Kepler (part 7) Hartig derivation

This is a derivation of Kepler's laws from a class handout I found on the web for Math 304 by Hartig.

1

We start by defining M as the mass of the sun and m as the mass of the planet and $\mathbf{r}$ as the position vector from the sun to the planet. Combining Newton's second law and the inverse square law of gravitation we have that

$$\mathbf{F} = m\mathbf{a} = -\frac{GMm}{r^2}\ \frac{\mathbf{r}}{r}$$

$$\mathbf{a} = -\frac{GM}{r^2}\ \frac{\mathbf{r}}{r}$$

We take $\mathbf{u_r}$ as a unit vector in the same direction as $\mathbf{r}$. I write $\hat{\mathbf{u}}_\mathbf{r}$ without its hat ($\hat{}$) as $\mathbf{u_r}$ so as not to confuse it with the derivative $\dot{\mathbf{u}}_\mathbf{r}$.

$$\mathbf{r} = r\mathbf{u_r}$$

$$\mathbf{a} = -\frac{GM}{r^2}\ \mathbf{u_r}$$

The acceleration is along the line of the radial vector, pointing toward the sun.

The velocity is the time-derivative of the position vector $\mathbf{r}$.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

and the acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{r}}$$

3 Feynman's dots, again

We set up the angular momentum as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$$

For a unit mass this is

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}}$$

We compute the time-derivative

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}})$$

by the standard vector application of the product rule which we've looked at above, this is equal to

$$= \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}}$$

and this is equal to zero, since any vector's cross product with itself is zero, including a reversed version of itself, as in the second term. We define a constant vector $\mathbf{h}$ such that

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$$

Since $\mathbf{h}$ is a constant, unchanging in both direction and magnitude, it defines a normal vector to the plane containing $\mathbf{r}$ and $\dot{\mathbf{r}}$. Align $\mathbf{h}$ with the z-axis so all the motion occurs in the xy-plane. Note that

$$h = |\mathbf{h}| = |\mathbf{r} \times \dot{\mathbf{r}}| = rv \sin\theta$$

where these are all scalar quantities and θ is the angle between $\mathbf{r}$ and $\dot{\mathbf{r}} = \mathbf{v}$.

4 Equal area

We consider the triangle formed by the position vector before and after a short period of time Δt, and the vector $\Delta\mathbf{r}$ connecting these two positions, where

$$\Delta\mathbf{r} \approx \dot{\mathbf{r}}\Delta t$$

The little bit of area ΔA that is swept out during this time is

$$\Delta A \approx \frac{1}{2}\ |\mathbf{r} \times \dot{\mathbf{r}}\Delta t|$$

$$\Delta A = \frac{h}{2}\ \Delta t$$

So we have that

$$\frac{\Delta A}{\Delta t} \approx \frac{h}{2}$$

and in the limit as $\Delta t \to 0$

$$\frac{dA}{dt} = \frac{h}{2}$$

(Note a difference with Feynman. He uses A for the area, but never actually computes its value $|\mathbf{r} \times \dot{\mathbf{r}}|$. Here, dA/dt is the area and it's the second derivative d^2A/dt^2 that is equal to zero. Which is another way of saying that $\mathbf{h}$ is constant).

5 Manipulating $\mathbf{a} \times \mathbf{h}$

The crucial step is to prove that

$$\mathbf{a} \times \mathbf{h} = GM\dot{\mathbf{u}}_\mathbf{r}$$

This takes a bit of work, so I'd like to defer it until the end. We'll just assume it for now. Take the equality and integrate with respect to time, obtaining

$$\int \mathbf{a} \times \mathbf{h} = \int GM\dot{\mathbf{u}}_\mathbf{r}$$

$$\dot{\mathbf{r}} \times \mathbf{h} = GM\mathbf{u}_\mathbf{r} + \mathbf{d}$$

where $\mathbf{d}$ is a constant *vector* of integration.

6 Dot product

We're almost there now. Take the left-hand side from above and form the dot product

