Laurent series

The Laurent series of a complex function $f(z)$ is a representation of that function as a power series which includes terms of negative degree (wikipedia).

For a function expanded about a point z_0 the series is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

To determine a particular coefficient a_k, multiply both sides of the above expression by $1/(z - z_0)^{k+1}$ and integrate around a closed path

$$\oint \frac{f(z)}{(z - z_0)^{k+1}} \, dz = \sum_{n=-\infty}^{\infty} \oint a_n \frac{(z - z_0)^n}{(z - z_0)^{k+1}} \, dz$$

Recall that all the a_n are constants and can come outside the integrals.

$$= \sum_{n=-\infty}^{\infty} a_n \oint \frac{(z - z_0)^n}{(z - z_0)^{k+1}} \, dz$$

Only one term in the infinite series on the right is non-zero. That is the term where $n = k$

$$= a_n \oint \frac{1}{(z - z_0)} \, dz$$

We know this one, it is:

$$= a_n \; 2\pi i$$

(If you doubt this, substitute $w = z - z_0$, so $dw = dz$ and we have

$$\oint \frac{dw}{w} = 2\pi i$$

Thus

$$2\pi i \; a_n = \oint \frac{f(z)}{(z - z_0)^{n+1}} \; dz$$

$$2\pi i \; a_{-1} = \oint \frac{f(z)}{z - z_0} \; dz$$