Cauchy integral theorem

Cauchy' First Integral Theorem

Cauchy 1 is a theorem that says the integral of an analytic function over a closed path (over a region without a singularity), is equal to zero.

$$\oint_C f(z)\ dz = 0$$

We proved this in the last part, so ssume that the theorem is correct.

We will integrate the function $f(z) = z$ over a rectangle $(R = [0, a] \times [b, 0]$. Write

$$z = x + iy$$

$$dz = dx + idy$$

$$f(x, y) = u(x, y) + iv(x, y)$$

Our integral is

$$\int z\ dz = \int (u + iv)\ (dx + idy)$$

$$= \int u\ dx - \int v\ dy + i \int v\ dx + i \int u\ dy$$

Since the whole thing is equal to zero over our closed path, both parts are equal to zero:

$$\int u\ dx - \int v\ dy = 0$$

$$\int v\ dx + \int u\ dy$$

Does this look familiar??

application of Cauchy 1

The function we'll be working with is the one we introduced before:

$$u(x, y) = e^{-x^2} e^{y^2} \cos 2xy$$

$$v(x, y) = e^{-x^2} e^{y^2} (-\sin 2xy)$$

Everything will simplify pretty quickly. Divide the path into its four parts and compute each separately: Over $C1$, $y = 0$ and $dy = 0$ so we have:

$$\int_{C1} = \int u\ dx = \int_0^a e^{-x^2} e^0 \cos 0\ dx = \int_0^a e^{-x^2}\ dx$$

C2 ($x = a$, $dx = 0$):

$$\int_{C2} = -\int_0^b e^{-a^2} e^{y^2} (-\sin 2ay)\ dy$$

C3 ($y = a$, $dy = 0$):

$$\int_{C3} = \int_a^0 e^{-x^2} e^{b^2} (\cos 2bx)\ dx$$

C4 ($x = 0$, $dx = 0$):

$$\int_{C4} = \int_b^0 e^{y^2} (-\sin 0)\ dy = 0$$

So all together:

$$\int_0^a e^{-x^2}\ dx - \int_0^b e^{-a^2} e^{y^2} (-\sin 2ay)\ dy + \int_a^0 e^{-x^2} e^{b^2} \cos 2bx\ dx = 0$$

$$\int_0^a e^{-x^2}\ dx = e^{-a^2}\int_0^b e^{y^2}(-\sin 2ay)\ dy + e^{b^2}\int_0^a e^{-x^2}\cos 2bx\ dx$$

Let $a \to \infty$. Then

$$e^{-a^2} \to 0$$

so the first term on the right side goes to zero and we have:

$$\int_0^\infty e^{-x^2}\ dx = e^{b^2}\int_0^\infty e^{-x^2}\cos 2bx\ dx$$

But we know the value of the left-hand side, it is

$$\int_0^\infty e^{-x^2}\ dx = \frac{\sqrt{\pi}}{2}$$

so

$$\int_0^\infty e^{-x^2}\cos 2bx\ dx = \frac{\sqrt{\pi}}{2}\ e^{-b^2}$$

The Gaussian that we knew, is a special case of this general form.