# The Binomial Series of Isaac Newton

In 1661, the nineteen-year-old Isaac Newton read the *Arithmetica Infinitorum* and was much impressed. In 1664 and 1665 he made a series of annotations from Wallis which extended the concepts of interpolation and extrapolation. It was here that Newton first developed his binomial expansions for negative and fractional exponents and these early papers of Newton are the primary source for our next discussion (Newton, 1967a, Vol. 1, p 89-142).

Newton made a series of extensions of the ideas in Wallis. He extended the tables of areas to the left to include negative powers and found new patterns upon which to base interpolations. Perhaps his most significant deviation from Wallis was that Newton abandoned the use of ratios of areas and instead sought direct expressions which would calculate the area under a portion of a curve from the value of the abscissa. Using what he knew from Wallis he could write down area expressions for the integer powers. Referring back to Figure 1, we have:

$$\frac{\text{Area under x}^n}{\text{Area of containing rectangle}} = \frac{1}{n+1}\text{, and}$$

$$\text{Area of the containing rectangle} = x \cdot x^n = x^{n+1}\text{, hence}$$

$$\text{Area under the curve } x^n = \frac{x^{n+1}}{n+1}$$

Using this form combined with binomial expansions, Newton wrote down progressions of expressions which calculated the area under curves in particular families. For example, he considered the positive and negative integer powers of $1+x$, i.e. the series of curves :

$$\ldots \quad , \quad y = \frac{1}{1+x}, \quad y = 1, \quad y = 1+x, \quad y = (1+x)^2, \quad y = (1+x)^3, \quad y = (1+x)^4, \quad \ldots$$

He was particularly interested in the hyperbola and wanted to find its area expression by interpolation after having failed to obtain its area by purely geometric considerations (Newton, 1967a, Vol. 1, p 94).

Newton drew the following graph of several members of this family of curves (Figure 4). Appearing in the graph are a hyperbola, a constant, a line, and a parabola, i.e. the first four curves in the progression.

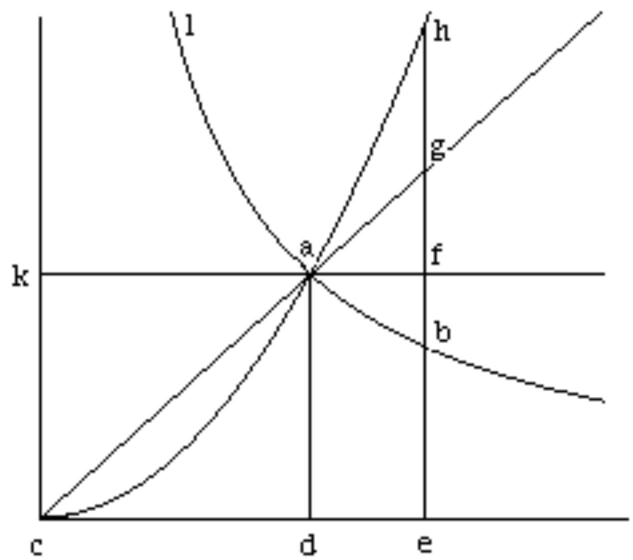


Fig. 4

Letting $ck = cd = 1$ and $de = x$, the ordinates here are: $eb = \frac{1}{1+x}$, $ef = 1$, $eg = 1 + x$, and $eh = (1+x)^2$. He then wrote down a series of expressions which calculate the areas under the curves over the segment $de = x$ as:

$$\text{Area}(afed) = x, \quad \text{Area}(aged) = x + \frac{x^2}{2}, \quad \text{Area}(ahed) = x + \frac{2x^2}{2} + \frac{x^3}{3}.$$

The third one is obtained by first expanding $(1+x)^2$ as $1 + 2x + x^2$ (see Appendix 3). Although the higher power curves did not appear in the graph, Newton went on to write down more area expressions for curves in this family. For the positive integer powers 3, 4, and 5 of $1+x$ he obtained the following area expressions by first expanding and then finding the area term by term.

third power: $x + \frac{3x^2}{2} + \frac{3x^3}{3} + \frac{x^4}{4}$

fourth power: $x + \frac{4x^2}{2} + \frac{6x^3}{3} + \frac{4x^4}{4} + \frac{x^5}{5}$

fifth power: $x + \frac{5x^2}{2} + \frac{10x^3}{3} + \frac{10x^4}{4} + \frac{5x^5}{5} + \frac{x^6}{6}$

At this point Newton wanted to find a pattern which would allow him to extend his calculations to include the areas under the negative powers of $1+x$. He noticed that the denominators form an arithmetic sequence while the numerators follow the binomial patterns. This binomial pattern in the numerators is not so surprising, given that they came from expansions. He then made the following table of the area expressions for $(1+x)^p$ (see Table 4), where each column represents the numbers in the numerators of the area function. The question then becomes: how can one fill in the missing entries? He began by assuming that the top row remains constant at the value 1.

