The Best of Calculus

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March 22, 2019

Contents

Part I

Archimedes

Chapter 1

Introduction

This was originally going to be a short book, an exploration of problems like the volume of the cone and sphere, or even just the area of a circle, with some simple physics thrown in. These questions contain within them the heart of calculus: infinities both large and small. I imagine myself looking over Archimedes' shoulder as he explains it to me.

I wrote most of the early chapters originally as short explanations for my son Sean as he studied calculus in high school. It bothers me that so often the good stuff gets left out — the ideas which make you go ... wow. More than five years later, I still find a lot of pleasure in trying to understand what Kepler and Newton did. It took a genius to figure it out the first time, but it is within anyone's grasp to appreciate what they found.

Then I thought, why not include other favorite problems like the area of the ellipse, the "headlight" problem for the parabola, or the reflective property of the ellipse, and the length and area under the cycloid curve (the "light on a bicycle wheel"). These are problems where calculus easily produces answers that can be checked by more elaborate geometric arguments. So here we are, with a somewhat longer book.

In the introduction to his book *Calculus*, Morris Kline says

> Anyone who adds to the plethora of introductory calculus texts owes an explanation, if not an apology, to the mathematical community.

I think of this book as a form of ultralight backpacking. We shed weight so as to ascend peaks rapidly, skimming the best of calculus — focusing on geometry and physics, and slinging differentials with abandon. Epsilon appears mainly as the physics constant ϵ_o. Starting with an intuitive notion of adding up many small pieces, we put integrals to work early solving problems.

Going fast allows time to get a view of sophisticated topics, among others, line integrals for work and flux, Newton's proof that a spherical mass acts as a point mass, and integration of a parametrized surface like the torus. Not to mention Kepler's Laws, and a derivation of the Gaussian distribution from first principles.

We do not disdain proof. On the contrary, *interesting* proofs are much to be admired. We prove the Pythagorean Theorem, and the quotient rule for derivatives, as well as Green's Theorem. There is a fun chapter on induction.

My favorite authors are Morris Kline, Richard Hamming, and Gil Strang. Sylvanus Thompson's simple book is my favorite first text, and it's even a Project Gutenberg project:

https://www.gutenberg.org/files/33283/33283-pdf.pdf

Having said what I like, briefly, here are some things I don't like.

The rigorous approach to calculus pioneered by Cauchy in the 1820's and exported to American schools by Richard Courant in the 1940's is a bad idea. We must motivate rigorous proof by demonstrating utility first, as Stewart says, "proofs come *after* understanding." Courant's way is the way to teach the subject the second *or third* time through.

Thompson:

> You don't forbid the use of a watch to every person who does not know how to make one. You don't object to the musician playing on a violin that he has not himself constructed. You don't teach the rules of syntax to children until they have already become fluent in the use of speech. It would be equally absurd to require general rigid demonstrations to be expounded to beginners in the calculus.

A second thing I dislike is calculus problems that are mostly arithmetic. Calculus comes from bright ideas, not complicated ones; if the math is gratuitously difficult, it's usually *not* a good problem. And a good problem often is one with a physical or practical foundation. Having said that, if a course could integrate elementary programming with calculus, I would be very happy.

I express my sincere thanks to the authors of my favorite books, which are listed in the references and mentioned at various places in the text. Almost everything in here was appropriated from them, and styled to my taste. I offer my profound thanks also to Eugene Colosimo, S.J. He was, for me, the best of a bunch of very special teachers.

If I stole your figure off the internet, I'm sorry. I intended to redraw it but have not yet found the time.

We start with my favorite mathematician, Archimedes.

[Update: Over time, I have added quite a bit of geometry. At this point, the title might as well be *Best of Calculus and Geometry.*]

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You can find the book on github here:

https://github.com/telliott99/Tex/tree/master/calculus_book

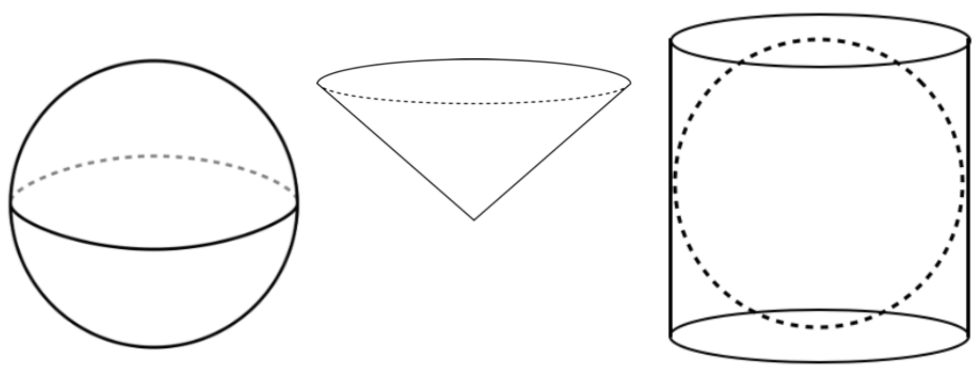
Chapter 2

Archimedes and the sphere

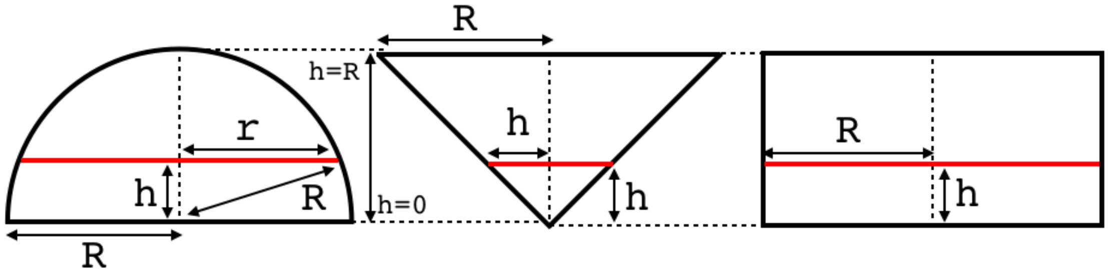
volume of the sphere: geometry

The very first derivation of the volume of a sphere was discovered by Archimedes. The following is his "simple" but subtle argument.

We compare a half-sphere and an inverted cone to a cylinder.



Below is a diagram showing a vertical cross-section through the center of each solid so we can visualize the geometry. The radius R is the same for all three. In addition, the cone has overall height equal to R.



Now we imagine making a horizontal slice through each solid at a height h, shown by the red lines. We will choose different values of h later and compare the results, the one shown here is arbitrary.

If you visualize this you should be able to see that each of these red slices is actually a circle. Any cross-section of a sphere is a circle. For the cylinder and cone, cross-sections perpendicular to the axis are circles as well.

The question we ask is: **what is the area for each horizontal slice**?

We need to determine the radius for each red circle. Moving right-to-left, the radius of the cylinder is just R. For the cone, the radius at each height h is equal to h (by similar triangles). And for the sphere, we use the Pythagorean theorem to find that

$$r^2 + h^2 = R^2$$

$$r^2 = R^2 - h^2$$

For more on this theorem see **here**.

The first insight of the proof is to recognize that the radius squared for the sphere's slice (r^2), plus the radius squared for the cone (h^2) is equal to R^2, the radius squared for the cylinder.

Since the areas are proportional to to the radius squared (namely $A =$

πr^2 and so on) and

$$\pi r^2 + \pi h^2 = \pi R^2$$

So the areas add too: **sphere plus cone equals cylinder**.

The second insight of the proof is to recognize that this property is invariant, it does not depend on which height we choose to make the slice. The three slices obtained at any height h add up like this. So if we imagine making a bunch of slices for each solid and adding them all up to find the volume, the volumes will add too.

This idea is now called Cavalieri's principle, though it was called the "method of indivisibles" before that.

The volume of the cylinder is simply πR^3. The volume of the cone is known to be one-third the area of the base times the height, or $1/3\ \pi R^3$. (See later for a derivation).

We subtract to find that the area of the half-sphere is $2/3\ \pi R^3$, and therefore the volume of the whole sphere is

$$V_{\text{sphere}} = \frac{4}{3}\pi R^3$$

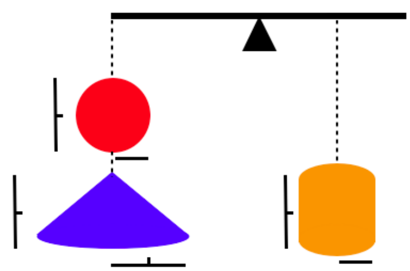
There is a bit of a trick here to hide the idea introduced in calculus, which makes this thinking rigorous. The sphere and cone have variable widths, which means that the radius will be different on the top of a slice compared to the bottom. Therefore, the slices have to be made very thin. In calculus they become infinitely thin, but we add up infinitely many of them.

Perhaps most interesting, Archimedes said that he discovered the correct result by balancing the three objects on a fulcrum.

According to Archimedes (in the Method, translation by Heath)

> For certain things which first became clear to me by a mechanical method had afterward to be demonstrated by geometry...it is of course easier, when we have previously acquired by the method some knowledge of questions, to supply the proof than it is to find the proof without any previous knowledge. This is a reason why, in the case of the theorems the proof of which Eudoxus was the first to discover, namely, that the cone is a third part of the cylinder, and the pyramid a third part of the prism, having the same base and equal height, we should give no small share of the credit to Democritus, who was the first to assert this truth...though he did not prove it.

I read somewhere that what Archimedes actually balanced is a set-up like that shown here



There are three factors that complicate our calculation: (i) we now have a single cone with radius $2r$ and height $2r$ (because it's doubled in both radius and height the cone's volume is increased by a factor of 2^3), (ii) the sphere and cone are twice as far from the fulcrum as the cylinder, and (iii) the cylinder is made out of something denser than the other objects (four times more dense).

Let πr^3 be one unit of volume, then the volumes are

$$
\begin{aligned}
\text{sphere} &= \frac{4}{3} \\
\text{cone} &= \frac{1}{3} \times 8 = \frac{8}{3} \\
\text{cylinder} &= 2
\end{aligned}
$$

That's $12/3 = 4$ for the sphere plus cone, and furthermore they count double since they are twice the distance from the fulcrum, giving 8 in our volume units. So the left side is $4\times$ the weight on the right side. However, we are told that the density of the material for the cylinder was four times that of the objects on the left. Hence, it should all balance.

I looked up some densities to try to guess what Archimedes used:

marble	2.56
sand	2.80
copper	8.63
silver	10.40
gold	19.30

How about marble and silver?

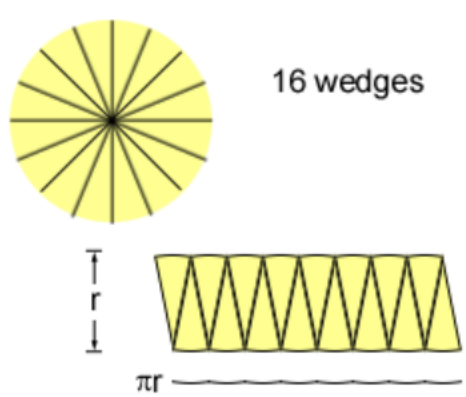
Chapter 3

Circle and cone

Area of the circle

This is a topic in geometry that comes even before the volume of the sphere, but I held off so as to start with Archimedes' most famous contribution.

We want the area of a circle. Imagine dividing a circle into wedges, like you might do with a pizza. Here, the pie has been divided into 16 parts.



Since the pieces are triangular, it is easy to stack them next to each other with the bases and tips alternating, as shown. Of course the bases are not straight, but have the same curvature as the edge of the circle.

The length of the short side is the radius, R, and the length of the long side is approximately one-half the circumference so

$$A = R \cdot \frac{1}{2} \cdot 2\pi R = \pi R^2$$

The trick is to imagine that we subdivide the circle into many slices. If there are infinitely many slices, the edges will be straight and this calculation becomes exact.

According to wikipedia

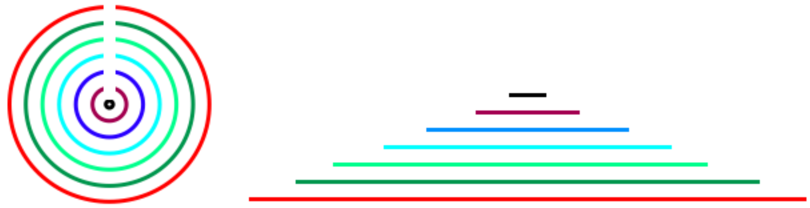
https://en.wikipedia.org/wiki/Area_of_a_circle

Eudoxus of Cnidus, born in the 5th century (408 BCE), proved that the area of a circle, like that of regular polygons, is proportional to both horizontal and vertical dimensions, and thus is proportional to the radius squared.

Somewhat later, it became clear that for a regular polyhedron, the area is equal to one-half the perimeter times the altitude from the center to each side (called the apothem). Allowing the polyhedron to achieve many, many sides, that formula gives $1/2 \cdot 2\pi r \cdot r = \pi r^2$.

The proof we gave above is very much like one attributed to Leonardo da Vinci, among others.

Another idea is to remove concentric strips from the edge and stack them.



We obtain a triangle of height r and base $2\pi r$ so its area is

$$A = \frac{1}{2}\, 2\pi r \cdot r = \pi r^2$$

This proof was given by Archimedes and is found in his *Measurement of a Circle*, proposition 1. However, many sources, including

`http://www.math.tamu.edu/~dallen/masters/Greek/eudoxus.pdf`

attribute the proof to Eudoxus, who was perhaps the second most famous mathematician of antiquity, and a colleague of Plato in Athens.

(The proof can be skipped if you get bogged down):

Let A be the area of the circle and T be the area of the triangle formed with base $2\pi r$ and height r. We will show that the following leads to a contradiction.

Assume $A > T$. That is, the difference $A - T$ is non-zero and positive: $A - T > 0$.

Using the methods described **here**, we know that it is possible to construct an inscribed polygon whose area differs from A by *as little as we please*. Call that area P. What we mean by *as little as we please* is that P can be made closer to A than T is.

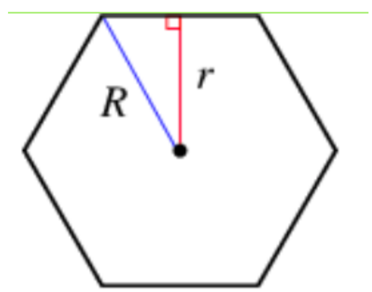
Since this is an *inscribed* polygon, we have

$$A - P < A - T$$

$$A + T < A + P$$

By this argument, we must have that $T < P$.

But for an inscribed polygon the perimeter is certainly less than the circumference of the circle and its apothem (the vertical to the sides of the polygon) is less than the radius of the circle



In the figure, it must be that $r < R$.

By this argument, we must have that $P < T$. However, we just got through showing that $T < P$. We have reached a contradiction.

Thus $A \not> T$.

A similar argument assuming $A < T$ also leads to a contradiction.

Since A is neither greater than nor smaller than T it must be equal to T.

$$A = T = \frac{1}{2} \cdot 2\pi R \cdot R = \pi R^2$$

The analysis is taken from Dunham's *Journey Through Genius*. Here is a quote he presents from Plutarch, talking about Archimedes:

> It is not possible to find in all geometry more difficult and intricate questions, or more simple and lucid explanations. Some ascribe this to his natural genius; while others think

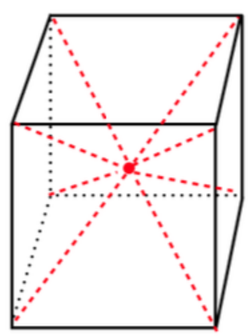
> that incredible effort and toil produced these, to all appearances, easy and unlabored results. No amount of investigation of yours would succeed in attaining the proof, and yet, once seen, you immediately believe you would have discovered it; by so smooth and so rapid a path he leads you to the conclusion required.

Volume of a cone

We needed the formula for the volume of a cone in the previous chapter. Let's start with something simpler, a pyramid.

Consider a cube with all eight edges having length s. So each of the six faces is a square with sides of length s and area s^2.

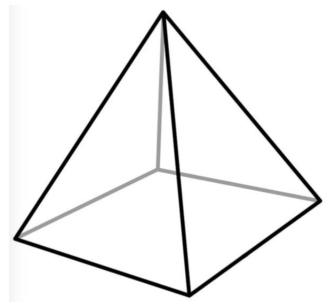
Label the central point inside the solid as P. Draw lines connecting each of the 8 external vertices to P, something like this.



Now we imagine slicing on planes that connect adjacent pairs of lines.

You can't do this in real life by slicing up a single cube or rectangular solid, because the cuts to form one surface would ruin some of the other pieces. The cuts must enter the solid at a corner and then pivot on a line ending at the exact center. (Perhaps you could do it with a "light saber" since the beam comes to a point).

The result is 6 identical pieces (square pyramids) looking something like this



This figure isn't quite accurate because our pyramids will have a height that is $s/2$, but just bear with me.

We started with a cube so that the six resulting solids would be identical. Unfortunately you can either have six pieces the same, or have some of the pieces with equal base and height, but you can't have both.

Let the six identical pyramid volumes each be V, their sum is equal to the volume that we started with. We have that

$$6V = s^3$$
$$V = \frac{1}{6}s^3$$

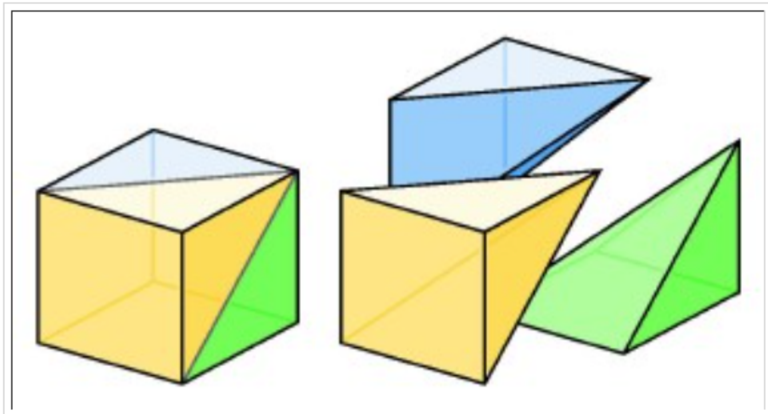
This is the volume for a pyramid with base area s^2 and height $s/2$.

The volume depends linearly on the height and the area of the base. The more general formula for a pyramid is really a linear function of h

$$V = \frac{1}{3}hs^2$$

and you can show this by starting with solids that are longer in one-dimension.

Here is an even better way to slice a cube..



Three congruent pyramids meet along a diagonal of a cube.

At first I thought it was a trick. But in fact, we have 3 identical right square pyramids.

The original cube has 12 edges. Each pyramid ends up getting three of those edges, all of them meeting at a vertex, plus it has two more edges along the base where there has been a cut, so the edge was shared.

In addition to those, there are two edges where a cut occurred along the diagonal of a face, and then finally the longest edge is (always the same) interior diagonal of the cube. The total number of edges is 8.

All three pyramids have a single one of the original external (square) bases, two faces that are one-half of an external face cut along the diagonal, and two faces that were originally internal. These latter two faces lie along the plane formed between the original interior diagonal axis and the diagonal cuts of the faces.

http://www.math.brown.edu/~banchoff/Beyond3d/chapter2/section02.html

Of course, a pyramid is not a cone. But an argument identical to the one we used for the sphere shows that the volume is independent of the shape of the base. It just depends on the area. So for a cone we finally obtain

$$V = \frac{1}{3}\pi r^2 h$$

This is Cavalieri's principle again. We will revisit this problem soon, to use our first bit of calculus. But before that, we need another very important result from geometry.

Part II

Some elements of classical geometry

Chapter 4

Euclid

Euclid and the postulates

Greek geometry starts hundreds of years before Euclid, who was born in 323 BC, the year that Alexander the Great died. But Euclid's book *Elements* is still an excellent place to begin surveying the foundations of geometry.

Euclid's geometry is mainly constructions (geometric figures) drawn with a pencil on a flat piece of paper, using a straight-edge or a compass or both.

Here are Euclid's first three postulates — statements that are assumed to be true:

◦ A straight line segment can be drawn joining any two points.

◦ Any straight line segment can be extended indefinitely in a straight line.

◦ Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

Let us assume these as well.

We ignore the difficulty in defining what is meant by *straight*.

If you've ever done any carpentry, for example, you probably know that unknown edges are determined to be straight by comparison with known straight edges. Saying that "a straight line is the shortest distance between two points", is not much practical help in drawing a straight line if you don't first have a straight edge.

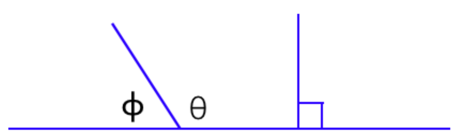
The fourth postulate is:

◦ All right angles are congruent, that is, equal to each other.

This one prompts a different question: what is a "right angle"?

If a line segment is drawn with one end on a line, let us refer to the two angles the line segment forms with the line as adjacent angles.

The definition of a right angle is this: if these two adjacent angles are equal, then they are both right angles.



On the left, one of the angles, ϕ, is smaller than the other one, θ.

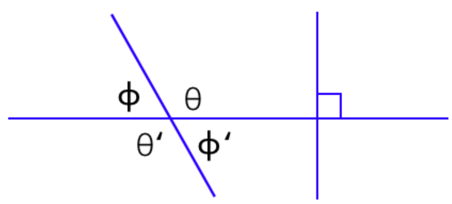
Alternatively, on the right, the two angles have equal measures. In this case we can conclude that both angles are right angles. A right angle is frequently designated by drawing a small square at the intersection. Since both angles are right angles, only one square is needed or usually drawn.

In all cases, the sum of the two angles $\phi + \theta$ is equal to two right angles or 180 degrees. There is nothing particularly special about 180 degrees for two right angles or 360 degrees for one whole turn.

Well, there is one thing: there are *approximately* 360 days in a year, which marks the sun's track across the sky. In his book, *Measurement*, Lockhart adopts the convention that one whole turn is equal to 1.

Later, we'll see that one whole turn is defined as 2π radians, and that convention turns out to be quite important.

Now, extend those lines below the horizontal



We said that the sum of the two angles $\phi + \theta$ is equal to two right angles, but so is the sum $\theta + \phi'$, for the same reason.

$$\phi + \theta = \theta + \phi'$$

We conclude that $\phi = \phi'$ and $\theta = \theta'$. On the right, if any one of the angles where two lines cross is a right angle, then all four are right angles.

This is called the vertical angle theorem.

There is a simple method to construct a line segment perpendicular to a line at a particular (given) point, or alternatively, through any point not on the line (see the video at the url):

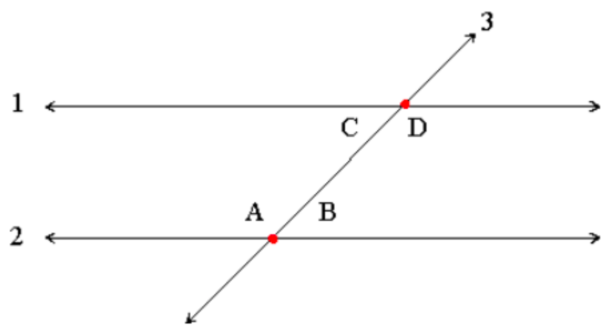
https://www.mathopenref.com/constperpextpoint.html

parallel postulate

All this seems rather obvious.

The fifth and final postulate is more subtle:

∘ If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.



Line 1 and line 2 are parallel, if and only if, $A + C = B + D = 180 = 2$ right angles. This postulate is equivalent to what is known as the parallel postulate.

http://mathworld.wolfram.com/EuclidsPostulates.html

Consider a familiar situation where this is not true. Suppose we are doing geometry on the surface of a sphere, such as the earth. Two adjacent lines of longitude can be drawn so as to cross the equator at right angles, and the lines are parallel there, but they meet (intersect) at the poles.

The parallel postulate only holds for geometry on a *flat* surface.

axioms

Euclid also lists five axioms. Here is are two examples:

◦ Things that are equal to the same thing are also equal to one another.

◦ If equals are added to equals, then the wholes are equal.

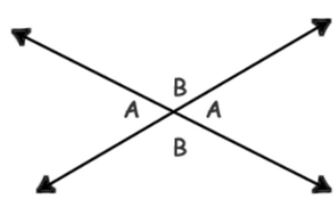
We will see how to proceed from the postulates and axioms to various proofs. *Given these assumptions*, we can prove theorems that must be true.

Thales

I'm a big fan of William Dunham's books — three of them are listed in the References.

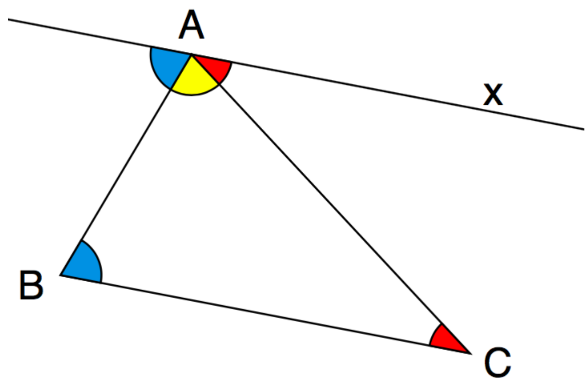
Dunham has written a lot about the history of mathematics in Greece, starting with Thales (624-546 BC), who was from a Greek town called Miletus on the coast of Asia Minor (modern Turkey). He lived long before Euclid. Although none of his writing survives, it is believed that Thales proved several early theorems including one we saw above

◦ The vertical angles formed by two straight lines crossing, are equal.



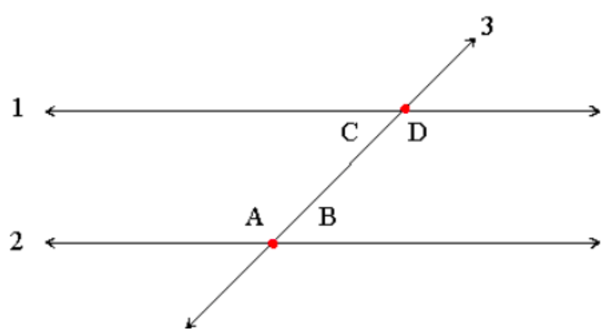
This theorem depends on a property of straight lines. In the proof, we used the axiom "equals added to equals are equal", alternatively "equals subtracted from equals are equal."

∘ The angle sum of a triangle is equal to two right angles.



This theorem depends in turn on a theorem which we laid the groundwork for above but did not state explicitly.

In the figure below, if 1 is parallel to 2, we said that $A + C = B + D =$ 180 degrees.



But we also know from the properties of two lines given above that $A + B = 180$ degrees. So

$$A + C = 180 = A + B$$

and then

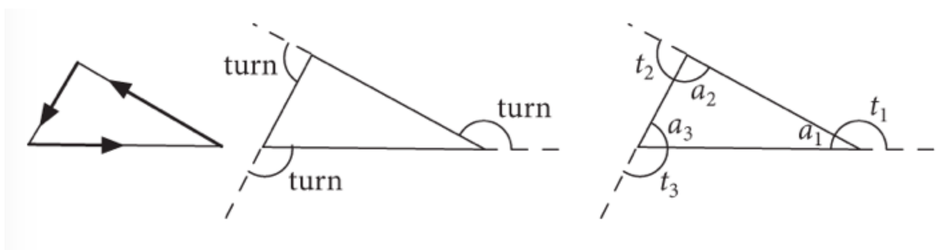
$$C = B$$

This is called the theorem on alternate interior angles. Given this, you can go back to the angles of a triangle problem and follow the colors to the proof.

another proof

Here is an alternative proof of the theorem on the sum of angles in a triangle adding to 180 degrees..

Imagine walking around the perimeter of a triangle in the counter-clockwise direction. At each vertex we turn left by a certain number of degrees, t, called the exterior angle. After passing through all three vertices, we must end up facing in the same direction as we started.



$$t_1 + t_2 + t_3 = 360$$

In addition, for each vertex, the interior angle a_i plus the exterior angle t_i add up to 180 degrees. If we add up all three pairs, we obtain

$$(t_1 + a_1) + (t_2 + a_2) + (t_3 + a_3) = 3 \cdot 180 = 540$$

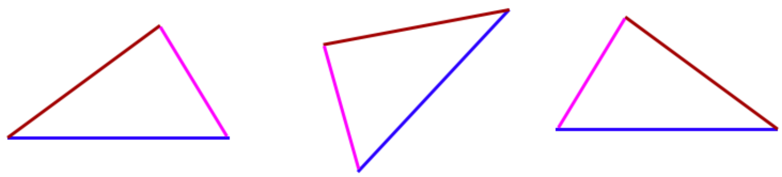
By subtraction

$$a_1 + a_2 + a_3 = 180$$

Congruence and similarity of triangles

◦ Two triangles are *congruent* if and only if they have the same three side lengths. This is often abbreviated SSS (side-side-side).

By this definition, a triangle and its mirror image are congruent. The three triangles shown below are all congruent, even though the first is flipped (it is the mirror image of the other two).



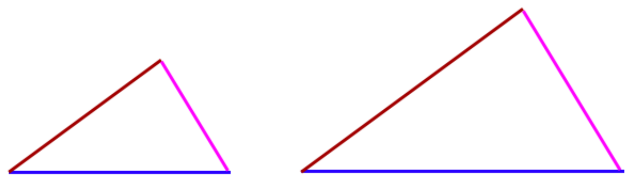
Having the same three sides means that the shape is the same, and all three angles are the same — the shapes are superimposable, with the proviso that we allow the shape to be flipped over.

Some triangles are *similar* but not congruent, with all three angles the same but of different overall sizes. We could call this AAA (angle-angle-angle). For similar triangles, the three corresponding pairs of sides are in the same proportions, but scaled differently.

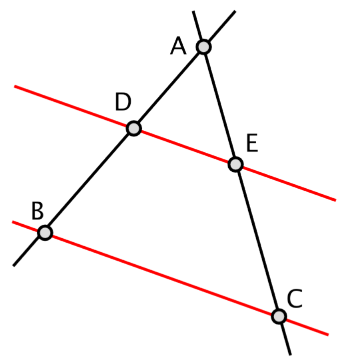
◦ Two triangles are similar if they have the same three angles.

Because of the angle sum theorem, if any two angles of a pair of triangles are known to be equal, then the third one must be equal as well.

Similar triangles have their sides in the same proportions.



Given any triangle, draw a line parallel to one side, which also joins the other two sides. The new triangle is similar to the given triangle. Similarity means that all the angles are equal. This is easily proved using the theorem on alternate interior angles.



In this example, these ratios are all equal

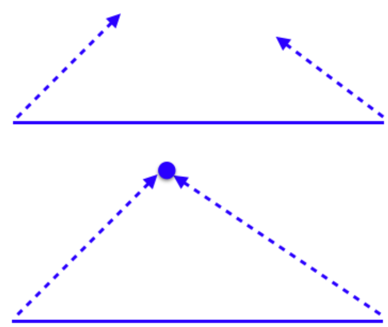
$$\frac{AD}{DB} = \frac{AE}{EC} = \frac{DE}{BC} =$$

In addition to SSS (side-side-side), there are other conditions that lead to congruence of two triangles when they are satisfied, namely

- ∘ SAS (side-angle-side)
- ∘ ASA (angle-side-angle)
- ∘ AAS (angle-angle-side)

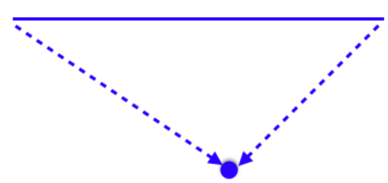
constructions

Again, the way I think about these conditions is to imagine trying to construct a triangle from the given information, and ask whether it is uniquely determined. Suppose we know ASA. The situation is thus:



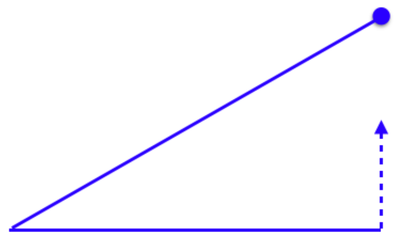
Plot the known side and start two other sides from the ends of that side containing the known angles. They must cross at a unique point.

But... actually, if we start the two lines pointing below the horizontal, there is another solution, the mirror image. This triangle is also congruent to the one above.



Alternatively, knowing two angles means we also know the third, because they must add to 180 degrees. For this reason, ASA and AAS imply that we have exactly the same information, because we know all three angles and (this part is important) we also know *which* two angles flank the known side.

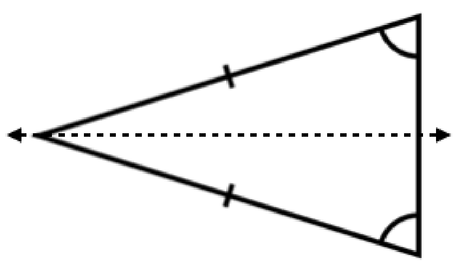
For a right-triangle, if the hypotenuse and one leg are equal, the two triangles are congruent.



In the figure, imagine the hypotenuse swinging on the hinge of its vertex with the horizontal base. There is only one angle where it will terminate on the vertical side with the correct length. This determines the angle between the known sides, or alternatively, the length of the third side.

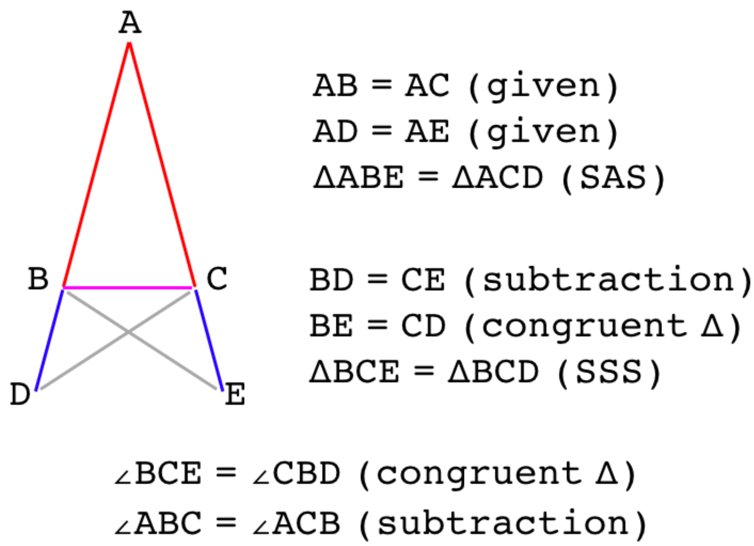
another theorem from Thales

◦ The base angles of an isosceles triangle are equal.



My favorite proof of this theorem is from symmetry. Draw a line from the vertex between the two equal sides to the midpoint of the base opposite. If you turn the triangle over along this axis, we obtain the same triangle back again. Alternatively, just say "side-side-side."

Euclid's proof is here:

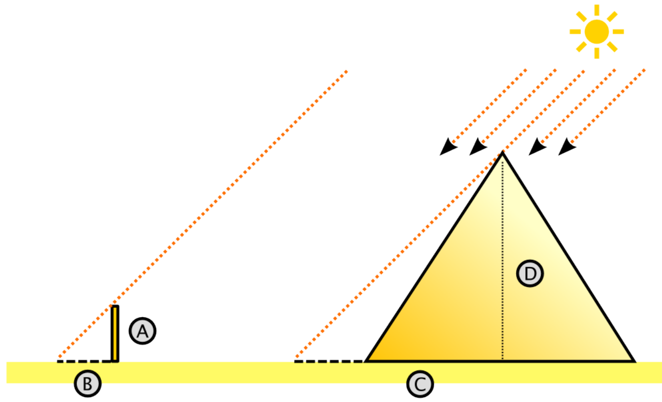


pyramid height

As we said, Thales was from Miletus and he lived around 600 BC. Thales is believed to have traveled extensively around the Mediterranean and was probably of Phoenician heritage, famous sailors.

During his travels, he went to Egypt, home to the great pyramids at Giza, which were already ancient then. They were built just around around 2560 BC (dated by reference to Egyptian kings) and were already 2000 years old at that time!

The story is that Thales asked the Egyptian priests about the height of the Great Pyramid of Cheops, and they would not tell him. So he set about measuring it himself. The current height is 480 feet. He used similar triangles.



platonic solids

https://en.wikipedia.org/wiki/Platonic_solid

> In three-dimensional space, a Platonic solid is a regular, convex polyhedron. It is constructed by congruent (identical in shape and size) regular (all angles equal and all sides equal) polygonal faces with the same number of faces meeting at each vertex. Five solids meet these criteria.

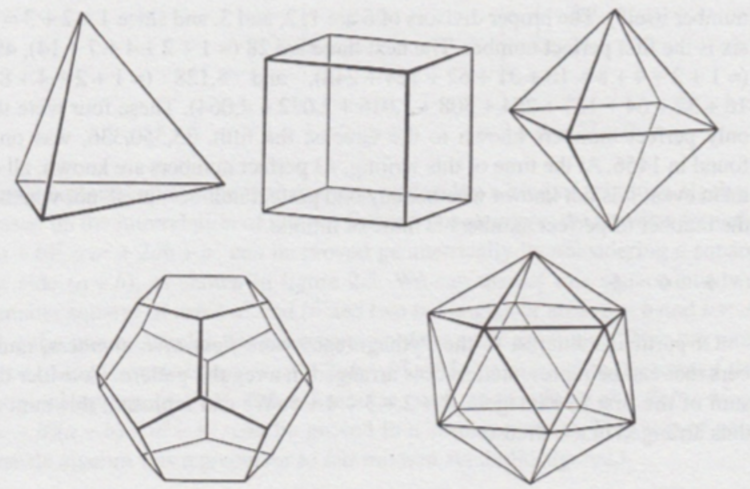


Figure 2.3. The five regular or Platonic solids

These are: (i) tetrahedron, (ii) cube, (iii) octagon, (iv) dodecagon, and (v) icosahedron.

There is a wonderful, simple proof that there are only five of them. Any solid requires at least three sides meeting at each vertex, otherwise the joint between two sides can just flap, like a hinge. Furthermore, the total of all the vertex angles added up must be less than 360 degrees.

So, three equilateral triangles total $60 \times 3 = 180$, four total $60 \times 4 = 240$ and five total $60 \times 5 = 300$. Six would be a hexagon lying in the plane. Three squares total $90 \times 3 = 270$, while four give a square array in the plane. Finally, three pentagons give $108 \times 3 = 324$. And that's it. Three hexagons would give $120 \times 3 = 360$, which gives an array in the plane.

Proving that all the angles and side lengths come out correctly, so that the possible solids actually can be constructed is another matter, however. Euclid devotes book XIII of *The Elements* to this:

https://mathcs.clarku.edu/~djoyce/elements/bookXIII/bookXIII.

html#props

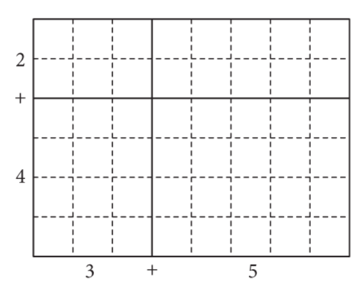
Chapter 5

Area

One aspect of calculus will be to determine the area of figures in the plane, particularly figures bounded by curves, as well as volumes in space. This is the magic of calculus, that we can make curves conform to rectilinear concepts of area and volume.

Since this introductory section is about Euclidean geometry, let's just say a few words about the area of a triangle. But we'll start with the rectangle.

To find the area of a rectangle, we must first fix a unit length. Then multiply the width by the height.



This particular figure (from Lockhart) shows the distributive law in action:

$$
\begin{aligned}
&(3+5)\cdot(4+2)\\
&= 3\cdot 4 + 3\cdot 2 + 5\cdot 4 + 5\cdot 2\\
&= 48
\end{aligned}
$$

Any combination of numbers that add up to 8, times any combination of numbers that add up to 6, gives the same result.

The next figure is a parallelogram, a four-sided figure whose two pairs opposite sides are parallel (left panel). As a consequence of the theorems we saw previously, the opposing angles are equal, and the adjacent angles add up to 180 degrees.

To find the area, we cut off a right triangle from the left and re-attach it on the right. The angles add up to form a straight line along the base and a right triangle at the upper right. The area is clearly $h \times b$.

What about triangles? Well, every triangle can be turned into a parallelogram, by attaching a rotated image of itself, like this:

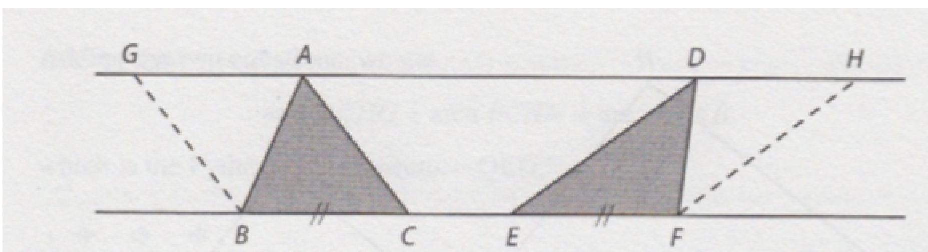
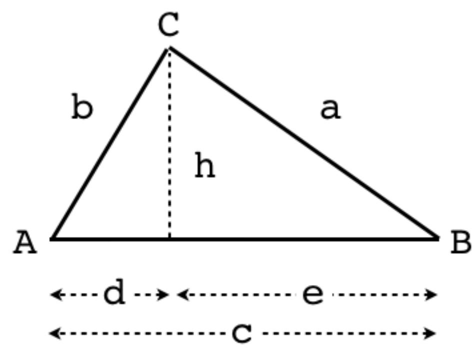


Figure 3.1. Proposition I 38: Triangles *ABC* and *DEF* have the same area

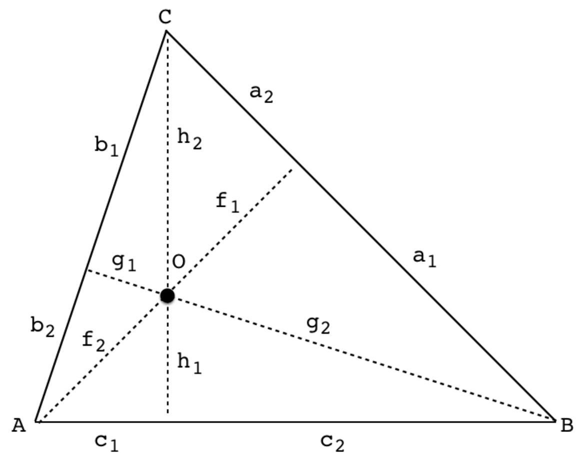
An acute triangle is on the left and an obtuse triangle on the right. Since the area of each triangle is one-half that of its corresponding parallelogram (because we added the same area to make the parallelogram), the area of a triangle is one-half the base times the height.



Here, the area is $hc/2$.

We could choose any side of the triangle to be the base and then multiply $1/2 \times$ base $\times$ height to get the area. We must always get the same answer!

If you accept the argument about the parallelogram above, it must be true, because the area of the triangle has to be the same no matter how you calculate it. Here's a proof:



In $\triangle ABC$ with sides a, b, c, drop the three altitudes from each of the three vertices to form right angles on the opposing sides. Ceva's theorem says that these altitudes cross at a single point (we will prove this later). Label the parts of the sides and the altitudes as shown in the diagram.

The area of the whole $\triangle ABC$ is equal to the sum

$$\triangle BOC + \triangle AOC + \triangle AOB$$

Using the rule, *twice* the area is

$$2A = af_1 + bg_1 + ch_1$$

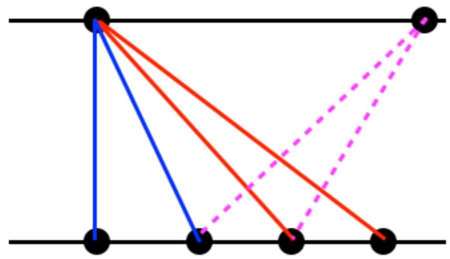
But each of these smaller areas can be computed in different ways. In particular $\triangle BOC$ can be viewed as having base g_2 and height b_1, while $\triangle AOB$ can be viewed as having base b_2 and height g_2, so (twice) the total area is also

$$\begin{aligned} 2A &= b_1 g_2 + b_2 g_2 + b g_1 \\ &= b g_2 + b g_1 = bg \end{aligned}$$

Similar calculations can be carried out for the other two sides. Hence the area is the same regardless of which side is chosen as the base.

□

A corollary is that all triangles with the same base and height have the same area.



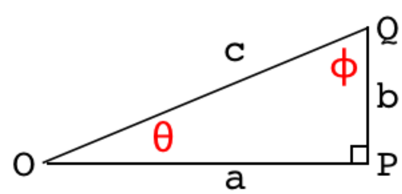
Chapter 6

Angle bisector

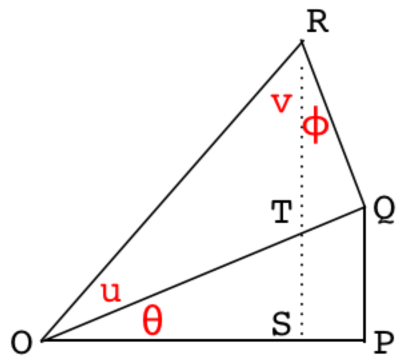
The main result we are headed for is the Pythagorean Theorem. Before we get there, however, it is worthwhile to continue our development of basic geometry with a discussion about right angles and right triangles.

A right triangle is a triangle containing one right angle. Right angles (and right triangles) are special. We saw previously that the definition of a right angle is that two of them add up to one straight line or 180 degrees. Since we proved that the sum of the three angles in any triangle is equal to one straight line, by extension, the sum of angles in any triangle is also equal to two right angles.

In the figure below, the angle at vertex P is a right angle. It is common to mark a right angle with a little square, as shown, but these are a pain to draw, so I will not usually do that. The side opposite P, namely c, is the hypotenuse.



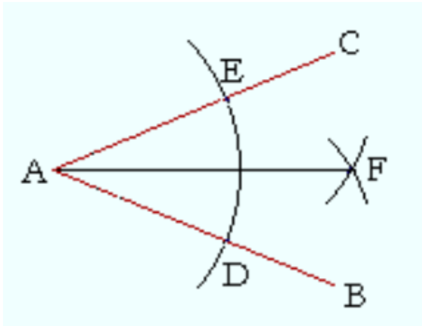
Since the sum of angles in a triangle is equal to two right angles, the sum of the angles θ and ϕ above is also equal to a right angle, or 90 degrees. Angles θ and ϕ are said to be complementary. This fact is often exploited in proofs. Here is an example we will see later on:



Suppose we are given that $\angle OPQ$ and $\angle OQR$ are right angles. We draw the altitude RS and observe that the angle at vertex S is a right angle. Therefore, in triangle ORS, the sum $\theta + u + v$ is equal to one right angle. At the same time, in triangle OQR, the sum $u + v + \phi$ is also equal to one right angle. Therefore, $\theta = \phi$. Further, $\triangle QRT$ and $\triangle OPQ$ are similar triangles.

angle bisector

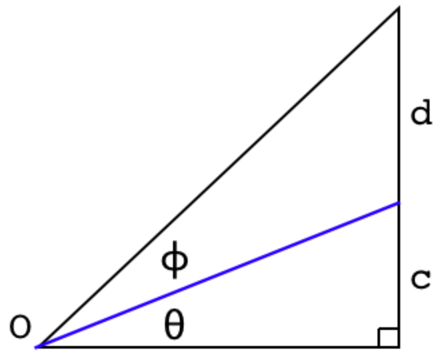
With that background, we now consider a classic problem: involving angle bisectors. Actually, before we do that, let's just show a method for constructing an angle bisector



To bisect angle $\angle BAC$ use the compass to mark off equal segments AD and AE and then mark off equal segments DF and EF. The line segment AF bisects the angle.

Proof: $\triangle ADF$ is congruent to $\triangle AEF$ by SSS. Therefore, $\angle CAF = \angle BAF$.

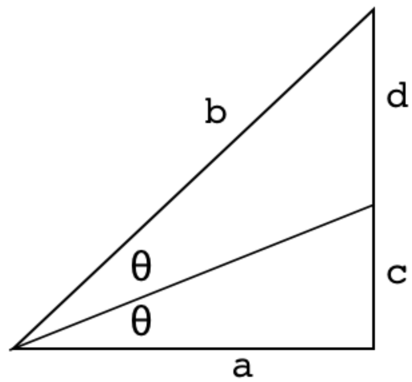
Now, back to our problem, and the diagram below.



Suppose we are given that the large triangle, and the bottom of the two smaller triangles are both right triangles.

We draw a line joining the vertex O on the left with the side opposite. This line could in general be drawn anywhere, however two interesting cases are when the angle at O is bisected, or when the side opposite is bisected.

These cases are different.



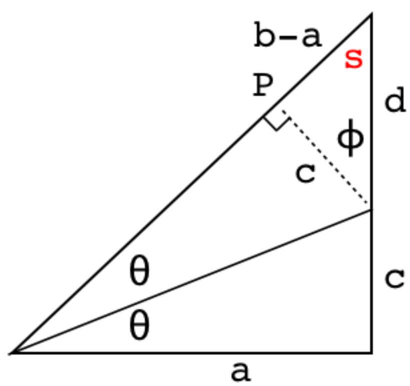
Here we have chosen the first possibility. We are in a position to prove an important theorem.

angle bisector theorem

With reference to the figure above, we are to prove that

$$\frac{d}{b} = \frac{c}{a}$$

Draw an altitude for the upper of the two small triangles, meeting the side of length b at point P.



By congruent triangles (the two triangles each with vertex angle θ), the altitude has length c.

By the rules for complementary angles discussed above:

$$2\theta + s = 90 = s + \phi$$

Hence, $2\theta = \phi$. We conclude that the smallest triangle at the top right of the figure is similar to the original. By similar triangles, we form the equal ratios of the hypotenuse to the adjacent angle (either ϕ or 2θ):

$$\frac{d}{c} = \frac{b}{a}$$

This is rearranged simply to give

$$\frac{d}{b} = \frac{c}{a}$$

which is what we were asked to prove.

$\square$

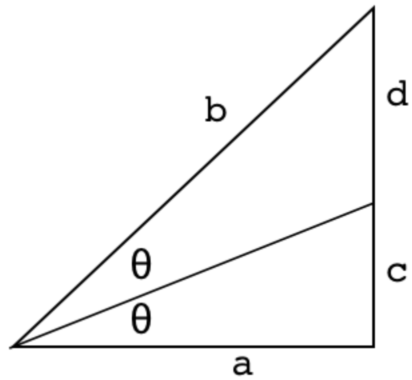
The result can be pushed a little further:

$$\frac{a}{b} = \frac{c}{d}$$

Here's the key point

$$\frac{a+b}{b} = \frac{c+d}{d}$$

$$\frac{a+b}{c+d} = \frac{b}{d} = \frac{a}{c}$$

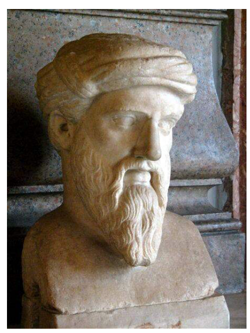


which is a surprising result and becomes important in looking at Archimedes method for approximating the value of π.

Chapter 7

Pythagoras

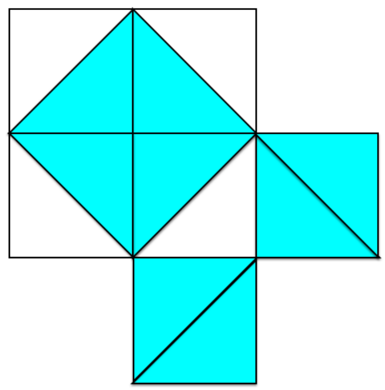
The most famous theorem of Greek geometry is also the most useful in Calculus.



Pythagoras (c.570-c.495 BC) was much younger than Thales but may have encountered him as a young man. Like many other Greek mathematicians, Pythagoras was not from the mainland, but from one of the islands, in his case, Samos, which is not far from Miletus, where Thales lived.

Pythagoras was famous as a philosopher as well as a mathematician. In fact, he founded a famous "school" and it is not sure now which of the theorems developed by this school are due to Pythagoras, and which to his disciples. It is not even clear whether the Pythagorean theorem, as we know it, was known to Pythagoras.

However, it's pretty certain that they knew something. The $3, 4, 5$ right triangle and many other Pythagorean triples (see below) had been known for a thousand years (since 1500 BC). Here is a special case, easily proved, for an isosceles right triangle.

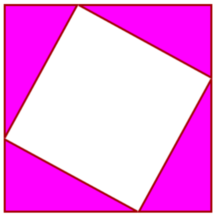


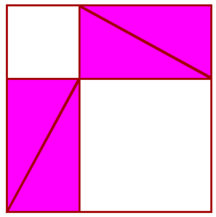
The area of the square on the hypotenuse is equal to twice the area on each side.

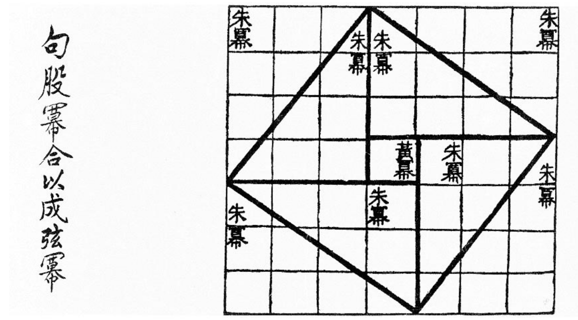
There are literally hundreds of proofs of the general theorem, that if a and b are the shorter sides of a right triangle and c is the hypotenuse, then

$$a^2 + b^2 = c^2$$

This one is sometimes called the "Chinese proof." I can easily imagine proceeding from the figure above to this one by simply rotating the inner square.







It really needs no explanation, but ..

In the left panel we have a large square box that contains within it a white square, whose side is also the hypotenuse of the four identical right triangles contained inside. Altogether the four triangles plus the white area add up to the total.

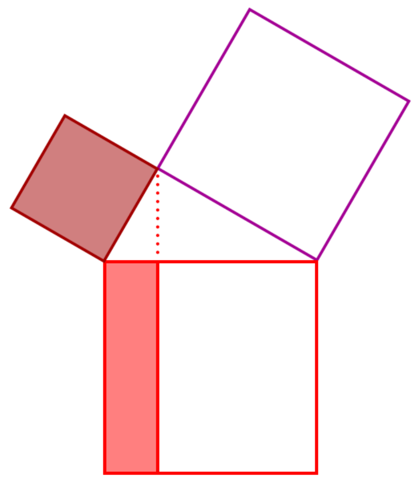
We simply rearrange the triangles. Now we evidently have the same area left over from the four triangles, because they still have the same area and the surrounding box has not changed.

But clearly, now the white area is the sum of the squares on the second and third sides of the triangles. Hence the two white squares on the right are equal in area to the large white square on the left. □

This proof is contained in the Chinese text Zhoubi Suanjing (right panel, above).

https://en.wikipedia.org/wiki/Zhoubi_Suanjing

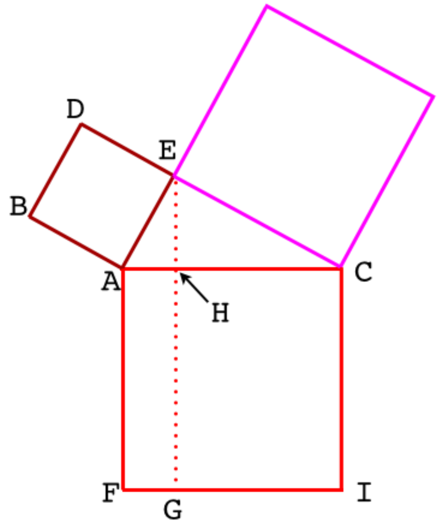
Euclid's proof



My favorite proof relies on the construction above (Euclid $I.47$, sometimes called the "bridal chair" or the "windmill"), where the central triangle is a right triangle, and the other constructions are squares. It is a bit more detailed, but it is the one place in the book that we actually show a proof from Euclid, which is my justification for including it.

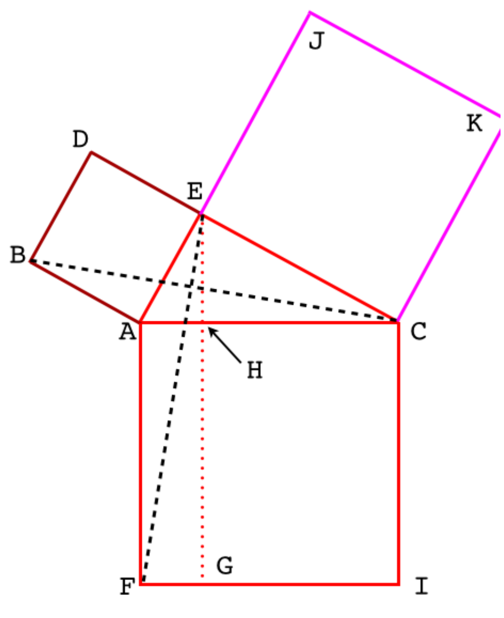
What we will show is that the part of the large square in red is equal in area to the entire small square, in maroon.

We label some points as shown:



First, drop a vertical line EHG, constructing the rectangle $AFGH$.

Finally, sketch dotted lines for the long sides of two triangles:



The crucial point is this: we will show that triangle ΔABC is congruent to triangle ΔAEF.

Use "side-angle-side". The two sets of sides are evidently equal

$$AB = AE, \quad AC = AF$$

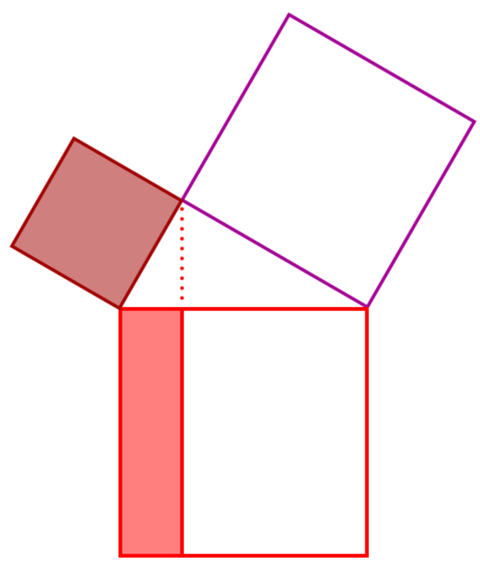
because these are given as sides of two squares.

What about the included angle? Both angle $\angle BAC$ and $\angle EAF$ contain a right angle plus the shared angle $\angle EAC$. So they are themselves equal, and thus we have proved the congruence relationship:

$$\Delta ABC = \Delta AEF$$

The next part of the proof is to tilt triangle ΔABC to the left and see that it has base AB and altitude AE so its area is one-half that of the small square $ABDE$. On the other hand triangle ΔAEF has base AF and altitude AH (as well as FG) so its area is one-half that of the rectangle $AFGH$.

Hence we have proved that the two colored areas in this figure are equal:

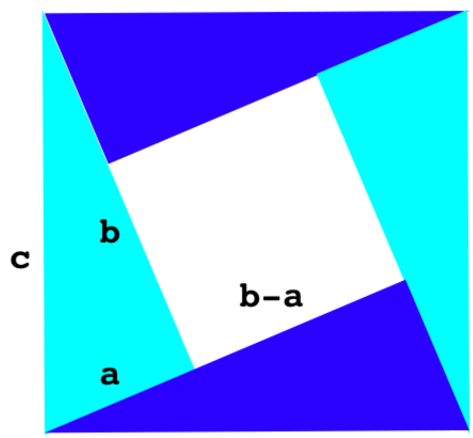


Finally, we could proceed to do the same thing on the right side of the figure, but we just appeal to symmetry. All the equivalent relationships will hold. $\square$

algebraic proofs

The following proofs are algebraic ones. Not so pretty, but fast.

Arrange 4 identical right triangles as shown in the figure below. The four triangles plus a small central square form a larger quadrilateral which is also a square.



The angles at the corners of the quadrilateral, at the points flanking the hypotenuse c, are right angles, because they are formed by addition of two complementary angles of congruent right triangles. Since the quadrilateral has four internal right angles and equal length sides, it is a square.

Now just calculate the area of the parts. We have four identical right triangles with sides a and b, plus the central square with sides $b - a$.

The area is

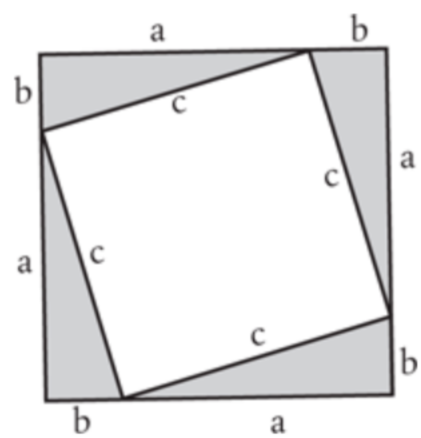
$$A = 4 \cdot \frac{1}{2}ab + (b-a)^2$$
$$= b^2 + a^2$$

But the area is also the square of side c.

$\square$

We have used various properties proved earlier, e.g. that the sum of the angles of any triangle is 180 degrees.

Here is a very similar proof:

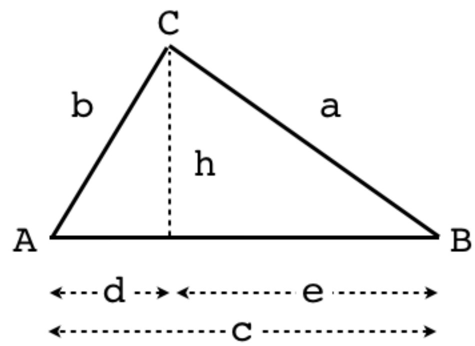


In this figure, the right triangles are aligned so that the big square has sides which combine the lengths $a+b$ and have area $(a+b)^2$. But we can also calculate the area as the sum of its components, namely, central tilted square plus the four triangles:

$$(a+b)^2 = c^2 + 4 \cdot \frac{ab}{2}$$
$$a^2 + b^2 + 2ab = c^2 + 2ab$$
$$a^2 + b^2 = c^2$$

□

For the third algebraic proof, divide a right triangle into two smaller ones by dropping an altitude, which meets the base at a right angle.



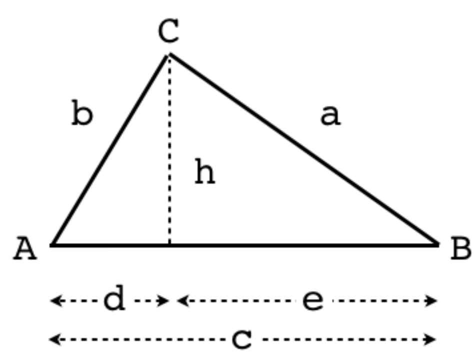
By complementary angles, these three triangles are all similar (e.g., the angle between sides b and h is equal to that between sides a and c). So we can construct ratios of sides that are equal.

We need a relationship involving a^2.

$$\frac{h}{b} = \frac{e}{a} = \frac{a}{c}$$

We choose the ones involving c and e:

$$a^2 = ce$$



and also a relationship involving b^2:

$$\frac{b}{d} = \frac{c}{b}$$
$$b^2 = cd$$

Putting the two together:

$$a^2 + b^2 = ce + cd$$

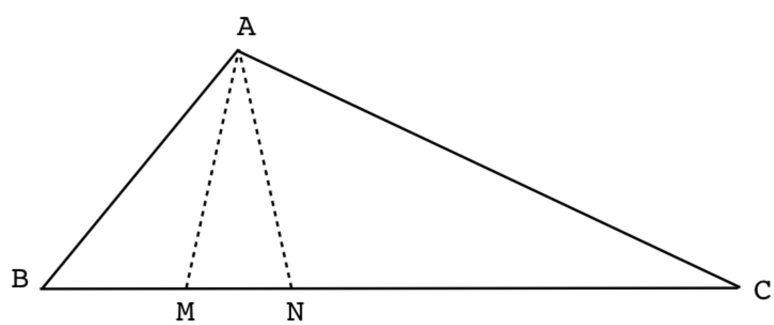
$$= c(d + e) = c^2$$

Which is what we wanted to prove. $\square$

There are more than 300 proofs of this theorem, including one by a President of the United States (hint: it's not 45).

Corollary

There are several corollaries of the Pythagorean theorem. We'll see a very important one later, called the law of cosines. Here is one from the Islamic geometer Ibn Quorra, who brought algebraic techniques, shunned by the Greeks, to geometry.



Let $\triangle ABC$ be *any* triangle (here it is obtuse). Draw AM and AN so that the new angles $\angle AMB$ and $\angle ANC$ are equal to $\angle A$. The corresponding triangles are similar to the original, because they share the angle of measure A plus one other from the original triangle.

Then

$$BM : AB = AB : BC$$

Thus, $AB^2 = BM \times BC$. Similarly

$$NC : AC = AC : BC$$

So $AC^2 = NC \times BC$ Therefore

$$AB^2 + AC^2 = BM \times BC + NC \times BC$$

$$= (BM + NC) \times BC$$

In the case where the angle at vertex A is a right angle, then M coincides with N, and $BM + NC = AC$, and this reduces to the Pythagorean theorem.

Pythagorean triples

The simplest right triangle with integer sides is $3, 4, 5$:

$$3^2 + 4^2 = 5^2$$

any multiple n will work

$$(3n)^2 + (4n)^2 = (5n)^2$$

but that's not so interesting. The triples which are not multiples of another triple are called *primitive*.

To go further, we can use Euclid's formula. For every integer m, n, with $m > n$, a Pythagorean triple is given by

$$a = m^2 - n^2 \qquad b = 2mn \qquad c = m^2 + n^2$$

It is better to choose m and n of opposite parity (one even and one odd). Otherwise, a, b and c will all be even and the triple won't be primitive.

It is easy to see why this works:

$$a^2 + b^2 = (m^2 - n^2)^2 + (2mn)^2$$

$$= m^4 - 2m^2n^2 + n^4 + 4m^2n^2$$

$$= m^4 + 2m^2n^2 + n^4$$

$$= (m^2 + n^2)^2 = c^2$$

Suppose $a = 5$. The two squares with a difference of 5 are

$$3^2 - 2^2 = 5$$

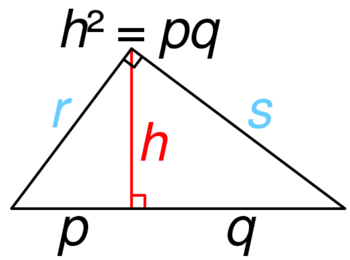
So $b = 2mn = 12$ and $c = 3^2 + 2^2 = 13$. And indeed $5^2 + 12^2 = 13^2$.

https://en.wikipedia.org/wiki/Pythagorean_triple#Enumeration_of_primitive_Pythagorean_triples

A thousand years before Pythagoras, the Babylonians knew the triple $4601, 4800, 6649$. It seems unlikely that they found this by random search.

geometric mean

As a slight detour from calculus, but on the topic of this chapter



According to the figure, the altitude of a right triangle is related to the two segments along the base by

$$h^2 = pq$$

$$h = \sqrt{pq}$$

That is, h is the geometric mean of these two values p and q.

The proof is simple. Using the Pythagorean theorem with the two small triangles (also right triangles), we obtain:

$$h^2 + p^2 = r^2$$

$$h^2 + q^2 = s^2$$

Summing

$$2h^2 + p^2 + q^2 = r^2 + s^2$$

Using the theorem with the big triangle:

$$r^2 + s^2 = (p + q)^2$$

$$= p^2 + 2pq + q^2$$

Equating the two expressions for $r^2 + s^2$ we get:

$$2h^2 + p^2 + q^2 = p^2 + 2pq + q^2$$

$$h^2 = pq$$

According to wikipedia:

https://en.wikipedia.org/wiki/Geometric_mean

The fundamental property of the geometric mean is that (letting m be the *geometric mean* here):

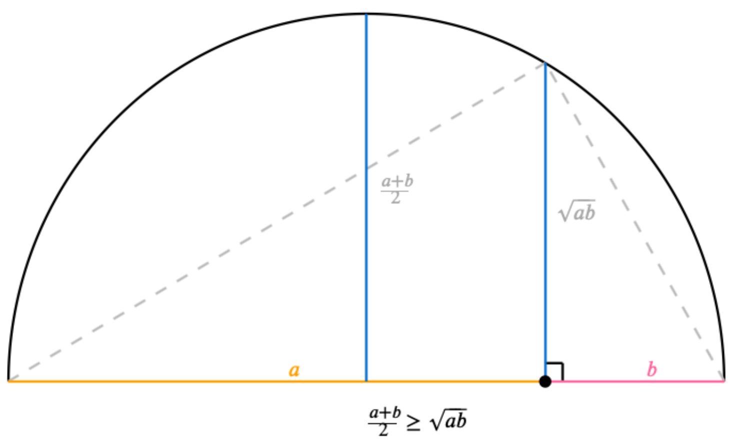
$$m\ [\ \frac{x_i}{y_i}\ ]\ = \frac{m(x_i)}{m(y_i)}$$

and one consequence is that

> This makes the geometric mean the only correct mean when averaging normalized results; that is, results that are presented as ratios to reference values.

A number of examples are given in the article.

Last: a proof-without-words that the geometric mean is always less than or equal to the arithmetic mean.



Chapter 8

Arcs of a circle

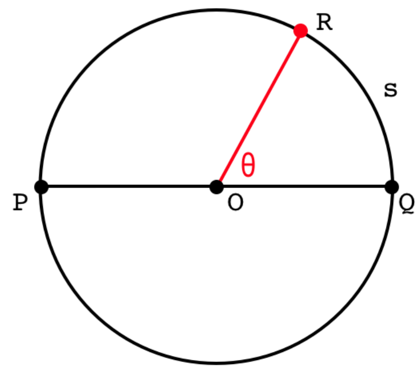
From a previous chapter, Euclid's third postulate was:

◦ Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center. The tool to do this is called a compass:

https://en.wikipedia.org/wiki/Compass_(drawing_tool)

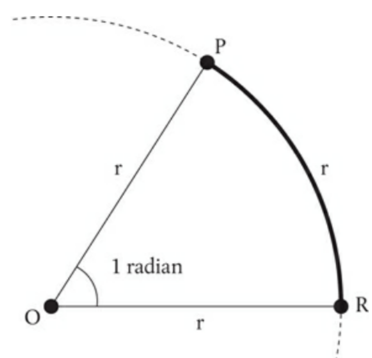
If the radius is extended so that it cuts the circle at two points, it is called a diameter. We saw previously that one can construct a line perpendicular to any given line. If that line is constructed perpendicular to the diameter at the point where it meets the circle, the new line is called a tangent line. By definition, the tangent line touches the circle at a single point.

arcs of a circle



In calculus and analytical geometry angles are defined in terms of radians of arc. For a unit circle with radius = 1, the total circumference is 2π, so the arc swept out by the angle θ is in the same ratio to 2π as the ratio of the angle's measure in degrees to $360°$.

It seems natural then to adopt the arc length as a measure of the angle, where $360°$ is equal to 2π *radians*, and an angle of $90°$, for example, a right angle, is equal to $\pi/2$ radians.



72. Definition of a radian.

We say that the angle θ is equal to the arc it sweeps out on the circumference, in radians.

$$s = \theta$$

To convert some more measures of angles in degrees to radians:

$$180^\circ = \pi,\ 90^\circ = \frac{\pi}{2}$$

$$60^\circ = \frac{\pi}{3},\ 45^\circ = \frac{\pi}{4},\ 30^\circ = \frac{\pi}{6}$$

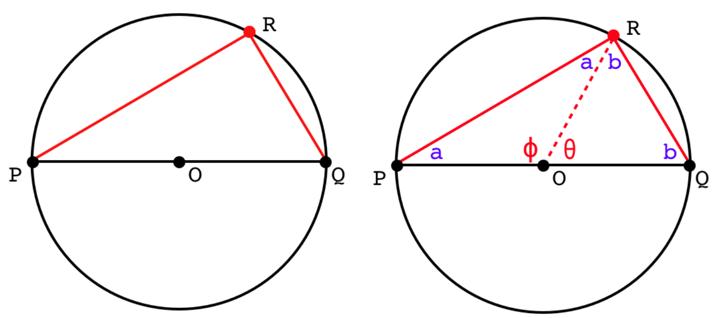
theorem

In this chapter, we introduce a few more theorems concerning circles, starting with the last of Thales' theorems:

∘ Any angle inscribed in a semicircle is a right angle.

Now, think of three points on the circumference of the circle as forming a triangle. If two points are on a diameter of the circle, the angle formed at any arbitrary but distinct third point is always a right angle.

To prove: $\angle PRQ$ is a right angle.



Solution: Draw the radius OR. Notice that $\triangle OPR$ and $\triangle OQR$ are both isosceles.

Label the respective base angles a and b. By considering that together the sum of the angles of $\triangle PQR$ can be written:

$$2a + 2b = \pi$$

$$a + b = \frac{\pi}{2}$$

But this is the measure of $\angle PRQ$.

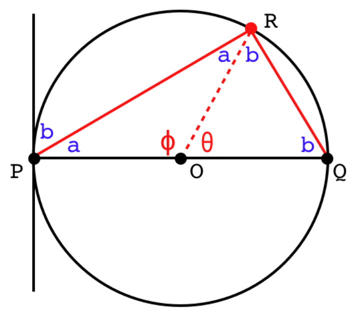
In addition, the arcs swept out by angles a and b (OPR and OQR on the diameter) clearly add up to π. This suggests that:

$$a = \frac{\theta}{2}$$
$$b = \frac{\phi}{2}$$

Proof:

$$2a + 2b = \pi = 2a + \phi$$
$$\phi = 2b$$

Consider the chord PR and draw the tangent at P.



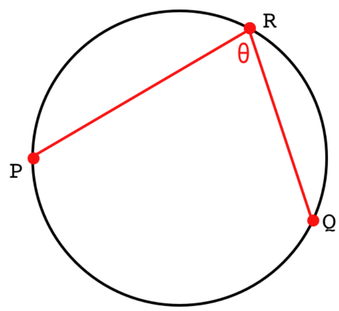
The arc between the tangent and the chord equals 2b because it is the same arc as cut off by $\angle\ PQR$ (which is $\angle\ b$).

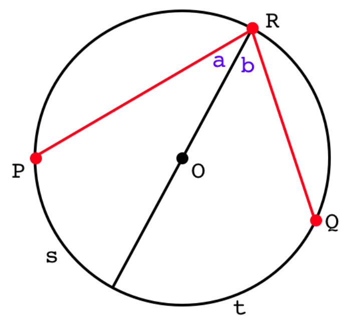
Take a chord of the circle, draw the diameter and the tangent. The same rule applies to both angles: one between the chord and the diameter, and the second between the chord and the tangent. The arc is twice the measure of the angle.

Generalized arc

Having established these basic facts we can do a bit more. We will use some of these results later on.

One is to generalize the result for all arcs. The examples so far contain the diameter in some way. Consider the arc swept out by the angle θ in this figure.





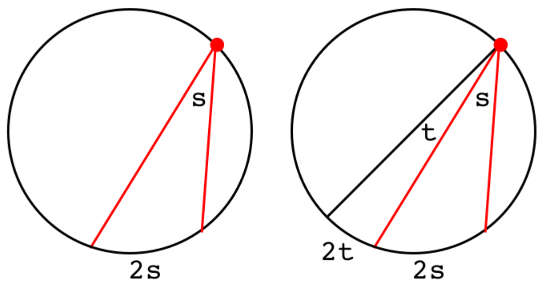
We can prove that the measure of the angle θ is equal to the 1/2 the arc swept out between P and Q. For a simple proof, draw the diameter: By our previous work:

$$b = \frac{t}{2}$$

$$a = \frac{s}{2}$$

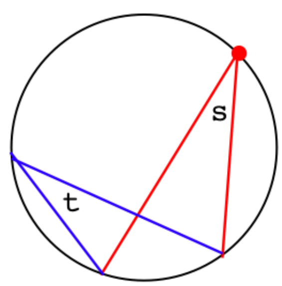
$$\theta = a + b = \frac{s + t}{2}$$

We have proved the theorem for two cases: where the diameter is one line segment flanking the angle, and where the angle includes the diameter. However, the theorem is true even if the angle does not include the diameter.



On the right, draw the diameter. Notice that we have two arcs which include the diameter: one with angle t and one with angle $s+t$. We obtain the generalized arc with angle s by subtraction.

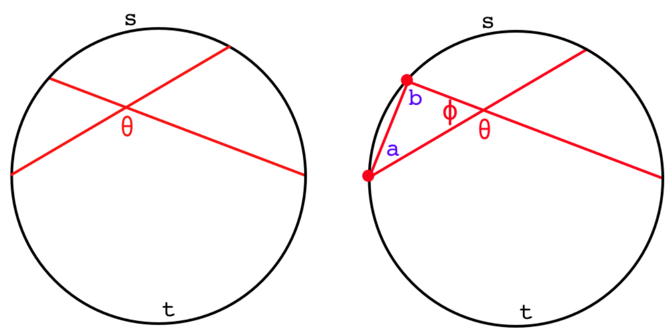
As a corollary, any two angles with vertexes on the circle that cut off the same arc are equal. In the figure, $s = t$. Also the triangles are similar triangles.



Intersecting chords

Given two chords, to prove:

$$\theta = 1/2(s+t)$$



θ is the average of the two arc lengths. Solution: Draw a triangle.

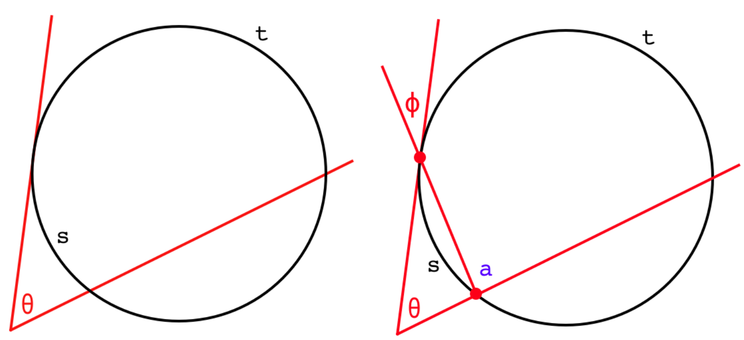
$$a = \frac{s}{2}$$

$$b = \frac{t}{2}$$

$$a + b = \theta = \frac{s + t}{2}$$

Tangent and secant

Rather than having all three points on the circle, one is now outside. We have the same arc swept out by the endpoints (t), but the included angle is now smaller, and there is a new small piece of arc length s.



To prove:

$$\theta = \frac{t-s}{2}$$

Solution: Draw the triangle. By our previous work (and supplementary angles):

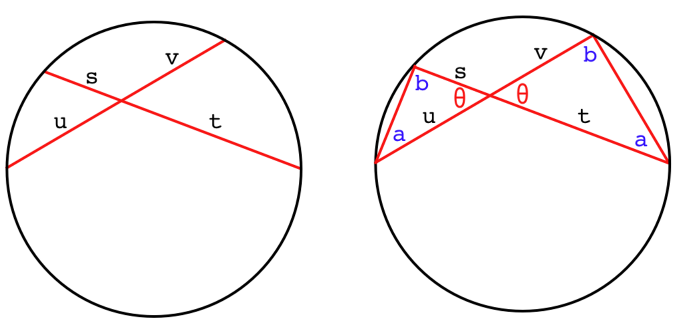
$$\phi = \frac{s}{2}$$
$$a = \frac{t}{2}$$

by supplementary angles:

$$\theta + \phi = a$$
$$\theta = \frac{t}{2} - \frac{s}{2}$$
$$= \frac{t-s}{2}$$

Chord segments

Finally, there is a simple algebraic relationship between chord segments. Draw two chords of the circle and label the lengths of the segments as shown (note: s and t do not refer to arcs any more).



Draw the two triangles. Notice that the two angles labeled a are equal because they sweep out the same arc of the circle, and similarly for the two angles labeled b. By similar triangles:

$$s/u = v/t$$

$$st = uv$$

Chapter 9

Eratosthenes

This part of the book is focused on geometry, and we take a look at Eratosthenes in this chapter as an important Greek scholar.

The widely held theory, that the ancient world believed the earth to be flat, is just wrong. People with any level of sophistication not only knew the earth is roughly spherical but also knew its size within a few percent of the true value.

One likely basis is the false story that Columbus had trouble getting financing for his proposed trip to China because everyone thought he would fall off the edge of the earth. This was a tall tale invented by Washington Irving, who also made up several remarkable fables about George Washington.

The real reason the Italians and the Portuguese thought Columbus would fail is that they had a pretty good idea of the size of the spherical earth and thus of the distance to China, while the over-optimistic Columbus believed it was about half the true value. The prospective financiers knew that he was not able to carry the supplies necessary for a trip of this length.

Morris Kline (*Mathematics and the Physical World*) says that the error is due to geographers after Eratosthenes, who reduced the estimated circumference from 24,000 to 17,000 miles.

Eratosthenes

Views of the Greek philosophers on the earth and its sphericity are detailed here

https://www.iep.utm.edu/thales/#SH8d

Here is a partial quotation:

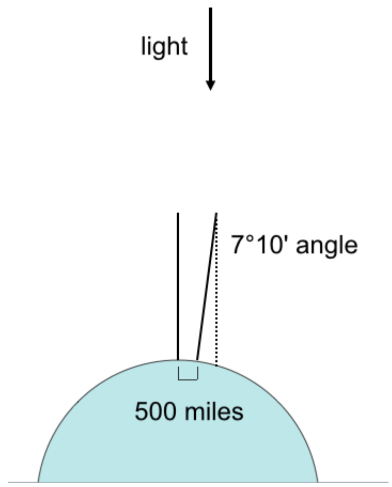
> There are several good reasons to accept that Thales envisaged the earth as spherical. Aristotle used these arguments to support his own view [...] . First is the fact that during a solar eclipse, the shadow caused by the interposition of the earth between the sun and the moon is always convex; therefore the earth must be spherical. In other words, if the earth were a flat disk, the shadow cast during an eclipse would be elliptical. Second, Thales, who is acknowledged as an observer of the heavens, would have observed that stars which are visible in a certain locality may not be visible further to the north or south, a phenomen[on] which could be explained within the understanding of a spherical earth.

https://en.wikipedia.org/wiki/Eratosthenes

Eratosthenes (ca. 276 - 195 BCE) measured the circumference of the earth from this observation: at high noon on June 21st there was no shadow seen at Syene, e.g., allegedly from a stick in the ground. Some people say it was a deep well, where the bottom was illuminated at midday.

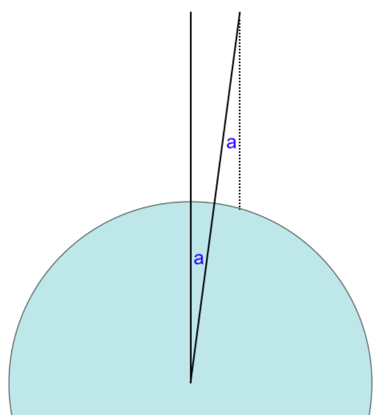
Syene is presently known as Aswan. It is on the Nile about 150 miles upstream of Luxor, which includes the famous site called the Valley of the Kings. At 24.1 degrees north latitude, Aswan or Syene is close enough to having the sun directly overhead on June 21. (The "Tropic of Cancer" is at 23 degrees, 26 minutes north).

This news about the lack of a shadow at Syene reached Alexandria, a famous center of learning of the ancient world. Alexandria lies on the Mediterranean some 500 miles north of Syene, and anyone there who was looking could observe that at high noon on June 21st there *was a shadow*. This shadow Eratosthenes measured to be some 7 degrees and a bit (7 degrees and 10 minutes).



A full 360 degrees divided by 7 degrees and a bit is approximately 50. So we can calculate on this basis that the circumference of the earth is about $50 \times 500 = 25000$ miles. That's pretty close to the correct value.

For this calculation, we assume that the sun's rays are effectively parallel (not a bad assumption given a distance of 93 million miles). Then we just use this:



an application of the alternate-interior-angles theorem.

It is curious how the distance from Alexandria to Syene was calculated [Kline]. "Camel trains, which usually traveled 100 stadia a day, took

50 days to reach Syene. Hence the distance was 5000 stadia...It is believed that a stadium was 157 meters.” We obtain

$$157 \times 5000 \times 50 = 39,250 \text{ km}$$

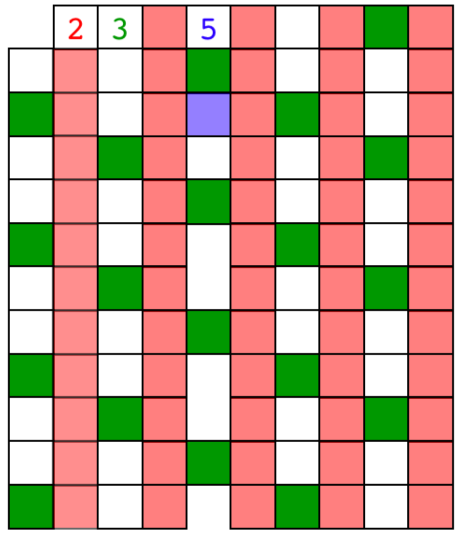
That’s a much better estimate than a method that relies on camels really deserves.

The sieve of Eratosthenes

Eratosthenes is famous in mathematics for his ”sieve” which allows one to compute the prime numbers in an economical fashion. This next section really has nothing to do with calculus, but it is inspired mathematics.

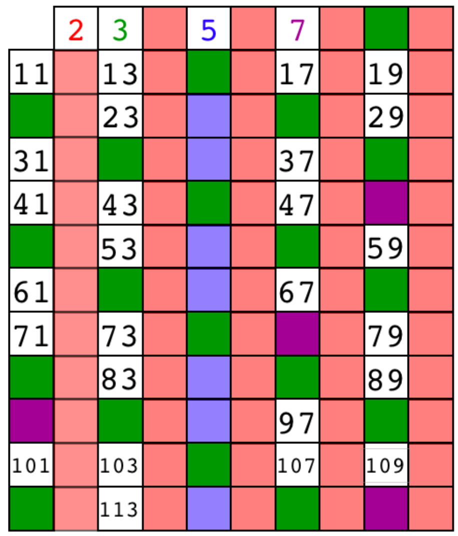
The sieve is operated by first enumerating all the integers to some upper limit (here 120). To do things manually it is convenient to use rows with 10 values, so there are 12 of them in all. Most of the boxes have not yet been numbered.

Starting with the first prime number, 2 (red), eliminate all the numbers divisible by 2 (all the even numbers). Here this has been done by coloring red all of the squares in the even numbered columns (all numbers ending in $2, 4, 6, 8, 0$).



Next, do the same thing with 3 (green). 6 was already eliminated previously, but all odd multiples of 3 like 9 and 15 go away at this step.

The next larger number that still has a white square is 5. The only squares eliminated are the white ones in the fifth row. The first value specifically eliminated at the 5 step is 25. Continue with 7, eliminating $49, 77, 91$ and 119.



The sieve ends when the number for the beginning of the next round, the smallest number not yet eliminated, is greater than the square root of the upper limit (here $\sqrt{120}$). So 7 is used for the last round, because after that round the smallest remaining integer is 11, but we terminate since $11^2 > 120$.

The graphic shows all the numbers which have yet to be eliminated after the round of 7. All of these numbers, 11, 13, 17, and so on, as well as those used as divisors for each round of the sieve $(2, 3, 5, 7)$, are prime numbers.

By testing for division by $2, 3, 5$ and 7, we have found the first 30 prime numbers.

From a performance standpoint, it is important that we do not need to carry out division. All that is really needed is repeated addition. Coding this algorithm in, say, Python is a good challenge. A bigger challenge is to come up with a method to *grow* the list of primes on demand. This can be done by keeping track of the first value to be tested above the limit, for each prime in the current list.

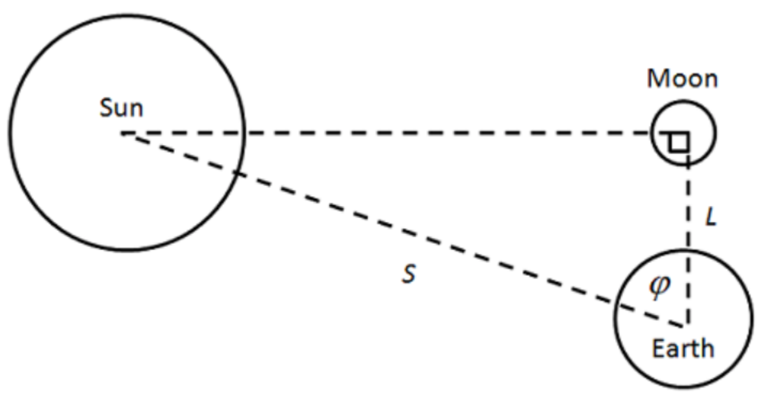
Aristarchus

Aristarchus of Samos (310-230 BCE) wrote a famous book in which he calculated the relative sizes of the sun and the moon and their distances from earth.

One straightforward observation is that the apparent size of the sun and moon in the sky is about the same. This can be seen during a solar eclipse, or observed at any other time by holding a disk up at a fixed distance from the eye, (while taking care to block most of the sun's rays). The value is approximately one-half degree.

Since the distance to the sun is much greater than that to the moon (see below), we can infer that the sun is much larger than the moon.

The central idea of Aristarchus is that, at half moon, the geometry of the three orbs is like this:



In other words, when the phase is half moon and that moon is exactly overhead, the sun has not yet set, but is a bit above the horizon.

If S is the distance to the sun and L is that to the moon, he estimated that

$$18 < \frac{S}{L} < 20$$

with the same ratio for their sizes. Unfortunately, this is not a particularly good estimate. The true value is about 390. Aristarchus obtained a value of 20 for the Earth-Moon distance in Earth radii. The correct value is about 60. Much better estimates were obtained later, by Hipparchus and Ptolemy.

However, Aristarchus made up for this by being the first person to propose a heliocentric theory of the solar system: that the earth and planets rotate around the sun.

https://en.wikipedia.org/wiki/On_the_Sizes_and_Distances_(Aristarchus)

quick estimate

Here is an estimate for the earth-moon distance based on a lunar eclipse.

One measures the time it takes for a complete, total eclipse. From the first shadow of the earth on the moon to the last, that time is about 3 hr. The moon has moved approximately 1 earth diameter in its orbit in that time.

However, we must correct for the fact that the first and last shadows occur on opposite edges of the moon. It was noted that the shape of the eclipse suggests the earth's diameter (at that distance) is about 2.5 moon diameters. So the moon has actually moved (2.5 + 1.0)/2.5 = 1.4 earth diameters in the given time. The relevant time becomes 2.14 hr.

Any correction for the true size of the earth's diameter is minimal because the earth-moon system is so far from the source of illumination.

The other piece of information we need is the time for a full revolution, one lunar cycle. This part is tricky. Naively, you'd look for the moon to be in the same place against the fixed stars (27 days, c. 8 hr). This is off because the earth has moved in the meantime — there is a parallax error. As a rough correction, mutliply by 360/330 degrees. The result in hours is 715.

The circumference of the orbit is then

$$715/2.143 = 333$$

earth diameters.

This gives a radius of 53 earth diameters, which is not too far from 60.

Chapter 10

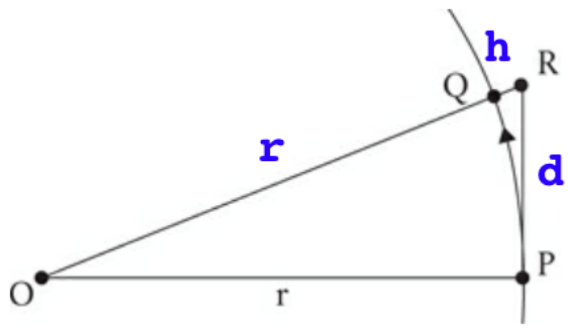
Circular orbits

Pythagoras and Newton

A previous chapter looked in detail at Pythagoras' Theorem, which is used incessantly from here on out. Here, we explore one use of the Pythagorean theorem and provide a taste of orbital mechanics, which is a particular focus of calculus. Newton made early calculations similar to these, which increased confidence about his famous inverse-square law and inspired the mathematics that led to the explanation of elliptical orbits.

Although the orbits of the planets around the sun are ellipses, they are very nearly circular and we will make that approximation for what follows here.

We use the Pythagorean Theorem to make another approximation. Using r for the (fixed) radius of the orbit for the moment, because the construction has capital letters for the points, including the symbol R:



$$r^2 + d^2 = (r + h)^2 = r^2 + 2rh + h^2$$
$$d^2 = 2rh + h^2$$

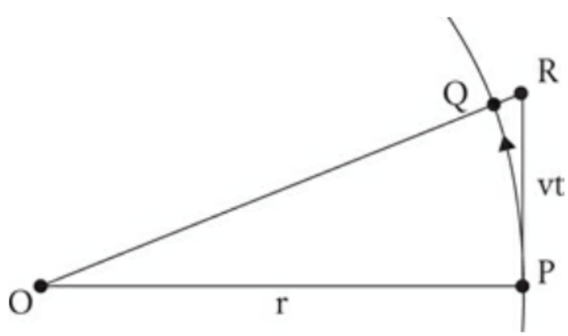
If $h << r$ then we can ignore the very small quantity h^2 and obtain

$$d^2 = 2rh$$
$$r = \frac{d^2}{2h}, \qquad h = \frac{d^2}{2r}$$

If the planet were not accelerated, then it would move from P to R, a distance d, and this is equal to the velocity $\times$ time:

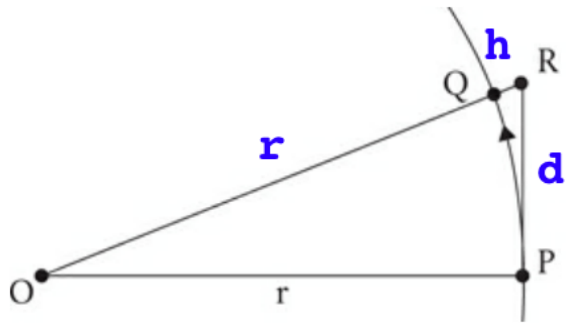
$$d = vt$$

At this point, we use an idea from calculus. *For a small enough segment of the orbit*, this distance PR is the same as the arc length PQ.



So we substitute for $d^2 = (vt)^2$ into the equation from above

$$h = \frac{d^2}{2r} \approx \frac{(vt)^2}{2r}$$



Also, for a small enough part of the orbit (again), h and d are perpendicular to each other as well.

At this point we use the additional assumption that the force is directed toward the sun. We might say that the distance *fallen* by the planet in this short time is h.

By the standard equation of motion, under gravitational acceleration g is related to h and the time t by this equation:

$$h = \frac{1}{2} g t^2$$

We combine the two different expressions for h

$$h = \frac{1}{2} g t^2 \approx \frac{(vt)^2}{2r}$$

$$g \approx \frac{v^2}{r}$$

Note: we have not covered this yet. If this idea (dependence on t^2) is completely new to you, you may want to come back to this part after going through the first **chapter** on calculus.

The equation $a = v^2/r$ comes even more easily with a little bit of calculus and the use of vectors. See **here**.

Kepler's Third Law

The famous mathematician Johannes Kepler (of whom much more later also), working with observational data from Tycho Brahe, had the following values for the radius R of the (assumed circular) orbit and the period T (time for completion of one orbit), for five planets.

Orbital data for the six planets known in Kepler's time

	$\bar{r}$ (units of $\bar{r}_{\text{Earth}}$)	T (years)
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.524	1.881
Jupiter	5.203	11.862

On the basis of this data, Kepler published his **third law** (in 1619, about 10 years after the first two). K3 states that

$$T^2 = kR^3$$

The square of the period is proportional to the cube of the radius of the orbit. The data in the table has been scaled so that $k = 1$.

For a circular orbit, the orbital speed, the magnitude of the velocity $v = |\mathbf{v}|$, is constant.

The period times the speed is equal to the circumference.

$$vT = C = 2\pi R$$
$$T = \frac{2\pi R}{v}$$

K3 above says that

$$R^3 = T^2$$
$$= \frac{(2\pi)^2 R^2}{v^2}$$

Hence

$$v^2 \approx \frac{1}{R}$$

We showed above that the acceleration for a circular orbit is

$$a = \frac{v^2}{R} = v^2 \cdot \frac{1}{R}$$

so we conclude that that

$$g = a \approx \frac{1}{R} \cdot \frac{1}{R} = \frac{1}{R^2}$$

if the acceleration of gravity g is directed toward the sun, with a magnitude that is inversely proportional to the square of the distance, then we can explain Kepler's third law by running this chain of reasoning in reverse.

comparing the moon to an apple

Earlier we worked out that the acceleration is

$$a = \frac{v^2}{R}$$

Let's figure out the acceleration of the moon. We make a decision to work in English units for this one.

The moon averages about 237 thousand miles from earth (221.5 - 252.7 thousand miles). The earth's circumference is about 24.9 thousand miles so its radius is about 3.96 miles. Thus, the ratio of the moon's distance to the center of the earth, compared to my distance to the center of the earth, is about 60 : 1 (ranging between 56-64).

What is the moon's velocity? The distance it travels in one complete orbit (in feet) is:

$$2\pi \cdot 2.4 \times 10^5 \cdot 5280$$

The time that takes in seconds is

$$28 \cdot 24 \cdot 3600$$

$$v = \frac{2\pi \cdot 2.4 \times 10^5 \cdot 5280}{28 \cdot 24 \cdot 3600}$$

The acceleration is v^2/R so we square everything except the radius.

$$a = \frac{(2\pi)^2 \cdot 2.4 \times 10^5 \cdot 5280}{(28 \cdot 24 \cdot 3600)^2} = 0.0085$$

That's in feet per second.

We compare this value to the acceleration measured at the surface of the earth, which is 32.2 in the same units. The ratio is 3788, which is just over $(61.5)^2$.

Newton:

> I began to think of gravity extending to the orb of the Moon . . . and computed the force requisite to keep the Moon in her Orb with the force of gravity at the surface of the earth . . . & found them answer pretty nearly. All this was in the

two plague years of 1665-1666. For in those days I was in the prime of my age for invention & minded mathematicks and Philosophy more than at any time since.

Part III

Analytic geometry and trigonometry

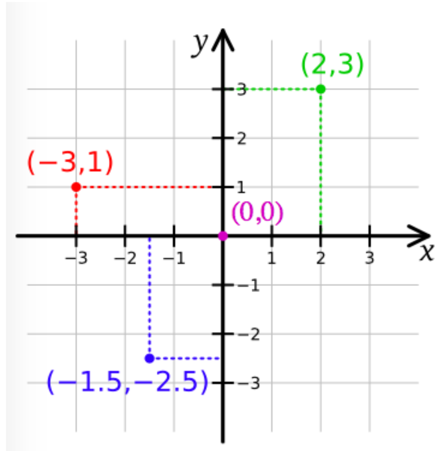
Chapter 11

Analytic geometry

It is difficult today to put ourselves in the places of those who tried to reason about mathematics through the ages. For example, the Greeks lacked algebra, and although the Romans worked with numbers they lacked decimal notation. The concept of 0 came much later (from India), and even in the Middle Ages there was as yet no such thing as the equals sign =, which dates from 1557.

https://en.wikipedia.org/wiki/Table_of_mathematical_symbols_by_introduction_date

The invention of analytic geometry is often ascribed solely to Descartes, but Fermat also had a big role. There are two fundamental ideas.



The first is to orient two number lines on a piece of paper, at right angles, and then consider pairs of numbers (x, y) in the 2D plane. Such pairs or tuples are called points.

Descartes published this idea in 1637. The presentation would be difficult to recognize as our current system, but the germ is there: axes where the position of a variable could be marked. Only the positive numbers would be shown, and the axes not necessarily perpendicular. As to the proofs, here is wikipedia on the subject:

> His exposition style was far from clear, the material was not arranged in a systematic manner and he generally only gave indications of proofs, leaving many of the details to the reader. His attitude toward writing is indicated by statements such as "I did not undertake to say everything," or "It already wearies me to write so much about it," that occur frequently. In conclusion, Descartes justifies his omissions and obscurities with the remark that much was deliberately omitted "in order to give others the pleasure of discovering [it] for themselves."

The second idea of analytic geometry is to plot all the points that satisfy some mathematic relationship between x and y, for example the parabola $y = x^2$.

To do this, pick a few values of x and calculate the corresponding values of y. For example: $(0,0), (\pm 1, 1), (\pm 2, 4), \ldots$. Plot these points, and then finally, sketch the graph of the curve, without actually trying to plot *all* of the individual points (of which there is an infinite number). We make the assumption here that the function being plotted is continuous, so that the sketch of a curve between two points that are close enough together will be fairly smooth and for x-values close to the plotted x, have corresponding y-values not too different from the plotted y.

formulas for a line

Suppose we pick two points (x_1, y_1) and (x_2, y_2), plot them on a graph, and then draw the line that connects them. Recall Euclid's first two postulates:

◦ A straight line segment can be drawn joining any two points.

◦ Any straight line segment can be extended indefinitely in a straight line.

Now we want to derive an equation that describes (is valid for) all the points or pairs of values (x, y) on this line. A general approach is to say that the line has some slope m, which is defined as the rate of change of y, called Δy, divided by the rate of change of x:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

This is the *point-slope equation.*

The value of m might be zero, for a horizontal line, where all the values of y are the same. Or it might be undefined, for a vertical line, where all the values of x are identical.

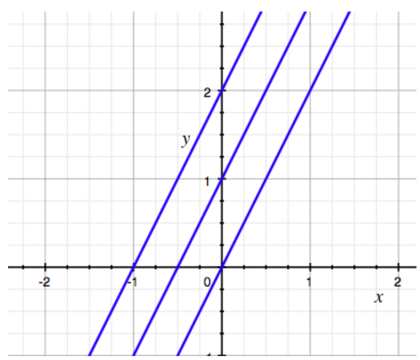
In most cases, however, $m \neq 0$ and $m \in (-\infty, \infty)$. That is, $|m|$ is not infinite but also non-zero.

Except in the case of the vertical line, we can write

$$y = mx + b$$

for any two points x and y on a given line. b is called the y-intercept, it is the value of y obtained when $x = 0$. This is the *slope-intercept equation* of the line.

Notice that the equation of a line is not determined just by the slope. One can draw a whole family of parallel lines with the same slope and different y-intercepts. For example here are three lines $y = 2x + b$ for $b = \{0, 1, 2\}$.



The value of x corresponding to $y = 0$ is the x intercept

$$x = -\frac{b}{m}$$

The first equation is easily derived from the second. Plugging in specific values of x and y we have

$$y_1 = mx_1 + b$$

$$y_2 = mx_2 + b$$

Subtracting

$$y_2 - y_1 = m(x_2 - x_1)$$

which rearranges to give the desired result.

formula for a circle

A circle can be defined as all the points at the same distance from a central point, let us label that point (h, k). The distance from the points to the center is the radius, denoted r.

Using the Pythagorean theorem, we can calculate the square of the distance from the origin as

$$r^2 = (x - h)^2 + (y - k)^2$$

The simplest circles are those whose central point is the origin of the coordinate system. In that case the equation simplifies to

$$r^2 = x^2 + y^2$$

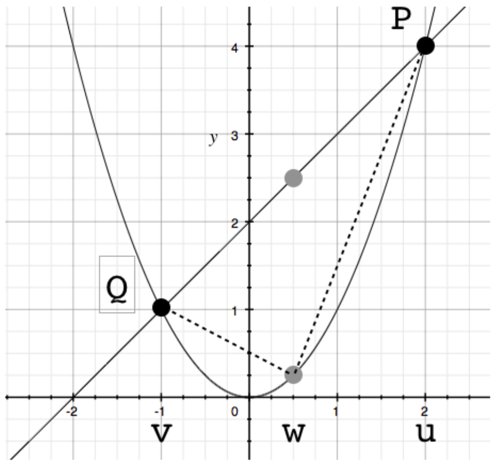
Usually, we know the value of r and we want to write an equation for y in terms of x. Then

$$y^2 = r^2 - x^2$$

$$y = \sqrt{r^2 - x^2}$$

example

Here is a moderately complicated example we will see much later in the book (**here**).



It shows the two axes (horizontal and vertical lines with distances marked), and four points plotted plus a line, two line segments, and a parabola. We can also "read" other points off the graph, such as the vertex of the parabola $(0, 0)$, or the intersections of the line with the axes at $(-2, 0)$ and $(0, 2)$.

It is assumed you've studied analytic geometry before, so we won't say much more now than what is in this chapter.

Much of a standard course in analytic geometry is concerned with the conic sections: circle, ellipse, parabola and hyperbola. Those are all wonderful topics, but we'll wait to explore them in detail in later chapters, where we can use a bit of calculus and also linear algebra.

For now, let's continue with trigonometry before getting back to the main subject.

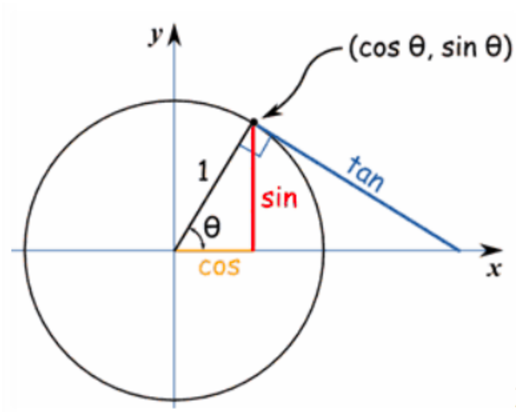
Chapter 12

Six functions

The most elementary trigonometric functions are sine and cosine.

basic definitions

The "unit circle" is a circle of radius 1 with its center positioned at the origin of coordinates, the place where the x and y axes cross. From the diagram you can see that any point (x, y) on the unit circle can be described in radial coordinates as $(\cos\theta, \sin\theta)$.



That is:

$$x = \cos\theta \qquad y = \sin\theta$$

If the circle has radius r then

$$x = r\cos\theta \qquad y = r\sin\theta$$

The tangent is

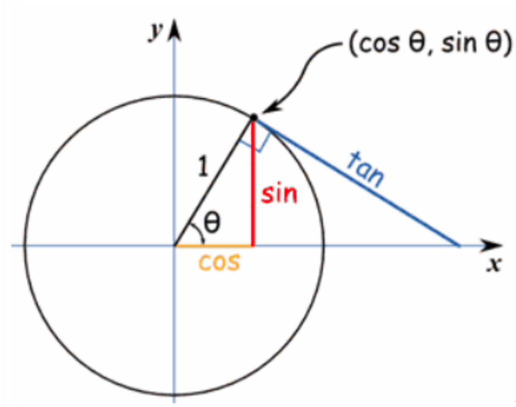
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

In the diagram, all three right triangles are similar.

Thus, by similar triangles, the blue side has this relationship

$$\frac{\text{blue side}}{1} = \frac{\sin\theta}{\cos\theta}$$

which explains why it is labeled as it is.



Stewart:

> The mathematicians of ancient India built on the Greek work to make major advances in trigonometry. They [used] the sine (sin) and cosine (cos) functions, which we still do today. Sines first appeared in the Surya Siddhanta, a series of Hindu astronomy texts from about the year 400, and were

developed by Aryabhata in Aryabhatiya around 500. Similar ideas evolved independently in China.

The other functions are the inverses of sine, cosine and tangent, namely: cosecant, secant and cotangent. The secant (inverse cosine) comes up, but the other two are not especially important in calculus. However, they do come up in one context that we will look at, Archimedes determination of the value of π. The crucial step in that approach will turn out to be the calculation of the cotangent of the half-angle $\theta/2$ given the values of cotangent and cosecant for angle θ.

The main relationship or identity is derived from the Pythagorean theorem. We had above that for a unit circle

$$x = r\cos\theta \qquad y = r\sin\theta$$

Since x and y are the sides of a right triangle whose hypotenuse is r

$$x^2 + y^2 = r^2$$

and for a unit circle

$$\cos^2\theta + \sin^2\theta = 1$$

which is usually written

$$\sin^2\theta + \cos^2\theta = 1$$

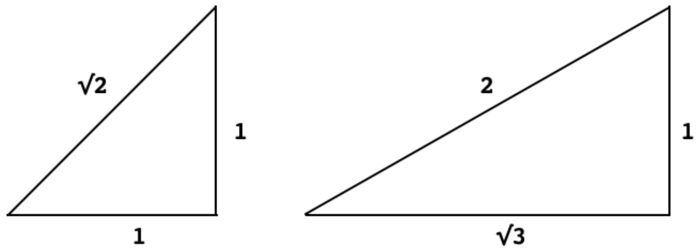
and transformed to

$$1 + \tan^2\theta = \sec^2\theta$$

It is assumed you've studied trigonometry before.

We can easily determine the values for these functions for three special cases.

The first is the angle 45 degrees or $\pi/4$. Draw an isosceles right triangle with sides of length 1 (left panel).



Then the hypotenuse has length $\sqrt{2}$ (from Pythagoras) and the values are

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$$

$$\tan\frac{\pi}{4} = 1$$

For the other two, bisect an equilateral triangle and erase one half (right panel). The smaller angle is 30 degrees or $\pi/6$ and its complement is 60 degrees or $\pi/3$.

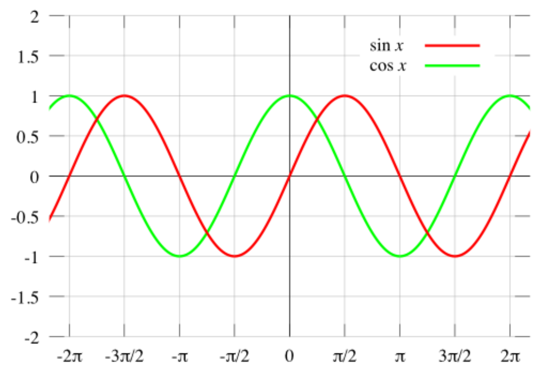
The values are

$$\sin\frac{\pi}{6} = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$$

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

graph

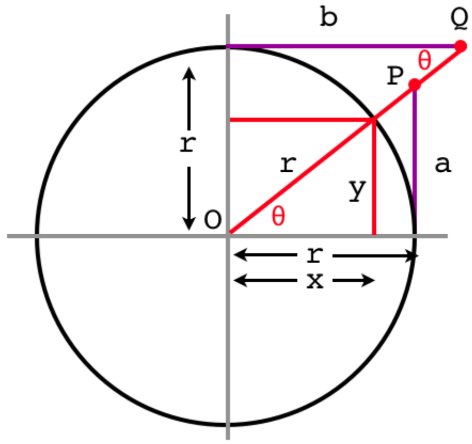


Savov:

> The sine function represents a fundamental unit of vibration. The graph of $\sin(x)$ oscillates up and down and crosses the x-axis multiple times. The shape of the graph of $\sin(x)$ corresponds to the shape of a vibrating string.

visualization of all six functions

Consider a unit circle. Extend the radius with the angle θ and then draw the vertical and horizontal tangents to the circle a and b.



The original triangle with sides x, y, r is similar to the triangle with sides r, a, OP, and both are similar to the triangle with sides b, r, OQ.

$$x, y, r \sim r, a, OP, \sim b, r, OQ$$

By similar $\triangle$

$$\frac{a}{r} = \frac{y}{x} = \tan\theta$$

But $r = 1$ so

$$a = \tan\theta$$

If you imagine a point moving around the circle a will get very large as $\theta \to \pi/2$, and in fact, approaches ∞ there (becomes undefined).

The segment OP is (by similar $\triangle$) to r as

$$\frac{OP}{r} = \frac{r}{x}$$

$$OP = \frac{1}{\cos\theta} = \sec\theta$$

The horizontal from the y-axis to Q is b. Consider θ near the top of the figure. By similar $\triangle$, the relations we had were

$$r/b = y/x = \tan\theta$$

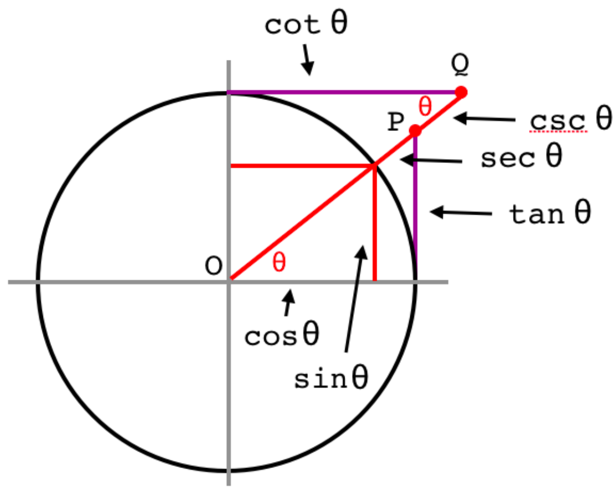
since $r = 1$

$$b = \frac{r}{\tan\theta} = \frac{1}{\tan\theta} = \cot\theta$$

Finally

$$r/OQ = 1/OQ = \sin\theta$$

$$OQ = \frac{1}{\sin\theta} = \csc\theta$$



Chapter 13

Sum of angles

cosine of a sum

The sum of angle formulas (i.e. formulas for the sine and cosine of the sum or difference of two angles) are used often in calculus, not only for working problems, but even in the derivation of the "derivative" of sine and cosine.

You really must know them. I think it's so important that we will show three ways of finding these formulas — not all in this chapter. The easiest way to remember them uses Euler's equation, and we won't be ready for that until later. See **here**.

There are four equations: $\sin s \pm t$ and $\cos s \pm t$.

I've memorized only this one:

$$\cos s - t = \cos s \cos t + \sin s \sin t$$

By $\cos s - t$ we mean $\cos(s - t)$, but have left off the parentheses.

Say "cos cos" and then recall the difference in sign.

check

I like this version because it can be checked easily. Set $s = t$:

$$\cos s - t = \cos 0 = 1 = \cos^2 s + \sin^2 s$$

which is our favorite trigonometric identity and obviously correct.

change signs

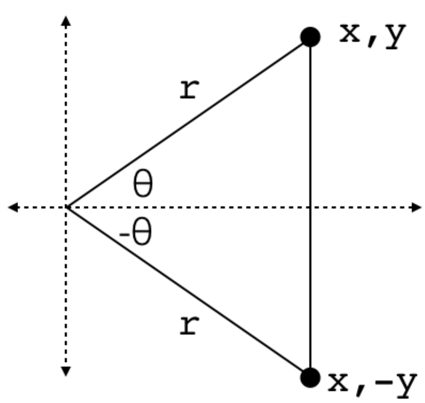
For $\cos s + t$ flip the sign on the second term.

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

This is simply a result of the fact that

$$\cos -\theta = \cos \theta$$

$$\sin -\theta = -\sin \theta$$



The diagram shows the reason: $\cos \theta = \cos -\theta = x/r$ while $\sin \theta = y/r = -(\sin -\theta) = -(-y/r)$.

Proof:

$$\cos(s - (-u)) = \cos s \cos(-u) + \sin s \sin(-u)$$

Since $\cos -x = \cos x$ and $\sin -x = -\sin x$:

$$\cos(s + u) = \cos s \cos u - \sin s \sin u$$

But u is just a dummy variable (it could be any symbol), so

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

sine of a sum

We will look at the proof for the sine formula later, for now just write it:

$$\sin s + t = \sin s \cos t + \sin t \cos s$$

Say "sin cos" and then, that here + goes with +. Like most things having to do with sine and cosine, there is a change of sign when changing from one to the other.

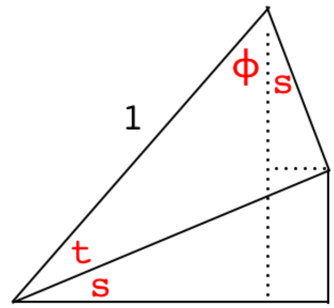
For $\sin s - t$, flip the sign on the second term, as before.

proof

Here is a geometric proof of both of the sum of angles formulas, using similar triangles. The key is to draw an inspired diagram.

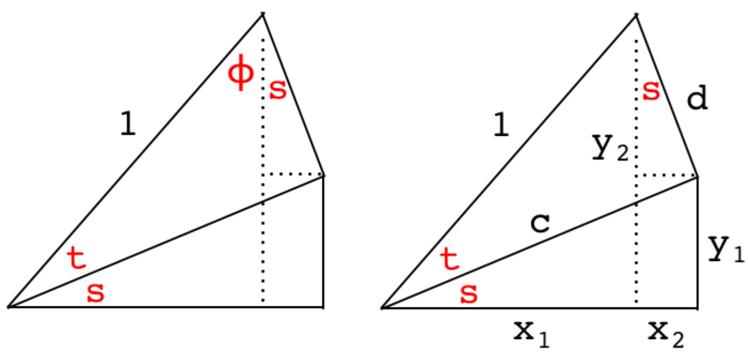
Consider a right triangle, with one of the angles labeled s. Construct another right triangle containing angle t, and scale it so that the base adjacent to angle t is just as long as the hypotenuse of the triangle containing angle s, and draw them one on top of the other as shown:

Scale the joined triangles so that the hypotenuse of the second triangle has unit length. Our crucial insight is to draw vertical and horizontal dotted lines as shown below.



The angle s is part of a right triangle with angle t adjacent, where the third acute angle is ϕ. But ϕ is also part of a second right angle containing t plus the angle adjacent to ϕ. Therefore, that adjacent angle is also equal to angle s.

We add some labels to the sides of the triangles and calculate the sine and cosine of s, t and $s + t$:



Since I already know the result I am looking for, I write what we had before

$$\cos s \cos t - \sin s \sin t$$

From the figure

$$\cos s = \frac{x_1 + x_2}{c}; \quad \cos t = \frac{c}{1}; \quad \cos s \cos t = x_1 + x_2$$

The sine of s is a little trickier, look at the small right triangle at the top of the figure

$$\sin s = \frac{x_2}{d}; \quad \sin t = \frac{d}{1}; \quad \sin s \sin t = x_2$$

The difference is

$$\cos s \cos t - \sin s \sin t = x_1$$

but from the diagram it's clear that

$$\cos s + t = x_1$$

□

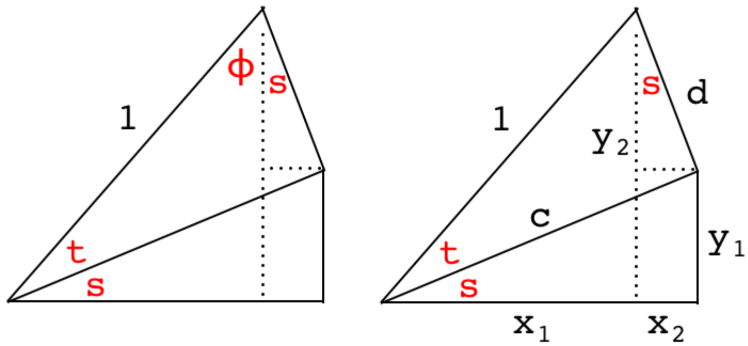
As a quick check we can ask what happens to the formula

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

when $t = 0$. Then the first term is the cosine of s, and the second term is equal to 0. The formula is symmetrical with respect to s and t.

extension to sine

Referring back to the diagram (and again, with our goal clearly in mind)



$$\sin s = \frac{y_1}{c}; \qquad \cos t = \frac{c}{1}; \qquad \sin s \cos t = y_1$$
$$\sin t = \frac{d}{1}; \qquad \cos s = \frac{y_2}{d}; \qquad \sin t \cos s = y_2$$

But

$$\sin s + t = y_1 + y_2 = \sin s \cos t + \sin t \cos s$$

Using the even/odd function rules, we get

$$\sin s - t = c + d = \sin s \cos t - \sin t \cos s$$

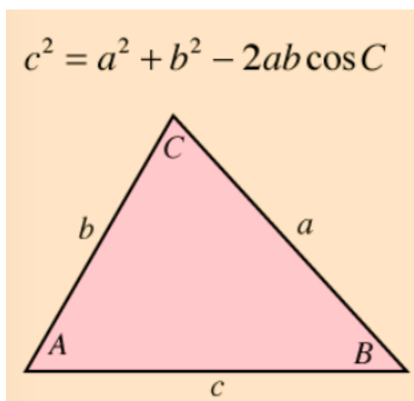
And that's all four of them.

Chapter 14

Law of cosines

Law of cosines

Designate the lengths of a triangle's sides as a, b, c and the angle between sides a and b as C (because it is opposite side c). The law of cosines says that



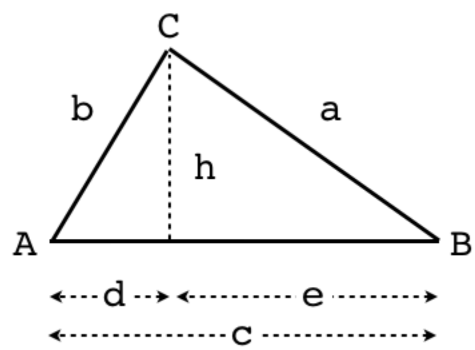
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Lockhart calls this the "generalized" Pythagorean theorem. We can view the term $-2ab\cos C$ as a correction term which disappears in the case where $\angle C$ is 90 degrees.

derivation

The result follows from the Pythagorean Theorem. (In fact, we can reuse the same diagram that was shown for the algebraic proof of the theorem).

For a triangle with sides a, b and c and angles opposite those sides A, B and C, divide the third side into two lengths $c = d + e$ using the vertical altitude from vertex C.



$$a^2 - e^2 = h^2 = b^2 - d^2$$

So

$$a^2 = e^2 + b^2 - d^2$$

Since $d = c - e$ and thus $d^2 = c^2 - 2ce + e^2$:

$$a^2 = e^2 + b^2 - (c^2 - 2ce + e^2)$$

$$= b^2 - c^2 + 2ce$$

but $e = a \cos B$ so

$$a^2 = b^2 - c^2 + 2ac \cos B$$

rearrange to give a more familiar form (this is the law of cosines)

$$b^2 = a^2 + c^2 - 2ac\cos B$$

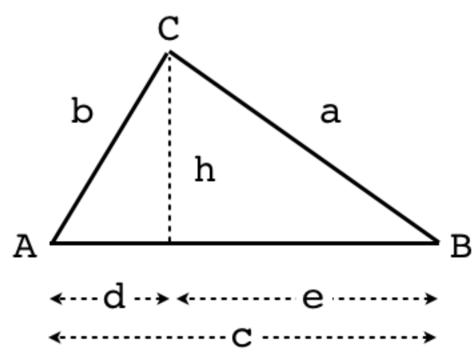
Any side of a triangle can be expressed in terms of the other two and the cosine of the angle between them. Thus, for example

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Law of sines

I'll just mention that there is another law called the law of sines. In contrast to the law of cosines, it is fairly trivial.



$$\frac{h}{b} = \sin A \qquad \frac{h}{a} = \sin B$$

Therefore

$$h = b\sin A = a\sin B$$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

We could do the same construction and argument with A and C or B and C. Therefore

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Part IV

Two basic operations in calculus

Chapter 15

Simple slopes

To introduce the two fundamental ideas in calculus, consider two measuring devices used while driving a car. Most good drivers look fairly often at the speedometer, which measures speed or velocity, or how fast you're going.

On the other hand, if someone gives you directions like "go three and a half miles and then turn left (where the old gas station used to be)" you will be watching your odometer.

Velocity times time = distance. We can think of speed and velocity as the same for now. Distance divided by time is velocity.

Velocity is the *rate of change* of distance with time, it has units of distance divided by time (say, miles per hour).

In calculus we say that the velocity is the **derivative** of the distance with respect to time, and the distance is the **integral** of the velocity with respect to time.

We can speak of velocity at a particular time t, as in "our current velocity is 60 miles per hour." But the distance, the integral, must be evaluated between appropriate starting and stopping points for the time. In our example, you must first look at your odometer *before* you start on that 3.5 mile drive.

time-dependence

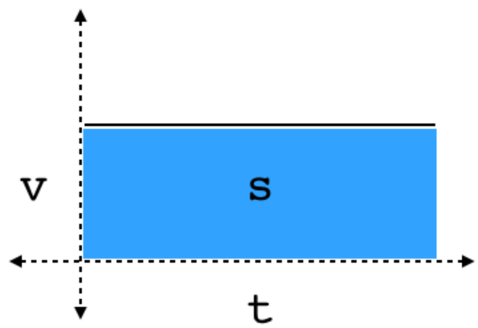
Distance equals velocity times time.

This is easy if the velocity is constant. Travel west on the interstate at exactly 60 miles per hour for 2 hours and your distance will be 120 miles from where you started (provided you don't start in Los Angeles). It is standard to use s to refer to the distance traveled and v for velocity. If the velocity is constant then:

$$s = vt$$

According to the internet, s is from the Latin "spatium", for "space, room, or distance."

Suppose we plot velocity as a *function of time* with v on the y-axis and t on the x-axis.



Since the velocity is constant, the result is a straight horizontal line. Furthermore, the distance traveled is the *area under the curve* (and above the x-axis) which is the area of a rectangle with sides v and t and as we said

$$s = vt$$

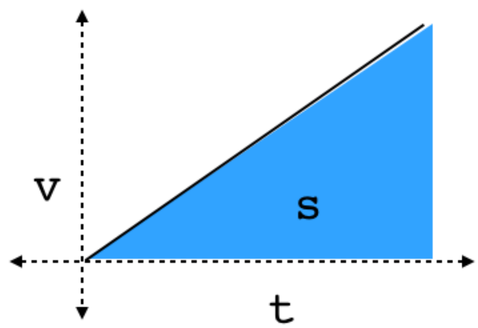
However, for most interesting problems velocity is not constant.

Imagine maintaining pressure on the gas pedal in the car steadily so that, starting from a stop at zero time, after 1 second your velocity is 10 mph, after 2 seconds it is 20 mph, after 3 seconds, 30 mph. If we continue at the same rate of acceleration, we'll go from 0 to 60 mph in 6 seconds, which is quite a respectable time.

This example has constant acceleration. Here, we say that v is a constant function of time, and write

$$v = at$$

where a is the acceleration.



What about the distance?

If a is not zero then v changes with time. If a is non-zero and constant, then v changes at a constant rate. Starting from 0, the final velocity will be $v = at$, but the distance traveled is no longer the product

$$s = v \times t \stackrel{?}{=}$$

because this v is the final velocity and that is not the correct v to use. For variable velocity, the distance traveled is the *average* velocity times the time. For smooth (constant) acceleration from zero to v, the average velocity is the average of the initial and final velocities:

$$v_{\text{avg}} = \frac{1}{2}\ (v_i + v_f) = \frac{1}{2}\ v$$

So the correct equation is:

$$s = v_{\text{avg}}\ t = \frac{1}{2}\ v \cdot t$$

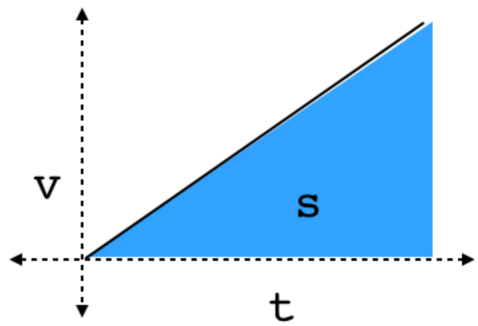
and since $v = at$

$$s = \frac{1}{2} a t^2$$

In this case, if we plot velocity as a function of time, we obtain a straight line that extends diagonally up with respect to the x-axis.

The distance traveled is the area under the curve, below the line and above the x-axis.

The shape whose area is needed is a triangle. This also accounts for the factor of 1/2.



You probably know that if a mass m is dropped from a tall building like the Tower of Pisa, then the distance it has fallen goes like the square of the time. The equation is:

$$s = \frac{1}{2} g t^2$$

where g is the acceleration due to gravity.

