Parametrizing Surfaces (Paul)

When we parametrized curves, we were able to work with curves where x,y and z were functions of a single parameter, which we usually call t. For surfaces, we will usually have two variables u and v. For example, on the globe we locate our position by longitude and latitude.

As an example, consider

$$\mathbf{r}(u,v) = u\hat{i} + u\ cos\ v\hat{j} + u\ sin\ v\hat{k}$$

We have three parametric equations, one for each variable

$$x = u$$
$$y = u\ cos\ v$$
$$z = u\ sin\ v$$

So what this looks like is that we'll have a circle parallel to the yz-plane with radius u, where $u = x$. We notice that if we square all the terms

$$x^2 = u^2$$
$$y^2 + z^2 = u^2 \cos^2\ v + u^2 \sin^2\ v = u^2 = x^2$$

Going from the equations back to the surface is fairly unusual. Typically we do things the other way around.

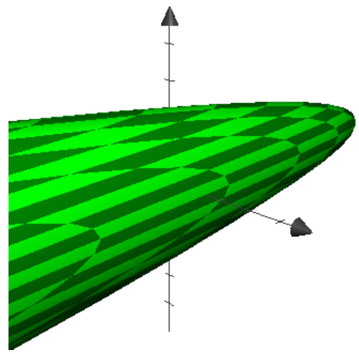
Paraboloid

Suppose we start with the elliptic paraboloid

$$x = 5y^2 + 2z^2 - 10$$

To visualize this, think about what happens when $x = 0$. Then we have an ellipse parallel to the yz-plane. At $x = -10$, $y = z = 0$; and x must be at least -10, because otherwise the squared terms would have to sum to less than zero, which is impossible with real numbers.

If we plot this



the paraboloid is symmetric around the x-axis (the cross-sections really are ellipses), and the vertex is at $x = -10$. Here, we can pick y and z to be anything (there is a point on the surface for every y, z pair). x is given by the equation above so

$$\mathbf{r}(y, z) = (5y^2 + 2z^2 - 10)\hat{i} + y\hat{j} + z\hat{k}$$

Sphere

$$x^2 + y^2 + z^2 = 16$$

This is just a sphere of radius 4. The parametric representation uses the two angles ϕ and θ.

$$x = \rho \ sin\phi \ cos\theta$$
$$y = \rho \ sin\phi \ sin\theta$$
$$z = \rho \ cos\phi$$

where ϕ is the angle with respect to the positive z-axis. So in the xy-plane

$$\phi = \pi/2, \quad cos\phi = 0, \quad z = 0, \quad x = \rho \ cos\theta, \quad y = \rho \ sin\theta$$

and we just have a circle of radius ρ. In fact, for any $z < \rho$, the angle ϕ is determined, which then makes the radius of the circle traced out by x and y smaller.

$$\mathbf{r}(\theta, \phi) = \rho \ sin\phi \ cos\theta \ \hat{i} + \rho \ sin\phi \ sin\theta \ \hat{j} + \rho \ cos\phi \ \hat{k}$$
$$0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi$$

Cylindrical coordinates

$$x^2 + y^2 = 25$$

Now we have no restrictions on z.

$$x = rcos\theta, \quad y = rsin\theta, \quad z = z$$

$$\mathbf{r}(\theta, \phi) = rcos\theta\ \hat{i} + rsin\theta\ \hat{j} + z\hat{k}$$

$$0 \leq \theta \leq 2\pi$$

Tangent plane

A crucial idea is the method to find the equation of the tangent plane to a surface at a point.

$$\mathbf{r}(u, v) = x(u, v)\ \hat{i} + y(u, v)\ \hat{j} + z(u, v)\ \hat{k}$$

The method is to find these two vectors

$$\mathbf{r_u}(u, v) = \frac{\partial x}{\partial u}(u, v)\ \hat{i} + \frac{\partial y}{\partial u}(u, v)\ \hat{j} + \frac{\partial z}{\partial u}(u, v)\ \hat{k}$$

$$\mathbf{r_v}(u, v) = \frac{\partial x}{\partial v}(u, v)\ \hat{i} + \frac{\partial y}{\partial v}(u, v)\ \hat{j} + \frac{\partial z}{\partial v}(u, v)\ \hat{k}$$

Now just form the cross product

$$\mathbf{r_u} \times \mathbf{r_v} = \mathbf{n}$$

Provided $\mathbf{n} \neq 0$, this is the normal vector to the surface.