Table 4

| p / term | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{x}{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\frac{x^2}{2}$ | | | | **?** | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $\frac{x^3}{3}$ | | | | | 0 | 0 | 1 | 3 | 6 | 10 | 15 |
| $\frac{x^4}{4}$ | | | | | 0 | 0 | 0 | 1 | 4 | 10 | 20 |
| $\frac{x^5}{5}$ | | | | | 0 | 0 | 0 | 0 | 1 | 5 | 15 |
| $\frac{x^6}{6}$ | | | | | 0 | 0 | 0 | 0 | 0 | 1 | 6 |
| $\frac{x^7}{7}$ | | | | | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

This binomial table is different from Wallis' table in that the rows are all nudged successively to the right so that the diagonals of the Wallis table become the columns of Newton's table. The binomial pattern of formation is now such that each entry is the sum of the entry to the left of it and the one above that one. Using this rule backwards as a difference we find, for example, that the **?** must be equal to -1. Each new diagonal to the left is the sequence of differences of the previous diagonal. This was Newton's first use of difference tables. Continuing on in a similar manner Newton filled in the table of coefficients for the area expressions under the curves $(1+x)^p$ as follows:

Table 5

| term \ p | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{x}{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\frac{x^2}{2}$ | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $\frac{x^3}{3}$ | 10 | 6 | 3 | 1 | 0 | 0 | 1 | 3 | 6 | 10 | 15 |
| $\frac{x^4}{4}$ | -20 | -10 | -4 | -1 | 0 | 0 | 0 | 1 | 4 | 10 | 20 |
| $\frac{x^5}{5}$ | 35 | 15 | 5 | 1 | 0 | 0 | 0 | 0 | 1 | 5 | 15 |
| $\frac{x^6}{6}$ | -56 | -21 | -6 | -1 | 0 | 0 | 0 | 0 | 0 | 1 | 6 |
| $\frac{x^7}{7}$ | 84 | 28 | 7 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

At this point Newton could write down the area under the hyperbola: $y = \frac{1}{1+x}$, (i.e. what we now call the natural logarithm of $1+x$) (see Figure 4) as:

(6) $$\text{Area}(abed) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} \ldots$$

He then made several detailed calculations using the first 25 terms of this series to compute hyperbolic areas to more than 50 decimal places. Newton later became aware that this function displayed logarithmic properties and could be used to create a table of common logarithms (Edwards, 1979, p. 160).

Newton repeatedly returned to the table of characteristic ratios made by Wallis (Table 3). As discussed previously, Newton abandoned Wallis' use of area ratios and set out to make a table of coefficients for a sequence of explicit expressions for calculating areas. He used the same set of curves whose characteristic ratios Wallis had tabulated in the row q=1/2, but Newton let r=1 (in the circle case r is the radius). Hence he considered the areas (over the segment de=x) under the following sequence of curves (see Figure 5):

$$\ldots\ ,\quad y=1,\quad y=\sqrt{1-x^2},\quad y=1-x^2,\quad y=(1-x^2)\sqrt{1-x^2},\quad y=(1-x^2)^2,\quad \ldots$$

These are the powers of $1-x^2$ at intervals of $1/2$. In this early manuscript Newton did not write fractions directly as exponents, but when he later announced the results of his researches in a series of letters he did, thus $y=(1-x^2)\sqrt{1-x^2}$ would become $y=(1-x^2)^{3/2}$. Several times Newton drew graphs of these curves inside the unit square (see Figure 5). (Newton's original manuscript containing one of these graphs is included with this paper.)

