Sum and product rule

Assume that

$$\lim_{x \to c} f(x) = L$$
$$\lim_{x \to c} g(x) = M$$

We want to show that

$$\lim_{x \to c} f(x) + g(x) = L + M$$

The limit of the sum is the sum of the limits.

Let $\epsilon > 0$ be arbitrary.

Then the existence of the limits means that

$$\forall\ \epsilon, \exists\ \delta_1 > 0 \mid \forall\ x,\ 0 < |x - c| < \delta_1 \to |f(x) - L| < \epsilon/2$$

and

$$\forall\ \epsilon, \exists\ \delta_2 > 0 \mid \forall\ x,\ 0 < |x - c| < \delta_2 \to |g(x) - M| < \epsilon/2$$

Let

$$\delta = \ \min\ (\delta_1, \delta_2)$$

Now for $|x - c| < \delta$:

$$|f(x) - L + g(x) - M| < \epsilon$$

But by the triangle inequality the left-hand side is

$$|f(x) - L| + |g(x) - M| < \epsilon$$

which proves the theorem.

proof of the product rule for limits

Assume that

$$\lim_{x\to c} f(x) = L$$
$$\lim_{x\to c} g(x) = M$$

We want to show that

$$\lim_{x\to c} f(x)\cdot g(x) = LM$$

The limit of the product is the product of the limits.

We need to show that

$$f(x)\cdot g(x) - LM$$

is small.

Subtract $Lg(x)$ and add it back

$$\begin{aligned} f(x)\cdot g(x) - LM &= f(x)\cdot g(x) - Lg(x) + Lg(x) - LM \\ &= (f(x)-L)g(x) + L(g(x)-M) \end{aligned}$$

Take the absolute value on both sides

$$|f(x)\cdot g(x) - LM| = |(f(x)-L)\cdot g(x) + L\cdot(g(x)-M)|$$

Use the triangle inequality to split up the sum:

$$\leq |(f(x)-L)\cdot g(x)| + |L\cdot(g(x)-M)|$$

This can be further massaged to

$$= |f(x)-L|\cdot|g(x)| + |L|\cdot|g(x)-M|$$

Write the whole thing:

$$|f(x)\cdot g(x) - LM| \leq |(f(x)-L)||g(x)| + |L||(g(x)-M)|$$

Now, play the epsilon-delta game: you pick ϵ and then I concentrate on a region so close to c that

$$|f(x) - L| < \epsilon$$

and

$$|g(x) - M| < \epsilon$$

If your epsilon is too large it would mess things up (why?), so in that case I will pick $|g(x) - M| = 1$.

Then I have

$$|f(x) - L| < \epsilon$$
$$|g(x) - M| < \epsilon$$
$$|g(x)| < |M| + 1$$

Go back to the equation we obtained above

$$|f(x) \cdot g(x) - LM| \leq |(f(x) - L)||g(x)| + |L||(g(x) - M)|$$

substitute on the right-hand side

$$|(f(x) - L)||g(x)| + |L||(g(x) - M)|$$
$$\leq \epsilon\ (|M| + 1) + |L|\ \epsilon$$
$$\leq \epsilon\ (|M| + |L| + 1)$$

That is:

$$|f(x) \cdot g(x) - LM| \leq \epsilon\ (|M| + |L| + 1)$$

as Adrian Banner says in *Calculus Lifesaver*:

> That's almost what I want! I was supposed to get ϵ on the right-hand side, but I got an extra factor of $|M|+|L|+1$. This is no problem—you just have to allow me to make my move

again, but this time I'll make sure that $|f(x) - L|$ is no more than $\epsilon/2(|M| + |L| + 1)$, and similarly for $|g(x) - M|$. Then when I replay all the steps, ϵ will be replaced by $\epsilon/(|M| + |L| + 1)$, and at the very last step, the factor $|M| + |L| + 1$ will cancel out and we'll just get our ϵ. So we have proved the result.

More formal proof of the product rule

Suppose that

$$\lim_{x \to c} f(x) = L$$
$$\lim_{x \to c} g(x) = M$$

To prove:

$$\lim_{x \to c} f(x) \cdot g(x) = LM$$

Proof

Let $\epsilon > 0$. By the definition of limits we can find three numbers δ_1, δ_2 and δ_3 such that

if $0 < |x - c| < \delta_1$:

$$|f(x) - L| < \frac{\epsilon}{2(1 + |M|)}$$

if $0 < |x - c| < \delta_2$:

$$|g(x) - M| < \frac{\epsilon}{2(1 + |L|)}$$

and third, if $0 < |x - c| < \delta_3$:

$$|g(x) - M| < 1$$

Now write

$$|g(x)| = |g(x) - M + M|$$

use the triangle inequality

$$|g(x)| \leq |g(x) - M| + |M|$$

Then according to (3), if if $0 < |x - c| < \delta_3$:

$$|g(x) \leq 1 + |M|$$

Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.

Then if $0 < |x - c| < \delta$:

$$|f(x) \cdot g(x) - LM| = |f(x) \cdot g(x) - L \cdot g(x) + L \cdot g(x) - LM|$$

by the triangle inequality (again)

$$\leq |f(x) \cdot g(x) - L \cdot g(x)| + |L \cdot g(x) - LM|$$

The next step is to factor (see below):

$$\leq |g(x)||f(x) - L| + |L||g(x) - M|$$

$$< (1 + |M|)\ \frac{\epsilon}{2(1 + |M|)} + (1 + |L|)\ \frac{\epsilon}{2(1 + |L|)}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$|f(x) \cdot g(x) - LM| < \epsilon$$

which completes the proof.

Factoring absolute values

We used another theorem about the absolute value function above:

$$|ab| = |a| \cdot |b|$$

This is certainly true for $a > 0$ and $b > 0$.

For $a < 0$ and $b < 0$, suppose $m > 0$ and $n > 0$ and $a = -m$ and $b = -n$, then

$$|ab| = |(-m)(-n)| = |mn| = mn$$

$$|a| \cdot |b| = |-m| \cdot |-n| = mn$$

Finally, for $a > 0$ and $n > 0$ and $b = -n$

$$|ab| = |a\ (-n)| = |-an| = an$$

$$|a| \cdot |b| = |a| \cdot |-n| = an$$

which completes the proof.