minimum/maximum

For a function $f(x, y)$, if $f_x = f_y = 0$ when evaluated at some point $P = (x_0, y_0)$, then we are at a *critical point* of f, which may be a maximum, a minimum, or neither.

One (sophisticated) way to see that (x_0, y_0) is a critical point of f is to consider the slopes of cross-sections of the surface. If we take the cross-section of the surface in the direction where y is constant, that slope is f_x, and similarly, when x is constant, we have f_y.

Two vectors in these directions are $\mathbf{u} =< 1, 0, f_x >$, and $\mathbf{v} =< 0, 1, f_y >$. At any point on the surface the normal vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} =< -f_x, -f_y, 1 >$$

If for some (x_0, y_0), $f_x = f_y = 0$ then $\mathbf{n} =< 0, 0, 1 >$. The normal points straight up, so the tangent plane is horizontal. This could be a minimum or a maximum of the function.

example

Verify that $f(x, y) = x^2 + y^2$ has a minimum at $(0, 0)$. By considering the cross-sections where $x = 0$ and $y = 0$ we can see that those curves are both parabolas opening up, so we have a *paraboloid*. The surface has radial symmetry, and for each value of $z > 0$, $z = x^2 + y^2$ is a constant, forming a circle.

$$f_x = 2x, \quad f_y = 2y,$$

These are both zero only at $(0,0)$, which is a critical point.

This function has continuous second derivatives

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = f_{yx} = 0$$

The second derivative test says to compute

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

If $D > 0$ and $f_{xx} > 0$, then the given point is a minimum.

Going through the same arguments for $f(x,y) = -(x^2+y^2)$, we obtain that $D > 0$ and $f_{xx} < 0$, so the point $(0,0)$ is a maximum.

example

An alternative surface is given by $f(x,y) = x^2 - y^2$. By considering the cross-sections where $x = 0$ and $y = 0$ we can see that both curves are parabolas, but one opens up and the other down. This is called a *hyperbolic paraboloid*.

In this case

$$f_x = 2x, \quad f_y = -2y,$$

Again, the only critical point is $(x_0, y_0) = (0,0)$

$$f_{xx} = 2, \quad f_{yy} = -2, \quad f_{xy} = f_{yx} = 0$$

$$D = f_{xx} f_{yy} - (f_{xy})^2 < 0$$

In this case, the point is a *saddle point*.

It might be interesting to look at some more unusual surfaces including the *volcano*

$$z = (x^2 + y^2)e^{-x^2+y^2}$$

and the *sombrero*

$$z = \frac{\sin \pi r}{\pi r}$$

but maybe that's enough for now

example

Find the minimum distance from the origin to a point on the plane $x + y + z = 3$.

We must minimize the distance

$$d = \sqrt{x^2 + y^2 + z^2}$$

or, to make things simpler, we minimize the square of the distance (since the distance then will be a minimum as well).

$$d^2 = x^2 + y^2 + z^2$$

Substituting for z

$$f(x, y) = d^2 = x^2 + y^2 + (3 - x - y)^2$$

$$= x^2 + y^2 + 9 - 3x - 3y - 3x + x^2 + xy - 3y + xy + y^2$$

$$= 2x^2 + 2y^2 - 6x - 6y + 2xy + 9$$

From the geometry it is clear that there is a single critical point and it is a minimum:

$$f_x = 4x - 6 + 2y = 0$$

$$f_y = 4y - 6 + 2x = 0$$

Multiply the first equation by -2 and add to the second:

$$-6x + 6 = 0$$

$$x = 1$$

Substitute for x in either equation:

$$y = 1$$

Substitute $x = y = 1$ into the equation of the plane:

$$1 + 1 + z = 3$$

$$z = 1$$

Note that, of course, the shortest distance of a point to a plane is obtained by moving along the normal vector to the plane, which is indeed $< 1, 1, 1 >$.

example

Find the minimum distance from the point $(1, 2, 0)$ to the cone $z^2 = x^2 + y^2$.

For any point (x, y, z) in space the square of the distance to $(1, 2, 0)$ is

$$d^2 = (x-1)^2 + (y-2)^2 + (z-0)^2$$

$$= x^2 - 2x + 1 + y^2 - 4y + 4 + z^2$$

Use the constraint that the point must lie on the cone:

$$z^2 = x^2 + y^2$$

so

$$d^2 = x^2 - 2x + 1 + y^2 - 4y + 4 + x^2 + y^2$$

$$= 2x^2 + 2y^2 - 2x - 4y + 5$$

The first derivatives:

$$f_x = 4x - 2 = 0$$

$$f_y = 4y - 4 = 0$$

Hence

$$x = \frac{1}{2}, \quad y = 1$$

$$z^2 = x^2 + y^2 = \frac{5}{4}$$

The distance is

$$d = \sqrt{(\frac{1}{2} - 1)^2 + (1 - 2)^2 + (\frac{5}{4})^2}$$

which is about 1.58.

example

A rectangular box, open at the top is to hold some constant volume V of cat food. Find the dimensions for which the surface area is a minimum.

Let x and y be the lengths of the sides of the base. Then the height is

$$z = \frac{V}{xy}$$

Two sides have area

$$xz = \frac{V}{x}$$

Two have area

$$yz = \frac{V}{y}$$

and the bottom has area xy so the total surface area is

$$A = \frac{2V}{x} + \frac{2V}{y} + xy$$

we must have

$$A_x = -\frac{2V}{x^2} + y = 0$$

$$A_y = -\frac{2V}{y^2} + x = 0$$

Rearranging the first equation:

$$y = \frac{2V}{x^2}$$

$$\frac{1}{y^2} = \frac{x^4}{4V^2}$$

Substitute into the second:

$$-\frac{x^4}{2V} + x = 0$$

$$\frac{x^4}{2V} = x$$

One root is $x = 0$ which we discard because that is not a relevant solution. So

$$\frac{x^3}{2V} = 1$$

So

$$x = (2V)^{1/3}$$

Now you can go back and solve the same equations for y but I just note that the equations are symmetric in x and y so $x = y$. The area of the base is then

$$A = xy = (2V)^{2/3}$$

and the height is

$$h = \frac{V}{A} = \frac{V}{(2V)^{2/3}} = 2^{-2/3}V^{1/3}$$

This looks a little weird but if we form the ratio

$$\frac{h}{x} = \frac{2^{-2/3}V^{1/3}}{(2V)^{1/3}} = 2^{-1} = \frac{1}{2}$$

The height is one-half either of the sides.