Functions of a complex variable

My motivation for learning about complex functions is two-fold: the theory is often described as being very beautiful, and second, I'd like to understand how more difficult integrals can be solved. Marsden gives three examples that he says are either very difficult or impossible if we are restricted to just the real numbers:

$$\int_0^\infty \frac{\sin^2 x}{x^2}\ dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\ dx = \frac{\pi}{\sin \alpha\pi}$$

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = \frac{2\pi}{\sqrt{a^2-1}}$$

Quick review

Everyone knows that complex numbers $\mathbb{C}$ arise in the context of finding solutions to the equation:

$$x^2+1=0$$

which has no solution among the real numbers (since x^2 is always ≥ 0), so we introduce $x = i = \sqrt{-1}$ which is then one solution ($-i$ is another).

Then, any square root like $\sqrt{-a} = \sqrt{a}\ \sqrt{-1} = i\sqrt{a}$.

However, let's just note in passing that the reverse is not necessarily true. Consider

$$i^2 = \sqrt{-1} \cdot \sqrt{-1} \stackrel{?}{=} \sqrt{(-1) \cdot (-1)} = \sqrt{1} = 1$$

The equality with a question mark is *not* valid, which explains why this "proof" is erroneous.

Thus, it may not be true that

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

when considering "imaginary" numbers.

We consider complex numbers z as combinations like $z = a + ib$ where a, b are both real numbers and b is called the imaginary part of the complex number z.

Some useful identities with i

$$i^2 = -1, \quad i = -\frac{1}{i}, \quad \frac{1}{i} = -i$$

For much of what is done with complex numbers the fact that i equals $\sqrt{-1}$ is irrelevant. Instead, we simply have ordered pairs of numbers (a, b) and the i is a bookkeeping device or marker to remind us that when we multiply

$$(a + ib)(c + id) = ac - bd + i(ad + bc)$$

the result of multiplying $ib \cdot id$ is a real number with the sign flipped, while a real number a times an imaginary number id is equal to iad.

Another way to keep track of the same information is in matrix form, namely:

$$a + ib = (a, b) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Such matrices can be added and multiplied in the normal way and give the desired results for complex numbers. Thus:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \times \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac+bd & -ad-bc \\ ad+bc & ac+bd \end{bmatrix}$$

A final way to write complex numbers is to use the complex plane, where points are plotted with the real part along the horizontal axis and the imaginary part along the vertical axis. r is the distance of the point from the origin and θ is the angle the ray makes with the positive x-axis in a CCW direction. Thus

$$a = r\cos\theta$$

$$b = r\sin\theta$$

$$a + ib = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta) = re^{i\theta}$$

where the last part makes use of Euler's famous equation. r is called the **modulus** and θ is called the **phase**.

Looking ahead, if

$$z = x + iy$$

then the complex conjugate of z is

$$z* = x - iy$$

and the *length* of z squared is

$$zz* = (x+iy)(x-iy)$$

$$= x^2 + y^2$$

$$= r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

.

Calculations may be easier in one form than another. Addition is better with $a + ib$ (the Cartesian format) and multiplication is better with the polar format, while matrices are fine for both addition and multiplication.

Here is multiplication in polar coordinates

$$r_1 e^{i\theta_1} \; r_1 e^{i\theta_1} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

and

$$(re^{i\theta})^2 = r^2 e^{i2\theta}$$

roots

Consider the value $\sqrt{z}$. Obviously, $\sqrt{r}\,\sqrt{r}$ is equal to r and then what we need to do is find an angle that is one-half of the original one, which leads to

$$\sqrt{re^{i\theta}} = \sqrt{r}\; e^{i\theta/2}$$

However, remember how in trigonometry we say that $\phi + 2k\pi$ is the same angle as ϕ for integer k? The same thing is true here. It is allowed that $\theta = (2k\pi + \theta)$ for $k \in \{0, \pm 1, \pm 2 \dots\}$, since it's the same point in the plane, which means that a second solution to the square root problem is

$$\sqrt{re^{i\theta}} = \sqrt{r}\; e^{i(\pi+\theta/2)}$$

because

$$\pi + \theta/2 + \pi + \theta/2 = 2\pi + \theta$$

which is the same angle as θ.

For the square root, there is only one additional distinct solution, since one-half of $4\pi + \theta = 2\pi + \theta/2$ is no different than $\theta/2$, but the cube root has 3 solutions and in general the n^{th} root has n solutions.

To keep things simpler, suppose we are on the unit circle with $r = 1$ and

$$\theta = \pi/2$$

so

$$z = e^{i\pi/2} = i$$

$\sqrt{e^{i\pi/2}} = \sqrt{i}$ has two possible values. One is

$$\sqrt{e^{i\pi/2}} = e^{i\pi/4}$$

and the other is

$$\sqrt{e^{i\pi/2}} = e^{i\ 5/4\pi}$$

Let's just check. The first one is the point at a distance 1 from the origin and angle $\theta = \pi/4$:

$$x = y = \cos\theta = \frac{1}{\sqrt{2}}$$

So we have that the square is:

$$(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})^2 = \frac{1}{2} - \frac{1}{2} + 2i\frac{1}{2} = i$$

and the second one is

$$x = -y = \cos\theta = -\frac{1}{\sqrt{2}}$$

The square is the same except the first term is $(-1/\sqrt{2})^2$, with the result unchanged. It's a bit counter-intuitive that squaring a number may possibly reduce the phase angle. Suppose