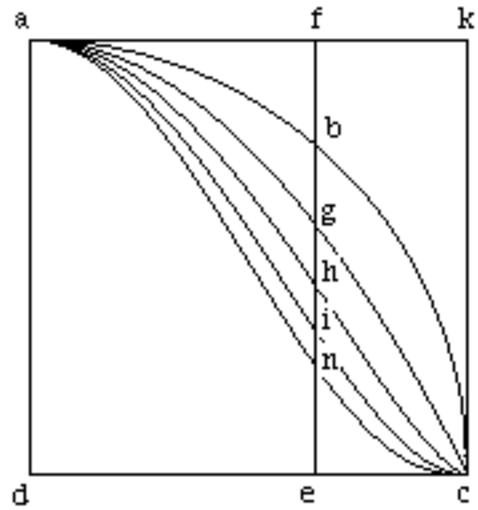


Fig. 5

He let $ad=dc=1$ and $de=x$ ; $ef, eb, eg, eh, ei, en, \ldots$ are then the ordinates of his series of curves respectively. Note that the curve abc is a circle, and $agc$ is a parabola.

For the <u>integer</u> powers of $1-x^2$, Newton could write down the areas in his graph (Figure 5) as:

$$\text{Area}(afed)=x, \quad \text{Area}(aged)=x-\frac{1}{2}x^3, \quad \text{Area}(aied)=x-\frac{2}{3}x^3+\frac{1}{5}x^5$$

As before, these are obtained by first expanding the binomials and then writing down the area expressions term by term. Once again he applied the characteristic ratios of Wallis to each separate term in the expansion (see Appendix 3). Although the higher powers no longer appeared in his graph, Newton continued this sequence of area expressions for $(1-x^2)^p$ as follows:

$$p=3: \quad x-\frac{3}{3}x^3+\frac{3}{5}x^5-\frac{1}{7}x^7$$

$$p=4: \quad x-\frac{4}{3}x^3+\frac{6}{5}x^5-\frac{4}{7}x^7+\frac{1}{9}x^9$$

$$p=5: \quad x-\frac{5}{3}x^3+\frac{10}{5}x^5-\frac{10}{7}x^7+\frac{5}{9}x^9-\frac{1}{11}x^{11}$$

etc.

Once again he saw that the denominators formed an arithmetic sequence and that the numerators followed a binomial pattern. As before Newton made a table of these results including an extension into the negative powers. Table 6 is a table of coefficients of the expressions which compute the area under the curves $y=(1-x^2)^p$.

Table 6

| term \ p | -1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ | 2 | $\frac{5}{2}$ | 3 | $\frac{7}{2}$ | 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{x}{1}$ | 1 | | 1 | | 1 | | 1 | | 1 | | 1 |
| $-\frac{x^3}{3}$ | -1 | | 0 | | 1 | | 2 | | 3 | | 4 |
| $\frac{x^5}{5}$ | 1 | | 0 | | 0 | | 1 | | 3 | | 6 |
| $-\frac{x^7}{7}$ | -1 | | 0 | | 0 | | 0 | | 1 | | 4 |
| $\frac{x^9}{9}$ | 1 | | 0 | | 0 | | 0 | | 0 | | 1 |
| $-\frac{x^{11}}{11}$ | -1 | | 0 | | 0 | | 0 | | 0 | | 0 |
| $\frac{x^{13}}{13}$ | 1 | | 0 | | 0 | | 0 | | 0 | | 0 |

It now remained to find a way to interpolate the missing entries for the fractional powers. In this table each entry is the sum of the entry two spaces to the left and the entry directly above that one. The entries above the diagonal of 1's had already been interpolated by Wallis in Table 3, and from these one could complete the table by differences as in Table 5. One could also have used the polynomials that appeared in the margins of Wallis' Table 3 to fill in this table. Newton, however, devised his own system of interpolation which he could check against these others. Instead of forming polynomial expressions for the interpolation of each row, Newton used the known entries to generate a system of linear equations whose solution would determine the missing entries.

He first noted that integer binomial tables obey the following additive pattern of formation (Table 7).

Table 7

| | | | | |
|---|---|---|---|---|
| a | a | a | a | a |
| b | a+b | 2a+b | 3a+b | 4a+b |
| c | b+c | a+2b+c | 3a+3b+c | 6a+4b+c |
| d | c+d | b+2c+d | a+3b+3c+d | 4a+6b+4c+d |
| e | d+e | c+2d+e | b+3c+3d+e | a+4b+4c+4d+e |

This pattern is formed by starting with a constant sequence (a,a,a,...) and an arbitrary left hand column (a,b,c,d,...); and then forming each entry as the sum of the one to the left and the one above that. This, as it stands, would not work for the completion of the fractional interpolated tables, because the entries in the top row must all be 1 in all the interpolated tables (i.e. a=1), but this would force the increment of the second row also to be one. To get around this difficulty, Newton rewrote this pattern so as to unlink the rows of Table 7. That is to say, he preserved the pattern within each individual row but he changed the names of the variables so that each variable appeared in only one row. As you move down the rows each new row can be described using successively one more variable. Changing the names of variables so that each row is independent of the others, the pattern now becomes Table 8.

