Quadratic

We want to solve

$$y = ax^2 + bx + c$$

We suppose that $a \neq 0$ (since this is a quadratic, after all), so we can divide by a on both sides:

$$\frac{y}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$$

Rearranging a bit

$$\frac{y}{a} - \frac{c}{a} = x^2 + \frac{b}{a}x$$

Our particular interest is in the value(s) for x when $y = 0$ so that leaves:

$$-\frac{c}{a} = x^2 + \frac{b}{a}x$$

The crucial insight is to see that the right-hand side is nearly a perfect square

$$x^2 + \frac{b}{a}x + ??$$

similar to

$$(x + m)^2 = x^2 + x(2m) + m^2$$

The coefficient of x on the right-hand side is $2m$.

In our problem we have b/a so we decide to try $2m = b/a$, which works

$$(x + \frac{b}{2a})^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$$

Now that we know what to do, go back to the original equation

$$-\frac{c}{a} = x^2 + \frac{b}{a}x$$

add the same term (what we called m^2) to both sides

$$-\frac{c}{a} + \frac{b^2}{4a^2} = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$$

write our perfect square

$$-\frac{c}{a} + \frac{b^2}{4a^2} = (x + \frac{b}{2a})^2$$

Almost there. Multiply both sides by $4a^2$

$$-4ac + b^2 = 4a^2(x + \frac{b}{2a})^2$$

That should look familiar.

Switch the order of terms on the left and take (both negative and positive) square roots:

$$\sqrt{b^2 - 4ac} = \pm\ 2a(x + \frac{b}{2a})$$

$$\pm\ \sqrt{b^2 - 4ac} = 2a(x + \frac{b}{2a})$$

$$-\frac{b}{2a} \pm \ \frac{\sqrt{b^2 - 4ac}}{2a} = x$$

$$x = \frac{-b \pm\ \sqrt{b^2 - 4ac}}{2a}$$

version 2

Another approach that I've seen which is counterintuitive (at least to start with). Start by saying explicitly we want the zeroes:

$$ax^2 + bx + c = 0$$
$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Make the inspired substitution:

$$x = t - \frac{b}{2a}$$

Then

$$x^2 = t^2 - \frac{b}{a}t + \frac{b^2}{4a^2}$$
$$\frac{b}{a}x = \frac{b}{a}t - \frac{b^2}{2a^2}$$

So we have then:

$$t^2 - \frac{b}{a}t + \frac{b^2}{4a^2} + \frac{b}{a}t - \frac{b^2}{2a^2} = -\frac{c}{a}$$

One set of cancellations

$$t^2 + \frac{b^2}{4a^2} - \frac{b^2}{2a^2} = -\frac{c}{a}$$

and another, with rearrangement

$$t^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Place over a common denominator and take the square roots

$$t^2 = \frac{4ac - b^2}{4a^2}$$

$$t = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Reverse the substitution:

$$x = \pm\frac{\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a}$$

Finally

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

version 3

Here is a third approach. We wish to find the values of x that make this statement true:

$$ax^2 + bx + c = 0$$
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Suppose that $x = m$ and $x = n$ are the two solutions. That means:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = (x - m)(x - n)$$

Now, expand the right-hand side

$$x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - (m + n)x + mn$$

it is clear that the coefficients of x on each side must be equal to each other:

$$-(m + n) = \frac{b}{a}$$
$$m + n = -\frac{b}{a}$$

Next we work on the constant term mn. A little algebra gives:

$$(m+n)^2 = m^2 + 2mn + n^2$$

$$(m-n)^2 = m^2 - 2mn + n^2$$

Subtract

$$(m+n)^2 - (m-n)^2 = 4mn$$

$$mn = (m+n)^2 - (m-n)^2 - 3mn$$

Going back to what we had above

$$x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - (m+n)x + mn$$

We see that

$$\frac{c}{a} = mn = (m+n)^2 - (m-n)^2 - 3mn$$

So

$$(m-n)^2 = \frac{c}{a} - (m+n)^2 + 3mn$$

But $c/a = mn$ so

$$(m-n)^2 = 4\frac{c}{a} - (m+n)^2$$

Recall that $m + n = -b/a$

$$(m-n)^2 = 4\frac{c}{a} - (\frac{b}{a})^2$$

$$m - n = \sqrt{4\frac{c}{a} - (\frac{b}{a})^2}$$

Together with what we found above

$$m + n = -\frac{b}{a}$$

When solved simultaneously, these two equations give the result we seek. Addition gives

$$2m = \sqrt{4\frac{c}{a} - (\frac{b}{a})^2} - \frac{b}{a}$$

Subtraction gives

$$2n = -\frac{b}{a} - \sqrt{4\frac{c}{a} - (\frac{b}{a})^2}$$

which has just flipped the sign on the square root term. Thus m and n differ only in the sign of that term.

Let's clean up the result

$$2m = \sqrt{\frac{4ac}{a^2} - (\frac{b}{a})^2} - \frac{b}{a}$$

Factor out the $\sqrt{1/a^2}$

$$2m = \frac{\sqrt{4ac - b^2}}{a} - \frac{b}{a}$$

$$m = \frac{-b + \sqrt{4ac - b^2}}{2a}$$