Algebra of quadratic equations

A **polynomial** is a sum containing various powers of an independent variable, which is usually given as x. For example:

$$y = c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$$

The powers must be positive integers or zero: $n \in \{0, 1, 2 \dots\}$.

Each power x^n is multiplied by its corresponding constant c_n. The original equation might contain multiple coefficients for a given power x^n that are then combined to form the constant.

Each c_n is some real number. It is usual in examples for these constants to be integers, but this is by no means a requirement.

A **quadratic** is a polynomial that contains a term with x^2 but no higher powers of x:

$$y = c_2x^2 + c_1x + c_0$$

This is usually written as

$$y = ax^2 + bx + c$$

where a, b, and c are constants.

A quadratic may or may not contain lower powers of x. That is, either or both of b and c might be equal to zero. All of these are quadratics:

$$y = ax^2$$

$$y = ax^2 + bx$$
$$y = ax^2 + c$$

In general, the roles the constants a and c play in the graph of a quadratic are fairly obvious, while that of b is more subtle.

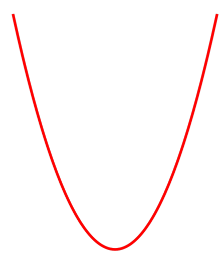
Changing the value of c shifts the graph up or down by the amount added. Comparing

$$y = ax^2$$
$$y = ax^2 + c$$

the second graph will be identical to the first, simply shifted up by c.

shape factor

a is called the **shape factor**. If a is positive, then the two "arms" of the parabola open up, and the **vertex** is the minimum value of the graph of the function.



If a is negative, then the graph opens down, and the vertex is the maximum.

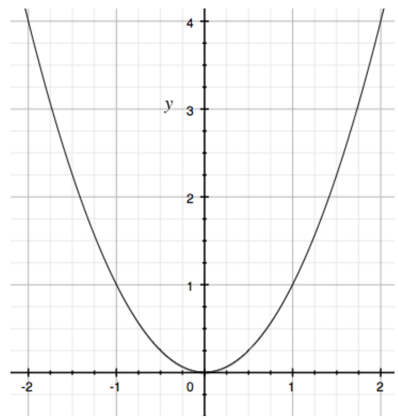
Any quadratic is symmetrical about the vertical axis that goes through its minimum (or maximum) point, the vertex.

For the rest of this discussion, we will only consider $a > 0$. If $a < 0$, the whole graph is flipped upside down. In that case, everywhere I say "minimum", we have a maximum instead.

Suppose

$$y = ax^2$$

x^2 is always greater than or equal to zero ($x^2 \geq 0$), therefore so is the value of this function, $y \geq 0$. The minimum value is the vertex at $(x = 0, y = 0)$.



The graph is symmetrical about the y-axis. This is also evident because $y = ax^2$ is an even function: $f(x) = f(-x)$.

Consider the values of y corresponding to $x = \{1, 2, 3, 4\}$. These are $y = \{a, 4a, 9a, 16a\}$. They increase like the square of x, but *linearly* with a.

If we plot $y = x^2$ and compare it to $y = 4x^2$, every y value in the second plot can be taken from the first one, just multiplied by 4. a stretches the plot linearly in the y-direction.

A substitution of variables $v = y/a$ turns $y = ax^2$ into $v = x^2$.

vertex

Every parabola with the same value of a has exactly the same shape. For the same a, they may differ in the position of the vertex, depending on the other constants, b and c. The graph of a parabola depends only on the shape factor and the position of the vertex.

The coordinates at the vertex are usually given as constants (h, k). Suppose the vertex of the parabola is at (h, k) with $a = 1$. Then the equation of the parabola is

$$(y - k) = (x - h)^2$$

It may seem counterintuitive that we subtract the value of the variable at the vertex, but this is consistent across all of the conic sections.

Rearranging

$$y = (x - h)^2 + k$$

we see that $y - k$ is like adding k to the constant c, it moves the graph up the page.

