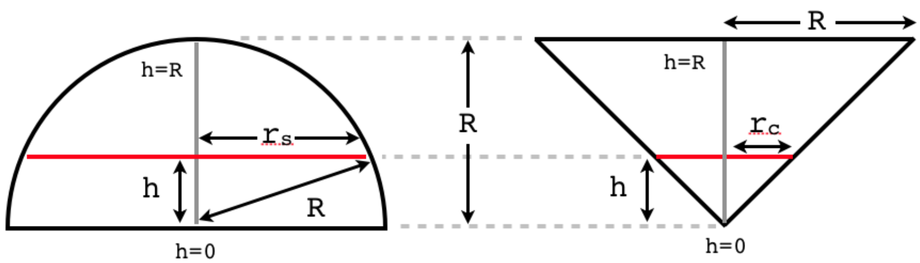
Cone and Sphere by Disks and Shells



Here is a figure that I used previously in the write-up about Archimedes, who found a formula for the volume of a sphere by subtracting a cone (or two cones) from a cylinder. The volume of a cone was previously known to be

$$V_{cone} = \frac{1}{3}\pi R^2 H, \quad \text{for } H = R, \quad V_{cone} = \frac{1}{3}\pi R^3$$

for a cone with base radius R and height H. Any solid of this type has the formula 1/3 base area $\times$ height.

Disks

We can use calculus to derive this formula. We think of slicing the volume horizontally into a series of disks. If the height at any point is h, with total height H and radius R, then by similar triangles the radius (right hand panel above) is

$$r = h\frac{R}{H}$$

the area of each disk is

$$A = \pi r^2 = \pi \frac{R^2}{H^2} h^2$$

and what we need to do is to add up all the disks for $h = 0 \rightarrow h = H$

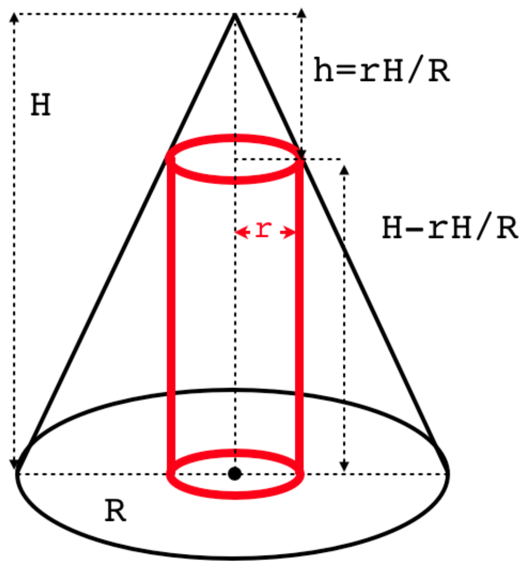
$$V = \int A \ dh = \int_{h=0}^{h=H} \pi \frac{R^2}{H^2} h^2 \ dh$$

$$= \frac{1}{3}\pi \frac{R^2}{H^2}\ h^3 \Big|_0^H = \frac{1}{3}\pi R^2 H$$

$$\boxed{V = \frac{1}{3}\pi R^2 H} \tag{1}$$

Shells

There is another way to "slice" the figure, which is called the method of shells.



We think of the volume as constructed from a series of concentric cylinders. Let's use the same letters we had previously, H for total height and R for base radius. At a height h measured down from the top, the radius r is, as before

$$r = h\frac{R}{H}$$

We have a cylinder whose circumference is

$$C = 2\pi r = 2\pi h\frac{R}{H}$$

The height of the cylinder is $H - h$, and the lateral surface area of the shell is

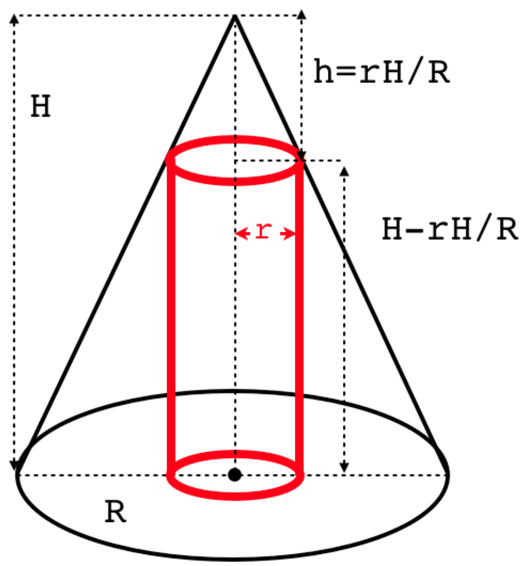
$$SA = C(H - h) = 2\pi h\frac{R}{H}(H - h) = 2\pi\frac{R}{H}(Hh - h^2)$$

