Sequences: analysis definitions

A sequence of numbers (a_n) starts at a specific place, with a particular first term a_1 and then goes on forever. Usually, there will be some kind of pattern:

$$(a_n) = a_1, a_2, a_3, \ldots a_n, \ldots$$

$$1, 2, 3 \ldots$$

or whatever. Note the difference between notation for the nth term—a_n, and the sequence itself. Technically a sequence is a function from the natural numbers to the reals($f : \mathbb{N} \to \mathbb{R}$).

monotonic sequences

- The sequence (a_n) is *increasing* $\iff \forall\, n \in N, \quad a_{n+1} \geq a_n$

Note the $\geq$. An increasing sequence need not be strictly increasing.

- A sequence is *monotonic* $\iff$ it is increasing or decreasing.

So there are two different types of monotonic sequences, those that go up and those that go down. For the most part, monotonic sequences that are increasing are mirror images of the ones that decrease, so we can concentrate on the first class.

A monotonic sequence that increases need not be *strictly increasing.*

bounded sequences

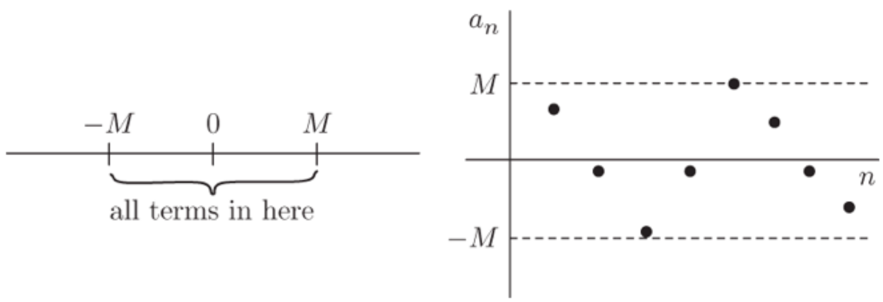
• The sequence (a_n) is *bounded above* $\iff$

$$\exists\ M \in \mathbb{R} \mid \forall\ n \in \mathbb{N}, a_n \leq M$$

If we can find a number M which is never exceeded by any term in the sequence, then it is bounded above. Any sequence that has an upper bound has many upper bounds (e.g. $M + 1$ is also an upper bound).

• The sequence (a_n) is *bounded* $\iff$ $\exists\ M > 0 \mid \forall\ n \in \mathbb{N}, |a_n| \leq M$

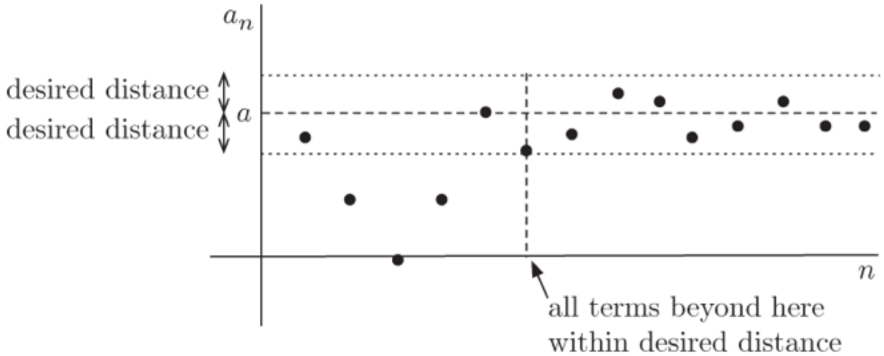
A sequence is bounded if it is bounded above *and* bounded below.



convergent sequences

• The sequence (a_n) *converges* to a $\iff$

$$\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N} \mid \forall\ n > N, |a_n - a| < \epsilon$$



The early behavior of a sequence is irrelevant. The requirement is that at some point (for index $n > N$), all the terms after a_n must be closer than ϵ to the value a: that is $|a_n - a| < \epsilon$. The terms may be either $> a$ or $< a$.

ϵ can be chosen as small as we like, and the sequence must get as close as ϵ to the value a.

• Every bounded monotonic sequence is convergent.

Here are some other statements that may or may not be true:

∘ Every bounded sequence is convergent.

∘ Every convergent sequence is bounded.

∘ Every monotonic sequence is convergent.

∘ Every convergent sequence is monotonic.

∘ Every monotonic sequence is bounded.

∘ Every bounded sequence is monotonic.

• Every convergent sequence is bounded.

If (a_n) converges then

$$\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N} \mid \forall\ n > N, |a_n - a| < \epsilon$$

There is an index in the sequence (N) where every term after that differs from a by as little as we please. So certainly, all these terms are less than $a+1$. Then we just find the maximum term in the sequence up to a_n. The maximum of that term and $a+1$ is our bound. Since the sequence is finite and no terms approach infinity, there is such a bound.

rest

- The sequence (a_n) *tends to* $\infty \iff$

$$\forall\ C > 0,\ \exists N \in \mathbb{N} \mid \forall\ n > N, a_n > C$$

- The sequence (a_n) is a *Cauchy* sequence $\iff$

$$\forall\ \epsilon > 0\ \exists\ N \in \mathbb{N} \mid \forall\ n, m > N, |a_n - a_m| < \epsilon$$

Infinity

Here is a complicated definition: the sequence (a_n) tends to infinity $(a_n \to \infty)$ if

$$\forall\ C > 0,\ \exists N \in \mathbb{N} \mid \forall\ n > N,\ a_n > C$$

Notice: for all $C > 0$. Pick any value C, and the sequence after some N must have terms all larger than C.