$$z = a^n$$

$$a = r(\cos\theta + i\sin\theta)$$

$$z = a^{1/n}$$

$$= r^{1/n} \cdot \ (\cos\theta + i\sin\theta)^{1/n}$$

$$(\cos\theta + i\sin\theta)^{1/n} = \cos\phi + i\sin\phi$$

where

$$\phi = \frac{\theta + 2k\pi}{n}, \quad k = 0, 1, 2 \ldots n-1$$

complex conjugate

The complex conjugate of z is written as $z*$ or $\bar{z}$. If

$$z = a + ib$$

then

$$z* = a - ib$$

That is, the point corresponding to $z*$ in the complex plane is the same as z, but reflected across the x-axis. In polar coordinates, if $z = re^{i\theta}$ then $z* = re^{i(-\theta)}$. Notice that

$$zz* = re^{i\theta} \ re^{i-\theta} = r^2 e^0 = r^2$$

Thus the value of z multiplied by its complex conjugate $z*$ is equal to the square of the length r.

We confirm that

$$zz* = (a + ib)(a - ib) = a^2 + b^2$$

Multiplication by $z*$ makes the product entirely real. Using the complex conjugate makes it easy to deal with

$$\frac{1}{z} = \frac{z*}{zz*} = \frac{a - ib}{a^2 + b^2}$$

or in polar notation:

$$= \frac{re^{-i\theta}}{r^2} = \frac{1}{r} \ e^{-i\theta}$$

Let's think about what this means for different z. If $r = 1$. In every case, the point is reflected across the x-axis (the ray has an angle $-\theta$ with the x-axis. There is no change in length for $r = 1$. But if say

$$z = 1 + i = (1, 1) = \sqrt{2}\ e^{i\ \pi/4}$$

then the new point has $r = \frac{1}{\sqrt{2}}$ and it is at

$$z = \frac{1}{\sqrt{2}} e^{-i\ \pi/4} = (\frac{1}{2}, -\frac{1}{2}) = \frac{1}{2} - i\frac{1}{2}$$

Actually, for that matter, addition of the complex conjugate also gives an entirely real result:

$$z + z* = x + iy + x - iy = 2x$$

exponential

Finally, we simply note some complex functions:

$$e^z = e^{x+iy} = e^x(e^{iy})$$

$$= e^x(\cos y + i \sin y)$$

$$= e^x \cos y + ie^x \sin y$$

Let n be an integer:

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

Suppose $r = 1$:

$$z^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

trig functions

Define the complex counterparts of the real trigonometric functions similarly to their real equivalents by

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

The usual properties hold, for example:

$$\frac{d}{dz}\ \sin z = \frac{1}{2i}(ie^{iz} + ie^{-iz})$$

$$= \frac{1}{2}(e^{iz} + e^{-iz}) = \cos z$$

and

$$\frac{d}{dz}\ \cos z = \frac{1}{2}(ie^{iz} + ie^{-iz})$$

$$= \frac{i}{2}(e^{iz} + e^{-iz})$$

$$= -\frac{1}{2i}(e^{iz} + e^{-iz}) = -\cos z$$

We can decompose, e.g. $\sin z$ like this

$$\sin z = \sin(x + iy) = \sin x \cos iy + \sin iy \cos x$$

Now

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\cos iy = \frac{1}{2}(e^{-y} + e^{y})$$

$$= \cosh y$$

Similarly

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\sin iy = \frac{1}{2i}(e^{-y} - e^{y})$$

$$\sin iy = -\frac{1}{2i}(e^{y} - e^{-y})$$

$$= -\frac{1}{i}\ \sinh y$$

$$= i \sinh y$$

Going back to the decomposition:

$$\sin z = \sin(x + iy) = \sin x \cos iy + \sin iy \cos x$$

$$= \sin x \cosh y + i \sinh y \cos x$$

Similarly one can show that

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

and

$$\sin^2 z + \cos^2 z = 1$$

If we say that Euler's formula is also good for a complex number z:

$$e^{iz} = \cos z + i \sin z$$

$$e^{-iz} = \cos -z + i \sin -z = \cos z - i \sin z$$

So

$$e^{iz} + e^{-iz} = 2 \cos z$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

and

$$e^{iz} - e^{-iz} = 2i \sin z$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Also

$$\cos iz = \frac{e^{-z} + e^{z}}{2} = \cosh z$$

$$\sin iz = \frac{e^{-z} - e^{z}}{2i}$$

$$i \sin iz = \frac{-1}{i} \sin iz = \frac{e^{-z} - e^{z}}{2} = \sinh z$$

logarithm

Nearly everything works for z similarly to the real numbers, except for the issue of multiple phase angles. For example

$$\ln z = \ln re^{i\theta} = \ln r + i\theta$$

but we may have any multiple of $2k\pi$ added to θ:

$$\ln z = \ln re^{i\theta + 2k\pi} = \ln r + i(\theta + 2k\pi), \quad k \in \{0, 1, 2, \dots\}$$

The value with $k = 0$ is called the **principal value** of $\ln z$, and it's taken from $-\pi \leq \theta \leq \pi$.

onward

Now we get to the point of this write-up. These are my first notes on the calculus of complex functions.

The sources include the beginning of Chapter 6 in Shankar's Basic Training book, Boas Chapter 14, and Nahin's *An imaginary tale: the story of* $\sqrt{-1}$, as well as a set of notes by Michael Alder that I found online. I have not worked through all of these, by any means. I am just trying to solidify my understanding by writing things out in my own format, if not my own words entirely.

