Euler's Gem

Here are sketches of two different derivations of Euler's famous formula, both following Dunham's book about Euler*

$$e^{i\theta} = \cos\theta + i\sin\theta$$

And of course, if $\theta = \pi$, we have

$$e^{i\pi} = -1 + 0$$
$$e^{i\pi} + 1 = 0$$

(what Feynman called "our jewel").

Using x is a bit simpler notation, so that's what I'll do here

$$e^{ix} = \cos x + i\sin x$$

preliminary

Start with the definition of i

$$i = \sqrt{-1}$$

Having i gives us new factorizations like

$$a^2 + b^2 = (a + bi)(a - bi)$$

since the terms with $\pm\ abi$ cancel and $-i^2 = 1$. So

$$1 = \cos^2 x + \sin^2 x$$

$$1 = (\cos x + i \sin x)(\cos x - i \sin x)$$

(Of course, we could switch sine and cosine here, but this is the convention.

Below, we will need the above plus one more identity involving i:

$$-i^2 = 1$$

so

$$u = -i^2 u$$
$$\frac{u}{i} = -iu$$

number one

Start with the inverse sine function:

$$x = \sin^{-1} y$$

$$y = \sin x$$

$$dy = \cos x \ dx$$

Then the side adjacent to x is $\sqrt{1-y^2}$ and so

$$\cos x = \sqrt{1-y^2}$$

We're interested in the integral

$$\int \frac{1}{\sqrt{1-y^2}} \ dy$$

which is just

$$= \int \frac{1}{\cos x} \ \cos x \ dx = x$$

Now, Euler makes a complex change of variable

$$y = iz$$

$$\frac{1}{1-y^2} = \frac{1}{1+z^2}$$

$$x = \int \frac{1}{1-y^2} \ dy$$

$$= \int \frac{1}{\sqrt{1+z^2}} \ i \ dz$$

we have converted the integral to having a plus sign under the square root and the answer is

$$= i \ln \left(\sqrt{1+z^2} + z\right)$$

(I will justify this elsewhere—it's a standard trig substitution but a bit complicated).

Now, just undo the substitution:

$$z = \frac{y}{i} = \frac{\sin x}{i}$$

$$\sqrt{1+z^2} = \sqrt{1-y^2} = \cos x$$

Hence our previous result

$$x = i \ln \left(\sqrt{1+z^2} + z\right)$$

is equivalent to

$$x = i \ln \left(\cos x + \frac{\sin x}{i}\right)$$

Recall our two identities involving i. The first one was

$$\frac{u}{i} = -iu$$

So when we had:

$$x = i \ln \left(\cos x + \frac{\sin x}{i}\right)$$

$$x = i \ln \left(\cos x - i \sin x\right)$$

$$ix = -\ln \left(\cos x - i \sin x\right)$$

$$= \ln \frac{1}{(\cos x - i \sin x)}$$

again

$$\frac{1}{\cos u - i \sin u} = \cos u + i \sin u$$

so

$$ix = \ln \frac{1}{(\cos x - i \sin x)} = \ln \left(\cos x + i \sin x\right)$$

Just exponentiate:

$$e^{ix} = \cos x + i \sin x$$

number two

Suppose we try this multiplication:

$$(\cos s + i \sin s)(\cos t + i \sin t)$$

$$= \cos s \cos t + i \sin s \cos t + i \cos s \sin t - \sin s \sin t$$

$$= (\cos s \cos t - \sin s \sin t) + i(\sin s \cos t + \cos s \sin t)$$

$$= \cos(s + t) + i \sin(s + t)$$

set $s = t$

$$(\cos s + i \sin s)^2 = \cos 2s + i \sin 2s$$

In fact, Euler showed that it works for fractional n but I'll assume that part.

$$(\cos s + i \sin s)^n = \cos ns + i \sin ns$$

Now multiply the difference rather than the sum:

$$(\cos s - i \sin s)(\cos t - i \sin t)$$

$$= (\cos s \cos t - \sin s \sin t) - i(\sin s \cos t + \sin t \cos s)$$

$$= \cos(s + t) - i(\sin(s + t))$$

again, with $s = t$

$$(\cos s - i \sin s)^2 = \cos 2s - isin2s$$

$$(\cos s - i \sin s)^n = \cos ns - isinns$$

Restate the two results:

$$(\cos s + i \sin s)^n = \cos ns + i \sin ns$$

$$(\cos s - i \sin s)^n = \cos ns - isinns$$

Add them

$$2 \cos ns = (\cos s + i \sin s)^n + (\cos s - i \sin s)^n$$

where the magic happens

Let

$$s = \frac{x}{n}$$

As $n \to \infty$, $s \to 0$, and

$$\sin s \to s$$

$$\cos s \to 1$$

$$\cos x = \cos ns$$
$$= \frac{1}{2}\ [\ (\cos s + i\sin s)^n + (\cos s - i\sin s)^n\ ]$$
$$= \frac{1}{2}\ [\ (1+is)^n + (1-is)^n\ ]$$
$$= \frac{1}{2}\ [\ (1+\frac{ix}{n})^n + (1-\frac{ix}{n})^n\ ]$$

but

$$e^{ix} = (1+\frac{ix}{n})^n$$

hence

$$\cos x = \frac{1}{2}\ [\ e^{ix} + e^{-ix}\ ]$$

By very similar manipulation to what's in the first part we can also obtain an expression for the sine:

$$2i\ \ \sin(ns) = (\cos s + i\sin s)^n - (\cos s - i\sin s)^n$$

which will lead to

$$\sin x = \frac{1}{2i}\ (e^{ix} - e^{-ix})$$

Adding together

$$2(\cos x + i\sin x) = e^{ix} + e^{-ix} + e^{ix} - e^{-ix}$$

$$\cos x + i\sin x = e^{ix}$$

check

Before quitting, we should check the formula. One way is to notice the connection between infinite series expansions for e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots$$

and sine:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

and cosine:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots$$

These can almost be added together to give what we seek, except for the problem of the alternating signs. What happens with e^{ix}?

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} \dots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} \dots \end{aligned}$$

The pattern is

$$\sum_{n=0}^{\infty} i^n = 1 + i - 1 - i + 1 \dots$$

And we're there. We just have to recognize that the pattern with e^{ix} has $i \sin x$ so as we said

$$e^{ix} = \cos x + i \sin x$$