$$\mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{h})$$

Use another vector identity to switch it around

$$= (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \mathbf{h}$$

But $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$ so

$$= \mathbf{h} \cdot \mathbf{h} = h^2$$

7 conic sections

What we've shown is that

$$h^2 = \mathbf{r} \cdot (GM\mathbf{u}_\mathbf{r} + \mathbf{d})$$

$$= r(GM + d\cos\theta)$$

$$= rGM(1 + \frac{d}{GM}\cos\theta)$$

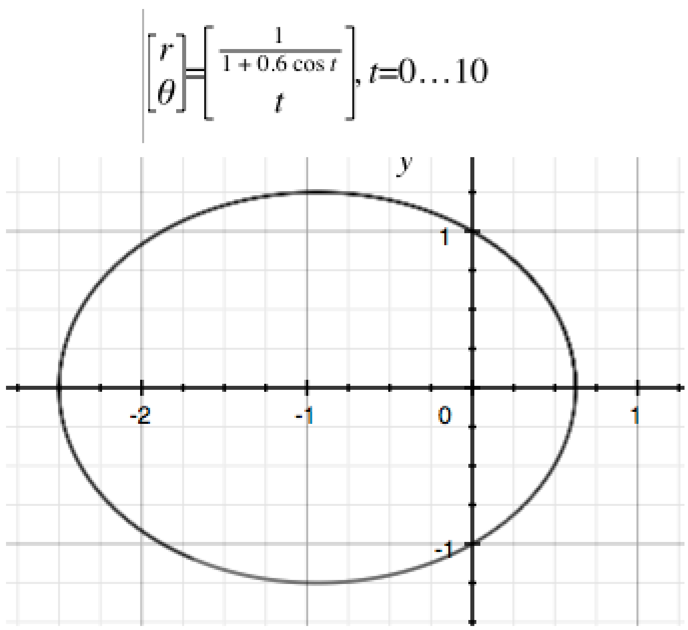
Define $k = h^2/GM$ and $e = d/GM$. Then

$$k = r(1 + e\cos\theta)$$

This is the equation of a conic section. In particular, if $e < 1$, then

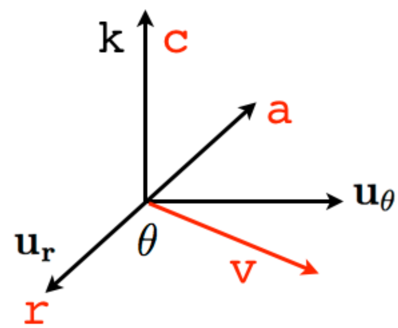
$$r = \frac{k}{1 + e\cos\theta}$$

is the equation of an ellipse. Here is an example with $k = 1$ and $e = 0.6$



8 Cleaning up

Here is a sketch of the situation



As we've said all along, $\mathbf{u_r}$ is a unit vector in the $\mathbf{r}$ direction, so that $\mathbf{r} = r\mathbf{u_r}$. By the central force hypothesis, the acceleration $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$ is in the $-\mathbf{u_r}$ direction. The source of all our complexity is that $\mathbf{v} = \dot{\mathbf{r}}$ is not perpendicular to $\mathbf{u_r}$ but forms an angle θ with it.

Also, we defined

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}$$

and aligned $\mathbf{h}$ with the $\hat{\mathbf{k}}$ direction. We analyzed $\mathbf{r} \times \mathbf{v}$ to show that $\mathbf{h}$ is a constant vector. $\mathbf{u}_\theta$ is the unit vector orthogonal to $\mathbf{u_r}$.

According to Hartig, what we have to prove is that

$$\mathbf{a} \times \mathbf{h} = GM\dot{\mathbf{u}}_\mathbf{r}$$

Go back to basic definitions.

$$\mathbf{r} = r\mathbf{u_r}$$

$$\mathbf{v} = \dot{r}\mathbf{u_r} + r\dot{\mathbf{u}}_\mathbf{r}$$

Recall that $\dot{\mathbf{u}}_\mathbf{r} = \dot{\theta}\mathbf{u}_\theta$ so

$$\mathbf{v} = \dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta$$

$$\begin{aligned} \mathbf{h} = \mathbf{r} \times \mathbf{v} &= r\mathbf{u_r} \times (\dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta) \\ &= r^2\dot{\theta}\hat{\mathbf{k}} \end{aligned}$$

The acceleration is

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{u_r}$$

So

$$\begin{aligned}\mathbf{a} \times \mathbf{h} &= -\frac{GM}{r^2}\mathbf{u_r} \times r^2\dot{\theta}\hat{\mathbf{k}} \\ &= -GM\dot{\theta}(-\mathbf{u}_\theta) \\ &= GM\dot{\theta}\mathbf{u}_\theta\end{aligned}$$

Again, recall that $\dot{\mathbf{u}}_\mathbf{r} = \dot{\theta}\mathbf{u}_\theta$ so

$$\mathbf{a} \times \mathbf{h} = GM\dot{\mathbf{u}}_\mathbf{r}$$

Now, integrate

$$\int \mathbf{a} \times \mathbf{h} = \int GM\dot{\mathbf{u}}_\mathbf{r}$$

$$\mathbf{v} \times \mathbf{h} = \dot{\mathbf{r}} \times \mathbf{h} = GM\mathbf{u_r}$$