Integration—practice problems

We can construct some good problems of our own, looking at what we get by differentiating various products, using the "product rule". The first one is particularly enlightening.

$$\begin{aligned} \frac{d}{dx}\ \sin x \cos x &= (\sin x \cos x)' \\ &= \sin x \cos x' + \sin x' \cos x \\ &= -\sin^2 x + \cos^2 x \end{aligned}$$

Starting with the identity $\sin^2 x + \cos^2 x = 1$, we can rearrange to obtain:

$$\sin^2 x + \cos^2 x - 1 = 0$$
$$-\sin^2 x - \cos^2 x + 1 = 0$$

Using these, the derivative above can be rearranged in various ways (by addition of the second identity):

$$\frac{d}{dx}\ \sin x \cos x = 1 - 2\sin^2 x$$

and (by addition of the first identity):

$$\frac{d}{dx}\ \sin x \cos x = 2\cos^2 x - 1$$

A third way uses the formula for addition of cosines:

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

$$\cos 2s = \cos^2 s - \sin^2 s = 2\cos^2 s - 1$$

Thus, a third version of the derivative is

$$(\sin x \cos x)' = \cos 2x$$

Summarizing, the equivalent forms are:

$$\begin{aligned}(\sin x \cos x)' &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \\ &= \cos 2x\end{aligned}$$

Using the second version

$$\sin x \cos x)' = 2\cos^2 x - 1$$

and integrating

$$\int (\sin x \cos x)' \ dx = \int 2\cos^2 x \ dx - \int 1 \ dx$$

$$\sin x \cos x = 2 \int \cos^2 x \ dx - x$$

$$\int cos^2 x \ dx = \frac{1}{2} \ [\ x + \sin x \cos x \ ] + C$$

And since (by the addition formula for sine)

$$\sin(s+t) = \sin s \cos t + \cos s \sin t$$

if $s = t$ then

$$\begin{aligned}\sin 2s &= 2 \sin s \cos s \\ \frac{1}{2} \sin 2s &= \sin s \cos s\end{aligned}$$

Plugging in above

$$\int cos^2x\ dx = \frac{1}{2}\ [\ x + \sin x \cos x\ ]$$

$$= \frac{1}{2}\ [\ x + \frac{1}{2} \sin 2x\ ]$$

On the other hand, above we had

$$(\sin x \cos x)' = \cos 2x = 1 - 2\sin^2 x$$

Integrating a rearranged version of the middle and right-hand terms, we obtain

$$\int 2\sin^2 x\ dx = \int 1\ dx - \int \cos 2x\ dx$$

$$\int sin^2x\ dx = \frac{1}{2}\ [\ x - \frac{1}{2} \sin 2x\ ] + C$$

However, it is always simpler to go from $\int \cos^2$ to $\int \sin^2$ by starting with the identity $\sin^2 x + \cos^2 x = 1$:

$$\sin^2 x + \cos^2 x = 1$$

So integrate

$$\int \sin^2 x\ dx + \int \cos^2 x\ dx = x$$

Plugging in from before, let's just check that last one:

$$\frac{1}{2}\ [\ x - \frac{1}{2} \sin 2x\ \ + \frac{1}{2}\ [\ x + \frac{1}{2} \sin 2x\ ] =?\ x$$

Looks like it checks.

At the end of all of this, the formulas to remember are the sum formulas (and how to derive the double-angle formulas) and the main result:

$$\int cos^2x\ dx = \frac{1}{2}\ [\ x + \frac{1}{2} \sin 2x\ ] + C$$

Actually, I can't remember this any more so I have to figure it out fresh. But if you you do have the formula, be sure to check it by differentiating:

$$\frac{d}{dx}\,\frac{1}{2}\,[\ x + \frac{1}{2}\sin 2x\ ] + C$$
$$= \frac{1}{2}(\ 1 + \cos 2x)$$

Then you still have to use the identity $\cos 2x = 1 - 2\sin^2 x$ so the derivative above is (remember, we are trying to get back to $\cos^2 x$):

$$= \frac{1}{2}(\ 1 + \cos 2x)$$
$$= \frac{1}{2}(\ 1 + 1 - 2\sin^2 x)$$
$$= 1 - \sin^2 x = \cos^2 x$$

OK!

example 2

$$\frac{d}{dx}\sin^2 x$$
$$= 2\sin x\ \frac{d}{dx}\sin x$$
$$= 2\sin x \cos x$$

Turning this around

$$\int \sin x \cos x\ dx = \frac{1}{2}\sin^2 x + C$$

On the other hand

$$\frac{d}{dx}\cos^2 x$$

$$= 2\cos x(-\sin x)$$

Turning this around

$$\int \sin x \cos x \; dx = -\frac{1}{2}\cos^2 x$$

Here, the constant is essential to save us from error when we equate the two. Write

$$\int \sin x \cos x \; dx = -\frac{1}{2}\cos^2 x = \frac{1}{2}\sin^2 x + C$$

Rearranging

$$\sin^2 x + \cos^2 x = 2C = 1$$

example 3

Another interesting one is

$$\frac{d}{dx}\; x \ln x = x\frac{1}{x} + \ln x = 1 + \ln x$$

Thus

$$x \ln x = x + \int \ln x \; dx$$

$$\int \ln x \; dx = x \ln x - x + C$$

If you don't remember this, use integration by parts with $u = \ln x$ and $dv = dx$, then $v = x$ and

$$\int \ln x \; dx = \int u \; dv = uv - \int v \; du$$

$$= \ln x \; x - \int x \; \frac{1}{x} \; dx = \ln x \; (x) - x$$

example 4

$$\frac{d}{dx} \ln\ (\cos x) = \frac{1}{\cos x}(-\sin x) - \tan x$$

Turning this around

$$\int \tan x\ dx = -\ln\ (\cos x) + C$$

example 5

$$\begin{aligned} \frac{d}{dx}\ \ln\ (\sec x + \tan x) &= \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x) \\ &= \sec x\ \frac{\tan x + \sec x}{\sec x + \tan x} \\ &= \sec x \end{aligned}$$