Notice that this is the same equation as we had earlier. The reason is that g is approximately constant near the surface of the earth. g is about 10 in units of m/s^2. A fall of four seconds is about 80 meters.

Galileo knew this formula (at least, he knew the t^2 part of it), which he obtained not from experiments at the Tower of Pisa, but by timing the descent of balls down an inclined plane.

initial position and velocity

If you want to be more complete and say that the starting point is not necessarily the origin of the coordinate system, add a constant s_0 to describe the initial distance from the origin and obtain:

$$s = vt + s_0$$

and similarly, a constant v_0 to describe the initial velocity as shown above.

The full equation of motion is

$$s = \frac{1}{2}at^2 + v_0t + s_0$$

We'll say much more about this later.

power rule

We will introduce the theory of calculus more formally in the next section of the book. For now, we just talk about a simple rule called the power rule.

Switching notation to y and x, suppose that y is a *function* of x and write $y = f(x)$.

Here are three types of dependency (with c as a constant), with three corresponding types of graph.

$$y = c$$

$$y = cx$$

$$y = cx^2$$

These are (respectively) the equations of: (i) a horizontal line, since y is constant, (ii) any other non-vertical line (y is proportional to x), and (iii), a parabola.

We ask "what happens if we change x a little bit" and use the notation dx to refer to this little bit of x.

What happens to y? y will usually change by a small amount. Call that amount dy.

However, in the first case, $y = c$, y does not actually depend on x at all. The result (dy, the change in y for a change in x, dx) is zero.

$$y = c, \qquad dy = 0 \cdot dx$$

The ratio dy/dx is the slope of the curve formed by plotting y against x. We call that slope the *derivative* of the function $f(x)$.

Divide both sides by dx and rewrite the above as:

$$\frac{dy}{dx} = 0$$

The plot is a horizontal line with slope 0.

In the second case, y is a linear function of x, the change in y, dy is the change dx multiplied by c:

$$y = cx, \qquad dy = c \cdot dx$$

rearranging.

$$\frac{dy}{dx} = c$$

In analytical geometry, we calculate the slope of a line as $\Delta y/\Delta x$.

For a line, the slope is constant and so it doesn't matter which two points with coordinates $(x_1, y_1), (x_2, y_2)$ we choose for the calculation. The following is true for *any* two pairs (x, y) on the line:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Above we had the example where $v = at$ with constant a. Then $dv/dt = a$.

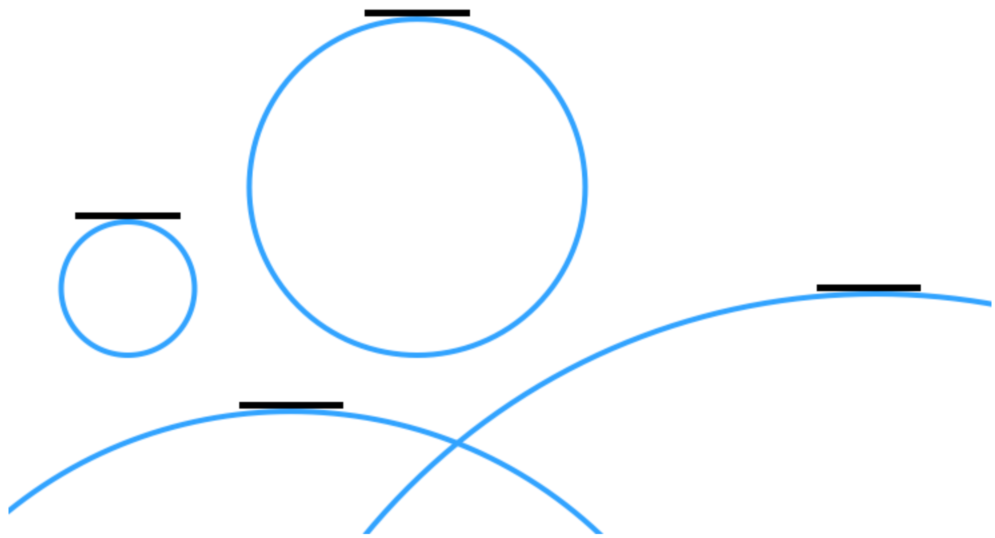
The third case is different.

$$y = cx^2$$

For a parabola, the slope of the curve at a point (the slope of the tangent to the curve $y = cx^2$) depends on the choice of x. The slope is steeper the further out you go in a positive direction on the x-axis.

It seems impossible to compute the slope of this curve in the standard way, by picking a second point near (x, y) and then calculating $\Delta y/\Delta x$, because the slope changes as we go out along the curve.

The key insight is that if x_1 is sufficiently close to x_2 the slope is constant. It's like saying that the earth is flat *locally*. If you detect any curvature, just zoom in a bit. In the figure



the line has constant length, but the distance to the circle from the end of the line decreases as we increase the size of the circle. In reality, we keep the curve the same size and decrease the length of the line, and then magnify the whole picture, until we get what you see.

Just zoom in until the line is a good enough approximation to the shape of the circle, if the curve doesn't look flat enough, zoom in some more.

As we are accelerating in the car, with constantly changing velocity, we can still have a unique velocity at a particular instant in time.

Or, put still another way, for a very small change Δx in either direction from x, we can get the same slope:

$$m = \frac{y + \Delta y}{x + \Delta x} = \frac{y - \Delta y}{x - \Delta x}$$

if Δx is small enough. If it's not, we can always make it smaller. That's the beauty of the real numbers.

Since the changes in x and y are so small, we use the new nomenclature: dy and dx.

power rule

To actually calculate slopes for curves (and straight lines), use the power rule.

For a horizontal line with zero slope:

$$y = c$$

$$\frac{dy}{dx} = 0$$

For a line with a slope c:

$$y = cx$$

$$\frac{dy}{dx} = c$$

For the parabola, the rule says that if $y = cx^2$, the slope or derivative is

$$\frac{dy}{dx} = 2cx$$

We've been writing c as the constant, so as not to confuse it with a, the acceleration. In analytic geometry, a parabola is usually written with a constant a, called the shape factor:

$$y = ax^2$$

Then, the slope is $2ax$.

If we had

$$y = ax^2 + bx + c$$

with a, b, c all constant, then the slope would be $2ax + b$.

The above uses our three rules from above, plus one more, that when taking the derivative of a polynomial, the derivative of the whole is simply the summed derivatives for each term.

For the equation of motion under gravity

$$s = \frac{1}{2}at^2 + v_0 t + s_0$$

$$v = \frac{ds}{dt} = at + v_0$$

$$\frac{dv}{dt} = a$$

Notice how the 1/2 and the 2 cancel in the second equation.

Continuing to the cubic, if y depends on x^3 like

$$y = cx^3$$

then

$$\frac{dy}{dx} = 3cx^2$$

The general form of the power rule is that if

$$y = x^n$$

then

$$\frac{dy}{dx} = nx^{n-1}$$

The exponent has been reduced by 1 power, and the value of that exponent applied as a factor in front of the expression.

This rule had already been discovered before Newton. It's a toss-up whether Fermat or Cavalieri was first. We will prove this later, but for now we just want to introduce the idea and practice using it.

note

If you already know some calculus you're probably jumping out of your chair while reading this chapter because you've had it pounded into you that dy/dx is not a quotient and believe that you can't simply multiply both sides of the equation by dx.

Well, you can. And I'll explain why as we go along.

Chapter 16

Easy pieces

Integration

Differentiation breaks things up into small pieces dx or dr. Integration adds up many little pieces. The symbol for integration is a relaxed S that stands for summation: $\int$.

As Thompson says

> The word integral simply means the whole. If you think of the duration of time for one hour, you may (if you like) think of it as cut up into 3600 little bits called seconds. The whole of the 3600 little bits added up together make one hour.

We boldly claim that from the point of view of problem-solving, integration is simply the inverse of differentiation.

Mathematicians hate this kind of talk, because it trivializes a profound statement, the fundamental theorem of calculus.

But for practical problem-solving our counter-claim is that this profundity *doesn't matter*. It is also likely to confuse the beginning student, another reason to put it aside for the time being. We'll return to this issue later, when we cover the theory of the subject very lightly.

The sum of a bunch of small pieces dy is equal to the sum of a bunch of small pieces dx times cx, when $dy/dx = cx$ describes how y changes with small changes in x at any particular point.

The key idea is *at any point*. The relationship between dy and dx depends on where you are on the curve. That's why we need integration.

Write

$$dy = f(x)\ dx$$

We want to solve

$$\int dy = \int f(x)\ dx$$

The sum of all the little pieces dy is just y

$$y = \int f(x)\ dx$$

Now, this surely sounds a little vague. But it will turn out that

$$F(x) = \int f(x)\ dx = y$$

exactly when the derivative of $F(x)$ is $f(x)$:

$$\frac{dF}{dx} = F'(x) = f(x)$$

This is the first of two bright ideas we need to solve an equation like $\int f(x)\ dx$. Just find $F(x)$ such that the derivative of $F(x)$ is $f(x)$.

Area of the circle

Let's spend some time analyzing the area of a circle. This provides crucial insight into what integral calculus can do.

Integration is used to compute areas and volumes, and other sums, by adding up many little pieces.

To calculate the area of a circle, we find the pieces we will use with one of three basic strategies: rings, slices of pie, or rectangles of area underneath the function obtained by solving $x^2 + y^2 = R^2$ (using the positive square root). These three approaches are illustrated in the figure above.

rings

In the first approach (left panel), we imagine the area being computed by adding up the individual areas of a series of very thin, concentric rings.

The total area to be computed is that of a circle of a definite, fixed size, and we denote the radius of this circle by capital R, a constant. On the other hand, the series of rings ranges from the origin of the circle to the circumference of the outmost ring. Each one of this progression of rings has a radius, so we use the lowercase r to describe them, with r being a variable—r varies from 0 at the origin to R at the outside of the circle.

Think about an individual ring, for example the outermost ring, which is similar to the circular peel or rind surrounding a thin slice of lemon.

We are working with areas here, in two dimensions, so the slice we imagine to be infinitely thin, and we are working with it as a cross-section or ring.

The area of the ring is the length times the width. The length is the circumference, $2\pi R$ for the outermost ring, but in general, for any of the inner rings it is $2\pi r$. The length is multiplied by the width of the slice, which is a small element of radius, dr. The small element of area contributed by an individual ring is dA:

$$dA = 2\pi r \ dr$$

Another way to explain this equation is to ask the question:

how does area change with increasing radius?

If we take a circle and increase its radius by a little bit, how does the area change? The answer is, it changes in proportion to the circumference, $2\pi r$.

Another way to say the same thing is that the derivative is

$$\frac{dA}{dr} = 2\pi r$$

Proceeding from the first equation, the total area is the sum of the areas for the series of rings.

$$A = \int dA = \int_0^R 2\pi r \ dr$$

It's worth emphasizing how this view is different than the examples of integration one usually sees first in a calculus book: these pieces of area are not rectangles but circles. But it poses most clearly the question we are trying to answer, "how does area change as r changes"?

In order to actually determine a value for the area we need two principles. The first is, as we mentioned before, that the solution to

$$\int f(x)\ dx$$

is $F(x)$ if and only if the derivative of $F(x)$ is equal to $f(x)$.

Continuing with our problem

$$\int 2\pi r\ dr = 2\pi \int r\ dr$$

In this step we used a fundamental rule that a constant can come "out from under" the integral sign. That's not surprising. We already know that (at least in the power rule) the derivative of a constant times some function is that constant times the derivative of the function. We will show that is a general rule later.

Now, we need to find a function whose derivative is r.

$$2\pi \int r\ dr$$

We know that function, it is r^2, with an extra factor of 1/2.

$$= 2\pi\ [\ \frac{1}{2}\ r^2\ ]\ = \pi r^2$$

Combining all the coefficients we have $\int 2\pi r\ dr = \pi r^2$ precisely because the derivative of πr^2 is just $2\pi r$.

The second principle we need comes from the Fundamental Theorem of Calculus, which takes account of the bounds on the integral (in this case 0 and R). The bounds are written attached to the integral as

$$\int_0^R$$

and on the expression to be evaluated attached to a vertical bar

$$\Big|_{r=0}^{r=R}$$

like this

$$2\pi \int_{r=0}^{r=R} r\ dr = \pi\ r^2\ \Big|_{r=0}^{r=R}$$

We say that the answer is this function, "evaluated between the bounds 0 and R."

The value of such a definite integral is $F(x)$ evaluated at the upper limit minus the value of $F(x)$ evaluated at the lower limit:

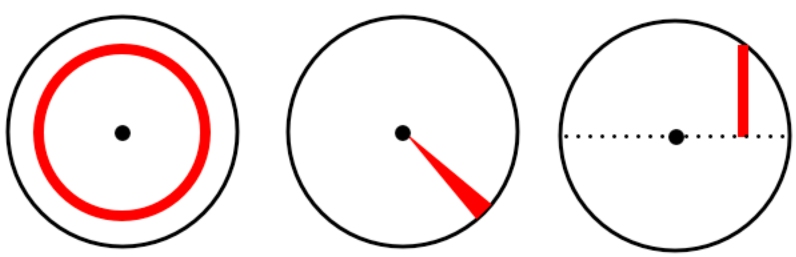
$$= \pi R^2 - \pi(0)^2 = \pi R^2$$

which appears to be correct.

Note in passing that the lower bound doesn't have to be 0, it could be some $\rho < R$. Then we'd have the area of a ring rather than a circle. And another thing, it's not uncommon to leave out the variable from the bounds, and write it like this:

$$2\pi \int_0^R r\ dr$$

wedges



In the second method (middle panel), we need to first find the area of a wedge. For a thin enough slice, this is a triangle, with a familiar formula: one-half the base times the height. The height is R, the radius of the circle.

For the base we need the length of a piece of arc of a circle. Recall that by definition, if we have a unit circle, then the angle of a wedge is equal to the arc it cuts out, and vice-versa, the arc is equal to the angle. (Thus, the total length if we go all the way around the unit circle is 2π).

For a circle with radius R, the length going all the way around is $2\pi R$, and the length of arc for any angle θ is θ times R.

The area we want is built up of a series of wedges that are almost infinitely slender, with angle $d\theta$, so these wedges have bases measuring $R\ d\theta$. The area of each triangular wedge is one-half the height times the base or

$$dA = \frac{1}{2} R\ R\ d\theta$$

For the total area

$$A = \int dA = \int \frac{1}{2} R\ R\ d\theta$$

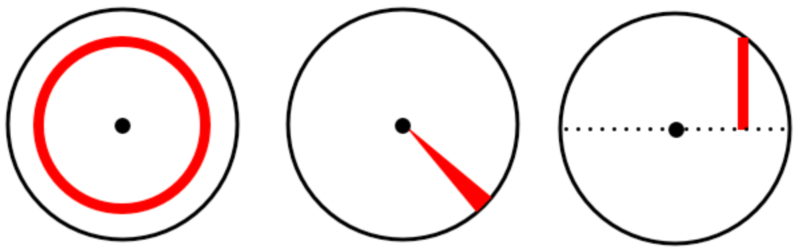
again we see that constants can come outside the integral

$$= \frac{1}{2} R^2 \int_{\theta=0}^{\theta=2\pi} d\theta$$

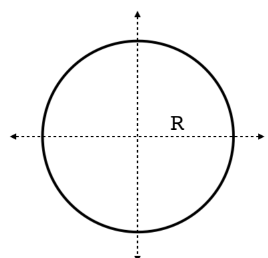
$$= \frac{1}{2} R^2 \theta \ \Big|_{\theta=0}^{\theta=2\pi}$$

$$= \pi R^2$$

area under the curve



The third view (right panel) is the most familiar, but has a somewhat harder calculation. We calculate the area under the positive square root in the equation for a circle (right panel), lying above the x-axis, and then multiply by two to get the whole thing.



$$x^2 + y^2 = R^2$$
$$y = f(x) = \sqrt{R^2 - x^2}$$

To get the area, we need to integrate:

$$\int y \ dx = \int_{-R}^{R} \sqrt{R^2 - x^2} \ dx$$

We will work through this problem **later**, after we review a few more techniques that are useful in doing integration problems.

Of course, the answer will turn out to be just what you'd expect. In fact, this must be so. If we solve the same problem by correctly using two different techniques and get different answers, then at least one of the techniques is wrong.

The area beneath the circle $y = \sqrt{R^2 - x^2}$ and above the x-axis is

$$\frac{1}{2}\pi R^2$$

which is multiplied by 2 to get the area of the whole circle.

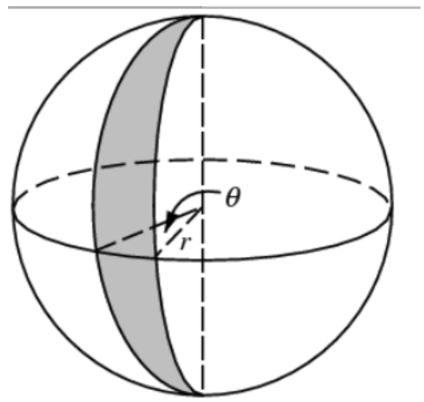
Volume of the sphere

We think about how the volume of the sphere depends on r ($r = 0 \rightarrow R$). An incremental change dr changes the volume by adding a thin shell of volume equal to the surface area of the sphere ($4\pi r^2$) times dr. That is

$$dV = 4\pi r^2 \ dr$$
$$V = \int dV = \int_0^R 4\pi r^2 \ dr$$
$$= 4\pi \ \ \frac{1}{3}r^3 \ \Big|_0^R = \frac{4}{3}\pi R^3$$

It's really as simple as that. Of course, you need to know the formula for the surface area to do it that way. Alternatively, if you know the volume of the sphere, taking the derivative is an easy way to get a formula for the surface area.

The image shows a "spherical lune", or segment of the surface of the sphere, as an aid to visualizing the whole surface.



We'll say a lot more about the volume of the sphere **later**.

technical note

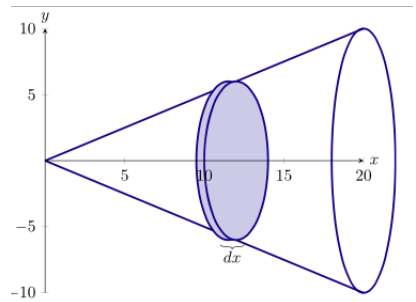
We should point out that this connection between volume and surface area is not true for *every* solid.

As an example, the surface area of a cube of side s is $6s^2$, which would have volume $2s^3$ if the relationship were always correct. In fact, there is something special about the *radial symmetry* of circles and spheres, and their lack of sharp corners and edges.

Here is one more example, to calculate the volume of a cone.

volume of a cone

We lay a cone along the x-axis with its vertex at the origin, opening to the right.



The cone is three-dimensional with the third axis (z) coming up out of the page. The intersection with the xy-plane is a triangle.

Can you see that in the xy-plane y is a linear function of x, i.e. $y = kx$ where k is a constant. The constant k is actually the ratio of the radius R to the height H. That is equal to $\Delta y/\Delta x$.

$$y = \frac{R}{H}x$$

If we slice the cone into thin sections perpendicular to the x-axis, each little piece is a circle with radius y and area πy^2. For a thin enough slice, the volume is that area times the width of the slice:

$$dV = \pi y^2 \ dx$$

Finding the volume of an individual piece is the important part of the calculus argument.

Now we just substitute the value of y in terms of x

$$dV = \pi \ [\frac{R}{H}]^2 x^2 \ dx$$

add up all the little volumes by setting up the integral

$$V = \int dV = \int \pi \ [\frac{R}{H}]^2 x^2 \ dx$$

We apply the basic rule that constant terms can move "out from under" the integral sign:

$$= \pi \ [\frac{R}{H}]^2 \ \int x^2 \ dx$$

This is a corollary of the result that constants are just carried through in taking the derivative.

We recognize that the value x lies in the interval between 0 and H, $[0, H]$, so these are the "bounds" on the integral, which we write as $\int_0^H$:

$$= \pi \ [\frac{R}{H}]^2 \int_0^H x^2 \ dx$$

and then just follow the rule for doing a problem like this: $\int x^2 = x^3/3$. So

$$= \pi \ [\frac{R}{H}]^2 \ [\frac{x^3}{3}] \ \Big|_0^H$$

$$= \frac{1}{3} \pi R^2 H$$

This is the answer precisely because the derivative of the result $(x^3/3)$ is equal to the integrand we started with (x^2).

Once again, we obtain the formula of one-third times the area of the base times the height. No matter what the shape of the base is, the area of each slice will be proportional to x^2 and we will end up with a formula involving one-third at the end.

We will see several other methods for obtaining this result.

Note in passing that we can obtain the volume of a fustum (a cone whose top has been cut off) as

$$= \pi \ [\frac{R}{H}]^2 \ [\frac{x^3}{3}] \ \Big|_{h_1}^{h_2}$$

$$= \pi \ [\frac{R}{H}]^2 \ [\frac{{h_2}^3}{3} - \frac{{h_1}^3}{3} \ ]$$

The geometers have given us an even more elegant formula (**here**).

Part V

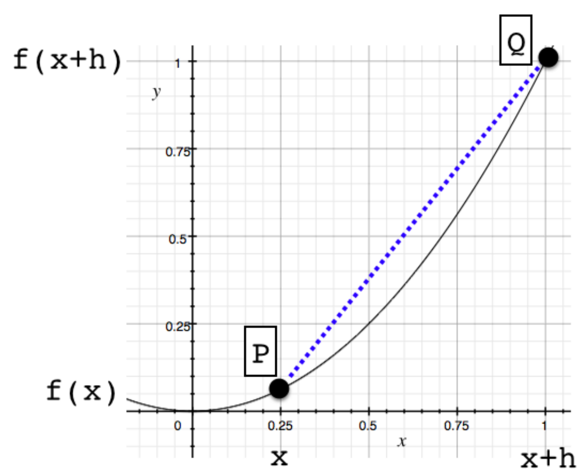
A little theory

Chapter 17

Difference quotients

In this chapter we look at the geometric interpretation of the derivative — which is the traditional way to begin calculus. The general approach was developed by Fermat.

Think for a minute about a curve such as the one shown in the figure, corresponding to some unspecified function $f(x)$.



At an arbitrary point P on the curve, for some value of x, we plot $y = f(x)$. This is Descartes' genius idea. The point on the graph of

$f(x)$ at x has coordinates $P = (x, f(x))$.

Now consider a point Q near P but also on the curve, For the x-coordinate of Q, a small amount is added to x. We might call that small amount Δx, but many authors use h, a simpler notation, and we will do so as well. The value of the function at $x + h$ is $f(x + h)$ and so Q has coordinates $Q = (x + h, f(x + h))$.

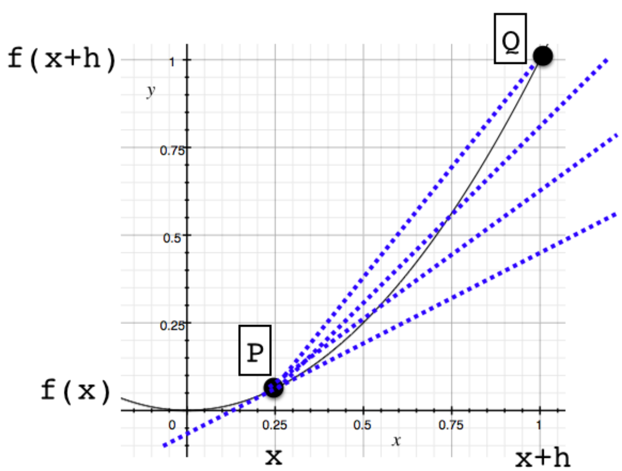
The slope of the (secant) line connecting Q and P is

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h}$$

This is a famous quantity, it's called the **difference quotient**.

The goal of differential of calculus is to find the slope of the *tangent* to the curve at the point P. What we have is an expression for the slope of the secant line PQ, which is close but not quite the same thing.

To go from the secant to the tangent, we ask "what happens to this expression as h gets smaller and smaller and approaches zero." The second point where the secant meets the curve comes closer and closer to the first one.



In mathematical language, we say the slope of the tangent is equal to the limit of the difference quotient as h tends to 0:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We'll say a bit more about limits in the next chapter, but for the moment you can think about

$$\lim_{h \to 0}$$

as meaning, "substitute $h = 0$ and see what happens to the expression of interest."

x squared

Let's try a couple of examples and look for a pattern.

$$f(x) = x^2$$

For this function, we write that the difference quotient is

$$\frac{(x+h)^2 - x^2}{h}$$

$$= \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \frac{2xh + h^2}{h}$$

Now divide by the denominator h

$$= 2x + h$$

Finally, to get the slope of the tangent, we evaluate the limit

$$\lim_{h \to 0} 2x + h = 2x$$

In evaluating the limit, we ask: what happens to this expression as h approaches 0. In this case, it cannot actually reach zero, because then our previous step of dividing by h would not be allowed. But we let h become really really small, and take advantage of the property of the limit which says that an expression can have a limit at c even if it can't be evaluated at c itself.

At every point on the curve $y = x^2$, the slope of the tangent line to the curve is $2x$. So the slope at $x = 0$ is 0, and the slope at $x = 2$ is 4, and so on.

This process of computing the difference quotient and then finding the limit as $h \to 0$ is called "taking the derivative." It produces an expression which is called the derivative of y with respect to x, in this case

$$\frac{dy}{dx} = 2x$$

and we can interpret this as the slope of the tangent to the curve of $f(x)$ at the point x.

Another useful shorthand uses the f from $f(x)$. We adopt the convention that the derivative of $f(x)$ can be written $f'(x)$.

$$f'(x) = 2x$$

To be even more succinct we might write y' for $f'(x)$.

If we repeat this exercise with a leading constant a (that is, for $f(x) = ax^2$), we find that every term in the numerator of the difference quotient will contain a, and the final result will be $2ax$. Constants just get carried through.

square root

Now look at the square root:

$$f(x) = \sqrt{x}, \quad (x \geq 0)$$

The difference quotient for this function is

$$\frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Clean up the numerator by multiplying by the conjugate

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} \quad \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{x + h - x}{h\ (\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{h}{h\ (\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

We evaluate the limit

$$\frac{dy}{dx} = \lim_{h \to 0} \ \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

inverse

Consider the inverse function

$$f(x) = 1/x, \quad (x \neq 0)$$

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Clean up the numerator

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h} \quad \frac{(x)(x+h)}{(x)(x+h)}$$

$$= \frac{x - (x+h)}{h\ (x)\ (x+h)}$$

$$= \frac{-h}{h\ (x)\ (x+h)}$$

$$= -\frac{1}{(x)\ (x+h)}$$

We evaluate the limit:

$$\lim_{h \to 0}\ -\frac{1}{(x)\ (x+h)}$$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

There's a pattern here. We will use the notation $f'(x)$ to indicate the slope of the curve $f(x)$ at x

$$f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x$$

$$f(x) = \sqrt{x} = x^{1/2} \quad \Rightarrow \quad f'(x) = \frac{1}{2}x^{-1/2}$$

$$f(x) = \frac{1}{x} = x^{-1} \quad \Rightarrow \quad f'(x) = -\frac{1}{x^2} = -x^{-2}$$

The general formula is

$$f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1}$$

This is easily proved (for integer n) using the binomial expansion for $(x+h)^n$ for integral n ($n \in 1, 2, \dots$). We need only the first three terms:

$$(x+h)^n = x^n + nx^{n-1}h + n\frac{(n-1)}{2}x^{n-2}h^2 + \dots$$

The key point is that the last term shown and all subsequent terms contain powers of h^2 or higher.

After division by h, for each of these terms there will remain one or more terms of h, and in the limit $\lim_{h \to 0}$ these become zero.

$$\lim_{h \to 0} \ \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \ \frac{x^n + nx^{n-1}h + n\frac{(n-1)}{2} \ x^{n-2}h^2 + \dots - x^n}{h}$$

$$= \lim_{h \to 0} \ \frac{nx^{n-1}h + n\frac{(n-1)}{2} \ x^{n-2}h^2 + \dots}{h}$$

$$= \lim_{h \to 0} \ nx^{n-1} + n\frac{(n-1)}{2} \ x^{n-2}h + \dots$$

$$= nx^{n-1}$$

Another question is what to do with a sum or difference of polynomials, such as

$$f(x) + g(x)$$

If you write out the difference quotient

$$\frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

everything can be exactly as before, just grouping all terms with $f(x)$ and those with $g(x)$ separately.

$$[f(x) + g(x)]' = f'(x) + g'(x)$$

We showed above by computing the difference quotient directly that

$$f(x) = \sqrt{x}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Here is another approach to the same problem. Consider

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

Solve for x as a function of y:

$$x = \sqrt{y}$$

We can do algebra with *differentials* (with some constraints):

$$\frac{dy}{dx}\,\frac{dx}{dy} = 1$$

$$2x\,\frac{dx}{dy} = 1$$

$$\frac{dx}{dy} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}$$

In observing the inverse relationship, remember that x and y are related by the equation $y = x^2$. For example, when $x = 2$, $dy/dx = 2x = 4$.

Using the relationship $f(x)$, when $x = 2$, $y = 4$, and $dx/dy = 1/2\sqrt{y} = 1/2\sqrt{4} = 1/4$, which is indeed the inverse of 4.

In this last section, after solving for x as a function of y, y is the *independent* variable. We can switch back to our usual notation:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

problem

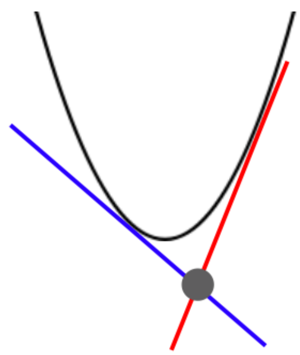
I found the following problems on the web. They are great practice and show what kinds of problems this approach of differentiation can solve. To prove:

Let $(a, f(a))$ and $(b, f(b))$ be two distinct points on the graph of a differentiable function f. Suppose that the tangent lines of f at these two points intersect, and call the point of intersection (c, d). Verifying the following facts is elementary.

1. If $f(x) = x^2$, then $c = (a + b)/2$, the arithmetic mean of a and b.
2. If $f(x) = \sqrt{x}$, then $c = \sqrt{ab}$, the geometric mean of a and b.
3. If $f(x) = 1/x$, then $c = 2ab/(a + b)$, the harmonic mean of a and b.

1

Here is a diagram for the first one:



The claim is that the x-coordinate of the point will be half-way between the x-coordinates for the two points on the parabola. We have:

$$y = f(x) = x^2$$

$$y' = f'(x) = 2x$$

At $x = a$, the slope is $2a$ and the equation of a line through the point

(a, a^2) is

$$y - a^2 = 2a(x - a)$$

At $x = b$, the equation is

$$y - b^2 = 2b(x - a)$$

To see where the lines cross, we set the y's to be equal, and solve for x:

$$\begin{aligned} 2a(x - a) + a^2 &= 2b(x - b) + b^2 \\ 2ax - a^2 &= 2bx - b^2 \\ 2x(a - b) &= a^2 - b^2 \\ &= (a + b)(a - b) \\ x &= \frac{1}{2}(a + b) \end{aligned}$$

2

We have:

$$\begin{aligned} y &= f(x) = \sqrt{x} \\ y' &= f'(x) = \frac{1}{2\sqrt{x}} \end{aligned}$$

At $x = a$, the slope is $1/2\sqrt{a}$ and the equation of a line through the point $(a, \sqrt{a})$ is

$$y - \sqrt{a} = \frac{1}{2\sqrt{a}}\ (x - a)$$

At $x = b$, the equation is

$$y - \sqrt{b} = \frac{1}{2\sqrt{b}}\ (x - b)$$

We set the y's to be equal

$$\frac{1}{2\sqrt{a}}\ (x - a) + \sqrt{a} = \frac{1}{2\sqrt{b}}\ (x - b) + \sqrt{b}$$

and solve for x. Multiply by $2\sqrt{a}\sqrt{b}$

$$(x-a)\sqrt{b} + 2a\sqrt{b} = (x-b)\sqrt{a} + 2b\sqrt{a}$$

Multiply through and cancel

$$x\sqrt{b} + a\sqrt{b} = x\sqrt{a} + b\sqrt{a}$$

$$x(\sqrt{b} - \sqrt{a}) = b\sqrt{a} - a\sqrt{b}$$

$$= \sqrt{a}\sqrt{b}(\sqrt{b} - \sqrt{a})$$

$$x = \sqrt{ab}$$

3

We have:

$$y = f(x) = \frac{1}{x}$$

$$y' = f'(x) = -\frac{1}{x^2}$$

At $x = a$, the slope is $-1/a^2$ and the equation of a line through the point $(a, 1/a)$ is

$$y - 1/a = -\frac{1}{a^2}\ (x-a)$$

At $x = b$, the equation is

$$y - 1/b = -\frac{1}{b^2}\ (x-b)$$

We set the y's to be equal

$$-\frac{1}{a^2}\ (x-a) + 1/a = -\frac{1}{b^2}\ (x-b) + 1/b$$

and solve for x:

$$(\frac{1}{b^2} - \frac{1}{a^2})x = 2(\frac{1}{b} - \frac{1}{a})$$

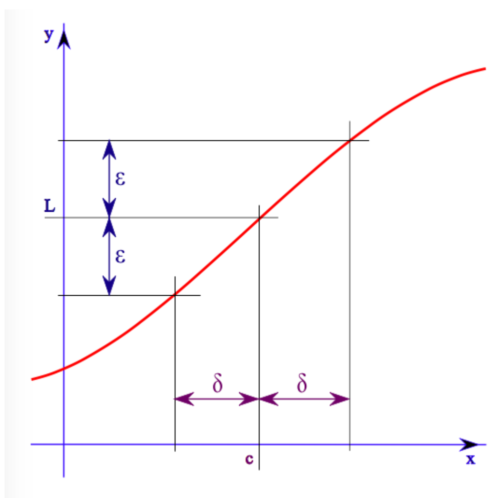
$$(\frac{1}{b}+\frac{1}{a})x = 2$$
$$(a+b)x = 2ab$$
$$x = \frac{2ab}{a+b}$$

Chapter 18

Limit concept

Limit concept

Consider the graph of a function $f(x)$. We might choose a power of x similar to $y = x^2$ or $y = x^3 - x$, which affirmatively has two properties that are of interest here: continuity and differentiability (we'll get to those ideas in a bit). Let's just say $y = f(x)$ is a "good" function. The functions we deal with in this book are all "good."



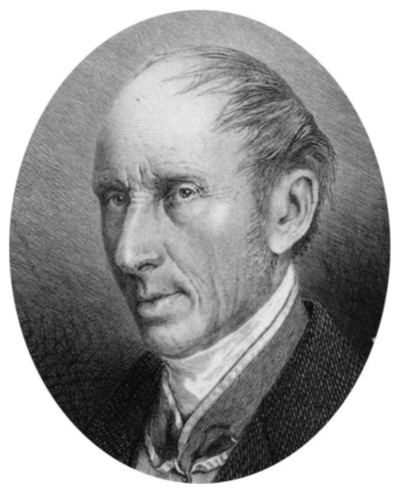
Focus on the neighborhood of a point on the x-axis, $x = c$.

By inspection of the graph, for points near c, the value of f at those points is not too different from L.

(It is also true here that the value of $f(x)$ at c is equal to L. This matters for continuity but not for limits).

We would like to say that the *limit* of $f(x)$ as *x approaches c* is equal to L. The idea is that we can make $f(x)$ as close to L as we please, provided we choose x sufficiently close to c.

> When the values successively attributed to a variable approach indefinitely to a fixed value, in a manner so as to end by differing from it by as little as one wishes, this last is called the limit of all the others. —Cauchy



Modern mathematicians don't like that word "approach", which conjures up movement and the involvement of time.

They also don't like reasoning from what they see in a graph, in part because no graph can show the whole function for the general case. To free ourselves from graphs and pictures, we will use an algebraic method from the formal apparatus of calculus.

There are two equivalent approaches, neighborhoods, and epsilon-delta formalism. Let's look at neighborhoods briefly.

neighborhoods

First, an *interval* between two real numbers a and b ($a < b$) contains every real number $a < x < b$.

$$(a, b) = x \mid a < x < b$$

The "|" means x "such that" the condition $a < x < b$ holds.

A *closed* interval $[a, b]$ includes the endpoints, $a \leq x \leq b$, while an *open* interval (a, b) excludes them. Half-open intervals like $[a, b)$ may be defined, and an interval with $\pm\ \infty$ as an endpoint is always open on that end, for example: $[a, \infty)$, because infinity *is not a number.*

Any open interval with a point p as its midpoint is called a *neighborhood* of p. Let r be the distance from p to the boundary of a particular neighborhood; r may be large or very very small. We denote a neighborhood of p as $N(p)$. $N(p)$ consists of all those values of x such that

$$|x - p| < r$$

.

which we would write more formally as

$$N(p) = x \mid |x - p| < r$$

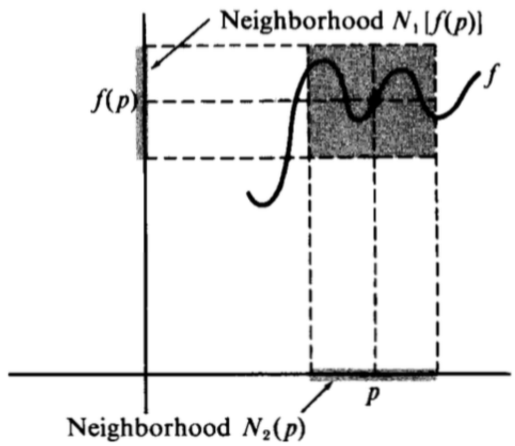
.

To say that the limit $f(x) \rightarrow L$ exists, we mean that for every neighborhood $N_1(L)$, no matter how small, there exists some neighborhood $N_2(p)$ such that $f(x)$ is contained within $N_1(L)$, written as

$$f(x) \in N_1(L)$$

whenever $x \in N_2(p)$.

If $N_1(L)$ is very small, then $N_2(p)$ may need to be very small as well, to guarantee that $f(x)$ is contained within N_1. Here is an example where this condition is satisfied.



The idea of a neighborhood is a nice abstraction to hide the apparatus of modern calculus, which we save for the Addendum.

An important fact about limits has to with the case where $x = p$. It is *not* necessary that $f(p) = L$. This relaxed condition is in fact crucial for calculus.

example 1

Limits can be easy or hard, depending on the problem. Here is one found in the previous chapter on difference quotients:

$$\lim_{h \to 0} \frac{2xh + h^2}{h}$$

When you see something like this, what you are supposed to do is reason about what happens as the variable h approaches 0 (gets smaller and smaller). The first step in that is to figure out what would happen if h actually would become zero.

Here, each term has a limit of 0 when h is zero, so we will have $0/0$. The zero on the bottom is trouble, it means that the expression becomes undefined.

However, suppose we first cancel h on top and bottom to obtain

$$\lim_{h \to 0} \frac{2x + h}{1} = \lim_{h \to 0} \; 2x + h$$

Now, the answer is just $2x$. This is valid as long as h approaches zero but is never actually equal to it.

Recall that we can have a limit for $f(x)$ as x approaches c, even if $f(c)$ does not exist.

example 2

Here is another important expression. What is the value of $f(x)$ as h approaches zero?

$$\cos h < f(x) < \frac{1}{\cos h}$$

Since $\cos 0 = 1$, the two outside terms both approach 1 in the limit as h approaches zero. Since $f(x)$ lies between them, it must also approach 1.

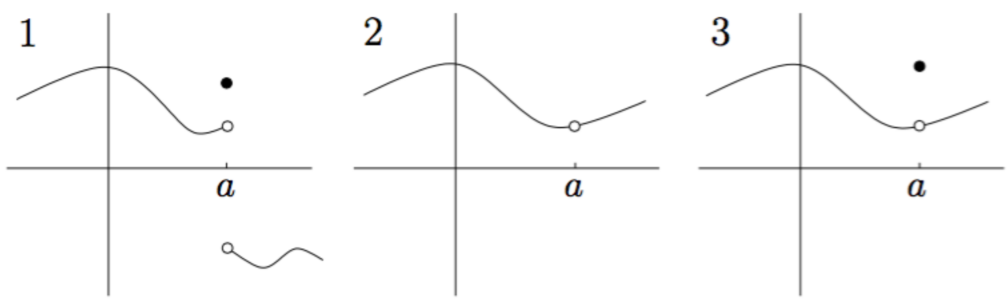
This is called the *squeeze theorem*.

The magical thing is that this is true even if, when $h = 0$, $x = 0/0$. We'll see this when we look at calculus of sine and cosine.

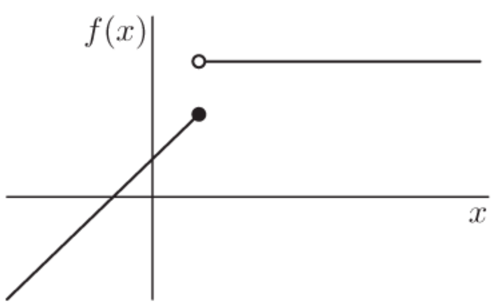
Continuity

Continuity has an intuitive definition: as Euler said, if we can graph a function *without lifting our pencil from the paper*, then the function is continuous.

Here are some graphs showing examples of how continuity can fail.



A filled circle means that the function yields that y-value for the corresponding x-value of the point, while an open circle means it does not. The function may yield some other value, or simply be undefined.



$$f(x) = \begin{cases} x+1 & \text{if } x \leq 1 \\ 3 & \text{if } x > 1 \end{cases}$$

For a function to be continuous at a point $x = c$, we imagine that if we vary x in neighborhood of c, then $f(x)$ should not change in value by too much.

Again, we will call that value L, the limit of $f(x)$ as $x \to c$. For L to exist we require that the two one-sided limits be equal. If we approach c from the high side ($x > c$) or the low side ($x < c$), the limit must be the same.

Very important: continuity requires, in addition, that $f(c)$ be equal to L.

Differentiability

For a function to be differentiable, we require that the limit

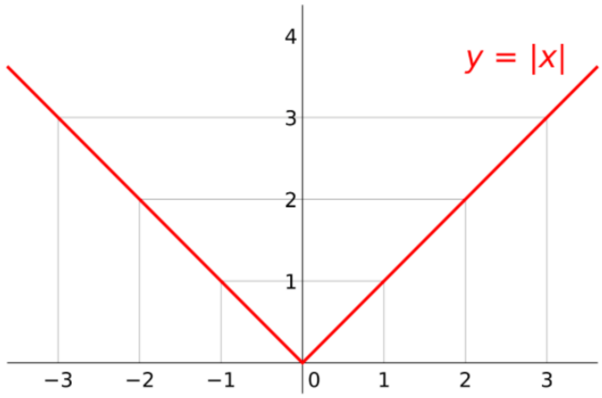
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. An example of a function that is continuous but not differentiable at a particular point is the absolute value function.

example: absolute value

An algebraic definition of the absolute value function is piecewise:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



The function $f(x) = |x|$ is continuous at $x = 0$ because the two one-sided limits exist and are equal to each other. They are also equal to $f(0) = 0$.

However, there is no defined slope at $x = 0$. The difference quotient gives different results for positive Δx (positive slope) than for negative Δx (negative slope).

Without getting too technical

> Note that the graph of the absolute value function is "all in one piece", but has a "sharp point" at the origin. We will not attempt to make these descriptions precise, other than to say that the fact that the graph comes "all in one piece" is a feature of continuity, and that graphs of differentiable functions are "smooth" in that they do not have "sharp points." The unambiguous and demonstrably true statement here is that the absolute value function is continuous at 0 but is not differentiable at 0.

https://oregonstate.edu/instruct/mth251/cq/Stage5/Lesson/diffVsCont.html

Chapter 19

Higher derivatives

We have defined the derivative of a function $f(x)$ as a limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

It is the limit of the difference quotient as $h \to 0$.

We introduced the power rule to obtain the derivative of integer powers of x. In addition, we said that the derivative of a sum of two or more functions is the sum of the derivatives.

The derivative is just a function itself. Consider a quadratic like

$$y = ax^2 + bx + c$$

The derivative is

$$y' = 2ax + b$$

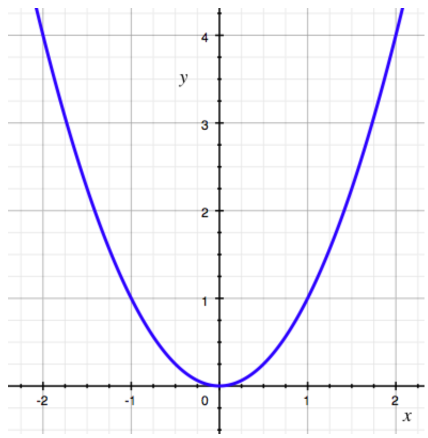
Since the derivative is just a function, why not take the derivative of the derivative, which is called

$$\frac{d^2y}{dx^2}$$

or more compactly, y double prime:

$$y'' = 2a$$

What is the meaning of the second derivative? It gives the slope of the slope, or how the slope is changing with a change in x. For a parabola that opens up, like this one



the second derivative is positive. This means that the slope continues to increase as x increases.

There is nothing to stop us from taking more derivatives. For a quadratic $y''' = 0$, which is not very interesting, but consider the cubic:

$$y = x^3$$

$$y' = 3x^2$$

$$y'' = 6x$$

$$y''' = 6$$

extrema

A very important use of the derivative is to find a maximum or minimum of a function. At such a point the slope is zero because the curve is headed sideways, just for a moment. For the quadratic, the slope is zero at the vertex:

$$y' = 0 = 2ax + b$$

$$x = -\frac{b}{2a}$$

You should recognize this equation from geometry. Without getting into details, it is obtained there by completing the square. Or alternatively, write the equation of a parabola whose vertex is (h, k)

$$(y - k) = a(x - h)^2$$

multiply out

$$y = ax^2 - 2ahx + h^2 + k$$

By comparison with the standard form

$$y = ax^2 + bx + c$$

it's clear that

$$-2ahx = bx$$

so the x-coordinate of the vertex is

$$h = -\frac{b}{2a}$$

The vertex is the maximum or minimum of a quadratic, depending on the sign of a. We can tell the difference by looking at the second derivative again:

$$y'' = 2a$$

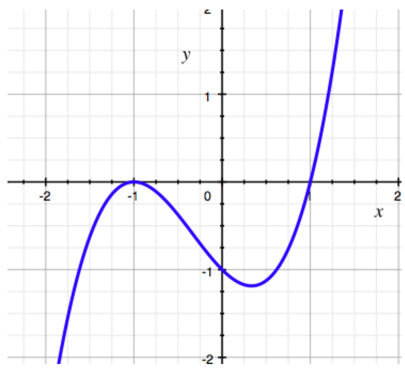
If $a > 0$ (a parabola that opens up), then the second derivative is positive and we have found a minimum value. If $y'' < 0$, we have a maximum.

Consider the cubic

$$
\begin{aligned}
y &= (x+1)(x+1)(x-1) \\
&= (x^2+2x+1)(x-1) \\
&= x^3+x^2-x-1
\end{aligned}
$$

From the first form, we can easily see that the roots are $x = \pm 1$. These are the values where the function crosses the y-axis, that is, where its value is zero.

The graph looks like this:



The first derivative is

$$
\begin{aligned}
y' &= 3x^2+2x-1 \\
&= (3x-1)(x+1)
\end{aligned}
$$

This expression is zero when $x = -1$ or $x = 1/3$. That does match the places where the curve is horizontal, as we can see.

The second derivative is

$$y'' = 6x + 2$$

For the first value $x = -1$, the second derivative is negative, and this corresponds to a local maximum for the function. For $x = 1/3$, the second derivative is positive, and this is a minimum. We say "local" because there may be more extreme values, as is the case here.

A maximum corresponds to a negative value for the slope of the slope because the slope is first positive, then zero, then negative. Its change with increasing x is negative.

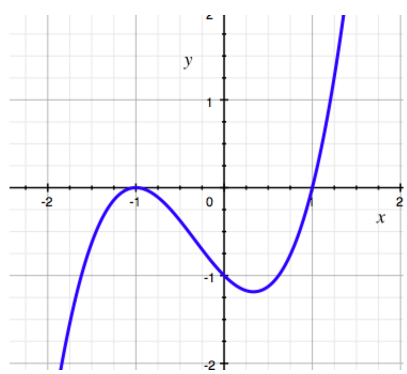
Again, the second derivative of our cubic is

$$y'' = 6x + 2$$

Setting this equal to zero, we obtain

$$x = -\frac{1}{3}$$

This point on the curve is an *inflection point*. It is a point (actually the only point for this curve) where the rate of change of the slope, which is negative to the left $x = -1/3$, changes to positive, and there is an instant where it is zero.



We will see this in connection with the Gaussian or normal curve. It's an interesting fact that the first standard deviation corresponds to the inflection point of the curve. At that point the second derivative of the function is equal to zero.

Rectangular area

Here is a classic problem.

We wish to construct a rectangle with the maximum area *for a fixed perimeter* (without the second statement the area would be infinite). Let's call the sides x and y, and the semi-perimeter S (constant) and so our constraint is that

$$S = x + y$$

The area is then

$$A = xy$$

substitute

$$\begin{aligned} A &= x(S - x) \\ &= Sx - x^2 \end{aligned}$$

Take the first derivative and set it equal to zero:

$$A' = S - 2x = 0$$

$$x = \frac{S}{2} = y$$

A square has the maximum area for a given perimeter constructed with right angles, as expected. We'll see many challenging problems of this type later on.

Chapter 20

Differentials

infinitesimals

We say that the derivative dy/dx is the slope of the tangent to the curve $y = f(x)$ at some particular point (x, y); it is the slope of a line that just touches the curve.

And it is frequently called the slope of the curve at the point (x, y).

We saw a formal definition for the derivative in terms of an expression for Δy as a function of Δx, which is then divided by Δx. We determine what is called the "limit" as Δx approaches zero.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

There appears to be a contradiction here. On one hand, we're saying that Δx must approach zero. On the other hand, it cannot really be zero, because division by zero is not defined. So how close does it have to be to zero?

Are dy and dx small, really small, really really small, or almost zero?

The official answer requires a cumbersome apparatus of limits, and it would say that dy/dx is not a quotient at all, but rather a single entity, the limit of a quotient, as we just said:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

However, our simple answer is that dx and dy are separable entities and they are just as small as they need to be. Take the example we used previously:

$$\frac{dy}{dx} = 3cx^2$$

Put the dx on the right-hand side:

$$dy = 3cx^2 \ dx$$

What this expression says is that for a small change in x which we call dx, we will obtain a small change in y called dy with the given relationship. $3cx^2$ gives the proportionality between dx and dy.

Suppose we evaluate $3cx^2$ at some particular x_0. Then it is a number, it has a fixed value depending on where we are on the curve. So write it as $k = 3cx_0^2$ and then:

$$dy = k \ dx$$

When we write this, we are making a *linear approximation* to the quadratic function. dy is not exactly equal to $k \ dx$, for most situations we are ignoring quadratic and higher terms.

Here, we treat dy and dx as very small but non-zero quantities. If there should ever be a problem because we've chosen Δx too large, just reduce it by some factor ($1/10$, 10^{-6}, 1/googol), whatever is needed,

https://en.wikipedia.org/wiki/Googol

and try again until the problem disappears (it will). Make dx and dy really really small. If that's not small enough, try making them smaller still.

By this trick, we free ourselves from limits. If you want to multiply by dx on both sides of an equality

$$\frac{dy}{dx} = nx^{n-1}$$

$$dx \cdot \frac{dy}{dx} = dy = nx^{n-1}\ dx$$

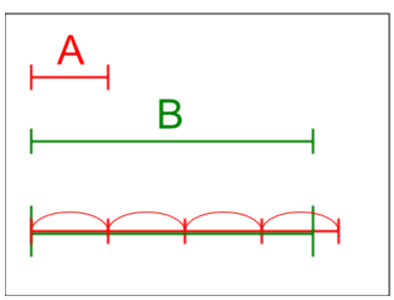
Feel free, go ahead and do it.

fancy

If you want to read more about the derivative as a limit, why dy/dx is not a quotient, and so on, you can look at any standard calculus book. Or start here

https://math.stackexchange.com/questions/21199/is-frac-textrmdy-textrmdx-not-a-ratio

The bottom line is that we want dy and dx to be very small compared to y and x, but one of the properties of the real numbers is that no matter how small small we choose A (or dx), there exists a positive integer n such that $n \cdot A > B$ (or $n \cdot dx > x$). This is called the Archimedean property of the real numbers.



Effectively what limits and neighborhoods do is to say, OK smart guy, you go first. Pick n. Then once you've picked n very large, we can always find dx very very small so that $n \cdot dx$ is still small compared with x. That's the whole trick.

However, in practice none of this is a problem because we view dy and dx as very small. Although often we only care about their ratio, some times we will need to separate them. This is legal, trust me.

Chapter 21

Fundamental theorem of calculus

> Calculus has a long history. Although Newton and Leibniz are credited with the invention of calculus in the late 1600s, almost all the basic results predate them. One of the most important is what is now called the Fundamental Theorem of Calculus (ftc), which relates derivatives to integrals.

https://mathcs.clarku.edu/~ma120/FTC.pdf

The usual way to begin the study of calculus is to think about the slope of the tangent to a curve at a point. If the point is some particular value of x, say $x = a$, then this is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

For any x in the domain of $y = f(x)$, we say that this slope is the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We may call this construct by various names such as the derivative of y with respect to x, dy/dx, or $f'(x)$.

The notation dy/dx is due to Liebnitz, and $f'(x)$ is due to Lagrange. For situations where x is a function of time t, Newton would write $\dot{x}$, $\ddot{x}$ and so on. Newton called his method "fluxions", concentrating on derivatives with respect to time.

$$y = f(x)$$
$$\frac{dy}{dx} = f'(x)$$

Evaluation of this limit for $f(x) = x^n$, $f(x) = e^x$, and $f(x) = \sin(x)$ then follows.

Some rules (product rule, chain rule and so on) will be introduced that allow us to calculate derivatives of more complicated functions. We also learn to keep note of various functions and their derivatives because it is essential to be able to "go backwards."

The inverse of differentiation is integration. By definition

$$y = f(x) = \int dy$$

Now

$$\frac{dy}{dx} = f'(x)$$
$$dy = f'(x)\ dx$$
$$y = f(x) = \int dy = \int f'(x)\ dx$$

There is an idea in integration which is really profound. We already introduced it in previous chapters by considering a ball or solid sphere in 3D space and the outer surface of the ball (technically, that *is* the sphere, but no matter). As Archimedes showed 2200 years ago, the volume of the sphere is this function of the cube of the radius.

$$V = f(r) = \frac{4}{3}\pi r^3$$

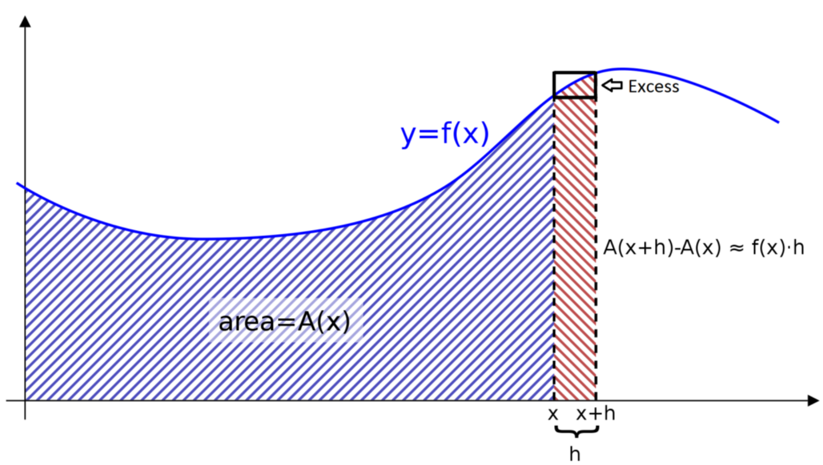
The question was: how does the volume change when the radius changes by a little bit? The big idea is to realize that the answer is exactly the same as the limit we gave above, it is the derivative of V with respect to r.

$$\frac{d}{dr}V = \frac{dV}{dr} = \frac{d}{dr}\frac{4}{3}\pi r^3 = 4\pi r^2$$

It is no accident that this result is the same as the formula for the surface area of the sphere. Increasing the radius r by a little bit dr, the new volume that is added is approximately the surface area $4\pi r^2$ times dr, the volume of the shell at the radius r.

$$dV = 4\pi r^2 \; dr$$

This is a completely general idea. The way it is usually introduced is to consider the graph of a function $f(x)$ in the plane.



We think not just about $f(x)$ itself, but the total area underneath the curve. Area is a function. Its value depends on the bounds. Let us fix the left-hand boundary (say, at $x = a$), but leave the right-hand bound as a variable, call it x.

The big idea is that the total area under the curve (the region in blue) is some as yet unknown function of x, $A = F(x)$.

The way we find $F(x)$ is to ask the question: how does the area $F(x)$ change when x changes by a little bit, say h? If you look at the figure it is clear that the answer is the area of the red rectangle, the added area is just $f(x)$ times h. If h is small enough, the answer is exact.

Recast in mathematical terms

$$\lim_{h \to 0} \ f(x) \ h = \lim_{h \to 0} \ F(x+h) - F(x)$$

We just divide by h

$$\lim_{h \to 0} \ f(x) = \lim_{h \to 0} \ \frac{F(x+h) - F(x)}{h}$$

and recognize that the term on the left does not depend any longer on h so

$$f(x) = \lim_{h \to 0} \ \frac{F(x+h) - F(x)}{h}$$

$f(x)$ is the derivative of $F(x)$:

$$f(x) = F'(x)$$

To find the area function F, we just need to find a function that, when we differentiate it, gives us $f(x)$.

The fundamental theorem of calculus states this principle. Usually, a new variable is introduced to remove any confusion that might arise with respect to x, which in the discussion above, we actually used in two different ways. Write

$$F(x) = \int_a^x f(t)\ dt$$

In the equation above, the real variable is x. t is what's called a dummy variable, since it might replaced with any other symbol without changing anything.

We have two different functions F and f. The value for each will vary with the value of x (since we evaluate at the right-hand bound $t = x$). The value of F depends also on the left-hand bound a. Anyway, having written

$$F(x) = \int_a^x f(t)\ dt$$

The Fundamental Theorem of Calculus (FTC) states:

$$F'(x) = f(x)$$

which is what we figured out before.

The FTC has a second part, which is

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

This gives us the way in which areas (and volumes and so on) are actually calculated. Start with the function $f(x)$. We find $F(x)$, and then just evaluate it at the endpoints a and b. The difference is the area under the curve $f(x)$ between the two bounds a and b.

The proof of this second part is usually done with what are called Riemann sums. We look at those a bit later in the book.

Part VI

Three rules for differentiating

Chapter 22

Chain rule

Here's a classic problem leading to the next idea. Temperature in the atmosphere depends on the altitude, decreasing about 3 degrees F for each 1000 feet increase in altitude above sea level.

$$T = T_0 - 3h$$

$$\frac{dT}{dh} = -3$$

(In degrees F per thousand feet).

Suppose we're ascending a mountain road at a rate of 500 feet per minute.

$$\frac{dh}{dt} = 0.5$$

(In thousands of feet per minute).

What is the rate of change of temperature with time? It will turn out that

$$\frac{dT}{dt} = \frac{dT}{dh} \cdot \frac{dh}{dt} = -3 \times 0.5 = -1.5$$

(In degrees F per minute).

chain rule

A compound function $f(g(x))$ means that we apply the function g to the input x, then feed the output of that to the function f. An example would be $\sin 2x$.

The chain rule says that if we have a compound function then

$$\frac{d}{dx}\, f(g(x)) = f'(g(x)) \cdot g'(x)$$

If we break this down a bit we can write:

$$t = g(x)$$
$$y = f(t) = f(g(x))$$

Then

$$y' = f'(t) \cdot g'(x) = f'(g(x)) \cdot g'(x)$$

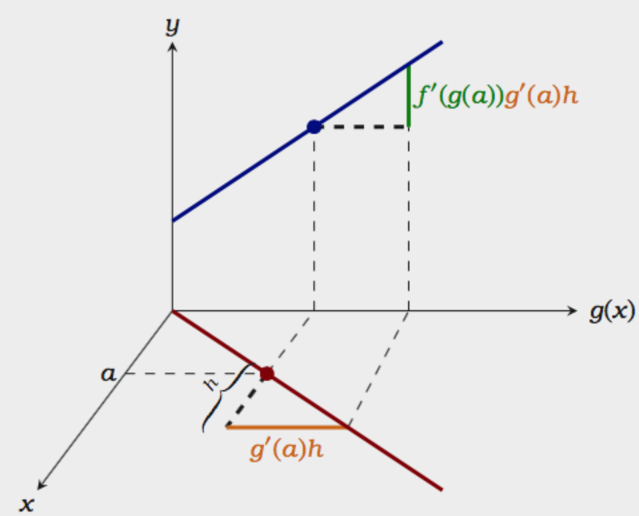


Figure 6.1: A geometric interpretation of the chain rule. Increasing a by a "small amount" h, increases $f(g(a))$ by $f'(g(a))g'(a)h$. Hence,

$$\frac{\Delta y}{\Delta x} \approx \frac{f'(g(a))g'(a)h}{h} = f'(g(a))g'(a).$$

For example

$$\frac{d}{dx}\sqrt{1-x^2} = \frac{1}{2}\,\frac{1}{\sqrt{1-x^2}}\,(-2x)$$

$$= -\frac{x}{\sqrt{1-x^2}}$$

What we've done is to treat $1-x^2$ as one whole expression. The factor of $-2x$ comes from the term $g'(x)$ in the chain rule.

We can do the same problem more slowly and clearly by substituting a new variable

$$t = 1 - x^2$$

Take the derivative:

$$\frac{dt}{dx} = -2x$$

Now rewrite the original $f(x)$ as $f(t)$:

$$y = f(x) = \sqrt{1-x^2}$$

$$y = f(t) = \sqrt{t}$$

$$\frac{dy}{dt} = \frac{1}{2}\,\frac{1}{\sqrt{t}}$$

What we really wanted to know was dy/dx. The chain rule says that

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx}$$

(you can think of the factors of dt as canceling, I won't tell anyone).

$$= \frac{1}{2}\,\frac{1}{\sqrt{t}}\,(-2x)$$

$$= -\frac{x}{\sqrt{t}}$$

$$= -\frac{x}{\sqrt{1-x^2}}$$

common roots

There are two common roots that we will run into in this book. The first is the one given above where we might also have a constant a^2 and then

$$\frac{d}{dx}\sqrt{a^2 - x^2} = -x \ \frac{1}{\sqrt{a^2 - x^2}}$$

The power is $1/2$ which gives a factor of $1/2$ from the power rule. The new power is then $1/2 - 1 = -1/2$, and we pick up a factor of $-2x$ from the chain rule. The 2's cancel.

The second is

$$\frac{d}{dx}(a^2 - x^2)^{3/2} = -3x \ \sqrt{a^2 - x^2}$$

The power is $3/2$ which gives a factor of $3/2$ from the power rule. The new power is then $3/2 - 1 = 1/2$, and we pick up a factor of $-2x$ from the chain rule. The 2's cancel.

We'll see both of these again a number of times.

proofs

Proofs of the chain rule and the theorems in the next chapter can be found in almost any standard calculus text, but are really beyond the level of information we are targeting here. The proof of the chain rule above basically depends on a property of limits. If

$$\lim_{x \to c} f(x) = L$$

$$\lim_{x \to c} g(x) = M$$

then

$$\lim_{x \to c} f(x) \ g(x) = LM$$

If you'd like to know more, there is an extensive chapter on Limits in the Addendum.

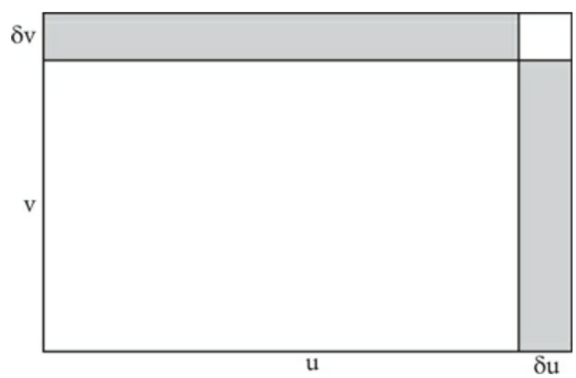
Chapter 23

Product and quotient rules

The product rule says that if we have functions $u(x)$ abbreviated as u, and $v(x)$ as v then

$$(uv)' = u'v + uv'$$

Here is one pictorial explanation:



If we have

$$(u + \delta u)(v + \delta v)$$

The product is

$$uv + u\delta v + v\delta u + \delta u \cdot \delta v$$

The difference quotient loses the first term by subtracting uv. The last term is much smaller than the others because it is $\delta \times \delta$ and can be neglected.

Here is another view:

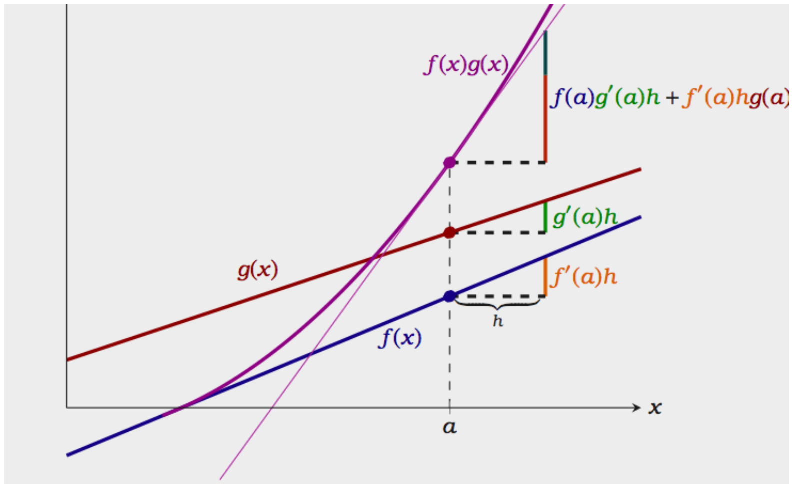


Figure 5.1: A geometric interpretation of the product rule. Since every point on $f(x)g(x)$ is the product of the corresponding points on $f(x)$ and $g(x)$, increasing a by a "small amount" h, increases $f(a)g(a)$ by the sum of $f(a)g'(a)h$ and $f'(a)hg(a)$. Hence,

We can check this in the simplest possible ways

$$(x)' = (1 \cdot x)' = 0 \cdot x + 1 \cdot 1 = 1$$

$$(x^2)' = (x \cdot x)' = 1 \cdot x + x \cdot 1 = 2x$$

$$(x^4)' = (x^3 \cdot x)' = x^3 \cdot 1 + 3x^2 \cdot x = 4x^3$$

or

$$(x^4)' = (x^2 \cdot x^2)' = x^2 \cdot 2x + 2x \cdot x^2 = 4x^3$$

and so on.

It is worth pointing out that one can build the general rule

$$(x^n)' = nx^{n-1}$$

by a chain of induction based on the product rule.

notes

There is a story that Leibnitz got this rule wrong at first — the claim is that he thought the answer was $f'(x)g'(x)$.

However, there isn't much evidence to support this. It seems to be a slander of Leibnitz by an English supporter of Newton, as part of the wars over who invented calculus. The supposition was shown in an unpublished notebook belonging to Liebnitz, as an example of an error.

https://mathoverflow.net/questions/181422/did-leibniz-really-get-the-leibniz-rule-wrong

The other observation is that the rule

$$(uv)' = u'v + uv'$$

was classically recited as "this times the derivative of that, plus that times the derivative of this." I learned it that way originally. But if you look closely at the equation you will see that I write the reverse these days.

The reason is that the version given above makes it easier to remember the quotient rule.

inverse

The product rule says that if we have functions $u(x)$ (abbreviated u), and $v(x)$ then

$$(uv)' = u'v + uv'$$

Remember that if u is not a simple function of x but a compound function like $f(g(x))$, then u' needs to take account of the inside function $g(x)$. By the chain rule

$$u' = f'(g(x)) \cdot g'(x)$$

As an example, when using the power rule might write

$$(\frac{1}{v})' = (v^{-1})' = -v^{-2} = -\frac{1}{v^2}$$

and this is fine so far as it goes.

But suppose v is a compound function, like $v(t)$, the correct result is really

$$(\frac{1}{v(t)})' = -\frac{1}{v^2}\ v'(t)$$

by the chain rule.

quotient rule

Accepting this result, use it and the product rule:

$$(u \cdot \frac{1}{v})' = u'\ \frac{1}{v} + u\ (\frac{1}{v})'$$

$$= u'\ \frac{1}{v} + u(-\frac{1}{v^2})v'$$

$$= \frac{u'v - uv'}{v^2}$$

The is known as the quotient rule.

As mentioned, take care to write the product rule as $u'v + uv'$ so as to get the quotient rule by just flipping the sign of the second term and then dividing by v^2.

You may find yourself wondering whether you have the sign right. Test with these two examples:

$$\frac{d}{dx}\ \frac{x}{1} = \frac{1 \cdot 1 - x \cdot 0}{1^2} = 1$$
$$\frac{d}{dx}\ \frac{1}{x} = \frac{1 \cdot 0 - 1 \cdot 1}{x^2} = \frac{-1}{x^2}$$

That looks correct. If you have the sign wrong this won't work out.

Strang

Gil Strang has many wonderful takes on calculus that I haven't seen in other books. For example:

Use the product rule here:

$$v \cdot \frac{1}{v} = 1$$

Take d/dx of both sides. By the product rule

$$\frac{dv}{dx} \cdot \frac{1}{v} + v \cdot \frac{d}{dx}\ (\frac{1}{v}) = 0$$

rearrange

$$v \cdot \frac{d}{dx}\ (\frac{1}{v}) = -\frac{dv}{dx} \cdot \frac{1}{v}$$

so

$$\frac{d}{dx} \cdot (\frac{1}{v}) = -\frac{1}{v^2} \cdot \frac{dv}{dx}$$

That's what we said!

This result extends the power rule to negative integers:

$$\frac{d}{dx}(x^{-n}) = \frac{d}{dx} \cdot (\frac{1}{x^n})$$

$$\frac{d}{dx} \cdot (\frac{1}{x^n}) = -\frac{1}{(x^n)^2}\ nx^{n-1}$$

$$= -nx^{-n-1}$$

Another one: we establish the power rule, by induction:

$$\frac{d}{dx}(u^{n+1})$$

$$= \frac{d}{dx}(u^n \cdot u)$$

$$= (nu^{n-1} \cdot \frac{du}{dx}) \cdot u + u^n \cdot \frac{du}{dx}$$

$$= (n+1)u^n \cdot \frac{du}{dx}$$

Since we used induction, the result applies only to integer n. Later we will see proofs that extend to rational and even irrational but real numbers n using implicit and logarithmic differentiation.

triple product

You don't see it much, but there is a triple product rule:

$$(uvw)' = u'vw + uv'w + uvw'$$

One might guess this is true by symmetry. Also

$$(uv \cdot 1)' = u'v \cdot (1) + uv' \cdot (1) + uv \cdot (1)'$$

The third term is zero, so we obtain just

$$u'v + uv'$$

Part VII

Standard integrals

Chapter 24

Powers and Polynomials

In this chapter we review briefly the most common integrals. Additional depth is given in a later chapter on **Techniques of integration**.

powers of x

The first **differentiation** that is done in basic calculus is

$$\frac{d}{dx}(x^n) = [x^n]' = nx^{n-1}$$

This answer is correct regardless of whether n is positive or negative, an integer or a rational number, or even irrational.

The proof for positive integers $n \in \{2, 3\}$ is simple. A general proof for positive integers is fairly easy using the binomial theorem.

Implicit differentiation will give a simple proof for all rational exponents. Still later we will have a proof for all real numbers. See **here**. We will finally find a function that yields x^{-1} as its derivative when we talk about the natural logarithm.

Ignoring some subtleties, we can **integrate** by reversing the process. We write

$$x^n = \int nx^{n-1} \, dx$$

A real math book will tell you that when the formula stands alone like this, we should add a constant of integration: $x^n + C$. But we are mostly using integrals to calculate areas and volumes, which means we determine the value of a definite integral by computing the difference of two expressions which both contain C, so it cancels.

If the factor of n is not given in the problem, we insert it as a divisor up front:

$$\int x^{n-1} \, dx = \frac{1}{n} x^n$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\int \sqrt{x} \, dx = \frac{2}{3} x^{3/2}$$

$$\int \frac{1}{\sqrt{x}} = 2\sqrt{x}$$

A common convention for notation is that $\int f(x) = F(x)$. Or, the other way around $f(x)$ is the derivative of $F(x)$, so $f(x) = F'(x)$. Let's make a table.

$f(x)$	$F(x)$
1	x
x	$x^2/2$
x^2	$x^3/3$
x^n	$x^{n+1}/n+1$
$\frac{1}{\sqrt{x}}$	$2\sqrt{x}$
$\sqrt{x}$	$\frac{2}{3}x^{3/2}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
e^x	e^x
$1/x$	$\ln x$
$a\cos ax$	$\sin ax$
$\cos ax$	$\frac{1}{a}\ \sin ax$
ae^{ax}	e^{ax}
e^{ax}	$\frac{1}{a}\ e^{ax}$

We will look at proofs for these trig functions and the exponential and logarithm in some detail in the next few chapters. For now, let's accept them provisionally.

fractional powers

Two additional integrals that we here add come from differentiations as seen in the chapter on the chain rule. With a^2 a constant:

$$\frac{d}{dx}\sqrt{a^2 - x^2} = -x \ \frac{1}{\sqrt{a^2 - x^2}}$$

$$\frac{d}{dx}(a^2 - x^2)^{3/2} = -3x \ \sqrt{a^2 - x^2}$$

To integrate, we reverse directions. The first one is

$$\int x \ \frac{1}{\sqrt{a^2 - x^2}} \ dx = - \ \sqrt{a^2 - x^2}$$

To check this, do the differentiation. The factor of $1/2$ on the right-hand side from the power and the leading -1 are canceled by the factor of -2 from the chain rule. Those are easy. The important thing is that we have that x under the integral sign.

The second one is

$$\int 2x \ \sqrt{a^2 - x^2} \ dx = -\frac{2}{3} \ (a^2 - x^2)^{3/2}$$

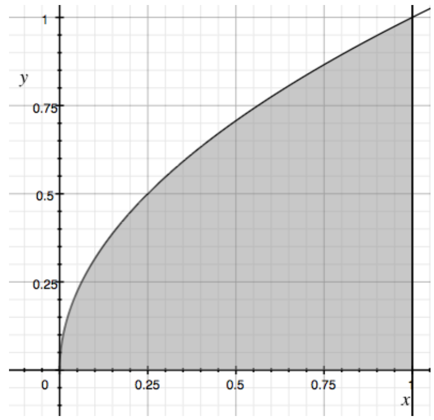
Again, check by differentiating the right-hand side. We get a factor of $3/2$ from the power, which cancels the leading $2/3$, then we get $-2x$ from the chain rule and cancel the minus sign. Factors of 2 and minus signs are easy, but we must have that x on the left, otherwise the answer will be much different.

square root

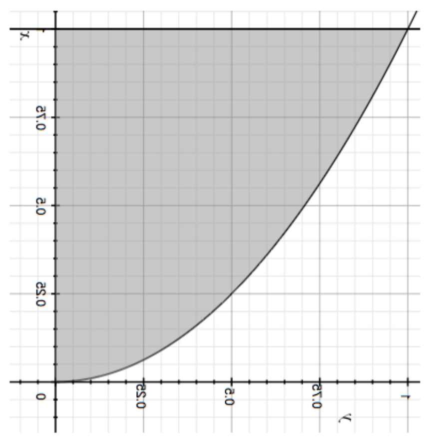
We will show that the integral of the square root gives the correct answer. Suppose we plot

$$y = \sqrt{x}$$

over the interval $[0, 1]$. The area under this curve is the shaded region.



The area seems to be more than $1/2$, but how much more? Now, take the very same plot, flip it over and rotate 90 degrees. We obtain something that is more familiar, namely the plot of $y = x^2$. The area under this curve is the white region



Clearly

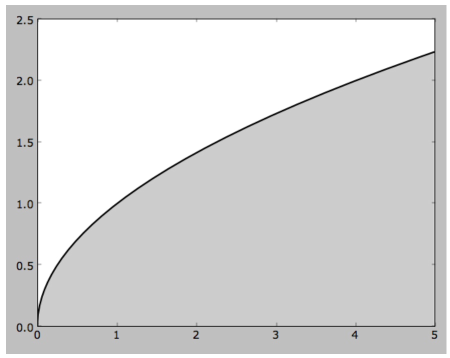
$$\int_0^1 \sqrt{x}\ dx + \int_0^1 x^2\ dx = 1$$

If you look in the list of formulas above (or just calculate) you will find that the first integral is equal to $2/3$ and the second one is equal to $1/3$.

This result holds for other intervals, not just $[0, 1]$.

Let's integrate to find the area under the curve in the interval $[0, b^2]$ (shaded gray below). We use b^2 for reasons that will be obvious in a minute.

$$A = \int_0^{b^2} \sqrt{x}\ dx = \frac{2}{3} x^{3/2}\ \Big|_0^{b^2} = \frac{2}{3} b^3$$



Now, consider x as a function of y:

$$x = y^2$$

Integrate to find the area between the curve and the y-axis (shaded white above). The only trick is that the bounds are $[0, b]$ because we want to add the two areas together to form a rectangle and the point on the curve at the upper bound will be $x = b^2, y = b$.

$$\int_0^b y^2\ dy = \frac{1}{3} y^3\ \Big|_0^b = \frac{b^3}{3}$$

Adding the results together, we obtain b^3 as the area of a box with width $\Delta x = b^2$ and height $\Delta y = b$, which is correct.

reversing the product rule

Playing around with the product rule can lead to the discovery of other important integrals that we will encounter later. The most important is

$$(\sin x \cos x)' = \cos x \cos x - \sin x \sin x$$
$$= \cos^2 x - \sin^2 x$$

Recall the most basic trig identity:

$$\sin^2 x + \cos^2 x = 1$$

$$-\sin^2 x = \cos^2 x - 1$$

so

$$(\sin x \cos x)' = 2\cos^2 x - 1$$

integrating

$$\sin x \cos x = 2 \int \cos^2 x \ dx - x$$

$$\int \cos^2 x \ dx = \frac{1}{2}(x + \sin x \cos x)$$

That is one we'll want to remember. We will use several techniques of integration to derive it from first principles, but this is the easiest way. See **here**.

Here are a few others:

$$(x \ln x)' = \ln x + \frac{x}{x} = \ln x + 1$$

$$\int \ln x \ dx = x \ln x - x$$

$$(x^2 \ln x)' = 2x \ln x + x$$

$$\int 2x \ln x = x^2 \ln x - \frac{x^2}{2}$$

$$(xe^x)' = e^x + xe^x$$

$$\int xe^x \ dx = xe^x - e^x$$

Stirling's approximation

For an application, consider

$$\int \ln x \ dx = x \ln x - x$$

Computing $n!$ gets unwieldy for large n (at least, without computers). It comes up in probability and other places. There is a famous formula for $n!$ called Stirling's approximation which comes in a more precise version

$$n! \approx \sqrt{2\pi n} \ (\frac{n}{e})^n$$

and a less precise one

$$\ln n! \approx n \ln n - n$$

The latter is easily derived:

$$\ln n! = \ln 1 + \ln 2 + \ldots \ln n$$

$$= \sum_{k=1}^{n} \ln k$$

$$\approx \int_1^n \ln x \ dx$$

which we showed is

$$
\begin{aligned}
&= x\ln x - x \,\Big|_1^n \\
&= (n\ln n - n) - (1\ln 1 - 1) \\
&= n\ln n - n + 1 \\
&\approx n\ln n - n
\end{aligned}
$$

Chapter 25

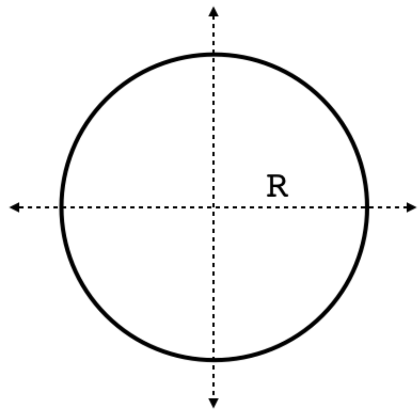
Shells and disks

This chapter is more challenging that the previous ones. But it's worth it, we will show the real power of calculus in finding volumes easily.

volume of the sphere by disks

We already saw how Archimedes found the volume of the sphere. Here we repeat the calculation using calculus of a single variable. This is the first example of what is called a "solid of revolution." We imagine revolving a curve (here, the top half of a circle), around an axis (the x-axis). Revolving the curve forms a surface, and we will consider the volume contained inside that surface.

Draw a circle of radius R, centered at the origin.



Points on the circle obey the standard equation

$$x^2 + y^2 = R^2$$

Rearrange and take the square root to get y as a function of x

$$y = f(x) = \sqrt{R^2 - x^2}$$

For this *function*, we would emphasize that we must choose one root, typically the positive one corresponding to the upper half-circle. The reason is that we need to have each x correspond to a *unique* value of y. If a vertical line cuts through two different curves for a single x, then we don't have a function.

Here, though, it doesn't matter, because are interested in vertical slices through the whole sphere. We will use the area of each slice, with radius y and area πy^2. Squaring, which makes the square root in $y = \sqrt{R^2 - x^2}$ go away, simplifies the integrals in this chapter a lot.

Move from left to right, from $x = -R$ to $x = R$ and add up the areas of all those slices:

$$V = \int_{-R}^{R} \pi y^2 \ dx$$

$$= \pi \int_{-R}^{R} (R^2 - x^2) \ dx$$

$$= \pi(R^2 x - \frac{x^3}{3}) \ \Big|_{-R}^{R}$$

At the upper bound, the term in parentheses is $2/3R^3$, and at the lower bound it is $-2/3R^3$, so subtracting we obtain

$$V = \frac{2}{3}\pi R^3 - (-\frac{2}{3}\pi R^3)$$

$$= \frac{4}{3}\pi R^3$$

as expected.

If you find yourself saying, that's just like what Archimedes did, well, yeah...

Evaluation of the lower bound is confusing (I find it so), so we could note that here the integrand $(R^2 - x^2)$ is an *even* function of x. What that means is

$$f(x) = f(-x)$$

and so

$$\int_{-R}^{R} f(x) \ dx = 2 \int_{0}^{R} f(x) \ dx$$

Take the integral from 0 to R and multiply by two. At this lower bound the value of the integral is zero.

There are several other derivations that use only one-variable calculus. My favorite of all is to integrate the surface area as a function of r, the radius.

We revisit this problem in some detail later, using multi-variable approaches (**here**).

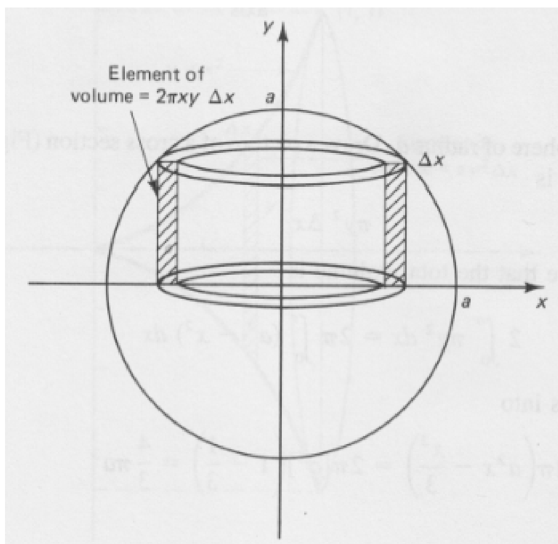
Separately, we will use the theorems of Pappus to find the volume of a "solid of revolution", namely rotation of a half-circle with its base on the x-axis, around that same axis. You can read about it here

http://mathworld.wolfram.com/PappussCentroidTheorem.html

We look at this toward the end of the book (**here**).

volume of the sphere by shells

Here is a picture of what we're doing, from Hamming's *Calculus*. The notation is different but the idea is the same.



We'll work with the hemisphere, above the xy-plane.

Let's divide the sphere up into concentric cylinders or shells, and let r vary from $0 \to R$. The circumference of the shell at each point is

$$C = 2\pi r$$

and the height of each is

$$h = \sqrt{R^2 - r^2}$$

The volume of each very thin cylinder is

$$dV = Ch\ dr = 2\pi r \sqrt{R^2 - r^2}\ dr$$

and we want

$$\int_0^R 2\pi r \sqrt{R^2 - r^2}\ dr$$

$$= -\frac{2}{3}\pi (R^2 - r^2)^{3/2} \Big|_0^R$$

We saw how this works in a previous chapter. To check

$$\frac{d}{dx}\ -\frac{2}{3}\ (R^2 - r^2)^{3/2} = \sqrt{R^2 - r^2}\ (2r)$$

That looks correct. Evaluate the result:

$$= -\frac{2}{3}\pi\ [\ -(R^2)^{3/2}\ ]$$

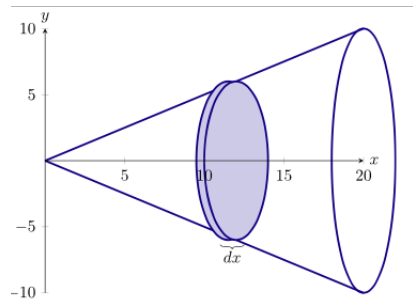
$$= \frac{2}{3}\pi R^3$$

Multiply by two to obtain the total volume.

That integral is one a little more challenging than most that we've had so far. Luckily there are not to many like it, and you will see the same one again and again.

volume of the cone by disks

We did this earlier. It's repeated here for comparison with the other approach.



We have a cone oriented so that it is symmetric about the x-axis, with its vertex at the origin and oriented so that it gets larger as we head to the right.

The equation for the line along the edge of the cone is that the y corresponding to each x is proportional to x with proportionality constant R/H. y is like the radius of the cone and x is like the height.

$$y = \frac{R}{H}x$$

When $x = 0$, $y = 0$, and when $x = H$, $y = R$. Since the formula is linear and it checks at two places, it is correct everywhere.

At each x we slice perpendicular to the x-axis obtaining a circle of radius y and area πy^2. We sum the areas of all these circles

$$V = \int_0^H \pi y^2 \; dx$$

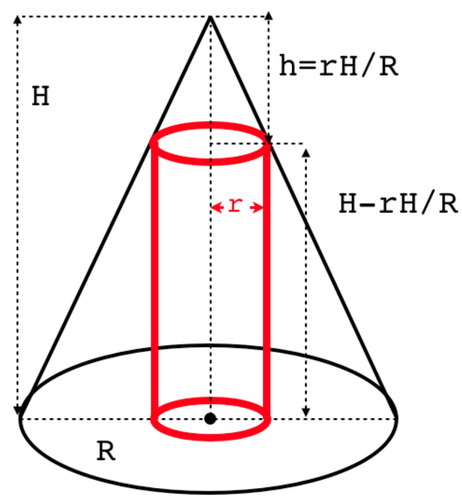
$$= \pi \int_0^H (\frac{R}{H}x)^2 \; dx$$

$$= \pi \, \frac{R^2}{H^2} \, [\frac{x^3}{3} \,] \Big|_0^H$$

$$= \frac{1}{3}\pi H R^2$$

shells

There is another way to "slice" the figure, which is the method of shells.



We think of the volume as constructed from a series of concentric cylinders. Let's use the same letters we had previously, H for total height and R for base radius. At a height h measured down from the top, the radius r is

$$r = h\frac{R}{H}$$

You can check this by similar triangles, or by calculation at two points, as we did above.

Each cylinder has circumference

$$C = 2\pi r = 2\pi h\frac{R}{H}$$

The height of the cylinder is $H - h$, and the lateral surface area of the shell is

$$SA = C(H - h)$$

$$= 2\pi h \frac{R}{H}(H - h)$$

$$= 2\pi \frac{R}{H}(Hh - h^2)$$

We add up all the shells for $h = 0 \to h = H$

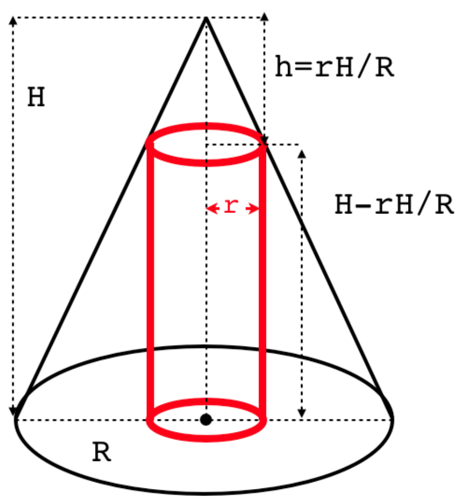
$$V = \int A\ dh = \int_0^H 2\pi \frac{R}{H}(Hh - h^2)$$

$$= 2\pi \frac{R}{H}(\frac{1}{2}Hh^2 - \frac{1}{3}h^3)\Big|_0^H$$

$$= 2\pi \frac{R}{H}(\frac{1}{6}H^3)$$

$$= \frac{1}{3}\pi R^2 H$$

Varying r instead of h



In the previous section we used h as the variable of integration, but we might just as well use r. In that case, r will vary from $r = 0 \to r = R$.

At each value, the circumference will be

$$C = 2\pi r$$

and the height of the cylinder will be

$$H - \frac{H}{R}r$$

The volume is the sum of all the little pieces of cylinder volume

$$\begin{aligned}
V &= \int_{r=0}^{r=R} 2\pi r(H - \frac{H}{R}r)\ dr \\
&= 2\pi H \int_{r=0}^{r=R} r - \frac{1}{R}r^2\ dr \\
&= 2\pi H\ (\frac{r^2}{2} - \frac{1}{R}\frac{r^3}{3})\ \Big|_0^R \\
&= 2\pi H(\frac{1}{6}R^2) = \frac{1}{3}\pi R^2 H
\end{aligned}$$

Part VIII

Sine and cosine

Chapter 26

A famous limit

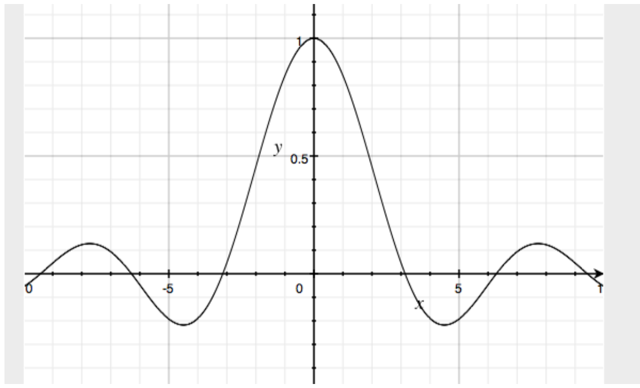
In this unit, we emphasize trigonometric and exponential functions. These are called transcendental functions because they "transcend algebra", meaning that they cannot be expressed as finite polynomials. We will see, however, that they can be expressed as *infinite* polynomials or series.

A famous limit

The fundamental results of calculus with respect to trigonometric functions depend on the value of this limit

$$\lim_{x \to 0} \frac{x}{\sin x}$$

The limit of the ratio of the angle to its sine as the angle gets very small is equal to 1. One way to explore this is to use a plotting application:



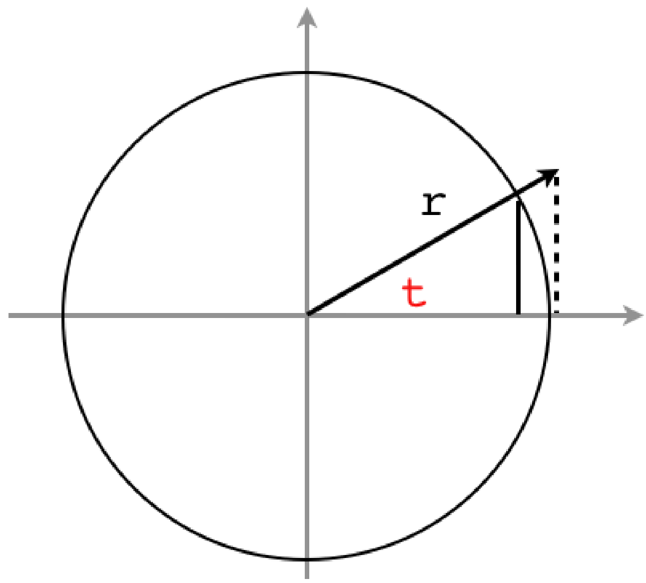
or a calculator such as that embedded in Python

```
>>> for i in range(1,100):
...     f = 1.0/i
...     print i, sin(f)/f
...
1 0.841470984808
..
97 0.999982286557
98 0.99998264621
99 0.999982995019
>>>
```

but these are (to be honest) cheating because when they calculate the sine of the angle they use a shortcut based on calculus.

Here is an actual proof that the ratio is equal to 1.

About notation: in the section above we used x as the variable name for an angle. Historically, of course, Greek letters were favored: θ, ϕ and so on. I use these occasionally, but most frequently I use s or both s and t. I prefer them because I feel like θ takes me a moment longer to process each time I see it than t does. Perhaps more important, t is a bit easier to typeset.



Consider the right triangle with radius r (the one that lies entirely inside the circle). Its base is $r\cos t$ and its height is $r\sin t$, so its area is

$$\begin{aligned} A &= \frac{1}{2} \cdot r\cos t \cdot r\sin t \\ &= \frac{1}{2} r^2 \sin t \cos t \end{aligned}$$

Consider next the sector of the circle (piece shaped like a slice of pie) containing the same angle, t. Recall that t is the length of the portion of the circumference along this sector (if t is measured in radians). If the circle is not a unit circle, then multiply by the radius.

t is some fraction of the total angular measure of the circle, namely $t/2\pi$, and we multiply by the total area of the circle to get the area of the sector:

$$A = \frac{t}{2\pi} \pi r^2 = \frac{1}{2} r^2 t$$

Finally, consider the right triangle containing the dotted line, whose base has length r. Because it is a similar triangle with the first one, its height (that dotted line) is in the same ratio to r, the base of the triangle, as $\sin t$ is to $\cos t$. Thus, its length is $r \tan t$.

The area of this triangle is

$$\begin{aligned} A &= \frac{1}{2} \cdot r \cdot r \tan t \\ &= \frac{1}{2} r^2 \, \frac{\sin t}{\cos t} \end{aligned}$$

Since the first triangle is smaller than the sector, and the sector is smaller than the second triangle, *no matter how small* t becomes:

$$\frac{1}{2} r^2 \sin t \cos t < \frac{1}{2} r^2 t < \frac{1}{2} r^2 \, \frac{\sin t}{\cos t}$$

Now cancel $r^2/2$

$$\sin t \cos t < t < \frac{\sin t}{\cos t}$$

and divide by $\sin t$

$$\cos t < \frac{t}{\sin t} < \frac{1}{\cos t}$$

As $t \to 0$, both $\cos t$ and $1/\cos t$ approach the same limit, 1. Therefore the ratio gets squeezed, and it approaches the same limit as well.

$$\lim_{x \to 0} \frac{x}{\sin x} = 1$$

Since the limit is 1, the inverse approaches the same limit. We have proved the basic limit:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

□

Chapter 27

Sine and cosine

Difference quotient for sine

The limit just obtained allows us to find the derivatives of sine and cosine.

Set up the difference quotient for sine:

$$\frac{\sin(x+h) - \sin x}{h}$$

Using the addition of angles formula:

$$= \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

Group the terms containing $\sin x$ and $\cos x$ separately

$$= \sin x \ \frac{(\cos h - 1)}{h} + \cos x \ \frac{\sin h}{h}$$

Evaluating the limit as $h \to 0$, the second term is

$$\cos x \ \lim_{h \to 0} \frac{\sin h}{h}$$

We can pull $\cos x$ out of the limit, because it does not depend on h. By the main result above (our "famous" limit), the limit part is equal to 1, so the whole expression is just equal to $\cos x$.

We will show in just a minute that the first term is zero, which means that we have in the end:

$$\frac{d}{dx} \sin x = \cos x$$

The derivative of the sine is the cosine.

second limit

We massage that left-hand term from above as follows. $\sin x$ can come out because it does not depend on h. We must then evaluate

$$\lim_{h \to 0} \frac{(\cos h - 1)}{h}$$

Now

$$\frac{\cos h - 1}{h} = \frac{\cos h - 1}{h} \cdot \frac{\cos h + 1}{\cos h + 1}$$

The numerator on the right is

$$(\cos h - 1)(\cos h + 1) = \cos^2 h - 1$$

$$= -\sin^2 h$$

so each term on the right gets one copy of $\sin h$:

$$-\frac{\sin h}{h} \cdot \frac{\sin h}{\cos h + 1}$$

The limit as $h \to 0$ of the first factor is equal to 1 as we saw before, but the second one is $0/2 = 0$, so the whole thing is zero.

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

as promised.

Derivative of the cosine

Set up the difference quotient for cosine:

$$\frac{\cos(x+h) - \cos x}{h}$$

Using addition of angles

$$\frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

Grouping like terms

$$= \cos x \ \frac{(\cos h - 1)}{h} - \sin x \ \frac{\sin h}{h}$$

But we just showed that

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

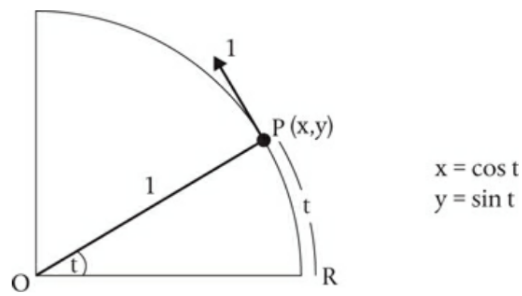
so the first term is zero. By the original limit derived above, the second term is

$$\lim_{h \to 0} \left[\ -\sin x \ \frac{\sin h}{h} \ \right] \ = -\sin x$$

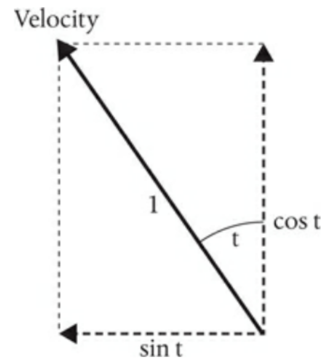
The derivative of the cosine is minus the sine.

$$\frac{d}{dx} \ \cos x = -\sin x$$

Another view



74. Finding the rates of change.



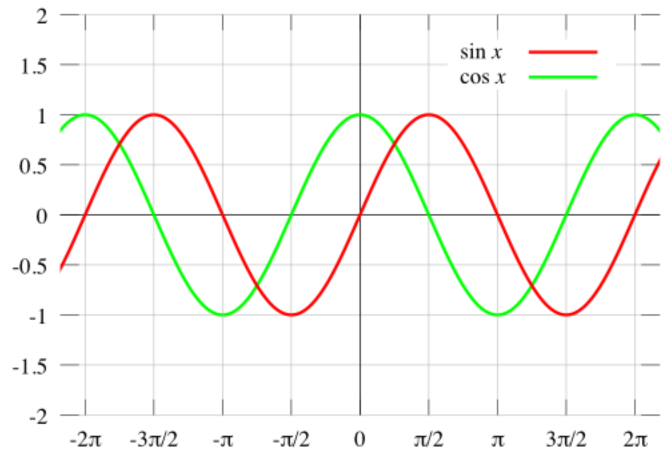
75. The velocity components.

Above are two figures from Acheson:

> One merit of radian measure — together with a unit radius — is that the distance travelled, PR, is not just proportional to the angle POR — it is actually equal to it, and will therefore be t.
>
> So P travels a distance t in time t and therefore goes round and round the circle at unit speed. Its velocity at any moment is therefore 1, directed along the tangent.
>
> And because the tangent is perpendicular to the radius OP, this direction of motion makes an angle t with the y-axis (right panel).
>
> Now, moving with speed 1 in the direction shown is equivalent to moving in the negative x-direction with speed sin t, at the same time as moving in the y-direction with speed cos t.

That is, the x-component, $\cos t$, has a velocity of $-\sin t$, which is its derivative. And the y-component, $\sin t$, has a velocity of $\cos t$.

graphs



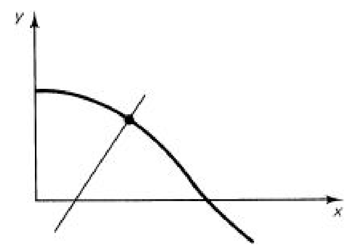
By examining the graphs of sine and cosine, we can see that the results obtained above make sense. The maximum slope of the sine function occurs when $\theta = 0$ because $\cos\theta = \cos 0 = 1$ there, and that matches the plot.

At the top of the arc for sine, when $\theta = \pi/2$, then $\sin\theta = 1$. The curve changes from increasing to decreasing at the very peak, and just for an instant, the slope is zero and the curve is horizontal. The corresponding value for $\cos\theta = \cos\pi/2 = 0$.

Furthermore, the slope of sine is positive when cosine is positive, while the slope of cosine is positive when sine is negative.

example

Here is a problem from Hamming that we can solve using these results.



The curve is $\cos x$ and the question is, if we form the norm (perpendicular) to the curve, does it ever pass through the origin?

The slope of the cosine is $-\sin x$. The slope we seek is perpendicular to that, so its product must yield -1. Thus $m = 1/\sin x$.

We use the point slope formula:

$$\frac{y - y_0}{x - x_0} = \frac{1}{\sin x}$$

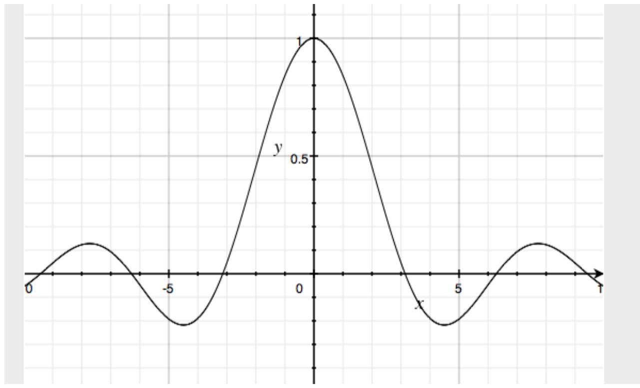
At the point $(x_0, y_0) = (0, 0)$

$$\frac{y}{x} = \frac{1}{\sin x}$$

But $y = \cos x$ so

$$\sin x \cos x = x$$
$$\frac{1}{2}\sin 2x = x$$
$$\sin 2x = 2x$$

One solution is $(0, 0)$. Are there any more? One way to answer this is to go back to the figure we started with where we plot $\sin x/x$:



No. $\sin x = x$ happens only at $x = 0$. In the figure, $\sin x/x = 1$ happens only at $x = 0$.

For an analytical proof, we observe that the slope of $\sin x$ is equal to $\cos x$ which is equal to 1 at $x = 0$ and for every x after that

$$0 < \cos x < 1, \qquad 0 < x < \frac{\pi}{2}$$

the slope of $\sin x$ is less than 1, while the slope of $y = x$ is equal to 1 everywhere.

The result is that the x rises faster than $\sin x$ everywhere after $x = 0$.

Other trig functions

We use the quotient rule, described **here**.

$$\frac{u'}{v} = \frac{u'v - uv'}{v^2}$$

Check that we've remembered it correctly:

$$[\ \frac{x}{1}\ ]' = \frac{1 \cdot 1 - x \cdot 0}{1} = 1$$

The derivative of the tangent is

$$[\ \frac{\sin x}{\cos x}\ ]' = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

and the secant (inverse cosine):

$$[\ \frac{1}{\cos x}\ ]' = \frac{-(-\sin x)}{\cos^2 x}$$

$$= \sec x \tan x$$

To round out the trig functions we have the cosecant and the cotangent. They aren't seen often, except as part of a strategy of making calculus more difficult than it needs to be.

The cosecant is the inverse of the sine:

$$\frac{d}{dx}\ \frac{1}{\sin\theta} = -\frac{1}{\sin^2\theta}\ \cos\theta$$

$$= -\csc\theta\cot\theta$$

And the cotangent is of course

$$\frac{d}{dx}\ \frac{\cos\theta}{\sin\theta} = \frac{-\sin\theta\sin\theta - \cos\theta\cos\theta}{\sin^2\theta}$$

$$= -\frac{1}{\sin^2\theta} = -\csc^2\theta$$

Notice the similarity to the secant and tangent, with a change of sign as well as substitution of the cotangent and cosecant.

differentiation trick

We used the sum of angles formulas (derived **here**) above. Here are two simple tricks requiring knowledge of the derivatives that help in the finding the sum of angles.

First, if we treat t as a constant and differentiate with respect to s (or vice-versa) we can go between formulas pretty easily:

Start from

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$

Differentiate

$$-\sin(s+t)\ ds = -\sin s \cos t\ ds - \cos s \sin t\ ds$$

$$\sin(s+t) = \sin s \cos t + \cos s \sin t$$

Or start from

$$\sin(s+t) = \sin s \cos t + \cos s \sin t$$

Differentiate

$$\cos(s+t)\ ds = \cos s \cos t\ ds - \sin s \sin t\ ds$$

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$

offsets to cosine

To rearrange the sum of angles formula above, we also need expressions for $\sin\theta$ in terms of cosine with an offset.

I find a simple approach is to take the derivative. We have the offset to sine to find cosine as

$$\cos\theta = \sin(\theta + \frac{\pi}{2})$$

the derivative is

$$-\sin\theta = \cos(\theta + \frac{\pi}{2})$$

$$\sin\theta = -\cos(\theta + \frac{\pi}{2}) = \cos(\theta - \frac{\pi}{2})$$

Part IX

Exponential and logarithm

Chapter 28

Exponential and logarithm

Principal and interest

Suppose I put 100 dollars in the bank, and the people at the bank say that after one year, they will give me an additional \$10 at that time. We say that they are paying 10% interest for the year on the principal P of \$100.

However, suppose I bargain with them. I get them to promise to pay me half the interest (5%) at the six-month mark, and the rest after one year. My account will hold \$105 after six months, and the interest due for the second half will be 5% of \$105, which is \$5.25 for a total of \$10.25.

The equation to describe this situation is that if the rate of interest for the year is r and the year is broken up into n periods when interest will be paid, the total amount at the end will be:

$$A = P(1 + \frac{r}{n})^n$$

In the example, we have $r = 0.10$ and $n = 2$ so

$$A = 100(1 + 0.05)^2 = 110.25$$

This is compound interest. If there are additional years t, the exponent will be nt rather than n.

And now we start wondering what happens if the bank pays every month so that $n = 12$ or every day so $n = 365$ or even every second. What happens if the interest is compounded *continuously*?

$$A = \lim_{n\to\infty} P \; [\; (1 + \frac{r}{n})^n \;]$$

Now it turns out that in the limit as n approaches ∞ these two expressions are equal

$$(1 + \frac{r}{n})^n = \; [\; (1 + \frac{1}{n})^n \;]^r$$

The same factor r can be either in the numerator of the second term inside or up in the exponent outside.

A quick proof is:

$$\lim_{n\to\infty} (1 + \frac{r}{n})^n$$

$$= \lim_{n\to\infty} (1 + \frac{r}{n})^{(n/r)r}$$

Define $m = n/r$ and so as $n \to \infty$, so does $m \to \infty$ and then we have

$$\lim_{m\to\infty} (1 + \frac{1}{m})^{(m)r}$$

and the r is outside. m is just a dummy variable so we write:

$$\lim_{n\to\infty} \; [\; (1 + \frac{1}{n})^n \;]^r$$

Therefore, going back to what we were working on, let us bring out the factor r and obtain

$$A = P(1 + \frac{1}{n})^{nr}$$

$$A = P \left[\ (1 + \frac{1}{n})^n \ \right]^r$$

Thus, the important question is, what is the value of this expression?

$$A = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

It does not depend on r. It will turn out that this limit is equal to the number e.

$$e = 2.71828\ 18284\ 59045 \ldots$$

First, though, let us review some properties of logarithms.

working with logarithms

The logarithm and exponential functions are inverses. If we have that

$$y = b^x$$

for some $b > 0, b \neq 1$, then we say that

$$x = \log_b \ y$$

Putting them together

$$y = b^{\ \log_b y}$$

The usual bases are 10 (common logarithm, $\log_{10}$, or just log), e (natural logarithm or ln), and 2 (binary logarithm or $\log_2$).

The rules for exponents are simple, if p and q are two numbers and we know the logarithms of p and q to base b

$$p = b^u; \quad q = b^v$$

then their product can be computed as:

$$pq = b^u \cdot b^v = b^{u+v}$$

It helps if we can actually compute b^{u+v}. In the old days there were tables of logarithms, so you just looked up the answer in the table.

The second rule is that:

$$(b^u)^v = b^{uv}$$

And in terms of logarithms we write

$$\log_b (b^u)^v = \log b^{uv} = v \log_b (b^u)$$

For example

$$2^2 = 2 \times 2 = 4$$

$$2^3 = 2 \times 2 \times 2 = 8$$

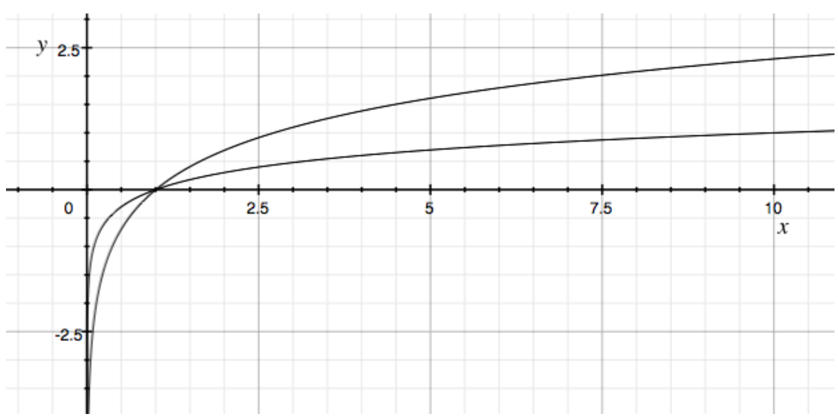
$$4 \times 8 = 2^2 \times 2^3 = 2^{2+3} = 2^5$$

$$= 2 \times 2 \times 2 \times 2 \times 2 = 32$$

and

$$(2^2)^3 = 4^3 = 64 = 2^6 = 2^{2 \times 3}$$

Here is a plot of $\log_{10}(x)$ and $\ln x$:



The first function reaches the value 1 when $x = 10$ and the second reaches the value 1 when $x = e$. Both have the value 0 at $x = 1$

because $b^0 = 1$ for any base, so the logarithm to any base of 1 is equal to 0.

It turns out that if we take the logarithm of x (where x is any number > 1) to two *different* bases, the ratio of the logarithms is a constant, independent of the value of x.

change of bases

This is nicely shown by the change of bases formula.

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Start with an expression with b as the base:

$$y = b^x$$

and by the definition of the logarithm

$$x = \log_b y$$

To derive the formula, take the logarithm to the base a on both sides of the first expression:

$$\log_a y = \log_a(b^x)$$

Now, just invoke the second rule on the right-hand side

$$= x \log_a b$$

and substitute for x from the second expression above

$$= \log_b y \log_a b$$

We're basically done.

y can be any value, so replace it by x

$$\log_a x = \log_b x \log_a b$$

Rearranging:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

One way I remember this is that first the logarithms to different bases are connected by some constant k

$$\log_b x = k \log_a x$$

and we substitute for k the inverse of the log to the *same* base as we have in the numerator:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

that is, I remember that we want $\log_a$ something *over* $\log_a$ something on the right.

Alternatively, you might look at the other formula

$$\log_a x = \log_a b \log_b x$$

and imagine the b's canceling in some way.

One other thing we can do is to set $x = a$ in the above formula. We start from

$$\log_b x = \frac{\log_a x}{\log_a b}$$

then with $x = a$

$$\log_b a = \frac{\log_a a}{\log_a b}$$

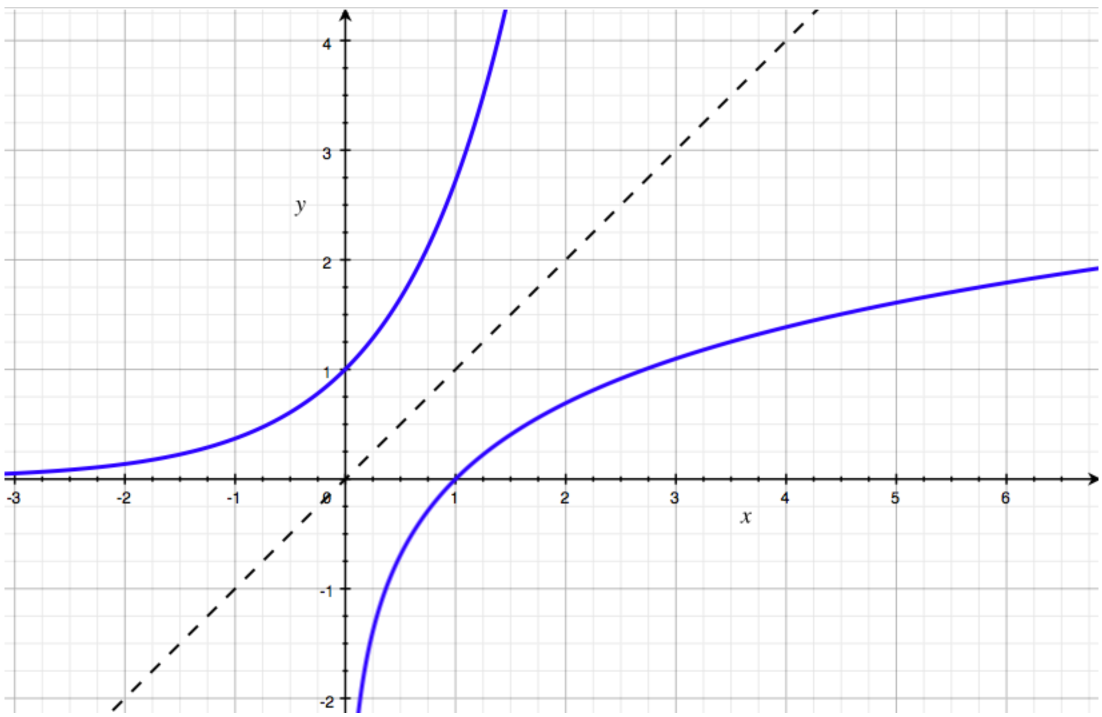
but $\log_a a = 1$ so

$$\log_b a = \frac{1}{\log_a b}$$

And that makes perfect sense. If we multiply by some factor k to convert from the logarithm in base a to base b, we must multiply by the inverse of the same factor to convert back again.

For the figure above of the common log (base 10) and the natural logarithm, $\ln 10 = 2.303$, and that looks about right, when $x = 10$ the first function is 1.0 and the second one is about 2.3.

The logarithm and the exponential are inverse functions, we can see that if we plot them together:



The upper curve is $y = e^x$ and the lower one is $y = \ln x$. As inverse functions, they are symmetric about the line $y = x$.

fractional exponents

The introduction above dealt mainly with integer exponents, but of course you know that the practical use of logarithms depends on fractional values. The simplest way to see how this works is to consider the square root.

$$\sqrt{2} \times \sqrt{2} = 2$$

If we think about what the exponent u to the base 2 would be such that

$$2^u = \sqrt{2}$$

We observe that by the rules for exponents

$$\sqrt{2} \times \sqrt{2} = 2^u \times 2^u = 2^{u+u} = 2^1$$

That is

$$u + u = 1$$

so $u = 1/2$. By the same logic the n^{th} root of b is $b^{1/n}$. And of course

$$(b^2)^{1/2} = b^{2 \times 1/2} = b^1$$

Feynman has a nice description of how logarithms were calculated (see Lectures, volume 1, Chapter 22, Algebra;

http://www.feynmanlectures.caltech.edu/I_22.html)

The basic idea is to take repeated square roots of the base (10), and then combine those to form the required value.

Less than 1

Fractional exponents leads to consideration of $0 < x < 1$. Write

$$x \, \frac{1}{x} = 1$$

Take the logarithm of both sides

$$\log(x\ \frac{1}{x}) = \log 1 = 0$$

$$= \log x + \log \frac{1}{x}$$

Thus

$$\log \frac{1}{x} = -\log x$$

Chapter 29

Half life

A sample of phosphate containing some percentage of the radioactive isotope ^{32}P emits beta particles (energetic electrons of about 0.5 MeV), which are emitted as the nucleus decays into ^{32}S. These can be measured with a Geiger counter, by scintillation counting or by other means (like exposure to X-ray film).

I used a device very similar to this. We would count radioactivity by holding up tubes at some distance like 10 cm, and evaluating how loud the device "screamed". Not quantitative, just trying to find the peak of activity eluted off a chromatographic column.

The measured activity decreases or decays with a half-life of just over 14 days (14.29, to be more precise).

If you want to solve a problem with a given amount of the radioisotope N_o at time-zero ($t = 0$), and you are asked for the amount at time t, what do you do?

If the time is an even multiple of the half-life, it's easy. For one half-life, multiply by $\frac{1}{2}$, for two half-lives multiply by $\frac{1}{4}$, for n half-lives multiply by $(\frac{1}{2})^n$.

If the time is not an even multiple of the half-life, you need the following equation (where T is the half-life and k is a rate constant)

$$N = N_o\, e^{-kt}$$

If you are given the half-life, you will also need that

$$kT = \ln 2 = 0.693$$

I want to show where this equation comes from to motivate our future discussion of the exponential. We will learn two equivalent definitions. The first is that

$$\frac{d}{dx}\, e^x = e^x$$

The second is that

$$\frac{d}{dx}\, \ln(x) = \frac{1}{x}$$

or

$$\int \frac{1}{x}\, dx = \ln(x)$$

In radioactive decay, each atom has a fixed, characteristic probability of disintegrating in the next short time interval dt, although it is impossible to tell in advance *which* nuclei will decay.

The probability varies for different types of radioactive atom (^{3}H, ^{14}C, ^{238}U, etc.), but for each phosphorus atom of this isotope in our sample of ^{32}P it is the same.

As a result, the number of atoms dN that will disintegrate or decay in the short time dt is proportional to N, the number currently present. A fixed fraction of all the atoms will be transformed. We write

$$dN = kN\ dt$$

Slinging differentials, we rearrange and integrate

$$\int \frac{dN}{N} = \int k\ dt$$

The answer is just

$$\ln(N) = kt + C_0$$

Form the exponential on both sides

$$N = Ce^k t$$

$(C = e^{C_0})$. We evaluate the constant C by setting $t = 0$ and find that $C = N_o$ so

$$N = N_o\ e^{kt}$$

Finally, in decay problems it is usual to let k be positive and introduce a minus sign

$$N = N_o\ e^{-kt}$$

As we said

$$kT = \ln 2 = 0.693$$

This is very useful to remember, because frequently we are given a half-life T and asked to compute using the equation with e^{-kt}. It will save time to convert T to k quickly.

The derivation is as follows. By definition, after one half-life has elapsed, when the time $t = T$, $N = N_o/2$.

$$\begin{aligned} \frac{N_o}{2} &= N_o\ e^{-kT} \\ \frac{1}{2} &= \ e^{-kT} \\ 2 &= \ e^{kT} \\ \ln 2 &= \ kT \end{aligned}$$

Equations for the growth of populations work similarly, with

$$N = N_o\ e^{kt}$$

and again, if the problem gives you an even number of doublings, or generations, just use that. For growth equations it is usual to use another symbol like g for the number of generations, where

$$N = 2N_o\ e^{kg}$$

but the same relation holds between g and k (because of the switched minus sign in e^{kt}).

$$\ln 2 = \ kg$$

guy from Philadelphia

Here is a crime scene example. According to Newton's Law of Cooling

$$T(t) = T_e + (T_0 - T_e)e^{-kt}$$

Suppose Tony Soprano whacks some wise guy from Philly at time-zero, and immediately drags the body into the meat cooler (or better yet, lures him in there first).

The temperature of the stiff at time t is given by the above equation. We'll say very roughly that $T_0 = 37$ (Celsius, formerly Centrigrade) and the environmental temperature $T_e = -3$ so the temperature as a function of time is given by

$$T = 40e^{-kt} - 3$$

$$3 + T = 40e^{-kt}$$

As the crime scene investigator, you need a value for k in order to find the time of death. One way to obtain that is to determine the temperature at two different times. You take the temperature of the body on arrival at the crime scene (an unknown time τ after death) and obtain a value of 27 degrees C. One hour later ($\tau+1$), after photographs have been taken and the scene dusted for fingerprints, you measure it to be 17 C. We have

$$\log\frac{30}{40} = -k\tau$$

$$\log\frac{20}{40} = -k(\tau+1)$$

Subtracting

$$\begin{aligned} k &= \log\frac{30}{40} - \log\frac{20}{40} \\ &= \log(\frac{30}{40}\cdot\frac{40}{20}) \\ &= \log 1.5 = 0.40 \end{aligned}$$

Since $\log\frac{30}{40} = -0.288$, we have from the first equation that

$$-0.288 = -k\tau = -0.40\tau$$

$$\tau \approx 0.72$$

It was only about 45 minutes after the hit when you first arrived.

Chapter 30

Another famous limit

Consider the function of the variable x defined by the power b^x, where b is some arbitrary constant.

$$f(x) = b^x$$

We will show later that the difference quotient whose limit defines $f'(x)$ can be manipulated to give the following form

$$b^x \lim_{h \to 0} \frac{b^h - 1}{h}$$

For this reason, we are interested in the properties of the limit

$$\lim_{h \to 0} \frac{b^h - 1}{h}$$

First, it is clear that the limit does not depend on x, so it is just a number.

working with the constant

We can easily calculate the value of

$$\lim_{h \to 0} \frac{b^h - 1}{h}$$

Set $h = 0.00001$ and use Python. For the following values of b I get:

- ○ $b = 2$ gives 0.693
- ○ $b = 2.71828 \cdots = e$ gives 1.000
- ○ $b = 10$ gives 2.303

This gives us one *definition* of e. It is the value of b for which this limit is equal to 1.

It will not surprise you to learn that these values correspond to the natural logarithm of the base.

playing with the limit expression

So starting from the definition that e is the value for which

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

We can rearrange the equation as follows:

$$\lim_{h \to 0} e^h = \lim_{h \to 0} 1 + h$$

$$\lim_{h \to 0} e = \lim_{h \to 0} (1 + h)^{1/h}$$

$$e = \lim_{h \to 0} (1 + h)^{1/h}$$

since e is just a constant (we haven't demonstrated that it is legal to manipulate limits in this way, but you can).

We can also substitute $nh = 1$ and so $n = 1/h$ and the expression can be written as

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

which is the first expression that we introduced for e.

All three limits are equivalent:

$$\begin{aligned} e &= \lim_{n \to \infty} (1 + \frac{1}{n})^n \\ &= \lim_{h \to 0} (1 + h)^{1/h} \end{aligned}$$

and

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

We can calculate e from this equation as well, but it doesn't converge so fast. For $n = 100000$, using Python, I got $e = 2.718168$, which is only correct to 3 decimal places.

On the other hand, one can use several approaches including the binomial theorem to evaluate

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

and show that it is equivalent to

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

which is yet another one of the many definitions for e.

The first three terms are $1 + 1 + 1/2$, which is reasonably close. After six terms, we have $e = 2.718$. The series converges rapidly because the inverse factorials get small very quickly. We calculate e as about 2.71828.

the function

In general we are interested in the *function* e^x and not just the number e.

Start with a general exponential (base b) and consider the problem of finding its slope. We want to see what happens to the exponential function when we vary x by a small amount h.

We want the difference quotient

$$\lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$$

for some unspecified base b. We can rewrite this as

$$\lim_{h \to 0} \frac{b^x(b^h - 1)}{h}$$

The great insight is to note that b^x does not depend on h, so it doesn't change as we go to the limit, and thus we have

$$b^x \lim_{h \to 0} \frac{b^h - 1}{h}$$

the limit is not dependent on x, so it is a constant. Therefore, the slope of the exponential

$$y = b^x$$

for some base b is

$$y' = cb^x$$

The slope is the same as the function, up to a constant.

the exponential function

The number e is such that if $b = e$, then $c = 1$ and we will have:

$$y = e^x$$

$$y' = e^x$$

The function e^x is its own derivative, which leads to all of its amazing properties.

Write

$$e^x = \lim_{n\to\infty} [\ (1 + \frac{1}{n})^n\ ]^x$$

$$e^x = \lim_{n\to\infty} (1 + \frac{1}{n})^{nx}$$

It turns out this is exactly the same as

$$= \lim_{n\to\infty} (1 + \frac{x}{n})^n$$

which seems counterintuitive.

But the result follows from

$$(1 + b)^{mn} = (1 + mb)^n$$

a quick proof

We saw this proof in the previous chapter. We want to show that

$$\lim_{n\to\infty} (1 + \frac{r}{n})^n = \lim_{n\to\infty} [\ (\frac{1}{n})^n\ ]^r$$

Start with

$$\lim_{n\to\infty} (1 + \frac{r}{n})^n = \lim_{n\to\infty} (1 + \frac{r}{n})^{(n/r)r}$$

Define $m = n/r$ and so as $n \to \infty$, so does $m \to \infty$ and then we have

$$\lim_{m\to\infty} (1 + \frac{1}{m})^{(m)r}$$

and the r is outside. m is just a dummy variable so we write:

$$\lim_{n\to\infty} [\ (1 + \frac{1}{n})^n\ ]^r$$

$\square$

proof using the binomial

Write the binomial expansion for both versions. For the first, we have

$$(1+b)^{mn} = 1+(mn)b+\frac{(mn)(mn-1)}{2!}b^2+\frac{(mn)(mn-1)(mn-2)}{3!}b^3+\dots$$

In the limit as $n \to \infty$, the subtracted values of $k = 1, 2, 3\dots$ are negligible and we have

$$= 1 + (mnb) + \frac{1}{2!}(mnb)^2 + \frac{1}{3!}(mnb)^3 + \dots$$

For the second we get

$$(1 + mb)^n = 1 + n(mb) + \frac{n(n-1)}{2}(mb)^2 + \frac{n(n-1)(n-2)}{3!}(mb)^3 + \dots$$

and with the same limit, we get the identical formula.

Going back to

$$e^x = \lim_{n\to\infty} (1 + \frac{x}{n})^n$$

$$= \frac{1}{0!}(\frac{x}{n})^0 + +\frac{n}{1!}(\frac{x}{n})^1 + \frac{n(n-1)}{3!}(\frac{x}{n})^2 + \frac{n(n-1)(n-2)}{3!}(\frac{x}{n})^3 + \cdots$$

The n's cancel and we have

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

This approach assumes that the binomial applies to fractional exponents, which is a bit complicated to show.

□

sketch of a proof using Taylor series

A more sophisticated way to do this is to use Taylor series, together with the fact that the derivative of e^x is equal to e^x.

We can find the value of e^x near $x = 0$ as

$$e^x = \sum_{n=0}^{n=\infty} \frac{f^n(0)}{n!} x^n$$

In particular, all the derivatives are just $e^x = 1$ at $x = 0$ so

$$e^x = \sum_{n=0}^{n=\infty} \frac{x^n}{n!}$$

$$= \frac{x^0}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$$

and if $x = 1$ we have that

$$e^1 = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} \dots$$

The exponential is its own derivative

As we've emphasized, the most important fact about the exponential function $f(x) = e^x$ is that this function is its own derivative.

