Guided Discovery: Proof of the Boundedness Theorem

Boundedness Theorem:
Let $f:[a,b]\to\mathbb{R}$ be continuous. Then f is bounded.
*"A **continuous** function on a **closed bounded** interval is bounded."*

1. Proof (by contradiction):
Let $f:[a,b]\to\mathbb{R}$ be continuous.
First, suppose f is not bounded above.

Then, for every $n\in\mathbb{N}$, there exists $x_n\in[a,b]$ such that $f(x_n)>n$.

- *Sketch a diagram showing possible locations of x_n and $f(x_n)$ for n = 10, 50, 100, 200.*

Since $[a,b]$ is bounded, the sequence (x_n) is bounded.
By the Bolzano-Weierstrass theorem, ______________________________

. . . (x_n) has a convergent subsequence, (x_{n_k}).

Let $\lim(x_{n_k})=L$.

- *What can we say about the possible location of L in the diagram?*

Since $x_{n_k}\in[a,b]$ $\forall k$, *we know that* $L\in[a,b]$.

- *Do we know whether the sequence $(f(x_{n_k}))$ is bounded or not?*

Since $f(x_{n_k})>n_k$ $\forall k\in\mathbb{N}$ (why?) $(f(x_{n_k}))$ is unbounded.

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We now have the following:

$f:[a,b]\to\mathbb{R}$ is continuous at $L\in[a,b]$ and

$x_{n_k}\in[a,b]\ \forall k$ and $\lim(x_{n_k})=L$.

- *What major theorem can now be invoked?*
- *What conclusion does this theorem give us?*

By the Sequential Criterion for Continuity (an "if and only if" theorem), we conclude that $\lim(f(x_{n_k}))=f(L)$.

This result contradicts the fact that ________________________________

. . . $(f(x_{n_k}))$ is unbounded.

Thus, the assumption that _________________________________ is false.

We conclude that f is bounded above.

(The proof where f is not bounded below is similar.) **Q.E.D.**

2. We have just shown that a **continuous** function on a **closed, bounded** interval is bounded."

- *Where does the proof use the hypothesis that the interval is* ***bounded****?*

Solution:

Boundedness of the interval guarantees that (x_n) is bounded so that the Bolzano-Weierstrass Theorem can be invoked to obtain a convergent subsequence.

- *Where does the proof use the hypothesis that the interval is **closed**?*

Solution:
Closure of the interval guarantees that the limit of the subsequence is contained in the domain, where the function is continuous.

- *Where does the proof use the hypothesis that the function is **continuous**?*

Solution:
Continuity of f at L is required by the Sequential Criterion to guarantee that the sequence $(f(x_{n_k}))$ converges. This convergence is incompatible with the fact that $(f(x_{n_k}))$ is unbounded, yielding the contradiction. ■

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