As Shankar says, just as we can associate with points on the x-axis a function $f(x)$, so we may associate with each point (x, y) in the complex plane a function

$$f(x, y) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real functions of two real variables i.e.

$$f : \mathbb{R}^2 \to \mathbb{R}^1$$

How to define continuity at a point for such a function? Well, first the function must be defined *at* the point. And then the function must approach that value as the limit when approaching the point.

The new twist is that since we are in $\mathbb{R}^2$ there is an infinite number of directions from which we can approach, rather than just two as in $\mathbb{R}^1$.

Example: the function

$$f(x, y) = \frac{x^2}{x^2 + y^2}$$

has several problems: it is not defined at the origin $(0, 0)$ but also, as we approach the origin along the x-axis and the y-axis we get different limiting values, namely

$$f(x, 0) = \frac{x^2}{x^2} = 1$$

$$f(0, y) = \frac{0}{y^2} = 0$$

Rewriting it in polar coordinates ($x = r\cos\theta, r^2 = x^2 + y^2$):

$$f(r, \theta) = \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$$

Shankar says: the function f is generally a function of *two* complex variables:

$$z = x + iy$$

$$z* = x - iy$$

which can be written in terms of x and y as

$$x = \frac{z + z*}{2}$$

$$y = \frac{z - z*}{2i}$$

Generally, the value of f depends on both z and $z*$, but we will be very interested in functions which depend only on z and not $z*$. The reason for this is that only such functions have the property that the derivative at a point does not depend on the direction from which we approach that point.

Consider the function:

$$f(x, y) = x^2 - y^2$$

$$= \frac{(z + z*)^2}{4} + \frac{(z - z*)^2}{4}$$

$$= \frac{1}{4} \left[\ z^2 + 2zz * + z *^2 + z^2 - 2zz * + z*^2 \ \right]$$

$$= \frac{z^2 + z*^2}{2}$$

This function is not a function only of z but of both z and $z*$.

We say that f is an *analytic* function of z if it does not depend on $z*$. Shankar says this means that "x and y enter f *only* in the combination $x + iy$".

The famous Cauchy-Riemann Equations (CRE) are true for $f \iff f$ is an analytic function of z.

For:

$$f(x,y) = u(x,y) + iv(x,y)$$

The CRE conditions are:

$$u_x = v_y$$

$$u_y = -v_x$$

I didn't quite follow the derivation he gives (see below) but let's just try them out.

Consider:

$$f(x,y) = x^2 - y^2 + 2ixy$$

CRE requires

$$u_x = 2x \stackrel{?}{=} v_y = 2x$$

$$v_x = 2y \stackrel{?}{=} -u_y = 2y$$

The function is analytic. As Shankar says, this is expected because:

$$x^2 - y^2 + 2ixy = (x+iy)(x+iy) = z^2$$

Consider:

$$f(x,y) = \cos y - i \sin y$$

CRE requires:

$$u_x = 0 \stackrel{?}{=} v_y = -\cos y$$

$$v_x = 0 \stackrel{?}{=} -u_y = -\sin y$$

This is "impossible" since there is no y that satisfies both of the conditions. And it's not surprising since

$$y = \frac{z - z*}{2i}$$

Consider:

$$f(x,y) = x^2 + y^2$$

CRE requires:

$$u_x = 2x \stackrel{?}{=} v_y = 2y$$

$$u_y = 0 \stackrel{?}{=} -v_x$$

CRE are only satisfied if $x = y$. Also not surprising since

$$x^2 + y^2 = zz*$$

Consider:

$$f(x, y) = x^2 - y^2$$

CRE requires:

$$u_x = 2x \stackrel{?}{=} v_y = -2y$$

which is true if $x = y$.

$$u_y = 0 \stackrel{?}{=} -v_x = 0$$

But "no importance is given to functions which obey the CRE only at isolated points or on lines."

Consider:

$$f(x, y) = e^x \cos y + ie^x \sin y$$

CRE requires:

$$u_x = e^x \cos y \stackrel{?}{=} v_y = e^x \cos y$$

$$u_y = -e^x \sin y \stackrel{?}{=} -v_x = -\sin ye^x$$

Both are true, so this one does satisfy CRE.

As an aside, Shankar doesn't mention it here but the last function is special, it is $f(z) = e^z$:

$$e^x \cos y + ie^x \sin y$$

$$= e^x(\cos y + i \sin y)$$

$$= e^x e^{iy}$$
$$= e^{x+iy}$$
$$= e^z$$

For functions of interest, it may often be true that CRE fails at particular points called *singularities*.

Consider:

$$f(x,y) = \frac{1}{z} = \frac{z*}{zz*} = \frac{x - iy}{x^2 + y^2}$$

We need:

$$u_x = \frac{d}{dx}\ \frac{x}{x^2+y^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$v_y = \frac{d}{dy}\ (-\frac{y}{x^2+y^2}) = -\frac{x^2-y^2}{(x^2+y^2)^2} = u_x$$
$$u_y = 0 = v_x$$

But the function blows up at the origin. This described by saying it has a pole at the origin. The function

$$f(z) = \frac{c}{z}$$

where c is a constant, also blows up at the origin. We say that the *residue* of the pole at the origin is c.

