Archimedean property

This simple idea can be stated in a variety of equivalent forms.

The simplest is that the real numbers are not bounded above in $\mathbb{N}$. No matter how large a real number x that we take, we can always find an integer that is larger.

Formal statements of the theorem all start like this: for any (arbitrary) real number x

$$\forall\ x \in \mathbb{R}$$

we can find a natural number n such that

$$\exists\ n \in \mathbb{N} \mid$$

- $n > x$

proof

By contradiction. Suppose that no integer n exceeds x. Then x is an upper bound for $\mathbb{N}$.

By the completeness axiom (more about this ahead), $\mathbb{N}$ must have a real least upper bound or supremum. Let β be this number.

$\beta - 1$ is not a bound for $\mathbb{N}$ (because β is the least upper bound). So there must be a positive integer $n_0 > \beta - 1$. But then $n_0 + 1$ is $\in \mathbb{N}$ but also $n_0 + 1 > \beta$, so β is not an upper bound for $\mathbb{N}$.

$\square$

An equivalent statement is that for any real a, however small, and any real x, however large, we can find

- $na > x$.

In the immortal words of somebody-or-other: if we have a bathtub full of water and a teaspoon, we can empty the bathtub (given enough time).

If you prefer a small real number. For any real number ϵ, however small, $(\forall\ \epsilon \in \mathbb{R})$

- $\frac{1}{n} < \epsilon$

Beck says:

> Theorem 7.6 (the Archimedean property) essentially says that **infinity is not part of the real numbers**... The Archimedean Property underlies the construction of an infinite decimal expansion for any real number, while the Monotone Sequence Property shows that any such infinite decimal expansion actually converges to a real number.

For example, when we are writing the decimal expansion of $\sqrt{2}$, we must stop somewhere. The Archimedean Property says that regardless of your specification of the difference between the "true value" $\sqrt{2}$ and the value of the truncated expansion, we can find a rational number $\frac{1}{n} < \epsilon$.

The axiom of completeness guarantees that this sequence converges, and we define $\sqrt{2}$ as the limit of the convergent sequence. (Coming below).

Apostol and Stewart

Apostol goes through this development:

• The set **P** of positive integers is *unbounded above*. The proof is to assume that P is bounded above. Then there is a largest element n of **P** which is less than the bound.

But by definition $n + 1$ is $\in \mathbf{P}$.

• For every real x there exists a positive integer n such that $n > x$. Proof: if this were not so, then x would be an upper bound for **P**.

Now, simply replace x with y/x:

• For every real y/x there exists a positive integer n such that $n > y/x$. Thus $nx > y$.

Apostol:

> Geometrically it means that any line segment, no matter how long, may be covered by a finite number of line segments of a given positive length, no matter how small. In other words, a small ruler used often enough can measure arbitrarily large distances. Archimedes realized that this was a fundamental property of the straight line and stated it explicitly as one of the axioms of geometry.

Stewart's definition is:

Given a real number $\epsilon > 0$, there exists a positive integer n such that

$$\frac{1}{10^n} < \epsilon$$

This is certainly compatible with the other definitions. If n is an integer than so is 10^n. So ϵ is Apostol's (small) positive length and if we can choose N so that $N\epsilon$ is as large as we please, we can certainly choose it so that $N\epsilon > 1$.

I interpret this as follows: in distinguishing two real numbers a and b (say, by trying to find another number that lies between them), if $a - b = \epsilon$ is the distance between them, we can always find

$$\frac{1}{10^n} < \epsilon$$

and so always find another number (either real or rational) that lies between a and b.

examples

∘ $(1, \frac{1}{2}, \frac{1}{3}, \dots)$ converges to 0.

proof

You tell me how close you want the values to get to zero, say, within ϵ of zero. Given ϵ, the Archimedean property guarantees we can find $1/N < \epsilon$. Then for all $n > N$ we have

$$\frac{1}{n} < \frac{1}{N} < \epsilon$$

∘ The sequence $(1, 2, 3, \dots)$ is not bounded. Any unbounded sequence fails to converge.

∘ The sequence

$$1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \dots$$

is the sequence of partial sums of the harmonic series, which does not converge.