Quick Review

Cauchy Integral theorem

If we can write an integral in this form

$$I = \int_C \frac{f(z)}{z - z_0}\ dz$$

then the theorem says that

$$I = 2\pi i f(z_0)$$

examples

$$\int_C \frac{1}{z}\ dz = 2\pi i$$

since the function is $f = 1$ (not $1/z$), and $f(0) = 1$.

$$\int_C \frac{1}{z^2 + 1}\ dz$$
$$= \int_C \frac{1}{(z - i)(z + i)}\ dz$$
$$= \frac{1}{2i} \int_C \frac{1}{z - i} - \frac{1}{z + i}\ dz$$

If the contour includes either one of the points i or $-i$, then there is a contribution of $2\pi i$ toward the value of the integral from that point. For example, the unit circle centered at $(0, i)$ gives $I = \pi$.

residues

Another approach to the previous example is to say that the value of the integral is the $2\pi i$ times the sum of the residues. For a simple pole of order 1, the residue is

$$\text{Res} = \lim_{z \to z_0} (z - z_0) f(z)$$

So in the above example where only $z = i$ is included in the contour

$$\begin{aligned} \text{Res} &= \lim_{z \to i} (z - i) \frac{1}{(z-i)(z+i)} \\ &= \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i} \end{aligned}$$

times $2\pi i$ gives $I = \pi$.

derivatives

Since

$$2\pi i f(z_0) = \int_C \frac{f(z)}{z - z_0}\, dz$$

write

$$2\pi i f'(z - z_0) = \int_C \frac{f(z)}{(z - z_0)^2}\, dz$$

$$2\pi i f^n(z - z_0) = n! \int_C \frac{f(z)}{(z - z_0)^n}\, dz$$

examples

$$f(z) = \frac{1}{z(z-2)^2}$$

The first reside is just

$$\text{Res}(0) = \lim_{z \to 0} (z) \frac{1}{z(z-2)^2}$$

$$= \lim_{z \to 0} (z) \frac{1}{(z-2)^2} = \frac{1}{4}$$

for the second pole, we need the derivative version. First multiply by $(z-2)^2$

$$(z-2)^2 f(z) = \frac{1}{z}$$

Compute the $N-1$ derivative of what's left

$$-\frac{1}{z^2}$$

Evaluate the limit as $z \to 2$

$$-\frac{1}{4}$$

The residue is this result multiplied by $n! = 1$, and the value of the integral is $2\pi i$ times the sum of the residues. The result in this case is zero.

Laurent series

Yet another way to work these problems is Laurent series. That's our next stop. But we can do a quick example. If we have the series, we can just integrate the term with $(z - z_0)^{-1}$ because all the other terms are equal to zero.

examples

Suppose

$$f(z) = \frac{e^z}{z^2}$$

Since we know that

$$e^z = 1 + z + z^2 \dots$$

$$f(z) = \frac{1}{z^2} + \frac{1}{z} + 1 + \dots$$

$$\int_C f(z)\ dz = \int_C \frac{1}{z}\ dz = 2\pi i$$

By the method of derivatives we would first multiply by z^2

$$e^z$$

Compute the $N-1$ derivative of what's left

$$e^z$$

Evaluate the limit as $z \to 0 = 1$ and finally, multiply by $2\pi i$. The result is the same.

Consider

$$\exp\ (\frac{1}{z^2})$$

We use the series for e^z and substitute z^2

$$= 1 + \frac{1}{z^2} + \frac{1}{2!z^4} + \dots$$

There is no term containing z^{-1} and therefore the residue is zero, even though there is a pole (the function is undefined at $z = 0$). Thus, even though the function is not analytic in a region containing the origin, the integral is zero. This is an example of a "removable singularity."

$$I = \int_C z^2 \sin(1/z)\ dz$$

The Maclaurin series for sine is

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

$$\sin 1/z = 1/z - \frac{1}{3!\ z^3} + \frac{1}{5!\ z^5} - \dots$$

Multiplying by z^2:

$$I = \int_C z - \frac{1}{3!\ z} + \frac{1}{5!\ z^3} - \dots$$

Here we do have a term with $1/z$ and its coefficient is 1. So the residue is $-1/3!$ and the value of the integral is $-\pi i/3$.

An example from Brown is

$$\frac{1}{z(z-2)^4}$$

Since their example is about Laurent series, they do the following. Rewrite

$$\frac{1}{z} = \frac{1}{2 + (z-2)}$$

to go with the other factor. Then

$$= \frac{1}{2} \cdot \frac{1}{1 - (-\frac{z-2}{2})}$$

So we have a factor of $1/2(z-2)^4$ multiplying something that can be expanded using the geometric series to give

$$1 + (-\frac{z-2}{2}) + (-\frac{z-2}{2})^2 + (-\frac{z-2}{2})^3 \dots$$

The only term that gives us anything like z^{-1} after multiplying by $1/2(z-2)^4$ is the last one:

$$\frac{1}{2} \cdot \frac{1}{(z-2)^4} \cdot (-\frac{z-2}{2})^3 = -\frac{1}{16} \frac{1}{(z-2)}$$

The coefficient is $-1/16$ which multiplies $2\pi i$ to give $-\pi i/8$.

We did this problem by the method of residues at the end of Chapter 19 in the main write-up.

Quotients