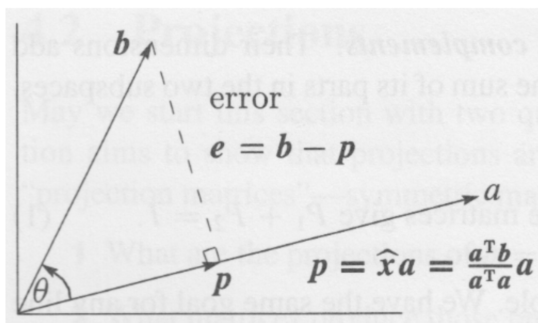
Projections

We have two vectors $\mathbf{a}$ and $\mathbf{b}$ and we wish to find the component of $\mathbf{b}$ that is parallel to $\mathbf{a}$ (the projection of $\mathbf{b}$ on $\mathbf{a}$), as well as $\mathbf{e}$, the part that is perpendicular. In this short writeup we will derive and use the equation, but let me just put it up here first:

$$\mathbf{p} = \hat{x}\ \mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\ \mathbf{a}$$

The projection of $\mathbf{b}$ on $\mathbf{a}$ is just "$\mathbf{a}$ dot $\mathbf{b}$ over $\mathbf{a}$ dot $\mathbf{a}$, times $\mathbf{a}$."



We say that $\mathbf{p}$ is some multiple $\hat{x}$ of $\mathbf{a}$

$$\mathbf{p} = \hat{x}\ \mathbf{a}$$

and that $\mathbf{b}$ is just the sum of $\mathbf{p}$ and $\mathbf{e}$

$$\mathbf{p} + \mathbf{e} = \mathbf{b}, \qquad \mathbf{e} = \mathbf{b} - \mathbf{p} = \mathbf{b} - \hat{x}\ \mathbf{a}$$

and then use perpendicularity

$$\mathbf{a}\cdot\mathbf{e} = 0 = \mathbf{a}\cdot(\mathbf{b} - \hat{x}\ \mathbf{a})$$

$$\hat{x} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}$$

We can divide by $\mathbf{a} \cdot \mathbf{a}$ since it is just a number. Finally

$$\mathbf{p} = \hat{x}\ \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a}$$

Notice that the factor $\mathbf{a} \cdot \mathbf{a} = 1$ if $\mathbf{a}$ is a unit vector. It's just a scaling factor.

Example 1. Vector $b = < u, v >$, what is its projection on $i = < 1, 0 >$?

$$\hat{x} = \frac{a \cdot b}{a \cdot a} = \frac{1 \times u + 0 \times v}{1 \times 1 + 0 \times 0} = u$$

So $p = ui = < u, 0 >$, as expected.

$$e = a - p = < 0, v >$$

Example 2. Vector $b = < 3, 1 >$, what is its projection on $a = < 1, 1 >$?

$$\hat{x} = \frac{a \cdot b}{a \cdot a} = \frac{1 \times 3 + 1 \times 1}{1 \times 1 + 1 \times 1} = \frac{4}{2} = 2$$

$$p = 2\ \ < 1, 1 > = < 2, 2 >$$

The projection is $< 2, 2 >$.

$$e = b - p = < 3, 1 > - < 2, 2 > = < 1, -1 >$$

$$p \cdot e = < 2, 2 > \cdot < 1, -1 > = 0$$

Notice that the lengths work for a right triangle:

$$|b|^2 = |p|^2 + |e|^2$$

$$10 = 8 + 2$$

This works for vectors in $\mathbb{R}^3$ as well.

