Two Determinant Proofs

In this short write-up we will prove two basic identities for determinants:

$$|\mathbf{AB}| = |\mathbf{A}|\ |\mathbf{B}|$$
$$|\mathbf{A}| = |\mathbf{A}^T|$$

$|\mathbf{AB}| = |\mathbf{A}|\ |\mathbf{B}|$

Strang does something interesting here (proving the $n \times n$ case by algebra is complicated). If $\mathbf{B}$ is zero, this equation is certainly true.

If $\mathbf{B}$ is non-zero then consider this ratio:

$$D(A) = \frac{|\mathbf{AB}|}{|\mathbf{B}|}$$

Does $D(A)$ have the first three properties that define the determinant of A? If $A = I$ then this becomes

$$D(A) = \frac{|\mathbf{B}|}{|\mathbf{B}|} = 1$$

For property 2, if two rows of A are exchanged, then so are the same two rows of AB and therefore $|AB|$ changes sign, and so does the ratio $D(A)$.

Finally property 3 (linearity) also holds (check for yourself).

Since the ratio $D(A)$ has the same three properties that define $|A|$, it equals $|A|$.

Note that if this is given then:

$$
\begin{gathered}
AA^{-1} = I \\
\big|A\big|\,\big|A^{-1}\big| = \big|I\big| = 1 \\
|A| = \frac{1}{|A^{-1}|}
\end{gathered}
$$

$|\mathbf{A}| = |\mathbf{A}^T|$

The simplest way to see this is use the rules for computing determinants of e.g. 3×3

$$
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - hf) - b(di - fg) + c(dh - eg)
$$

Now, the transpose is

$$
\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = a(ei - hf) - d(bi - ch) + g(bf - ec)
$$

All the same terms are there. But we haven't proved these rules yet

To Prove: $|\mathbf{A}| = |\mathbf{A}^T|$

Either $\mathbf{A}$ is *singular* with determinant zero (and so is its transpose), or it can be factored into triangular matrices $\mathbf{L}$ and $\mathbf{U}$.

To do this, the rows of $\mathbf{A}$ may need to be reordered first by multiplication with a permutation matrix $\mathbf{P}$. Thus

$$\mathbf{PA} = \mathbf{LU}$$

Let's start with the permutation matrix. $\mathbf{P}$ has determinant either ± 1, by rule 2 (from the previous write-up).

For any permutation matrix, the transpose is also the inverse

$$\mathbf{P}^T = \mathbf{P}^{-1}$$

so

$$\mathbf{PP}^T = \mathbf{I}$$

(since any row or column of either has at most a single 1, and those match up in the row by column multiplication because of the transpose.

We proved above that

$$|\mathbf{AB}| = |\mathbf{A}|\ |\mathbf{B}|$$

Let's call this the product rule. By using this rule and then that the determinant of the identity matrix is 1

$$\left|\mathbf{P}\right| \left|\mathbf{P}^T\right| = \left|\mathbf{PP}^T\right| = \left|\mathbf{I}\right| = 1$$

Now, either $\left|\mathbf{P}\right| = 1$ and so, by the last equation, $\left|\mathbf{P}^T\right| = 1$, or else both determinants are minus 1.

In either case

$$\left|\mathbf{P}\right| = \left|\mathbf{P}^T\right|$$

Moving to the right-hand side

$$\left|\mathbf{U}\right| = \begin{vmatrix} p & a & b \\ 0 & q & c \\ 0 & 0 & r \end{vmatrix}$$

By rule 7 (previous write-up), $|\mathbf{U}| = pqr$. Transposition doesn't change the values on the diagonal, so the determinant of a triangular matrix is not changed by the transpose.

$$\left|\mathbf{U}\right| = \left|\mathbf{U}^T\right|$$

There is a particular decomposition into **LU** in which **L** has 1's on the diagonal, although that's not required for what comes next. But in that case, $|\mathbf{L}| = 1$. Otherwise

$$\left|\mathbf{L}\right| = \begin{vmatrix} u & 0 & 0 \\ x & v & 0 \\ y & z & w \end{vmatrix} = 1$$

and so (again, since transposition doesn't change the diagonal entries), $|L| = |L^T|$.

putting it all together

We have that

$$\mathbf{PA} = \mathbf{LU}$$

and we have shown that individually, each of **P**, **L** and **U** has the same determinant as the respective transpose. Further, we have the product rule, $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.

Now since

$$\mathbf{PA} = \mathbf{LU}$$

$$|\mathbf{PA}| = |\mathbf{LU}|$$

and

$$(\mathbf{PA})^T = (\mathbf{LU})^T$$

$$|(\mathbf{PA})^T| = |(\mathbf{LU})^T|$$

Looking first at the right hand side, by the property of the transpose

$$|(\mathbf{LU})^T| = |\mathbf{U}^T\mathbf{L}^T|$$

by the product rule

$$|\mathbf{U}^T\mathbf{L}^T| = |\mathbf{U}^T||\mathbf{L}^T|$$

by the identities above

$$= |\mathbf{U}||\mathbf{L}|$$

$$= |\mathbf{L}||\mathbf{U}|$$

by the product rule

$$= |\mathbf{LU}|$$

In summary

$$|(\mathbf{PA})^T| = |\mathbf{LU}|$$

by our initial formulation

$$|\mathbf{PA}| = |\mathbf{LU}|$$

Thus, we have shown that

$$|\mathbf{PA}| = |(\mathbf{PA})^T|$$

but by the product rule

$$|\mathbf{PA}| = |\mathbf{P}||\mathbf{A}|$$

and by the property of the transpose

$$(\mathbf{PA})^T = \mathbf{A}^T\mathbf{P}^T$$

so

$$|(\mathbf{PA})^T| = |\mathbf{A}^T\mathbf{P}^T|$$

by the product rule

$$= |\mathbf{A}^T||\mathbf{P}^T|$$

Thus

$$|\mathbf{PA}| = |(\mathbf{PA})^T|$$

$$|\mathbf{P}||\mathbf{A}| = |\mathbf{A}^T||\mathbf{P}^T|$$

but

$$|\mathbf{P}| = |\mathbf{P}^T|$$

so

$$|\mathbf{A}| = |\mathbf{A}^T|$$