Above, we derived the famous series for

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots$$

Note that a good approximation for small x is

$$e^x \approx 1 + x$$

If you need more precision, you can add another term:

$$e^x \approx 1 + x + \frac{x^2}{2!}$$

It is easy to see that the derivative of this series is the series itself.

Take $\frac{d}{dx}$ of the last series.

$$\begin{aligned}\frac{d}{dx}\ e^x &= 0 + (1)\frac{x^{1-1}}{1!} + (2)\frac{x^{2-1}}{2!} + (3)\frac{x^{3-1}}{3!} + \cdots \\ \frac{d}{dx}\ e^x &= 0 + \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots \\ &= e^x\end{aligned}$$

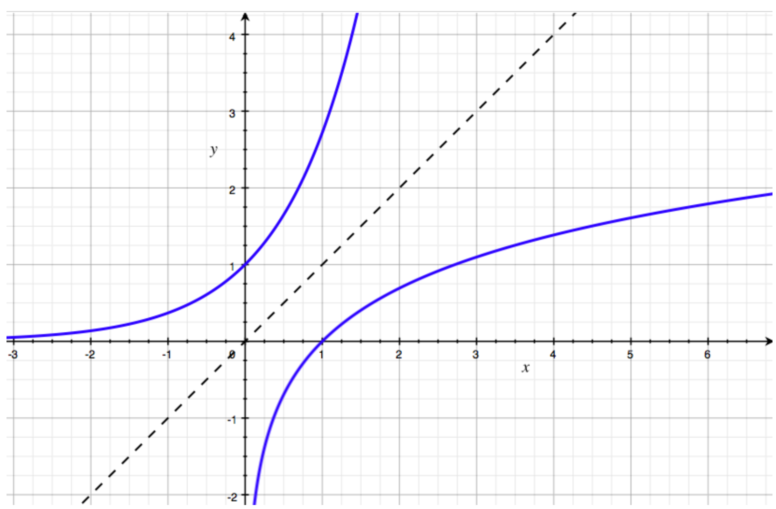
Each exponent n that comes down through the power rule, finds an n in $n! = n \times (n-1) \times (n-2) \ldots$ to cancel, leaving $n-1$ in the exponent as well as $(n-1)!$.

Chapter 31

Exponential to logarithm

The logarithm and the exponential function are inverses of each other. We will prove this assertion here.

It seems reasonable enough if we plot them together. The upper curve is $y = e^x$ and the lower one is $y = \ln x$.



As inverse functions, e^x and $\ln x$ are symmetric about the line $y = x$.

If we consider a particular x value, for example $x = 1$, then the slope of the curve $y = e^x = e$ at the point $(1, e)$ is the inverse of the slope of the curve $y = \ln x$ when $x = e$ (at the point $(e, 1)$).

from exponential to logarithm

The logarithm can be defined as follows:

$$\int \frac{1}{t}\ dt = \ln(t)$$

It is possible to prove this starting from the assumption that e is its own derivative, or that e is the limit we discussed above. Alternatively, we can start from this statement and derive the facts about e. We will do the latter in a later chapter.

Now, differentiate both sides to find the derivative of the logarithm. By the FTC:

$$\frac{d}{dt} \int \frac{1}{t}\ dt = \frac{1}{t} = \frac{d}{dt} \ln(t)$$

This is great because we never did generate x^{-1} by differentiating powers of x.

And now we know how to go back the other way, using the definition that the exponential function is its own derivative to establish the first statement above.

The proof is so simple that if you blink, you'll miss it.

$$y = e^x$$

$$\frac{dy}{dx} = e^x = y$$

Invert

$$\frac{dx}{dy} = \frac{1}{y}$$

$$dx = \frac{1}{y}\ dy$$

Integrate

$$\int dx = x = \int \frac{1}{y}\ dy$$

And what is x? It is $\ln y$!

$$\ln(y) = \int \frac{1}{y}\ dy$$

And since y is just a letter, we can write the same for x, or t

$$\ln x = \int \frac{1}{x}\ dx, \qquad \ln t = \int \frac{1}{t}\ dt$$

One of the prettiest things I've ever seen.

The math purists don't like it, but in general it is OK to do algebra with differentials. One important restriction is that dy/dx (and dx/dy) should not be equal to zero. And a second one is that we just remember we are never passing to the limit, just getting really, really, really close. (As close as you like).

Now that we have this definition, we have another way of estimating the value of e. We add up the areas for little slices under the function $y = 1/x$ until the total reaches 1. The corresponding value of $x = e$.

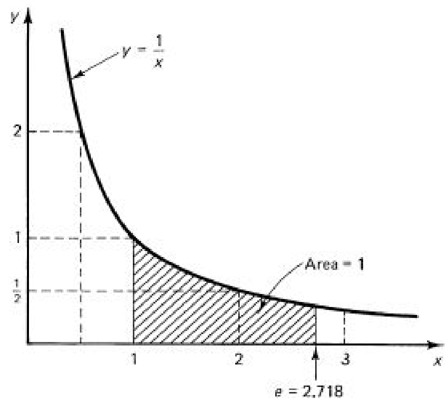


Figure 14.3-1 Estimate of e

(from Hamming).

For example, we might try intervals of 0.1 and do

$$0.1 \cdot \frac{1}{1} + 0.1 \cdot \frac{1}{1.1} + 0.1 \cdot \frac{1}{1.2} + \cdots + 0.1 \cdot \frac{1}{2.7}$$

reverse direction

Above we showed that $(e^x)' = e^x$ implies

$$\ln x = \int \frac{1}{x}\, dx$$

Differentiate both sides

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

We want to go backward now, to show that the derivative of the function $f(x) = e^x$ is itself.

Start with

$$\ln(e^x) = x$$

$$\frac{d}{dx}\ln(e^x) = \frac{d}{dx}x = 1$$

but using the property we just proved and the chain rule, this is also

$$\frac{d}{dx}\ln(e^x) = \frac{1}{e^x}\ \frac{d}{dx}e^x$$

so these two expressions are equal and

$$\frac{1}{e^x}\ \frac{d}{dx}e^x = 1$$

$$\frac{d}{dx}e^x = e^x$$

Magic.

This is really the primordial differential equation.

Strang's view of the series

Gil Strang has a nice introduction to the exponential (for real numbers). He says we want to "construct a function" for $y = e^x$. The first, amazing, property (I) is that the derivative of the function is equal to the function itself.

$$y = y'$$

The second property, a boundary condition, is that at $x = 0$, we want $y = 1$. That's because e is not a variable and we want $e^x = e^0$ to be equal to 1 like every other exponential we know. [A third property that will turn out to be true is $e^{x_1+x_2} = e^{x_1} \cdot e^{x_2}$.]

Using the second condition we write:

$$y(x) = 1$$

Whatever else is true, if there are no terms containing x (because $x = 0$) then $y = 1$. By I we must have that the derivative is equal to the function:

$$y(x) = 1$$
$$y'(x) = 1$$

But now, if we try to evaluate the derivative starting from $y(x)$, where does that 1 come from? It must come from

$$y(x) = 1 + x$$
$$y'(x) = 1$$

Now, though, it's no longer true that $y = y'$ so we fix that

$$y(x) = 1 + x$$
$$y'(x) = 1 + x$$

And where does that x come from in the derivative? It must come from

$$y(x) = 1 + x + \frac{x^2}{2}$$
$$y'(x) = 1 + x$$

And so on ... Following this procedure, we build up the definition

$$\begin{aligned} y(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\ &= \sum_0^\infty \frac{x^n}{n!} \end{aligned}$$

The test for convergence of this infinite series is to find the values of x such that:

$$\lim_{n\to\infty} \frac{|A_{n+1}|}{|A_n|} < 1$$

That is, we need

$$\lim_{n\to\infty} \frac{x}{n+1} < 1$$

But this is true for any x.

Chapter 32

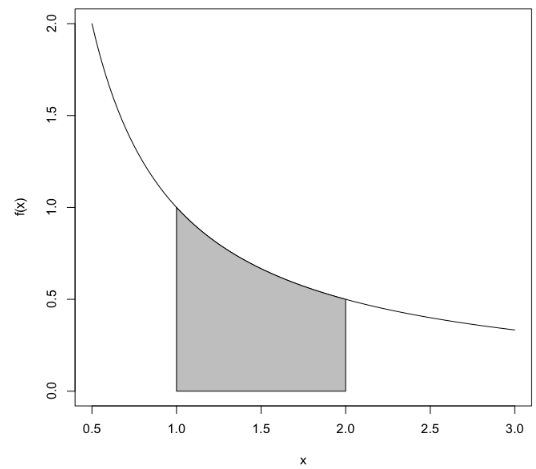
Logarithm to exponential

In more advanced treatments (analysis), what they do is to investigate a particular function which we will call L, and show that the function L has all the properties of the logarithm, *so it is the logarithm.*

We follow that path here, and then after that go backwards from the logarithm to the other properties of e, like its numerical value, and the fact that the derivative of e^x is e^x itself.

This approach comes straight from David Jerrison's lecture in Calculus 1 (MIT online course). We define the logarithm function as

$$L(x) = \int_1^x \frac{dt}{t}$$



For example, the logarithm of 2 is the area under the curve above, $f(x) = 1/x$, between $1 \leq x \leq 2$. Having defined

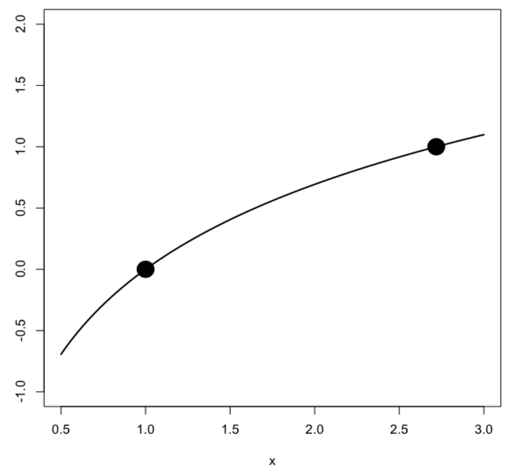
$$L(x) = \int_1^x \frac{dt}{t}$$

By the Fundamental Theorem of Calculus (part II) we have

- Property 1

$$L'(x) = \frac{1}{x}$$

The slope of the logarithm function is always positive ($x > 0$), but is undefined for $x = 0$



- Property 2

$$L(1) = \int_1^1 \frac{dt}{t} = 0$$

This property is by definition. It fits with our use of exponents, where $b^0 = 1$.

- Property 3

$$L''(x) = -\frac{1}{x^2}$$

Although the area under the curve $ln(x)$ is always increasing, so the slope is always positive, the rate of increase of the slope is always decreasing, so the shape is concave down.

- Property 4

$$L(e) = 1$$

This is by definition as well. In extending to exponents it means x is a single-valued function of y, so we can write $y = L(x) \iff e^y = x$.

- Property 5

$$L(ab) = L(a) + L(b)$$

To show that this last statement is true involves showing that

$$\int_1^{ab} \frac{dt}{t} = \int_1^{a} \frac{dt}{t} + \int_a^{ab} \frac{dt}{t}$$

is both true and equivalent to the first statement.

For the arguments a and ab we have

$$L(ab) = \int_1^{ab} \frac{dt}{t}$$

$$L(a) = \int_1^{a} \frac{dt}{t}$$

Both of these are true by definition. The one that takes a little work is

$$L(b) = \int_a^{ab} \frac{dt}{t}$$

We do a substitution. Let $au = t$, then $a\ du = dt$ and

$$L(b) = \int \frac{a\ du}{au} = \int \frac{du}{u}$$

At this point we run into a new idea. When we change the variable we also must change the bounds. At the first one, we have $t = a$ so

$$u = \frac{t}{a} = \frac{a}{a} = 1$$

At the upper bound, we have $t = ab$ so

$$u = \frac{t}{a} = \frac{ab}{a} = b$$

The integral $\int_a^{ab}$ becomes $\int_1^{b}$ and we have

$$L(b) = \int_1^{b} \frac{du}{u}$$

which is again, true by definition.

Thus, the function L has the property that

$$L(ab) = L(a) + L(b)$$

which is one of the two major properties of logarithms.

- Property 6

To see that the second property is also true, start with

$$L(a^r) = \int_1^{a^r} \frac{dt}{t}$$

Substitute $t = u^r$, so $dt = ru^{r-1}du$. The bounds are changed as follows (r can be anything):

$$t = 1, \quad t = u^r \to u = 1$$

$$t = a^r, \quad t = u^r \to u = a$$

This gives

$$\begin{aligned} L(a^r) &= \int_{t=1}^{t=a^r} \frac{dt}{t} \\ &= \int_{u=1}^{u=a} \frac{1}{u^r}(ru^{r-1})du \\ &= r\int_1^a \frac{du}{u} = rL(a) \end{aligned}$$

That is

$$L(a^r) = rL(a)$$

As Dunham says (using A for L)

> "these properties of the hyperbolic area—namely $A(ab) = A(a) + A(b)$ and $A(a^r) = rA(a)$—exactly mirror the corresponding properties of logarithms. Clearly something interesting is afoot."

Difference quotient for logarithm

As seen in Hamming, we can also go back to the definition of the logarithm as the inverse of the exponential

$$f(x) = \log_b x$$

write the difference quotient

$$f'(x) = \lim_{h \to 0} \frac{\log_b(x+h) - \log_b x}{h}$$

and then use the properties of logarithms to rearrange it as follows:

$$= \lim_{h \to 0} \frac{\log_b(\frac{x+h}{x})}{h}$$

$$= \lim_{h \to 0} \frac{\log_b(1 + \frac{h}{x})}{h}$$

$$= \lim_{h \to 0} \log_b \ [\ (1 + \frac{h}{x})^{1/h}\ ]$$

$$= \lim_{h \to 0} \frac{1}{x}\ [\ \log_b(1 + \frac{h}{x})^{x/h}\ ]$$

Finally

$$= \frac{1}{x}\ \lim_{h \to 0}\ [\ \log_b(1 + \frac{h}{x})^{x/h}\ ]$$

So it's clear that we will need to evaluate the term for which we are taking the logarithm, in the limit

$$= \lim_{h \to 0} (1 + \frac{h}{x})^{x/h}$$

Let $t = h/x$. Then this becomes

$$= \lim_{t \to 0} (1 + t)^{1/t}$$

which ought to look familar from previous chapters. It is one of the definitions of e. We have then that

$$\frac{d}{dx}\log_b x = \frac{1}{x}\ \log_b e$$

If we use the natural logarithm, then we have

$$\frac{d}{dx}\ln x = \frac{1}{x}\ \ln e = \frac{1}{x}$$

There is another derivation which is essentially identical to this one in videos on Khan Academy.

Chapter 33

Famous limit revisited

e as a limit

Here, we look at a more rigorous proof to demonstrate that the number e is equivalent to this limit:

$$\lim_{n \to \infty} (1 + \frac{1}{n})^n$$

http://aleph0.clarku.edu/~djoyce/ma122/elimit.pdf

So we start with a different definition of e and show that it is equivalent. We begin with this definition of the natural logarithm:

$$\ln x = \int_1^x \frac{1}{t} \ dt$$

We have worked through some consequences above, following David Jerrison's MIT lecture. It can be shown fairly easily that this function has all the properties of the natural logarithm.

In addition to that definition, we need two more properties, first:

$$\ln 1 = \int_1^1 \frac{1}{t} \ dt = 0$$

fairly obvious, since the upper and lower bounds are equal, and then second, our definition of e. It is the number such that

$$\ln e = \int_1^e \frac{1}{t}\ dt = 1$$

So here's the proof. Let t be any number in an interval $[1, 1+1/n]$. We're interested in what happens as n gets large. We have that

$$1 \le t \le 1 + \frac{1}{n}$$

If we invert each term, then $\le$ becomes $\ge$, but we will instead rearrange the terms:

$$\frac{1}{1+\frac{1}{n}} \le \frac{1}{t} \le 1$$

The only tricky step is this one: for each of the above, we integrate the variable t between the endpoints 1 and $1+1/n$, remembering that n is just a number and so is $1+1/n$, so we have

$$\int_1^{1+1/n} \frac{1}{1+\frac{1}{n}}\ dt \le \int_1^{1+1/n} \frac{1}{t}\ dt \le \int_1^{1+1/n} 1\ dt$$

The first integral is a constant times t evaluated between $1+1/n$ and 1 which is equal to the constant times $1/n$:

$$[\ \frac{1}{1+\frac{1}{n}}\ ]\ \frac{1}{n} = \frac{1}{1+n}$$

The second one is $\ln(1+1/n)$ by the definition of the logarithm, and the third is the same integral as the first but without the constant, so we have that:

$$\frac{1}{1+n} \le \ln(1+\frac{1}{n}) \le \frac{1}{n}$$

From here on, we just rearrange things a bit. Raising e to the power of each term doesn't change the inequality:

$$e^{\frac{1}{1+n}} \leq 1 + \frac{1}{n} \leq e^{1/n}$$

The left-hand inequality

$$e^{\frac{1}{1+n}} \leq 1 + \frac{1}{n}$$

can be raised to the power $(n+1)$ giving:

$$e \leq (1 + \frac{1}{n})^{n+1}$$

then divide by $(1 + \frac{1}{n})$

$$\frac{e}{1 + 1/n} \leq (1 + \frac{1}{n})^n$$

We notice that, in the limit as $n \to \infty$, this becomes

$$e \leq \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

Similarly for the right-hand inequality,

$$1 + \frac{1}{n} \leq e^{1/n}$$

raise to the power n giving:

$$(1 + \frac{1}{n})^n \leq e$$

and in the limit as $n \to \infty$, this becomes

$$\lim_{n \to \infty} (1 + \frac{1}{n})^n \leq e$$

Call the limit L.

The only way that $e \leq L$ and $L \leq e$ can both be true is if e is equal to the limit in question. This is the squeeze theorem. Hence

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

Part X

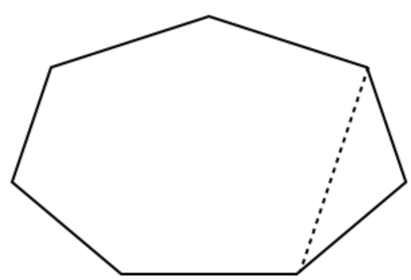
Mathematical induction

Chapter 34

Induction

In the figure below we have a polygon—an irregular heptagon. Actually, there are three polygons altogether, there is the heptagon with $n + 1$ sides, the hexagon with only n sides that would result from cutting along the dotted line, and the triangle that is cut off.

What we would like to do is to find a formula for the sum of the internal angles that depends only on the number of sides or vertices.



The first part of the answer is to guess. In the figure, you can see that by adding the extra vertex to go to the $n+1$-gon, we added a triangle, or perhaps you'd rather say than in going from $n + 1$ to n we lost a triangle.

In either case, the difference is 180°. The difference between having n

sides and $n+1$ sides is to add $180°$.

The second part of the argument is to suppose that $n = 3$, in that case we must have simply $180°$ degrees for a triangle. So we guess that the formula may be

$$(n-2)180° = S_n$$

where S is the sum of the angles in an n-gon.

We can use induction to prove that this formula is correct.

The proof has two parts. We must verify the formula for a base case like the triangle, which we've done. You may wish to check that it works for the square as well, but that's not strictly necessary.

The second part of the proof is to verify that in going from n to $n+1$, we add another $180°$.

$$(n-2)180° + 180° \stackrel{?}{=} ((n+1)-2)180°$$

On the left-hand side, we have the sum of angles for n sides, which we assume is correct, and then we just add $180°$ to it. On the right, we have substituted $n+1$ into the formula.

Now we need to show that these are equivalent. But of course

$$(n-2)x + x = ((n+1)-2)x$$

$$n - 2 + 1 = n + 1 - 2$$

□

That is the inductive proof of the formula.

We can visualize an inductive proof as a kind of chain. We showed that the "base case" is true, for n = 3. We also showed that if the formula works for n (when plugging into $n-2(180) = S$), it must work for n+1.

> Mathematical induction proves that we can climb as high as we like on a ladder, by proving that we can climb onto the bottom rung (the basis) and that from each rung we can climb up to the next one (the step).

- Graham, Knuth and Patashnik

sums of integers

To compute Riemann sums, we need to find formulas for several sums.

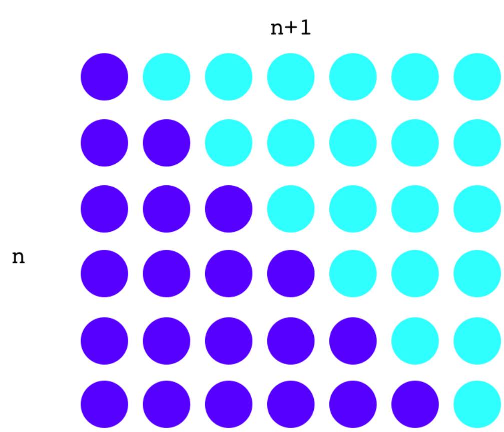
To keep it simple, let's start with finite sums like the integers from 1 to n

$$1 + 2 + 3 + \cdots + n$$

The numbers we seek are called the triangular numbers. These are

$$1, 3, 6, 10, 15 \cdots$$

Here is a striking "visual proof" of the formula to obtain T_n, the n^{th} such number. The total number of circles in the figure below is $n \times (n+1)$ and this is exactly two times the sum of the integers from 1 to n.



$$2S = n(n+1)$$

There is a famous story about Gauss that, as a schoolboy, he "saw" how to add the integers from 1 to 100 as two parallel sums

$$1 + \quad 2 + \quad 3 + \cdots + 99 + 100$$

$$100 + 99 + 98 + \cdots + 2 + 1$$

Added together horizontally, these two series must equal twice the sum of 1 to 100. But in the vertical, we notice that we have n sums, each of which is equal to $n+1$. So, again

$$2S = n(n+1)$$

$$S = \frac{1}{2}\, n(n+1)$$

For $n = 100$ the value of the sum is 5050. Another way of looking at this result is that between 1 and 100 there are 100 representatives of the "average" value in the sequence, which (because of the monotonic steps) is $(100+1)/2 = 50.5$.

Or alternatively, view the sum as ranging from 0 to 100 (with the same answer). Now there are 101 examples of the average value $(100{+}0)/2 = 50)$.

Proof of the formula $n(n+1)/2$ by induction

Returning to the sum of integers, one proof follows the method of induction. In this approach, however, one must first guess the correct formula. We guess $n(n+1)/2$, of course.

Now, we *assume* that the answer is correct for n.

$$S_n = \frac{n(n+1)}{2}$$

So clearly, if S_n is correct, then

$$S_{n+1} = S_n + (n+1)$$

Follow out the arithmetic:

$$\begin{aligned} &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

But this is precisely what we would obtain by using the formula, and substituting $n+1$ for n. Hence the formula gives the correct result for $n+1$, assuming that it gives the correct result for n. In turn, it gives the correct result for n, assuming it gives the correct result for $n-1$. Eventually, we reach the base case, where we can actually verify that the result is correct.

Try it on the first value in the sequence (the "base case").

$$\frac{1(1+1)}{2} = 1$$

That checks. So the whole chain of reasoning is correct. ✓

Derivation using sums

It seems a shame to spoil such a beautiful proof "without words" as the one above by saying anything more, but I can't resist.

You will notice that we were given the formula and only proved it true. As we quoted Archimedes in the first chapter

> it is of course easier, when we have previously acquired by the method some knowledge of questions, to supply the proof than it is to find the proof without any previous knowledge.

I'd like to derive the equation we have been using using algebra. The general method can be used to get the sum of the squares of integers, or their cubes, or even higher powers.

For any number k it is true that

$$(k+1)^2 = k^2 + 2k + 1$$

So consider what happens if we sum the values from $k = 1 \to n$ for each of these terms

$$\sum_{k=1}^{n}(k+1)^2 = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1$$

If the equation is valid for any individual k, then it is also true adding the equations for all k from 1 up to n.

Rearranging

$$\sum_{k=1}^{n}(k+1)^2 - \sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1$$

Now think about the left-hand side in our equation.

$$\sum_{k=1}^{n}(k+1)^2 - \sum_{k=1}^{n} k^2$$

If we count down rather than up, start with $k = n$. We have the following terms

$$k = n \;\text{ gives }\; (n+1)^2 - (n)^2$$

$$k = n-1 \;\text{ gives }\; (n)^2 - (n-1)^2$$

$$k = n - 2 \quad \text{gives} \quad (n-1)^2 - (n-2)^2$$

$$\cdots$$

$$k = 1 \rightarrow \quad (2)^2 - (1)^2$$

We must add all of these together. But notice how all the terms except the first and last cancel. For example we have $-(n)^2$ in the top line and $(n)^2$ in the second. This is called a "collapsing" or "telescoping" sum. We obtain

$$S = (n+1)^2 - 1$$

Bringing back the right-hand side we have

$$(n+1)^2 - 1 = \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1$$

We can bring the constant factor 2 out of the sum, and also, we recognize that the sum of the value 1 a total of n times is just n.

$$(n+1)^2 - 1 = 2\sum_{k=1}^{n} k + n$$

Subtract n from both sides. The left hand side is

$$(n+1)^2 - 1 - n = n^2 + n = n(n+1)$$

Finally, divide by 2:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

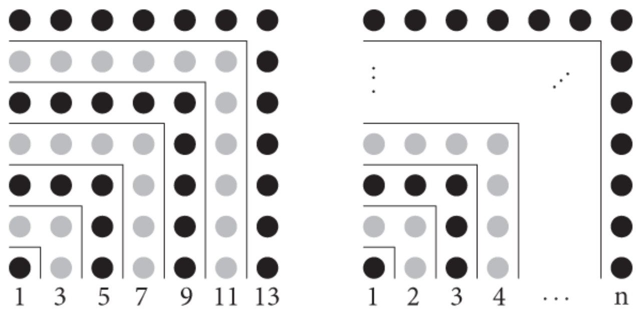
That's our formula.

There is much more (sum of squares, cubes and so on) in the next chapter.

Odd number theorem

Here is a simple but very useful inductive proof.

The *odd number theorem* says that the sum of the first n odd numbers is equal to n^2. Here is a "proof without words".



We prove this by induction.

$$(0 \times 2 + 1) + (1 \times 2 + 1) + (2 \times 2 + 1) + (\dots$$
$$\dots + (n-1) \times 2 + 1) = n^2$$

Notice that the nth odd number is $2 \times (n-1) + 1$.

Our formula says that

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

If you like the summation style:

$$\sum_{k=0}^{n} 2k - 1 = k^2$$

As an example, the first five odd numbers are

$$1 + 3 + 5 + 7 + 9 = 25 = 5^2$$

So, if we consider the next odd number, n changes to $n + 1$. The left-hand side gets another term: we add $2 \times (n + 1) - 1$ to it. That is equal to $2n + 1$.

To maintain the equality, add the same quantity to the right-hand side:

$$n^2 + 2n + 1 = (n + 1)^2$$

Rearrange the result, and that's our formula back again. We have proved the inductive step.

To finish, note that the base case is simply

$$1 = 1^2$$

$\square$

Chapter 35

Fibonacci sequence

Continuing with the topic of induction, let's introduce the Fibonacci numbers and Binet's formula.

These are numbers in a series formed by adding together the two previous numbers in the series:

$$F_{n+1} = F_n + F_{n-1}$$

It remains to choose the first two numbers, which are 1 and 1. Thus the first ten Fibonacci numbers are

$$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad 55$$

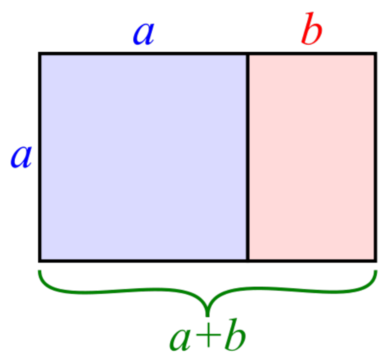
Binet's formula is an explicit formula for F_n which saves us from calculating all the intermediate numbers:

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}$$

where ϕ is the Golden Ratio $(1+\sqrt{5})/2$ and ψ is its conjugate $(1-\sqrt{5})/2$.

Golden ratio

The basic definition involves the following construction:

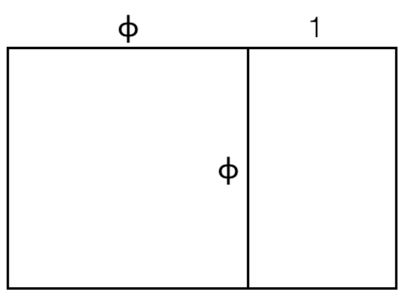


We start with a square of side length a and then extend one side by length b, forming two rectangles. When the ratios of side lengths for these two rectangles are the same, then that ratio is the golden ratio:

$$\phi = \frac{a}{b} = \frac{a+b}{a}$$

(ϕ is often written Φ).

Rescaling of the figure in both dimensions doesn't change the ratios, so let $b = 1$. Then (changing to the symbol ϕ):



$$\frac{\phi}{1} = \frac{\phi+1}{\phi}$$

$$\phi^2 = 1 + \phi$$

We will need only this last result below.

The equation can be solved numerically using the quadratic formula. Put everything on the left-hand side

$$\phi^2 - \phi - 1 = 0$$

Recall that the solutions are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We obtain

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \psi = \frac{1-\sqrt{5}}{2}$$

This gives values of $\phi \approx 1.61803$ and $\psi \approx -0.61803$.

It seems like $\phi = 1 - \psi$. Proof:

$$2\phi = 1 + \sqrt{5}$$

$$2\psi = 1 - \sqrt{5}$$

Adding them together

$$2(\phi + \psi) = 2$$

What a nice symmetry:

$$\phi + \psi = 1$$

$$\phi = 1 - \psi$$

$$\psi = 1 - \phi$$

Note that since ψ is a solution it is also true that

$$\psi^2 = 1 + \psi$$

An alternative proof of that is:

$$\psi^2 = (1 - \phi)^2$$

$$= 1 - 2\phi + \phi^2$$
$$= 1 - 2\phi + 1 + \phi$$
$$= 2 - \phi = 2 - (1 - \psi) = 1 + \psi$$

We can also do the arithmetic

$$\phi^2 = \frac{1+\sqrt{5}}{2} \cdot \frac{1+\sqrt{5}}{2} = \frac{6+2\sqrt{5}}{4} = 1 + \phi$$
$$\psi^2 = \frac{1-\sqrt{5}}{2} \cdot \frac{1-\sqrt{5}}{2} = \frac{6-2\sqrt{5}}{4} = 1 + \psi$$

back to the proof

Wikipedia says that you can prove Binet's formula using induction.

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}$$

https://en.wikipedia.org/wiki/Mathematical_induction#Example:_Fibonacci_numbers

That is an entertaining challenge. For induction we assume that the formula is correct for F_{n-1} and F_n and must prove that:

$$F_{n+1} = F_n + F_{n-1}$$
$$= \frac{\phi^n - \psi^n}{\phi - \psi} + \frac{\phi^{n-1} - \psi^{n-1}}{\phi - \psi}$$
$$= \frac{(\phi^n + \phi^{n-1}) - (\psi^n + \psi^{n-1})}{\phi - \psi}$$

Although one could imagine it is more complicated, the simple idea is to try to show that

$$\phi^{n+1} = \phi^n + \phi^{n-1}$$

and the same for ψ, and that will complete the proof of the inductive step.

Write

$$\phi^{n+1} = \phi^2 \ \phi^{n-1} = (1+\phi) \ \phi^{n-1} = \phi^{n-1} + \phi^n$$

similarly

$$\psi^{n+1} = \psi^2 \ \psi^{n-1} = (1+\psi)\psi^{n-1} = \psi^{n-1} + \psi^n$$

This shows that the inductive step is valid.

Now we just need to verify the base cases. We should check at least the first two of them, because there are two values in the recursion formula $F_n + F_{n-1}$.

It's a matter of convenience whether we consider the series to start with $n = 1$ or $n = 0$. If the latter, then the zeroth Fibonacci number is 0, and the first is 1 and we obtain the same series.

For $n = 0$ we have $\phi^0 - \psi^0$ which is just zero.

For $n = 1$, we have

$$\frac{\phi^1 - \psi^1}{\phi - \psi} = 1$$

We decide to continue with $n = 2$

$$\phi^2 = \phi + 1$$

$$\psi^2 = \psi + 1$$

so

$$\frac{\phi^2 - \psi^2}{\phi - \psi} = \frac{\phi + 1 - \psi - 1}{\phi - \psi} = 1$$

This completes the proof.

□

At this point we note the curious pattern

$$\phi^2 = \phi + 1$$

$$\phi^3 = \phi^2 + \phi = 2\phi + 1$$
$$\phi^4 = \phi^2 + 2\phi + 1 = 3\phi + 2$$
$$\phi^5 = 2\phi^2 + 3\phi + 1 = 5\phi + 3$$
$$\phi^6 = 4\phi^2 + 4\phi + 1 = 8\phi + 5$$

so it looks like

$$\phi^n = F_n \ \phi + F_{n-1}$$

The coefficients *are* the Fibonacci numbers.

The same is true for ψ since $\psi^2 = \psi + 1$.

We could prove this by induction, and it would be a proof of Binet's formula as well because the coefficient of ϕ^n or ψ^n is equal to F_n so

$$\phi^n = F_n \ \phi + F_{n-1}$$

$$\psi^n = F_n \ \psi + F_{n-1}$$

and then

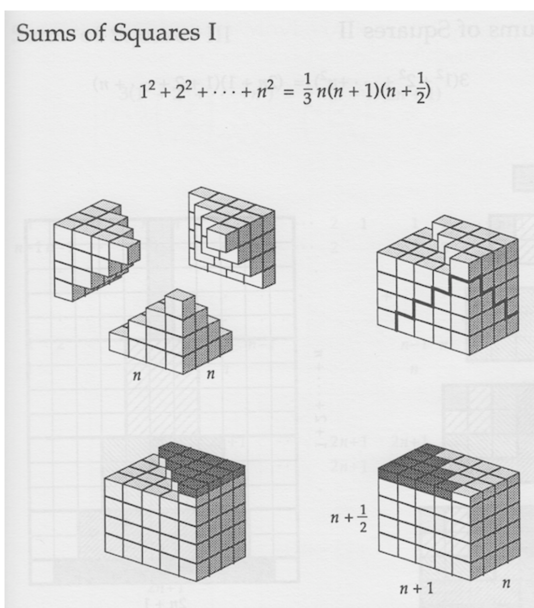
$$\frac{\phi^n - \psi^n}{\phi - \psi} = \frac{(F_n \ \phi + F_{n-1}) - (F_n \ \psi + F_{n-1})}{\phi - \psi}$$
$$= F_n \ \frac{\phi - \psi}{\phi - \psi} = F_n$$

Chapter 36

Integer sums

This chapter continues with a full treatment of formulas for sums of squares, cubes, and so on. It is somewhat detailed, and can easily be skipped without loss of continuity.

Sum of Squares



We want the sum of the squares of the first n integers. There is a visual proof for this one as well (above).

$$\sum_{k=1}^{n} k^2$$

We obtain (by methods we will see below) the formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)}{2}\frac{2n+1}{3}$$

$$= \frac{n(n+1)(2n+1)}{6}$$

This formula is also written as

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}\ (2n^3 + 3n^2 + 2n)$$

$$= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{3}$$

We can check it by induction. The base case is easy

$$\frac{1(2)(3)}{6} = 1$$

✓

Now for the induction step:

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n+1}{6}\ [\ (n)(2n+1) + 6(n+1)\ ]$$

Look at what's in the brackets

$$(n)(2n+1) + 6(n+1)$$

$$= 2n^2 + 7n + 6$$

$$= (n+2)(2n+3)$$

$$= (n+1+1)(2(n+1)+1)$$

So altogether we have

$$= \frac{(n+1)(n+1+1)(2(n+1)+1)}{6}$$

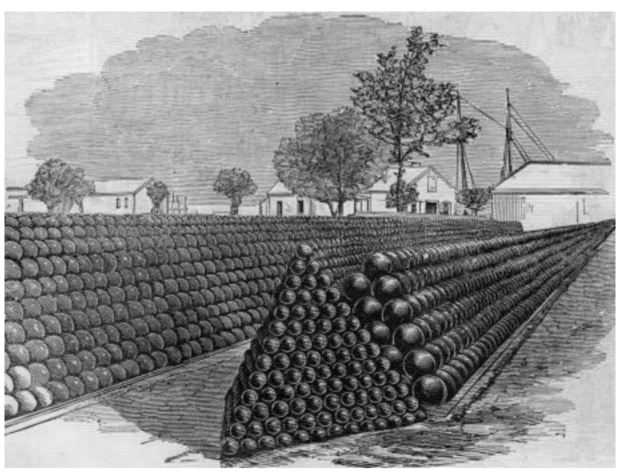
which indeed, is the formula we had above, substituting $n+1$ for n.

Strang's proof

Here are two more approaches. The first one is in Strang's *Calculus*. He says "the best place to start is a good guess". So again, our goal is to find a formula for:

$$S = \sum_{k=1}^{n} k^2$$

Perhaps we visualize a pile of cannonballs



Each layer contains a square number of cannonballs (1, then 4, then 9, etc.). The shape is a pyramid with dimensions $n \times n \times n$. We know the formula for the volume of a pyramid, and guess

$$S_n = \frac{1}{3}n^3$$

To test it, check whether this difference is n^2 (as it should be):

$$S_n - S_{n-1} = \frac{1}{3}n^3 - \frac{1}{3}(n-1)^3$$

Now

$$\begin{aligned}
(n-1)^2 &= n^2 - 2n + 1 \\
(n-1)^3 &= (n-1)(n^2 - 2n + 1) \\
&= n^3 - 3n^2 + 3n - 1
\end{aligned}$$

So

$$S_n - S_{n-1} = \frac{1}{3}(n^3 - n^3 + 3n^2 - 3n + 1)$$

We see that our guess is off by the residual terms

$$\begin{aligned}
&\frac{1}{3}(3n^2 - 3n + 1) \\
&= n^2 - n + \frac{1}{3}
\end{aligned}$$

Strang says: the guess needs *correction terms*. To cancel 1/3 in the difference, subtract $n/3$ from the sum. And to add back n in the difference, add back $1+2+\cdots+n(n+1)/2$ to the sum. Our new guess is

$$
\begin{aligned}
S_n &= \frac{1}{3}n^3 + \frac{n(n+1)}{2} - \frac{n}{3} \\
&= \frac{n}{6}(2n^2 + 3(n+1) - 2) \\
&= \frac{n}{6}(2n+1)(n+1) \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}
$$

which may be easier to remember as

$$S_n = \frac{n(n+1)}{2} \times \frac{2n+1}{3}$$

Derivation by collapsing sum

We proceed as we did in the case of the sum of integers. There we used $(k+1)^2$ and worked out the consequences. Here we use

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1$$

We sum everything.

$$\sum_{k=1}^{n}(k+1)^3 = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 3k^2 + \sum_{k=1}^{n} 3k + \sum_{k=1}^{n} 1$$

As before, subtract the first term on the right from the left-hand side, giving us a collapsing sum. We obtain

$$\sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n} k^3 = (n+1)^3 - 1$$
$$= n^3 + 3n^2 + 3n$$

Recognize that the last sum is just n

$$\sum_{k=1}^{n} 1 = n$$

subtract it from $3n$

$$= n^3 + 3n^2 + 2n$$
$$= n(n^2 + 3n + 2)$$
$$= n(n+1)(n+2)$$

Assembling everything

$$n(n+1)(n+2) = \sum_{k=1}^{n} 3k^2 + \sum_{k=1}^{n} 3k$$

We pull out the factor of 3 from the sums

$$n(n+1)(n+2) = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k$$

The first sum on the right is what we seek, the second one is what we obtained before

$$n(n+1)(n+2) = 3\sum_{k=1}^{n} k^2 + \frac{3}{2}\,n(n+1)$$

Multiply by 2

$$2n(n+1)(n+2) = 6\sum_{k=1}^{n} k^2 + 3n(n+1)$$

Rearrange

$$6\sum_{k=1}^{n} k^2 = 2n(n+1)(n+2) - 3n(n+1)$$

Factor out the $n(n+1)$

$$6\sum_{k=1}^{n} k^2 = n(n+1)\ [\ 2(n+2) - 3\ ]$$

$$6\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)}{2}\ \frac{(2n+1)}{3}$$

Sum of Cubes

Let's do one more. It will help in working out the Riemann Sum for n^3. We proceed exactly as before

$$(k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1$$

Sum each term from $k = 1 \to k = n$

$$\sum_{k=1}^{n}(k+1)^4 = \sum_{k=1}^{n} k^4 + \sum_{k=1}^{n} 4k^3 + \sum_{k=1}^{n} 6k^2 + \sum_{k=1}^{n} 4k + \sum_{k=1}^{n} 1$$

Rearrange and compute the collapsing sum.

$$\sum_{k=1}^{n}(k+1)^4 - \sum_{k=1}^{n} k^4 = \sum_{k=1}^{n} 4k^3 + \sum_{k=1}^{n} 6k^2 + \sum_{k=1}^{n} 4k + \sum_{k=1}^{n} 1$$

$$(n+1)^4 - 1 = \sum_{k=1}^{n} 4k^3 + \sum_{k=1}^{n} 6k^2 + \sum_{k=1}^{n} 4k + \sum_{k=1}^{n} 1$$

Substitute for the right-hand sum

$$(n+1)^4 - 1 = \sum_{k=1}^{n} 4k^3 + \sum_{k=1}^{n} 6k^2 + \sum_{k=1}^{n} 4k + n$$

Rearrange some more

$$\sum_{k=1}^{n} 4k^3 = (n+1)^4 - 1 - \sum_{k=1}^{n} 6k^2 - \sum_{k=1}^{n} 4k - n$$

Expand the term $(n+1)^4$ and pick up the $-1-n$:

$$(n+1)^4 - 1 - n$$

$$= n^4 + 4n^3 + 6n^2 + 4n + 1 - 1 - n$$
$$= n^4 + 4n^3 + 6n^2 + 3n$$

Factor out an n

$$= (n)(n^3 + 4n^2 + 6n + 3)$$

And another $n + 1$

$$= (n)(n + 1)(n^2 + 3n + 3)$$

Recall our previous results:

$$\sum_{k=1}^{n} 6k^2 = 6 \sum_{k=1}^{n} k^2$$
$$= 6 \frac{n(n+1)(2n+1)}{6}$$
$$= n(n+1)(2n+1)$$

Similarly

$$\sum_{k=1}^{n} 4k = 4 \sum_{k=1}^{n} k$$
$$= 4 \frac{n(n+1)}{2}$$
$$= 2n(n+1)$$

Substitute all three of these results (and pull out the factor of 4 from the sum):

$$4\sum_{k=1}^{n} k^3 = (n)(n+1)(n^2+3n+3) - n(n+1)(2n+1) - 2n(n+1)$$

Just a bit more algebra. See that we have $n(n+1)$ in each term. We have

$$\begin{aligned}
&= n(n+1)\ [\ (n^2+3n+3) - (2n+1) - 2\ ] \\
&= n(n+1)\ [\ n^2+3n+3-2n-1-2\ ] \\
&= n(n+1)\ [\ n^2+n\ ] \\
&= n(n+1)\cdot n(n+1)
\end{aligned}$$

So all together we have

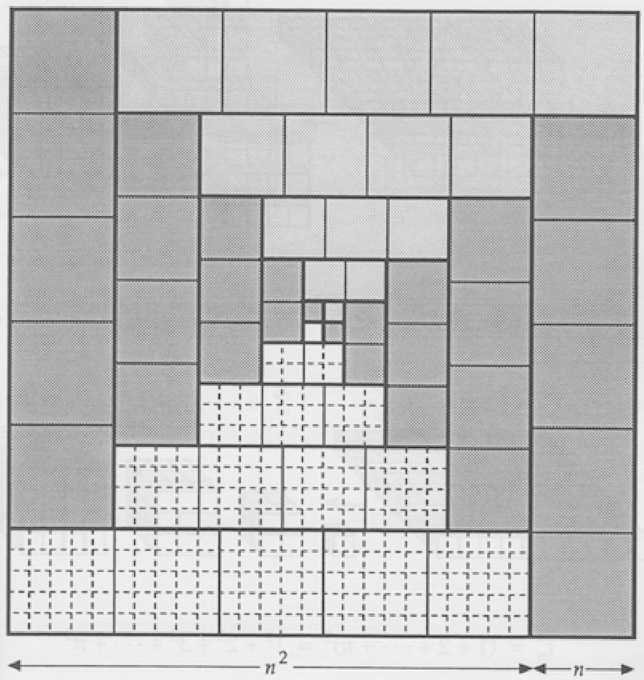
$$\begin{aligned}
4\sum_{k=1}^{n} k^3 &= n(n+1)\cdot n(n+1) \\
\sum_{k=1}^{n} k^3 &= \frac{n(n+1)}{2}\cdot\frac{n(n+1)}{2} \\
\sum_{k=1}^{n} k^3 &= \ [\ \frac{n(n+1)}{2}\ ]^2
\end{aligned}$$

A remarkable simplification!

I have some other write-up about these problems, which I appended below. But we could just look at another beautiful proof without words

Sums of Cubes IV

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}[n(n+1)]^2$$



—Antonella Cupillari and Warren Lushbaugh
(independently)

Analysis

We have shown in other write-ups, and you can easily verify by searching that the sum of the integers between 1 and n is

$$1 + 2 + \cdots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

The next one, the sum of the squares of the first n integers, is useful for certain derivations in calculus (e.g. the Riemann sum to integrate $y = x^2$)

$$1^2 + 2^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{3}$$

I was quite surprised to find that the sum of cubes is also simple and frankly, amazing

$$1^3 + 2^3 + \cdots + n^3 = \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{2^2} = [\ \frac{n(n+1)}{2}\ ]^2 = [\ \sum_{k=1}^{n} k\ ]^2$$

$$\boxed{\sum_{k=1}^{n} k^3 = [\ \sum_{k=1}^{n} k\ ]^2} \tag{36.1}$$

Let's just try to prove the last formula using induction.

The "base case" is pretty simple. For $n = 2$

$$1^3 + 2^3 = 1 + 8 = 9$$

and

$$\frac{n^2(n+1)^2}{2^2} = \frac{2^2(3^2)}{2^2} = 3^2 = 9$$

Check. Now for the induction step what we need to show is that what we get assuming the formula for n is correct and then adding the term $(n+1)^3$

$$\boxed{\frac{n^2(n+1)^2}{2^2} + (n+1)^3} \tag{36.2}$$

is equal to what we get by plugging $n+1$ into the formula.

$$\boxed{\frac{(n+1)^2(n+2)^2}{2^2}} \tag{36.3}$$

We need to show that eqn 2 is equal to eqn 3.

$$\frac{n^2(n+1)^2}{2^2} + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{2^2}$$

First, we can factor out and cancel $(n+1)^2$ from both sides. So then we have

$$\frac{n^2}{2^2} + (n+1) \stackrel{?}{=} \frac{(n+2)^2}{2^2}$$

$$n^2 + 4(n+1) \stackrel{?}{=} (n+2)^2$$

Sure, that looks correct! And we're done with the proof by induction, so we can put a little box.

□

Looking deeper

$$\sum_{k=1}^{n} k^3 = [\ \sum_{k=1}^{n} k\ ]^2$$

I wanted to try to understand something more about why this is true. A simple web search revealed the answer. Here's an interesting pattern for the cubes of integers

$$1^3 = 1$$

$$2^3 = 8 = 3 + 5$$

$$3^3 = 27 = 7 + 9 + 11$$

$$4^3 = 64 = 13 + 15 + 17 + 19$$

$$5^3 = 125 = 21 + 23 + 25 + 27 + 29$$

If you want a formula for n^3, notice that the first term is $n^2 - n + 1$ and the last term is $n^2 - n + 2n - 1$, and the number of terms for each sum equals n. (There are n odd numbers between 1 and $2n - 1$).

In other words, the sum of all the cubes of integers from 1^3 to n^3 is equal to the sum of all the odd numbers up to $n^2 - n + 2n - 1 = n^2 + n - 1$.

How many of these numbers are there? A little thought should convince you that the answer is $(n^2 + n)/2$. For example, with $n = 5$, our last odd number is $5^2 + 5 - 1 = 29$, and we have $(25 + 5)/2 = 15$ terms.

We want the sum of the first $(n^2 + n)/2$ odd numbers.

Let's look at another pattern

$$1 = 1$$

$$2^2 = 4 = 1 + 3$$

$$3^2 = 9 = 1 + 3 + 5$$

$$4^2 = 16 = 1 + 3 + 5 + 7$$

$$5^2 = 25 = 1 + 3 + 5 + 7 + 9$$

The *odd number theorem* says that the sum of the first n odd numbers is equal to n^2. We want the sum of the first $(n^2 + n)/2$ odd numbers, so that's $((n^2 + n)/2)^2$. And that's how we get our formula.

Part XI

Series

Chapter 37

Infinite series

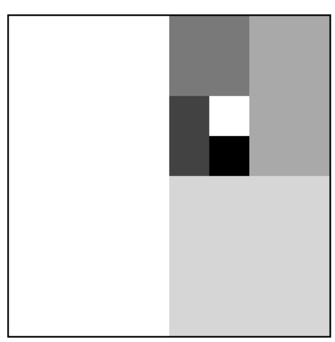
Strang says the most important series of all is the *geometric* series and I think that's right:

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

Probably the most well-known example has $x = 1/2$:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

Although the number of terms in this series is infinite, the sum is finite. Here is a "visual proof" for the sum starting at the second term:



In this figure, we see that $1/2+1/4+1/8+\cdots=1$. So the sum of the series written above is equal to $1+1=2$.

We'd like to determine algebraically or analytically what the sum is. There is a simple approach to this. Let

$$S = 1 + x + x^2 + x^3 + \dots$$

$$Sx = x + x^2 + x^3 + \dots$$

$$S - Sx = 1$$

$$= S(1-x)$$

$$S = \frac{1}{1-x}$$

This seems to work. For the first example, with $x = 1/2$, we obtain 2, as expected (starting the series at 1). Also, multiplying

$$(1-x)(1 + x + x^2 + x^3 + \dots)$$

apparently cancels every term after the first 1.

Unfortunately, there's a problem. Above, we assumed that S is a number. But infinity is *not* a number. For some values of x, the series is not finite.

Consider $x = 1$:

$$1 + x + x^2 + x^3 + \cdots = 1 + 1 + 1 + 1 + \dots$$

And when $x = -1$, the series oscillates:

$$1 - 1 + 1 - 1 + \dots$$

So the answer we obtained before is the value of the sum, but only when that value exists.

The careful way to analyze this says, let us look at sums when we stop the series early, after n terms. The partial sum S_n is

$$S_n = 1 + x + x^2 + x^3 + \cdots + x^n$$

This value, S_n is certain to be finite if x is finite. We do what we did before.

$$(1-x)S_n = S_n - xS_n$$

On the right-hand side, multiplication by $1-x$ produces a ”telescoping sum” so that

$$= 1 + x - x + x^2 - x^2 \cdots + x^n - x^n - x^{n+1}$$

$$= 1 - x^{n+1}$$

Division gives

$$S_n = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

To evaluate the infinite series, we ask what this limit is as n grows larger

$$\lim_{n\to\infty} \frac{x^{n+1}}{1-x}$$

If

$$x^{n+1} \to 0 \quad \text{as} \quad n \to \infty$$

then the series converges. This only happens when $|x| < 1$ and we call this range of values the ”radius of convergence” of the series.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \Longleftrightarrow \quad -1 < x < 1$$

For something like

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + \ldots$$

write

$$\sum_{n=0}^{\infty} ax^n = a\sum_{n=0}^{\infty} x^n = \frac{a}{1-x}$$

To compute the sum of a geometric series, use this formula:

$$\frac{\text{first term}}{1 - \text{common ratio}}$$

example

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

This is the classic geometric series. The common ratio is $1/2$. Since the ratio is between -1 and 1, the series converges. The first term is $1/2$ and the value of the sum is:

$$\frac{1/2}{(1-1/2)} = 1$$

Alternatively

$$\sum_{n=0}^{\infty} \frac{1}{2^n}$$

This is the same as before except the first term is 1 and the sum is

$$\frac{1}{(1-1/2)} = 2$$

Repeating decimals

Consider

$$\sum_{n=1}^{\infty} \frac{1}{10^n}$$

$$= \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots$$

This one is also a geometric series. The sum is obviously

$$= 0.1111\ldots$$

$$= \frac{1}{9}$$

But the formula works here too.

Other interesting repeating decimals are:

$$n = 0.012345679012345679012345679\ldots$$

To see this in a more familiar form, multiply by 10^9

$$1,000,000,000\ n = 12345679.012345679012345679 0\ldots$$

The difference is

$$999,999,999\ n = 12,345,679$$

$$\frac{12345679}{999,999,999} = \frac{1}{81}$$

or even

$$\frac{1}{243} = 0.004115226337448559670781893\ldots$$

The repeat has 25 digits.

http://mathworld.wolfram.com/243.html

Harmonic series

The harmonic series is

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

Obviously, n cannot be equal to 0 so we start at $n = 1$.

The harmonic series diverges. A classic proof is to group the terms:

$$= \frac{1}{1} + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \dots$$

Clearly, we can continue this grouping operation forever. The next group has denominators from $9 \dots 16$, and in general, from $2^{n-1} + 1$ up to 2^n.

The sum for each group is at least $1/2$, so the series is larger term by term than:

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

But this series is clearly divergent, so the harmonic series, whose sum is larger term by term, also diverges.

According to Acheson, this proof dates to 1350 and can be attributed to the French scholar, Nicole Oresme.

A fun fact about this series is that if we consider the related series of inverse primes, that is

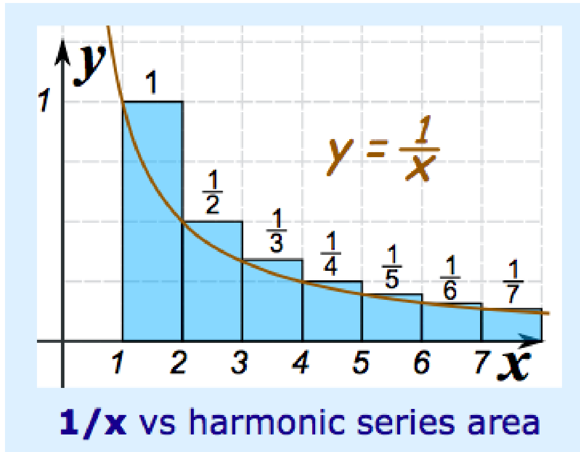
$$\sum_{n=2}^{\infty} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots$$

This sum *also* diverges (see Maor's book on infinity). Maor also says that the Nth partial sum of the harmonic series, the sum of the first N terms, obeys this inequality:

$$\ln N < S_N < \ln N + 1$$

From this, we deduce that the sum of the first one googol (10^{100}) terms of the series, is just a bit more than 230. Even so, the harmonic series diverges.

Here is another proof:



The area under the boxes is equal to the sum of the harmonic series. We will show that the integral, the area under the smooth curve, diverges to ∞. Since the harmonic series is larger than that, it also diverges.

The integral is

$$I = \int_1^\infty \frac{1}{x}\ dx$$

which we evaluate by substituting a finite upper limit

$$I = \int_1^a \frac{1}{x}\ dx$$

$$= \ln x \Big|_1^a = \ln a$$

But as $a \to \infty$, the logarithm also tends to ∞, so it diverges.

Related to geometric series

Again, the geometric series is:

$$S = 1 + x + x^2 + x^3 + \dots$$

Differentiate. The right-hand side is

$$1 + 2x + 3x^2 + 4x^3 + \dots$$

and the left-hand side is

$$\frac{d}{dx}\ [\ \frac{1}{1-x}\ ]\ = \frac{1}{(1-x)^2}$$

Try multiplying out

$$(1-x)(1 + 2x + 3x^2 + 4x^3 + \dots)$$
$$= 1 + 2x - x + 3x^2 - 2x^2 + 4x^3 - 3x^3 + \dots)$$
$$= 1 + x + x^2 + x^3 + \dots$$
$$= \frac{1}{1-x}$$

It checks. And since

$$(\frac{1}{1-x})(\frac{1}{1-x}) = \frac{1}{(1-x)^2}$$
$$(1 + x + x^2 + \dots)(1 + x + x^2 + \dots)$$
$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$$

Integrate

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$
$$\int \frac{1}{1-x}\ dx = \int 1 + x + x^2 + x^3 + \dots\ dx$$
$$-\ln|1-x| = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

Let $x = 1/2$

$$-\ln\frac{1}{2} = \ln 2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \ldots$$

We can calculate $\ln 2$.

This converges fairly slowly. Still 20 terms gives six places correct (0.693147).

Change variables

Start with the geometric series

$$1 + x + x^2 + x^3 + \ldots$$

Replace x by $-x^2$:

$$1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1+x^2}$$

Now, does that right-hand side look familiar? Perhaps not, if you haven't seen the trigonometric functions. Take it on faith that

$$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x$$

where $\tan^{-1}$ or arc tangent is the *inverse function* to the tangent. That is, if

$$x = \tan\theta$$

then

$$\theta = \tan^{-1} x$$

Integrate the left-hand side:

$$\int 1 - x^2 + x^4 - x^6 + \dots \; dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

Set $x = 1$. The angle with $\tan^{-1}\theta = 1$ is $\theta = \pi/4$. Thus:

$$\tan^{-1} 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

We have found a series that equals π! How cool is that?

This particular series converges extremely slowly. It can be improved in a couple ways. One way is by combining adjacent terms: $1/5 - 1/7 = 2/35; 1/9 - 1/11 = 2/99$ and so on.

Another way is to set $x = 1/\sqrt{3}$, then the angle with that tangent is $\pi/6$ (recall that $\sin \pi/6 = 1/2$ and $\cos \pi/6 = \sqrt{3}/2$).

So

$$\begin{aligned}
\frac{\pi}{6} &= \frac{1}{\sqrt{3}} - \frac{1}{3}\;(\frac{1}{\sqrt{3}})^3 + \frac{1}{5}\;(\frac{1}{\sqrt{3}})^5 - \frac{1}{7}\;(\frac{1}{\sqrt{3}})^7 + \dots \\
&= \frac{1}{\sqrt{3}}(1 - \frac{1}{3}\;(\frac{1}{3}) + \frac{1}{5}\;(\frac{1}{3})^2 + \frac{1}{7}\;(\frac{1}{3})^3 \dots) \\
&= \frac{1}{\sqrt{3}}(1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \frac{1}{729} - \frac{1}{2673})
\end{aligned}$$

```
>>> from math import sqrt
>>> r = 1/sqrt(3)
>>> r * (1 - 1.0/9 + 1.0/45 - 1.0/189 + 1.0/729 - 1.0/2673)
0.5235514642438139
>>>
```

$\pi/6$ is equal to 0.5235987755... So we have only first 4 places correct. This series converges fairly quickly and is actually pretty easy to compute.

There are numerous other series for π:

http://mathworld.wolfram.com/PiFormulas.html

another example

Alcock has an interesting series in *Mathematics Rebooted*

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

It's the harmonic series, except that every term is squared. Does this series converge?

We're tempted to compare it to the geometric series with $r = 1/2$. Unfortunately the terms of the geometric series ultimately become smaller than those of our new series, so that's no help.

This is true even if we pick r smaller, like $r = 1/10$. Eventually, our new series will have larger terms.

$$\text{harmonic: } 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \dots$$

$$\text{new: } 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} + \frac{1}{100} + \dots$$

$$\text{geometric: } 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \dots$$

The trick is to find another series which we can show converges (it's not one I've seen before).

Alcock gives the formula for the sum, and we need to prove this by induction. Leaving off the first term

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

As $n\to\infty$, this is equal to 1.

Before we get started, it is fair to ask where this formula comes from. On the one hand, we note that

$$\frac{1}{n(n+1)}>\frac{1}{(n+1)^2}$$

so it will clearly do the job for us, term by term. Alternatively, we could write

$$\frac{1}{n(n-1)}>\frac{1}{n^2}$$

We may have to drop a term or two at the very beginning to line things up, but this is never a problem.

Now, we notice that

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$$

In other words, any sum will be a telescoping sum. So the result is simple, just what we had above with a different symbol.

$$\sum_{n=1}^{n=k}\frac{1}{n}-\frac{1}{n+1}=1-\frac{1}{k+1}$$

Now for the proof by induction. Clearly this works for the base case. We need to add the next term (for $n+1$) from the left-hand side to the right-hand side and show that it reduces to the correct form.

$$(1-\frac{1}{n+1})+\frac{1}{(n+1)(n+2)}$$

$$= 1 - \ [\ \frac{1}{n+1} - \frac{1}{(n+1)(n+2)}\ ]$$

$$= 1 - \ [\ \frac{n+2-1}{(n+1)(n+2)}\ ]$$

$$= 1 - \ [\ \frac{1}{n+2}\ ]$$

which is indeed the sum on the right updated from n to $n+1$. So the solution is correct, the answer is finite, and thus the series does converge.

Then, comparison with our series shows the one which we just proved convergent, is bigger term by term. So our series converges as well.

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \dots$$

$$\frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \frac{1}{5\cdot 5} + \dots$$

The second series is clearly smaller than the first, which is equal to 1.

This is a bit of a sneaky example, because the sum (including the first term) is a famous series related to π, found in the reference at the end of the last section:

$$\frac{\pi^2}{6} = \sum_1^\infty \frac{1}{k^2}$$

The fly and the train

Two trains (or bicycles) are 20 miles apart and headed directly toward each other at a speed of 10 mph. A fly starts from the front of the train on the left, flies at a speed of 15 mph to the tip of the train on the right, then turns around immediately and flies back to the first. This cycle continues until the trains meet, ending everything.

How far does the fly fly?

http://mathworld.wolfram.com/TwoTrainsPuzzle.html

There is a hard way and an easy way to do this problem. The story was that being shown the problem, Johnny Von Neumann did it instantly, and when asked how he did it, said that he used infinite series (this is the hard way).

We will sum an infinite series. It takes a little time to set up, but we know how to solve it. Let's do one round to see what happens.

- ◦ initial distance between trains: 20 miles
- ◦ relative speed of fly to second train: 25 miles per hour
- ◦ time to meet: $20/25 = 0.8$ hour
- ◦ distance the fly travels: $0.8 \times 15 = 12$ miles
- ◦ each train travels $0.8 \times 10 = 8$ miles
- ◦ distance separating the trains after this round: 4 miles

The ratio $4/20 = 1/5$ miles allows us to set up the series:

$$12 + 12\frac{1}{5} + 12(\frac{1}{5})^2 + \dots$$

$$12 \; [\; 1 + \frac{1}{5} + (\frac{1}{5})^2 + \dots \;]$$

$$1/1 - r = 5/4$$

Total distance = 15 miles.

The easy way is that the closing speed of the two trains, 20 miles per hour, is the same as the initial separation of 20 miles, so the trains meet in 1 hour. The fly travels 15 miles in one hour.

We should be able to get the series from the train speeds, the fly speed and the initial distance. Let

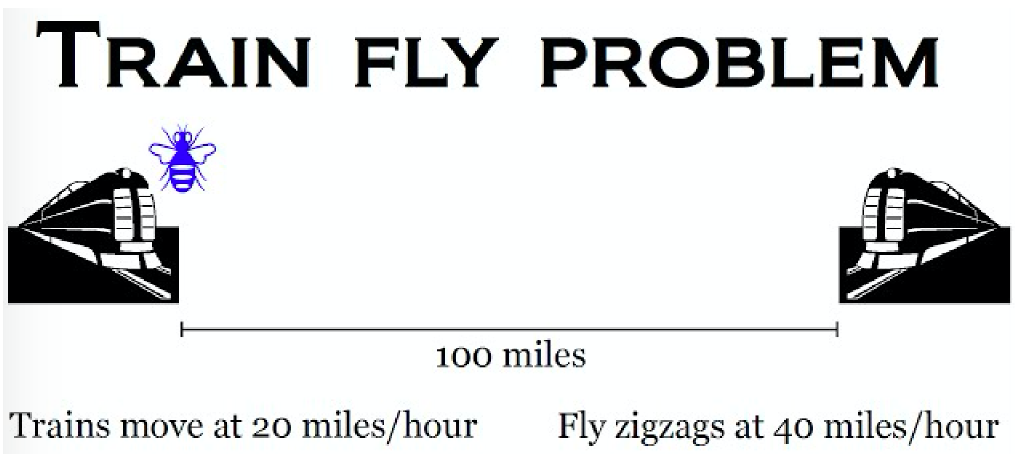
$$d = \text{ initial distance/(train speed + fly speed)} = \frac{20}{15+10}$$

The common ratio is

$$r = 1 - d = \frac{1}{5}$$

The initial value a is the fly speed times d.

Here is a nice graphic for a variant that I found on the web.



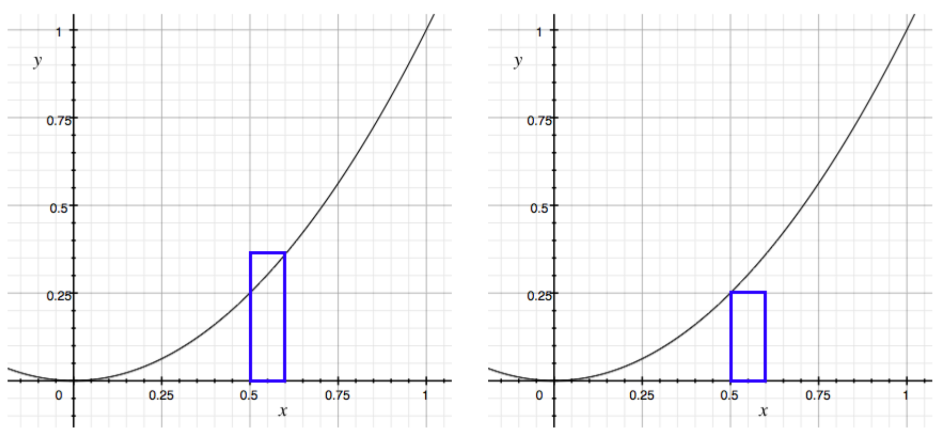
Chapter 38

Riemann sums

Riemann sums

We introduced integration as simply the reverse of differentiation. The fundamental theorem of calculus gives us a means of evaluating integrals between two bounds.

Starting with Courant, however, calculus courses have sought a more formal approach. The first (though not the only) method is to compute what are called Riemann sums for the the area bounded under a curve.

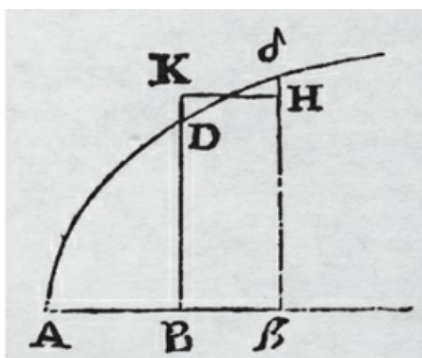


Our first example is to calculate the area under the curve $f(x) = x^2$. The areas of many small rectangles are added to form the sum.

The key is to set up a calculation or expression for the area in terms of a variable number of rectangles, N. Although any rectangle can only be an approximation to a curved surface, if we use many skinny rectangles the approximation will be very good, and *in the limit* as $N \to \infty$, as we use an infinite number of rectangles, it will be exactly right.

The proof that it *is* right is to show that the sum of a set of rectangles bounding the curve below, and the sum for a second set bounding the curve above, *converge to the same limit* as $N \to \infty$.

Another way to see this is to look at this diagram from Newton's book *De Analysi* (from Acheson's book *The Calculus Story*).



Newton's argument is that if we draw a box as illustrated, there must be one height, one point along the top of the rectangle where the area under the curve but not in the box, to the right, is exactly equal to the area over the curve and in the box, to the left. At that point, the errors exactly cancel.

If this point is always bounded by the y-values of the left and right sides of the box, then as the boxes get smaller and smaller, the balance point will always be included somewhere in the rectangle, so the answer will come out exactly right. (This latter assumption breaks down at maxima and minima, but only for finite boxes. Let's admire the diagram and move on to the actual method).

example

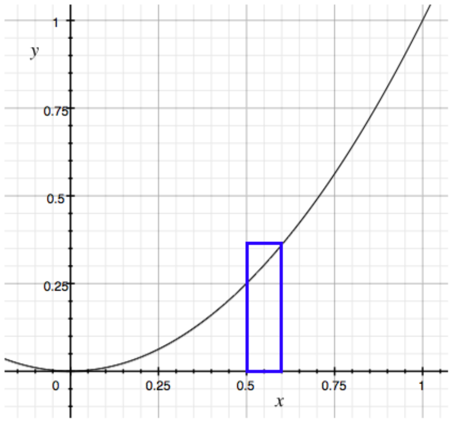
We start with x^2. It is the simplest power curve, and we actually know the answer due to work by Archimedes, called the **quadrature**.

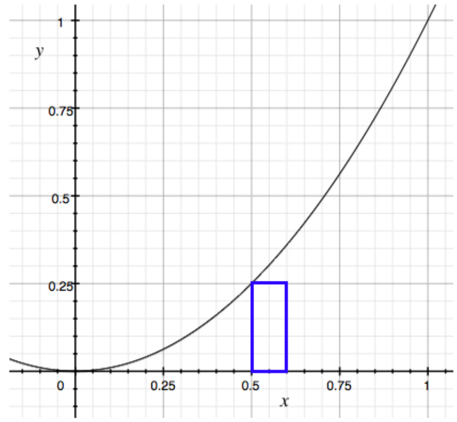
Consider the region bounded by the x-axis, the y-axis, the line $x = 1$, and the curve $y = x^2$.

Partition the region on the x-axis between $x = 0$ and $x = 1$ into N segments. Each segment will contain a tall, thin rectangle that extends from the x-axis up to the curve.

Here is a figure that illustrates the basic idea. The region between 0 and 1 is divided into 10 segments, so the width of each segment is 0.1.

In the left panel, the blue rectangle shown is the sixth segment; its left and right bounds are $x = 0.5$ and $x = 0.6$. The height is $x^2 = 0.6^2 = 0.36$. The right panel is the same, except the height corresponds to the value at the left-hand bound $x^2 = 0.5^2$.





We compute the area (using the first set of rectangles) as

$$
\begin{aligned}
A &= 0.1(0.1^2 + 0.2^2 + \cdots + 1.0^2) \\
&= 0.1(0.01 + 0.04 + \cdots + 1.0) \\
&= 0.1(3.85) = 0.385
\end{aligned}
$$

This is obviously an over-estimate of the area (as it will be for any function that increases over the interval), but the trick is that as the number of rectangles becomes very large, the result will converge to the exact area we want.

area as a limit

We divide the region into N intervals. Each interval has width $1/N$. The x-values of the right hand side of these boxes are

$$\frac{1}{N}, \frac{2}{N} \cdots \frac{N}{N}$$

We can rewrite this as

$$\sum_{k=1}^{N} \frac{k}{N}$$

We will first compute the sums for the set of boxes that is an overestimate, it uses the x-value of the right hand side of each box times the width of each box:

$$A = \sum_{k=1}^{N} f(\frac{k}{N}) \cdot \frac{1}{N}$$

Since the function is x^2 this is

$$A = \sum_{k=1}^{N} (\frac{k}{N})^2 \cdot \frac{1}{N}$$

And since N is a constant, it can be pulled out from the summation:

$$A = \frac{1}{N^3} \sum_{k=1}^{N} k^2$$

Now we need an expression for the sum of the squares of the first N integers. We will show below that the formula is $N \cdot (N+1)/2 \cdot (2N+1)/3$. Distributing the factor of $1/N^3$ over each term, we obtain

$$A = \frac{N}{N} \cdot \frac{N+1}{2N} \cdot \frac{2N+1}{3N}$$

If $N \to \infty$ and we take the limit, the result is the value of the integral

$$I = \lim_{N \to \infty} \frac{N}{N} \cdot \frac{N+1}{2N} \cdot \frac{2N+1}{3N}$$
$$= 1 \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

The reason is that as $N \to \infty$ the difference between N and $N+1$ (or $N-1$ and N becomes negligible compared to the size of N, therefore the ratio $(N+1)/2N$, for example is equal to $1/2$ in the limit. In other words, for, say

$$\lim_{N\to\infty} \frac{1}{2N}\ (N+1) = \lim_{N\to\infty} \frac{1}{2}(1+\frac{1}{N}) = \frac{1}{2}$$

We must still compute the sums for the set of boxes that is an under-estimate (for finite N), with the x-value used for the function taken from the left-hand side. This is the series

$$\frac{0}{N}, \frac{1}{N} \ldots \frac{N-1}{N}$$

If the function is $f(x) = x^2$ the sum is

$$A = \sum_{k=0}^{N-1} (\frac{k}{N})^2 \cdot \frac{1}{N}$$

Since the first term is zero, just remove it and start the index from 1

$$A = \sum_{k=1}^{N-1} (\frac{k}{N})^2 \cdot \frac{1}{N}$$

For the integral, we obtain

$$I = \lim_{N\to\infty} \frac{N-1}{N} \cdot \frac{N}{2N} \cdot \frac{2N-1}{3}$$

We obtain exactly the same result as before. This is really the proof that the method works. In the limit of infinite N, the methods which over-estimate and under-estimate the area converge to the same value, therefore the result is exactly correct.

integer sums

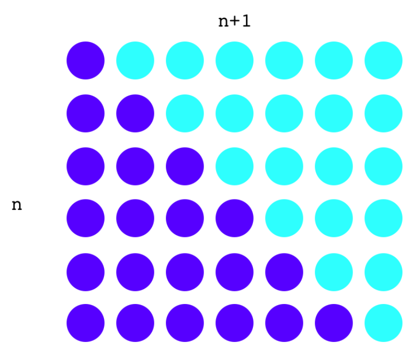
The first series that one usually sees of this type is

$$\sum_{k=1}^{n} k$$

The sum of the integers from $1 \to N$. There is a simple formula

$$\frac{n(n+1)}{2}$$

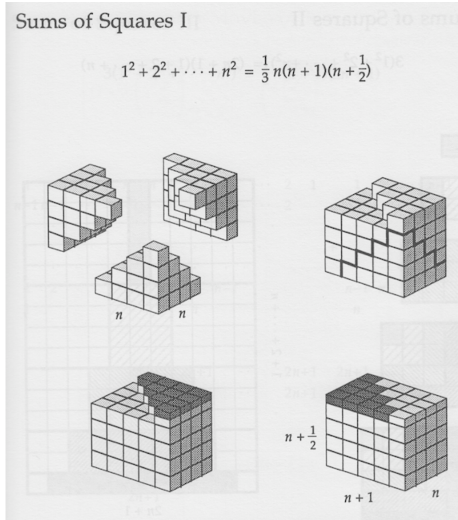
Here is a famous "proof without words":



For Riemann sums, we need a series one level higher. We need the sum of the squares of the first n integers. The answer is what we had before, multiplied by another term

$$\frac{n(n+1)}{2}\,\frac{2n+1}{3}$$

Here is another "proof without words."



For much more detail, see the chapter in the Addendum.

back to the Riemann sum

We plug that expression into the Riemann Sum:

$$= \frac{1}{N^3} \frac{N(N+1)}{2} \; \frac{2N+1}{3}$$

Each of the terms in $N(N+1)(2N+1)$ is grouped with one of the N's in the denominator at the left

$$= \frac{1}{6} \; \frac{N}{N} \; \frac{(N+1)}{N} \; \frac{(2N+1)}{N}$$

In the limit as N gets very large.

$$\lim_{N \to \infty} \frac{N}{N} = 1$$

$$\lim_{N \to \infty} \frac{N+1}{N} = \lim_{N \to \infty} 1 + \frac{1}{N} = 1$$

$$\lim_{N\to\infty} \frac{2N+1}{N} = \lim_{N\to\infty} 2 + \frac{1}{N} = 2$$

Thus, the final answer is 1/3, which agrees with Archimedes.

n cubed

The height of the first interval is $(1/N)^3$ and that of the kth interval is $(k/N)^3$. The total area is:

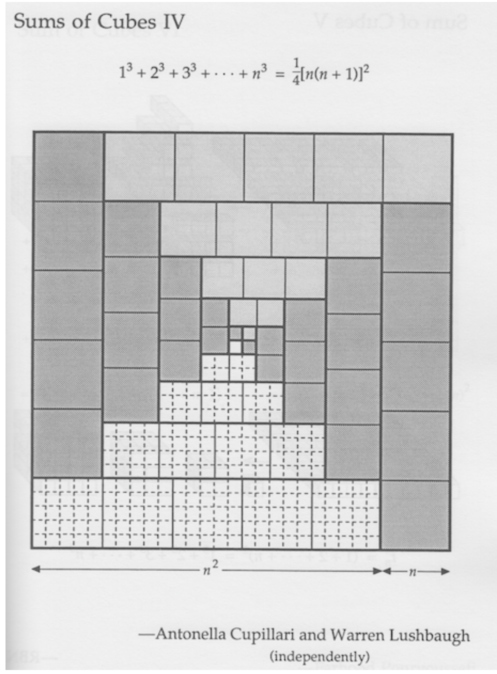
$$\sum_{k=1}^{N} (\frac{k}{N})^3 \times \frac{1}{N}$$

Since N is a constant, it can be pulled out from the summation:

$$\frac{1}{N^4} \sum_{k=1}^{N} k^3$$

So now we need an expression for the sum of the cubes of the first N integers.

Yet another ingenious "proof without words."



$$\sum_{k=1}^{N} k^3 = \frac{1}{4}\ [\ n(n+1)\ ]^2$$

$$= \frac{1}{4}\ N(N+1)N(N+1)$$

As before, each of the four factors of N in the denominator cancels an N or $N+1$ on top and we're left with just $1/4$.

It turns out that if you let the interval be $[0,b]$ or even $[a,b]$, we obtain the expressions you will be used to from integral calculus, namely

$$\int_a^b n^2 = \frac{n^3}{3}\ \Big|_a^b$$

and

$$\int_a^b n^3 = \frac{n^4}{4}\ \Big|_a^b$$

Chapter 39

Fermat area

In computing Riemann sums, it isn't required that the intervals have the same width, only that the width of the largest goes to zero in the limit.

Courant and John describe a variation on Riemann sums using intervals of unequal (but graduated) width. This "trick" allows them to derive the formula for

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\int_a^b x^n \, dx = \frac{b^{n+1} - a^{n+1}}{n+1}$$

for all natural numbers n first, and then with some elaborations, for real n except $n = -1$.

This result (for integers) is due to Fermat and was achieved about 1640 (i.e. 25 years before Newton). I found that proof on the web

http://fredrickey.info/hm/CalcNotes/Fermat-Integration.pdf and there is also a good discussion in Maor's *e, the Story of a Number*. We'll look at Fermat's proof here and save the other for the Addendum (**here**). Fermat's version achieves simplicity by using the interval $[0, b]$ with its lower bound at zero.

derivation

Let E be a positive constant less than 1. Divide the region $[0, b]$ into subintervals with boundaries

$$\dots bE^3,\ bE^2,\ bE,\ b$$

How do we get the width of the largest rectangle to decrease to zero? By taking the limit $E \to 1$.

Construct rectangles in the usual way that circumscribe the curve $y = x^n$ and add up their areas. For the ith rectangle, the width is

$$bE^i - bE^{i+1}$$

$(bE^{i+1} < bE^i)$, and the height is

$$(bE^i)^n$$

so the overall sum is

$$S = \sum_{i=0}^{\infty} (bE^i)^n\ (bE^i - bE^{i+1})$$

$$= b^{n+1} \sum_{i=0}^{\infty} (E^i)^n\ (E^i - E^{i+1})$$

$$= b^{n+1} \sum_{i=0}^{\infty} (E^i)^{n+1}\ (1 - E)$$

$$= b^{n+1}\ (1 - E)\ \sum_{i=0}^{\infty} (E^{n+1})^i$$

Since E is a positive constant less than 1, E^{n+1} is also. Let $q = E^{n+1}$. The sum becomes

$$\sum_{i=0}^{\infty} q^i = q^0 + q^1 + q^2 + q^3 + \ldots$$

Recall that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

for $|x| < 1$.

So, going back to E we have

$$S = b^{n+1}\ (1 - E)\ \frac{1}{1 - E^{n+1}}$$

and using the same identity again

$$1 - x = \frac{1}{1 + x + x^2 + x^3 + \ldots}$$

so

$$S = b^{n+1}\ \frac{1}{(1 + E + E^2 + E^3 + \ldots)(1 - E^{n+1})}$$

All the terms in the infinite series starting at E^{n+1} acquire counterparts with a minus sign, hence

$$= b^{n+1}\ \frac{1}{1 + E + E^2 + E^3 + \cdots + E^n}$$

Now take the limit as $E \to 1$. The fraction becomes just

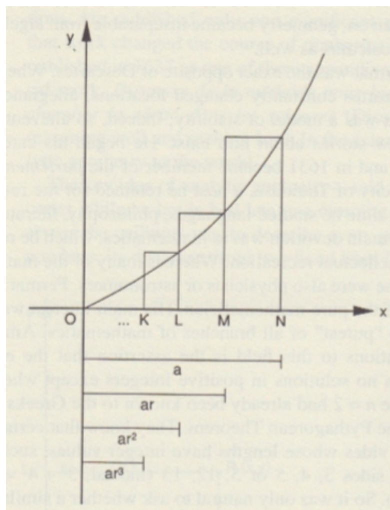
$$\frac{1}{1 + E + E^2 + E^3 + \cdots + E^n} = \frac{1}{n+1}$$

and we have

$$\int_0^b x^n = \frac{b^{n+1}}{n+1}$$

which is what we get when we evaluate x^{n+1} on the interval $[0, b]$ and then divide by $n + 1$.

This diagram from Maor uses slightly different nomenclature — the interval points are of the form $a, ar, ar^2 \ldots$ (moving from right to left).



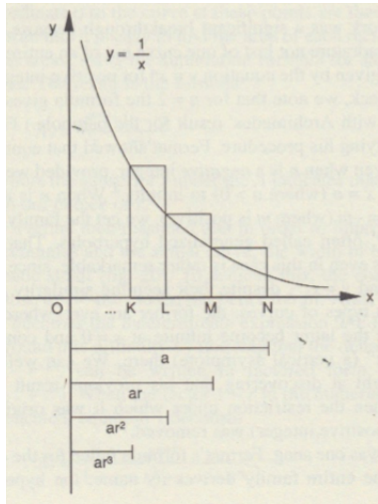
Saint Vincent

It turns out that the above analysis applies for integer $n < -1$,

All except for the most important case, the hyperbola with $n = -1$. The problem that we run into is division by zero.

However, a Belgian Jesuit named Saint Vincent noticed something about the intervals underneath the curve $y = 1/x$. This work was completed about 1631 and published in 1647.

Starting at N and moving backward, let's compute the width (h), height h and area A for each interval.



N:

$$w = a - ar = a(1 - r), \quad h = \frac{1}{a}, \quad A = 1 - r$$

M:

$$w = ar - ar^2 = ar(1 - r), \quad h = \frac{1}{ar}, \quad A = 1 - r$$

L:

$$w = ar^2 - ar^3 = ar^2(1 - r), \quad h = \frac{1}{ar^2}, \quad A = 1 - r$$

Maor:

> This means that as the distance from 0 grows geometrically, the corresponding areas grow in equal increments, that is, arithmetically, and this remains true even when we go to the limit ... But this in term implies that the relationship between area and distance is logarithmic.

The area under the curve 1/x is the logarithm. It took some time to figure out that the base of the logarithm was e.

Chapter 40

Numerical integration

It has been shown that some important integrals cannot be "solved" analytically (i.e. we cannot find $F(x)$). For example, the normal distribution (with mean and standard deviation both equal to 1) is described by this probability density function:

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

To find the expected value or probability that the value lies between bounds a and b we should compute:

$$\int_a^b f(x)\ dx$$

However, there is no function $F(x)$ such that $F'(x) = f(x)$, so we cannot solve the equation in the normal way by computing $F(b) - F(a)$.

To compute the integral, we fall back on Riemann sums. Divide the closed range $[a, b]$ into N rectangles whose individual height is the value $f(x)$ somewhere in the rectangle, and then compute the sum of

$\Delta x \times f(x)$ over the whole interval. A simple approach uses rectangles of constant width (equal to $b - a/N$).

Using Python:

https://gist.github.com/telliott99/5a1190217a130c7ee01dee17ea483f7b

I hope the flow is clear. The function `get_xvalues` generates a list of x-values starting from the middle of the first step past a and continuing to the last step just before b. `integrate` simply computes $f(x)$ for each x-value, sums all of those values, and adjusts for the width of the steps (rectangles).

We integrate $f(x) = x^2$ over the ranges $[0, 1]$ and $[1, 2]$ and obtain the expected results (1/3 and 7/3).

We also integrate the normal probability density function over ranges $[-2, 2]$ to $[-10, 10]$. With standard deviation equal to 1, we obtain the expected result that 95% of the density lies within two standard deviations of the mean. Essentially all of the density lies within four standard deviations of the mean.

And finding that the total area equals 1 confirms that the normalization factor $1/\sqrt{2\pi}$ is correct. That is, the value of the unnormalized integral is equal to $\sqrt{2\pi}$.

Refinements

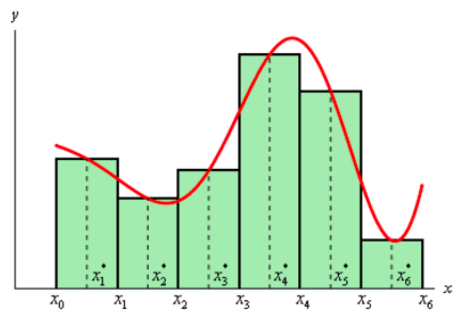
Classically, the major improvement to be made to this algorithm is to make a more accurate estimation of the area for each small rectangle. This is not so important with fast computers. For example, with a million steps rather than 100, I obtain

```
> python numerical_int.py
0.333333333333
```

```
2.33333333333
>
```

for the first two integrations, in about two seconds.

The calculation above uses the midpoint rule, where $f(x)$ is evaluated at the midpoint of the range.



The step size is computed and used to generate a list of values where each rectangle starts, then half the step is added to give the midpoint.

If we think of a and b as the bounds for each small rectangle, then the average of a and b is the midpoint:

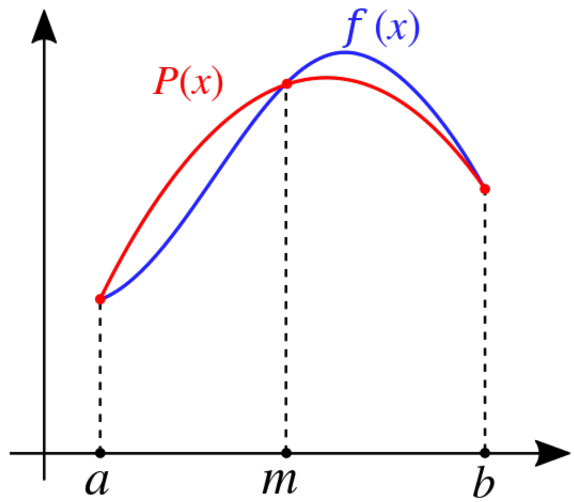
$$m = \frac{a+b}{2}$$

We evaluate $f(m)$, the function at the midpoint, and then multiply by the width:

$$M = f(m) \cdot (b-a)$$

Simpson's rule is a more sophisticated approach that uses:

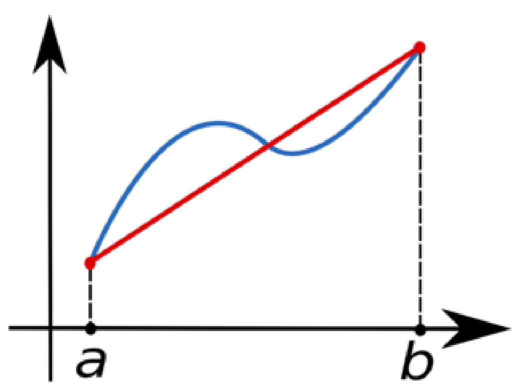
$$\frac{f(a) + 4f(m) + f(b)}{6} \cdot (b-a)$$



We sample once each from a and b, and four times from m, and average those samples.

The trapezoidal rule is

$$\frac{f(a)+f(b)}{2}\cdot(b-a)$$



Simpson's rule is really just a combination of the other two rules, namely, it is equal to $(2M+T)/3$. It weights the value at each endpoint as $1\times$ and then the midpoint as twice the combined values at the endpoints.

Essentially, this fits a parabola to the points a, m, b and then computes the area. It is really Archimedes' result (quadrature) in disguise.

Consider the parabola $y = -x^2 + 1$ between $x = -1 \to 1$ (the points where it crosses the x-axis on its way down). Our task is to find the correct y-value to use as the average height of the function in this region. Inverting the standard result for the area under $y = x^2$, the area under this parabola is $2/3$ of the area above it. The result we seek is $2/3$.

So a quadratic approximation to the area samples four times at the vertex plus once each at $x = \pm 1$. We sample because we understand that the curve is probably not exactly quadratic.

Chapter 41

Continuum of numbers

This chapter is a bit of a digression, and can easily be skipped, but it's fun and not too technical.

Our ideas about sets of numbers start with the counting numbers or positive integers

$$1, 2, 3 \dots$$

which is also known as the set of *natural* numbers, $\mathbb{N}$.

$\mathbb{N}$ is an infinite set.

Proof by contradiction: assume there is a greatest natural number. Add 1 to it.

$\square$

We extend $\mathbb{N}$ with the number 0, also known as the additive identity, plus the negatives or additive inverses of all the numbers in $\mathbb{N}$ where $n + (-n) = 0$:

$$\mathbb{Z} = \{\cdots - 3, -2, -1, 0, 1, 2, 3 \dots\}$$

$\mathbb{Z}$ is what they call a *ring*. The standard operations addition, subtraction and multiplication are defined and allowed, but not division. The allowed operations always give numbers as results which are contained in $\mathbb{Z}$.

Suppose we want division. We define the rational numbers $\mathbb{Q}$:

$$\mathbb{Q} = \frac{p}{q}, \qquad p \in \mathbb{Z}, q \in \mathbb{N}$$

The rational numbers have great properties. For example

∘ For any two rational numbers one can find another such number which lies between them. Here is one possible approach:

$$r = \frac{1}{2}\left[\,\frac{p_1}{q_1} + \frac{p_2}{q_2}\,\right]$$

r is a rational number and it lies between the two numbers we started since it is the average of the two.

Proof: relabel $s = p_1/q_1$ and $t = p_2/q_2$ thus

$$r = \frac{1}{2}\,[s + t\,]$$

$$2r = s + t$$

$$2r - 2s = t - s$$

suppose $s < t$, then $t - s > 0$ and so

$$r - s > 0$$

$$r > s$$

A similar argument will show that

$$r < t$$

thus

$$s < r < t$$

□

Thus, it is not unreasonable to have the idea that the number line can be divided into pieces as small as you like by finding rational numbers between rational numbers between rational numbers, and so on.

It sounds good, but there is a big problem which you probably know: some numbers cannot be expressed as the ratio of two integers, for example, $\sqrt{2}$.

Further, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, etc.. And don't forget π and e. Here is a proof that $\sqrt{2}$ is irrational

Suppose $\sqrt{2}$ is **rational.** That means it can be written as the ratio of two integers p and q

$$\sqrt{2} = \frac{p}{q} \tag{1}$$

where we may assume that p **and** q **have no common factors**. (If there are any common factors we cancel them in the numerator and denominator.) Squaring in (1) on both sides gives

$$2 = \frac{p^2}{q^2} \tag{2}$$

which implies

$$p^2 = 2q^2 \tag{3}$$

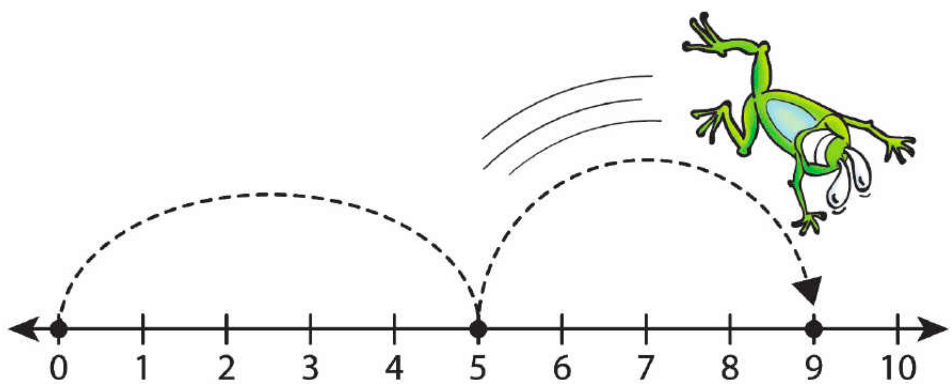
Thus p^2 is even. The only way this can be true is that p itself is even. But then p^2 is actually divisible by 4. Hence q^2 and therefore q must be even. So p and q are both even which is a contradiction to our assumption that they have no common factors. The square root of 2 cannot be rational!

To quote Hardy (*A Mathematician's Apology*):

> The proof is by reductio ad absurdum, and reductio ad absurdum, which Euclid loved so much, is one of a mathematicians finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.

The numbers like $\sqrt{2}$ are said to be *irrational* numbers and the set of these, plus all the other numbers is called the real numbers $\mathbb{R}$.

This led Dedekind to formulate the famous Dedekind cut. Visualize the standard number line as an infinite line on (an infinite) piece of paper.



Each real number corresponds to a cut, a knife-edge coming down somewhere on this number line. Every other number that is not equal to this one, is either $>$ or $<$ the number specified by the cut.

One position is $\sqrt{2}$, one is $7/4$ and so on.

We said that for any two rational numbers one can find a rational number which lies between them.

Three related statements are also true. We will show that

∘ for any two rational numbers one can find a real number which lies between them

∘ for any two real numbers one can find a rational number which lies between them

∘ for any two real numbers one can find a real number which lies between them

continuum

∘ Between any two *real* numbers it is always possible to find a rational number.

Proof: pick

$$N \in \mathbb{N} \text{ such that } N > \frac{1}{b-a}$$

Then

$$\frac{1}{N} < b - a$$

Define the set $\mathbf{A}$ as follows:

$$\mathbf{A} = \{ \frac{m}{N} : \ m \in \mathbb{N}\}, \quad \text{a subset of } \mathbb{Q}$$

The claim is that

$$\mathbf{A} \cap (a, b) \neq \emptyset$$

There do exist numbers within the open interval (a, b) that are in the set $\mathbb{Q}$.

The proof is by contradiction. Assume on the contrary that the set $\mathbf{A}$ does not contain a rational number lying inside this interval. In other words:

$$\mathbf{A} \cap (a, b) = \emptyset$$

Now, find the largest integer m_1 such that $m_1/N < a$ (it is OK if m_1 is equal to 0). Then the next rational number in $\mathbf{A}$ must be larger than b since the set intersection is empty:

$$\frac{m_1 + 1}{N} > b$$

But this implies that

$$\frac{m_1 + 1}{N} - \frac{m_1}{N} > b - a$$

$$\frac{1}{N} > b - a$$

which contradicts our condition on N above. Hence the assumption is false and so

$$\mathbf{A} \cap (a, b) \neq \emptyset$$

Thus there must exist a rational number r in $\mathbf{A}$ such that $a < r < b$.

example

Consider the open interval: $(\sqrt{2}, \sqrt{3})$.

$$a = \sqrt{2} \approx 1.414$$

$$b = \sqrt{3} \approx 1.732$$

$$b - a \approx 0.3178$$

$$\frac{1}{b - a} \approx 3.1462$$

Pick $N \geq 4$, for example

$$N = 4: \quad 1.414 < \frac{6}{4} = 1.5 < 1.732$$

$$N = 5: \quad 1.414 < \frac{8}{5} = 1.6 < 1.732$$
$$N = 6: \quad 1.414 < \frac{9}{6} = 1.5 < 1.732$$

(In this case $N = 2$ and $N = 3$ happen to work as well).

∘ Between any two rational numbers it is always possible to find a real number.

One proof consists of finding a *particular* irrational in the interval (a, b), where a and b are rational. For $a < b$, we simply add to the number a the following

$$c = \frac{\sqrt{2}}{2}(b - a)$$

c is smaller than $b - a$ (because $\sqrt{2}/2 < 1$) so the result $a + c$ lies between a and b. We also know that c is irrational, because $\sqrt{2}$ times any rational number is irrational. Finally, $a + c$ is irrational because adding $\sqrt{2}$ times a rational number to any rational number produces an irrational number.

Proof of the first preliminary requirement: $\sqrt{2}$ times a rational is irrational. Suppose for integer p, q, r, s we have

$$\sqrt{2}\,\frac{p}{q} = \frac{r}{s}$$

then

$$\sqrt{2} = \frac{rq}{ps}$$

But the right-hand side is rational, so this is a contradiction.

For the second requirement, again by contradiction suppose

$$\sqrt{2}\,\frac{p}{q} + \frac{s}{t} = \frac{u}{v}$$

for integer p, q, r, s, u, v. But the right-hand side of

$$\sqrt{2} = \frac{q}{p}(\frac{u}{v} - \frac{s}{t})$$

is rational, so this is a contradiction.

Note in passing that powers are different. What do you think about

$$r = \sqrt{2}^{\sqrt{2}}$$

You may think r is "likely" to be irrational. Just a mess. But how about

$$r^{\sqrt{2}}$$

Whether r is rational or irrational

$$r^{\sqrt{2}} = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{2} = 2$$

!!

○ Between any two real numbers it is always possible to find another real number. This one is subtle. Suppose the two real numbers are "really, really close."

We suppose that they are not equal, so they must be different, say $a < b$.

Since they are different, at some stage in the decimal expansions of a and b, there must be a first position at which a and b differ. If b does not have a 0 at the next position, terminate there and that will be c.

For example:

$$a = 1.23456789129..$$

$$b = 1.23456789133..$$

$$c = 1.23456789130..$$

b must have some digit following this first position where it does not match a, and which is also not equal to zero (otherwise it would be a terminating decimal and thus a rational number). So we can always find a place to terminate to form c.

Eternity is a very long time, especially towards the end.

(credited to Woody Allen)

variations of infinity

In other words there is *no least number* x such that $x > 0$, for example, and no greatest number x such that $x < 1$.

Proof: Assume that m is the smallest number > 0. The rational number $m/2 < m$ is also greater than zero, but smaller than m. Thus, m is not the smallest positive number.

In general, there is no number that is the closest number to another number.

That is actually OK. Here's what's really weird. Cantor proved that the set $\mathbb{Q}$ is *countably finite*. Each element in $\mathbb{Q}$ can be paired in order with a member of $\mathbb{N}$.

The idea of the proof is to show that one can set up a correspondence between $\mathbb{N}$ and $\mathbb{Q}$, assigning each number $r \in \mathbb{Q}$ in a particular order to $1, 2, 3, \ldots$. Here is the figure from Courant and John:

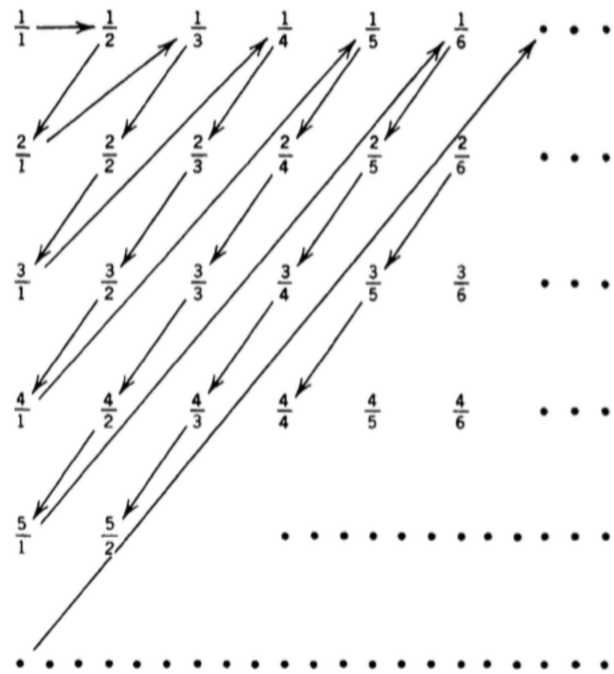


Figure 1.S.1 Denumerability of the positive rationals.

Basically, each row contains all the rational numbers with a particular numerator, and each column contains all the numbers with a particular denominator, arranged in strict increasing order.

Next, set up the sequence indicated by the arrows:

$$\frac{1}{1} \quad \frac{1}{2}, \frac{2}{1} \quad \frac{1}{3}, \frac{2}{2}, \frac{3}{1} \quad \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1} \quad \frac{1}{5}, \frac{2}{4}, \frac{3}{3}, \frac{4}{2}, \frac{5}{1} \quad \frac{1}{6}, \frac{2}{5}, \frac{3}{4}, \frac{4}{3}, \frac{5}{2}, \frac{6}{1} \cdots$$

Then remove all fractions that are duplicates because they are not in lowest terms.

$$\frac{1}{1} \quad \frac{1}{2}, \frac{2}{1} \quad \frac{1}{3}, \frac{3}{1} \quad \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1} \quad \frac{1}{5}, \frac{5}{1} \quad \frac{1}{6}, \frac{2}{5}, \frac{3}{4}, \frac{4}{3}, \frac{5}{2}, \frac{6}{1} \cdots$$

Finally, each r in this sequence is assigned to a natural number (in the sequence $\mathbb{N}$), establishing the denumerability property. 1/3 is paired with 4 and 3/1 is paired with 5, and so on.

Cantor showed that such a correspondence (which we just established for $\mathbb{Q}$), is impossible for $\mathbb{R}$. The proof of this is is not hard, but we will skip it here. You can check out the chapters on Georg Cantor in Dunham's *Journey Through Genius*.

Thus, the rational numbers are said to be "countably infinite", while the real numbers are not countable. (There is also a proof that the transcendental numbers are much more numerous than the non-transcendental ones).

So when we say that the set of numbers $r > 0$ has *no least element*, our problem is two-fold. We can pick a rational member in the interval $(0, r)$, but subsequently, we can always find a smaller rational element.

And once we get really close with the small rational element, there are infinitely more irrational than rational ones waiting beyond. And yet, given any such very close irrational number, we can always find a smaller rational number, still larger than the bound.

I told you it was weird.

This property of the real numbers, that there is no closest number to any given number, accounts for virtually all of the theoretical difficulties in calculus which are solved by the use of limits and the apparatus of δ and ϵ or alternatively, neighborhoods. For more, see the chapter on Limits and Continuity in the addendum.

Part XII

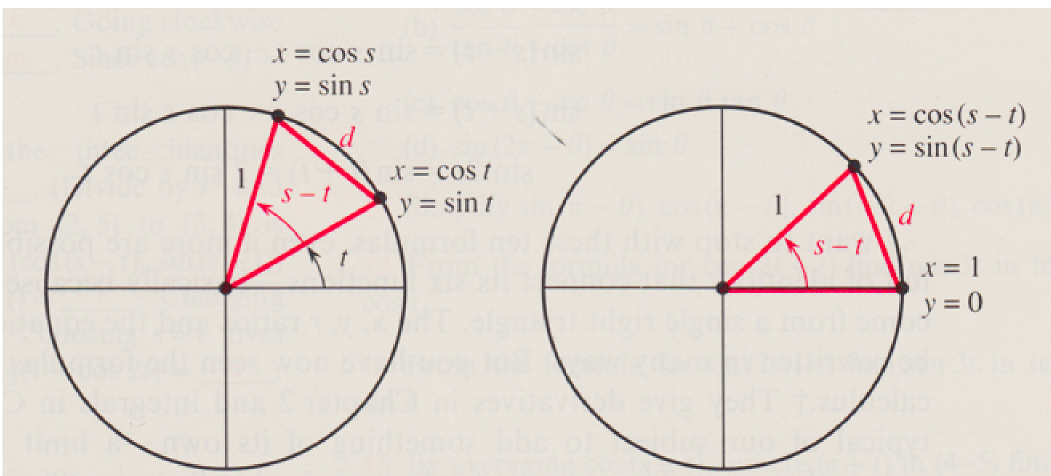
Sum of angles and an application

Chapter 42

Sum of angles again

Strang

For a geometric derivation of the sum of angles formula with minimal setup, I really like this figure from Strang



Distance between points and equal distances in two circles.

We have the same triangle in the two panels, just rotated clockwise on the right.

To compute the distance between two points in the plane we do

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(This is just the Pythagorean theorem in disguise).

We don't actually need to take the square root, let's stick with

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

In the first figure, t is the angle between the lower radius and the x-axis, s is the angle between the upper radius and the x-axis, and as labeled, $s - t$ is the angle between the two radii.

The distance d squared for the left panel is

$$d^2 = (\cos s - \cos t)^2 + (\sin s - \sin t)^2$$

Multiply out:

$$d^2 = \cos^2 s - 2\cos s\cos t + \cos^2 t + \sin^2 s - 2\sin s\sin t + \sin^2 t$$

We have two copies of $\sin^2 + \cos^2$, one for angle s and one for t

$$d^2 = 2 - 2\cos s\cos t - 2\sin s\sin t$$

In the right panel, the two radii have been rotated, preserving the same angle between them.

$$d^2 = (\cos(s - t) - 1)^2 + \sin(s - t)^2$$

(Don't forget the 1).

$$= \cos^2(s - t) - 2\cos(s - t) + 1 + \sin^2(s - t)$$

$$= 2 - 2\cos(s - t)$$

Because the included angle hasn't changed, neither has the distance, so we can equate the two expressions.

$$2 - 2\cos(s - t) = 2 - 2\cos s \cos t - 2\sin s \sin t$$

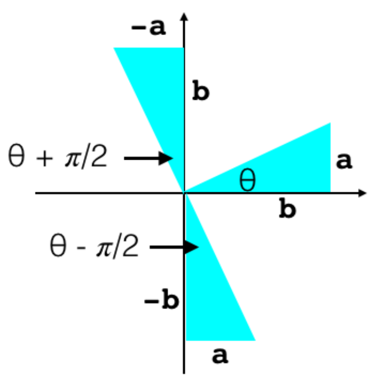
Subtract 2 from both sides, multiply by 1/2, and change sign to give

$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

This is our first formula, for the cosine of the difference of two angles.

getting to sine

Look at the relationships between sine and cosine for angles that are related by addition or subtraction of $\pi/2$.



In the figure, I have simply rotated the same triangle.

From the figure we can easily read off these four identities

$$\sin(\theta + \pi/2) = b = \cos\theta$$

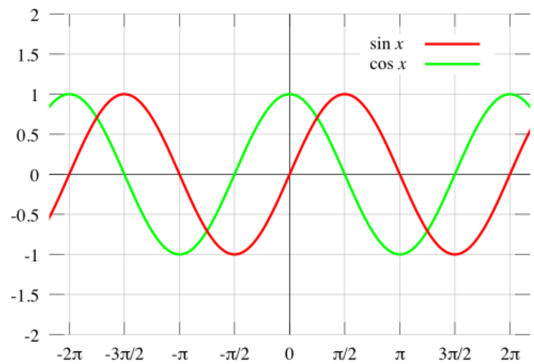
$$\cos(\theta + \pi/2) = -a = -\sin\theta$$

And

$$\sin(\theta - \pi/2) = -b = -\cos\theta$$

$$\cos(\theta - \pi/2) = a = \sin\theta$$

Here is an alternative derivation which proceeds from the graph of sine and cosine versus the angle.



Pick some angle (say $\theta = 0$), then $\cos\theta = 1$. What is the angle for which the sine gives the same result? The sine curve is exactly like the cosine, it is just shifted to the right by a *phase change* of $\pi/2$.

That angle is $\theta + \pi/2$. The phase change is added to the angle:

$$\cos\theta = \sin(\theta + \frac{\pi}{2})$$

Try the same reasoning in reverse.

The cosine curve is exactly like the sine, it is just shifted by a phase change of $-\pi/2$, i.e. to the left. Pick some angle (say $\theta = \pi/2$), then $\sin\theta = 1$. What is the value of the angle for which the cosine gives the same result?

It is $\theta - \pi/2$. The phase change is subtracted from the angle θ:

$$\sin\theta = \cos(\theta - \frac{\pi}{2})$$

In summary, switching sine for cosine gives a valid expression, but there is a difference of *sign* for the phase.

back to our task

So our sum of angles formula (well, really, the difference of angles) was

$$\cos(s-t) = \cos s \cos t + \sin s \sin t$$

Let

$$u = t - \frac{\pi}{2}$$

$$t = u + \frac{\pi}{2}$$

Substitute for t

$$\cos\ [\ s - (u + \frac{\pi}{2})\ ]\ = \cos s \cos(u + \frac{\pi}{2}) + \sin s \sin(u + \frac{\pi}{2})$$

Regroup the left-hand side

$$\cos\ [\ (s - u) - \frac{\pi}{2})\ ]\ = \cos s \cos(u + \frac{\pi}{2}) + \sin s \sin(u + \frac{\pi}{2})$$

Referring to the results we obtained above, cosine something minus $\pi/2$ is the sine of that something:

$$\sin(s-u) = \cos s \cos(u + \frac{\pi}{2}) + \sin s \sin(u + \frac{\pi}{2})$$

Cosine something plus $\pi/2$ is minus the sine:

$$\sin(s-u) = -\cos s \sin u + \sin s \sin(u + \frac{\pi}{2})$$

Sine something plus $\pi/2$ is cosine:

$$\sin(s-u) = -\cos s \sin u + \sin s \cos u$$

Rearrange:

$$\sin(s-u) = \sin s \cos u - \sin u \cos s$$

This is correct, but the path is fraught with error!

For now, memorize. Soon we will see a very simple and effective **aid to memory** due to Euler.

sum of tangents

It is also easy to derive the sum of tangents from the sum of sines and cosines.

$$\tan s+t = \frac{\sin s+t}{\cos s+t}$$
$$= \frac{\sin s\cos t + \cos s\sin t}{\cos s\cos t - \sin s\sin t}$$

Divide by $\cos s\cos t$

$$\tan s+t = \frac{\tan s + \tan t}{1 - \tan s\tan t}$$

$$\tan s-t = \frac{\tan s - \tan t}{1 + \tan s\tan t}$$

We will use these for a few problems later in the book.

Chapter 43

Double and half angles

We will find it useful in several problems to be able to compute the sine, cosine and tangent of angle 2θ, knowing the values for θ. These formulas can be rearranged to give the values of $\theta/2$ in terms of θ.

I can't remember these formulas, but derive them from the sum of angles when needed.

cosine

Start with our old friend:

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

Let $s = t$:

$$\cos 2s = \cos^2 s - \sin^2 s$$

Since $\sin^2 s + \cos^2 s = 1$, $-\sin^2 = \cos^2 -1$ so

$$\cos 2s = 2\cos^2 s - 1$$

We can use this formula to compute the value for $2s$ given that for s. To go from 2θ to θ:

$$\cos^2 s = \frac{1}{2}(1 + \cos 2s)$$
$$\cos s = \sqrt{\frac{1}{2}(1 + \cos 2s)}$$

sine

$$\sin s + t = \sin s \cos t + \cos t \sin s$$

Let $s = t$:

$$\sin 2s = 2 \sin s \cos s$$

Put the other way

$$\sin s = \frac{\sin 2s}{2 \cos s}$$

tangent

The formulas for the tangent are easily obtained by substitution. Let us simplify the notation a bit by setting $S = \sin 2t$ and $S' = \sin t$ and similarly for cosine and tangent. From above we have the basic relationships

$$S' = \frac{S}{2C'}$$

and

$$C' = \sqrt{\frac{1}{2}(1 + C)}$$
$$2[C']^2 = 1 + C$$

So the tangent ($T' = \tan s$) is:

$$T' = \frac{S'}{C'} = \frac{S}{2C'}\ \frac{1}{C'} = \frac{S}{2[C']^2}$$

$$= \frac{S}{1+C}$$

That's a fairly remarkable simplification!

Another way to say the same thing:

$$\frac{1}{T'} = \frac{1}{T} + \frac{1}{S}$$

This result can be massaged in various ways. Multiply on the top and bottom of the right-hand side by T

$$T' = \frac{ST}{S+T}$$

Also, since

$$T' = \frac{S}{1+C} = \frac{S'}{C'}$$

$$C' = \frac{S'(1+C)}{S}$$

In going from unprimed (2θ) to prime (θ), it seems that the most straightforward way is to compute

$$C' = \sqrt{\frac{1}{2}(1+C)}$$

$$T' = \frac{S}{1+C}$$

and then the sine last

$$S' = \frac{S}{2C'}$$

check

Let's try checking the results for a known angle

$$2\theta = \pi/3$$

$$S = \sin 2\theta = \frac{\sqrt{3}}{2}, \quad \cos 2\theta = \frac{1}{2}, \quad T = \tan 2\theta = \sqrt{3}$$

$$S' = \sin\theta = \frac{1}{2}, \quad \cos\theta = \frac{\sqrt{3}}{2}, \quad T' = \tan\theta = \frac{1}{\sqrt{3}}$$

Our first equation is

$$T' = \frac{ST}{S+T} = \frac{3/2}{(3/2)\sqrt{3}} = \frac{1}{\sqrt{3}}$$

That looks good. The second one is

$$S' = \sqrt{\frac{ST'}{2}}$$

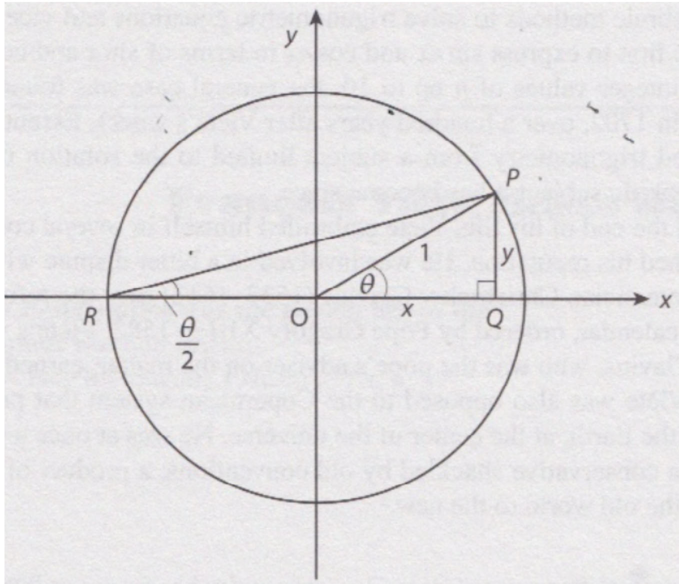
$$ST' = \frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}} = \frac{1}{2}$$

$$S' = \sqrt{\frac{1}{2}\ \frac{1}{2}} = \frac{1}{2}$$

These both look correct.

geometric approach

Here are two simple geometric derivations of the half angle formulas for sine and cosine.



For the first, draw an angle θ in a unit circle. We proved the theorem that the angle at the left (on the circle) is equal to $\theta/2$ **here**.

Algebraically, write

$$\cos\frac{\theta}{2} = \frac{1+x}{\sqrt{(1+x)^2+y^2}}$$

$$= \frac{1+x}{\sqrt{1+2x+x^2+y^2}}$$

$$= \frac{1+x}{\sqrt{2+2x}}$$

$$= \sqrt{\frac{1+\cos\theta}{2}}$$

To get the formula for the sine just use the identity

$$\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$$

$$\sin^2\frac{\theta}{2} = 1 - \frac{1+\cos\theta}{2} = \frac{1-\cos\theta}{2}$$

Using the prime notation from above

$$S' = \sqrt{\frac{1-C}{2}}$$

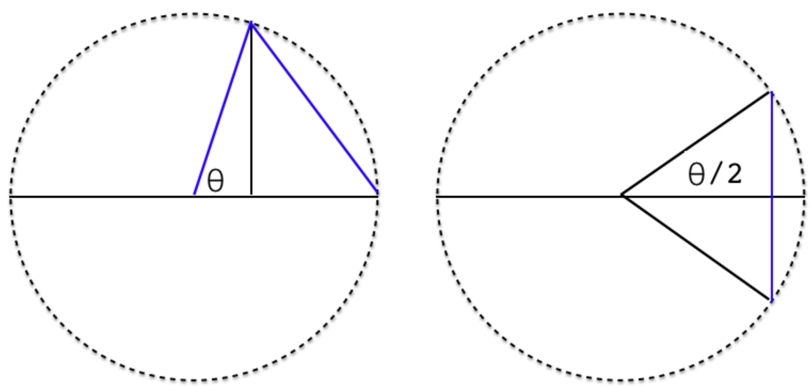
This is easily checked

$$S'^2 + C'^2 = 1$$
$$\frac{1-C}{2} + \frac{1+C}{2} = 1$$

which looks correct.

For the second derivation, again draw an angle θ in a unit circle. As usual, then, the base of the right triangle is $\cos\theta$ and the height is $\sin\theta$ (left panel, below).

Now, also draw the chord of the circle, corresponding to the arc θ.



Notice that the chord is the hypotenuse of a right triangle whose height is $\sin\theta$ and whose base is $1 - \cos\theta$. Therefore, the length of the chord is

$$c = \sqrt{(1-\cos\theta)^2 + \sin^2\theta} = \sqrt{2 - 2\cos\theta}$$

Now, rotate the chord as shown in the right panel. We see that one-half the chord is equal to the sine of $\theta/2$:

$$\sin\theta/2 = \frac{1}{2}\,\sqrt{2 - 2\cos\theta}$$

$$S' = \frac{1}{\sqrt{2}}\ \sqrt{1 - C}$$

and

$$S'^2 + C'^2 = 1$$

$$C'2 = 1 - S'^2 = 1 - \frac{1 - C}{2} = \frac{1 + C}{2}$$

$$C' = \frac{1}{\sqrt{2}}\ \sqrt{1 + C}$$

Chapter 44

Value of pi

Archimedes and π

Since Archimedes is a strong presence in this book, we will discuss his method for approximating the value of π, the ratio of the circumference of a circle to its diameter. The commonly cited result is

> The ratio of the circumference of any circle to its diameter is less than 3 1/7 but greater than 3 10/71.

In decimal that is $3.140845.. < \pi < 3.1428571$.

However, to some extent this misses the main idea, that Archimedes described an iterative procedure which can be used to calculate the value of π *to any desired accuracy.*

Although the idea is beautiful, his argument is somewhat unwieldy in detail, so instead we will use modern trigonometry to achieve the same result more economically.

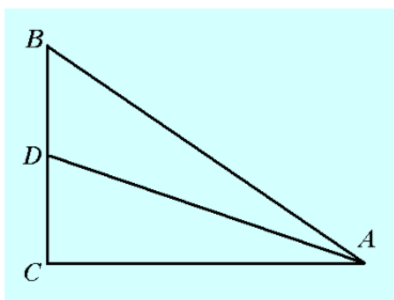
For a discussion of Archimedes actual method (based on a translation by Heath), see this web page https://itech.fgcu.edu/faculty/clindsey/mhf4404/archimedes/archimedes.html

and I have worked out the same proof in detail in this **chapter**.

In addition, we will connect the trigonometry to easy formulas for the perimeter and area of inscribed and circumscribed polygons. The first part is in this chapter, and the second part has been split out into another **chapter**, which is in the Addendum.

If this material is too esoteric, it can be skipped without loss of continuity in the rest of the book.

I should also point out that although we don't follow Archimedes exactly, a key element which he relies upon is the proof that, for an angle bisector in a right triangle, the adjacent sides are in the same proportion as the two segments formed where the bisector meets the other side.



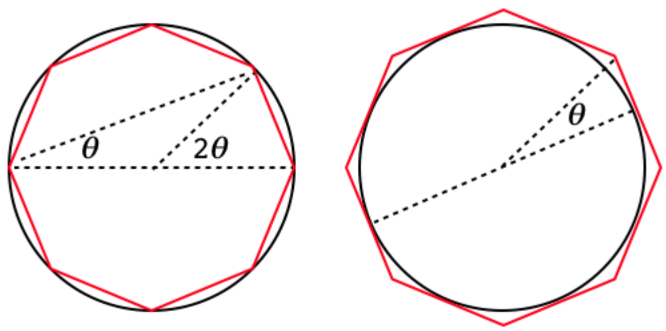
Here:

$$\frac{AB}{AC} = \frac{BD}{DC}$$

We showed a proof of this earlier (**here**).

the method

We will approximate the value of π by squeezing it between the perimeter of an inscribed polygon, which is less than the circumference of the circle, and the perimeter of a circumscribed polygon, which is greater than the circumference of the circle.



We use a circle of *diameter* equal to 1 (rather than the radius, which is more usual). The circumference of the circle is then equal to π, the value which gets squeezed between the two perimeters.

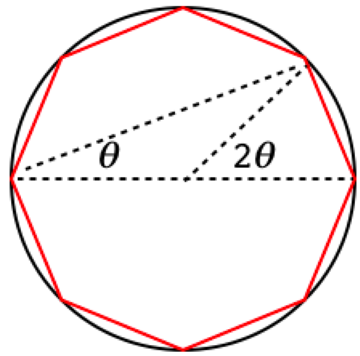
The figure shows a sketch of the polygons when $n = 8$. We will be increasing the number of sides by a factor of 2 at each step, so these are really 2^n-gons with $n = 3$ here.

Finding perimeters in terms of angle θ

For the left panel, we have 8 sides, so the central angle (marked 2θ) is equal to

$$\frac{2\pi}{8} = \frac{\pi}{4} = 45^\circ$$

and θ is one-half that.



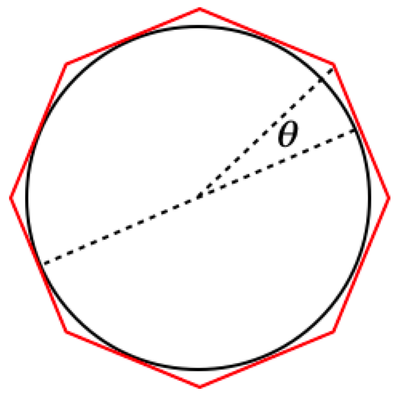
By a standard theorem (from Thales), the triangle above containing angle θ, with the diameter as one side, and two other vertices also on the circle, is a right triangle. The inscribed n-gon side of length S (shown in red) is equal to $\sin\theta$, since the hypotenuse of the triangle is the diameter of the circle, which is equal to 1.

The total perimeter is $8 \cdot S$.

[Alternatively, use half the angle at the center of the circle (i.e. θ). Then half the length of the red line $S/2$, divided by the radius ($r = 1/2$) gives $S = \sin\theta$, the same result.]

For the right panel, we have the same circle (now showing the outside polygon, circumscribing the circle), it is just rotated slightly.. One dashed line extends a bit further to the vertex of the n-gon outside. The angle marked θ is one-half the angle we marked as 2θ previously since now the diameter comes down to the middle of the side.

We compute the whole length of the side T as follows. The half-side is $T/2$ and the hypotenuse of the triangle is one-half the unit diameter, which is $1/2$, so $T = \tan\theta$. The total perimeter is $8 \cdot T$.



All of this gives us two simple equations for the two perimeters. At each stage there are 2^n sides, the length of each short side S on the inside equals $\sin\theta$ and the length of each short side on the outside T is equal to $\tan\theta$, where $\theta = 2\pi/2^n$.

The total length of the inside perimeter is $nS = n\sin\theta$ and that of the outside is $nT = n\tan\theta$. When we go from θ to $\theta/2$ and n to $2n$, we must compute the new values S' and T' from S and T using the half-angle formulas, and then also multiply by 2 to take account of the change from n to $2n$ for the total circumference.

The base case

If we go back to the square ($n = 2, 2^n = 4$), then the angle θ is $\pi/4$.

The tangent is $T = \tan\pi/4 = 1$ and the sine is $S = \sin\pi/4 = 1/\sqrt{2}$.

Our formulas say that on the inside, the perimeter is $4S = 4/\sqrt{2} = 2\sqrt{2}$ and on the outside, the perimeter is $4T = 4$.

From simple geometry, we can calculate that the circumscribing square has a side length which is twice the radius of the circle, that is, 1 for our circle with unit diameter, so its perimeter is 4, which checks.

Similarly, an inscribed square can be decomposed into four isosceles right triangles with sides of length $1/2$ and hypotenuse $1/\sqrt{2}$, so the total perimeter is $4/\sqrt{2}$, which also checks.

Now, what we are going to do is to increase n in steps of 1, that increases 2^n by a factor of $2^1 = 2$ each time. Doubling n halves the angle. So all we need is a way to compute trigonometric functions of $\theta/2$, knowing the values for θ, so we can calculate what happens to the perimeter. We already know how to do that.

Half angle formulas

We have derived these elsewhere. Refer to this **chapter**.

The unprimed values refer to angle θ, while the primed ones have angle $\theta/2$.

$$C' = \sqrt{\frac{1}{2}(1+C)}$$

This can be rearranged (e.g.) to give $2[C']^2 = 1 + C$, which we'll use in a second.

$$S' = \frac{S}{2C'}$$

$$T' = \frac{S'}{C'} = \frac{S}{2[C']^2}$$
$$= \frac{S}{1+C}$$

So, given S, C and T, first calculate C' and T' and then S'. To get the perimeters, remember that factor of two from doubling n, the number of sides.

another approach

This web page originally got me started with this derivation

http://personal.bgsu.edu/~carother/pi/Pi3d.html

(Unfortunately, the link is dead now, probably because the University took Dr. Carother's pages down when he died, idiots). It has been preserved by the wayback machine:

https://web.archive.org/web/20171024182015/http://personal.bgsu.edu/~carother/pi/Pi3d.html

On that page, there was given an arguably simpler pair of formulas listed, namely, for an inside perimeter p and an outside perimeter P

$$P' = \frac{2pP}{p+P}$$

$$p' = \sqrt{pP'}$$

The first equation can be rearranged to give

$$\frac{1}{P'} = \frac{1}{2}\,[\frac{1}{P} + \frac{1}{p}\,]$$

which is the definition of the harmonic mean of p and P, while the second equation is the geometric mean.

Since in our derivation p and P are the same multiple of S and T, it seems like the same relationships should hold for the sine and tangent, but we must remember the extra factor of 2.

From the half-angle formulas, we said that

$$T' = \frac{S}{1+C}$$

Multiply top and bottom on the right by T:

$$T' = \frac{ST}{T+S}$$

Recall that S is the same as p, within a factor of n, and that T is the same as P, within the same factor.

$$p = nS$$

$$P = nT$$

while

$$P' = 2nT'$$

Going back to

$$\begin{aligned} T' &= \frac{ST}{T+S} \\ 2nT' &= \frac{2 \cdot nS \cdot nT}{nT + nS} \\ P' &= \frac{2pP}{p+P} \end{aligned}$$

This is what was given.

For the second one

$$\begin{aligned} S' &= \frac{S}{2C'} \\ &= \frac{S}{2} \ \frac{T'}{S'} \end{aligned}$$

Then

$$4[S']^2 = S \cdot 2T'$$

$$[2nS']^2 = nS \cdot 2nT'$$

Changing variables, $p' = 2nS'$

$$[p']^2 = pP'$$

Finally

$$p' = \sqrt{pP'}$$

which matches what was given.

