Volumes by Integration

One of the great things about calculus is the discovery that one can use integration to compute not only the areas of plane figures, but also surface areas and volumes in 3D space ($\mathbb{R}3$).

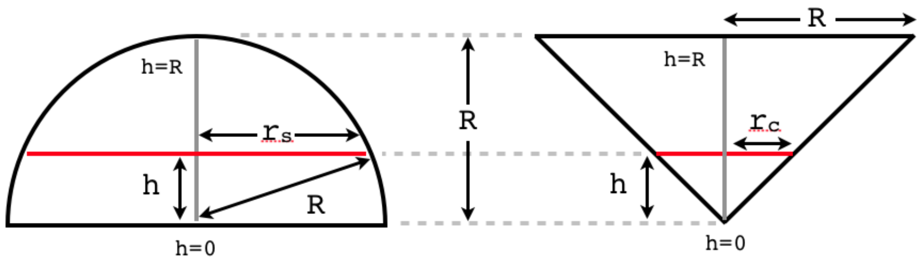
Let's start by thinking about the volume contained inside the sphere. The sphere is the locus of points at a distance R from the origin (i.e., it's hollow), and the volume inside is technically called a "ball". But I'm just going to use the phrase "volume of the sphere" here.

Archimedes

Archimedes was the first person to obtain the formula for the volume of a sphere. He did this by using a method of "slices", which is very close to what we think of as the method of integration.

Consider a hemisphere of radius R oriented with its base in the xy-plane. Position this next to a cone having the same radius R and the same height (also R). Kind of a squat-looking cone.

Now invert the cone, so that its tip is on the plane and the fattest part is farthest away.



What Archimedes found is that for *any* cross-section parallel to the xy-plane, at some height h, the slices have the property that the area of the sphere's cross-section plus the area of the cone's cross-section is a constant. Furthermore, together they equal the area of the cross-section of a cylinder with radius R (i.e. πR^2).

We calculate the radius of the cross-section of the sphere (which is a circle) at height h from the xy-plane as

$$r_s^2 + h^2 = R^2$$
$$r_s^2 = R^2 - h^2$$

Since the cone has total height and radius both equal to R, and since it's inverted, at any height h, the radius at that height is equal to h

$$r_c^2 = h^2$$

Addition of the areas of the two cross-sections gives

$$\pi r_s^2 + \pi r_c^2 = \pi(R^2 - h^2) + \pi(h^2) = \pi R^2$$

which is equal to the area of the cross-section of a cylinder with radius R.

When we add up all the slices from bottom to top, since the cross-sections add to πR^2 at any height, the total of all the cross-sections, which are volumes, add up too.

Hemisphere plus cone equals cylinder.

Since the volumes of the cylinder and cone are πR^3 and $\pi R^3/3$, respectively, the volume of the hemisphere is the difference, and the volume of the sphere is twice that, namely $4/3\pi R^3$.

Method of disks

We want to explore modern methods using integration to obtain this result.

Here is one from first-year calculus. Consider a function $f(x)$, and imagine that we rotate the graph of the function around the x-axis. The rotational symmetry allows us to calculate the volume as a single integral.

The method is to "slice" the volume using slices perpendicular to the x-axis. These cross-sections are circles, because of the rotation. Every slice has volume equal to the area of that slice times the width dx, so the total volume is obtained by adding up all of the slices:

$$V = \pi \int [f(x)]^2 \ dx$$

Our specific problem is to obtain a sphere, which we get by rotating the top half of the graph of a circle

$$f(x) = +\sqrt{R^2 - x^2}$$

The bounds are $x = -R \to R$. So the integral is just

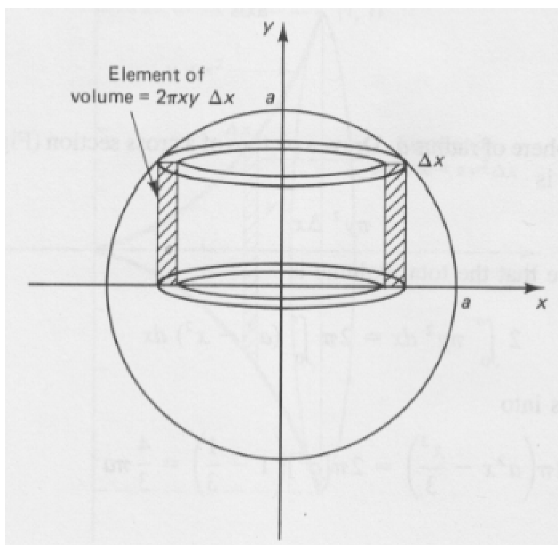
$$V = \pi \int_{-R}^{R} (R^2 - x^2)\ dx$$

$$= \pi\ [\ (R^2x - \frac{x^3}{3})\Big|_{-R}^{R}\ ]$$

$$= \pi\ [\ (R^3 - \frac{1}{3}R^3) - (-R^3 - -\frac{1}{3}R^3)\ ]\ = \frac{4}{3}\pi R^3$$

Method of shells

Here is a picture of what we're doing, from Hamming's Calculus text. The notation is different but the idea is the same.



