# Cauchy's Formula

**Quick review**

We showed before that if $f(z)$ is analytic everywhere in a domain and we integrate around a closed path enclosing that region

$$I = \oint f(z)\ dz = 0$$

On the other hand, if the region includes a singularity, the value of the integral is independent of the path (but non-zero because of the singularity).

We obtain the same value for the line integral around any path. If that path encloses a singularity, then the value of the integral is non-zero.

So consider any $z_0$ in the domain

$$\oint \frac{f(z)}{(z - z_0)}\ dz$$

Parametrize $z$ as a circle of radius $\rho$ centered on $z_0$

$$z = z_0 + \rho e^{i\theta}$$

$$dz = i\rho e^{i\theta}\ d\theta$$

then

$$I = \oint \frac{f(z)}{(z - z_0)}\ dz$$

$$= \oint \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}} \; i \; \rho e^{i\theta} \; d\theta$$

$$= i \oint f(z_0 + \rho e^{i\theta}) \; d\theta$$

Since we have the same value for *any* path, imagine that that $\rho \to 0$ so $z \to z_0$ and then in the limit we have

$$I = i \oint f(z_0) \; d\theta$$

but $f(z_0)$ is a constant value so it can come out from the integral

$$I = if(z_0) \oint \; d\theta = 2\pi i \; f(z_0)$$

$$\oint \frac{f(z)}{(z - z_0)} \; dz = 2\pi i \; f(z_0)$$

**example**

We can use the inverse function $(1/z)$ as an example. This function has a singularity at the origin. Compare with the form of Cauchy2:

$$\oint \frac{f(z)}{z - z_0} \; dz = 2\pi i f(z_0)$$

We can match this form if we set $f(z) = 1$ and $z_0 = 0$. The theorem says we can write the value of the integral as

$$I = 2\pi i f(z_0) = 2\pi i$$

This matches what we obtained by parametrizing the unit circle. There we had

$$z = e^{i\theta}, \quad \theta = 0 \to 2\pi$$

$$dz = iz \; d\theta$$

$$\oint \frac{1}{z}\ dz = \int_0^{2\pi} i\ d\theta = 2\pi i$$

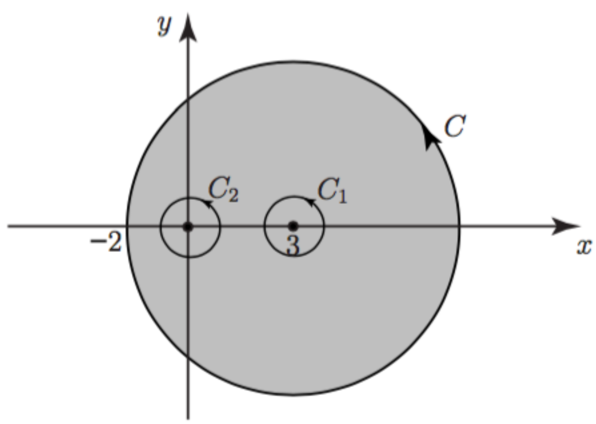
This is $2\pi i$ times the value of the function at $z_0 = 0$, which is 1.

**example**

Consider

$$\oint \frac{6}{z(z-3)}\ dz$$

As shown in the figure, we are supposed to take for the curve $C$ the set of points $|z-3| = 5$, which is a circle of radius 5 surrounding the point $z = 3 + 0i$.




This function does have two points of singularity, namely $z = 0$ and $z = 3$, so we expect that the value of the integral will not be zero. We use (4) above to write

$$\oint_C f(z)\ dz = \oint_{C_1} f(z)\ dz + \oint_{C_2} f(z)\ dz$$

We do not need to specify the curves because we will use (3) to calculate the values.

However, we do need to manipulate the function a bit. We use the method of partial fractions. Leaving aside the factor of 6 for a moment:

$$\frac{1}{z(z-3)} = \frac{A}{z} + \frac{B}{z-3} = \frac{A(z-3)+B(z)}{z(z-3)}$$

From looking at the numerator on the left- and right-hand sides, we see that $A = -B$ (because $Az + Bz = 0$), and that $-3A = 1$. Hence

$$A = -\frac{1}{3}, \quad B = \frac{1}{3}$$

Recall the factor of 6 and substitute for $A$ and $B$ in the middle expression to obtain:

$$f(z) = \frac{-2}{z} + \frac{2}{z-3}$$

So now we can split the integrals for each curve into two parts. We have:

$$I = \oint_{C_1} f(z)\ dz + \oint_{C_2} f(z)\ dz$$

$$= \oint_{C_1} \frac{-2}{z}\ dz + \oint_{C_1} \frac{2}{z-3}\ dz + \oint_{C_2} \frac{-2}{z}\ dz + \oint_{C_2} \frac{2}{z-3}\ dz$$

