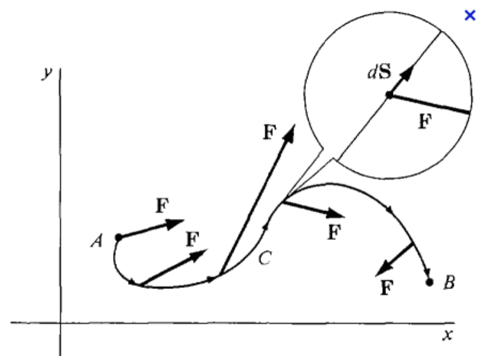
Auroux 19, Vector Fields and Line Integrals



Before I start notes on the lecture, I'd like to summarize where we're going. We imagine an object moving along a trajectory or curve C. It is moving in a vector field $F = f(x, y)$, which varies with position. At each point along the curve we want to compute $\mathbf{F} \cdot dr$, the dot product of $\mathbf{F}$ with the next little bit of our trajectory. We take the part of $\mathbf{F}$ in the same direction as $\mathbf{dr}$ and then sum up all those little bits to find the total work

$$\int \mathbf{F} \cdot \mathbf{dr} = W = \int \mathbf{F} \cdot \mathbf{v}\ dt = \int \mathbf{F} \cdot \hat{\mathbf{T}}\ ds$$

Now, $\mathbf{F} \cdot \mathbf{dr}$ looks a little overwhelming. How do you integrate a couple of vectors? Even $\mathbf{F} \cdot \mathbf{v}$ too, for that matter, and what is $\hat{\mathbf{T}}$? However, if we break this up into components, it will make more sense.

The way to understand $\hat{\mathbf{T}}$ is

$$\mathbf{dr} = ds\ \hat{\mathbf{T}}$$

What this means is that the vector $\mathbf{dr}$ is in the direction $\hat{\mathbf{T}}$, tangent to the curve, and it has magnitude ds, the little bit of arc length. One way to see this is to divide by dt So

$$\frac{\mathbf{dr}}{dt} = \mathbf{v} = \frac{\mathbf{dr}}{ds}\ \frac{ds}{dt} = \frac{ds}{dt}\ \hat{\mathbf{T}} = |v|\ \ \hat{\mathbf{T}}$$

So what we are doing here is, at each point along the curve there is the next little change in the **v** vector, $\Delta\mathbf{v}$ and we want

$$\lim_{\Delta\mathbf{v}\to 0}\sum_i \mathbf{F}\cdot\Delta\mathbf{dr} = \lim_{\Delta t\to 0}\sum_i \mathbf{F}\cdot\frac{\Delta r}{\Delta t}\Delta t = \int_C \mathbf{F}\cdot\mathbf{dr} = \int_{t=t_0}^{t=t_1}\mathbf{F}\cdot\frac{\mathbf{dr}}{dt}dt$$

Notice the change in limits that goes with the switch to dt. Now, in the end we will actually compute using a formula like this

$$\int_C \mathbf{F}\cdot\mathbf{dr} = \int < M\hat{\imath}, N\hat{\jmath} > \ \cdot < dx, dy >$$

with one important reservation. We need to simplify this integral to have a single variable. We can't really integrate $\int ..\ dx\ dy$. We can only do this because x and y are related by their trajectory. We may use a parameter t, or perhaps, just express y in terms of x.

$$\int_C \mathbf{F}\cdot\frac{\mathbf{dr}}{dt}\ dt = \int < M, N > \ \cdot < \frac{dx}{dt}, \frac{dy}{dt} > dt$$

But we have to express things in terms of a single variable. Then we will have a simple integral over a single variable.

Before we do that, let's review a couple of vector fields.

Field 1. $F =< i, j >$ This field is the same everywhere $< 1, 1 >$.

Field 2. $F =< xi, 0 >$ This field is proportional to x, points left and right ($\Delta y = 0$), and reverses sign at the y-axis.

Field 3. $F =< xi, yj >$ This is a radial field, with magnitude proportional to r.

Field 4. $F =< -yi, xj >$ This is a rotating field. At $< 1, 1 >$ the field is $< -1, 1 >$ and $\perp r$. The magnitude is proportional to r, so it's like a rotating disk.

Example 1.

$$F = -yi + xj$$

which is the rotating field. $r(t)$ is

$$x = t$$

$$y = t^2$$

$$0 < t < 1$$

The curve is just $y = x^2$, parametrized. We have

$$\int_{t=0}^{t=1} F \cdot \frac{dr}{dt}\ dt = \int_0^1 < -y, x > \ \cdot \ < \frac{dx}{dt}, \frac{dy}{dt} > \ dt$$

$$< -t^2, t > \ \cdot < 1, 2t > \ dt$$

$$\int_0^1 t^2\ dt = \frac{1}{3}$$

We could have said

$$\int_C M\ dx + N\ dy = -y\ dx + x\ dy$$

and expressed y in terms of x. This works out the same as before, because $y = x^2$, $dy = 2xdx$ so we have

$$\int_C M\ dx + N\ dy = -x^2\ dx + x\ 2xdx = \frac{1}{3}$$

with limits $x = 0$ and $x = 1$ (because we had $t = 0$ and $t = 1$ and $x = t$).

Example 2. The curve C is a circle of radius a centered at the origin, going ccw, and the field is $F =< xi, yj >$. Now you could set this up and solve it, but you can also notice that

$$\int \mathbf{F} \cdot \hat{\mathbf{T}}\ ds$$

at every point on the curve the radial vector for the field $< x, y > \perp \hat{\mathbf{T}}$, so the whole thing is is just 0.

Example 3. The curve C is again a circle of radius a centered at the origin, going ccw, and the field is the rotating one, $F =< -y, x >$. Now you can notice that

$$\int \mathbf{F} \cdot \hat{\mathbf{T}} = |F| \ = \sqrt{x^2 + y^2} = a$$

so this is just

$$\int_C \mathbf{F} \cdot \hat{\mathbf{T}} ds = \int_C a\ ds = 2\pi a^2$$

if you fail to see this, then you can say we have

$$\int_C M\ dx + N\ dy = \int_C -y\ dx + y\ dy$$

and

$$x = a\ cos\theta, \quad dx = -a\ sin\theta\ d\theta$$

$$y = a\ sin\theta, \quad dy = a\ cos\theta\ d\theta$$

The first term becomes $a^2\ sin^2\theta\ d\theta$ and the second is $a^2\ cos^2\theta\ d\theta$ and so

$$\int_C a^2 d\theta = 2\pi a^2$$