# Quick Double Integrals

In this short write-up I want to take a very quick look at double integrals. The basic principle of calculus of several variables is that we look at the effect of small changes in *one variable at a time*. So we write

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

where, for example, $\partial f/\partial x$ is the derivative of $f$ with respect to $x$, treating $y$ as a constant.

For integration, consider a rectangle where $x$ varies between $0 \to 2$, while $y = 0 \to 3$. To compute the area of this rectangle, we write

$$A = \int_{x=0}^{2} \int_{y=0}^{3} \, dy \; dx$$

There is no $f(x, y)$ inside the integral sign (except implicitly $f(x, y) = 1$), and the result will be the area. In the same way $\int dx = x_f - x_i$, the length of $x$.

To evaluate the integral, look at the part inside, the inner integral

$$\int_{y=0}^{3} \, dy$$

We need a function whose derivative is $dy$, holding $x$ constant. In this case, that's just $y$, evaluated between the limits. We obtain

$$\int_{y=0}^{3} \, dy = y \, \Big|_0^3 = 3$$

So we proceed to evaluate the outer integral

$$A = \int_{x=0}^{2} 3\ dx = 3 \int_{x=0}^{2}\ dx = 3 \times 2 = 6$$

**something more ambitious**

Now consider a circle. We're going to integrate the function $f(x, y) = 1$ over a circle of radius $R$. For simplicity, we do only the first quadrant, and multiply at the end by 4.

$$A = \int_{x=0}^{R} \int_{y=0}^{\sqrt{R^2-x^2}}\ dy\ dx$$

Since $y$ changes as a function of $x$, we need to account for that. That's why we have the upper limit on $y$. The inner integral is easy

$$\int_{y=0}^{\sqrt{R^2-x^2}}\ dy = \sqrt{R^2 - x^2}$$

But now the outer integral is

$$A = \int_{x=0}^{R} \sqrt{R^2 - x^2}\ dx$$

This presents a bit of a problem because we don't have the derivative for what's inside the square root. We continue anyway, with a trig substitution.

$$x = R \sin\theta$$

$$dx = R\cos\theta\ d\theta$$

$$\sqrt{R^2 - x^2} = R\cos\theta$$

So the integral is

$$\int R^2 \cos^2\theta\ d\theta$$

I've done this before. The answer is

$$= R^2 \ \frac{1}{2}(\theta + \sin\theta\cos\theta)$$

The really tricky part is the limits. We could switch back to $x$ or we can recognize that when $x = R$, $\sin\theta = 1$ and when $x = 0$, $\sin\theta = 0$ so we evaluate between $\theta = 0 \to \pi/2$. That makes $\sin\theta\cos\theta$ go away. The result is just

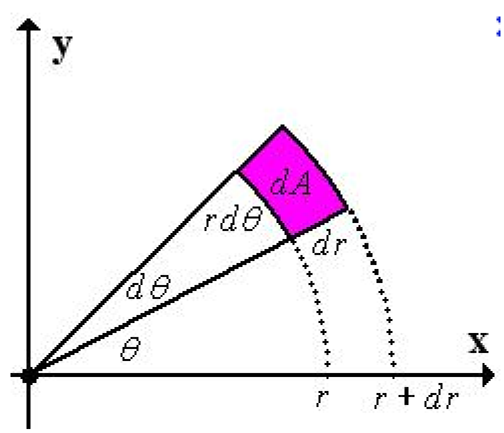
$$= R^2 \ \frac{1}{2}(\frac{\pi}{2})$$

Multiply by 4 to obtain $\pi R^2$.

**an easier way**

There is another approach, which is the reason for this particular write-up. The "area element" in 2D using polar coordinates is

$$dA = r \ dr \ d\theta$$




The figure shows a rationale for this. $\theta$ or $d\theta$ is not a length, $r \ d\theta$ is a length. What we're saying is that the area element in the plane is

$$dA = dy \ dx = r \ dr \ d\theta$$

and for any particular region, we can pick the one that's easier. Clearly, for a circle, the second way is easier. We set up the integral

$$A = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} r \ dr \ d\theta$$

The inner integral is

$$\int_{r=0}^{R} r \ dr = \frac{r^2}{2} \Big|_{r=0}^{R} = \frac{R^2}{2}$$

The outer integral is

$$A = \int_{\theta=0}^{2\pi} \frac{R^2}{2} \ d\theta = \frac{R^2}{2} \ 2\pi = \pi R^2$$

**application**

We can apply what we've learned to the following problem. In the Gaussian distribution and also in the physics of molecular velocities we run into an integral like

$$I = \int_{-\infty}^{\infty} e^{-x^2} \ dx$$

There is a great solution to this. Write

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} \ dx \int_{-\infty}^{\infty} e^{-y^2} \ dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \ dx \ dy$$

But we can change this to polar coordinates

$$= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} \ r \ dr \ d\theta$$

The inner integral is just

$$\int_0^\infty e^{-r^2}\ r\ dr = -\frac{1}{2}e^{-r^2}\ \Big|_0^\infty = \frac{1}{2}$$

So then we have

$$I^2 = \int_0^{2\pi} \frac{1}{2}\ d\theta = \pi$$

To put this another way

$$I = \sqrt{\pi}$$

$$I = \int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$$