Turning this around

$$\int \sec x\ dx = \ln\ (\sec x + \tan x) + C$$

example 6

$$\begin{aligned} &\frac{d}{dx}\ \ln(\ln\ \ x) \\ &= \frac{1}{\ln x}\ \frac{1}{x} \\ &= \frac{1}{x\ \ln x} \end{aligned}$$

Turning this around

$$\int \frac{1}{x\ \ln x}\ dx = \ln(\ln\ \ x) + C$$

example 7

What is

$$\int \tan^2 x \; dx$$

We never saw $\tan^2 x$ as a simple derivative. However, we have the identity

$$1 + \tan^2 = \sec^2$$

And we *did* obtain $\sec^2 x$ as the derivative of $\tan x$!

$$\int \tan^2 x \; dx$$

$$= -\int 1 + \sec^2 x \; dx$$

$$= -x + \tan x + C$$

example 8

How about

$$\frac{d}{dx} \ln \sec x$$

$$= \frac{1}{\sec x} \; (\sec x \;\; \tan x) = \tan x$$

Turning this around

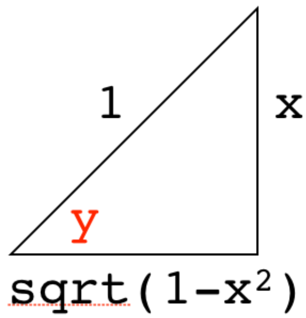
$$\int \tan x \; dx = \ln \sec x + C$$

inverse sine

The two most important inverse trig functions are $\sin^{-1}$ and $\tan^{-1}$. Rather than start with the integration problems, why not start by differentiating

$$\frac{d}{dx}\ \sin^{-1} x\ dx$$

We visualize a right triangle with y as the angle, x as the side opposite and 1 as the hypotenuse. Then the side adjacent is $\sqrt{1-x^2}$.



$$y = \sin^{-1} x$$
$$\frac{dy}{dx} = ?$$

Write something we *do* know

$$x = \sin y$$
$$\frac{dx}{dy} = \cos y$$
$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

Well alrighty then. Now we just turn it into an integration problem:

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x + C$$

inverse tangent

Visualize a different right triangle, again with y as the angle, and x as the side opposite but now 1 is the side adjacent and $\sqrt{1+x^2}$ is the hypotenuse.

$$y = \tan^{-1} x$$
$$\frac{dy}{dx} = ?$$

And again, write something we *do* know

$$x = \tan y$$
$$\frac{dx}{dy} = \sec^2 y$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y = \frac{1}{\sqrt{1+x^2}}$$

As before, we just turn it into an integration problem:

$$\int \frac{1}{\sqrt{1+x^2}}\, dx = y = \tan^{-1} x + C$$

There is a simple relationship between inverse sine and cosine:

$$\sin^{-1} x + \cos^{-1} = \frac{\pi}{2}$$
$$\frac{d}{dx}\sin^{-1} x + \frac{d}{dx}\cos^{-1} = 0$$
$$\frac{d}{dx}\sin^{-1} x = -\frac{d}{dx}\cos^{-1}$$

which is why we don't spend much time worrying about $\cos^{-1}$.

inverse secant

The last useful one of these is the inverse secant.

We put x as the hypotenuse, and 1 as the side adjacent to the angle y. Then the side opposite is $\sqrt{x^2 - 1}$.

$$y = \sec^{-1} x$$
$$\frac{dy}{dx} = ?$$

And yet again, write something we *do* know

$$x = \sec y$$
$$\frac{dx}{dy} = \sec y \tan y$$
$$\frac{dy}{dx} = \frac{1}{\sec y} \frac{1}{\tan y}$$
$$= \frac{1}{x} \frac{1}{\sqrt{x^2 - 1}}$$

Just turn it into an integration problem:

$$\int \frac{1}{x \sqrt{x^2 - 1}} \, dx = y = \sec^{-1} x + C$$

Examples 8 to 10 can all be re-written with a constant a substituted for 1. In each case, the answer changes slightly (to have a function like $\tan^{-1} \frac{x}{a}$), and in the case of the last example, there is another factor of $1/a$ out in front.

One can handle these by re-drawing the triangle substituting a for 1, or one can factor the expression to obtain something like this:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\ dx$$

$$= \int \frac{1}{a}\ \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}}\ dx$$

Now substitute $t = x/a$ and $dt = dx/a$ so this is just

$$= \int\ \frac{1}{\sqrt{1 - t^2}}\ dt$$

$$= \sin^{-1} t + C$$

$$= \sin^{-1} \frac{x}{a} + C$$