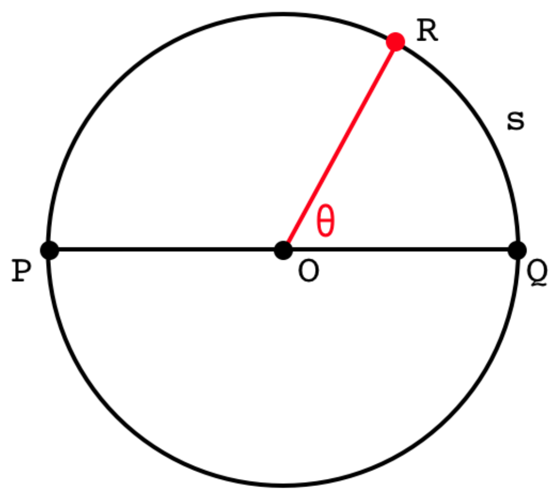
Arcs of a circle



In calculus and analytical geometry angles are defined in terms of radians of arc. For a unit circle with radius $= 1$, the total circumference is 2π, so the arc swept out by the angle θ is in the same ratio to π as the ratio of the angle's measure in degrees to $180°$. We say that the angle θ in this figure is equal to the arc it sweeps out on the circumference.

$$s = \theta$$

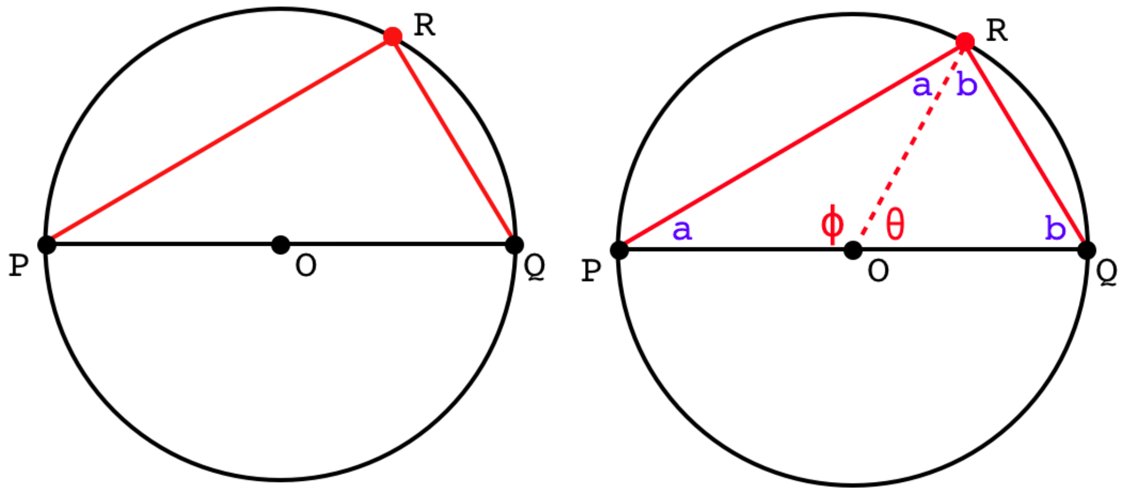
We substitute measures of angles in radians for the ones in degrees:

$$180° = \pi,\ 90° = \frac{\pi}{2}$$

$$60° = \frac{\pi}{3},\ 45° = \frac{\pi}{4},\ 30° = \frac{\pi}{6}$$

Now, think of the same points on the circumference of the circle as forming a triangle. If two points are on a diameter of the circle, the angle at any third point is always a right angle.

To prove: $\angle PRQ$ is a right angle.



Solution: Draw the radius OR. Notice that $\triangle OPR$ and $\triangle OQR$ are both isosceles. Label the respective base angles a and b. By considering that they comprise the angles of $\triangle PQR$:

$$2a + 2b = \pi$$

$$a + b = \frac{\pi}{2}$$

In addition, the arcs swept out by angles a and b (OPR and OQR on the diameter) clearly add up to π. This suggests that:

$$a = \frac{\theta}{2}$$

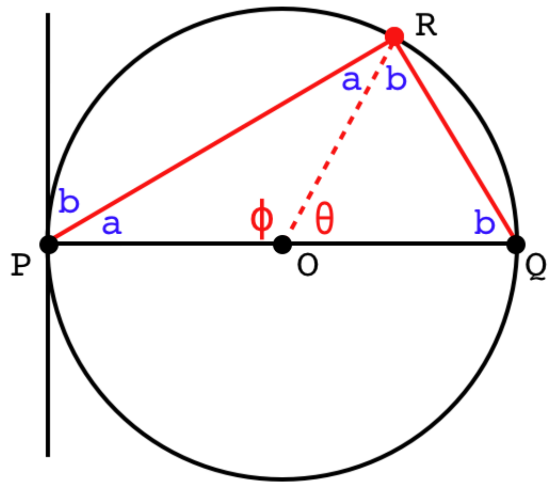
$$b = \frac{\phi}{2}$$

Solution:

$$2a + 2b = \pi = 2a + \phi$$

$$\phi = 2b$$

Consider the chord PR and draw the tangent at P.

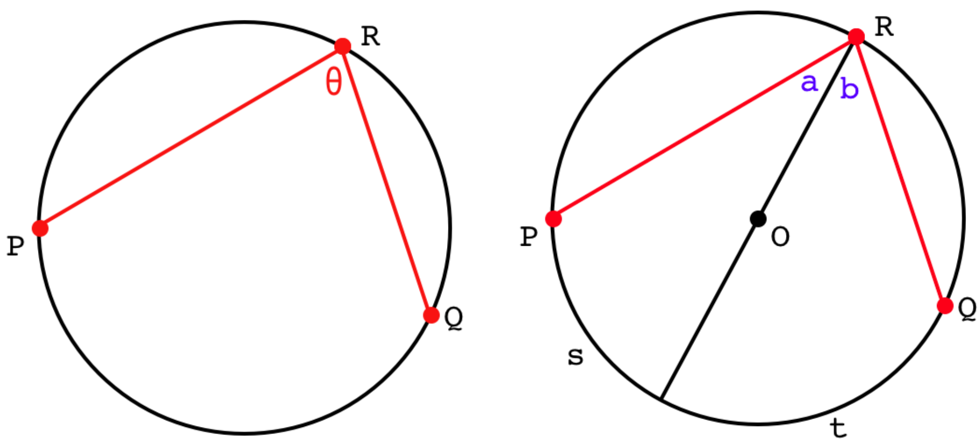


The arc between the tangent and the chord equals 2b because it is the same arc as cut off by $\angle\, PQR$ (which is $\angle\, b$).

Take a chord of the circle, draw the diameter and the tangent. The same rule applies to both angles: one between the chord and the diameter, and the second between the chord and the tangent. The arc is twice the measure of the angle.

Generalized arc

Having established these basic facts we can do a bit more. One is to generalize the result for all arcs. The examples so far contain the diameter in some way. Consider the arc swept out by the angle θ in this figure.



We can prove that the measure of the angle θ is equal to the 1/2 the arc swept out between P and Q. For a simple proof, draw the diameter: By our previous work:

$$b = \frac{t}{2}$$

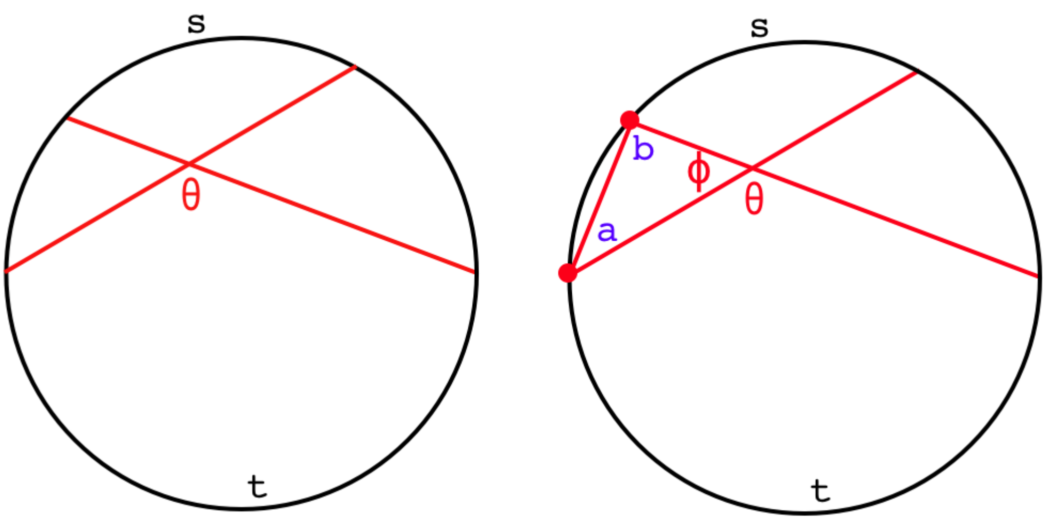
$$a = \frac{s}{2}$$

$$\theta = a + b = \frac{s+t}{2}$$

Intersecting chords

Given two chords, to prove:

$$\theta = 1/2(s+t)$$



θ is the average of the two arc lengths. Solution: Draw a triangle.

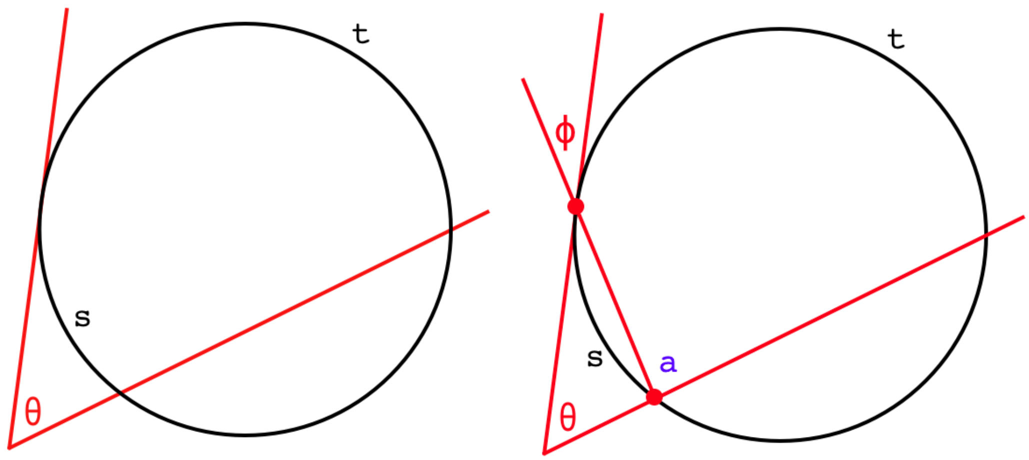
$$a = \frac{s}{2}$$

$$b = \frac{t}{2}$$

$$a + b = \theta = \frac{s+t}{2}$$

Tangent and secant

Rather than having all three points on the circle, one is now outside. We have the same arc swept out by the endpoints (t), but the included angle is now smaller, and there is a new small piece of arc length s.



To prove:

$$\theta = \frac{t-s}{2}$$

Solution: Draw the triangle. By our previous work (and supplementary angles):

$$\phi = \frac{s}{2}$$

$$a = \frac{t}{2}$$

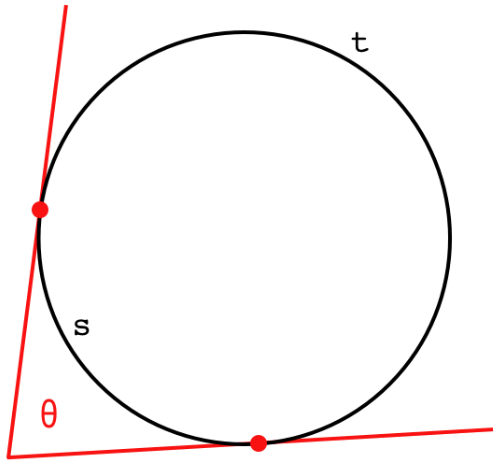
by supplementary angles:

