Continued fractions

$$\sqrt{N} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

According to Problem 64 of Project Euler, any square root can be written as a continued fraction, in the form shown above. We'll try first with $\sqrt{23}$ and see if we can find a pattern.

step 0

The first part of the method is as follows: find the next smallest perfect square, in this case $4 \times 4 = 16$. Now do:

$$\sqrt{23} = 4 + \sqrt{23} - 4 = 4 + \frac{1}{\frac{1}{\sqrt{23}-4}}$$

So $a_0 = 4$, and we are ready to start the next stage with the inverse of the fraction we obtained above

$$\frac{1}{\sqrt{23} - 4}$$

If we generalize this as

$$\frac{n_i}{\sqrt{23} - d_i}$$

We will be looking for a pattern connecting n_i, d_i and n_{i+1}, d_{i+1}.

$$a_0 = 4,\ n_0 = 1,\ d_0 = 4$$

step 1

At this point, we start the repeating process. We need to rationalize the denominator

$$\frac{1}{\sqrt{23}-4} = \frac{1}{\sqrt{23}-4}\,\frac{\sqrt{23}+4}{\sqrt{23}+4} = \frac{\sqrt{23}+4}{23-16} = \frac{\sqrt{23}+4}{7}$$

and simplify. In this case we have $4/7 < 1$ and so we do:

$$\frac{\sqrt{23}+4}{7} = \frac{\sqrt{23}+4+3-3}{7} = 1 + \frac{\sqrt{23}-3}{7} = 1 + \frac{1}{\frac{7}{\sqrt{23}-3}}$$

If we just focus on what is under the 1 in the second fraction we need to work on

$$\frac{7}{\sqrt{23}-3}$$

$$a_1 = 1,\ n_1 = 7,\ d_1 = 3$$

step 2

$$\frac{7}{\sqrt{23}-3} = \frac{7}{\sqrt{23}-3}\,\frac{\sqrt{23}+3}{\sqrt{23}+3} = \frac{7(\sqrt{23}+3)}{14}$$

This is where the simplification gets a little trickier, we have another factor in the numerator, f. My hypothesis is that we will always have that f evenly divides into the denominator, as it does here, so..

$$\frac{7(\sqrt{23}+3)}{14} = \frac{\sqrt{23}+3}{2}$$

Now we need to convert this into a form with $\sqrt{23}-n$. Here is the step I don't understand. while we could pull out a 2 here, in the example, they do 3

$$\frac{\sqrt{23}+3}{2} = \frac{\sqrt{23}+3-3+3}{2} = 3 + \frac{\sqrt{23}-3}{2}$$

Perhaps it's as simple as using $+3-3$ because we had 3 there already. Finally, we form the continued fraction part (inverting) and have this to go forward with

$$\frac{2}{\sqrt{23}-3}$$

At this stage we have:

$$a_0 = 4,\ n_0 = 1,\ d_0 = 4$$
$$a_1 = 1,\ n_1 = 7,\ d_1 = 3$$
$$a_2 = 3,\ n_2 = 2,\ d_2 = 3$$

step 3

Rationalize

$$\frac{2}{\sqrt{23}-3} = \frac{2}{\sqrt{23}-3}\,\frac{\sqrt{23}+3}{\sqrt{23}+3} = \frac{2(\sqrt{23}+3)}{14} = \frac{\sqrt{23}+3}{7}$$

Simplify

$$\frac{\sqrt{23}+3}{7} = \frac{\sqrt{23}+3+4-4}{7} = 1 + \frac{\sqrt{23}-4}{7}$$

$$a_3 = 1,\ n_3 = 7,\ d_3 = 4$$

Invert

$$\frac{7}{\sqrt{23}-4}$$

step 4

Rationalize

$$\frac{7}{\sqrt{23}-4}\frac{\sqrt{23}+4}{\sqrt{23}+4} = \frac{7(\sqrt{23}+4)}{7} = \sqrt{23}+4$$

Simplify

$$\sqrt{23}+4 = \sqrt{23}+4-4+4 = 8+\sqrt{23}-4$$

Invert

$$\frac{1}{\sqrt{23}-4}$$

$$a_4 = 8,\ n_4 = 1,\ d_4 = 4$$

Summarizing

We have

$$a_0 = 4,\ n_0 = 1,\ d_0 = 4$$

$$a_1 = 1,\ n_1 = 7,\ d_1 = 3$$

$$a_2 = 3,\ n_2 = 2,\ d_2 = 3$$

$$a_3 = 1,\ n_3 = 7,\ d_3 = 4$$

$$a_4 = 8,\ n_4 = 1,\ d_4 = 4$$

At this point, we have the same fraction to work on as what we started with, so we will just repeat the same steps over again.