Fundamental Theorem of Calculus - Proof

David Joyce has proofs of the two statements of the FTC

`http://aleph0.clarku.edu/~ma120/FTCproof.pdf`

which we will follow here.

The two statements are often called FTC-1 and FTC-2. Citing historical precedent, Joyce calls the second one the FTC and the first its inverse, or FTC^{-1}.

FTC

The FTC is what we use when we evaluate definite integrals. If F is an antiderivative of f, then:

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

We will require that f be *continuous* on $[a, b]$. Strictly speaking, this isn't necessary, but it makes the proof simpler. For a function with a finite number of discontinuities, one can just chop up the integral into its component pieces.

For the inverse statement (FTC^{-1}), we require again that f be continuous on $[a, b]$ and F be the accumulation function defined by

$$F(x) = \int_a^x f(t)\ dt$$

Then the theorem is that F is differentiable on $[a, b]$ and its derivative is f. That is

$$F'(x) = f(x) \quad \text{for } x \in [a, b]$$

This is usually written

$$\frac{d}{dx} \int_a^x f(t)\ dt = f(x)$$

We have adopted the "dummy" variable t to avoid confusion.

proof of the inverse FTC

We start with the inverse theorem. First of all, since f is continuous, it is integrable, so we know that the integral

$$F(x) = \int_a^x f(t)\ dt$$

actually exists.

We need to show that $F'(x) = f(x)$.

We go back to the definition of the derivative

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\ [\ F(x+h) - F(x)\ ]$$

$$= \lim_{h \to 0} \frac{1}{h}\ [\ \int_a^{x+h} f(t)\ dt) - \int_a^x f(t)\ dt\ ]$$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\ dt$$

We will show that this limit equals $f(x)$ We will only prove the case where $h > 0$. The other proof is similar but has minus signs in various places.

On the interval $[x, x+h]$, $f(t)$ has a minimum value m and a maximum value M (by the extreme value theorem). So

$$m \leq f(t) \leq M$$

for every $x \in [a, b]$, and when we integrate each term of the inequality we get

$$\int_x^{x+h} m \; dt \leq \int_x^{x+h} f(t) \; dt \leq \int_x^{x+h} M \; dt$$

Since m and M are constants and $\int dt = h$ between these limits:

$$hm \leq \int_x^{x+h} f(t) \; dt \leq hM$$

dividing through by h

$$m \leq \frac{1}{h} \int_x^{x+h} f(t) \; dt \leq M$$

Now, as $h \to 0$, all values of f on the interval $[x, x+h]$ approach the same value, and in particular, $m \to f(x)$ and $M \to f(x)$. Being squeezed between them

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) = f(x)$$

$\square$

proof of the FTC

Let

$$G(x) = \int_a^x F'(t) \; dt$$

Take derivatives on both sides

$$G'(x) = \frac{d}{dx} \int_a^x F'(t)\ dt$$

so

$$G'(x) = F'(x)$$

by the theorem we just proved.

Therefore $G(x)$ and $F(x)$ differ at most by a constant

$$G(x) = F(x) + C$$

for $x \in [a, b]$.

In particular, at $x = a$ we have

$$G(a) = F(a) + C$$

but

$$G(a) = \int_a^a F'(t)\ dt = 0$$

Hence

$$F(a) = -C$$

At $x = b$ we have

$$G(b) = F(b) + C$$

but $C = -F(a)$ so

$$G(b) = F(b) - F(a)$$

By the original definition of G

$$G(b) = \int_a^b F'(t)\ dt$$

Hence

$$\int_a^b F'(t)\ dt = F(b) - F(a)$$

□