# The Remainder Term and Error Estimation

If the Taylor series for a function $f(x)$ is truncated at the $n^{\text{th}}$ term, what is the difference between $f(x)$ and the value given by the $n^{\text{th}}$ Taylor polynomial? That is, what is the error involved in using the Taylor polynomial to approximate the function?

Suppose you expand f around c, and that f is $(n+1)$ -times continuously differentiable on an open interval containing c. Taylor's Theorem with the Remainder Term says that if x is another point in this interval, then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1},$$

where z is a number in the open interval between x and c.

$p_n(x; c) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$ is the $n^{\text{th}}$ degree Taylor polynomial at c. The other term on the right is called the **Lagrange remainder term**:

$$R_n(x; c) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

The appearance of z, a point between x and c, and the fact that it's being plugged into a derivative suggest that there is a connection between this result and the Mean Value Theorem. In fact, for $n = 0$ the result says

$$f(x) = f(c) + f'(z) \cdot (x-c),$$

where z is between x and c. This is the Mean Value Theorem.

On the one hand, this reflects the fact that Taylor's theorem is proved using a generalization of the Mean Value Theorem. On the other hand, this shows that you can regard a Taylor expansion as an *extension* of the Mean Value Theorem.

There is also an expression for the error which involves an integral. I won't discuss it here.

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**Example.** Compute the Remainder Term $R_3(x; 1)$ for $f(x) = \sin 2x$ .

For the *third* remainder term, I need the *fourth* derivative:

$$f'(x) = 2\cos 2x, \quad f''(x) = -4\sin 2x, \quad f'''(x) = -8\cos 2x, \quad f^{(4)}(x) = 16\sin 2x.$$

The Remainder Term is

$$R_3(x; 1) = \frac{16\sin 2z}{4!}(x-1)^4,$$

where z is a number between x and 1. □

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**Example.** Compute the Remainder Term $R_n(x;3)$ for $f(x) = e^{4x}$ .

Since I want the $n^{\text{th}}$ Remainder Term, I need to find an expression for the $(n+1)^{\text{st}}$ derivative. I'll compute derivative until I see a pattern:

$$f'(x) = 4e^{4x}, \quad f''(x) = 4^2e^{4x}, \quad f'''(x) = 4^3e^{4x}.$$

Notice that it's easier to see the pattern if you don't multiply out the power of 4.

Thus,

$$f^{(n)}(x) = 4^n e^{4x}, \quad \text{so} \quad f^{(n+1)}(x) = 4^{n+1}e^{4x}.$$

The Remainder Term is

$$R_n(x;3) = \frac{4^{n+1}e^{4z}}{(n+1)!}(x-3)^{n+1},$$

where z is a number between x and 3. □

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There are several things you might do with the Remainder Term:

1. Estimate the error in using $p_n(x;c)$ to estimate $f(x)$ on a given interval $(c-r, c+r)$ . (The interval and the degree n are fixed; you want to find the error.)

2. Find the smallest value of n for which $p_n(x;c)$ approximates $f(x)$ to within a given error ("tolerance") on a given interval $(c-r, c+r)$ . (The interval and the error are fixed; you want to find the degree.)

3. Find the largest interval $(c-r, c+r)$ on which $p_n(x;c)$ approximates $f(x)$ to within a given error ("tolerance"). (The degree and the error are fixed; you want to find the interval.)

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**Example.** The Maclaurin series for $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

What is the largest error which might result from using the first three terms of the series to approximate $\ln(1+x)$ , if $0 \le x \le 1$ ?

The remainder term is

$$R_n(x;0) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1},$$

where $0 < z < x$ . I want to estimate the maximum size of $|R_3(x;0)|$ . I take absolute values, because I don't care whether the error is positive or negative, only how large it is.

$f(x) = \ln(1+x)$ , and you can check by taking derivatives that $f^{(4)}(x) = \dfrac{-6}{(1+x)^4}$ . Thus, $f^{(4)}(z) = \dfrac{-6}{(1+z)^4}$ . So

$$|R_3(x;0)| = \left| \frac{\frac{-6}{(1+z)^4}}{4!}(x-0)^4 \right| = \frac{1}{4}\frac{1}{(1+z)^4}|x|^4.$$

Since I want the largest possible error, I want to see how large the terms $\dfrac{1}{(1+z)^4}$ and $|x|^4$ could be.