Table 8

| | | | | |
|---|---|---|---|---|
| a | a | a | a | a |
| b | c+b | 2c+b | 3c+b | 4c+b |
| d | e+d | f+2e+d | 3f+3e+d | 6f+4e+d |
| g | h+g | i+2g+h | k+3i+3h+g | 4k+6i+4h+g |
| l | m+l | n+2m+l | p+3n+3m+l | q+4p+6n+6m+l |

Using Table 8, if any entry in the first row is known the whole row is known. If any two entries in the second row are known then one can solve for b and c and fill in the entire row. If any three entries in the third row are known one can solve for d, e and f and fill in the entire row. Thus with a sufficient number of known values in a given row one could solve a system of linear equations for all the variables in that row. Newton solved sets of linear equations to find these values and that allowed him to fill in the interpolated table. This method allowed him not only to interpolate the binomial table at increments of 1/2, but at any increment, for example, thirds.

He then completed Table 6. Let us complete the third row, for example, using the known values 0, ?, 0, ?, 1, ? We obtain d=0, f+2e+d=0, and 6f+4e+d=1. Thus d=0, $f = \frac{1}{4}$, and $e = -\frac{1}{8}$. We can now complete the entire row using these values, but it should be noted here that although we used three equations to find d, e, and f there are actually an infinite number of equations involving these three variables. One might ask if this set of equations is consistent. They are, but Newton did not address this issue. He is satisfied because the values he finds agree with Wallis and with the additive pattern of table formation. With the completion of Table 6, Newton will also obtain a new way to calculate π which will validate his method in a geometric representation. Table 6 now becomes:

Table 9

| p / term | -1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ | 2 | $\frac{5}{2}$ | 3 | $\frac{7}{2}$ | 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{x}{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\frac{-x^3}{3}$ | -1 | $\frac{-1}{2}$ | 0 | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ | 2 | $\frac{5}{2}$ | 3 | $\frac{7}{2}$ | 4 |
| $\frac{x^5}{5}$ | 1 | $\frac{3}{8}$ | 0 | $\frac{-1}{8}$ | 0 | $\frac{3}{8}$ | 1 | $\frac{15}{8}$ | 3 | $\frac{35}{8}$ | 6 |
| $\frac{-x^7}{7}$ | -1 | $\frac{-5}{16}$ | 0 | $\frac{3}{48}$ | 0 | $\frac{-1}{16}$ | 0 | $\frac{5}{16}$ | 1 | $\frac{35}{16}$ | 4 |
| $\frac{x^9}{9}$ | 1 | $\frac{35}{128}$ | 0 | $\frac{-15}{384}$ | 0 | $\frac{3}{128}$ | 0 | $\frac{-5}{128}$ | 0 | $\frac{35}{128}$ | 1 |
| $\frac{-x^{11}}{11}$ | -1 | $\frac{-63}{256}$ | 0 | $\frac{105}{3840}$ | 0 | $\frac{-3}{256}$ | 0 | $\frac{3}{256}$ | 0 | $\frac{-7}{256}$ | 0 |
| $\frac{x^{13}}{13}$ | 1 | $\frac{231}{1024}$ | 0 | $\frac{-945}{46080}$ | 0 | $\frac{7}{1024}$ | 0 | $\frac{-5}{1024}$ | 0 | $\frac{7}{1024}$ | 0 |

The column $p=\frac{1}{2}$ gives an infinite series which calculates the area under any portion of a circle (see Figure 5). That is to say, that $\text{Area}(abed)$ is given by (7), where $de = x$.

(7) $$x-\frac{1}{2}\frac{x^3}{3}-\frac{1}{8}\frac{x^5}{5}-\frac{3}{48}\frac{x^7}{7}-\frac{15}{384}\frac{x^9}{9}-\frac{105}{3840}\frac{x^{11}}{11}-\dots$$

Letting $x=1$ in this series calculates the area of one quarter of the circle and thus yields a new calculation of π:

(8) $$\frac{\pi}{4}=1-\frac{1}{6}-\frac{1}{40}-\frac{1}{112}-\frac{5}{1152}-\frac{7}{2816}-\dots$$

Checking that this series does agree with the value of π obtained from geometrical arguments like those of Archimedes, as well as the infinite product of Wallis; provided Newton with a validation of this interpolation in alternate representations.