We'll work with the hemisphere, above the xy-plane.

Let's divide the sphere up into concentric cylinders or shells, and let r vary from $0 \to R$. The circumference of the shell at each point is

$$C = 2\pi r$$

and the height of each is

$$h = \sqrt{R^2 - r^2}$$

The volume of each very thin cylinder is

$$dV = Ch\ dr = 2\pi r\sqrt{R^2 - r^2}\ dr$$

and we want

$$\int_{r=0}^{r=R} 2\pi r\sqrt{R^2 - r^2}\ dr$$

$$= -\frac{2}{3}\pi(R^2 - r^2)^{3/2}\Big|_0^R = -\frac{2}{3}\pi\ [\ -(R^2)^{3/2}\ ] = \frac{2}{3}\pi R^3$$

Multiply by two to obtain the total volume.

Double integral

We now move to multi-variable calculus, and consider the sphere as a function $f(x, y)$ of points in the xy-plane. The graph of this function is plotted on the z-axis, above the points. (Here we consider one octant of the hemisphere, the part where $x, y, z > 0$). The equation is

$$R^2 = x^2 + y^2 + z^2$$
$$z = f(x, y) = \sqrt{R^2 - x^2 - y^2}$$

The "shadow" of the sphere is a quadrant of the circle of radius R (since $z = 0$ in the xy-plane). For each little area element dA inside the shadow, we find the distance up to the graph of the function as $f(x, y)$.

It's going to be an adventure, but suppose we try using Cartesian coordinates, and integrate first with respect to y and then with respect to x. We will have

The volume is this double integral

$$V = \int_{x=0}^{x=R} \int_{y=0}^{\sqrt{R^2-x^2}} \sqrt{R^2 - x^2 - y^2}\ dy\ dx$$

This looks rather forbidding, but it won't be too bad. Let us first, for the moment, substitute $a = \sqrt{R^2 - x^2}$ and keep in mind that this is a constant for the inner integral. So that part is now

$$\int_{y=0}^{a} \sqrt{a^2 - y^2}\ dy$$

A little investigation reveals that the right approach to this is a trig substitution, namely

$$y = a \sin\theta$$
$$dy = a \cos\theta\ d\theta$$

We will have

$$\int \sqrt{a^2 - a^2 \sin^2\theta}\ a\cos\theta\ d\theta$$
$$\int a^2 \sqrt{1 - \sin^2\theta} \cos\theta\ d\theta$$
$$\int a^2 \cos^2\theta\ d\theta$$

The crucial thing is to change the limits. We need to find the value of θ at the old limits for y, namely

$$y = 0, \quad 0 = a\sin\theta, \quad \theta = 0$$

$$y = a, \quad a = a \sin\theta, \quad \theta = \frac{\pi}{2}$$

So

$$\int_0^{\pi/2} a^2 \cos^2\theta \; d\theta$$

$$= a^2 \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta) \; d\theta$$

$$= a^2 \; \frac{1}{2} \; (\theta + \frac{1}{2}\sin 2\theta) \; \Big|_0^{\pi/2}$$

$$= \frac{a^2}{2}(\frac{\pi}{2} + 0 - 0 - 0)$$

$$= a^2\frac{\pi}{4} = (R^2 - x^2)\frac{\pi}{4}$$

So then finally the outer integral is

$$\int_{x=0}^{x=R} (R^2 - x^2)\frac{\pi}{4} \; dx$$

$$= \frac{\pi}{4}(R^2 x - \frac{1}{3}x^3) \; \Big|_0^R$$

$$= \frac{\pi}{4} \; \frac{2}{3}R^3 = \frac{1}{6}\pi R^3$$

Recall that this is for one octant, so for the whole sphere we multiply by 8 and obtain:

$$= \frac{8}{6}\pi R^3 = \frac{4}{3}\pi R^3$$

Triple integral

For the triple integral, we are integrating the volume element over the entire range of values inside the sphere.

$$V = \iiint dV$$

We can do this integral relatively easily in both cylindrical and spherical coordinates. For completeness, I'm going to try it in Cartesian coordinates as well (without a net).

Consider the one-eighth of the sphere that has $x > 0, y > 0, z > 0$.

