Trig substitutions

Kline has a discussion of rotatation of axes (Chapter 7 A1). It leads to a method for finding the angle by which to rotate in order to eliminate the xy term in the general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Start with the rotation formulas. We rotate the xy-axes to $x'y'$ by an angle θ. Then we consider a point $P = (x, y)$

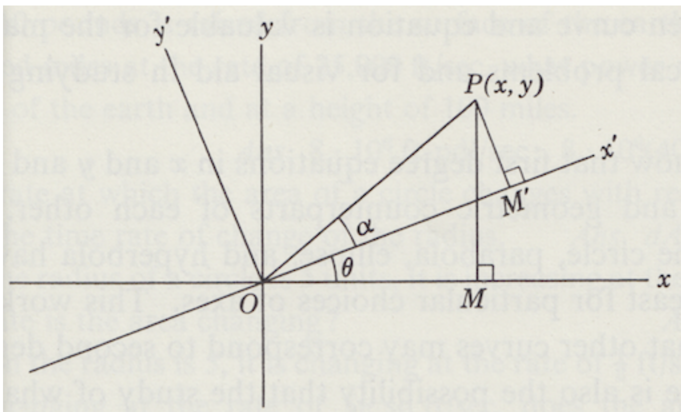

Figure 7A-1

Drop a perpendicular to the $x'y'$-axes and form the angle α. We have used a construction like this to derive the sum of angles formulas. Here, we go over the same ground, but in reverse:

$$\cos\theta + \alpha = \cos\theta\cos\alpha - \sin\theta\sin\alpha$$

$$x = OP(\cos\theta + \alpha)$$

$$= OP(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$
$$= x'\cos\theta - y'\sin\theta$$

Using the other formula

$$\sin\theta + \alpha = \sin\theta\cos\alpha + \cos\theta\sin\alpha$$

$$y = OP(\sin\theta + \alpha)$$
$$= OP(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$
$$= x'\sin\theta + y'\cos\theta$$

Substituting into the general equation, we obtain a large number of terms including:

$$Ax^2 = A(x'^2\sin^2\theta - 2x'y'\sin\theta\cos\theta + y'^2\cos^2\theta)$$
$$Bxy = B(x'^2\sin\theta\cos\theta - x'y'\sin^2\theta + x'y'\cos^2\theta - y'^2\sin\theta\cos\theta)$$
$$Cy^2 = C(x'^2\sin^2\theta + 2x'y'\sin\theta\cos\theta + y'^2\cos\theta)$$

plus other terms that do not contain $x'y'$. Gather together those $x'y'$ terms and set the sum equal to zero to make them vanish:

$$-2Ax'y'\sin\theta\cos\theta - Bx'y'\sin^2\theta + Bx'y'\cos^2\theta + 2Cx'y'\sin\theta\cos\theta = 0$$

$$-2A\sin\theta\cos\theta - B\sin^2\theta + B\cos^2\theta + 2C\sin\theta\cos\theta = 0$$

Employ the sum of angles formulas in a different guise:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

We plug into the $x'y'$ formula to obtain:

$$(C - A)\sin 2\theta + B\cos 2\theta = 0$$

$$\tan 2\theta = \frac{B}{A - C}$$

A reassuring simplification. As an example, for the hyperbola

$$xy = 1$$

$$xy - 1 = 0$$

$B = 1$ and $A = C = 0$, for which we simply invert the formula to find that

$$\cot 2\theta = 0 = \cos 2\theta$$

Hence $2\theta = \pi/2$ and $\theta = \pi/4$. For the transformed equation, most terms are zero since $A = C = D = E = 0$. We are left with only

$$B(x'^2 \sin\theta\cos\theta - y'^2 \sin\theta\cos\theta) - 1 = 0$$

$$\frac{x'^2}{2} - \frac{y'^2}{2} = 1$$

which is the rotated hyperbola.