Analysis: density

Numbers between numbers

• Between any two rational numbers one can always find another rational number.

Suppose we have p/q and r/s both $\in \mathbb{Q}$ with

$$p/q < r/s$$

then

$$p/q + p/q < p/q + r/s$$

and

$$p/q + r/s < r/s + r/s$$

so

$$p/q < \frac{p/q + r/s}{2} < r/s$$

The average of a and b is greater than a and less than b. It is also a rational number since it is equal to

$$\frac{1}{2}\ \frac{sp + rq}{qs}$$

• Between any two rational numbers it is always possible to find a real number.

$$\forall\ a, b \in \mathbb{Q}\ \exists\ c \in \mathbb{R} \mid c \in (a, b)$$

Courant says it "is not so obvious,; we shall accept it as a basic axiom."

One proof consists of finding a *particular* irrational in the interval (a, b), where a and b are rational. For $a < b$, we simply add to the number a the following

$$c = \frac{\sqrt{2}}{2}(b - a)$$

This added part c is smaller than $b - a$ (because $\sqrt{2}/2 < 1$) so $a + c$ lies between a and b. We also know that c is irrational, because $\sqrt{2}$ times any rational number is irrational. Finally, $a + c$ is irrational because adding $\sqrt{2}$ times a rational number to any rational number produces an irrational number.

Proof of the first preliminary requirement: $\sqrt{2}$ times a rational is irrational. Suppose for integer p, q, r, s we have

$$\sqrt{2}\,\frac{p}{q} = \frac{r}{s}$$

then

$$\sqrt{2} = \frac{rq}{ps}$$

But the right-hand side is rational, so this is a contradiction.

For the second requirement, again by contradiction suppose

$$\sqrt{2}\,\frac{p}{q} + \frac{s}{t} = \frac{u}{v}$$

for integer p, q, r, s, u, v. But the right-hand side of

$$\sqrt{2} = \frac{q}{p}(\frac{u}{v} - \frac{s}{t})$$

is rational, so this is a contradiction.

• Between any two *real* numbers it is always possible to find a rational number.

$$\forall\ a, b \in \mathbb{R}\ \exists\ r \in \mathbb{Q} \mid r \in (a, b)$$

Proof: pick

$$N \in \mathbb{N} \text{ such that } N > \frac{1}{b-a}$$

Then

$$\frac{1}{N} < b - a$$

Let the set

$$\mathbf{A} = \{\ \frac{m}{N} :\ m \in \mathbb{N}\}, \quad \text{a subset of } \mathbb{Q}$$

The claim is that

$$\mathbf{A} \cap (a, b) \neq \emptyset$$

There do exist numbers within the open interval (a, b) that are in the set $\mathbb{Q}$.

The proof is by contradiction. Assume on the contrary that the set $\mathbf{A}$ does not contain a rational number lying inside this interval. In other words:

$$\mathbf{A} \cap (a, b) = \emptyset$$

Now, find the largest integer m_1 such that $m_1/N < a$ (it is OK if m_1 is equal to 0). Then the next rational number in $\mathbf{A}$ must be larger than b since the set intersection is empty:

$$\frac{m_1 + 1}{N} > b$$

But this implies that

$$\frac{m_1 + 1}{N} - \frac{m_1}{N} > b - a$$

$$\frac{1}{N} > b - a$$

which contradicts our condition on N above. Hence the assumption is false and so

$$\mathbf{A} \cap (a, b) \neq \emptyset$$

tin Thus there must exist a rational number r in $\mathbf{A}$ such that $a < r < b$.

• Between any two real numbers it is always possible to find another real number.

$$\forall\ a, b \in \mathbb{R}\ \exists\ c \in \mathbb{R} \mid c \in (a, b)$$

Suppose the two real numbers are "really, really close." They are not equal, so they must be different, say $a < b$.

Since they are different, at some stage in the decimal expansions of a and b, there must be a first position at which a and b differ. If b does not have a 0 at the next position, terminate there and that will be c.

For example:

$$a = 1.23456789129..$$

$$b = 1.23456789133..$$

$$c = 1.23456789130..$$

b must have some digit following this first position where it does not match a, and which is also not equal to zero (otherwise it would be a terminating decimal and thus a rational number). So we can always find a place to terminate to form c.

Suppose we said to find the first digit where a and b differ and add 1 to the following digit. When would this fail? when we have something like this:

$$a = 1.2345678912999999\ldots$$

$$b = 1.2345678913000000\ldots$$

with the decimal expansions continuing forever like this. But this means that b is a rational number. Or alternatively we have that $a = b$ in the limit as the decimal expansion of a continues forever.