Integrating z*

If the contour (curve) of integration C is parametrized in terms of t, then

$$\int_C f(z)\ dz = \int_a^b f[z(t)]\ z'(t)\ dt$$

Let's look at $f(z) = z*$ (note that this function is *not* analytic).

Take our curve to be the circle of radius 2 centered at the origin, and proceed halfway around, between the endpoints $z = -i \rightarrow i$.

On this curve it is natural to parametrize in terms of θ

$$z = 2e^{i\theta}$$

we have

$$dz = 2ie^{i\theta}\ d\theta$$

In radial coordinates the complex conjugate is particularly simple

$$z* = 2e^{-i\theta}$$

Notice that

$$zz* = 2e^{i\theta}\ 2e^{-i\theta} = 4 = |z|^2$$

For the integral, we have

$$\int z*\ dz = \int 2e^{-i\theta}\ 2ie^{i\theta}\ d\theta$$

$$= 4i \int_{-\pi/2}^{\pi/2} d\theta = 4\pi i$$

A closed contour where we go all the way around would also have a non-zero integral, namely 2π times the radius squared, times i.

Alternatively

$$zz* = 4$$

$$\int z* \; dz = 4 \int \frac{1}{z} \; dz$$

Again

$$z = 2e^{i\theta}$$

$$dz = 2ie^{i\theta} \; d\theta = iz \; d\theta$$

So the integral is just

$$\int z* \; dz = 4 \int \frac{1}{z} \; dz$$

$$= 4 \int \frac{1}{z} \; iz \; d\theta$$

$$= 4i \int d\theta = 4\pi i$$