Plane and a point

Consider the plane containing three points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Find two vectors in the plane by subtracting the second and third from the first.

$$u = (1,0,0) - (0,1,0) = <1,-1,0>$$

$$v = (1,0,0) - (0,0,1) = <1,0,-1>$$

Obtain the normal vector by computing the cross product

$$N = u \times v \Rightarrow \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 1i + 1j + 1k = <1,1,1>$$

One equation of the plane is then

$$N \cdot w = 0$$

for any vector w in the plane.

Consider a fixed point in the plane (x_0, y_0, z_0). Then any other point in the plane (x, y, z) yields a vector from the fixed point which, dotted with n, yields 0

$$<x - x_0, y - y_0, z - z_0> \ \cdot \ <1,1,1> = 0$$

$$x - x_0 + y - y_0 + z - z_0 = 0$$

$$x + y + z = x_0 + y_0 + z_0 = d$$

Plugging in one of the points yields

$$x + y + z = 1$$

Consider any point in space, e.g. $P = (3,4,6)$. Find the point Q on the plane which is closest to P, the point we arrive at by subtracting some fraction of N from P. We have a point and a vector

$$Q = P - tN$$

$$Q = (3,4,6) \ - \ t < 1,1,1 >$$

Since Q is in the plane, its components x, y, z satisfy $x + y + z = 1$! So

$$(3 - t) + (4 - t) + (6 - t) = 1$$

$$13 - 3t = 1$$

$$t = 4$$

$$Q = (-1, 0, 2)$$

Check that Q is in the plane

$$-1 + 0 + 2 = 1$$

and $P - Q$ is parallel to N

$$P - Q = < 4, 4, 4 >$$

which is definitely a multiple of N.

Where does the vector w that goes from the origin to point $P = (3, 4, 6)$ hit the plane? Call that point R. Again we have a point and a vector

$$R = (0,0,0) + tw = (0,0,0) + t < 3,4,6 >$$

And again, since R is in the plane, its components x, y, z satisfy $x + y + z = 1$. So

$$3t + 4t + 6t = 1$$

$$t = \frac{1}{13}$$

$$R = (\frac{3}{13}, \frac{4}{13}, \frac{6}{13})$$

Notice that the vector $Q - R$ is in the plane, as it should be

$$(Q - R) \cdot N = ((-1, 0, 2) - (\frac{3}{13}, \frac{4}{13}, \frac{6}{13})) \cdot \ < 1, 1, 1 >$$

$$= < \frac{-16}{13}, \frac{-4}{13}, \frac{20}{13} > \ \cdot \ < 1, 1, 1 > = 0$$

And, adding the horizontal and vertical components together

$$Q - R + P - Q = P - R = (3, 4, 6) - (\frac{3}{13}, \frac{4}{13}, \frac{6}{13})$$

$$= (\frac{36}{13}, \frac{48}{13}, \frac{72}{13})$$

the result is parallel to w.