Divergence

Green's Theorem is

$$\oint_C \mathbf{F} \cdot \mathbf{dr} = \iint\limits_R curl(\mathbf{F})\ dA$$

$$\oint_C M\ dx + N\ dy = \iint\limits_R (N_x - M_y)\ dA$$

It can be used for finding area by computing a line integral. The trick is to imagine an M and N such that

$$N_x - M_y = 1$$

because then we have

$$\oint_C M\ dx + N\ dy = \iint\limits_R\ dA = Area$$

An especially common choice is:

$$M = -\frac{y}{2}, \quad N = \frac{x}{2}$$

So we have that

$$A = \frac{1}{2} \oint_C x\ dy - y\ dx$$

Let's try this out on a circle. What is the parametrization of C? We need x and y as functions of t:

$$x = a\ cos\ t$$

$$dx = -a\ sin\ t\ dt$$

$$y = a\ sin\ t$$

$$dy = a\ cos\ t\ dt$$

$$A = \frac{1}{2}\int_{t=0}^{t=2\pi} a^2 cos^2 t \ dt + a^2 sin^2 t \ dt$$
$$= \frac{1}{2} \ a^2 \ 2\pi = \pi a^2$$

What about an ellipse? What is the parametrization of C? We need x and y as functions of t:

$$x = a \ cos \ t,$$
$$dx = -a \ sin \ t \ dt$$
$$y = b \ sin \ t$$
$$dy = b \ cos \ t \ dt$$

To see that the above is correct, do this

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cos^2 \ t + sin^2 \ t = 1$$

Clearly the formula of an ellipse. Now plug into the integral

$$A = \frac{1}{2}\int_{t=0}^{t=2\pi} ab \ cos^2 t \ dt + ab \ sin^2 t \ dt$$
$$\frac{1}{2} \ ab \ 2\pi = \pi ab$$

Finally, what seems a very simple example: the part of the rectangle $[0,1] \times [0,2]$ below the line $y = 2x$, is actually complicated because there are three segments of the curve:

$$y = 2x$$
$$dy = 2 \ dx$$
$$A = \frac{1}{2}\oint_C x \ dy - y \ dx$$
$$C1 : x = 0 \to 1, \ \ y = 0$$
$$C2 : x = 1, \ \ y = 0 \to 2$$
$$C3 : y = 2x, \ \ x = 1 \to 0$$
$$M = -\frac{y}{2}, \quad N = \frac{x}{2}$$
$$A_1 = \frac{1}{2}\oint_C x \ dy - y \ dx = \frac{1}{2}\oint_C x \ 0 - 0 \ dx = 0$$

$$A_2 = \frac{1}{2}\oint_C x\ dy - y\ dx = \frac{1}{2}\oint_C 1\ dy - y\ 0 = y\Big|_0^2 = 1$$

$$A_3 = \frac{1}{2}\oint_C x\ dy - y\ dx = \frac{1}{2}\oint_C x\ 2\ dx - 2x\ dx = 0$$

The total area is just 1.

divergence example

Verify the divergence theorem for a hemisphere of radius a with $\mathbf{F} = \langle x, y, z\rangle$. Restate the theorem

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = \iiint_D \nabla\cdot\mathbf{F}$$

Noting that the field is given in x, y, z-coordinates, recall that

$$\hat{\mathbf{n}}\ dS = a^2\ \langle x, y, z\rangle\ \sin\phi\ d\phi\ d\theta$$

So on the left side

$$\mathbf{F}\cdot\hat{\mathbf{n}}\ dS = \langle x, y, z\rangle \cdot a^2\langle x, y, z\rangle = a^3$$

and

$$= \iint_S a^3\ \sin\phi\ d\phi\ d\theta$$

$$= \int_S a^3\ [\ -\cos\phi\ \Big|_0^{\pi/2}\ ]\ d\theta$$

$$= \int_S a^3\ d\theta = 2\pi a^3$$

Don't forget the bottom surface. In this problem, there is a component of the field in the z direction

$$\mathbf{F}\cdot\hat{\mathbf{n}}\ dS = \langle x, y, z\rangle\cdot\langle 0, 0, -1\rangle\ dx\ dy = -z\ dx\ dy$$

however, the value of this field on the xy-plane is $z = 0$ so there is no flux.

