Kepler (part 8) Fitzpatrick derivation

This is a derivation of Kepler's laws from a book I found on the web for Fitzpatrick's course on Mechanics.

1

He starts by establishing unit vectors in polar coordinates as $\mathbf{e_r}$ and $\mathbf{e}_\theta$ and then parametrically

$$\mathbf{e_r} = \langle \cos\theta, \sin\theta \rangle$$

$$\mathbf{e}_\theta = \langle -\sin\theta, \cos\theta \rangle$$

$$\mathbf{e}_\theta \perp \mathbf{e_r}$$

So

$$\dot{\mathbf{e}}_\mathbf{r} = \dot{\theta}\mathbf{e}_\theta$$

$$\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e_r}$$

Writing the position vector as

$$\mathbf{r} = r\ \mathbf{e_r}$$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e_r} + r\dot{\mathbf{e}}_\mathbf{r} = \dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e}_\theta$$

For the acceleration

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d}{dt}\ (\dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e}_\theta)$$

$$
\begin{aligned}
&= \ddot{r}\mathbf{e_r} + \dot{r}\dot{\mathbf{e}}_\mathbf{r} + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta \\
&= \ddot{r}\mathbf{e_r} + \dot{r}\dot{\theta}\mathbf{e}_\theta + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e_r} \\
&= (\ddot{r} - r\dot{\theta}^2)\mathbf{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta
\end{aligned}
$$

As before we recognize the coefficient for $\mathbf{e}_\theta$ as

$$
\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})
$$

and this term is also equal to zero because the acceleration is all radial and so the term in parentheses must be zero and so

$$
2r\dot{r}\dot{\theta} + r\ddot{\theta} = 0
$$

if we integrate

$$
\int 2r\dot{r}\dot{\theta} + r\ddot{\theta} = r^2\dot{\theta} = h
$$

where h is a constant.

The physical interpretation comes from angular momentum, which is defined as

$$
\begin{aligned}
\mathbf{l} &= m\mathbf{r} \times \dot{\mathbf{r}} \\
&= m(r\mathbf{e_r} \times (\dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e}_\theta) \\
&= mr^2\dot{\theta}\ \hat{\mathbf{k}}
\end{aligned}
$$

That is,

$$
mh \ = |\mathbf{l}|
$$

At this point he goes through the standard analysis to obtain that the area swept out in a small time $\delta A/\delta t = h/2$. I think we can skip this part.

2

This derivation has an unusual approach to using the information from the inverse square law. Define a new radial variable, the inverse of r

$$r = \frac{1}{u}$$

Differentiate with respect to time

$$\dot{r} = -\frac{\dot{u}}{u^2}$$

obviously. But what is $\dot{u}$?

$$\dot{u} = \frac{du}{dt} = \frac{du}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\ \frac{du}{d\theta}$$

So

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{\dot{\theta}}{u^2}\frac{du}{d\theta}$$

Recall $r^2\dot{\theta} = \dot{\theta}/u^2 = h$ so

$$= -h\ \frac{du}{d\theta}$$

Differentiate again with respect to time

$$\ddot{r} = -h\frac{d}{dt}\ (\frac{du}{d\theta}) = -h\dot{\theta}\ \frac{d^2u}{d\theta^2}$$

but $\dot{\theta} = hu^2$ so

$$= -h^2u^2\ \frac{d^2u}{d\theta^2}$$

Now, go back to our previous expression for the acceleration, it is

$$-\frac{GM}{r^2} = \ddot{r} - r\dot{\theta}^2$$

Plug in for $\ddot{r}$ and multiply everything by -1:

$$\frac{GM}{r^2} = h^2u^2\ \frac{d^2u}{d\theta^2} + r\dot{\theta}^2$$

Rearrange ($ru = 1$):

$$\frac{GM}{h^2} = \frac{d^2u}{d\theta^2} + \frac{r^3}{h^2}\dot{\theta}^2$$

but $h = r^2\dot{\theta}$ and $h^2 = r^4\dot{\theta}^2$ so

$$\frac{GM}{h^2} = \frac{d^2u}{d\theta^2} + \frac{1}{r}$$

$$\frac{GM}{h^2} = \frac{d^2u}{d\theta^2} + u$$

How about that? Now we have a basic differential equation in u

We guess the solution has, say $\cos\theta$ and constants A and C.

$$u = A\cos\theta + C$$

because

$$\frac{d^2u}{d\theta^2} = -A\cos\theta$$

So

$$C = \frac{GM}{h^2}$$

$$u = A\cos\theta + \frac{GM}{h^2}$$

Technically, we should have θ_0 in the solution, but we can just set that equal to zero, since we don't care about where we start. Go back to r

$$1 = r(A\cos\theta + \frac{GM}{h^2})$$

$$\frac{h^2}{GM} = r(A\frac{h^2}{GM} + A\cos\theta)$$

Define

$$e = A = \frac{GM}{h^2}$$

so now we have

$$\frac{h^2}{GM} = r(1 + e\cos\theta)$$

which is exactly what we had with Varberg.