For a function to be analytic it must be true that

$$f_x = -i f_y$$

(see below) where

$$f_x = \frac{\partial f}{\partial x}$$
$$f_y = \frac{\partial f}{\partial y}$$

Thus

$$f_x = u_x + iv_x$$
$$f_y = u_y + iv_y$$
$$-if_y = -i(u_y + iv_y) = v_y - iu_y$$

Since both real and imaginary parts must be equal, we obtain the CRE.

Proof of CRE

Recall that

$$f = f(z)$$
$$z = x + iy$$
$$\frac{\partial z}{\partial x} = 1, \quad \frac{\partial z}{\partial y} = i$$

By the standard rules of partial differentiation from multivariable calculus:

$$\frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz} \cdot 1$$
$$\frac{\partial f}{\partial y} = \frac{df}{dz}\frac{\partial z}{\partial y} = \frac{df}{dz} \cdot i$$

But we also have that

$$z = u(x) + iv(y)$$

Unfortunately u and v are difficult to distinguish in this font.

So let me rewrite them as g and h:

$$z = g(x) + ih(y)$$
$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} + i\frac{\partial h}{\partial x}$$
$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} + i\frac{\partial h}{\partial y}$$

Combining the first and third equations:

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} + i\frac{\partial h}{\partial x}$$

At this point, I want to abuse the notation somewhat by substituting $\partial g/\partial x = g_x$ and so on. Thus

$$\frac{df}{dz} = g_x + ih_x$$

Combining the second and fourth equations:

$$i\frac{df}{dz} = \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} + i\frac{\partial h}{\partial y}$$

Similar to above, rewrite this as

$$i\frac{df}{dz} = g_y + ih_y$$

Multiply by $-i$:

$$\frac{df}{dz} = -ig_y + h_y$$

So now we have two expressions for df/dz, which are, of course, equal and in particular, both the real and the imaginary parts must be equal.

$$g_x + ih_x = h_y - ig_y$$

Hence:

$$g_x = h_y$$

$$g_y = -h_x$$

And now we should go back to the standard u and v:

$$u_x = v_y$$

$$u_y = -v_x$$

which are the CRE.

It is worth taking a breath for a moment and repeating what we just said: the derivative of an analytic (that is, differentiable) complex function z is

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

$$= u_x + iv_x = -i(u_y + iv_y) = v_y - iu_y$$

a second derivation

Because this is all still quite new it may be useful to work through an apparently different derivation of the CRE, as presented in Nahin. Consider how to define the derivative of a complex function. Suppose we define:

$$\frac{df}{dz} = f'(z_0)$$

$$= \lim_{\Delta z \to 0} f(z_0 + \Delta z) - f(z_0)/\Delta z$$

Now

$$z = f(x, y) = x + iy$$

So

$$f'(z_0) = \lim_{\Delta x, \Delta y \to 0} f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)/\Delta x + i\Delta y$$

And now we insist that the value of the derivative should be independent of the direction of approach to the origin. So, out of all the possible approaches, come along the x-axis or along the y-axis. In the first case, $\Delta y = 0$ and we have

$$f'(z_0) = \lim_{\Delta x \to 0} f(x_0 + \Delta x, y_0) - f(x_0, y_0)/\Delta x$$

Expand in u and v

$$= \lim_{\Delta x \to 0} u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - ivu(x_0, y_0)/\Delta x$$

$$= \lim_{\Delta x \to 0} u(x_0 + \Delta x, y_0) - u(x_0, y_0) + iv(x_0 + \Delta x, y_0) - ivu(x_0, y_0)/\Delta x$$

$$= u_x + iv_x$$

Similarly, approaching along the y-axis ($\Delta x = 0$):

$$f'(z_0) = \lim_{\Delta y \to 0} f(x_0, y_0 + \Delta y) - f(x_0, y_0)/i\Delta y$$

Expand in u and v

$$= \lim_{\Delta y \to 0} u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - ivu(x_0, y_0)/i\Delta y$$

$$= \lim_{\Delta y \to 0} u(x_0, y_0 + \Delta y) - u(x_0, y_0) + iv(x_0, y_0 + \Delta y) - ivu(x_0, y_0)/i\Delta y$$

$$= \frac{1}{i}(u_y + iv_y)$$

$$= v_y - iu_y$$

Insisting that these two values for the derivative must be equal:

$$u_x + iv_x = v_y - iu_y$$

Since both real and imaginary parts must be equal, we have the CRE.

A third, very simple proof is given in Alder:

Suppose $f : C \to C$ is a function, taking $x + iy$ to $u(x, y) + iv(x, y)$, then the derivative is a matrix of partial derivatives:

$$\begin{matrix} u_x & u_y \\ v_x & v_y \end{matrix}$$

.. the above matrix is the two dimensional version of the slope of the tangent line in dimension one. It gives the linear part (corresponding to the slope) of the affine map which best approximates f at each point.

.. if f is differentiable in the *complex* sense, this must be just a linear complex map, i.e. it multiplies by some complex number. So the matrix must be in our set of complex numbers. In other words, for every value of x it looks like

$$\begin{matrix} a & -b \\ b & a \end{matrix}$$

for some real numbers a, b, which change with x.

Of course, this constraint leads directly to the CRE.

harmonics

Boas says this about analytic functions (that satisfy the CRE).

"If $f(z) = u = iv$ is analytic in a region, then u and v satisfy Laplace's equation, that is, u and v are harmonic functions..."

