Electric dipole

Suppose we consider an electric dipole with two charges $+q$ and $-q$, each a distance a from the origin along the x-axis (so their separation is $2a$). We wish to calculate the electric field along two special lines, namely, the x and y-axes.

For the first one, suppose the test charge lies to the right, so that it is closer to the positive charge $+q$. The electric field is the sum of the fields for each of the two charges. We write

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \; (\frac{1}{(r-a)^2} - \frac{1}{(r+a)^2}) \; \hat{\mathbf{i}}$$

We can manipulate the part in the parentheses as follows:

$$\frac{1}{(r-a)^2} - \frac{1}{(r+a)^2} = \frac{(r+a)^2 - (r-a)^2}{(r+a)^2(r-a)^2}$$

$$= \frac{(r^2 + 2ar + a^2) - (r^2 - 2ar + a^2)}{(r+a)^2(r-a)^2}$$

$$= \frac{4ar}{(r+a)^2(r-a)^2}$$

We will combine a factor of $2a$ with q as the dipole moment

$$p = 2aq$$

leaving another $2r$ to account for. If we expand the denominator we obtain

$$(r+a)^2(r-a)^2$$

$$= (r^2 + 2ar + a^2)(r^2 - 2ar + a^2)$$
$$= r^4 - 2r^3a + r^2a^2 + 2ar^3 - 4r^2a^2 + 2ra^3 + a^2r^2 - 2ra^3 + a^4$$

We are interested in $r \gg a$ so we will only keep terms in a but not a^2

$$= r^4 - 2r^3a + 2ar^3 = r^4$$

Thus

$$\mathbf{E} = \frac{p}{2\pi\epsilon_0 r^3}\,\hat{\mathbf{i}}$$

For the other one, along the y-axis, suppose the separation from the origin is again r. The vertical component of the field from $+q$ (in the positive $\hat{\mathbf{j}}$ direction), is just canceled by the vertical component of the field from $-q$. What remains are two identical components that point in the $-\hat{\mathbf{i}}$ direction with magnitude

$$\frac{q}{4\pi\epsilon_0}\,\frac{1}{(a^2 + r^2)}\cos\phi$$

The sum is

$$\mathbf{E} = -\frac{q}{2\pi\epsilon_0}\,\frac{r}{(a^2 + r^2)^{3/2}}$$

Approximation for this??