes to 0.