Introduction to Taylor Series

In this short write-up I want to introduce the Taylor Series. Rather than try to derive it, I will just write out the definition

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

where $f'(a)$ is the first derivative of $f(x)$ evaluated at a, and so on. This is a (possibly) infinite series and the nth term is

$$\frac{f^{(n)}(a)}{n!}(x-a)^n$$

This says that the Taylor series of the function $f(x)$ "near" a point $x = a$ is equal to this sum.

The beauty of Taylor Series (despite its complexity) is that it turns any differentiable function into a polynomial. Polynomials are easy to integrate and work with.

The first thing to say about Taylor Series is they give the correct answer for functions that we know. For example, suppose we have

$$f(x) = ax^2 + bx + c = 1$$

We get the derivatives and evaluate them "near" the point $x = 0$.

$$f(x) = ax^2 + bx + c = c$$

$$f'(x) = 2ax + b = b$$

$$f''(x) = 2a$$

The series is then

$$f(x) = c + b(x) + \frac{21}{2!}(x)^2 + \cdots$$

But there are no more terms. That's it. And this is just

$$f(x) = c + bx + ax^2$$

Let's take a second example. Suppose $f(x) = e^x$ and again, we evaluate "near" $x = 0$. We have

$$f(x) = e^x = 1$$
$$f'(x) = e^x = 1$$
$$f''(x) = e^x = 1$$

The series is

$$f(x) = e^x = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots$$
$$f(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$
$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Which matches what we already know about e^x. For example, it is obvious that

$$\frac{d}{dx}e^x = e^x$$

Let's try to find something new. Suppose we expand $f(x) = cos\ x$ near $x = 0$

$$f(x) = cos\ x = cos\ 0 = 1$$
$$f'(x) = -sin\ x = -sin\ 0 = 0$$
$$f''(x) = -cos\ x - cos\ 0 = -1$$
$$f'''(x) = sin\ x = sin\ 0 = 0$$
$$f''''(x) = cos\ x = cos\ 0 = 1$$

and this continues in a cycle with period 4. The series is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$
$$f(x) = cos\ x = 1 - \frac{1}{2!}(x-0)^2 + \frac{1}{4!}(x-0)^4 + \cdots$$
$$f(x) = cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Similarly, for $f(x) = sin\ x$ near $x = 0$

$$f(x) = sin\ x = 0$$

$$f'(x) = cos\ x = 1$$
$$f''(x) = -sin\ x = 0$$
$$f'''(x) = -cos\ x = -1$$
$$f''''(x) = sin\ x = 0$$

The series is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$
$$f(x) = sin\ x = x - \frac{1}{3!}(x-0)^3 + \frac{1}{5!}(x-0)^5 + \cdots$$
$$f(x) = sin\ x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

Other examples..