# Sum of cosines

Consider two points on the unit circle, the ray to one forms an angle $s$ with the positive $x$-axis, and similarly the other forms an angle $t$, with $s > t$. The two points have coordinates

$$x_1, y_1 = \cos t, \sin t$$

$$x_2, y_2 = \cos s, \sin s$$

The square of the distance separating the two points is

$$(x_2 - x_1)^2 + (y_2 - y_1)^2$$

write that out

$$(\cos s - \cos t)^2 + (\sin s - \sin t)^2$$

$$= \cos^2 s - 2\cos s \cos t + \cos^2 t + \sin^2 s - 2\sin s \sin t + \sin^2 t$$

$$= 2 - 2\cos s \cos t - 2\sin s \sin t$$

Starting to look familiar? Now, rotate the construction so the angle that was previously $t$ is equal to zero. One of the rays now lies along the $x$-axis, and the point on the unit circle is $(1, 0)$. The angle formed by the other ray is $s - t$ and that point is $(\cos s - t, \sin s - t)$. The square of the distance between them is

$$(\cos s - t - 1)^2 + (\sin s - t - 0)^2$$

$$= \cos^2 s - t - 2\cos s - t + 1 + \sin^2 s - t$$

$$= 2 - 2\cos s - t$$

The distances are equal, so the squares are equal, so

$$2 - 2\cos s \cos t - 2\sin s \sin t = 2 - 2\cos s - t$$

$$\cos s \cos t + \sin s \sin t = \cos s - t$$

Let $-u = t$

$$\cos s \cos(-u) - \sin s \sin(-u) = \cos s + u$$

But $\cos u = \cos -u$ while $\sin -u = -\sin u$ so

$$\cos s \cos u - \sin s \sin u = \cos s + u$$

And since $u$ and $t$ are just "dummy" variables

$$\cos s - t = \cos s \cos t + \sin s \sin t$$

$$\cos s + t = \cos s \cos t - \sin s \sin t$$