Laplace's equation is:

$$\nabla^2 f = 0$$

Consider the function

$$u(x,y) = x^2 - y^2$$
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
$$= 2 - 2 = 0$$

To find the function $v(x,y)$ such that $z = u + iv$ is analytic, use the CRE:

$$u = x^2 - y^2$$

$$v_y = u_x = 2x$$
$$v_x = -u_y = 2y$$

So it looks like $2xy$ will work. In particular

$$z = x^2 - y^2 + i2xy + \text{constant}$$

But of course

$$x^2 - y^2 + i2xy = (x + iy)^2 = z^2$$

which does not depend on $z*$.

To explore why this is true, take the second derivatives of the CRE:

$$u_x = v_y$$

$$u_y = -v_x$$

$$u_{xx} = v_{yx}$$
$$u_{xy} = v_{yy}$$
$$u_{yx} = -v_{xx}$$
$$u_{yy} = -v_{xy}$$

But the mixed partials must be equal so

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy}$$
$$u_{xx} + u_{yy} = 0$$
$$v_{xx} = -u_{yx} = -u_{xy} = -v_{yy}$$
$$v_{xx} + v_{yy} = 0$$

analyticity

According to Nahin:

Every polynomial of z is analytic. If $f(z)$ and $g(z)$ are analytic functions then so are $f(z)+g(z)$, $f(z) \cdot g(z)$, $f(z)/g(z)$ where $g(z) \neq 0$, and $f(g(z))$. Accordingly, if a function is a polynomial or can be expanded as a polynomial it is analytic:

$$f(z) = z^2 = x^2 - y^2 + i2xy$$

$$f(z) = e^z$$

$$f(z) = \frac{z^2}{z^2+1}$$

The last one has singularities at $z = \pm i$.

Here is one more which we will use in to explore contour integrals later.

$$f(z) = e^{-z^2}$$

$$= e^{-(x^2-y^2+i2xy)}$$

$$= e^{-x^2+y^2-i2xy}$$

I wrote this as

$$re^{i\theta}$$

where

$$r = e^{-x^2+y^2} = e^{-x^2} e^{y^2}$$

$$\theta = -2xy$$

Then I say (using $\cos\theta = \cos -\theta$):

$$u(x,y) = e^{-x^2} e^{y^2} \cos\theta = e^{-x^2} e^{y^2} \cos 2xy$$

$$v(x,y) = e^{-x^2} e^{y^2} \sin\theta = e^{-x^2} e^{y^2} (-\sin 2xy)$$

conformal mapping

At this point let's just remind ourselves how the visual representation of a complex function differs from the more familiar case of a function $f : \mathbb{R}^1 \to \mathbb{R}^1$.

In the real case, the first dimension is the independent variable x and the second is $y = f(x)$ and the derivative is the *slope* of the curve produced by plotting pairs of $x, f(x)$.

In the complex case, our numbers z are points in the complex plane. They are *mapped* to other complex numbers in a different complex plane, which is often called w, where $w = f(z) = u(x, y) + iv(x, y)$.

The derivative does not have any notion of slope. Our requirement for differentiability included the constraint that the derivative of the function at a point z_0 must be the same no matter from what direction we approach that point. This leads to the CRE and the study of only analytic functions. Many such functions may have isolated points at which they are not defined, and that will still be OK.

In the figure, is shown the mapping corresponding to the complex function $w = f(z) = z^2$.

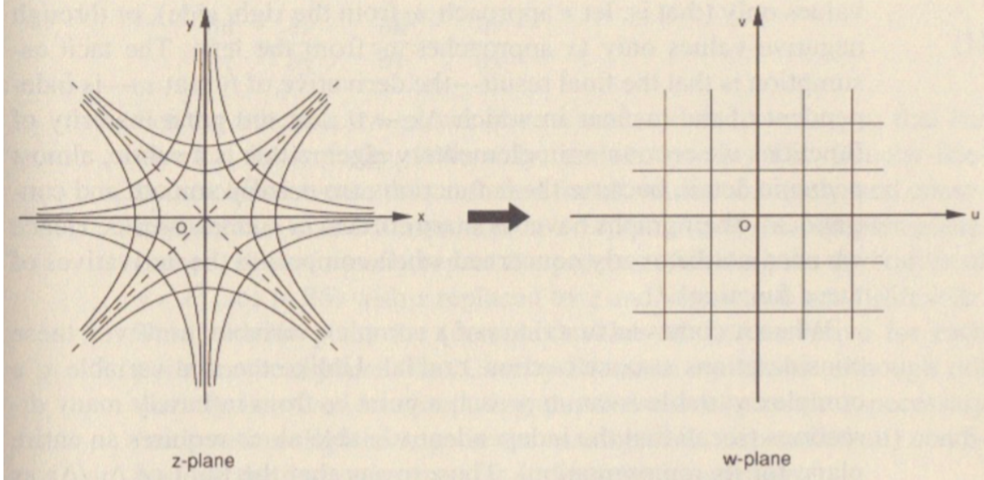


FIG. 71. Mapping by the complex function $w = z^2$.

$$z = x + iy$$
$$z^2 = x^2 - y^2 + i2xy$$

The functions $u = x^2 - y^2$ and $v = 2xy$, they are both hyperbolas. Consider what happens in the case where $u = c$ where c is a constant. The values of x and y that satisfy this constrain lie on hyperbolas in the z-plane. For example, the points on the curve $xy = 1/2$ correspond to the points on the curve $v = 1$, which is a straight vertical line in the w plane.