We add up all the shells for $h = 0 \to h = H$

$$V = \int A \; dh = \int_0^H 2\pi \frac{R}{H}(Hh - h^2)$$

$$= 2\pi \frac{R}{H}(\frac{1}{2}Hh^2 - \frac{1}{3}h^3)\Big|_0^H = 2\pi \frac{R}{H}(\frac{1}{6}H^3) = \frac{1}{3}\pi R^2 H$$

Varying r instead of h



In the previous section we used h as the variable of integration, but we might just as well have used r. In that case, r will vary from $r = 0 \to r = R$. At each value, the circumference will be

$$C = 2\pi r$$

and the height of the cylinder will be

$$H - \frac{H}{R}r$$

The volume is the sum of all the little pieces of cylinder volume

$$V = \int_{r=0}^{r=R} 2\pi r(H - \frac{H}{R}r) \; dr$$

$$= 2\pi H \int_{r=0}^{r=R} r - \frac{1}{R}r^2 \; dr = 2\pi H \; (\frac{r^2}{2} - \frac{1}{R}\frac{r^3}{3}) \; \Big|_0^R = 2\pi H(\frac{1}{6}R^2) = \frac{1}{3}\pi R^2 H$$

Lateral surface area

We can use a similar method for the surface area (not counting the base). We go back to the picture with slices. Each slice has a circumference of $2\pi r$.

If we look at the slices, the height of each is dh, but the actual length of the area element is elongated because of the slanted side. If we call the angle between the slanted side and the horizontal θ, and the length of the slanted side is S

$$\frac{H}{S} = sin\ \theta$$

$$H = S\ sin\ \theta$$

$$dh = ds\ sin\ \theta$$

so the element of surface area is $dA = C\ ds$. We add them all up

$$SA = \int C\ ds = \int 2\pi r\ ds$$

$$= \int 2\pi r \frac{1}{sin\ \theta}\ dh = \frac{2\pi}{sin\ \theta} \int r\ dh$$

Now substitute for r

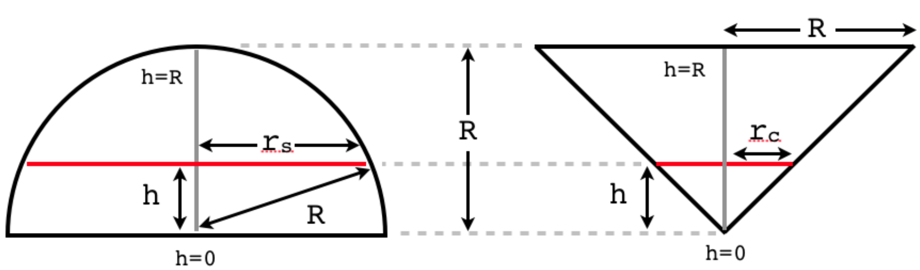
$$= \frac{2\pi}{sin\ \theta} \int_{h=0}^{h=H} \frac{R}{H} h\ dh$$

$$= \frac{2\pi}{sin\ \theta} \frac{R}{H} \left[\frac{h^2}{2}\right]\Big|_0^H$$

$$= \frac{\pi}{sin\ \theta} RH$$

$$\boxed{SA = \pi RS} \tag{2}$$

Volume of the sphere by disks



Going back to the figure with the hemisphere in it, we will slice this into horizontal disks of radius r. If we label the height h as shown in the figure, increasing from 0 at the bottom to $h = R$ at the top, then for the slice at each position h we have a circle with area

$$A = \pi r^2 = \pi(R^2 - h^2)$$

The little bit of volume is

$$dV = A\ dh = \pi(R^2 - h^2)\ dh$$

We integrate (add up all these slices)

$$\int_{h=0}^{h=R} \pi(R^2 - h^2)\ dh = \pi\ [\ R^2 h - \frac{h^3}{3}\ ]\ \Big|_0^R = \frac{2}{3}\pi R^3$$

Since this is for the hemisphere, the sphere is twice the value or $(4/3)\pi R^3$, as expected.

