Problems in integrating volumes

The first example is a cone. We're working with xxx cones where the radius of the base is R and the height of the cone is H. We can orient the cone however we like, for example along the z-axis is the vertex at the origin and opening up.

At any height z, the radius of the cross-section at that height is

$$r = \frac{R}{H} z$$

and the cross-sections are circles so

$$x^2 + y^2 = r^2 = \frac{R^2}{H^2} z^2$$

The equation of the surface is then

$$z = \frac{H}{R} \sqrt{x^2 + y^2}$$

The volume is

$$V = \iiint dV = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{R^2-x^2} \int_{H/R\sqrt{x^2+y^2}}^{H} dz \ dy \ dx$$

The sensible thing is to do this integral in cylindrical coordinates, so

$$V = \int \int \int dz \ r \ dr \ d\theta$$

What are the bounds on z? Simply that $z = H/R\ r \to z = H$.

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \int_{Hr/R}^{R} dz\ r\ dr\ d\theta$$

So the middle integral is

$$\int_{r=0}^{R} (R - \frac{H}{R}\ r)\ r\ dr$$

$$= \frac{1}{2} R^3 - \frac{H}{R} \frac{1}{3} R^3$$