# Kepler (part 4): Area

At this point, we have almost all the tools we need to follow the derivation of Kepler's laws. We just need a bit more discussion of area "swept out" by a planet in a short time. Our approach is based on Varberg *Calculus* (online version only, Chapter 14).

We revisit the triangle formed by the motion of the planet, and confirm that twice the area of the triangle is equal to

$$h = r^2 \ \frac{d\theta}{dt}$$

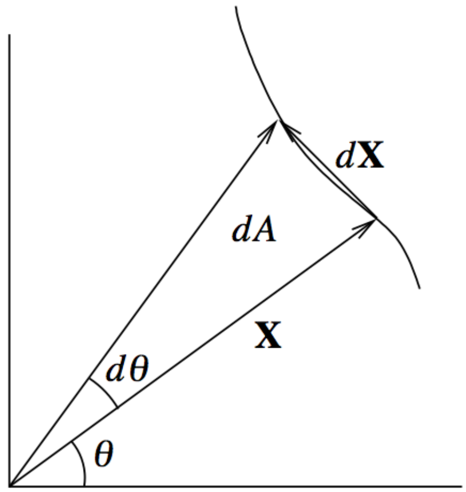
The thing that makes it confusing is that whereas Feynman used the velocity for one of the sides of the triangle and showed that

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$$

is a constant, Varberg use $d\mathbf{r}$ for their triangle. It leads to some confusing aspects in the presentation, which I want to work through since I like everything else about their derivation.

Here is their diagram

Figure 14.3




They use $\mathbf{X}$ for the position vector, but I will label it as $\mathbf{r}$, following Feynman. This vector "stays always in the $xy$-plane." Also, I will use $\mathbf{u_r}$ for the vector they call $\mathbf{L}$ and similarly $\mathbf{u}_\theta$ for the vector they call $\mathbf{L}^\perp$.

We are asked to show that

$$2\ \frac{dA}{dt} = \ |\frac{d\mathbf{r}}{dt} \times \mathbf{r}|$$

This presentation is a bit different than Feynman's, and it confused me for a while, because, just for starters, we proved before that $dA/dt = 0$ but that will not be the case here, because now $dA/dt$ means something different.

What I'm going to do is to change the notation a bit and say that in a short time $\Delta t$, the area that is swept out is $\Delta A$, corresponding to a length $d\mathbf{r} = \mathbf{v}\Delta t$, and that by the geometry we have

$$2\ \Delta A = |\mathbf{r} \times \mathbf{v}\Delta t|$$

I assert that it is OK to bring $\Delta t$ out of the cross product (by the rule of scalar multiplication), since it is a scalar quantity, and is a constant at any stage of its future journey to the limit when $\Delta t \to 0$, so I write

$$2 \ \Delta A = |\mathbf{r} \times \mathbf{v}| \Delta t$$

Now we have

$$2 \ \frac{\Delta A}{\Delta t} = |\mathbf{r} \times \mathbf{v}|$$

and in the limit

$$2 \ \frac{dA}{dt} = |\mathbf{r} \times \mathbf{v}|$$

Now this is not quite what we were asked to prove, but recall that

$$\frac{d\mathbf{r}}{dt} \times \mathbf{r} = \dot{\mathbf{r}} \times \mathbf{r} = -\mathbf{r} \times \dot{\mathbf{r}}$$

so the absolute values are the same. Again, the result is that

$$2 \ \frac{dA}{dt} = |\dot{\mathbf{r}} \times \mathbf{r}| = |\mathbf{r} \times \dot{\mathbf{r}}| = |\mathbf{h}| = h$$

If you're not completely happy with the argument allowing this step:

$$2 \ \Delta A = |\mathbf{r} \times \mathbf{v} \Delta t| = |\mathbf{r} \times \mathbf{v}| \Delta t$$

recall that

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$$

so

$$|\mathbf{r} \times \mathbf{v} \Delta t| = |\mathbf{c} \Delta t| = h \Delta t$$

Varberg *et al.* also give a second argument which which we will go through because it gives us the term

