Sums of integers

In what follows we are looking for simple formulas to calculate the value of expressions of the form

$$\sum_{k=1}^{n} f(k)$$

For simplicity we suppress the indices and write

$$\sum f(k)$$

sum of integers

The first example is a formula for the sum of the integers $1 \ldots n = \sum k$.

There's a trick! Write:

$$\sum (k+1)^2 = \sum \; [\; k^2 + 2k + 1 \;]$$

A fundamental rule for sums is that they can be broken up (by associativity):

$$\sum (k+1)^2 = \sum k^2 + \sum 2k + \sum 1$$

telescoping sum

Now, move the first term on the right-hand side over to the left and consider:

$$\sum (k+1)^2 - \sum k^2$$

Write out the individual terms of each sum

$$= [\ 2^2 + 3^2 + \cdots + n^2 + (n+1)^2\ ]$$
$$-\ [\ 1^2 + 2^2 + 3^2 + \cdots + n^2\ ]$$

Every term in the first set of brackets has a counterpart in the second one to cancel it, except $(n+1)^2$. Similarly, every term in the second part except 1^2 cancels.

So the end result is

$$= (n+1)^2 - 1$$

And after expanding the square we can cancel the 1

$$= n^2 + 2n$$

This is called a *telescoping* sum.

algebra

We now have

$$n^2 + 2n = \sum 2k + \sum 1$$

Factor out the 2

$$n^2 + 2n = 2\sum k + \sum 1$$

$\sum k$ is what we seek. The other term is

$$\sum 1 = 1 + 1 + \cdots + 1$$

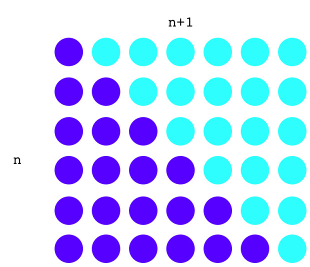
In the sum on the right there are n terms, so $\sum 1 = n$

$$n^2 + 2n = 2\sum k + n$$

$$\frac{n^2 + n}{2} = \sum k$$

$$\sum k = \frac{n(n+1)}{2}$$

A familiar result.



sum of integers, squared

Write:

$$\sum (k+1)^3 = \sum \; [\; k^3 + 3k^2 + 3k + 1 \;]$$

Move the first term on the right to the left-hand side, and we have another telescoping sum, which after the subtraction, gives us

$$(n+1)^3 - 1 = \sum 3k^2 + \sum 3k + \sum 1$$

Factor out the 3

$$(n+1)^3 - 1 = 3\sum k^2 + 3\sum k + \sum 1$$

$\sum k^2$ is what we seek.

$\sum k$ is what we obtained in the previous section and $\sum 1 = n$, as before:

$$(n+1)^3 - 1 = 3\sum k^2 + \frac{3n(n+1)}{2} + n$$

algebra

Expand the left-hand side and cancel the ± 1

$$n^3 + 3n^2 + 3n = 3\sum k^2 + \frac{3n(n+1)}{2} + n$$

Multiply by 2

$$2n^3 + 6n^2 + 6n = 6\sum k^2 + 3n(n+1) + 2n$$

$$2n^3 + 3n^2 + n = 6\sum k^2$$

$$n(2n^2 + 3n + 1) = 6\sum k^2$$

$$n(n+1)(2n+1) = 6\sum k^2$$

$$\sum k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2} \cdot \frac{(2n+1)}{3}$$

sum of integers, cubed

Write:

$$\sum (k+1)^4 = \sum \; [\; k^4 + 4k^3 + 6k^2 + 4k + 1 \;]$$

$$\sum (k+1)^4 = \sum k^4 + \sum 4k^3 + \sum 6k^2 + \sum 4k + \sum 1 \;]$$

Move the first term on the right to the left-hand side, and we have another telescoping sum, which after the subtraction, gives us

$$(n+1)^4 - 1 = 4\sum k^3 + 6\sum k^2 + 4\sum k + \sum 1$$

First, expand the left-hand side and cancel the 1's:

$$n^4 + 4n^3 + 6n^2 + 4n = 4\sum k^3 + 6\sum k^2 + 4\sum k + \sum 1$$

$\sum k^3$ is what we seek.

