Power series: convergence

Consider

$$\sum_0^\infty \frac{x^n}{n!}$$

For real x:

$$e^x = \sum_0^\infty \frac{x^n}{n!}$$

This may be used as the definition of the exponential for real x. Now we wish to look at complex z. It turns out to be true that

$$e^z = \sum_0^\infty \frac{z^n}{n!}$$

Following Kharkar, we look at the closely related series

$$\sum_0^\infty (-1)^n \frac{z^n}{n!}$$

We want to determine the radius of convergence of the series. We use the ratio test to see what happens to successive numbers in the sequence as n gets large.

The ratio test is to look at the ratio of the $n+1$ term to the n term using moduli:

$$\lim_{n\to\infty} |\frac{(-1)^{n+1}\ z^{n+1}}{(n+1)!}\ \frac{n!}{(-1)^n\ z^n}|$$

We can move the absolute value signs to each term, and when we do that the terms with -1 go away so

$$= \lim_{n\to\infty} \frac{|z^{n+1}|}{|(n+1)!|} \frac{|n!|}{|z^n|}$$

$$= \lim_{n\to\infty} \frac{|z|}{|n+1|} = 0$$

Since the ratio tends to zero in the limit, the series is convergent, and it is absolutely convergent since we used the modulus. Also, there is no z in the limit, so this result is true for any z, and the ratio of convergence R is infinite.

So the power series for complex z is good anywhere and it is:

$$e^z = \sum_0^\infty \frac{z^n}{n!}$$

Recall that complex sine and cosine are:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}\ z^{2n+1}}{(2n+1)!}$$

By the same ratio test as before this also converges absolutely for all z. Also, notice that $|\sin z|$ is just every other term from e^z, so comparing term by term it converges by that test as well, and it has the same radius of convergence.

The same goes for cosine:

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^{2n}\ z^{2n}}{(2n)!}$$