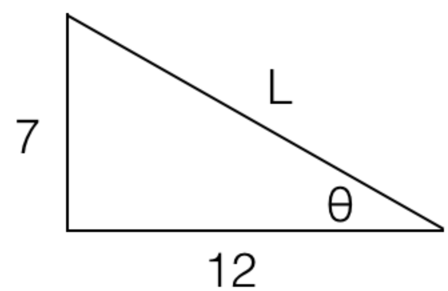
Charles's birdhouse

Suppose we are constructing a dormer on a roof. Here is the main roof.



The pitch of a roof is usually given as rise over run (e.g. 7 in 12 or 7/12). We can calculate the length of the third side (the hypotenuse) using Pythagoras

$$7^2 + 12^2 = L^2$$

$L = 14.07$. or we can say that 7/12 is the tangent of the angle θ.

$$\tan\theta = 7/12 = 0.583$$

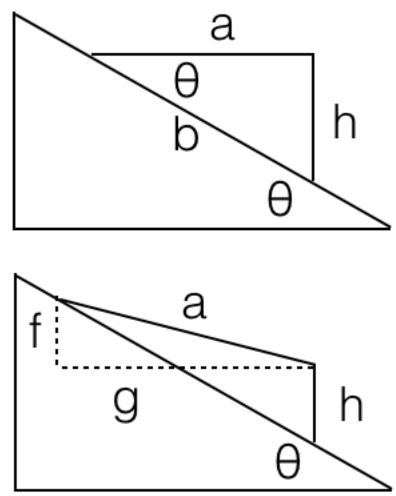
$$\theta = \tan^{-1} 0.583 = 0.528$$

That's in radians. To convert to degrees, multiply by 57.3 (degrees per radian).

$$\theta = 0.528 \times 57.3 = 30.25$$

which seems correct since if L were exactly twice 7, then the angle would be exactly 30 degrees.

dormer

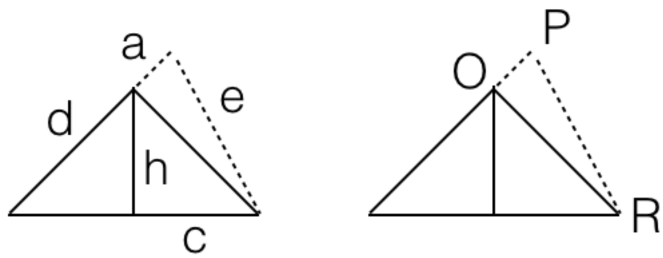


Now for the dormer. Although we might possibly have a dormer that has a horizontal ridge, the more general case does not. So we choose the dimensions of the dormer as, say

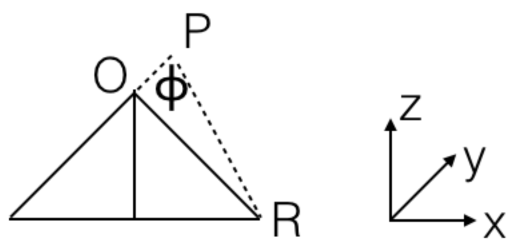
- the lowest point on the roof
- the highest point on the roof
- the height of the dormer, h

Then we know the dimensions a, g, f. If you want we could calculate them based on θ, h and the angle of the dormer, but I assume it is more natural to pick f and g or the high and low points on the roof, than to determine the exact angle of the dormer roof.

We also need a shape for the profile of the dormer. For example, we might choose $c = h$ in the figure below. This determines d, using Pythagoras.



At this point, we want to calculate the shape of the decking of the dormer roof. We are assuming that we have two of the sides a, and d, based on what we've said so far. We do not know the third side, e, nor do we know any of the angles.



One way to approach this is using vectors. If we choose the origin O as shown and x, y, z oriented in the usual way, then we can write:

$$\mathbf{OR} = \langle\ c, 0, -h\ \rangle$$

$$\mathbf{OP} = \langle\ 0, g, f\ \rangle$$

$$\mathbf{PR} = \langle\ c, -g, -(f+h)\ \rangle$$

So then the lengths are easily calculated, e.g.

$$d^2 = |\mathbf{OR}|^2 = c^2 + h^2$$

$$a^2 = |\mathbf{OP}|^2 = g^2 + f^2$$

We don't need vectors for these first two, we can get them from Pythagoras. But notice

$$e^2 = |\mathbf{PR}|^2 = c^2 + g^2 + (f+h)^2$$
$$= a^2 + d^2 + 2fh$$

It cannot be an accident that this is almost Pythagoras (the triangle we are interested in has sides a,d,e but is not a right triangle), and $f+h$ is the height of the dormer. We will see later why this works.

