Integrating z

Suppose $f(z) = z$, so we want

$$f(z) = u(x, y) + iv(x, y) = x + iy$$

Write

$$I = \int u\ dx - \int v\ dy + i\ [\ \int v\ dx + \int u\ dy\ ]$$

$$I = \int x\ dx - \int y\ dy + i\ [\ \int y\ dx + \int x\ dy\ ]$$

Suppose the path proceeds from the origin to the point $(1, 1)$ either directly (C_1) or first horizontally out to $(1, 0)$ (C_2) and then vertically (C_3).

Along C_1 we parametrize as follows: $y = x = t$, $t = 0 \to 1$, so $dy = dx = dt$ and we have

$$I = \int t\ dt - \int t\ dt + i\ [\ \int t\ dt + \int t\ dt\ ]$$

$$= 2i\ \int_0^1 t\ dt = 2i\ \frac{t^2}{2}\ \Big|_0^1 = i$$

Along C_2, $y = 0$, $dy = 0$, $x = 0 \to 1$ so

$$= \int x\ dx - \int y\ dy + i\ [\ \int y\ dx + \int x\ dy\ ]$$

$$= \int x \; dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2}$$

Along C_3, $x = 1$, $dx = 0$, $y = 0 \to 1$ so

$$= \int x \; dx - \int y \; dy + i \; [\int y \; dx + \int x \; dy \;]$$

$$= - \int y \; dy + i \; \int 1 \; dy$$

$$= -\frac{y^2}{2} + iy \Big|_0^1 = -\frac{1}{2} + i$$

Notice that

$$I_{C1} = I_{C2} + I_{C3} = i$$

and a closed path where we return to the origin would be zero.

Also, since $f(z)$ is analytic, we can just do

$$\int_{0+0i}^{1+i} z \; dz = \frac{z^2}{2} \Big|_{0+0i}^{1+i}$$

$$= \frac{1}{2} \; (1+i)^2 = \frac{1}{2} \; (1+i)(1+i) = \frac{1}{2} \; (1 - 1 + 2i) = i$$

Let's try another path: the unit circle, going counter-clockwise. Write:

$$z = e^{i\theta}, \quad dz = ie^{i\theta} d\theta$$

$$\oint f(z) \; dz = i \; \int_0^{2\pi} e^{i2\theta} \; d\theta$$

$$= i \; \frac{1}{2i} e^{i2\theta} \Big|_0^{2\pi} = \frac{1}{2} \; (1 + i0 - 1 - i0) = 0$$