Stokes problems (math24.net)

Stokes Theorem is:

$$\oint_C \mathbf{F}\cdot\mathbf{r} = \iint_R (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\; dS$$

Problem 1

Given

$$\mathbf{F} = \langle yz, xz, xy\rangle$$

Show that the integral

$$\oint_C yz\ dx + xz\ dy + xy\ dz = 0$$

over *any* closed curve C.

One way to do this is to guess the potential function for which $\mathbf{F} = \nabla f$.

$$f(x,y,z) = xyz$$

fulfills this criterion. Since this is true, the curl of $\mathbf{F}$ must be zero. By Stokes theorem, the integral is zero for any closed curve C.

A second approach is to actually calculate the curl

$$\nabla\times\mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$$

$$= \langle x - x, y - y, z - z\rangle = \langle 0,0,0\rangle$$

and the dot product with *any* $\hat{\mathbf{n}}$ is zero.

Problem 2

Evaluate

$$\oint_C (y+2z)dx + (x+2z)dy + (x+2y)dz$$

where C is the intersection of the unit sphere $x^2+y^2+z^2=1$ with the plane $x+2y+2z=0$. This looks fairly hard at first. How to parameterize this curve? But we start by calculating

$$\nabla \times \mathbf{F} = \langle 2-2, 2-1, 1-1 \rangle = \langle 0,1,0 \rangle$$

What is $\hat{\mathbf{n}}\ dS$? Our surface is a part of the plane. Notice that $(0,0,0)$ is a solution of the equation for the plane, so it goes through the origin. Therefore, the intersection is a circle of radius 1. The plane has normal vector $\mathbf{n} = \langle 1,2,2 \rangle$ and *unit normal* $\hat{\mathbf{n}} = 1/3\ \mathbf{n}$ so

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} = \frac{2}{3}$$

Thus we have

$$\iint_R (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\ dS = \iint_R \frac{2}{3}\ dS$$

which is just two-thirds the area of the unit circle, or $4/3\pi$.

Problem 3

Evaluate

$$\oint_C y^3\ dx - x^3\ dy + z^3\ dz$$

where C is the intersection of the cylinder $x^2+y^2=a^2$ and the plane $x+y+z=b$.

The normal vector to the plane is $\mathbf{n} = \langle 1,1,1 \rangle$. We could certainly parametrize the curve in terms of the angle θ going around the cylinder. z would move from a minimum at $\theta = \pi/4$ to a maximum on the other side of the circle.

Let's try the curl first.

$$\begin{aligned} \mathbf{F} &= \langle y^3, -x^3, z^3 \rangle \\ \nabla \times \mathbf{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \langle 0, 0, -3x^2 - 3y^2 \rangle \end{aligned}$$

Using the equation of the surface $z = b - x - y$, we get that $f_x = -1 = f_y$ so

$$\hat{\mathbf{n}}\ dS = \langle -f_x, -f_y, 1\rangle\ dx\ dy$$

and

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\ dS = -3x^2 - 3y^2\ dx\ dy$$

$$\begin{aligned}
\iint_R (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\ dS &= \iint_R -3x^2 - 3y^2\ dx\ dy \\
&= -3 \iint_R x^2 + y^2\ dx\ dy
\end{aligned}$$

We need to integrate this over a circle of radius a, so switch to polar coordinates

$$= -3 \int \int r^2\ r\ dr\ d\theta$$

$$= -3\ \int \frac{1}{4} a^4\ d\theta$$

$$= -\frac{3}{2}\pi a^4$$