Expand

$$(y - k) = a(x - h)^2$$

$$y = ax^2 - 2ahx + h^2 + k$$

and compare with the canonical representation

$$y = ax^2 + bx + c$$

The coefficients of corresponding powers must be equal so

$$-2ah = b$$

$$h = -\frac{b}{2a}$$

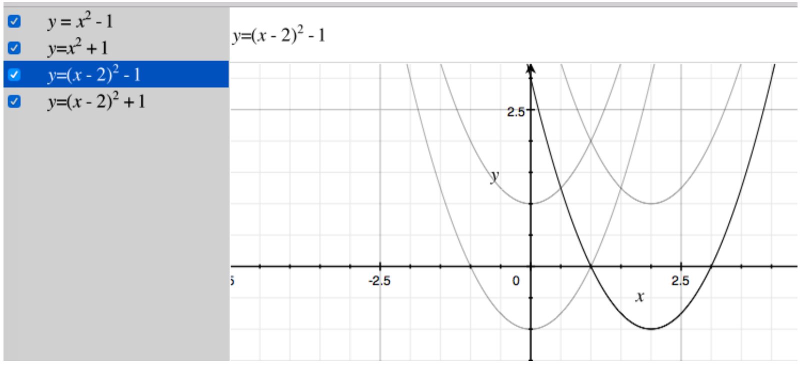
The x-value of the vertex is $x = -b/2a$. Also

$$h^2 + k = c$$

The y-value of the vertex is

$$k = c - h^2 = c - b^2/4a$$

Here are four plots all with shape factor $a = 1$. They differ in the position of the vertex :



The two with the vertex below the x-axis have $c = -1$.

The two with the vertex on the y-axis have $h = 0$, the others have $h = 2$. To write the latter two in the canonical form, expand $(x - h)^2 - 2 = x^2 - 2xh + h^2 - 2$.

roots

The roots of a quadratic are those values of x for which the corresponding value of y is zero.

For example, if $x = 0$ then

$$y = x^2 = 0^2 = 0$$

$x = 0$ is the only value that yields this result, since if $x \neq 0$, then $y = x^2 > 0$.

Suppose we shift the graph of the equation *down*, by subtracting 1, i.e. letting $c = -1$:

$$y = x^2 - 1$$

From examining the plot of this function you will observe that there are two points where the graph of the function crosses the x-axis (where

$y = 0$). We can guess them from the plot as $x = \pm 1$, and confirm that result directly by substituting into the equation and checking that we get $y = 0$.

$$y = (1)^2 - 1 = 0$$
$$y = (-1)^2 - 1 = 0$$

Another way to get this answer is to factor the original equation.

$$y = (x + 1)(x - 1)$$

Now it is obvious that either $x = \pm 1$ gives $y = 0$ as the result.

If a quadratic can be factored to the form

$$y = (x - r_1)(x - r_2)$$

then r_1 and r_2 are the roots, because if $x = r_1$ or $x = r_2$, then $y = 0$.

On the other hand, suppose we shift the graph *up*

$$y = x^2 + 1$$

Since x^2 is always positive or zero, there is no value of x which gives $y = 0$. We say that such an equation has no (real) roots. An equivalent statement or observation is that its graph does not cross the x-axis.

Again, if a quadratic can be factored to the form

$$y = (x - r_1)(x - r_2)$$

then

$$y = x^2 - (r_1 + r_2)x + r_1 r_2$$

In the canonical representation

$$ax^2 + bx + c$$

c is the product of the roots, and b is the negative of the sum.

For the example

$$y = x^2 - 1$$

the sum of the roots is 0 and their product is -1. From the first fact:

$$r_2 = -r_1$$

substituting into the second.

$$r_1^2 = 1$$

If you've had practice factoring quadratics with integer roots, you should be very familiar with this fact: c is the product of the roots, and b is the negative of the sum.

three types

In summary, we can classify parabolas into three types.

The first one has a graph that does not cross the x-axis. It has no real roots.

The second type has a graph that does cross the x-axis and has two distinct real roots of the form

$$y = (x - r_1)(x - r_2)$$

The third one has repeated roots

$$y = (x - r)(x - r)$$

This type has a single value of x that yields $y = 0$. This happens when the graph of the parabola just touches the x-axis — the vertex is on the x-axis.

The example given above $y = x^2$ is a special case of this type where $r = 0$.

quadratic formula

The quadratic formula gives the roots of any quadratic. In the case where there are no real roots, the results from the quadratic formula are complex numbers of the form $p \pm q\sqrt{-1}$, (where p and q are real numbers).

The formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You should memorize this.