Let's work from the outside in. We will do x last. The limits on x are $x = 0 \to R$. Simple enough. The shadow of the sphere in the xy-plane is a circle with radius R and so the limits on y are $y = 0 \to \sqrt{R^2 - x^2}$.

And then, naturally enough, the limits on z are $z = 0 \to \sqrt{R^2 - x^2 - y^2}$.

So our integral is

$$\int_{x=0}^{R} \int_{y=0}^{\sqrt{R^2-x^2}} \int_{z=0}^{\sqrt{R^2-x^2-y^2}} dz\ dy\ dx$$

The inner integral is trivial. The middle integral is then

$$\int_{y=0}^{\sqrt{R^2-x^2}} \sqrt{R^2 - x^2 - y^2}\ dy$$

And this is where it gets interesting. Notice that x is constant here. So, setting $a^2 = R^2 - x^2$, the integral is of the form

$$\int \sqrt{a^2 - y^2}\ dy$$

And (you can look it up) this actually does have an analytical solution, namely

$$\frac{a^2}{2}\sin^{-1}\frac{y}{a} + \frac{y\sqrt{a^2 - y^2}}{2}$$

Plugging in the upper limit $y = \sqrt{R^2 - x^2}$ we obtain

$$\frac{R^2 - x^2}{2}\sin^{-1}\frac{\sqrt{R^2 - x^2}}{\sqrt{R^2 - x^2}} + \frac{\sqrt{R^2 - x^2}\sqrt{R^2 - x^2 - (R^2 - x^2)}}{2}$$

Luckily, the second term is zero. And since $\pi/2 = \sin^{-1}(1)$, for this part we have just

$$= \frac{R^2 - x^2}{2} \cdot \frac{\pi}{2}$$

At the lower, limit ($y = 0$), the first term includes $\sin^{-1}(0)$, which equals zero, and the second term has a factor of y, so the whole thing is just zero.

Finally, we come to the outer integral, which is

$$\begin{aligned} &\frac{\pi}{4}\int_0^R R^2 - x^2\ dx \\ &= \frac{\pi}{4} \cdot \frac{2}{3}R^3 \\ &= \frac{\pi}{6}R^3 \end{aligned}$$

Since there are eight such volumes in the whole sphere, we obtain the familiar answer.

Triple integral in cylindrical coordinates

The equation of the sphere in Cartesian coordinates is:

$$R^2 = x^2 + y^2 + z^2$$

Converting to polar coordinates we have

$$\begin{aligned} r^2 &= x^2 + y^2 \\ R^2 &= z^2 + r^2 \\ z &= \sqrt{R^2 - r^2} \end{aligned}$$

Recall that the area element in polar coordinates is $dA = r\ dr\ d\theta$.

If we integrate first with respect to z we have

$$\int \int \int dz\ r\ dr\ d\theta$$

The limits on r and θ are as usual:

$$\int_0^{2\pi} \int_0^R \int dz\ r\ dr\ d\theta$$

In particular, we integrate over the entire shadow of the sphere on the xy-plane. Now, for each value of r and θ we must find the limits on z. From above, we get

$$\int_0^{2\pi} \int_0^R \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} dz \ r \ dr \ d\theta$$

The inner integral is just

$$z \ \Big|_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} = 2(\sqrt{R^2-r^2})$$

