Examples for Green's theorem (work)

State the theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{r} = \iint_R \nabla \times \mathbf{F} \; dA$$

$$\int_C M \; dx + N \; dy = \iint_R (N_x - M_y) \; dx \; dy$$

To start with, if $\mathbf{F}$ is the gradient of some function, we call such a function the potential, and the integral of the work over a closed path is just zero.

$$\mathbf{F} = < y, x >$$

$$f = xy$$

Suppose we take the line integral of $\mathbf{F} \cdot \mathbf{r}$ over the unit square $(0, 0)$ to $(0, 1)$, etc.

$$\oint_C y \; dx + x \; dy =$$

I get $0 + 1 - 1 + 0 = 0$.

A sign change can make all the difference.

$$\mathbf{F} = < -y, x >$$

$$N_x - M_y = 1 - -1 = 2 \neq 0$$

A common field with zero curl in 3D is

$$\mathbf{F} = < yz, xz, xy >$$

$$\nabla \times \mathbf{F} = < R_y - Q_z, P_z - R_x, Q_x - P_y >$$

$$= x - x, y - y, z - z > = \mathbf{0}$$

Auroux

Suppose

$$\mathbf{F} = < ye^{-x}, \frac{1}{2}x^2 - e^{-x} >$$

And suppose further that the region is the unit disk centered at $(2, 0)$ The line integral does not look like fun, and the region is no help, but

$$N_x - M_y = x + e^{-x} - e^{-x} = x$$

So the integral of the curl is just

$$\iint_R x \ dA$$

which is $\overline{x}$ on this disk, which is just equal to 2 by symmetry.

Paul

Given

$$\mathbf{F} = < xy, x^2y^3 >$$

The curl is

$$N_x - M_y = 2xy^3 - x$$

If the region is the triangle $(0, 0) \to (1, 0) \to (1, 2) \to (0, 0)$ then

$$\int_0^1 \int_0^{2x} 2xy^3 - x \ dy \ dx$$

inner

$$= \frac{1}{2}xy^4 - xy \ \Big|_0^{2x} = 8x^5 - 2x^2$$

outer

$$\int_0^1 8x^5 - 2x^2 \ dx = \frac{8}{6}x^6 - \frac{2}{3}x^3\Big|_0^1 = \frac{8}{6} - \frac{2}{3} = \frac{2}{3}$$

Try the line integral to check it.

ellipse

Of course, my favorite example is the area of the ellipse. Suppose $N_x - M_y = 1$. Then the curl integral is the area of the region. If the components of $\mathbf{F}$ are $N = x/2$ and $M = -y/2$, this condition holds. Parametrize the ellipse.

$$x = a\cos\theta$$

$$y = b\sin\theta$$

So, for the left hand side we have

$$\int_C M\ dx + N\ dy = \int_C -\frac{1}{2}y\ dx + \frac{1}{2}x\ dy$$

$$= \int_0^{2\pi} (-\frac{1}{2})(b\sin\theta)\ (-a\sin\theta)\ d\theta\ \ + (\frac{1}{2})(a\cos\theta)\ (b\cos\theta)\ d\theta$$

$$= \frac{1}{2}ab \int_0^{2\pi} \sin^2\theta\ d\theta + \int_0^{2\pi} \cos^2\theta\ d\theta$$

$$= \frac{1}{2}ab \int_0^{2\pi}\ d\theta = \pi ab$$

You may wonder why we chose $\mathbf{F} = \ \langle -y/2, x/2\rangle$ since there are many other values that would work. The reason is that the integral is particularly easy. Let's try one other choice:

$$\mathbf{F} = \ \langle 0, x\rangle$$

We use the same parametrization from above. The left hand side is:

$$\int_C M\ dx + N\ dy = \int_C 0\ dx + x\ dy$$

$$= \int_0^{2\pi} a\cos\theta\ b\cos\theta\ d\theta$$

$$= ab \int_0^{2\pi} \cos^2\theta\ d\theta$$

$$= \frac{1}{2}ab\ (\theta + \sin\theta\cos\theta)|\Big|_0^{2\pi})$$

$$= \pi ab$$