Surfaces the easy way

Let's see if we can figure out the basic facts about surface integrals in the simplest possible way.

For a point on a surface, we use the tangent plane approximation. Imagine that the surface is tiled, composed of numerous tiny planes. At any point, the surface is locally flat, having uniform slope and constant normal vector. We will need to find the angle θ this plane makes with the horizontal. We use that angle for this calculation:

$$dA = \cos\theta \ dS$$
$$dS = \frac{1}{\cos\theta} \ dA$$

A little piece of area on the surface is made smaller in its projection to the xy-plane by this factor.

Like any plane, we describe a tangent plane by its normal vector, $\mathbf{n}$. You should be able to see that $\mathbf{n}$ makes the same angle θ with the z-axis or $\hat{\mathbf{k}}$ unit vector, so that

$$\mathbf{n} \cdot \hat{\mathbf{k}} = |\mathbf{n}||\hat{\mathbf{k}}| \ \cos\theta$$

Since $\hat{\mathbf{k}}$ is a unit vector, $|\hat{\mathbf{k}}| = 1$

$$\mathbf{n} \cdot \hat{\mathbf{k}} = |\mathbf{n}| \ \cos\theta$$

$$\frac{1}{\cos\theta} = \frac{\mathbf{n} \cdot \hat{\mathbf{k}}}{|\mathbf{n}|}$$

As always, we find $\mathbf{n}$ by first finding two vectors in the plane and then forming the cross-product. We will use the basis vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$, so we think about a cross-section of the plane at the point of interest, parallel to the xz-plane.

The surface at the cross-section is just a line with some tilt to it, with slope f_x. For each unit of change in x or $\hat{\mathbf{i}}$ direction, there is a change of f_x in the $\hat{\mathbf{k}}$ direction and none in the $\hat{\mathbf{j}}$ direction. So one vector in the plane is

$$\langle 1, 0, f_x \rangle$$

The second is, of course,

$$\langle 0, 1, f_y \rangle$$

The cross-product is

$$\langle -f_x, -f_y, 1 \rangle = \mathbf{n}$$

We have formulated things so that $\mathbf{n} \cdot \hat{\mathbf{k}}$ is 1! Our currency exchange rate for area in the plane compared to area on the sloped tangent plane is just

$$\frac{1}{\cos\theta} = \frac{\mathbf{n} \cdot \hat{\mathbf{k}}}{|\mathbf{n}|} = \frac{1}{|\mathbf{n}|} = \frac{dS}{dA}$$
$$dS = |\mathbf{n}| \; dA$$

and

$$|\mathbf{n}| = \sqrt{1 + f_x^2 + f_y^2}$$

Hence

$$\iint_S dS = \iint \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy$$

How about an example? The hemisphere of radius R

$$x^2 + y^2 + z^2 = R^2$$
$$z = \sqrt{R^2 - x^2 - y^2}$$
$$f_x = -\frac{x}{z}; \quad f_y = -\frac{y}{z}$$
$$|\mathbf{n}| = \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + (\frac{x}{z})^2 + (\frac{y}{z})^2}$$
$$= \frac{1}{z}\sqrt{z^2 + x^2 + y^2} = \frac{R}{z}$$

Hence

$$SA = \iint |\mathbf{n}| \, dx \, dy = \iint \frac{R}{z} \, dx \, dy$$
$$= R \iint \frac{1}{\sqrt{R^2 - r^2}} \, r \, dr \, d\theta$$
$$-2\pi R \left[\sqrt{R^2 - r^2} \right] \Big|_0^R = 2\pi R^2$$

and twice that for the sphere.

3