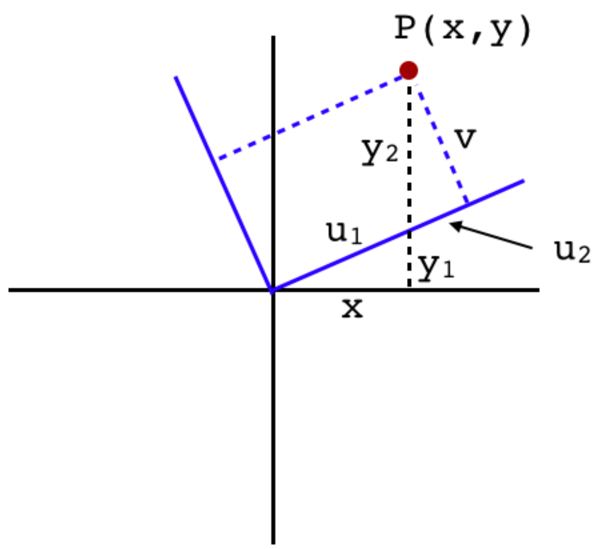
Rotation of coordinates

I wanted to work on a derivation of the rotation of coordinates formulas. I've done it before in different ways, but I always have difficulties remembering those proofs. So here we will just fool around with the algebra a bit.



The setup is the usual one. We have a point $P = (x, y)$ and we want to express the same point in terms of rotated coordinate axes. This means that we need to find the distances u and v that are the projections of the point onto the new axes. In the diagram, I express some distances in terms of two components: $y = y_1 + y_2$ and $u = u_1 + u_2$.

The angle isn't marked explicitly, but let's call it θ. We can write immediately that

$$x = u_1 \cos\theta$$

$$y_1 = u_1 \sin\theta$$

Notice also that the second triangle with labeled sides is similar to the first. So we can also write

$$u_2 = y_2 \sin\theta$$

$$v = y_2 \cos\theta$$

Now we just play with the equations. We want a function giving u

$$u = f(x, y, \theta)$$

Start with

$$u = u_1 + u_2$$

We have two equations involving u_1. Multiply the first by $\cos\theta$

$$x\cos\theta = u_1 \cos^2\theta$$

Multiply the second one by $\sin\theta$

$$y_1 \sin\theta = u_1 \sin^2\theta$$

If we add and then factor out $\cos^2\theta + \sin^2\theta = 1$ we obtain:

$$u_1 = x\cos\theta + y_1 \sin\theta$$

We have that $u_2 = y_2 \sin\theta$ so

$$u = u_1 + u_2$$

$$u = x\cos\theta + y_1 \sin\theta + y_2 \sin\theta$$

Factor out $y_1 + y_2 = y$

$$u = x\cos\theta + y\sin\theta$$

$$v = f(x, y, \theta)$$

$$v = y_2 \cos\theta$$
$$v = (y - y_1)\cos\theta$$
$$v = y\cos\theta - y_1\cos\theta$$

Go back to

$$x = u_1 \cos\theta$$
$$y_1 = u_1 \sin\theta$$
$$y_1 = \frac{x}{\cos\theta}\sin\theta$$

So

$$v = y\cos\theta - y_1\cos\theta$$
$$v = y\cos\theta - \frac{x}{\cos\theta}\sin\theta\cos\theta$$
$$v = y\cos\theta - x\sin\theta$$

We can use the same equations to get expressions for x and y:

$$x = f(u, v, \theta)$$

$$u = u_1 + u_2$$
$$u = \frac{x}{\cos\theta} + y_2\sin\theta$$
$$u = \frac{x}{\cos\theta} + \frac{v}{\cos\theta}\sin\theta$$

We multiply by $\cos\theta$:

$$u\cos\theta = x + v\sin\theta$$
$$x = u\cos\theta - v\sin\theta$$

and we have an expression for x in terms of u and v and θ.

$y = f(u, v, \theta)$

$$y = y_1 + y_2$$
$$y = u_1 \sin\theta \; + \ldots$$

Go back to the two equations involving y_2

$$u_2 = y_2 \sin\theta$$

$$v = y_2 \cos\theta$$

multiply by either $\sin\theta$ or $\cos\theta$ to obtain:

$$u_2 \sin\theta = y_2 \sin^2\theta$$

$$v \cos\theta = y_2 \cos^2\theta$$

Add

$$u_2 \sin\theta + v\cos\theta = y_2(\sin^2\theta + \cos^2\theta) = y_2$$

Hence

$$y = u_1 \sin\theta + y_2$$
$$y = u_1 \sin\theta + u_2 \sin\theta + v\cos\theta$$
$$y = u\sin\theta + v\cos\theta$$

And we have an expression for y in terms of u and v and θ.

Notice the difference rotating from xy to uv

$$u = x\cos\theta + y\sin\theta$$

$$v = -x\sin\theta + y\cos\theta$$

while rotating from uv to xy

$$x = u\cos\theta - v\sin\theta$$

$$y = u\sin\theta + v\cos\theta$$

The difference is a switch from minus to plus and plus to minus on the $\sin\theta$ term. And the reason is simple, think of the latter rotation as being through the angle $-\theta$, then

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

So in comparing rotations by angle θ in the counter-clockwise and clockwise directions, the difference is just a change in sign for the sine term.

We said the difference rotating from xy to uv

$$u = x\cos\theta + y\sin\theta$$

$$v = -x\sin\theta + y\cos\theta$$

As a check on our work, consider rotation by 90 degrees, $\theta = \pi/2$. We have $\cos\theta = 0$ and $\sin\theta = 1$, so

$$u = y$$

$$v = -x$$

which is indeed a counter-clockwise rotation, matching the direction of rotation of the x, y-axes to u, v in our diagram.