Calculation

Let's run a simulation to see what kind of numbers we get. Start with the square ($n = 2$, $2^n = 4$) Previously we found that $S = 1/\sqrt{2}$ and $T = 1$ so

$$p = 2^n S = \frac{4}{\sqrt{2}} = 2.8284$$

$$P = 2^n T = 4$$

Let's try a script to calculate this to larger n.

https://gist.github.com/telliott99/19f521c807210171a4847b319104b3df

Output:

```
> python pi.py
 2 2.8284271247  4.0000000000
 3 3.0614674589  3.3137084990
 4 3.1214451523  3.1825978781
 5 3.1365484905  3.1517249074
 6 3.1403311570  3.1441183852
 7 3.1412772509  3.1422236299
 8 3.1415138011  3.1417503692
```

```
 9 3.1415729404  3.1416320807
10 3.1415877253  3.1416025103
11 3.1415914215  3.1415951177
12 3.1415923456  3.1415932696
13 3.1415925766  3.1415928076
14 3.1415926343  3.1415926921
15 3.1415926488  3.1415926632
16 3.1415926524  3.1415926560
17 3.1415926533  3.1415926542
18 3.1415926535  3.1415926537
19 3.1415926536  3.1415926536
>
```

That looks pretty good to me, although it's a bit slow to converge.

This is really quite amazing. Archimedes has not only calculated π to 3 significant figures. More important, he has provided us with an iterative procedure that can be used to calculate the value to *any precision we desire*. As an engineer, Archimedes knew that 3.1416 is precise enough, so he stopped.

After all, no one wants to be William Shanks, or one of these guys:

https://en.wikipedia.org/wiki/Chronology_of_computation_of_

Quote:

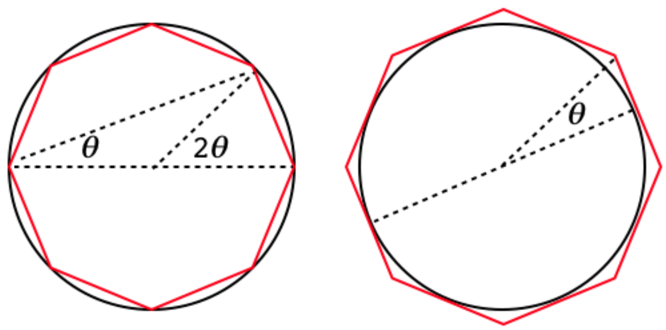
> [He] calculated pi to [n] digits, but *not all were correct.*

There is an additional **chapter** which substantially extends the above discussion, showing a geometric derivation of the basic relationships and developing new formulas involving the areas as well as the perimeters of the sectors of inscribed and circumscribed polygons.

area

I became aware later that there is yet another way to apply the method, and that is to calculate the *areas* of inscribed and circumscribed polygons. We'll go through this briefly.

For this approach we use a unit circle (radius 1) rather than a diameter of 1, as we did above. As before, we define θ to be the central angle of the half-sector (i.e. $\theta = 2\pi/2n$).



Rather than draw an entirely new figure, just imagine in the left panel that we draw the angle bisector of angle 2θ. The area of each new triangle is then $\sin\theta\cos\theta/2$ and the total area of the inner polygon is

$$a = n\sin\theta\cos\theta = nSC$$

in the notation we adopted previously in this chapter. And, as before, to progress to a' we have a factor of 2 as well as the new values S' and C':

$$a' = 2nS'C'$$

For the circumscribed or outer polygon, we just have what we had before, that the side length of the triangle in the right panel is $\tan\theta$ so the total area is

$$A = nT$$

Bring in the half-angle formulas as follows:

$$a' = 2nS'C' = 2n \cdot \frac{S}{2C'} \cdot C' = nS$$

That is slick, but we need an expression for nS:

$$aA = nSC \cdot n\frac{S}{C} = [nS]^2$$

$$aA = [a']^2$$
$$a' = \sqrt{aA}$$

This is like, and yet subtly different than what we had when calculating the perimeter.

Since

$$A = nT$$

and

$$\begin{aligned} A' &= 2nT' \\ &= 2n\frac{ST}{S+T} = 2\frac{nS \cdot nT}{nS + nT} \\ A' &= 2\frac{a'A}{a' + A} \end{aligned}$$

Compare

$$a' = \sqrt{aA} \qquad A' = 2\frac{a'A}{a' + A}$$

$$p' = \sqrt{pP'} \qquad P' = 2\frac{pP}{p + P}$$

However, it turns out that when you take account of the differing size of the circle for perimeter and area methods, and thus the initial values of p, P, a and A, the different order of operations results in precisely the same calculation.

Part XIII

Euler

Chapter 45

Euler's equation

According to the historians, Newton came up with a series for e^x about 1669 in his book *De analysi per aequationes numero terminorum infinitas*, although he didn't call e by that name or anything.

Recall that one definition of the exponential is

$$y = e^x = y'$$

We try to *approximate* this function by a series. At $x = 0$, $e^x = e^0 = 1$ so

$$y = 1$$

So far so good. However, the derivative y' is then zero. How do we get a 1 into the derivative? By adding a term of x

$$y = 1 + x$$

Now the derivative matches in its first term

$$y' = 1$$

But we also need

$$y' = y = 1 + x$$

and that x has to come from somewhere in the expression for y. So add $x^2/2$ because its derivative is just x and we obtain

$$y = 1 + x + \frac{x^2}{2}$$

and again we need

$$y' = y = 1 + x + \frac{x^2}{2}$$

so

$$y = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$$

Well, you get the idea. This continues forever.

There are also infinite series for sine and cosine, of course. The have traditionally been attributed to Newton, who described them in the work cited earlier.

According to Stewart (*Significant Figures*), they were known much earlier, discovered by the Indian mathematician, Madhava (approximately 1350-1425 C.E.).

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$

Clearly, the derivative of the first series for sine is equal (term by term) to the series for cosine. And the derivative of the sine is minus the cosine. It all checks.

Actually, we can use what we know about the derivatives of sine and cosine to follow an approach similar to that for the exponential, to come up with these series ourselves.

In particular, we know that the second derivative is minus the function, and the fourth derivative is the function itself.

$$\sin x =$$

The first term can't be 1 because $\sin 0 = 0$ so write

$$\sin x = x + \dots$$

Now, we want $\sin'' x = -\sin x$ so

$$\sin x = x - \frac{x^3}{3!} + \dots$$

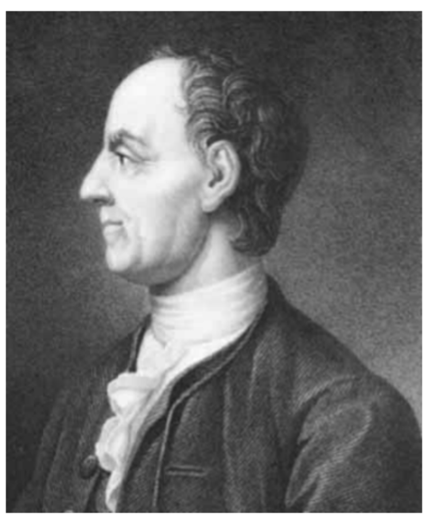
does that. Then $\sin'''' x = \sin x$ so

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

Then just take the derivative of this to get the series for cosine.

noticing a connection

Here is a portrait of Euler where he is not wearing that silly hat.



Euler said, let us consider the complex number

$$z = \cos\theta + i\sin\theta$$

How in the world did he come up with this? Perhaps because he already knew the answer.

My guess is that he looked at the formula

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots$$

and said: it cannot be an accident that the sine and cosine series look so similar, in fact when added together they have exactly the same terms, just with some periodically occuring minus signs.

$$\sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \dots$$

Where would I get alternating plus and minus signs from? Maybe if I substitute ix for x?

$$e^{ix} = \sum_{n=0}^{\infty} \frac{ix^n}{n!} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} \dots$$

Clearly, part of this is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots$$

while the rest is

$$ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} \dots = i\sin x$$

quick and dirty derivation

Let's just see what we can do.

If we assume that calculus is legal with complex numbers (assuming that i is just a number), we can do the following

$$z = \cos\theta + i\sin\theta$$

$$\frac{dz}{d\theta} = -\sin\theta + i\cos\theta$$

(It turns out that it is legal, but the argument requires the fact these functions are all *analytical*, which is beyond our scope here).

And since $i^2 = -1$

$$\frac{dz}{d\theta} = i^2\sin\theta + i\cos\theta$$

$$\frac{dz}{d\theta} = i(i\sin\theta + \cos\theta) = iz$$

Rearrange

$$\frac{1}{z}\ dz = i\ d\theta$$

$$\int \frac{1}{z}\ dz = \int i\ d\theta$$

$$\ln z = i\theta$$

Exponentiate:

$$z = e^{i\theta}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Euler's famous result in a few lines, which he proved more rigorously (but not completely so) by other approaches explored **here**.

addendum

We've really gone fast and loose in this section. One thing that makes series hard is that a series is sometimes valid only for a certain range of x. Consider

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 \dots$$

The equality is easily verified if you multiply both sides by $1 - x$. You get the right-hand side, plus a version of the right hand side containing every term except the first, all with minus signs. It all adds up to 1.

The problem is, the series is obviously not valid for $x = 1$, nor for $x > 1$. Consider $x = 2$. (It gets worse. Consider $x = -1$). In fact, it turns out to be valid only for $|x| < 1$, which is called the radius of convergence.

Luckily, the exponential series and those for sine and cosine *are* valid for all x.

According to Nahin (*An Imaginary Tale*), both Abraham de Moivre and Roger Cotes knew Euler's identity decades before he published it. Which may be a good example of

https://en.wikipedia.org/wiki/Stigler%27s_law_of_eponymy

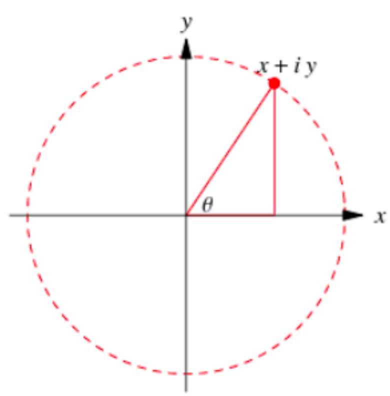
Chapter 46

Euler sum of angles

Euler's equation says that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

One way of thinking about the equation is to view complex numbers as points in the plane. Complex numbers are composed of a real part (say x) and a complex part (iy), where $i = \sqrt{-1}$.



It turns out that $e^{i\theta}$ corresponds to the specification of such a point in radial coordinates in the *Argand* or complex plane.

Later, we will look at a quick derivation of Euler's equation.

One valid proof is to simply plug into the series for the exponential

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

giving

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \dots \\ &= \cos x + i \sin x \end{aligned}$$

which, I realize, we also haven't seen yet. So, take it on faith for the moment.

We can use Euler's equation to get something extremely useful for calculus.

using Euler

Switch notation to s and t

$$e^{is} = \cos s + i \sin s$$

$$e^{it} = \cos t + i \sin t$$

$$e^{i(s+t)} = \cos(s+t) + i \sin(s+t)$$

But

$$e^{i(s+t)} = e^{is} e^{it}$$

$$= (\cos s + i \sin s)\ (\cos t + i \sin t)$$

$$= \cos s \cos t + \cos s\ i \sin t + i \sin s \cos t - \sin s \sin t$$

$$= (\cos s \cos t - \sin s \sin t) + i(\sin s \cos t + \cos s \sin t)$$

We have an equality between two complex numbers, both equal to $e^{i(s+t)}$. For this to be true, both the real and imaginary parts must be equal.

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$
$$\sin(s+t) = \sin s \cos t + \cos s \sin t$$

The addition formulas for sine and cosine.

I find that this is an extremely useful way of remembering how these formulas can be derived.

Chapter 47

deMoivre

The formula says that for integer n

$$[\ \cos x + i \sin x\ ]^n = \cos nx + i \sin nx$$

If we know Euler's formula the derivation is trivial:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$(e^{i\theta})^n = \ [\ \cos x + i \sin x\ ]^n$$
$$= e^{i\theta n} = e^{in\theta} = \cos n\theta + i \sin n\theta$$

induction

We can also prove it by induction. Multiply the first formula above by $(\cos x + i \sin x)$. The left-hand side is the form we seek. The right-hand side is

$$(\cos nx + i \sin nx) \cdot (\cos x + i \sin x)$$

$$= \cos nx \cos x - \sin nx \sin x + i(\sin nx \cos x + \cos x \sin nx)$$

Using the sum of angles formulas we obtain

$$= \cos(nx + x) + i \sin(nx + x)$$

$$= \cos(n + 1)x + i \sin(n + 1)x$$

which completes the inductive step.

The base can be chosen as $n = 1$.

example

Let $n = 3$. Then

$$[\ \cos x + i \sin x\ ]^3 = \cos 3x + i \sin 3x$$
$$= (\cos^2 x - \sin^2 x + 2i(\sin x \cos x)) \cdot (\cos x + i \sin x)$$

Taking the real part of the last expression we have

$$\cos 3x = \cos^3 x - \sin^2 x \cos x - 2 \sin^2 x \cos x$$

$$= \cos^3 x - 3 \sin^2 x \cos x$$

This can be massaged

$$\cos 3x = \cos x(\cos^2 x - 3 \sin^2 x)$$

$$= \cos x(\cos^2 x - 3(1 - \cos^2 x))$$
$$= 4 \cos^3 x - 3 \cos x$$

standard formula

This agrees with the standard formula:

$$\cos 3x = \cos 2x + x$$

$$\begin{aligned}
&= \cos 2x \cos x - \sin 2x \sin x \\
&= (\cos^2 x - \sin^2 x) \cos x - 2 \cos x \sin x \sin x \\
&= \cos^3 x - \cos x \sin^2 x - 2 \cos x \sin^2 x \\
&= \cos^3 x - 3 \cos x \sin^2 x
\end{aligned}$$

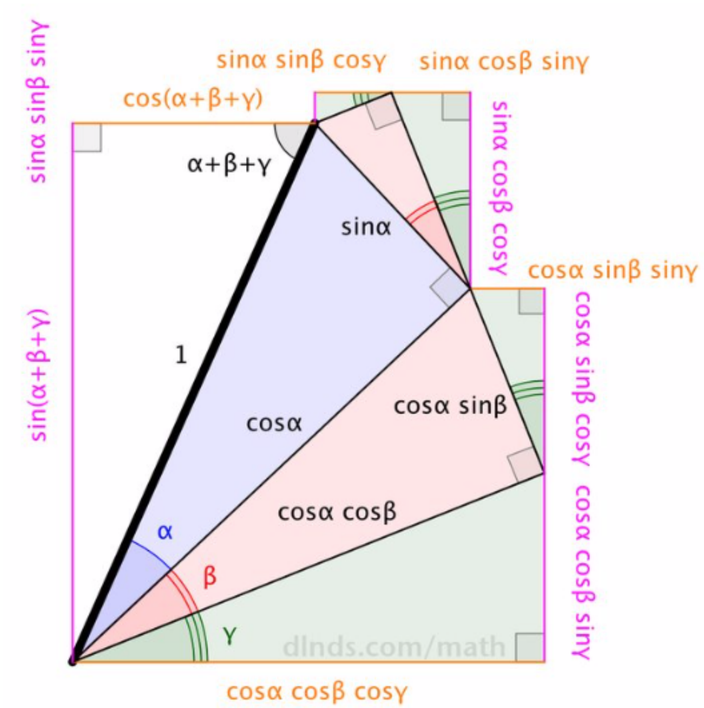
From here

https://math.stackexchange.com/questions/852122/picture-intuitive-proof-of-cos3-theta-4-cos3-theta-3-cos-theta

we get a nice geometric derivation.

17

Enhancing my diagram for the angle-sum formula (currently featured in Wikipedia) to use three angles will get you pretty close ...



Note: the formula is not valid for non-integer powers.

https://en.wikipedia.org/wiki/De_Moivre%27s_formula#Roots_of_complex_numbers

Chapter 48

e is irrational

e is irrational

I found a nice proof of the irrationality of e in the calculus text by Courant and Robbins. It is a proof by contradiction. We start by assuming that e is rational.

$$e = \frac{p}{q}, \quad p, q \in \mathbb{N}$$

We make use of the infinite series representation of e

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

From this, it is obvious that $e > 2$. If you're interested, there is a proof that $e < 3$ in the book.

Equating the series representation to the rational fraction p/q:

$$\frac{p}{q} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Multiply both sides by $q!$. For the left-hand side, we have

$$e\ q! = \frac{p}{q}\ q! = p(q-1)!$$

We won't need to do anything more with this, but note that since $e\ q!$ is equal to $p(q-1)!$, we can see that the left-hand side, $e\ q!$, is clearly an integer. Therefore, the right-hand side must also be an integer. This is the series

$$q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{(q-1)!} + \frac{q!}{q!} + \frac{q!}{(q+1)!} + \cdots$$

Now, $q!$ is obviously an integer. And for every integer $k < q$, $k!$ divides $q!$ evenly

$$\frac{q!}{k!} = q \times (q-1) \times (q-2) \cdots \times (q-k+1)$$

In our series

$$q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{(q-1)!} + \frac{q!}{q!} + \frac{q!}{(q+1)!} + \cdots$$

all the terms to the left of $q!/(q-1)!$ are integers, as is $q!/(q-1)! = q$ and $q!/q! = 1$.

So now our concern is with the fractions that follow. We will show that these sum up to something less than 1. We have

$$\frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots$$

Since $q >= 2$

$$\frac{1}{(q+1)} <= \frac{1}{3}$$

$$\frac{1}{(q+1)(q+2)} <= (\frac{1}{3})^2$$

and so on, and the entire remaining series of fractions is less than or equal to

$$\frac{1}{3} + (\frac{1}{3})^2 + (\frac{1}{3})^3 + \cdots$$

This is the geometric series with $r = 1/3$ and first term equal to r, and the sum is known to be

$$\frac{1}{3}(1/(1-\frac{1}{3})) = \frac{1}{2}$$

Since the right-hand side is equal to an integer plus something "less than or equal to $\frac{1}{2}$", it is not an integer, and cannot be equal to the left-hand side, which is equal to an integer. We have reached a contradiction. Therefore e cannot be equal to p/q, for $p, q \in \mathbb{N}$.

Part XIV

Serious integration

Chapter 49

Techniques of integration

This chapter surveys some useful techniques for integration. There are four approaches commonly given in introductory calculus:

- u substitution
- trigonometric substitution
- integration by parts (IBP)
- partial fractions

We'll take a quick look at all of these.

Here is an example for substitution.

$$\int \tan t \ dt$$

No doubt, when students first see this, they are mystified. Have I ever seen any $f(x)$ that gives the tangent as its derivative?

Insight comes from writing the two parts of the tangent:

$$\int \frac{\sin t}{\cos t} \ dt$$

We recognize that

$$\frac{d}{dt}\cos t = -\sin t$$

Let

$$u = \cos t$$

$$du = -\sin t \ dt$$

the integral is

$$-\int \frac{1}{u} \ du = -\ln u$$

$$= -\ln \cos t$$

Since the cosine can take on values where the logarithm is not defined (i.e. < 0) the answer is usually given as

$$= -\ln |\cos t|$$

Another classic one is the secant

$$\int \sec x \ dx$$

There is also a trick to this one, multiply top and bottom by $\sec x + \tan x$

$$\int \sec x \ \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

You see that $\sec^2 x$ is the derivative of $\tan x$ and $\sec x \tan x$ is the derivative of $\sec x$ so this is just

$$\int \frac{1}{u} \ du$$

again, namely

$$\int \sec x \ dx = \ln |\sec x + \tan x \ | + C$$

Now compare these two integrals

$$\int x \sqrt{1-x^2}\ dx$$

$$\int \sqrt{1-x^2}\ \ dx$$

The extra value of x makes a big difference in the first one. I look at the x and know we have the derivative of what is inside the square root. So let $u = 1-x^2$ and then $du = -2x\ dx$ and the integral becomes

$$\int (-\frac{1}{2})\ \sqrt{u}\ du = -\frac{1}{3}u^{3/2}$$

$$= -\frac{1}{3}(1-x^2)^{3/2}$$

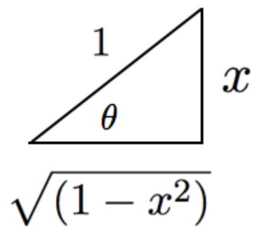
This integral comes up in the problem of finding the average value of x over the unit circle (or half-circle). Since it's an even function of x, the result is zero, if the bounds are centered around zero like as $[-b, b]$.

With practice you will not need to write the substitution. Just say, OK, I know I have the x. Now the integral of the square root is $(1-x^2)^{3/2}$. I need one factor of $2/3$ to neutralize the exponent, and another factor of $-1/2$ for $-x^2$ so I write the answer, and then check by differentiating.

For the second integral, we do not have the derivative of what's inside the square root.

$$\int \sqrt{1-x^2}\ \ dx$$

Nevertheless, this one can be solved using what is called a trig substitution. Consider this figure



We draw a generic right triangle. The figure labels that angle as θ (I often use t just because it's easier to type). Since we have $\sqrt{1-x^2}$, we know we will need x for the side opposite the angle and 1 for the hypotenuse. So let

$$x = \sin\theta$$

$$dx = \cos\theta \ d\theta$$

And from Pythagoras

$$\sqrt{1-x^2} = \cos\theta$$

Take a look at

$$\int \sqrt{1-x^2} \ dx$$

The square root is on the top. If it were on the bottom, the cosines would cancel (we will see that problem later). We get

$$\int \cos^2\theta \ d\theta$$

This is an integral that comes up a lot, and there are several ways to do it. We will solve this **soon**.

For now, the answer is:

$$\int \cos^2\theta \ d\theta = \frac{1}{2} \ [\ \theta + \sin\theta\cos\theta \ ]$$

This is easily verified by differentiating:

$$\cos^2\theta = \frac{1}{2}\,[\ 1 - \sin^2\theta + \cos^2\theta\ ]$$

$$= \frac{1}{2}\,[\ \cos^2\theta + \cos^2\theta\ ]$$

Although people always do this problem as described, it's worth pointing out that we could do a different trigonometric substitution. There is no reason not to write

$$x = \cos\theta$$

$$dx = -\sin\theta\ d\theta$$

$$\sqrt{1-x^2} = \sin\theta$$

and so

$$\int \sqrt{1-x^2}\ dx = -\int \sin^2\theta\ d\theta$$

Again, the answer is:

$$\int \sin^2\theta\ d\theta = \frac{1}{2}\,[\ \theta - \sin\theta\cos\theta\ ]$$

Of course this is true since

$$\sin^2\theta + \cos^2\theta = 1$$

$$\int \sin^2\theta\ d\theta + \int \cos^2\theta\ d\theta = \int 1\ d\theta = \theta$$

If you add the two answers given for $\sin^2$ and $\cos^2$ you'll get the same result.

change of variables

u-substitution and trigonometric substitution, or just substitution in general, has one subtle aspect, which is that if you change the variable, you must also change the bounds. Either that, or change back to the original variable at the end. For example, above we had

$$\int x \sqrt{1-x^2}\ dx$$

We let $u = 1 - x^2$ and then $du = -2x\ dx$ and the integral becomes

$$\int (-\frac{1}{2})\ \sqrt{u}\ du = -\frac{1}{3}u^{3/2}$$

Suppose the bounds on x were $[0,1]$

$$\int_0^1 x \sqrt{1-x^2}\ dx$$

The bounds on the integral in u must be adjusted:

$$x = 0, \qquad u = 1$$

$$x = 1, \qquad u = 0$$

so

$$-\frac{1}{3}u^{3/2}\ \Big|_1^0 = \frac{1}{3}u^{3/2}\ \Big|_0^1 = \frac{1}{3}$$

Alternatively, switch back to x

$$-\frac{1}{3}u^{3/2} = -\frac{1}{3}(1-x^2)^{3/2}\ \Big|_0^1$$

$$= -\frac{1}{3}(0-1) = \frac{1}{3}$$

Our second example above was

$$\int \sqrt{1-x^2}\ dx = \frac{1}{2}\ [\ \theta + \sin\theta\cos\theta\ ]$$

Suppose the bounds on x were $[0,1]$. The first substitution we tried was

$$x = \sin\theta$$

To find the bounds on θ, we must ask, what value of θ gives the correspond value of x? We obtain

$$x = \sin\theta = 0, \qquad \theta = 0$$

$$x = \sin\theta = 1, \qquad \theta = \pi/2$$

So

$$\frac{1}{2}\ [\ \theta + \sin\theta\cos\theta\ ]\ \Big|_0^{\pi/2}$$

The second term is 0 at both bounds, and the first is $\pi/2$, which is our answer.

Alternatively, we can switch the variable back to x

$$x = \sin\theta$$

$$\theta = \sin^{-1} x$$

so the answer is

$$\frac{1}{2}\ [\ \theta + \sin\theta\cos\theta\ ]\ \Big|_0^{\pi/2} = \frac{1}{2}\ [\ \sin^{-1} x + x\sqrt{1-x^2}\ ]\ \Big|_0^{1}$$

And again, the second term is zero at both bounds. The first term is just $\pi/2$, which is the same as we obtained before.

integration by parts (IBP)

Another approach to the integral uses integration by parts.

IBP is a reversal of the product rule. Consider two functions of x, $u(x)$ and $v(x)$. (Drop the (x) notation for the moment).

$$(uv)' = v\ du + u\ dv$$

The idea is that when we integrate this we will have

$$\int (uv)' = uv = \int v\ du + \int u\ dv$$

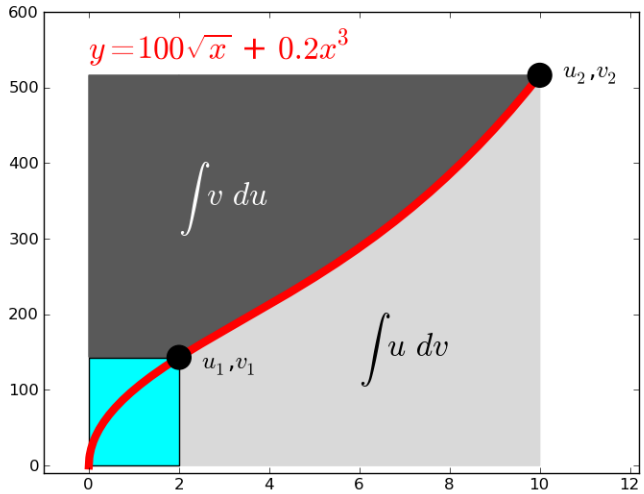
Rearranging, we obtain

$$\int u\ dv = uv - \int v\ du$$

Just to be clear, with an explicit independent variable x this is:

$$\int u \frac{dv}{dx}\ dx = uv - \int v \frac{du}{dx}\ dx$$

Here is a figure from Strang:



examples

As a first example, take

$$\int x\ e^x\ dx$$

If this were

$$\int x\ e^{x^2}\ dx$$

there would be no problem, because upon differentiating x^2 using the chain rule, we will get the x that we see in the example. For the first problem, write

$$u = x$$

$$du = dx$$

$$dv = e^x\ dx$$

this is what we have. Now let's see how this simplifies things

$$v = \int e^x\ dx = e^x$$

Using the formula

$$\int u \ dv = uv - \int v \ du$$

$$uv = xe^x$$

$$\int v \ du = \int e^x \ dx = e^x$$

So

$$\int xe^x \ dx = xe^x - e^x$$

Check

$$\frac{d}{dx} \ [\ xe^x - e^x \ ] = xe^x$$

It is clear that the extra term in the answer is there to eliminate an extra term in the derivative. This pattern is generally true with integration by parts.

The problem we had above, $\cos^2 x$, can be solved by this method. But let's wait and give it its own section.

exponential

Consider the question: what is the average value of x for the negative exponential function over the interval $[0, \infty)$?

Start with

$$\int_0^\infty \ e^{-kx} \ dx$$

$$= -\frac{1}{k} e^{-kx} \ \Big|_0^\infty$$

Here we are faced with the problem of evaluating a function at the point $x = \infty$ but *infinity is not a number*. The approach is to evaluate the expression at some very large bound b, and then ask what happens if $b \to \infty$. (See **here**).

At the upper bound we get zero, and at the lower bound we are subtracting $-1/k$ so

$$\int_0^\infty e^{-kx}\ dx = \frac{1}{k}$$

Now we want

$$\int_0^\infty x\ e^{-kx}\ dx$$

Let

$$u = x$$

$$du = dx$$

$$dv = e^{-kx}\ dx$$

$$v = -\frac{1}{k}e^{-kx}$$

So IBP says

$$\int u\ dv = uv - \int v\ du$$

$$\int x\ e^{-kx}\ dx = -\frac{x}{k}e^{-kx} + \int \frac{1}{k}\ e^{-kx}\ dx$$

$$= \ [\ -\frac{x}{k}e^{-kx} - \frac{1}{k^2}\ e^{-kx}\ ]\ \Big|_0^\infty$$

At the lower limit the first term is zero and the second $-1/k^2$ which we change to $1/k^2$ by subtraction. At the upper limit, the second term is a negative exponential so that goes to zero as $x \to \infty$.

The first term is more complicated.

$$\lim_{x\to\infty}\ xe^{-kx} = ?$$

Rewrite it as a ratio:

$$= \lim_{x \to \infty} \frac{x}{e^{kx}}$$

We get infinity for both top and bottom and so invoke **L'Hospital**:

$$= \lim_{x \to \infty} \frac{1}{ke^{kx}} = 0$$

The final answer here is

$$\int_0^\infty x\ e^{-kx}\ dx = \frac{1}{k^2}$$

In fact, we can solve this integral for any power x^n in the numerator. Just invoke L'Hospital n times. We will see this elsewhere (**here**).

By now, you should have a good handle on the integral as a measure of the area between the curve $f(x)$ and the x-axis.

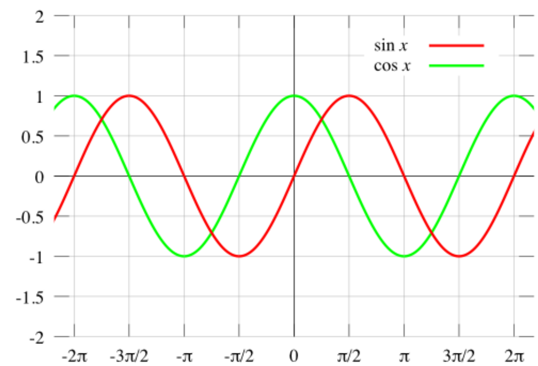
below the x-axis

One question we haven't dealt with is what happens when the curve dips below the x-axis. Consider the line $y = x - 1$ which crosses the x-axis at $x = 1$ and the y-axis at $y = -1$.

$$\int_0^1 x - 1\ dx = \frac{x^2}{2} - x\ \Big|_0^1$$

$$= \frac{1}{2} - 1 = -\frac{1}{2}$$

The absolute value of the area is correct but the sign is negative. When we integrate a function that dips below the x-axis, the result will be negative.



A further demonstration of this is the trigonometric functions $\sin x$ and $\cos x$:

They apparently spend as much of the time below the x-axis as above it. These functions repeat with a period of 2π. A consequence of this is that the integral of sine or cosine over a period of exactly 2π is always equal to zero, regardless of the exact starting point.

$$\int_0^{2\pi} \sin x \; dx = -\cos x \, \Big|_0^{2\pi} = -(-1) - 1 = 0$$

This is true for *any* bounds whose difference is 2π.

area between two curves

A common application of integrals is to give the area between two curves, which may be obtained simply by subtracting one function from another and integrating the difference.

$$A = \int f(x) - g(x) \; dx$$

an easy problem

Consider the two curves $y = \sqrt{x}$ and $y = x^2$. These curves cross at

$$\sqrt{x} = x^2$$

We can see by inspection that this happens at $x = 0$ and $x = 1$.

In this region $(0 \leq x \leq 1)$ the square root is larger than the square.

Furthermore, we already calculated that the part below $y = x^2$ is $1/3$ of the area of the rectangle with corners $(0, 0)$ and (x, y). The same is true for the area above the square root. From this we can predict what the result will be.

$$A = \int f(x) - g(x) \ dx$$

For this problem

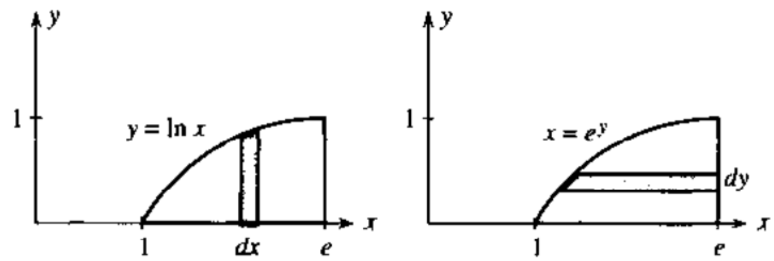
$$A = \int_0^1 \sqrt{x} - x^2 \ dx$$

$$= \frac{2}{3}x^{3/2} - \frac{x^3}{3} \ \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

a problem done two ways

Consider $y = \ln x$. This can be done two ways, as the figure shows. We can use x as the variable or y.



The standard approach would be x. We want

$$\int_1^e \ln x \; dx$$

We recall fooling around with the derivatives of products to pull this up:

$$= x \ln x - x \;\Big|_1^e$$

$$= \; [\; e - e \;] \; - \; [\; 0 - 1 \;] \; = 1$$

It's the other way that we subtract $f(x) - g(x)$:

$$= \int (e - e^y) \; dy$$

$$= ey - e^y \;\Big|_0^1$$

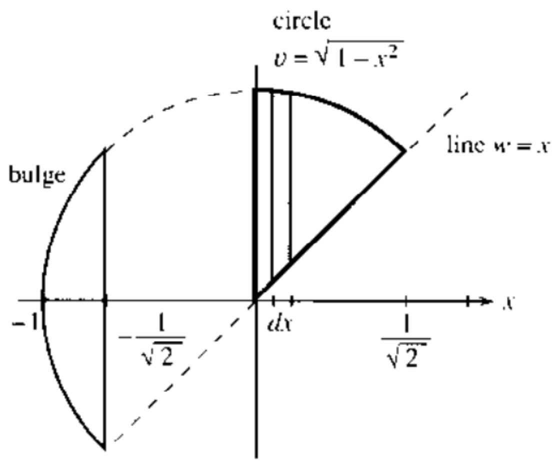
$$= \; [\; e - e^1 \;] - \; [\; 0 - 1 \;]$$

$$= 1$$

That looks very smooth but it's got a lot of calculus in it!

a more complicated problem

Consider the two curves $y = x$ and the unit circle $y = \sqrt{1 - x^2}$.



We see that the circle lies above the line for most of its length. However, it's complicated. Part of the figure is below the x-axis, and there is a bulge on the left end.

The key is that we must appreciate the relationship (know which is on top) in order to do integrals of differences between curves properly.

First solve for the points where the two curves cross:

$$y = x = \sqrt{1 - x^2}$$

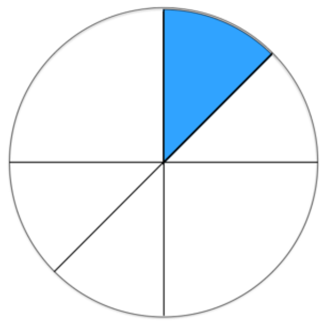
$$2x^2 = 1$$

$$x = \pm\sqrt{1/2} \; = \pm\frac{1}{\sqrt{2}}$$

$$y = \pm\frac{1}{\sqrt{2}}$$

The area of the sector in the first quadrant is pretty easy.

$$A = \int_0^{1/\sqrt{2}} \sqrt{1 - x^2} - x \; dx$$



We referred to the solution of $\int \sqrt{1-x^2}\ dx$ previously in this chapter, and we will actually finally solve it in the next. For now, just assume the result:

$$= \frac{1}{2}\ [\ \sin^{-1} x + x\sqrt{1-x^2}\ ]\ \ - \frac{x^2}{2}\ \Big|_0^{1/\sqrt{2}}$$

At the lower bound everything including $\sin^{-1} x$ is zero, and at the upper bound we have

$$= \frac{1}{2}\ [\ \frac{\pi}{4} + \frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\ ] - \frac{1}{4} = \frac{\pi}{8}$$

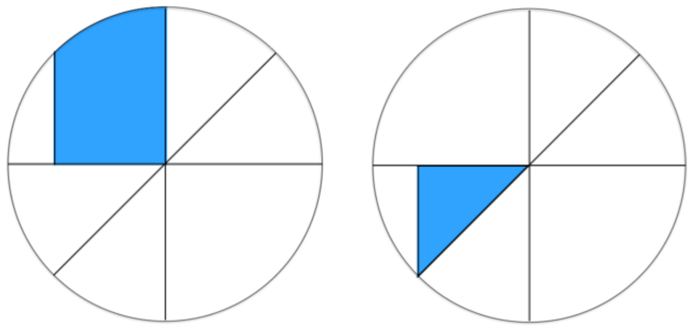
which we confirm from elementary geometry is just a slice of the pie.

For the area to the left of the y-axis we must think a little more. The line $y = x$ dips below the x-axis, so the integral of the second part $g(x)$

$$\int_{-1/\sqrt{2}}^{0} x\ dx$$

is negative.

It's probably less confusing to just do the parts above and below the x-axis separately!



The area above the x-axis is

$$A = \frac{1}{2}\ [\ \sin^{-1} x + x\sqrt{1-x^2}\ ]\ \Big|_{-1/\sqrt{2}}^{0}$$

At the upper bound, we have again zero, and at the lower bound

$$\frac{1}{2}\ [\ -\frac{\pi}{4} + (-\frac{1}{\sqrt{2}})\ \frac{1}{\sqrt{2}}\ ]$$

which must be subtracted, so we change signs and obtain

$$\frac{\pi}{8} + \frac{1}{4}$$

This corresponds to a slice of the pie plus the triangle beneath it, which has two sides of length $1/\sqrt{2}$ and area $1/4$.

For the area below the x-axis:

$$\int_{-1/\sqrt{2}}^{0} x\ dx$$

$$= \frac{x^2}{2}\ \Big|_{-1/\sqrt{2}}^{0} = -\frac{1}{4}$$

The area below the x-axis is *minus* the result of the integral (the integral yields a negative area).

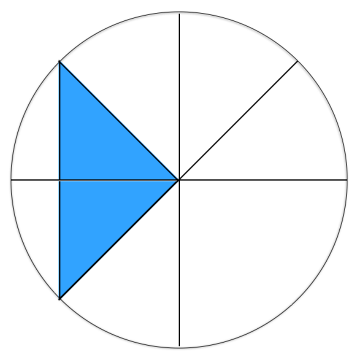
So lose the minus sign

$$A = \frac{1}{4}$$

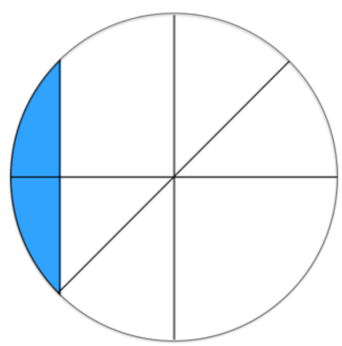
The total for this region (between $-1/\sqrt{2}$ and 0) is then just

$$\frac{\pi}{8} + \frac{1}{4} + \frac{1}{4} = \frac{\pi}{8} + \frac{1}{2}$$

This is effectively a slice of pie plus a triangle of base $2/\sqrt{2}$ and height $1/\sqrt{2}$.



The third part is the bulge to the left of $x = -1/\sqrt{2}$.



We will calculate the area above the x-axis and multiply by two:

$$A = 2\int_{-1}^{-1/\sqrt{2}} \sqrt{1 - x^2}\ dx$$

$$= 2\ \frac{1}{2}\ [\ \sin^{-1} x + x\sqrt{1-x^2}\ ]\ \Big|_{-1}^{-1/\sqrt{2}}$$

The leading factors cancel.

$$= \sin^{-1} x + x\sqrt{1-x^2}\ ]\ \Big|_{-1}^{-1/\sqrt{2}}$$

At the upper bound we have (as we found before)

$$= -\frac{\pi}{4} - \frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}} = -\frac{\pi}{4} - \frac{1}{2}$$

At the lower bound, $\sin^{-1} x = -\pi/2$ and $x\sqrt{1-x^2}$ is zero.

We're subtracting, so we have

$$= -\frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{2}$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

Adding all the pieces together

$$A = \frac{\pi}{8} + \frac{\pi}{8} + \frac{1}{2} + \frac{\pi}{4} - \frac{1}{2} = \frac{\pi}{2}$$

which is obviously correct for one-half the unit circle.

Chapter 50

Inverse sine

Inverse sine

This is our first of the inverse trigonometric functions (there are only two really important ones).

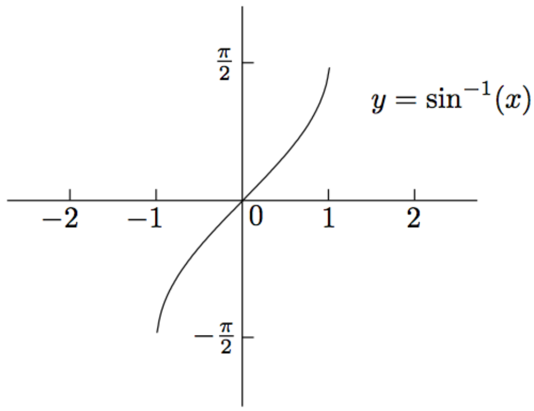
$$y = \sin^{-1} x$$

Read this as y is the arc sine or inverse sine of x.

More usefully, we can say that y is the angle whose sine is x. This means the same thing, we have just solved the first equation for x:

$$x = \sin y$$

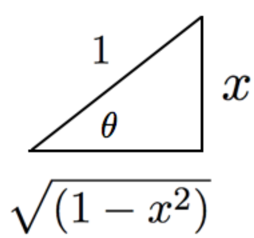
Plot the angle as a function of its sine.



The graph is the same shape as the standard sine curve, but flipped and rotated 90 degrees.

We need to be careful about the interval on which we're working, since if we go too far we will duplicate values and thus, no longer have a *function*. You can see from the plot that the range of y should be $[-\pi/2, \pi/2]$.

We can derive something useful from this with a trig substitution:



Basic trigonometry says that if

$$x = \sin\theta$$

$$\theta = \sin^{-1} x$$

then

$$\cos\theta = \sqrt{1-x^2}$$

so differentiating

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\sin\theta = \cos\theta = \sqrt{1-x^2}$$

Inverting

$$\frac{d\theta}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Furthermore, integrating:

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \int d\theta = \theta$$

$$= \sin^{-1} x$$

and

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

This integral arises in many problems.

Let's switch to using t for θ. Another derivation is to say

$$x = \sin t$$

$$dx = \cos t\ dt$$

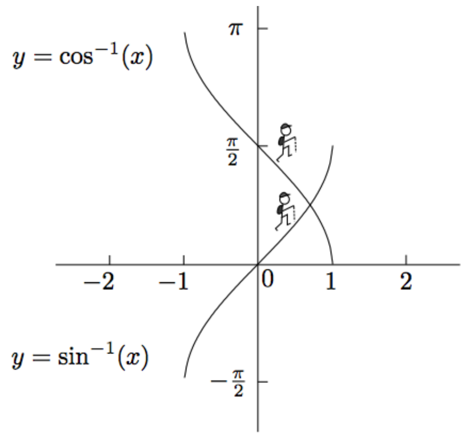
so the integral is

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \int \frac{1}{\cos t}\cos t\ dt = t$$

The complementary angles of a right triangle add up to $\pi/2$, thus

$$\sin^{-1} t + \cos^{-1} t = \pi/2$$

Here's a picture of the two functions together. It is not hard to believe they add up to a constant.



We find the inverse cosine easily by differentiating:

$$\sin^{-1} t + \cos^{-1} t = \pi/2$$

$$\frac{d}{dt} \sin^{-1} t + \frac{d}{dt} \cos^{-1} t = 0$$

$$\frac{d}{dt} \sin^{-1} t = -\frac{d}{dt} \cos^{-1} t$$

where we had

$$\frac{d}{dt} \sin^{-1} t = \frac{1}{\sqrt{1 - x^2}}$$

The inverse cosine isn't seen that much because its the very same problem as the inverse sine.

inverse tangent

However, the inverse tangent is important. Let

$$x = \tan t$$

$$t = \tan^{-1} x$$

Differentiating the first equation

$$\frac{dx}{dt} = \sec^2 t$$

If you haven't seen this before, try using the quotient rule on $\sin t / \cos t$. Inverting

$$\frac{dt}{dx} = \cos^2 t$$

Basic trigonometry will show that if

$$x = \tan t$$

then the hypotenuse must be $\sqrt{1 + x^2}$ so

$$\cos t = \frac{1}{\sqrt{1 + x^2}}$$

and then

$$\frac{dt}{dx} = \cos^2 t = \frac{1}{1 + x^2}$$

Integrate

$$\int dt = \int \frac{1}{1 + x^2}\ dx$$

$$t = \int \frac{1}{1 + x^2}\ dx$$

But $t = \tan^{-1} x$ so

$$\tan^{-1} x = \int \frac{1}{1 + x^2}\ dx$$

We will show elsewhere that

$$(1 + x^2) \cdot (1 - x^2 + x^4 - x^6 + \dots) = 1$$

You can check by multiplying it out.

Rearrange and integrate

$$\int \frac{1}{1+x^2}\ dx = \int 1 - x^2 + x^4 - x^6 + \dots\ dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Combining with what's above

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

The angle whose tangent is 1 is $\pi/4$ so

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

This is the first of a number of series that yield π.

Chapter 51

Cosine squared

This is where we solve some integrals that we put off earlier. We already laid the groundwork by talking about trig substitutions (**here**) and about the inverse sine function, a related topic (**above**).

Let's begin with the integral of $\sqrt{1-x^2}$.

We saw this in the problem of the area of the circle (**here**):

$$\int \sqrt{1-x^2}\ dx$$

We also saw it in the denominator when talking about the inverse sine

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x$$

To solve the first one, we use a trig substitution to give:

$$x = \sin t$$

$$dx = \cos t\ dt$$
$$\sqrt{1-x^2} = \cos t$$

So

$$\int \sqrt{1-x^2}\ dx = \int \cos^2 t\ dt$$

We will finally solve the cosine squared in a second, but just note that the more general version of this problem has a constant, let's call it a:

$$\int \sqrt{a^2-x^2}\ dx$$

To deal with that, factor out an a

$$= a\sqrt{1-(x/a)^2}\ dx$$

Then substitute

$$au = x$$

so

$$a\ du = dx$$

and we obtain

$$= a\sqrt{1-u^2}\ a\ du$$
$$= a^2\sqrt{1-u^2}\ du$$

An equivalent solution is to do it during the trig substitution.

Rather than use 1 for the hypotenuse, use the constant. For the circle we had

$$\int \sqrt{R^2-x^2}\ dx$$

Let

$$x = R\sin t$$

$$dx = R\cos t\ dt$$

$$\sqrt{R^2-x^2} = R\cos t$$

So the integral is

$$R^2 \int \cos^2 t \ dt$$

And if you look back at that problem, I promised that we would come up with a factor of R^2 to make the area come out right. Well there it is. We still need something involving π.

It turns out the answer is

$$\int \cos^2 t \ dt = \frac{1}{2} \ [\ t + \sin t \cos t \ ]$$

We can integrate from $x = [-R, R]$. In that case $t = [-\pi/2, \pi/2]$. Or if we use 0 for the lower bound on x (remembering that we'll need a factor of 2 at the end), we will have $t = 0$ at the lower bound as well. No matter, either way the second term with $\sin t \cos t$ will be zero.

In the first case we'll get $\pi/2$, and in the second case we get $\pi/4$ and then multiply by 2 to get the same thing. Since we only did the top half of the circle we pick up another factor of 2 for that.

Cosine squared

Enough fooling around. We want to find the integral of

$$\int \cos^2 x \ dx$$

which as we've seen is very common in problems using trig substitution and otherwise. The first thing to note is that

$$\int \sin^2 x \ dx$$

is the same problem, because

$$\sin^2 x + \cos^2 x = 1$$

so

$$\int \sin^2 x \; dx + \int \cos^2 x \; dx = \int 1 \; dx = x$$

method 0

I call this method 0 because it's not really methodical, we just guess. If you play around differentiating products of functions (like e^x, $\ln x$, $\sin x$, $\cos x$ and x), you will soon discover that

$$\frac{d}{dx} \; [\; \sin x \cos x \;] \; = \cos^2 x - \sin^2 x$$

which can be manipulated (using $\sin^2 x + \cos^2 x = 1$) to give either

$$\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$$

or

$$\cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

Integrating both sides of the $\cos^2$ form we obtain

$$\sin x \cos x = 2 \int \cos^2 x \; dx - x$$

and rearranging:

$$\int \cos^2 x \; dx = \frac{1}{2}(x + \sin x \cos x)$$

method 1

There are two other systematic approaches that can be contrasted. The first, which is arguably the simpler one, is to remember the addition formula for cosine

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

As mentioned earlier, the trick I use to remember these formulas is to work out the consequences for this one:

$$\cos(s-t) = \cos s \cos t + \sin s \sin t$$

This makes perfect sense since if $s = t$ then we get

$$\cos 0 = \cos^2 s + \sin^2 s = 1$$

which we know is correct. So

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$

If $s = t$ then (changing to x)

$$\cos 2x = \cos^2 x - \sin^2 x$$

which we saw above is equal to

$$= 2\cos^2 x - 1$$

The "double angle" formula is then

$$2\cos^2 x = 1 + \cos 2x$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Integrating

$$\int \cos^2 x \ dx = \int \frac{1}{2}(1 + \cos 2x) \ dx$$

$$\frac{1}{2}(x + \frac{1}{2}\sin 2x)$$

We check by differentiating. Leaving the factor of $1/2$ out, we obtain for d/dx:

$$1 + \cos 2x$$

which, as we saw above, is equal to $2\cos^2 x$. Remembering the factor of $1/2$, we obtain the expected result.

Comparing our results so far, we have obtained two different answers, namely

$$\int \cos^2 x \; dx = \frac{1}{2}(x + \sin x \cos x)$$

$$\int \cos^2 x \; dx = \frac{1}{2}(x + \frac{1}{2}\sin 2x)$$

which indicates (if there is no mistake), that

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

to see that this is correct, recall the addition formula for sine:

$$\sin(s + t) = \sin s \cos t + \sin t \cos s$$

then if $s = t$

$$\sin 2s = 2 \sin s \cos s$$

with a slight rearrangement, this is indeed what we had.

method 2

In the second method, we do a substitution to take advantage of the integration by parts formula

$$\int u \; dv = uv - \int v \; du$$

Let $u = \cos x$, so $du = -\sin x \; dx$, and let $dv = \cos x \; dx$ so $v = \sin x$, so

$$\int \cos^2 x \; dx = \sin x \cos x + \int \sin^2 x \; dx$$

This still seems like not much progress since (as we saw) $\int \sin^2 x \ dx$ is really the same problem as $\int \cos^2 x \ dx$

$$\int \sin^2 x \ dx = \int (1 - \cos^2 x) dx = \int dx - \int \cos^2 x dx$$

but, forging ahead, we combine the two results

$$\int \cos^2 x \ dx = \sin x \cos x + x - \int \cos^2 x dx$$

Rearranging:

$$\int \cos^2 x \ dx = \frac{1}{2} \ [\ \sin x \cos x + x \ ]$$

which is what we had before.

Integration by parts where the result is a related integral can be applied to the general case of $\int \cos^n x \ dx$ with even n, as well as many interesting and more advanced problems. It's worth remembering that it is in our toolbox.

Geometric significance

We ran into the integral of cosine squared in looking at the area of the circle. We'll use a unit circle to make it a bit simpler.

The integral we obtained was

$$\int \cos^2 \theta \ d\theta = \frac{1}{2} \ [\ \theta + \sin \theta \cos \theta \ ]$$

which can be evaluated in two ways. We can figure out the bounds on the substituted variable θ. Since $x = \sin \theta$:

$$x = 0 \Rightarrow \theta = 0$$

$$x = 1 \Rightarrow \theta = \frac{\pi}{2}$$

thus

$$\frac{1}{2} \; [\; \theta + \sin\theta \cos\theta \;] \; \Big|_0^{\pi/2}$$

and what's nice about this is the second term is zero at both the upper and lower bound, so we end up with $\pi/4$, which is correct for the quarter circle.

Or we can switch back to x. We had

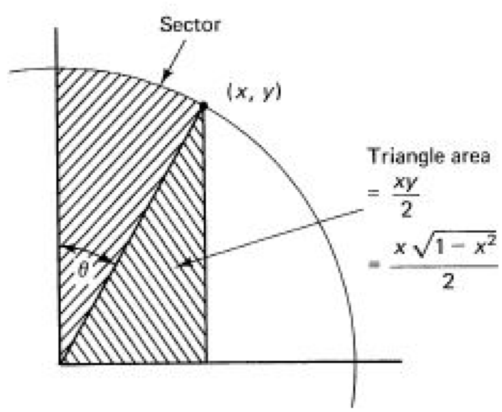
$$\frac{1}{2} \; [\; \theta + \sin\theta \cos\theta \;]$$

Since $x = \sin\theta$, $\theta = \sin^{-1} x$ (the arc sine of x) so

$$\frac{1}{2} \; [\; \sin^{-1} x + x\sqrt{1 - x^2} \;]$$

With this approach, we can place the upper bound for this anywhere in $[0, 1]$ without any trouble.

And here is where the geometry is nice. Notice that the area under the curve can be constructed in two parts by drawing the radius to the point (x, y).



There is a sector of the circle and a triangle. These correspond to the two terms of

$$\frac{1}{2}\ [\ \sin^{-1} x + x\sqrt{1-x^2}\ ]$$

I find that very illuminating.

alternative parametrization

By the way, there is another approach which gets at the close relationship between $\sin^2\theta$ and $\cos^2\theta$.

For a parametrized unit circle:

$$x = \cos\theta$$

$$dx = -\sin\theta\ d\theta$$

$$y = \sin\theta$$

so

$$\int y\ dx = \int -\sin^2\theta\ d\theta$$

using the formula relating these that we had above

$$= \int \cos^2\theta\ d\theta - \int d\theta$$

I see this and I think, uh oh, we have an extra term here. How will we possibly obtain the same result as before? But it works out!

$$= \frac{1}{2}\ [\ \theta + \sin\theta\cos\theta\ ]\ - \theta$$

$$= \frac{1}{2}\ [\ \sin\theta\cos\theta - \theta\ ]$$

There is another difference. For the trig substitution we had $x = \sin\theta$, but here we have $x = \cos\theta$. That means

$$x = 0 \Rightarrow \theta = \frac{\pi}{2}$$

$$x = 1 \Rightarrow \theta = 0$$

so when we evaluate the result at these bounds we get

$$= \frac{1}{2} \; [\; \sin\theta\cos\theta - \theta \;] \; \Big|_{\pi/2}^{0}$$

switching bounds means multiplying the expression by -1

$$= \frac{1}{2} \; [\theta - \sin\theta\cos\theta \;] \; \Big|_{0}^{\pi/2}$$

We have switched the sign on the second term $(\sin\theta\cos\theta)$ — compare with the previous answer. But it doesn't matter because it is zero at both of the extreme bounds on the interval. We end up with $\pi/4$, as before.

In 16.4 Hamming turns this argument around. He starts with the picture of the area and works backward to show that $\cos\theta$ is the derivative of $\sin\theta$ without using the limit of $\sin\theta/\theta$, which we worked out **here**.

Chapter 52

Improper integrals

Generally speaking, an integral is improper when one of three conditions holds: (i) the upper bound is $\lim x \to +\infty$, (ii) the lower bound is $\lim x \to -\infty$, or (iii) at one of the bounds, the value of the function is undefined $\to \pm\infty$.

If the function's value becomes undefined in the middle of an interval, first break the integral into pieces.

Then, integrate the function anyway, and if, when we evaluate it at that problematic bound, the result is finite (often 0), then we can use the result.

$$\int_1^\infty \frac{1}{x^2}\ dx$$
$$= -\frac{1}{x}\ \Big|_1^\infty$$

The upper bound is ∞, but the value of the there is zero. So

$$= -(-\frac{1}{1}) = 1$$

On the other hand, the same integral with bounds $[0, 1]$ blows up at the lower bound. That area is infinite.

Here is a second example. Compare:

$$\int_0^1 \frac{1}{x}\ dx$$

$$\int_0^1 \frac{1}{\sqrt{x}}\ dx$$

For both, the value of $f(x)$ becomes infinitely large (the limit does not exist) as $x \to 0$. Nevertheless, the area under the second curve is finite, while that under the first is not.

Informally, the way we roll here is to substitute another bound, like a, which is very small but not zero:

$$\int_a^1 \frac{1}{x}\ dx$$

$$= \ln x \Big|_a^1$$

and now we ask, what happens if we plug in 0 for a? The value of the integral "blows up" at the lower bound. $\ln 0$ doesn't exist and the logarithm of a very small number approaches $-\infty$. So this integral can't be evaluated.

If we think of it as a Riemann sum with rectangles of width 1, it is like the harmonic series (with a lower bound of 1)

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

We know this diverges.

On the other hand:

$$\int_a^1 \frac{1}{\sqrt{x}}\ dx$$

$$= 2\sqrt{x}\ \Big|_a^1$$

Now, when we evaluate at the lower bound, with 0 for a, we get 0. Therefore, the value of the integral is just:

$$= 2\sqrt{1} - 0 = 2$$

negative exponential

Here is the negative exponential function:

$$\int_0^\infty e^{-x}\ dx$$

$$= -e^{-x}\ \Big|_0^\infty$$

$$= -(0 - 1) = 1$$

And a variation:

$$\int_0^\infty 2\pi e^{-r^2}\ r\ dr$$

Letting $t = r^2$, this is what we just did:

$$\pi \int_0^\infty -e^{-t}\ dt = \pi$$

The negative exponential often appears with a constant factor, traditionally denoted by λ:

$$\int_0^\infty e^{-\lambda x}\ dx$$

$$= -\frac{1}{\lambda}\ e^{-\lambda x}\ \Big|_0^\infty$$

$$= -\frac{1}{\lambda}\ (0 - 1) = \frac{1}{\lambda}$$

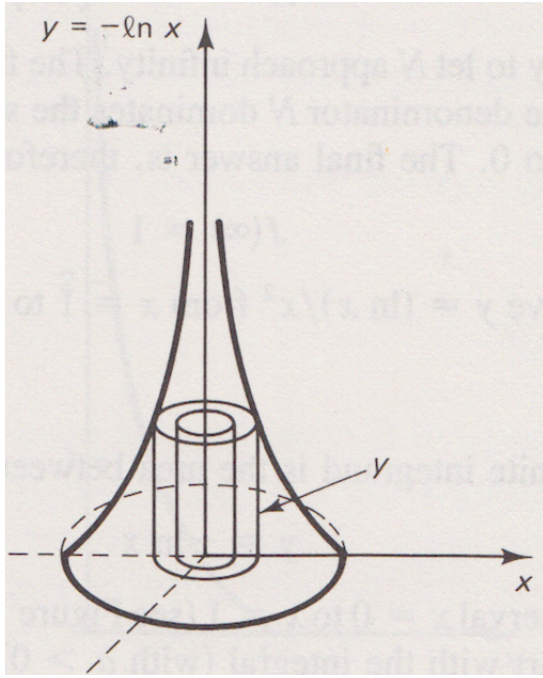
This form of the negative exponential $e^{-\lambda x}$ is a valid (and famous) probability density function if the total value of the integral is equal to 1. We achieve this by "normalizing" it, multiplying by λ:

$$p(x) = \lambda \int e^{-\lambda x}\ dx$$

We'll see all of these again.

log 1/x

The inverse log function provides a fun example of an improper integral with a very simple and finite result. The example (and the figure) are from Hamming.



We consider the function

$$y = -\ln x = \ln \frac{1}{x}$$

The minus sign is used so we are working with positive y, recognizing that the function is not defined at $x = 0$. Rotate the curve around the y axis and find the volume.

To use the method of cylinders, we consider a series of concentric cylindrical surfaces of width dx, ranging from $x = 0$ to $x = 1$. For each value of x, the surface area is the height of the function $h = -\ln x$ times the circumference $2\pi x$ to give a volume for each element of

$$\begin{aligned} dV &= 2\pi x \; y \; dx \\ &= 2\pi x \; (-\ln x) \; dx \end{aligned}$$

We get the total volume by integrating over the interval $[0, 1]$

$$V = \int_0^1 2\pi x \ (-\ln x) \ dx$$
$$= -2\pi \int_0^1 x \ln x \ dx$$

Store the factor of $-\pi$ for second; we will need the 2 sooner. What is

$$\int 2x \ln x \ dx$$

The systematic approach is to use integration by parts, but let's just guess. If we had $F(x) = x^2 \ln x$ then part of the derivative $f(x) = F'(x)$ would be what we want:

$$\frac{d}{dx} x^2 \ln x = 2x \ln x + \frac{x^2}{x}$$
$$= 2x \ln x + x$$

to cancel the extra x, we need another term, namely $-x^2/2$

$$F(x) = x^2 \ln x - \frac{x^2}{2}$$
$$F'(x) = 2x \ln x + x - x$$
$$= 2x \ln x$$

So

$$\int 2x \ln x \ dx = x^2 \ln x - \frac{x^2}{2}$$

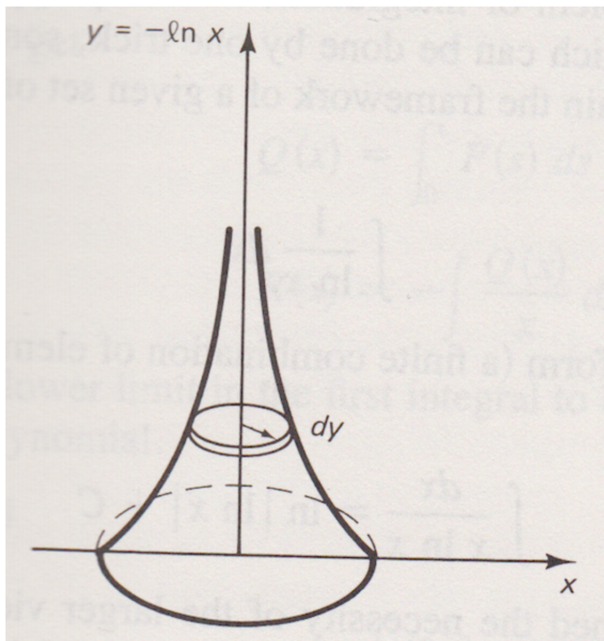
The volume V is equal to $-\pi$ times

$$x^2 \ln x - \frac{x^2}{2} \Big|_0^1$$

At the upper limit, $\ln 1 = 0$ so this is $-1/2$, and the question becomes, what happens to $x^2 \ln x$ as x approaches 0? We guess that x^2 must become small faster than $\ln x$ approaches $-\infty$. So at the lower limit, we will get zero, and the whole thing is

$$V = -\pi \ (-\frac{1}{2}) = \frac{\pi}{2}$$

Let's try by the method of disks and then come back to this limit. We also have a figure for the second approach



Again

$$f(x) = -\ln x$$

As y ranges from 0 to ∞, each disk has a width dy and an area equal to πx^2 so the volume element is

$$dV = \pi x^2 \ dy$$

The next part is really interesting. We follow Hamming. Rather than flip the figure, we can change variables and write

$$y = -\ln x$$

$$dy = -\frac{1}{x}\ dx$$

so the volume element becomes

$$= -\pi x^2\ \frac{1}{x}\ dx$$

$$= -\pi x\ dx$$

The limits also change. Before we had $y = 0 \to \infty$ and now we have $x = 1 \to 0$ (because $y = -\ln x$), so

$$V = \int_1^0 -\pi x\ dx$$

$$= \pi \int_0^1 x\ dx$$

$$= \frac{\pi}{2} x^2 \Big|_0^1 = \frac{\pi}{2}$$

So it looks like we were right about that limit.

But what is the formal method for evaluating

$$\lim_{x \to 0} x^2 \ln x$$

We convert this to a fraction

$$\lim_{x \to 0} \frac{\ln x}{1/x^2}$$

Since both numerator and the denominator go to ∞ as $x \to 0$, this is an indeterminate form, and we can use L'Hospital's rule (**here**).

We need derivatives of the numerator and the denominator. The numerator gives $1/x$ and the denominator gives $-2/x^3$ so we have

$$\frac{1}{x}\ \frac{1}{-2/x^{-3}} = \frac{x^3}{x}(-\frac{1}{2}) = -\frac{x^2}{2}$$

which in the limit as $x \to 0$, also goes to 0.

Chapter 53

Partial fractions

Strang gives this example:

$$\int \frac{1}{x-2} + \frac{3}{x+2} - \frac{4}{x}\ dx$$
$$= \ln|x-2| + 3\ln|x+2| - 4\ln|x|$$

That seems straightforward enough. Which function would produce that sum?

$$\frac{1}{x-2} + \frac{3}{x+2} - \frac{4}{x}$$
$$= \frac{(x+2)(x) + 3(x-2)(x) - 4(x-2)(x+2)}{(x-2)(x+2)(x)}$$
$$= \frac{x^2 + 2x + 3x^2 - 6x - 4x^2 + 16}{x^3 - 4x}$$
$$= \frac{-4x + 16}{x^3 - 4x}$$

We call this form P/Q, and it's the type of problem we are trying to solve using *partial fractions*. We start by factoring Q (although sometimes, the factors are given).

$$Q = x^3 - 4x = x(x^2 - 4)$$

$$= x(x-2)(x+2)$$

Let's try with a different numerator to see how it works. We write:

$$\frac{P}{Q} = \frac{3x^2 + 8x - 4}{(x-2)(x+2)(x)}$$

$$= \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x}$$

where A,B and C are constants.

We recognize that we can put these three fractions over Q as the common denominator.

These are the partial fractions that add up to P/Q. We need to find the values of A,B and C.

Here are two methods (the first one is slower):

Do what we just said. Put the right-hand side over the common denominator Q:

$$\frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x}$$

$$= \frac{A(x+2)(x) + B(x-2)(x) + C(x-2)(x+2)}{(x-2)(x+2)(x)}$$

Now make the numerators match:

$$3x^2 + 8x - 4 = A(x+2)(x) + B(x-2)(x) + C(x-2)(x+2)$$

$$= Ax^2 + 2Ax + Bx^2 - 2Bx + Cx^2 - 4C$$

We actually have three equations:

$$Ax^2 + Bx^2 + Cx^2 = 3x^2$$

$$2Ax - 2Bx = 8x$$

$$-4C = -4$$

From the last one $C = 1$. From the first one we have:

$$A + B + C = 3$$

$$A + B = 2$$

and then

$$A - B = 4$$

Add them together to get $2A = 6, soA = 3$ and then $B = -1$. We obtain finally

$$\frac{P}{Q} = \frac{3}{x-2} + \frac{-1}{x+2} + \frac{1}{x}$$

second method

The second approach is called the "cover-up method." We have:

$$\frac{3x^2 + 8x - 4}{(x-2)(x+2)(x)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x}$$

Multiply by $(x-2)$

$$\frac{3x^2 + 8x - 4}{(x+2)(x)} = (\frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x})(x-2)$$

$$= A + \frac{B(x-2)}{x+2} + \frac{C(x-2)}{x}$$

Now evaluate at $x = 2$

$$\frac{3(2)^2 + 8(x) - 4}{(2+2)(2)} = \frac{12 + 16 - 4}{8} = 3 = A$$

Notice that we do not need to actually write

$$A + \frac{B(x-2)}{x+2} + \frac{C(x-2)}{x}$$

Nor, in calculating B, do we need to write

$$\frac{A(x+2)}{x-2} + B + \frac{C(x+2)}{x}$$

since we will pick x to zero out those terms, instead, just substitute $x = -2$ into

$$\frac{3x^2+8x-4}{(x-2)(x)}$$
$$= \frac{3(-2)^2+8(-2)-4}{(-2-2)(-2)}$$
$$B = \frac{12-16-4}{8} = \frac{-8}{8} = -1$$

For C multiply the left-hand side by x and evaluate at $x = 0$ (to make the A and B terms go away):

$$\frac{3x^2+8x-4}{(x-2)(x+2)} = \frac{-4}{-4} = 1 = C$$

same degree

How about

$$\int \frac{3x^2+2x+7}{x^2+1}\,dx$$

To use the method, P must be of a lower degree than Q, but here they both contain multiples of x^2 (degree two). We separate off the term of $3x^2$ by finding another 3:

$$\frac{3x^2 + 2x + 7}{x^2 + 1} = \frac{3x^2 + 3 + 2x + 4}{x^2 + 1}$$

$$= 3 + \frac{2x + 4}{x^2 + 1}$$

Now we just have to solve:

$$\int 3 + \frac{2x}{x^2 + 1} + \frac{4}{x^2 + 1}\ dx$$

$$= 3x + \ln(x^2 + 1) + 4\tan^{-1} x + C$$

repeated factor

$$\frac{2x + 3}{(x - 1)^2}$$

We have two factors of $x - 1$. Solution: use $(x - 1)^2$ for one of the fractions:

$$\frac{2x + 3}{(x - 1)^2} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2}$$

$$2x + 3 = A(x - 1) + B$$

set $x = 1$, then

$$B = 2(1) + 3 = 5$$

and

$$2x + 3 = Ax - A + 5$$

$A = 2$ solves this.

more examples

These few examples are from wikipedia. We would like to simplify

$$\frac{3x+5}{(1-2x)^2}$$

We suppose that this fraction can be decomposed as follows

$$\frac{3x+5}{(1-2x)^2} = \frac{A}{(1-2x)^2} + \frac{B}{(1-2x)}$$

We multiply by the term with B to put everything over a common denominator:

$$\frac{A}{(1-2x)^2} + \frac{B}{(1-2x)}$$
$$= \frac{A}{(1-2x)^2} + \frac{B(1-2x)}{(1-2x)^2}$$

Getting rid of the denominators altogether

$$3x+5 = A + B(1-2x)$$

Now both the constant terms and the terms in x must be equal:

$$-2Bx = 3x$$
$$B = -\frac{3}{2}$$
$$A + B = 5$$
$$A = \frac{13}{2}$$

And so

$$\frac{3x+5}{(1-2x)^2} = \frac{13/2}{(1-2x)^2} + \frac{-3/2}{(1-2x)}$$

To integrate, we would do this

$$\int \frac{3x+5}{(1-2x)^2}\, dx = \int \frac{13/2}{(1-2x)^2}\, dx + \int \frac{-3/2}{(1-2x)}\, dx$$

$$= \frac{13/4}{(1-2x)} + (3/4)\ln(1-2x)$$

Example 2.

$$f(x) = \frac{1}{x^2+2x-3} = \frac{1}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

$$A(x-1) + B(x+3) = 1$$

$$Ax + Bx = 0$$

$$A = -B$$

$$-B + 3B = 1$$

$$B = \frac{1}{4}$$

$$f(x) = \frac{1}{4}(\frac{-1}{x+3} + \frac{1}{x-1})$$

Chapter 54

Revolution in the air

Surface area

The next topic really is an exciting step forward in calculus. We start looking at surface area using geometric arguments as well as results from calculus of one variable.

I will use S for the surface area. Sometimes for brevity I might write area instead of surface area.

Suppose a function $y = f(x)$ is revolved around the x-axis. Imagine slicing it into disks in the usual way, moving along the x-axis in increments dx.

Now, rather than compute the volume, we want the surface area of the solid. We might try adding up the perimeter of all the disks.

Suppose we start with the simple cone with $R = H$. The cone opens out to the right, with the vertex at the origin.

What we have is the function

$$y = x$$

The circumference at any point x is

$$2\pi y = 2\pi x$$

And the surface area is

$$A = \int x \ dx$$

(this has a subtle error that we will fix).

$$= \int_0^H 2\pi x \ dx$$

$$= \pi x^2 \Big|_0^H$$

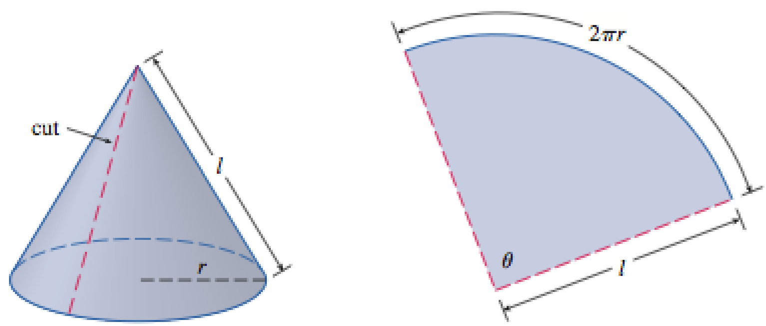
$$= \pi x^2 = \pi H^2$$

and since $H = R$

$$= \pi R H$$

Now, this is obviously not the correct answer.

For the geometry, imagine cutting the surface of a cone directly along the slant and then opening the surface and laying it flat out flat. We end up with a part (a sector) of a circle.



The radius of that circle is the slant height of the cone. The slant is labeled l in the figure (not mine) and the radius is r. Let's use L and R (capital letters for constants) in what follows:

$$L = \sqrt{R^2 + H^2}$$

The total circumference of the circle flat in the plane would be $2\pi L$.

However, the arc length along the sector that we actually used in the previous calculation is the circumference of the base of the cone, which is $2\pi R$.

So the total area of the sector (equivalent to the surface area of the cone) is the total area of the circle, times the ratio of the sector circumference to the total circumference.

$$S = \pi L^2 \, \frac{2\pi R}{2\pi L} = \pi R L$$

The error in our application of calculus to this problem is a factor of L/R, the ratio of the slant height to the radius of the base.

It turns out that what we did wrong was to multiply the circumference at each point by dx. What we should have done is to multiply it by the little increment of slant instead. This is called the path element for the curve ds.

Since

$$L = \sqrt{R^2 + H^2}$$

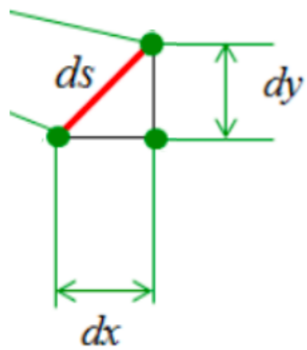
and in this problem

$$R = H$$

$$L = \sqrt{2}R$$

So the answer we obtained was πR^2, whereas the correct answer is $\pi R L$, and in this problem $L = \sqrt{2}R$, so the correct answer is $\sqrt{2}\pi R^2$.

path element



For the surface area of a "volume of revolution", instead of dx we need the actual length of the path element along the curve.

From Pythagoras we have

$$ds^2 = dx^2 + dy^2$$
$$= (1 + \frac{dy^2}{dx^2})\ dx^2$$
$$ds = \sqrt{1 + f'(x)^2} \cdot dx$$

We'll use this many times, both for surfaces of volumes of revolution and also for line integrals.

sphere: surface area

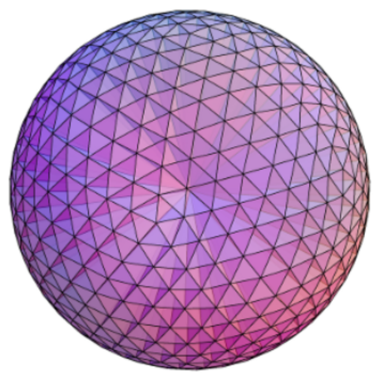
Calculus provides a simple proof for the surface area of a sphere, starting from the formula for the volume of a sphere

$$V = \frac{4}{3}\pi R^3$$

Suppose we take a sphere of radius r. (I use r here because now the radius will be a variable). If we increase the radius by a little bit dr, then how does the volume change? It changes exactly like the surface area! That is

$$dV = S \ dr$$
$$S = \frac{d}{dr} \ V = \frac{d}{dr} \ \frac{4}{3}\pi r^3 = 4\pi r^2$$

Another way to see this is to break up the entire surface area of the sphere into small cones. According to Acheson, this argument is due to Johannes Kepler.



If the number of cones is very large, the base of each one is almost flat. Call the area of the base dS and the height is of course R.

The volume of a single cone is

$$\frac{1}{3}R \ dS$$

If we add up the volumes of all the very thin cones from the entire sphere we will have the volume of the sphere

$$\frac{1}{3}R \ S$$

but we already know this is just $4/3\pi R^3$, so clearly

$$\frac{1}{3}R \cdot S = \frac{4}{3}\pi R^3$$
$$S = 4\pi R^2$$

slices

Another approach is to make a volume of revolution and add up the surface part by the method of slices. This is similar to the volume calculation we did before, except this time, for each slice we need to use the path element ds rather than dx.

Consider a sphere of radius R centered at the origin and make slices perpendicular to the x axis. We have that

$$y = \sqrt{R^2 - x^2}$$

The circumference for each slice is then $2\pi y$.

When we did the volume integral for a sphere in one dimension it was

$$V = \int_{-R}^{R} \pi y^2 \, dx$$

Here we are looking for the surface area, and adding up a bunch of small strips from the perimeter, but the differential is not dx In other words, we can't just do

$$S = \int 2\pi y \, dx$$

For the surface area of a "volume of revolution", instead of dx we need the actual length of the path element along the curve.

$$ds = \sqrt{1 + f'(x)^2} \cdot dx$$

What is the slope of a circle?

$$y = \sqrt{R^2 - x^2}$$

$$\frac{dy}{dx} = f'(x) = -\frac{x}{\sqrt{R^2 - x^2}} = -\frac{x}{y}$$

so

$$ds = \sqrt{1 + f'(x)^2} \cdot dx$$
$$= \sqrt{1 + \frac{x^2}{y^2}} \ dx$$

We want

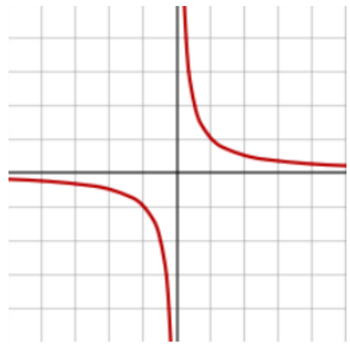
$$S = \int 2\pi y \ ds$$
$$= \int 2\pi y \ \sqrt{1 + \frac{x^2}{y^2}} \ dx$$
$$= 2\pi \int \sqrt{y^2 + x^2} \ dx$$
$$= 2\pi R \int \ dx$$
$$= 2\pi R \ x \ \Big|_{-R}^{R} = 4\pi R^2$$

That simplified beautifully. Usually integrals with the path element get messy.

Gabriel's horn

The inverse function is

$$f(x) = \frac{1}{x}$$
$$f'(x) = -\frac{1}{x^2}$$



We consider the curve from $x = 1 \to \infty$.

To get the surface area

$$S = \int 2\pi y \; ds$$

$$= 2\pi \int \frac{1}{x} \sqrt{1 + f'(x)^2} \; dx$$

$$= 2\pi \int \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \; dx$$

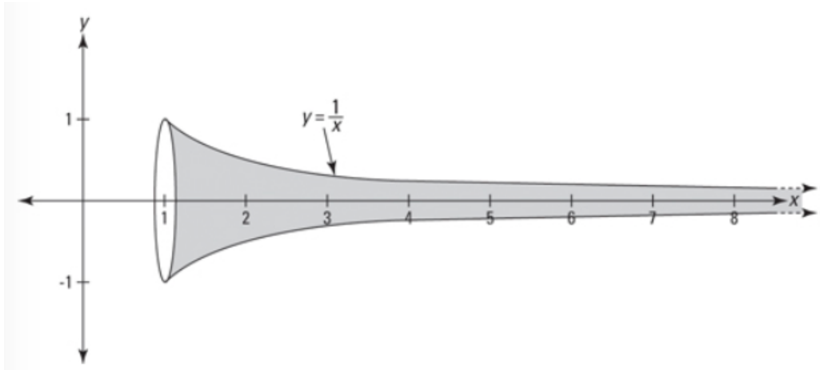
This looks hard! However, we notice that the factor

$$\sqrt{1 + \frac{1}{x^4}} > 1$$

So, if the integral without this factor diverges, the one with it diverges too. And

$$\int_1^\infty \frac{1}{x} \; dx = \ln x \, \Big|_1^\infty$$

certainly diverges at the upper limit.



Why is it so surprising that the surface area of this horn is infinite? It is surprising because the volume is finite.

The volume is

$$\int \pi y^2 \ dx$$

$$= \int \pi \frac{1}{x^2} \ dx$$

$$= \pi \ [-\frac{1}{x} \ \Big|_1^\infty \ ] = \pi$$

Wait for the inevitable joke: doesn't that blow your mind?

Volumes

A solid of revolution is formed by revolving a curve around a central axis, typically, the x-axis. We can get the volume of the solid by slicing it into disks.

In **Sphere and cone**, we revolved a half-circle to obtain the volume of a sphere. The integral was

$$V = \int_{-R}^{R} \pi y^2 \ dx$$

We also did the cone:

$$V = \pi \int_0^H (\frac{R}{H}x)^2 \ dx$$

However, this approach can be used with any curve or pair of curves.

We also found just above, the volume (and surface area) of Gabriel's horn, using the curve $y = 1/x$.

The volume of the solid formed by rotating the curves $f(x)$ and $g(x)$ around the x-axis on the interval $[a, b]$ ($f(x) < g(x)$ everywhere) is:

$$V = \int_a^b f(x)^2 - g(x)^2 \ dx$$

For $g(x) = 0$ this resolves to the familiar form.

If the curve is given in parametric form (both x and y as a function of t), then

$$V_x = \int_a^b \pi y^2 \ \frac{dx}{dt} \ dt$$
$$V_y = \int_a^b \pi x^2 \ \frac{dy}{dt} \ dt$$

where V_x is revolved around the x-axis, and so on.

The corresponding surface areas are

$$A_x = \int_a^b 2\pi y \ \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \ dt$$
$$A_y = \int_a^b 2\pi x \ \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \ dt$$

Torus

Consider a circle of radius R, displaced upward from the x-axis. The distance from the origin to the center of the circle is a.

The equation of the upper half of this circle is

$$y = \sqrt{R^2 - x^2} + a$$

So

$$y^2 = R^2 - x^2 + a^2 + 2a\sqrt{R^2 - x^2}$$

The equation of the bottom half of the circle is almost identical

$$y = -\sqrt{R^2 - x^2} + a$$

So

$$y^2 = R^2 - x^2 + a^2 - 2a\sqrt{R^2 - x^2}$$

Subtracting the bottom from the top, the first three terms cancel and we have

$$y_{\text{top}}^2 - y_{\text{bottom}}^2 = 4a\sqrt{R^2 - x^2}$$

Don't forget to multiply by π:

$$A = 4\pi a\sqrt{R^2 - x^2}$$

Adding up the area of each slice of the donut

$$V = \int_{-R}^{R} 4\pi a\sqrt{R^2 - x^2}\ dx$$

We need a trig substitution:

$$x = R\sin t$$

$$dx = R\cos t \; dt$$
$$\sqrt{R^2 - x^2} = R\cos t$$

So the integral is $4\pi a$ times

$$R^2 \int \cos^2 t \; dt$$

As always:

$$\int \cos^2 t \; dt = \frac{1}{2} \; [t + \sin t \cos t \;]$$

And if the bounds are right, that second term will disappear. Let's see.

$$x = [-R, R]$$
$$t = [-\pi/2, \pi/2]$$

The second term does go away, giving π for what's in the brackets, and we obtain finally

$$V = 4\pi a \cdot R^2 \cdot \frac{\pi}{2}$$
$$= 2\pi a \cdot \pi R^2$$

This is the same as obtained by multiplying the area of the cross-section of the torus by the distance traveled by the center in revolving around the x-axis. The general principle is named after Pappus.

Chapter 55

Powers of sine and cosine

Occasionally, we run into higher powers of the sine and or the cosine.

This chapter explains how to figure these out using integration by parts, but it is complicated enough that if you can, it is probably better to look up the answer in a table of integrals.

If the integrand contains only integral powers like $\sin^m x$ and $\cos^n x$, then there are two cases. If both m and n are even (or only one is present and that power is even), you can use the formula we will develop here. Otherwise, simply separate out one factor of $\sin x$ or $\cos x$ to combine with dx, like this:

$$
\begin{aligned}
&\int \sin^5 x \cos^4 x \; dx \\
&= \int \sin^4 x \cos^4 x \sin x \; dx \\
&= \int (1-\cos^2 x)^2 \cos^4 x \sin x \; dx \\
&= \int (\cos^4 x - 2\cos^6 x + \cos^8 x) \sin x \; dx
\end{aligned}
$$

This is just

$$-\int u^4 - 2u^6 + u^8 \; du$$

To handle the case with m and n both even, first use the identity $\sin^2 + \cos^2 = 1$ to convert to all sine or all cosine. Converting the function with the smaller exponent will give the least complicated expression. Now we have an integral of only sine or only cosine raised to an even power. We will use n to represent that power. Let's do cosine first:

$$\int \cos^n x \; dx$$

We use integration by parts:

$$u = \cos^{n-1} x$$

$$du = -(n-1)\cos^{n-2} x \sin x \; dx$$

$$dv = \cos x \; dx$$

$$v = \sin x$$

So $\int \cos^n x \; dx$ is $\int u \; dv$ and this is equal to $uv - \int v \; du$. We have (reversing terms and writing vu):

$$\sin x \cos^{n-1} x + \int \sin x (n-1) \cos^{n-2} x \sin x \; dx$$

The last term is

$$\sin x (n-1) \cos^{n-2} x \sin x$$

$$= (n-1)\cos^{n-2} x \sin^2 x$$

$$= (n-1)\cos^{n-2} x \; (1 - \cos^2 x)$$

$$= (n-1)\cos^{n-2} x - (n-1)\cos^n x$$

The trick is that although we have produced $-(n-1)\cos^n x$ on the right-hand side, this can be moved to the left-hand side, and added to the expression we started with. Placing the above result under the integral and moving all of the $\cos^n x$ terms to the left-hand side, we obtain

$$n \int \cos^n x \; dx = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \; dx$$

We divide by n to obtain the general formula.

$$\boxed{\int \cos^n x \; dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \; dx}$$

A few specific examples:

$$\int \cos^2 x \; dx = \frac{1}{2} \sin x \cos x + \frac{1}{2} \int \cos^0 x \; dx$$

$$= \frac{1}{2}(\sin x \cos x + x)$$

$$\int \cos^4 x \; dx = \frac{1}{4} \sin x \cos^3 x + \frac{3}{4} \int \cos^2 x \; dx$$

$$= \frac{1}{4} \sin x \cos^3 x + \frac{3}{8}(\sin x \cos x + x)$$

$$\int \cos^6 x \; dx = \frac{1}{6} \sin x \cos^5 x + \frac{5}{6} \int \cos^4 x \; dx$$

$$= \frac{1}{6} \sin x \cos^5 x + \frac{5}{24} \sin x \cos^3 x + \frac{5}{16}(\sin x \cos x + x)$$

We will work out the formula for sine below, but notice:

$$\sin^2 x + \cos^2 x = 1$$

$$\int \sin^2 x\ dx + \int \cos^2 x\ dx = \int dx = x$$

Looking at the formula for cosine squared

$$\int \cos^2 x\ dx = \frac{1}{2}(\sin x \cos x + x)$$

it should be clear that we will end up with the same formula for sine squared, but just flip the sign on the term $\sin x \cos x$ to make it go away in the sum. Let's see:

$$\int \sin^n x\ dx$$

We use integration by parts:

$$u = \sin^{n-1} x$$

$$du = (n-1) \sin^{n-2} x \cos x\ dx$$

$$dv = \sin x\ dx$$

$$v = -\cos x$$

For $uv - \int v\ du$ we have:

$$\sin^{n-1} x(-\cos x) - \int (-\cos x)(n-1) \sin^{n-2} x \cos x\ dx$$

Just as before, the last term is

$$\sin^{n-2} x \cos^2 x$$

$$= \sin^{n-2} x(1 - \sin^2 x)$$

$$= \sin^{n-2} x - \sin^n x$$

So the whole thing is:

$$\int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^4 x\,dx$$

$$n\int \sin^n x\ dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\ dx$$

$$\boxed{\int \sin^n x\ dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\ dx}$$

So for $\sin^2 x$:

$$\int \sin^2 x\ dx = -\frac{1}{2}\sin^{n-1} x \cos x + \frac{1}{2}\int\ dx$$

$$\int \sin^2 x\ dx = \frac{1}{2}(-\sin^{n-1} x \cos x + x)$$

As predicted, we have simply switched the sign on the first term.

Part XV

Hyperbolics

Chapter 56

Hyperbolic functions

The hyperbolic functions are defined to be:

$$2\cosh x = e^x + e^{-x}$$

$$2\sinh x = e^x - e^{-x}$$

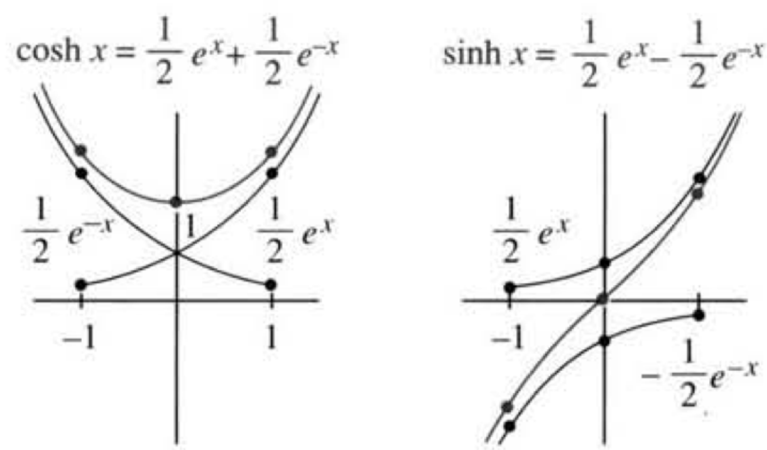


Fig. 6.18 Cosh x and sinh x. The hyperbolic functions combine $\frac{1}{2}e^x$ and $\frac{1}{2}e^{-x}$.

When we work through Euler's formula

$$e^{ix} = \cos x + i\sin x$$

we will find that

$$e^{ix} + e^{-ix} = 2\cos x$$
$$e^{ix} - e^{-ix} = -2\ i\sin x$$

which is in a way, parallel to the hyperbolic definitions.

The difference of squares has a simple value:

$$\cosh^2 t - \sinh^2 t = 1$$

Everything about the hyperbolic sine is reminiscent of the regular trig functions but with a sign change.

A plot of $\sinh t$ on the x-axis and $\cosh t$ on the y-axis yields a hyperbola in the same way the $y^2 - x^2 = 1$ does.

derivatives

$$\frac{d}{dx} 2\sinh x = \frac{d}{dx}(e^x - e^{-x}) = e^x + e^{-x} = 2\cosh x$$
$$\frac{d}{dx} 2\cosh x = \frac{d}{dx}(e^x + e^{-x}) = e^x - e^{-x} = 2\sinh x$$

Also, note that:

$$2\sinh x + 2\cosh x = 2e^x$$
$$e^x = \sinh x + \cosh x$$

Because of this, and by symmetry, we expect that the series should be

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

The values of the functions at zero are

$$\sinh 0 = 0$$

$$\cosh 0 = 1$$

relativity

The hyperbolic functions come into the mathematics of relativity, where for an observer in a moving reference frame, the following equations hold:

$$x' = \frac{x - vt}{\sqrt{1 - v^2}}$$
$$t' = \frac{t - vx}{\sqrt{1 - v^2}}$$

The quantity s^2 is invariant where

$$s^2 = t^2 - x^2$$

Proof:

$$x'^2 = \frac{x^2 - 2xvt + v^2t^2}{1 - v^2}$$
$$t'^2 = \frac{t^2 - 2xvt + v^2x^2}{1 - v^2}$$
$$t'^2 - x'^2 = \frac{(t^2 - x^2) - v^2(t^2 - x^2)}{1 - v^2}$$
$$= t^2 - x^2$$

The hyperbolic functions come in by defining a parameter θ (the "rapidity")

$$\cosh \theta = \frac{1}{\sqrt{1 - v^2}}$$

Then

$$\sinh^2\theta = \cosh^2\theta - 1 = \frac{1}{1-v^2} - 1 = \frac{v^2}{1-v^2}$$
$$\sinh\theta = \frac{v}{\sqrt{1-v^2}}$$

So we can rewrite

$$x' = \frac{x - vt}{\sqrt{1-v^2}} = x\cosh\theta - t\sinh\theta$$
$$t' = \frac{t - vx}{\sqrt{1-v^2}} = t\cosh\theta - x\sinh\theta$$

$\tanh\theta$

We had

$$\sinh\theta = \frac{v}{\sqrt{1-v^2}}$$
$$\cosh\theta = \frac{1}{\sqrt{1-v^2}}$$

so

$$\tanh\theta = v$$

leading us to explore the properties of the hyperbolic tangent. Going back to the beginning:

$$2\sinh\theta = e^{\theta} - e^{-\theta}$$
$$2\cosh\theta = e^{\theta} + e^{-\theta}$$
$$\tanh\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$$

The derivative is (by the quotient rule):

$$\frac{d}{d\theta}\ \tanh\theta = \frac{\cosh^2\theta - \sinh^2\theta}{\cosh^2\theta}$$
$$= \frac{1}{\cosh^2\theta}$$

Chapter 57

Hyperbolic substitution

I came across this integral:

$$\int \sqrt{x^2 - a^2}\ dx$$

with an unusual substitution method to solve it.

We've had $\sqrt{a^2 - x^2}$ as the integrand before, with the circle. If we manipulate this new expression:

$$y = \sqrt{x^2 - a^2}$$

$$y^2 = x^2 - a^2$$

$$x^2 - y^2 = a^2$$

You can see where the hyperbolic connection comes from.

the answer is

The best way to solve this is to look it up:

$$I = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\ \ln(x + \sqrt{x^2 - a^2})$$

https://www.quora.com/How-do-you-evaluate-the-integral-int-sqrt-x-2-a-2-dx

We shouldn't just accept this of course, but check by differentiating. Make our lives simpler by reserving the factor of 1/2 from both terms.

We will need

$$[\sqrt{x^2 - a^2}\ ]' = \frac{x}{\sqrt{x^2 - a^2}}$$

The first term is then

$$[x\sqrt{x^2 - a^2}\ ]' = \sqrt{x^2 - a^2} + \frac{x^2}{\sqrt{x^2 - a^2}}$$

$$= \frac{2x^2 - a^2}{\sqrt{x^2 - a^2}}$$

And the second term is a^2 times

$$[\ln(x + \sqrt{x^2 - a^2})\ ]' = \frac{1}{x + \sqrt{x^2 - a^2}}(1 + \frac{x}{\sqrt{x^2 - a^2}})$$

$$= \frac{1}{x + \sqrt{x^2 - a^2}}(\frac{\sqrt{x^2 - a^2} + x}{\sqrt{x^2 - a^2}})$$

$$= \frac{a^2}{\sqrt{x^2 - a^2}}$$

where we have brought back the factor of a^2 in the last step. Subtracting the second term from the first

$$= \frac{2x^2 - 2a^2}{\sqrt{x^2 - a^2}}$$

$$= 2\sqrt{x^2 - a^2}$$

and then finally, recall the factor of $1/2$, which yields the desired result.

hyperbolic substitution

To solve the integral, we use a hyperbolic substitution, which is the interesting part. It will make the integral trivial, however, getting back to the original variable x will be a challenge. Let

$$x = \frac{a}{2}(e^t + e^{-t})$$

If you look again at the hyperbolic functions, you'll see that this is just $x = a\cosh t$. Let's work through the substitution:

$$x^2 = \frac{a^2}{4}\ (e^{2t} + 2 + e^{-2t})$$

$$\sqrt{x^2 - a^2} = \sqrt{\frac{a^2}{4}\ (e^{2t} + 2 + e^{-2t}) - a^2}$$

$$= \sqrt{\frac{a^2}{4}\ (e^{2t} - 2 + e^{-2t})}$$

a bit tricky, but it works.

$$= \frac{a}{2}\ (e^t - e^{-t})$$

$$= a\sinh t$$

So that's going to be a big help. We can get dx by differentiation of $\cosh t$ directly, or work through the exponentials:

$$x = \frac{a}{2}(e^t + e^{-t})$$

$$dx = \frac{a}{2}(e^t - e^{-t})\ dt = a\sinh t\ dt$$

integral

So after the substitution, the integral is

$$\int a^2 \sinh^2 t \ dt$$

which turns out to be easy in terms of the exponentials

$$= \int \frac{a^2}{4} \ (e^t - e^{-t})^2 \ dt$$

$$= \int \frac{a^2}{4} \ (e^{2t} - 2 + e^{-2t}) \ dt$$

$$= \frac{a^2}{4} \ (\frac{e^{2t}}{2} - 2t - \frac{e^{-2t}}{2})$$

That was easy, which is the whole point.

reversing the substitution: term 2

This is the tricky part. We have the integral in terms of t. For a nice value of x we could maybe figure out the new bounds for t but in general we have to go back to x. So

$$x = \frac{a}{2}(e^t + e^{-t})$$

Let $e^t = z$ and $t = \ln z$ so

$$x = \frac{a}{2}(z + \frac{1}{z})$$

$$z^2 - \frac{2x}{a}z + 1 = 0$$

$$z = \frac{(2x/a) \pm \ \sqrt{(2x/a)^2 - 4}}{2}$$

taking only the positive root

$$= \frac{x}{a} + \sqrt{(x/a)^2 - 1}$$

$$= \frac{1}{a}(x + \sqrt{x^2 - a^2})$$

So

$$t = \ln z = \ln |x + \sqrt{x^2 - a^2}| - \ln |a|$$

Recall that the solution to the integral had one term with t that we can now write:

$$-\frac{a^2}{4}\ 2t = -\frac{a^2}{2}\ [\ \ln |x + \sqrt{x^2 - a^2}| - \ln a\ ]$$

The latter part of this gets folded into the constant of integration so

$$= -\frac{a^2}{2}\ \ln |x + \sqrt{x^2 - a^2}| + C$$

which matches the second half of the answer we were given.

reversing the substitution: term 1

The other part is

$$= \frac{a^2}{4}\ (\frac{e^{2t}}{2} - \frac{e^{-2t}}{2})$$

We know the answer so let's work backwards from that

$$x = \frac{a}{2}(e^t + e^{-t})$$

$$\sqrt{x^2 - a^2} = \frac{a}{2}\ (e^t - e^{-t})$$

So

$$x\ \sqrt{x^2 - a^2} = \frac{a^2}{4}(e^t + e^{-t})(e^t - e^{-t})$$

$$= \frac{a^2}{4}(e^{2t} - e^{-2t})$$

Multiply the last expression by 1/2 and we obtain

$$= \frac{a^2}{4}\ (\frac{e^{2t}}{2} - \frac{e^{-2t}}{2})$$

Therefore this expression is equal to

$$\frac{1}{2}x\ \sqrt{x^2 - a^2}$$

which completes the proof.

□

I put this problem in the book because of the interesting substitution, and to give a little more exposure to the hyperbolic functions. It is harder than your typical integral.

I would prefer (with Strang) to just admit that there are a lot of hard integrals that can be solved, and then move on. We have more interesting things to do with our time.

It also helps to remember that for real problems, they usually cannot be solved in "closed form," so learning a bunch of integral manipulations is not so helpful in the real world.

Chapter 58

Hanging chain

The hanging chain, or catenary, is a famous problem solved by Johann Bernoulli about 1700.

https://en.wikipedia.org/wiki/Johann_Bernoulli

solution

This is quite a challenging problem. See

https://gordma.wordpress.com/2014/03/26/we-know-the-lion-by-his-claw/

Bernoulli posed the problem as a public challenge, and said of Newton's anonymous submission: "tanquam ex ungue leonem" ("we know the lion by his claw").

We imagine an ideal hanging cable (like the cable for a suspension bridge, without the vertical cables or bridge deck), although its ends might be at different heights. The cable is ideal: it will not stretch or contract, is perfectly flexible, and has constant mass per unit length.

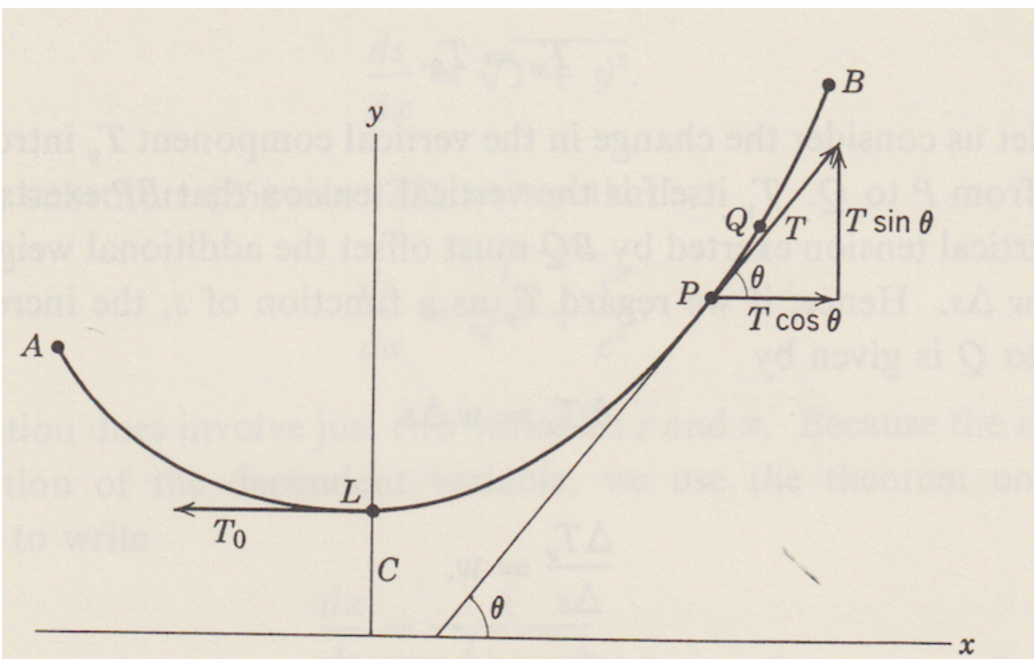


Figure 16-10

Consider a position P on the chain. The part of the chain below P

pulls down on it, because of its weight, and the part above pulls up. This force is called tension, T, and since the cable isn't moving, these forces are equal and opposite for every position on the chain.

The tension is in the direction of the angle θ tangent to the curve. By convention it is positive upward.

We can decompose the tension T into components in the x and y directions:

$$T = T_x + T_y$$
$$T_x = T\cos\theta$$
$$T_y = T\sin\theta$$

The curve traced out by the chain is described by $y = f(x)$, and it has a tangent at each point which is the ratio T_y/T_x.

moving to Q

Now consider moving from P up to the nearby point Q. The weight acting at Q is greater than that at P. If the arc length of L to P is s, we add an additional small length Δs to it in moving from P to Q. The weight of that little segment is the weight per unit length w, $\times \Delta s$.

The key point in the solution to this problem is that the additional force from the added weight changes T_y (because gravity points downward), but does not affect T_x. T_x is the same at every point in the chain, including at the lowest point, which has no weight acting on it. At this point, L, the tension is labeled as T_0, but T_x is equal everywhere.

$$T_x = T_0$$

The change in T_y when moving from P to Q is:

$$\Delta T_y = w \Delta s$$

So

$$\frac{\Delta T_y}{\Delta s} = w$$

and as we pass to the limit:

$$\frac{dT_y}{ds} = w$$

Since w is a constant, this means that

$$T_y = ws + D$$

where D is a constant of integration. But at L, both $s = 0$ and $T_y = 0$, so $D = 0$, and so:

$$T_y = ws$$

This difference in T_y at each point is what is responsible for the change in θ as we trace out the curve of the cable or chain.

Tangent

From above:

$$\frac{T_y}{T_x} = \frac{ws}{T_0}$$

but

$$\frac{T_y}{T_x} = \frac{T\sin\theta}{T\cos\theta}$$

so

$$\frac{ws}{T_0} = \tan\theta$$

And $\tan\theta$ is the slope of the curve so

$$y' = \tan\theta$$

Let $c = T_0/w$ so

$$y' = \frac{ws}{T_0} = \frac{s}{c}$$

Getting to x

The second point of real insight required for this problem is that we now have a differential equation relating y' to s, but we would like to have y (and y') as a function of x.

So finally, we need to relate s to x and y. Recall that s is arc length. We know a formula for that:

$$ds^2 = dx^2 + dy^2$$

and

$$\begin{aligned}\frac{ds}{dx} &= \sqrt{1 + y'^{\,2}} \\ &= \sqrt{1 + s^2/c^2}\end{aligned}$$

Hence

$$dx = \frac{1}{\sqrt{1 + s^2/c^2}}\, ds$$

and our integral is

$$\int \frac{s}{c} \frac{1}{\sqrt{1+s^2/c^2}} \ ds$$

Integration

We do two substitutions. First, let $u = s/c$ and then $c \ du = ds$ so

$$dx = c \frac{1}{\sqrt{1+u^2}} \ du$$

Second, because we have $\sqrt{1+u^2}$, do a trig substitution with $u/1 = \tan t$. Then $1/\sqrt{1+u^2} = \cos t$, $\sqrt{1+u^2} = \sec t$ and $du = \sec^2 t \ dt$ so

$$\begin{aligned} \int dx &= c \int \cos t \ \sec^2 t \ dt \\ &= c \int \sec t \ dt \end{aligned}$$

Integrate:

$$\begin{aligned} x &= c \ln |\sec t + \tan t| \\ &= c \ln |\sqrt{1+u^2} + u| \\ &= c \ln |\frac{s}{c} + \sqrt{1+s^2/c^2}| \end{aligned}$$

Solve for s

Exponentiate:

$$e^{x/c} = \frac{s}{c} + \sqrt{1 + s^2/c^2}$$

Let $z = s/c$. Then

$$(e^{x/c} - z)^2 = 1 + z^2$$

$$e^{2x/c} - 2ze^{x/c} + z^2 = 1 + z^2$$

$$e^{2x/c} - 2ze^{x/c} = 1$$

$$e^{x/c}(e^{x/c} - 2z) = 1$$

$$e^{x/c} - 2z = e^{-x/c}$$

$$z = \frac{1}{2}(e^{x/c} - e^{-x/c})$$

$$s = c\ \frac{e^{x/c} - e^{-x/c}}{2}$$

Back to y

Finally, recall that

$$y' = \frac{s}{c}$$

So

$$y' = \frac{e^{x/c} - e^{-x/c}}{2}$$

Integrate:

$$y = \frac{1}{2}\ c(e^{x/c} + e^{-x/c})$$

This is the hyperbolic cosine.

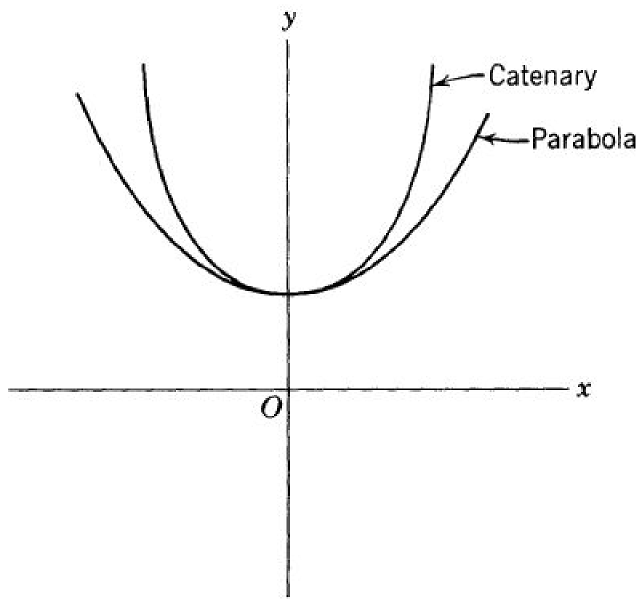
$$y = c\ \cosh\frac{x}{c}$$

And

$$s = c \ \sinh \frac{x}{c}$$

Recall that $c = T_0/w$. There is more to the problem (see Kline, for example), we need to figure out how T_0 depends on the geometry of the problem. But, this seems a good place to stop.

Let's just plot it:



It looks like a parabola but it's steeper.

Part XVI

Fancy differentiation

Chapter 59

Implicit differentiation

In this section we take a look at implicit differentiation, a remarkably powerful technique.

We will prove that the power rule $d/dx\ x^n = nx^{n-1}$ applies not only for integer n, both positive and negative, but also for rational n like $\sqrt{x}$ or $1/x^{3/2}$.

Implicit differentiation

Suppose we have a circle of radius R with equation

$$x^2 + y^2 = R^2$$

$$y = \sqrt{R^2 - x^2}$$

We'd like to find the derivative. It is not too hard by the standard method. By the chain rule

$$y' = \frac{1}{2}\ \frac{1}{\sqrt{R^2 - x^2}}(-2x) = -\frac{x}{y}$$

But there is another way.

$$x^2 + y^2 = R^2$$

Take d/dx of all the terms on both sides. The right-hand side is a constant so

$$2x + \frac{d}{dx}\ y^2 = 0$$

By the chain rule:

$$2x + 2y\ \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

We can also imagine a new variable, call it a parameter t where

$$x = t$$

Think of t like ticks of a clock and x has been scaled to follow t exactly.

Since y is a function of x and x is a function of t, y is also a function of t.

$$y = f(t)$$

Now differentiate with respect to t, using the chain rule:

$$x^2 + y^2 = R^2$$

$$2x\ \frac{dx}{dt} + 2y\ \frac{dy}{dt} = 0$$

It is OK to multiply by dt

$$x\ dx + y\ dy = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

If you find the idea of a parametric equation confusing, read ahead to **this chapter**.

After a while you will not need the parameter or even the chain rule. Just say

$$x^2 + y^2 = R^2$$

$$2x\ dx + 2y\ dy = 0$$

and so on.

Equivalent statements:

$$2x + 2y\ \frac{dy}{dx} = 0$$

$$2x + 2y\ y' = 0$$

derivative of the cosine

To obtain

$$\frac{d}{dx}\ \cos x$$

we originally set up a difference quotient. Here is another way using implicit differentiation and the chain rule.

Start from

$$\sin^2 x + \cos^2 x = 1$$

$$2 \sin x\ (\frac{d}{dx} \sin x) + 2 \cos x\ (\frac{d}{dx} \cos x) = 0$$

Plugging in for the derivative of $\sin x$:

$$2 \sin x \cos x + 2 \cos x (\frac{d}{dx} \cos x) = 0$$

$$\sin x + (\frac{d}{dx} \cos x) = 0$$

Rearranging, we obtain

$$\frac{d}{dx} \cos x = -\sin x$$

power rule

We can use implicit differentiation to prove that the power rule is correct for rational exponents:

$$y = x^{m/n}$$

$$y^n = x^m$$

Differentiate implicitly:

$$ny^{n-1}\ dy = mx^{m-1}\ dx$$

$$\frac{dy}{dx} = \frac{mx^{m-1}}{ny^{n-1}}$$

$$= \frac{m}{n}\ \frac{x^{m-1}}{(x^{m/n})^{n-1}}$$

$$= \frac{m}{n}\ \frac{x^{m-1}}{x^{m-m/n}}$$

Add up the exponents of x:

$$(m-1) - (m - \frac{m}{n}) = \frac{m}{n} - 1$$

The result is the power rule.

$$\frac{d}{dx}\ x^{m/n} = \frac{m}{n}\ x^{m/n-1}$$

Later we will use logarithmic differentiation to show that it applies for *any real number* n. That's quite impressive.

example

Consider the hyperbola

$$xy = c$$

where c is a constant. The standard approach would be

$$y = \frac{c}{x}$$

$$\frac{dy}{dx} = y' = -\frac{c}{x^2}$$

Implicit differentiation:

$$\frac{d}{dx}\ xy = \frac{d}{dx}\ c = 0$$

$$(xy)' = x\ y' + y\ x' = 0$$

$$xy' + y = 0$$

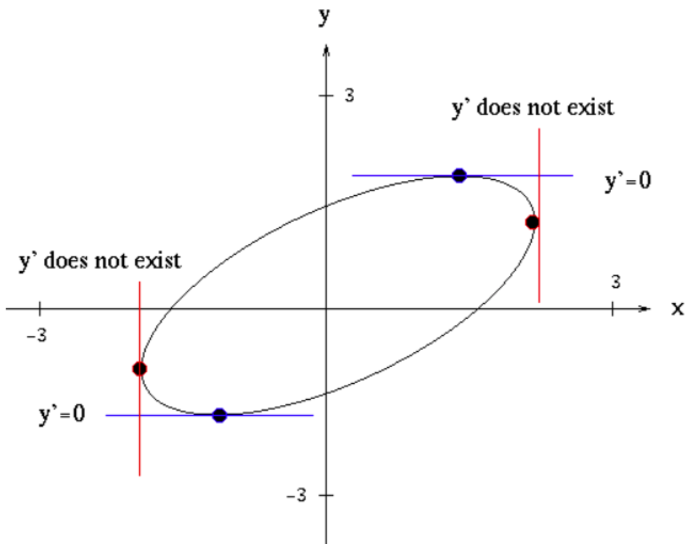
$$y' = -\frac{y}{x}$$

$$= -\frac{c}{x^2}$$

example

Suppose we have the equation of a tilted ellipse:

$$x^2 - xy + y^2 = 3$$



The problem asks us to find the extreme values of x and y on this ellipse. We take derivatives implicitly using the product rule on the second term:

$$x^2 - xy + y^2 = 3$$
$$2x\ dx - (x\ dy + y\ dx) + 2y\ dy = 0$$
$$2x\ dx - y\ dx = -2y\ dy + x\ dy$$
$$\frac{dy}{dx} = \frac{2x - y}{-2y + x}$$

Set y' equal to 0:

$$-\frac{2x - y}{2y - x} = 0$$

The extremes of y occur when $y' = 0$, that is when

$$y = 2x$$

Substituting $y = 2x$ into the original equation we obtain

$$x^2 - 2x^2 + 4x^2 = 3$$

$$x^2 = 1$$

$$x = \pm\ 1$$

Substituting again ($x = \pm\ 1$) into the original equation

$$1 - y + y^2 = 3$$

$$y^2 - y - 2 = 0$$

$$(y-2)(y+1) = 0$$

$$y = 2, -1$$

On the other hand, the maximum values of x occur when y' does not exist:

$$2y = x$$

By symmetry, I claim that this occurs when $y = \pm\ 1$, with analogous solutions for x.

example

Hamming asks us to find the angles at which two curves meet in the first quadrant.

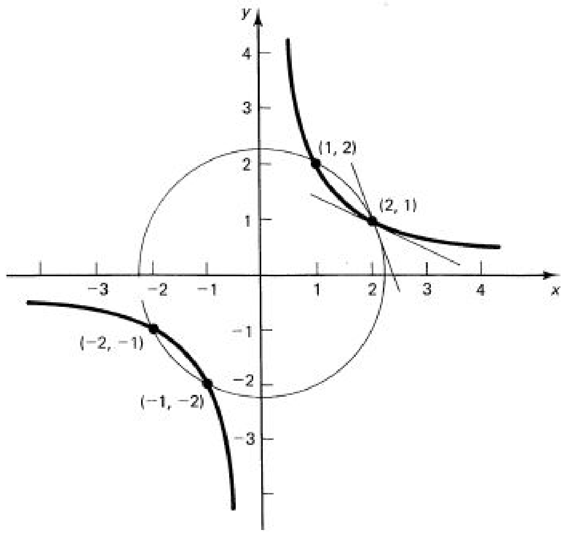


Figure 8.3-1 Angle between a circle and a hyperbola

The first step is to find the points where the curves meet.

$$x^2 + y^2 = 5$$
$$xy = 2$$

so

$$x^2 + (\frac{2}{x})^2 = 5$$
$$x^4 + -5x^2 + 4 = 0$$
$$x^2 = \frac{5 \pm \sqrt{25 - 16}}{2}$$
$$= \frac{5 \pm 3}{2} = 1, 4$$
$$x = 1, 2$$

The corresponding points are $(1,2)$ and $(2,1)$.

We have missed two other solutions. This happened when we took the positive square root of x at the last step. However, we are only interested in the first quadrant, so we can ignore those two for now.

Hamming solves this a bit differently, instead, he says to add and subtract $2xy = 4$ from the equation of the circle

$$x^2 + 2xy + y^2 = 9$$

$$(x+y)^2 = 9$$

$$x + y = \pm\, 3$$

and

$$x^2 - 2xy + y^2 = 1$$

$$(x-y)^2 = 1$$

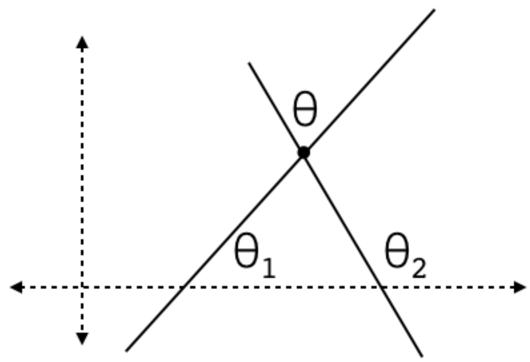
$$x - y = \pm\, 1$$

This leads to the same two answers in the first quadrant: $(2,1)$ and $(1,2)$.

We're not done yet. We need the angles.

angle between two lines

Consider two lines with slopes m_1 and m_2 which make angles θ_1 and θ_2 with the positive x-axis. They meet at a point, forming the angle θ, which we take to be the angle through which the first line is rotated counter-clockwise to meet the second.



You should be able to show that $\theta_1 + \theta = \theta_2$.

Look at the triangle formed by the two lines and the x-axis. The (unlabeled) upper angle is equal to θ, by the vertical angles theorem. Therefore $\theta + \theta_1$ plus the (unlabeled) angle at the right side of the triangle add up to 180 degrees. But θ_2 plus this angle is also equal to 180 degrees, by the supplementary angles theorem.

We have then

$$\theta_1 + \theta = \theta_2$$

$$\theta = \theta_2 - \theta_1$$

The slope of a line is equal to the tangent of the angle the line makes with the x-axis.

We'll see the connection in a bit, for now, we just convert the above relationship between angles with what we know in the original problem, slopes, by taking the tangent of both sides:

$$\tan\theta = \tan(\theta_2 - \theta_1)$$

We need the sum (difference) of angles formula for the tangent.

sum of angles, tangent

From the sum of angles for sine and cosine we can write

$$\tan s + t = \frac{\sin s + t}{\cos s + t} = \frac{\sin s \cos t + \cos s \sin t}{\cos s \cos t - \sin s \sin t}$$

Dividing by $\cos s \cos t$ we obtain

$$\tan s + t = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$

Exchanging $-t$ for t changes the sign of the tangent

$$\tan s - t = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

We had

$$\tan \theta = \tan(\theta_2 - \theta_1)$$

so

$$\tan \theta = \tan(\theta_2 - \theta_1) = \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_1 \tan \theta_2}$$

finishing up

We said that the slope of a line is equal to the tangent of the angle the line makes with the x-axis. Therefore, we can substitute slopes for tangents

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

And now we're finally ready to solve the problem we were originally given. We use implicit differentiation to obtain the slopes of the circle and the ellipse, both of which we did earlier.

Suppose we call the hyperbola curve 1

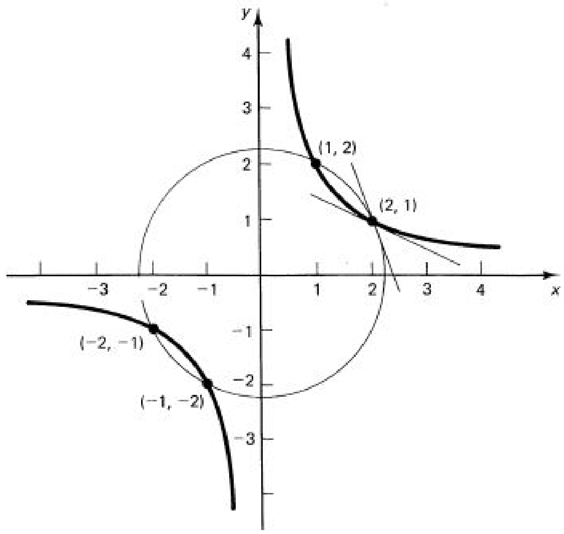


Figure 8.3-1 Angle between a circle and a hyperbola

It has slope (at the point $(2,1)$):

$$m_1 = y' = -\frac{2}{x^2} = -\frac{1}{2}$$

The circle at the point $(2,1)$ has slope:

$$m_2 = y' = -\frac{x}{y} = -2$$

This is quite reasonable, if you look at the diagram. So then

$$\tan\theta = \frac{-2 - -1/2}{1+1} = -\frac{3}{4}$$

That's about $-37°$.

The minus sign comes about because we rotate curve 1 (the hyperbola) *clockwise* to meet curve 2 (the circle).

Chapter 60

Logarithmic differentiation

Now that we know the definition

$$\ln x = \int \frac{1}{x}\ dx + C$$

by the fundamental theorem of calculus:

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

We can extend this to a general $f(x)$ using the chain rule

$$\frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)}\ f'(x)$$

Rearrange:

$$\frac{f'(x)}{f(x)} = (\ln(f(x))'$$

or

$$\frac{y'}{y} = \ [\ln y]'$$

Check this with a simple example

$$y = x^2$$

$$y' = 2x$$

$$2x = x^2 \ [\ \frac{d}{dx} \ln x^2 \ ]$$

$$2x = x^2 \ [\ \frac{1}{x^2}\ 2x \ ]$$

Slightly different statement

Write

$$y = f(x)$$

$$\ln y = \ln f(x)$$

Now differentiate implicitly. Imagine that x (and y) are functions of t and differentiate with respect to t. By the chain rule:

$$\frac{1}{y}\ \frac{dy}{dt} = \frac{1}{f(x)}\ f'(x)\ \frac{dx}{dt}$$

Lose the dt:

$$\frac{1}{y}\ dy = \frac{1}{f(x)}\ f'(x)\ dx$$

$$(\ln y)' = \frac{1}{f(x)}\ f'(x)$$

This is a slight variation of what we had above. We can see that it works by a bit of manipulation. We had

$$\frac{1}{y}\ dy = \frac{1}{f(x)}\ f'(x)\ dx$$

$$\frac{1}{y}\ dy = \frac{1}{y}\ y'\ dx$$

$$\frac{1}{y}\ \frac{dy}{dx} = \frac{1}{y}\ y'$$

$$\frac{dy}{dx} = y'$$

But of course.

example

A new problem using this result is to find the derivative of x^x, where the standard methods don't work.

Take the logarithm:

$$\ln(x^x) = x \ln x$$

Now, take the derivative of the logarithm of $f(x)$:

$$1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

Use the rule:

$$\begin{aligned} y' &= y \ \frac{d}{dx} \ln y \\ &= x^x \ (\ln x + 1) \end{aligned}$$

example

In general, any time we have a power of x that is itself a function, we need logarithmic differentiation.

$$\begin{gathered} y = x^{\cos x} \\ \ln y = \cos x \ln x \\ \frac{1}{y} \ dy = \ [\ \frac{\cos x}{x} - \sin x \ln x \ ] \ dx \\ \frac{dy}{dx} = x^{\cos x} \ [\ \frac{\cos x}{x} - \sin x \ln x \ ] \end{gathered}$$

power rule

An important result is to prove the power rule for all real x. We want

$$\frac{d}{dx}x^n$$

where n is not just a positive or negative integer ($\neq -1$), but can be any real number, like π or e or $\sqrt{2}$. Straight-up implicit differentiation is easier for all but the irrational numbers, but it's a subtle thing to talk about real numbers as the limits of rational numbers. We combine logarithmic and implicit differentiation

Write

$$y = x^n$$
$$\ln y = n \ \ln x$$

Imagine both x and y as functions of t with $x = t$, then

$$\frac{d}{dt} \ \ln y = \frac{d}{dt} \ n \ \ln x$$

But n is now *just a constant* so

$$\frac{d}{dt} \ \ln y = n\frac{d}{dt} \ \ln x$$
$$\frac{1}{y} \ \frac{dy}{dt} = n\frac{1}{x} \ \frac{dx}{dt}$$
$$\frac{1}{y} \ dy = n\frac{1}{x} \ dx$$
$$\frac{dy}{dx} = n\frac{y}{x}$$
$$= n\frac{x^n}{x}$$

$$= nx^{n-1}$$

Did you see what just happened! We proved the power rule for real n in a few simple lines. Wow.

example

$$\frac{d}{dx}x^e = e\ x^{e-1}$$
$$\frac{d}{dx}x^\pi = \pi\ x^{\pi-1}$$

example

The following problem is an example of turning calculus into arithmetic, an approach in high school courses which I really dislike. But it's really common to see such problems there so we might as well do one.

Differentiate

$$y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$$

We could use the product, quotient and chain rules for this (and we can be sure it will be messy in the end). An alternative is to use the properties of the logarithm to break the right-hand side into a polynomial. Our formula from above was

$$\frac{y'}{y} = \ [\ln y]'$$

Take the logarithm of both sides for the problem

$$\ln y = \ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)$$

$$\ln y = \ln(x^5) - \ln(1-10x) - \ln(\sqrt{x^2+2})$$

Now, when we differentiate, it is really implicit differentiation. The three terms are

$$\frac{1}{x^5} 5x^4 \ dx = \frac{5}{x} \ dx$$

$$-\frac{1}{(1-10x)}(-10) \ dx = \frac{10}{(1-10x)} \ dx$$

$$-\frac{1}{\sqrt{x^2+2}} \cdot \frac{1}{2\sqrt{x^2+2}} 2x \ dx = -\frac{x}{x^2+2} \ dx$$

On the left we get (including the dx from the terms on the right-hand side):

$$\frac{1}{y}\frac{dy}{dx} = \frac{y'}{y}$$

$$= \frac{5}{x} + \frac{10}{(1-10x)} - \frac{x}{x^2+2}$$

To finish the problem, we need to multiply through by y

$$y' = \left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)\left(\frac{5}{x} + \frac{10}{(1-10x)} - \frac{x}{x^2+2}\right)$$

which I won't try to simplify.

Chapter 61

L'Hospital

origin

The rule discussed below is attributed to l'Hospital, which is also often written l'Hôpital. The latter spelling is modern French usage, while the one we use (because it is easier to typeset in titles) is English usage. It also happens to be the way l'Hospital spelled his name himself, back in the day.

In any event, this rule was actually discovered by Johann Bernoulli. According to Stewart, these two mathematicians had entered into a business arrangement whereby the Marquis de l'Hospital bought the rights to Bernoulli's mathematical discoveries.

http://www.stewartcalculus.com/data/ESSENTIAL%20CALCULUS%20Early%20Transcendentals/upfiles/projects/ecet_wp_0307_stu.pdf

According to wikipedia,

> In the 17th and 18th centuries, the name was commonly spelled "l'Hospital", and he himself spelled his name that way. However, French spellings have been altered: the silent 's' has been removed and replaced with the circumflex over

the preceding vowel (l'Hôpital). The former spelling is still used in English where there is no circumflex.

limit of a quotient

We are trying to determine the limit of the quotient of two functions

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

and suppose that we run into trouble because both functions have problems at c. If

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$$

or

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \infty$$

or

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = -\infty$$

and if

$$\lim_{x \to c} \frac{f'(x)}{g'(x)}$$

exists

Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Here's a simple example. What is

$$\lim_{x \to 0} \frac{\sin x}{x}$$

We *know* this. it is equal to 1. But $\sin 0 = 0$ so take derivatives

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

If the limit is still of the form 0/0 just repeat the process.

basic proof

If

- f and g are well-behaved (continuously differentiable)
- The first differentiation yields a finite limit as $x \to c$ and
- The form of the quotient is 0/0, with $f(c) = g(c) = 0$

then

$$\begin{aligned}
&\lim_{x\to c} \frac{f(x)}{g(x)} \\
&= \lim_{x\to c} \frac{f(x) - f(c)}{g(x) - g(c)} \\
&= \lim_{x\to c} \frac{[\ f(x) - f(c)\ ]/(x-c)}{[\ g(x) - g(c)\ ]/(x-c)} \\
&= \frac{\lim_{x\to c} [\ f(x) - f(c)\ ]/(x-c)}{\lim_{x\to c} [\ g(x) - g(c)\ ]/(x-c)} \\
&= \frac{f'(x)}{g'(x)}
\end{aligned}$$

application to the exponential

We will develop a proof that

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

using l'Hospital's Rule.

