Boundedness

1 f is bounded above

A continuous function on a closed bounded interval is bounded.

theorem

If f is continuous on $[a, b]$ then f is bounded above on $[a, b]$, that is, there is some number M such that $f(x) \leq M$ for all x in $[a, b]$.

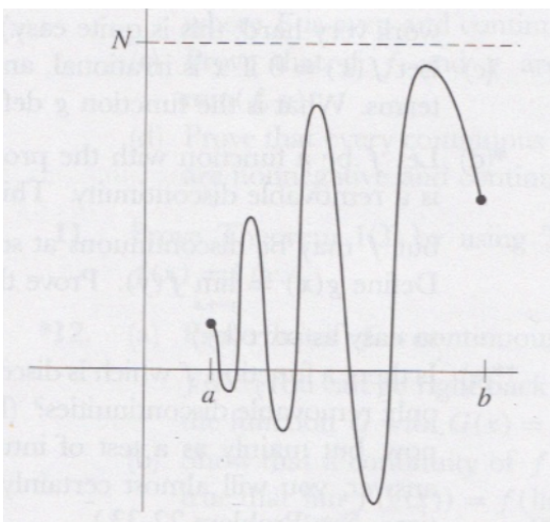

FIGURE 2

www-history.mcs.st-and.ac.uk/~john/analysis/Lectures/L21.html

restatement

A continuous function on a closed bounded interval is bounded and achieves its bounds.

2 Spivak

preliminary theorem

If f is continuous at a, then there is a $\delta > 0$ such that f is bounded above on the interval $[a - \delta, a + \delta]$.

proof

By the definition of continuity, for every $\epsilon > 0$, there is a $\delta > 0$ such that if

$$|x - a| < \delta$$

then

$$|f(x) - f(a)| < \epsilon$$

Pick any ϵ, for example $\epsilon = 1$. Then we have that if $|x - a| < \delta$

$$|f(x) - f(a)| < 1$$

$$-1 < f(x) - f(a) < 1$$

$$f(a) - 1 < f(x) < f(a) + 1$$

On the interval $(a - \delta, a + \delta)$, $f(x)$ is bounded above by $f(a) + 1$.

$\square$

Define the set A as all those values in the interval from a to the "right" for which the function $f(x)$ is bounded above:

$A = \{x : a \leq x \leq b$, and f is bounded above on $[a, x]\}$

Clearly, $A \neq \varnothing$, since $a \in A$.

A is bounded above (by b) so A has a least upper bound, α.

(Note: the set A is bounded above, but the theorem is about the function f, that is, it applies to $\{f(y) : a \leq y \leq x\}$).

We claim that $\alpha = b$.

Note first that α cannot be equal to a, since a is in A and since there is a δ such that f is bounded on $[x : a \leq x \leq a + \delta]$.

Suppose instead that $\alpha < b$ (it cannot be greater).

By the preliminary theorem, there exists a $\delta > 0$ such that f is bounded on $[a, a + \delta]$.

Since α is the *least* upper bound of A, there is some x_0 in A satisfying $\alpha - \delta < x_0 < \alpha$. This means that f is bounded on $[a, x_0]$.

But if x_1 is any number with $\alpha < x_1 < \alpha + \delta$, then f is also bounded on $[x_0, x_1]$. Then f is bounded on $[x_0, x_1]$. Therefore, f is bounded on $[a, x_1]$, so x_1 is in A, which contradicts the fact that α is an upper bound for A.

This contradiction shows that $\alpha = b$.

There is an additional small bit extending the interval to include b.

3 other people

proof

Suppose f is defined and continuous at every point of the interval $[a, b]$.

If f were not bounded above, we could find a point x_1 in $[a, b]$ with $f(x_1) > 1$, a point x_2 with $f(x_2) > 2$, and so on.

Consider the sequence (x_n). By the Bolzano-Weierstrass Theorem, it has a subsequence (x_{ij}) which converges.

Call that point $\alpha \in [a, b]$.

By our construction, $f(x_{ij})$ is unbounded.

But by the continuity of f, this sequence should converge to $f(\alpha)$, and we have a contradiction.

To show that f attains its bounds, take M to be the least upper bound of the set $X = \{\ f(x) \mid x \in [a, b]\ \}$. We need to find a point $\beta \in [a, b]$ with $f(\beta) = M$.

To do this we construct a sequence in the following way: for each $n \in \mathbb{N}$, let x_n be a point for which $|M - f(x_n)| < 1/n$.

Such a point must exist because otherwise $M - 1/n$ would be an upper bound of X.

Some subsequence of $(x_1, x_2 \dots)$ converges to β (say) and

$$(f(x_1), f(x_2), \dots) \to M$$

and so by continuity $f(\beta) = M$, as required.

proof 2

Let $\mathbf{A} = [a, b]$.

Clearly, $\mathbf{A} \neq \varnothing$ ($\mathbf{A}$ is not empty), since a is in $\mathbf{A}$, and $\mathbf{A}$ is bounded above by b. So, $\mathbf{A}$ has a least upper bound, α.

Assume that f is *not* bounded above.

What does that mean? It means that no matter what $n \in \mathbb{N}$ we choose, we can find $x_n \in \mathbf{A} : f(x_n) > n$.

Since $\mathbf{A}$ is bounded, and the x_n are in $\mathbf{A}$, the sequence (x_n) is bounded. By the Bolzano-Weierstrass theorem, (x_n) has a convergent subsequence $(x_n k)$. So it has a limit, L.

Since all the terms of $(x_n k)$ are in $\mathbf{A}$, so is L.

But $f((x_n k))$ is unbounded.

By the Sequential Criterion for Continuity (an if and only if theorem), we conclude that $\lim f((x_n k)) = f(L)$.

This is a contradiction. Therefore, f is bounded above.