In this sense, the hyperbolic curves in the z-plane shown in the figure are mapped into rectangular grid in the w-plane. An important note here is that the angles where these curves meet are the same in both the z-plane and the w-plane. In both cases the lines meet at right angles.

According to wolfram

`http://mathworld.wolfram.com/ConformalMapping.html`

A conformal mapping, also called a **conformal map**, conformal transformation, angle-preserving transformation, or biholomorphic map, is a transformation that preserves local angles. An analytic function is conformal at any point where it has a nonzero derivative.

looking ahead

As motivation to do the work that is coming, consider these statements from the summary article in wikipedia:

One of the central tools in complex analysis is the line integral. The line integral around a closed path of a function that is holomorphic everywhere inside the area bounded by the closed path is always zero, which is what the Cauchy integral theorem states. The values of such a holomorphic function inside a disk can be computed by a path integral on the disk's boundary, as shown in (Cauchy's integral formula).

Path integrals in the complex plane are often used to determine complicated real integrals, and here the theory of residues among others is applicable (see methods of contour integration). A "pole" (or isolated singularity) of a function is a point where the function's value becomes unbounded, or "blows up". If a function has such a pole, then one can compute the function's residue there, which can be used to compute path integrals involving the function; this is the content of the powerful residue theorem.

Cauchy' First Integral Theorem

Cauchy 1 is a theorem that say the integral of an analytic function over a closed path (over a region without a singularity), is equal to zero.

$$\oint_C z\ dz$$

Assume for the moment that the theorem is correct. We will integrate over a rectangle $(R = [0, a] \times [b, 0]$. Write

$$z = x + iy$$

$$dz = dx + idy$$

$$f(x, y) = u(x, y) + iv(x, y)$$

Our integral is

$$\int z\ dz = \int (u + iv)\ (dx + idy)$$

$$= \int u\ dx - \int v\ dy + i \int v\ dx + i \int u\ dy$$

Since the whole thing is equal to zero over our closed path, both parts are equal to zero:

$$\int u\ dx - \int v\ dy = 0$$

The function we'll be working with is the one we introduced before:

$$u(x, y) = e^{-x^2} e^{y^2} \cos 2xy$$

$$v(x, y) = e^{-x^2} e^{y^2} (-\sin 2xy)$$

Everything will simplify pretty quickly. Divide the path into its four parts and compute each separately: Over $C1$, $y = 0$ and $dy = 0$ so we have:

$$\int_{C1} = \int u\ dx = \int_0^a e^{-x^2} e^0 \cos 0\ dx = \int_0^a e^{-x^2}\ dx$$

C2 ($x = a$, $dx = 0$):

$$\int_{C2} = -\int_0^b e^{-a^2} e^{y^2} (-\sin 2ay)\ dy$$

C3 ($y = a$, $dy = 0$):

$$\int_{C3} = \int_a^0 e^{-x^2} e^{b^2} (\cos 2bx)\ dx$$

C4 ($x = 0$, $dx = 0$):

$$\int_{C4} = \int_b^0 e^{y^2} (-\sin 0)\ dy = 0$$

So all together:

$$\int_0^a e^{-x^2}\ dx - \int_0^b e^{-a^2} e^{y^2} (-\sin 2ay)\ dy + \int_a^0 e^{-x^2} e^{b^2} \cos 2bx\ dx = 0$$

$$\int_0^a e^{-x^2}\ dx = e^{-a^2} \int_0^b e^{y^2} (-\sin 2ay)\ dy + e^{b^2} \int_0^a e^{-x^2} \cos 2bx\ dx$$

Let $a \to \infty$. Then

$$e^{-a^2} \to 0$$

so the first term on the right side goes to zero and we have:

$$\int_0^\infty e^{-x^2}\ dx = e^{b^2} \int_0^\infty e^{-x^2} \cos 2bx\ dx$$

But we know the value of the left-hand side, it is

$$\int_0^\infty e^{-x^2}\ dx = \frac{\sqrt{\pi}}{2}$$

so

$$\int_0^\infty e^{-x^2} \cos 2bx\ dx = \frac{\sqrt{\pi}}{2}\ e^{-b^2}$$

The Gaussian that we knew, is a special case of this general form.

proof of Cauchy 1

As above, we write:

$$z = x + iy$$

$$dz = dx + idy$$

$$f(x, y) = u(x, y) + iv(x, y)$$

Our integral is

$$\int z \ dz = \int (u + iv) \ (dx + idy)$$

$$= \int u \ dx - \int v \ dy + i \int v \ dx + i \int u \ dy$$

And we are going around a closed path so

$$= \oint u \ dx - \oint v \ dy + i \oint v \ dx + i \oint u \ dy$$

Back in vector calculus we proved Green's theorem, which says that for two real functions of x and y: $M(x, y)$ and $N(x, y)$:

$$\oint_C M dx + N dy = \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) \ dx \ dy$$

Back then, M and N were components of a vector field

$$= \iint_R \nabla \times \mathbf{F} \ dA$$

but the important thing is just that they are real functions of two variables $f : \mathbb{R}^2 \to \mathbb{R}^2$.

In terms of u and v we have for the real part of Cauchy's Theorem that $M = u$ and $N = -v$ (notice the minus sign!). So:

$$\oint u \ dx - \oint v \ dy = \iint_R (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \ dx \ dy$$

$$= -\iint_R (v_x + u_y)\ dx\ dy$$

But, according to the CRE

$$u_y = -v_x$$

Hence, the integral is zero.

