Video 22: Morera and L

Morera's Theorem

This is really the converse of Cauchy's Theorem that the line integral of an analytic function $f(z)$ around a closed "nice" curve is equal to zero.

Suppose f is continuous on a domain D. If

$$\int_\gamma f(z)\; dz = 0 \quad \forall \text{ triangles } \gamma \text{ lying in } D$$

then f is analytic in D.

proof

Let $z_0 \in D$ and Ω be the disk $\{z : |z - z_0| < r\}$, with $r > 0$ and small enough so that Ω is contained in D.

Along a closed curve of three line segments joining $z_0 \to z$, $z \to z + h$ and then $z + h \to z_0$, we have that, for z also in Ω:

$$\int_{z_0}^{z} f(w)\; dw + \int_{z}^{z+h} f(w)\; dw + \int_{z+h}^{z_0} f(w)\; dw = 0$$

Moving the third term to the right-hand side, and reversing the path then subtracting the first term, we obtain:

$$\int_{z}^{z+h} f(w)\; dw = \int_{z_0}^{z+h} f(w)\; dw - \int_{z_0}^{z} f(w)\; dw$$

Switch the left- and right-hand sides and using the following new definition:

$$F(z) = \int_{z_0}^{z} f(w)\ dw$$

We obtain:

$$F(z+h) - F(z) = \int_{z}^{z+h} f(w)\ dw$$

Now, suppose h is a small complex number and we divide both sides by h:

$$\frac{F(z+h) - F(z)}{h} = \int_{z}^{z+h} \frac{f(w)}{h}\ dw$$

and subtract $f(z)$

$$\frac{F(z+h) - F(z)}{h} - f(z) = \int_{z}^{z+h} \frac{f(w)}{h}\ dw - f(z)$$

If we consider the line integral of the function 1 along a path going from $z \to z+h$ we can write

$$\int_{z}^{z+h}\ dw = h$$

so

$$-f(z) = \frac{-f(z)}{h} \int_{z}^{z+h}\ dw$$

Both $f(z)$ and h are constants so

$$-f(z) = -\int_{z}^{z+h} \frac{f(z)}{h}\ dw$$

Substitute this into what we had above

$$\begin{aligned}\frac{F(z+h) - F(z)}{h} - f(z) &= \int_{z}^{z+h} \frac{f(w)}{h}\ dw - \int_{z}^{z+h} \frac{f(z)}{h}\ dw \\ &= \int_{z}^{z+h} \frac{f(w) - f(z)}{h}\ dw\end{aligned}$$

epsilon-delta

With $\epsilon > 0$ given, we can choose δ small enough that

$$|f(w) - f(z)| < \epsilon$$

when

$$|w - z| < \delta$$

Also choose $|h| < \delta$.

Then

$$|\int_z^{z+h} (f(w) - f(z))\ dw\ | \leq |\epsilon| \int_z^{z+h} dw = \epsilon\ |h|$$

So

$$|\int_z^{z+h} \frac{f(w) - f(z)}{h}\ dw| \leq \epsilon$$

and

$$|\frac{F(z+h) - F(z)}{h} - f(z)| = |\int_z^{z+h} \frac{f(w) - f(z)}{h}\ dw| \leq \epsilon$$

So now as $\delta \to 0$, then $\epsilon \to 0$ and $h \to 0$ and so

$$\lim_{h \to 0} |\int_z^{z+h} \frac{f(w) - f(z)}{h}\ dw\ | = 0$$

So we have finally:

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} - f(z) = 0$$

That is:

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = F'(z) = f(z)$$

So $F(z)$ is differentiable. And since $F(z)$ is analytic, its derivative $f(z)$ is also analytic.