Examples:

$y = x^2$

$$x = \frac{-(0) \pm \sqrt{0^2 - 4(1)(0)}}{2(1)} = 0$$

$y = x^2 - 1$

$$x = \frac{-(0) \pm \sqrt{0^2 - 4(1)(-1)}}{2(1)} = \pm\frac{\sqrt{4}}{2} = \pm 1$$

$y = x^2 + 1$

$$x = \frac{-(0) \pm \sqrt{0^2 - 4(1)(1)}}{2(1)} = \pm\frac{\sqrt{-4}}{2} = ??$$

The result of the last calculation is a pair of complex numbers. A complex number is a number of the form $p + q\sqrt{-1}$, often written $p + iq$.

We may write the expression under the square root as

$$D = b^2 - 4ac$$

where D stands for discriminant. If $D < 0$ then the result is a pair of complex numbers and we say there are no real roots. If $D = 0$ then there is a single (repeated) root.

Although the existence of complex roots means the graph does not cross the x-axis and there is no x such that $f(x) = 0$, nevertheless these roots do have physical meaning.

The two complex roots are related, they are called complex conjugates. Label them as $z = p \pm q\sqrt{-1}$ and plug them into the factored quadratic

$$y = (x - z_1)(x - z_2)$$

$$\begin{aligned} y = &\ [\ x - (p + q\sqrt{-1})\ ]\ [\ x - (p - q\sqrt{-1})\ ] \\ = &\ (x - p - q\sqrt{-1})(x - p + q\sqrt{-1}) \end{aligned}$$

Multiplying this out, the square roots will (always) disappear:

$$\begin{aligned} y = x^2 - px + xq\sqrt{-1} - px + p^2 - pq\sqrt{-1} - xq\sqrt{-1} + pq\sqrt{-1} + q^2 \\ = x^2 - 2px + p^2 + q^2 \\ = (x - p)^2 + q^2 \end{aligned}$$

The meaning of this is the following: even with complex roots, the factored form gives a real result when multiplied out. The real part of $z = p \pm q\sqrt{-1}$ is p, and this is the value of x at the minimum.

q^2 is the displacement of the vertex up from the x-axis at the minimum.

more about b

Consider the basic equation

$$y = ax^2 + bx + c$$

$$\frac{y-c}{a} = x^2 + \frac{b}{a}x$$

By judicious manipulation we can make the last term $b/a \cdot x$ go away (this is *always* true). The general procedure is called completing the square. Write

$$x^2 + \frac{b}{a}x + \_\_\_ = (x + \_\_\_)^2$$

We seek two values to substitute for the spaces $\_\_\_$.

Now, of course, the $\_\_\_$ on the left-hand side is the square of the second one, on the right.

But there is another constraint. Namely, the second $\_\_\_$ is related to the cofactor of x on the left-hand side.

Recall that

$$x^2 + 2mx + m^2 = (x+m)^2$$

Compare that with

$$x^2 + \frac{b}{a}x + \_\_\_ = (x + \_\_\_)^2$$

Can you see that b/a must be equal to $2m$ and so $m = b/2a$? We need two copies of the second term in the binomial expansion (m), to put as the cofactor of x in the term $2mx$. Since the standard form in the last expression has b/a equivalent to $2m$, we get $b/2a$ equivalent to m.

If that's not clear, just verify that the following works. Write:

$$x^2 + \frac{b}{a}x + \_\_\_ = (x + \frac{b}{2a})^2$$

$$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = (x + \frac{b}{2a})^2$$

Now that we know what is needed to complete the square, go back to our problem.

$$\frac{y-c}{a} = x^2 + \frac{b}{a}x$$

To make the perfect square, we add $(b/2a)^2$ to the right-hand side, and to maintain the equality add the same thing to the left-hand side:

$$\frac{y-c}{a} + (\frac{b}{2a})^2 = x^2 + \frac{b}{a}x + (\frac{b}{2a})^2$$

As we saw above, the right-hand side is also $(x+b/2a)^2$ so we can write

$$\frac{y-c}{a} + (\frac{b}{2a})^2 = (x + \frac{b}{2a})^2$$

Finally, multiply through by a and rearrange slightly

$$y = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

We can get several things from this.

First, the minimum value of y occurs when the squared term is equal to zero, that is when

$$x + \frac{b}{2a} = 0$$

$$x = -\frac{b}{2a}$$

Therefore, the vertex of the parabola is at this value of x. We found this earlier by writing $(y-k) = a(x-h)^2$ and multiplying out.

When $x = -b/2a$, the corresponding value of y is

$$y = a(-\frac{b}{2a} + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

$$= c - \frac{b^2}{4a}$$

This also matches what we had before.

