Euler's identity and complex numbers

Taylor series

The way most people first see a connection demonstrated between exponential and trigonometric functions is through the Taylor Series, which allows construction of a series that approximates a function provided the derivatives are known. (For a different approach, see Chapter 22 in Feynman's Lectures of Physics). Near $x = 0$, the approximation is

$$f(x) \approx f(0) + f^1(0)\frac{x}{1!} + f^2(0)\frac{x^2}{2!} + f^3(0)\frac{x^3}{3!} \dots$$

where $f^n(0)$ is the nth derivative of f evaluated at 0. Applied to the exponential function e^x, it's simple, since the function is its own derivative, and at zero, is equal to 1. Thus

$$e^x \approx 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$$

The series also provides a method for finding the numerical value of e, by just plugging in $x = 1$.

I won't do it here, but it turns out that if you apply this method to the sine and cosine, and then to the complex sine function $\sin ix$, it is easy to show that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

radial coordinates

Instead of x, y ccordinates, a different way to represent complex numbers uses the notion of an angle θ and radius r in the complex plane. This idea has in the past been credited to a guy named Argand (the Argand plane). However, another mathematician named Wessel wrote about it earlier, but his paper was apparently not noticed.

$$a + bi = r\cos\theta + ri\sin\theta$$

We use Euler's famous identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$a + bi = re^{i\theta}$$

The x-coordinate is a and the y-coordinate is b and so the length of the hypotenuse is

$$r = \sqrt{x^2 + y^2}$$

Compute θ by finding the angle whose tangent is what we need:

$$\theta = tan^{-1}(\frac{y}{x})$$

We gain some feeling for what θ means in the symbol $e^{i\theta}$. It is the angle a vector from the origin to the complex number makes with the positive x-axis.

derivation

It's fun to look at a quick derivation of Euler's equation. It may lack something in mathematical rigor, but it's like lightning.

Suppose we imagine a point on a circle of radius $r = 1$ in the complex plane.

$$z = \cos\theta + i\sin\theta$$

Now manipulate in the way that Euler was always ready to do:

$$\begin{aligned} \frac{dz}{d\theta} &= -\sin\theta + i\cos\theta \\ &= i^2\sin\theta + i\cos\theta \\ &= i(i\sin\theta + \cos\theta) \\ &= iz \end{aligned}$$

So rearrange and integrate

$$\begin{aligned} \frac{1}{z}\ dz &= i\ d\theta \\ \int \frac{1}{z}\ dz &= \int i\ d\theta \\ \ln z &= i\theta \\ z = e^{i\theta} &= \cos\theta + i\sin\theta \end{aligned}$$

Amazing.

exponential forms of sine and cosine

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Now

$$\begin{aligned} e^{-i\theta} &= \cos -\theta + -i\sin\theta \\ &= \cos\theta + -i\sin\theta \end{aligned}$$

By addition, then

$$\begin{aligned} e^{i\theta} + e^{-i\theta} &= 2\cos\theta \\ \frac{1}{2}(e^{i\theta} + e^{-i\theta}) &= \cos\theta \end{aligned}$$

And by subtraction

$$e^{i\theta} - e^{-i\theta} = 2i\cos\theta$$

$$\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \sin\theta$$

All of the familiar relationships hold. In particular

$$\frac{d}{d\theta}\ \cos\theta = \frac{d}{d\theta}\ \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$= \frac{1}{2}i(e^{i\theta} - e^{-i\theta}) = -\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = -\sin\theta$$

$$\frac{d}{d\theta}\ \sin\theta = \frac{d}{d\theta}\ \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

$$= \frac{1}{2}(e^{\theta} + e^{-i\theta}) = \cos\theta$$

Finally, (since $e^{i\theta}e^{-i\theta} = 1$):

$$\sin^2\theta = -\frac{1}{4}(e^{i\theta} - e^{-i\theta})^2 = -\frac{1}{4}(e^{2i\theta} - 2 + e^{-2i\theta})$$

$$\cos^2\theta = \frac{1}{4}(e^{i\theta} + e^{-i\theta})^2 = \frac{1}{4}(e^{2i\theta} + 2 + e^{-2i\theta})$$

If you follow the signs carefully, you will see that summing the last two equations just gives 1.

more on Euler

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Multiply by $e^{i\phi}$

$$e^{i\theta}\ e^{i\phi} = e^{i(\theta+\phi)}$$

So the right-hand side is

$$\cos(\theta + \phi) + i\sin(\theta + \phi)$$

But the left-hand side is

$$= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$

$$= \cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \sin\phi\cos\theta)$$

So, as we said before, if two complex numbers are equal, both the real and the imaginary parts are equal.

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$$

That should look familiar. When mathematics works out like that, not only is it sure to be right, but it should fill you with awe.