Derivative of trigonometric functions

The two most basic trigonometric derivatives are:

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

sine

The first of these comes from working with the difference quotient

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

using the sum formula for sine:

$$\sin(x+h) = \sin x\cos h + \sin h\cos x$$

we obtain

$$\lim_{h\to 0}\frac{\sin x\cos h + \sin h\cos x - \sin x}{h}$$

rearranging

$$\lim_{h\to 0}\frac{\sin x(\cos h - 1)}{h} + \frac{\sin h\cos x}{h}$$

and since

$$\lim_{h\to 0}\frac{\sin h}{h} = 1$$

the limit of the right-hand term is $\cos x$. On the other hand the limit of $\cos h - 1/h$ is 0, so the left-hand term has a limit of 0.

We've worked out these limits elsewhere. The first one is quite famous, I'm sure you know it (or should know it). The second can actually be derived from the first:

$$\lim_{h \to 0} \frac{\cos h - 1}{h} \times \frac{\cos h + 1}{\cos h + 1}$$

$$= \lim_{h \to 0} \frac{\cos^2 h - 1}{h \cos h}$$

$$= \lim_{h \to 0} \frac{-\sin h}{h} \times \frac{\sin h}{\cos h}$$

The left-hand term is -1 (as above) and the right-hand term is $0/1 = 0$.

cosine

The derivative of the cosine can also be obtained by using the difference quotient, but why don't we just try implicit differentiation plus the previous result:

$$\sin^2 x + \cos^2 x = 1$$

$$\frac{d}{dx}(\sin^2 x + \cos^2 x) = 0$$

$$2 \sin x \frac{d}{dx} \sin x + 2 \cos x \frac{d}{dx} \cos x = 0$$

$$2 \sin x \cos x + 2 \cos x \frac{d}{dx} \cos x = 0$$

$$\sin x + \frac{d}{dx} \cos x = 0$$

You can see how this turns out!

tangent and secant

What about other functions like $tan\ x$ and $sec\ x$? We use the quotient rule.

$$(\frac{u}{v})' = \frac{vu' - uv'}{v^2}$$

I check this mentally by considering $1/x$. We pick up a minus sign from $-uv'$, which is what we want, so this is correct. For $tan\ x$ we have:

$$\frac{u}{v} = \frac{sin\ x}{cos\ x}$$

$$(\frac{u}{v})' = \frac{cos^2 x + sin^2 x}{cos^2 x} = \frac{1}{cos^2 x} = sec^2 x$$

Probably the most important of the rest is the secant, because of its connection with tangent (above)

$$\frac{d}{dx}\frac{1}{\cos x} = \frac{-(-\sin x)}{\cos^2 x}$$

$$= \sec\ x \tan\ x$$

the rest of them

We can use the quotient rule for these too, or we can just memorize them by their similarity with what we had already.

$$\frac{d}{dx}\sec x = \sec\ x \tan\ x$$

$$\frac{d}{dx}\csc x = -\csc\ x \cot\ x$$

and

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

These are (perhaps) too hard to remember, but they are easy enough to derive with the quotient rule, and they will come in handy for evaluating various kinds of integrals.

inverse sine

Suppose we label a right-triangle with sides x (opposite the angle y) and 1 for the hypotenuse. Then

$$\frac{x}{1} = \sin y$$

$$y = \sin^{-1} x$$

y is *the angle whose sine is* x. Where can we go from this? Well, Pythagoras says that the adjacent side is $\sqrt{1-x^2}$, which is also $\cos y$. What is the derivative of the inverse sine? A trick is to start with the straightforward sine function:

$$x = \sin y$$

$$\frac{dx}{dy} = \cos y$$

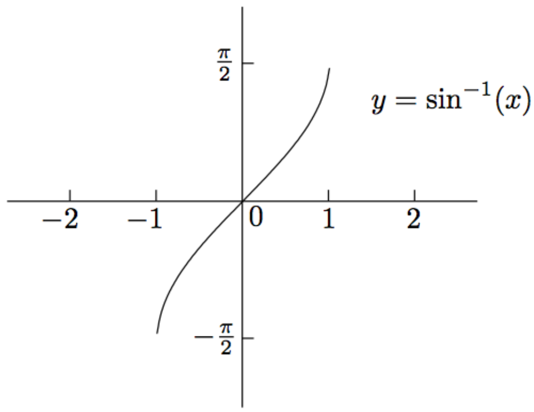
but

$$\cos y = \sqrt{1-x^2}$$

Now, we want the derivative of the inverse sine, that is dy/dx:

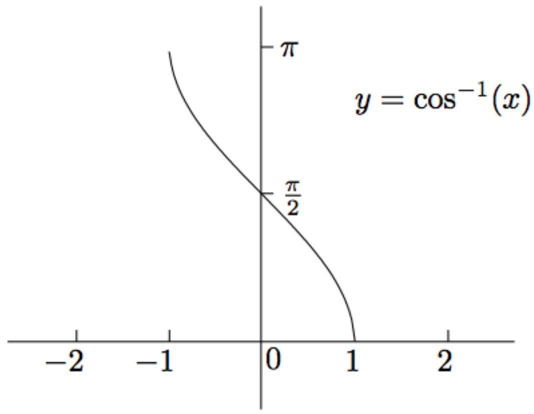
$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

All this assumes that we limit the range of the inverse sine to the appropriate interval (domain: $-1 \le x \le 1$; range: $-\pi/2 \le y \le \pi/2$).



