Lines in R^3

One way of specifying a line in 3D-space is as the intersection of two planes. Another way is by giving a vector and a point in space. Let's look at these in turn. Suppose we have the following two planes:

$$x + y - z = 7$$

$$2x - 3y + z = 3$$

Since the x, y, z terms are not related by a multiplicative constant, the planes are not parallel, so they will meet in a line, and the solutions consist of all the points on the line. Let's find one solution, at $x = 0$. Then

$$y - z = 7$$

$$-3y + z = 3$$

Adding

$$-2y = 10$$

$$y = -5$$

$$z = y - 7 = -12$$

Our solution $P_0 = (0, -5, -12)$. Now find a second solution, at $z = -3$

$$x + y = 4$$

$$2x - 3y = 6$$

Solving

$$x = 4 - y$$

$$2(4 - y) - 3y = 6$$

$$8 - 5y = 6$$

$$y = \frac{2}{5}$$

$$x = \frac{18}{5}$$

The second point is $P_1 = (18/5, 2/5, -3)$. Now we have two points on the line. Its equation is

$$L = P_0 + t(P_1 - P_0)$$

$$L = (0, -5, -12) + t(\frac{18}{5}, \frac{27}{5}, 9)$$

We can re-scale the vector that multiplies t to have integer components (or length 1, or whatever we wish). Why not multiply by 5/9?

$$L = (0, -5, -12) + t(2, 3, 5)$$

There is another way to do this problem that might be a little easier. Consider that the equation of the first plane gives its normal vector n_1 as

$$n_1 = < 1, 1, -1 >$$

Similarly the normal vector to the second plane is n_2

$$n_2 = < 2, -3, 1 >$$

Now, the vector that is parallel to the line of intersection is orthogonal to both n_1 and n_2 (Do you see why?) So we compute the cross-product:

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= -2i - 3j - 5k$$

Multiplying by -1 gives what we obtained above.