Nahin Imaginary Tale Ch. 1

Nahin gives this problem from Diophantus: a right triangle has area 7 and perimeter 12. Find the lengths of the two sides.

From these constraints, write:

$$ab = 14$$

$$a + b + \sqrt{a^2 + b^2} = 12$$

analysis

Suppose the triangle is isosceles. Then $a = b$ and

$$a^2 = 14$$

$$a = \sqrt{14} \approx 3.74$$

Then we must have

$$c^2 = a^2 + b^2 = 14 + 14 = 28$$

$$c = 5.29$$

The perimeter would be approximately

$$3.74 + 3.74 + 5.29 = 12.77$$

This perimeter is too long. But an isosceles triangle has the *shortest* perimeter for a given area. Hence, there is no solution.

my solution

$$a + b + \sqrt{a^2 + b^2} = 12$$

Solve this algebraically. First, remove the square root.

$$a^2 + b^2 = (12 - a - b)^2$$

$$= 12^2 - 12a - 12b - 12a + a^2 + ab - 12b + ab + b^2$$

Cancel a^2 and b^2

$$0 = 12^2 - 12a - 12b - 12a + ab - 12b + ab$$

$$= -24a - 24b + 2ab + 12^2$$

$$= -12a - 12b + ab + 72$$

Substitute for b from $ab = 14$:

$$= -12a - 12\frac{14}{a} + 14 + 72$$

$$-12a^2 + 86a - 168 = 0$$

Factor out -2

$$6a^2 - 43a + 84 = 0$$

$$a = \frac{43 \pm \sqrt{43^2 - 4 \cdot 6 \cdot 84}}{12}$$

$$= \frac{43 \pm \sqrt{1849 - 2016}}{12}$$

$$= \frac{43 \pm \sqrt{-167}}{12}$$

There is no real solution.

Diophantus solution

Nahin says a bright idea is to substitute:

$$a = \frac{1}{x}$$

$$b = 14x$$

so then

$$a + b + \sqrt{a^2 + b^2} = 12$$

$$\frac{1}{x} + 14x + \sqrt{\frac{1}{x^2} + 14^2x^2} = 12$$

Nahin says this is **easily** put in the form

$$172x = 336x^2 + 24$$

Easily? We have the same problem with the square root as before. Multiply by x

$$1 + 14x^2 + \sqrt{1 + 14^2x^4} = 12x$$

Now isolate the square root and square the terms on the other side

$$(-14x^2 + 12x - 1)^2$$

$$= 14^2x^4 - (14)12x^3 + 14x^2 - (14)12x^3 + 144x^2 - 12x + 14x^2 - 12x + 1$$

The two terms from squaring $\sqrt{1 + 14^2x^4}$ disappear, leaving:

$$0 = -(14)12x^3 + 14x^2 - (14)12x^3 + 144x^2 - 12x + 14x^2 - 12x$$

$$= -2(14)12x^3 + 28x^2 + 144x^2 - 24x$$

$$-336x^3 + 172x^2 - 24x = 0$$

Divide by x

$$-336x^2 + 172x - 24 = 0$$

Make all terms positive

$$336x^2 + 24 = 172x$$

This matches what Nahin has.

Remove a factor of 4

$$-84x^2 + 43x - 6 = 0$$

We have still to solve for x and then go back to a and b. But we cannot solve for x, because the discriminant is

$$43^2 - 4 \cdot 84 \cdot 6 = 1849 - 2016 = -167$$

that is, less than zero. So again, there are no real solutions.

compare solutions

Suppose we do write

$$x = \frac{-43 \pm \sqrt{-167}}{2(-84)}$$

and compare with

$$a = \frac{43 \pm \sqrt{-167}}{12}$$

How does this make sense in light of the substitution $a = 1/x$? How can we have the same value in the numerator of both expressions, when they are inverses?

Recall that for complex numbers

$$\frac{1}{z} = \frac{z*}{zz*}$$

Taking the positive root of x

$$x = \frac{-43 + \sqrt{-167}}{2(-84)}$$

$$= \frac{43}{168} - \frac{\sqrt{-167}}{168}$$

$$x* = \frac{43}{168} + \frac{\sqrt{-167}}{168}$$

and

$$xx* = \frac{43^2}{168^2} + \frac{167}{168^2}$$

Hence

$$\frac{1}{x} = \frac{x*}{xx*} = \frac{43/168 + \sqrt{-167}/168}{43^2/168^2 + 167/168^2}$$

$$= \frac{43 + \sqrt{-167}}{43^2/168 + 167/168}$$

The numerators match now. The denominator in the last expression is

$$\frac{43^2}{168} + \frac{167}{168}$$

$$43^2 + 167 = 1849 + 167 = 2016$$

But

$$12 \times 168 = 2016$$

So the denominators match, as well.