Auroux 20, FTC

At the end, we'll get to the fundamental theorem of calculus for line integrals, which is that

$$\int_{P1}^{P2} \nabla \mathbf{F} \cdot d\mathbf{r} = f_{P2} - f_{P1} \iff \mathbf{F} = \nabla f$$

$$\int_{P1}^{P2} f_x \ dx + f_y \ dy = f(P2) - f(P1)$$

Example 1. We usually have x and y as functions of a parameter t. Also we will have a vector field F where

$$F = < M, N >$$

$$F = < P, Q, R >$$

and we are interested in the integral along the curve

$$\int_C F \cdot dr = \int_C F \cdot T ds = \int_C P dx + Q dy + R dz$$

Suppose

$$F = < x, y, z >$$

and we have equations for $x(t), y(t), z(t)$

$$x = t, \quad y = t, \quad z = 2t^2$$

$$\frac{dr}{dt} = < \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} > = < 1, 1, 4t >$$

$$\int_C F \cdot dr = \int_C < t, t, 2t^2 > \ \cdot < 1, 1, 4t > dt = \int_C (2t + 8t^3) dt = t^2 + 2t^4$$

Evaluate from say, $t = 0$ to $t = 1$

$$t^2 + 2t^4 = 3$$

Example 2. Suppose F is $< y, x >$ and C is a sector of the unit circle between $0 <= \theta <= \frac{\pi}{4}$, so that we start at the origin and go out along the radius, along the circle, and then come back to the origin. Break the curve up into three parts.

$$\int_{C1} = \int_0^1 Mdx + Ndy = \int_0^1 ydx + xdy = 0$$

Notice that both $y = 0$ and $dy = 0$, since we're going out along y = 0. Also, notice that F is $< 0, x >$, so that $F \perp dr$ and so $F \cdot dr = 0$. For C2 we are on the unit circle going from $(0, 1)$ to $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

$$\int_C ydx + xdy$$

The natural thing to do here is to change variables

$$x = cos\ \theta, \quad y = sin\ \theta$$

$$dx = -sin\ \theta, \quad dy = cos\ \theta$$

$$\int_{C2} ydx + xdy = \int_{C2} (-sin^2\ \theta + cos^2\ \theta)\ d\theta$$

Perhaps you can recognize the double-angle formula?

$$\int_{C2} cos\ 2\theta\ d\theta = \frac{1}{2} sin\ 2\theta$$

$$[\ \frac{1}{2} sin\ 2\theta\ ]\Big|_0^{\pi/4} = \frac{1}{2}$$

For C3 we are moving back along the radius from $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ to $(0, 0)$

$$\int_C ydx + xdy$$

Notice that we are moving along the line $y = x$ so $dy = dx$ and

$$\int_C ydx + xdy = 2 \int_C xdx = x^2\ \Big|_{\frac{1}{\sqrt{2}}}^0 = -\frac{1}{2}$$

The total integral is the sum which is $0 + \frac{1}{2} - \frac{1}{2} = 0$. The reason for this special result is that F is the gradient of a potential function. We are just going to guess what that function is

$$f(x, y) = xy$$

The gradient of f is

$$\nabla f = < f_x, f_y > = < y, x >$$

The fundamental theorem of calculus for line integrals with a conservative vector field is

$$\int_C F \cdot dr = f(P_1) - f(P_2)$$

The example is a closed curve ($P_1 = P_2$) so the difference is just 0.

The question you might ask is how do we know that $f(x, y) = xy$ is a potential function? Answer: a conservative vector field has zero curl: $N_x = M_y$.

To restate

$$\int_{P1}^{P2} \nabla \mathbf{F} \cdot d\mathbf{r} = f_{P2} - f_{P1} \iff \mathbf{F} = \nabla f$$

Here's a proof

$$\int_C \nabla \mathbf{F} \cdot d\mathbf{r} = \int_C f_x \; dx + f_y \; dy$$

$$x = x(t), \quad dx = x'(t) \; dt$$

$$y = y(t), \quad dy = y'(t) \; dt$$

$$\int_{t_0}^{t_1} (f_x \; \frac{dx}{dt} + f_y \; \frac{dy}{dt}) \; dt = \int_{t_0}^{t_1} \frac{df}{dt} \; dt = \int_{t_0}^{t_1} df = f(t_1) = f(t_0)$$

Remember the field in example 2: $< y, x >$. Can we think of a function f whose $df/dx = y$ and $df/dy = x$? How about $f = xy$! Repeat:

Section 1:

$$P_0 = (0, 0); \quad P_1 = (1, 0); \quad f(P_1) - f(P_0) = 0 - 0 = 0$$

Section 2:

$$P_0 = (1, 0); \quad P_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}); \quad f(P_1) - f(P_0) = \frac{1}{2} - 0 = \frac{1}{2}$$

Section 3:

$$P_0 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}); \quad P_1 = (0, 0); \quad f(P_1) - f(P_0) = 0 - \frac{1}{2} = -\frac{1}{2}$$

And the sum is, of course, 0.

Consider $F = < -y, x >$. F is not the gradient of some function, because $N_x \neq M_y$. Very important: Curl(F) $= N_x - M_y$. For a conservative vector field, the curl is zero.

(1) if F is conservative, $\int_C F \cdot dr = 0$ for all closed paths; (2) $\int_C F \cdot dr$ is path-independent; (3) $F = \nabla f$ and (4) $M \; dx + N \; dy$ is an *exact* differential: $df = f_x \; dx + f_y \; dy$.