Imaginary roots

One can tell at least roughly where the roots of a polynomial are from the graph of the equation — they are simply the points where the graph crosses the x-axis.

We can find them algebraically.

quadratic

The general form of the quadratic is

$$y = f(x) = ax^2 + bx + c$$

The roots are those values of x that give $f(x) = 0$

$$0 = ax^2 + bx + c$$

$$-\frac{c}{a} = x^2 + \frac{b}{a}x$$

The quadratic formula (which gives the roots) is obtained by completing the square.

We guess the correct amount to add to both sides

$$(\frac{b}{2a})^2 - \frac{c}{a} = x^2 + \frac{b}{a}x + (\frac{b}{2a})^2$$

This helps because the right-hand side is now a perfect square

$$(\frac{b}{2a})^2 - \frac{c}{a} = (x + \frac{b}{2a})^2$$

Multiply the c/a term on top and bottom by $4a$:

$$(\frac{b}{2a})^2 - \frac{4ac}{(2a)^2} = (x + \frac{b}{2a})^2$$

Rearrange

$$(x + \frac{b}{2a})^2 = (\frac{b}{2a})^2 - \frac{4ac}{(2a)^2}$$

Put the right-hand side over a common denominator

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{(2a)^2}$$

Take the square root

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The first term of this looks familiar. Going back to

$$y = f(x) = ax^2 + bx + c$$

Take the derivative and set it equal to zero

$$y' = 0 = 2ax + b$$

$$x = \frac{-b}{2a}$$

This is the value of x at the minimum (or maximum) value for the graph of $f(x)$.

Plugging that into the standard equation:

$$y = a(\frac{-b}{2a})^2 + b(\frac{-b}{2a}) + c$$

$$= \frac{b^2/2 - b^2}{2a} + c = \frac{-b^2}{4a} + c$$

discriminant

The quantity under the square root is called the discriminant

$$D = b^2 - 4ac$$

If $D > 0$, there are two roots, both real.

For example, a quadratic equation with $a > 0$ has a graph that opens up. It has two real roots if the vertex is below the x-axis. The condition for this is $D > 0$.

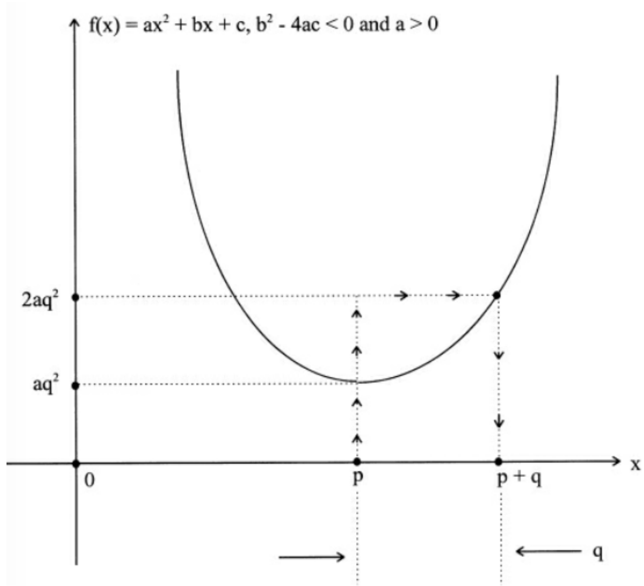
If the graph just touches the x-axis, there is a repeated root. This happens when $D = 0$, and we can see from the quadratic formula above that the root is also the minimum.

There are no real roots if $D < 0$, because of the negative square root. Graphically, this happens when the curve is shifted up (by making c larger) so that the value of the function at the vertex is positive.

imaginary roots

Nahin tells us that we can determine something about complex roots from the graph as well. I'd never heard of that before.

Consider the plot of a general quadratic with $a > 0$ and negative discriminant ($b^2 - 4ac < 0$).



The roots are complex conjugates of the form $p \pm iq$, and can be obtained from the quadratic equation.

Write $f(x)$ in *factored* form:

$$f(x) = a\ [x - (p + iq)\ ]\ [\ x - (p - iq)\ ]$$
$$= a\ [\ (x - p - iq)(x - p + iq)\ ]$$

Multiplying out

$$= a\ [\ x^2 - px - iqx - px + p^2 + ipq + iqx - ipq + q^2\ ]$$
$$= a\ [\ x^2 - 2px + p^2 + q^2\ ]$$
$$= a\ [\ (x - p)^2 + q^2\ ]$$

This expression is entirely real.

The value of $f(x)$ is clearly a minimum when $x = p$.

p is also the real part of the complex roots. At the minimum $x = p$ and the y-value is equal to aq^2.

When $x = p + q,\ f(x) = 2aq^2$.

Geometrically, one can think about measuring the displacement from the x-axis at the minimum.

Then, find the point where the displacement is twice that, and find the corresponding change in x from the vertex.

All it takes is a ruler and pencil.

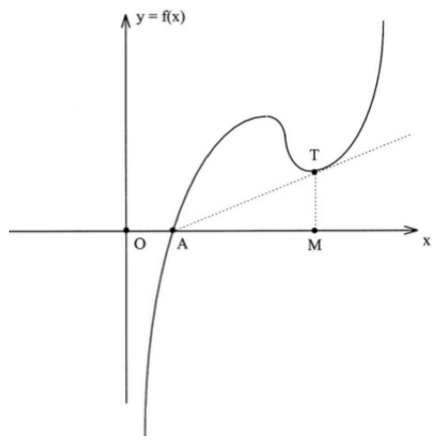
cubic

Suppose we have a cubic with only one real root, $x = k$, and a pair of conjugate roots $x = p \pm iq$. The factored form of the cubic is

$$f(x) = (x - k)(x - p + iq)(x - p - iq)$$

We already did this exact multiplication, above

$$= (x - k)(x^2 - 2xp + p^2 + q^2)$$



The equation of a line going through the real root at $(k, 0)$ is

$$\frac{\Delta y}{\Delta x} = \frac{y}{x - k} = \lambda$$

where λ is the slope. So

$$y = \lambda(x - k)$$

We are interested in one particular line, one with slope such that it just touches the curve to be a tangent, at T above.

At the point of tangency we have the displacement of x from the root as $x = \hat{x}$ and write

$$\lambda(\hat{x} - k) = (\hat{x} - k)(\hat{x}^2 - 2p\hat{x} + p^2 + q^2)$$

Since $(\hat{x} - k) \neq 0$, we can divide to give:

$$\lambda = \hat{x}^2 - 2p\hat{x} + p^2 + q^2$$

$$\hat{x}^2 - 2p\hat{x} + p^2 + q^2 - \lambda = 0$$

This is a quadratic in $\hat{x}$. Moreover, since the line just touches the curve, we are at the point where the discriminant is equal to zero. That is

$$4p^2 - 4(p^2 + q^2 - \lambda) = 0$$

That is

$$-4(q^2 - \lambda) = 0$$

$$\lambda = q^2$$

The tangent line has slope $\lambda = q^2$.

The value of $\hat{x}$ can then be computed from

$$\hat{x}^2 - 2p\hat{x} + p^2 + q^2 - \lambda = 0$$

Since $\lambda = q^2$:

$$\hat{x}^2 - 2p\hat{x} + p^2 = 0$$

$$(\hat{x} - p)^2 = 0$$

$$\hat{x} = p$$

Geometrically, use a straight-edge to construct the line and find the tangent point. q is the square root of the slope of this line. This is the quantity TM/AM.

The x displacement of T from the real root is p. This is the quantity AM.