Power series

$$f(z) = \sum_{0}^{\infty} a_n \ (z - z_0)^n$$

If $f(z)$ has a positive (or infinite) radius of convergence, then inside the disk $|z - z_0| < R$, $f(z)$ is infinitely differentiable, and each derivative is given by a power series:

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) \ a_n \ (z - z_0)^{n-k}$$

Consider as an example this power series with $a_n = n$:

$$f(z) = \sum_{0}^{\infty} n \ z^n = 0 + z + 2z^2 + 3z^3 \dots$$

Differentiate term by term:

$$f'(z) = 1 + 2^2 z + 3^2 z^2 + \dots + n^2 z^{n-1} + \dots$$

Does this match the definition?

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) \ a_n \ (z - z_0)^{n-k}$$

The first derivative is for $k = 1$ so

$$n \dots (n - k + 1) = n$$

and $z_0 = 0$ so

$$f'(z) = \sum_{n=1}^{\infty} n\ a_n\ z^{n-1}$$

Recall that $a_n = n$

$$= \sum_{n=1}^{\infty} n^2\ z^{n-1}$$

which indeed matches

$$f'(z) = 1 + 2^2 z + 3^2 z^2 + \dots + n^2 z^{n-1} + \dots$$

Go back to

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)\ a_n\ (z - z_0)^{n-k}$$

If $z = z_0$ then

$$f^{(k)}(z_0) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)\ a_n\ 0^{n-k}$$

Now you might think that all these terms would be zero, and they are provided that $n - k \neq 0$, but there:

$$f^{(k)}(z_0) = \sum_{n=k}^{\infty} n(n-1)\dots 1\ a_n\ 0^0$$

$$f^{(k)}(z_0) = n!\ a_n$$

$$a_n = \frac{1}{n!}\ f^{(k)}(z_0)$$

And the reason is that

$$0^0 = \lim_{n \to 0} n^n = 1$$

`http://mathforum.org/dr.math/faq/faq.0.to.0.power.html`

exponential

Write the exponential as

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3} + \dots$$

We could use the formula for the derivative, or just differentiate term by term and obtain that

$$f(z) = f'(z)$$

Now write (by the chain rule)

$$[\, e^{-z} f(z)]' = -e^{-z} f(z) + e^{-z} f'(z)$$

and since $f(z) = f'(z)$

$$= -e^{-z} f'(z) + e^{-z} f'(z)$$

But this is just 0. So that means the original compound function is a constant, since its derivative is zero:

$$e^{-z} f(z) = C$$

and this is true for any z so

$$e^0 f(0) = f(0) = C$$

so going back to the series, when $z = 0$

$$f(0) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{0^n}{n!}$$

Now, the numerator is zero for all terms except $n = 0$ where it is equal to 1 (see above), and that denominator is $0! = 1$, so

$$f(0) = 1 = C$$

Thus

$$e^{-z} f(z) = 1$$
$$f(z) = e^z$$