Orthogonality of sine and cosine: summary

This is a summary of the results for integrating products of sine and cosine in preparation for Fourier series. In the derivation I did (elsewhere) I followed Paul in integrating $\int \cos n\pi x/L \cos m\pi x/L$ over $x = -L \to L$, etc.

Other people do it more simply over the interval $x = -\pi \to \pi$, and obtain his results by a change of variable. So here I will use the easier form. See Wolfram for details about the change of variable. As they say, for periodic $f(x)$ with period $2L$, any interval $[x_0, x_0 + 2L]$ will work.

cosine times cosine

$$\int_{-\pi}^{\pi} \cos mx \cos nx \; dx = \pi \delta_{mn}$$

δ_{mn} is the Kronecker delta:

$$\delta_{mn} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

I got this from Wolfram but actually it's not quite right. If $n = m = 0$ then the result should be 2π.

sine times sine

$$\int_{-\pi}^{\pi} \sin mx \sin nx \ dx = \pi \delta_{mn}$$

sine times cosine

$$\int_{-\pi}^{\pi} \sin mx \cos nx \ dx = \frac{\sin^2}{2} \Big|_{-\pi}^{\pi} = 0$$

This is easy to prove. For integer $k \in 0, \pm 1, \pm 2 \ldots$, the sine of $k\pi$ is zero. Two other important results:

$$\int_{-\pi}^{\pi} \sin nx \ dx = 0$$

$$\int_{-\pi}^{\pi} \cos nx \ dx = 0$$

cofactors

Let's write the formulas for the cofactors and then we'll look at how we got there. The series for $f(x)$ is

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \ dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \ dx$$

evaluation

To evaluate the cofactor a_1, multiply both sides by dx and integrate term by term. Every term like $\cos nx\ dx$ is really something else in disguise, namely

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos nx\ \ \cos mx\ dx$$

with $m = 0!$. And as we saw, that integral is just 0. We are left with

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{2}\ a_0\ \frac{1}{2\pi}\int_{-\pi}^{\pi} dx = \frac{1}{2}\ a_0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$$

All other terms like a_1 are found by multiplying the series by $\cos x$. The only survivor is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos x\ dx = a_1\ \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 x dx = \frac{1}{2}\ a_1$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\ dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$$