Fourier series, introduction

If we consider the series $\cos kx$ and $\sin kx$ for $k \in \mathbb{N} \quad (1, 2, 3 \ldots)$:

$$\cos x, \cos 2x, \cos 3x \ldots \cos nx$$

$$\sin x, \sin 2x, \sin 3x \ldots \sin nx$$

Any function in this set is *orthogonal* to each of the others, by which we mean that:

$$\int f(x)g(x)\ dx = 0$$

Thus, for $m, n \in \mathbb{N}$:

$$\int \cos mx \cos nx\ dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \\ 2\pi, & m = n = 0 \end{cases}$$

The usual bounds to think about are $\int_0^{2\pi}$ and $\int_{-\pi}^{\pi}$, i.e. bounds that are multiples of π and are separated by 2π.

$\int \sin mx \sin nx\ dx$ is almost the same, the only non-zero case is $m = n \neq 0$, where the result is π.

$$\int \sin mx \sin nx\ dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \\ 0, & m = n = 0 \end{cases}$$

And finally, for $\int \sin mx \cos nx\ dx$ all cases are zero. We will use these facts to develop Fourier series that approximate $f(x)$.

cosine-cosine

Before we do that, let's look at these integrals in a bit more detail.

Recall the cosine addition formulas:

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$
$$\cos(s-t) = \cos s \cos t + \sin s \sin t$$

Adding them together and moving a factor of 2:

$$\cos s \cos t = \frac{1}{2} \; [\; \cos(s+t) + \cos(s-t) \;]$$

So then

$$\int \cos mx \cos nx \; dx$$
$$= \frac{1}{2} \int \cos(m+n)x + \cos(m-n)x \; dx$$

For the first case, $m = n = 0$, we have

$$\frac{1}{2} \int \cos 0 + \cos 0 \; dx = \int dx$$

and so we obtain just x evaluated between the limits.

We will choose limits $\int_a^b$ separated by 2π. But no matter what limits we choose, whether $\int_0^{2\pi}$ or any $\int_{-\pi}^{\pi}$ or $\int_a^{a+2\pi}$, we obtain 2π for the result.

If $m = n \neq 0$, then in exactly the same way, we obtain the value of π as the result from the second term (remember the factor of $\frac{1}{2}$). The first term is zero, as follows:

For any non-zero integer k, whether $k = m + n$ or $k = m - n$ (as we will have below):

$$\int_a^b \cos kx \; dx = \frac{\sin kx}{k} \; \Big|_a^b$$

whether we choose bounds $\int_0^{2\pi}$ or $\int_{-\pi}^{\pi}$ or some other $\int_a^b$, with $b - a = 2\pi$, we obtain 0 for the result. Graph the sine function between any a and $a + 2\pi$ the area of the function above zero is equal to the area below zero no matter what the value of a. This is true for cosine as well.

sine-sine

$$\int \sin mx \sin nx \ dx$$

Go back to the sum of cosines above and subtract the first equation from the second to obtain

$$\sin s \sin t = \frac{1}{2}(\cos s - t - \cos s + t)$$

Hence

$$\int \sin mx \sin nx \ dx$$

$$= \frac{1}{2} \int \cos(m-n)x - \cos(m+n)x \ dx$$

This is very similar to what we had before. The difference is the minus sign. Here, if $m = n = 0$ the two terms are identical $(\cos 0)$ and they cancel. On the other hand, if $m = n \neq 0$ we get a value of π from the first term as before.

For any non-zero k in the argument to the cosine, we have

$$\int_a^b \cos kx \ dx = \frac{\sin kx}{k} \Big|_a^b$$

which, again as we saw before is zero for any $\int_a^{a+2\pi}$.

sine-cosine

One way to do this is to remember the addition formula for sine:

$$\sin(s+t) = \sin s \cos t + \sin t \cos s$$

$$\sin(s-t) = \sin s \cos t - \sin t \cos s$$

$$\sin s \cos t = \frac{1}{2}\ [\ \sin(s+t) + \sin(s-t)\ ]$$

So we have

$$\int \sin mx \cos nx\ dx$$

$$= \frac{1}{2} \int \sin(m+n)x + \sin(m-n)x\ dx$$

Here, the case with $m = n = 0$ is $\int \sin 0$ which is just 0. For $m = n \neq 0$, the second term is zero and the first is

$$\int_a^b \sin kx\ dx = -\frac{\cos kx}{k}\ \Big|_a^b$$

which is zero for any $\int_a^{a+2\pi}$. Lastly, for $m \neq n$ we obtain

$$\int_a^b \sin(m+n)x + \sin(m-n)x\ dx$$

$$= -\frac{\cos(m+n)x}{m+n} - \frac{\cos(m-n)x}{m-n}\ \Big|_a^b$$

which is zero for any $\int_a^{a+2\pi}$ for the same reason. Hence for sine-cosine all cases are zero.