Acceleration and gravity

This is a simple explanation of how to treat motion near the earth's surface, where the acceleration due to gravity is a constant, g. The velocity of an object in the vertical direction is

$$v = v_0 - gt$$

That is, up is taken to be positive and down is negative. If an object starts out with velocity v_0 (in the y-direction), after time t its velocity will be given by this equation. This can be checked by taking the derivative

$$\frac{d}{dt}\,v = \frac{dv}{dt} = -g$$

The derivative of velocity with respect to time is acceleration, so this checks. Here, it is just $-g$. Similarly distance is also a function of time. That function is

$$h = h_0 + v_0 t - \frac{1}{2} g t^2$$

This expression can be checked by differentiating.

$$\frac{d}{dt}\,h = \frac{dh}{dt} = v = v_0 - gt$$

The result of taking the derivative explains why the distance equation has $-\frac{1}{2}gt^2$ while the velocity equation has just $-gt$.

Example 1. A ball is thrown so that it goes upward with a velocity of 16 m/s. If $g = 32\ ft/s^2$, what is the position of the ball at time t?

We have the distance equation

$$h = h_0 + v_0 t - \frac{1}{2} g t^2$$

We set $h_0 = 0$, $v_0 = 16$ and $g = 32$

$$h = 16t - 16t^2$$

We wish to know when $h = 0$

$$0 = 16t(1 - t)$$

$t = 0$ is a solution, which is obviously correct. The ball starts with $h = 0$ at $t = 0$. The other solution is $t = 1$. The ball returns to $h = 0$ at $t = 1$.

Notice also that

$$v = v_0 - gt = 16 - 32t$$

so when $v = 0$

$$0 = v_0 - gt = 16 - 32t$$

$$16 = 32t$$

and $t = 1/2$. The trajectory of this ball is a parabola. It reaches its vertex when the upward velocity is zero ($t = 1/2s$). It returns to the earth in a time equal to that which was needed for its ascent.

Example 2. Find t if a ball is dropped from a height = 392 feet, for $h_0 = 392$ and $v_0 = 0$ The distance equation is

$$h = h_0 + v_0 t - \frac{1}{2} g t^2$$

We have $h_0 = 392$ and $v_0 = 0$

$$0 = 392 - \frac{1}{2} g t^2$$

$$784 = 16t^2$$

$$\frac{784}{16} = 49 = t^2$$
$$t = 7$$

Example 3. A ball is thrown up in the air making an angle θ with respect to the horizontal. What value of θ will give the maximum horizontal distance?

$$x(t) = v_x t$$
$$y(t) = v_y t - \frac{1}{2} g t^2$$

$$v_x = v \cos\theta$$
$$v_y = v \sin\theta$$

We find the time t when $y = 0$ and the ball has come back down to earth. We can remove one factor of t from each term on the right (we lose a possible solution but it's the one we already know, $y = 0$ at $t = 0$).

$$y(t) = 0 = v_y t - \frac{1}{2} g t^2$$
$$0 = v_y - \frac{1}{2} g t$$
$$t = \frac{2}{g} v_y$$

Substitute for t in the equation for $x(t)$ above, converting it to $x(\theta)$

$$x(t) = v_x t = v_x \frac{2}{g} v_y$$
$$x(\theta) = v \cos\theta \; (\frac{2}{g}) \; v \sin\theta$$
$$= \frac{2v^2}{g} \sin\theta \cos\theta$$

Remembering the sum of angles formula ($\sin 2s = 2 \sin s \cos s$):

$$= \frac{v^2}{g} \sin 2\theta$$

This is a maximum (for fixed v) when $\sin 2\theta$ is a maximum (equal to 1, so $\theta = \pi/4$. Alternatively

$$\frac{dx}{d\theta} = 0 = \frac{d}{dx}(\frac{2v^2}{g})\ sin\theta\ cos\theta$$

$$0 = (\frac{2v^2}{g})\ [\ -sin^2\theta + cos^2\theta\ ]$$

Eliminate the constants in front and then we have

$$0 = -sin^2\theta + cos^2\theta$$

$$sin\theta = cos\theta$$

$$\theta = \tan^{-1} 1 = \frac{\pi}{4} = 45^\circ$$