Example 3. Vector $b = < 3, 4, 4 >$ and $a = < 2, 2, 1 >$.

$$\hat{x} = \frac{a \cdot b}{a \cdot a} = \frac{2 \times 3 + 2 \times 4 + 1 \times 4}{2 \times 2 + 2 \times 2 + 1 \times 1} = \frac{18}{9} = 2$$

$$p = 2\ \ < 2, 2, 1 > = < 4, 4, 2 >$$

$$e = b - p = < 3, 4, 4 > - < 4, 4, 2 > = < -1, 0, 2 >$$

$$e \cdot p = -4 + 4 = 0$$

Now suppose that we add a second vector $a_2 = < 1, 0, 0 >$ (renaming the first one a_1). And we ask to project b onto the plane formed by a_1 and a_2. Put the two vectors into a matrix A

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$$

The projection is some combination of the two vectors a_1 and a_2.

$$p = \hat{x}_1 a_1 + \hat{x}_2 a_2 = A\hat{x}$$

We know that $e = b - p = b - A\hat{x}$ is $\perp$ to the plane.

$$a_1^T(b - A\hat{x}) = 0 = a_2^T(b - A\hat{x})$$

Put them into a matrix

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^T(b - A\ \hat{x}) = 0$$

$$A^T A\ \hat{x} = A^T b$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

The fundamental equation is then

$$p = A\hat{x} = A(A^T A)^{-1} A^T b$$

Compare with the one-dimensional case

$$a \frac{a^T b}{a^T a}$$

Let's work the example

$$A^T = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix}$$

$$(A^TA)^{-1}A^T = \frac{1}{5}\begin{bmatrix} 0 & 2 & 1 \\ 5 & -4 & -2 \end{bmatrix}$$

And finally

$$P = A(A^TA)^{-1}A^T = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$p = Pb = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 15 \\ 24 \\ 12 \end{bmatrix}$$

$$e = b - p = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 15 \\ 24 \\ 12 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 \\ -4 \\ 8 \end{bmatrix}$$

We should confirm both $a_1 \perp e$ and $a_2 \perp e$.

$$< 2, 2, 1 >< 0, -\frac{4}{5}, \frac{8}{5} > = 0$$

$$< 1, 0, 0 >< 0, -\frac{4}{5}, \frac{8}{5} > = 0$$

This problem can be made easier by using an orthonormal basis for the plane. One such basis is obtained by finding the projection of a_1 on a_2 and subtracting that from a_1. Since a_2 is a unit vector, the projection p is just $< 2, 0, 0 >$ and the new basis vector a_3 (before normalization) is $< 0, 2, 1 >$. The unit vector is $u = < 0, 2/\sqrt{5}, 1/\sqrt{5} >$.

Then

$$A = \begin{bmatrix} 0 & 1 \\ 2/\sqrt{5} & 0 \\ 1/\sqrt{5} & 0 \end{bmatrix}$$

and

$$AA^T = P = \begin{bmatrix} 0 & 1 \\ 2/\sqrt{5} & 0 \\ 1/\sqrt{5} & 0 \end{bmatrix}\begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 1 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & 2/5 \\ 0 & 2/5 & 1/5 \end{bmatrix}$$

We easily confirm that

$$Pa_3 = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 4/5 \\ 2/5 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 2/5 \\ 1/5 \end{bmatrix} = a_3$$

And for that matter $Pa_1 = a_1$; $Pa_2 = a_2$; and $Pu = u$.

another way

It is worth pointing out that there is another way to do this problem which avoids the matrix multiplications required above. We first use the two vectors in the plane to find the normal vector, then find the projection of vector b on it. That is e. Then we find p by subtraction.

Remember the "determinant" trick for doing the cross-product

$$\begin{bmatrix} i & j & k \\ 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$n = < 0, 1, -2 >$$

Check that $n \perp$ to both a_1 and a_2. Now recall $b = < 3, 4, 4 >$ and do

$$e = \frac{b \cdot n}{n \cdot n} n = -\frac{4}{5} \ < 0, 1, -2 > = \frac{1}{5} \ < 0, -4, 8 >$$

which matches what we got before!

last point

Finally, note that the vectors we started with, $a_1 = < 2, 2, 1 >$ and $a_2 = < 1, 0, 0 >$ are eigenvectors of the projection matrix P with eigenvalue $\lambda = 1$, since

$$Pa_1 = a_1$$

$$Pa_2 = a_2$$

In fact, *any* vector that is a linear combination of a_1 and a_2 (that lies in the plane), has this property. Furthermore,

$$Pv = PPv$$

for all v. Once a vector is projected into the plane, another projection doesn't change it.