Newton later became aware that the interpolation procedure based on the patterns of Table 8 was equivalent to the assumption that rows of this table could be interpolated using polynomial equations of increasing degree. That is to say, the first row is constant, the second row is linear, the third row is quadratic, and so on. This is consistent with the method used by Wallis, and would suggest to Newton a general procedure for the interpolation of data which we will describe in the next section.

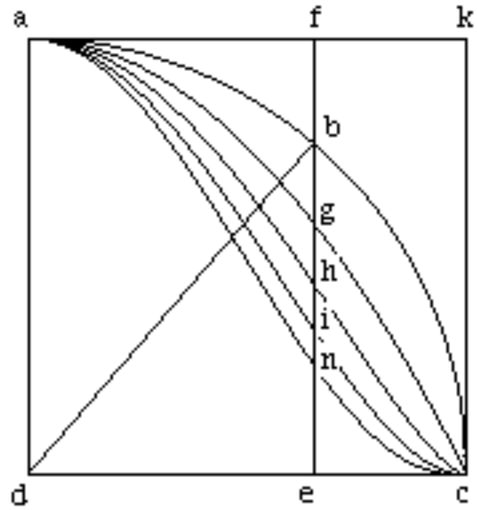


Fig. 6

Newton also pointed out that this series allowed him to compute $\arcsin(x)$. By adding a line from $d$ to $b$ in Figure 5 (see Figure 6), and subtracting the area of $\Delta dbe$ from $\text{Area}(abed)$, one obtains the area of the circular sector $(abd)$. Since this is the circle of radius one, twice the area of $\text{sector}(abd)$ equals $\text{arclength}(ab)$ (when r=1, area=π, and circumference=2π). The triangle $(dbe)$ to be subtracted from the series has area equal to $\frac{1}{2}x\sqrt{1-x^2}$ .

Satisfied with his interpolation methods Newton began searching for a pattern in the columns of his table which would allow him to continue each series without having to repeat his tedious interpolation procedure row by row. Note that some of the fractions in Table 9 are not reduced. In earlier tabulations Newton did reduce the fractions but he soon became aware that this would only obscure any possible patterns in their formations. Following the example set by Wallis, he sought a pattern of continued multiplication of arithmetic sequences. Since the circle was so important to him he studied the $p=\frac{1}{2}$ column first. Factoring the numbers in these fractions he found that they could be produced by continued multiplication as:

(9) $$\frac{1}{1}\cdot\frac{1}{2}\cdot\frac{-1}{4}\cdot\frac{-3}{6}\cdot\frac{-5}{8}\cdot\frac{-7}{10}\cdot\frac{-9}{12}\cdot\frac{-11}{14}\cdot\ldots$$

Similarly, the entries in the $p=\frac{3}{2}$ column could be produced by continued multiplication as:

(10) $$\frac{1}{1}\cdot\frac{3}{2}\cdot\frac{1}{4}\cdot\frac{-1}{6}\cdot\frac{-3}{8}\cdot\frac{-5}{10}\cdot\frac{-7}{12}\cdot\frac{-9}{14}\cdot\ldots$$

In order to further investigate these patterns, Newton carried out an interpolation of the binomial table at intervals of 1/3. Using the patterns from Table 8 and solving the systems of equations for the variables in each row he produced the following interpolated Table 10. Note that at this point he does not write down the terms in the expansions for which these numbers are coefficients. Newton never mentions an explicit context of area calculations for which Table 10 was intended. At this point he is working solely within a table representation in order to find an explicit formula for the fractional binomial numbers whose patterns began revealing themselves in (9) and (10). After another long round of solving systems of linear equations, Newton arrived at:

Table 10

| 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | 1 | $\frac{4}{3}$ | $\frac{5}{3}$ | 2 | $\frac{7}{3}$ | $\frac{8}{3}$ | 3 | $\frac{10}{3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | 1 | $\frac{4}{3}$ | $\frac{5}{3}$ | 2 | $\frac{7}{3}$ | $\frac{8}{3}$ | 3 | $\frac{10}{3}$ |
| 0 | $\frac{-1}{9}$ | $\frac{-1}{9}$ | 0 | $\frac{2}{9}$ | $\frac{5}{9}$ | 1 | $\frac{14}{9}$ | $\frac{20}{9}$ | 3 | $\frac{35}{9}$ |
| 0 | $\frac{5}{81}$ | $\frac{4}{81}$ | 0 | $\frac{-4}{81}$ | $\frac{-5}{81}$ | 0 | $\frac{14}{81}$ | $\frac{40}{81}$ | 1 | $\frac{140}{81}$ |
| 0 | $\frac{-10}{243}$ | $\frac{-7}{243}$ | 0 | $\frac{5}{243}$ | $\frac{5}{243}$ | 0 | $\frac{-7}{243}$ | $\frac{-10}{243}$ | 0 | $\frac{25}{243}$ |
| 0 | $\frac{22}{729}$ | $\frac{14}{729}$ | 0 | $\frac{-8}{729}$ | $\frac{-7}{729}$ | 0 | $\frac{7}{729}$ | $\frac{8}{729}$ | 0 | $\frac{-14}{729}$ |