Remember that z is between 0 and x, and $0 \le x \le 1$ . So

$$0 < z < x \le 1.$$

First, $0 \le x \le 1$ means that

$$|x|^4 \le 1^4 = 1.$$

How large can $\dfrac{1}{(1+z)^4}$ be, given that $0 < z < 1$ ? As z goes from 0 to 1, $\dfrac{1}{(1+z)^4}$ *decreases*, so it is largest if $z = 0$ . This means that

$$\frac{1}{(1+z)^4} \le 1.$$

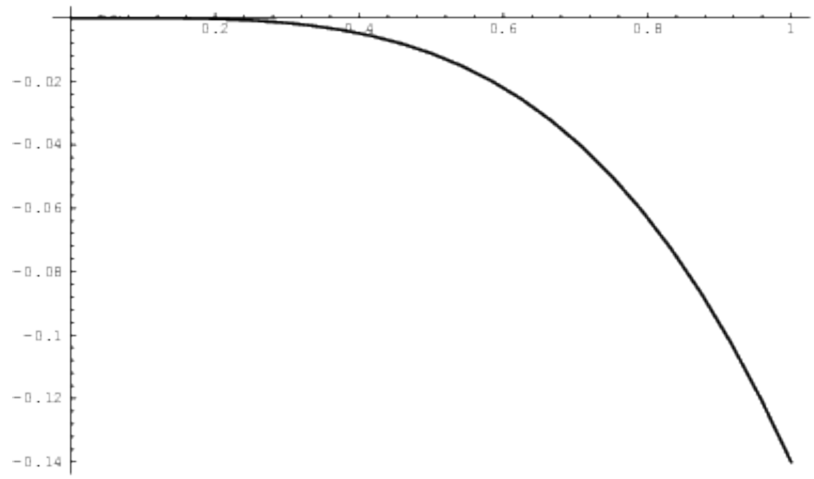
In general, to estimate the z-term you'd have to find the absolute max on the interval for z. If you know that the z-term is either increasing or decreasing, you can check its value at the interval endpoints, and take the largest.

Using the estimates for $\dfrac{1}{(1+z)^4}$ and $|x|^4$ , I have

$$|R_3(x;0)| \le \frac{1}{4}\cdot 1 \cdot 1 = \frac{1}{4}.$$

The error is no greater than $\dfrac{1}{4}$ .

I can check this by plotting the difference between the $3^{\mathrm{rd}}$ degree Taylor polynomial and $\ln(1+x)$ .




From the picture, it looks as though the maximum error is around 0.15 (in absolute value). The estimated error was pretty conservative. □

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**Example.** (a) Compute $R_3(x;0)$ for $f(x) = \dfrac{1}{2+x}$ , and express $f(x)$ using $p_3(x)$ and the remainder term.

Since I want $R_3(x;0)$ , I need the fourth derivative:

$$f'(x) = \frac{-1}{(2+x)^2}, \quad f''(x) = \frac{2}{(2+x)^3}, \quad f'''(x) = \frac{-6}{(x+x)^4}, \quad f^{(4)}(x) = \frac{24}{(2+x)^5}.$$

Thus,

$$R_3(x;0) = \frac{24}{(2+z)^5} \cdot \frac{1}{4!}x^4 = \frac{x^4}{(2+z)^5}.$$

Now

$$\frac{1}{2+x} = \frac{1}{2} \cdot \frac{1}{1-\left(-\dfrac{x}{2}\right)} = \frac{1}{2} \cdot \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots\right).$$

Therefore,

$$\frac{1}{2+x} = \frac{1}{2} \cdot \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8}\right) + \frac{x^4}{(2+z)^5},$$

where z is between 0 and x. □

(b) Use $R_3(x;0)$ to approximate the largest error that occurs in using $p_3(x)$ to approximate $\dfrac{1}{2+x}$ for

$0 \leq x \leq 1$ .

I have

$$|R_3(x; 0)| = \frac{1}{(2+z)^5} \cdot |x|^4.$$

I'll estimate the z and x-terms one at a time.

Since $0 \leq x \leq 1$ , I have

$$|x|^4 \leq 1^4 = 1.$$

Since $0 \leq x \leq 1$ and z is between 0 and x, it follows that $0 \leq z \leq 1$ . On this interval, $\dfrac{1}{(2+z)^5}$ decreases, so it attains its largest value at $z = 0$ . Therefore,

$$\frac{1}{(2+z)^5} \leq \frac{1}{(2+0)^5} = \frac{1}{32}.$$

Thus,

$$|R_3(x; 0)| \leq \frac{1}{32} \cdot 1 = \frac{1}{32}.$$

The error is no greater than $\dfrac{1}{32}$ . □

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**Example.** Find the smallest value of n for which the $n^{\text{th}}$ degree Taylor series for $f(x) = e^{2x}$ at $c = 0$ approximates $e^{2x}$ on the interval $0 \leq x \leq 1$ with an error no greater than $10^{-6}$ .

