Chain Rule and Quotient Rule

Using $f'(x)$ is sometimes not as nice as the $\frac{dy}{dx}$ differential notation, because the latter allows us to do algebraic manipulation of the differentials. If $y = x^2$ (for example), we write

$$y = f(x) = x^2$$

Then

$$\frac{dy}{dx} = f'(x) = 2x$$

It is perfectly OK to move the dx to the other side

$$\frac{dy}{dx}dx = dy = 2x\ dx$$

Imagine that x is some function of another variable, like time t. Then y will also be a function of t (through its dependence on x), and it will also be true that

$$\frac{dy}{dt} = \frac{dy}{dx}\ \frac{dx}{dt}$$

To convince yourself that this is true, just cancel the dt on top and bottom. So if we divide the previous equation on both sides by dt, we see that the result makes sense:

$$\frac{dy}{dt} = (2x)\ \frac{dx}{dt}$$

because

$$\frac{dy}{dx} = 2x$$

Suppose we are given $y = f(x)$, and $x = f(t)$ and asked to calculate $\frac{dy}{dt}$. The way we get it is:

$$\frac{dy}{dt} = \frac{dy}{dx}\ \frac{dx}{dt}$$

Here is an example

$$y = x^2, \qquad x = 3t,$$

$$\frac{dy}{dx} = 2x, \qquad \frac{dx}{dt} = 3, \qquad \frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt} = 6x = 18t$$

which we can easily confirm by just substituting into $y = x^2$

$$y = x^2 = (3t)^2 = 9t^2, \qquad \frac{dy}{dt} = 18t$$

This is what's called the chain rule. Another example will give a glimpse of how useful it can be. Suppose we are given

$$y = \sqrt{(1 + x^2)}$$

What is dy/dx? We make a "substitution"

$$u = 1 + x^2, \qquad \frac{du}{dx} = 2x$$

$$y = \sqrt{u}, \qquad \frac{dy}{du} = \frac{1}{2\sqrt{u}}$$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = \frac{1}{2\sqrt{u}}2x = \frac{x}{\sqrt{(1+x^2)}}$$

There is one more rule to cover, and that is the quotient rule. If we have

$$\frac{f(x)}{g(x)}$$

or more simply

$$\frac{u}{v}$$

what is

$$\frac{d}{dx}(\frac{u}{v}) = ?$$

We use the known value for $\frac{d}{dx}v^{-1}$, the product rule and the chain rule

$$\frac{d}{dx}(\frac{u}{v}) = \frac{du}{dx}\;(\frac{1}{v}) + \frac{d}{dx}(\frac{1}{v})\;u = \frac{u'}{v} - \frac{v'}{v^2}\;u = \frac{u'v}{v^2} - \frac{v'u}{v^2} = \frac{u'v - v'u}{v^2}$$

We check by using it on a simple known example

$$y = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{(0)(x) - (1)(1)}{x^2} = -\frac{1}{x^2}$$

Perhaps that was too easy. How about

$$y = \frac{x}{x^2}$$

$$\frac{dy}{dx} = \frac{(1)(x^2) - (2x)(x)}{x^4} = -\frac{x^2}{x^4} = -\frac{1}{x^2}$$