Example

Suppose we have

$$\mathbf{r}(u, v) = u\ \hat{i} + 2v^2\ \hat{j} + (u^2 + v)\ \hat{k}$$

and we're looking at the point $(2, 2, 3)$. We find the two tangent vectors using the equation above

$$\mathbf{r_u} = \hat{i} + 2u\ \hat{k} = <1, 0, 2u>$$

$$\mathbf{r_v} = 4v\ \hat{j} + \hat{k} = <0, 4v, 1>$$

The cross product is set up like this

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 4v & 1 \end{bmatrix}$$

so

$$\mathbf{n} = -8uv\ \hat{i} - \hat{j} + 4v\ \hat{k} = < -8uv, -1, 4v >$$

So far so good, but now we need to solve for u and v. We plug the point we were given into the parametric equations

$$x = u, \quad y = 2v^2, \quad z = u^2 + v$$

$$2 = u, \quad 2 = 2v^2, \quad 3 = u^2 + v$$

From the first two equations we get

$$u = 2, v = \pm 1$$

And the third equation restricts v further

$$v = -1$$

So then the normal vector is

$$\mathbf{n} = < 16, -1, -4 >$$

and the tangent plane is just

$$16(x-2) - (y-2) - 4(z-3) = 0$$

$$16x - y - 4z = 18$$

That was fairly painless!

Surface Area

The surface area is given by

$$A = \int\int_D \|\mathbf{r_u} \times \mathbf{r_v}\|\ dA$$

Consider a sphere of radius a

$$x^2 + y^2 + z^2 = a^2$$

If we parametrize this as before

$$x = \rho\ sin\phi\ cos\theta = a\ sin\phi\ cos\theta$$

$$y = \rho\ sin\phi\ sin\theta = a\ sin\phi\ sin\theta$$

$$z = \rho\ cos\phi = a\ cos\phi$$

We can either remember that the normal vector at any point on this sphere is

$$\mathbf{n} = < x, y, z >$$

or we can write

$$\mathbf{r}(\theta, \phi) = a\ sin\phi\ cos\theta\ \hat{i} + a\ sin\phi\ sin\theta\ \hat{j} + a\ cos\phi\ \hat{k}$$

$$\mathbf{r}_\theta = -a\ sin\phi\ sin\theta\ \hat{i} + a\ sin\phi\ cos\theta\ \hat{j} + 0\ \hat{k}$$

$$\mathbf{r}_\phi = a\ cos\phi\ cos\theta\ \hat{i} + a\ cos\phi\ sin\theta\ \hat{j} - a\ sin\phi\ \hat{k}$$

And

$$\mathbf{N} = \mathbf{r}_\theta \times \mathbf{r}_\phi = a^2 \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -sin\phi\ sin\theta & sin\phi\ cos\theta & 0 \\ cos\phi\ cos\theta & cos\phi\ sin\theta & -sin\ \phi \end{bmatrix}$$

the components are (all multiplied by a^2) and then we have

$$-sin^2\phi\ cos\ \theta\ \hat{i}$$

$$-sin^2\phi\ sin\ \theta\ \hat{j}$$

$$-sin\phi\ cos\phi\ sin^2\theta - sin\phi\ cos\phi\ cos^2\theta\ \hat{k} = -sin\phi\ cos\phi\ \hat{k}$$

$$\mathbf{N} = a^2\ sin\phi\ \ < -sin\phi\ cos\theta, -sin\phi\ sin\theta, -cos\phi >$$

$$\|\mathbf{N}\| = a^2\ sin\phi\ \sqrt{sin^2\phi\ cos^2\theta + sin^2\phi\ sin^2\theta + cos^2\phi}$$

$$\|\mathbf{N}\| = a^2\ sin\phi\ \sqrt{sin^2\phi + cos^2\phi} = a^2\ sin\phi$$

$$\mathbf{n} = \mathbf{N}/\|\mathbf{N}\| = < -sin\phi\ cos\theta, -sin\phi\ sin\theta, -cos\phi >$$

This is the unit normal at the point

$$(a\ sin\phi\ cos\theta, a\ sin\phi\ sin\theta, a\ cos\phi) = (x, y, z)$$

The sign of the normal vector is negative. By convention $\mathbf{n}$ points into the sphere.

