Inverse functions

For two functions $f(x)$ and $g(x)$, we say that f and g are inverse functions if application of first g and then f in series, called the composition of f and g, gives x back again. The converse is also true if f and g are inverses. Sometimes there are restrictions on the domain for one function but not the other.

$$f \circ g(x) = f(\ g(x)) \ = g(\ f(x)) = g \circ f(x)$$

Some simple examples are

$$f(x) = x + 1, \quad g(x) = x - 1$$

$$f(x) = cx, \quad g(x) = \frac{1}{c}x$$

Familiarity with analytical geometry will be enough to recognize that the product of the slopes of f and g is equal to 1 for these first two equations, at least. This statement that slopes of inverse functions are multiplicative inverses hides a subtle difficulty, however.

For example, what about

$$f(x) = x^2, \quad g(x) = \sqrt{x}$$

$$f'(x) = 2x, \quad g'(x) = \frac{1}{2\sqrt{x}}, \qquad 2x\ \frac{1}{2\sqrt{x}} \neq 1$$

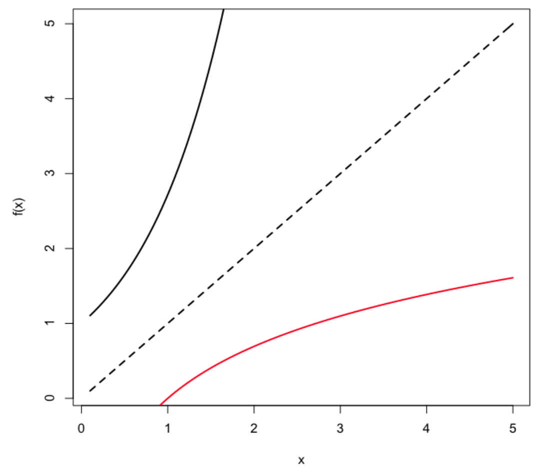
Or

$$f(x) = e^x, \quad g(x) = ln(x)$$
$$e^{ln(x)} = x, \quad ln(e^x) = x$$

But

$$f'(x) = e^x, \quad g'(x) = \frac{1}{x}, \qquad e^x\ \frac{1}{x} \neq 1$$

Still, a quick graph looks like these functions are symmetric about y = x.



The R code was

```
f <- function(x) { x }
x1 = 0.1
x2 = 5
plot(f,from=x1,to=x2, col='black',lwd=2,lty=2)
plot(log,from = x1,to = x2,col='red', lwd=2,add = T)
plot(exp,from = x1,to = x2,col='black', lwd=2,add = T)
```

But as we saw above, the slopes do not multiply to give 1, and the symmetry about $y = x$ is close but not exactly right. There is something else going on.

What is happening is that although we've written $f(x)$ and $g(x)$ as functions of x, which is perfectly valid, when we are composing to apply the inverse after the forward operation we need to write something slightly different:

$$y = f(x), \qquad x = g(y)$$

So, when we multiply the slopes together to test equality with 1, we must evaluate the expressions for different inputs! Going back to the square root

$$f'(x) = 2x, \qquad g'(y) = \frac{1}{2\sqrt{y}}$$

If we evaluate $f(x) = x^2$ at $x = 5$ and obtain a slope of $f'(x) = 2x = 10$, we must evaluate $g'(y)$ at 25, then we have $g'(y) = 1/(2\sqrt{25})$ and obtain the correct result. Similarly, for

$$f'(x) = e^x, \qquad g'(y) = \frac{1}{y}$$

if we evaluate $f(x) = e^x$ at $x = 2$ we obtain a slope of e^2; we must evaluate $g'(y)$ at $y = e^2$ giving slope $= 1/e^2$. Then the two slopes together give 1.