An alternative method to calculate the distance e squared is to find the total displacement in each of the three directions between P and R, square each of those and add them (which is just saying find the length of the vector in other language).

$$\Delta x = c$$
$$\Delta y = g$$
$$\Delta z = h + f$$

Summing the squares:

$$e^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$
$$= c^2 + g^2 + (h+f)^2$$
$$= c^2 + g^2 + h^2 + 2hf + f^2$$

but $g^2 = a^2 - f^2$

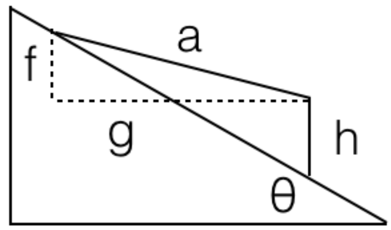
$$e^2 = c^2 + a^2 - f^2 + h^2 + 2hf + f^2$$
$$= c^2 + a^2 + h^2 + 2hf$$

and $h^2 = d^2 - c^2$

$$e^2 = c^2 + a^2 + d^2 - c^2 + 2hf$$

$$= a^2 + d^2 + 2hf$$

And that is the same is what we had above. Recall what a, f and h are:

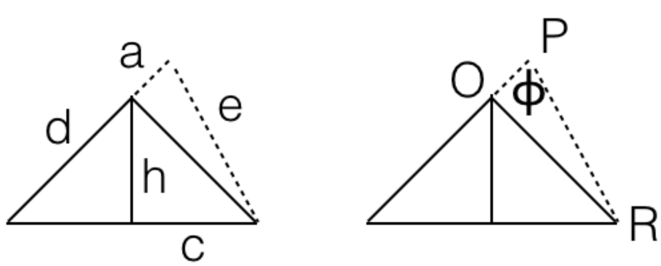


So now we have e^2 and thus e.

Practically speaking we could just stop here, knowing all three sides a,d and e. We lay out the triangle with a pair of dividers or a big meter stick, and just cut it.

But we set up the vectors in order to use the properties of the *dot product.*

We labeled the vertices O, P,R as shown. If the angle at vertex P is labeled as ϕ



The dot product of two vectors is equal to the product of the length of each vector times the cosine of the angle between them.

The dot product is

$$\mathbf{PR} \cdot \mathbf{PO} = \mathbf{PR} \cdot (-\mathbf{OP})$$

$$= \langle\ 0, g, f\ \rangle \cdot -\langle\ c, -g, -(f+h)\ \rangle$$
$$= g^2 + f(f+h)$$

and that is equal to the product of the length of each vector times the cosine of the angle between them

$$ae\cos\phi$$

So

$$\cos\phi = \frac{g^2+f^2+fh}{ae}$$

and a similar computation can be made for any angle of the triangle.

law of cosines

An alternative formulation that uses the lengths of the sides is the law of cosines, which says that if vertex ϕ is the angle opposite side d while the other two sides a and e flank angle ϕ, then

$$d^2 = a^2 + e^2 - 2ae\cos\phi$$
$$\cos\phi = \frac{a^2+e^2-d^2}{2ae}$$

(If ϕ were 90 degrees this would be Pythagoras).

Compared to what we had before:

$$\cos\phi = \frac{g^2+f^2+fh}{ae}$$

For these to be equivalent statements we must show that

$$2(g^2+f^2+fh) = a^2+e^2-d^2$$

Let's start with the right-hand side

$$a^2+e^2-d^2$$

we can substitute $e^2 = a^2 + d^2 + 2hf$ from before so we have

$$2a^2 + d^2 + 2hf - d^2$$

$$= 2a^2 + 2hf$$

and $f^2 + g^2 = a^2$ so

$$= 2(g^2 + f^2) + 2hf$$

Looks like we did it! These are equivalent computations:

$$\cos\phi = \frac{a^2 + e^2 - d^2}{2ae}$$

$$\cos\phi = \frac{g^2 + f^2 + fh}{ae} = \frac{a^2 + fh}{ae}$$