Center of Mass

As you know, in single variable calculus we can interpret

$$\int_a^b f(x)dx$$

as the area underneath the curve $y = f(x)$ between the lines $x = a$ and $x = b$ (our limits). In multi-variable calculus we compute the double integral over the same region as follows

$$\int_{x=a}^{x=b} \int_{y=0}^{y=f(x)} dy\ dx = \int_{x=a}^{b} y\Big|_0^{f(x)}\ dx = \int_a^b f(x)\ dx$$

To be more general, we'd just say that we compute the double integral over the region R

$$\iint\limits_R dx\ dy$$

with the understanding that we can compute the inner integral with respect to either x or y, whichever is more convenient. Another difference from the single-variable approach is that we can extend this approach by computing

$$\iint\limits_R g(x,y)\ dx\ dy$$

Suppose, for example, that $g(x,y)$ is a function that gives the density of a flat object for each coordinate x, y. In this case we usually use the label $\rho(x,y)$. This integral gives the total mass of the object:

$$M = \iint\limits_R \rho(x,y)\ dx\ dy$$

To find the center of mass we compute

$$M_x = \iint\limits_R \rho(x,y)\ y\ dx\ dy$$

$$M_y = \iint_R \rho(x,y)\ x\ dx\ dy$$

And then finally

$$\bar{x} = \frac{M_x}{M}$$

$$\bar{y} = \frac{M_y}{M}$$

Let's do a simple example. Suppose our region is a rectangle with the origin as one corner and the point $(1, 2)$ as the opposite corner. It's just a 2D box of width 1 and height 2. And let's say our density function is $\rho(x, y) = xy$. Then

$$M = \iint_R \rho(x,y)\ dx\ dy = \int_{y=0}^{y=2} \int_{x=0}^{x=1} xy\ dx\ dy$$

The inner integral is

$$\frac{1}{2}x^2 y \Big|_0^1 = \frac{1}{2}y$$

and the rest is

$$M = \int_{y=0}^{y=2} \frac{1}{2}y\ dy = \frac{1}{4}y^2 \Big|_0^2 = 1$$

Now

$$M_x = \int_{y=0}^{y=2} \int_{x=0}^{x=1} xy^2\ dx\ dy$$

The inner integral is

$$\frac{1}{2}x^2 y^2 \Big|_0^1 = \frac{1}{2}y^2$$

and the rest is

$$M_x = \int_{y=0}^{y=2} \frac{1}{2}y^2\ dy = \frac{1}{6}y^3 \Big|_0^2 = \frac{4}{3}$$

Last, M_y can be done in the same order

$$M_y = \int_{y=0}^{y=2} \int_{x=0}^{x=1} x^2 y\ dx\ dy$$

The inner integral is

$$\frac{1}{3}x^3y \Big|_0^1 = \frac{1}{3}y$$

and the rest is

$$M_y = \int_{y=0}^{y=2} \frac{1}{3}y \; dy = \frac{1}{6}y^2 \Big|_0^2 = \frac{2}{3}$$

Thus our center of mass is at the point $2/3, 4/3$. If it had made our lives easier, either integral could be computed with respect to y before x.