Volume of the sphere by shells

For all of these examples, we can let either r or h be the variable of integration, since there is a simple relationship between r and h. For the cylinder it involves the ratio R/H, and for the sphere

$$h^2 + r^2 = R^2$$

when $h = 0$ at the "fat end" of the solid. A different but still simple formula can be found when $h = 0$ at the tip of the solid.

Let's divide the sphere up into concentric cylinders or shells, and let r vary from $0 \to R$. The circumference of the shell at each point is

$$C = 2\pi r$$

and the height of each is

$$h = \sqrt{R^2 - r^2}$$

The volume of each very thin cylinder is

$$dV = Ch\ dr = 2\pi r\sqrt{R^2 - r^2}\ dr$$

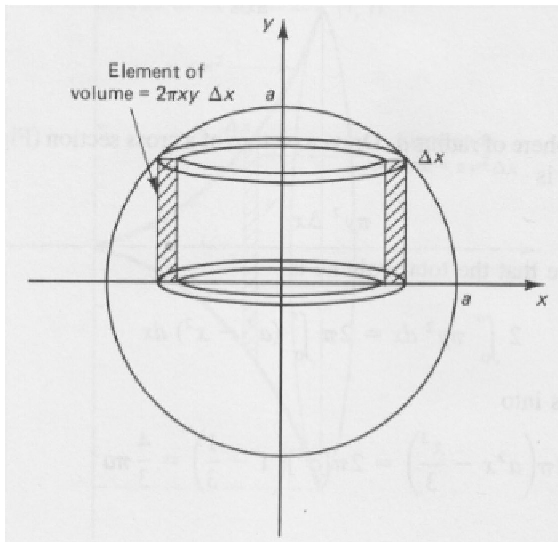
and we want

$$\int_{r=0}^{r=R} 2\pi r\sqrt{R^2 - r^2}\ dr$$

$$= -\frac{2}{3}\pi(R^2 - r^2)^{3/2}\Big|_0^R = -\frac{2}{3}\pi\ [\ -(R^2)^{3/2}\ ] = \frac{2}{3}\pi R^3$$

as before.

Here is a picture of what we're doing, from Hamming's Calculus text. The notation is different but the idea is the same.



Surface area of a sphere

My favorite way of doing this is the simplest. Suppose we have a sphere with radius r and we increase r by a little bit dr. What is the change in volume, dV? It's the surface area times dr

$$dV = SA\ dr$$

$$SA = \frac{dV}{dr} = \frac{d}{dr}\ (\frac{4}{3}\pi r^3) = 4\pi r^2$$

A bit fancier method is to imagine we revolve a function $y = f(x)$ around the x-axis. To compute the surface area of the solid, we slice it into disks in the usual way, but moving along the x-axis in increments dx. Then we need to find the surface area of the disk. In this situation, the elements of surface area ds are given by:

$$ds = \sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

After setting up ds, we will integrate

$$\int 2\pi y\ ds$$

If we have a circle with radius R centered at the origin.

$$x^2 + y^2 = R^2; \quad y = f(x) = \sqrt{R^2 - x^2}$$

Using implicit differentiation, it is easy to show that

$$2x\ dx + 2y\ dy = 0$$

$$\frac{dy}{dx} = -x/y$$

Then

$$ds = \sqrt{1 + \frac{x^2}{y^2}}\ dx = \sqrt{1 + \frac{x^2}{(R^2 - x^2)}}\ dx$$

And

$$SA = 2\pi \int y\ ds = 2\pi \int \sqrt{(R^2 - x^2)}\ \sqrt{1 + \frac{x^2}{(R^2 - x^2)}}\ dx$$

$$2\pi \int \sqrt{R^2 - x^2 + x^2}\ dx = 2\pi \int R\ dx = 2\pi R x$$

I do like the way that simplifies!. Now evaluate from $x = -R \rightarrow R$, giving:

$$SA = 4\pi R^2$$

The expected result.