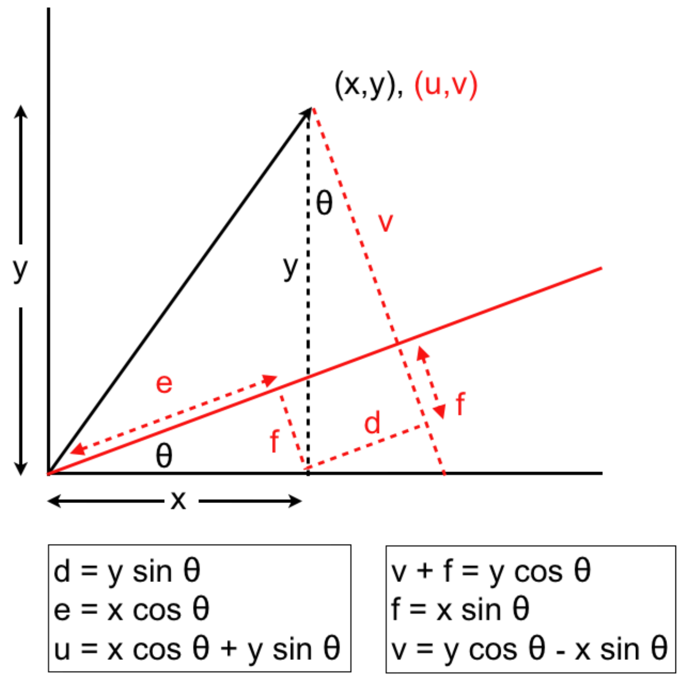
Rotation and reflection matrices for two dimensions

rotation

We begin by looking at the geometry of rotation of a point counter-clockwise.



Looking at the diagram, we obtain equations for the new coordinates u,v in terms of the old coordinates x,y. Rewriting this result using vector notation

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

If we write the multiplication explicitly we have

$$u = x\cos\theta + y\sin\theta$$

$$v = -x\sin\theta + y\cos\theta$$

The matrix for clockwise rotation can also be obtained by a geometric construction, but since we already have the equations just mutliply u and v by sine and cosine as follows

$$u\sin\theta = x\sin\theta\cos\theta + y\sin^2\theta$$

$$v\cos\theta = -x\sin\theta\cos\theta + y\cos^2\theta$$

$$y = u\sin\theta + v\cos\theta$$

By a similar process we obtain

$$x = u\cos\theta - v\sin\theta$$

Since x, y, u, and v are just "dummy" variables (we could use any letters), switch back to having x, y as the first vector and in vector notation we have (for clockwise rotation)

$$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}u \\ v\end{bmatrix}$$

To check this, set the angle to 90°, then the matrix for clockwise rotation is

$$\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}$$

This matrix multiplied by x, y produces -y, x, which is correct. We can check this a little further by computing the matrices for $\pi/3$, $\pi/4$ and $\pi/6$. They are

$$R_{\pi/3} = \begin{bmatrix}1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2\end{bmatrix}, \quad R_{\pi/4} = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}, \quad R_{\pi/6} = \begin{bmatrix}\sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2\end{bmatrix}$$

Notice that

$$\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} \\ 1/\sqrt{2}\end{bmatrix}$$

Which is a 45° ccw rotation. And that 30° (right matrix) followed by 60° is indeed 90°.

$$\begin{bmatrix}1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2\end{bmatrix} \times \begin{bmatrix}\sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2\end{bmatrix} = \begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}$$