Conservation of Energy

We're going to compute the time-derivative of the energy E, where E is the sum of kinetic and potential energy. We start with this basic identity

$$\mathbf{v} = \dot{\mathbf{r}}$$

Here $\dot{\mathbf{r}}$ is the derivative of the position vector $\mathbf{r}$ with respect to time. It is equal to the velocity, and its magnitude is equal to the speed. Now we write an expression for the kinetic energy ($1/2 \times$ mass $\times$ velocity squared).

$$K = \frac{1}{2}m\ \|\mathbf{v}\|^2 = \frac{1}{2}m\ \|\dot{\mathbf{r}}\|^2$$

Note that for any vector, its magnitude squared is equal to the dot product with itself

$$\|\dot{\mathbf{r}}\|^2 = \|\dot{\mathbf{r}}\|\|\dot{\mathbf{r}}\| = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$$

$$K = \frac{1}{2}m\ \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$$

Taking the time-derivative of the kinetic energy

$$\frac{d}{dt}K = \frac{1}{2}m\ \frac{d}{dt}(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) = m\ \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}$$

The last part comes from the product rule. We have two identical terms of $\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}$, so we cancel a factor of 2 to obtain the result above.

We recall Newton's second law

$$\mathbf{F} = m\mathbf{a} = m\frac{d^2}{dt^2}\mathbf{r} = m\ddot{\mathbf{r}}$$

$$\boxed{\frac{d}{dt}K = \dot{\mathbf{r}} \cdot \mathbf{F}} \tag{1}$$

In the next part, we look at the potential energy. Recall that V is a function of $\mathbf{r}$

$$V = V(\mathbf{r})$$

and so the gradient of V is

$$\nabla V(\mathbf{r}) = < \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} >$$

while $\dot{\mathbf{r}}$ is just

$$\dot{\mathbf{r}} = < \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} >$$

We work backward from the result to the time-derivative of the potential energy.

$$\nabla V \cdot \dot{\mathbf{r}} = < \frac{\partial V}{\partial x}\frac{dx}{dt}, \frac{\partial V}{\partial y}\frac{dy}{dt}, \frac{\partial V}{\partial z}\frac{dz}{dt} = \frac{d}{dt}V$$

$$\boxed{\nabla V \cdot \dot{\mathbf{r}} = \frac{d}{dt}V} \tag{2}$$

Now we just combine our results

$$\frac{d}{dt}E = \frac{d}{dt}K + \frac{d}{dt}V = \dot{\mathbf{r}} \cdot \mathbf{F} + \nabla V \cdot \dot{\mathbf{r}}$$

But

$$\boxed{\mathbf{F} = -\nabla V(\mathbf{r})} \tag{3}$$

so

$$\frac{d}{dt}E = \dot{\mathbf{r}} \cdot -(\nabla V) + \nabla V \cdot \dot{\mathbf{r}} = 0$$