This interesting approach is from the book *Mooculus*. Remember that n is a variable in this section.

The first thing is to rewrite this as an exponential, first for the term in parentheses:

$$1 + \frac{1}{n} = e^{\ln(1+1/n)}$$

and then for the whole thing

$$(1 + \frac{1}{n})^n = (e^{\ln(1+1/n)})^n$$

$$= \ e^{n \ln(1+1/n)}$$

To evaluate the limit, we need to evaluate the limit of the exponent

$$\lim_{n \to \infty} n \ln(1 + \frac{1}{n})$$

At first, it doesn't look like we can use the rule (there is no quotient), but there is a standard trick for these situations. Just rearrange to divide by the inverse

$$= \lim_{n \to \infty} \frac{\ln(1 + \frac{1}{n})}{\frac{1}{n}}$$

Both the top and the bottom limits are easily evaluated to be equal to 0, so we can apply the rule.

We need to evaluate

$$= \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

The derivative of the numerator is (by the chain rule)

$$f'(n) = \frac{-n^{-2}}{1+1/n}$$

while the denominator is just

$$g'(x) = -n^{-2}$$

The factor of $-n^{-2}$ cancels from both top and bottom, leaving us with

$$\lim_{n\to\infty} \frac{1}{1+1/n} = 1$$

That is:

$$\lim_{n\to\infty} n \ \ln(1+\frac{1}{n}) = 1$$

Going back to the original problem then

$$\lim_{n\to\infty} (1+\frac{1}{n})^n = \lim_{n\to\infty} \ e^{n \ln(1+1/n)}$$

$$= e^1 = e$$

exponential function

A parallel argument can be used to prove that

$$e^x = \lim_{n\to\infty} (1+\frac{1}{n})^{nx} = \lim_{n\to\infty} (1+\frac{x}{n})^n$$

with the assistance again of l'Hospital's Rule.

What we need to show is that we can bring x inside the parentheses

$$e^x = \lim_{n\to\infty} (1 + \frac{x}{n})^n$$

Taking logarithms, this is the same as saying that

$$x = \lim_{n\to\infty} n \cdot \ln(1 + \frac{x}{n})$$

As before, we form the quotient

$$\frac{\log(1 + x/n)}{1/n}$$

Use the chain rule to obtain the derivative of the numerator — remember that n is the variable:

$$\frac{1}{1 + x/n} \ (-\frac{x}{n^2})$$

The derivative of the denominator is $-1/n^2$ so we can cancel

$$\frac{f'(x)}{g'(x)} = \frac{1}{1 + x/n} \ (x)$$

We take the limit

$$\lim_{n\to\infty} \frac{1}{1 + x/n} \ (x) = x$$

Part XVII

Applications

Chapter 62

Maximum likelihood

A nice application of logarithmic differentiation is of a set of Bernoulli trials, like a series of coin flips where the coin isn't fair, but instead has a probability p of coming up heads (H) and $1 - p$ of coming up tails (T).

Note that this classic example is actually impossible to achieve. One cannot "weight" a coin to do this, though it is easy with a die (singular of dice).

http://www.stat.columbia.edu/~gelman/research/published/diceRev2.pdf

Now, p is unknown, but we have some data about how the coin performs, and we wish to use the data to estimate p by the method of maximum likelihood. We observe this sequence of trials:

$$HTHHTTTHTHHH$$

Theory says that the probability of observing this sequence of events is dependent on p in the following way:

$$p(1-p)pp(1-p)(1-p)(1-p)p(1-p)ppp = p^7(1-p)^5$$

We call the probability of observing this data, given some underlying probability model p, the likelihood L:

$$L(p) = p^7(1-p)^5$$

(It's called the likelihood, because the total probability is not equal to 1).

In general, since each trial is independent and identically distributed

https://en.wikipedia.org/wiki/Independent_and_identically_distributed_random_variables

we can write that for n trials and k successes we would have

$$L(p) = p^k(1-p)^{n-k}$$

Here, n and k are constants for any particular sequence, but we would like to have the general formula.

To find the maximum for L we differentiate and set that equal to 0.

$$\frac{d}{dp}L = 0$$

However, we note that since $\ln L$ increases and decreases along with L, the value of p that gives a maximum for L also gives a maximum for $\ln L$. So we will take the logarithm of L and set that equal to zero:

$$\frac{d}{dp}\ln L = 0$$

$$\ln L = k \ln p + (n-k)\ln(1-p)$$

Take the derivative d/dp of both sides (we get a minus sign from the chain rule):

$$\frac{d}{dp}\ln L = 0 = \frac{k}{p} - \frac{n-k}{1-p}$$

$$\frac{k}{p} = \frac{n-k}{1-p}$$

Multiply through by $1-p$ and also by $1/k$:

$$\frac{1-p}{p} = \frac{n-k}{k}$$

$$\frac{1}{p} = \frac{n}{k}$$

$$p = \frac{k}{n}$$

As we might have guessed, the maximum likelihood estimate of p is simply the ratio of the observed number of successes to the number of trials: k/n.

Chapter 63

Optimization of area

A general optimization problem expresses some dependence as a function, e.g. $A = f(x)$, where $f(x)$ is moderately complicated. We wish to find the value of x that gives a maximum (or a minimum) for A, perhaps within some limited domain of x, or sometimes over all possible values of x.

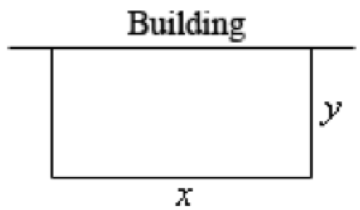
Usually, the first part is to construct the function $f(x)$, using some constraint that is given in the problem statement. Then, the basic method is to find the first derivative A' and set it equal to zero, and solve for x.

There is no shortage of fun (and challenging) problems of this type. We did the rectangular area problem earlier in the chapter on higher derivatives.

In this chapter we do problems involving area or volume. The next chapter has some additional examples.

Three-sided fence

We wish to build a fence, using an existing barn for one of the sides, so we need fencing only on three sides. The total length of available fencing is 500 ft. This is the constraint.



The hard way to solve this is to use the constraint to express y in terms of x:

$$500 = 2y + x$$
$$y = \frac{500 - x}{2}$$

Now write the area as

$$A = xy = 250x - \frac{1}{2}x^2$$

Take the first derivative and set it equal to zero:

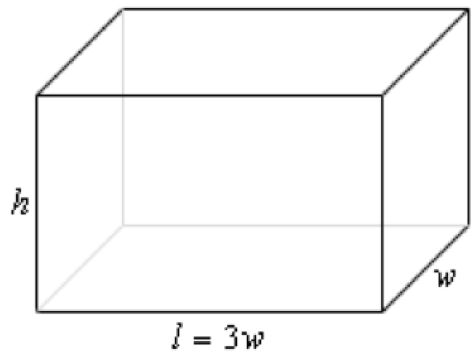
$$A' = 250 - x = 0$$

Clearly, $x = 250$ and $y = 125$.

The easy way (Nahin) is to imagine that we enclose *another* identical rectangular area on the other side of the barn. From the first problem it is known that the two areas together should form a square. Hence, we obtain the half-square as the answer.

Box with an expensive top

We wish to build a box. The box is unusual in that the cost of the top and bottom is more than the sides (10 v. 6 per unit area — let's say it is in square feet but that doesn't really matter).



As noted in the figure, the length of the side is three times the width. We wish to minimize the cost.

The constraint is that the total volume of the box is 50 cubic feet. Using the constraint, we can solve for h in terms of w

$$50 = 3wwh$$
$$h = \frac{50}{3w^2}$$

The cost C is

$$C = 2\ [\ 10 \cdot 3w^2 + 6 \cdot wh + 6 \cdot 3wh\ ]$$
$$= 2\ [30w^2 + 24wh\ ]$$
$$= 60w^2 + \frac{800}{w}$$

Take the first derivative and set it equal to zero:

$$C' = 120w - \frac{800}{w^2} = 0$$
$$w^3 = \frac{800}{120}$$
$$w = (\frac{800}{120})^{1/3} = 1.88$$

Box with maximum volume

This problem features a box with a square base and a total surface area of S. We wish to maximize the volume of the box.

The constraint is the surface area (plus the fact of the square base). If b is the base length and h is the height, we have that

$$S = 2b^2 + 4bh$$
$$h = \frac{S - 2b^2}{4b}$$

The volume is

$$V = b^2 h$$
$$= b^2 \ \frac{S - 2b^2}{4b}$$
$$= \frac{S}{4}b - \frac{1}{2}b^3$$

Take the first derivative and set it equal to zero:

$$V' = \frac{S}{4} - \frac{3}{2}b^2 = 0$$
$$S = 6b^2$$

$$b = \sqrt{\frac{S}{6}}$$

We can also find h

$$S = 2b^2 + 4bh$$

$$S = 2\frac{S}{6} + 4h\sqrt{\frac{S}{6}}$$

$$\frac{1}{3}S = 2h\sqrt{\frac{S}{6}}$$

$$h = \frac{1}{\sqrt{6}}\sqrt{S} = b$$

Since $b = h$, what we have is a cube. Not that surprising.

Cylindrical can with maximum volume for its surface area

A cylindrical can is to be formed with a volume of 1.5 cubic liters. What are the dimensions if we wish to minimize the surface area (materials used for construction)?

Suppose that the radius is r and the height is h. The formula for volume tells us that

$$V = 1.5 = \pi r^2 h$$

(Note: the linear dimensions of this volume, and of r, are in tenths of a meter, since a liter is a cubic decimeter—one-tenth of a meter).

The surface area is

$$A = 2\pi r^2 + 2\pi rh$$

Substituting for h

$$A = 2\pi r^2 + 2\pi r(\frac{1.5}{\pi r^2})$$

$$= 2\pi r^2 + \frac{3}{r}$$

Take the first derivative and set equal to zero:

$$A' = 4\pi r - \frac{3}{r^2} = 0$$

$$r^3 = \frac{3}{4\pi}$$

$$r = (\frac{3}{4\pi})^{1/3} = 0.620$$

$$h = \frac{1.5}{\pi r^2} = 1.24$$

Multiply by 10 to get the r and h in centimeters.

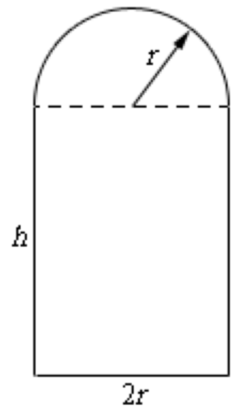
$$r = 6.2 \text{ cm}$$

$$h = 12.4 \text{ cm}$$

It's no accident that $h = 2r$.

Fancy window

We wish to construct this window, with a semi-circular arch added on top of the rectangular region below.



The quantity to maximize is the area of the window. The amount of framing for the window is the constraint, with a total length equal to 12. Strangely, the framing does not include the dotted line. Using the constraint, we can solve for h in terms of r:

$$2h + 2r + \pi r = 12$$
$$h = \frac{12 - (2+\pi)r}{2}$$

The area is

$$A = 2rh + \frac{1}{2}\pi r^2$$
$$= 12r - (2+\pi)r^2 + \frac{1}{2}\pi r^2$$
$$= 12r - 2r^2 - \frac{1}{2}\pi r^2$$

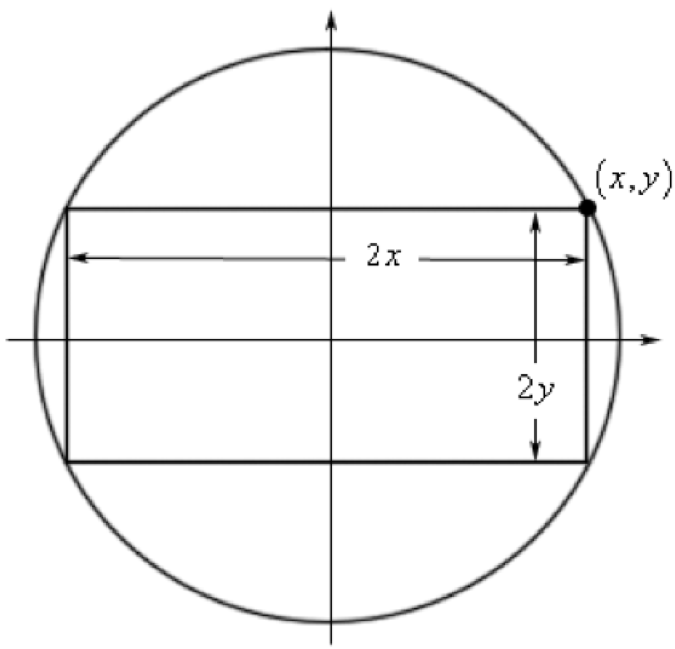
We take the first derivative and set it equal to zero:

$$A' = 12 - 4r - \pi r = 0$$
$$r = \frac{12}{4+\pi} = 1.68$$
$$h = \frac{12 - (2+\pi)r}{2} = 1.68$$

That's an interesting result! We should probably revise our drawing, and the client should probably think of a different kind of window.

Rectangle in a circle

Suppose we have a circle of fixed radius R, centered at the origin. Pick a value for x such that $0 \leq x \leq R$. Form the rectangle with all four vertices on the circle. That is, for the y value corresponding to that x, the vertices are $\pm\, x, \pm\, y$.



What this means is that given a particular x

$$y = \sqrt{R^2 - x^2}$$

and therefore the sides of the rectangle are $2x$ and $2\sqrt{R^2 - x^2}$, with area

$$A = xy = 4x\sqrt{R^2 - x^2}$$

We wish to find the value of x which gives the maximum area. We will take the first derivative and set it equal to zero. But, to begin with

$$\frac{d}{dx}\ \sqrt{R^2 - x^2} = -\frac{1}{2}\frac{2x}{\sqrt{R^2 - x^2}}$$

$$= -\frac{x}{\sqrt{R^2 - x^2}}$$

so, using the product rule:

$$A' = (4x)(-\frac{x}{\sqrt{R^2 - x^2}}) + 4(\sqrt{R^2 - x^2}) = 0$$

$$= -\frac{x^2}{\sqrt{R^2 - x^2}} + \sqrt{R^2 - x^2} = 0$$

$$x^2 = R^2 - x^2$$

and since $x^2 + y^2 = R^2$

$$x^2 = x^2 + y^2 - x^2 = y^2$$

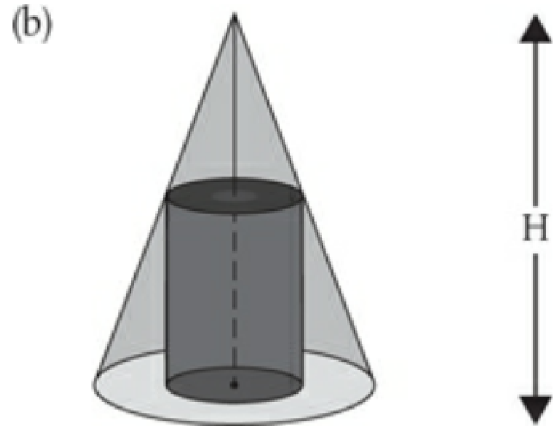
Thus, $x = y$. So the maximum area is for a square. No longer a surprise, I trust.

Mrs. Sidway's problem

This actually a volume problem, but whatever.

In Acheson we find a problem published in a 1714 edition of *The Ladies Diary: or The Woman's Almanack*, posed by a Mrs. Sidway (as a gardening problem, no less).

66. (a) The *Ladies Diary*. (b) Mrs. Sidway's problem.



If a cylinder lies contained within a cone (with its upper edge lying along the surface of the cone), how large should the radius be in order for the cone to have the maximum volume?

I find it easier to think about this problem by first placing the origin at the left of the figure, before moving it to the center for the second part.

The cone has fixed height H and radius R. If we place the origin at the left edge and allow a variable x to range from this origin to the right up to a distance R, you should be able to see first, that for any x

$$\frac{h}{x} = \frac{H}{R}$$

by similar triangles. This is a standard relationship for the cone.

But the radius of the cylinder, measured from the central axis of both cylinder and cone is

$$r = R - x = R - h\frac{R}{H}$$

Rearranging

$$h = \frac{H}{R}(R - r)$$

At this point, we compute the volume as

$$V = Ah = 2\pi r^2 \cdot \frac{H}{R}(R - r)$$

V depends on r and we can take the derivative with respect to r and then set it equal to zero in order to find the maximum.

When we do this, the constants (2π) will carry through into the first derivative and then disappear since we set the result equal to zero. Thus, we can ignore them

$$\begin{aligned} V &\approx r^2 \cdot \frac{H}{R}(R - r) \\ &= r^2 \cdot (H - \frac{H}{R} r) \\ &= Hr^2 - \frac{H}{R} r^3 \end{aligned}$$

The derivative is

$$\begin{aligned} 0 &= 2Hr - 3\frac{H}{R} r^2 \\ 2r &= 3\frac{r^2}{R} \\ \frac{2}{3}R &= r \end{aligned}$$

The maximum volume is obtained with r equal to two-thirds the radius of the cone at the bottom.

Chapter 64

Other optimizations

Projectile range

Suppose we fire a cannon where the ball has velocity v at an angle θ with the horizon (straight up would be $\pi/2$ radians). We wish to determine the angle that will give the maximum range. We did this problem in a physics section, but it won't hurt to look at it again here.

The problem has a trick, namely that the distance in the horizontal or x-direction depends on the time (and therefore distance) in the y-direction, since when $y = 0$, the cannonball will fall to earth and not move any more.

If you draw a diagram you will see that

$$v_y = v \sin\theta$$

where v_y is the initial velocity in the y-direction.

The basic equation of motion under gravity is that

$$y = v_y t - \frac{1}{2} g t^2$$

with $g = 32$ so

$$y = v_y t - 16t^2$$

At the point of interest $y = 0$ so

$$0 = v_y t - 16t^2$$

$$v_y t = 16t^2$$

This has two solutions, namely $t = 0$ (not what we are interested in) and

$$\begin{aligned} v_y &= 16t \\ t &= \frac{v_y}{16} \\ &= \frac{v \sin\theta}{16} \end{aligned}$$

On the other hand, the quantity we are really interested in is the distance in the x-direction. Similarly to v_y,

$$\begin{aligned} v_x &= v \cos\theta \\ x = v_x t &= v \cos\theta \ \frac{v \sin\theta}{16} \\ &= \frac{v^2}{16} \ \cos\theta \sin\theta \end{aligned}$$

This looks a little strange but all it really says is that the range is a function of the angle θ (and also of the square of the velocity). If you do dimensional analysis at this point you might also be confused unless you remember that the factor of 16 has units of meters per second squared.

We take the first derivative and set it equal to zero:

$$x' = \frac{v^2}{16}\,(\cos^2\theta - \sin^2\theta) = 0$$

$v = 0$ gives a solution to this equation, but it's not the solution we want. We want the solution given by

$$\cos^2\theta - \sin^2\theta = 0$$

The velocity and the gravitational constant have dropped out, which makes sense. It makes intuitive sense that the angle for maximum range (given a velocity), should not depend on that velocity.

The above expression is zero when $\cos\theta = \sin\theta$. If you don't see this you can say:

$$\begin{aligned}
1 - \sin^2\theta - \sin^2\theta &= 0 \\
\sin^2\theta &= \frac{1}{2} \\
\sin\theta &= \frac{1}{\sqrt{2}} \\
\theta &= \frac{\pi}{4}
\end{aligned}$$

An elevation of 45 degrees gives the maximum range.

Closest point to a parabola

Suppose we consider the simple parabola

$$y = x^2$$

Our problem is to find the point(s) (x, y) on the parabola that have the shortest distance to $P = (0, 1)$.

One possibility is that $(0, 0)$ is the minimum. But it will turn out that it is not, and so there will be two such points, which are symmetrical about the $y-$axis. Therefore, we consider only $x \geq 0$.

The distance from any point (x, y) to $P = (0, 1)$ is

$$d = \sqrt{(0 - x)^2 + (1 - y)^2}$$

It is the case that if we minimize d^2, we also minimize d, so let's rewrite the equation as

$$\begin{aligned} D &= (0 - x)^2 + (1 - y)^2 \\ D &= x^2 + 1 - 2y + y^2 \end{aligned}$$

Now, the constraint is that $y = x^2$ so plugging in we get

$$\begin{aligned} D &= y + 1 - 2y + y^2 \\ &= 1 - y + y^2 \end{aligned}$$

Take the first derivative (with respect to y) and set it equal to zero:

$$\begin{aligned} D' &= -1 + 2y = 0 \\ y &= \frac{1}{2} \\ x &= \frac{1}{\sqrt{2}} \end{aligned}$$

Check the actual distance:

$$
\begin{aligned}
d &= \sqrt{(0-x)^2 + (1-y)^2} \\
&= \sqrt{(\frac{1}{\sqrt{2}})^2 + (1-\frac{1}{2})^2} \\
&= \sqrt{\frac{1}{2} + \frac{1}{4}} \\
&= \frac{\sqrt{3}}{2} \\
&= 0.866
\end{aligned}
$$

$$
(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{2})
$$

Note that $(1/\sqrt{2}, 1/2)$ is closer to $(0, 1)$ than is $(0, 0)$, as we said.

The slope of the line from $(0, 1)$ to our point $(1/\sqrt{2}, 1/2)$ is

$$
\frac{\Delta y}{\Delta x} = \frac{1 - 1/2}{1/\sqrt{2}} = -\frac{1}{\sqrt{2}}
$$

The slope of the tangent to the parabola is $2x$, and at $(1/\sqrt{2}, 1/2)$ it is

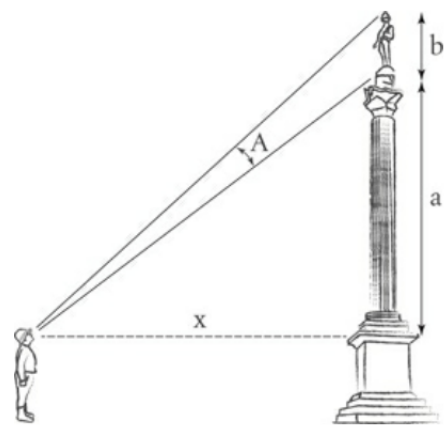
$$
m = 2x = 2\frac{1}{\sqrt{2}} = \sqrt{2}
$$

Since the product of the slopes is -1, the line corresponding to the minimum distance is perpendicular to the tangent.

The next two problems are about maximizing angles. They are variants with a slight twist.

Nelson's column

Nelson's column is a column with a statue of Nelson on top, naturally. The hero of Trafalgar is honored at London's Trafalgar square. Here is Acheson's sketch:



Somehow we need to maximize the angle A as a function of x, but we are given values that form the tangent of two angles.

However, we know that for θ in the half-open interval $[0, \pi/2)$, as θ increases so does $\tan\theta$. Therefore, if we maximize $\tan\theta$, θ will also be a maximum. This is a standard trick to remember.

Let's use s for the entire angle and t for the lower triangle, then

$$A = s - t$$

and the lengths we are given can be used to express the tangents of s and t.

We derived $\tan s - t$ before, and can do it again:

$$\tan s - t = \frac{\sin s - t}{\cos s - t}$$

$$= \frac{\sin s \cos t - \cos s \sin t}{\cos s \cos t + \sin s \sin t}$$
$$= \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

Plugging in:

$$\tan A = \frac{\frac{a+b}{x} - \frac{a}{x}}{1 + \frac{a(a+b)}{x^2}}$$

The numerator simplifies to b/x, so let's keep the b on top and multiply on the bottom by x

$$\tan A = \frac{b}{x + a(a+b)/x}$$

I got a bit of a mess trying to work with this as it is, so I multiplied by x on top and bottom a second time:

$$\tan A = \frac{bx}{x^2 + a(a+b)}$$

Use the familiar quotient rule to take the derivative and set it equal to zero:

$$0 = \frac{b\ [\ x^2 + a(a+b)\ ] - bx(2x)}{[\ x^2 + a(a+b)\ ]^2}$$

This occurs when the numerator is zero so discard the denominator!

$$0 = b\ [\ x^2 + a(a+b)\ ] - bx(2x)$$

Factor out the b, put the two terms on opposite sides and obtain

$$x^2 + a(a+b) = 2x^2$$

$$x^2 = a(a+b)$$

This is the solution given in Acheson. Furthermore, the statue is fairly small compared to the column's height. If we let $b << a$ then

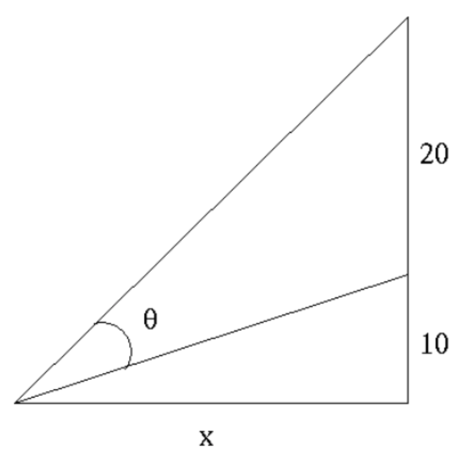
$$x^2 \approx a^2$$

$$x \approx a$$

The appropriate viewing angle is 45 degrees and since the statue is about 169 feet high that would put you in the middle of traffic unless you're quite careful.

movie screen

A movie screen on a wall is 20 feet high and 10 feet above the floor. At what distance x from the front of the room should you position yourself so that the viewing angle θ of the movie screen is as large as possible ?



$$\tan s - t = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

Plugging in the values provided:

$$\tan \theta = \tan s - t$$

$$= \frac{30/x - 10/x}{1 + 300/x^2}$$

$$= 20 \ \frac{x}{x^2 + 300}$$

We take the derivative and set it equal to zero. We can ignore the leading factor of 20, obtaining

$$0 = \frac{x^2 + 300 - 2x^2}{(x^2 + 300)^2} = \frac{-x^2 + 300}{(x^2 + 300)^2}$$

This is equal to zero when the numerator is zero, that is, when

$$x = \pm\sqrt{300}$$

Since x is a distance we take the positive square root.

$$x = \sqrt{300} = 10\sqrt{3}$$

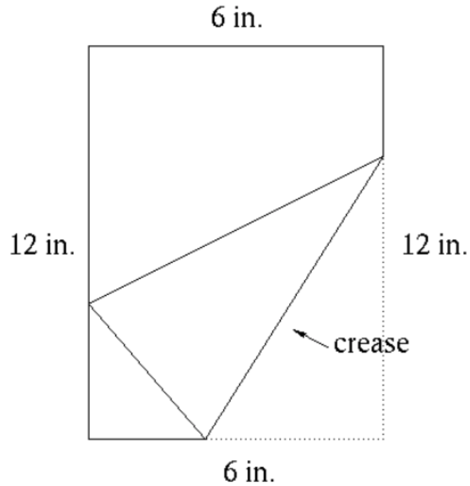
The angles are worth working out. The tangent of the lower angle t is $1/\sqrt{3}$. This is a right triangle with hypotenuse equal to 2 and sine equal to $1/2$. Therefore $t = \pi/6$.

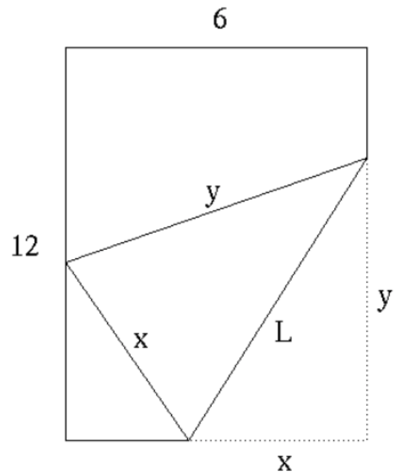
The tangent of the entire angle s is equal to $3/\sqrt{3} = \sqrt{3}$. This is a right triangle with hypotenuse equal to 2 and cosine equal to $1/2$. Therefore $s = \pi/3$.

Therefore the angle to the screen $\theta = s - t$ at the maximum is $\pi/6$ or 30 degrees.

folded paper

Consider a piece of paper with the dimensions 6×12. We pick up the lower right-hand corner and place it against the left side, folding to form a crease.

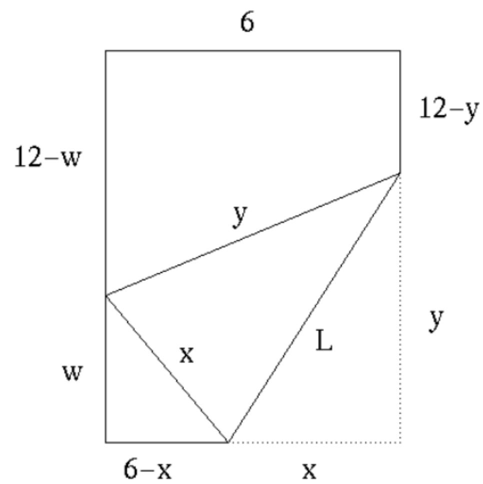




The possible positions on the left-hand side to place that corner range from the bottom up to a distance 6 inches above the bottom. The length of the crease is a variable, and and we wish to find the crease with the minimum length.

The variable distances can be labeled as shown on the right. The length $L^2 = x^2 + y^2$.

I notice that the drawing is not properly scaled (the long dimension is too short). In fact, the angle that the folded part makes on the left-hand side is a right angle. After all, it is a corner of the original sheet, but also we have two congruent triangles, they must both be right triangles.



At this point, I notice that the problem can be re-scaled so the length of the paper is 2 and the width is 1, in order to simplify the arithmetic. I have not re-done the drawings to reflect that yet, but our math will take advantage of it.

The range of the distance x is $[1/2, 1]$, while that for $y = [1, 2]$.

We need to find a relationship between x and y. Start by labeling another variable distance w, as shown above.

The connection that we need can be found by relating x, y and w to the total area of the paper. We have two right triangles with sides x and y and total area xy.

The other triangle has

$$
\begin{aligned}
w^2 &= x^2 - (1-x)^2 \\
&= 2x - 1 \\
w &= \sqrt{2x-1}
\end{aligned}
$$

and area (we leave w as it is for now).

$$\frac{1}{2}(1-x)w$$

We will need it later, so let's get the derivative of w with respect to x

$$\frac{dw}{dx} = \frac{1}{\sqrt{2x-1}} = \frac{1}{w}$$

Last, we have a rhombus. The average of the two vertical sides is

$$\frac{1}{2}(2 - w + 2 - y) = 2 - \frac{1}{2}(w + y)$$

and since the horizontal side is 1, this is also equal to the area.

area calculation

From the dimensions of the paper, the total area is 2 and this is equal to the three triangles and the rhombus added together

$$2 = xy + \frac{1}{2}(1-x)w + 2 - \frac{1}{2}(w + y)$$

$$4 = 2xy + (1-x)w + 4 - w - y$$

Cancel the 4 and gather terms with y. The left-hand side is

$$y - 2xy = -y(2x-1) = -yw^2$$

So

$$-yw^2 = (1-x)w - w$$

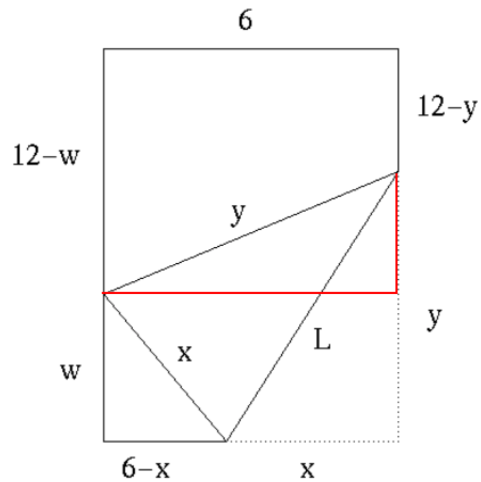
Factor out one w

$$-yw = (1-x) - 1 = -x$$

$$y = \frac{x}{w}$$

That's a nice simplification. We can check that this is correct at one extreme. When $x = w = 1$ the ratio is $1 = y$ and we see that is correct for the fold at 45 degrees. At the other extreme we have $w = 0$ and the ratio is undefined.

And only now do I see that this calculation was unnecessary. Draw the horizontal



Can you see the similar triangles? The angle between w and x is rotated by 90 degrees counter-clockwise to form the angle between the horizontal of length 1 and y, so $w/x = 1/y$, which is exactly what we said.

Now, minimize L

$$L = x^2 + y^2$$
$$= x^2 + \frac{x^2}{w^2}$$

Take the derivative:

$$\frac{dL}{dx} = 2x + \frac{2xw^2 - 2wx^2 \cdot 1/w}{w^4}$$

$$\frac{dL}{dx} = 2x + \frac{2x(2x-1) - 2x^2}{(2x-1)^2}$$

Factor out $2x$ and set equal to zero:

$$0 = 1 + \frac{(2x-1) - x}{(2x-1)^2}$$

$$0 = (2x-1)^2 + x - 1$$

$$4x^2 - 3x = 0$$

Factor out another x

$$4x - 3 = 0$$

$$x = \frac{3}{4}$$

The minimum crease length occurs when x is halfway along its range.

I found the last problem here:

https://www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/maxmindirectory/MaxMin.html

Chapter 65

Related rates

One simple form of related rates problem has two objects moving at right angles from each other, with positions and speeds given in terms of the origin.

For example: "A moves west at x miles per hr, his current position is x_0 miles west of the origin, while B moves south at y miles per hr, and his current position is y_0 miles south of the origin. At what rate are they moving apart?"

For the distance, we use Pythagoras:

$$h^2 = x^2 + y^2$$

All three values are functions of time so

$$2hh' = 2xx' + 2yy'$$

$$h' = \frac{1}{h}(xx' + yy')$$

We will have to calculate h from x_0 and y_0.

Another simple related rates problems involves two quantities with an equation relating the two quantities, e.g. the volume and radius of a sphere, where the sphere is a "balloon being inflated" or something

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

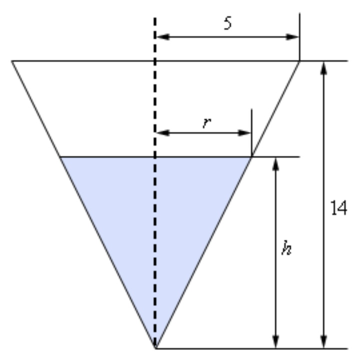
or as usually stated in these problems

$$V' = 4\pi r^2 \, r'$$

If we know V' and r we can calculate r'. Usually, rather than give you r they will give you V, so then

$$r = (\frac{4}{3}V)^{1/3}$$

A related problem :) is where the object is a cone (maybe inverted) and it's filling up with a fluid.



Here, the formula for the volume of a cone is

$$V = \frac{1}{3}\pi r^2 h$$

The problem is that since r and h depend on each other, we can't simply do

$$V' = \frac{1}{3}\pi r^2 h'$$

(this is wrong!)

In this case it's important to realize that the radius r and the height h of the fluid at its current level have the same ratio as the radius R and height H of the container.

$$\frac{r}{h} = \frac{R}{H}$$

$$r = \frac{R}{H}h$$

$$h = \frac{H}{R}r$$

so we can substitute using the relationship between r and h

$$V = \frac{1}{3}\pi r^2 \frac{H}{R} r = \frac{1}{3}\pi \frac{H}{R} r^3$$

Alternatively, we can eliminate r

$$V = \frac{1}{3}\pi \frac{R^2}{H^2} h^3$$

For example, with the figure above, ($R = 5$ and $H = 14$ feet), and given water is draining from the tank at $V' = -2ft^3$ per hour

"At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?"

$$V = \frac{1}{3}\pi\frac{R^2}{H^2}h^3$$
$$V' = \pi\frac{R^2}{H^2}h^2h'$$

We're given V' and h, H and R, so can solve for h'.

The second question is "At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?"

We need r' given h (and R, H, and V')

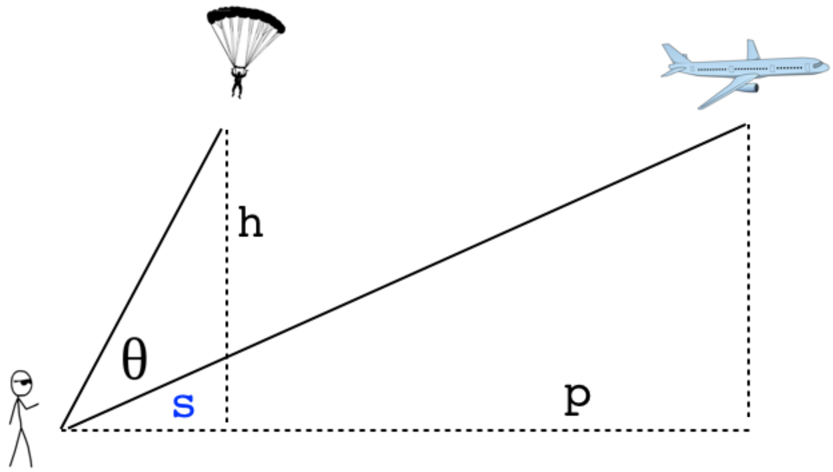
$$V = \frac{1}{3}\pi\frac{H}{R}r^3$$
$$V' = \pi\frac{H}{R}r^2r'$$

$$r = \frac{R}{H}h$$
$$r^2 = (\frac{R}{H})^2h^2$$

Plugging in

$$V' = \pi\frac{R}{H}h^2r'$$

Here is another related rates problem. An airplane and a parachutist are at the same height currently, and in the same direction as you look at them.



The airplane moves away from you at 500 ft/s. The parachutist is floating downward at -10 ft/s and will land 1000 ft away from you. The current value of $h = 2000$ ft. The current value of $p = 8000$ ft. Find $d\theta/dt$.

$$p(t) = p_0 + 500t$$
$$h(t) = h_0 - 10t$$
$$p' = 500$$
$$h' = -10$$

Find equations for the angles involved

$$\tan s = \frac{2000}{p}$$

we use the constant value of 2000 rather than h, which will vary.

$$u = s + \theta$$
$$\tan u = \frac{h}{1000}$$

Take the derivatives. For the airplane

$$\tan s = \frac{2000}{p}$$

$$\frac{d}{dt}\tan s = \sec^2 s\frac{ds}{dt} = \frac{d}{dt}\frac{2000}{p} = -2000\frac{1}{p^2}\frac{dp}{dt}$$

$$\frac{ds}{dt} = -2000\frac{1}{p^2}\frac{dp}{dt}\cos^2 s$$

In the above equation, we know $p = 8000$ and $dp/dt = 500$. We have to find the cosine. If $\tan s = 1/4$ then $\cos s = \sqrt{16/17}$.

$$\frac{ds}{dt} = -2000\ \frac{1}{8000^2}\ 500\ \frac{16}{17}$$

$$\frac{ds}{dt} = -\frac{1}{4}\frac{1}{16}\frac{16}{17} = -\frac{1}{68} = -0.0147$$

For the parachutist

$$\frac{d}{dt}\tan u = \sec^2 u\frac{du}{dt} = \frac{d}{dt}\frac{h}{1000} = \frac{1}{1000}\frac{dh}{dt}$$

$$\frac{du}{dt} = \frac{1}{1000}\frac{dh}{dt}cos^2 u$$

In the above equation, we know $dh/dt = -10$. We have to find the cosine. If $\tan u = 2$ then $\cos u = 1/\sqrt{5}$. So

$$\frac{du}{dt} = 0.001\ (-10)\ \frac{1}{5} = -0.002$$

Since $\theta = u - s$

$$\frac{d\theta}{dt} = \frac{du}{dt} - \frac{ds}{dt} = -0.002 + 0.0147 = 0.0127$$

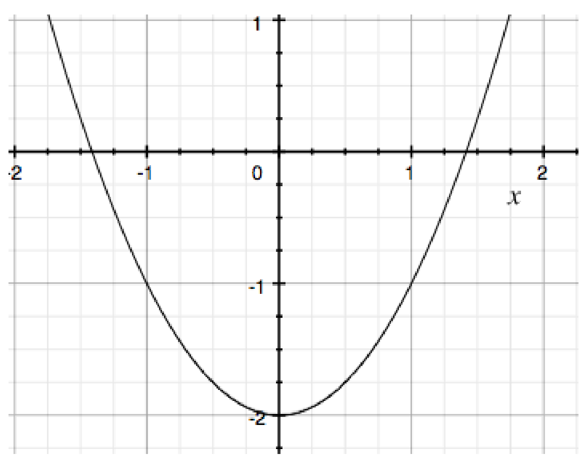
The angle θ between plane and parachutist is *increasing* with time (about 3/4 of a degree per second).

Chapter 66

Newton square root

Here is a method for finding roots of equations quickly, often called Newton's method, or the Newton-Raphson method. As an example, here is a plot of the function

$$f(x) = x^2 - 2$$



which is equal to zero when $x = \pm\sqrt{2}$. That is, we want the roots of the following equation

$$x^2 - 2 = 0$$

and more generally

$$x^2 - N = 0$$

to find the square root of some other number.

Pick a point g (for guess). Then we need to construct the line tangent to the curve at that point, with slope $m = f'(g)$ and ask, where does this line intercept the x-axis?

The slope is $\Delta y / \Delta x$.

$$\frac{f(g) - 0}{g - r} = f'(g)$$

with r being the x-coordinate at the intercept. Rearrange

$$\frac{f(g)}{f'(g)} = g - r$$

$$r = g - \frac{f(g)}{f'(g)}$$

square root problem

For this particular problem, we have

$$f(g) = g^2 - N$$

$$f'(g) = 2g$$

$$r = g - \frac{g^2 - N}{2g} = \frac{1}{2}(g + \frac{N}{g})$$

In other words, r is the average of g and N/g.

Now set $g = r$ and repeat.

This can be encapsulated into the following algorithm:

- Make a guess g and compute N/g

- Average g and N/g to find a new guess
- Repeat until satisfied

The algorithm converges rapidly for most problems.

```
2
1.5
1.4166666666666665
1.4142156862745097
1.4142135623746899
1.414213562373095
```

notes

It is worthwhile to try to make a good first guess. If the method goes wrong (which it can, when equations have bumps or other issues), the problem can be fixed by making a better guess.

You can read more about the method here:

http://www.math.ubc.ca/~anstee/math104/newtonmethod.pdf

For the particular problem that we worked, finding the square root of N, the equation says

$$g' = \frac{1}{2}(g + \frac{N}{g})$$

find your next guess by averaging the current guess g and N/g.

This equation goes back at least as far as Heron of Alexandria (10-70 AD).

Part XVIII

More series

Chapter 67

Binomial distribution

You are likely familiar with the binomial distribution from algebra. This distribution gives both the coefficients and the powers for each term in an expansion of the form

$$(a+b)^n$$

where n is a positive integer. The first four examples are:

$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

and in general

$$(a+b)^n = c_0a^n + c_1a^{n-1}b + c_2a^{n-2}b^2 + \cdots + c_{n-1}ab^{n-1} + c_nb^n$$

where each of the c_k may be different (though they are inversely symmetric, with $c_0 = c_n$ and $c_1 = c_{n-1} \ldots$).

Ignoring the powers $a^{n-k}b^k$ for the moment, the coefficients are

$$1 \quad 1$$

1 2 1

1 3 3 1

1 4 6 4 1

As you know, this pattern is called Pascal's Triangle. One striking thing about the triangle is that each value which is not on an edge can be formed by adding together the two values that lie directly above it.

The values running down the left and right sides are always 1.

Another approach is that, rather than formatting as a triangle, some people just make vertical columns:

1 1

1 2 1

1 3 3 1

1 4 6 4 1

The first column contains only 1's, the second column is the natural numbers (or positive integers), and the third is the "triangular" numbers, where the difference between successive numbers is the sequence of natural numbers.

We will be very interested in the general case, and therefore we need to set a convention for going across a row n using the index k. Notice that the fourth row $n = 4$ has five terms. It is convenient to let the index run from $k = 0$ to $k = n$.

We can formalize the observation that a value is the sum of the two values above it by saying that the coefficient at position k of row n is the sum of the values at positions k plus position $k-1$ of the preceeding row $n - 1$. For example, for $n = 4$, the $k = 2$ value is 6. 6 is the sum of the values at $k = 2$ and $k = 1$ of the row for $n = 3$. Continuing:

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

The corresponding expansions are:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

The power terms (leaving the coefficients out for the moment) are written in order as decreasing powers of a, starting from n, and increasing powers of b, starting from 0:

$$a^nb^0 + a^{n-1}b^1 + a^{n-2}b^2 + \cdots + a^2b^{n-2} + ab^{n-1} + a^0b^n$$

which is the same as

$$a^n + a^{n-1}b^1 + a^{n-2}b^2 + \cdots + a^2b^{n-2} + ab^{n-1} + b^n$$

understanding the pattern

In multiplying

$$(a+b)(a+b) = a^2 + 2ab + b^2$$

we can think of a generative procedure that goes term-by-term through the second multiplicand, multiplying first by a on the left

$$a \cdot (a+b) = a^2 + ab$$

and then by b on the right

$$(a+b) \cdot b = ab + b^2$$

Finally, add the two results together:

$$a \cdot (a+b) + (a+b) \cdot b$$
$$= a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2$$

We see that the two terms with the same power a^1b^1 come from multiplying $a \cdot (b)$ on the first pass, then adding $(a) \cdot b$ from the second.

This seems quite obvious, but it sets the stage for looking at a more complicated case.

In the expansion for $(a+b)^5$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

the coefficient of the $n = 5$, $k = 2$ term a^3b^2 is 10. We can get that 10 by taking two terms from the expansion for $(a+b)^4$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Namely, the $n = 4$, $k = 2$ term is multiplied by b

$$(4a^3b) \cdot b$$

and the $n = 4$, $k = 3$ term is multiplied by a

$$a \cdot (6a^2b^2)$$

when the two results are added together we obtain

$$(4a^3b) \cdot b + a \cdot (6a^2b^2) = 10a^3b^2$$

For the general case, we see that in writing the expansion for row $n+1$, at the kth term we form the power

$$a^{n+1-k}b^k$$

We have two contributions from the previous row. One comes from multiplying

$$a^{n+1-k}b^{k-1} \cdot b$$

and the second comes from

$$a \cdot a^{n-k}b^k$$

The term that is multiplied by a, namely $a^{n-k}b^k$, is found at the same column, immediately above $a^{n+1-k}b^k$.

The other one, $a^{n-k+1}b^{k-1}$ (the same as $a^{n+1-k}b^{k-1}$), is the term that precedes it. (Recall that powers of a get larger as we go to the left, and powers of b get smaller).

This explains why we obtain the pattern seen in Pascal's triangle as the coefficients for each expansion. As we said before: the coefficient at the position k of row n is the sum of the values at positions k and $k-1$ of row $n-1$.

choose

There is a very useful formula that allows one to find the coefficients of each term in the binomial expansion without actually filling out the triangle. It is called "n choose k".

The official way to write this expression is

$$\binom{n}{k}$$

The formula is:

$$\binom{n}{k} = \frac{n!}{(n-k!)\ k!}$$

This gives the number of different combinations of k objects one can choose from n total objects. For example, suppose you have 5 Bob Marley CD's, and you want to pick 2 of them to take to a party, that is "5 choose 2" and the formula is

$$\frac{5!}{3!\ 2!} = \frac{5 \times 4 \times 3 \times 2}{3 \times 2 \times 2} = \frac{120}{12} = 10$$

This is also the coefficient in row n, position k of Pascal's Triangle (indexing from $k = 0$).

For our example, that is row 5 position 2.

1 5 10 10 5 1

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

which is indeed 10.

Notice a cancelation we can do in our formula. We we rewrite the numerator with $n!$:

$$\frac{n \times (n-1) \cdots \times (n-k+1) \times (n-k)! \cdots}{(n-k)!\ k!}$$

We can cancel the factor of $(n-k)!$ and then have

$$\frac{n(n-1)\cdots(n-k+1)}{k!}$$

In the example:

$$\frac{5 \times 4 \times 3 \times 2}{3 \times 2 \times 2} = \frac{5 \times 4}{2} = \frac{20}{2} = 10$$

It is a good idea to memorize the combinations formula. It comes up repeatedly, in the derivations of calculus, and in probability.

quick derivation of the choose formula

Suppose you have n CD's and you want to pick 2 of them to take with you. There are n choices for the first one—maybe you choose *Natty Dread*. And there are $n-1$ choices for the second one, you pick *Kaya*.

That would give $n \cdot (n-1)$ here, and in the general case $n \cdot (n-1) \cdot \ldots (n-k+1)$. However, the order is not important, you might just as well have picked *Kaya* and then *Natty Dread*. That's why we have to correct for the over-counting. In this particular case we divide by 2, and in the general case divide by k!.

Binomial theorem: proof

A concise statement of the binomial formula is that the general term is of the form

$$\binom{n}{k} \, a^{n-k} \, b^k$$

and the whole sum is

$$\sum_{k=0}^{n} \binom{n}{k} \, a^{n-k} \, b^k$$

where (from the theory of combinations):

$$\binom{n}{k} = \frac{n!}{(n-k)! \; k!}$$

To do an actual calculation, we would first cancel the factor of $(n-k)!$ on top and bottom yielding

$$\binom{n}{k} = \frac{n \times (n-1) \cdots \times (n-k+1)}{k!}$$

Pascal's Lemma

In order to prove the theorem (using induction) we will need the following result:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Here is a simple proof of this result, from the theory of combinations. Imagine that we are considering how many ways there are of forming a committee of k members from a total of n people.

We know that the number of ways of doing this is of course just

$$\binom{n}{k}$$

Now, suppose that among these n people we focus on one particular person, call her Alice. Then there are two types of committees in our collection of combinations: those in which Alice is a member, and those in which she is not.

For all committees of the first type, in addition to Alice, the other $k-1$ members must be drawn from $n-1$ people. The number of ways of doing this is

$$\binom{n-1}{k-1}$$

For the second case, where Alice is not a member, we must recruit all k members from $n-1$ people, since we are leaving Alice out. The number of ways of doing this is

$$\binom{n-1}{k}$$

But putting them together, these must be equal to the total number obtained by the standard analysis, and hence we have that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

This is effectively what we said near the beginning of the introduction to the binomial theorem: in computing the kth coefficient in the nth row, we add the k and $k-1$ values from the preceding row.

This preliminary result is called Pascal's Lemma and it is really the heart of the proof.

In using it below, we will alter its form slightly by substituting $n+1$ for n. Thus:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Induction

If we look at one more term in the expansion for $(a+b)^n$, writing the term preceding the one given above, we have

$$(a+b)^n = \cdots + \binom{n}{k-1}\ a^{n-k+1}\ b^{k-1} + \binom{n}{k}\ a^{n-k}\ b^k + \dots$$

Notice that the exponent decreases by one for a as we move to the right, while it increases by one for b.

When we form the new general term in the expansion for $(a+b)^{n+1}$, as we said before, we multiply the first term by b and the second one by a, obtaining

$$\cdots + b\binom{n}{k-1}\ a^{n-k+1}\ b^{k-1} + a\binom{n}{k}\ a^{n-k}\ b^k + \dots$$

$$= \cdots + \binom{n}{k-1} \; a^{n-k+1} \; b^k + \binom{n}{k} \; a^{n-k+1} \; b^k + \dots$$

But these two powers are the same, and since

$$n - k + 1 = (n+1) - k$$

their sum is the general term in the expansion for $(a+b)^{n+1}$

$$= \cdots + C \; a^{(n+1)-k} \; b^k + \dots$$

In adding them, we add their coefficients:

$$\cdots + \; [\; \binom{n}{k-1} + \binom{n}{k} \;] \; a^{(n+1)-k} \; b^k + \dots$$

Referring back to Pascal's Lemma, we substitute

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

yielding

$$\cdots + \; \binom{n+1}{k} + a^{(n+1)-k} \; b^k + \dots$$

We obtain the general term for the $n+1$ expansion. This completes the inductive part of the proof. It remains to check the binomial formula for a base case like $n = 1$ or $n = 2$, which I invite you to do.

$\square$

Courant's proof

Courant gives a proof by induction which I've transcribed and annotated separately. It duplicates significant parts of what we've already gone through, but there is some new material, which I've added in here.

Pascal's Lemma

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

was "proved" above by an argument about combinations which do or do not include a person named Alice. In Courant's version, the factorials are manipulated to provide a direct mathematical proof.

simplification

As we saw before, in the choose formula, there are terms that cancel. We expand $n!$ partially

$$n! = n \cdot (n-1) \ldots (n-k+1) \cdot (n-k)!$$

The last term is also present in the denominator of our formula

$$C(n,k) = \frac{n!}{k!\ (n-k)!}$$

so we can simplify

$$C(n,k) = \frac{n \cdot (n-1) \ldots (n-k+1)}{k!}$$

We will be interested in coefficients with $k+1$, so let's take a look at n "choose" $k+1$. The original definition would give:

$$C(n,k+1) = \frac{n!}{(k+1)!\ (n-(k+1))!}$$

Remove one set of parentheses

$$= \frac{n!}{(k+1)!\ (n-k-1)!}$$

By the same argument, expand $n!$

$$= \frac{n \cdot (n-1) \dots (n-k+1) \cdot (n-k) \cdot (n-k-1)!}{(k+1)!\ (n-k-1)!}$$

$$= \frac{n \cdot (n-1) \dots (n-k+1) \cdot (n-k)}{(k+1)!}$$

You should convince yourself that the last term in the numerator is correct. It seems somewhat counterintuitive that for $k+1$ the last term is $(n-k)$ rather than say, $(n-k+2)$, as I thought at first.

induction

Let us examine the general statement

$$C(n,k) + C(n,k+1)$$

Rewriting it as the factorial using the simplification we found above

$$\frac{n \cdot (n-1) \dots (n-k+1)}{k!} + \frac{n \cdot (n-1) \dots (n-k+1) \cdot (n-k)}{(k+1)!}$$

We can factor out $(n-k)/(k+1)$ from the second term:

$$= (\frac{n-k}{k+1}) \ \cdot \frac{n \cdot (n-1) \dots \ (n-k+1)}{k!}$$

and we see that the second term is that factor multiplied by the first term.

Hence, the complete sum becomes

$$[\ 1 + (\frac{n-k}{k+1})\ ] \ \cdot \frac{n \cdot (n-1) \dots \ (n-k+1)}{k!}$$

Take the leading factor and put it over a common denominator

$$\frac{(k+1)+(n-k)}{k+1} = \frac{n+1}{k+1}$$

so the expression now becomes

$$= (\frac{n+1}{k+1}) \cdot \frac{n \cdot (n-1) \ldots\ (n-k+1)}{k!}$$

$$= \frac{(n+1) \cdot n \cdot (n-1) \ldots\ (n-k+1)}{(k+1)!}$$

rearranging the last term in the numerator slightly

$$= \frac{(n+1) \cdot n \cdot (n-1) \ldots\ [\ (n+1)-k\ ]}{(k+1)!}$$

$$= C(n+1, k+1)$$

This is the correct expression for $n+1$ (because it has $n+1$ in the right places), and it is the correct expression for $k+1$ because it ends with $n+1$ minus k (rather than $k+1$).

recap

We assume that $C(n,k)$ is the correct coefficient for $a^{n-k}b^k$ in the expansion of $(a+b)^n$, and that $C(n,k+1)$ is the correct coefficient for the succeeding term $a^{n-(k+1)}b^{k+1}$ in the same expansion. Multiplication of the first by b and the second by a leads to:

$$[\ C(n,k) + C(n,k+1)\ ]\ a^{n-k}b^{k+1}$$

We need to tweak one exponent slightly when considering this as part of the expansion for $(a+b)^{n+1}$. The n in the exponent for a should be expressed in terms of n referring to the incremented value $n+1$

so it needs to step down one unit, becoming a^{n-k-1} which is equal to $a^{n-(k+1)}$. Thus we have

$$[\ C(n,k) + C(n,k+1)\ ]\ a^{n-(k+1)} b^{k+1}$$

We showed that

$$C(n,k) + C(n,k+1) = (1 + \frac{n-k}{k+1}) \cdot C(n,k)$$

$$= \frac{n+1}{k+1}) \cdot C(n,k)$$

$$= C(n+1,k+1)$$

which is what the binomial theorem gives. This completes the proof by induction. □

probability

The binomial theorem is also used extensively in working with discrete probability.

A simple example is to consider a standard (6-sided), fair die. We ask:

What is the probability that in rolling six such dice (or one die six times), we will observe some distribution of results, like one or more 6's?

There is an easy way to do this. We calculate the probability that no 6 is observed on one roll as 5/6 and on six rolls as

$$(5/6)^6 = 15625/46656 = 0.334898$$

The outcome of at least one 6 is the complement of this so its probability is 0.665102.

The other way to do it is to use the formula that gives the distribution over all possible outcomes:

$$C(n,k)p^k(1-p)^{n-k}$$

In this case, we have

```
C(6,0) = 1
C(6,1) = 6!/(1! 5!) = 6
C(6,2) = 6!/(2! 4!) = 15
C(6,3) = 6!/(3! 3!) = 720/36 = 20
C(6,4) = 15
C(6,5) = 6
C(6,6) = 1

P(0 6's) = 1          (5/6)^6 = 0.334898
P(1 6's) = 6  (1/6)   (5/6)^5 = 0.401878
P(2 6's) = 15 (1/6)^2 (5/6)^4 = 0.200939
P(3 6's) = 20 (1/6)^3 (5/6)^3 = 0.053584
P(4 6's) = 15 (1/6)^4 (5/6)^2 = 0.008038
P(5 6's) = 6  (1/6)^5 (5/6)^1 = 0.000643
P(6 6's) = 1  (1/6)^6         = 0.000021
```

The total is 1.000001, or 1, within the error introduced by truncation.

Suppose instead that we flip a (fair) coin 100 times. What will be the distribution then? Since p = 1 - p = 0.5, the power terms are all the same, namely

$$0.5^{100} = 7.8886e - 31$$

But the factorials are kind of awkward. For example, what is

$$100!/50!50! = ?$$

100! has 158 digits in it. Python can do this calculation, the result is $1.008913e+29$ so we end up with $P = 0.0796$.

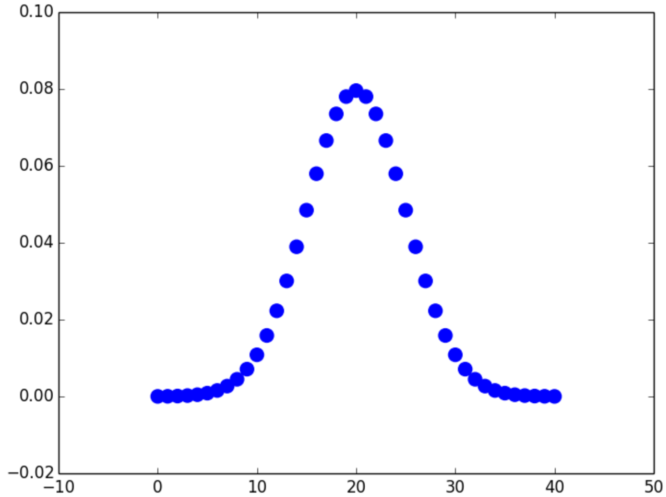
We use Python to compute the factorials in an efficient way, and tabulate the values for the probability from 35-50

```
35 0.000863855665742
36 0.00155973939648
37 0.00269792760472
38 0.00447287997624
39 0.00711073226993
40 0.0108438667116
41 0.0158690732365
42 0.0222922695466
43 0.0300686426442
44 0.0389525597891
45 0.0484742966264
46 0.0579583981403
47 0.066590499991
48 0.0735270104067
49 0.0780286641051
50 0.0795892373872
```

The distribution is symmetrical about the midpoint.

Adding up values gives the result that 99.8% of the total probability lies between 35-65 inclusive.

Here is a plot of the values from 30-70.



Look familiar?

Chapter 68

Taylor series

Suppose we have a function $f(x)$, but

Shankar:

> "imagine that you don't have access to the whole function. You cannot see the whole thing. You can only zero-in on a tiny region."

around $f(0)$, where you know the value. So the question is, what do we guess the function will do near $f(0)$?

The first approximation is that

$$f(x) \approx f(0)$$

We really can't say anything more. $f(0)$ is the best guess for what the value of the function is (we're talking about continuous and continuously differentiable functions).

Now suppose we know the slope of the function at 0, $f'(0)$. Then, since

$$\Delta y = f'(0)\Delta x = f'(0)(x - 0)$$

we can get a better approximation as the linear approximation:

$$f(x) \approx f(0) + f'(0)\ x + \dots$$

For most functions, there will be more terms. If f is not a linear function, then the slope won't be constant. So

> "the rate of change itself has a rate of change .. the second derivative."

The term we are going to add is

$$f''(0)\ \frac{x^2}{2}$$

so

$$f(x) \approx f(0) + f'(0)\ x + f''(0)\ \frac{x^2}{2} + \dots$$

A simple way to see why we have $x^2/2$ is to take derivatives on both sides. The terms like $f'(0)$ and $f''(0)$ are constants, they have been evaluated at $x = 0$. The first derivative is

$$f'(x) \approx f'(0) + f''(0)\ x + \dots$$

We evaluate at $x = 0$ and the term $f''(0)\ x$ goes away because of the $x = 0$ multiplying the constant $f''(0)$. So we have just

$$f'(x) \approx f'(0)$$

and that matches. Now take the second derivative

$$f''(x) \approx f''(0)$$

and that matches too. We can see a pattern here.

The fourth term is

$$f(x) \approx f(0) + f'(0)\ x + f''(0)\ \frac{x^2}{2!} + f'''(0)\ \frac{x^3}{3!} + \dots$$

You might not be expecting the factorial which I snuck in there. But if you go back to the exercise above, where we evaluated derivatives, you can see why it works. When we take the first derivative

$$\frac{d}{dx}(f'''(0)\ \frac{x^3}{3!}) = f'''(0)\ \frac{x^2}{2!}$$

the 3 comes down from the power and then turns 3! in the denominator into 2!. The next derivative will bring down the 2. So everything cancels properly.

If you like Σ notation, we can write

$$f(x) = \sum_{n=0}^{\infty} f^n(0)\frac{x^n}{n!}$$

with the understanding that $0! = 1$. The approximation is better the closer x is to 0, and the more terms the better as well.

There is one final wrinkle to this derivation. The series can be modified deal with x near any value a, not just near 0. The modification is

$$f(x) = \sum_{n=0}^{\infty} f^n(a)\frac{(x-a)^n}{n!}$$

This is the Taylor series. The series near $a = 0$ is known as the Maclaurin series.

1/1-x

The first example is

$$f(x) = \frac{1}{1-x}$$

We know the answer to this.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3$$

Proof:

$$1 = (1-x)(1+x+x^2+x^3)$$

Multiplying by 1, the second term x is matched by $-x$ from the first term in the multiplication by $-x$, and so on. The whole thing vanishes, leaving just 1.

We want to evaluate $f(x)$ near 0, let's say, at $x = 0.1$. The correct value of the function is

$$f(x) = \frac{1}{0.9} = 1.11111\ldots$$

Let's try to approximate using the series. We need derivatives

$$f(x) = \frac{1}{1-x}$$

$$f'(x) = \frac{1}{(1-x)^2} = (1-x)^{-2}$$

$$f'(0) = 1$$

so the linear approximation is

$$f(x) \approx 1 + 1x = 1.1$$

For the next term we obtain

$$f''(x) = 2(1-x)^{-3}$$

The 2 is cancelled by the 2! in the denominator, so this cofactor is 1 and we're left with

$$f''(0)\ \frac{x^2}{2} = x^2 = 0.01$$

And I think we can see where this one is going.

However, you probably remember that this series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

diverges for $|x| \geq 1$, and the Taylor series does too.

The morale of the story is that for some series, there is a radius of convergence and the series is only valid for x within that radius.

binomial

Another very useful series is the binomial.

$$f(x) = (1+x)^n$$

$$f(0) = 1$$
$$f'(0) = n(1+x)^{n-1} = n$$
$$f''(0) = n(n-1)(1+x)^{n-2} = n(n-1)$$

So the series is

$$(1+x)^n \approx 1 + nx + n(n-1)\frac{x^2}{2}$$

We use this one a lot.

A nice application is relativistic energy

$$E = mc^2 f$$

$$f = 1/\sqrt{1 - \frac{v^2}{c^2}}$$

This is, in disguise, a binomial with $n = -1/2$ and $x = -v^2/c^2$ so the expansion is

$$f \approx 1 + nx = 1 + \frac{v^2}{2c^2}$$

so the energy is

$$E \approx mc^2(1 + \frac{v^2}{2c^2})$$

And we see that the second term is just the kinetic energy, $mv^2/2$.

polynomials

The beauty of Taylor Series (despite its complexity) is that it turns any differentiable function into a polynomial. Polynomials are easy to integrate and work with.

The first thing to say about Taylor Series is they give the correct answer for functions that we know. For example, suppose we have

$$f(x) = ax^2 + bx + c = 1$$

We get the derivatives and evaluate them "near" the point $x = 0$.

$$f(x) = ax^2 + bx + c = c$$

$$f'(x) = 2ax + b = b$$

$$f''(x) = 2a$$

The series is then

$$f(x) = c + b(x) + \frac{21}{2!}(x)^2 + \cdots$$

But there are no more terms. That's it. And this is just

$$f(x) = c + bx + ax^2$$

exponential, sine and cosine

Suppose $f(x) = e^x$ and again, we evaluate "near" $x = 0$. We have

$$f(x) = e^x = 1$$

$$f'(x) = e^x = 1$$

$$f''(x) = e^x = 1$$

The series is

$$f(x) = e^x = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots$$

$$f(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Which matches what we already know about e^x. For example, it is obvious that

$$\frac{d}{dx}e^x = e^x$$

Let's try to find something new. Suppose we expand $f(x) = cos\ x$ near $x = 0$

$$f(x) = cos\ x = cos\ 0 = 1$$

$$f'(x) = -sin\ x = -sin\ 0 = 0$$

$$f''(x) = -cos\ x - cos\ 0 = -1$$

$$f'''(x) = sin\ x = sin\ 0 = 0$$

$$f''''(x) = cos\ x = cos\ 0 = 1$$

and this continues in a cycle with period 4. The series is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$f(x) = cos\ x = 1 - \frac{1}{2!}(x-0)^2 + \frac{1}{4!}(x-0)^4 + \cdots$$

$$f(x) = cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Similarly, for $f(x) = sin\ x$ near $x = 0$

$$f(x) = sin\ x = 0$$

$$f'(x) = cos\ x = 1$$

$$f''(x) = -sin\ x = 0$$

$$f'''(x) = -cos\ x = -1$$

$$f''''(x) = sin\ x = 0$$

The series is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$f(x) = sin\ x = x - \frac{1}{3!}(x-0)^3 + \frac{1}{5!}(x-0)^5 + \cdots$$

$$f(x) = sin\ x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

funny series

In Strogatz book (*The Joy of x*), he gives the following series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

and he says that the sum of the series is equal to the natural logarithm of 2:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

with the provision that you have to calculate the sum in the order given.

For example, the second, third and fourth partial sums are:

$$S_2 = \frac{1}{2}; \quad S_3 = \frac{5}{6}; \quad S_4 = \frac{14}{24}; \quad S_5 = \frac{94}{120}$$

with $S_4 = 0.583$ and $S_5 = 0.783$. For any partial sum S_n and the previous sum S_{n-1} the value of the series will be bounded by the two sums.

I thought I would try to show that $\ln 2$ is the correct value for series, by using a Taylor series for the logarithm.

Taylor says we can write a function $f(x)$ (near the value $x = a$) as an infinite sum

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

where f^n means the nth derivative of f and f^0 is just f, and these derivatives are to be evaluated at $x = a$. Near $a = 0$ this simplifies to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n$$

Let's calculate the derivatives of the logarithm:

$$f^0 = \ln x; \quad f^1 = \frac{1}{x} = x^{-1}; \quad f^2 = -x^{-2}; \quad f^3 = 2x^{-3}; \quad f^4 = -3!\, x^{-4}$$

The first thing I notice is that we can't use $a = 0$, since $f^1 = 1/x$ is undefined there. So, let's try $a = 1$. Then (evaluated at $a = 1$)

$$f^0 = \ln x = 0; \quad f^1 = \frac{1}{x} = 1; \quad f^2 = -x^{-2} = -1; \quad f^3 = 2; \quad f^4 = -3!$$

Going back to the definition

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

I get the following series near $a = 1$:

$$\ln x = \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \cdots$$

For the special value $x = 2$, all the terms $(x-1)^n$ go away (which confirms that $a = 1$ is an excellent choice!). We have then

$$\begin{aligned} \ln x &= \frac{0}{0!} + \frac{1}{1!} - \frac{1}{2!} + \frac{2}{3!} - \frac{3!}{4!} + \cdots \\ &= 0 + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \end{aligned}$$

which is what was to be proved.

Chapter 69

Series convergence

Tests for convergence

Series like the Taylor series can be very helpful in approximating a function. Here we review some common tests to see whether the sum of an infinite series converges to a finite limit, or instead diverges.

Start with the geometric series

$$\sum_{k=0}^{\infty} x^k$$

Suppose we compute the sum of a number of terms n

$$s_n = x^0 + x^1 + x^2 + \cdots + x^n$$

Since this is a finite series, the sum exists so

$$xs_n = x^1 + x^2 + \cdots + x^{n+1}$$

Then

$$s_n - xs_n = (1-x)s_n = 1 - x^{n+1}$$

$$s_n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}, \quad x \neq 1$$

Clearly, if $x > 1$ then $x^{n+1} \to \infty$ as n gets large, and this is true even if $x = 1$. If $x < -1$ then the second term alternates in sign and its absolute value gets very large as n gets large.

Only for $|x| < 1$, as $n \to \infty$, the second term vanishes and we have

$$s_n = \frac{1}{1-x}$$

For example,

$$\sum_{k=0}^{\infty} (\frac{1}{2})^k = \frac{1}{1-1/2} = 2$$

$$\sum_{k=0}^{\infty} (\frac{1}{3})^k = \frac{1}{1-1/3} = \frac{3}{2}$$

and so on.

A second famous series is the harmonic series

$$\sum_{k=0}^{\infty} \frac{1}{k^p}$$

especially with $p = 1$

$$\sum_{k=0}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

This series diverges.

One proof is the following. Assume that the series converges. Then its sum has a limit which we can call L.

$$L = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \dots$$

$$
\begin{aligned}
&> \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} \dots \\
&= 1 + \frac{1}{2} + \frac{1}{3} \dots \\
&= L
\end{aligned}
$$

a contradiction. Therefore, the harmonic series diverges.

Let us look at some tests of convergence.

Divergence test

The first test requires that the limit of the individual terms a_k must tend to zero

$$\lim_{k \to \infty} a_k = 0$$

if not, then the sum diverges. For example,

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots$$

This clearly diverges, since

$$\lim_{k \to \infty} a_k = 1$$

Or

$$
\begin{aligned}
&\lim_{k \to \infty} \frac{k}{2k+1}, \quad k \in \{1, 2, \dots\} \\
&= \lim_{k \to \infty} \frac{1}{2 + 1/k} = \frac{1}{2} \neq 0
\end{aligned}
$$

Another example is the harmonic series

$$\lim_{k \to \infty} \frac{1}{k} = 0, \quad k \in \{1, 2, \dots\}$$

Despite passing this limit test, the harmonic series diverges. Thus, a pass is necessary but not sufficient.

Integral test

The integral test says that a (well-behaved) function $f(x)$

$$\int_1^\infty f(x)\ dx$$

converges $\iff$

$$\sum_{k=1}^\infty f(k)$$

also converges. The function must be continuous and integrable, etc.

Let's apply this test to the harmonic series

$$\sum_{k=0}^\infty \frac{1}{k}$$

We have

$$\int_1^\infty \frac{1}{x}\ dx = \ln|x|\ \Big|_1^\infty$$

but the upper bound has the limit

$$\lim_{k\to\infty} \ln|k| = \infty$$

In general, for

$$\sum_{k=0}^\infty \frac{1}{k^p}$$

if $p > 1$, the sum converges, but not otherwise:

$$\int_1^\infty x^{-p}\ dx = \frac{1}{1-p} x^{1-p}\ \Big|_1^\infty$$

For $p > 1$

$$\lim_{x\to\infty} x^{1-p} = 0$$

On the other hand

$$\int_1^\infty \frac{1}{n^2}\ dn = -\frac{1}{n}\ \Big|_1^\infty = 0 - -1 = 1$$

so the $\sum 1/n^2$ converges.

Comparison test

If we compare a series and a convergent series and the test series is smaller term-by-term, then it also converges. Similarly, if a series is larger than a divergent series when compared term-by-term, it also diverges. Any finite number of terms from the beginning of a series may be disregarded before starting the comparison.

Since

$$\sum_{k=0}^{\infty} \frac{1}{k^2}$$

converges, so does

$$\sum_{k=0}^{\infty} \frac{1}{k^2 + 10}$$

And since

$$\sum_{k=0}^{\infty} \frac{1}{k}$$

diverges, so does

$$\sum_{k=0}^{\infty} \frac{1}{\ln|k+1|}$$

since for $k > 2$

$$\ln|k+1| < k$$

so

$$\frac{1}{\ln|k+1|} > \frac{1}{k}$$

Ratio test

Consider
$$\sum_{k=0}^{\infty} a_k, \quad a_k > 0$$
$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L$$

$$\begin{cases} L < 1 & \text{: converges} \\ L > 1 & \text{: diverges} \\ L = 1 & \text{: inconclusive} \end{cases}$$

As an example
$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

Check
$$\lim_{k\to\infty} \frac{1/(k+1)!}{1/k!} = \frac{1}{k+1} < 0$$

This one is also easily checked by the comparison test since

$$k! > k^2, \quad k > 3$$

Since $1/k^2$ converges, so does $1/k!$.

Or
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$
$$\lim_{k\to\infty} \frac{1/n+1}{1/n} = \frac{n}{n+1} = 1$$

so the test is inconclusive.

a bit more

In the previous chapter, we looked at Taylor series and showed that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

Actually, this series is really interesting from the point of view of convergence. An infinite series is convergent if all of the terms add up to something finite, like $\ln 2$.

In this case, the terms alternate sign, which makes me wonder. We have the positive terms

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \ldots$$

The two terms

$$\frac{1}{5} + \frac{1}{7} = \frac{12}{35} = 0.343 > \frac{1}{3}$$

The next four terms

$$\frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15}$$

$$= \frac{715}{6435} + \frac{585}{6435} + \frac{495}{6435} + \frac{429}{6435} = \frac{2224}{6435}$$

$$= 0.346 > \frac{1}{3}$$

Notice that the decimal sum is increasing.

This is sorta like the harmonic series. It diverges to $+\infty$.

The negative terms are

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \cdots$$

$$= (-\frac{1}{2}) \cdot (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

which *is* the harmonic series. It diverges to $-\infty$.

So two series which separately diverge to $+\infty$ and $-\infty$ add to something finite!

Another amusing rearrangement of the $\ln 2$ series (Acheson) is:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{7} + \frac{1}{8} + \cdots$$

$$= (1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + (\frac{1}{5} - \frac{1}{10}) - \frac{1}{12} + \dots$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \dots$$

$$= \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots)$$

How can the series be equal to one-half of itself?

These groupings of terms with alternating signs are illegal. They do not yield correct values for the sum. However, an adjacent grouping does work, namely:

$$(1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + (\frac{1}{7} - \frac{1}{8}) + \cdots$$

$$= \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \frac{1}{56} + \dots$$

which, as you can see, converges rather slowly.

Chapter 70

Newton binomial

standard binomial

As you know, the binomial expansion for the first few positive integers n is

$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

$$(a+b)^n = \sum_{k=0}^{k=n} c_k \; a^{n-k}b^k$$

The coefficients c_k are given by Pascal's triangle or by computing

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

For

$$k=0, \qquad \frac{n!}{0!(n-0)!} = 1$$

$$k = 1, \qquad \frac{n!}{1!(n-1)!} = n$$

$$k = 2, \qquad \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2!}$$

$$k = 3, \qquad \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{3!}$$

This can also be written as

$$= \frac{n}{1} \cdot \frac{(n-1)}{2} \cdot \frac{(n-2)}{3}$$

Let us just consider binomials of the form $a = 1$, so then substitute x for b

$$(1+x)^n = \sum_{k=0}^{k=n} c_k \ x^k$$

The expansion becomes ($x^0 = 1$)

$$1 + n \cdot x + n \cdot \frac{(n-1)}{2} \cdot x^2 + \frac{n}{1} \cdot \frac{(n-1)}{2} \cdot \frac{(n-2)}{3} \cdot x^3 + \dots$$

This series terminates when $n = k$, and the coefficients are symmetric about $k = n/2$. If n is even, the two middle terms of the sequence are equal.

Now, the natural question is, what will happen if we substitute r for n, where is r is a rational exponent, or may even be negative?

Newton

Newton wrote the binomial expansion in this way

$$(P + PQ)^{m/n} = P^{m/n} + \frac{(m)}{(n)} P^{m/n} Q + \frac{(m)}{(n)} \frac{(m-n)}{(2n)} P^{m/n} Q^2 + \cdots$$

$$+\frac{(m)}{(n)}\frac{(m-n)}{(2n)}\frac{(m-2n)}{(3n)}P^{m/n}Q^3+\cdots$$

This looks a little strange to modern eyes, but it's actually the same as the standard binomial.

Notice that we can factor out $P^{m/n}$ so

$$(1+Q)^{m/n}=1+\frac{(m)}{(n)}Q+\frac{(m)}{(n)}\frac{(m-n)}{(2n)}Q^2+\frac{(m)}{(n)}\frac{(m-n)}{(2n)}\frac{(m-2n)}{(3n)}Q^3+\cdots$$

Then just bring n up into the numerator

$$(1+Q)^{m/n}=1+\frac{m/n}{1}Q+\frac{(m/n)}{1}\frac{(m/n-1)}{2}Q^2+$$

$$+\frac{(m/n)}{1}\frac{(m/n-1)}{2}\frac{(m/n-2)}{3}Q^3+\cdots$$

Substitute r for m/n

$$(1+Q)^r=1+rQ+r\frac{(r-1)}{2}Q^2+r\frac{(r-1)}{2}\frac{(r-2)}{3}Q^3+\cdots$$

This is the binomial with $x=Q$.

One difference is that Newton used it for negative integers and even for a fractional power. A key change is that in these cases the series becomes infinite.

usage

We can use this to compute roots. Suppose $Q=1$ and $r=1/3$, so we are looking for the cube root of 2. The terms of the series are $1+1/3+\ldots$

$$\frac{1}{3} \cdot \frac{-2/3}{2} = -\frac{1}{3^2}$$
$$-\frac{1}{3^2} \cdot \frac{-5/3}{3} = \frac{5}{3^4}$$
$$\frac{5}{3^4} \cdot \frac{-8/3}{4} = -\frac{10}{3^5}$$
$$-\frac{10}{3^5} \cdot \frac{-11/3}{5} = \frac{22}{3^6}$$
$$\frac{22}{3^6} \cdot \frac{-14/3}{6} = \frac{154}{3^8}$$
$$..$$

$$= 1 + \frac{1}{3} - \frac{1}{9} + \frac{5}{81} - \frac{10}{243} + \frac{22}{729} - \frac{154}{6561} \dots$$

which seems pretty close (but not that close) to 1.259921.

Another use is to obtain a series for $1/(1+x)$.

$$(1+Q)^r = 1 + rQ + r\frac{(r-1)}{2}Q^2 + r\frac{(r-1)}{2}\frac{(r-2)}{3}Q^3 + \cdots$$
$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots$$

Newton checked this by multiplying:

$$1 = (1+x)(1 - x + x^2 - x^3 + x^4 + \cdots)$$

And if you know that the area under this curve is the logarithm, you can integrate the series for $1/(1+x)$ to obtain

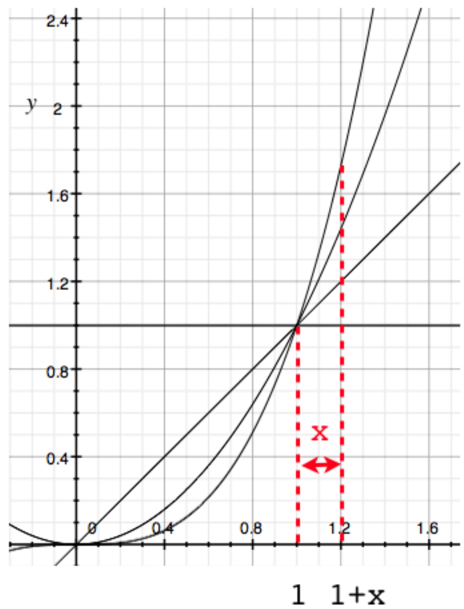
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

The logarithms obtained by this method can be easily verified.

derivation

So far as I know Newton did not provide a proof of his version of the binomial theorem. I found a discussion of how he came to it here:

http://www.quadrivium.info/MathInt/Notes/NewtonBinomial.pdf



Following Wallis, Newton studied expressions for the area under various curves.

powers of x

Above: $y = x^n, n = \{0, 1, 2\}$. These areas were apparently known prior to Newton, and I am not sure exactly how this was done, but I will use results obtained by integration.

It's perhaps a little confusing, but we use x here for two purposes. First, the curves are $f(x) = x^n$. Second, we seek the area under each of these curves between the endpoints 1 and $1 + x$. So for each n, we will compute the integral of x^n and evaluate that between the limits 1 and $1+x$.

- For $y = x^0 = 1$, the area under the curve is just $x \cdot 1 = x$.
- For $y = x$, the area under the curve is

$$\frac{x^2}{2}\Bigg|_1^{1+x} = \frac{(1+x)^2-1}{2} = \frac{2x+x^2}{2} = x + \frac{x^2}{2}$$

- For $y = x^2$, the area under the curve is

$$\frac{x^3}{3}\Bigg|_1^{1+x} = \frac{(1+x)^3-1}{3} = \frac{3x+3x^2+x^3}{3}$$

$$= x + x^2 + \frac{x^3}{3}$$

- For $y = x^3$, the area under the curve is

$$\frac{x^4}{4}\Bigg|_1^{1+x} = \frac{(1+x)^4-1}{4} = \frac{4x+6x^2+4x^3+x^4}{4}$$

$$= x + \frac{3}{2}x^2 + x^3 + \frac{x^4}{4}$$

which we can write as

$$= x + \frac{3}{2}x^2 + \frac{3}{3}x^3 + \frac{x^4}{4}$$

- For $y = x^4$, the area under the curve is

$$\frac{x^5}{5}\Bigg|_1^{1+x} = \frac{(1+x)^5-1}{5}$$

$$= \frac{5x+10x^2+10x^3+5x^4+x^5}{5}$$

$$= x + 2x^2 + 2x^3 + x^4 + \frac{x^5}{5}$$

which we can write as

$$x + \frac{4}{2}x^2 + \frac{6}{3}x^3 + \frac{6}{4}x^4 + \frac{1}{5}x^5$$

∘ For $y = x^5$, we would find

$$x + \frac{5}{2}x^2 + \frac{10}{3}x^3 + \frac{10}{4}x^4 + \frac{5}{5}x^5 + \frac{1}{6}x^6$$

If we look carefully at what we've obtained, we see that there is a sum of terms like x^p/p times a cofactor which goes like Pascal's triangle or a standard binomial expansion (and indeed, that's where it came from). Newton organized the cofactors into a table.

table

p	0	1	2	3	4	5
$x/1$	1	1	1	1	1	1
$x^2/2$	0	1	2	3	4	5
$x^3/3$	0	0	1	3	6	10
$x^4/4$	0	0	0	1	4	10
$x^5/5$	0	0	0	0	1	5
$x^6/6$	0	0	0	0	0	1
$x^7/7$	0	0	0	0	0	0

That is, we have for x^2 that the area is

$$(1)x + (2)\frac{1}{2}x^2 + (1)\frac{1}{3}x^3 = x + x^2 + \frac{1}{3}x^3$$

Newton noticed (as did Pascal) that the pattern of coefficients can be generated by addition. For example, the 6 in the column for $n = 4$

is generated by adding together the entry to its immediate left (3), plus the entry above that (also 3). Thus, having the first row (all 1), and the column under $n = 0$, one can generate the rest of the table mechanically.

Now, Newton says, what happens if we add an additional column for $n = -1$, and we make a rule that the entry in the first row must be 1, because all the other columns have this value.

p	-1	0	1	2	3	4	5
$x/1$	1	1	1	1	1	1	1
$x^2/2$	.	0	1	2	3	4	5
$x^3/3$	.	0	0	1	3	6	10
$x^4/4$	.	0	0	0	1	4	10
$x^5/5$	.	0	0	0	0	1	5
$x^6/6$	.	0	0	0	0	0	1
$x^7/7$	.	0	0	0	0	0	0

How do we fill in the missing entries? By using the addition rule! The first missing value must be a -1, so that it plus the 1 above add together to give the 0 to its right.

p	-1	0	1	2	3	4	5
$x/1$	1	1	1	1	1	1	1
$x^2/2$	-1	0	1	2	3	4	5
$x^3/3$	.	0	0	1	3	6	10
$x^4/4$	.	0	0	0	1	4	10
$x^5/5$	.	0	0	0	0	1	5
$x^6/6$	.	0	0	0	0	0	1
$x^7/7$	.	0	0	0	0	0	0

He filled out the rest of the column for $n = 0$ using this idea.

p	-1	0	1	2	3	4	5
$x/1$	1	1	1	1	1	1	1
$x^2/2$	-1	0	1	2	3	4	5
$x^3/3$	1	0	0	1	3	6	10
$x^4/4$	-1	0	0	0	1	4	10
$x^5/5$	1	0	0	0	0	1	5
$x^6/6$	-1	0	0	0	0	0	1
$x^7/7$	1	0	0	0	0	0	0

This gives the series for $1/(1+x)$ that we have above.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \ldots$$

One can check that it's correct by multiplying out.

$$1 = 1 - x + x - x^2 + x^2 - x^3 + x^3 - x^4 + x^4 + \cdots = 1$$

Using this idea, one can fill in the table for the negative integers. In particular, the column for -2 is

p	-2	-1	0	1	2	3	4	5
$x/1$	1	1	1	1	1	1	1	1
$x^2/2$	-2	-1	0	1	2	3	4	5
$x^3/3$	3	1	0	0	1	3	6	10
$x^4/4$	-4	-1	0	0	0	1	4	10
$x^5/5$	5	1	0	0	0	0	1	5
$x^6/6$	-6	-1	0	0	0	0	0	1
$x^7/7$	7	1	0	0	0	0	0	0

so the series is

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$$

Multiply by $1 + x$. We do this by first multiplying by x

$$x - 2x^2 + 3x^3 - 4x^4 + \dots$$

and then adding to the series itself to obtain

$$1 - x + x^2 - x^3 + \dots$$

This confirms that

$$\frac{1}{(1+x)} = (1+x)(1 - 2x + 3x^2 - 4x^3 + \dots)$$

rational powers

What about fractional powers?

p	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$x/1$	1	1	1	1	1	1	1	1	1
$x^2/2$	-1	.	0	.	1	.	2	.	3
$x^3/3$	1	.	0	.	0	.	1	.	3
$x^4/4$	-1	.	0	.	0	.	0	.	1
$x^5/5$	1	.	0	.	0	.	0	.	0
$x^6/6$	-1	.	0	.	0	.	0	.	0

After a lot of work, Newton comes to two simple ideas.

First, the addition rule remains in place, for entries that are separated by a whole unit. The missing entries in the table above depend not on the entries to the immediate left, but one more column over.

Given just one of these missing entries, the entire row can be filled in. Let's suppose that $1/2$ is the entry between 0 and 1. We use the addition rule to fill in the rest of that row:

p	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$x/1$	1	1	1	1	1	1	1	1	1
$x^2/2$	-1	$-1/2$	0	1/2	1	3/2	2	5/2	3
$x^3/3$	1	.	0	.	0	.	1	.	3
$x^4/4$	-1	.	0	.	0	.	0	.	1
$x^5/5$	1	.	0	.	0	.	0	.	0

So then the question is, where does the first entry of $-1/2$ or $1/2$ come from in the row for x^2?

Newton came up with the following pattern, which is just a generalization of the rule: add up the entry to the left plus the entry above it.

$$\begin{array}{ccccc}
a & a & a & a & a \\
b & a+b & 2a+b & 3a+b & 4a+b \\
c & b+c & a+2b+c & 3a+3b+c & 6a+4b+c \\
d & c+d & b+2c+d & a+3b+3c+d & 4a+6b+4c+d \\
.. & & & &
\end{array}$$

However, this won't work for the fractional tables, because $a = 1$ and the increment between whole values in the second row is now $2a = 2$, whereas it should be 1.

To solve this problem, we use the same pattern, but *decouple* the rows from each other by changing the names of the variables.

$$\begin{array}{ccccc}
a & a & a & a & a \\
b & c+b & 2c+b & 3c+b & 4c+b \\
d & e+d & f+2e+d & 3f+3e+d & 6f+4e+d \\
g & h+g & i+2g+h & k+3i+3h+g & 4k+6i+4h+g \\
.. & & & &
\end{array}$$

Consider the first pattern

$$b \quad c+b \quad 2c+b \quad 3c+b \quad 4c+b$$

applied to the row for x^2 where we have, starting with the column 0:

$$0 \quad ? \quad 1 \quad ? \quad 2$$

We can write two equations in two unknowns, namely

$$b = 0$$

$$2c + b = 1$$

so $c = 1/2$. Now, fill in the fractional columns as

$$c + b = 1/2$$

$$3c + b = 3/2$$

For the second pattern

$$d \quad e+d \qquad f+2e+d \quad 3f+3e+d \quad 6f+4e+d$$

applied to the third row (x^3) where we have starting with the column 0

$$0 \quad ? \quad 0 \quad ? \quad 1$$

We conclude that

$$d = 0$$

$$f + 2e + d = 0$$

$$6f + 4e + d = 1$$

Solving for $e = -f/2$ and substituting in the last equation:

$$6f - 2f = 1$$

We have $f = 1/4$ and $e = -1/8$ and then fill in the fractional columns as

$$e + d = -1/8$$

$$3f + 3e + d = 3/4 - 3/8 = 3/8$$

This allows us to fill in the rest of the third row.

p	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$x/1$	1	1	1	1	1	1	1	1	1
$x^2/2$	-1	$-1/2$	0	$1/2$	1	$3/2$	2	$5/2$	3
$x^3/3$	1	$3/8$	0	$-1/8$	0	$3/8$	1	$15/8$	3
$x^4/4$	-1	.	0	.	0	.	0	.	1
$x^5/5$	1	.	0	.	0	.	0	.	0

At this point, we can extend the second row to the left indefinitely $(-3/2, -2\ldots)$, use the pattern to find a single entry in any given row, then use the addition rule to generate all the other entries in that row.

We will skip the solution and just fill in one of the columns, for the $1/2$ power

p	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$x/1$	1	1	1	1	1	1	1	1	1
$x^2/2$	-1	$-1/2$	0	$1/2$	1	$3/2$	2	$5/2$	3
$x^3/3$	1	$3/8$	0	$-1/8$	0	$3/8$	1	$15/8$	3
$x^4/4$	-1	.	0	$3/48$	0	.	0	.	1
$x^5/5$	1	.	0	$-15/384$	0	.	0	.	0

Here is the final table:

Table 9

p / term	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$\frac{x}{1}$	1	1	1	1	1	1	1	1	1	1	1
$\frac{-x^3}{3}$	-1	$\frac{-1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$\frac{x^5}{5}$	1	$\frac{3}{8}$	0	$\frac{-1}{8}$	0	$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6
$\frac{-x^7}{7}$	-1	$\frac{-5}{16}$	0	$\frac{3}{48}$	0	$\frac{-1}{16}$	0	$\frac{5}{16}$	1	$\frac{35}{16}$	4
$\frac{x^9}{9}$	1	$\frac{35}{128}$	0	$\frac{-15}{384}$	0	$\frac{3}{128}$	0	$\frac{-5}{128}$	0	$\frac{35}{128}$	1
$\frac{-x^{11}}{11}$	-1	$\frac{-63}{256}$	0	$\frac{105}{3840}$	0	$\frac{-3}{256}$	0	$\frac{3}{256}$	0	$\frac{-7}{256}$	0
$\frac{x^{13}}{13}$	1	$\frac{231}{1024}$	0	$\frac{-945}{46080}$	0	$\frac{7}{1024}$	0	$\frac{-5}{1024}$	0	$\frac{7}{1024}$	0

Looking at these values, Newton came up with his version of the binomial, which can generate them, and is where we started.

Pi

Computation of π

The series generated under the 1/2 power is

$$x - \frac{1}{4}\,\frac{x^2}{2} + \frac{3}{16}\,\frac{x^3}{3} - \frac{15}{96}\,\frac{x^4}{4} + \cdots$$

if $x = \sqrt{1-u^2}$, this is

$$x - \frac{1}{4}\ \frac{x^2}{2} + \frac{3}{16}\ \frac{x^3}{3} - \frac{15}{96}\ \frac{x^4}{4} + \cdots$$

and it's the area under the quarter-circle ($\pi/4$), when $x = 1$.

$$\frac{\pi}{4} = 1 - \frac{1}{4}\ \frac{1}{2} + \frac{3}{16}\ \frac{1}{3} - \frac{15}{96}\ \frac{1}{4}$$

$$= 1 - \frac{1}{8} + \frac{3}{48} - \frac{15}{384} + \cdots$$

which is a reasonable series for $\pi/4$.

Taylor series

It's worth mentioning how we would get this series by a modern approach. Write the general Taylor series

$$\sum f^n(x-a)\frac{(x-a)^n}{n!}$$

near $a = 0$

$$\sum f^n(x)\frac{x^n}{n!}$$

The function is

$$f(x) = \frac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$$

The derivatives are

$$f'(x) = -\frac{1}{2}(1+x)^{-3/2} = -\frac{1}{2}$$

$$f''(x) = \frac{3}{4}(1+x)^{-5/2} = \frac{3}{4}$$

$$f'''(x) = -\frac{15}{8}(1+x)^{-7/2} = -\frac{15}{8}$$

$$f''''(x) = \frac{105}{16}(1+x)^{9/2} = \frac{105}{16}$$

The series is then

$$1 - \frac{1}{2}x + \frac{3}{4}\frac{x^2}{2} - \frac{15}{8}\ \frac{x^3}{3!} + \frac{105}{16}\ \frac{x^4}{4!} + \dots$$

$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 + \dots$$

which matches the table, above.

Part XIX

Practice problems

Chapter 71

Integration problems

This chapter and the next contain a number of problems which you may find challenging and instructive. We have reached the end of our discussion of the theory of calculus at this point, though we will see many more applications later.

Definite integrals

$$\int_0^{\pi/2} \cos x \ dx$$

$$\int_0^{\pi/2} \cos\theta \ d\theta$$

$$\int_{-\pi/2}^{\pi/2} \sin x \ dx$$

$$\int_0^1 x^2 \ dx$$

$$\int_0^1 \sqrt{x} \ dx$$

$$\int_1^3 \frac{1}{x}\ dx$$

$$\int_1^2 3x^2 + 2x + 1\ dx$$

$$\int_0^8 x^{2/3}\ dx$$

$$\int_0^{\pi/4} \cos 2t\ dt$$

$$\int_1^3 \frac{1}{x^2}\ dx$$

$$\int_0^3 \frac{2t}{1+t^2}\ dt\ ;\quad \text{hint: watch the bounds}$$

$$= \int_{u=1}^{u=10} \frac{1}{u}\ du$$

$$= \ln u \Big|_{u=1}^{u=10} = \ln 10 - \ln 1 = \ln 10$$

$$\int_0^1 (3x-2)^3\ dx$$

$$\int_0^1 xe^{-x^2}\ dx$$

$$\int_{-\pi/4}^{\pi/4} \cos 2x \ dx$$

$$\int_{-1}^{1} xe^x \ dx$$

$$\int_{0}^{1/2} \frac{1}{\sqrt{1-x^2}} \ dx$$

$$\int_{0}^{e-1} \ln(x+1) \ dx; \quad \text{hint: what is } \frac{d}{dx} x \ln x \ ?$$

$$\int_{\pi/3}^{\pi/2} \tan\frac{\theta}{2} \ \sec^2\frac{\theta}{2} \ d\theta$$

$$\int_{\pi/2}^{x} \cos t \ dt$$

$$\int_{0}^{\ln 2} e^{2x} \ dx$$

$$\int_{0}^{2} (x^3+k) \ dx = 10 \ ; \quad \text{find } k$$

$$\int_{0}^{\ln 2} e^{2x} \ dx$$

$$\int_{1}^{e} \frac{\ln t}{t} \ dt$$

$$\int_{0}^{1} xe^{x^2+1} \ dx$$

Indefinite integrals

$$\int \tan x \ dx$$

$$\int \ln x \; dx$$

$$\int \sec^2 x \; dx$$

$$\int \csc^2 \theta \; d\theta$$

$$\int \tan \theta \sec \theta \; d\theta$$

$$\int (\sqrt{x} + \frac{1}{x^3}) \; dx$$

$$\int \frac{3x^2 + x - 1}{x^2} \; dx$$

$$\int \frac{1}{u - 3} \; du$$

$$\int \cos^2(2x) \;\; \sin 2x \; dx$$

$$\int \frac{x}{\sqrt{3 - 4x^2}} \; dx$$

$$\int \frac{1}{\sqrt{9 - x^2}} \; dy$$

$$\int \frac{x}{(2 - x^2)^3} \; dx$$

$$\int \frac{e^x}{1 - 2e^x} \; dx$$

$$\int e^{x + e^x} \; dx$$

$$\int (x^3 - \sin 2x)\ dx$$

$$\int \frac{e^{3x}}{e^x}\ dx$$

$$\int \frac{z}{1-4z^2}\ dz$$

$$\int \frac{5}{1+x^2}\ dx$$

$$\int \frac{\cos x}{\sin^2 x}\ dx$$

$$\int \tan^4 t\ \ \sec^2 t\ dt$$

$$\int e^x \cos(e^x)\ dx$$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\ dx$$

$$\int \frac{x+1}{x^2+1}\ dx$$

$$\int \frac{x}{x+a}\ dx\ ;\quad \text{hint:}\ \ = \int \frac{x+a-a}{x+a}\ dx$$

$$\int a^u\ du\ ;\quad a = const$$

$$\int e^{4-\ln x}\ dx$$

$$\int x\,\sqrt{x+2}\;dx\ ; \quad \text{hint: } u = x+2$$

$$\int \frac{x}{\sqrt{x+3}}\;dx\ ; \quad \text{hint: } u = x+3$$

$$\int \frac{1+x}{\sqrt{x}}\;dx$$

$$\int \frac{1}{x^2+2x+5}\;dx$$

$$\int \frac{x^2+3}{x-1}\;dx\ : \quad \text{hint: make top divisible by } x-1$$

$$\int \frac{\ln x}{3x}\;dx$$

$$\int \frac{e^x}{e^x+1}\;dx$$

$$\int \frac{1+x}{\sqrt{x}}\;dx$$

$$\int \sin\theta\cos\theta\;d\theta$$

for the last, give both versions of the answer and show they are equal

$$\int (\sin x+\cos x)^2\;dx$$

$$\int (1+\tan x)^2\;dx$$

$$\int \frac{\cos^2 x}{1+\sin x}\;dx$$

$$\int \frac{\sin x}{1+\sin x}\ dx$$

$$\int \sin^3\ dx$$

$$\int \sec^2 x\ \sqrt{5+\tan x}\ dx$$

$$\int \cos x\ e^{1+\sin x}\ dx$$

$$\int e^x \cos(e^x)\ dx$$

$$\int x \sin x\ dx$$

$$\int e^x \sin x\ dx$$

$$\int \frac{\sin x+\cos x}{e^{-x}+\sin x}\ dx\ ;\quad \text{hint: multiply by } e^x/e^x$$

$$\int \frac{2^{\ln x}}{x}\ dx$$

$$\int \frac{1}{x\ln x}\ dx$$

$$\int \frac{\ln\sqrt{x}}{x}\ dx$$

For absolute value problems, recall that

$$|x| = \begin{cases} x & : x \geq 0 \\ -x & : x < 0 \end{cases}$$

The method is to find the place where the expression inside the absolute value symbols is equal to zero, then integrate piecewise, substituting as shown above.

$$\int_0^2 |t-1| \; dt$$

Since $t-1=0$ when $t=1$ this is

$$\int_0^1 -(t-1) \; dt + \int_1^2 (t-1) \; dt$$

$$= (-\frac{1}{2}t^2+t) \Big|_0^1 + (\frac{1}{2}t^2-t) \Big|_1^2$$

$$= (-\frac{1}{2}+1-0+0)+(2-2-\frac{1}{2}+1)=1$$

Tricky to evaluate.

FTC

There is a perverse desire to make sure you understand the FTC (part 1).

If $F(x)$ is "nice" and

$$F(x) = \int_a^x f(t) \; dt$$

then..

$$F'(x) = \frac{d}{dx} \; F(x) = \frac{d}{dx} \; \int_a^x f(t) \; dt = f(x)$$

Problems: for each $G(x)$ below, find $G'(x)$

$$G(x) = \int_1^x 2t \ dt$$

$$G(x) = \int_0^x (2t^2 + \sqrt{t}) \ dt$$

$$G(x) = \int_0^x \tan t \ dt$$

$$G(x) = \int_{x^2}^x \frac{t^2}{1+t^2} \ dt$$

The last problem needs first to be manipulated into a (sum of) integrals between a constant (0) on the lower bound and x above, and then the one with x^2 must take account of the fact that if $t = -x^2$ then $dt = -2x \ dx$.

$$G(x) = -\int_0^{-x^2} \frac{t^2}{1+t^2} \ dt + \int_0^x \frac{t^2}{1+t^2} \ dt$$

$$G'(x) = -\frac{x^4}{1+x^4} \ (-2x) + \frac{x^2}{1+x^2}$$

hard one

Here is a problem involving the actual integral we had above. I didn't know how to solve it completely, but I found the answer on the web and can work backward and see that it's correct. Call it a challenge. It looks simple enough:

$$\int \frac{\sqrt{x}}{1-x} \ dx$$

substitute

$$u = \sqrt{x}, \quad u^2 = x, \quad 2u\ du = dx$$

we obtain

$$\int \frac{u}{1-u^2}\ 2u\ du$$

$$2\int \frac{u^2}{1-u^2}\ du$$

If this were x in the numerator rather than x^2, it would be simple. Still, it looks like it ought to be easy, somehow. The answer is here:

(http://integrals.wolfram.com/index.jsp)

Let's change to x:

$$\int \frac{x^2}{1-x^2}\ dx$$

The first part of the answer is a useful trick for many problems. If the numerator is the same as the denominator, within a constant, then:

$$= -\int \frac{1-x^2-1}{1-x^2}\ dx$$

$$= -\int 1\ dx - \int \frac{1}{1-x^2}\ dx$$

Now the real trick is that the second part can be re-worked because it is a difference of squares

$$(1-x)(1+x) = 1-x^2$$

$$\int \frac{1}{1-x^2}\ dx = \frac{1}{2}\int (\frac{1}{1+x} + \frac{1}{1-x})\ dx$$

If we put the two terms on the right over the common denominator $1-x^2$, then for the numerator we have $1-x+1+x=2$. !! So the whole integral is

$$\begin{aligned}&\int \frac{x^2}{1-x^2}\ dx\\ &= -x - \frac{1}{2}\ [\ (\ln(1+x) - \ln(1-x))\ ]\ + C\\ &= -x - \frac{1}{2}\ \ln\frac{(1+x)}{(1-x)} + C\end{aligned}$$

I'll leave it to you to work out the answer to the original problem with $\sqrt{x}$.

another hard one

$$\int \frac{\sqrt{x^2+1}}{x}\ dx$$

Substitution: let $x = \tan t$. So opp $= x$, adj $= 1$, hyp $= \sqrt{1+x^2}$.

$$\begin{aligned}x &= \tan t\\ dx &= \sec^2 t\ dt\\ \sqrt{1+x^2} &= \sec t\end{aligned}$$

So the integral is

$$\begin{aligned}&\int \frac{\sec t}{\tan t}\ \sec^2 t\ dt\\ &= \int \frac{\sec t}{\tan t}(1+\tan^2 t)\ dt\end{aligned}$$

The first term is

$$\int \frac{1}{\sin t}\ dt$$

and the second is

$$\begin{aligned} &\int \sec t \tan t\ dt \\ &= \int \frac{\sin t}{\cos^2 t}\ dt \end{aligned}$$

The second part is easy $(1/\cos t)$. But the first requires more work. Let

$$\begin{aligned} u &= \cos t \\ du &= -\sin t\ dt \end{aligned}$$

We rewrite the integral as

$$\int \frac{\sin t}{\sin^2 t}\ dt$$

$$\begin{aligned} &= -\int \frac{1}{1-u^2}\ du \\ &= -\frac{1}{2}\int \frac{1}{1+u} + \frac{1}{1-u}\ du \\ &= -\frac{1}{2}(\ln(1+u) - \ln(1-u)) \end{aligned}$$

So, in terms of t we have (combining)

$$\frac{1}{\cos t} - \frac{1}{2}(\ln(1 + \cos t) - \ln(1 - \cos t))$$

In order to substitute back to x, we recall that

$$\frac{1}{\sqrt{1 + x^2}} = \cos t$$

and I think we'll just leave it right there. Well, in the original problem we had a definite integral with limits $\sqrt{15}$ and $\sqrt{3}$, so that $\cos t = 1/4$ at the high end and $\cos t = 1/2$ at the low end which makes it considerably easier to evaluate.

$$\begin{aligned} &= 4 - \frac{1}{2}(\ln 5/4 - \ln 1/2) - 2 + \frac{1}{2}(\ln 3/2 - \ln 1/2) \\ &= 2 - \frac{1}{2}(\ln 5/2 + \ln 3) \end{aligned}$$

Chapter 72

Table of integrals

I thought we could work on the derivations of some of the integrals shown in standard tables.

Here is what I have so far. Each one of the following will come in three versions [A: (x^2+1), B: (x^2-1), C: $(1-x^2)$]. Let's group them as follows:

$$\int \frac{1}{\sqrt{x^2+1}}\,dx \tag{72.1}$$

$$\int \sqrt{x^2+1}\,dx \tag{72.2}$$

$$\int \frac{1}{x^2+1}\,dx \tag{72.3}$$

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx \tag{72.4}$$

$$\int \frac{1}{x\sqrt{x^2+1}}\,dx \tag{72.5}$$

$$\int \frac{1}{x^2\sqrt{x^2+1}}\,dx \tag{72.6}$$

In every case, we can also deal with a similar integral containing x^2+a^2 (i.e. a^2 substituted for 1). This is done either by factoring the a part out or by setting up a trig substitution a little differently. For the most part, we'll just keep it simple, but we'll look at the effect of having a^2 in place of 1 for a few examples as shown below.

Let's see how far we get.

1A $\int 1/\sqrt{x^2+1}\ dx$

This one is fairly simple, a "trig substitution". Since we have $\sqrt{x^2+1}$, this suggests a right triangle with two sides x and 1, and hypotenuse $\sqrt{x^2+1}$. We will have a right triangle with angle t, and then imagine that we have opposite and adjacent sides:

$$\text{opp} = x, \quad \text{adj} = 1$$

and hypotenuse

$$\text{hyp} = \sqrt{x^2+1}$$

and, in terms of the trig functions, we obtain:

$$x = \tan t, \quad dx = \sec^2 t\ dt, \quad 1/\sqrt{x^2+1} = \cos t$$

This gives:

$$= \int \cos t \sec^2 t\ dt$$

$$= \int \sec t \; dt$$

$$= \ln|\; \sec t + \tan t \;| + c$$

Substitute back with x and finally we have, in summary:

$$\int \frac{1}{\sqrt{x^2+1}} \; dx = \ln|\; x + \sqrt{x^2+1} \;| + C$$

check

$$\frac{d}{dx} \ln|\; x + \sqrt{x^2+1} \;|$$

$$= \frac{1}{x + \sqrt{x^2+1}}(1 + \frac{x}{\sqrt{x^2+1}})$$

$$= \frac{1}{x + \sqrt{x^2+1}}(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}})$$

$$= \frac{1}{\sqrt{x^2+1}}$$

dealing with a^2

Let's think about how we'd have to change things for $x^2 + a^2$. One approach is to change the trig functions. Change the side with length 1 to have length a, then with hypotenuse $\sqrt{x^2+a^2}$, we have

$$\frac{x}{a} = \tan t, \quad dx = a \sec^2 t \; dt, \quad a/\sqrt{x^2+a^2} = \cos t$$

$$\int \frac{1}{\sqrt{x^2+a^2}} \; dx = \int \frac{1}{a} \cos t \; a \sec^2 t \; dt = \int \sec t \; dt$$

We have the same integral and obtain almost the same answer as before

$$= \ln | \ \sec t + \tan t \ | + c$$
$$= \ln | \ \frac{1}{a}(\sqrt{x^2 + a^2} + x) \ | \ + C$$

check

We should check this result.

$$\frac{d}{dx} \ln \ | \ \frac{1}{a}(\sqrt{x^2 + a^2} + x) \ |$$

$$= \frac{a}{x + \sqrt{x^2 + a^2}} \cdot \frac{1}{a}(\frac{x}{\sqrt{x^2 + a^2}} + 1)$$

$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \frac{x + \sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}} = \frac{1}{\sqrt{x^2 + a^2}}$$

factoring method

A second approach is to factor out the a, obtaining

$$\int \frac{1}{\sqrt{x^2 + a^2}} \ dx = \frac{1}{a} \int \frac{1}{\sqrt{(x/a)^2 + 1}} \ dx$$

I think of this as multiplying "on the top and on the bottom" by $1/a$, with the factor out in front being the multiplication on the top. Now, substitute

$$u = x/a, \quad a\ du = dx$$

$$\frac{1}{a}\int \frac{1}{\sqrt{(x/a)^2+1}}\ dx = \int \frac{1}{\sqrt{u^2+1}}\ du$$

$$\begin{aligned} &= \ln|\ \sqrt{u^2+1} + u\ | + C \\ &= \ln\ |\ \sqrt{(x/a)^2+1} + x/a\ | + C \\ &= \ln\ |\ \frac{1}{a}\sqrt{x^2+a^2} + \frac{1}{a}x\ | + C \\ &= \ln|\ \frac{1}{a}(\sqrt{x^2+a^2} + x)\ |\ + C \end{aligned}$$

as we had before.

inverse hyperbolic sine

There is actually *another* version of the answer to this problem. It involves the hyperbolic sine, which is defined as follows

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} \\ \cosh x &= \frac{e^x + e^{-x}}{2} \end{aligned}$$

The answer is

$$\int \frac{1}{\sqrt{x^2+a^2}}\ dx = \sinh^{-1}\frac{x}{a}$$

Proof. Represent the value of the integral as z, then

$$z = \sinh^{-1} \frac{x}{a}$$
$$\frac{x}{a} = \sinh z$$
$$\frac{x}{a} = \frac{1}{2}(e^z - e^{-z})$$

Solve for z. Let $u = e^z$ so we have

$$\frac{2x}{a} = u - \frac{1}{u}$$
$$au^2 - 2xu - a = 0$$

The roots are

$$u = \frac{2x \pm \sqrt{4x^2 + 4a^2}}{2a}$$
$$u = \frac{x \pm \sqrt{x^2 + a^2}}{a}$$

For real z, $e^z > 0$ so we need the positive root

$$u = \frac{x + \sqrt{x^2 + a^2}}{a}$$

Go back to z

$$e^z = \frac{x + \sqrt{x^2 + a^2}}{a}$$
$$z = \ln|x + \sqrt{x^2 + a^2}| - \ln a$$

1B $\int 1/\sqrt{x^2-1}\ dx$

This one seems a bit easier. Again, do a trig substitution.

$$\text{hyp} = x, \quad \text{adj} = 1, \quad \text{opp} = \sqrt{x^2-1}$$

Now, $x = \sec t$, $dx = \sec t \tan t\ dt$, and $\sqrt{x^2-1} = \tan t$

$$= \int \frac{1}{\tan t} \sec t \tan t\ dt$$
$$= \int \sec t\ dt$$

The same as for # 1A except for the minus sign under the square root.

$$= \ln|\sec t + \tan t| + c = \ln|x + \sqrt{x^2-1}| + C$$

check

$$\frac{1}{x+\sqrt{x^2-1}}\ (1 + \frac{x}{\sqrt{x^2-1}})$$
$$= \frac{1}{x+\sqrt{x^2-1}}\ (\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}) = \frac{1}{\sqrt{x^2-1}}$$

dealing with a^2

$$\int \frac{1}{\sqrt{x^2-a^2}}\ dx$$

Our substitution is now

$$\text{hyp} = x, \quad \text{adj} = a, \quad \text{opp} = \sqrt{x^2 - a^2}$$

with $x = a\sec t$, $dx = a\sec t \tan t \ dt$, and $\sqrt{x^2 - a^2} = a\tan t$. Then the integral becomes

$$\int \frac{1}{a\tan t} \ a\sec t \tan t \ dt$$

The integral comes out just as before: $\ln \mid \sec t + \tan t \mid$, but the substitution back to x gives an extra factor of $1/a$ for both terms:

$$= \ln \ \mid \frac{x}{a} + \frac{1}{a}\sqrt{x^2 - a^2} \mid$$

These two results (A and B) are usually combined and written

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} \ dx = \ln \mid \frac{1}{a}\sqrt{x^2 \pm a^2} + \frac{1}{a}x \mid \ + C$$

1C $\int 1/\sqrt{1 - x^2} \ dx$

This one is just $\sin^{-1} x$. We get that by

$$\text{opp} = x, \quad \text{hyp} = 1, \quad \text{adj} = \sqrt{1 - x^2}$$

So

$$x = \sin t$$

$$t = \sin^{-1} x$$

$$\frac{dx}{dt} = \cos t = \sqrt{1 - x^2}$$
$$\frac{dt}{dx} = \frac{1}{\cos t} = \frac{1}{\sqrt{1 - x^2}}$$

The answer is

$$\int \frac{1}{\sqrt{1 - x^2}} = \sin^{-1} x$$

dealing with a^2

It's easy to see that we will have

$$\int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

2A $\int \sqrt{x^2 + 1}\ dx$

Multiply top and bottom by what's on the top

$$= \int \frac{x^2 + 1}{\sqrt{x^2 + 1}}\ dx$$
$$= \int \frac{x^2}{\sqrt{x^2 + 1}} + \frac{1}{\sqrt{x^2 + 1}}\ dx$$

The term on the right is # 1A from above. The answer is:

$$\ln |\sqrt{x^2 + 1} + x| + C$$

Now for the term on the left. That is actually # 4A. The answer (still with an integral in it) is:

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx = x\sqrt{x^2+1} - \int \sqrt{x^2+1}\,dx + C$$

We've come full circle. Appearing in the answer is minus the same integral that we started with. It's OK. We assemble everything, grouping the two identical terms on the left, divide by 2, and obtain:

$$\int \sqrt{x^2+1}\,dx = \frac{1}{2}\,x\sqrt{x^2+1} + \frac{1}{2}\ln|\,\sqrt{x^2+1}+x\,| + C$$

check

Leave the factor of 1/2 aside for the moment. The derivative of the first term is

$$\sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}}$$

We checked the second term above in # 1A, it is just

$$\frac{1}{\sqrt{x^2+1}}$$

So now we have

$$\sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}$$

$$= \frac{x^2+1}{\sqrt{x^2+1}} + \frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}$$
$$= \frac{2(x^2+1)}{\sqrt{x^2+1}}$$

Remember the factor of 1/2, and simplify

$$= \sqrt{x^2+1}$$

Trig substitution

This one can also be done by a trig substitution: $x/a = \tan t$. Then $\sqrt{x^2+a^2}/a = \sec t$, and $dx = \sec^2 t \ dt$ so we have

$$\int \sec^3 t \ dt = \frac{1}{2}(\sec t \tan t + \ln |(\sec t + \tan t)| + C$$

We pick up some factors of a in substituting back to x

$$= \frac{1}{2}(\frac{\sqrt{x^2+a^2}}{a} \ \frac{x}{a} + \ln |\frac{1}{a}(\sqrt{x^2+a^2} + x)| + C$$

2B $\int \sqrt{x^2-1} \ dx$

Use a trig substitution.

$$\text{hyp} = x, \quad \text{adj} = 1, \quad \text{opp} = \sqrt{x^2-1}$$
$$x = \sec t, \quad dx = \sec t \tan t \ dt, \quad \sqrt{x^2-1} = \tan t$$

We have:

$$\int \sqrt{x^2-1}\ dx = \int \sec t \tan^2 t\ dt$$
$$= \int \sec t + \sec^3 t\ dt$$

We've done $\sec^3$ elsewhere:

$$= \ln|\sec t + \tan t\ | + \frac{1}{2}\sec t \tan t + \frac{1}{2}\ln|\sec t + \tan t\ |C$$
$$= \frac{3}{2}\ln|x+\sqrt{x^2-1}| + \frac{1}{2}x\sqrt{x^2-1} + C$$

check

Do it term by term. The derivative of the first term is

$$= \frac{3/2}{x+\sqrt{x^2-1}} \cdot (1 + \frac{x}{\sqrt{x^2-1}})$$
$$= \frac{3/2}{x+\sqrt{x^2-1}} \cdot (\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}) = \frac{3/2}{\sqrt{x^2-1}}$$

The second part is

$$= \frac{1}{2}(\sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}})$$
$$= \frac{1}{2}(\frac{x^2-1}{\sqrt{x^2-1}} + \frac{x^2}{\sqrt{x^2-1}})$$

Putting it all together we have

$$= \frac{3/2 + x^2 - 1/2}{\sqrt{x^2 - 1}} = \sqrt{x^2 - 1}$$

2C $\int \sqrt{1 - x^2}\ dx$

Multiply top and bottom by $\sqrt{1 - x^2}$

$$= \int \frac{1 - x^2}{\sqrt{1 - x^2}}$$

$$= \int \frac{1}{\sqrt{1 - x^2}}\ dx - \frac{x^2}{\sqrt{1 - x^2}}\ dx$$

The first term is just $\sin^{-1} x$ and the second one is # 4C. Substituting the answer from there:

$$\int \sqrt{1 - x^2}\ dx = \sin^{-1} x + x\sqrt{1 - x^2} - \int \sqrt{1 - x^2}\ dx$$

Two copies of our integral, so:

$$2\int \sqrt{1 - x^2}\ dx = \sin^{-1} x + x\sqrt{1 - x^2}$$

$$\int \sqrt{1 - x^2}\ dx = \frac{1}{2}\sin^{-1} x + \frac{x}{2}\sqrt{1 - x^2}$$

check

We differentiate term by term. Remember the factor of $1/2$. The first term is just

$$\frac{1}{\sqrt{1-x^2}}$$

while the derivative of $x\sqrt{1-x^2}$ is

$$\begin{aligned} &\frac{-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} \\ &= \frac{-x^2+1-x^2}{\sqrt{1-x^2}} \end{aligned}$$

Adding the terms together we obtain

$$= \frac{1}{\sqrt{1-x^2}} + \frac{-x^2+1-x^2}{\sqrt{1-x^2}}$$

Recall the factor of $1/2$

$$= \frac{1-x^2}{\sqrt{1-x^2}} = \sqrt{1-x^2}$$

3A $\int 1/(x^2+1)\ dx$

This one is just $\tan^{-1} x$. We derive this as follows

$$\text{opp} = x, \quad \text{adj} = 1$$

then the hypotenuse is

$$\text{hyp} = \sqrt{x^2+1}$$

We have

$$x = \tan t$$

$$t = \tan^{-1} x$$

$$\frac{dx}{dt} = \sec^2 t = 1 + x^2$$

$$\frac{dt}{dx} = (\tan^{-1} x)' = \frac{1}{1+x^2}$$

Summarizing:

$$\int \frac{1}{x^2+1}\ dx = \tan^{-1} x + C$$

If we started with

$$\int \frac{1}{x^2+a^2}\ dx = \frac{1}{a^2} \int \frac{1}{(x/a)^2+1}\ dx$$

then we pick up a factor of a from the substitution ($u = x/a$, $a\ du = dx$) and end up with

$$= \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

3B $\int 1/(x^2-1)\ dx$

Ingenious trick:

$$\begin{aligned}
&\int \frac{1}{x^2-1}\ dx \\
&= \int \frac{1}{(x+1)(x-1)}\ dx \\
&= -\frac{1}{2}\int \frac{1}{x+1} - \frac{1}{x-1}\ dx \\
&= \frac{1}{2}\int \frac{1}{x-1} - \frac{1}{x+1}\ dx \\
&= \frac{1}{2}(\ln\ |\ x-1\ |\ -\ln\ |\ x+1\ |\ ) + C
\end{aligned}$$

check

Leave aside the factor of 1/2.

$$\begin{aligned}
&\frac{d}{dx}\ \ln\ |\ x-1\ |\ -\ln\ |\ x+1\ |\ \ | \\
&= \frac{1}{x-1} - \frac{1}{x+1} \\
&= \frac{x+1-x+1}{x^2-1}
\end{aligned}$$

Recall the factor of 1/2, and we're done.

3C $\int 1/(1-x^2)\ dx$

Ingenious trick, again:

$$= \frac{1}{2} \int \frac{1}{1-x} + \frac{1}{1+x}\ dx$$

$$= \frac{1}{2} - \ln|1-x| + \ln|1+x| + C$$

$$= \frac{1}{2} \ln|\ \frac{1+x}{1-x}\ |$$

check

Leave aside the factor of 1/2.

$$\frac{d}{dx}\ \ln|\ 1+x\ \ - \ln|\ 1-x\ |$$

$$= \frac{1}{1+x} + \frac{1}{1-x}$$

$$= \frac{1-x+1+x}{1-x^2}$$

recall the factor of 1/2, and we're done.

4A $\int x^2/\sqrt{x^2+1}\ dx$

We'll try integration by parts (IBP). Let $u = x$ and $du = dx$, then

$$dv = \frac{x}{\sqrt{x^2+1}}\,dx$$
$$v = \sqrt{x^2+1}$$

We have then

$$= x\sqrt{x^2+1} - \int \sqrt{x^2+1}\,dx$$

and the right-hand term is # 1A.

Recall:

$$\int \sqrt{x^2+1}\,dx = \frac{1}{2}\,[\;x\sqrt{x^2+1} + \ln|\sqrt{1+x^2}+x| + C$$

So ???

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\,[\;x\sqrt{x^2+1} + \ln|\sqrt{1+x^2}+x| + C$$

4B $\int x^2/\sqrt{x^2-1}\,dx$

Use IBP. Let $u = x$, $du = dx$,

$$dv = \frac{x}{\sqrt{x^2-1}}\,dx$$
$$v = \sqrt{x^2-1}$$

So the integral is

$$= \frac{3}{2}x\sqrt{x^2-1} - \frac{3}{2}\ln|x+\sqrt{x^2-1}| + C$$

4C $\int x^2/\sqrt{1-x^2}\ dx$

Use IBP. Let $u = x$, $du = dx$,

$$dv = \frac{x}{\sqrt{1-x^2}}\ dx$$
$$v = -\sqrt{1-x^2}$$

So the integral is

$$= -x\sqrt{1-x^2} + \int \sqrt{1-x^2}\ dx$$

which is # 1C. Kind of circular!

$$= -x\sqrt{1-x^2} + \sin^{-1} x + \frac{x}{2}\sqrt{1-x^2}$$

$$= \sin^{-1} x - \frac{x}{2}\sqrt{1-x^2}$$

5A $\int 1/x\sqrt{x^2+1}\ dx$

$$\text{opp} = x, \quad \text{adj} = a, \quad \text{hyp} = \sqrt{x^2+a^2}$$

$$x = a\tan t$$
$$dx = a\sec^2 t\ dt$$
$$\frac{a}{\sqrt{x^2+a^2}} = \cos t$$

So

$$\int \frac{1}{x\sqrt{x^2+a^2}}\ dx$$
$$= \int \frac{1}{a\tan t}\ \cos t\ a\sec^2 t\ dt$$
$$= \int \frac{1}{\sin t}\ dt$$

It's easy to forget, but we've seen this one. It is just

$$\int \csc t\ dt$$
$$= -\ln\ |\ \csc t + \cot t\ |$$
$$= -\ln\ |\ \csc t + \cot t\ |$$
$$= -\ln\ |\ \frac{\sqrt{x^2+a^2}}{x} + \frac{a}{x}\ |$$

check1

Remember the factor of -1. The first part of the derivative is just

$$\frac{1}{\frac{\sqrt{x^2+a^2}}{x} + \frac{a}{x}}$$

Multiply top and bottom by x

$$\frac{x}{\sqrt{x^2+a^2}+a}$$

Now we need

$$\frac{d}{dx} \ \frac{\sqrt{x^2+a^2}}{x} + \frac{a}{x}$$

The first term is

$$= (\frac{x^2}{\sqrt{x^2+a^2}} - \sqrt{x^2+a^2})\frac{1}{x^2}$$

Combined with the second term we have

$$\begin{aligned}
&= (\frac{x^2}{\sqrt{x^2+a^2}} - \sqrt{x^2+a^2})\frac{1}{x^2} - \frac{a}{x^2} \\
&= (\frac{x^2 - x^2 - a^2}{\sqrt{x^2+a^2}})\frac{1}{x^2} - \frac{a}{x^2} \\
&= \frac{-a^2}{x^2\sqrt{x^2+a^2}} - \frac{a}{x^2}
\end{aligned}$$

Finally, multiply by what we got from the logarithm:

$$= \frac{x}{\sqrt{x^2+a^2}+a} \ [\ \frac{-a^2}{x^2\sqrt{x^2+a^2}} - \frac{a}{x^2}\ ]$$

$$= \frac{1}{\sqrt{x^2+a^2}+a} \ [\ \frac{-a^2}{x\sqrt{x^2+a^2}} - \frac{a}{x}\ ]$$

Factor out $-a$ and recall the -1 so it's just a to remember

$$= \frac{1}{\sqrt{x^2+a^2}+a} \ [\ \frac{a}{x\sqrt{x^2+a^2}} + \frac{1}{x}\ ]$$

I don't know what to do with that factor of $\sqrt{x^2+a^2}+a$.

check2

Remember the factor of -1. The derivative of the rest is weird looking for sure. Do it in the variable t! It's just

$$\csc t = \frac{\sqrt{x^2+a^2}}{x}$$

times the derivative of $\csc t$. What is that? It is (using x):

$$(\frac{x^2}{\sqrt{x^2+a^2}} - \sqrt{x^2+a^2})\frac{1}{x^2}$$

For this part I get (by the usual trick)

$$\frac{1}{x^2}\frac{a^2}{\sqrt{x^2+a^2}}$$

needs more work!

5B $\int 1/x\sqrt{x^2-1}\ dx$

This one is the third important inverse trig function $\sec^{-1} x$.

$$\int \frac{1}{x\sqrt{x^2-1}}\ dx = \sec^{-1} x + C$$

How do we get this?

$$\text{hyp} = x, \quad \text{adj} = 1, \quad \text{opp} = \sqrt{x^2-1}$$

We have

$$x = \sec t$$
$$t = \sec^{-1} x$$

$$\frac{dx}{dt} = \sec t \tan t = x\sqrt{x^2 - 1}$$
$$\frac{dt}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$

5C $\int 1/x\sqrt{1 - x^2}\ dx$

Recall

$$\frac{1}{\sqrt{1 - x^2}} = \frac{1}{(1 + x)(1 - x)}$$
$$= \frac{1}{2}(\frac{1}{1 + x} + \frac{1}{1 - x})$$

What now?