For the imaginary part:

$$\oint v\ dx + \oint u\ dy = \iint_R (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})\ dx\ dy$$

$$= \iint_R (u_x - v_y)\ dx\ dy$$

But, according to the CRE

$$u_x = v_y$$

And again, the integral is zero. Therefore:

$$= \oint u\ dx - \oint v\ dy + i \oint v\ dx + i \oint u\ dy = 0$$

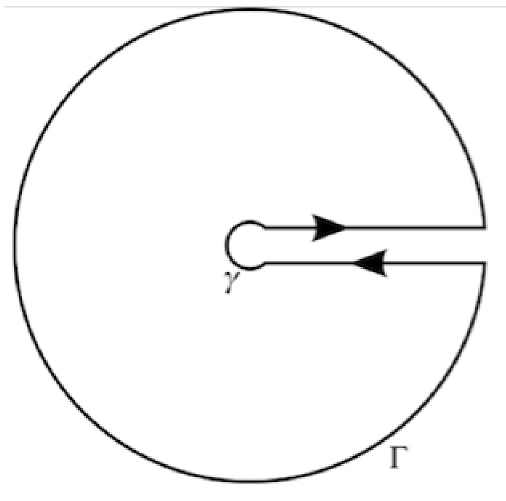
derivation of Cauchy 2

Suppose $f(z)$ is analytic everywhere within some region *except* at a singularity, z_0. For example, suppose we have

$$\frac{f(z)}{z - z_0}$$

and suppose we integrate this around a special closed path in the region of analyticity:

$$\oint \frac{f(z)}{z - z_0}\ dz$$



It's not labeled but the singularity z_0 is at the center of the two concentric circles. The "keyhole" excludes z_0 so f is analytic everywhere in the region enclosed by the path, and the total integral is zero by Cauchy's first theorem. The straight line segments are so close as to be equal, but traversed in opposite directions, so the contribution from them is zero. Thus we have that the integral around the outer ring counter-clockwise + the integral around the inner ring clockwise add up to zero.

Reversing the direction of integration on the inner ring changes the sign of the value, hence we have that

$$\oint_{C \text{ outer}} \frac{f(z)}{z - z_0}\, dz - \oint_{C \text{ inner}} \frac{f(z)}{z - z_0}\, dz = 0$$

But we haven't said anything about the radius of these rings.

What this means is that the value of the integral around a ring enclosing a singularity is not zero, but it has the same value for a ring of *any* radius. (It is independent of the radius).

$$\oint_{C \text{ outer}} \frac{f(z)}{z - z_0}\, dz = \oint_{C \text{ inner}} \frac{f(z)}{z - z_0}\, dz$$

We can parametrize this path by saying that each point on the curve is given by

$$z = z_0 + \rho e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

$$dz = \rho i e^{i\theta} \; d\theta$$

$$\oint \frac{f(z)}{z - z_0} \; dz = \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}} \; \rho i e^{i\theta} \; d\theta$$

$$= i \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) \; d\theta$$

$$= i \int_0^{2\pi} f(z) \; d\theta$$

We may choose ρ as small as we like, and so we choose it very small $(\rho \to 0)$ so

$$f(z) \to f(z_0) = \text{constant}$$

and since it's constant we can bring it out of the integral!

$$\oint \frac{f(z)}{z - z_0} \; dz = f(z_0) i \int_0^{2\pi} d\theta$$

$$= 2\pi i f(z_0)$$

using Cauchy 2

Example: consider a semicircle of radius R lying in the first two quadrants with its diameter on the real x-axis, and a function. We wish to evaluate:

$$\oint_C f(z) \; dz$$

$$= \oint_C \frac{e^{iaz}}{b^2 + z^2} \; dz$$

where a and b are positive constants. $f(z)$ has singularities at $z = \pm ib$, where $b < R$. In particular, we are interested in what happens as $R \to \infty$.

Along the x-axis, we have $y = 0$ and $dz = dx$ and

$$\int_{C1} = \int_{-R}^{R} \frac{e^{iax}}{b^2 + x^2}\ dx$$

Along the semi-circular arc, we have $r = R$ and $\theta = 0 \to \pi$ and

$$z = Re^{i\theta}$$

$$dz = iRe^{i\theta}\ d\theta$$

$$\int_{C2} = \int_{0}^{\pi} \frac{e^{ia(Re^{i\theta})}}{b^2 + R^2 e^{i2\theta}}\ iRe^{i\theta}\ d\theta$$

Thus

$$\oint_C z\ dz = \oint_C \frac{e^{iaz}}{b^2 + z^2}\ dz$$

$$= \int_{-R}^{R} \frac{e^{iax}}{b^2 + x^2}\ dx + \int_{0}^{\pi} \frac{e^{ia(Re^{i\theta})}}{b^2 + R^2 e^{i2\theta}}\ iRe^{i\theta}\ d\theta$$

Rewriting the integrand for the integral on the left-hand side:

$$\frac{e^{iaz}}{b^2 + z^2} = \frac{e^{iaz}}{(z + ib)(z - ib)} = \frac{e^{iaz}}{i2b}(\frac{1}{z - ib} - \frac{1}{z + ib})$$

Having factored out $1/i2b$, rewrite the integral as

$$\frac{1}{i2b}(\oint \frac{e^{iaz}}{z - ib}\ dz - \oint \frac{e^{iaz}}{z + ib}\ dz) = \int_{-R}^{R} \frac{e^{iax}}{b^2 + x^2}\ dx + \int_{0}^{\pi} \frac{e^{ia(Re^{i\theta})}}{b^2 + R^2 e^{i2\theta}}\ iRe^{i\theta}\ d\theta$$

That's quite a mouthful!

