General proof

$$\int_0^{\pi/4} \sin^4 x \cos^2 x \; dx$$

Forget the bounds for now. Convert to all sine:

$$= \int \sin^4 x \; dx - \int \sin^6 x \; dx$$

To do this problem, I will derive the general formula for

$$\int \sin^n x \; dx$$

Use integration by parts:

$$u = \sin^{n-1} x$$

$$dv = \sin x \; dx$$

$$du = (n-1)\sin^{n-2} x \cos x \; dx$$

$$v = -\cos x \; dx$$

So the integral is

$$\int \sin^n x \; dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x \; dx$$

Now

$$\sin^{n-2} x \cos^2 x = \sin^{n-2} x(1-\sin^2 x) = \sin^{n-2} x - \sin^n x$$

Plug into the integral

$$\int \sin^n x \; dx = -\sin^{n-1} x \cos x + (n-1)\int (\sin^{n-2} x - \sin^n x) \; dx$$

$$n\int \sin^n x \; dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \; dx$$

$$\int \sin^n x \; dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \; dx$$

$$\int \sin^6 x \; dx = -\frac{1}{6}\sin^5 x \cos x + \frac{5}{6}\int \sin^4 x \; dx$$

$$\int \sin^4 x \; dx = -\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}\int \sin^2 x \; dx$$

$$\int \sin^2 x \; dx = -\frac{1}{2}\sin x \cos x + \frac{1}{2}x$$

back to our problem

$$\int \sin^4 x \; dx - \int \sin^6 x \; dx$$

$$= \int \sin^4 x \; dx + \frac{1}{6}\sin^5 x \cos x - \frac{5}{6}\int \sin^4 x \; dx$$

$$= \frac{1}{6}\sin^5 x \cos x + \frac{1}{6}\int \sin^4 x \; dx$$

$$= \frac{1}{6}\sin^5 x \cos x + \frac{1}{6}\;[\; -\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}\int \sin^2 x \; dx \;]$$

$$= \frac{1}{6}\sin^5 x \cos x + \frac{1}{6}\;[\; -\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}(-\frac{1}{2}\sin x \cos x + \frac{1}{2}x) \;]$$

Now we just have to multiply out

$$= \frac{1}{6}\sin^5 x \cos x - \frac{1}{24}\sin^3 x \cos x - \frac{1}{16}\sin x \cos x + \frac{1}{16}x$$

Now evaluate from $x = 0 \Rightarrow x = \pi/4$. Recall that $\sin n\pi = 0$ for integer n so at the lower bound all terms are zero. At the upper bound we have that $\sin \pi/4 = \cos \pi/4 = 1/\sqrt{2}$.

$$= \frac{1}{6}\;\frac{1}{8} - \frac{1}{24}\;\frac{1}{4} - \frac{1}{16}\;\frac{1}{2} + \frac{1}{16}\;\frac{\pi}{4}$$

$$= \frac{1}{48} - \frac{1}{96} - \frac{1}{32} + \frac{\pi}{64} = \frac{3\pi - 4}{192}$$