Summary of trig and other integrals

To make things simpler, I'm going to use the convention that the function $F(x)$ has as its derivative $f(x)$.

$$\frac{d}{dx}\ F(x) = f(x)$$

In that case, we write, alternatively

$$\int f(x)\ dx = F(x) + C$$

For a *definite* integral, we evaluate F at the two endpoints of the interval $[a, b]$

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

but for the *indefinite* integral we need to remember to write the constant C.

basic trig derivatives

You've seen differentiation of the 6 basic trig functions before, namely:

$$F(x) = \cos x\ ; \quad f(x) = -\sin x$$
$$F(x) = \sin x\ ; \quad f(x) = \cos x$$
$$F(x) = \tan x\ ; \quad f(x) = \sec^2 x$$

$$F(x) = \sec x \ ; \quad f(x) = \sec x \ \tan x$$
$$F(x) = \csc x \ ; \quad f(x) = -\csc x \ \cot x$$
$$F(x) = \cot x \ ; \quad f(x) = -\csc^2 x$$

Our focus is on integration, so when faced with a problem like

$$\int \sin x \ dx = -\cos x + C$$

you just find $f(x) = -\sin x$ in the list, read off $F(x)$, and take account of the minus sign.

I'm sure you've noticed the symmetry between the "co-" functions $\csc x$ and $\cot x$ and the ones above. Just remember the minus sign.

And don't forget, if you see this problem you should be able to give the solution:

$$\int \sec^2 x \ dx = \tan x + C$$

going from $f(x)$ to $F(x)$

Now, looking at the list of functions $f(x)$ above, which trig functions are missing? We need to be able to solve problems like:

$$\int \tan x \ dx$$
$$= \int \frac{\sin x}{\cos x} \ dx$$

Substitute $u = \cos x$ and notice that we have $-du = \sin x\ dx$ as well. That is,

$$\int \frac{\sin x}{\cos x}\ dx$$

$$= -\int \frac{1}{u}\ du$$

$$= -\ln\ |u| + C$$

$$= -\ln\ |\cos x| + C$$

And by the usual symmetry we have a similar result for the cotangent.

$$\int \cot x\ dx = \ln\ |\sin x\ |$$

What about the secant?

$$\int \sec x\ dx$$

This involves a more subtle trick.

$$= \int \sec x\ \frac{\sec x + \tan x}{\sec x + \tan x}\ dx$$

$$= \int\ \frac{\sec^2 x + \sec x\ \ \tan x}{\sec x + \tan x}\ dx$$

Notice that if $u = \sec x + \tan x$, then $du = \sec^2 x + \sec x\ \ \tan x\ dx$

So, substituting, all we have is just:

$$= \int \frac{1}{u}\ du$$

$$= \ln\ |u| + C$$

$$= \ln\ |\sec x + \tan x\ | + C$$

Fill out our table:

$$\begin{aligned}
F(x) &= \sin x\ ; \quad f(x) = -\cos x \\
F(x) &= \cos x\ ; \quad f(x) = \sin x \\
F(x) &= \tan x\ ; \quad f(x) = -\ln\ |\cos x\ | \\
F(x) &= \sec x\ ; \quad f(x) = \ln\ |\sec x + \tan x\ | \\
F(x) &= \csc x\ ; \quad f(x) = -\ln\ |\csc x + \cot x\ | \\
F(x) &= \cot x\ ; \quad f(x) = \ln\ |\sin x|
\end{aligned}$$

cosine squared

One more trig function that comes up regularly is $\cos^2 x$. I've worked it elsewhere so here, let me just list the answer:

$$\int \cos^2\ x\ dx = \frac{1}{2}(x + \sin x\ \cos x) + C$$

It is easily checked by differentiating $F(x)$. There is an alternative formula:

$$\int \cos^2\ x\ dx = \frac{1}{2}(x + \frac{1}{2}\ \sin\ 2x) + C$$

The two are related by the double-angle formula.

logarithms and exponentials

You know (and we used it above) that

$$\int \frac{1}{x}\; dx = \ln x + C$$

But what is

$$f(x) = \ln x \;; \quad F(x) = ??$$

Let's try differentiating

$$\frac{d}{dx}\; x \ln x$$

$$= x\frac{1}{x} + \ln x = 1 + \ln x$$

So..

$$\frac{d}{dx}\; (x \ln x - x) = \ln x$$

Our table should then have

$$f(x) = \frac{1}{x} \;; \quad F(x) = \ln x$$

$$f(x) = \ln x \;; \quad F(x) = \; x \ln x - x$$

We can also try differentiating another product

$$\frac{d}{dx}\; xe^x$$

$$= xe^x + e^x$$

So

$$\frac{d}{dx}\; (xe^x - e^x) = xe^x$$

And our table will include

$$f(x) = xe^x \;; \quad F(x) = xe^x - e^x$$

inverse trig functions

Inverse trig functions as the result of integration:

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x$$
$$\int \frac{1}{1+x^2}\ dx = \tan^{-1} x$$
$$\int \frac{1}{|x|\ \sqrt{1+x^2}}\ dx = \sec^{-1} x$$

A moderately complicated one is

$$\int \frac{1}{\sqrt{1+x^2}}\ dx = \ln(\sqrt{1+x^2}+x)$$

Check it by differentiating. We obtain:

$$\frac{d}{dx}\ \ln(\sqrt{1+x^2}+x)$$

$$= (\frac{1}{\sqrt{1+x^2}+x})(\frac{2x}{2\sqrt{1+x^2}}+1)$$
$$= (\frac{1}{\sqrt{1+x^2}+x})(\frac{x}{\sqrt{1+x^2}}+\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}})$$
$$= (\frac{1}{\sqrt{1+x^2}+x})(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}})$$

which does indeed simplify to

$$= \frac{1}{\sqrt{1+x^2}}$$

hyperbolic trig functions

Without getting into the theory, the fundamental result about $\cosh x$ is

$$\int \sinh x \; dx = \cosh x$$

$$\int \cosh x \; dx = \sinh x$$

Notice the lack of a minus sign. The reason for the name "hyperbolic" is this:

$$\cosh^2 x - \sinh^2 x = 1$$