Quick review

We do a quick review with differentiation of basic complex functions of z: z, z^2, $\sqrt{z}$, $1/z$, $\mathrm{Log}(z)$, e^z, $\cos z$, and $\sin z$.

z

$$z = x + iy = u(x, y) + iv(x, y)$$
$$u_x = 1 = v_y$$
$$u_y = 0 = v_x$$

I guess this qualifies as meeting CRE.

$$(z)' = u_x + iv_x = 1$$

Note: for $\bar{x} = x* = x - iy$, $u_x \neq v_y$ and CRE are *not* satisfied.

z^2

$$z^2 = (x + iy)(x + iy) = x^2 - y^2 + i2xy$$
$$u_x = 2x = v_y$$
$$u_y = -2y = -v_x$$
$$(z^2)' = u_x + iv_x = 2x + i2y = 2z$$

$\sqrt{z}$

This one is easiest in polar coordinates. Take the principal value:

$$f(z) = \sqrt{z} = \sqrt{r}e^{i\theta/2}$$

We need to rewrite this as $z = u(r,\theta) + iv(r,\theta)$

$$u(r,\theta) = \sqrt{r}\cos\theta/2$$

$$v(r,\theta) = \sqrt{r}\sin\theta/2$$

So

$$u_r = \frac{1}{2\sqrt{r}}\cos\theta/2$$

$$u_\theta = \sqrt{r}\ \frac{1}{2}\ (-\sin\theta/2)$$

$$v_r = \frac{1}{2\sqrt{r}}\sin\theta/2$$

$$v_\theta = \sqrt{r}\ \frac{1}{2}\ \cos\theta/2$$

We see that

$$ru_r = v_\theta$$

$$rv_r = -u_\theta$$

The polar CRE are satisfied and

$$(\sqrt{z})' = e^{-i\theta}(u_r + iv_r)$$

$$= e^{-i\theta}(\frac{1}{2\sqrt{r}}\cos\theta/2 + i\frac{1}{2\sqrt{r}}\sin\theta/2$$

$$= e^{-i\theta}\frac{1}{2\sqrt{r}}e^{i\theta/2} = \frac{1}{2\sqrt{r}}e^{-i\theta/2}$$

$$= \frac{1}{2\sqrt{z}}$$

For an explanation of where that $e^{-i\theta}$ comes from in the polar form of the derivative, see the chapter on the polar CRE in our write-up on complex analysis.

1/z

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$

Every partial derivative will have a factor of $(x^2+y^2)^2$ in the denominator. It is easier if we just remember that and don't write it explicitly:

$$u_x = x^2 + y^2 - 2x^2 = y^2 - x^2$$

$$u_y = -2xy$$

$$v_x = 2xy$$

$$v_y = x^2 + y^2 - 2y^2 = y^2 - x^2$$

So So $u_x = v_y$ and $u_y = -v_x$ and CRE are satisfied.

$$[\ \frac{1}{z}\ ]' = u_x + iv_x = \frac{y^2-x^2}{(x^2+y^2)^2} + i\ \frac{2xy}{(x^2+y^2)^2}$$

We feel that this derivative should be equal to $-1/z^2$. So let's try to work backward, leaving the minus sign out for now.

$$\frac{1}{z^2} = \frac{1}{z}\ \frac{1}{z} = \frac{z*}{zz*}\ \frac{z*}{zz*}$$

So the denominator is clearly correct.

$$(zz*)^2 = (x^2+y^2)^2$$

On top we have

$$(z*)^2 = (x - iy)(x - iy) = x^2 - y^2 - i2xy$$

??? But remember the minus sign, we really have

$$(z*)^2 = y^2 - x^2 + i2xy$$

This matches what we have from $u_x + iv_x$.

$$[\ \frac{1}{z}\ ]' = -\frac{1}{z^2}$$

1/z in polar form

****

Log(z)

$$\mathrm{Log}(z) = \mathrm{Log}(re^{i\theta}) = \ln r + i\theta$$

Now rewrite in terms of x and y:

$$= \ln\sqrt{x^2 + y^2} + i\tan^{-1}\frac{y}{x}$$

$$= \frac{1}{2}\ln x^2 + y^2 + i\tan^{-1}\frac{y}{x}$$

$$u_x = \frac{1}{2}\frac{1}{(x^2 + y^2)}\ 2x = \frac{x}{x^2 + y^2}$$

$$u_y = \frac{y}{x^2 + y^2}$$

$$v_x = \frac{1}{1 + (y/x)^2)}\ (-\frac{y}{x^2}) = -\frac{y}{x^2 + y^2}$$

$$v_y = \frac{1}{1 + (y/x)^2)}\ 1/x = \frac{x}{x^2 + y^2}$$

So $u_x = v_y$ and $u_y = -v_x$ and CRE are satisfied.

