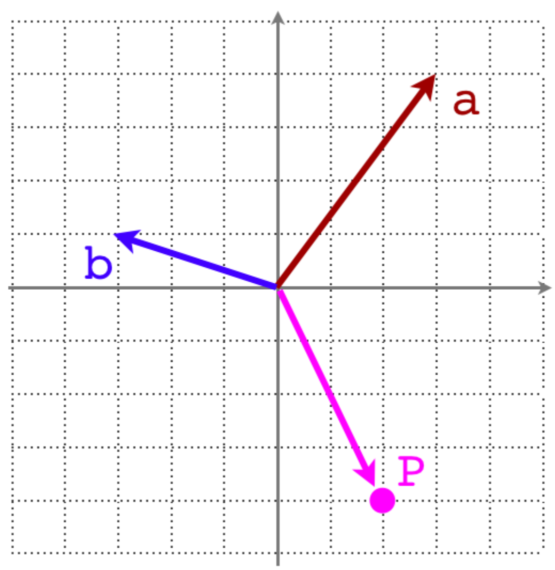
Nullspace of a matrix

Let's consider two vectors in R2.

$$\mathbf{a} = < 3, 4 > = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\mathbf{b} = < -3, 1 > = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

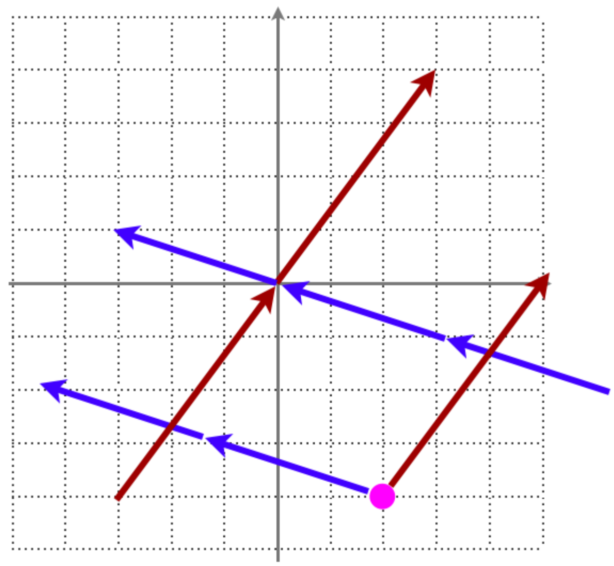
We can see from a plot that these two vectors are obviously not pointing in the same direction.



We say that $\mathbf{a}$ is not a multiple (or a linear combination) of $\mathbf{b}$, because there is no constant k such that $k \times \mathbf{a} = \mathbf{b}$. Now consider a point P

anywhere in R2, say $(2, -4)$.

We can construct a linear combination of **a** and **b** that reaches this P or any other point. Simply place one of the vectors at the origin and move along it (perhaps in a negative direction), and place the second vector at P and move along it, and find where the two lines meet.



Call the coordinates of the point where the lines cross x and y. There are actually two possibilities depending on which vector we choose for each role. Suppose we move in the reverse direction from the origin along **a** (in quadrant III) and from P to the left along **b**. From the point-slope equation we know that:

$$(0-y)/(0-x) = 4/3$$

$$y = \frac{4}{3}x$$

because the slope of **a** is 4/3, and

$$(P_y - y)/(P_x - x) = -\frac{1}{3}$$

because the slope of $\mathbf{b}$ is $-1/3$. Plugging in $P = (2, -4)$, we have

$$\frac{-4 - y}{2 - x} = -\frac{1}{3}$$

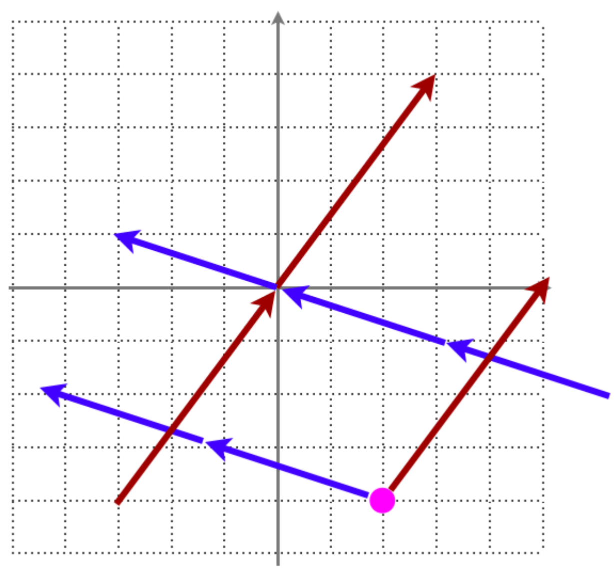
$$12 + 3y = 2 - x$$

$$x = -10 - 3y = -10 - 3\ \frac{4}{3}x = -10 - 4x$$

$$x = -2$$

$$y = -\frac{8}{3}$$

The other solution is symmetrical (we're dealing with a parallelogram). We would need to go $+2$ across and $+8/3$ up from P to reach $x, y = (4, -4/3)$. Suppose we want to know the actual multipliers for $\mathbf{a}$ and $\mathbf{b}$, i.e. the fraction of the length of a and b that we travel along each vector. From the figure, we can estimate that the values will be about -0.7 and -1.25.



Call these multipliers u and v. In matrix language, we arrange our two vectors **a** and **b** in a matrix like this

$$M = \begin{bmatrix}\mathbf{ab}\end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 4 & 1 \end{bmatrix}$$

We set up a multiplication

$$\begin{bmatrix} 3 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

One way we can solve the system (the Algebra 2 way) is to do the multiplication

$$3u - 3v = 2$$

$$4u + v = -4$$

Solve the second equation for v and plug into the first

$$3u - 3(-4 - 4u) = 2$$

$$15u = -10$$

$$u = -\frac{2}{3}$$

$$v = -4 - 4(-\frac{2}{3}) = -\frac{4}{3}$$

In matrix language we would say that

$$\begin{bmatrix} 3 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -2/3 \\ -4/3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Now, consider P as not just a point but a vector **c**. Then our multiplication is u times the first column (vector **a**) + $v \times \mathbf{b}$ is equal to vector $\mathbf{c} = < 2, -4 >$.

$$-\frac{2}{3} < 3, 4 > + -\frac{4}{3} < -3, 1 > = < -2 + 4, -\frac{8}{3} - \frac{4}{3} > = < 2, -4 >$$

Since we could have chosen P to be any point, $\mathbf{c}$ can be any vector in the x,y-plane and it can be constructed as a linear combination of $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{c} = u\ \mathbf{a} + v\ \mathbf{b}$$

We say that $\mathbf{c}$ is in the *column space* of the matrix whose columns are the vectors $\mathbf{a}$ and $\mathbf{b}$.

$$M = \begin{bmatrix}\mathbf{ab}\end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 4 & 1 \end{bmatrix}$$

because we can obtain $\mathbf{c}$ as a linear combination of the columns of M.

$$u\ \mathbf{a} + v\ \mathbf{b}\ = \mathbf{c}$$

$$M \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix}\mathbf{ab}\end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{c}$$

Using just the two vectors $\mathbf{a}$ and $\mathbf{b}$, there is no linear combination that gives the zero vector except

$$0\ \mathbf{a} + 0\ \mathbf{b}\ = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

But of course, since

$$u\ \mathbf{a} + v\ \mathbf{b}\ = \mathbf{c}$$

$$u\ \mathbf{a} + v\ \mathbf{b}\ - \mathbf{c} = \mathbf{0}$$

$$u\ \mathbf{a} + v\ \mathbf{b}\ + w\ \mathbf{c} = \mathbf{0}$$

$(w = -1)$. But once we add $\mathbf{c}$ to the matrix M' there is a way to do it.

$$M' = \begin{bmatrix}\mathbf{abc}\end{bmatrix} = \begin{bmatrix} 3 & -3 & 2 \\ 4 & 1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 & 2 \\ 4 & 1 & -4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 3 & -3 & 2 \\ 4 & 1 & -4 \end{bmatrix} \begin{bmatrix} -2/3 \\ -4/3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The combination $u = -2/3, v = -4/3, w = -1$ solves this equation and brings us back to zero. This new vector, call it $\mathbf{x}$

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -2/3 \\ -4/3 \\ -1 \end{bmatrix}$$

is said to be in the *nullspace* of M' because $M'\mathbf{x} = \mathbf{0}$.

For any matrix M, if there exists a non-zero solution $\mathbf{x}$ to

$$M\mathbf{x} = \mathbf{0}$$

it's the same thing as saying we can find u, v, w such that

$$u\ \mathbf{a} + v\ \mathbf{b}\ + w\ \mathbf{c} = \mathbf{0}$$

and then among the consequences are these two: that the columns of M are not linearly independent

$$u\ \mathbf{a} + v\ \mathbf{b}\ = -w\ \mathbf{c}$$

and the determinant of M is equal to 0, because we can "zero out" one of the columns of M.