$$r^2 \ \frac{d\theta}{dt} = h$$

By the geometry of the triangle, the area is

$$2 \ dA = r \ r d\theta = r^2 d\theta$$

where $r$ is $|\mathbf{r}|$. And then they say

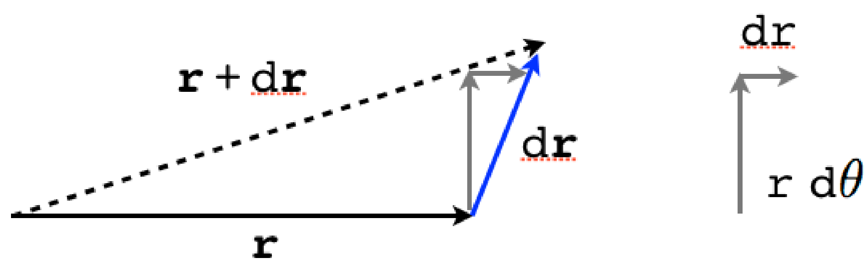
$$2\ \frac{dA}{dt} = r^2\ \frac{d\theta}{dt}$$

They do this without comment, but this result assumes that $r$ does not vary with time, although clearly it does (that's the whole point of everything we are doing here). The product rule would give us:

$$\frac{d}{dt}\ r^2 d\theta = 2r\ d\theta\ \frac{dr}{dt} + r^2\ \frac{d\theta}{dt}$$

This looks a little weird because of the single differential $d\theta$, but I think what it means is that in the limit, the first term goes to zero.

Another way of explaining this is to break the area into two parts.




The first part is $(1/2)r$ times $r\ d\theta$, the length of (almost straight) arc perpendicular to $\mathbf{r}$. This is the part we get by assuming that $r$ is constant. And in the limit as $t \to 0$, the resulting $d\theta/dt$ has some value, namely, the angular velocity.

The second part is $(1/2)r\ d\theta$ times $dr$. This is the part that accounts for the change in $r$, but it contains two differentials, only one of which gets rescued into some quantity by $dt$. The other just goes to zero, so the whole thing is zero.

Anyway, let's continue with the argument.

Go back to the right-hand side of what we were asked to prove above

$$2\ \frac{dA}{dt} = |\frac{d\mathbf{r}}{dt} \times \mathbf{r}|$$

and show that it turns into $r^2 d\theta/dt$. Using $\mathbf{u_r}$ for the unit vector in the $\mathbf{r}$ direction, we have

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\mathbf{u_r}) = \frac{dr}{dt}\mathbf{u_r} + r\dot{\mathbf{u}}_\mathbf{r}$$

where the first part is just separating the scalar and unit vector part of $\mathbf{r}$ and the rest is from the product rule. At this point we recall the result that $\dot{\mathbf{u}}_\mathbf{r} = d\theta/dt\ \mathbf{u}_\theta$, so we have

$$= \frac{dr}{dt}\mathbf{u_r} + r\frac{d\theta}{dt}\ \mathbf{u}_\theta$$

So now this is what we need to cross with $\mathbf{r}$, also known as $r\mathbf{u_r}$. We write

$$(\frac{dr}{dt}\mathbf{u_r} + r\frac{d\theta}{dt}\ \mathbf{u}_\theta) \times r\mathbf{u_r}$$
$$= (\frac{dr}{dt}\mathbf{u_r} \times r\mathbf{u_r}) + (r\frac{d\theta}{dt}\ \mathbf{u}_\theta \times r\mathbf{u_r})$$

The first term is zero (the cross-product of $\mathbf{u_r}$ with itself), and because these are unit vectors the absolute value of the second term's vector cross-product is 1

$$|r\frac{d\theta}{dt}\ \mathbf{u}_\theta \times r\mathbf{u_r}| = r^2\frac{d\theta}{dt}\ |\mathbf{u}_\theta \times \mathbf{u_r}| = r^2\frac{d\theta}{dt}$$

So what we've shown is that

$$2\,\frac{dA}{dt} = |\mathbf{r} \times \dot{\mathbf{r}}|$$

and

$$2\,\frac{dA}{dt} = r^2\,\frac{d\theta}{dt}$$

This term ($r^2\ d\theta/dt$) is what Hartig calls $c$ and the other guys call $h$. As the vector $\mathbf{h}$, it points in the $\hat{\mathbf{k}}$ direction and is the angular momentum but without the mass component.