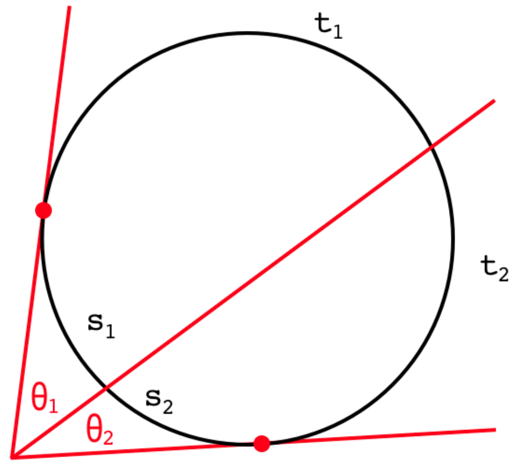
$$\theta + \phi = a$$

$$\theta = \frac{t}{2} - \frac{s}{2}$$

$$= \frac{t-s}{2}$$

Two tangents





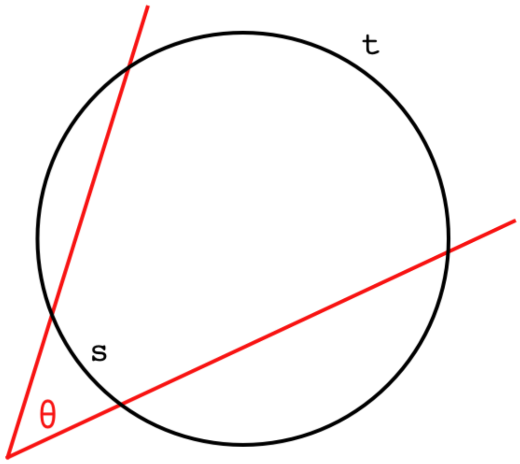
Draw a secant line By our previous work:

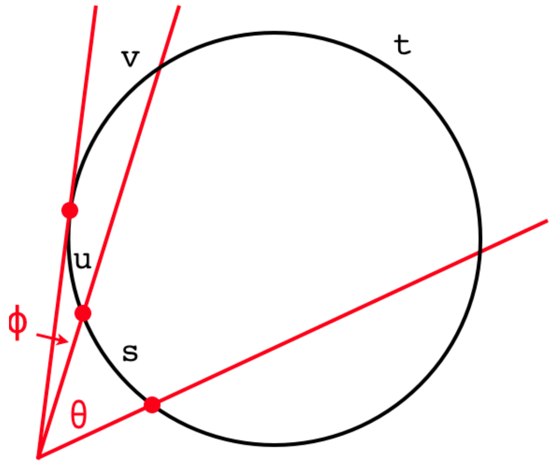
$$\theta_1 = \frac{t_1 - s_1}{2}$$
$$\theta_2 = \frac{t_2 - s_2}{2}$$

By addition:

$$\theta = \frac{t - s}{2}$$

Two secants





Draw a tangent line. By our previous work:

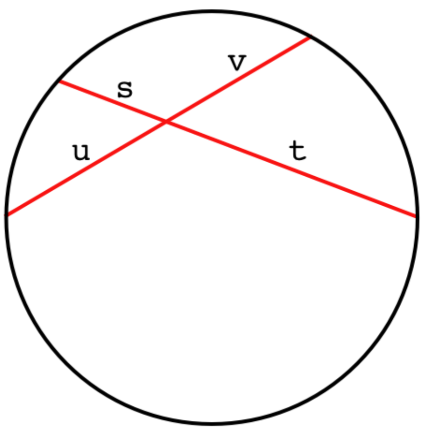
$$\theta + \phi = \frac{t-s}{2} + \frac{v-u}{2}$$
$$\phi = \frac{v-u}{2}$$

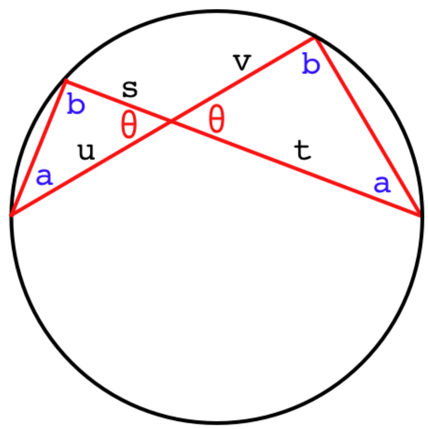
By subtraction:

$$\theta = \frac{t-s}{2}$$

Chord segments

Finally, there is a simple algebraic relationship between chord segments. Draw two chords of the circle and label the lengths of the segments as shown (note: s and t do not refer to arcs any more).





To prove:

$$st = uv$$

Solution: Draw the two triangles. Notice that the two angles labeled a are equal because they sweep out the same arc of the circle, and similarly for the two angles labeled b. By similar triangles:

$$s/u = v/t$$
$$st = uv$$