So the middle integral is then

$$\int_0^R 2(\sqrt{R^2-r^2}) \ r \ dr$$

$$= -\frac{2}{3}(R^2-r^2)^{3/2} \ \Big|_0^R = \frac{2}{3}R^3$$

The outer integral is trivial

$$\int_0^{2\pi} \frac{2}{3}R^3 \ d\theta = \frac{4}{3}\pi R^3$$

Triple integral in spherical coordinates

Here is the figure from *How to Ace Calculus*

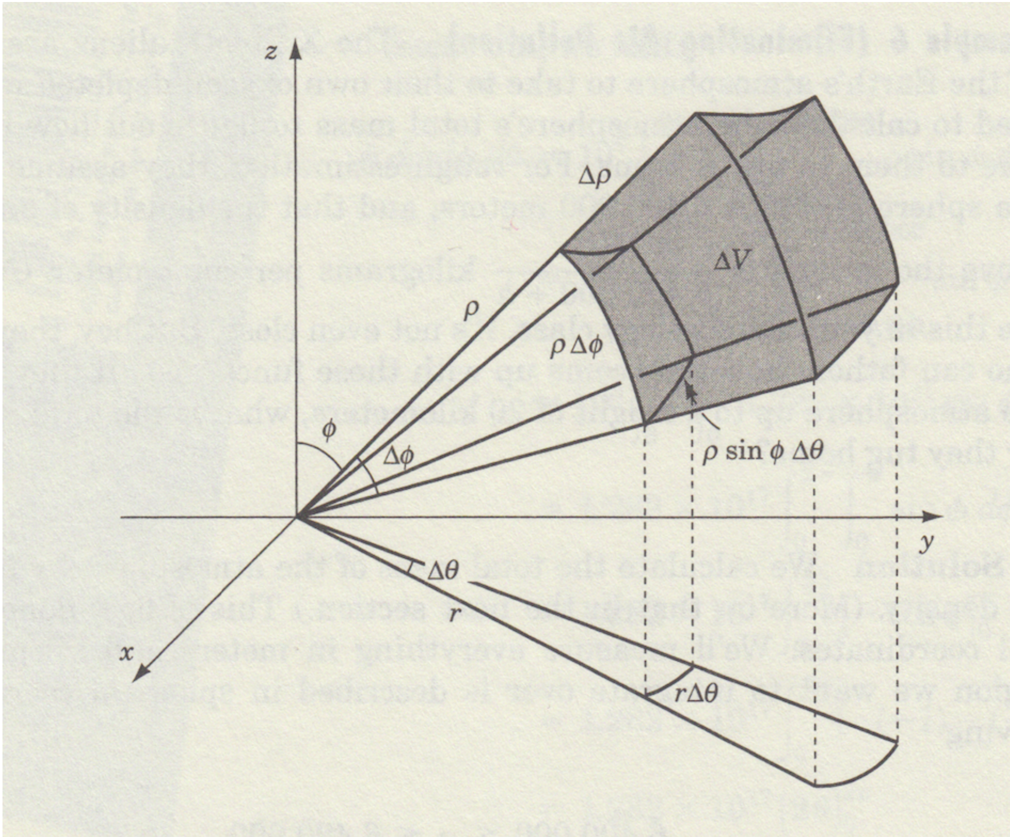


Figure 9.23 Varying each coordinate in spherical coordinates.

In spherical coordinates, we see that the volume element is

$$dV = \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$

And having used that, our work is essentially done. We just set up the integral

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$

The peculiarity of this is that the limits on ρ do not depend on the angles ϕ and θ. So now the inner integral is just

$$= \int_0^R \rho^2 \sin\phi \; d\rho = \frac{1}{3} R^3 \sin\phi$$

And the term $R^3/3$ is a constant. So we put that aside and evaluate the middle integral

$$\int_0^\pi \sin\phi \; d\phi$$

$$= -\cos\phi \; \Big|_0^\pi = -(-2) = 2$$

So in the end we have

$$2\pi \cdot 2 \cdot \frac{1}{3} R^3 = \frac{4}{3} \pi R^3$$

Method of spheres

The last method is my favorite. I'm not sure what it's official name is, so I just made one up.

This is an integral in a single variable. We think about how the volume of the sphere depends on r ($r = 0 \to R$). An incremental change dr changes the volume by adding a thin shell (sphere?) of volume equal to the surface area of the sphere ($4\pi r^2$) times dr. That is

$$dV = 4\pi r^2 \; dr$$

$$V = \int dV = \int_0^R 4\pi r^2 \; dr$$

$$= 4\pi \;\; \frac{1}{3} r^3 \; \Big|_0^R = \frac{4}{3} \pi R^3$$

Again.

Deriving the formula we used

Up above in the first section of volume integrals (Cartesian coordinates) we used this formula

$$\int \sqrt{a^2 - y^2}\ dy = \frac{a^2}{2}\sin^{-1}\frac{y}{a} + \frac{y\sqrt{a^2 - y^2}}{2}$$

Let me re-write it in a more familiar but equivalent form

$$\int \sqrt{a^2 - x^2}\ dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2 - x^2}}{2}$$

We use a simple trig substitution.

$$\frac{x}{a} = \sin t$$

$$x = a\ \sin t$$

$$dx = a\ \cos t\ dt$$

Using Pythagoras

$$\sqrt{a^2 - x^2} = a \cos t$$

Substituting

$$\int \sqrt{a^2 - x^2}\ dx$$

$$= a \cos t\ a \cos t\ dt$$

$$= a^2 \cos^2 t\ dt$$

An old friend! I'm not going to work this one out from scratch, we've seen it in other write-ups. One of the equivalent forms for this integral is

$$\int \cos^2 t \; dt = \frac{1}{2}(t + \sin t \cos t)$$

Picking up the outside factor of a^2 and substituting we obtain

$$\begin{aligned} &= a^2 \; \frac{1}{2}(\sin^{-1}\frac{x}{a} + \frac{x}{a}\frac{\sqrt{a^2 - x^2}}{a}) \\ &= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2 - x^2}}{2} \end{aligned}$$