Second and more generally, the roots occur when

$$y = 0 = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

$$a(x + \frac{b}{2a})^2 = \frac{b^2}{4a} - c$$

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which is the quadratic formula.

Third, any quadratic can be rewritten as

$$y = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

$$y = a(x - h)^2 + k$$

translation

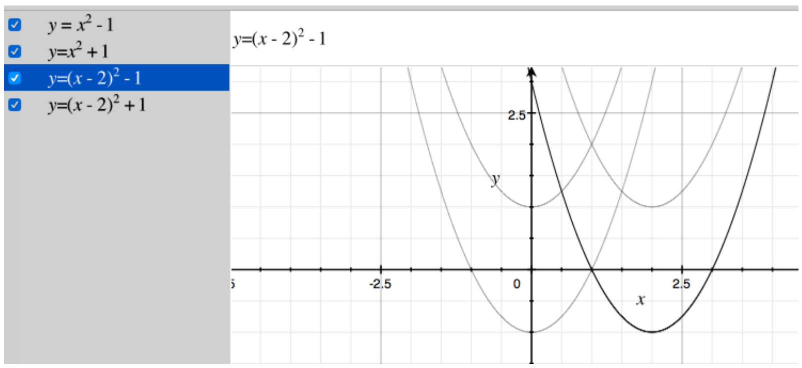
The basic shape depends only on a.

b (combined with $2a$) moves the vertex right or left from the y-axis.

a, b, c all together combine to move it up and down from the x-axis.

If you play around with a plotting application and change b you will find that the shape stays the same, but both the x and the y-values of the vertex will change as b changes.

Here are the four plots again.



The two with the vertex on the y-axis have $h = 0$, the others have $h = 2$.

The plots with real roots have $k = -1$ and $(y - k) = y + 1$, the others have $k = 1$.

For example, the lower right-hand plot is

$$(y + 1) = (x - 2)^2 = x^2 - 4x + 4$$

$$y = x^2 - 4x + 3$$

This can be factored easily:

$$y = (x - 1)(x - 3)$$

The roots are at $x = 1, x = 3$, which checks.

rotation

You might ask, what about rotation? For example, if we rotate $y = x^2$ by 45 degrees clockwise, what would be the equation to describe it?

The short answer is that such equations do exist, and they have terms like xy or uv in them. They are not polynomials of the type we've been describing.

As an example to rotate through 45°, replace x and y by u and v with

$$x = u\cos\theta - v\sin\theta = \frac{u}{\sqrt{2}} - \frac{v}{\sqrt{2}}$$
$$x = ku - kv$$

$$y = u\sin\theta + v\cos\theta = \frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}$$
$$y = ku + kv$$

Substitute for x and y in the standard equation:

$$y = ax^2 + bx + c$$

$$ku + kv = a(ku - kv)^2 + b(ku - kv) + c$$
$$u + v = ak(u^2 - 2uv + v^2) + b(u - v) + \frac{c}{k}$$

Notice the term $-2akuv$ that mixes u and v.

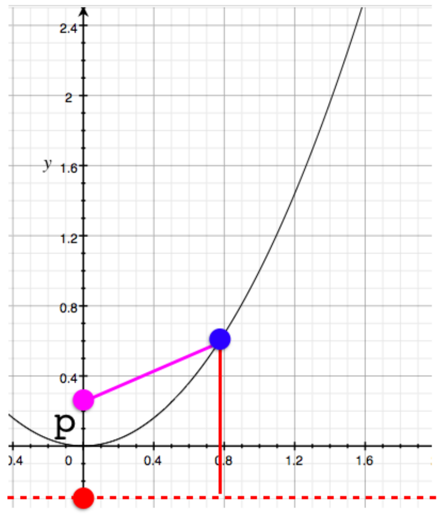
This is a more advanced topic than we can deal with here.

focus and directrix

There is a geometric definition of the parabola. Based on what we said above, without loss of generality, we can translate any parabola to the origin of coordinates, with equation $y = ax^2$.

Now, pick a point on the y-axis a distance p up from the origin, colored magenta in the figure. This point is called the focus.

Then draw a line parallel to the x-axis which intersects the y-axis the same distance p below the origin. This line is called the directrix. It is colored red and is dashed.



The parabola consists of all those points whose distance to the focus is equal to the vertical distance to the directrix.