The second term on the left-hand side has a singularity at $z = -ib$, which is *outside* the region and hence by Cauchy 1 that integral is zero.

So now

$$\frac{1}{i2b}\oint\frac{e^{iaz}}{z-ib}\,dz=\int_{-R}^{R}\frac{e^{iax}}{b^2+x^2}\,dx+\int_0^{\pi}\frac{e^{ia(Re^{i\theta})}}{b^2+R^2e^{i2\theta}}\,iRe^{i\theta}\,d\theta$$

Looking at the other contour integral

$$\frac{1}{i2b}\oint\frac{e^{iaz}}{z-ib}\,dz$$

we have a singularity at $z=z_0=ib$, which, as R becomes large, is inside the semicircular region and thus by Cauchy 2 the integral is equal to $2\pi i\ f(z_0)$ where

$$f(z_0)=e^{iaz_0}=e^{iaib}=e^{-ab}$$

and so we have

$$\begin{aligned}\frac{1}{i2b}\oint\frac{e^{iaz}}{z-ib}\,dz&=\frac{1}{i2b}2\pi ie^{-ab},\quad R>b\\&=\frac{\pi}{b}e^{-ab}\end{aligned}$$

Putting it all together

$$\frac{\pi}{b}e^{-ab}=\int_{-R}^{R}\frac{e^{iax}}{b^2+x^2}\,dx+\int_0^{\pi}\frac{e^{ia(Re^{i\theta})}}{b^2+R^2e^{i2\theta}}\,iRe^{i\theta}\,d\theta$$

Nahin shows that the second integral on the right-hand side vanishes as $R\to\infty$. This has gotten a bit out of hand so I will just assume that part and say then that:

$$\frac{\pi}{b}e^{-ab}=\int_{-\infty}^{\infty}\frac{e^{iax}}{b^2+x^2}\,dx$$

$$\frac{\pi}{b}e^{-ab}=\int_{-\infty}^{\infty}\frac{\cos(ax)}{b^2+x^2}\,dx+i\int_{-\infty}^{\infty}\frac{\sin(ax)}{b^2+x^2}\,dx$$

The imaginary part of the left-hand side is zero, so we have

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{b^2 + x^2}\ dx = 0$$

"which is no surprise since the integrand is an odd function of x". But the other result is:

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{b^2 + x^2}\ dx = \frac{\pi}{b} e^{-ab}$$

In the special case $a = b = 1$ we obtain

$$\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2}\ dx = \frac{\pi}{e} = 1.15572735$$

which is not only an integral we didn't know how to do before, but a remarkable fraction as the result.

Numerical integration from $0 \to b$ where $1 \le b \le 100$, and then multiplying by 2 gives values around 1.15, but there is a bit of instability and it starts not to work at all above 80 or so. Probably an overflow issue. Increasing b doesn't make the approximation better.

residues

Consider

$$f(z) = \frac{1}{z} = \frac{z*}{zz*} = \frac{x - iy}{x^2 + y^2}$$

This function is analytic because it depends only on z and not on $z*$. If you insist, we can calculate the derivatives (note that all four derivatives have $(x^2 + y^2)^2 = d$ for the denominator. The numerators are:

$$u_x :\ x^2 + y^2 - 2x^2 = y^2 - x^2$$

$$u_y :\ -2yx$$

$$v_x: \; -(-2xy) = -u_y$$
$$v_y: \; -(x^2 - y^2) = y^2 - x^2 = u_x$$

However $1/z$ "blows up" at the origin, and we say that it has a singularity or pole at the origin and the residue at the origin is equal to 1.

The definition of the residue R at $z = z_0$ where z_0 is a pole of $f(z)$:

$$R(z = z_0) = \lim_{z \to z_0} \; (z - z_0) \; f(z)$$

In the case of $f(z) = 1/z$

$$R(z = z_0) = \lim_{z \to z_0} \; \frac{z - z_0}{z}$$
$$= \lim_{z \to 0} \; \frac{z - 0}{z} = 1$$

If the function were c/z with c a constant, then the residue would be

$$\lim_{z \to z_0} \; (z - z_0) \; \frac{c}{z} = c$$

As a second example, consider

$$f(z) = \frac{1}{z+i} \cdot \frac{1}{z-i}$$

This function has poles at two points, $z = \pm i$. As we approach $z = i$ we obtain

$$= \frac{1}{2i} \; \frac{1}{z-i}$$

the residue at this pole is $1/2i$.

In this problem we can do:

$$f(z) = \frac{1}{z+i} \cdot \frac{1}{z-i} = \frac{1}{2i} \; (\frac{1}{z-i} - \frac{1}{z+i})$$

To see that this is true put the fractions on the right-hand side over a common denominator:

$$= \frac{1}{2i} \, (\frac{(z+i)-(z-i)}{(z-i)(z+i)})$$

so

$$R(z=i) = \lim_{z \to i} \, (z-i) f(z)$$

$$= \lim_{z \to i} \, (z-i) \, \frac{1}{2i} \, (\frac{1}{z-i} - \frac{1}{z+i})$$

$$= \frac{1}{2i} \, (1-0) = \frac{1}{2i}$$

The theory of residues is rich and (forgive me) complex, so I think I'll stop here and go study up some more.

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