Newton to Kepler

The last part of Varberg's derivation is about K3. To prove:

$$T^2 = \frac{(2\pi)^2}{GM}\ a^3$$

where T is the period, GM is our constant from before, and a is the length of the half-major axis of the ellipse. In other words, the period of an orbit is the 3/2 power of the "radius", technically the semi-major axis of the ellipse.

Start with K2

$$2\ \frac{dA}{dt} = h$$

Integrate with respect to time over one revolution obtaining an ellipse with area πab and the period T for the time

$$2\pi ab = hT$$

$$T^2 = (\frac{2\pi ab}{h})^2$$

Now, go back to the equation for the orbit

$$r(1 + e\cos\theta) = \frac{h^2}{GM}$$

Consider one-half an orbit between $\theta = 0 \to \theta = \pi$. The length of the axis is $2a$, equal to $2r$ for this orbit, so

$$2a = \frac{h^2}{GM(1 + e\cos\pi)} + \frac{h^2}{GM(1 + e\cos 0)}$$

$$= \frac{h^2}{GM}\left(\frac{1}{1-e} + \frac{1}{1+e}\right)$$

$$= \frac{h^2}{GM} \; \frac{2}{1-e^2}$$

So

$$a = \frac{h^2}{GM} \; \frac{1}{1-e^2}$$

For an ellipse

$$\frac{b^2}{a^2} = 1 - e^2$$

so

$$a = \frac{h^2}{GM} \; \frac{a^2}{b^2}$$

$$b^2 = \frac{ah^2}{GM}$$

We had

$$T^2 = \left(\frac{2\pi ab}{h}\right)^2$$

$$= \left(\frac{2\pi a}{h}\right)^2 \; \frac{ah^2}{GM}$$

$$= \frac{(2\pi)^2}{GM} a^3$$

which is K3. GM is the gravitational constant times the mass of the sun. The term due to the angular momentum h has dropped out.

Note that we can get an estimate for GM from observation of the orbits of the planets, and that G can be determined very simply, allowing us to find M, and "weigh the sun".