Product, Chain and Quotient Rules

In previous work, we found out how to differentiate (find the derivative of) polynomial functions of x. The power rule is:

$$f(x) = x^n, \qquad f'(x) = nx^{n-1}$$

We also saw that constants just come along with the result

$$f(x) = ax^n, \qquad f'(x) = anx^{n-1}$$

We didn't bother to show it, but you already know that

$$f(x) = a, \qquad f'(x) = 0$$

The slope of $y = \textit{constant}$ is just 0. And we saw that the power rule extends to negative and rational exponents.

$$f(x) = \frac{1}{x}, \qquad f'(x) = -\frac{1}{x^2}$$

$$f(x) = \sqrt{x}, \qquad f'(x) = \frac{1}{2\sqrt{x}}$$

Finally, we saw that for a sum of two functions

$$f(x) + g(x), \qquad (f(x) + g(x))' = f'(x) + g'(x)$$

This leads naturally to the question, what about a product of functions?

$$f(x)\ g(x), \qquad (f(x)\ g(x))' = ?$$

I won't go through a proof of this, but the product rule is

$$(f(x)\ g(x))' = f(x)\ g'(x) + f'(x)\ g(x)$$

Written with differentials and different letters u and v this is the same as

$$u = f(x), v = g(x)$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

which is frequently abbreviated as

$$(uv)' = uv' + vu'$$

I find this form particularly easy to remember, but don't forget what v' means. It means $\frac{dv}{dx}$ (or some other variable). Sometimes the product rule is given as "this times the derivative of that plus that times the derivative of this", where "this" is u and "that" is v.

Let's do some examples

$$f(x) = x^2 = x\ x, \quad (x\ x)' = x(1) + (1)x = 2x$$

The answer must match what we already know to be true!

$$f(x) = \sqrt{x}\sqrt{x}, \quad (\sqrt{x}\sqrt{x})' = \frac{1}{2\sqrt{x}}\sqrt{x} + \sqrt{x}\frac{1}{2\sqrt{x}} = 1$$

$$f(x) = ax, \quad (ax)' = a\ (1) + (0)x = a$$

A last one that we already know. Suppose

$$f(x) = (x+3)(2x-2) = 2x^2 + 4x - 6$$

by the power rule:

$$f'(x) = \frac{d}{dx}2x^2 + 4x - 6 = 4x + 4$$

by the product rule:

$$f'(x) = \frac{d}{dx}(x+3)(2x-2) = (x+3)(2) + (2x-2)(1) = 4x + 4$$

It is worth while to play around with some examples, once we know the trig function derivatives, and add the exponentials when we get there.

$$f(x) = x\ sin\ x, \quad f'(x) = x\ cos\ x + sin\ x$$

$$f(x) = sinx\ cos\ x, \quad f'(x) = -sin^2x + cos^2x = 2cos^2x - 1$$

Later, when we want to go backward from the derivative cos^2x to the original function (i.e. integration), this second result will become very useful.