Pick an arbitrary point on the parabola (in blue), with coordinates (x, ax^2). The squared distance to the focus (magenta point) is

$$\Delta y^2 + \Delta x^2 = (ax^2 - p)^2 + x^2$$

while the squared distance to the directrix (red line) is $(ax^2 + p)^2$.

For the correct choice of p these distances are to be equal:

$$(ax^2 - p)^2 + x^2 = (ax^2 + p)^2$$

$$a^2x^4 - 2apx^2 + p^2 + x^2 = a^2x^4 + 2apx^2 + p^2$$

Canceling two terms on each side

$$-2apx^2 + x^2 = +2apx^2$$

Divide by x^2

$$-2ap + 1 = 2ap$$
$$4ap = 1$$
$$p = \frac{1}{4a}$$

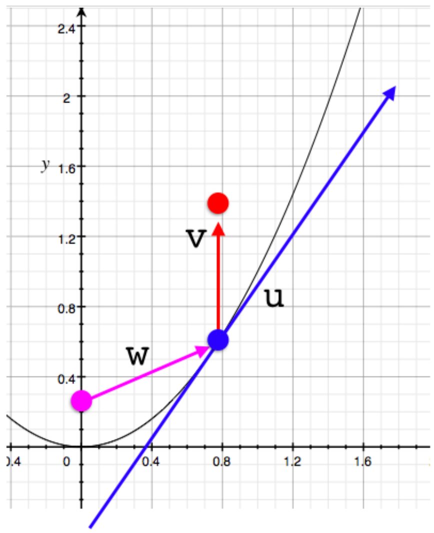
The shape factor a determines the distance of the focus from the origin, which is p, and from the directrix, which is $2p$.

headlight problem

The reflective property of the parabola asserts that if a light ray emitted from the focus bounces off any point of the parabola, it then heads off in the vertical direction.

Snell's law says that the angle of incidence and reflection to the inside surface of the parabola must be equal.

Applying that law to this problem, the angle of incidence is the angle of the magenta vector $\mathbf{w}$ with the tangent vector $\mathbf{u}$. This is equal to the angle of reflection, the angle of the tangent $\mathbf{u}$ with the vertical vector $\mathbf{v}$.



We assert that there exists a point on the y-axis (the focus, colored magenta), with the property that when we draw a vector to any point on the parabola, the angle that this vector makes with the tangent to the parabola is equal to the angle the tangent makes with the vertical.

Let the distance of this point from the origin be p. Then

$$\mathbf{w} = \langle x, ax^2 - p \rangle$$

The tangent has slope $2ax$ so

$$\mathbf{u} = \langle 1, 2ax \rangle$$

Scale the vertical to be a unit vector

$$\mathbf{v} = \langle 0, 1 \rangle$$

By the definition of the dot product, the cosine of the angle between $\mathbf{w}$ and $\mathbf{u}$ is

$$\frac{\mathbf{w} \cdot \mathbf{u}}{u\ w}$$

By the equal angle constraint, this is equal to the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$

$$\frac{\mathbf{u} \cdot \mathbf{v}}{u\ v} = \frac{\mathbf{w} \cdot \mathbf{u}}{u\ w}$$

Since $v = 1$ we have

$$w\ (\mathbf{u} \cdot \mathbf{v}) = \mathbf{w} \cdot \mathbf{u}$$

That's the important logic of the solution.

Now it's just algebra: The length of $\mathbf{w}$ is

$$w = \sqrt{x^2 + (ax^2 - p)^2}$$

while

$$\mathbf{u} \cdot \mathbf{v} = 2ax$$

$$\mathbf{w} \cdot \mathbf{u} = x + 2ax(ax^2 - p)$$

So

$$w\ (\mathbf{u} \cdot \mathbf{v}) = \mathbf{w} \cdot \mathbf{u}$$

$$\sqrt{x^2 + (ax^2 - p)^2}\ (2ax) = x + 2ax(ax^2 - p)$$

Divide by $2ax$:

$$\sqrt{x^2 + (ax^2 - p)^2} = \frac{1}{2a} + (ax^2 - p)$$

Square both sides

$$x^2 + (ax^2 - p)^2 = \frac{1}{(2a)^2} + \frac{1}{a}(ax^2 - p) + (ax^2 - p)^2$$

A nice cancelation:

$$x^2 = \frac{1}{(2a)^2} + \frac{1}{a}(ax^2 - p)$$

We can also cancel the x^2:

$$0 = \frac{1}{(2a)^2} + \frac{1}{a}(-p)$$

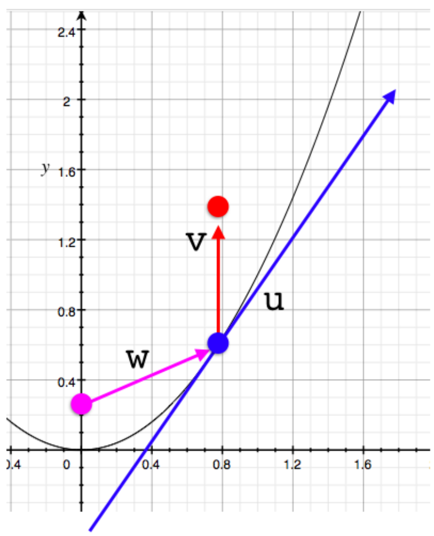
and finally cancel an a:

$$0 = \frac{1}{4a} - p$$

$$p = \frac{1}{4a}$$

The point $(0, 1/4a)$ is, as we saw before, the focus of the parabola.

Since p is independent of x, this property holds for every point on the parabola.



An alternative, more geometric approach is to note that the angle the vector $\mathbf{u}$ makes with the vertical at (x, ax^2) is equal to the angle $\mathbf{u}$ makes with the y-axis (just off the image to the bottom).

This angle is equal to the angle between $\mathbf{w}$ and $\mathbf{u}$ if and only if the triangle is isosceles, that is, if length of the vector $\mathbf{w}$ is equal to the distance between $(0, p)$ and the intersection of $\mathbf{u}$ with the y-axis.

We start by exploring the properties of a line through the point (x, ax^2) with slope equal to $2ax$.

From this point on, the point on the parabola is *fixed*. We want to write an equation for a line with the same slope as the parabola at this point, the same slope as the vector **u**.

We will be re-using x as a variable. To reduce confusion, label the fixed value at the point as $\hat{x}$, so then $\hat{y} = a\hat{x}^2$, and the slope is $2a\hat{x}$.

The point-slope formula for the line is

$$2a\hat{x} = \frac{\Delta y}{\Delta x} = \frac{y = \hat{y}}{x - \hat{x}} = \frac{y - a\hat{x}^2}{x - \hat{x}}$$

The intersection with the y-axis occurs at $y = 0$ so there

$$\begin{aligned} 2a\hat{x} &= \frac{-a\hat{x}^2}{x - \hat{x}} \\ 2 &= \frac{-\hat{x}}{x - \hat{x}} \\ 2x - 2\hat{x} &= -\hat{x} \\ x &= \frac{\hat{x}}{2} \end{aligned}$$

The intersection of **u** with the x-axis is at $\hat{x}/2$.

For the intersection with the y-axis, $x = 0$ and then

$$\begin{aligned} 2a\hat{x} &= \frac{y - a\hat{x}^2}{-\hat{x}} \\ -2a\hat{x}^2 &= y - a\hat{x}^2 \\ y &= -a\hat{x}^2 \end{aligned}$$

What we've discovered is that the point of intersection is the same distance below the x-axis as our point on the parabola $(\hat{x}, a\hat{x}^2)$ is above it. We could have used congruent triangles proceeding from the discovery above that the intersection of with the x-axis is at $\hat{x}/2$.

Our goal is to show that the triangle is isosceles:

$$a\hat{x}^2 + p = w$$

$$a\hat{x}^2 + p = \sqrt{\hat{x}^2 + (a\hat{x}^2 - p)^2}$$

$$(a\hat{x}^2 + p)^2 = \hat{x}^2 + (a\hat{x}^2 - p)^2$$

Continuing

$$a^2\hat{x}^4 + 2ap\hat{x}^2 + p^2 = \hat{x}^2 + a^2\hat{x}^4 - 2ap\hat{x}^2 + p^2$$

Does this look familiar?

Cancel two terms

$$2ap\hat{x}^2 = \hat{x}^2 - 2ap\hat{x}^2$$

Divide by $\hat{x}^2$:

$$2ap = 1 - 2ap$$

$$4ap = 1$$

$$p = \frac{1}{4a}$$

And we already proved this is true, if the magenta point we start from is the focus.

Hence the lengths are equal, the triangle is isosceles, and the corresponding angles are equal. The point we've been using is just the focus.