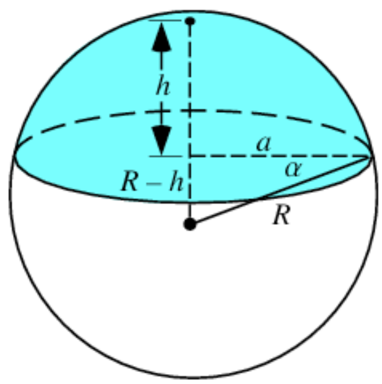
Spherical Cap - part I

In this short write-up I want to derive the formula for the volume of a spherical cap. This is the solid obtained by slicing off a part of a sphere with a plane.



The formula we will derive is

$$\boxed{V_{cap} = \frac{1}{3}\pi h^2(3R - h)} \tag{1}$$

or equivalently

$$V = \pi(Rh^2 - \frac{1}{3}h^3)$$

We can see that this equation makes sense for the extreme case where $h = R$. We get

$$V = \frac{1}{3}\pi R^2(3R - R) = \frac{2}{3}\pi R^3$$

Surface area of the sphere

Let's begin by remembering the formula for the surface area of the sphere

$$A = 4\pi R^2$$

Archimedes has a derivation of this in *On the Sphere and Cylinder* which is explained in Dunham's book *The Mathematical Universe*. I'd prefer not to take that detour here, but note that calculus provides a simple proof, starting from the formula for the volume of a sphere

$$V = \frac{4}{3}\pi R^3$$

Suppose we take a sphere of radius r. (I use r here because for just this part, the radius will be a variable). If we increase the radius by a little bit dr, then how does the volume change? It changes exactly like the surface area! That is

$$dV = A\ dr$$

$$A = \frac{d}{dr}\ V = \frac{d}{dr}\ \frac{4}{3}\pi r^3 = 4\pi r^2$$

Another way to see this is to break up the entire surface area of the sphere into small cones, each with area dA (almost flat) and height R. The volume of one cone is

$$\frac{1}{3}R\ dA$$

If we add up the volumes of all the little cones from the entire sphere we will have the volume of the sphere

$$\frac{1}{3}R\ A$$

but we already know this is just $4/3\pi R^3$, so clearly

$$\frac{1}{3}RA = \frac{4}{3}\pi R^3$$

$$A = 4\pi R^2$$

Volume of the sphere using calculus

A modern way to do this is by integration of slices (from Strang)

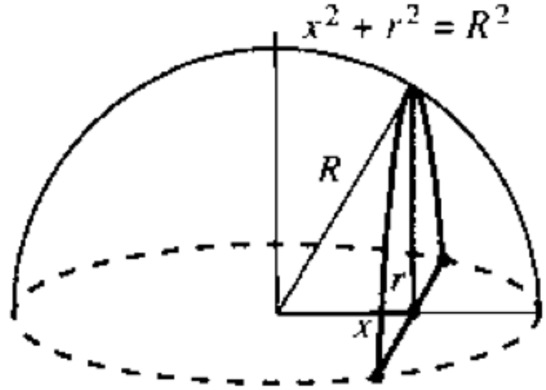


Fig. 8.4 **A half-sphere**

At each value of x, the cross-section of the hemisphere (radius R) is a half-circle with radius r such that

$$x^2 + r^2 = R^2$$

the area of this hemisphere cross-section is

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(R^2 - x^2)$$

For the whole sphere, each cross-section is a circle with area

$$A = \pi r^2 = \pi(R^2 - x^2)$$

For the volume, we just add up all these slices. To make it simple, take x from $x = -R \to x = R$

$$V = \int_{-R}^{R} \pi(R^2 - x^2)\ dx$$

$$V = \pi \int_{-R}^{R} (R^2 - x^2)\ dx$$

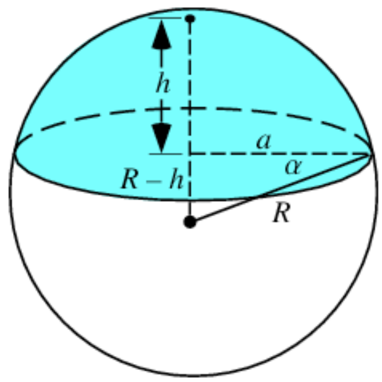
$$= \pi \left[\ R^2 x - \frac{1}{3}x^3\ \right] \Big|_{-R}^{R}$$

$$= \pi\ (R^3 - \frac{1}{3}R^3 - (-R)^3 + \frac{1}{3}(-R)^3$$

$$= \frac{4}{3}\pi R^3$$

Volume of the spherical cap

For the cap, just change the lower limit of integration to $x = R - h$!!



We need to evaluate

$$V = \pi \ [\ R^2 x - \frac{1}{3} x^3 \ ] \Big|_{R-h}^{R}$$

Leave aside the factor of π, and break the expression into two parts

$$R^2 x \Big|_{R-h}^{R} - \frac{1}{3} x^3 \Big|_{R-h}^{R}$$

For the left term we get

$$R^3 - R^3 + R^2 h = R^2 h$$

For the right side we get

$$-\frac{1}{3} R^3 + \frac{1}{3} (R - h)^3)$$

$$= -\frac{1}{3} R^3 + \frac{1}{3} R^3 - R^2 h + R h^2 - \frac{1}{3} h^3$$

Adding left and right terms together, the $R^2 h$ cancel, and we have finally

$$V = \pi (R h^2 - \frac{1}{3} h^3)$$

Factoring out $\frac{1}{3} h^2$

$$V = \frac{1}{3} \pi h^2 (3R - h)$$

which is the formula we gave at the top.

Volume of a spherical belt

We can calculate the volume of any spherical belt by using the appropriate limits of integration. For example, the belt from $r = 0 \to r = R - h$ has volume

$$V = \pi \; [\; R^2 x - \frac{1}{3} x^3 \;] \;\Big|_0^{R-h}$$

Leaving the π aside for now

$$R^2(R-h) - \frac{1}{3}(R-h)^3$$

$$R^3 - R^2 h - \frac{1}{3}(R^3 - 3R^2 h + 3Rh^2 - h^3)$$

$$\frac{2}{3}R^3 - Rh^2 + \frac{1}{3}h^3$$

With the factor of π

$$V = \pi(\frac{2}{3}R^3 - Rh^2 + \frac{1}{3}h^3)$$

Adding the cap and the belt together:

$$V_{tot} = \pi(Rh^2 - \frac{1}{3}h^3 + \frac{2}{3}R^3 - Rh^2 + \frac{1}{3}h^3)$$

Almost everything cancels

$$V_{tot} = \pi(\frac{2}{3}R^3)$$

The cap and the belt together make up a hemisphere.