On the sphere

To repeat the beginning of the write-up on parametrization of the sphere, we said that:

A sphere centered at the origin is defined as the set of points x, y, z at a distance ρ away from $(0, 0, 0)$, leading to the equation $x^2 + y^2 + z^2 = \rho^2$. We are looking for a "parametrization" or relationship between x, y, z coordinates and spherical coordinates in terms of one radial and two angular variables. These are usually called ρ, θ, and ϕ.

If we think of the vector $< x, y, z >$ to a point on the sphere, then θ is the angle it makes going ccw from the positive x-axis and ranges from $0 \leq \theta \leq 2\pi$. ϕ is the "polar" angle that the same vector makes with the positive z-axis and ranges from $0 \leq \phi \leq \pi$.

The projection of ρ in the xy-plane is r.

$$r = \rho \cos(\frac{\pi}{2} - \phi) = \rho \sin \phi$$

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

$$z = \rho \sin(\frac{\pi}{2} - \phi) = \rho \cos \phi$$

We went on to analyze the Jacobian and generate this equation for the volume element

$$\rho^2 \sin \phi \ d\rho \ d\phi \ d\theta$$

and for the volume as a triple integral

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{a} \rho^2 \sin \phi \ d\rho \ d\phi \ d\theta$$

Using the figure from *How to ace calculus*

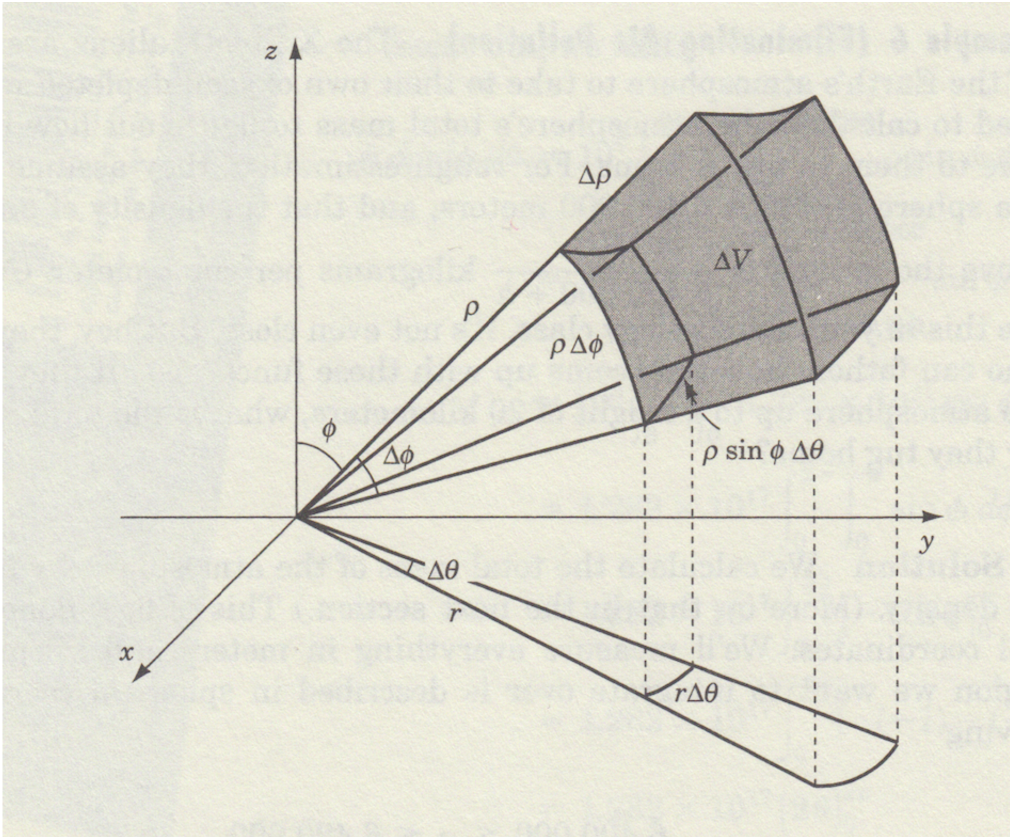

Figure 9.23 Varying each coordinate in spherical coordinates.

We can visualize where these come from. In particular, the vertical sides of the volume element have length $\rho \; d\phi$, while the horizontal sides are $\rho \; \sin\phi \; d\theta$. A good way to think about this is that the vertical slices include the vertical axis of the sphere and are "great circles" of radius ρ, while the horizontal slices are smaller. In fact, if we look at the projection onto the xy-plane, we have a circle of radius r, where $r = x^2 + y^2$, but also $r = \rho \; \sin\phi$.

We can also parametrize the sphere with cylindrical coordinates (r, θ, and z). In setting up an integral for the triple volume, we will do z first. The bounds on z are crucial. If the radius of the sphere is R, then

$$R^2 = x^2 + y^2 + z^2$$

but since $r^2 = x^2 + y^2$

$$R^2 = z^2 + r^2$$

so $z = -\sqrt{R^2 - r^2} \to \sqrt{R^2 - r^2}$ and and for the volume as a triple integral

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{R} \int_{z=-\sqrt{R^2-r^2}}^{z=\sqrt{R^2-r^2}} dz \ r \ dr \ d\theta$$

The inner integral is just

$$2\sqrt{R^2 - r^2}$$

The middle integral is

$$\int_{r=0}^{R} 2\sqrt{R^2 - r^2} \ r \ dr$$

$$= -\frac{2}{3}(R^2 - r^2)^{3/2} \Big|_{r=0}^{R}$$

$$= \frac{2}{3}R^3$$

We pick up a factor of 2π from the outer integral, giving the familiar answer.

Cylindrical coordinates gives a nice approach to the problems of the spherical cap and the "cored" apple.

For the spherical cap, we change the lower bound of integration for z to $R - h$ (h being the height of the cap)

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{R} \int_{z=R-h}^{z=\sqrt{R^2-r^2}} dz \ r \ dr \ d\theta$$

The inner integral is

$$\sqrt{R^2 - r^2} - (R - h)$$

The middle integral is

$$\int_{r=0}^{R} (\sqrt{R^2 - r^2} - (R - h)) \ r \ dr$$

$$= -\frac{1}{3}(R^2 - r^2)^{3/2} - \frac{1}{2}(R - h)r^2 \Big|_{r=0}^{R}$$

$$= \frac{1}{3}R^3 - \frac{1}{2}(R - h)R^2$$

The final answer is multiplied by 2π. For the latter problem, we simply change the lower bound of integration for r to $r = a$.

The middle integral is

$$\int_{r=a}^{R} 2\sqrt{R^2 - r^2}\ r\ dr$$

$$= -\frac{2}{3}(R^2 - r^2)^{3/2}\ \Big|_{r=a}^{R}$$

$$= \frac{2}{3}(R^2 - a^2)^{3/2}$$

And the final answer is

$$\frac{4}{3}\pi(R^2 - a^2)^{3/2}$$