For the divergence,

$$\nabla\cdot\mathbf{F} = 1 + 1 + 1 = 3$$

which is pretty easy!. Now, integrate

$$\iiint_D 3\ dV$$

Well, the volume is $\frac{2}{3}\pi a^3$ so we obtain $2\pi a^3$.

OSU example

A problem from OSU asks us to verify the divergence theorem for

$$\mathbf{F} = \ \langle y, x, z \rangle$$

where the region is

$$0 \leq z \leq 16 - x^2 - y^2$$

The graph of $z = 16 - x^2 - y^2$ is a paraboloid which opens downward and has its vertex at $z = 16$. When $z = 0$ we have a circle of radius $r = 4$.

Recall that

$$\hat{\mathbf{n}}\ dS = \langle -f_x, -f_y, 1 \rangle\ dA$$

so for this paraboloid surface we have

$$z = f(x, y) = 16 - x^2 - y^2$$

$$\hat{\mathbf{n}}\ dS = \ \langle 2x, 2y, 1 \rangle\ dA$$

This corresponds to $\hat{\mathbf{n}}$ pointing out of the surface. Then

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\ dS = \iint_R 4xy + z\ dA$$

$$= \int_{-4}^{4} \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} 4xy + 16 - x^2 - y^2\ dx\ dy$$

xy-coordinates are not a good way to do this problem. Convert to polar coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$dA = r\ dr\ d\theta$$

$$\iint_R (4r^2 \sin\theta\cos\theta + 16 - r^2)\ r\ dr\ d\theta$$

The region of integration is the disk of radius $r = 4$

$$\int_0^{2\pi} \int_0^4 (4r^2 \sin\theta\cos\theta + 16 - r^2)\ r\ dr\ d\theta$$

The inner integral is

$$\int_0^4 4r^3 \sin\theta\cos\theta + 16r - r^3\ dr$$

$$r^4 \sin\theta\cos\theta + 8r^2 - \frac{1}{4}r^4 \Big|_0^4$$

$$= 256\sin\theta\cos\theta + 128 - 64$$

$$= 256\sin\theta\cos\theta + 64$$

The outer integral is

$$\int_0^{2\pi} 64 + 256\sin\theta\cos\theta\ d\theta$$

$$= 128\pi + 256\sin^2\theta\Big|_0^{2\pi}$$

$$= 128\pi$$

There is another part of our solid. That is the disk in the xy-plane. For this disk, the unit normal (pointing out) is just $\langle 0, 0, -1\rangle$.

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = -\iint_R z\ dA$$

but remember that we're on the xy-plane so $z = 0$ and the whole integral is 0.

We're not done yet! We still have to compute

$$\iiint_R \nabla\cdot\mathbf{F}$$

$$= \iiint_R P_x + Q_y + R_z\ dV$$

since $\mathbf{F} = \langle y, x, z\rangle$ this is just equal to 3. So we need

$$3\iiint_R dV$$

If we convert to cylindrical coordinates, we will integrate over the disk of radius $r = 4$. What is the upper bound on z?

$$z = 16 - x^2 - y^2 = 16 - r^2$$

So we have

$$\int_0^{2\pi} \int_0^4 \int_0^{16-r^2} dz\ r\ dr\ d\theta$$

The inner integral is just $16 - r^2$. The middle integral is

$$\int_0^4 16r - r^3\ dr$$

$$= 8r^2 - \frac{1}{4}r^4 \Big|_0^4$$

$$= 128 - 64 = 64$$

Finally, we pick up 2π from the outer integral for a final result of 128π, which matches what we had above.

Or, we could have just said that the solid is a hemisphere of radius 4 so the volume is a standard formula. Again, we need

$$3 \iiint_R dV$$

so

$$3V = 3\ \frac{1}{2}\ \frac{4}{3}\pi 4^3$$

$16^2 = 256$ and one-half of that is 128, times π.