Notice that

$$f'(x) = 2e^{2x}, \quad f''(x) = 2^2 e^{2x}, \quad f^{(3)}(x) = 2^3 e^{2x}, \quad \ldots, \quad f^{(n)}(x) = 2^n e^{2x}.$$

So

$$|R_n(x; 0)| = \left|\frac{2^{n+1} e^{2z}}{(n+1)!} x^{n+1}\right| = \frac{2^{n+1} e^{2z}}{(n+1)!} |x|^{n+1} \quad \text{for} \quad 0 \leq z \leq x \leq 1.$$

First, I'll estimate how large the z and x-terms can be. Since $0 \leq z \leq 1$ and since $e^{2z}$ is an increasing function, I have

$$e^{2z} \leq e^2.$$

Since $0 \leq x \leq 1$ and $x^n$ is an increasing function, I have

$$|x|^{n+1} \le 1^{n+1} = 1.$$

Thus,

$$|R_n(x;0)| \le \frac{2^{n+1}e^2}{(n+1)!}.$$

Therefore, I want the smallest n for which

$$\frac{2^{n+1}}{(n+1)!} < 10^{-6}.$$

I can't solve this inequality algebraically, so I'll have to use trial-and-error:

| n | $\dfrac{2^{n+1}e^2}{(n+1)!}$ |
|---|---|
| 1 | 14.7781121978613 |
| 2 | 9.852074798574201 |
| 3 | 4.9260373992871 |
| 4 | 1.97041495971484 |
| 5 | 0.65680498657161 |
| 6 | 0.18765856759189 |
| 7 | 0.046914641897972 |
| 8 | 0.010425475977327 |
| 9 | 0.0020850951954654 |
| 10 | $3.7910821735735262 \cdot 10^{-4}$ |
| 11 | $6.3184702892892108 \cdot 10^{-5}$ |
| 12 | $9.7207235219833994 \cdot 10^{-6}$ |
| 13 | $1.3886747888547714 \cdot 10^{-6}$ |
| 14 | $1.8515663851396951 \cdot 10^{-7}$ |

The smallest value of n is $n = 14$ . □

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You can also use the Remainder Term to estimate the error in using a Taylor polynomial to approximate an integral.

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**Example.** Calvin wants to impress Phoebe Small by using the MacLaurin series for $e^{2x}$ to approximate

$\int_0^{0.5} xe^{2x}\,dx$ to within 0.0001. How many terms of the series should he use?

The Maclaurin series for $e^{2x}$ is

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}.$$

(Substitute $u = 2x$ in the standard series for $e^u$ .) I want to know how many terms of the series to use to approximate the integral.

Since $f(x) = e^{2x}$ , $f'(x) = 2e^{2x}$ , $f''(x) = 2^2 e^{2x}$ , and in general, $f^n(x) = 2^n e^{2x}$ . Therefore,

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(z)(x-c)^{n+1} = \frac{1}{(n+1)!} \cdot 2^n \cdot e^{2z} \cdot x^{n+1}.$$

In the integral, x goes from 0 to 0.5, and z is a number between 0 (the expansion point) and x. Therefore, I know that z is a number between 0 and 0.5. Taking the worst possible case, the largest $e^{2c}$ could be is $e^{2 \cdot 0.5} = e$ . Replace $e^{2z}$ with e to obtain

$$R_n(x) \le \frac{1}{(n+1)!} \cdot 2^n \cdot e \cdot x^{n+1}.$$

Insert this into the integral (remembering to multiply by x):

$$\text{error} \le \int_0^{0.5} \frac{1}{(n+1)!} \cdot 2^n \cdot e \cdot x^{n+2}\,dx = \frac{1}{(n+1)!} 2^{n+1} \cdot e \cdot \frac{1}{n+3} \cdot (0.5)^{n+3}.$$

I want the smallest value of n for which this ugly mess is less than 0.0001. The easiest way to do this is by trial: Plug in successive values of n until you discover that $n = 6$ is the smallest value that works. □

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