Searching, as before, for a pattern of repeated multiplication of arithmetic sequences that would generate the columns of this table, Newton discerned the following pattern for the column $p = \frac{1}{3}$.

(11) $$\frac{1}{1} \cdot \frac{1}{3} \cdot \frac{-2}{6} \cdot \frac{-5}{9} \cdot \frac{-8}{12} \cdot \frac{-11}{15} \cdot \frac{-14}{18} \cdot \frac{-17}{21} \cdot \ldots$$

Here the sequence of numerators and denominators both change by increments of 3 (ignoring the first term), the former going down while the later go up. In (9) and (10) the same thing happened but by increments of 2. At this point Newton wrote down an explicit formula for the binomial numbers in an arbitrary column $p = \frac{x}{y}$.

(12) $$\frac{1}{1} \cdot \frac{x}{y} \cdot \frac{x-y}{2y} \cdot \frac{x-2y}{3y} \cdot \frac{x-3y}{4y} \cdot \frac{x-4y}{5y} \cdot \frac{x-5y}{6y} \cdot \ldots$$

This formula makes perfect sense given the form of the examples from which it was constructed. If the y's in the denominators are taken into the numerators it becomes the formula for the binomial coefficient that is familiar to a modern reader. Dropping the first term and letting $n = \frac{x}{y}$, as Newton would do in his later letters, (12) becomes:

(13) $$\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} \cdot \frac{n-5}{6} \cdot \ldots$$

The binomial series became the engine which generated a wealth of examples from which Newton would later build his version of calculus. Once he had written down (12) and later (13), he began a long series of experiments and checks to convince himself of its validity. For example, by using synthetic division one can write:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \ldots$$

By computing the area of this series term by term one could then arrive at the series for hyperbolic areas (6). It is important to note here that this is not how Newton first constructed (6). (Several history books give that impression, as does one of Newton's own accounts). He noticed this later and saw it as an important algebraic confirmation of the validity of his table construction. Thus his extension of the binomial table could be checked against both geometric areas, and algebraic generalizations of arithmetic.

His original interpolations were designed to calculate areas under families of curves but Newton soon saw that by changing the terms to which the coefficients were applied he could use these numbers to calculate the points on the curve as well. This was particularly useful for root extractions. He simply had to replace the area terms $\frac{x^{n+1}}{n+1}$ with the original terms $x^n$ from which they came. The coefficients in the tables remain the same. For example, the $p = \frac{1}{2}$ column of Table 9 can be used to calculate square roots as:

(14) $$(1-x^2)^{1/2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{1}{128}x^8 - \ldots$$

(15) $$(1-x)^{1/2} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{1}{128}x^4 - \ldots$$

The $p = \frac{1}{3}$ column of Table 10 can be used to calculate cube roots as:

(16) $$(1-x^2)^{1/3} = 1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 - \frac{5}{81}x^6 - \frac{10}{243}x^8 - \ldots$$

(17) $$(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 - \ldots$$

These series appear in this form in the letters (1676) to Oldenburg in which Newton explained his binomial series at the request of Leibniz (Callinger, 1982; Struik, 1986). He checked the consistency of these series in many ways. Various geometric methods for finding square roots were known against which (14) and (15) could be checked. The series (15) can be multiplied by itself term by term to arrive at 1+x , all other terms canceling out. Newton never gave anything like a formal proof of the validity of these generalized binomial expansions. His approach was always empirical. He tried them in various contexts and they worked. If certain values were needed for which a particular series diverged, he just rewrote it in another form until he found one that converged. Questions concerning convergence were treated empirically for over a century after Newton. As we shall see in the next discussion, Newton did not distinguish between the interpolation of scientific data and the continuous calculation of mathematically defined curves.

**References** cited in the text can be found at
http://www.quadrivium.info/MathInt/Notes/WallisNewtonRefs.pdf