Paraboloid

Our first example was the paraboloid

$$z = f(x, y) = 2 - x^2 - y^2$$

The vertex of this solid is at $(x = 0, y = 0, z = 2)$.

We can see where the function gets its name. Visualize the intersection with the yz-axis ($x = 0$). There, $z = 2 - y^2$, and we have a standard parabola opening down, with vertex $z = 0$.

The same thing happens at the intersection with the xz-axis (where $y = 0$ and we have $z = 2 - x^2$).

Finally, the intersection with the xy-axis is $z = 0$ so

$$x^2 + y^2 = 2$$

and this is a circle with radius $\sqrt{2}$.

If we view the surface of the paraboloid as enclosing a volume, we can calculate that volume by a double integral of the function over its "shadow" in the xy-plane.

$$V = \iint_R f(x, y)\ dy\ dx$$

We need to figure out the limits on x and y. The outer integral (in x) ranges over the whole diameter of the circle from $x = -\sqrt{2} \to \sqrt{2}$. For any fixed value of x, $y = \sqrt{2 - x^2}$, so the integral can be set up as

$$\int_{x=-\sqrt{2}}^{x=\sqrt{2}} \int_{y=-\sqrt{2-x^2}}^{y=\sqrt{2-x^2}} 2 - x^2 - y^2 \; dy \; dx$$

Over the first quadrant

$$\int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} 2 - x^2 - y^2 \; dy \; dx$$

The inner integral is

$$\int_0^{\sqrt{2-x^2}} 2 - x^2 - y^2 \; dy$$

$$= 2 - x^2 y - \frac{y^3}{3} \Big|_0^{\sqrt{2-x^2}}$$

$$= 2 - x^2\sqrt{2 - x^2} - \frac{(2 - x^2)^{3/2}}{3}$$

$$= 2 - \sqrt{2 - x^2}\; (x^2 + \frac{(2 - x^2)}{3})$$

$$= 2 - \frac{2}{3}\sqrt{2 - x^2} - \frac{2}{3}x^2 \; \sqrt{2 - x^2}$$

And this is going to awkward. So instead, we go back and use the obvious symmetry and do this in cylindrical coordinates