6A $\int 1/x^2\sqrt{x^2 + 1}\ dx$

$$\int \frac{1}{x^2\sqrt{x^2 + 1}}\ dx$$

I struggled with this so, finally, I looked it up in Strang. It turned out I had the right answer, my check by differentiation had a mistake.

$$\int \frac{1}{x^2\sqrt{x^2 \pm a^2}} \ dx = \mp \frac{\sqrt{x^2+a^2}}{a^2 x} + C$$

And it's easy enough to check when you know it's right. Use the first case $(x^2 + a^2)$. Save the factor of $-1/a^2$ for later.

Recall the quotient rule $(u'v - uv'/v^2)$. The derivative is

$$[\ \frac{x^2}{\sqrt{x^2+a^2}} - \sqrt{x^2+a^2}\ ]\ \frac{1}{x^2}$$

$$[\ \frac{x^2 - x^2 - a^2}{\sqrt{x^2-a^2}}\ ]\ \frac{1}{x^2}$$

Do the subtraction, cancel using the factor of $-1/a^2$ and we're done. So now, try to derive it (first case) from the integral.

$$\text{opp} = x, \quad \text{adj} = a, \quad \text{hyp} = \sqrt{x^2+a^2}$$

$$x = a \tan t$$

$$dx = a \sec^2 t \ dt$$

$$\frac{a^2}{x^2} = \frac{1}{\tan^2 t} = \frac{\cos^2 t}{\sin^2 t}$$

$$\frac{1}{\sqrt{x^2+a^2}} = \cos t$$

Substituting:

$$= \int \frac{\cos^2 t}{a^2 \sin^2 t} \cos t \frac{a}{\cos^2 t} \ dt$$

$$= \frac{1}{a}\int \cot t \csc t \; dt$$
$$= -\frac{\csc t}{a} + c$$
$$= -\frac{\sqrt{x^2+1}}{ax}$$

6B $\int 1/x^2\sqrt{x^2-1} \; dx$

See # 6A.

6C $\int 1/x^2\sqrt{a^2-x^2} \; dx$

$$\text{opp} = x, \quad \text{adj} = \sqrt{a^2-x^2}, \quad \text{hyp} = a$$

$$x = a \sin t$$
$$dx = a \cos t \; dt$$
$$\frac{a}{\sqrt{a^2-x^2}} = \sec t$$

$$\int \frac{1}{x^2\sqrt{a^2-x^2}} \; dx$$
$$= \int \frac{1}{a^2 \sin^2 t} \; \frac{1}{a} \sec t \; a \cos t \; dt$$
$$= \frac{1}{a^2}\int \frac{1}{\sin^2 t} \; dt$$

The derivative of $\tan t$ is $\sec^2 t\ dt$, so the derivative of $\cot t$ is $-\csc^2 t\ dt$, and our integral is then $-\cot t$ so

$$= -\frac{1}{a^2}\frac{\sqrt{a^2 - x^2}}{x} + C$$

check

$$\frac{d}{dx}\quad -\frac{1}{a^2}\frac{\sqrt{a^2 - x^2}}{x}$$

Save the factor of $-1/a^2$. Recall the quotient rule (above). We have

$$(\frac{-x^2}{\sqrt{a^2 - x^2}} - \sqrt{a^2 - x^2})\ \frac{1}{x^2}$$

$$(\frac{-x^2 - a^2 + x^2}{\sqrt{a^2 - x^2}})\ \frac{1}{x^2}$$

Do the cancellation in the numerator, recall the factor of $-1/a^2$, and we're done.

Chapter 73

Pi and 22 over 7

An unusual wikipedia article within a collection of calculus topics contains a proof that $22/7 > \pi$.

https://en.wikipedia.org/wiki/List_of_calculus_topics#Integral_calculus

This isn't too hard to establish if you are calculating π by one of the many known methods or simply remember

$$\pi = 3.14159265\ldots$$

since $22/7 = 3\ 1/7$ and

$$1/7 = 0.142857\ldots$$

repeating.

What's nice about this problem is there is a complicated-looking integral involved:

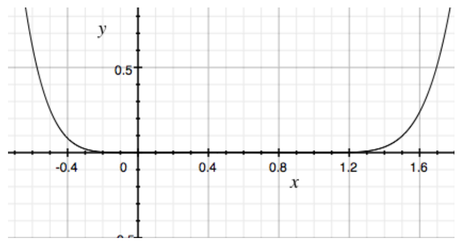
$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\ dx$$

I have no idea where this expression came from yet, but it looks like we should be able to solve it.

proof

The proof is the following:

First, the value of the integrand is zero at the bounds $[0, 1]$ and very small but positive everywhere between (because of the powers x^2 and x^4).



Since the integral is positive we can say that

$$I > 0$$

In fact, the integral can be solved, and it turns out to be equal to

$$I = \frac{22}{7} - \pi$$

We obtain

$$\frac{22}{7} - \pi > 0$$

which yields the promised inequality.

$$\pi < \frac{22}{7}$$

algebra

How to solve this? Basically, what we do is multiply and divide to get a bunch of terms that we can integrate one by one. The first step is to write the terms of

$$(1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4$$

from the binomial theorem, substituting $-x$ for x.

Now multiply by x^4 to obtain

$$x^4 - 4x^5 + 6x^6 - 4x^7 + x^8$$

The somewhat tricky part is the division by $1 + x^2$. The first factor is x^6, since

$$x^6 \cdot x^2 = x^8$$

after subtracting that, we also need to subtract x^6 (obtained by multiplying by the 1 in $1 + x^2$), leaving

$$x^4 - 4x^5 + 5x^6 - 4x^7$$

Next, we get a factor of $-4x^5 \cdot x^2 = -4x^7$, so then we also need to subtract $-4x^5$, leaving

$$x^4 + 5x^6$$

$5x^4 \cdot x^2 = 5x^6$, so then we need to subtract $5x^4$, leaving $-4x^4$

$$-4x^4$$

$-4x^2 \cdot x^2 = -4x^4$, so then we need to subtract $-4x^2$, leaving

$$4x^2$$

$4 \cdot x^2 = 4x^2$, so then we need to subtract 4, leaving

$$-4$$

This last -4 is the remainder, still over $(1 + x^2)$.

Combining all these factors we have

$$x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$$

Check by multiplying out by $1 + x^2$. I get

$$x^6 - 4x^5 + 5x^4 - 4x^2 + 4 + x^8 - 4x^7 + 5x^6 - 4x^4 + 4x^2 - 4$$

$$= x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$$

which seems correct.

integrating

$$\int_0^1 x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$$

The first five terms are easy:

$$\frac{1}{7} - 4\ \frac{1}{6} + 5\ \frac{1}{5} - 4\ \frac{1}{3} + 4$$

$$= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4$$

$$= 3\ \frac{1}{7} = \frac{22}{7}$$

The last term is

$$\int_0^1 -\frac{4}{1+x^2} = -4\tan^{-1} x\ \Big|_0^1 = -4 \cdot \frac{\pi}{4} = -\pi$$

Thus, finally we obtain

$$\frac{22}{7} - \pi$$

as promised.

Part XX

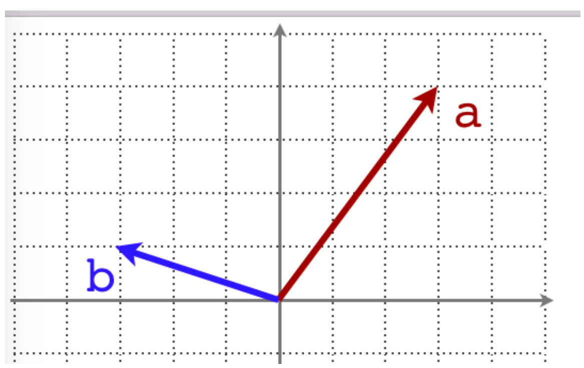
Vectors

Chapter 74

Vector dot product

In this chapter, we look at a few useful properties and operations of vectors in two- and three-dimensional space. I assume that you have already encountered vectors before, so this is not totally new.

From a geometrical point of view, a vector is a mathematical object that has both magnitude and direction. For example, in the standard 2D-coordinate system, the (maroon) vector $\langle\ 3, 4\ \rangle$ goes out from the origin three units in the x-direction and four units in the y-direction.



Vectors are written in bold type:

$$\mathbf{a} = \langle\ 3, 4\ \rangle$$

$$\mathbf{b} = \langle\ -3, 1\ \rangle$$

A vector has one property of a line, slope, but the fixed magnitude means that a vector does not extend to infinity as a line does. The squared length of a vector can be computed as the sum of the squares of its components, according to Pythagoras.

$$(\text{length } \mathbf{a})^2 = |\mathbf{a}|^2 = 3^2 + 4^2$$

By convention, we allow vectors to move about in space. We mean that two vectors of the same length, and pointing in the same direction are considered to be the same object, regardless of where they are located in space. (Some physics problems don't allow this, but in math it's the usual case).

So if we have the vector $\mathbf{v} = \langle\ 1, 1\ \rangle$ starting at the origin $(0, 0)$ and ending at the point $(1, 1)$, and compare it to a second vector $\mathbf{u}$ that starts from $(2, 0)$ and ends at $(3, 1)$, those are considered to be the same vector.

As you might guess, the vector that connects two points (x_1, y_1) and (x_2, y_2) is

$$\mathbf{p} = \langle\ x_2 - x_1, y_2 - y_1\rangle$$

If we do the subtraction in reverse we have

$$\mathbf{q} = \langle\ x_1 - x_2, y_1 - y_2\rangle$$

$$\mathbf{p} = -\mathbf{q}$$

Vectors add by adding their components:

$$\mathbf{a} = \langle\ 3, 4\ \rangle$$

$$\mathbf{b} = \langle\ -3, 1\ \rangle$$

$$\mathbf{a} + \mathbf{b} = \langle\ 0, 5\ \rangle$$

Subtraction works the same way.

From a linear algebra point of view, a vector is simply an ordered collection of numbers

$$\mathbf{u} = \ \langle\ u_1, u_2, \cdots\ u_n\rangle$$

where n could be very large, even infinite.

However, a lot of work is done in two or three dimensions (officially $\mathbb{R}^2$ and $\mathbb{R}^3$), and the principles developed there carry over nicely into n-dimensional space. So let's start by thinking about a two-dimensional vector

$$\mathbf{u} = \ \langle\ u_1, u_2\ \rangle$$

As I've said, the vector $\mathbf{u}$ can be thought of as an arrow that goes from the origin to the point (u_1, u_2). It has both length and direction, with the length given by

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2}$$

and its direction is

$$\frac{u_2}{u_1} = \tan\ \theta, \qquad \theta = \tan^{-1}\ \frac{u_2}{u_1}$$

where θ is the angle the vector makes (rotating counter-clockwise) from the positive x-axis.

Any vector can be converted into a *unit vector*, a vector of length one, by dividing by its length. For example if $\mathbf{u} = \ \langle\ 1, 2\ \rangle$ then

$$\hat{\mathbf{u}} = \frac{1}{|\mathbf{u}|}\ \mathbf{u} = \ \langle\ \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle$$

$\hat{\mathbf{u}}$ is a unit vector pointing in the same direction as $\mathbf{u}$.

The line through the origin with slope $m = u_2/u_1$ and equation

$$y = mx$$

can be thought of as the extension of vector $\mathbf{u}$ obtained by multiplying some t times $\mathbf{u}$ for all $t \in \mathbb{R}$. We have stretched the vector to infinity, and beyond!

The standard unit vectors point in the direction of the x, y and z axes.

$$\begin{aligned} \hat{\mathbf{i}} &= \langle\ 1, 0, 0\ \rangle \\ \hat{\mathbf{j}} &= \langle\ 0, 1, 0\ \rangle \\ \hat{\mathbf{k}} &= \langle\ 0, 0, 1\ \rangle \end{aligned}$$

We can write the vector using these unit vectors as

$$\mathbf{a} = \langle\ 3, 4\ \rangle = 3 \cdot \hat{\mathbf{i}} + 4 \cdot \hat{\mathbf{j}}$$

Dot product

We now introduce a procedure for multiplying two vectors, the *dot product*, and derive the relationship between the dot product of two vectors and the angle between them. Suppose we have two vectors

$$\begin{aligned} \mathbf{a} &= \langle a_1, a_2 \rangle \\ \mathbf{b} &= \langle b_1, b_2 \rangle \end{aligned}$$

Geometrically, we might think of these as being one vector extending from the origin in the x, y-plane to the point (a_1, a_2), and the other vector extending from the origin to (b_1, b_2). The dot product is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

We can extend this to a pair of vectors in n-dimensional space

$$\mathbf{a} = \langle a_1, a_2, \dots a_n \rangle$$

$$\mathbf{b} = \langle b_1, b_2, \dots b_n \rangle$$

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=0}^{n} a_i b_i$$

The two vectors being multiplied (whose dot product is computed) must have the same dimension, the same n. Also, the result of the multiplication—the dot product—is a number. This is in contrast to another form of vector multiplication (the cross-product) which yields a vector as the result.

notation

The dot $(\cdot)$ in the dot product may also used to set apart two multiplicands in scalar multiplication, to increase clarity. So, you ask, how can we tell what is meant? Well, consider

$$v \cdot \frac{1}{v}$$

$$\mathbf{a} \cdot \mathbf{b}$$

It's a dot product if the two objects are vectors, otherwise it's multiplication.

Some properties

The dot product obeys the usual rules: it is associative, commutative and distributive.

The commutative property of the dot product:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

follows from the same property for multiplication of real numbers, since

$$\mathbf{a} \cdot \mathbf{b} = \sum_n a_n b_n$$

$$= \sum_n b_n a_n = \mathbf{b} \cdot \mathbf{a}$$

For the distributive property, suppose

$$\mathbf{b} = \mathbf{c} + \mathbf{d}$$

Then

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d}$$

You can easily verify this by computing each term of the respective products.

$$\mathbf{b} = \ \langle b_1, b_2 \rangle \ = \mathbf{c} + \mathbf{d} = \ \langle c_1 + d_1, c_2 + d_2 \rangle$$

$$\mathbf{a} \cdot \mathbf{b} = a_1(c_1 + d_1) + a_2(c_2 + d_2)$$

$$= a_1 c_1 + a_1 d_1 + a_2 c_2 + a_2 d_2$$

$$= a_1 c_1 + a_2 c_2 + a_1 d_1 + a_2 d_2$$

$$= \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d}$$

Another example that we will need below is

$$(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b}$$

by the commutative property

$$= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\ \mathbf{a} \cdot \mathbf{b}$$

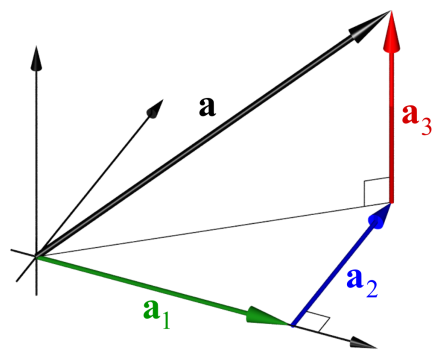
Length of a vector

As we said, the length of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$, designated $|\mathbf{a}|$, is computed by a straightforward application of the Pythagorean Theorem:

$$|\mathbf{a}|^2 = a_1^2 + a_2^2$$

We leave the result as the square for simplicity.

This is easily extended to more dimensions by sequential application of the same method.



In $\mathbb{R}^3$:

$$|\mathbf{a}|^2 = a_1^2 + a_2^2 + a_3^2$$

In $\mathbb{R}^n$:

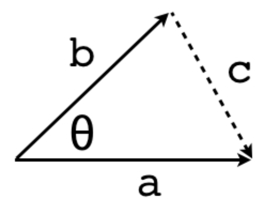
$$|\mathbf{a}|^2 = a_1^2 + a_2^2 + \cdots + a_n^2$$

Notice that

$$|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$$

Relation to θ

Now we are ready for the main idea. Suppose we draw two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^2$ with their tails at the same point. Designate the angle between them as θ and the vector representing the side opposite as $\mathbf{c}$.



The orientation of **c** doesn't matter for the argument that follows. As shown

$$\mathbf{b} + \mathbf{c} = \mathbf{a}$$

$$\mathbf{c} = \mathbf{a} - \mathbf{b}$$

Compute the dot product of **c** with itself

$$\mathbf{c} \cdot \mathbf{c} = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$$

Recalling the result from above, this is

$$\mathbf{c} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\ \mathbf{a} \cdot \mathbf{b}$$

Since

$$|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$$

and so on, we have that

$$\mathbf{c} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\ \mathbf{a} \cdot \mathbf{b}$$

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\ \mathbf{a} \cdot \mathbf{b}$$

Does this remind you of the **law of cosines**?

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Comparing the two equations, we see that

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \cos\theta$$

This relationship is extremely useful because it allows us to compute the cosine of the included angle via the dot product.

Even more important, two vectors which are perpendicular will have $\cos\theta = 0$, so their dot product is zero. Two vectors in pointed in the same direction have $\cos\theta = 1$ so it's just the product of the magnitudes.

This result extends to vectors in $\mathbb{R}^n$. Proof: choose a coordinate system where the two vectors lie in the same plane. Then apply the standard method.

For example, suppose I have the vector

$$\mathbf{u} = \langle p, q \rangle$$

Find a vector $\mathbf{v}$ perpendicular to $\mathbf{u}$.

$$\mathbf{v} = \langle q, -p \rangle$$

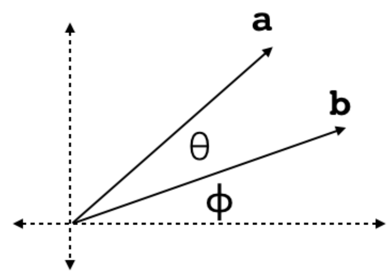
$\mathbf{v}$ is perpendicular to $\mathbf{u}$ because

$$\mathbf{u} \cdot \mathbf{v} = pq + q(-p) = 0$$

How to find a vector in $\mathbb{R}^5$ perpendicular to $\langle 1, 1, 1, 1, 0 \rangle$? Any vector of the form $\langle 0, 0, 0, 0, k \rangle$ will do, where k is some real number.

Alternate derivation

Here is another approach which doesn't depend on knowing the law of cosines, but uses the addition rule for cosine instead.



Vector **a** forms an angle θ with vector **b**. **b** forms an angle ϕ with the x-axis, so the angle between **a** and the x-axis is $\theta + \phi$.

Find the dot product using components. If $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$ then

$$a_x = a\cos(\theta + \phi)$$

$$b_x = b\cos\phi$$

$$a_y = a\sin(\theta + \phi)$$

$$b_y = b\sin\phi$$

So

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y$$

$$= ab\ [\ \cos(\theta + \phi)\cos\phi + \sin(\theta + \phi)\sin\phi\ ]$$

Using the rule

$$\cos s - t = \cos s \cos t + \sin s \sin t$$

the part in parentheses is

$$\cos(\theta + \phi)\cos\phi + \sin(\theta + \phi)\sin\phi$$

$$= \cos(\theta + \phi - \phi) = \cos\theta$$

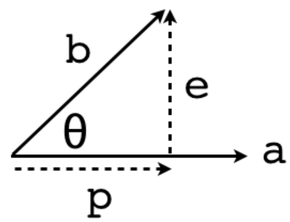
Another important property is that the value of the dot product is *independent* of the coordinate system chosen, because rotation or translation cannot change the lengths of the vectors nor the angle between them.

Projection

If $|\mathbf{a}| = 1$ we say that $\mathbf{a}$ is a *unit vector*. In that case

$$\mathbf{b} \cdot \mathbf{a} = |\mathbf{b}| \cos\theta$$

Looking at the figure, $|\mathbf{b}| \cos\theta$ is the length of the *projection* of $\mathbf{b}$ on $\mathbf{a}$. (Recall that the dot product is a scalar—a number—and not a vector).



The result, $\mathbf{b} \cdot \mathbf{a} = |\mathbf{b}| \cos\theta$, is the length of the part of $\mathbf{b}$ that extends in the same direction as $\mathbf{a}$. The corresponding vector is

$$\mathbf{p} = (\mathbf{b} \cdot \mathbf{a})\ \mathbf{a}$$

The other component of $\mathbf{b}$ is the part that is perpendicular to $\mathbf{p}$

$$\mathbf{p} + \mathbf{e} = \mathbf{b}$$

We compute $\mathbf{e}$ as the difference $\mathbf{b} - \mathbf{p}$. $\mathbf{e}$ is the part of $\mathbf{b}$ that is perpendicular to the projection. As a final note, the formula given here is a simplification for the situation in which $\mathbf{a}$ is a unit vector. If not, the complete formula is:

$$\mathbf{p} = \frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\ \mathbf{a}$$

Vectors allow simple proofs for some geometric theorems such as Ceva's theorem and the law of cosines.

example

Here is a problem from Nahin:

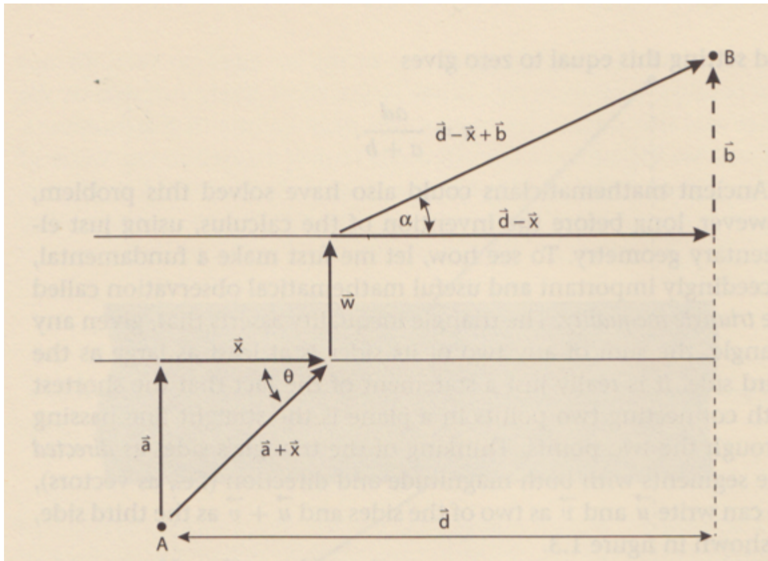


FIGURE 1.4. Bridge geometry in vector notation.

Two towns are on opposite sides of a river at points A and B. It is desired to choose the site of a bridge so as to minimize the distance between the two towns when traveling over the bridge. The problem can be set up algebraically and solved by differential calculus. However, the vector approach is more fun, and allows us to introduce the important *triangle inequality.*

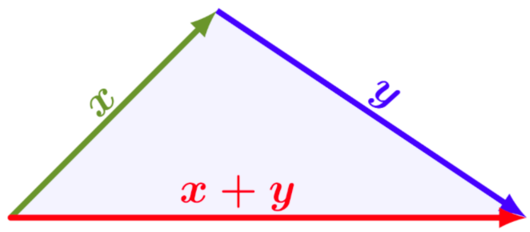
Vectors are shown in the figure: $\mathbf{a}$ is the perpendicular distance from A to the river, and similarly for $\mathbf{b}$. $\mathbf{x}$ determines the placement of the bridge. If the horizontal distance between A and B is $\mathbf{d}$, then $\mathbf{d} - \mathbf{x}$ is the horizontal distance between B and the bridge. The distance across the bridge is $\mathbf{w}$, which cannot be changed. Its length will just be added onto our shortest path.

We want to choose $\mathbf{x}$ so that the path from A to B is the shortest.

The path from A to the bridge is $\mathbf{a}+\mathbf{x}$, that from the bridge to B is $\mathbf{b}+\mathbf{d}-\mathbf{x}$ so all together we have (taking the lengths of the vectors)

$$L = |\mathbf{a}+\mathbf{x}| + |\mathbf{b}+\mathbf{d}-\mathbf{x}|$$

The triangle inequality says that the lengths of two sides of a triangle add to be larger than or equal to the length of the third side.



$$|\mathbf{x}| + |\mathbf{y}| \geq |\mathbf{x}+\mathbf{y}|$$

The rule is that the minimal value for the sum $|\mathbf{x}| + |\mathbf{y}|$ occurs when they point in the same direction.

In our problem, the minimum length occurs when $\mathbf{a}+\mathbf{x}$ and $\mathbf{b}+\mathbf{d}-\mathbf{x}$ point in the same direction. In other words, when $\theta = \alpha$.

Then, by similar triangles,

$$\frac{x}{a} = \frac{d-x}{b}$$

$$bx = ad - ax$$

$$x = \frac{ad}{a+b}$$

example

Here is one from Strang.

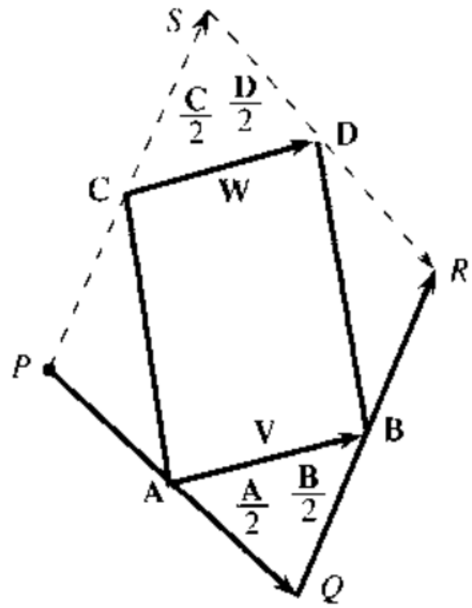


Fig. 11.4 Four midpoints

Consider *any* four-sided figure in space, such as $PQRS$ in the figure. (Note: $|\mathbf{A}| \neq |\mathbf{B}|$, and so on, and S is not co-planar with P, Q, R. I claim that the midpoints of the sides form a parallelogram $ABCD$.

We will prove that $\mathbf{V} = \mathbf{W}$.

The figure makes it almost obvious.

$$\mathbf{V} = \frac{\mathbf{A}}{2} + \frac{\mathbf{B}}{2}$$

$$\mathbf{W} = \frac{\mathbf{C}}{2} + \frac{\mathbf{D}}{2}$$

The segment from P to R can be covered in two ways

$$\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$$

Divide both sides by 2 and obtain

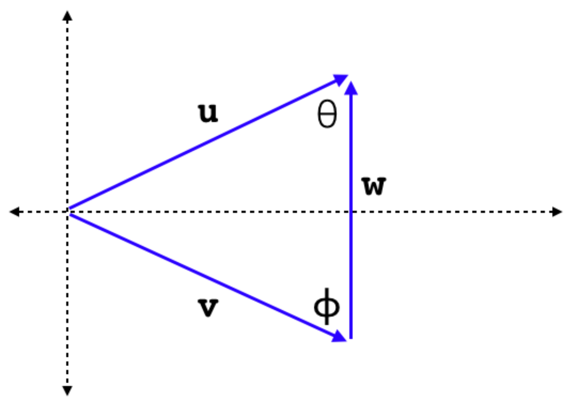
$$\frac{\mathbf{A}}{2} + \frac{\mathbf{B}}{2} = \frac{\mathbf{C}}{2} + \frac{\mathbf{D}}{2}$$

$$\mathbf{V} = \mathbf{W}$$

□

example

And here is one from Euclid:



We are given a triangle with two sides the same length (isosceles). Without loss of generality, draw the triangle with its vertex at the origin and the midpoint of the third side on the x-axis.

To prove: $\theta = \phi$.

Let

$$\mathbf{u} = \langle a, b \rangle$$

$$\mathbf{v} = \langle a, -b \rangle$$

$$\mathbf{w} = \langle 0, 2b \rangle$$

We compute the dot products so that the angle between the vectors is acute and the dot product is > 0.

$$\mathbf{u} \cdot \mathbf{w} = 2b^2$$

$$= |\mathbf{u}||\mathbf{w}| \cos\theta = \sqrt{a^2 + b^2}\ 2b \cos\theta$$

$$\cos\theta = \frac{b}{\sqrt{a^2 + b^2}}$$

which is also obvious from the figure. We didn't need vectors for this.

$$(-\mathbf{w}) \cdot \mathbf{v} = 2b^2$$

$$= \sqrt{a^2 + b^2}\ 2b \cos\phi$$

$$\cos\phi = \frac{b}{\sqrt{a^2 + b^2}}$$

We obtain the same result for $\cos\phi$ as for $\cos\theta$ and then finally

$$\theta = \phi$$

Chapter 75

Vector cross product

Suppose we have two ordinary vectors $\mathbf{u}$ and $\mathbf{v}$. These must be in $\mathbb{R}3$ because the cross-product is only defined for vectors in $\mathbb{R}3$.

Their respective lengths are u and v.

We write the cross-product as

$$\mathbf{u} \times \mathbf{v} = \mathbf{w}$$

The simplest definition is that the magnitude of $\mathbf{w}$ is

$$w = uv \ \sin\theta$$

The symmetry with the dot product is obvious. Also

$$|\mathbf{u} \times \mathbf{v}|^2 + |\mathbf{u} \cdot \mathbf{v}|^2 = (uv)^2$$

The direction is defined by saying that $\mathbf{w}$ is orthogonal to the plane which contains both $\mathbf{u}$ and $\mathbf{v}$, and its sign is given by the right-hand rule. Curl the fingers of your right hand around in the direction from $\mathbf{u}$ to $\mathbf{v}$. Your thumb points in the same direction as $\mathbf{w}$.

The term $\sin\theta$ means that the cross-product of any vector with itself is zero.

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

To make the notation simpler, we define

$$\mathbf{u} = \langle p, q, r \rangle$$

$$\mathbf{v} = \langle x, y, z \rangle$$

and in order to compute the cross product, we form what looks like a really weird matrix

$$\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ p & q & r \\ x & y & z \end{bmatrix}$$

and write its "determinant"

$$\mathbf{u} \times \mathbf{v} = (qz - ry)\ \hat{\mathbf{i}} + (rx - pz)\ \hat{\mathbf{j}} + (py - qx)\ \hat{\mathbf{k}}$$

We can show that the resulting vector is orthogonal to the two starting vectors, $\mathbf{u}$ and $\mathbf{v}$. Test that by forming the dot product with $\mathbf{u}$.

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = p(qz - ry) + q(rx - pz) + r(py - qx)$$

The first and fourth terms cancel, the second and fifth terms cancel, and the third and sixth terms also cancel.

So $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$, and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ as well.

In fact, a very common use for the cross-product is to find the normal vector to a plane in vector calculus.

As an aside, we could have skipped this calculation. The following rule holds for vectors:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

(we will explore triple products below). So

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{u}) \cdot \mathbf{v} = 0$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = -\mathbf{v} \cdot (\mathbf{v} \times \mathbf{u}) = -(\mathbf{v} \times \mathbf{v}) \cdot \mathbf{u}) = 0$$

About the angle

How to show that

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \ \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

According to wikipedia, this is the *definition* of the cross-product, and from this one can derive the expression that we got by setting up our matrix and computing its "determinant." So that is what we are going to do.

I am going to go back to the notation we had before, rather than use subscripts like a_x, etc.

$$\mathbf{u} = \langle p, q, r \rangle$$

$$\mathbf{v} = \langle x, y, z \rangle$$

We proceed from the "determinant" definition of the cross product and show that the length of that vector squared plus the square of the dot product is equal to u^2v^2. By the argument we made above, the magnitude of the cross product is then equal to $uv \sin\theta$.

$$\mathbf{u} \times \mathbf{v} = (qz - ry)\hat{\mathbf{i}} + (rx - pz)\hat{\mathbf{j}} + (py - qx)\hat{\mathbf{k}}$$

$$|\mathbf{u} \times \mathbf{v}|^2 = (qz - ry)^2 + (rx - pz)^2 + (py - qx)^2$$

$$= (qz)^2 - 2qryz + (ry)^2 + (rx)^2 - 2prxz + (pz)^2 + (py)^2 - 2pqxy + (qx)^2$$

$$\mathbf{u} \cdot \mathbf{v} = px + qy + rz$$

$$(\mathbf{u} \cdot \mathbf{v})^2 = (px)^2 + (qy)^2 + (rz)^2 + 2pqxy + 2prxz + 2qryz$$

When we add these together, all the terms with cofactor 2 cancel so that leaves

$$|\mathbf{u} \times \mathbf{v}|^2 + (\mathbf{u} \cdot \mathbf{v})^2$$

$$= (qz)^2 + (ry)^2 + (rx)^2 + (pz)^2 + (py)^2 + (qx)^2 + (px)^2 + (qy)^2 + (rz)^2$$

rearranging terms

$$= (px)^2 + (py)^2 + (pz)^2 + (qx)^2 + (qy)^2 + (qz)^2 + (rx)^2 + (ry)^2 + (rz)^2$$

$$= (p^2 + q^2 + r^2)(x^2 + y^2 + z^2)$$

$$= |\mathbf{u}|^2|\mathbf{v}|^2$$

That was tedious, but it we made it.

All of these properties of the cross-product are connected.

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$$

$$\begin{aligned}\mathbf{a}\times\mathbf{b} = & \ \langle qu - rt, rs - pu, pt - qs\rangle\end{aligned}$$

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

$$|\mathbf{a}\times\mathbf{b}|^2 + (\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2$$

Triple products

Suppose we have

$$\begin{aligned}\mathbf{a} = & \ \langle p, q, r\rangle \\ \mathbf{b} = & \ \langle s, t, u\rangle \\ \mathbf{c} = & \ \langle x, y, z\rangle\end{aligned}$$

And

$$\begin{aligned}\mathbf{a}\times\mathbf{b} = & \ \langle qu - rt, rs - pu, pt - qs\rangle \\ \mathbf{b}\times\mathbf{c} = & \ \langle tz - uy, ux - sz, sy - tx\rangle \\ \mathbf{a}\times\mathbf{c} = & \ \langle qz - ry, rx - pz, py - qx\rangle\end{aligned}$$

Algebraically

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= p(tz - uy) + q(ux - sz) + r(sy - tx) \\ \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) &= s(ry - qz) + t(pz - rx) + u(qx - py) \\ \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) &= x(qu - rt) + y(rs - pu) + z(pt - qs)\end{aligned}$$

So

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

The way to remember this is that these are all the same cyclic permutation.

A much simpler proof is to remember that the cross-product $\mathbf{a}\times\mathbf{b}$ is the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$ and the *scalar* triple product is the signed volume of the parallelipiped formed by the three vectors. Signed meaning that $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$ so the area may come out negative, if we order $\mathbf{a}$ and $\mathbf{b}$ differently.

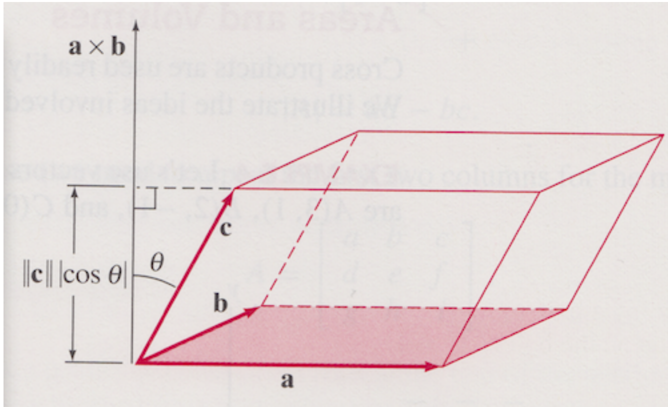


Figure 1.58 The parallelepiped determined by **a**, **b**, and **c**.

Recall that the direction of $\mathbf{a} \times \mathbf{b}$ is perpendicular to both vectors. If we are careful to write the cross-product in the correct order using the right-hand rule, the result of the dot product will always be positive, with the projection of $\mathbf{c}$ onto the cross-product equal to the height of the solid. In particular, for this arrangement, we must write $\mathbf{a} \times \mathbf{b}$, $\mathbf{b} \times \mathbf{c}$, or $\mathbf{c} \times \mathbf{a}$.

It doesn't matter which two vectors we choose as the base of our solid, the volume must come out the same.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

Chapter 76

Point and plane

Construct a plane containing 3 points

Consider three points

$P = (1, 0, 0)$, $Q = (0, 1, 0)$, and $R = (0, 0, 1)$.

Find two vectors in the plane by subtracting the second and third from the first.

$$\begin{aligned} \mathbf{u} &= (1, 0, 0) - (0, 1, 0) \\ &= \langle 1, -1, 0 \rangle \\ \mathbf{v} &= (1, 0, 0) - (0, 0, 1) \\ &= \langle 1, 0, -1 \rangle \end{aligned}$$

Obtain the normal vector by computing the cross product

$$\mathbf{N} = \mathbf{u} \times \mathbf{v} \Rightarrow \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 1\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 1\hat{\mathbf{k}} = \langle 1, 1, 1 \rangle$$

One equation of the plane is then

$$\mathbf{N} \cdot \mathbf{w} = 0$$

for any vector **w** in the plane.

Consider a fixed point in the plane (x_0, y_0, z_0). Then any other point in the plane (x, y, z) yields a vector from the fixed point which, dotted with **N**, yields 0

$$\langle x - x_0, y - y_0, z - z_0\rangle \ \cdot \ \langle 1, 1, 1\rangle \ = 0$$

$$x - x_0 + y - y_0 + z - z_0 = 0$$

$$x + y + z = x_0 + y_0 + z_0 = d$$

Plugging in any one of the points yields

$$x + y + z = d = 1$$

find the closest point in the plane

Consider any point in space, e.g. $P = (3, 4, 6)$.

Find the point Q on the plane which is closest to P, the point we arrive at by subtracting some fraction of N from P.

We have a point and a vector

$$Q = P - t\mathbf{N}$$

$$Q = (3, 4, 6) \ - \ t\langle 1, 1, 1\rangle$$

Since Q is in the plane, its components x, y, z satisfy $x + y + z = 1$!

So

$$(3 - t) + (4 - t) + (6 - t) = 1$$

$$13 - 3t = 1$$

$$t = 4$$

$$Q = (-1, 0, 2)$$

Check that Q is in the plane

$$-1 + 0 + 2 = 1$$

and $P - Q$ is parallel to $\mathbf{N}$

$$P - Q = \langle 4, 4, 4 \rangle$$

that is definitely a multiple of $\mathbf{N}$.

point where a vector crosses the plane

Where does the vector $\mathbf{w}$ that goes from the origin to point $P = (3, 4, 6)$ hit the plane? Call that point R. Again we have a point and a vector

$$R = (0, 0, 0) + t\mathbf{w} = (0, 0, 0) + t \langle 3, 4, 6 \rangle$$

And again, since R is in the plane, its components x, y, z satisfy $x + y + z = 1$. So

$$3t + 4t + 6t = 1$$
$$t = \frac{1}{13}$$
$$R = (\frac{3}{13}, \frac{4}{13}, \frac{6}{13})$$

Notice that the vector $Q - R$ is in the plane, as it should be

$$(Q - R) \cdot \mathbf{N} = ((-1, 0, 2) - (\frac{3}{13}, \frac{4}{13}, \frac{6}{13})) \cdot \langle 1, 1, 1 \rangle$$
$$= \langle \frac{-16}{13}, \frac{-4}{13}, \frac{20}{13} > \cdot \langle 1, 1, 1 \rangle = 0$$

And, adding the horizontal and vertical components together

$$Q - R + P - Q = P - R = (3, 4, 6) - (\frac{3}{13}, \frac{4}{13}, \frac{6}{13})$$
$$= (\frac{36}{13}, \frac{48}{13}, \frac{72}{13})$$

the result is parallel to $\mathbf{w}$.

Lines in space

One way of specifying a line in 3D-space is as the intersection of two planes. Another way is by giving a vector and a point in space. Let's look at these in turn. Suppose we have the following two planes:

$$x + y - z = 7$$

$$2x - 3y + z = 3$$

Since the x, y, z terms are not related by a multiplicative constant, the planes are not parallel, so they will meet in a line, and the solutions consist of all the points on the line. Let's find one solution, at $x = 0$. Then

$$y - z = 7$$

$$-3y + z = 3$$

Adding

$$-2y = 10$$

$$y = -5$$

$$z = y - 7 = -12$$

Our solution $P_0 = (0, -5, -12)$. Now find a second solution, at $z = -3$

$$x + y = 4$$

$$2x - 3y = 6$$

Solving

$$x = 4 - y$$

$$2(4 - y) - 3y = 6$$

$$8 - 5y = 6$$

$$y = \frac{2}{5}$$

$$x = \frac{18}{5}$$

The second point is $P_1 = (18/5, 2/5, -3)$. Now we have two points on the line. Its equation is

$$L = P_0 + t(P_1 - P_0)$$

$$L = (0, -5, -12) + t(\frac{18}{5}, \frac{27}{5}, 9)$$

We can re-scale the vector that multiplies t to have integer components (or length 1, or whatever we wish). Why not multiply by 5/9?

$$L = (0, -5, -12) + t(2, 3, 5)$$

There is another way to do this problem that might be a little easier. Consider that the equation of the first plane gives its normal vector n_1 as

$$n_1 = < 1, 1, -1 >$$

Similarly the normal vector to the second plane is n_2

$$n_2 = < 2, -3, 1 >$$

Now, the vector that is parallel to the line of intersection is orthogonal to both n_1 and n_2 (Do you see why?) So we compute the cross-product:

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= -2i - 3j - 5k$$

Multiplying by -1 gives what we obtained above.

Chapter 77

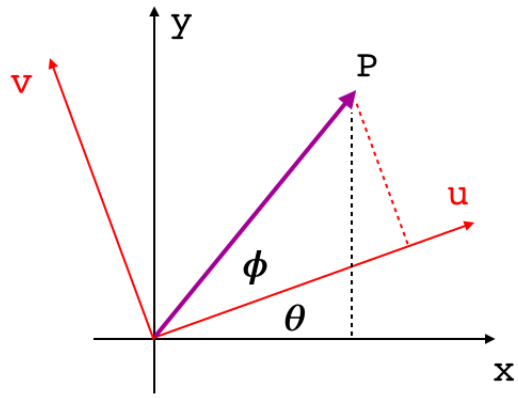
Geometric rotation

Our goal here is to find the equations for rotation of coordinates. We want to be as simple as we can, so that we can (at least try to) remember how the derivation works.

In this chapter we look at geometric approaches. Vector methods are simpler, in my opinion, but they require knowing the idea of matrix multiplication. The geometry can be confusing to set up, but once that's done you can just read the answer off the diagram.

Stewart

I found a nice method in Stewart that depends on knowing the sum of angles formulas, which I urged you to memorize.



Here, we have the standard xy-coordinates in black. The rotated uv-coordinate system is in red, with a rotation angle θ in the counter-clockwise direction.

The ray to the point P has length r. Notice that the coordinates of point P in the u, v system are naturally expressed in terms of ϕ:

$$u = r\cos\phi$$

$$v = r\sin\phi$$

while x and y are naturally expressed in terms of the combined angle $\theta + \phi$.

$$x = r\cos(\theta + \phi)$$

Now, use the sum formula for cosine:

$$x = r\cos\theta\cos\phi - r\sin\theta\sin\phi$$

But we have from above that

$$u = r\cos\phi$$

$$v = r\sin\phi$$

So

$$x = u\cos\theta - v\sin\theta$$

It's as easy as that.

In the same way:

$$
\begin{aligned}
y &= r\sin(\theta+\phi) \\
&= r\sin\theta\cos\phi + r\cos\theta\sin\phi \\
&= u\sin\theta + v\cos\theta
\end{aligned}
$$

Solve for u and v

To convert these formulas to functions $u = f(x, y)$ and $v = f(x, y)$, there is a hard way and an easy way. We do the hard way first:

$$x = u\cos\theta - v\sin\theta$$

so

$$x\cos\theta = u\cos^2\theta - v\sin\theta\cos\theta$$

and

$$y = u\sin\theta + v\cos\theta$$

so

$$y\sin\theta = u\sin^2\theta + v\sin\theta\cos\theta$$

adding:

$$x\cos\theta + y\sin\theta = u$$

similarly:

$$x\sin\theta = u\sin\theta\cos\theta - v\sin^2\theta$$

$$y\cos\theta = u\sin\theta\cos\theta + v\cos^2\theta$$

add minus the first to the second:

$$-x\sin\theta + y\cos\theta = v$$

It *is* easier when you know where you're going. In summary:

$$u = x\cos\theta + y\sin\theta$$

$$v = -x\sin\theta + y\cos\theta$$

and the original pair:

$$x = u\cos\theta - v\sin\theta$$

$$y = u\sin\theta + v\cos\theta$$

The difference is the sign of the sine.

The easy way is to switch x, y for u, v and at the same time, substitute $-\theta$ for θ. What this amounts to is relabeling our diagram with x, y being the rotated axes, and then rotating in the opposite direction (cw rather than ccw).

$$x = u\cos\theta - v\sin\theta$$

switch

$$u = x\cos-\theta - y\sin-\theta$$

$$= x\cos\theta + y\sin\theta$$

(Recall that $\cos -x = \cos x$ and $\sin -x = -\sin x$).

For y

$$y = u\sin\theta + v\cos\theta$$

switch

$$v = x\sin-\theta + y\cos-\theta$$

$$v = -x\sin\theta + y\cos\theta$$

A small test

Rotation of coordinates counter-clockwise (x, y to u, v) gives $-\sin$ in the formula for the vertical component v), whereas clockwise rotation (u, v to x, y) gives $-\sin$ in the formula for the horizontal component x.

One way to see that this is correct is to substitute from the u, v formulas into the x, y ones:

$$x = (x\cos\theta + y\sin\theta)\cos\theta - (-x\sin\theta + y\cos\theta)\sin\theta$$

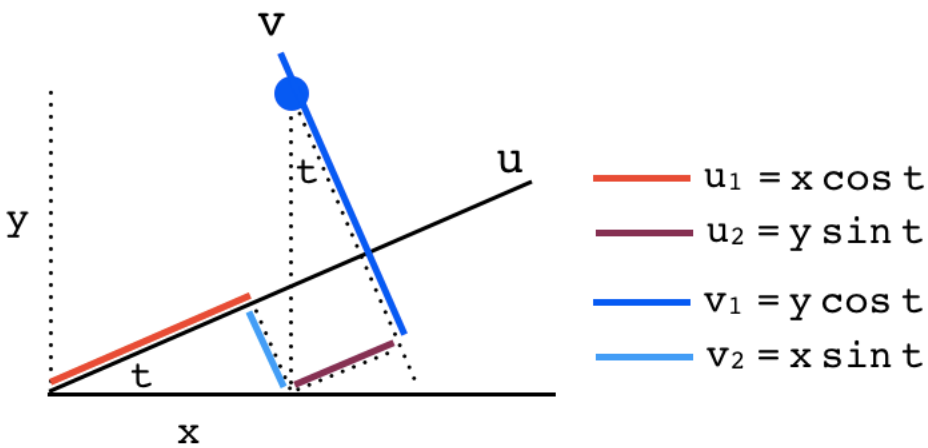
Can you see that if this is multiplied out, we will get $x(\cos^2\theta + \sin^2\theta) = x$ and the terms with y will just cancel?

A similar thing happens with the other one:

$$y = (x\cos\theta + y\sin\theta)\sin\theta + (-x\sin\theta + y\cos\theta)\cos\theta = y$$

standard derivation

Here is a second derivation with a sort of minimalist diagram of rotation



We draw the horizontal x-axis and the rotated u-axis. The angle between them is t. We plot our point and then draw perpendiculars to both axes. To finish the set-up, we draw perpendiculars from the point $(x, 0)$ as shown.

Once the drawing is rendered, we are almost done. You will know that you've done it right if you have both x and y as the *hypotenuse of a right triangle*. Now we just work our way through

$$u_1 = x \cos t$$
$$u_2 = y \sin t$$

So

$$u = x \cos t + y \sin t$$

(All the triangles in the diagram are similar, with small angle t. Can you prove it?)

$$v_1 = y \cos t$$
$$v_2 = x \sin t$$
$$v = -x \sin t + y \cos t$$

Shankar derivation

It seems that the more you talk about rotation, the less clear things become. Nevertheless, I will show one more from Shankar's book, and then work through a calculation.

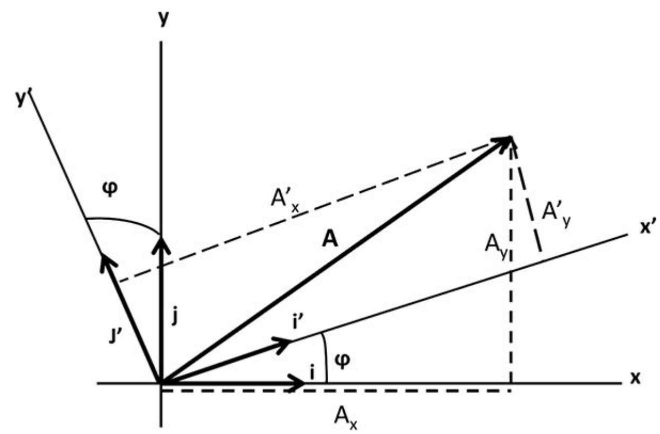


Figure 2.4 The same vector $\mathbf{A}$ is written as $\mathbf{i}A_x + \mathbf{j}A_y$ in one frame and as $\mathbf{i}'A_x' + \mathbf{j}'A_y'$ in the other. The dotted lines indicate the components in the two frames.

Write $\hat{\mathbf{i}}'$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ as follows.

To reach the end of $\hat{\mathbf{i}}'$ we go out in the $\hat{\mathbf{i}}$ direction. How far? Call it x. In terms of the new unit vector $\hat{\mathbf{i}}'$ we have

$$\frac{x}{|\hat{\mathbf{i}}'|} = \cos\phi$$

But $|\hat{\mathbf{i}}'| = 1$ so

$$x = \cos\phi$$

x is shorter than $\hat{\mathbf{i}}$ by the factor of $\cos\phi$.

We also need to go up in the $\hat{\mathbf{j}}$ direction. Use the other triangle with $\hat{\mathbf{j}}$ rotated to $\hat{\mathbf{j}}'$ (which is similar to the first triangle).

$$\frac{y}{|\hat{\mathbf{j}}'|} = \sin\phi$$

But $|\hat{\mathbf{j}}'| = 1$ so

$$y = \sin\phi$$

y is shorter than $\hat{\mathbf{j}}$ by the factor of $\sin\phi$.

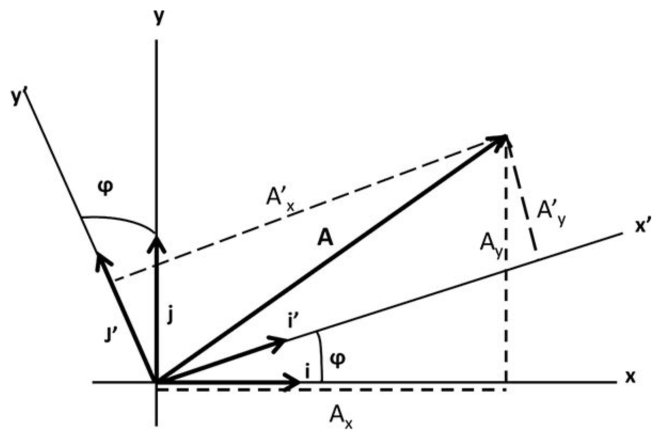


Figure 2.4 The same vector $\mathbf{A}$ is written as $\mathbf{i}A_x + \mathbf{j}A_y$ in one frame and as $\mathbf{i}'A_x' + \mathbf{j}'A_y'$ in the other. The dotted lines indicate the components in the two frames.

So to construct $\hat{\mathbf{i}}'$ we go out in the $\hat{\mathbf{i}}$ direction a distance of $\cos\phi$ and up in the $\hat{\mathbf{j}}$ direction a distance of $\sin\phi$:

$$\hat{\mathbf{i}}' = \hat{\mathbf{i}}\cos\phi + \hat{\mathbf{j}}\sin\phi$$

A similar process will yield

$$\hat{\mathbf{j}}' = \hat{\mathbf{j}}\cos\phi - \hat{\mathbf{i}}\sin\phi$$

Now write the vector $\mathbf{A}$ in two ways:

$$\mathbf{A} = A_x'\,\hat{\mathbf{i}}' + A_y'\,\hat{\mathbf{j}}'$$

$$= (\hat{\mathbf{i}} \cos\phi + \hat{\mathbf{j}} \sin\phi) A'_x + (\hat{\mathbf{j}} \cos\phi - \hat{\mathbf{i}} \sin\phi) A'_y$$
$$= \hat{\mathbf{i}} (\cos\phi A'_x - \sin\phi A'_y) + \hat{\mathbf{j}} (\sin\phi A'_x + \cos\phi A'_y)$$

or in the usual way

$$\mathbf{A} = A_x \, \hat{\mathbf{i}} + A_y \, \hat{\mathbf{j}}$$

So

$$A_x = \cos\phi \; A'_x - \sin\phi \; A'_y$$
$$A_y = \sin\phi \; A'_x + \cos\phi \; A'_y$$

Finally compute the length of $\mathbf{A}$ (squared):

$$A_x^2 = \cos^2\phi \; {A'_x}^2 + \sin^2\phi \; {A'_y}^2 - 2\cos\phi \; A'_x \; \sin\phi \; A'_y$$
$$A_y^2 = \sin^2\phi \; {A'_x}^2 + \cos^2\phi \; {A'_y}^2 + 2\sin\phi \; A'_x \; \cos\phi \; A'_y$$

Add together and notice the cancellation of the mixed A'_x and A'_y terms:

$$A_x^2 + A_y^2 = \cos^2\phi \; {A'_x}^2 + \sin^2\phi \; {A'_y}^2 + \sin^2\phi \; {A'_x}^2 + \cos^2\phi \; {A'_y}^2$$
$$= {A'_x}^2 + \sin^2\phi \; {A'_y}^2 + \cos^2\phi \; {A'_y}^2$$
$$= {A'_x}^2 + {A'_y}^2$$

As we should expect, the length of a vector such as $\mathbf{A}$ is *invariant*, it does not depend on the choice of coordinate system.

Chapter 78

Vector rotation

If you know something about matrix multiplication, just remember this result:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If not, read on.

Our goal here is to find the equations for rotation of coordinates. We want to be as simple as we can, so that we can remember how the derivation works. We will need a couple of preliminary facts, however.

- a procedure to compute the dot product of two vectors
- two vectors are $\perp$ (orthogonal) $\iff$ the dot product is zero
- the projection of a vector $\mathbf{a}$ onto *any* other **unit** vector $\hat{\mathbf{e}}$ is just the dot product $\mathbf{a} \cdot \hat{\mathbf{e}}$
- a rotated set of coordinates is a set of orthogonal unit vectors (in whatever direction we choose)

Recall that the dot product is a number computed from the components of any two vectors $\mathbf{a}$ and $\mathbf{b}$ by the following procedure:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Some examples:

$$\langle 3, 2 \rangle \cdot \langle 2, 5 \rangle = 3 \times 2 + 2 \times 5 = 16$$

$$\langle 1, 0 \rangle \cdot \langle 0, 1 \rangle = 0$$

$$\langle \cos\theta, \sin\theta \rangle \cdot \langle -\sin\theta, \cos\theta \rangle = 0$$

As we might expect, the unit vector along the x-axis, usually called $\hat{\mathbf{i}}$, and the unit vector along the y-axis, $\hat{\mathbf{j}}$, are perpendicular to each other (second example, above).

Similarly, for any angle θ, the given vectors $\langle \cos\theta, \sin\theta \rangle$ and $\langle -\sin\theta, \cos\theta \rangle$ are perpendicular. These two examples should suggest to you a general method for finding a second vector orthogonal to one you are given.

The length of a vector $\mathbf{a}$ is represented as $|\mathbf{a}|$, or even just a, and the length squared is

$$a^2 = \mathbf{a} \cdot \mathbf{a}$$

With respect to the projection, an example should also make that clearer. Suppose we are working with two-dimensional vectors and we decide that our new x-axis should be in the direction of the vector $3, 4$. The first thing to do is to re-scale this to be a unit vector. The length squared is

$$\langle 3, 4 \rangle \cdot \langle 3, 4 \rangle = 9 + 16 = 25$$

Hence the length is 5 and our new unit vector $\hat{\mathbf{u}}$ is

$$\hat{\mathbf{u}} = \langle 3/5, 4/5 \rangle$$

We also need a unit vector $\hat{\mathbf{v}}$ such that $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = 0$. We obtain

$$\hat{\mathbf{v}} = \langle -4/5, 3/5 \rangle$$

or

$$\hat{\mathbf{v}} = \langle 4/5, -3/5 \rangle$$

These two vectors are the same vector, just pointing in opposite directions (which is in the same direction, for the purpose of vectors).

Then for *any* vector $\mathbf{a}$, we can compute the same vector in a set of rotated coordinates based on $\mathbf{u}$ and $\mathbf{v}$ as

$$a_u = \mathbf{a} \cdot \hat{\mathbf{u}}$$

$$a_v = \mathbf{a} \cdot \hat{\mathbf{v}}$$

derivation

All we have to do is to think about rotation of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ through an angle θ counter-clockwise.

Start with $\hat{\mathbf{i}}$. The new vector we seek is still a unit vector, but rotated so that it forms an angle θ with the positive x-axis.

The new vector has both $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components. Projection onto $\hat{\mathbf{i}}$ gives a vector with unit length times $\cos\theta$ or just $\cos\theta$, and similarly, the projection onto $\hat{\mathbf{j}}$ gives a length $\sin\theta$. Clearly the squared length is $\cos^2\theta + \sin^2\theta$, so this is a unit vector.

In vector notation we would say that

$$\langle 1, 0 \rangle \ \Rightarrow \ \langle \cos\theta, \sin\theta \rangle$$

In matrix language the two vectors are related in this way:

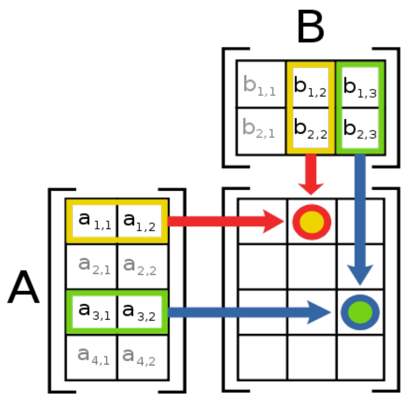
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \cos\ \theta \\ \sin\ \theta \end{bmatrix}$$

where $\Rightarrow$ refers to rotation.

So the question is, what matrix will multiply $\hat{\mathbf{i}}$ to give this result?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\ \theta \\ \sin\ \theta \end{bmatrix}$$

Recall that matrix multiplication works like this



For a matrix times a vector, B would have only a single column.

So going back to this:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\ \theta \\ \sin\ \theta \end{bmatrix}$$

I hope it's pretty clear that $a = \cos\theta$ and $c = \sin\theta$:

$$\begin{bmatrix} \cos\theta & b \\ \sin\theta & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\ \theta \\ \sin\ \theta \end{bmatrix}$$

Can you see why this is true?

On the other hand, rotation of the unit $\hat{\mathbf{j}}$ vector by θ should give

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\ \theta \\ \cos\ \theta \end{bmatrix}$$

The minus sign comes because the new unit vector is now sticking out into the second quadrant.

Again, it should be clear that

$$\begin{bmatrix} a & -\sin\ \theta \\ c & \cos\ \theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\ \theta \\ \cos\ \theta \end{bmatrix}$$

Now, just put them together:

$$R_{ccw} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

In particular, a rotation of 90° ccw goes like this

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\hat{\mathbf{i}}$ is rotated to become $\hat{\mathbf{j}}$.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$\hat{\mathbf{j}}$ is rotated to become $-\hat{\mathbf{i}}$.

I claim that since the matrix we found works for both of the unit vectors it will work for any vector, since any vector can be written as a linear combination of the unit vectors

$$\mathbf{a} = a_1\,\hat{\mathbf{i}} + a_2\,\hat{\mathbf{j}}$$

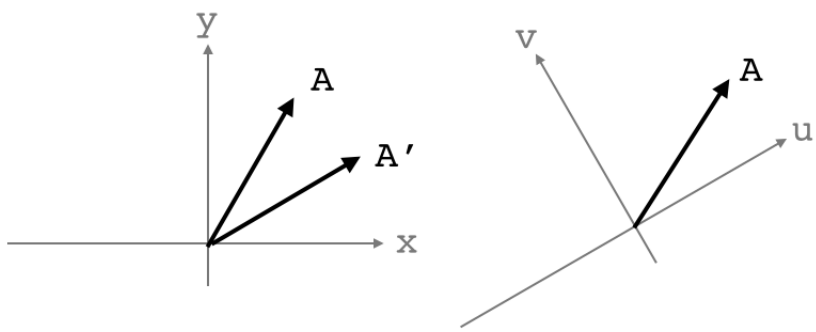
The inverse of the matrix we derived would be used for clockwise rotation and it is just

$$R_{cw} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

You can verify this by remembering the rule for 2×2 or by multiplication

$$R_{cw}\ R_{ccw} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Don't be confused when someone talks about rotation of the coordinate system. Here, the coordinate system stayed fixed but we rotated the vector counter-clockwise. We achieve the same thing (and use the same equation) for a *clockwise* rotation of the coordinate system through an angle θ.



If you really want to rotate the coordinate system counter-clockwise, rotate the vector clockwise.

consequence

One other neat thing comes out of this when we ask about rotation by an angle $s + t$. We can write two equivalent expressions, one by substituting $\theta = s + t$, and the other by doing two sequential applications of the matrix. That is:

$$\begin{bmatrix} \cos(s+t) & -\sin(s+t) \\ \sin(s+t) & \cos(s+t) \end{bmatrix} = \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

Look at the term on the upper-left, $\cos(s+t)$. Sound familiar? Carry out the matrix multiplication on the right for that element

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$

We have derived the cosine addition formula. Similarly, the bottom-left term is for the sine

$$\sin(s+t) = \sin s \cos t + \cos s \sin t$$

yet another way

We can look at this in still a different way. Write

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In this representation, the vector $\langle x, y \rangle$ is a linear combination of the unit vectors $\hat{\mathbf{i}} = \langle 1, 0 \rangle$ and $\hat{\mathbf{j}} = \langle 0, 1 \rangle$.

To rotate the point, we just want to use a different set of unit vectors. The new unit vectors (for the rotated axes) are $\langle \cos\theta, \sin\theta \rangle$ and $\langle -\sin\theta, \cos\theta \rangle$.

If you compute their lengths, it is clear that they are, in fact, unit vectors.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = x \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + y \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Written as a matrix multiplication, this is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Chapter 79

Parametric equations

Suppose we are working in $\mathbb{R}^2$. Consider two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. The vector $\mathbf{v}$ which goes in the direction PQ can be obtained as

$$\mathbf{v} = \langle\ x_2 - x_1, y_2 - y_1 \rangle$$

Written like that as just $\mathbf{v}$ by itself, the default would be that we consider that vector to lie with its tail at the origin. Multiplying by a real number t will extend the vector to any point lying on the line with slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

that goes through the origin.

If $t < 0$ then the vector goes backward.

To start from some other position, add the vector which extends to that point. The position vector is commonly called $\mathbf{r}$. So the complete specification of a line in $\mathbb{R}^2$ would be something like:

$$l = \mathbf{r} + t\mathbf{v}$$

This is called the *parametric* representation of the line.

As a second example, consider a circle of radius R with its center at the origin. We can write the equation of this circle in several ways. In Cartesian coordinates:

$$x^2 + y^2 = R^2$$

In polar coordinates:

$$r = R$$

or parametrically

$$x = R\cos\theta$$

$$y = R\sin\theta$$

where θ takes on values in the interval $[0, 2\pi]$.

With the appropriate units for t, we can write

$$x = R\cos t$$

$$y = R\sin t$$

where t is often understood to be the time, although it doesn't have to be. (Alternatively, we might use the time and an angular velocity ω in the combination as ωt).

We can write something equivalent using only one variable, for the position vector

$$\mathbf{r}(t) = \langle\ R\cos t, R\sin t\rangle = R\ \langle\cos t, \sin t\rangle$$

In three dimensions, we just add another component to $\mathbf{r}$ for z. Perhaps

$$\mathbf{r}(t) = \ \langle\cos t, \sin t, t\rangle$$

which traces out a spiral whose shadow in the xy-plane is the unit circle.

We can do calculus with vector functions of this type: both differentiating and integrating.

The rule is simple: the dimensions are independent. We differentiate each one separately. For example:

$$\mathbf{r} = \langle \cos t, \sin t \rangle$$
$$\frac{d}{dt}\,\mathbf{r} = \langle -\sin t, \cos t \rangle$$

but

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \mathbf{v}$$

the time-derivative of position is velocity.

So, we observe that, for motion on the unit circle, the velocity is perpendicular to the position vector because

$$\mathbf{r} \cdot \mathbf{v} = \langle\ \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t \rangle = 0$$

By the same logic the acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{r}} = \langle -\cos t, -\sin t \rangle$$

The acceleration vector **a** points in the same direction as the position vector **r**, with opposite sign.

Observe that the magnitude of the velocity vector is

$$|\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

This magnitude of **v** is unchanging in time. So on the circle, there is acceleration even though the speed is constant

tangent and normal

We observe that the velocity vector at a point is tangent to the curve. So, if we need a unit tangent vector

$$\hat{\mathbf{T}} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

All of these are functions of t. We could have written:

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

But we resist the urge to do that.

The normal vector $\hat{\mathbf{n}}$ to the curve is perpendicular to $\hat{\mathbf{T}}$. We construct $\hat{\mathbf{n}}$ in the usual way, by taking the x and y components of $\hat{\mathbf{T}}$ and writing:

$$\hat{\mathbf{n}} = \ \langle -y, x \rangle$$

or $\langle y, -x \rangle$. In either case, the dot product with $\mathbf{v}$ will be zero.

For an ellipse, we can parametrize like this:

$$\mathbf{r} = \ \langle a \cos t, b \sin t \rangle$$

$$\mathbf{v} = \ \langle -a \sin t, b \cos t \rangle$$

Now, it is no longer true that the tangent and the position vector are orthogonal. The ellipse is more interesting than the circle in this respect, as we will see.

I would just mention that for a surface, we need two variables in the parametrization. For example, to parametrize the surface of the sphere, we might use the polar angle ϕ and the radial angle θ. Any position on the globe can be specified with its longitude and latitude. We'll see a lot more about this as well.

time-derivative of products

As we mentioned above, to take the derivative with respect to the parameter (such as time), we just go through each component of a vector

$$\mathbf{r} = \langle x, y, z \rangle$$

$$\frac{d\mathbf{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$$

The question arises, what about products? Is there a product rule for vectors? Here's an example

$$\frac{d}{dt}\ \mathbf{r} \cdot \mathbf{v}$$

It turns out that there is.

$$\frac{d}{dt}\ \mathbf{r} \cdot \mathbf{v} = \frac{d\mathbf{r}}{dt} \cdot \mathbf{v} + \mathbf{r} \cdot \frac{d\mathbf{v}}{dt}$$

The reason is that the individual components of the dot product are simple functions of t, and our rule is to differentiate one component at a time.

Let's use Newton's dot notation for d/dt:

$$\mathbf{r} = \langle x, y, z \rangle$$

$$\dot{\mathbf{r}} = \mathbf{v} = \langle \dot{x}, \dot{y}, \dot{z} \rangle$$

$$\mathbf{r} \cdot \dot{\mathbf{r}} = x\dot{x} + y\dot{y} + z\dot{z}$$

The derivative is

$$\begin{aligned} \frac{d}{dt}\mathbf{r} \cdot \dot{\mathbf{r}} &= \frac{d}{dt}(x\dot{x} + y\dot{y} + z\dot{z}) \\ &= \dot{x}\dot{x} + x\ddot{x} + \dot{y}\dot{y} + y\ddot{y} + \dot{z}\dot{z} + z\ddot{z} \\ &= \dot{x}\dot{x} + \dot{y}\dot{y} + \dot{z}\dot{z} + x\ddot{x} + y\ddot{y} + z\ddot{z} \\ &= \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \mathbf{r} \cdot \ddot{\mathbf{r}} \end{aligned}$$

The same is true of the cross-product. The torque is $\mathbf{F} \times \mathbf{r}$ Let's take the derivative:

$$\frac{d}{dt}\ [\ \mathbf{F} \times \mathbf{r}\ ]\ =$$

Let's write $\mathbf{F} = \langle M, N, P \rangle$, then the cross-product gives a vector with components

$$[\ Nz - Py\ ]\ \hat{\mathbf{i}} + \ [\ Px - Mz\ ]\ \hat{\mathbf{j}} + \ [\ My - Nx\ ]\ \hat{\mathbf{k}}$$

where both M, N, P and x, y, z are *functions* of time.

The time derivative is obtained by the product rule. Again, I will use dots, and here we separate the components onto different lines:

$$\begin{gathered} [\ \dot{N}z + N\dot{z} - \dot{P}y - P\dot{y}\ ]\ \hat{\mathbf{i}} + \\ +\ [\ \dot{P}x + P\dot{x} - \dot{M}z - M\dot{z}\ ]\ \hat{\mathbf{j}} + \\ +\ [\ \dot{M}y + M\dot{y} - \dot{N}x - N\dot{x}\ ]\ \hat{\mathbf{k}} \end{gathered}$$

but this is just two different cross-products added together. The first one is

$$\begin{gathered} [\ \dot{N}z - \dot{P}y\ ]\ \hat{\mathbf{i}} + \ [\ \dot{P}x - \dot{M}z\ ]\ \hat{\mathbf{j}} + \ [\ \dot{M}y - \dot{N}x\ ]\ \hat{\mathbf{k}} \\ = \dot{\mathbf{F}} \times \mathbf{r} \end{gathered}$$

and the second is:

$$\begin{gathered} [\ N\dot{z} - P\dot{y}\ ]\ \hat{\mathbf{i}} + \ [\ P\dot{x} - M\dot{z}\ ]\ \hat{\mathbf{j}} + \ [\ M\dot{y} - N\dot{x}\ ]\ \hat{\mathbf{k}} \\ = \mathbf{F} \times \dot{\mathbf{r}} \end{gathered}$$

Putting it all together

$$\frac{d}{dt}\ [\ \mathbf{F} \times \mathbf{r}\ ]\ = \dot{\mathbf{F}} \times \mathbf{r} + \mathbf{F} \times \dot{\mathbf{r}}$$

The product rule for differentiation holds for both the dot product and the cross-product.

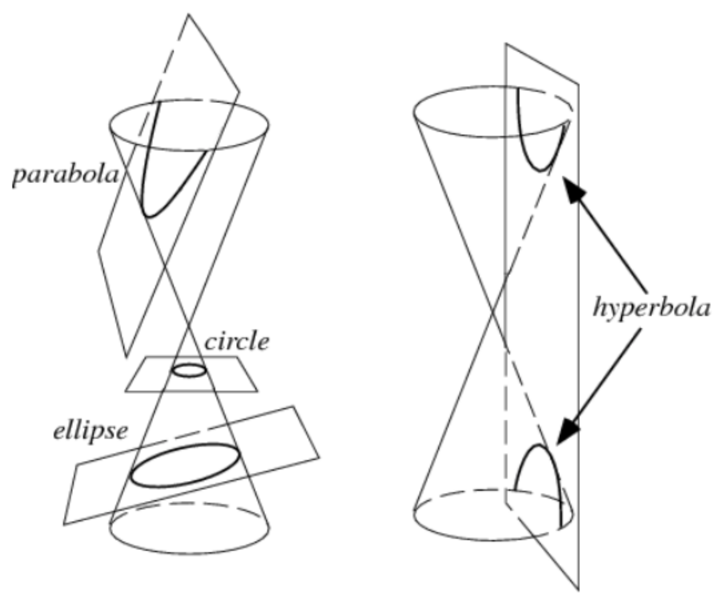
Part XXI

Conic Sections

Chapter 80

Circle

We now consider what are called quadratic forms, as distinguished from linear equations (i.e., for lines). The quadratics contain a squared term (or a term that mixes x and y).



The simplest example is the equation for a unit circle centered at the origin:

$$x^2 + y^2 = 1$$

Pythagoras tells us that for a point (x, y), the square of the distance from the origin is $x^2 + y^2$. This equation describes all the points whose distance from the origin is equal to $\sqrt{1} = 1$. But all the points equidistant to a point form a circle. We generalize

$$x^2 + y^2 = r^2$$

It is clear that when $y = 0$, $x = \pm\, r$. r is the radius of the circle.

Now, what happens if we displace the unit circle from the origin so its center is at $(1, 0)$? What this amounts to is adding 1 to the x value of every point. If we solve for x

$$x = \sqrt{1 - y^2}$$

and then add 1

$$x = \sqrt{1 - y^2} + 1$$
$$(x - 1)^2 = 1 - y^2$$
$$(x - 1)^2 + y^2 = 1$$

Or, more generally

$$(x - h)^2 + (y - k)^2 = r^2$$

where the origin of the circle is at (h, k).

Multiplying out:

$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$$

$$x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2)$$

Comparing to the most general form for a quadratic

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

We see that

$$A = 1, \quad B = 1, \quad C = 0$$

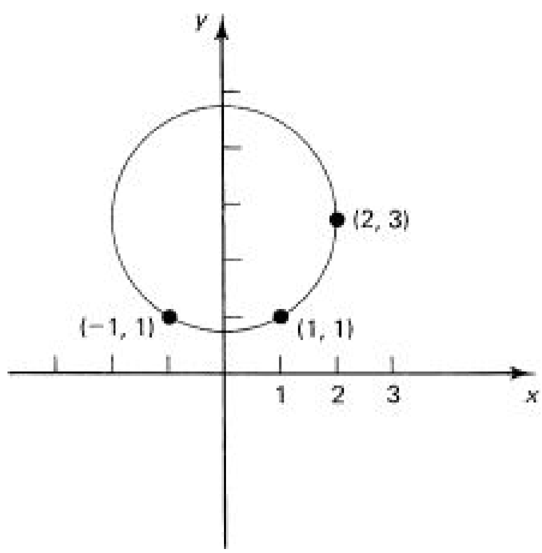
and in fact, this is true for all circles. (If $A = B \neq 1$, just divide all the terms by A).

Moreover

$$D = -2h, \quad E = -2k, \quad F = h^2 + k^2 - r^2$$

This equation can help us solve the following problem from Hamming: find the equation of the circle that passes through the following three points:

$$(-1, 1), (1, 1), (2, 3)$$



We write

$$x^2 + y^2 + Dx + Ey + F = 0$$

From the values of x and y at each of the three points we get

$$1 + 1 - D + E + F = 0$$

$$1+1+D+E+F=0$$
$$4+9+2D+3E+F=0$$

Three equations in three unknowns. We can do that.

Adding the first two equations together:

$$4+2(E+F)=0$$

so $E+F=-2$.

Subtracting the first two equations (or substituting the result for $E+F$) tells us that $D=0$.

Adding (-3) times the second equation to the third gives:

$$1+6-D-2F=0$$

$$7-2F=0$$

$F=7/2$, and since $E+F=-2$, $E=-11/2$.

So the solution is

$$x^2+y^2-\frac{11}{2}y+\frac{7}{2}=0$$

You can check that it works for all three points:

$$(-1,1),(1,1),(2,3)$$

The first two are easy, while the third gives

$$4+9-\frac{11}{2}3+\frac{7}{2}=0$$

$$8+18-33+7=0$$

which looks correct.

completing the square

We can improve this by completing the square. We see that

$$y^2 - \frac{11}{2}y + (\frac{11}{4})^2 = (y - \frac{11}{4})^2$$

We must add that back to the right-hand side of the original to obtain:

$$x^2 + (y - \frac{11}{4})^2 = (\frac{11}{4})^2 - \frac{7}{2}$$

The center is at $(0, 11/4)$. The radius doesn't come out cleanly but r^2 is

$$\frac{121}{16} - \frac{56}{16} = \frac{65}{16}$$

so r is slightly more than 2.

Or recall that we had:

$$D = -2h, \qquad E = -2k, \qquad F = h^2 + k^2 - r^2$$

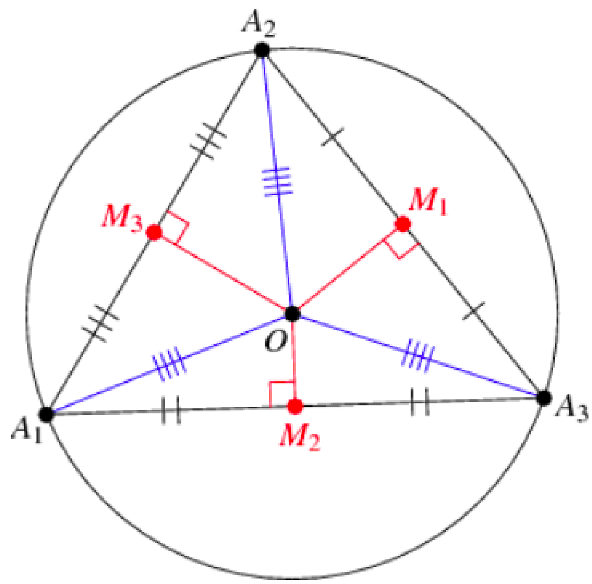
From this, we have that $h = 0$ and $k = -E/2 = 11/4$, and the radius is more complicated, as we said.

plane geometry

We can check our work by solving the problem using a technique from plane geometry. Again, we want the circle passing through three points:

$$(-1, 1), (1, 1), (2, 3)$$

Take two of the points to be placed on a circle and construct the line segment joining them (a chord of the circle). Find the midpoint of the chord and erect a perpendicular bisector through the midpoint. Now, every point lying on the bisector is equidistant from the two starting points. Proof: draw the two triangles including that point, the two starting points and the midpoint of the bisector. The two triangles are congruent. Here is the general picture.

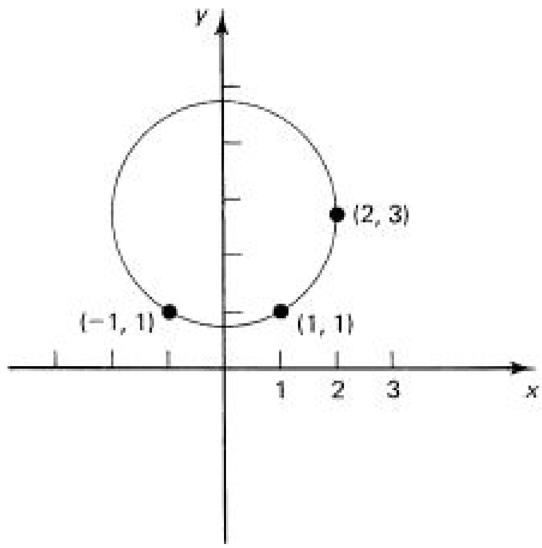


It's a bit trickier to prove that *every* point that is equidistant from the two points lies on the bisector. We assume that.

Since every point that is equidistant from the two points lies on the bisector, the radius of the circle lies on the bisector.

Then, erect a perpendicular bisector of a chord joining another pair chosen from the three points. This new bisector and the first one meet at the center of the circle.

In our case two points $(-1, 1), (1, 1)$ are symmetric about the y-axis. Therefore it is clear that the perpendicular bisector for these two points is the y-axis.



For the second bisector, form the vector between $(1,1)$ and $(2,3)$ as $\mathbf{v} = \langle 1, 2 \rangle$. The midpoint is at $(1,1) + \mathbf{v}/2 = (3/2, 2)$.

The slope of the bisector is the negative inverse of the slope for the chord which is $-1/2$ so the equation of the bisector is

$$y - y_0 = -\frac{1}{2}(x - x_0)$$

Plugging in the point that we know, we obtain

$$y - 2 = -\frac{1}{2}(x - 3/2)$$

We want to solve for y when $x = 0$, crossing the first bisector, the y-axis

$$y - 2 = -\frac{1}{2}(-3/2)$$
$$y = \frac{11}{4}$$

So the center is at $(0, 11/4)$, which matches what we had before. We compute the distance to one of the points $(1,1)$ as

$$d = \sqrt{1^2 + (11/4 - 1)^2} = \sqrt{1 + 49/16}$$

which also matches our previous result.

quadratics

The technique of completing the square comes from the standard equation

$$(x+p)^2 = x^2 + 2px + p^2$$

We run into problems where we have the $2px$ but not the p^2. For example

$$x^2 + y^2 + Dx + F = 0$$

Focus on

$$x^2 + Dx$$

we want to turn this into

$$(x + \text{something})^2$$

if D is like $2p$ we need to add something like p^2:

$$x + Dx + \frac{D^2}{4}$$

$$= (x + \frac{D}{2})^2$$

Since we added $D^2/4$ on the left, we must also add it on the right. We obtain

$$(x + \frac{D}{2})^2 + y^2 + F = \frac{D^2}{4}$$

You don't believe me? Multiply it out

$$x^2 + Dx + \frac{D^2}{4} + y^2 + F = \frac{D^2}{4}$$

To form $(x + D/2)^2$ on the left-hand side, we added $D^2/4$ (what we needed) to both sides.

Again, the general equation for a quadratic is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

In starting to work with one of these, the first thing to do is to see if there is a term which "mixes" x and y, that is, whether there is some term like Bxy. If there is, we might think about rotating the curve so that it is in a standard orientation.

We'll talk about how to do that **here**, in the context of the ellipse. However, the approach is general.

Let us assume we've done that, we relabel the new A, C etc. and assume here that $B = 0$.

Once in standard orientation, the next thing we might do is to translate the quadratic so that it is centered on the origin. We do that by completing the square for both x and y. We did some of that in this chapter, and we'll talk more about it **here** in the context of the parabola. Once again, however, the approach is general.

Cases:

- Both A and C present, and $F < 0$. If
  - A and C are both > 0: it's an ellipse.
  - A and C are of opposite signs: it's a hyperbola.
- A, C and F are all negative: it's imaginary.

• Only one squared term is present, but we still have the other variable $Ax^2 + Ey + F = 0$: it's a hyperbola.

Not every quadratic equation gives a conic. Some are "degenerate". For example, having done all the right manipulations, we might end up with something like

$$A'(x-h)^2 + B'(y-k)^2 = 0$$

which has only $x = h$ and $y = k$ as a solution. It's a point.

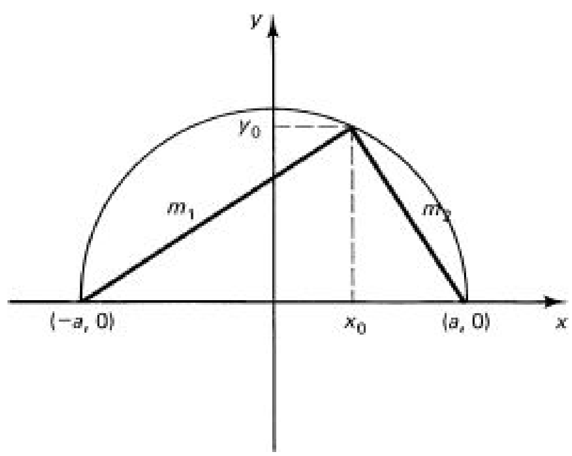


Figure 6.2-3 Angle in a semicircle

Here is another problem from Hamming. We need to prove that the angle above is a right angle. Suppose the equation of the circle is

$$x^2 + y^2 = a^2$$

The point on the circle is (x_0, y_0).

Our first solution uses slopes and points. The line from $(-a, 0)$ to

(x_0, y_0) has slope

$$m_1 = \frac{y_0}{x_0 + a}$$

The line from $(a, 0)$ to (x_0, y_0) has slope

$$m_2 = \frac{y_0}{a - x_0}$$

Two lines meet at a right angle if the product of their slopes is equal to -1.

$$m_1 m_2 = \frac{y_0}{x_0 + a} \, \frac{y_0}{a - x_0}$$

$$= \frac{y_0^2}{a^2 - x_0^2} = \frac{y_0^2}{x_0^2 + y_0^2 - x_0^2} = -1$$

This was not pretty, it's just good exercise.

And here is a proof using vectors and the dot product. Consider the semicircle centered on the origin with radius a, so the ends of the diameter are at $(x = \pm\, a, 0)$.

Form the vectors from those ends to an arbitrary point (x, y) on the perimeter:

$$\mathbf{u} = \ \langle x + a, y \rangle$$

$$\mathbf{v} = \ \langle x - a, y \rangle$$

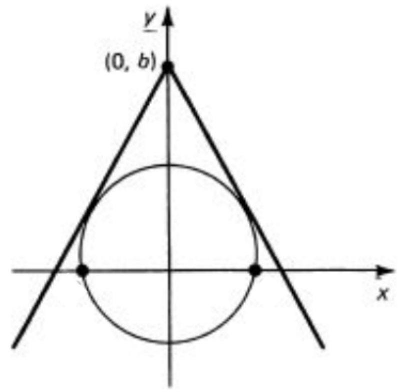
Notice that

$$\mathbf{u} \cdot \mathbf{v} = (x + a)(x - a) + y^2$$

$$= x^2 - a^2 + y^2 = 0$$

because $x^2 + y^2 = a^2$ for any point on the circle.

As our last example, consider the problem of finding the equation of a line tangent to a circle that goes through some arbitrary point b.



We take the circle to have radius a and be centered at the origin. We take the point b to be on the y-axis. The equation of the line on the right side is

$$\frac{y - y_0}{x - x_0} = m = \frac{y - b}{x}$$
$$y = mx + b$$

(well, of course).

For the point or points where the line intersects the circle we also have

$$y = \sqrt{a^2 - x^2}$$
$$\sqrt{a^2 - x^2} = mx + b$$
$$a^2 - x^2 = m^2x^2 + 2bmx + b^2$$
$$(m^2 + 1)x^2 + 2bmx + b^2 - a^2 = 0$$

From the quadratic equation:

$$x = \frac{-2bm \pm \sqrt{4b^2m^2 - 4(m^2 + 1)(b^2 - a^2)}}{2(m^2 + 1)}$$

We are looking for the case where there is a single solution so the discriminant under the square root must be equal to zero:

$$4b^2m^2 = 4(m^2 + 1)(b^2 - a^2)$$

$$
\begin{aligned}
m^2b^2 &= m^2b^2 - m^2a^2 + b^2 - a^2 \\
0 &= -m^2a^2 + b^2 - a^2 \\
m &= \pm\frac{\sqrt{b^2 - a^2}}{a}
\end{aligned}
$$

This makes sense since if $a = b$ the single tangent should be horizontal with zero slope. Notice that if $a^2 > b^2$ there is no real solution. This corresponds to having b inside the circle.

Chapter 81

Curvature

Curvature of curves

We would like to have a way to describe the curvature of a curve at a point. The way this is done is to fit a circle to the curve, to construct a circle that has the same first and second derivatives as the curve at the point of interest. (This section is taken from Hamming).

The general equation of a circle is

$$(x-h)^2 + (y-k)^2 = r^2$$

Implicit differentiation gives

$$2(x-h) + 2(y-k)\ y' = 0$$

$$(x-h) + (y-k)\ y' = 0$$

Differentiate again (using the product rule)

$$1 + (y')^2 + (y-k)\ y'' = 0$$

The known values in these equations are the point (x, y), the slope y' and the second derivative y'', taken from the curve we want to analyze.

The unknowns are the parameters of the circle that we're trying to find: h, k, and r.

Now, solve the last equation for

$$y - k = -\frac{1 + (y')^2}{y''}$$

and solve the second equation for

$$x - h = -(y - k)\ y'$$

From these two equations we can find h and k, the center of the circle.

Substituting for $x - h$ into the general equation of the circle we obtain

$$(y - k)^2\ (y')^2 + (y - k)^2 = r^2$$
$$(y - k)^2\ [\ (y')^2 + 1\ ] = r^2$$

Now substitute for $y - k$

$$[\ \frac{1 + (y')^2}{y''}]^2\ [\ (y')^2 + 1\ ] = r^2$$
$$\frac{[\ 1 + (y')^2\ ]^3}{(y'')^2} = r^2$$

Take the square root to find the radius.

$$r = \frac{[\ 1 + (y')^2\ ]^{3/2}}{y''}$$

If the original curve had been a straight line or the second derivative zero at that point, we would face the problem of dividision by zero.

For this reason, it makes sense to define the *curvature* as the inverse of the radius of the fitted circle. κ is used for the curvature:

$$\kappa = \frac{1}{r} = \frac{y''}{[\ 1 + (y')^2\ ]^{3/2}}$$

We write this as the absolute value:

$$\kappa = \frac{1}{r} = |\frac{y''}{[\ 1 + (y')^2\ ]^{3/2}}|$$

because r is always a positive quantity.

So, for example, any straight line (which has $y'' = 0$), has zero curvature.

test with a known circle

Try using a circle centered at $(0, 0)$, as the curve to be fitted. We should get back $h = 0, k = 0, r = r$.

$$x^2 + y^2 = r^2$$

$$2x + 2y\ y' = 0$$

$$y' = -\frac{x}{y}$$

The second derivative is

$$y'' = -\ [\frac{y - y'x}{y^2}\ ]$$

Combined with the previous result

$$y'' = -\ [\frac{y - (-x/y) \cdot x}{y^2}\ ]$$

$$= -\frac{y^2 + x^2}{y^3} = -\frac{r^2}{y^3}$$

Calculate the three values:

$$\kappa = |\frac{y''}{[\ 1 + (y')^2\ ]^{3/2}}|$$
$$= \frac{r^2}{y^3} \cdot \frac{1}{(1 + x^2/y^2)^{3/2}}$$
$$= r^2 \cdot \frac{1}{[\ y^2(1 + x^2/y^2)\ ]^{3/2}}$$
$$= r^2 \cdot \frac{1}{[r^2\ ]^{3/2}}$$
$$= \frac{1}{r}$$

That's exactly what we want. $\kappa = 1/r$ corresponds to a circle of radius r. To find the center of the circle:

$$y - k = -\frac{1 + (y')^2}{y''}$$

$$= \frac{1 + x^2/y^2}{r^2/y^3}$$

$$= \frac{y^2 + x^2}{r^2/y}$$
$$= y$$

For x

$$x - h = -(y - k)\ y'$$
$$= -(y - k)(-\frac{x}{y})$$

Since we found that $k = 0$

$$x - h = -y(-\frac{x}{y}) = x$$

Everything looks correct. Having carried out all of this preparation, let's do a real problem.

parabola

Consider a simple parabola

$$y = x^2 \quad\quad y' = 2x \quad\quad y'' = 2$$

The radius is

$$r = |\frac{[\ 1 + (y')^2\ ]^{3/2}}{y''}|$$

$$= \frac{[\ 1 + 4x^2\ ]^{3/2}}{2}$$

The y-coordinate of the origin of the circle, k, is

$$k = y + \frac{1 + (y')^2}{y''}$$

$$= y + \frac{1 + 4x^2}{2}$$

and h is

$$h = x + (y - k)\ 2x$$

If we decide to fit the circle to the parabola at the point $(0, 0)$ we have

$$r = \frac{1}{2} \quad\quad k = \frac{1}{2} \quad\quad h = 0$$

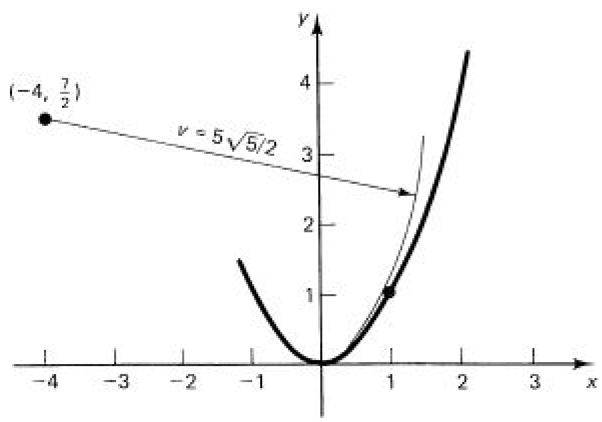
Perhaps more interesting, for the point $(1,1)$ we have

$$r = \frac{5^{3/2}}{2}$$

$$\begin{aligned} k &= y + \frac{1+4x^2}{2} \\ &= 1 + \frac{5}{2} = \frac{7}{2} \end{aligned}$$

$$\begin{aligned} h &= x + (y-k)\ 2x \\ &= 1 + (1-7/2)2 = -4 \end{aligned}$$

Which seems to be correct.



Find the slope of the line from the center of the circle to $(1,1)$:

$$\begin{aligned} m &= \frac{1-k}{1-h} \\ &= \frac{1-7/2}{1-(-4)} = \frac{-5/2}{5} = -\frac{1}{2} \end{aligned}$$

Recall that slope of the parabola at $(1,1)$ is $y' = 2x = 2$. The slope of the line perpendicular to the tangent to the curve at the point $(1,1)$ is the negative of the reciprocal, which is just what we obtained.

The squared distance between the center and the point should be equal to the radius squared from above:

$$(1 - 7/2)^2 + (1 - (-4))^2$$

$$= \frac{1}{4} \cdot 5^2 + 5^2 = \frac{5}{4} \cdot 5^2 = \frac{5^3}{4}$$

Taking the square root, we obtain what we had above:

$$r = \frac{5^{3/2}}{2}$$

Notice that the point $(1,1)$ is 5 units horizontally across from the center and $5/2$ units down. If we were to translate the whole thing to the origin and then find the slope of the tangent to the circle at that point it would be

$$-\frac{x}{y} = -\frac{5}{5/2} = -2$$

which is the same as the slope of the parabola at that point, within sign.

The second derivative is

$$y'' = -\frac{r^2}{y^3}$$

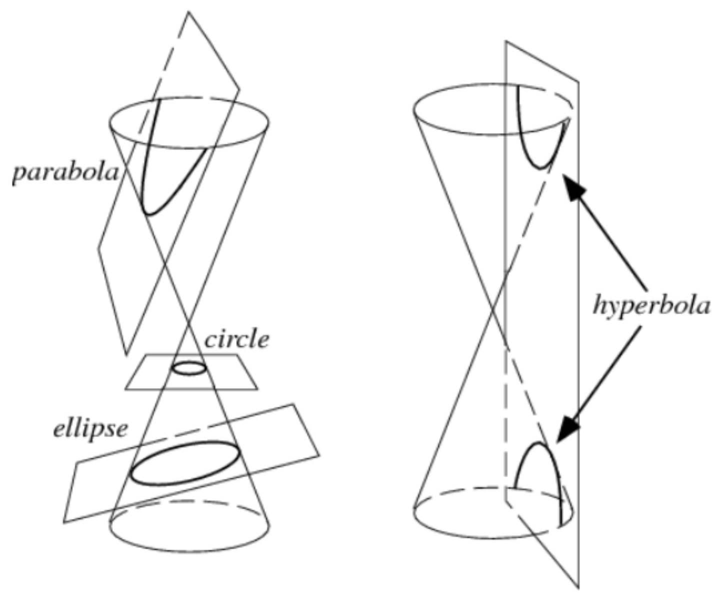
$$= -\frac{5^3/4}{(5/2)^3} = -2$$

which is also the same, within sign. The difference in sign comes about because we have not adjusted the equation for the circle. This point is in the fourth quadrant for the version moved to the origin.

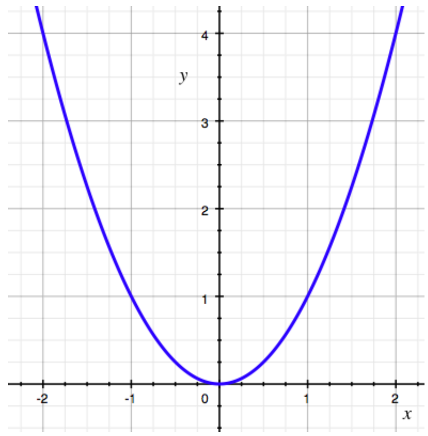
Chapter 82

Parabola

You probably know that the *conic sections* are the parabola, circle and ellipse, and hyperbola.



After the circle, perhaps the most familiar of these is the parabola. Consider a parabola in standard orientation, opening up and with its axis of symmetry pointed straight up and down.



A parabola of this orientation whose vertex is located at the origin and with axis of symmetry $x = 0$ is described by the very simple equation

$$y = ax^2$$

For the one shown above, a is equal to 1, giving $y = x^2$. For example, the points $(1, 1)$ and $(2, 4)$ lie on the curve.

polynomials and quadratics

A **polynomial** is a sum containing various powers of an independent variable, which is usually given as x. For example:

$$y = c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$$

The powers must be positive integers or zero: $n \in \{0, 1, 2 \dots\}$.

Each power x^n is multiplied by its corresponding constant c_n. The original equation might contain multiple coefficients for a given power x^n that are then combined to form the constant.

Each c_n is some real number. It is usual in examples for these constants to be integers, but this is by no means a requirement.

A **quadratic** is a polynomial that contains a term with x^2 but no higher powers of x:

$$y = c_2x^2 + c_1x + c_0$$

This is usually written as

$$y = ax^2 + bx + c$$

where a, b, and c are constants.

A quadratic may or may not contain lower powers of x. That is, either or both of b and c might be equal to zero. All of these are quadratics:

$$y = ax^2$$

$$y = ax^2 + bx$$

$$y = ax^2 + c$$

In general, the roles the constants a and c play in the graph of a quadratic are fairly obvious, while that of b is more subtle.

Changing the value of c shifts the graph up or down by the amount added. Comparing

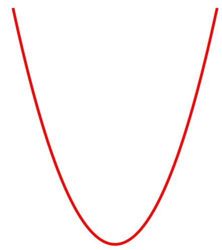
$$y = ax^2$$

$$y = ax^2 + c$$

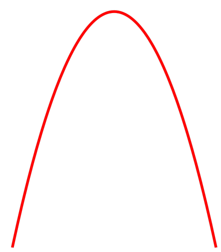
the second graph will be identical to the first, simply shifted up by c.

shape factor

a is called the **shape factor**. If a is positive, then the two "arms" of the parabola open up, and the **vertex** is the minimum value of the graph of the function.



If a is negative, then the graph opens down, and the vertex is the maximum.



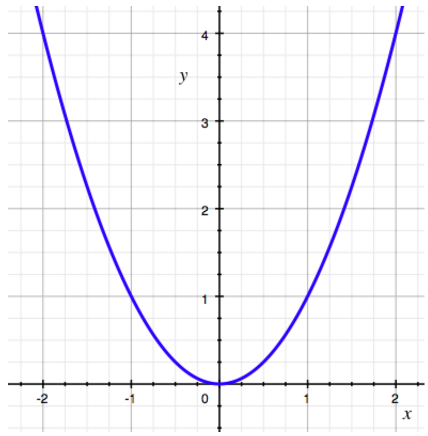
Any quadratic is symmetrical about the vertical axis that goes through its minimum (or maximum) point, the vertex.

For the rest of this discussion, we will only consider $a > 0$. If $a < 0$, the whole graph is flipped upside down. In that case, everywhere I say "minimum", we have a maximum instead.

Suppose

$$y = ax^2$$

x^2 is always greater than or equal to zero ($x^2 \geq 0$), therefore so is the value of this function, $y \geq 0$. The minimum value is the vertex at $(x = 0, y = 0)$.



The graph is symmetrical about the y-axis. This is also evident because $y = ax^2$ is an even function: $f(x) = f(-x)$.

Consider the values of y corresponding to $x = \{1, 2, 3, 4\}$. These are $y = \{a, 4a, 9a, 16a\}$. They increase like the square of x, but *linearly* with a.

If we plot $y = x^2$ and compare it to $y = 4x^2$, every y value in the second plot can be taken from the first one, just multiplied by 4. a stretches the plot linearly in the y-direction.

A substitution of variables $v = y/a$ turns $y = ax^2$ into $v = x^2$.

vertex

Every parabola with the same value of a has exactly the same shape. For the same a, they may differ in the position of the vertex, depending on the other constants, b and c. The graph of a parabola depends only on the shape factor and the position of the vertex.

The coordinates at the vertex are usually given as constants (h, k). Suppose the vertex of the parabola is at (h, k) with $a = 1$. Then the equation of the parabola is

$$(y-k) = (x-h)^2$$

It may seem counterintuitive that we subtract the value of the variable at the vertex, but this is consistent across all of the conic sections.

Rearranging

$$y = (x-h)^2 + k$$

we see that $y-k$ is like adding k to the constant c, it moves the graph up the page.

Expand

$$(y-k) = a(x-h)^2$$

$$y = ax^2 - 2ahx + h^2 + k$$

and compare with the canonical representation

$$y = ax^2 + bx + c$$

The coefficients of corresponding powers must be equal so

$$-2ah = b$$

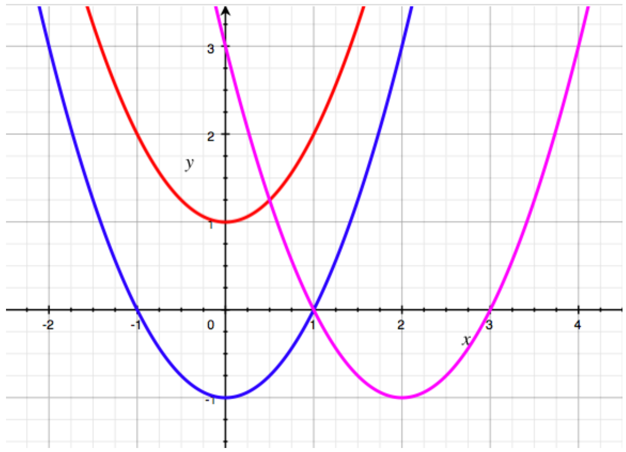
$$h = -\frac{b}{2a}$$

The x-value of the vertex is $x = -b/2a$. Also

$$h^2 + k = c$$

The y-value of the vertex is

$$k = c - h^2 = c - b^2/4a$$



Here are three plots, all with shape factor $a = 1$, which differ only in the position of the vertex: $y = x^2 - 1$ (blue), $y = x^2 + 1$ (red), and $y = (x-2)^2 - 1$ (magenta).

The two with the vertex below the x-axis have $c = -1$, while the one with the vertex above the x-axis has $c = 1$.

Notice that the magenta one has $h = 2$. To write it in canonical form, expand

$$\begin{aligned} y &= (x-h)^2 - 2 \\ &= x^2 - 2xh + h^2 - 2 \\ &= x^2 - 4x + 2 \end{aligned}$$

Even though $c > 0$ for this one, the vertex is below the x-axis. That's because $b = -2h$ also affects the placement of the vertex. We see that effect here as $c = h^2 - 2$.

roots

The roots of a quadratic are those values of x for which the corresponding value of y is zero.

For example, if $x = 0$ then

$$y = x^2 = 0^2 = 0$$

$x = 0$ is the only value that yields this result, since if $x \neq 0$, then $y = x^2 > 0$.

Suppose we shift the graph of the equation *down*, by subtracting 1, i.e. letting $c = -1$:

$$y = x^2 - 1$$

From examining the plot of this function you will observe that there are two points where the graph of the function crosses the x-axis (where $y = 0$). We can guess them from the plot as $x = \pm 1$, and confirm that result directly by substituting into the equation and checking that we get $y = 0$.

$$y = (1)^2 - 1 = 0$$
$$y = (-1)^2 - 1 = 0$$

Another way to get this answer is to factor the original equation.

$$y = (x + 1)(x - 1)$$

Now it is obvious that either $x = \pm 1$ gives $y = 0$ as the result.

If a quadratic can be factored to the form

$$y = (x - r_1)(x - r_2)$$

then r_1 and r_2 are the roots, because if $x = r_1$ or $x = r_2$, then $y = 0$.

On the other hand, suppose we shift the graph *up*

$$y = x^2 + 1$$

Since x^2 is always positive or zero, there is no value of x which gives $y = 0$. We say that such an equation has no (real) roots. An equivalent statement or observation is that its graph does not cross the x-axis.

Again, if a quadratic can be factored to the form

$$y = (x - r_1)(x - r_2)$$

then

$$y = x^2 - (r_1 + r_2)x + r_1 r_2$$

In the canonical representation

$$ax^2 + bx + c$$

c is the product of the roots, and b is the negative of the sum.

For the example

$$y = x^2 - 1$$

the sum of the roots is 0 and their product is -1. From the first fact:

$$r_2 = -r_1$$

substituting into the second.

$$r_1^2 = 1$$

If you've had practice factoring quadratics with integer roots, you should be very familiar with this fact: c is the product of the roots, and b is the negative of the sum.

three types

In summary, we can classify parabolas into three types.

The first one has a graph that does not cross the x-axis. It has no real roots.

The second type has a graph that does cross the x-axis and has two distinct real roots of the form

$$y = (x - r_1)(x - r_2)$$

The third one has repeated roots

$$y = (x - r)(x - r)$$

This type has a single value of x that yields $y = 0$. This happens when the graph of the parabola just touches the x-axis — the vertex is on the x-axis.

The example given above $y = x^2$ is a special case of this type where $r = 0$.

quadratic formula

The quadratic formula gives the roots of any quadratic. In the case where there are no real roots, the results from the quadratic formula are complex numbers of the form $p \pm q\sqrt{-1}$, (where p and q are real numbers).

The formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You should memorize this.

Examples:

$y = x^2$

$$x = \frac{-(0) \pm \sqrt{0^2 - 4(1)(0)}}{2(1)} = 0$$

$y = x^2 - 1$

$$x = \frac{-(0) \pm \sqrt{0^2 - 4(1)(-1)}}{2(1)} = \pm\frac{\sqrt{4}}{2} = \pm 1$$

$y = x^2 + 1$

$$x = \frac{-(0) \pm \sqrt{0^2 - 4(1)(1)}}{2(1)} = \pm\frac{\sqrt{-4}}{2} = ??$$

The result of the last calculation is a pair of complex numbers. A complex number is a number of the form $p + q\sqrt{-1}$, often written $p + iq$.

We may write the expression under the square root as

$$D = b^2 - 4ac$$

where D stands for discriminant. If $D < 0$ then the result is a pair of complex numbers and we say there are no real roots. If $D = 0$ then there is a single (repeated) root.

Although the existence of complex roots means the graph does not cross the x-axis and there is no x such that $f(x) = 0$, nevertheless these roots do have physical meaning.

The two complex roots are related, they are called complex conjugates.

Label them as $z = p \pm q\sqrt{-1}$ and plug them into the factored quadratic

$$y = (x - z_1)(x - z_2)$$

$$\begin{aligned} y = &\ [\ x - (p + q\sqrt{-1})\ ]\ [\ x - (p - q\sqrt{-1})\ ] \\ &= (x - p - q\sqrt{-1})(x - p + q\sqrt{-1}) \end{aligned}$$

Multiplying this out, the square roots will (always) disappear:

$$y = x^2 - px + xq\sqrt{-1} - px + p^2 - pq\sqrt{-1} - xq\sqrt{-1} + pq\sqrt{-1} + q^2$$

$$= x^2 - 2px + p^2 + q^2$$
$$= (x-p)^2 + q^2$$

The meaning of this is the following: even with complex roots, the factored form gives a real result when multiplied out. The real part of $z = p \pm q\sqrt{-1}$ is p, and this is the value of x at the minimum.

q^2 is the displacement of the vertex up from the x-axis at the minimum.

more about b

Consider the basic equation

$$y = ax^2 + bx + c$$
$$\frac{y-c}{a} = x^2 + \frac{b}{a}x$$

By judicious manipulation we can make the last term $b/a \cdot x$ go away (this is *always* true). The general procedure is called completing the square. Write

$$x^2 + \frac{b}{a}x + \_\_\_ = (x + \_\_\_)^2$$

We seek two values to substitute for the spaces ___.

Now, of course, the ___ on the left-hand side is the square of the second one, on the right.

But there is another constraint. Namely, the second ___ is related to the cofactor of x on the left-hand side.

Recall that

$$x^2 + 2mx + m^2 = (x+m)^2$$

Compare that with

$$x^2 + \frac{b}{a}x + \_\_\_ = (x + \_\_\_)^2$$

Can you see that b/a must be equal to $2m$ and so $m = b/2a$? We need two copies of the second term in the binomial expansion (m), to put as the cofactor of x in the term $2mx$. Since the standard form in the last expression has b/a equivalent to $2m$, we get $b/2a$ equivalent to m.

If that's not clear, just verify that the following works. Write:

$$x^2 + \frac{b}{a}x + \_\_\_ = (x + \frac{b}{2a})^2$$

$$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = (x + \frac{b}{2a})^2$$

Now that we know what is needed to complete the square, go back to our problem.

$$\frac{y-c}{a} = x^2 + \frac{b}{a}x$$

To make the perfect square, we add $(b/2a)^2$ to the right-hand side, and to maintain the equality add the same thing to the left-hand side:

$$\frac{y-c}{a} + (\frac{b}{2a})^2 = x^2 + \frac{b}{a}x + (\frac{b}{2a})^2$$

As we saw above, the right-hand side is also $(x+b/2a)^2$ so we can write

$$\frac{y-c}{a} + (\frac{b}{2a})^2 = (x + \frac{b}{2a})^2$$

Finally, multiply through by a and rearrange slightly

$$y = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

We can get several things from this.

First, the minimum value of y occurs when the squared term is equal to zero, that is when

$$x + \frac{b}{2a} = 0$$

$$x = -\frac{b}{2a}$$

Therefore, the vertex of the parabola is at this value of x. We found this earlier by writing $(y - k) = a(x - h)^2$ and multiplying out.

When $x = -b/2a$, the corresponding value of y is

$$y = a(-\frac{b}{2a} + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

$$= c - \frac{b^2}{4a}$$

This also matches what we had before.

Second and more generally, the roots occur when

$$y = 0 = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

$$a(x + \frac{b}{2a})^2 = \frac{b^2}{4a} - c$$

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which is the quadratic formula.

Third, any quadratic can be rewritten as

$$y = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

$$y = a(x - h)^2 + k$$

translation

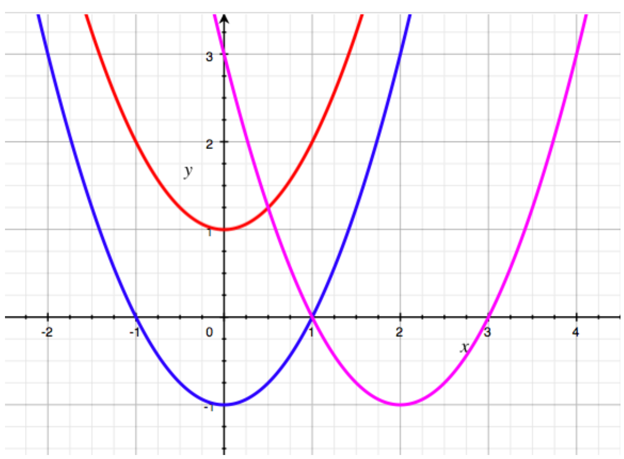
The basic shape depends only on a.

b (combined with $2a$) moves the vertex right or left from the y-axis.

a, b, c all together combine to move it up and down from the x-axis.

If you play around with a plotting application and change b you will find that the shape stays the same, but both the x and the y-values of the vertex will change as b changes.

Here are the three plots again.



$y = x^2 - 1$ (blue), $y = x^2 + 1$ (red), and $y = (x - 2)^2 - 1$ (magenta).

The two with the vertex on the y-axis have $h = 0$, the other has $h = 2$.

The plots with real roots have $k = -1$ and $(y - k) = y + 1$, the other has $k = 1$.

For example, the lower right-hand plot is

$$(y + 1) = (x - 2)^2 = x^2 - 4x + 4$$

$$y = x^2 - 4x + 3$$

This can be factored easily:

$$y = (x - 1)(x - 3)$$

The roots are at $x = 1, x = 3$, which checks.

rotation

You might ask, what about rotation? For example, if we rotate $y = x^2$ by 45 degrees clockwise, what would be the equation to describe it? The short answer is that such equations do exist, and they have terms like xy or uv in them. They are not polynomials of the type we've been describing.

As an example to rotate through 45°, replace x and y by u and v with

$$x = u \cos\theta - v \sin\theta = \frac{u}{\sqrt{2}} - \frac{v}{\sqrt{2}}$$

$$x = ku - kv$$

$$y = u \sin\theta + v \cos\theta = \frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}$$

$$y = ku + kv$$

Substitute for x and y in the standard equation:

$$y = ax^2 + bx + c$$

$$ku + kv = a(ku - kv)^2 + b(ku - kv) + c$$
$$u + v = ak(u^2 - 2uv + v^2) + b(u - v) + \frac{c}{k}$$

Notice the term $-2akuv$ that mixes u and v.

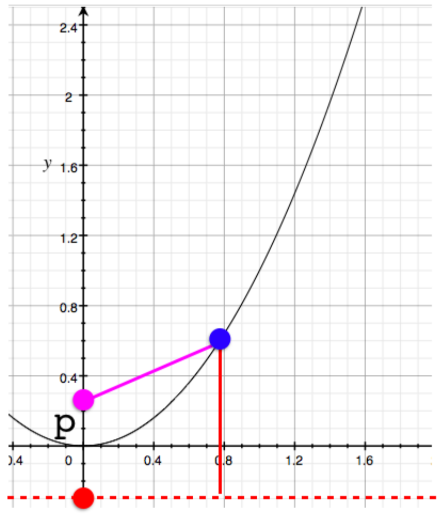
This is a more advanced topic than we can deal with here.

focus and directrix

There is a geometric definition of the parabola. Based on what we said above, without loss of generality, we can translate any parabola to the origin of coordinates, with equation $y = ax^2$.

Now, pick a point on the y-axis a distance p up from the origin, colored magenta in the figure. This point is called the focus.

Then draw a line parallel to the x-axis which intersects the y-axis the same distance p below the origin. This line is called the directrix. It is colored red and is dashed.



The parabola consists of all those points whose distance to the focus is equal to the vertical distance to the directrix.

Pick an arbitrary point on the parabola (in blue), with coordinates (x, ax^2). The squared distance to the focus (magenta point) is

$$\Delta y^2 + \Delta x^2 = (ax^2 - p)^2 + x^2$$

while the squared distance to the directrix (red line) is $(ax^2 + p)^2$.

For the correct choice of p these distances are to be equal:

$$(ax^2 - p)^2 + x^2 = (ax^2 + p)^2$$

$$a^2x^4 - 2apx^2 + p^2 + x^2 = a^2x^4 + 2apx^2 + p^2$$

Canceling two terms on each side

$$-2apx^2 + x^2 = +2apx^2$$

Divide by x^2

$$-2ap + 1 = 2ap$$

$$4ap = 1$$
$$p = \frac{1}{4a}$$

The shape factor a determines the distance of the focus from the origin, which is p, and from the directrix, which is $2p$.

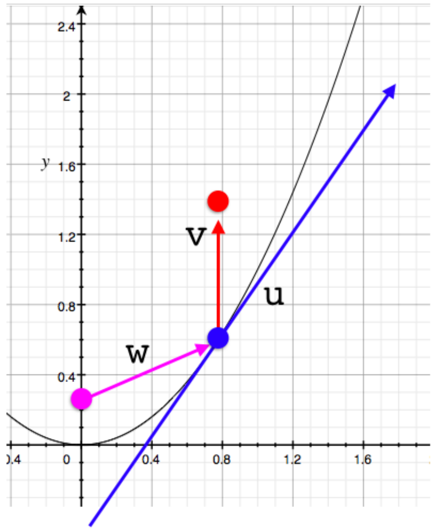
Chapter 83

Headlight problem

The reflective property of the parabola asserts that if a light ray emitted from the focus bounces off any point of the parabola, it then travels off in the vertical direction.

Snell's law for reflection says that the angle of incidence and reflection to the inside surface of the parabola must be equal. It is curious that this law has Snell's name on it, since the fact was known to Euclid, and Heron had a proof of it. The proof depends on the assumption that light travels the shortest path between two points.

In any case, applying that law to this problem, the angle of incidence is the angle of the magenta vector $\mathbf{w}$ with the tangent vector $\mathbf{u}$. This is equal to the angle of reflection, the angle of the tangent $\mathbf{u}$ with the vertical vector $\mathbf{v}$.



We assert that there exists a point on the y-axis (the focus, colored magenta), with the property that when we draw a vector to any point on the parabola, the angle that this vector makes with the tangent to the parabola is equal to the angle the tangent makes with the vertical.

Let the distance of this point from the origin be p. Then

$$\mathbf{w} = \langle x, ax^2 - p \rangle$$

The tangent has slope $2ax$ so

$$\mathbf{u} = \langle 1, 2ax \rangle$$

Scale the vertical to be a unit vector

$$\mathbf{v} = \langle 0, 1 \rangle$$

By the definition of the dot product, the cosine of the angle between $\mathbf{w}$ and $\mathbf{u}$ is

$$\frac{\mathbf{w} \cdot \mathbf{u}}{u \; w}$$

By the equal angle constraint, this is equal to the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$

$$\frac{\mathbf{u} \cdot \mathbf{v}}{u\ v} = \frac{\mathbf{w} \cdot \mathbf{u}}{u\ w}$$

Since $v = 1$ we have

$$w\ (\mathbf{u} \cdot \mathbf{v}) = \mathbf{w} \cdot \mathbf{u}$$

That's the important logic of the solution.

Now it's just algebra: The length of $\mathbf{w}$ is

$$w = \sqrt{x^2 + (ax^2 - p)^2}$$

while

$$\mathbf{u} \cdot \mathbf{v} = 2ax$$

$$\mathbf{w} \cdot \mathbf{u} = x + 2ax(ax^2 - p)$$

So

$$w\ (\mathbf{u} \cdot \mathbf{v}) = \mathbf{w} \cdot \mathbf{u}$$

$$\sqrt{x^2 + (ax^2 - p)^2}\ (2ax) = x + 2ax(ax^2 - p)$$

Divide by $2ax$:

$$\sqrt{x^2 + (ax^2 - p)^2} = \frac{1}{2a} + (ax^2 - p)$$

Square both sides

$$x^2 + (ax^2 - p)^2 = \frac{1}{(2a)^2} + \frac{1}{a}(ax^2 - p) + (ax^2 - p)^2$$

A nice cancelation:

$$x^2 = \frac{1}{(2a)^2} + \frac{1}{a}(ax^2 - p)$$

We can also cancel the x^2:

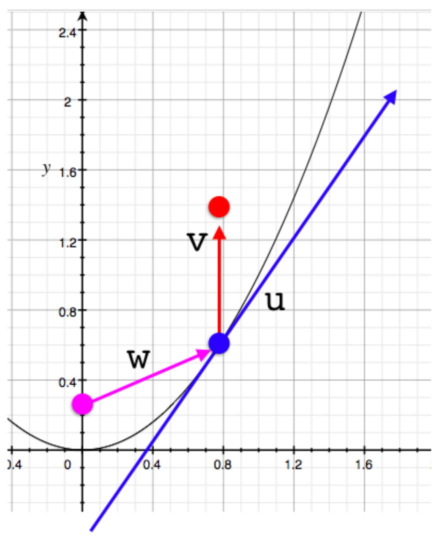
$$0 = \frac{1}{(2a)^2} + \frac{1}{a}(-p)$$

and finally cancel an a:

$$0 = \frac{1}{4a} - p$$
$$p = \frac{1}{4a}$$

The point $(0, 1/4a)$ is, as we saw before, the focus of the parabola.

Since p is independent of x, this property holds for every point on the parabola.



An alternative, more geometric approach is to note that the angle the vector $\mathbf{u}$ makes with the vertical at (x, ax^2) is equal to the angle $\mathbf{u}$ makes with the y-axis (just off the image to the bottom).

This angle is equal to the angle between $\mathbf{w}$ and $\mathbf{u}$ if and only if the triangle is isosceles, that is, if length of the vector $\mathbf{w}$ is equal to the distance between $(0, p)$ and the intersection of $\mathbf{u}$ with the y-axis.

We start by exploring the properties of a line through the point (x, ax^2) with slope equal to $2ax$.

From this point on, the point on the parabola is *fixed*. We want to write an equation for a line with the same slope as the parabola at this point, the same slope as the vector $\mathbf{u}$.

We will be re-using x as a variable. To reduce confusion, label the fixed value at the point as $\hat{x}$, so then $\hat{y} = a\hat{x}^2$, and the slope is $2a\hat{x}$.

The point-slope formula for the line is

$$2a\hat{x} = \frac{\Delta y}{\Delta x} = \frac{y - \hat{y}}{x - \hat{x}} = \frac{y - a\hat{x}^2}{x - \hat{x}}$$

The intersection with the y-axis occurs at $y = 0$ so there

$$\begin{aligned} 2a\hat{x} &= \frac{-a\hat{x}^2}{x - \hat{x}} \\ 2 &= \frac{-\hat{x}}{x - \hat{x}} \\ 2x - 2\hat{x} &= -\hat{x} \\ x &= \frac{\hat{x}}{2} \end{aligned}$$

The intersection of $\mathbf{u}$ with the x-axis is at $\hat{x}/2$.

For the intersection with the y-axis, $x = 0$ and then

$$\begin{aligned} 2a\hat{x} &= \frac{y - a\hat{x}^2}{-\hat{x}} \\ -2a\hat{x}^2 &= y - a\hat{x}^2 \\ y &= -a\hat{x}^2 \end{aligned}$$

What we've discovered is that the point of intersection is the same distance below the x-axis as our point on the parabola $(\hat{x}, a\hat{x}^2)$ is above it.

We could have used congruent triangles proceeding from the discovery above that the intersection of with the x-axis is at $\hat{x}/2$.

Our goal is to show that the triangle is isosceles:

$$a\hat{x}^2 + p = w$$

$$a\hat{x}^2 + p = \sqrt{\hat{x}^2 + (a\hat{x}^2 - p)^2}$$

$$(a\hat{x}^2 + p)^2 = \hat{x}^2 + (a\hat{x}^2 - p)^2$$

Continuing

$$a^2\hat{x}^4 + 2ap\hat{x}^2 + p^2 = \hat{x}^2 + a^2\hat{x}^4 - 2ap\hat{x}^2 + p^2$$

Does this look familiar?

Cancel two terms

$$2ap\hat{x}^2 = \hat{x}^2 - 2ap\hat{x}^2$$

$$4ap\hat{x}^2 = \hat{x}^2$$

$$4ap = 1$$

$$p = \frac{1}{4a}$$

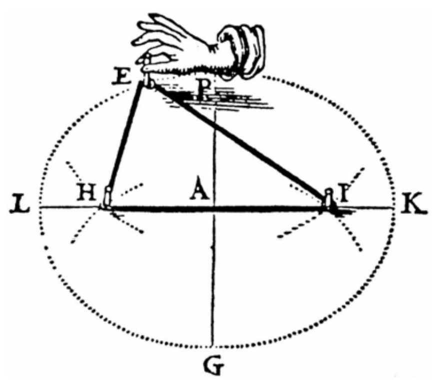
And we already proved this is true, if the magenta point we start from is the focus.

Hence the lengths are equal, the triangle is isosceles, and the corresponding angles are equal. The point we've been using is just the focus.

Chapter 84

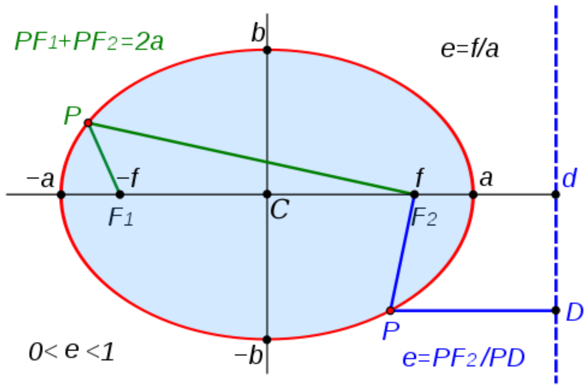
Ellipse

construction



Learning how to draw an ellipse using two pins and a circular piece of string holding a pencil is an early adventure in mathematics. The ellipse is the set of all points whose combined distance to the two pins (foci) is the same.

The drawing is reproduced from a 17th century book in Acheson (see the References).



The pin positions with respect to the origin or center are called the foci, lying at the points shown in the figure as $(\pm f, 0)$.

We will use the notation c: the focus in the first quadrant is at the point $(c, 0)$.

The lengths of the axes (called semi-major and semi-minor) are usually labeled a and b.

Consider the situation when the pencil is at the point $P = (0, a)$. The distance to the left focus is $c + a$, so the length L of the string is twice that

$$L = 2(c + a)$$

The combined distance from each point on the ellipse to the two foci is the length of the string minus the distance between the two foci

$$L - 2c = 2(c + a) - 2c = 2a$$

standard equation

We learn in algebra that the equation for an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We will derive this equation below.

The relation of a and c to b can be seen from the point $Q = (0, b)$ (see previous figure) where the combined distance to the two foci is just

$$QF_1 + QF_2$$

From what we said above the distance is $2a$, but Pythagoras also gives us

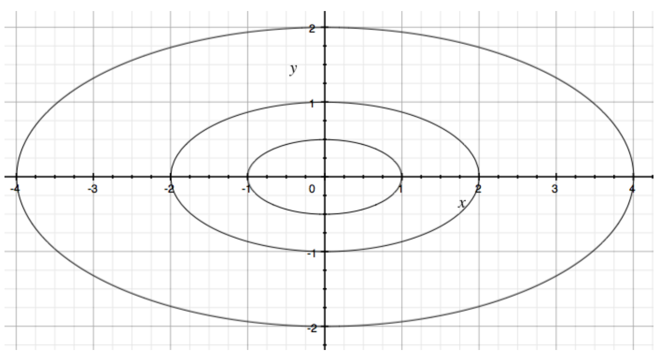
$$QF_1 + QF_2 = 2a = 2\sqrt{b^2 + c^2}$$

so

$$b^2 + c^2 = a^2$$
$$c^2 = a^2 - b^2$$

Given a and b one can then find c easily.

Here are three ellipses drawn with the same center.



The difference is an adjustment in the value on the right-hand side of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$$

where $r = \{1/2, 1, 2\}$. This is equivalent to scaling both a and b by the same factor of r

$$\frac{x^2}{(ra)^2} + \frac{y^2}{(rb)^2} = 1 = (\frac{x/a}{r})^2 + (\frac{y/b}{r})^2$$

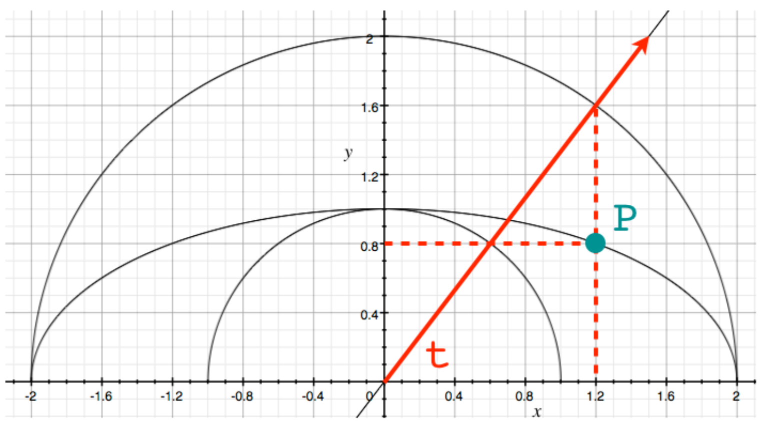
When $r = 2$ we need to make the string a bit less than twice as long, because the length c is also involved:

$$\frac{L_2}{L_1} = \frac{ra + c}{a + c}$$

parametrization

An alternative view is the one below, which shows (black curves) the upper half of two circles of radius $r = 1$ and $r = 2$ and an ellipse whose equation is

$$\frac{x^2}{2^2} + \frac{y^2}{1} = 1$$



Here $a = 2$ and $b = 1$.