How do we integrate this to get the surface area? Aren't we mixing up x, y, z with θ, ϕ? The answer is that in this formula

$$A = \int\int_D \|\mathbf{r}_\theta \times \mathbf{r}_\phi\|\ dA = \int\int_D \|\mathbf{N}\|\ dA = \int\int_D a^2\ sin\phi\ dA$$

$dA = \ d\theta\ d\phi$. It is not really the area element, because we need to correct by using the Jacobian, which is what we really just computed above.. The area element is actually $a^2\ sin\phi\ d\theta\ d\phi$.

$$A = \int\int_D a^2\ sin\phi\ d\theta\ d\phi$$

$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi$$

$$A = \int_0^{\pi}\int_0^{2\pi} a^2\ sin\phi\ d\theta\ d\phi$$

The inner integral is

$$\int_0^{2\pi} a^2\ sin\phi\ d\theta = 2\pi a^2\ sin\phi$$

and the outer integral is then

$$\int_0^{\pi} 2\pi a^2\ sin\phi\ d\phi = 2\pi a^2(-1)(cos\phi)\Big|_0^{\pi} = 2\pi a^2(-1)(-2) = 4\pi a^2$$

which is correct.

Formulas (Marsden)

We don't actually have to set up the cross product as the determinant, because we are going to get the norm (magnitude) of it right away.

$$\|\mathbf{r}_\theta \times \mathbf{r}_\phi\| = \sqrt{(\frac{\partial(x,y)}{\partial(u,v)})^2 + (\frac{\partial(y,z)}{\partial(u,v)})^2 + (\frac{\partial(x,z)}{\partial(u,v)})^2}$$

where

$$\frac{\partial(y,z)}{\partial(u,v)}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

and so on. This is our old friend, the Jacobian. Thus, the area formula becomes

$$A = \int\int_R \sqrt{(\frac{\partial(x,y)}{\partial(u,v)})^2 + (\frac{\partial(y,z)}{\partial(u,v)})^2 + (\frac{\partial(x,z)}{\partial(u,v)})^2}\ du\ dv$$

One more

I have one more example from Paul. He says: find the surface area of the portion of the sphere of radius 4 that lies inside the cylinder $x^2 + y^2 = 12$ and above the xy-plane.

This shape is called a "spherical cap." The problem should be pretty easy because we just worked out the area element for the sphere above. We have

$$A = \int \int_D a^2 \ sin\phi \ d\theta \ d\phi$$

The trick is to find out where the cylinder and the sphere intersect. We have

$$x^2 + y^2 + z^2 = 16$$

$$x^2 + y^2 = 12$$

These are both true when $z^2 = 4, z = \pm\ 2$. What we need to do is to find the angle ϕ that this corresponds to. Since we know that

$$z = a \ cos\phi, \quad a = 4, \quad z = 2$$

$$cos\phi = \frac{1}{2}, \quad \phi = \frac{\pi}{3}$$

So we have that the range of ϕ is

$$0 \le \phi \le \frac{\pi}{3}$$

Remember that $\phi = 0$ at the top of the sphere, and that's the part we want, above the circle formed by the intersection of the cylinder and the sphere.

$$A = \int_0^{\pi/3} \int_0^{2\pi} a^2 \ sin\phi \ d\theta \ d\phi$$

The inner integral is

$$\int_0^{2\pi} a^2 \ sin\phi \ d\theta = 2\pi a^2 \ sin\phi$$

and the outer is

$$\int_0^{\pi/3} 2\pi a^2 \ sin\phi \ d\phi$$

$$2\pi a^2 \ (-cos\phi) \ \Big|_0^{\pi/3} = 2\pi a^2(-\frac{1}{2} + 1) = \pi a^2 = 16\pi$$