Archimedean property

This property can be stated in a variety of forms. One statement is that the real numbers are not bounded above in $\mathbb{N}$. No matter how large a real number x that we take, we can always find an integer that is larger.

Statements of the theorem all start with this: for any real number x ($\forall x \in \mathbb{R}$), we can find an integer n such that ($\exists\ n \in \mathbb{N} \mid\ $):

- $n > x$

Or, for any real *a however small* we can find

$na > x$.

In the immortal words of xxx: if we have a bathtub full of water and a teaspoon, we can empty the bathtub (given enough time).

If you prefer a small real number like ϵ ($\forall \epsilon \in \mathbb{R}$)

- $\frac{1}{n} < \epsilon$

Beck says:

> Theorem 7.6 (the Archimedean property) essentially says that **infinity is not part of the real numbers**... The Archimedean Property underlies the construction of an infinite decimal expansion for any real number, while the Monotone Sequence Property shows that any such infinite decimal expansion actually converges to a real number.

Apostol goes through this development:

• The set **P** of positive integers is *unbounded above.* The proof is to assume that P is bounded above. Then there is a largest element n of **P** which is less than the bound.

But by definition $n + 1$ is $\in \mathbf{P}$.

• For every real x there exists a positive integer n such that $n > x$. Proof: if this were not so, then x would be an upper bound for **P**.

Now, simply replace x with y/x:

• For every real y/x there exists a positive integer n such that $n > y/x$. Thus $nx > y$.

Apostol:

> Geometrically it means that any line segment, no matter how long, may be covered by a finite number of line segments of a given positive length, no matter how small. In other words, a small ruler used often enough can measure arbitrarily large distances. Archimedes realized that this was a fundamental property of the straight line and stated it explicitly as one of the axioms of geometry.

Stewart's definition is:

Given a real number $\epsilon > 0$, there exists a positive integer n such that

$$\frac{1}{10^n} < \epsilon$$

This is certainly compatible with the other definitions. If n is an integer than so is 10^n. So ϵ is Apostol's (small) positive length and if we can choose N so that $N\epsilon$ is as large as we please, we can certainly choose it so that $N\epsilon > 1$.

I interpret this as follows: in distinguishing two real numbers a and b (really, trying to find another number in the gap between them), if $a - b = \epsilon$ is the distance between them, we can always find

$$\frac{1}{10^n} < \epsilon$$

and so always find another real number that lies between a and b.