The standard parametrization of the ellipse is

$$x = a \cos t$$

$$y = b \sin t$$

which I had trouble visualizing, until I drew the picture. The thing is that the parameter t is *not* the angle that a ray to P makes with the x-axis, as it is for the circle. Instead, to find the x value of P corresponding to t, we extend the ray with angle t to the larger circle, with radius a, where we read off the x-value as

$$x = a\cos t$$

We go back to find the intersection of the same ray with the small circle to get

$$y = b\sin t$$

The algebraic way to do this is to show that the parametrization is equivalent to the original formulation

$$x^2 = a^2\cos^2 t$$

$$y^2 = b^2\sin^2 t$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$$

as expected.

Derivation of the equation of the ellipse

Although it is a bit tedious, it's a reasonable exercise to derive the equation of the ellipse from the geometric constraint. Recall that a is the length of the semi-major axis and c the distance to each of the foci from the origin.

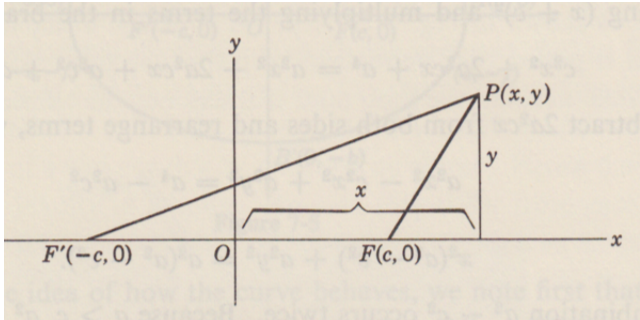


Figure 7-4

For any point x, y on the ellipse, the distance to the focus in the first quadrant is

$$\sqrt{(x-c)^2+y^2}$$

and combined distances to both foci are equal to $2a$ so

$$2a = \sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2}$$

Now we just do some algebra. Pick one square root and rearrange

$$\sqrt{(x-c)^2+y^2} = 2a - \sqrt{(x+c)^2+y^2}$$

Square both sides

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + (x+c)^2 + y^2$$

Cancel y^2

$$(x-c)^2 = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + (x+c)^2$$

But

$$(x+c)^2 - (x-c)^2 = 4xc$$

so

$$0 = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + 4xc$$
$$a^2 + xc = a\sqrt{(x+c)^2+y^2}$$

Square again

$$a^4 + 2a^2xc + x^2c^2 = a^2(x^2 + 2xc + c^2 + y^2)$$

$$a^4 + 2a^2xc + x^2c^2 = a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2$$

Cancel $2a^2xc$

$$a^4 + x^2c^2 = a^2x^2 + a^2c^2 + a^2y^2$$

Gather terms

$$a^4 - a^2c^2 = a^2x^2 - x^2c^2 + a^2y^2$$

$$a^2(a^2 - c^2) = x^2(a^2 - c^2) + a^2y^2$$

Recall that $b^2 = a^2 - c^2$

$$b^2a^2 = b^2x^2 + a^2y^2$$

Divide by a^2b^2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

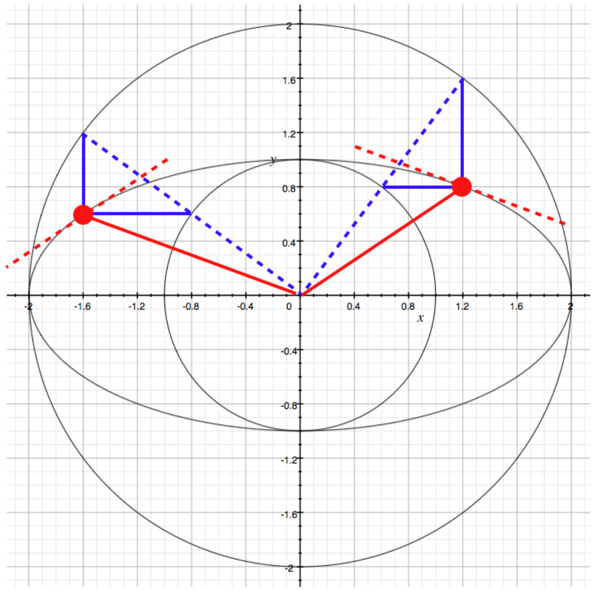
rotation

Let's return to the diagram of the ellipse with two bounding circles of radius a and radius b. There is a new diagram below. Consider the coordinates of the point $P = (x, y)$ (the red dot in the first quadrant) as functions of the angle t. As we said, t is *not* the angle of a ray from the origin to P.

Draw a ray (blue dotted line) from the origin makes an angle t with the x-axis. As before, extend the ray to the outer circle. The radius is a, the angle is t, and

$$a \cos t = x$$

This is the parametrization of the ellipse introduced previously.



The ray drawn with angle t has the same x-intercept with the outer circle as our point P on the ellipse. Similarly, the intercept of the ray with the inner circle has the same y-value as the point P on the ellipse.

We estimate the point $P = (1.2, 0.8) = (6/5, 4/5)$. Using our algebraic equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Recall that $a = 2$ and $b = 1$ so

$$x^2 + 4y^2 = 4$$

Plugging in for x^2 and y^2 we get

$$\frac{36}{25} + 4\,(\frac{16}{25}) = \frac{100}{25} = 4$$

as expected. Reading off the intercepts for the ray with angle t (dotted blue line) with the outer circle, we have the point $(1.2, 1.6)$ at a distance

2 from the origin. Thus,

$$\frac{1.2}{2} = 0.6 = \cos t$$

$$t \approx 0.927 \text{ rad} \approx 53^\circ$$

Looking again at the figure, we want to consider what happens for the angle $u = t + \pi/2$. This is the dotted blue ray in the second quadrant.

We might calculate the values of sine and cosine for u, but notice that if we view u as a vector, its *dot product* with t must be equal to zero. The coordinates of the intercept of the rotated vector with the outer circle are $(-1.6, 1.2)$, so the cosine of the angle u is

$$\cos u = -0.8$$

$$u \approx 2.498 = t + \frac{\pi}{2} \text{ rad} \approx 143^\circ$$

We confirm that

$$2.498 - 0.927 = 1.57 = \frac{\pi}{2}$$

The coordinates of the point on the ellipse are $(-1.6, 0.6)$, which we check against the formula

$$x^2 + 4y^2 = 4$$

$$(1.6)^2 + 4(0.6)^2 = 2.56 + 4(0.36) = 4$$

(no clean fractions this for this one).

tangent

Finally, and this is really the crucial result:

the vector to the point, call it Q, on the ellipse (red dot in the second quadrant) is the *tangent to the ellipse* for the point P in the first quadrant.

How did this happen? Recall what we did. We had

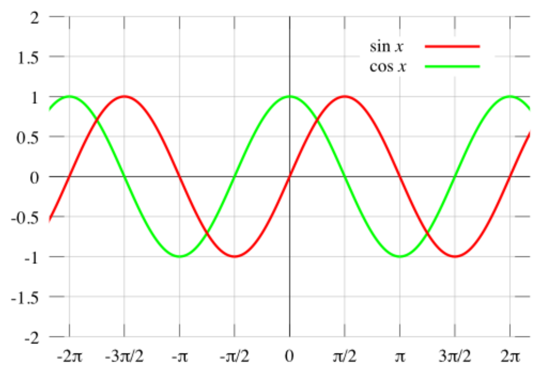
$$x = a\cos t$$

$$y = b\sin t$$

The rotated point $Q = (x', y')$ is

$$x' = a\cos(t + \frac{\pi}{2})$$

$$y' = b\sin(t + \frac{\pi}{2})$$



Sine is like cosine, but shifted to the right by $\pi/2$

$$\cos\theta = \sin(\theta + \frac{\pi}{2})$$

$$\sin\theta = -\cos(\theta + \frac{\pi}{2})$$

So

$$x' = a\cos(t + \frac{\pi}{2}) = -a\sin t$$

$$y' = b\sin(t + \frac{\pi}{2}) = b\cos t$$

Let's look at the position vector, which can be written $\mathbf{r}(t)$, since it's a function of the angle t or the time, but we will just use $\mathbf{r}$. It has components x and y.

$$\mathbf{r} = \ \langle x, y \rangle \ = \ \langle a\cos t, b\sin t \rangle$$

Now, the tangent to the ellipse is precisely the direction in which a particle at (x, y) is currently moving on the ellipse. The tangent vector points in the same direction as the velocity vector, but $\mathbf{v}$ is just the time-derivative of the position vector.

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \ \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle \\ &= \ \langle -a\sin t, b\cos t \rangle \\ &= \ \langle x', y' \rangle \end{aligned}$$

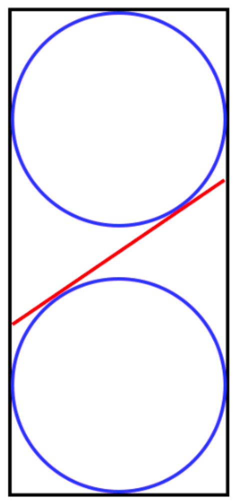
These two methods — using the time-derivative as the tangent, and rotation of t by $\pi/2$ — generate the same vector. And that's the point. :)

Starbird

Here is a neat approach to the ellipse that I saw in one of Michael Starbird's lectures.

Imagine a glass cylinder, shown here in cross-section and colored black. The cylinder has been sliced through at an angle by a plane, and we suppose a flat piece of glass in the shape of an ellipse is glued between the two halves.

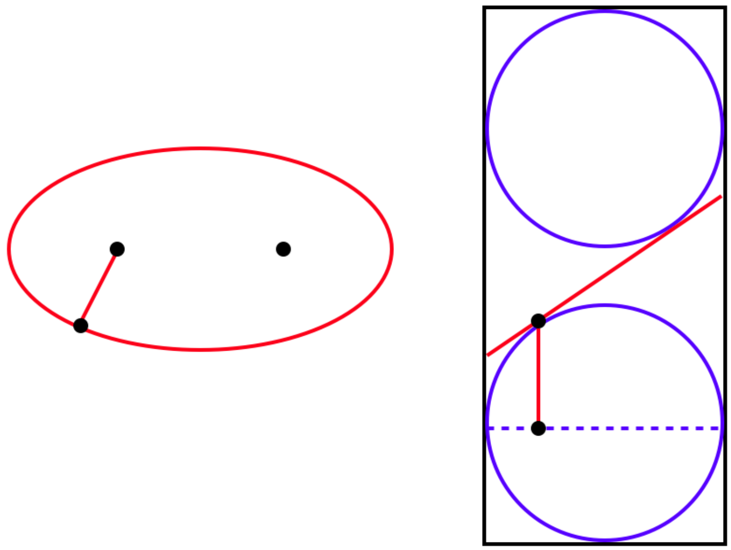
The elongated region in red (formed at the plane of the cut) is the ellipse, and the cylinder is oriented so that at each horizontal position going across the page, the two points on the ellipse are at the same vertical position. We see the plane of the cut edge-on.



Two spheres that fit snugly inside the cylinder lie above and below the ellipse, just touching it. The planar surface of the ellipse is tangent to the spheres, touching each one at a single point.

We claim that the points where the spheres touch the ellipse are the foci of the ellipse.

By the nature of the construction, the two spheres just fit inside the cylinder. That means the intersection where the spheres touch the cylinder is a circle, the lower one is shown with a dotted blue line in the next figure.

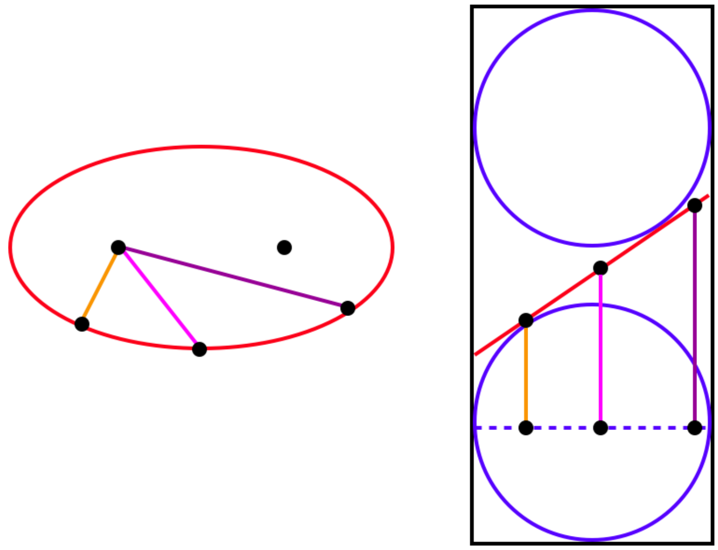


Now consider any point on the ellipse. On the left, we see one point on the ellipse together with two interior points we claim are foci, with a line drawn from our point to one of the foci.

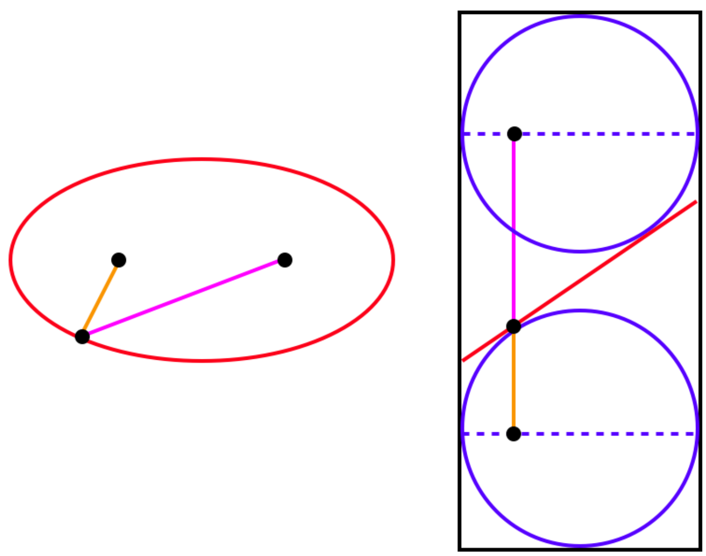
We said that this point is the point where the ellipse touches the lower sphere. We conclude that the line we've drawn from the edge of the ellipse to the focus is a tangent to the sphere.

A second tangent of interest is the perpendicular dropped vertically down the surface of the cylinder, shown in the right panel. Since they are both tangents, this line is the same length as line to the focus.

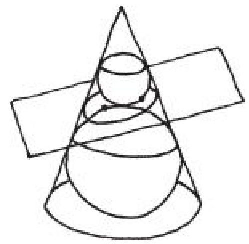
But the construction, and this equality, holds for any point on the ellipse, as shown in the next figure.



Finally, this is true for both spheres (below). The sum of the perpendicular tangents for any point is a constant.



Thus, the points where the spheres touch the ellipse are its foci, because the sum of the distances to any point on the ellipse, which is equal to the sum of the vertical tangents, is a constant.



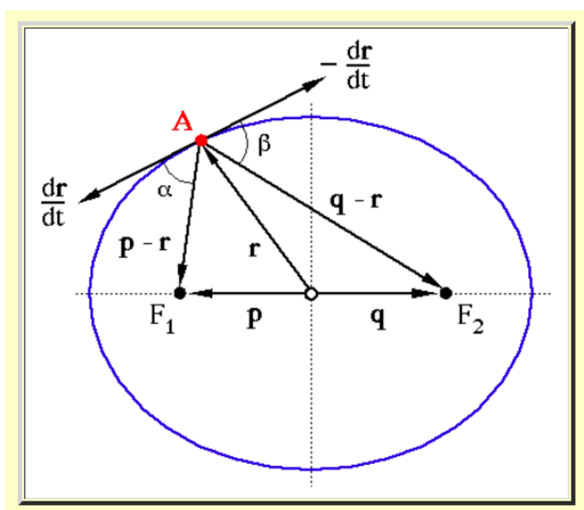
According to Lockhart, the same argument can be used to prove that the cross sections of a cone are ellipses (which seems strange at first since we've been demonstrating that the cross-sections of cylinders are also ellipses).

Chapter 85

Ellipse reflected rays

In any ellipse, the segments from the foci to any point on the ellipse make equal angles with the tangent. This means that light rays emitted from one focus and striking anywhere on the ellipse will pass through the other focus upon reflection. It is the principle behind "whispering galleries."

Here is a vector proof. A simple geometric proof follows.



http://163.178.103.176/Fisiologia/renal/objetivo_1/Medical_

Lithotripsy.htm

We have seen previously that

$$\mathbf{r} = \langle x, y\rangle = \langle a\cos t, b\sin t\rangle$$

$$\mathbf{v} = \dot{\mathbf{r}} = \langle -a\sin t, b\cos t\rangle$$

$\mathbf{v}$ points in the same direction as the tangent.

Now construct a vector from the origin to the focus F as

$$\mathbf{q} = \langle c, 0\rangle$$

and the vector to F' is $\mathbf{p}$.

The vector corresponding to PF going toward the focus is

$$\mathbf{q} - \mathbf{r}$$

(since $\mathbf{r} + PF$ gets to the same place as $\mathbf{q}$).

and the one corresponding to PF' is

$$\mathbf{p} - \mathbf{r}$$

By the standard definition of an ellipse, the sum of the lengths is a constant

$$|\mathbf{q} - \mathbf{r}| + |\mathbf{p} - \mathbf{r}| = 2a, \quad \text{constant}$$

Lemma about a time-derivative

$\mathbf{r}$ is a vector function of time. We need to establish a property of the time-derivative of such a function. As an example use an arbitrary vector function of time, $\mathbf{w}$.

$$\frac{d}{dt}|\mathbf{w}| = \frac{d}{dt}\sqrt{\mathbf{w} \cdot \mathbf{w}}$$

by the chain rule:

$$= \frac{1}{2\sqrt{\mathbf{w} \cdot \mathbf{w}}} \ \frac{d}{dt}(\mathbf{w} \cdot \mathbf{w})$$

by the product rule

$$= \frac{1}{2\sqrt{\mathbf{w} \cdot \mathbf{w}}} \ (\frac{d\mathbf{w}}{dt} \cdot \mathbf{w} + \mathbf{w} \cdot \frac{d\mathbf{w}}{dt})$$

$$= \frac{1}{|\mathbf{w}|} \ (\frac{d\mathbf{w}}{dt} \cdot \mathbf{w})$$

Hence we have

$$\frac{d}{dt}|\mathbf{w}| = \frac{d\mathbf{w}}{dt} \cdot \frac{\mathbf{w}}{|\mathbf{w}|}$$

The rate of change of the magnitude of $\mathbf{w}$ is equal to a part of the rate of change of $\mathbf{w}$ itself, namely that part which points in the same direction as $\mathbf{w}$ itself. Effectively, what we've done is to decompose a differential change in $\mathbf{w}$ with time into two parts, one parallel to $\mathbf{w}$ and one perpendicular to it. The latter does not contribute to a change in the length.

back to the proof

We had

$$|\mathbf{q} - \mathbf{r}| + |\mathbf{p} - \mathbf{r}| = 2a$$

where $2a$ is a constant, so the time-derivative of the left-hand side is also zero.

$$\frac{d}{dt} \ (|\mathbf{q} - \mathbf{r}| + |\mathbf{p} - \mathbf{r}|) = 0$$

Now using the lemma

$$\frac{d}{dt}|\mathbf{q}-\mathbf{r}| = \frac{d}{dt}(\mathbf{q}-\mathbf{r}) \cdot \frac{\mathbf{q}-\mathbf{r}}{|\mathbf{q}-\mathbf{r}|}$$

and since $\mathbf{q}$ is constant:

$$= -\frac{d\mathbf{r}}{dt} \cdot \frac{\mathbf{q}-\mathbf{r}}{|\mathbf{q}-\mathbf{r}|}$$

So for the whole thing we have (rearranging terms):

$$\frac{d\mathbf{r}}{dt} \cdot \frac{\mathbf{q}-\mathbf{r}}{|\mathbf{q}-\mathbf{r}|} = -\frac{d\mathbf{r}}{dt} \cdot \frac{\mathbf{p}-\mathbf{r}}{|\mathbf{p}-\mathbf{r}|}$$

Take a look at that!

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \mathbf{v}$$

so we actually have

$$\mathbf{v} \cdot \frac{\mathbf{q}-\mathbf{r}}{|\mathbf{q}-\mathbf{r}|} = -\mathbf{v} \cdot \frac{\mathbf{p}-\mathbf{r}}{|\mathbf{p}-\mathbf{r}|}$$

The terms dotted with $\mathbf{v}$ are *unit vectors* from the point to the two foci.

Using the definition of the dot product, for a unit vector $\hat{\mathbf{b}}$

$$\mathbf{a} \cdot \hat{\mathbf{b}} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|\cos\theta$$

this

$$\mathbf{v} \cdot \frac{\mathbf{q}-\mathbf{r}}{|\mathbf{q}-\mathbf{r}|} = -\mathbf{v} \cdot \frac{\mathbf{p}-\mathbf{r}}{|\mathbf{p}-\mathbf{r}|}$$

is the same as

$$|\mathbf{v}|\cos\alpha = |-\mathbf{v}|\cos\beta$$

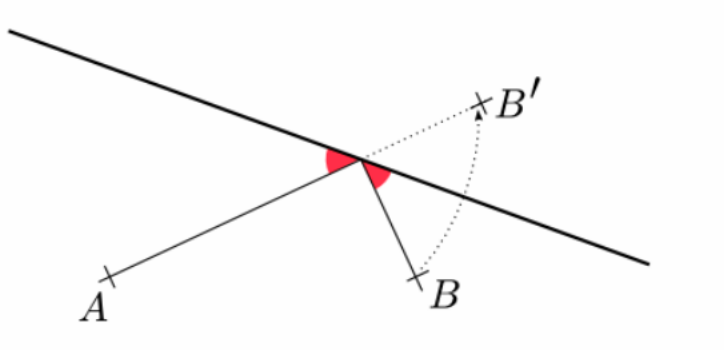
so

$$\alpha = \beta$$

□

geometry

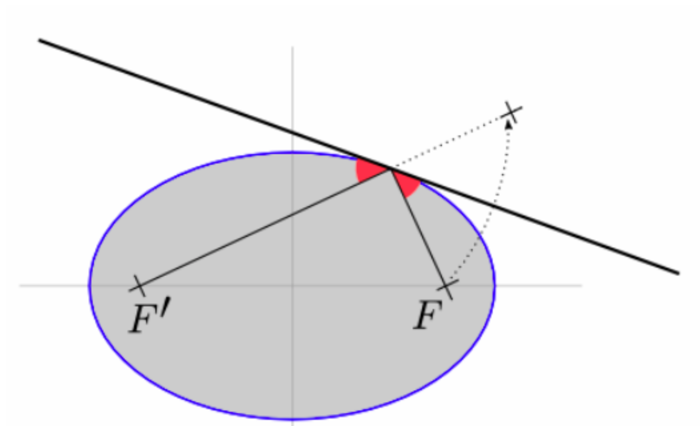
The geometric proof is even simpler. In another **chapter**, we considered the problem of the "shortest path."



The problem is to go from A to the line and then back to B by the shortest path. The clever solution is to place B' on the other side of the line at the same distance away. By definition (see Euclid) the shortest path A to B' is a straight line.

We can use vertical angles (or supplementary angles twice) and then similar triangles to prove that the two angles colored red are equal.

Now consider an enhanced diagram of the same situation:



We draw the tangent to the ellipse. By definition, the tangent has only a single point on the curve. This point lies at a distance $2a$ from the combined foci. All other points on the line are farther away from the two foci than the point of intersection. (You would have to make the string bigger to draw the ellipse that goes through any of those points).

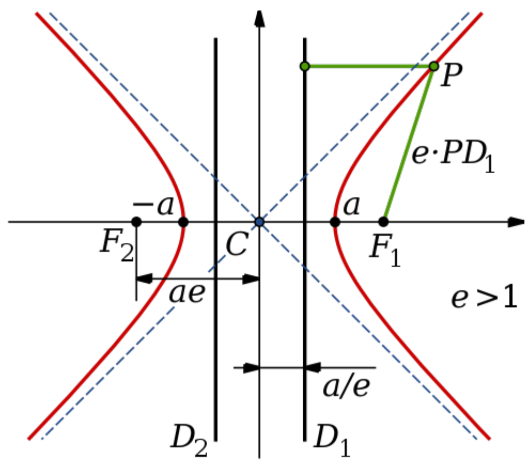
Therefore, the path shown is the shortest path from F' to the tangent and then to F. But we know that for the shortest path the angles colored red are equal.

http://math.stackexchange.com/questions/1063977/how-to-geometrically-prove-the-focal-property-of-ellipse

Chapter 86

Hyperbola

Here is a hyperbola as shown in the wikipedia article on the subject.



Hyperbolas of this type (that open "east-west") have equations of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Rearranging

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2}$$

so the minimum value of x occurs when $y = 0$ and $x = a$.

The *conjugate* hyperbola of this one is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

or equivalently

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

opens "north-south."

And, although I will wait to deal with this complication, we have to mention another very common hyperbola

$$xy = c$$

where it must be true that $x \neq 0$ and $y \neq 0$.

Another feature of hyperbolas is the asymptote, the straight line which is approached when $x, y >> a, b$. In the case of the first example

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

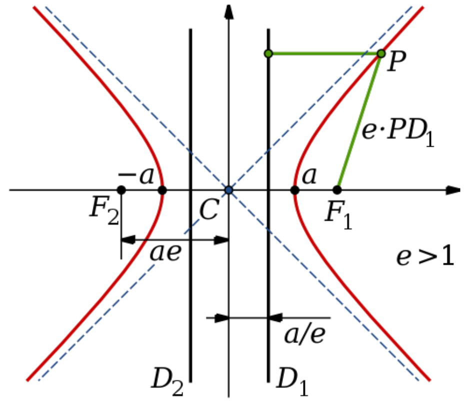
$$y^2 = \frac{b^2}{a^2}x^2 - \frac{1}{a^2}$$

but for large x and y this approaches

$$y^2 = \frac{b^2}{a^2}x^2$$

$$y = \pm\frac{b}{a}x$$

As the diagram suggests:



The following diagram gives geometric meaning to the b coefficient which really derives from the slope of the asymptotic line. We go vertically up from $x = a$ to the asymptote and then go left to the $y-$axis, that intercept is b.

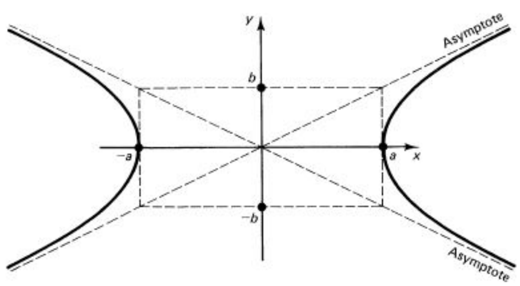


Figure 6.6-1 Hyperbola

geometry

Kline gives the following string and pencil construction for the hyperbola.

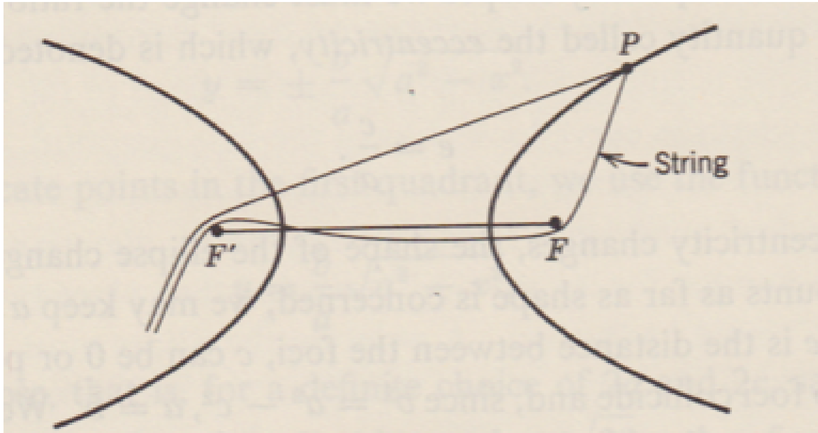


Figure 7-9

Pick two foci F and F' and loop a long piece of string around them, holding it tight. Then place the pencil at some point P on a line between the two foci, at a fixed position in the upper loop.

Now let the string slowly slip up past F' in both directions, increasing the length of PF and PF' by the same amount for each small slip. What this amounts to is that the difference $PF - PF'$ is constant.

If we place the origin halfway between F and F' then

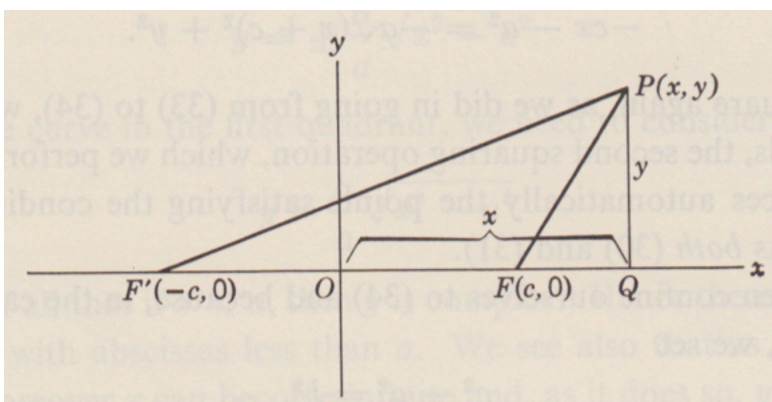


Figure 7-10

$$PF = \sqrt{(x-c)^2 + y^2}$$

$$PF' = \sqrt{(x+c)^2 + y^2}$$

and the difference $PF' - PF$ is

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}$$

and if the constant distance

$$PF' - PF = 2a$$

then

$$\sqrt{((x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

Now we repeat the approach we took for the ellipse:

$$\sqrt{((x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$$

Square

$$(x+c)^2 + y^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x-c)^2 + y^2$$

Cancel y^2

$$(x+c)^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x-c)^2$$

Since

$$(x+c)^2 - (x-c)^2 = 4cx$$

we have

$$4cx = 4a^2 + 4a\sqrt{(x+c)^2 + y^2}$$
$$cx - a^2 = a\sqrt{(x+c)^2 + y^2}$$
$$c^2x^2 - 2ca^2x + a^4 = a^2(x+c)^2 + a^2y^2$$
$$c^2x^2 - 2ca^2x + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2$$
$$(c^2 - a^2)x^2 - a^2y^2 = (c^2 - a^2)a^2$$

Define b^2 slightly differently here

$$b^2 = c^2 - a^2$$

so

$$b^2x^2 - a^2y^2 = b^2a^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

which looks familiar.

Part XXII

More geometry

Chapter 87

Rotation

Yet another thing that can happen to make life complicated is rotation.

Consider the rotated parabola. You are probably used to seeing examples where it opens to the right or left. These are obtained by having an equation like

$$a(y-k)^2 = (x-h)$$

with $x = g(y)$

Then, the easiest thing to do is to switch x for y, solve the problem, and switch back at the end.

But it is also possible to rotate through a different angle, like 45°. What happens then? Well, basically we replace x and y by u and v with

$$x = u\cos\theta - v\sin\theta$$

$$y = u\sin\theta + v\cos\theta$$

(I derived these **here**). For 45°, $\sin\theta = \cos\theta = 1/\sqrt{2}$. Let

$$k = \sin\theta = \cos\theta = 1/\sqrt{2}$$

Substitute for x and y as given above

$$y = ax^2 + bx + c$$

$$ku + kv = a(ku - kv)^2 + b(ku - kv) + c$$

$$u + v = ak(u^2 - 2uv + v^2) + b(u - v) + \frac{c}{k}$$

Now, we might attempt to solve this for v in terms of u, but there is a new term $-2uv$ which mixes up u and v. That is what gives a parabola that is not symmetric with respect to either the x-axis or the y-axis.

general problem

The most general equation for a parabola, ellipse or hyperbola is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

This includes rotated versions of all three.

Kline says (Chapter 7) to consider a rotation through an angle θ. I will use t for θ We wrote above

$$x = u\ cost - v\ sint$$

$$y = u\ sint + v\ cost$$

First compute the products:

- $x^2 = u^2 \cos^2 t - 2uv \sin t \cos t + v^2 \sin^2 t$
- $xy = u^2 \sin t \cos t + uv \cos^2 t - uv \sin^2 t - v^2 \sin t \cos t$
- $y^2 = u^2 \sin^2 t + 2uv \sin t \cos t + v^2 \cos^2 t$

Now try substituting into the general equation (I know, it's a mess). We collect the coefficients for all the terms u^2, uv, v^2, etc., separately:

- $[A\cos^2 t + B\sin t\cos t + C\sin^2 t\ ]\ u^2$
- $[-2A\sin t\cos t + B\cos^2 t - B\sin^2 t + 2C\sin t\cos t\ ]\ uv$
- $[A\sin^2 t - B\sin t\cos t + C\cos^2 t\ ]\ v^2$
- $[D\cos t + E\sin t\ ]\ u$
- $[-D\sin t + E\cos t\ ]\ v$

The insight is this: we must choose t so as to eliminate the coefficient of the term that mixes u and v: namely uv.

$$-2A\sin t\cos t + B\cos^2 t - B\sin^2 t + 2C\sin t\cos t = 0$$

Recall those sum of angles formulas!

$$\cos^2 t - \sin^2 t = \cos 2t$$

$$2\sin t\cos t = \sin 2t$$

So

$$-A\sin 2t + B\cos 2t + C\sin 2t = 0$$

giving

$$\tan 2t = \frac{B}{A-C}$$

example

Consider

$$xy = 1$$

Here A and C are zero, while $B = 1$. What angle's tangent is not defined? $\pi/2$. As $2t$ approaches $\pi/2$, its tangent approaches ∞. So the value of t we seek is $t = \pi/4$.

We go back and compute the coefficients for all the other terms. Since only $B \neq 0$ and since $\sin \pi/4 = \cos \pi/4 = 1/\sqrt{2}$, we get

$$[\ \frac{A}{2} + \frac{B}{2} + \frac{C}{2}\ ]\ u^2 +\ [\ \frac{A}{2} - \frac{B}{2} + \frac{C}{2}\ ]v^2 = 1$$

$$= \frac{u^2}{2} - \frac{v^2}{2} = 1$$

which is the equation of a rectangular hyperbola opening left and right.

test

Suppose you run into a general conic equation with some version of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Ask these questions to decide what you have:

- **Are both variables squared?**

  No: It's a parabola.
  Yes: Go to the next test....

  - **Do the squared terms have opposite signs?**

    Yes: It's an hyperbola.
    No: Go to the next test....

    - **Are the squared terms multiplied by the same number?**

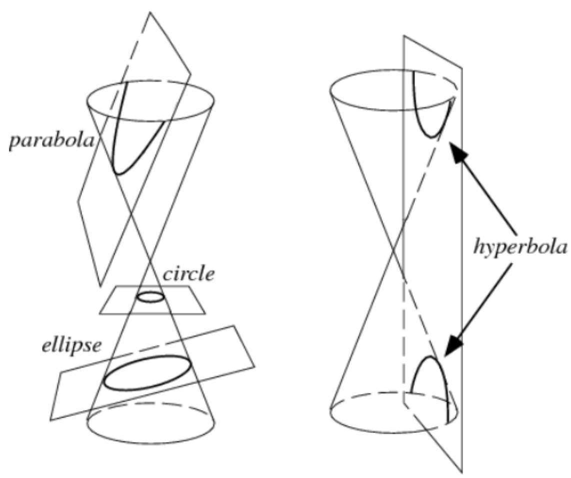
      Yes: It's a circle.
      No: It's an ellipse.

Kline goes through the effort of showing that, after rotating to a standard orientation, *every* equation of the general form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be translated to the origin to give a standard parabola, circle, ellipse or hyperbola.

conic sections



Everyone learns in high school that the conic sections can be obtained by slicing a double cone with a plane and taking the points that belong to both. Here is a simple example.

The level curves of a cone are

$$x^2 + y^2 = r^2$$

and the equation of the cone is $z = kr$ where $k = H/R$ is a constant. Suppose $k = 1$.

Suppose we also have a plane like

$$z = y + 1$$

This plane has normal vector

$$\mathbf{N} = \langle 0, -1, 1 \rangle$$

so the x-axis lies in the plane because $\mathbf{N} \cdot \hat{\mathbf{i}} = 0$. Another vector orthogonal to both and also in the plane is $\langle 0, 1, 1 \rangle$.

The normal vector to the cone depends on where you are, but if you are at $x = 0, y = 1$ then it would be

$$\mathbf{N} = \langle 0, 1, -1 \rangle$$

In the yz-plane it points down at a 45 degree angle. The two normal vectors are the same (within sign), so if there is a solution it should be a parabola.

We can see that there should be a solution, because the plane intersects the y-axis at $y = -1$ $(z = 0)$ and the z-axis at $z = 1$. If you draw a sketch, one point is outside the cone and the other inside it, so the plane must cut the cone.

Every point on the intersection of the plane and the cone satisfies both equations:

$$\sqrt{x^2 + y^2} = 1 + y$$
$$x^2 + y^2 = 1 + 2y + y^2$$
$$x^2/2 = y + 1/2$$

This is a parabola but it is *not* the parabola formed by the intersection. It is the projection of that intersection onto the xy-plane.

Such projections are linear transformations, which simply amount to rescaling of the variables x to x' and y' to y' (in this case only the latter) without changing the nature of the curve—a parabola is still a parabola.

However, an ellipse may become a circle, and vice-versa.

In this case, the normal vector forms an angle of 45 degrees with the vertical z-axis since

$$\cos\theta = \frac{\langle 0, -1, 1\rangle \cdot \langle 0, 0, 1\rangle}{\sqrt{1+1}\,\sqrt{1}} = \frac{1}{\sqrt{2}}$$

This is the factor by which the actual curve is stretched compared to the projection in the plane.

For the general problem, we would need to rotate all the points on the curve using angles obtained from the normal vector. We want to tilt $\mathbf{N}$ so that it points straight up and has its magnitude unchanged.

`https://en.wikipedia.org/wiki/Rotation_matrix`

In 3D we we could rotate points (or the coordinate system) first with respect to the xy-plane (ignoring z) using the standard transformation with this matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This is the same rotation that we had before — it leaves the z-coordinate unchanged. The relevant t is calculated from the x and y components of $\mathbf{N}$ using $t = \tan^{-1} y/x$. Then use the given matrix, or perhaps switch the signs on the sine.

After $\mathbf{N}$ has been rotated so that it lies along either the x- or y-axis, then rotate in the xz-plane or yz-plane until N is vertical.

Having done this, I believe there should be no mixed terms containing xy, so we won't need to rotate to remove those, as done before.

Chapter 88

Hippocrates

Hippocrates of Chios (470-410 BC) was a major figure in Greek geometry. (Not to be confused with the physician of the same name, from Kos). Hippocrates focused on quadrature, the process of constructing (with straight-edge and compass) a square with area equal to a given geometric figure, particularly curved figures, called lunes. Here is one of the first of these—construction of the square equivalent to a given rectangle.

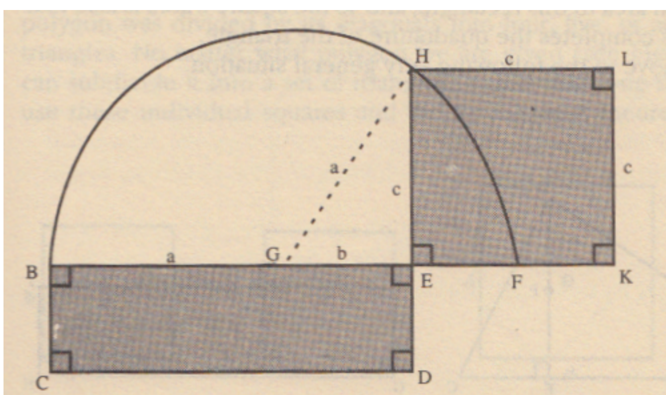


FIGURE 1.7

The construction says to: (i) extend BE horizontally, (ii) mark off the same distance as DE to construct EF, (iii) find the midpoint G of

BF, (iv) draw the half-circle of radius BG, (v) extend DE up to meet the circle at H, construct the square of side the same length as EH.

As suggested by the dotted line in the figure, the proof invokes the Pythagorean theorem. The long side of the rectangle is $a+b$, while its short side is $a-b$, so the area is

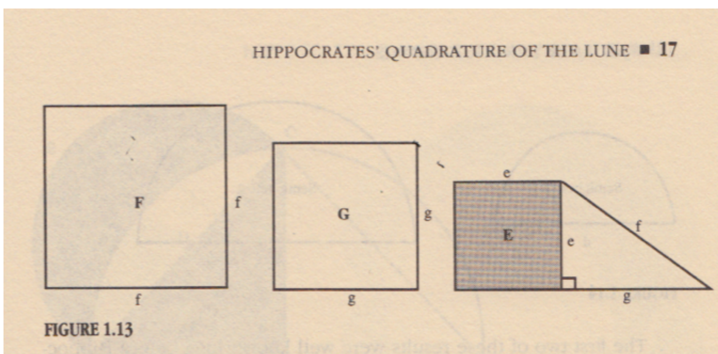
$$A = (a+b)(a-b) = a^2 - b^2$$

but Pythagoras says that is equal to c^2. $\square$

The side of the square, c is the geometric mean of the sides of the rectangle.

$$c = \sqrt{(a+b)(a-b)}$$

Hippocrates "squared" rectangles, triangles and polygons. A lot of his constructions depended on Pythagoras as suggested by this figure:



where two squares resulting from manipulation of part of a polygon need to be subtracted to obtain the final result.

Hippocrates moved on to curves, trying to find squares with area equal to that under or between two curves. That turns out to be a class of problems where few have solutions (in fact, only five, according to Dunham). Famously, it is impossible to square the circle. However, here is one that is possible, it is an example of (the) quadrature of the lune.

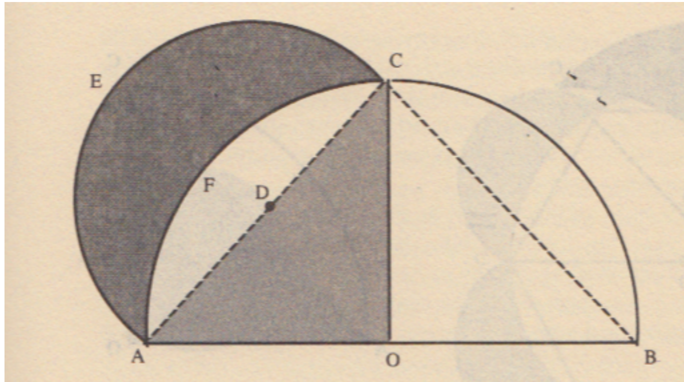


FIGURE 1.15

We will prove that the two shaded regions are equal in area.

Consider the smaller semicircle with base ADC, which is also the hypotenuse of the right triangle. Let radius AD be equal to r. Let the large semicircle have radius AO equal to R. Pythagoras tells us that

$$R^2 + R^2 = (2r)^2$$

$$R^2 = 2r^2$$

Let the area of the triangle be T. (Its value is $R^2/2$ but that's not needed).

The segment of the larger semicircle (shaded light) is the area of the quadrant minus the area of the triangle

$$\pi\frac{R^2}{4} - T$$

The area of the dark shaded lune is the area of the small semicircle minus the light-shaded area

$$\pi\frac{r^2}{2} - \ [\ \pi\frac{R^2}{4} - T\ ]$$

$$= \pi\frac{r^2}{2} - \pi\frac{2r^2}{4} + T$$

$$= \pi\frac{r^2}{2} - \pi\frac{r^2}{2} + T$$
$$= T$$

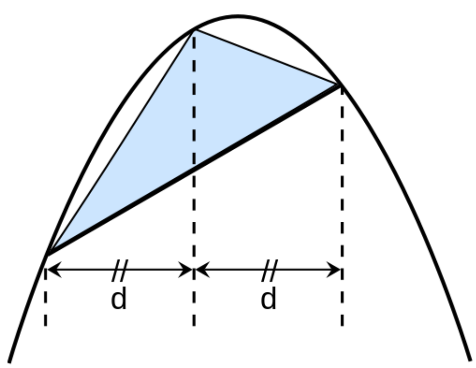
Which is just the area of the triangle shaded gray.

Chapter 89

Archimedes and quadrature

Let's talk about Archimedes, and parabolas.

Here is a figure from wikipedia, showing a parabola and a chord of the parabola, which might be drawn between any two points. A triangle is constructed from the chord in the following way: the point dividing the horizontal distance in half is found and that is used for the x-value of the third point.



The Greek genius Archimedes showed that the total area underneath the curve, between the two outside vertices of the triangle, is 4/3 times the area of the triangle shown in blue. The method he used is called the "quadrature of the parabola" and it is (from our modern perspective) a relatively simple though still revolutionary idea.

One simple and very interesting consequence is that the slope of the tangent to the parabola at this midway point is equal to the slope of the chord.

The general equation of a parabola is

$$y = ax^2 + bx + c$$

But for any given parabola, we can translate it to the origin and the parabola at the origin with the same shape is

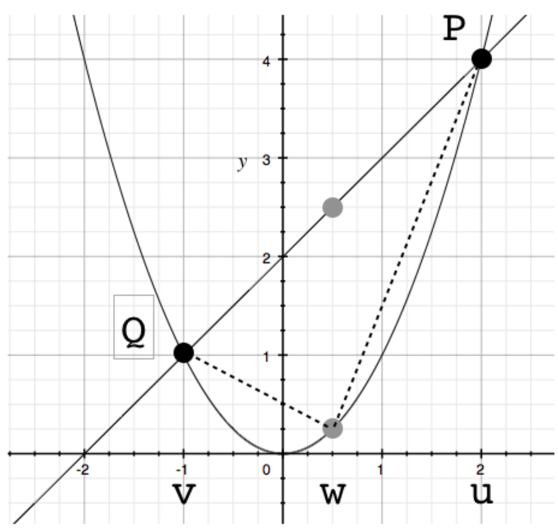
$$y = ax^2$$

This can be demonstrated by completing the square.

If we pick two points on the parabola at $x = u$ and $x = v$, then the corresponding coordinates are

$$P = (u,\ au^2)$$

$$Q = (v,\ av^2)$$



P is the right-hand point in the figure. Let us say that $au^2 > av^2$ and and the slope m of the chord that connects them is

$$m = \frac{au^2 - av^2}{u - v} = \frac{a(u^2 - v^2)}{u - v} = \frac{a(u - v)(u + v)}{u - v}$$

so

$$m = a(u + v)$$

We can see that this formula gives the correct answer for $u = -v$, since the slope at the vertex is 0. Now label the midpoint $x = w$

$$w = \frac{1}{2}(u + v)$$

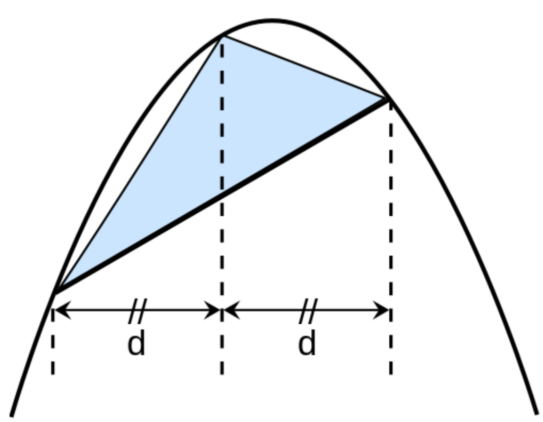
And the slope at w (from calculus) is

$$f'(w) = 2aw == 2a\ \frac{1}{2}(u + v) = a(u + v)$$

So the proposition is correct.

Quadrature

Another interesting thing about this figure is that the area of the triangle can be found from the length of the vertical coming down from the top.



If we simply turn the graph sideways in our mind, then the two small triangles share the part of this line within the blue region, which is their "base", b. And they both have "height" d, since w was chosen as half way between u and v, so their areas are equal, and the total area of the two together is

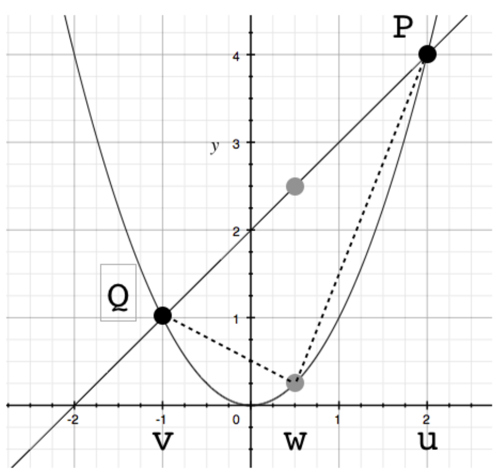
$$A = bd$$

We want to find an expression for the area only in terms of u and v (no b or d). Let's look at the second version of the figure again below.

To repeat, what we found above is that the slope at the point on the parabola corresponding to $x = w$ is equal to the slope of the line that connects v and u, and more important to us now, that the area of the combined triangle (vertices u, v, w) is

$$A = (u - w)\ b = \frac{1}{2}(u - v)\ b$$

where b is the distance parallel to the y-axis between the two points marked in gray.



The length of the "base" b is the average of the y-values for $x = u$ and

$x = v$, minus aw^2.

$$b = \frac{1}{2}(au^2 + av^2) - aw^2$$

and from before

$$w = \frac{1}{2}(u + v)$$

so we have

$$b = \frac{1}{2}(au^2 + av^2) - a\ [\ \frac{1}{2}(u + v)\ ]^2$$

Factor out $a/4$

$$= \frac{1}{4}a\ [\ 2u^2 + 2v^2 - (u + v)^2\ ]$$

$$= \frac{1}{4}a\ [\ 2u^2 + 2v^2 - u^2 - 2uv - v^2\ ]$$

$$= \frac{1}{4}a\ [\ u^2 - 2uv + v^2\ ]$$

$$b = \frac{1}{4}a\ (u - v)^2$$

The area is

$$\boxed{A = bd = \frac{1}{8}a\ (u - v)^3} \tag{89.1}$$

check

We'll check three cases to see if this makes sense. First if

$$u = v$$

then the area is zero and $w = u = v$, so that's good. Second, if

$$u = -v$$

then

$$A = \frac{1}{8}a\ (u - v)^3 = \frac{1}{8}a\ (2u)^3 = au^3$$

We compare this result with a direct computation by geometry. In the figure we have two symmetric triangles with individual area

$$\frac{1}{2}u \ au^2$$

The total area is twice that, so it checks. Finally, suppose we have $v = 0$

$$A = \frac{1}{8}a \ (u - v)^3$$

This one is harder to see, but we have that

$$d = \frac{1}{2}(u - v) = \frac{1}{2}u$$

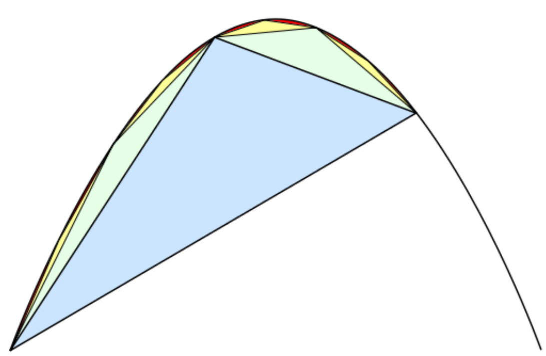
b is the distance between the average y-value which is $(1/2)au^2$ and $aw^2 = a(u/2)^2$

$$b = a \ (\frac{1}{2}u)^2 - \frac{1}{2} \ [\ au^2 - 0 \ ] \ = \ \frac{1}{4}a \ u^2$$

$$A = bd = \frac{1}{8}au^3$$

so they all check.

Quadrature of the parabola



The reason for the whole preceding argument is this. The area formula is

$$A = bd = \frac{1}{8}a\ (u - v)^3 = k(u - v)^3, \quad k = const$$

It is solely a function of $u - v$. Suppose we draw two new triangles (in light green, above). For each of these triangles the distance between the new vertices is one-half what we had before. So everything that we have for the big blue triangle is also true for these two new ones, just adjusted by a factor of $u' - v' = (1/2)(u - v)$.

What this means is that the area of each light green triangle is in the ratio to the blue one of $(1/2)^3 = 1/8$. But there are two of these new triangles, so the new area we added is in the ratio $1/4$.

Suppose we do it again, constructing the yellow triangles. The new area of each is in the ratio $(1/4)^3 = 1/64$ but there are now 4 of these yellow triangles so the total area is in the ratio $1/16 = (1/4)^2$

If we call the area of the original triangle T, that of the blue plus the light green is

$$A = T + \frac{1}{4}T$$

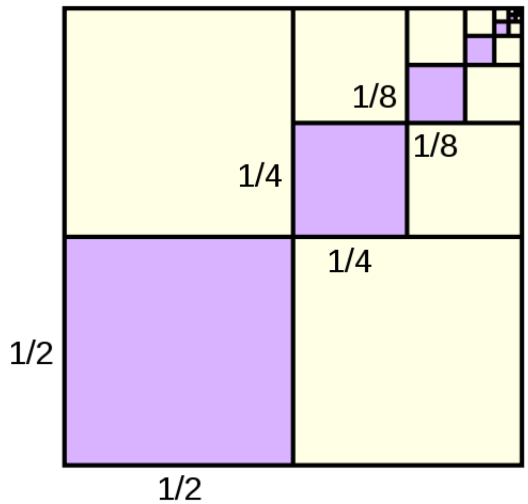
and with the addition of the yellow it is

$$A = T + \frac{1}{4}T + \frac{1}{16}T$$

so, as an infinite series it is

$$A = T(1 + \frac{1}{4} + \frac{1}{16} + \cdots)$$

Here is Archimedes' proof that the sum of this series (not counting the first term) is $1/3$.



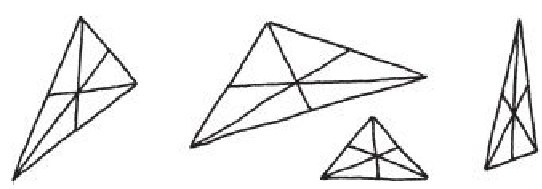
So the total is 4/3, and the complete area under the parabola is 4/3 the area of the triangle drawn as we described!

This called the "method of exhaustion", and not just because it's a lot of work.

Chapter 90

Ceva's theorem

Ceva's Theorem says that if we start with a triangle and draw line segments connecting each vertex with the midpoint of the opposite side, then the three line segments cross at a single, unique point. The question is: how do we know this? We can obviously draw a line to the midpoint of the opposing side from two vertices, and these will cross at some point.

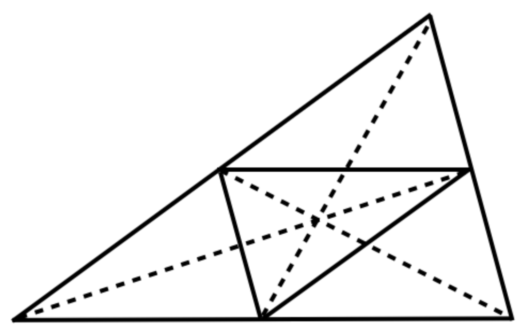


Then, we can either draw to the midpoint of the opposing side from the third vertex, and ask whether that line goes through the central point, or we can draw a line to the point and extend it to the opposing side, and then ask whether it bisects the side. That's the question. We will show that the answer is yes.

Furthermore, it is possible to show that for any of these line segments, this point (called the *centroid*) lies one-third of the length from the side, and two-thirds of the length from the vertex. We will use calculus to compute the centroid later in the book (and get the same answer).

In other write-ups I've shown one proof of this using similar triangles. The first approach shown here is one outlined in Lockhart's book *Measurement*. And the second one uses vectors.

Lockhart's proof



The idea is to connect the midpoints of the sides. If we do that, the construction results in four smaller triangles. It is easy to show that these triangles are congruent, and are similar to the large one we started with. (By similar triangles, a short side opposite a midpoint/vertex is parallel to the side containing the midpoint. I'm sure you can finish the proof).

Because of the congruent triangles, we also have three congruent parallelograms, and these have rotational symmetry around the centroid. Therefore the centroid is a single point.

If you don't like that argument, notice that the dotted lines play the same role for each of the small triangles, extending from a vertex to the midpoint of the opposite side.

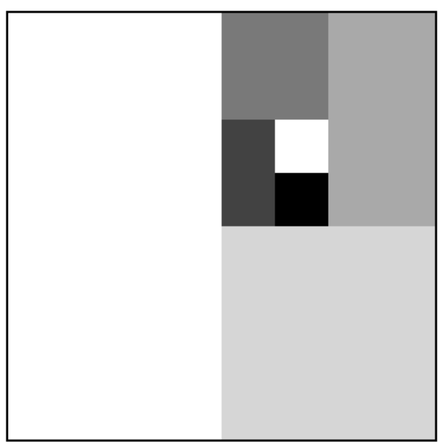
What this means is that *if* the centroid is a single point, then centroids of the larger triangle and the small central triangle are the same point. But we can just continue in the same way, inscribing a new, even smaller, triangle, using the midpoints of the small central triangle, and this process can be extended *ad infinitum*. Hence, in the limit, we will reach a single point.

We can locate the centroid by imagining that we find successive midpoints of a length from opposite ends left and right. The first point is at $1/2$ of the length (from the left), the second comes back from 1 by $1/4$ so is at 0.75 (at the right), the third is at $0.5 + 1/8$ (from the left), so every second round we get closer to the centroid by advancing from the left by

$$S = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots$$

Now, we can either assume this sum is finite (for now) or recognize that it is certainly smaller than

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$$



So if

$$S = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots$$

then

$$2S = 1 + \frac{1}{4} + \frac{1}{16} + \ldots$$

and

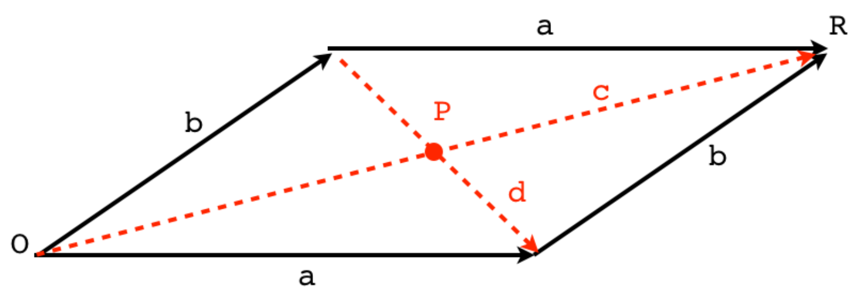
$$3S = 1S + 2S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

That is, $3S = 1 + 1$, so $S = 2/3$.

using vectors

Ceva's Theorem says that if we start with a triangle and draw the line segments connecting each vertex with the midpoint of the opposite side, the three line segments cross at a single, unique point. Furthermore, it is possible to show that for any of these line segments, the *centroid* lies one-third of the length from the side, and two-thirds of the length from the vertex.

Using vectors makes everything particularly simple. As a warmup, let's start by looking at the midpoint of the diagonals for a parallelogram including the triangle of interest. We will prove that the two diagonals cross at their mid-points (at P).



by construction:

$$\mathbf{c} = \mathbf{a} + \mathbf{b}$$

$$\mathbf{b} + \mathbf{d} = \mathbf{a} \ \Rightarrow \ \mathbf{d} = \mathbf{a} - \mathbf{b}$$

Let's define P as the point we reach by going halfway along $\mathbf{c}$

$$\mathbf{c}/2 = (\mathbf{a} + \mathbf{b})/2$$

What we need to show is that if we do

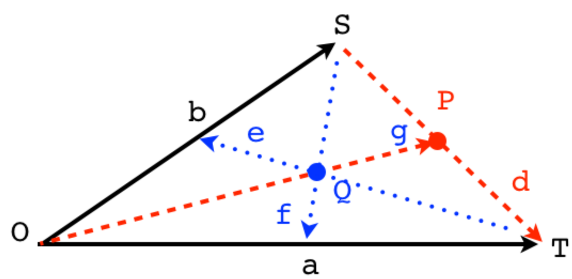
$$\mathbf{b} + \mathbf{d}/2$$

we arrive at P. Since $\mathbf{d} = \mathbf{a} - \mathbf{b}$

$$\mathbf{b} + \mathbf{d}/2 = \mathbf{b} + (\mathbf{a} - \mathbf{b})/2 = (\mathbf{a} + \mathbf{b})/2$$

$\square$

Vectors make that pretty easy. We will use this result below. Now, here is the triangle.



Construct a line to the midpoint of the opposing side

$$\mathbf{b} + \mathbf{f} = \mathbf{a}/2 \;\Rightarrow\; \mathbf{f} = \mathbf{a}/2 - \mathbf{b}$$

$$\mathbf{a} + \mathbf{e} = \mathbf{b}/2 \;\Rightarrow\; \mathbf{e} = \mathbf{b}/2 - \mathbf{a}$$

Refer to the first diagram for $\mathbf{c} = (\mathbf{a} + \mathbf{b})$. The halfway point is:

$$\mathbf{a}/2 + \mathbf{b}/2 = \mathbf{g}$$

By the property proved in the first part, we know that this bisects the side $\mathbf{d}$.

It makes it a little easier when we know that Q is two-thirds of the way along the line segment. We have three paths to move to Q. The first two are constructed using the property that the vector bisects the opposing side.

from S:

$$\mathbf{b} + \frac{2}{3}\mathbf{f} = \mathbf{b} + \frac{2}{3}(\mathbf{a}/2 - \mathbf{b}) = (\mathbf{a} + \mathbf{b})/3$$

from T:

$$\mathbf{a} + \frac{2}{3}\ \mathbf{e} = \mathbf{a} + \frac{2}{3}\ (\mathbf{b}/2 - \mathbf{a}) = (\mathbf{a} + \mathbf{b})/3$$

from O:

$$\frac{2}{3}\mathbf{g} = \frac{2}{3}\ \frac{1}{2}\mathbf{c} = (\mathbf{a} + \mathbf{b})/3$$

□

By moving two-thirds of the way along **g**, one-third of the way along **c**, we arrive at the same point defined to be two-thirds of the way along the bisectors of opposing sides drawn from S and T.

factor

How would we find the factor of 2/3 if we didn't already know? Here is one way. Call that unknown factor r. By symmetry it is the same for all three lines.

$$r\ (\mathbf{a} + \mathbf{b})/2 + (1 - r)\ \mathbf{e} = \mathbf{b}/2$$

$$r\ (\mathbf{a} + \mathbf{b})/2 + (1 - r)\ (\mathbf{f} = \mathbf{a}/2$$

Substitute for **e** and **f**:

$$r\ (\mathbf{a} + \mathbf{b})/2 + (1 - r)\ (\mathbf{b}/2 - \mathbf{a}) = \mathbf{b}/2$$

$$r\ (\mathbf{a} + \mathbf{b})/2 + (1 - r)\ (\mathbf{a}/2 - \mathbf{b}) = \mathbf{a}/2$$

add

$$r\ (\mathbf{a}+\mathbf{b})+(1-r)\ (-\mathbf{b}/2-\mathbf{a}/2)=\mathbf{b}/2+\mathbf{b}/2$$

$$\frac{3}{2}\ r\ (\mathbf{a}+\mathbf{b})=\mathbf{a}+\mathbf{b}$$

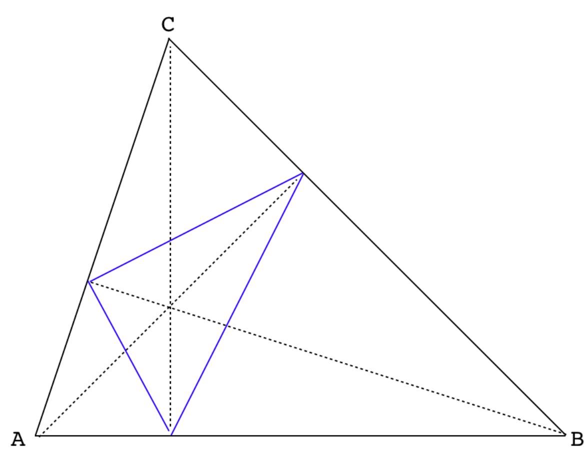
$$r=2/3$$

Chapter 91

Orthocenter

An altitude of a triangle is a line extended from a vertex so as to form a right angle with the opposing side. The orthocenter is the point where the three altitudes of a triangle meet.

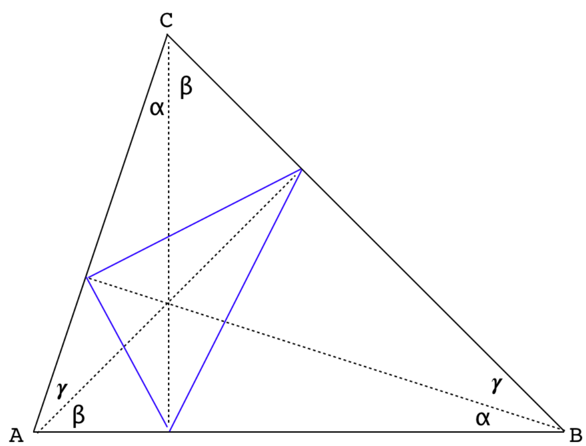
Assume for now that the three altitudes *do* meet at a single point, we will come back to this question later.



The altitudes are drawn as dotted lines in the figure above. The points where these lines meet their opposite sides have also been connected, forming another triangle outlined in blue.

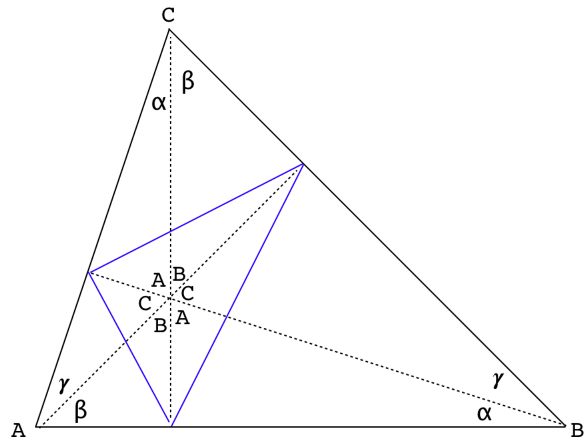
The same construction for the centroid, formed from lines that bisect the opposing sides, gives four small triangles which are all congruent. In the case of the orthocenter, we will show that the three outer triangles are similar (though obviously not congruent), while the one in the center is different.

The first observation is that the angles A, B, and C are divided by the altitudes into two parts with the measures repeated as shown below.



Proof: there are two right triangles formed that include A as a base angle. The corresponding complementary angles must be equal, and these are labeled α. A similar argument gives β and γ.

Switching our attention to the central angles, we can show that these have measures equal to the vertices.

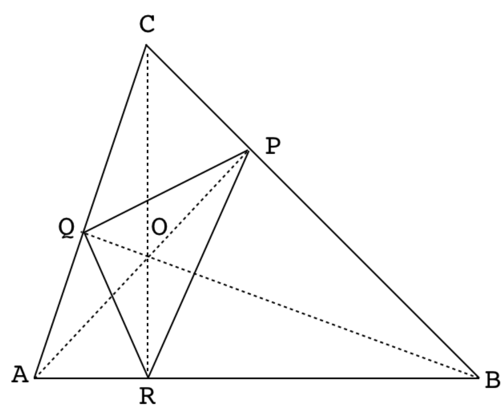


The argument builds on the one above, each central angle is part of a right triangle with one of α, β, or γ as the complementary angle.

angle bisectors

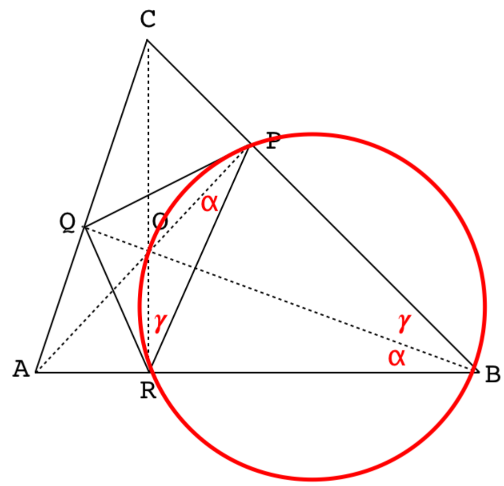
The next fact we need is that the altitudes are angle bisectors for the inscribed triangle.

Here is a neat and quick proof from Courant and Robbins.



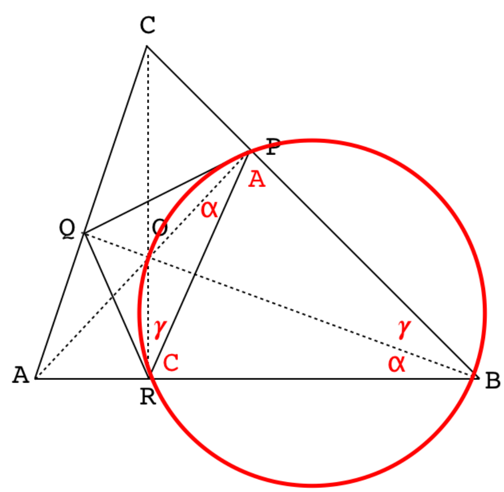
Label the vertices of the altitudes as P, Q, R and the orthocenter as O. Since $\angle OPB$ is a right angle and so is $\angle ORB$, the quadrilateral

$OPBR$ containing both can be inscribed into a circle with OB as the diameter.

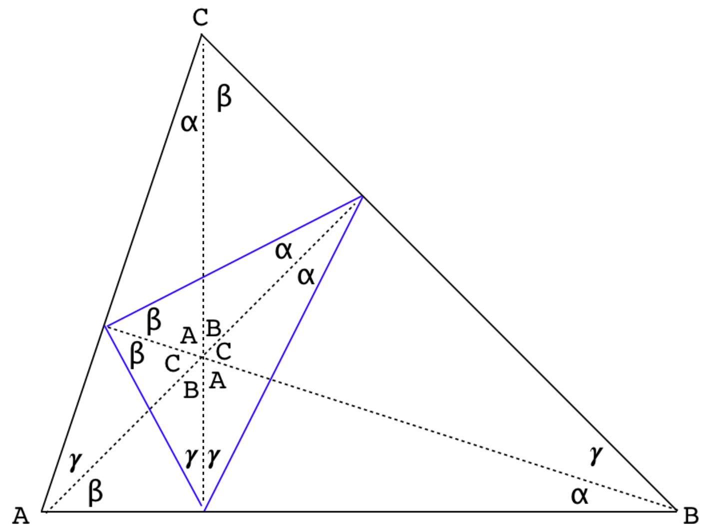


Now, we use the theorem that if two angles on the circumference of a circle sweep out the same arc, they are equal to each other This allows labeling of α and γ as shown.

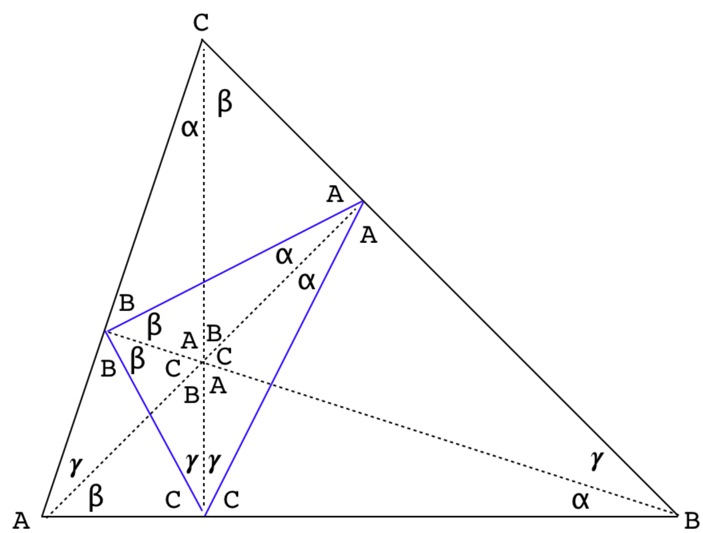
Then, since A is the complement of α, and C is the complement of γ, we obtain



We could draw two more circles, but instead just invoke symmetry:



So finally, we have



Thus, we have established that the altitudes are angle bisectors for the included triangle, and that the three small outer triangles are congruent, by AAA.

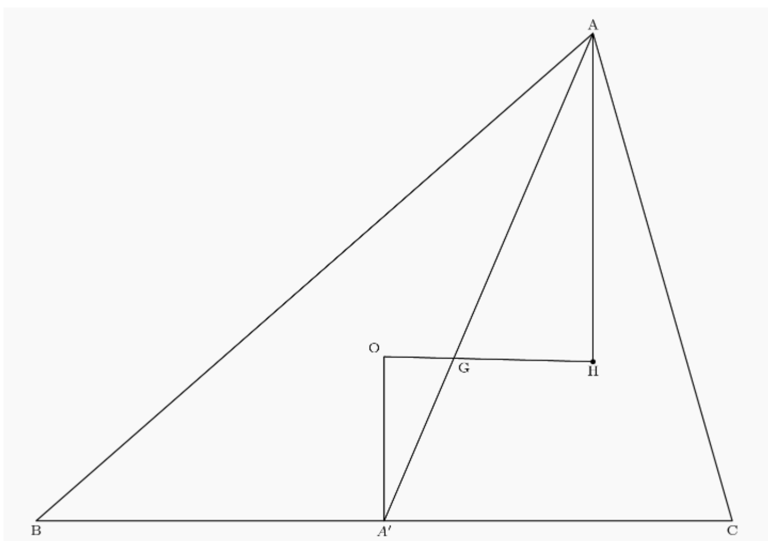
Orthocenter exists

For the above derivation, we assumed that the three altitudes do indeed meet at a single point. This is a direct consequence of Ceva's Theorem, which we've seen before.

Below we give an alternative proof, due to Euler, which is stunning, following

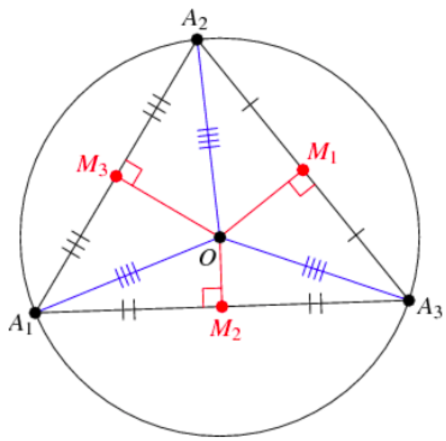
https://artofproblemsolving.com/wiki/index.php/Orthocenter

Borrowing their figure:



The orientation is reversed from what we had above. First, the point O is the circumcenter of the triangle: the center of the circle which contains all three vertices of the triangle.

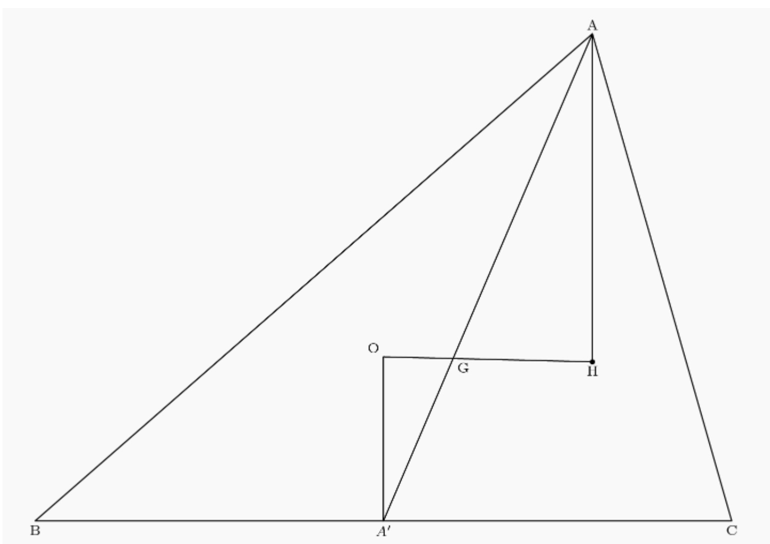
Clearly, this circle has a center. The classic construction is to bisect each side (here BC is bisected at A'), and erect a perpendicular. The point where the three perpendiculars cross is the circumcenter, which is the center of the circle.



So, assume we have done this and that point is O.

The next point, G, is the centroid. One way to find this point is to draw all three lines connecting vertices with the midpoints of the opposite side (AA'). However, if you recall, the distance from the vertex A to G is twice the distance from the midpoint A' to G. Hence we draw point G using arithmetic.

Now, extend OG by twice its length, to H. ($2OG = GH$).



Because AG is twice $A'G$ and GH is twice OG and the two triangles share both angle $\angle OGA'$ (equal to $\angle AGH$), they are similar triangles.

Since $\angle A'OG$ is a right angle, therefore so is $\angle AHG$. This means that AH is perpendicular to BC. Thus, AH is a part of the altitude from A to BC (the whole altitude is not shown).

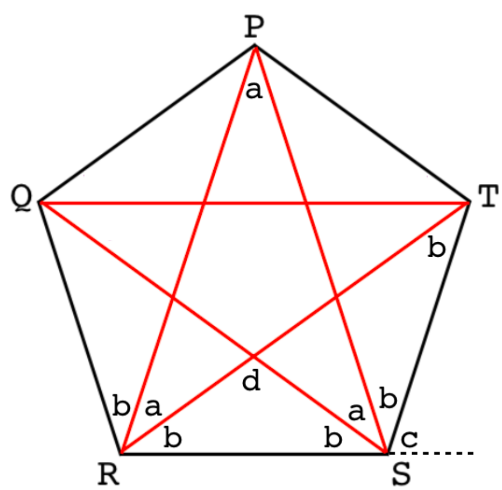
The same construction could be done for the other two vertices, each time ending at H. This shows that H is unique, and that H is on all three altitudes.

This proof also demonstrates that the orthocenter, centroid and circumcenter lie on a single line, and that the distance from centroid to orthocenter is twice that from centroid to circumcenter.

Chapter 92

Pentagon

In this short write-up we explore some properties of a regular pentagon. First, draw all of the internal chords of the figure and label some angles. It will turn out that all the angles have one of only three different measures in the figure, but for now we label them as a-d.



One way to start thinkgin about the angles is to imagine that we walk along RS and then make a left-turn at S, to face T. The angle through which we turn is c. In going around the whole perimeter to return to

R (and face horizontally again) we turn 5 times. Hence $5c = 2\pi$ or

$$c = 2 \cdot \frac{\pi}{5}$$

It will turn out that all the angles are multiples of $\pi/5$.

Next, observe that the angles labeled a are equal, by the five-fold rotational symmetry of the figure. There are 5 such angles in total.

The measure of the whole angle at each vertex of the pentagon is $b + a + b$. The equality of b on the left and right sides of a follows, again, from rotational symmetry.

The vertex angle is $b + a + b$, that angle plus c is

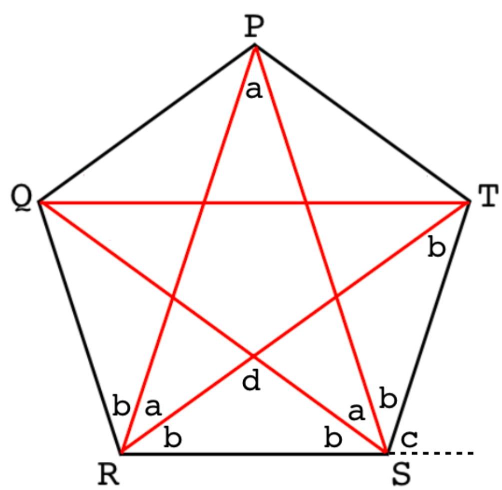
$$b + a + b + c = \pi$$

and since $c = 2 \cdot \pi/5$, we compute that $b + a + b = 3 \cdot \pi/5$.

Now consider $\triangle RST$. The sum of the angles in this triangle is

$$b + b + a + b + b = \pi$$

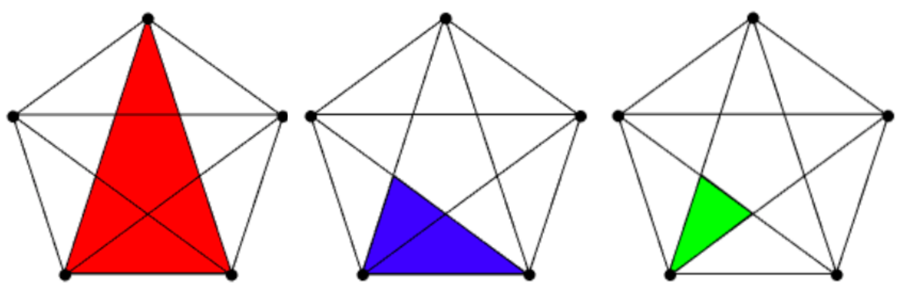
Using the previous computation for $b + a + b$, we easily find that $2b = 2 \cdot \pi/5$ and thus $b = \pi/5$. Reusing the result for $b + a + b = 3 \cdot \pi/5$, we find that $a = b = \pi/5$.



Comparing $\triangle RST$ to the small triangle with R and S as vertices (or simply computing with the value $b = \pi/5$), we see that d is equal to $3 \cdot \pi/5$. Therefore the inner pentagon is also a regular pentagon, which we could have deduced from the rotational symmetry alone.

Now observe the angle that QT makes with ST. This angle is equal to c (that is, $2 \cdot \pi/5$), and therefore these are alternate interior angles of two parallel lines. Thus QT is parallel to RS. (Or simply add the included angles $a + b + b + a + b = \pi$).

One can draw two types of isosceles triangles using the chords and sides of the pentagon. One is short and fat, the other, tall and skinny. The first class have base angles equal to $2 \cdot \pi/5$ and the second, base angles equal to $\pi/5$. Here are three examples of tall and skinny:



If we take the side length of the pentagon to be 1, then the long side length of the red triangle is 1 plus some other value, call that x. x is also the length of the short side or base for the blue triangle. We use the fact that red and blue are similar and form the ratio ϕ of the long side to the base (red on the left, blue on the right):

$$\phi = \frac{1+x}{1} = \frac{1}{x}$$

Rearrange:

$$x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1+4}}{2}$$

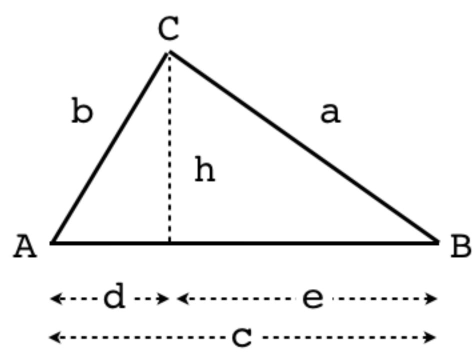
Of course ϕ is the golden ratio where we have taken the positive branch of the square root:

$$\phi = 1 + x = \frac{1 + \sqrt{5}}{2}$$

Chapter 93

Heron's formula

Heron's Formula can be used to compute the area of a triangle from the lengths of its sides. It is a simple formula that does not explicitly include the altitude h or the parts of side c.



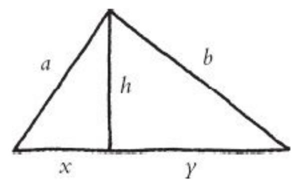
If s is the semi-perimeter

$$s = \frac{1}{2}(a + b + c)$$

then

$$A = \sqrt{s + (s - a) + (s - b) + (s - c)}$$

Lockhart's version



Here the triangle is labeled slightly differently than the one above. The bottom side c is split into x and y. We can write three equations:

$$x^2 + h^2 = a^2$$
$$y^2 + h^2 = b^2$$
$$x + y = c$$

Our ultimate objective is an equation that contains only a, b and c. Lockhart gives us a target for the first part of the derivation:

$$2xc = c^2 + a^2 - b^2$$

Let's just start manipulating equations to get there. Subtract the second from the first:

$$x^2 - y^2 = a^2 - b^2$$

Square the third

$$x^2 + 2xy + y^2 = c^2$$

Add the two new equations

$$2x^2 + 2xy = c^2 + a^2 - b^2$$

Substitute for y

$$2x^2 + 2x(c - x) = c^2 + a^2 - b^2$$

$$2xc = c^2 + a^2 - b^2$$

Finally a slight rearrangement:

$$x = \frac{c^2 + a^2 - b^2}{2c} = \frac{c}{2} + \frac{a^2 - b^2}{2c}$$

This says that to find the point where c is divided into x and y, we move from the center $c/2$ a distance of $(a^2 - b^2)/2c$.

The corresponding equation for y is

$$y = \frac{c}{2} - \frac{a^2 - b^2}{2c}$$

which is easily checked by adding together the final two equations, obtaining $x + y = c$.

For the area, we will need h somehow. It is easier to use h^2.

$$\begin{aligned} h^2 &= a^2 - x^2 \\ &= a^2 - \frac{(c^2 + a^2 - b^2)^2}{(2c)^2} \end{aligned}$$

The area squared is

$$\begin{aligned} A^2 &= \frac{1}{4}c^2h^2 \\ &= \frac{1}{4}c^2a^2 - \frac{1}{4}c^2\frac{(c^2 + a^2 - b^2)^2}{(2c)^2} \end{aligned}$$

Lockhart:

> the algebraic form of this measurement is aesthetically unacceptable. First of all, it is not symmetrical; second, it's hideous. I simply refuse to believe that something as natural as the area of a triangle should depend on the sides in such an absurd way. It must be possible to rewrite this ridiculous expression...

Here's a start:

$$16A^2 = (2ac)^2 - (c^2 + a^2 - b^2)^2$$

This is much better, notice that we have only a, b and c.

We will now go through two difference of squares manipulations. First

$$\begin{aligned}
16A^2 = &\ [\ 2ac + (c^2 + a^2 - b^2)\ ]\ [\ 2ac - (c^2 + a^2 - b^2)\ ] \\
= &\ [\ (a + c)^2 - b^2)\ ]\ [\ b^2 - (a - c)^2\ ] \\
= &\ (a + c + b)(a + c - b)(b + a - c)(b - a + c)
\end{aligned}$$

So

$$A = \sqrt{\frac{a + b + c}{2} \cdot \frac{a + c - b}{2} \cdot \frac{a + b - c}{2} \cdot \frac{-a + b + c}{2}}$$

At this point, we recognize the semi-perimeter $s = (a + b + c)/2$ and then we see that each of the other terms is s minus one of the sides

$$A = \sqrt{s \cdot (s - a) \cdot (s - b) \cdot (s - c)}$$

check

As a simple example, if we have a right triangle with sides 3,4,5, then the area is one-half of 3 times 4 = 6. The semi-perimeter is s

$$s = \frac{(3 + 4 + 5)}{2} = \frac{12}{2} = 6$$

We have

$$A = \sqrt{6(6 - 5)(6 - 4)(6 - 3)} = \sqrt{6(1)(2)(3)} = 6$$

Part XXIII

Gravitation

Chapter 94

Falling bodies

The simplest kind of motion is movement in a single dimension. To analyze it, the first thing we need to do is to pick an origin for our coordinate system, the place where x or $y = 0$. For a gravity problem the only dimension is up and down, and usually it's most convenient to pick the ground as the origin. It is conventional to use y as the variable.

Next, we need to decide when to start the clock, to say when is $t = 0$.

Such problems often have constant acceleration due to gravity, with $\mathbf{a} = -9.8$ m/sec^2. For simplicity we can take $\mathbf{a} = -10$ m/sec^2. The minus sign indicates that our coordinate system assigns positive numbers to positions further above us than ground-level, while the acceleration due to gravity points toward the earth. In 1D, we need not worry about vectors and direction, we just have to remember the convention about sign.

Velocity is the derivative of position with respect to time.

$$v = \frac{dy}{dt}$$

and acceleration is the second derivative

$$a = \frac{d^2y}{dt^2} = \frac{dv}{dt}$$

The two pieces of information with which we usually start the analysis are the initial position y_0 and the initial velocity v_0. We seek an equation that will tell us the current position, $y(t)$, given these two values plus the acceleration. Rather than do a formal integration, we make a a guess based on the relationship to the second derivative above:

$$y(t) \approx at^2 + C$$

We need to adjust the guess to get rid of the 2 that will come down when we take the first derivative.

$$y(t) = \frac{1}{2}at^2 + C$$

We also realize that the constant C can include terms of two types. First, anything like t times something will go away when we take the second derivative, so we should write

$$y(t) = \frac{1}{2}at^2 + C_1t + C_2$$

(where C_1 and C_2 are now different constants of integration). If we take the first derivative we have

$$v(t) = \frac{dy}{dt} = at + C_1$$

and we recognize that C_1 is just the initial velocity

$$v(0) = v_0 = 0 + C_1$$

so we have

$$y(t) = \frac{1}{2}at^2 + v_0t + C_2$$

Finally, we realize that, for $t = 0$ we have

$$y(0) = C_2 = y_0$$

so, finally

$$y(t) = \frac{1}{2}at^2 + v_0 t + y_0$$

which should be very familiar. With this equation in hand, knowing a, v_0 and t_0, we have only two variables, y and t. Given t it is easy to solve for y. Similarly, we can solve for v given t using

$$v(t) = at + v_0$$

It can be a little awkward to find t corresponding to a given y, but the last equation provides a nice trick for this. Solve for t (simplify the notation by writing v for $v(t)$):

$$v - v_0 = at$$

$$t = \frac{v - v_0}{a}$$

Now, given v (for example $v = 0$ at the top of the curve) we can find t. We can also plug this into the other equation and get something simple and useful. Rather than write $y(t)$ just write y so

$$y = \frac{1}{2}at^2 + v_0 t + y_0$$

$$2(y - y_0) = at^2 + 2v_0 t$$

$$2(y - y_0) = a\ (\frac{v - v_0}{a})^2 + 2v_0\ (\frac{v - v_0}{a})$$

$$2a(y - y_0) = (v - v_0)^2 + 2v_0(v - v_0)$$

$$= v^2 - 2vv_0 + v_0^2 + 2v_0 v - 2v_0^2$$

$$= v^2 - v_0^2$$

So finally,

$$2a(y - y_0) = v^2 - v_0^2$$
$$v^2 = v_0^2 + 2a(y - y_0)$$

Rather than find the time first and plug into the standard equation to get the velocity, we can go directly between velocity and position or vice versa.

Here are our equations re-written for gravity problems:

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$$
$$v - v_0 = -gt$$
$$v^2 = v_0^2 - 2g(y - y_0)$$

general examples

One simple question is: what is the time to fall from a given height h.

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$$

The initial velocity is zero, so

$$y - y_0 = -\frac{1}{2}gt^2$$
$$= 0 - h = -h$$
$$t = \sqrt{\frac{2h}{g}}$$

And given this time, the terminal velocity is:

$$v = v_0 - gt = 0 - g\sqrt{\frac{2h}{g}} = -\sqrt{2gh}$$

The second simple situation is to fire a projectile up with initial velocity v_0. Then we ask, what is the maximum height. This is one way

$$v^2 = v_0^2 - 2g(y - y_0)$$

At the maximum height $v = 0$:

$$0 = v_0^2 - 2gh$$

$$h = \frac{v_0^2}{2g}$$

Another way is to first get the time:

$$v - v_0 = -gt$$

$$-v_0 = -gt$$

$$t = \frac{v_0}{g}$$

Now the height is

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$$

$$h = -\frac{1}{2}g(\frac{v_0}{g})^2 + v_0\frac{u}{g}$$

$$h = \frac{v_0^2}{2g}$$

The object returns to earth when the height is equal to zero

$$0 = -\frac{1}{2}gt^2 + v_0 t$$

$$u = \frac{1}{2}gt$$

$$t = \frac{2v_0}{g}$$

One-half the time is spent going up, and the other half coming down. It's worth pointing out that the change in potential energy at height h is equal to

$$U = mgh = mg\frac{v_0^2}{2g} = \frac{1}{2}mv_0^2$$

which is equal to the kinetic energy at launch.

clever derivation

Shankar offers this derivation in his first Physics lecture. Write

$$\frac{dv}{dt} = a$$

Multiply both sides by v

$$v\frac{dv}{dt} = av$$

The first key step is to recognize that the left-hand side is equal (by the chain rule) to

$$v\frac{dv}{dt} = \frac{d}{dt}(\frac{v^2}{2})$$

So rewrite what we had including this and on the right-hand side use the definition $v = dx/dt$

$$\frac{d}{dt}(\frac{v^2}{2}) = a\ \frac{dx}{dt}$$

The second key is to recognize that we can get rid of dt and just think about this as an equality between differentials

$$d(\frac{v^2}{2}) = a\ dx$$

Now integrate

$$\int d(\frac{v^2}{2}) = \int a\ dx$$

for the constants of integration use the initial values

$$\frac{v^2}{2} - \frac{v_0^2}{2} = a(x - x_0)$$

That should look familiar:

$$v^2 = v_0^2 + 2a(x - x_0)$$

numerical examples

Suppose you are on the roof of a building of height $y_0 = 15$ m and throw a rock upward with velocity $v_0 = 10$ m/s. We find the maximum height as the position where $v = 0$. From the second equation

$$t = \frac{v - v_0}{a} = \frac{0 - 10}{-10} = 1 \text{ s}$$

$$y = \frac{1}{2}at^2 + v_0 t + y_0 = (-5)t^2 + (10)t + 15 = 20 \text{ m}$$

How fast is it going when it hits the ground? From the third equation

$$v^2 = v_0^2 + 2a(y - y_0) = (10)^2 + 2(-10)(-15) = 400$$

$$v = \sqrt{v^2} = \sqrt{400} = \pm\ 20 \text{ m/s}$$

There are two solutions, the one with negative value corresponds to the rock hitting the ground at the end of the throw. The positive velocity is the same rock being thrown from the ground upward with velocity 20 m/s. This will also hit the ground with velocity −20 m/s.

To find the time when the rock hits the ground, from the first equation

$$y(t) = \frac{1}{2}at^2 + v_0 t + y_0 = 0 = (-5)t^2 + 10t + 15$$

$$t^2 - 2t - 3 = 0 = (t - 3)(t + 1) = 0$$

So either $t = 3$ or $t = -1$ seconds. The first solution is the one we thought we wanted, the second corresponds to the positive velocity situation we had above. For a throw from the ground up, the trajectory is symmetric, two seconds going up and two seconds coming back down again. And reusing the second equation, for the part coming down $v - v_0 = at$, so $v = at = -20$ m/s.

example

A ball is thrown so that it goes upward with a velocity of 16 m/s. If $g = 32\ ft/s^2$, what is the position of the ball at time t?

We have the distance equation

$$h = h_0 + v_0 t - \frac{1}{2} g t^2$$

We set $h_0 = 0$, $v_0 = 16$ and $g = 32$

$$h = 16t - 16t^2$$

We wish to know when $h = 0$

$$0 = 16t(1 - t)$$

$t = 0$ is a solution, which is obviously correct. The ball starts with $h = 0$ at $t = 0$. The other solution is $t = 1$. The ball returns to $h = 0$ at $t = 1$.

Notice also that

$$v = v_0 - gt = 16 - 32t$$

so when $v = 0$

$$0 = v_0 - gt = 16 - 32t$$

$$16 = 32t$$

and $t = 1/2$. The trajectory of this ball is a parabola. It reaches its vertex when the upward velocity is zero ($t = 1/2s$). It returns to the earth in a time equal to that which was needed for its ascent.

example

Find t if a ball is dropped from a height = 392 feet, for $h_0 = 392$ and $v_0 = 0$ The distance equation is

$$h = h_0 + v_0 t - \frac{1}{2} g t^2$$

We have $h_0 = 392$ and $v_0 = 0$

$$0 = 392 - \frac{1}{2} g t^2$$

$$784 = 16t^2$$
$$\frac{784}{16} = 49 = t^2$$
$$t = 7$$

maximum range

Here is a problem in 2D. A ball is thrown making an angle θ with respect to the horizontal. What value of θ will give the maximum horizontal distance?

$$x(t) = v_x t$$
$$y(t) = v_y t - \frac{1}{2} g t^2$$

$$v_x = v \cos\theta$$

$$v_y = v \sin\theta$$

We find the time t when $y = 0$ and the ball has come back down to earth. We can remove one factor of t from each term on the right (we lose a possible solution but it's the one we already know, $y = 0$ at $t = 0$).

$$y(t) = 0 = v_y t - \frac{1}{2} g t^2$$

$$0 = v_y - \frac{1}{2} g t$$

$$t = \frac{2}{g} v_y$$

Substitute for t in the equation for $x(t)$ above

$$x(t) = v_x t = v_x \frac{2}{g} v_y$$

converting it to $x(\theta)$

$$x(\theta) = v \cos\theta \ (\frac{2}{g}) \ v \sin\theta$$

$$= \ \frac{2v^2}{g} \sin\theta \cos\theta$$

Remembering the sum of angles formula ($\sin 2s = 2 \sin s \cos s$):

$$= \ \frac{v^2}{g} \sin 2\theta$$

This is a maximum (for fixed v) when $\sin 2\theta$ is a maximum (equal to 1, so $\theta = \pi/4$. Alternatively

$$x(\theta) = \frac{2v^2}{g} \sin\theta \cos\theta$$

$$\frac{dx}{d\theta} = (\frac{2v^2}{g})\ [\ \cos^2\theta - \sin^2\theta\ ]$$

Set the first derivative equal to zero. Eliminate the constants in front:

$$0 = -\sin^2\theta + \cos^2\theta$$

$$\sin\theta = \cos\theta$$

$$\theta = \tan^{-1} 1 = \frac{\pi}{4} = 45^\circ$$

Chapter 95

Escape from the earth

In this chapter, we will calculate the energy required to move a body which is initially at some distance r_1 from the center of the earth (say, R, the earth's radius), to a position at some other distance r_2.

Eventually, we will consider problems related to the orbits of planets around the sun. But let's start with idealized circular orbits and consider escape velocity which does not require vector calculus to solve.

gravitational potential

First, suppose you stand on the earth's surface and throw a ball straight up into the air with a velocity v. It reaches some maximum height where its vertical velocity is zero. It has traded kinetic energy for potential energy. Be sure to move out of its way as it comes down.

Potential energy due to gravity increases with height above the earth.

Suppose we climb the steps up to the top of the leaning Tower of Pisa and then drop a marble over the edge. At the start the marble has zero velocity and at the end it has some velocity which we can calculate, neglecting air resistance.

We have the basic equation of motion with acceleration

$$y = \frac{1}{2}at^2 + v_0 t + y_0$$

and its time-derivative

$$v = \frac{dy}{dt} = at + v_0$$

We can obtain a formula that does not involve the time

$$v^2 - v_0^2 = 2a(y - y_0)$$

Do this by starting with equation 1 and rearranging:

$$2(y - y_0) = at^2 + 2v_0 t$$

then solve equation 2 for $t = (v - v_0)/a$ and substitute

$$\begin{aligned} 2(y - y_0) &= a\frac{(v - v_0)^2}{a^2} + 2v_0\frac{v - v_0}{a} \\ 2a(y - y_0) &= (v - v_0)^2 + 2v_0(v - v_0) \\ &= v^2 - v_0^2 \end{aligned}$$

For gravity

$$v^2 - v_0^2 = -2g(y - y_0)$$

We write the sign of the acceleration as negative, that is, $-g$, where it's understood that $g > 0$.

The usual choice is to have the coordinate system point up. If the ball is over my head, then $|v| < |v_0|$ (both going up and coming down), so the left-hand side is negative, which matches $y > y_0$ only if the sign on the acceleration is minus.

Later, when we calculate work, in some cases, the relevant force will be the one which we have applied to oppose gravity, and that force points up.

Call the distance $y - y_0 = h$. Then

$$v^2 - v_0^2 = -2gh$$

At the top, $v = 0$

$$v_0^2 = 2gh$$

This gives

$$\frac{1}{2}mv_0^2 = mgh$$

Which you probably recognize. At a height h above the ground there is a potential energy difference of mgh. After dropping through a height h, all this potential energy is converted to kinetic energy $mv^2/2$.

$$v^2 = 2gh$$

$$v = \sqrt{2gh}$$

escape velocity

Now, imagine an object starting from rest on the surface of the earth and then giving it enough velocity in an idealized trajectory (simply the vertical direction) so that it can move far enough away to be free from gravity altogether. In this problem the force decreases as the object moves away.

A simple approach is to use the principle of conservation of energy: we will impart enough kinetic energy so that the increase in potential energy after the motion is just balanced. The question is how to compute the potential energy.

Later on, in vector calculus we will use this expression for work

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

We must use an integral because the force is a function of r, and we use the dot product because the force and the changing position vector are not always aligned.

The work done over the course of the motion (force times distance) is equal to the energy added to the object. The vector equation says that only the component of the force in the same direction as the motion contributes.

Luckily, we don't need to use a vector equation for this problem. Just place the center of the earth at the origin, and treat all of the mass of the earth as being at that point. (We will develop Newton's proof of this later, see **here**).

Then, consider the force and motion as occurring only in one direction. While x or y could be used for the variable, the conventional choice is r. All of the (scalar) force and motion is in the r direction.

In the vector approach, we learn that the force $\mathbf{F}$ is the gradient of a scalar function (called the potential energy)

$$\mathbf{F} = -\nabla U$$

In 1D this just amounts to doing the scalar integral.

The usual derivation is that the change in potential energy in going from configuration a to b, is minus the work done in going from a to b where

$$W_{ab} = \int_a^b F(r)\ dr$$

and then

$$\Delta U = U_b - U_a = -W_{ab}$$

For the first example above, we have $F = -mg$ and so

$$W = \int_0^h -mg\ dy = -mgh$$

(Gravity does work on an object when it falls in the gravitational field, it does negative work on the object as it rises). Then

$$\Delta U = mgh$$

For the second case, where the change is large enough that the force is not constant, we write $F(r)$

$$F(r) = -\frac{GmM}{r^2}$$

$$\begin{aligned} W_{ab} &= \int_a^b -\frac{GmM}{r^2}\ dr \\ &= \frac{GmM}{r}\ \Big|_a^b \\ &= GmM\ [\frac{1}{r_b} - \frac{1}{r_a}\ ] \end{aligned}$$

then

$$\begin{aligned} \Delta U &= U_b - U_a = -W_{ab} \\ &= GmM\ [\frac{1}{r_a} - \frac{1}{r_b}\ ] \end{aligned}$$

We pick a convenient reference point, namely $b \to \infty$. The upper bound of ∞ makes this an *improper* integral. We say "oh we're not really going to infinity, just really far away, and then wonder, what would happen if we did go to infinity."

We also agree that this configuration is defined to have zero potential energy. Then

$$U_b - U_a = GmM \ [\frac{1}{r_a} - \frac{1}{r_b} \ ]$$

$$-U_a = \frac{GmM}{r_a}$$

$$U_a = -\frac{GmM}{r_a}$$

If we're thinking about the potential due to gravity, the force points toward the earth. The potential energy is larger the further away from earth you go, and is largest at infinity, where by definition it is equal to zero. Thus, all other potential energies are negative. To be on earth is, in that sense, to be trapped, since it requires an input of energy to lift you to some larger r and less negative U.

This is, in magnitude, equal to the work we would have to do pitting some force against gravity to lift a mass m to completely escape from the earth's gravity. The larger the starting radius the less work there is to do. Another way to state this is that the extra potential energy at some radius $r_1 > r_2$ is exactly equal to the work required to move between these points.

How much kinetic energy do we need to impart? Using the standard form for K, we obtain this expression, valid for the earth's surface:

$$\frac{1}{2}mv^2 = \frac{GM}{R}m$$

$$\frac{1}{2}v^2 = \frac{GM}{R}$$

$$v = \sqrt{\frac{2GM}{R}}$$

The velocity needed is independent of the object's mass (though of course it will take more energy to get a bigger object from zero up to a certain velocity). This velocity is called the escape velocity.

If we recall that the acceleration due to gravity at the surface of the earth is

$$g = \frac{GM}{R^2}$$

substituting

$$v = \sqrt{2gR}$$

We need to watch the units.

If we use English (Imperial) units, $a = 32$ feet per sec^2. $R = 3959$ miles, and in feet that is $3959 \cdot 5280$. Hence $v = \sqrt{2 \cdot 32 \cdot 3959 \cdot 5280} \approx 36500$ feet per second.

15 miles per hour is exactly equal to 22 feet per second.

$$15\ \frac{\text{miles}}{\text{hour}} \cdot 5280\ \frac{\text{feet}}{\text{mile}} \cdot \frac{1}{3600}\ \frac{\text{hour}}{\text{second}} = 22\ \frac{\text{feet}}{\text{second}}$$

The velocity works out to about 24,900 miles per hour. That's a lot. A point on the equator of the earth rotates 24,900 miles a day.

In MKS units, $a = 9.8$ meters per sec^2 and the earth's radius is 6371 km so

$$\begin{aligned} v &= \sqrt{2aR} \\ &= \sqrt{2 \cdot 0.0098 \cdot 6371} \\ &\approx 11.2\ \frac{\text{km}}{\text{sec}} \end{aligned}$$

The energy required to achieve a stable orbit around the earth has two parts: a radial part that gives the needed potential energy and a tangential part that gives the necessary orbital velocity. In the next chapter we will look at orbital velocities.

Kline

Morris Kline uses a different approach. We start with the acceleration due to gravity whose absolute value is

$$a = \frac{GM}{r^2}$$

You might think we should write this with a minus sign because a points down, while r increases going up. However, as we said before, this is not correct. The reason is that the force that must be exerted to escape from gravity points *up*.

Write this explicitly as the second derivative of position with respect to time

$$a = \frac{d^2r}{dt^2} = \frac{GM}{r^2}$$

Now, he says, we'd like to integrate both sides (with respect to t, naturally). That would give us $v(t)$ on the left. But what is written on the right is not explicitly a function of time. What to do?

Use the chain rule!

Manipulating the left-hand side:

$$\frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$$

Clever, eh? Hence

$$v\frac{dv}{dr} = \frac{GM}{r^2}$$

Moving dr to the right-hand side and integrating, we obtain

$$\frac{v^2}{2} = -\frac{GM}{r} + C$$

You may recognize a connection between kinetic and potential energy here, we are just missing the mass m.

We can evaluate the constant C by recognizing that we want $v = 0$ for some particular r that we're going to choose. It may be $r \to \infty$ as we had above, or it might be some other r.

What we could do is just to deal with the velocity and the radius at the two points, subtract, and then the constant goes away.

$$\frac{v_2^2}{2} - \frac{v_1^2}{2} = - \left[\ \frac{GM}{r_2} - \frac{GM}{r_1} \ \right]$$

Let's call the radius where $v = 0$, r_1. There

$$0 = \frac{GM}{r_1} + C$$

$$C = -\frac{GM}{r_1}$$

$$\frac{v^2}{2} = \frac{GM}{r} - \frac{GM}{r_1}$$

Now ask what happens in the limit as $r_1 \to \infty$, and the r of interest to us is the radius of the earth R

$$\lim_{r_1 \to \infty} \frac{GM}{r_1} = 0$$

$$\frac{v^2}{2} = \frac{GM}{R}$$

and this is the same result as we had before.

Kline II

Kline also uses a different argument to achieve the same goal.

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}$$

Multiply both sides by

$$\frac{dr}{dt} \cdot \frac{d^2r}{dt^2} = -\frac{GM}{r^2} \cdot \frac{dr}{dt}$$

The derivative is just a function. Pretend we don't know what it is, call it u, and substitute the left-hand side *only*.

$$u = \frac{dr}{dt}$$

$$u\ \frac{du}{dt} = -\frac{GM}{r^2} \cdot \frac{dr}{dt}$$

Now, multiply by dt and integrate. Or if you prefer, reverse the chain rule and integrate both sides with respect to t.

$$\frac{1}{2}u^2 = \frac{GM}{r} + C$$

Recall that u is really just v

$$v = \frac{dr}{dt} = u$$

$$\frac{1}{2}v^2 = \frac{GM}{r} + C$$

A bit of sleight of hand.

Chapter 96

Uniform circular motion

Here we want to think about some problems related to the orbits of planets. The first topic is uniform circular motion. In this analysis we will idealize the orbits of the planets as circles.

The eccentricity of an ellipse is defined so that

$$ea = f$$

where $f = \sqrt{a^2 - b^2}$ is the distance from the center to either focus. If $e = 0$ then the curve is a circle rather than an ellipse.

According to wikipedia, the eccentricity of orbit of the objects known to Kepler (6 planets plus the moon) are

Mercury	0.024
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.054
Moon	0.055

These vary with time.

Looking at Earth, the distance from the sun is very nearly 150×10^6 km ($147,098,074$ at perihelion and $152,097,701$ km at aphelion, with a mean value of 149.6×10^6 km).

That is plus or minus 2.5×10^6 km or about 200 earth diameters.

Besides the eccentricity of the orbit, there is also the complication that the earth does not "revolve around the sun" but instead both earth and sun revolve around their common center of mass.

However the mass of the sun is about 1.989×10^{30} kg, while that of the earth is only about 5.972×10^{24} kg. The ratio is about 3.3×10^5.

So the center of mass is displaced from the center of the sun by about

$$\frac{5.972 \times 10^{24}}{1.989 \times 10^{30}} \, 150 \times 10^6 \approx 450$$

That's 450 km. Since the sun's radius is about $695,000$ km, the center of mass is very close to the center of the sun.

basic equations

We should be quite familiar with polar coordinates, we write

$$x = R \cos \theta$$

$$y = R \sin \theta$$

(I'm going to use R for the radius since it's a constant in this analysis. Also, we will also have $\mathbf{r}(t)$, the position vector).

We are talking about objects that change position with time, so we need to introduce time here somehow. We will say that

$$\theta = \omega t$$

Our clock ticks in seconds, and ω (in units of radians per second) tells how θ is calibrated with respect to the clock.

So now we can write the components of

$$\begin{aligned} \mathbf{r}(t) &= \langle\ x(t), y(t)\ \rangle \\ &= R\ \langle \cos\theta, \sin\theta \rangle \\ &= R\ \langle \cos\omega t, \sin\omega t \rangle \end{aligned}$$

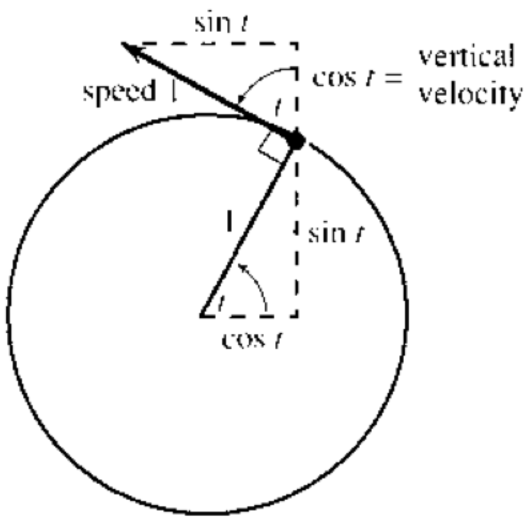
We have the magnitude R times a unit vector.

And then we just differentiate to find the velocity and acceleration

$$\begin{aligned} \mathbf{r}(t) &= R\ \langle \cos\omega t, \sin\omega t \rangle \\ \mathbf{v}(t) &= \omega R\ \langle -\sin\omega t, \cos\omega t \rangle \\ \mathbf{a}(t) &= -\omega^2 R\ \langle \cos\omega t, \sin\omega t \rangle \\ &= -\omega^2 R\ \mathbf{r}(t) \end{aligned}$$

Notice that $\mathbf{v}(t)$ is orthogonal to $\mathbf{r}(t)$ (compute the dot product to see this)

$$\mathbf{v}(t) \cdot \mathbf{r}(t) = 0$$



and the acceleration is on exactly the same line as $\mathbf{r}$ but points inward, toward the sun or origin of the system.

$$\mathbf{a}(t) = -\omega^2 R\ \mathbf{r}(t)$$

Perhaps if we adjusted our clock to have the appropriate units of time, we wouldn't need ω, but it gives the magnitude of the velocity.

$$\begin{gathered} v = |\mathbf{v}| \\ = |\omega R\ \langle -\sin\omega t, \cos\omega t\rangle| \\ = \sqrt{\omega^2 R^2 (\sin^2\omega t + \cos^2\omega t)} \\ = \sqrt{\omega^2 R^2 (\sin^2\theta + \cos^2\theta)} \\ = \omega R \end{gathered}$$

and

$$a = |\mathbf{a}| = \omega^2 R$$

We combine these to get an important identity

$$a = \omega^2 R = (\frac{v}{R})^2 R = \frac{v^2}{R}$$

$$aR = v^2$$

$$v = \sqrt{aR}$$

We can apply the formula to find the orbital velocity for an object going around the earth. The acceleration due to gravity is

$$a = \frac{GM}{R^2}$$

and then the orbital velocity is

$$v = \sqrt{aR} = \sqrt{\frac{GM}{R}}$$

So called low-earth orbits start at a height of about 160 km. We add that to the earth's radius (6371 km)

$$v = \sqrt{aR} = \sqrt{0.0098 \cdot 6531} = 8.00$$

in kilometers per second.

Compare this to the radial velocity at earth's equator which is about

$$40075/(24 \cdot 3600) = 4.63$$

in km per second. There's a reason why rockets are launched from Florida.

Of course, the radial velocity depends on latitude. One must multiply by the cosine of the latitude. The radial velocity at the north pole is zero.

In addition to the energy from the orbital velocity, we also need to add the potential energy to get to a particular orbit. We showed previously that

$$V = -GM/R$$

$$\Delta V = -GM(\frac{1}{R_2} - \frac{1}{R_1}) = GM(\frac{1}{R_1} - \frac{1}{R_2})$$

We can use these to look at some other orbits. Geostationary orbit 42,000 km Moon orbit avg = 385,000 km

The variation in satellite orbits is pretty extreme.

energy

We did this calculation in the previous chapter.

The force due to gravity is

$$\mathbf{F} = -\frac{GmM}{r^2}\hat{\mathbf{r}}$$

The potential is a function, which when we take

$$-\frac{d}{dr}U = \mathbf{F}$$

$$U = -\frac{GmM}{r} + C$$

Define

$$U_\infty = 0 \rightarrow C = 0$$

The total energy is

$$E = K + U = \frac{1}{2}mv^2 - \frac{GmM}{r}$$

if we want an object to just reach $r = \infty$ with zero energy, starting from $r = R$, then

$$0 = \frac{1}{2}mv^2 - \frac{GmM}{R}$$

$$v^2 = \frac{2GM}{R}$$

This is the escape velocity.

If the force is as given above, the magnitude of the acceleration is

$$a = \frac{GM}{r^2}$$

but for uniform circular motion we had

$$a = \frac{v^2}{r} = \frac{GM}{r^2}$$

$$v^2 = \frac{GM}{r}$$

or at the surface of the earth

$$v^2 = \frac{GM}{R}$$

This is the orbital velocity.

So, near the earth's surface, the escape velocity is approximately $\sqrt{2}$ times the orbital velocity.

Finally, we know that the velocity is distance divided by time, so for one full revolution it is

$$v = \frac{2\pi R}{T}$$

but

$$v^2 = \frac{GM}{R} = (\frac{2\pi R}{T})^2$$

Rearranging

$$T^2 = \frac{(2\pi)^2}{GM} R^3$$

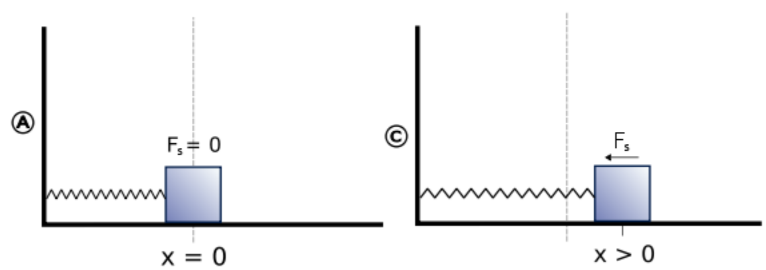
This is Kepler's Third Law: the square of the period is proportional to the cube of the orbit's radius.

The derivation is easy if we assume the orbits are circles, which is pretty close to being true. It is also true for an elliptical orbit. That is a bit harder calculation, which we will get to later.

Chapter 97

Harmonic oscillator

Here we look at the classical equations for oscillation, using as one of the sources the Physics lectures by Prof. Shankar. The simplest example is called the mass and spring system.



In the picture, the left panel is the equilibrium position, while the right view is the mass displaced to the right by some distance x. We pulled it there and have just released it. The force due to the spring is $F = -kx$. Not shown is the picture when the mass goes to the left of the equilibrium position and compresses the spring. To begin with, we will use a frictionless table. Without friction, the mass will oscillate back and forth forever.

We don't need vectors for this since we are working in one dimension. By experiment, we find that the force is proportional to the displacement and directed toward the equilibrium position.

$$F = -kx$$

Newton's Law says that $F = ma$ so

$$F = ma = m\ddot{x} = -kx$$

$$\ddot{x} + \frac{k}{m}x = 0$$

This equation is easy to solve. We need something whose second derivative is minus itself, within some constant. A good choice is

$$x(t) = A\cos\omega t$$

We choose $\cos t$ rather than $\sin t$ because we want $x(0) \neq 0$.

$$\ddot{x}(t) = -\omega^2 A\cos\omega t$$

so that

$$[\ -\omega^2 + \frac{k}{m}\ ]\ A\cos\omega t = 0$$

A corresponds to $x(0)$, which we want to be non-zero. This equation is zero (for all t) only when

$$\frac{k}{m} = \omega^2$$

$$\omega = \sqrt{k/m}$$

Observe that A can be anything we want. It corresponds to the maximum amplitude of the oscillation, determined by how far we pull out the mass to start things off.

In contrast, ω is determined by the characteristics of the spring, and is inversely proportional to the mass. ω is also related to the frequency of oscillation. If T is the time period for one complete oscillation, then $\omega T = 2\pi$, or if f is the frequency of oscillation, then $\omega = 2\pi f$.

phase

Finally, this solution assumes that $v_0 = 0$

$$\dot{x}(t) = -\omega A \sin \omega t$$

$$\dot{x}(0) = -\omega A \sin \omega(0) = 0$$

That is more restrictive than necessary. We can deal with this in various ways, one is to add a term containing the sine to $x(t)$

$$x(t) = A \cos \omega t + B \sin \omega t$$

$$\dot{x}(t) = -\omega A \sin \omega t + B \cos \omega t$$

$$\dot{x}(0) = B = v_0$$

Another way is to add a phase term to the angle, which doesn't change the derivatives

$$x(t) = A \cos \omega t + \phi$$

$$\dot{x}(t) = -\omega A \sin \omega t + \phi$$

$$\dot{x}(0) = v_0 = -\omega A \sin \phi$$

It is interesting to see why these amount to the same thing. Recall

$$\cos \omega t + \phi = \cos \omega t \cos \phi - \sin \omega t \sin \phi$$

But ϕ is a constant so if $-B = \sin \phi$ and $A = \cos \phi$

$$\cos \omega t + \phi = A \cos \omega t + B \sin \omega t$$

However, for now we will agree to set our clock to zero when the mass has just been released and has zero velocity, at the maximum amplitude of the oscillation.

Conservation of Energy

At this point, Prof. Shankar does this calculation

$$x(t) = A\cos\omega t$$

$$v = \dot{x} = -\omega A\sin\omega t$$

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$= \frac{1}{2}m\omega^2 A^2\sin^2\omega t + \frac{1}{2}kA^2\cos^2\omega t$$

But $\omega^2 = k/m$ so

$$= \frac{1}{2}kA^2\sin^2\omega t + \frac{1}{2}kA^2\cos^2\omega t$$

$$E = \frac{1}{2}kA^2$$

Not only is this independent of time, but we can write

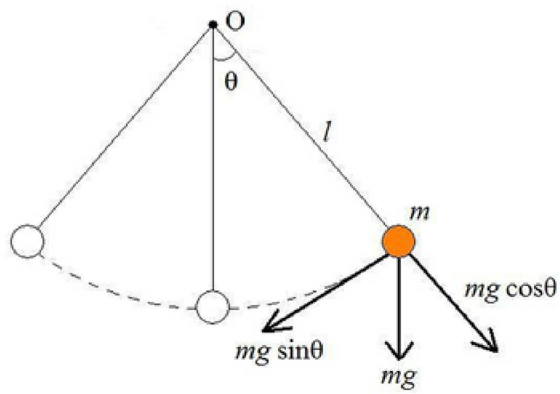
$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Given A and x, we can find v, and so on.

Chapter 98

Pendulum

A useful extension of the harmonic oscillator is to the problem of the pendulum.



The *torque* on the mass is computed from the component of the gravitational force perpendicular to the rod, which is $-mg\sin\theta$

The vector components are drawn a bit too long in the figure, the total force should be the hypotenuse of a right triangle with sides $F\sin\theta$ and $F\cos\theta$.

$$\tau = -mgL\sin\theta$$

We apply the small angle approximation $\sin x \approx x$ and obtain:

$$\tau = -mgL\theta$$

Torque is related to angular momentum the way force is related to momentum

$$\tau = I\ddot{\theta}$$

so

$$I\ddot{\theta} + mgL\theta = 0$$

This is exactly the equation we solved above. In particular

$$\omega = \sqrt{\frac{mgL}{I}} = \sqrt{\frac{mgL}{mL^2}} = \sqrt{\frac{g}{L}}$$

The period T times the angular frequency is 2π

$$T\omega = 2\pi$$

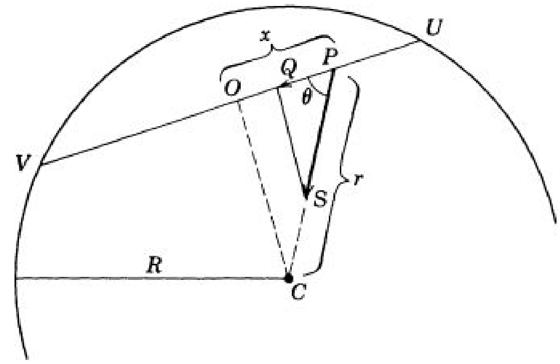
$$T = 2\pi\sqrt{\frac{L}{g}}$$

The period is independent of the mass.

Tunneling through the earth

Kline (Fig 10-18) has a fun problem.

Imagine that a tunnel has been bored straight through the earth between U and V in the figure and we consider the motion of a particle that is free to move through the tunnel — maybe a railroad car on some kind of track.



The car will move under the force of gravity. Kline does not derive it here, but the force inside the earth is variable depending on the position in the tunnel, partly due to the change in r, and partly due to the fact that the mass outside the current radius r does not generate any net force. Rather than the standard

$$F = \frac{GmM}{r^2}$$

the formula is

$$F = \frac{GmM}{R^3} r$$

Kline says, let $k = GM/R^3$ so

$$F = kmr$$

We choose the coordinate system as shown, with the origin at the midpoint of the tunnel. The effect of the force is to move the car along the tunnel, so we need to calculate the part of the force that points in that direction. It is simply $F \cos\theta$, where $\cos\theta = x/r$. Somewhat miraculously, with a nice cancelation, the force becomes

$$F = kmx$$

We take x positive to the right, while the force is in the negative x-direction so switch signs

$$F = -kmx$$

Apply Newton's second law, obtaining a differential equation

$$F = -kmx = m\ddot{x}$$

$$-kx = \ddot{x}$$

We know the solution to this one. The second derivative of the function is minus the function. A general solution is

$$x(t) = A\sin t + B\cos t$$

A factor of $\sqrt{k}$ is needed inside the trig functions:

$$x(t) = A\sin(\sqrt{k}\cdot t) + B\cos(\sqrt{k}\cdot t)$$

Check by differentiating. Now we use two initial conditions, one is that $v_0 = 0$.

$$v_0 = \dot{x}(0) = A\sqrt{k}\cos(\sqrt{k}\cdot 0) - B\sqrt{k}\sin(\sqrt{k}\cdot 0)$$

At $t = 0$, $\sin t = 0$, so we need the first term to be zero, with $A = 0$ and the equation reduces to

$$x(t) = B\cos(\sqrt{k}\cdot t)$$

The second is that $x(0) = x_0$ so $B = x_0$ and

$$x(t) = x_0\cos(\sqrt{k}\cdot t)$$

We have a periodic or oscillatory motion. The period is

$$T\sqrt{k} = 2\pi$$

$$T = \frac{2\pi}{\sqrt{k}}$$

We defined

$$k = GM/R^3$$

but recall that at the surface of the earth

$$32m = \frac{mMG}{R^2}$$

$$32 = \frac{MG}{R^2}$$

so

$$k = \frac{GM}{R^3} = \frac{32}{R}$$

and

$$T = 2\pi\sqrt{\frac{R}{32}}$$

With the radius measured in feet

$$R = 3959 \cdot 5280 = 20903520$$

$$\frac{1}{\sqrt{k}} = 808.2$$

$T = 5077$ seconds or about 85 minutes.

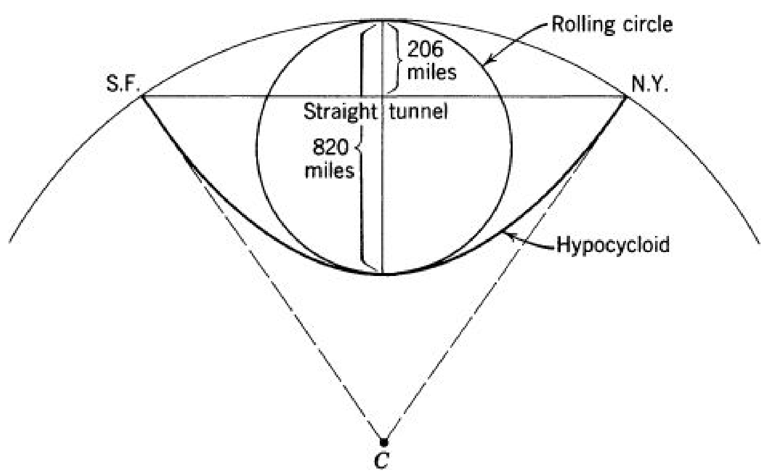
It is very interesting that this result does not depend on the location or the length of the tunnel.

The fastest time occurs for a hypocycloidal path.

Kline:

> The following data give some idea of what can be gained by using a hypocycloidal path. Suppose that a straight tunnel

is dug between New York and San Francisco. Along the surface of the earth the two cities are about 2,575 miles apart, but the tunnel would be 2,530 miles long. At the midpoint the tunnel would be 206 miles below the surface; the maximum velocity acquired by the object, which would be at the midpoint, would be 1.57 mi/ sec; and a one-way journey, according to (reference), would take about 42 minutes. The hypocycloidal path would be 2940 miles long. At the midpoint the tunnel would be 820 miles below the surface; the velocity of the object at that point would be 3 mi/ sec; and the one-way journey would take 26 minutes.



We can calculate the velocity for the straight line path.

$$\dot{x}(t) = -x_0\sqrt{k}\sin(\sqrt{k}\cdot t)$$

which has a maximum when the sine term is equal to 1 , at one-quarter and three-quarters of the period.

$$|v_{\max}| = x_0\sqrt{k} = 1265/808 = 1.57 \text{ mi/sec}$$

which matches the text.

We can check the calculation another way, by using the conservation of energy. We have that the potential energy lost is mgh (it shouldn't matter that g changes as we go inside the earth). This is equal to the kinetic energy gained:

$$\frac{1}{2}mv^2 = mgh$$
$$v = \sqrt{2 \cdot 32 \cdot 206 \cdot 5280} = 8343 \text{ ft/sec} = 1.58 \text{ mi/sec}$$

Another angle is to ask how things would differ on the moon? Given:

mass is 1/81; radius is 3/11; gravity is 1/6.

k goes like M/R^3 or $1/81 \times (11/3)^3 = 11^3/3^7$

$\sqrt{k}$ goes like

$$\frac{11\sqrt{11}}{3^3\sqrt{3}} = \frac{11\sqrt{11}}{27\sqrt{3}} = 0.78$$

Since T is proportional to the inverse it's about 1.28 times longer. Not as much different as you might expect.

