Minmax problems in multi-variable calculus

Two simple examples

Suppose that we have three variables x, y, z and we know that

$$x + y + z = 120$$

We want to find solve this subject to the constraint that xyz is a maximum. Now, this problem really has only two variables, because once we have picked x and y we know z

$$z = 120 - x - y$$

Our function $f(x, y)$ is then

$$P = f(x, y) = xyz = xy(120 - x - y)$$

We want to find the critical points of this function, and determine which is a maximum. To do this, we take the partial derivatives

$$f(x, y) = 120xy - x^2y - xy^2$$

$$f_x = 120y - 2xy - y^2$$

By symmetry

$$f_y = 120x - 2xy - x^2$$

We set $f_x = 0$ and $f_y = 0$

$$0 = 120y - 2xy - y^2$$

$$0 = 120x - 2xy - x^2$$

We notice that we can factor out a single x or y

$$y(120 - 2x - y)$$

$$x(120 - 2y - x)$$

So obviously $x = 0$ and $y = 0$ are critical points, but also we are at a critical point if

$$120 - 2y - x = 0$$

$$120 - 2x - y = 0$$

We can solve this system

$$120 - 2y - x = 120 - 2x - y$$

$$-2y - x = -2x - y$$

$$-y = -x$$

$$x = y$$

$$120 - 2x - x = 0$$

$$x = y = 40$$

$$z = 120 - x - y = 40$$

This appears to be a maximum because

$$x^3 > (x-1)(x+1)x = x^3 - x$$

for all $x > 0$

A problem that looks different at first but works in a similar way is the following. Consider the plane that goes through the x-axis at (6,0,0), the y-axis at (0,4,0) and the z-axis at (0,0,3). By substitution, we can see that the following equation is the equation of this plane.

$$2x + 3y + 4z = 12$$

Construct a rectangular box with one corner at the origin and the other corner at (x, y, z), subject to the constraint that x, y, z is in this plane. Pick x, y, z to give the maximum volume. We know that

$$z = \frac{1}{4}(12 - 2x - 3y)$$

$$V = f(x, y) = xyz = xy\ \frac{1}{4}(12 - 2x - 3y$$

$$= 3xy - \frac{1}{2}x^2y - \frac{3}{4}xy^2$$

This is quite similar to the first problem, except for the cofactors, which leads to a solution that is not symmetrical in x, y, z. Take the partial derivatives, factor if possible, and set them equal to 0.

$$f_x = 3y - xy - \frac{3}{4}y^2$$

$$= y(3 - x - \frac{3}{4}y) = 0$$

$$f_y = 3x - \frac{1}{2}x^2 - \frac{3}{2}xy$$

$$= x(3 - \frac{1}{2}x - \frac{3}{2}y) = 0$$

As before, one set of critical points is where $x = 0$ or $y = 0$ but these are minima. To solve, let's set the two derivatives (without the factors x or y) equal to each other

$$3 - x - \frac{3}{4}y = 3 - \frac{1}{2}x - \frac{3}{2}y$$

$$x + \frac{3}{4}y = \frac{1}{2}x + \frac{3}{2}y$$

$$\frac{1}{2}x = \frac{3}{4}y$$

$$x = \frac{3}{2}y$$

Back substituting

$$3 - \frac{1}{2}x - \frac{3}{2}y = 0$$

$$3 - \frac{3}{4}y - \frac{3}{2}y = 0$$

$$1 - \frac{3}{4}y = 0$$

$$y = \frac{4}{3}$$

$$x = \frac{3}{2}y = 2$$

$$z = \frac{1}{4}(12 - 2x - 3y) = \frac{1}{4}(12 - 4 - 4) = 1$$

The volume is

$$V = xyz = 2\frac{4}{3} = \frac{8}{3}$$