Intermediate Value Theorem

Bolzano's Theorem

We prove this as a preliminary to proof of the IVT.

If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$ then there is some x in $[a, b]$ such that $f(x) = 0$.

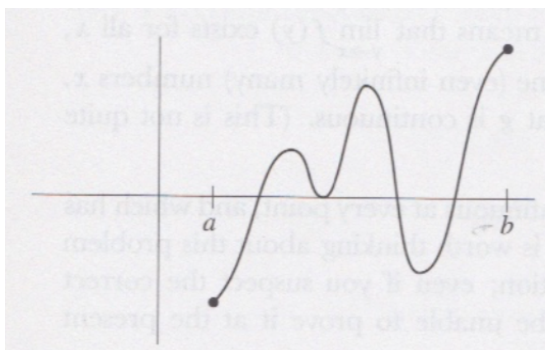

FIGURE 1

Proof

- Let $\mathbf{S}$ be the set of all $x \in [a, b]$ such that $f(x) \leq 0$.
- $\mathbf{S}$ is non-empty ($\mathbf{S} \neq \varnothing$) since $a \in \mathbf{S}$.
- Since $f(b) > 0$, $b \notin \mathbf{S}$, and since the relevant interval is $[a, b]$, b is larger than all members of $\mathbf{S}$, and so b is an upper bound of $\mathbf{S}$.

• Therefore, by **completeness**, there exists a least upper bound or supremum of $\mathbf{S}$.

• Define c to be that supremum. Since $f(x)$ is continuous, $\lim_{x \to c} f(x) = f(c)$.

One of three things is true: $f(c) > 0, f(c) < 0$ or $f(c) = 0$. We claim that $f(c) = 0$. The proof is by contradiction.

Suppose $f(c) > 0$.

• We define $\epsilon_1 = f(c)/2$. Then ϵ_1 is positive and $2\epsilon_1 = f(c) > 0$.

• By the definition of the limit, there exists δ_1, such that for all $0 < |x - c| < \delta_1$ it is true that

$$|f(x) - f(c)| < \epsilon_1$$

• Then (see the triangle inequality write-up)

$$-\epsilon_1 < f(x) - f(c) < \epsilon_1$$

$$-\epsilon_1 < f(x) - 2\epsilon_1 < \epsilon_1$$

$$\epsilon_1 < f(x) < 3\epsilon_1$$

so this implies that $f(x) > 0$ everywhere in the interval $c - \delta_1 < x < c + \delta_1$.

• It would appear that we have found a smaller upper bound for the set $\mathbf{S}$ in the interval $[c - \delta_1, c]$. But c is a least upper bound, so this is a contradiction.

We conclude that $f(c) > 0$ is impossible.

Suppose $f(c) < 0$.

• We can define $\epsilon_2 = -f(c)/2$. Then $\epsilon_2 > 0$ and $-f(c) = 2\epsilon_2$.

• By the definition of the limit, there exists δ_2, such that for all $0 < |x - c| < \delta_2$ it is true that

$$|f(x) - f(c)| < \epsilon_2$$

• Then

$$-\epsilon_2 < f(x) - f(c) < \epsilon_2$$

We have $-f(c) = 2\epsilon_2$

$$-\epsilon_2 < f(x) + 2\epsilon 2 < \epsilon_2$$

$$-3\epsilon_2 < f(x) < -\epsilon_2$$

which implies that $f(x) < 0$ everywhere in the interval $c - \delta_2 < x < c + \delta_2$.

• It would appear that we have found a value for $x < 0$ in the interval $[c, c + \delta_2]$. But c is a least upper bound for $\mathbf{S}$, there are not supposed to be any negative values of $f(x)$ to the right of c, so this is a contradiction.

We conclude that $f(c) < 0$ is impossible.

The last remaining possibility is that $f(c) = 0$.

Suppose that $f(b) < 0$ and $f(a) > 0$. Define $g(x) = -f(x)$. Note that $g(x)$ is continuous on the same interval, and repeat the argument. The conclusion does not depend on this assumption.

This completes the proof of Bolzano's Theorem.