$$[\ \mathrm{Log}(z)\ ]' = u_x + iv_x = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$= \frac{z*}{zz*} = \frac{1}{z}$$

exp(z)

$$e^z = e^{x+iy} = e^x\ e^{iy} = e^x(\cos y + i\sin y)$$

$$= e^x \cos y + ie^x\ \ \sin y$$

$$u_x = e^x \cos y$$

$$u_y = -e^x \sin y$$

$$v_x = e^x \sin y$$

$$v_y = e^x \cos y$$

So $u_x = v_y$ and $u_y = -v_x$ and CRE are satisfied.

$$(e^z)' = e^x \cos y + ie^x \sin y = e^z$$

cos z and sin z

To begin, we suppose that Euler's formula also works for complex numbers

$$e^{iz} = \cos z + i\sin z$$

$$e^{-iz} = \cos z - i\sin z$$

then

$$\cos z = \frac{1}{2}\ [\ e^{iz} + e^{-iz}\ ]$$

$$\sin z = \frac{1}{2i}\ [\ e^{iz} - e^{-iz}\ ]$$

We can take derivatives with respect to z easily:

$$(\cos z)' = \frac{1}{2}\ [\ ie^{iz} - ie^{-iz}\ ]$$

$$= -\frac{1}{2i}\ [\ e^{iz} - e^{-iz}\ ] = -\sin z$$

$$(\sin z)' = \frac{1}{2i}\ [\ ie^{iz} + ie^{-iz}\ ]$$

$$= \frac{1}{2}\ [\ e^{iz} + e^{-iz}\ ] = \cos z$$

There is another way to get this result which separates z into $u(x,y)$ and $v(x,y)$.

We need a preliminary result, consider a purely imaginary z:

$$z = iy$$

Then

$$e^{iz} = e^{iiy} = e^{-y}$$

$$e^{-iz} = e^{-iiy} = e^{y}$$

$$\cos iy = \frac{1}{2}\ [\ e^{iz} + e^{-iz}\ ] = \frac{1}{2}\ [\ e^{-y} + e^{y}\ ]$$

The term on the right may be familiar from calculus, it is $\cosh y$.

$$\cos iy = \cosh y$$

Similarly

$$\sin iy = \frac{1}{2i}\ [\ e^{iz} - e^{-iz}\ ] = \frac{1}{2i}\ [\ e^{-y} - e^{y}\ ]$$

Having i on the bottom is the same as $-i$ on the top. We use the factor of -1 to switch the order of terms:

$$\sin iy = i\frac{1}{2}\ [\ e^{y} - e^{-y}\ ] = i \sinh y$$

Now we proceed with the cosine first

$$\cos z = \cos x + iy$$

$$= \cos x \cos iy - \sin x \sin iy$$

$$= \cos x \cosh y - i \sin x \sinh y$$

For the sine:

$$\sin z = \sin x + iy$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

Note that we have separated $u(x, y)$ from $v(x, y)$. This allows us to test the CRE, cosine first.

Note that $\sinh y' = \cosh y$ and $\cosh y' = \sinh y$ so

$$u_x = -\sin x \cosh y$$

$$u_y = \cos x \sinh y$$

$$v_x = -\cos x \sinh y$$

$$v_y = -\sin x \cosh y$$

So $u_x = v_y$ and $u_y = -v_x$ and the CRE are satisfied.

Write the derivative.

$$[\cos z\ ]' = u_x + iv_x = -\sin x \cosh y - i \cos x \sinh y = -\sin z$$

For the sine:

$$u_x = \cos x \cosh y$$

$$u_y = \sin x \sinh y$$

$$v_x = -\sin x \sinh y$$

$$v_y = \cos x \cosh y$$

So $u_x = v_y$ and $u_y = -v_x$ and CRE are satisfied.

$$[\sin z\ ]' = u_x + iv_x = \cos x \cosh y - i \sin x \sinh y = \cos z$$