Riemann sums for positive integer powers

Courant and John describe a variation on Riemann sums using intervals of unequal (graduated) width. This "trick" allows them to derive the formula for

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\int_a^b x^n \, dx = \frac{b^{n+1} - a^{n+1}}{n+1}$$

for all natural numbers n first, and then with some elaborations, for real n except $n = -1$.

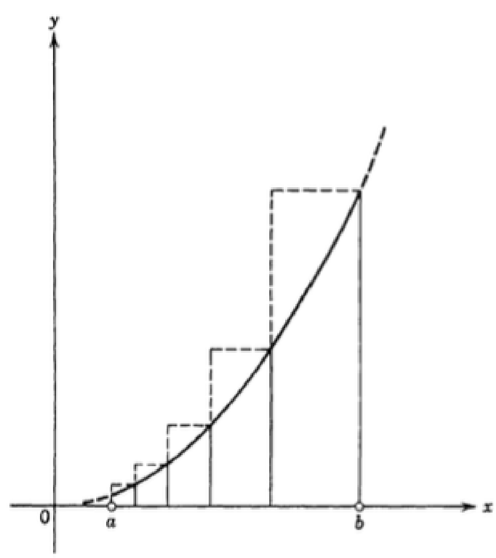

Figure 2.13 Area under a parabolic arc by geometric subdivision.

We subdivide the interval $[a, b]$ by points with spacing that increases by a factor of q at each step

$$a,\ aq,\ aq^2,\ \ldots\ aq^{n-1},\ aq^n$$

At the final, nth step we have

$$aq^n = b$$

Solving for the common ratio q we have

$$q = (b/a)^{1/n}$$

The points of division are

$$x_i = aq^i$$

The width of the ith rectangle is

$$\Delta x_i = aq^i - aq^{i-1} = aq^i\ (1 - \frac{1}{q})$$

$$= aq^i\ [\ \frac{q-1}{q}\ ]$$

The widest rectangle is the last one

$$\Delta x_n = aq^n\ [\ \frac{q-1}{q}\ ]$$

$$= b\ [\ \frac{q-1}{q}\ ]$$

In the usual way, we will let the number of rectangles $n \to \infty$.

At the same time, since

$$q = (b/a)^{1/n}$$

then $q \to 1$.

So then $\Delta x_n \to 0$, and so do all the rectangles, which are smaller.

The function of interest is to raise x to the positive integer power p. For each rectangle, the area is

$$A_i = x_i^p \ \Delta x_i$$

$$= (aq^i)^p \ aq^i \ [\ \frac{q-1}{q} \ ]$$

$$= a^{p+1} \ (q^i)^{p+1} \ [\ \frac{q-1}{q} \ ]$$

$$= a^{p+1} \ (q^{p+1})^i \ [\ \frac{q-1}{q} \ ]$$

For the integral, we need to add all these up (from $i = 1$ to $i = n$):

$$I = \sum_{i=1}^{n} a^{p+1} \ (q^{p+1})^i \ [\ \frac{q-1}{q} \ ]$$

We can take out values that don't depend on i from the summation:

$$I = a^{p+1} \ [\ \frac{q-1}{q} \ ] \ \sum_{i=1}^{n} (q^{p+1})^i$$

Recall that for a geometric series with common ratio r the nth sum (starting from $i = 0$) is

$$S_n = 1 + r + r^2 \cdots + r^n = \sum_{i=0}^{n} r^i$$

$$= \frac{1 - r^n}{1 - r} = \frac{r^n - 1}{r - 1}$$

Substituting q for r:

$$S_n = \frac{q^n - 1}{q - 1}$$

For the expression above

$$\sum_{i=1}^{n} (q^{p+1})^i$$

we factor out one q^{p+1} so as to start from $i = 0$

$$= q^{p+1} \ \sum_{i=0}^{n} (q^{p+1})^i$$

and then the common ratio is q^{p+1} and the sum is

$$\sum_{i=0}^{n} (q^{p+1})^i = \frac{(q^{p+1})^n - 1}{q^{p+1} - 1} = \frac{q^{n(p+1)} - 1}{q^{p+1} - 1}$$

The whole sum or integral I that we seek is

$$I = a^{p+1} \ [\ \frac{q-1}{q} \ ] \ q^{p+1} \ \frac{q^{n(p+1)} - 1}{q^{p+1} - 1}$$

$$= a^{p+1} \ (q-1) \ q^p \ \frac{q^{n(p+1)} - 1}{q^{p+1} - 1}$$

$$= a^{p+1} \ (q-1) \ q^p \ \frac{(b/a)^{p+1} - 1}{q^{p+1} - 1}$$

Since

$$a^{p+1} \ [\ (b/a)^{p+1} - 1 \ ] \ = b^{p+1} - a^{p+1}$$

we obtain

$$I = \ [\ b^{p+1} - a^{p+1} \ ] \ q^p \ \frac{q-1}{q^{p+1} - 1}$$

Referring to the sum for a geometric progression again, we have from above

$$S_n = \frac{q^n - 1}{q - 1}$$

So (for $q \neq 1$) and $n = p + 1$, the inverse of that is what we have for the right-hand term

$$\frac{q-1}{q^{p+1}-1} = \frac{1}{S_{p+1}}$$

where

$$S_{p+1} = 1 + q + q^2 + \cdots + q^{p+1}$$

Substituting

$$I = \ [\ b^{p+1} - a^{p+1}\ ]\ q^p\ \frac{1}{1 + q + q^2 + \cdots + q^{p+1}}$$

As we saw near the beginning, as $n \to \infty$, $q \to 1$, and so do all the powers of q so the term

$$q^p = 1$$

and also

$$1 + q + q^2 + \cdots + q^{p+1} = p + 1$$

so the fraction is just equal to $1/(p+1)$ and we have finally:

$$I = \ [\ b^{p+1} - a^{p+1}\ ]\ \frac{1}{p+1}$$

which is what we sought to prove.