We have previously derived expressions for the other sums, so substitute them

$$n^4 + 4n^3 + 6n^2 + 4n = 4\sum k^3 + 6\;[\frac{n(n+1)(2n+1)}{6}\;] \;+4\;[\;\frac{n(n+1)}{2}\;] \;+n$$

Reduce the fractions

$$n^4 + 4n^3 + 6n^2 + 4n = 4\sum k^3 + n(n+1)(2n+1) + 2n(n+1) + n$$

Cancel the solitary n

$$n^4 + 4n^3 + 6n^2 + 3n = 4\sum k^3 + n(n+1)(2n+1) + 2n(n+1)$$

At this point we could multiply out the right-hand side and cancel some terms, but notice that we can factor the left-hand side, finding just an n but also an $(n+1)$:

$$n(n^3 + 4n^2 + 6n + 3) = 4\sum k^3 + n(n+1)(2n+1) + 2n(n+1)$$

$$n(n+1)(n^2 + 3n + 3) = 4\sum k^3 + n(n+1)(2n+1) + 2n(n+1)$$

So then, gathering terms

$$4\sum k^3 = n(n+1)\;[\;n^2 + 3n + 3 - 2n - 1 - 2\;]$$

$$= n(n+1)\;[\;n^2 + n\;]$$

$$\sum k^3 = \frac{n(n+1)}{2}\;\frac{n(n+1)}{2}$$

induction

We check the first formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

by using induction to prove it.

The base case is $1 = 1(1+1)/2 = 1$. Good. Then we must show that

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1)$$

$$= \frac{(n)(n+1)}{2} + (n+1)$$

$$= \frac{(n)(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

which is exactly what we get by substituting $(n+1)$ for n in the formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$$

sum of integers, to the fourth power

One last time. The first expression looks pretty forbidding, a quintic:

$$\sum (k+1)^5 = \sum \ [\ k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1\ ]$$

Similar to what we did before,

$$\sum (k+1)^5 - \sum k^5 = (n+1)^5 - 1$$

On the right-hand side we have $5 \sum k^4$ which what we seek, and then

$$10 \sum k^3 + 10 \sum k^2 + 5 \sum k + \sum 1$$

$$= 10 \; [\; \frac{n(n+1)}{2} \; \frac{n(n+1)}{2} \;] \; + 10 \; [\; \frac{n(n+1)}{2} \cdot \frac{(2n+1)}{3} \;] + 5 \; [\; \frac{n(n+1)}{2} \;] \; + n$$

That looks like a mess, but let's move the trailing n to the left-hand side and then see if we can find a factor of $n(n+1)$ on the left, just like before:

$$(n+1)^5 - 1 - n$$

$$= n^5 + 5n^4 + 10n^3 + 10n^2 + 4n$$

Obviously, we have a factor of n

$$= n(n^4 + 5n^3 + 10n^2 + 10n + 4)$$

And just on faith, I know we can find a factor of $(n+1)$ as well:

$$= n(n+1)(n^3 + 4n^2 + 6n + 4)$$

You'll see it checks if you multiply out.

So now we write out the whole thing

$$5 \sum k^4 = n(n+1) \; [\; (n^3 + 4n^2 + 6n + 4) - \frac{10}{4} \; n(n+1) - \frac{10}{6} \; (2n+1) - \frac{5}{2} \;]$$

Multiply both sides by 6 and reduce the fractions

$$30 \sum k^4 = n(n+1) \; [\; (6n^3 + 24n^2 + 36n + 24) - 15n(n+1) - 10(2n+1) - 15 \;]$$

Let's try to work with what's in the brackets on the right

$$6n^3 + 24n^2 + 36n + 24 - 15n^2 - 15n - 20n - 10 - 15$$

$$6n^3 + 9n^2 + 11n - 1$$

I don't think that can be factored with integers.

$$30 \sum k^4 = n(n+1) \; [\; 6n^3 + 9n^2 + 11n - 1 \;]$$