Two of these four parts do not contain poles (the first and last), so those are just zero, and we have

$$I = \oint_{C_1} \frac{2}{z-3}\ dz + \oint_{C_2} \frac{-2}{z}\ dz$$

At this point we can use (3) from above, that

$$\oint_C \frac{f(z)}{z-z_0}\ dz = 2\pi i f(z_0)$$

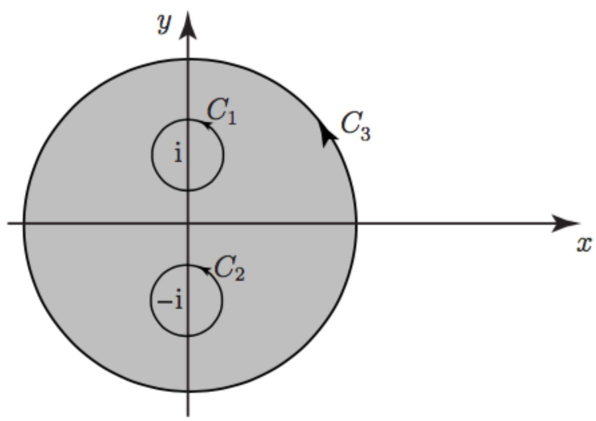
(recognizing that the denominator for the second integral can be written as $z-0$). So the result is $2\pi i f(z_0)$ for both integrals, but the value of the function is just 2 for the first term and $-2$ for the second term, which cancel. In this case, the total integral is just zero. Notice that the cancellation comes because $A = -B$, which we would obtain for any two factors $z$ and $z - a$ ($a \neq 0$), as long as the function itself is a constant

**example**

Consider

$$\oint \frac{z}{z^2 + 1}\ dz$$

This function has poles at $z = \pm i$.




We could find where the poles are by solving $z^2 + 1 = 0$ or we could factor

$$z^2 + 1 = (z + i)(z - i)$$

This leads us to the strategy of partial fractions as before

$$\frac{z}{z^2 + 1} = \frac{A}{z + i} + \frac{B}{z - i}$$

By inspection, $A = B = 1/2$ is a solution, so

$$\oint \frac{z}{z^2 + 1}\ dz = \oint \frac{1/2}{(z + i)} + \frac{1/2}{(z - i)}\ dz$$

As before, curve $C_1$ encloses a pole only for the second term, and $C_2$ for the first term. We use

$$\oint_C \frac{f(z)}{z - z_0}\ dz = 2\pi i f(z_0)$$

where the function is simply the value 1/2 at both points. So we obtain

$$2\pi i\ \frac{1}{2} = \pi i$$

for *each* pole.

Another way is to write

$$\frac{z}{z^2+1} = \frac{z}{(z+i)(z-i)} = \frac{z/z+i}{z-i}$$

For $C_1$, this has a pole at $z = i$, so the value of the integral is

$$2\pi i\ f(z_0) = 2\pi i\ \frac{i}{i+i} = \pi i$$

For $C_2$

$$\frac{z/z-i}{z+i}$$

At the pole $z = -i$, the value of $f(z) = z/z - i$ is again 1/2.

Yet another way to obtain this result, by analogy to calculus of real variables. Substitute

$$w = z^2 + 1$$

$$dw = 2z\ dz$$

$$\oint \frac{z}{z^2+1}\ dz = \frac{1}{2}\int \frac{1}{w}\ dw$$

$$= \frac{1}{2}\ \mathrm{Log}\ (w)$$

$$= \frac{1}{2}\ \mathrm{Log}\ (z^2+1)$$

$$\text{Log } z^2 = \text{Log } r^2 e^{i2\theta} = \ln r + 2i\theta$$

if we evaluate over a closed contour ($\theta = 0 \to 2\pi$) the terms with $\ln r$ vanish and we have then $4\pi i$ times one-half or $2\pi i$. (Problem with the sum +1?

**example**

$$I = \int \frac{z^2}{4 - z^2} \, dz$$

on the circle of radius 2 centered at $z_0 = -1 + 0i$

$$\gamma(\theta) = z_0 + 2e^{i\theta} = 1 + 2e^{i\theta}$$

Notice that the zeroes of the denominator occur at $z = \pm\, 2$ and that one of these is contained within our path of integration.

If we factor the denominator

$$\frac{1}{4 - z^2} = \frac{1}{4} \left[ \frac{1}{2 - z} + \frac{1}{2 + z} \right]$$

we can split the integral

$$I = \frac{1}{4} \int \frac{z^2}{2 - z} + \frac{z^2}{2 + z} \, dz$$

The first one is just zero, by Cauchy's theorem. The second term contains the singularity:

$$I = \frac{1}{4} \int \frac{z^2}{2 + z} \, dz$$

Here $f(z) = z^2$ and

$$z - z_0 = z + 2$$

so $z_0 = -2$ so the value of the manipulated integral is

$$2\pi i \; f(z_0) = 2\pi i \; 4 = 8\pi i$$

and the value of the original one is

$$I = \frac{1}{4} 8\pi i = 2\pi i$$