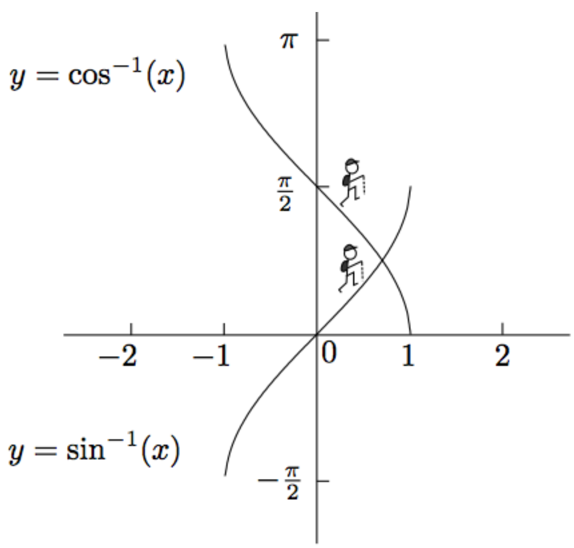
And above, we actually have to make a choice when we do $\sqrt{1 - x^2}$. We take the positive square root, and if you look at the figure, this is correct. The function increases continuously, $f'(x)$ is positive on this interval. (Except that it is not defined at the endpoints).

inverse cosine



As before, we limit the range of the inverse cosine to the appropriate interval (domain: $-1 \leq x \leq 1$), however, the range is different: $0 \leq y \leq \pi$). Also, if you look at the graph, the function is continuously decreasing over this interval (so we expect $f'(x)$ to be negative everywhere (except the endpoints, where it is not defined).

Now here is something curious: if we plot them both together



Can you believe that if we add the two functions together, the result is a constant? Actually

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

Differentiating:

$$\frac{d}{dx} \sin^{-1} x + \frac{d}{dx} \cos^{-1} x = 0$$

The derivative of inverse cosine is just the same as the derivative of the inverse sine, multiplied by -1!

$$y = \cos^{-1} x$$
$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

If you want to see an actual calculation:

$$y = \cos^{-1} x$$

$$x = \cos y$$
$$dx = -\sin y \; dy$$
$$\frac{dy}{dx} = -\frac{1}{\sin y} =$$

Now, of course $x = \sin y$ but that doesn't really help us here. Instead do this:

$$\sin^2 y + \cos^2 y = 1$$

$$x = \cos y$$
$$\sin^2 y + x^2 = 1$$

$$\sin y = \pm\sqrt{1-x^2}$$

so

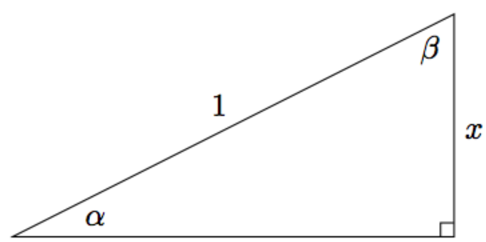
$$y = \cos^{-1} x$$
$$\frac{dy}{dx} = -\frac{1}{\sin y} = \pm\frac{1}{\sqrt{1-x^2}}$$

but remember that $f'(x) < 0$ everywhere so

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

Let's go back to that curious statement about the sum of inverse sine and cosine:

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{4}$$



We'll call the angle α (originally y). We have

$$\alpha = \sin^{-1} x$$

but

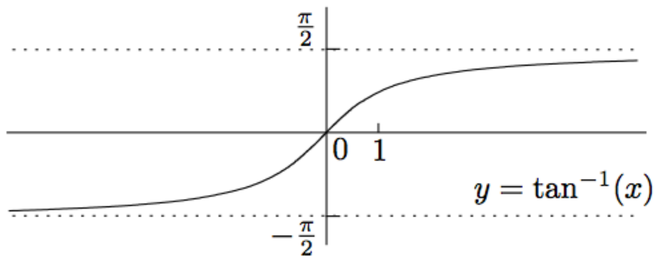
$$\beta = \cos^{-1} x$$

and

$$\alpha + \beta = \frac{\pi}{2}$$

Now, it's pretty easy to see how it works.

inverse tangent



Notice the domain and range, and that the slope is everywhere greater than 0, except at the extrema.

$$y = \tan^{-1} x$$

so

$$x = \tan y$$

$$dx = \sec^2 y \; dy$$

We want

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

since

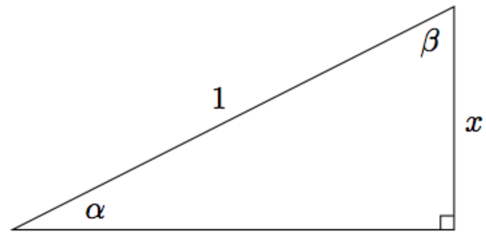
$$\sec^2 y = 1 + \tan^2 y$$

$$\tan y = x$$

$$\sec^2 y = 1 + x^2$$

$$\frac{d}{dx} \; \tan^{-1} x == \frac{1}{\sec^2 y} = \frac{1}{1 + x^2}$$

Remember the setup (this figure has the angle as α but we've gone back to using y).



Why is this wrong?

$$x = \sin y$$
$$\sqrt{1+x^2} = \cos y$$

So

$$\frac{d}{dx}\tan^{-1} x = \cos^2 y = 1 + x^2$$