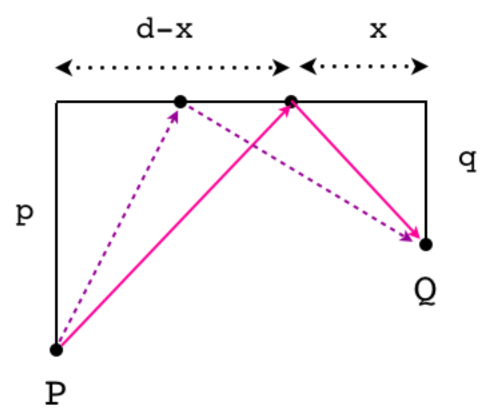
Part XXIV

Time

Chapter 99

Shortest path

As shown in the figure below, we have two points P and Q, which might be, say, the origin and destination of a journey that must also go to the river (horizontal line at the top). The point where we reach the river can be adjusted, and we seek the path that has the shortest overall distance.



There is a hard way to do this problem, and an easy way. The hard way involves a tiny bit of calculus (to do the minimization). I'll show that one first.

Depending on where we put the point on the river, we have a horizontal distance x to that point from Q, and a horizontal distance $d-x$ to the same point from P.

The path consists of two parts, from P to the river, and from the river back to Q. That distance is

$$\sqrt{p^2 + (d-x)^2} + \sqrt{q^2 + x^2}$$

We take the derivative with respect to x. It's a little tricky.

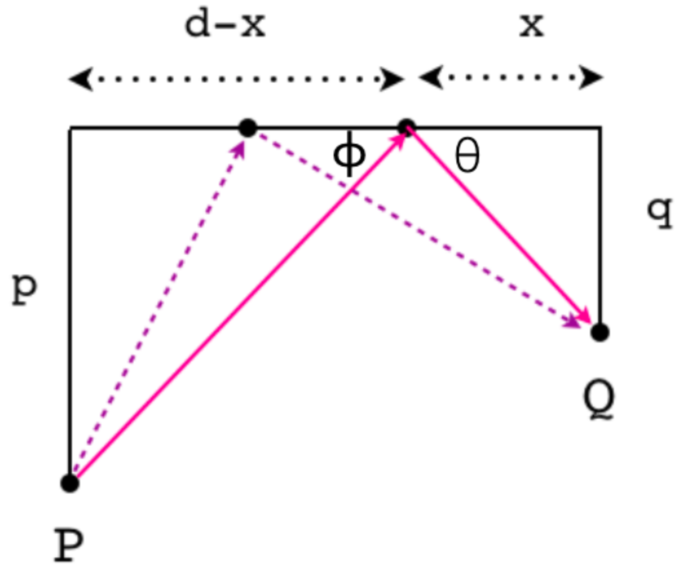
For the first term, we get the denominator from applying the power rule to the square root (plus a factor of $1/2$), then we apply the chain rule to $(d-x)^2$ and get $2(d-x)$ in the numerator, and finally a factor of -1 from $-x$. The twos cancel. The second term is similar.

Set the result equal to 0:

$$-\frac{d-x}{\sqrt{p^2 + (d-x)^2}} + \frac{x}{\sqrt{q^2 + x^2}} = 0$$

$$\frac{x}{\sqrt{q^2 + x^2}} = \frac{(d-x)}{\sqrt{p^2 + (d-x)^2}}$$

It's just algebra from this point. But notice that we don't need it. Both of these ratios correspond to the cosine of an angle.



$$\frac{x}{\sqrt{q^2 + x^2}} = \cos\theta$$

$$\frac{(d-x)}{\sqrt{p^2 + (d-x)^2}} = \cos\phi$$

Because of this, the angles are equal as well.

Here's the algebra:

$$\frac{x^2}{q^2 + x^2} = \frac{(d-x)^2}{p^2 + (d-x)^2}$$

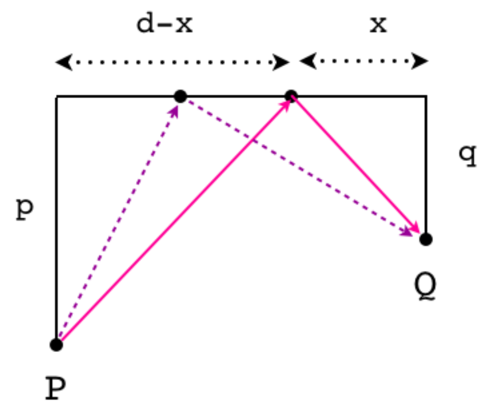
invert and simplify

$$\frac{q^2}{x^2} + 1 = \frac{p^2}{(d-x)^2} + 1$$

cancel $+1$ and take square roots

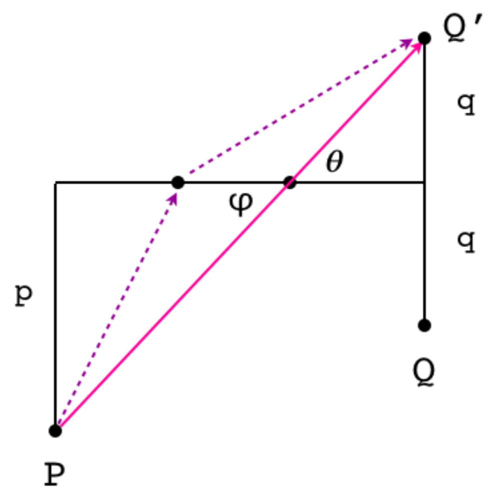
$$\frac{q}{x} = \frac{p}{(d-x)}$$

The result is that the two triangles should be similar (since their sides are in proportion and they are both right triangles).



Another way to say it is that the angle where we come from P to the river, and the angle by which we leave the river to Q should be equal.

Now for the easy way.



We draw a vertical up the same distance q to point Q'—the mirror image of Q. Minimizing the distance to Q' is the same problem because it's exactly the same distance.

What's the shortest distance between two points? A straight line from P to Q'. With a straight line then the two angles ϕ and θ are equal and the similarity of the triangles follows immmediately.

This is a famous result in physics. It's true for light rays, that when you shine a light from P at a mirror, the light rays arriving at Q come by the shortest path. The law about the angles being equal is called the "law of reflection" and it was known to Euclid.

Pool players who know nothing about about Euclid know this result, and they use it all the time in making bank shots.

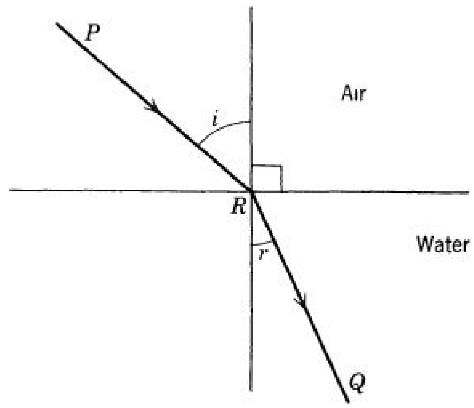
But this is only the beginning. The general principle is called "least action."

Snell's Law

A variation on the previous problem yields Snell's Law for refraction.

Consider the problem of a light ray passing from P to Q, where P is in air, and Q is in a medium like water, with a higher refractive index and lower speed of light.

Figure 8-9



The physical principle is that light takes the path of shortest time. We need to find the R that makes this true.

Suppose the total horizontal distance between P and Q is d. Let x be the horizontal distance from P to R, then $d - x$ is the horizontal distance from R to Q. The vertical distances are fixed, let's call them p (for PR) and q (for QR).

The time taken is the distance divided by the speed. Let the speed of light in air be u and the speed of light in water be v. There are two segments of the trip:

$$t_1 = \frac{\sqrt{x^2 + p^2}}{u}$$

$$t_2 = \frac{\sqrt{(d-x)^2 + q^2}}{v}$$

$$t = t_1 + t_2 = \frac{\sqrt{x^2 + p^2}}{u} + \frac{\sqrt{(d-x)^2 + q^2}}{v}$$

We have time t as a function of x and we take the first derivative and set it equal to zero:

$$\frac{x}{u\sqrt{x^2+p^2}} + \frac{-(d-x)}{v\sqrt{(d-x)^2+q^2}} = 0$$

$$\frac{x}{u\sqrt{x^2+p^2}} = \frac{(d-x)}{v\sqrt{(d-x)^2+q^2}}$$

Rather than fool with the square roots, notice that

$$\sqrt{x^2+p^2} = PR$$

and

$$\sqrt{(d-x)^2+q^2} = RQ$$

so

$$\frac{x}{u\sqrt{x^2+p^2}} = \frac{(d-x)}{v\sqrt{(d-x)^2+q^2}}$$

becomes

$$\frac{x}{u\ PR} = \frac{(d-x)}{v\ RQ}$$

Furthermore $x/PR = \sin\theta_i$ and $(d-x)/RQ = \sin\theta_r$ so

$$\frac{\sin\theta_i}{u} = \frac{\sin\theta_r}{v}$$

The sines of the angles for each side of the barrier are in the same ratio as the velocities in the respective medium.

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{u}{v}$$

$$\sin\theta_r = \frac{v}{u}\ \sin\theta_i$$

Since the speed of light in air is higher than in water $u > v$, $v/u < 1$ which means that $\sin\theta_r < \sin\theta_i$ and thus $\theta_r < \theta_i$.

We can also use the refractive index n which is proportional to the reciprocal of the speed.

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_r}{n_i}$$

Chapter 100

Across the river

The problem we want to solve is as follows. There is a (very smooth) river that flows with a speed u, while you and your twin brother each swim with a speed v. You have used your trigonometry skills to mark out a position which is up-river from the start point a distance equal to the width of the river, d.

Your brother wants to race (across and back versus up and down), and he's willing to let you pick which way you want to swim. Which should you choose?

To solve this problem, first recall that the time to travel any distance d at a constant speed v is

$$t = d/v$$

Also, the time to swim up-river (and back) depends on your relative speed with respect to the land, going up-river it is $v - u$ and coming down-river it is $v + u$. The total time for the first trip is

$$t_1 = \frac{d}{v-u} + \frac{d}{v+u} = d(\frac{v+u+v-u}{v^2-u^2}) = \frac{2dv}{v^2-u^2}$$

For the second trip, across and back, you need to swim upstream at some angle θ in order that the current will sweep you back down to a point on the line that goes directly across the river. The vectors make a little right triangle with hypotenuse v and far side u. The effective speed is $\sqrt{v^2-u^2}$. So, just as for the up and back trip, there is a requirement that $v > u$. You should probably calculate θ before leaving shore, $\theta = \sin^{-1}(u/v)$.

Having calculated the speed, the time is just

$$t_2 = \frac{2d}{\sqrt{v^2-u^2}}$$

The ratio of the two times is

$$\begin{aligned} \frac{t_1}{t_2} &= \frac{2dv}{v^2-u^2}\frac{\sqrt{v^2-u^2}}{2d} \\ &= \frac{v}{\sqrt{v^2-u^2}} \end{aligned}$$

Let's check this result quickly. Suppose $v = 5, u = 3$ (in m/s^2). Suppose we go up-river 7200 m at 2 m/s (taking 1 hr) and then back down-river 7200 m at 8 m/s for 0.25 hr or 5/4 hr total.

Pythagoras tells us that we go across the river at 4 m/s. If the one-way distance is 7200 m, then we do each half in $1800s = 0.5 + 0.5 = 1.0$ hr. Going across is faster.

Which is a bit surprising. The average speed up- and down-river is 5, yet we are slower than going across with a constant speed of 4. That's because we already spent a whole hour at 2 m/s going up-river, we still have to come back!

There is a very famous experiment which depends on this difference.

http://en.wikipedia.org/wiki/Michelson-Morley_experiment

Chapter 101

Relativity and time

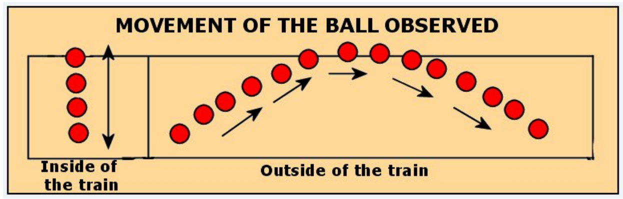
relative motion

Imagine that you are seated in a train looking out the window. Another train is just next to you, sliding past on the adjacent track, moving to your right.

All you can see out the window is the other train and what is happening inside. This is an idealized world (what Einstein called a Gedanken or thought experiment) so trains do not make noise or lurch from side to side as they move.

Galileo's classical **relativity** says that there is no physical test you can do from inside your train to decide which train is moving. It might be that the other train is moving or it could be that your train is moving. You can't tell.

vertical and horizontal independence



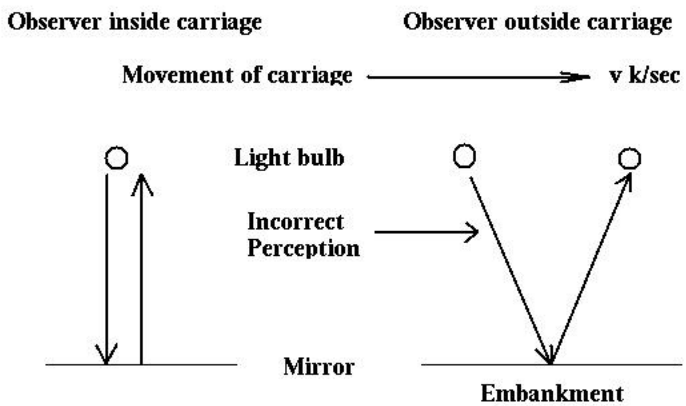
Suppose there is a guy sitting in the other train and he throws a ball straight up in the air and then it comes down. His view of things is shown on the left, and your view of things is on the right.

You see a horizontal movement that he does not see, because of the relative velocity of the trains.

However, movement in the vertical direction is independent of the horizontal direction and vice-versa, so the time each of you measures for the flight of the ball is the same, governed by its initial velocity upward and the downward force of gravity. So is the vertical velocity of the ball at each point, and your calculations for the acceleration due to gravity will be the same.

man with a flashlight

Now suppose instead that the man in the other train is holding a flashlight overhead, pointed at a mirror on the floor. He turns it on very briefly, emitting a light ray that travels (from his perspective) vertically.



He observes the time when the light returns back from the mirror. To make things simpler for our calculations, let us just consider the path from the flashlight to the floor. The light traveled down a distance e in time s.

He can calculate the *speed of light* as distance divided by time

$$c = \frac{e}{s}$$

your view

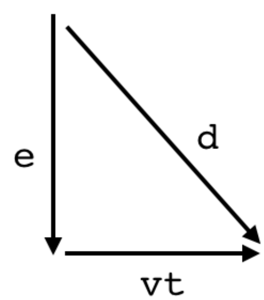
Suppose also that the light starts on its journey at precisely the same time as the man, the flashlight and that spot on the floor are all directly opposite you, in your train.

Considered from your perspective, the light ray will cover a longer distance, because of the relative velocity between you and the other train.

What you see is that the man turned on the flashlight while he was exactly opposite, but by the time the light ray hits the floor, the floor has moved to the right. So the light travels to the floor by an angled path with distance d, hitting the point directly below the other guy when that point is some distance off to your right (it is similar to the example with the thrown ball).

Of course, you must be going very fast for this to be obvious, because the speed of light is so large ($3 \cdot 10^8$ m/sec or about $186,300$ mph).

Suppose that by your watch, the time taken for this to happen is t. e is the vertical component of d, and the horizontal component is the relative speed of the two trains v multiplied by your own time t. Speed times time equals distance.



You can also calculate the speed of light. You saw it move a distance d in a time t so

$$c = \frac{d}{t}$$

Einstein's principle of relativity

One of Einstein's fundamental principles of relativity is that *the speed of light is the same for all observers.*

We must both obtain the *same* velocity for light:

$$c = \frac{e}{s} = \frac{d}{t}$$

Since the distances are clearly not equal ($d \neq e$), neither can the times be equal: $s \neq t$.

That is a basic paradox of relativity. There is no longer such a thing as absolute time. And if two observers who are in relative motion cannot agree on the time, they will not be able to agree on whether two events are simultaneous or not.

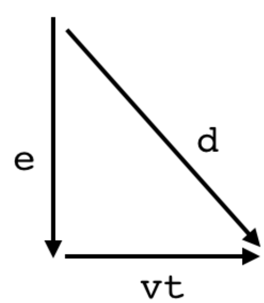
calculation

A little algebra remains.

We can rearrange what's above above slightly

$$e = cs, \quad d = ct$$

Since d is the hypotenuse of a right triangle with sides e and vt



Pythagoras tells us that

$$d^2 = e^2 + (vt)^2$$

Substitute $d = ct$ and $e = cs$:

$$(ct)^2 = (cs)^2 + (vt)^2$$

Isolate the terms containing t:

$$(ct)^2 - (vt)^2 = (cs)^2$$

Factor out t^2

$$[\ c^2 - v^2\ ]\ t^2 = c^2 s^2$$

Divide by c^2

$$[\ \frac{c^2 - v^2}{c^2}\ ]\ t^2 = s^2$$

$$[\ 1 - \frac{v^2}{c^2}\ ]\ t^2 = s^2$$

$$\sqrt{1 - \frac{v^2}{c^2}} \cdot t = s$$

Define

$$\gamma = \sqrt{1 - \frac{v^2}{c^2}}$$

so

$$\gamma\ t = s$$

The factor γ shows up often in relativity.

Look again

$$\gamma = \sqrt{1 - \frac{v^2}{c^2}}$$

If $v = 0$ then $\gamma = 1$, so nothing really changes if the other train is not moving.

As v gets larger, v^2/c^2 is always positive so we have the square root of something smaller than 1, and thus $\gamma < 1$.

Since

$$\gamma\ t = s$$

and $\gamma < 1$, it must be that $s < t$.

The moving observer's value for time, s, is smaller than what you measure, t.

If v gets larger, γ gets smaller.

If v were to become as large as c, then $\gamma \to 0$, and then, time would stand still.

Part XXV

Electric fields

Chapter 102

Field of an infinite wire

These are two classic problems: find the electric field $\mathbf{E}$ for an infinite wire charged with density λ or an infinite sheet charged with density σ (positive charge). Let's start with the wire.

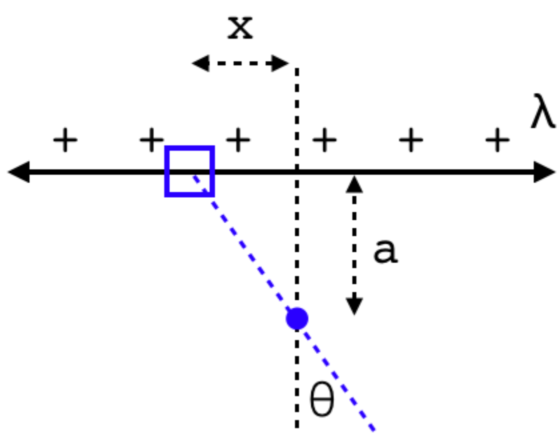
The force between two charges is given by Coulomb's Law:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \, \frac{q_1 q_2}{r^2} \, \hat{\mathbf{e}}_r$$

while the field is the force on a unit test charge due to a given charge.

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \, \frac{q}{r^2} \, \hat{\mathbf{e}}_r$$

infinite wire



Consider a point at a distance a from the wire. Call the nearest position on the wire $x = 0$. The force from a small element at position x points radially away from the element.

Break that force into two components, the component horizontal in the figure will be canceled by a component in the other direction from its counterpart at $-x$. The vertical component is $F \cos\theta$.

Coulomb says that

$$E = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}$$

If the small element has width dx and the density is λ, the total charge in that element is λdx. The distance from the element to the position we are evaluating is $\sqrt{x^2 + a^2}$. So the small part of the electric field is

$$dE = \frac{\lambda}{4\pi\epsilon_0}\frac{dx}{(a^2 + x^2)}$$

There is a further factor of $\cos\theta$ to account for the fact that only the vertical part of the force is not canceled. In terms of a and x:

$$\cos\theta = \frac{a}{\sqrt{a^2+x^2}}$$

$$dE = \frac{\lambda a}{4\pi\epsilon_0}\frac{dx}{(a^2+x^2)^{3/2}}$$

To integrate this, we need a trig substitution. Because of the a^2+x^2, choose the tangent. (Also, Shankar says, we will want to integrate to ∞ as a limit, so neither sine nor cosine will do).

$$x/a = \tan\theta$$

$$x = a\tan\theta$$

$$dx = a\sec^2\theta\ d\theta$$

We already have

$$a/\sqrt{a^2+x^2} = \cos\theta$$

$$\frac{1}{(a^2+x^2)^{3/2}} = \frac{\cos^3\theta}{a^3}$$

Thus

$$dE = \frac{\lambda a}{4\pi\epsilon_0} a\sec^2\theta\ \frac{\cos^3\theta}{a^3}\ d\theta$$

$$dE = \frac{\lambda}{4\pi\epsilon_0 a}\cos\theta\ d\theta$$

$$E = \int \frac{\lambda}{4\pi\epsilon_0 a}\cos\theta\ d\theta$$

$$= \frac{\lambda}{4\pi\epsilon_0 a}\ \sin\theta$$

The limits require care. The original limits on x were (for an infinite wire) $x = -\infty \to \infty$. Now $x = \tan\theta$, so $\theta = -\pi/2 \to \pi/2$. Evaluating $\sin\theta$ between those limits we get $1 - -1 = 2$.

$$E = \frac{\lambda}{2\pi\epsilon_0 a}$$

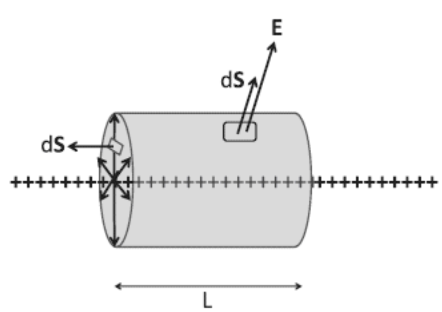
The field falls off like $1/a$ rather than $1/a^2$. We visualize the electric field lines as being perpendicular to the wire. If we take a cross-section in the perpendicular plane, the field lines spread like 1 over the distance. It is a one-dimensional spreading.

We can get the same result using Gauss's Law. We haven't introduced it properly yet, but just state it here:

$$\int_S \mathbf{E} \cdot \mathbf{r} = \frac{Q}{\epsilon_0}$$

The flux of the electric field across a closed surface is equal to the charge enclosed Q times the constant $1/\epsilon_0$, which is about 9×10^9.

Surround a section of wire of length L with a cylindrical Gaussian surface.



The amount of charge enclosed is λL. Gauss says

$$\int_S \mathbf{E} \cdot \mathbf{r} = \frac{Q}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$$

By symmetry, the field is radial, and so the dot product with the surface elements on the ends of the cylinder is zero. The dot product with the surface elements on the curved part is just $E\ dS$ so the whole integral is

$$\int_S \mathbf{E} \cdot \mathbf{dS} = E \int dS = E\ 2\pi a L$$

$$E\ 2\pi a L = \frac{\lambda L}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi a \epsilon_0}$$

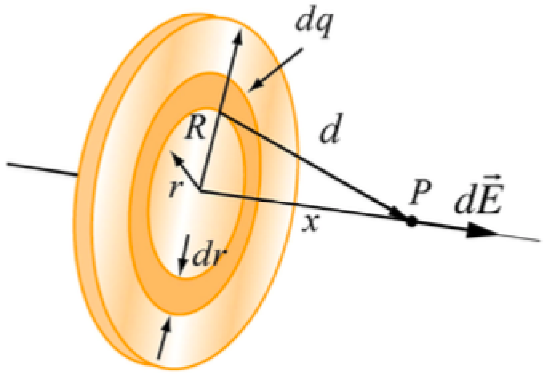
and that matches what we had before.

Chapter 103

Field of an infinite sheet

Here we evaluate the field at a point a distance a away from an infinite sheet. The charge density is σ.

We start by thinking about a ring with radius r centered on the position that is closest to the point where we're conducting the evaluation. The normal vector passes through the center of the ring and also through our point. The distance between them is x



The distance between points on the ring and the point where the field is being evaluated is d.

The force is directed from each little segment on the ring toward the point, but the part of the force that is not perpendicular is canceled in each case, by an opposite component coming from the piece of the ring on the other side. So again, we will have a factor of $\cos\theta$.

The relevant distance squared for Coulomb's Law is

$$d^2 = r^2 + x^2$$

the factor of $\cos\theta$ is

$$\cos\theta = \frac{x}{d} = \frac{x}{\sqrt{r^2 + x^2}}$$

So Coulomb says:

$$dE = \frac{dq}{4\pi\epsilon_0} \frac{x}{(r^2 + x^2)^{3/2}}$$

The ring has width dr and length $2\pi r$, so it has area $2\pi r\ dr$ and charge $2\pi r\sigma\ dr$. We have

$$dE = \frac{2\pi r\sigma}{4\pi\epsilon_0} \frac{x}{(r^2 + x^2)^{3/2}}\ dr$$

$$dE = \frac{x\sigma}{2\epsilon_0} \frac{r}{(r^2 + x^2)^{3/2}}\ dr$$

The denominator is similar to what we had in the first problem, but now we have $r\ dr$ up top.

The integral is

$$E = \int dE = \int \frac{x\sigma}{2\epsilon_0} \frac{r}{(r^2 + x^2)^{3/2}}\ dr$$

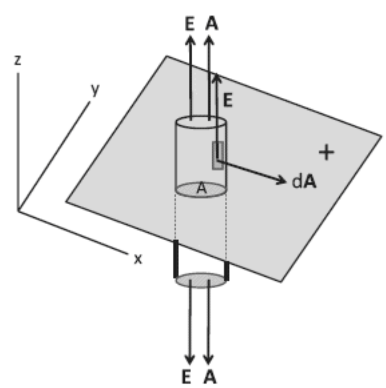
$$= \frac{x\sigma}{2\epsilon_0}\left(-\frac{1}{\sqrt{r^2 + x^2}}\right)$$

We evaluate between $r = 0 \to \infty$. The term in parentheses becomes $- - 1/\sqrt{x^2}$, so the answer is finally

$$E = \frac{\sigma}{2\epsilon_0}$$

Remarkably, the field is independent of x. If we take a cross-section in the perpendicular plane, the field lines do not spread out.

As before, we can also use Gauss's Law.



The Gaussian surface is a cylinder, as shown. The enclosed charge is σA. Gauss says

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \frac{Q}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

Only the end caps of the cylinder intercept any field lines (so the dot product on the curved parts is zero), and for these the field and the normal vector point in the same direction so

$$\int_S \mathbf{E} \cdot \mathbf{dS} = E \int dS$$

We must be careful here. There are *two* end caps

$$\int dS = 2A$$

$$= 2A\ E = \frac{\sigma A}{\epsilon_0}$$

$$E = \frac{\sigma}{2\epsilon_0}$$

sphere

The problem of the sphere comes up in gravitation as well as electrostatics. We will sidestep the determination of how a sphere or a solid ball behaves. (See the chapter on the shell theorem, due to Newton). We will instead use Gauss's Law, which says the total electric flux through a surface enclosing a charge Q is

$$\Phi_E = \frac{Q}{\epsilon_0}$$

The flux is defined to be the dot product of the electric field with the surface element, integrated over the entire surface.

$$\Phi_E = \iint_S \mathbf{E} \cdot d\mathbf{A}$$

If we have a sphere, it is radially symmetric. Therefore, we expect that the electric field will be perpendicular to the surface of the sphere, and to any (imaginary) sphere that we might draw around the sphere at a radius r from the center. This means that the dot product is just EdA, so

$$\Phi_E = \iint_S EdA = E \iint_S dA = 4\pi r^2 E$$

$$\frac{Q}{\epsilon_0} = 4\pi r^2 E$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\ \hat{\mathbf{r}}$$

$$\mathbf{F} = q\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\ \hat{\mathbf{r}}$$

Chapter 104

Field of a dipole

Consider a dipole with charge $+q$ on one end, at $(a, 0)$, and charge $-q$ on the other end $(-a, 0)$. We want to calculate the field due to the dipole at various points in the xy-plane. That gives us everything, since every other plane formed by rotation of this one around the x-axis has the same field.

This material comes from volume II of Shankar's Physics. The second part, using the potential, has the calculus. The first part, using the field, is there for contrast and because he uses various tricks to approximate and simplify the answer.

The relevant equation is a variation on Coulomb's Law:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_o} \cdot \frac{1}{r^2} \cdot \hat{\mathbf{e_r}}$$

starting with the field, E

For a general point (x, y) the vector from the positive pole to the point is

$$(x - a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

the squared distance is

$$(x-a)^2 + y^2$$

the unit vector is

$$\frac{(x-a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x-a)^2 + y^2)^{1/2}}$$

so

$$\mathbf{E}_+ = \frac{q}{4\pi\epsilon_o} \cdot \frac{(x-a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x-a)^2 + y^2)^{3/2}}$$

For the other pole, we get

$$\mathbf{E}_- = -\frac{q}{4\pi\epsilon_o} \cdot \frac{(x+a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x+a)^2 + y^2)^{3/2}}$$

The total field is the sum of these two.

$$\mathbf{E} = \frac{q}{4\pi\epsilon_o} \cdot \ [\frac{(x-a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x-a)^2 + y^2)^{3/2}} - \frac{(x+a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x+a)^2 + y^2)^{3/2}}\ ]$$

special solutions

On the $x-$axis, $y = 0$ so

$$\mathbf{E} = \frac{q}{4\pi\epsilon_o} \cdot \ [\frac{(x-a)\hat{\mathbf{i}}}{(x-a)^3} - \frac{(x+a)\hat{\mathbf{i}}}{(x+a)^3}\ ]$$

$$= \frac{q}{4\pi\epsilon_o} \cdot \ [\frac{(x+a)^2 - (x-a)^2}{(x-a)^2\ (x+a)^2}\ ]\ \hat{\mathbf{i}}$$

$$= \frac{q}{4\pi\epsilon_o} \cdot \ [\frac{4ax}{(x^2 - a^2)^2}\ ]\ \hat{\mathbf{i}}$$

Define the **dipole moment** as

$$\mathbf{p} = 2aq\ \hat{\mathbf{i}}$$

so

$$\mathbf{E} = \frac{\mathbf{p}}{4\pi\epsilon_o} \cdot \frac{2x}{(x^2 - a^2)^2}$$

For $x >> a$ (and since $y = 0$, $r = x$):

$$\mathbf{E} = \frac{1}{2\pi\epsilon_o} \cdot \frac{\mathbf{p}}{r^3}$$

On the $y-$axis, $x = 0$ so

$$\mathbf{E} = \frac{q}{4\pi\epsilon_o} \cdot \ [\frac{(-a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((-a)^2 + y^2)^{3/2}} - \frac{a\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((a)^2 + y^2)^{3/2}}\ ]$$

$$= \frac{q}{4\pi\epsilon_o} \cdot \ [\frac{2(-a\hat{\mathbf{i}})}{(a^2 + y^2)^{3/2}}\ ]$$

$$= -\frac{1}{4\pi\epsilon_o} \cdot \ \frac{\mathbf{p}}{(a^2 + y^2)^{3/2}}$$

For $y >> a$

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_o} \cdot \ \frac{\mathbf{p}}{r^3}$$

Compare with the x-axis:

$$\mathbf{E} = \frac{1}{2\pi\epsilon_o} \cdot \frac{\mathbf{p}}{r^3}$$

In both cases, the field falls like $1/r^3$, but numerically it is twice as large along the x-axis for a given r. Also, along the x-axis the field points to the right, while along the y-axis it points to the left.

general solution

Shankar does some slick thinking to simplify the general case. We have a complicated beast:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_o} \cdot \ [\ \frac{(x-a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x-a)^2+y^2)^{3/2}} - \frac{(x+a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x+a)^2+y^2)^{3/2}} \ ]$$

Shankar says "$\mathbf{E}$ vanishes when $a = 0$. The net field is non-zero only becauase $a \neq 0$, and the non-zero part will start out as the first power of a in a Taylor series expansion. To keep the dimension of the field the same, the extra a must really be a/r", and we saw that in the simple cases above because we had an extra $\mathbf{p}/r$ and $\mathbf{p}$ is proportional to a.

Then: (the equation) has two parts, each with a numerator divided by the denominator, or the numerator times the inverse denominator. We can get the single power of a from either term, and the a_0 term from the other. If we get a^1 from the numerator we may set $a = 0$ in the denominator and vice-versa.

Consider

$$\mathbf{E}_+ = \frac{q}{4\pi\epsilon_o} \cdot \frac{(x-a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x-a)^2+y^2)^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_o} \cdot \frac{\mathbf{r} - a\hat{\mathbf{i}}}{((x-a)^2+y^2)^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_o} \cdot \ [\ \frac{\mathbf{r}}{((x-a)^2+y^2)^{3/2}} - \frac{a\hat{\mathbf{i}}}{((x-a)^2+y^2)^{3/2}} \ ]$$

Expand

$$= \frac{q}{4\pi\epsilon_o} \cdot \ [\ \frac{\mathbf{r}}{((x^2-2ax+a^2+y^2)^{3/2}} - \frac{a\hat{\mathbf{i}}}{((x^2-2ax+a^2+y^2)^{3/2}} \ ]$$

$$\approx \frac{q}{4\pi\epsilon_o} \cdot \ [\ \frac{\mathbf{r}}{((x^2-2ax+y^2)^{3/2}} - \frac{a\hat{\mathbf{i}}}{((x^2+y^2)^{3/2}} \ ]$$

In the first term, we keep $\mathbf{r}$ on top (which is a^0), so we keep a^1 but not a^2 on the bottom. In the second term, we keep a^1 on top, so we don't need to keep any terms involving a on the bottom. Now substitute $r^2 = x^2 + y^2$:

$$\mathbf{E}_+ \approx \frac{q}{4\pi\epsilon_o} \cdot \ [\ \frac{\mathbf{r}}{(r^2 - 2ax)^{3/2}} - \frac{a\hat{\mathbf{i}}}{r^3} \ ]$$

In dealing with $\mathbf{E}_-$, we change the sign on both q and a

$$\mathbf{E}_- \approx -\frac{q}{4\pi\epsilon_o} \cdot \ [\ \frac{\mathbf{r}}{(r^2 + 2ax)^{3/2}} + \frac{a\hat{\mathbf{i}}}{r^3} \ ]$$

so the total field due to the dipole is

$$\mathbf{E} \approx \frac{q}{4\pi\epsilon_o} \cdot \ [\ \frac{\mathbf{r}}{(r^2 - 2ax)^{3/2}} - \frac{\mathbf{r}}{(r^2 + 2ax)^{3/2}} - 2\frac{a\hat{\mathbf{i}}}{r^3} \ ]$$

Then, bringing out $1/r^3$

$$\approx \frac{q}{4\pi\epsilon_o r^3} \cdot \ [\ \frac{\mathbf{r}}{(1 - 2ax/r^2)^{3/2}} - \frac{\mathbf{r}}{(1 + 2ax/r^2)^{3/2}} - 2a\hat{\mathbf{i}} \ ]$$

A last simplification comes from $(1+z)^n = 1 + nz + \ldots$

$$(1 - 2ax/r^2)^{-3/2} = 1 - \frac{-3}{2} \ \frac{2ax}{r^2}$$

$$= 1 + \frac{3ax}{r^2}$$

Hence

$$\mathbf{E} \approx \frac{q}{4\pi\epsilon_o r^3} \cdot \ [\ -2a\hat{\mathbf{i}} + \mathbf{r}(1 + \frac{3ax}{r^2}) - \mathbf{r}(1 - \frac{3ax}{r^2}) \ ]$$

$$\mathbf{E} \approx \frac{q}{4\pi\epsilon_o r^3} \cdot \ [\ -2a\hat{\mathbf{i}} + 3\mathbf{r}\frac{2ax}{r^2} \ ]$$

$$\approx \frac{1}{4\pi\epsilon_0 r^3}\ [\ \ -\mathbf{p} + 3\mathbf{r}\ \frac{(\mathbf{p}\cdot\mathbf{r})}{r^2}\ ]$$

since $\mathbf{p}\cdot\mathbf{r} = 2axq$.

starting with the potential, V

The potential is the integral of the field dotted with $\mathbf{r}$, going to some agreed-upon end point, like ∞, with zero potential. The basic equation is

$$V = \frac{q}{4\pi\epsilon_0 r}$$

In this case we add the signed contributions from each pole

$$V = \frac{q}{4\pi\epsilon_0}\ [\frac{1}{r_+} - \frac{1}{r_-}\ ]$$

That's it for the potential.

We will simplify by moving things to the numerator like so

$$V = \frac{q}{4\pi\epsilon_0}\ [\frac{r_- - r_+}{r_+ r_-}\ ]$$

But first, extend this by defining the position vector $\mathbf{r}$ relative to the center of the dipole so that in terms of vectors

$$\mathbf{r}_+ = \mathbf{r} - a\hat{\mathbf{i}}$$

$$\mathbf{r}_- = \mathbf{r} + a\hat{\mathbf{i}}$$

and

$$r_+ = \sqrt{\mathbf{r}_+\cdot\mathbf{r}_+}$$

$$= \sqrt{r^2 + a^2 - 2\mathbf{r}\cdot a\hat{\mathbf{i}}}$$

Approximate (for $r >> a$) by setting $a^2 = 0$ and pulling out r:

$$r_+ = r\sqrt{1 - 2\mathbf{r} \cdot a\hat{\mathbf{i}}/r^2}$$

$$r_- = r\sqrt{1 + 2\mathbf{r} \cdot a\hat{\mathbf{i}}/r^2}$$

Simplify using the binomial expansion

$$r_+ \approx r(1 - \mathbf{r} \cdot a\hat{\mathbf{i}}/r^2)$$

$$r_- \approx r(1 + \mathbf{r} \cdot a\hat{\mathbf{i}}/r^2)$$

That numerator will be

$$r_- - r_+ = r(1 + \mathbf{r} \cdot a\hat{\mathbf{i}}/r^2) - r(1 - \mathbf{r} \cdot a\hat{\mathbf{i}}/r^2)$$

$$= 2\mathbf{r} \cdot a\hat{\mathbf{i}}/r$$

So now, going back to

$$V = \frac{q}{4\pi\epsilon_0}\ [\frac{1}{r_+} - \frac{1}{r_-}\ ]$$

in the brackets, in the numerator we will have what we got in the line before and in the denominator we have

$$r_+r_- = r\sqrt{1 + 2\mathbf{r} \cdot a\hat{\mathbf{i}}/r^2} \cdot r\sqrt{1 - 2\mathbf{r} \cdot a\hat{\mathbf{i}}/r^2}$$

Approximating the square roots as just equal to 1 (since $a << r$), that gives

$$\approx r^2$$

which leaves us with

$$V = \frac{q}{4\pi\epsilon_0}\ [\frac{2\mathbf{r} \cdot a\hat{\mathbf{i}}/r}{r^2}\ ]$$

The dipole moment is

$$\mathbf{p} = 2aq\hat{\mathbf{i}}$$

so

$$V = \frac{1}{4\pi\epsilon_0} \ [\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \ ]$$

In terms of $\mathbf{r} = \langle x, y \rangle$

$$\mathbf{p}\cdot\mathbf{r} = 2aq\hat{\mathbf{i}}\cdot\mathbf{r} = px$$

and $r^2 = x^2 + y^2$ so

$$V = \frac{p}{4\pi\epsilon_0} \ [\frac{x}{(x^2+y^2)^{3/2}} \ ]$$

compute the field

$$\mathbf{E} = -\nabla V$$

$$E_x = -\frac{p}{4\pi\epsilon_0} \ \frac{\partial}{\partial x} \ [\ \frac{x}{(x^2+y^2)^{3/2}} \ ]$$

The partial derivative is $(u'v - uv')/v^2$ which is

$$= \frac{(x^2+y^2)^{3/2} - x\cdot 3/2(x^2+y^2)^{1/2}\cdot 2x}{[(x^2+y^2)^{3/2}]^2}$$

$$= \frac{1}{(x^2+y^2)^{3/2}} - \frac{3x^2}{(x^2+y^2)^{5/2}}$$

$$= \frac{1}{r^3} - \frac{3}{r^3} \ \frac{x^2}{r^2}$$

so

$$V = \frac{p}{4\pi\epsilon_0 r^3} \ (3\cos^2\theta - 1)$$

Similarly

$$E_y = -\frac{p}{4\pi\epsilon_0} \frac{\partial}{\partial y} [\frac{x}{(x^2+y^2)^{3/2}}]$$

The partial is

$$(-\frac{3}{2})x \frac{1}{(x^2+y^2)^{5/2}} 2y = \frac{-3xy}{r^5}$$

$$= -\frac{1}{r^3} 3\sin\theta\cos\theta$$

Combining these results, the field is

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3}(3\cos^2\theta\hat{\mathbf{i}} - \hat{\mathbf{i}} + 3\sin\theta\cos\theta\hat{\mathbf{j}}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3}(3p\cos^2\theta\hat{\mathbf{i}} - p\hat{\mathbf{i}} + 3p\sin\theta\cos\theta\hat{\mathbf{j}}$$

Since

$$\mathbf{p} = p\hat{\mathbf{i}}$$

The middle term is $-\mathbf{p}$.

The unit vector is

$$\hat{\mathbf{e}}_r = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$$

so the other two terms give

$$3p\cos\theta(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})$$

$$= 3p\cos\theta\ \hat{\mathbf{e}}_r$$

$$= (3\mathbf{p}\cdot\hat{\mathbf{e}}_r)\ \hat{\mathbf{e}}_r - \mathbf{p}$$

tacking on the part out front we have

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3} [(3\mathbf{p}\cdot\hat{\mathbf{e}}_r)\ \hat{\mathbf{e}}_r - \mathbf{p}]$$

which matches what we had above at the end of the section on the field

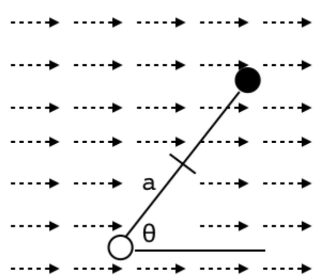
$$\approx \frac{1}{4\pi\epsilon_0 r^3} \left[\ -\mathbf{p} + 3\mathbf{r} \ \frac{(\mathbf{p}\cdot\mathbf{r})}{r^2} \ \right]$$

allowing for the fact that $\mathbf{r}/r = \hat{\mathbf{e}}_r$.

Chapter 105

Dipole in a field

Consider a dipole with a total separation $2a$ between the charges, $+q$ and $-q$. The dipole lies in a uniform electric field E (say it is horizontal) at an angle to the field lines of θ.



The forces on the two charges from the field cancel.

However, there are also two torques, which both tend to align the dipole with the field. For the charge $+q$, there is a force Eq. The torque is that component of the force perpendicular to the axis of the dipole, $F \sin\theta$, times the distance from the axis of rotation at the point where the force is applied, which is equal to a.

$$\tau = aEq\sin\theta$$

Since there are two charges, the torque is double:

$$\tau = 2aEq\sin\theta$$

Even though the second charge is negative, the force on it tends to rotate the dipole in the same direction as the force on the positive charge.

$2aq$ is equal to the dipole moment, p so

$$\tau = pE\sin\theta$$

but this is

$$\tau = \mathbf{p} \times \mathbf{E}$$

The order of the two terms in the cross-product is a matter of definition: $\tau = \mathbf{r} \times \mathbf{F}$. This order makes the resulting vector point into the page.

There is a potential energy when the dipole lies at an angle to the field. For work done by a force we would write

$$U = -W = \int \mathbf{F} \cdot d\mathbf{r}$$

The equivalent formulation for torque is

$$\int \tau \cdot d\theta = \int_{\theta_0}^{\theta} pE\sin\theta$$

$$= -pE\cos\theta \,\Big|_{\theta_0}^{\theta}$$

If we choose θ_0 corresponding to zero potential energy

$$U = pE\cos\theta = \mathbf{p} \cdot \mathbf{E}$$

When θ is 90 degrees and the dipole is perpendicular to the field, the potential energy is pE.

Chapter 106

Capacitor

A capacitor consists of two charged objects: parallel plates, or concentric sphere and shell. The plates are separated by a dielectric (insulator). A capacitor stores energy in the form of the electric field between the plates.

The potential difference is dependent on the amount of charge that is present, and physical characteristics like the size and geometry of the plates and the distance between them.

For the example of two (infinite) parallel plates, we verified elsewhere the formula for the electric field

$$E = \frac{\sigma}{\epsilon_0}$$

This result is easy to see if we visualize an electric field with lines of flux. By symmetry, there is nowhere to go for a line that leaves the positive plate but directly toward the negative plate. There is no space for the lines to spread out with distance, hence, no dependence on the separation.

For a finite capacitor, if we ignore edge effects, the same basic result holds.

The voltage difference does depend on the distance a separating the plates.

$$V = Ea = \frac{\sigma a}{\epsilon_0}$$

Now since $\sigma = Q/A$

$$V = \frac{Q}{A} \frac{\sigma a}{\epsilon_0}$$

We consolidate all the terms except the charge into a factor $C = \epsilon_0 A/a\sigma$ so that gives

$$V = Ea = \frac{Q}{C}$$

Capacitance C relates voltage to charge

$$CV = Q$$

The higher the capacitance, the smaller the voltage for a given amount of stored charge. The permittivity ϵ_0 changes to ϵ for other materials, where $\epsilon = k\epsilon_0$. For example, for paper, $k = 3.5$ or so.

The capacitance is defined to be the ratio of the electric charge on each conductor to the potential difference. The unit is the farad, which is equal to one coulomb per volt. Typical values might be microfarads μF.

A very useful property of capacitors is that they block direct current, yet allow alternating current to pass. Also, when combined in appropriate circuits, they can be used to tune a circuit to a particular resonant frequency (e.g. in a radio).

energy

If we consider a capacitor with a charge Q on it, work must be done to bring a small amount of positive charge dQ to the positively charged plate. Work is equal to force times distance.

The energy stored in a capacitor can be calculated as the work done in moving a small amount of charge "from" one plate "to" another.

$$dW = E\ dq\ a = V\ dq$$

$$W = \int dW = \int V\ dq = \int \frac{Q}{C}\ dq$$

Hence, in building up a charge Q on a capacitor the work done is just

$$W = \frac{Q^2}{2C} = \frac{(CV)^2}{2C} = \frac{1}{2}CV^2$$

The integral here takes account of the fact that the voltage at the time any small charge dq is transferred depends on how much charge is currently on the plates.

Since current does not actually flow across the capacitor, for each electron that leaves the positive plate, one must join the negative plate.

discharge circuit

Consider a resistor R and a capacitor C, and a switch. The capacitor is initially charged. Close the switch and what happens? Use the standard rules and go around the circuit looking at the voltage drops across the components. We get

$$\frac{Q}{C} - IR = 0$$

Define the current as $I = -dq/dt$, and then

$$\frac{Q}{C} = IR = -\frac{dq}{dt}R$$

$$\frac{dq}{Q} = -\frac{1}{RC}dt$$
$$Q = Q_0 e^{-t/RC}$$

The charge decays exponentially with time with characteristics governed by RC. In particular, if $Q = Q_0/2$ then

$$\ln 2 = \frac{T_{1/2}}{RC}$$

The current is the time derivative of the charge

$$I = -\frac{dq}{dt}$$

$$= \frac{1}{RC}Q_0 e^{-t/RC}$$
$$= \frac{1}{RC}I_0$$

In exactly the same way as for the charge

$$\frac{dI}{dt} == -\frac{1}{RC}I_0$$

$$RC\ \frac{dI}{dt} + I_0 = 0$$

When you hook up a voltage and a capacitor, there is an initial current, but as the capacitor plates acquire charge the current dies away exponentially.

energy

On the other hand,onsider a circuit containing a battery (or EMF $\mathcal{E}$), a resistor R and a capacitor C, and a switch. The capacitor is initially uncharged. Close the switch and what happens? Use the standard rules and go around the circuit looking at the voltage

series and parallel

Everyone has probably seen the equations for resistors.

$$V = IR$$

Put two resistors in series, and each one must carry the full current, but the voltage drop is distributed, part of it over each resistor separately. Hence:

$$\begin{aligned} V_{tot} &= V_1 + V_2 = IR_1 + IR_2 \\ &= I(R_1 + R_2) = IR_e \end{aligned}$$

where R_e is the equivalent resistance for the two resistors together.

$$R_1 + R_2 = R_e$$

In contrast, if the resistors are in parallel, they have the same voltage drop and the current is distributed.

$$I = \frac{V}{R}$$

$$I = I_1 + I_2 = \frac{V}{R_e}$$

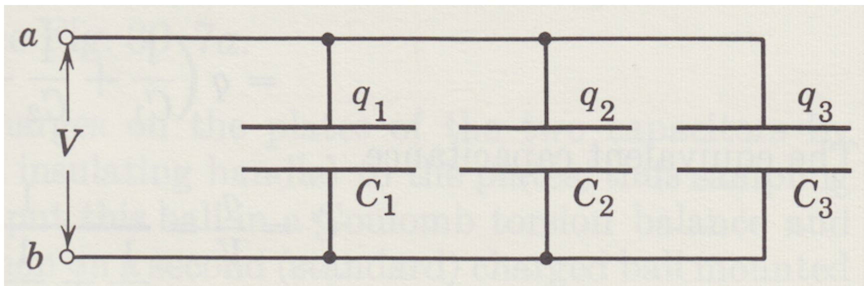
Thus

$$\frac{V}{R_e} = \frac{V}{R_1} + \frac{V}{R_2}$$

$$\frac{1}{R_e} = \frac{1}{R_1} + \frac{1}{R_2}$$

Resistances add in series, and their reciprocals add, in parallel.

Capacitors are just the opposite. Capacitance is additive in parallel, but the reciprocals add in series.



In parallel, both components have the same voltage, but the charge is additive.

$$V = \frac{Q_1}{C_1}$$

$$V = \frac{Q_2}{C_1}$$

$$Q_{tot} = C_e V = C_1 V + C_2 V$$

Thus

$$C_e = C_1 + C_2$$

In series, the charge is the same and the voltages add

$$V_{tot} = \frac{Q}{C_1} + \frac{Q}{C_2} = \frac{Q}{C_e}$$

$$\frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{C_e}$$

AC circuit

If we put a capacitor into an AC circuit, then the charge is

$$q(t) = CVe^{j\omega t}$$

where as before C is a number that describes the capacity of the capacitor (with units of coulombs/volt), and i is renamed to be j because the electrical engineers use i and I for current. Anyway the point is that the current across such a device is

$$i(t) = \frac{dq}{dt} = j\omega CVe^{j\omega t} = j\omega q(t)$$

To solve this differential equation, we need to find a function where

$$i = \frac{dq}{dt} = j\omega q$$

If the voltage goes like the cosine, then this is a problem.

What we are going to do is to write the voltage as

$$V(t) = V_o e^{j\omega t}$$

where $j = \sqrt{-1}$. Then

$$q(t) = CV_o e^{j\omega t}$$

so the current is

$$i(t) = j\omega CV_o e^{j\omega t}$$

so the equivalent of the resistance for the capacitor is

$$\begin{aligned} X_C &= \frac{1}{j\omega C} \\ &= -\frac{j}{\omega C} \end{aligned}$$

What does this mean? It means that when the voltage is at the peak of its cycle, the current through the capacitor is 90 degrees out of phase with it.

Chapter 107

Work, energy and potential

To keep things simple, in this chapter we will use only one dimension. The beauty of orthogonal directions (or *basis vectors* for space) is independence: the net force or displacement or whatever we care about is the sum of what happens in x, plus what happens in y, and the same for z.

The second nice feature is that when two vectors are not parallel, say the force and the distance, then we take the component which is in the same direction using the dot product: $\mathbf{F} \cdot d\mathbf{r} = F \cos \theta \times r$. If we adopt the convention that this has already been done implicitly, then we can just write force times distance or Fd.

The only thing left is to deal with the fact that in some cases the force might not be constant over distance (or time or whatever pieces we're adding up), so we set up the integral $\int F dx$ and add up for each piece the value of F for the corresponding position x_i.

work-energy

Previously, we **derived** this relationship for motion with acceleration a:

$$v^2 - v_0^2 = 2a(x - x_0)$$

starting from the basic equation of motion

$$x = \frac{1}{2}at^2 + v_0 t + x_0$$

and the fact that velocity is the time-derivative of position

$$v = \frac{dx}{dt} = 2a + v_0$$

Newton's second law says that $a = F/m$ so

$$v^2 - v_0^2 = 2a(x - x_0)$$

$$v^2 - v_0^2 = 2\frac{F}{m}(x - x_0)$$

Rearranging

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = F(x - x_0)$$

The term $\frac{1}{2}mv^2$ is called the kinetic energy. The change in kinetic energy is equal to the force, times the distance over which it operates.

$$K - K_0 = \Delta K = Fd$$

If the force is not constant then

$$\Delta K = \int F \ dx$$

We make another definition: Fd is the work done by the force

$$W = Fd = \Delta K$$

Suppose we take our marble back up to the leaning Tower of Pisa and drop it, then the work done by the force of gravity on the marble during its fall through height h is $W = Fd = mgh$, which is equal to the kinetic energy acquired during the fall. If the initial velocity is zero then

$$\frac{1}{2}mv^2 = mgh$$
$$v = \sqrt{2gh}$$

Work has a sign. If the work done is in the direction that the force acts, it's positive.

potential energy

If we consider the marble as it is balanced on the wall at the top of the tower (or for that matter while it is still in my pocket), we say that it has more potential energy than it does before I carry it up, or after its arrival back down at the bottom.

The change in potential energy is exactly equal to the change in kinetic energy.

There is a funny business about the sign. By definition:

$$\Delta K = K - K_0 = \int_{x_0}^{x} F \ dx$$

Suppose the function G is the integral of $F \ dx$ then

$$\Delta K = G(x) - G(x_0) = G - G_0$$

so

$$K - G = K_0 - G_0$$

We don't like those minus signs.

The solution is to define $U = -G$, where U is called the potential energy. Then

$$K + U = K_0 + U_0$$

This is the law of conservation of energy: the kinetic energy K plus the potential energy U is constant.

Going back to G we see that

$$G' = \frac{d}{dx} G = F$$

so

$$-\frac{d}{dx} U = F$$

Minus the derivative of the poential energy is equal to the force.

The vector form of this relationship is

$$-\nabla U = \mathbf{F}$$

visualization

The relationships derived above are quite easy to visualize. In practice there can be confusion about the sign, but the basic idea is simple.

Consider this 550 foot tall "mountain" on the island of Iwo Jima. (It has a special importance for me. The fact that I am here (or was) is due to something that *didn't* happen there but could have, in Feb-Mar 1945).

I found a topographical map on the web

The lines connect positions with equal height. They connection positions of equal potential, and are what Auroux calls level curves.

The lines shown on the map are roughly circular (at least near the bottom), and the smaller the circle the greater the height.

The potential energy U increases as the height increases. The gradient of U, ∇U points perpendicularly to the level curves.

∇U points in the direction of increasing height. If you happen to stumble and fall, however, the direction that gravity pushes you is downhill, and it will be exactly opposite to the direction in which ∇U points. Hence $\mathbf{F} = -\nabla U$.

There are, as we said issues about sign. For example, when work is done by a force that is increasing the potential energy, that force acts in the opposite direction from gravity or from the electric field. Also, in electricity, we must worry about the sign of the charge that is responding to the field.

mass and spring

In the mass and spring system, we find experimentally that

$$F = -kx$$

using the definitions above we find that

$$U = -\int F dx = \frac{1}{2}kx^2$$

and conservation of energy says that

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \text{const}$$

If the mass is pulled to the right for an initial displacement A and then let go, at any time afterward (idealizing with no friction)

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2$$

$$\frac{k}{m}(A^2 - x^2) = v^2$$

$$v = \sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}$$

We **solved** the differential equation of this system

$$F = -kx = ma = m\ddot{x}$$

$$\ddot{x} + \frac{k}{m}x = 0$$

One solution is

$$x(t) = A\cos\omega t$$

$$\ddot{x}(t) = -\omega^2 x(t) = -\frac{k}{m}x(t)$$

so

$$\omega^2 = \frac{k}{m}$$

$$\omega = \pm\ \frac{k}{m}$$

ω is the angular frequency (it is called that because one can view the oscillation as the projection of circular motion onto one dimension).

Going back to the equation for the velocity

$$v = \pm\ \omega\sqrt{A^2 - x^2}$$

If we want the velocity for $x = 0$ it is just $v = \pm\ A\omega$. It is plus or minus depending on the direction of travel.

If the time t is equal to one period $t = T$, then the product ωT is equal to 2π so $\omega = 2\pi/T$. The units are radians per second. One can also write $\omega = 2\pi f$ where f is the frequency.

potential

The last topic for this chapter is potential. Potential is not the same as potential energy. It is rather, that work done by a force, gravitational or electrical or whatever, *on a unit charge or mass.*

Let's start with the force itself. As our example, take the electric force.

The force exerted at a point on a unit test charge is called the electric field $\mathbf{E}$. Let's agree to work in one dimension so that $\mathbf{F} = F$ and $\mathbf{E} = E$ and so on. Then

$$F = qE$$

Now recall from above that the work $W = Fd$, work is the product of force times distance.

Potential is the work with the charge separated out, it is

$$W = qEd = qV$$

$$V = Ed$$

Potential is also called *voltage* and has units of volts.

Electric field strength is defined in terms of newtons per coulomb, which now makes sense. One volt is defined as one joule per coulomb.

Power is the time-derivative of the work

$$P = \frac{d}{dt}W = \frac{d}{dt}\ qEd$$

So, for a constant electric field, since current $i = dq/dt$

$$P = iEd = Vi$$

and if the resistor follows Ohm's Law ($i = V/R$) then

$$P = Vi = i^2R$$

example

A light bulb has a resistance of $R = 240\Omega$. If the circuit is $120V$, then the current will be $i = V/R = 0.5A$, one-half an amp, and the power will be

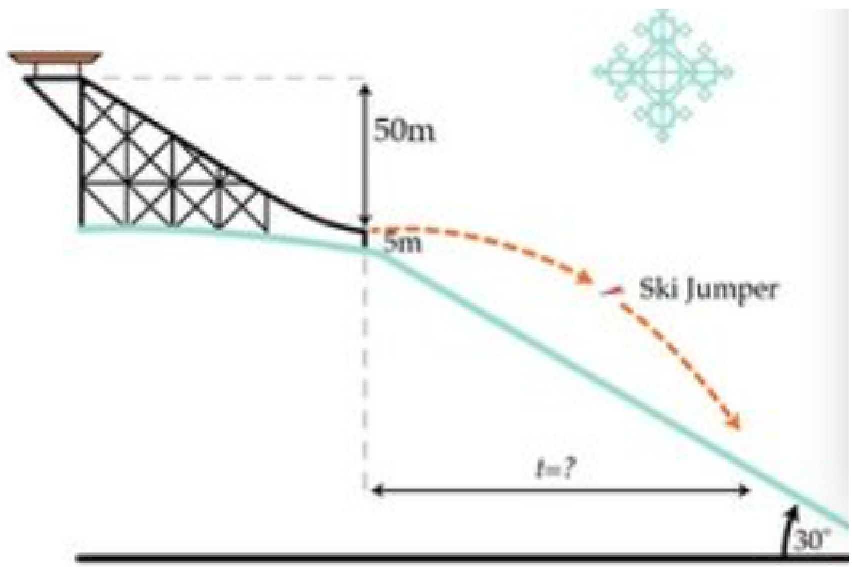
$$P = i^2 R = 60W$$

60 watts.

In my shop I have a tool with a $1 - 1/2$ hp (horsepower) motor. 1 hp $= 746W$. The current it draws is $P/V = 1119W/120V = 9.3A$. That should be OK for a standard 15 amp circuit.

ski jumper

This problem doesn't have any calculus in it but it's still fun.



A ski jumper goes down a ramp of 50 meters height which is the same angle as the slope, 30 degrees. At the very end of the ramp all velocity is converted into horizontal motion by a small lip. We may not neglect friction: the coefficient of kinetic friction $\mu = 0.05$.

Given some horizontal velocity v, the equations of motion (taking y

positive downward) are:

$$x = vt$$

$$y = \frac{1}{2}gt^2 - 5$$

(We take the origin of coordinates to be 5 m below the release point.) Solve for the time when

$$\frac{y}{x} = \tan\theta = \frac{1}{\sqrt{3}}$$

If we can ignore the extra 5 feet, we would have

$$\tan\theta = \frac{g}{2v}t$$

$$t = \frac{2v}{g\sqrt{3}}$$

$$x = v^2\frac{2}{g\sqrt{3}} = \frac{v^2}{g\cos\theta}$$

If we cannot neglect it we have

$$\tan\theta = \frac{g}{2v}t - \frac{5}{vt}$$

$$\frac{1}{2}gt^2 - v\tan\theta\ t - 5 = 0$$

Given v we can solve for t and then compute vt.

first part

The force of gravity is mg, reduced by the factor of $\cos\theta$. The force which determines friction is that pointed perpendicular into the slope, $mg\sin\theta$. Friction opposes gravity along the ramp, giving a net force of

$$mg\cos\theta - \mu mg\sin\theta$$

The work done is the force times the distance, the length of the ramp, which is $h \sin\theta$. We have

$$W = mgh(\cos\theta - \mu \sin\theta) \sin\theta$$

This is equal to the kinetic energy at take-off so

$$\begin{aligned} \frac{1}{2}\, mv^2 &= mgh(\cos\theta - \mu \sin\theta) \sin\theta \\ v^2 &= 2gh(\cos\theta - \mu \sin\theta) \sin\theta \end{aligned}$$

so finally

$$\begin{aligned} x &= \frac{1}{g\cos\theta}\; 2gh(\cos\theta - \mu \sin\theta) \sin\theta \\ &= 2h\; (1 - \mu \tan\theta) \sin\theta \\ &= 100\; (1 - \frac{0.05}{\sqrt{3}})\frac{1}{2} = 50 - \frac{2.5}{\sqrt{3}} \end{aligned}$$

Part XXVI

Theta and r

Chapter 108

Polar coordinates

In polar coordinates points are plotted in terms of distance from the origin, r and the angle θ that this ray makes with the positive x-axis. Converting from x, y to r, θ is pretty easy:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

To go the other way, use Pythagoras to write

$$x^2 + y^2 = r^2$$

$$\theta = \tan^{-1}(\frac{y}{x}), \quad x \neq 0$$

In polar coordinates, as in Cartesian (xy) coordinates, the equation of a circle depends on whether it is at the origin or not. If it is at the origin, then something like

$$r = 3$$

defines the graph. But if it's away from the origin, then the equations are of the form:

$$r = a\cos\theta + b\sin\theta$$

Let's do a derivation and then look at examples. We can manipulate these equations to go back to Cartesian coordinates.

$$r = 2h \cos\theta + 2k \sin\theta$$

The reason for choosing these particular coefficients will become clear shortly. Substitute x and y.

$$r = 2h \cdot \frac{x}{r} + 2k \cdot \frac{y}{r}$$

so

$$r^2 = 2hx + 2ky$$

$$x^2 + y^2 = 2hx + 2ky$$

$$[\ x^2 - 2hx\ ]\ +\ [\ y^2 - 2ky\ ]\ = 0$$

complete both squares

$$[\ x^2 - 2hx + h^2\ ]\ +\ [\ y^2 - 2ky + k^2\ ]\ = h^2 + k^2$$

$$(x-h)^2 + (y-k)^2 = h^2 + k^2$$

That is, for an equation of the form

$$r = 2h \cos\theta + 2k \sin\theta$$

the origin is at (h, k) and the radius is

$$r = \sqrt{h^2 + k^2}$$

If the equation contains only $\sin\theta$ then compute k equal to one-half the coefficient of $\sin\theta$, with the origin at $(0, k)$ and radius $r = k$.

Similarly, if the equation contains only $\cos\theta$ then compute h equal to one-half the coefficient of $\cos\theta$, with the origin at $(h, 0)$ and radius $r = h$.

examples

For example

$$r = 3\sin\theta$$

is a circle centered at $(0, 3/2)$, with a radius of $3/2$ (it passes through the origin and the point $(x = 0, y = 3)$. All such circles (with just one of $\sin\theta$ or $\cos\theta$) have this property.

And

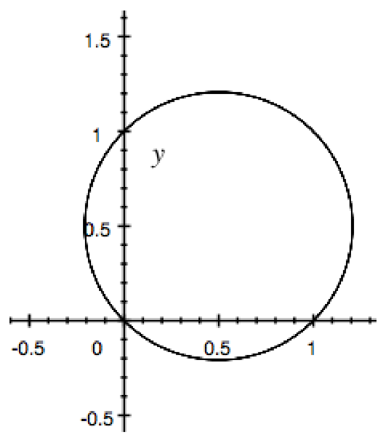
$$r = \sin\theta + \cos\theta$$

is a circle centered at $(1/2, 1/2)$ with a radius squared:

$$r^2 = h^2 + k^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}$$

$$r = \frac{1}{\sqrt{2}}$$

We see that all circles of this form pass through the origin.



other conic sections

Parabolas look like this:

$$r = \frac{1}{1 \pm a \sin\theta}$$
$$r = \frac{1}{1 \pm a \cos\theta}$$

The ones with $\sin\theta$ open up and down, the others open left and right. If the sign of the a term is negative, the parabola opens up or to the right.

The general formulas are

$$r = \frac{ep}{1 \pm e \sin\theta} = p \, \frac{1}{1/e + \sin\theta}$$
$$r = \frac{ep}{1 \pm e \cos\theta} = p \, \frac{1}{1/e + \cos\theta}$$

where e is the eccentricity ($e = 1$ for a parabola). If $e < 1$ then we have an **ellipse**.

parabola

Consider

$$r = \frac{2}{1 + \sin\theta}$$

Plot it to see. Or convert to Cartesian coordinates:

$$r + r\sin\theta = 2$$
$$\frac{y}{r} = \sin\theta$$
$$r + y = 2$$
$$r^2 = (2 - y)^2 = x^2 + y^2$$
$$4 - 4y + y^2 = x^2 + y^2$$
$$y - 1 = -\frac{1}{4}x^2$$

ellipse

If we measure r and θ from a focus, then

$$x = c + r\cos\theta$$

$$y = r\sin\theta$$

one can derive a formula for the ellipse in terms of r, θ

$$r = \frac{ep}{1 \pm e\cos\theta}$$

(We talk more about this **here**). For example if

$$r = \frac{1}{1 + e\cos\theta}$$

with $0 < e < 1$, we will get an ellipse.

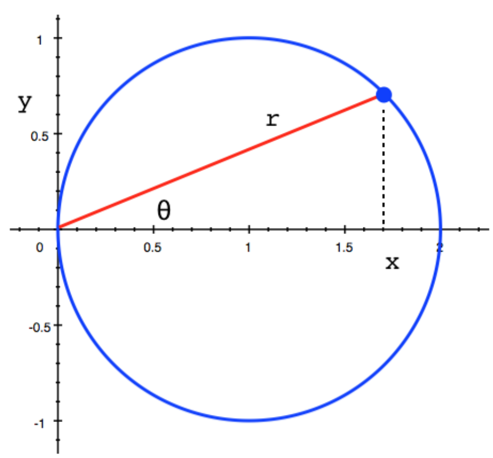
Chapter 109

Polar conics

circle

A very simple circle in polar coordinates is $r = a$. There is no θ-dependence when the circle has its center at the origin.

For a circle of radius a centered at $(a, 0)$ then



$$a^2 = (x - a)^2 + y^2$$
$$x^2 - 2ax + y^2 = 0$$

Always, $x = r\cos\theta$ and $y = r\sin\theta$ so

$$r^2(\sin^2\theta + \cos^2\theta) - 2ar\cos\theta = 0$$
$$r^2 - 2ar\cos\theta = 0$$
$$r = 2a\cos\theta$$

If the center of the circle is on the y-axis the equation is similar but with $\sin\theta$. A more general equation is

$$r = 2h\cos\theta + 2k\sin\theta$$

which is a circle that touches the origin, and has its center at (h, k).

The most general equation is with the circle anywhere in the plane. If we remember to specify the center at (s, ϕ) in *radial* coordinates, then the law of cosines easily yields

$$r^2 + s^2 - 2rs\cos(\theta - \phi) = a^2$$

reverse

Start from

$$r = 2h\cos\theta + 2k\sin\theta$$

Always, $x = r\cos\theta$ and $y = r\sin\theta$ so

$$r = 2h\frac{x}{r} + 2k\frac{y}{r}$$
$$r^2 = 2hx + 2ky$$
$$x^2 + y^2 = 2hx + 2ky$$

Easily rearrange and complete the square:

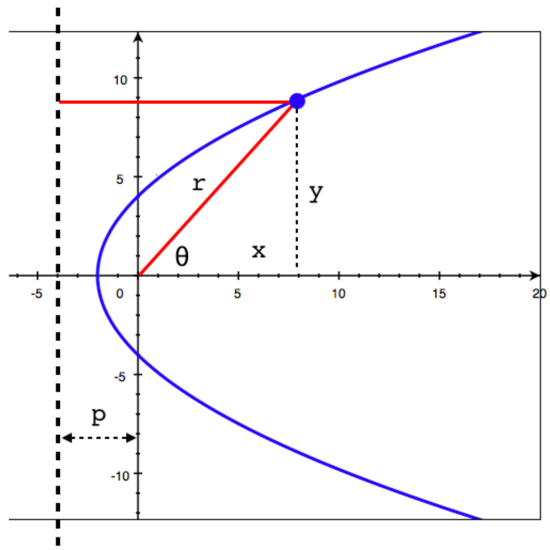
$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = h^2 + k^2$$
$$(x-h)^2 + (y-k)^2 = h^2 + k^2$$

For a circle touching the origin, $h^2 + k^2 = a^2$

$$(x-h)^2 + (y-k)^2 = a^2$$

parabola

To derive the equation for a parabola in polar coordinates it is convenient to rotate from the standard orientation by 90 degrees CW. In this way θ will have its usual relationship with the x-axis.



The origin of coordinates is placed at the focus, the distance to the vertex for this parabola is 2, and the distance from the origin to the directrix is $p = 4$.

Note that in Cartesian coordinates, this parabola will be of the form is $x = ay^2$ because of the rotation.

The distance from the focus to a general point (x, y) is just r. The distance from the directrix to the point is $p+x$. The geometric constraint gives simply:

$$r = p + x$$

We make the standard substitution $x = r\cos\theta$.

$$r = p + r\cos\theta$$

Some rearrangement gives the standard equation

$$r = \frac{p}{1 - \cos\theta}$$

For a vertically oriented parabola we would have $\sin\theta$ instead.

reverse

To go back to Cartesian coordinates, reverse the substitution for x:

$$r = \frac{p}{1 - x/r}$$

$$r - x = p$$

$$r^2 = (x + p)^2$$

Use $r^2 = x^2 + y^2$:

$$x^2 + y^2 = x^2 + 2px + p^2$$

$$y^2 = 2px + p^2$$

$$\frac{1}{2p}y^2 = x + \frac{p}{2}$$

This looks unusual. However, the equation that was actually plotted was $r = 4/(1 + \cos\theta)$ $(p = 4)$.

Note: here we have used p as the distance from the focus to the directrix, which is twice the distance to the vertex. If we call the latter distance c, the $p = 2c$. Previously we showed that $4ac = 1$, so $a = 1/4c$. Thus we obtain $a = 1/8$:

$$\frac{1}{8}y^2 = x + 2$$

This shape factor matches the plot (four units above the axis (at $x = 0$ is two units to the right of the vertex) and the vertex is at $(-2, 0)$. a is unusually small, the reason is so the parabola will open quickly, giving room to put all the labels in the diagram.

ellipse

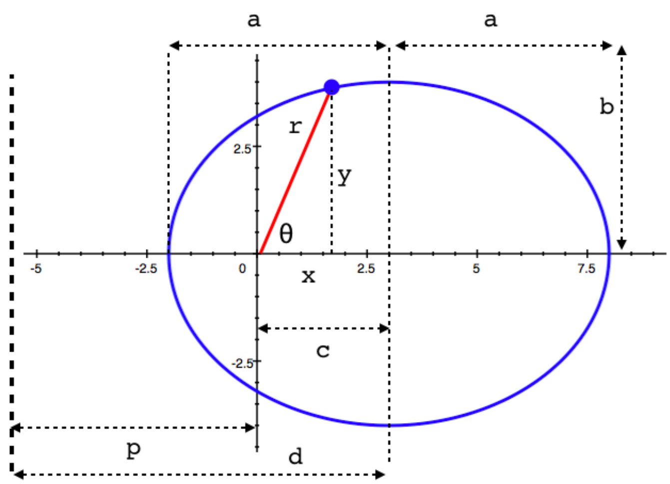
From the geometry of the ellipse, with the center at the origin, it is fairly easy to show that

$$a^2 = b^2 + c^2$$

and derive the equation of the ellipse in Cartesian coordinates:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

To derive the equation for an ellipse in polar coordinates it is convenient to shift the origin of coordinates to be the left focus of the ellipse at $(-c, 0)$.



The ellipse plotted here has $a = 5, b = 4$ and so $c = \sqrt{a^2 - b^2} = 3$. It has been shifted so the focus at $(-3, 0)$ is the origin of coordinates.

The eccentricity e is defined by the geometric constraint (next), which can be shown to be equivalent to $c/a = 0.6$.

Let p be the distance from the focus to the directrix, and let d be the distance from the directrix to the center of the ellipse.

The ellipse can be defined by its **geometric constraint**.

This says that for any point on the ellipse, the ratio of the distance from the focus (and here, the origin) to the point (that is, r), when divided by the distance from the point to the directrix, $x+p$, is equal to a constant, which we will call the eccentricity e.

$$\frac{r}{x+p} = \frac{r}{p + r\cos\theta} = e$$

$$r = e(p + r\cos\theta)$$

We simply rearrange to isolate r

$$r(1 - e\cos\theta) = ep$$

$$r = \frac{ep}{1 - e\cos\theta}$$

reverse

Going back is more complicated for the ellipse. Reverse the substitution $x/r = \cos\theta$.

$$r(1 - ex/r) = ep$$

$$r - ex = ep$$

$$r = ex + ep$$

There's a *magic* substitution that we will justify below:

$$ep = a(1 - e^2)$$

Using that, we have

$$r = ex + a(1 - e^2)$$

Use $r^2 = x^2 + y^2$:

$$x^2 + y^2 = e^2x^2 + 2exa(1 - e^2) + a^2(1 - e^2)^2$$

Combine cofactors for x^2, obtaining $(1-e^2)$ and then divide through by $(1-e^2)$:

$$x^2 + \frac{y^2}{1-e^2} = 2exa + a^2(1-e^2)$$

Complete the square for x by adding $(ea)^2$ to both sides

$$x^2 - 2exa + (ea)^2 + \frac{y^2}{1-e^2} = a^2(1-e^2) + (ea)^2$$

$$(x-ea)^2 + \frac{y^2}{1-e^2} = a^2(1-e^2) + (ea)^2$$

We asserted that $ea = c$. Simplify the right-hand side at the same time:

$$(x-c)^2 + \frac{y^2}{1-e^2} = a^2$$

This is great, because we need to shift the origin of coordinates back to the center of the ellipse by exactly this amount.

Unfortunately, I have not discovered any way to make that derivation simpler.

solve for $1-e^2$

To deal with $1-e^2$, recall that the basic geometry says

$$a^2 - c^2 = b^2$$

$$1 - (\frac{c}{a})^2 = \frac{b^2}{a^2}$$

Since $c = ea$

$$1 - e^2 = \frac{b^2}{a^2}$$

so what we had simplifies as the inverse of that times y^2

$$(x-c)^2 + \frac{a^2}{b^2}\, y^2 = a^2$$

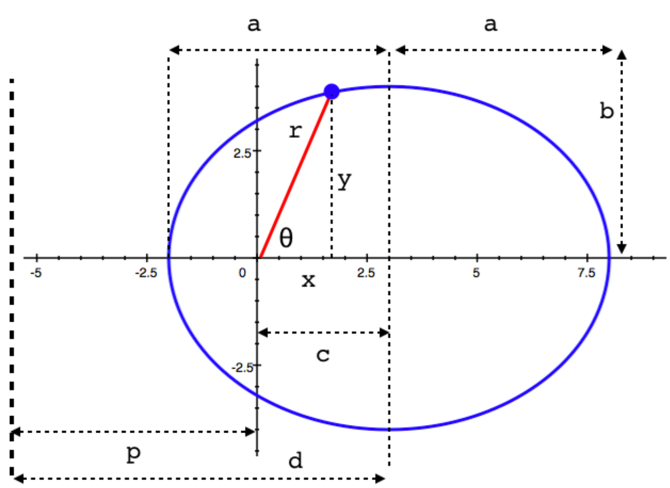
$$\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

Exactly what we want. Furthermore, we can view this derivation in reverse as a proof of $c = ea$ for an ellipse with this equation in Cartesian coordinates, shifted out to focus $(-c, 0)$.

Now let's explain the substitution:

$$ep = a(1-e^2)$$

It is easiest to start by finding an expression for d, then getting e and p.



solve for d and e

Applying the geometric constraint to the extreme left end:

$$\frac{a-c}{d-a} = e$$

At the very top of the ellipse the distance to the focus is $\sqrt{b^2+c^2}$ but this is also just a, which means

$$\frac{a}{d} = e = \frac{a-c}{d-a}$$

so

$$ad - a^2 = ad - cd$$

thus

$$d = \frac{a^2}{c}$$

One can also obtain this result by equating the ratios for the left and right ends of the ellipse.

Notice that the ratio a/d obeys the geometric constraint:

$$\frac{a}{d} = e = \frac{ac}{a^2} = \frac{c}{a}$$

We have proved $ae = c$, using only the geometry.

A longer, but pretty, proof is to start from the ratio for the extreme right end:

$$e = \frac{a+c}{d+a}$$

substitute $d = a^2/c$

$$= \frac{a+c}{a^2/c + a}$$

Multiply top and bottom by $1/a$

$$e = \frac{1 + c/a}{a/c + 1}$$

Put top and bottom over common denominators

$$e = \frac{(a+c)/a}{(a+c)/c} = \frac{c}{a}$$

solve for p

$$p = d - c = \frac{a^2}{c} - c$$
$$= \frac{a^2 - c^2}{c} = \frac{b^2}{c}$$

finally

$$pe = \frac{b^2}{c} \cdot \frac{c}{a} = \frac{b^2}{a}$$
$$= \frac{a^2 - c^2}{a}$$
$$= \frac{a^2 - e^2 a^2}{a}$$
$$= a(1 - e^2)$$

which is the special substitution we used.

summary of the summary

The circle (touching the origin), parabola (rotated to the right), and the ellipse are, in order:

$$r = 2h\cos\theta + 2k\sin\theta$$
$$r = \frac{p}{1 - \cos\theta}$$
$$r = \frac{ep}{1 - e\cos\theta}$$

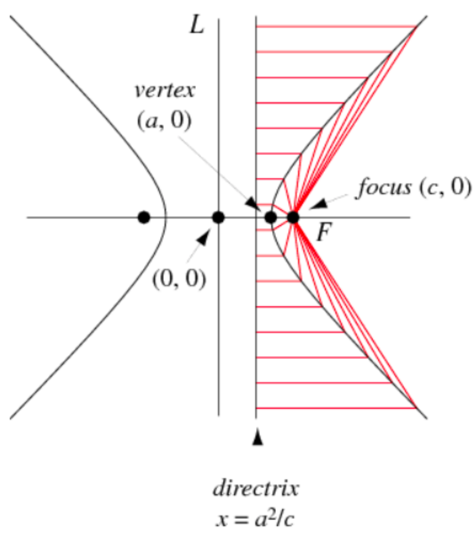
In the last case, note that $0 < e < 1$, so the parabola is the same but with $e = 1$.

Chapter 110

Polar hyperbola

We can also use a focus-directrix approach to the definition of the hyperbola. It will turn out to give the same formula as that of the ellipse and the parabola, but there are a few twists and turns along the way.

http://mathworld.wolfram.com/Hyperbola.html



The equation will turn out to be similar to that for the ellipse and parabola

$$r = \frac{ep}{1 - e\cos\theta}$$

except that $e > 1$, as we will see.

First, though, recall the equation that we derived before, namely

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

meaning of a and b

This gives a hyperbola of the type graphed above, opening "east-west". As we used in the derivation, c is the distance from the origin to each focus. Although the value of x is never 0, the value of y can be, and at those two points we have

$$x^2 = a^2$$

$$x = \pm a$$

So a is the horizontal distance to points on the curve along the x-axis.

As for b, we rearrange the basic equation

$$b^2x^2 - a^2y^2 = a^2b^2$$

asymptotes

Associated with the hyperbola is a pair of lines called asymptotes. Their equation is

$$b^2x^2 - a^2y^2 = 0$$

Factoring

$$(bx + ay)(bx - ay) =$$

which has solutions when

$$y = \pm \frac{b}{a} x$$

For a hyperbola in standard orientation like this, these lines go through the origin.

eccentricity

Notice that, unlike the ellipse, $a < c$. We define the eccentricity with the same equation as for the ellipse

$$ea = c$$

but now realize that for a hyperbola $e > 1$.

Recall that we defined

$$b^2 = c^2 - a^2, \qquad (c > a)$$

Hence

$$b^2 = (ea)^2 - a^2 = a^2(e^2 - 1)$$

whereas before for the ellipse we had

$$b^2 = a^2(1 - e^2)$$

directrix

For the directrix we will assume the answer, and then show that it leads to the desired properties. From the diagram above we read that on the directrix

$$x = \frac{a^2}{c}$$

That is, the distance d from the y-axis is equal to

$$d = \frac{a^2}{c}$$

and since $c = ea$

$$d = \frac{a}{e}$$

which is just what we had before with the ellipse:

$$ea = c$$

$$ed = a$$

(except that now with the hyperbola $e > 1$ and $c > a > d$ whereas before with the ellipse $e < 1$ and $d > a > c$).

On the x-axis the distance from the focus to the curve is $c - a$ and from the curve to the directrix is $a - d$. We consider the ratio of the two distances

$$\begin{aligned} \frac{c-a}{a-d} &= \frac{ea-a}{a-a/e} \\ &= \frac{a(e-1)}{a(1-1/e)} \\ &= \frac{(e-1)}{(1-1/e)} \end{aligned}$$

Multiply by e on top and bottom:

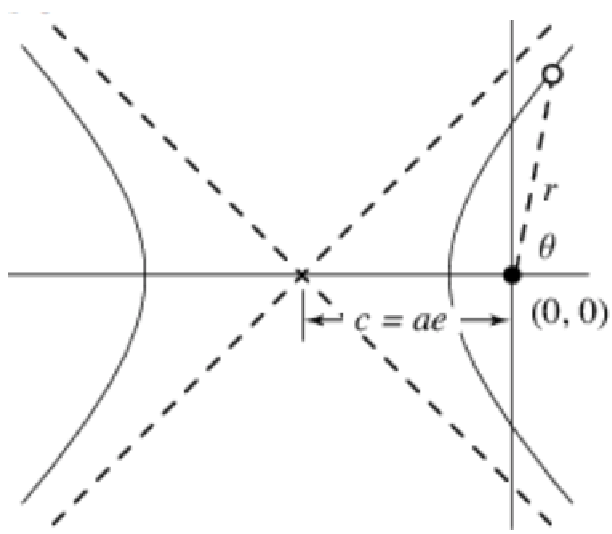
$$\begin{aligned} &= e\frac{(e-1)}{(e-1)} \\ &= e \end{aligned}$$

definition of p

As before, in a similar way to the directrix $d = a^2/c$, we define the focal parameter p as

$$\begin{aligned} p &= \frac{b^2}{c} \\ &= \frac{c^2 - a^2}{c} \\ &= \frac{e^2a^2 - a^2}{c} \\ &= \frac{a^2(e^2 - 1)}{c} \\ &= \frac{a(e^2 - 1)}{e} \end{aligned}$$

polar coordinates



The geometric constraint is

$$\frac{PF}{PD} = e$$

where PF is just r and the problem is to evaluate the length of PD.

$$PD = r\cos\theta + (c - d)$$

Hence we have that

$$e = \frac{r}{r\cos\theta + (c - d)}$$

$$er\cos\theta + e(c - d) = r$$

$$r(e\cos\theta - 1) = -e(c - d)$$

$$r = \frac{e(c - d)}{1 - e\cos\theta}$$

The numerator

$$e(c - d) = e(ea - \frac{a}{e})$$

$$a(e^2 - 1)$$

$$= ep$$

So finally we obtain:

$$r = \frac{ep}{1 - e\cos\theta}$$

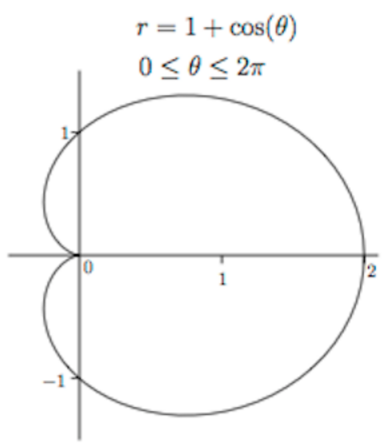
as the equation of a standard hyperbola in polar coordinates.

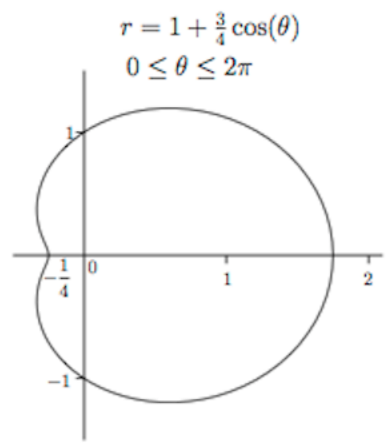
We recovered the same equation for all three conic sections: parabola, ellipse and hyperbola. The only difference is the value of e. Here $e > 1$, for the parabola $e = 1$ and for the ellipse $e < 1$.

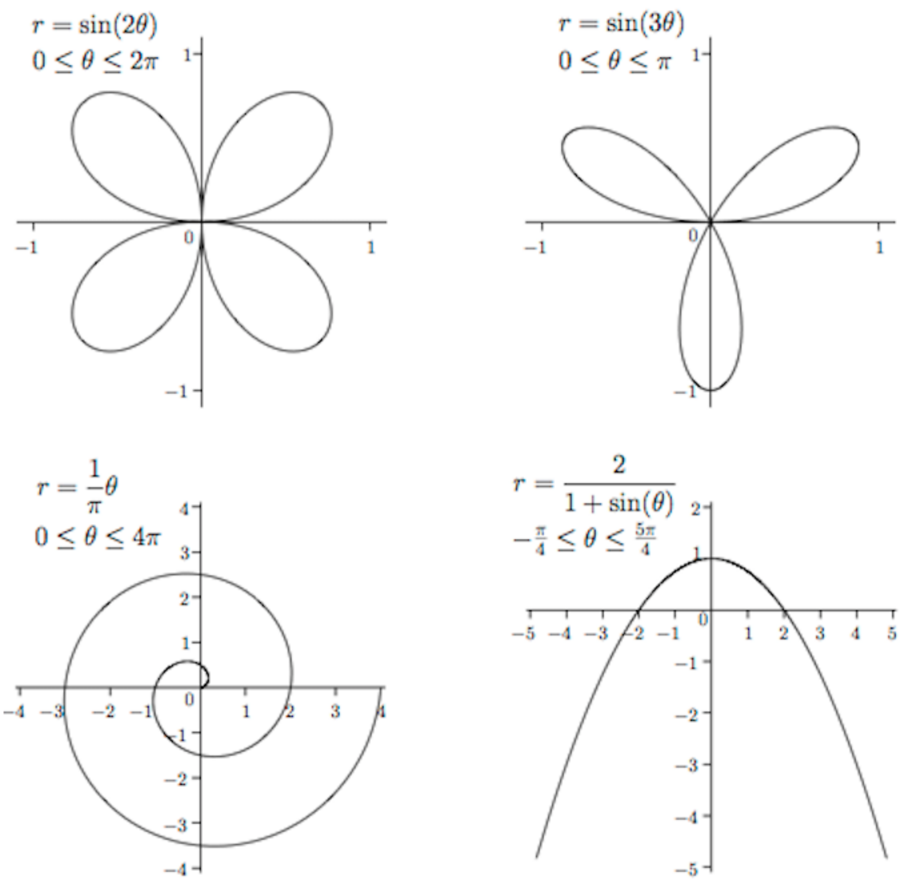
Chapter 111

Polar area

Here are some fancy examples of polar curves from *The Calculus Lifesaver*.







Integration to find areas

The idea for (one-dimensional) integration in polar coordinates is that we know r as a function of θ. For example, we had the circle centered at $(0, 3/2)$ given by

$$r = 3\sin\theta$$

We imagine dividing up the circle into little triangles, sectors where

$$\theta \to \theta + \Delta\theta$$

The sector is approximately a triangle with side r and base $r \times \Delta\theta$ (the latter is the length of the arc of the circle on its circumference).

The area of each little sector is

$$\frac{1}{2} r^2 d\theta$$

example

For this problem:

$$r = 3\sin\theta$$

The total area is then

$$\int_0^{2\pi} \frac{1}{2} r^2 d\theta$$
$$= \int_0^{2\pi} \frac{1}{2} (3\sin\theta)^2 d\theta$$
$$= \frac{9}{2} \int_0^{2\pi} \sin^2\theta d\theta$$

This looks hard but we've done it before. One way is to recall that

$$[\ \sin\theta\cos\theta\ ]' = -\sin^2\theta + \cos^2\theta$$
$$= 1 - 2\sin^2\theta$$

Integrate

$$\int [\ \sin\theta\cos\theta\ ]'\ d\theta = \int (1 - 2\sin^2\theta)\ d\theta$$

$$\sin\theta\cos\theta = \theta - 2\int \sin^2\theta \ d\theta$$

Hence

$$\int \sin^2\theta \ d\theta = \frac{1}{2}(\theta - \sin\theta\cos\theta)$$

So our answer is

$$= (\frac{9}{2}) \ \frac{1}{2}(x - \sin\theta\cos\theta) \ \Big|_0^{2\pi}$$

$$= (\frac{9}{2}) \ \frac{1}{2}(2\pi)$$

$$= \frac{9\pi}{4}$$

which is correct for a circle with radius 3/2.

example

The second example is from *How to ace the rest of calculus*. We have two circles, both of radius 1. The first one is centered at the origin. We are given the equation of the second in polar coordinates:

$$r = 2\cos\theta$$

Plugging in some values for θ: and calculating r:

$$\theta = 0 \to r = 2$$

$$\theta = \frac{\pi}{6} \to r = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{4} \to r = \sqrt{2}$$

$$\theta = \frac{\pi}{3} \rightarrow r = 1$$
$$\theta = \frac{\pi}{2} \rightarrow r = 0$$

We can also convert to x, y-coordinates. Multiply by r:

$$r^2 = 2r\cos\theta$$

Substituting $r^2 = x^2 + y^2$ and $x = r\cos\theta$:

$$x^2 + y^2 = 2x$$

Complete the square:

$$(x^2 - 2x + 1) + y^2 = 1$$
$$(x - 1)^2 + y^2 = 1$$

The second circle is centered at $(1, 0)$. Note that for this circle it is *not* true that $x^2 + y^2 = 1$.

Now, the problem given is to calculate the shaded area in the figure.

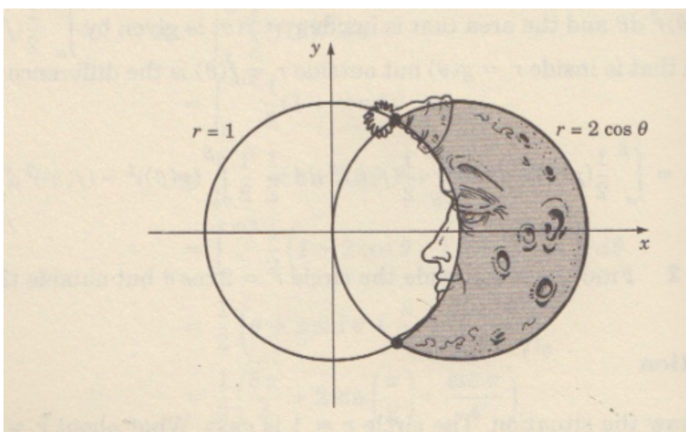


Figure 3.13 Area inside $r = 2\cos\theta$ but outside $r = 1$.

First, we must find the value of θ at the points of intersection between the two circles. We solve the two equations simultaneously:

$$y^2 = 1 - x^2$$

$$(x-1)^2 + y^2 = (x-1)^2 + 1 - x^2 = 1$$

$$-2x + 2 = 1$$

$$2x = 1$$

$$x = \frac{1}{2}$$

$$y = \sqrt{1 - x^2} = \pm\frac{\sqrt{3}}{2}$$

$$\theta = \tan^{-1}\frac{y}{x} = \pm\sqrt{3}$$

Look it up:

$$\theta = \pm\frac{\pi}{3}$$

or notice that we are on the unit circle so $\cos\theta = x = 1/2$, $\theta = \pm\pi/3$. That's the hard way. The easy way is

$$r = 1 = 2\cos\theta$$

$$\theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$$

The area of an arc of the unit circle is the r^2 times one-half the arc length in radians.

$$A = \frac{1}{2}\int r^2 \, d\theta$$

We will subtract the area of the inner arc from that covered by the outer one

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta)^2 - 1 \, d\theta$$

Recall that

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - 1 + \cos^2\theta$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

so

$$(2\cos\theta)^2 = 4\ \frac{1}{2}(1 + \cos 2\theta) = 2(1 + \cos 2\theta)$$

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta)^2 - 1\ d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 2\cos 2\theta + 1\ d\theta$$

$$= \frac{1}{2}\ [\ \sin 2\theta + \theta\ ]\ \Bigg|_{-\pi/3}^{\pi/3}$$

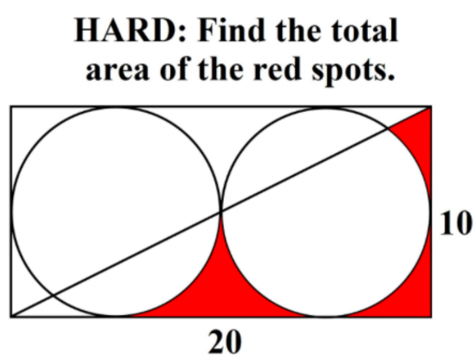
Since $\sin 2\pi/3 = \sqrt{3}/2$:

$$= \frac{1}{2}(\sqrt{3} + \frac{2\pi}{3}) = \frac{\sqrt{3}}{2} + \frac{\pi}{3}$$

Chapter 112

Circular segment

I found a hard geometry problem on the web:



We know it's hard because it says so!

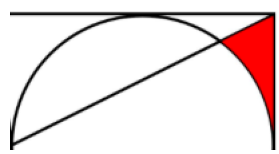
I liked it particularly because there are at least four different ways to calculate the answer using basic geometry and trigonometry, plus standard integration as well as polar integration. I got the same answer each time, so I have increased confidence in the result.

It is included here because it's a fun problem, and one of the methods was polar integration to find the area.

As a first step, we observe that the problem has been made artificially complicated by using these particular values for the side lengths. If both dimensions are scaled down by a factor of 5, then we obtain a rectangle with side lengths 4 and 2. The two circles become unit circles. We must just remember to re-scale to obtain the final answer, multiplying areas by 25.

The area of any arched corner segment is pretty easy, since 4 of them put together are equal to the difference between the area of a square with side length 2 and a unit circle: $4 - \pi$, so each one is $1 - \pi/4$.

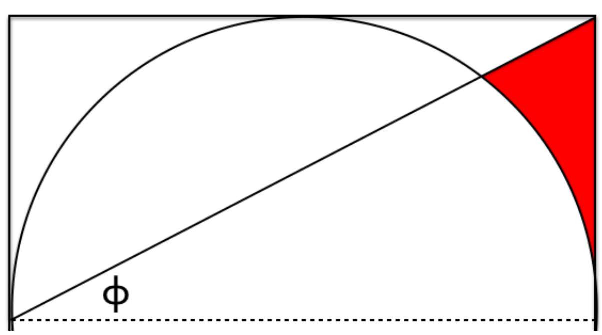
The real difficulty is the upper right-hand corner.



One of the arches is divided into two pieces, and we are supposed to only count the red part of the arch.

The basic right triangle that we see repeated in these images has side lengths in the ratio 1 : 2. Its area is just 1, and the smaller angle is

$$\phi = \tan^{-1} 0.5 \approx 0.4636 \text{ rad } \approx 26.565 \text{ deg}$$



That's not a nice round number, but OK.

My first thought was to calculate the area cut off by the chord of a circle, called a "circular segment". Then we could calculate the white part of the divided arch:

$$\text{triangle} \ - \ \text{segment} \ - \ \text{arch}$$

$$1 - \ \text{segment} \ - (1 - \pi/4)$$

and subtract that from one whole arch to get the red part.

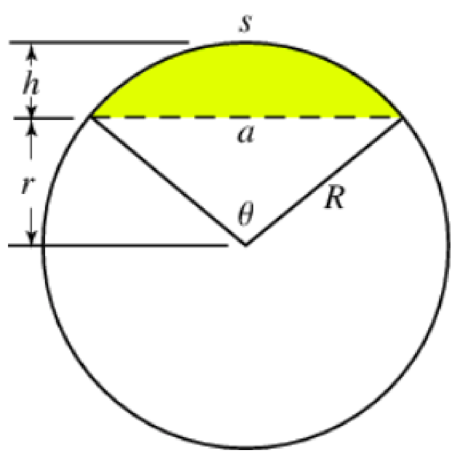
$$(1 - \pi/4) - \ [\ 1 - \ \text{segment} \ - (1 - \pi/4)\ ]$$

$$1 - \frac{\pi}{2} + \ \text{segment}$$

We will use this result at the end for our final answer.

There is an easier way, which we find by exploring this direction just a little further.

A circular segment is like a polar cap, but in two dimensions.



http://mathworld.wolfram.com/CircularSegment.html

We carefully distinguish between the circular segment, in yellow, and the circular sector, which is the area of that slice of the circular pie swept out by the angle θ.

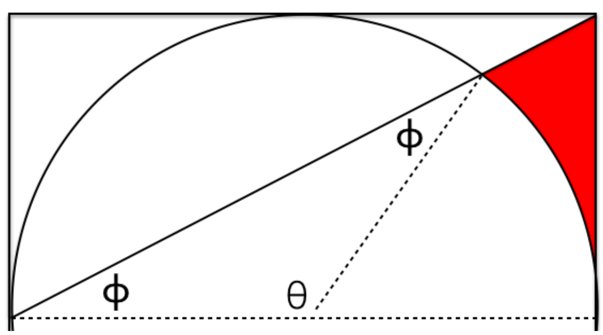
The area of the circular sector with central angle θ is the fraction of the total circular angle, times the area of a unit circle. The result is just half the angle.

$$\frac{\theta}{2\pi} \cdot \pi = \frac{\theta}{2}$$

For the actual calculation of a circular segment, we would need not only the angle θ, but also r and a, which we would need to derive from θ by applying the Pythagorean theorem and/or trigonometry. It can be done! However, we see a better way.

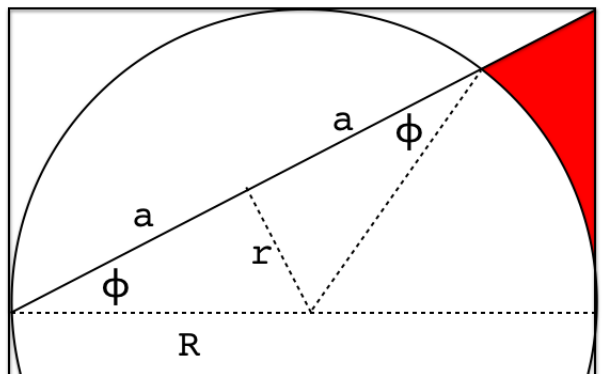
isosceles triangles

The first key idea to draw θ on our diagram, and realize that θ is the apex angle of an isosceles triangle. The two sides are both radii of our circle, and so are equal to each other! Therefore $\theta = \pi - 2\phi$.



So now we just calculate the circular sector swept out by θ and figure out the area of the isosceles triangle, subtract to find the spherical segment and go on from there. The beauty of this is we do not need to know the formulas for circular segments..

Let's continue by finding the lengths and areas of parts of the central isosceles triangle with angles ϕ-ϕ-θ.



The triangle area is easy because one-half of it is a triangle similar to the original. This means that $r/a = \tan\phi = 0.5$ so $a/r = 2$ and

$$a = 2r$$

Furthermore

$$r = R\sin\phi$$

$$a = R\cos\phi$$

The area of the entire isosceles triangle is two of these smaller ones:

$$A = ar = R^2\sin\phi\cos\phi$$

We can also do it solely in terms of r

$$A = 2r^2$$

$$= 2R^2\sin^2\phi$$

For this unit circle

$$A = 2\sin^2\phi$$

This works because

$$\cos\phi = 2\sin\phi$$

so

$$\sin\phi\cos\phi = 2\sin^2\phi$$

Because of the square we get an exact answer:

$$\sin\phi = \frac{1}{\sqrt{5}}$$

$$\sin^2\phi = \frac{1}{5}$$

$$A = 2\sin^2\phi = 0.4$$

Later, we will have a use for h:

$$h = R - r = R(1 - \sin\phi)$$

$$= 1 - \sin\phi$$

We could now move on to consideration of the circular sector with angle θ. But wait!

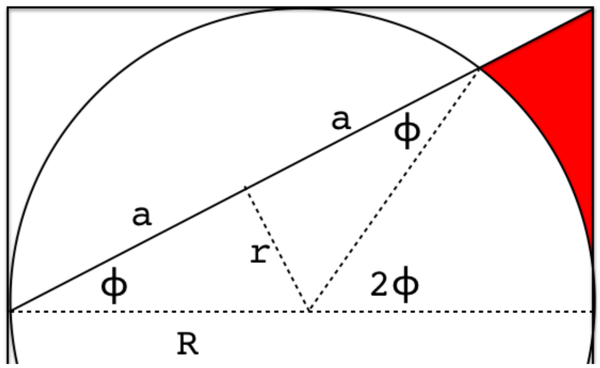
the other triangle

Notice at this point a different circular sector, just to the right of the isosceles triangle. Several different familiar theorems gives the angle as 2ϕ.

∘ $\theta + \phi + \phi = \pi$ but at the same time θ plus the unknown angle also equals π. Thus, the angle is 2ϕ.

∘ ϕ and the unknown angle both sweep out the same arc on the circle, but the unknown angle is at the origin of the circle, and ϕ is on the perimeter. A famous theorem says the the unknown angle is twice ϕ. The same argument in reverse is a *proof* of the theorem.

○ Draw a horizontal line (not shown) intersecting the angled line and the perimeter of the circle, at the upper-right, just above and to the right of the letter ϕ. The angle between the new horizontal and the angled line is ϕ, by another famous theorem (interior angles ...), and thus the unknown angle is twice ϕ, by that same famous theorem.



The area of the sector is, by the same calculation we did before, the ratio of the angle to 2π times the area of the unit circle.

$$\frac{2\phi}{2\pi}\,\pi = \phi$$

calculation 1: ignore the circular segment

So far we have that the area of the isosceles triangle is $2\sin^2\phi = \sin\phi\cos\phi$.

The circular sector with angle 2ϕ has area ϕ.

The part of the large lower triangle that is not in red is

$$\text{circular sector } + \text{ isosceles triangle}$$

$$\phi + 2\sin^2\phi$$

Then what we want is to subtract that from the large triangle:

$$A = 1 - \phi - 2\sin^2\phi$$

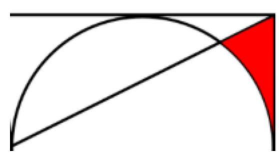
$$= 1 - \approx 0.46 - 0.4$$

I get ≈ 0.14, which seems reasonable.

Let's try to remember that: the red part of the arch is $1 - \phi - 2\sin^2\phi$.

calculation two: polar area

We can also use calculus, namely to compute a polar area. Look again at



What if we could get the area of the lower triangle minus the red part?

This is a pretty easy integral in polar coordinates. Since the angle is usually given as θ, for the moment we relabel ϕ as θ. We also reuse the variable a.

Set up a circle of radius a with its left edge at the origin, the equation of that circle in polar coordinates is

$$r = 2a\cos\theta$$

For example, radius $a = 1$ gives

$$\text{at } \theta = 0, \qquad r = 2$$

$$\text{at } \theta = \frac{\pi}{4}, \qquad r = \sqrt{2}$$

$$\text{at } \theta = \frac{\pi}{2}, \qquad r = 0$$

The arc of the upper semi-circle that we're seeing is $\theta = 0 \rightarrow \frac{\pi}{2}$

$$\text{at } \theta = \phi, \qquad r = 2\cos\phi$$

Since $\cos\phi = 2/\sqrt{5}$, $r = 4/\sqrt{5}$.

You don't believe me that this is correct? We have four equations. Always:

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Do not cancel the $r = 1$ yet.

We also have the equation of this circle. The general equation is

$$r = 2h\cos\theta + 2k\sin\theta$$

where (h, k) is the origin of the circle. Here, the origin is at $h = 1$ so

$$r = 2\cos\theta$$

Now comes the magic. Substitute for $\cos\theta$:

$$r = 2\frac{x}{r}$$

$$r^2 = 2x$$

but

$$r^2 = x^2 + y^2$$

We obtain

$$x^2 + y^2 = 2x$$

$$x^2 - 2x + y^2 = 0$$

Complete the square and add the same term on the right

$$(x-1)^2 + y^2 = 1$$

This is indeed a unit circle with origin at $(1,0)$.

polar area

The area is made up of many small wedges with angle θ, and $r = f(\theta) = 2\cos\theta$. The wedges are approximately triangles with area $1/2 \cdot r \cdot r\ d\theta$. So the total area is

$$A = \int \frac{1}{2}\ [\ f(\theta)\ ]^2\ d\theta$$

$$= \frac{1}{2}\ 4 \int \cos^2\theta\ d\theta$$

$$= 2\ [\ \frac{1}{2}(\theta + \sin\theta\cos\theta)\ ]\ \Big|_0^\phi$$

At the lower bound, this is zero so

$$A = \phi + \sin\phi\cos\phi$$

For this ϕ we have that:

$$A = \phi + 2\sin^2\phi$$

This should be equal to the area of the circular sector of angle 2ϕ plus the area of the isosceles triangle, and if you look back, you'll see that we have a match.

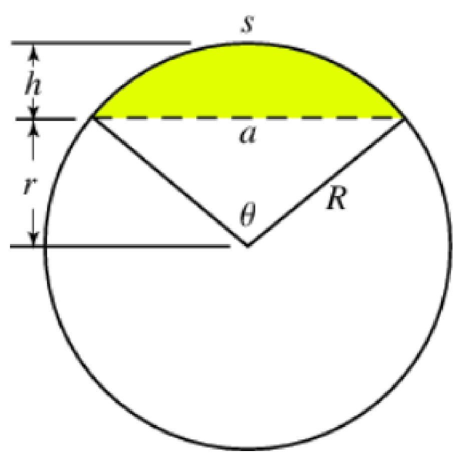
To get the red arch, we must subtract this from the triangle, which has unit area.

$$A = 1 - \phi - 2\sin^2\phi$$

That matches what we had before. To finish up, we do this two more ways, both requiring the area of the circular segment.

calculation 3: standard calculus

We find the circular segment by standard integral calculus:



Recall that the area of (the upper half of) a circle drawn in standard orientation (x horizontal) is

$$\int \sqrt{R^2 - x^2}\ dx$$

The bounds we need are $R - h$ to R.

Here we have rotated a quarter-turn and are integrating vertically, so we'll use y as the variable. The area of the half-circle to the right of the y-axis is

$$\int_{R-h}^{R} \sqrt{R^2 - y^2}\ dy$$

This is a unit circle so

$$A = \int_{1-h}^{1} \sqrt{1 - y^2}\ dy$$

The answer (done many times by now)

$$\frac{1}{2}\ [\ \sin^{-1} y + y\sqrt{1-y^2}\ ]\ \Big|_{1-h}^{1}$$

At the upper bound we get $\pi/4$. For the lower bound we need h

$$h = 1 - \sin\phi$$

so that bound is just $\sin\phi$. The first term in parentheses is

$$\sin^{-1}(\sin\phi)$$

The angle whose sine is $\sin\phi$ is just ϕ!

The second term is

$$(\sin\phi)\sqrt{1-(\sin\phi)^2}$$
$$= \sin\phi\cos\phi$$

Altogether, we have

$$\frac{\pi}{4} - \frac{1}{2}(\phi + \sin\phi\cos\phi)$$

$$= \frac{\pi}{4} - \frac{\phi}{2} - \sin^2\phi$$

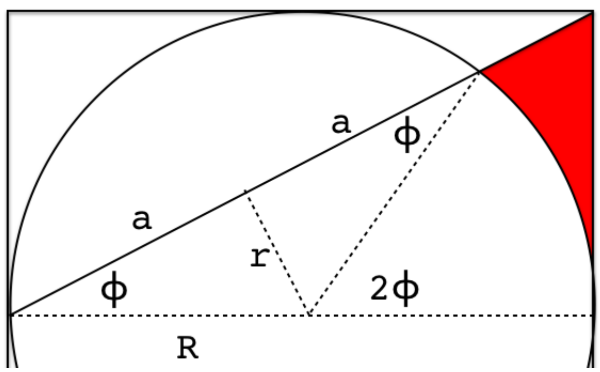
This is (almost) the circular segment.

It is off by a factor of 2. The reason is that the integral only gives the area of that part of the circle that is above the x-axis. We must multiply the provisional answer by 2.

$$= \frac{\pi}{2} - \phi - 2\sin^2\phi$$

Wait for the calculation of the red part of the arch, starting from the circular segment.

calculation four: geometry



We have that the area of the isosceles triangle is exactly

$$\sin\phi\cos\phi = 2\sin^2\phi = 0.4$$

The area of the circular sector with angle θ is

$$\frac{\theta}{2\pi}\,\pi = \frac{\theta}{2}$$

in terms of angle ϕ:

$$= \frac{1}{2}(\pi - 2\phi)$$
$$= \frac{\pi}{2} - \phi$$

The area of the circular segment is the area of the circular sector minus that of the isosceles triangle:

$$\frac{\pi}{2} - \phi - 2\sin^2\phi$$

This matches what we had for the third calculation, by integration of the equation of the circle.

the red part of the arch

At the very beginning we calculated the red part of the arch as:

$$1 - \frac{\pi}{2} + \text{ segment}$$

Now we have the circular segment with angle θ (in terms of ϕ) as

$$\frac{\pi}{2} - \phi - 2\sin^2\phi$$

So the answer for the red part of the arch is

$$1 - \frac{\pi}{2} + \frac{\pi}{2} - \phi - 2\sin^2\phi$$

$$1 - \phi - 2\sin^2\phi$$

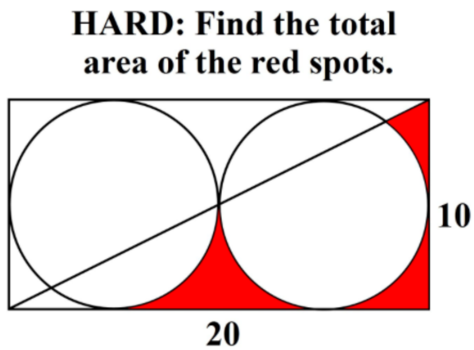
This matches parts one and two. All four methods give the same answer, which is quite a relief.

finishing up

To meet the problem statement, we must add this fractional arch to 3 complete copies, and then multiply the whole thing by the square of the scaling factor (because this is an area):

$$25\ [1 - \phi - 2\sin^2\phi\ \ + 3(1 - \pi/4)\ ]$$

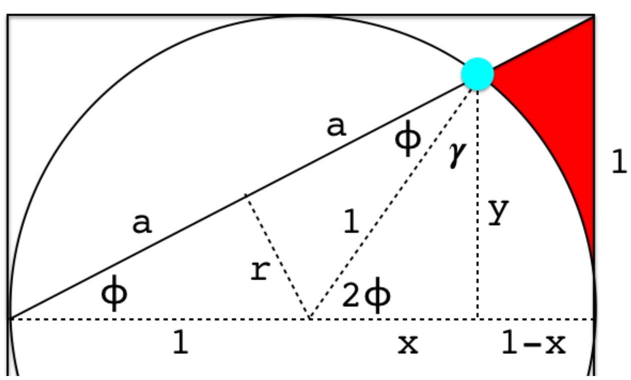
I'm too tired to calculate it.



A fifth way

Later on, I thought of another approach, also pretty simple. Recall the point described in words way back in the section on **the other triangle**.

Here, that point is shown in cyan:



From the diagram we can see that

$$\frac{y}{2a} = \sin\phi$$

$$y = 2a \sin \phi$$

To get x and y in terms of only ϕ we can proceed as follows. A basic result we obtained before was that

$$a = \cos \phi = 2 \sin \phi$$

so

$$\begin{aligned} y &= 2a \sin \phi \\ &= 4 \sin^2 \phi \\ &= \cos^2 \phi \\ &= (\frac{2}{\sqrt{5}})^2 = \frac{4}{5} \end{aligned}$$

x looks more complicated but since $x^2+y^2 = 1$, we can at least calculate $x = 3/5$.

Notice that

$$y = \sin 2\phi$$

and by the double-angle formula

$$y = 2 \sin \phi \cos \phi = \cos^2 \phi$$

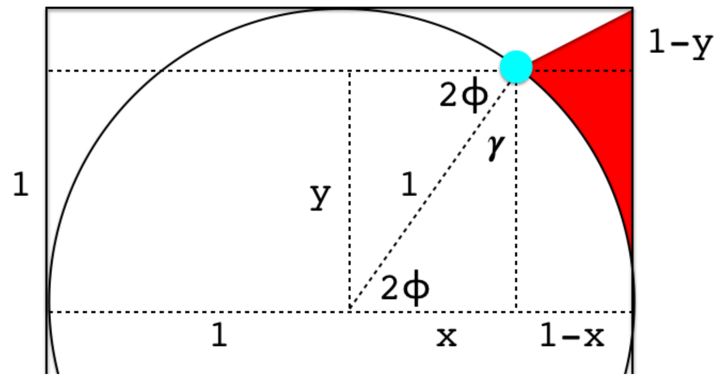
as before.

So for x we have

$$\begin{aligned} x &= \cos 2\phi \\ &= \cos^2 \phi - \sin^2 \phi \\ &= 2 \cos^2 \phi - 1 = 2y - 1 \end{aligned}$$

which is confirmed by doing the arithmetic.

Redraw the figure slightly



Notice that part of the red arch is the area to the right of the circle, and the rest is a triangle.

Let $x = f(y) = \sqrt{1-y^2}$ be the function and calculate the relevant area (the part of the circle in the first quadrant lying below the horizontal line at y) as

$$A = \int_0^y \sqrt{1-y^2}\ dy$$

$$= \frac{1}{2}\ [\sin^{-1} y + y\ \sqrt{1-y^2}\ ]\ \Big|_0^y$$

At the lower bound everything is zero and at the upper bound

$$A = \frac{1}{2}(\sin^{-1} y + y\ \sqrt{1-y^2})$$

From the figure, we can see that $2\phi = \sin^{-1} y$ so

$$A = \frac{1}{2}(2\phi + xy)$$

$$= \phi + \frac{xy}{2}$$

Leave this as it is for the moment.

The area of the red arch below the horizontal line is y minus this.

$$A = y - \phi - \frac{xy}{2}$$

Finally we must add the area of the triangle above. That area is

$$\frac{1}{2}(1-x)(1-y)$$

$$= \frac{1}{2}(1-x-y+xy)$$

Putting it all together

$$A = y - \phi - \frac{xy}{2} + \frac{1}{2}(1-x-y+xy)$$

The xy terms cancel.

$$A = y - \phi + \frac{1}{2}(1-x-y)$$

Recall that $x = 2y - 1$

$$A = y - \phi + \frac{1}{2}(1-2y+1-y)$$

$$= 1 - \phi - \frac{1}{2}y$$

And $y = 4\sin^2\phi$ so

$$A = 1 - \phi - 2\sin^2\phi$$

This matches what we had before.

We obtained the same area five ways (well, at least four ways, two are variants of each other).

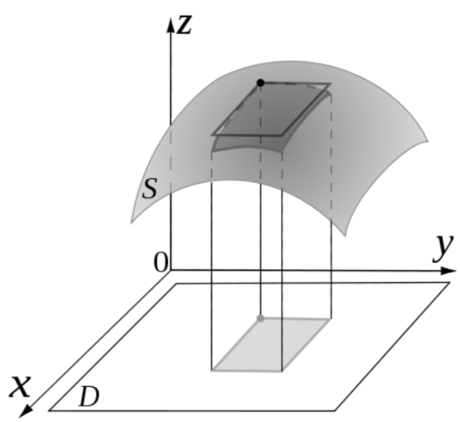
Part XXVII

Multiple Variables

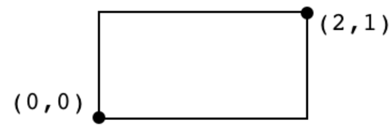
Chapter 113

Double integrals

Double integrals start with a function of two variables, say $f(x, y)$. Think of this as a surface with height $z = f(x, y)$, it could be a sloping roof or something more irregular, but still smooth.



Integration is performed over a region in the xy-plane that is the *shadow* of the surface. Consider a rectangular region with opposing corners at coordinates $(0, 0)$ and $(2, 1)$. The bounds might be written as $R = [0, 2] \times [0, 1]$.



We call the area element $dA = dx\ dy$ or $dy\ dx$, and write the integral as

$$\iint f(x,y)\ dA = \int \int f(x,y)\ dy\ dx$$

The double integral is the volume contained between the plane and the surface at z, within the bounds of the region R. Write the bounds explicitly

$$\iint_R f(x,y)\ dA = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\ dy\ dx$$

iterated integral

The idea is that if we slice the volume vertically perpendicular to the x axis, we have a standard integral to yield the area of the slice when integrating over dy parallel to the slice, and then in a second integral we add up all of these area slices over the range of x.

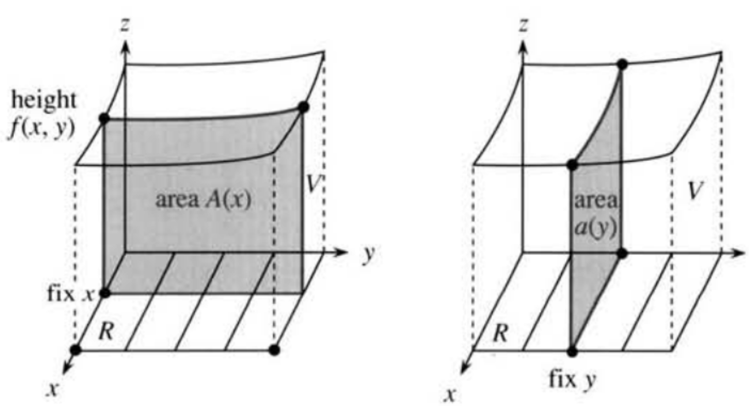


Fig. 14.2 A slice of V at a fixed x has area $A(x) = \int f(x,y)dy$.

This is known as an *iterated integral*. The *order of integration* refers to whether the slices are taken perpendicular to the x-axis ($\int\int f(x,y)\ dy\ dx$) or perpendicular to the y-axis ($\int\int f(x,y)\ dx\ dy$).

For this first example we sidestep one of the important complications of double integrals compared to single integrals. Because the area is a rectangle, the bounds of x and y don't depend on where we are in the box, they are the same for each slice.

We have

$$= \int_{x=0}^{2} \int_{y=0}^{1} f(x,y)\ dy\ dx$$

Suppose

$$f(x,y) = xy^2$$

For slices in the vertical direction (integrating over dy first and dx second), we have

$$\int_{x=0}^{x=2} \int_{y=0}^{y=1} xy^2\ dy\ dx$$

The notation says that we will do the integral with respect to y first. It is the *inner* integral.

$$\int_{y=0}^{y=1} xy^2\ dy$$

As usual for multivariable calculus, in this computation we treat one variable, namely x, as a constant. So we have

$$\int_{y=0}^{y=1} xy^2\ dy = \frac{1}{3}xy^3\ \Big|_{y=0}^{y=1} = \frac{1}{3}x$$

We plug this result into the outer integral

$$\int_{x=0}^{x=2} \frac{1}{3}x\ dx = \frac{1}{6}x^2\ \Big|_{x=0}^{x=2} = \frac{2}{3}$$

In general, you can switch the order of integration.

$$\int_{y_1}^{y_2} \int_{x_1}^{x_2} \ dx \ dy = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y) \ dy \ dx$$

There is a famous theorem due to Fubini that tells when this is allowed. It will always be OK for us.

Sometimes we will only be able to compute a double integral one way (because we can't find the anti-derivative), but here we can do it the other way to check our result.

$$\int_{y=0}^{y=1} \int_{x=0}^{x=2} xy^2 \ dx \ dy$$

Now, the inner integral is

$$\int_{x=0}^{x=2} xy^2 \ dx = \frac{1}{2}x^2y^2 \ \Big|_0^2 = 2y^2$$

and the outer integral is

$$\int_{y=0}^{y=1} 2y^2 \ dy = \frac{2}{3}y^3 \ \Big|_0^1 = \frac{2}{3}$$

The result is the same, so that looks good.

Let's do a second example. Suppose we have the surface $f(x,y) = x^2+y^2$, a paraboloid surface opening up. Our region R is the *Cartesian product*

$$R = [-1,1] \times [0,1]$$

This notation means that x lies on the interval $[-1,1]$ and y on $[0,1]$.

We have

$$\int \int x^2 + y^2 \ dx \ dy$$

We can do this in either order, so we'll do the inner integral as

$$\int_{-1}^{1} x^2 + y^2 dx = \frac{1}{3}x^3 + xy^2 \Big|_{-1}^{1}$$

$$= \frac{1}{3} + y^2 - (-\frac{1}{3} - y^2) = \frac{2}{3} + 2y^2$$

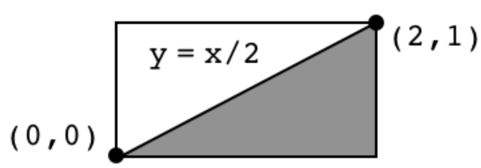
Now the outer integral is

$$\int_{0}^{1} \frac{2}{3} + 2y^2 \ dy = \frac{2}{3}y + \frac{2}{3}y^3 \Big|_{0}^{1}$$

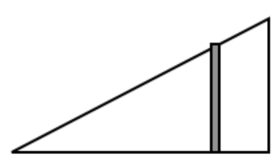
$$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

changed bounds

More commonly, the shadow in the xy-plane is not a rectangle but instead, x and y are related by some function $y = f(x)$ or $x = f(y)$.



Here, we use only half of the rectangle, the part that lies below the line $y = x/2$. Now the upper bound, (the value of y) at the top of each slice changes depending on the value of x.



When we integrate dy first, the bounds are $y = [0, x/2]$

$$\int_{x=0}^{x=2} \int_{y=0}^{y=x/2} xy^2 \; dy \; dx$$

The inner integral is

$$\int_{y=0}^{y=x/2} xy^2 \; dy = \frac{1}{3}xy^3 \; \Big|_{y=0}^{y=x/2}$$

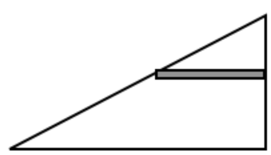
$$= \frac{1}{3}x(\frac{x}{2})^3 = \frac{1}{3}\;\frac{1}{8}\; x^4$$

and the outer integral is

$$\int_{x=0}^{x=2} \frac{1}{3}\;\frac{1}{8}x^4 \; dx = \frac{1}{3}\;\frac{1}{8}\frac{x^5}{5} \; \Big|_{x=0}^{x=2}$$

$$= \frac{1}{3}\;\frac{1}{5}\; 4 = \frac{4}{15}$$

On the other hand, if we integrate dx first, then the bounds for x are $x = [2y, 2]$ and y covers the entire range $[0, 1]$.



So we have

$$\int_{y=0}^{y=1} \int_{x=2y}^{x=2} xy^2 \; dx \; dy$$

The inner integral is

$$\int_{x=2y}^{x=2} xy^2 \; dy = \frac{1}{2}x^2y^2 \; \Big|_{x=2y}^{x=2} = 2y^2 - 2y^4$$

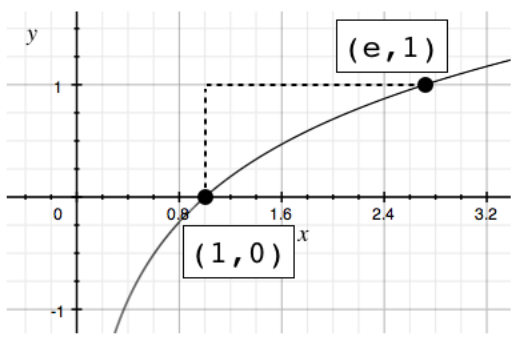
and the outer integral is

$$\int_{y=0}^{y=1} 2y^2 - 2y^4 \, dy = \frac{2}{3}y^3 - \frac{2}{5}y^5 \Big|_{y=0}^{y=1}$$

$$= \frac{2}{3} - \frac{2}{5} = \frac{10}{15} - \frac{6}{15} = \frac{4}{15}$$

Strange limits

Suppose we have a really simple function with a region whose boundary is something like $y = \ln x$. We are interested in the region above the curve $y = \ln x$ and below $y = 1$. We go from $x = 1$ (where $y = 0$) to $x = e$ (where $y = 1$).



Our simple function is just 1. When integrated over the region, this gives the area.

$$\int \int_R 1 \, dA = A$$

If we integrate dy first, our slices are vertical. The bounds on y are $[\ln x, 1]$. The bounds for x are $[1, e]$.

$$\int_{x=1}^{x=e} \int_{y=\ln x}^{y=1} dy \, dx$$

The inner integral is

$$y \Big|_{y=\ln x}^{y=1} = 1 - \ln x$$

and the outer integral is

$$\int_{x=1}^{x=e} 1 - \ln x \; dx = x - (x \ln x - x)$$

$$= 2x - x \;\ln x \Big|_{x=1}^{x=e} = 2e - e - 2 + 0 = e - 2$$

If we do the integral with dx first, we have

$$\int_{y=0}^{y=1} \int_{x=1}^{x=e^y} dx \; dy$$

The inner integral is

$$\int_{x=1}^{x=e^y} dx = x \Big|_{x=1}^{x=e^y} = e^y - 1$$

and the outer integral is

$$\int_{y=0}^{y=1} e^y - 1 \; dy = e^y - y \Big|_0^1$$

$$= e - 1 - 1 + 0 = e - 2$$

Only one way

Next, consider

$$\int \int_R e^{y^2} \; dA$$

We don't have a way to do dy first

$$\int e^{y^2}\ dy = ?$$

However

$$\int e^{y^2}\ dx = e^{y^2}\ x$$

Now with just the right limits, we might have $x = [0, y]$, then

$$\int_{x=0}^{x=y} e^{y^2}\ dx = e^{y^2}\ x\ \Big|_{x=0}^{x=y} = e^{y^2}\ y$$

We have the y that we need and the outer integral is

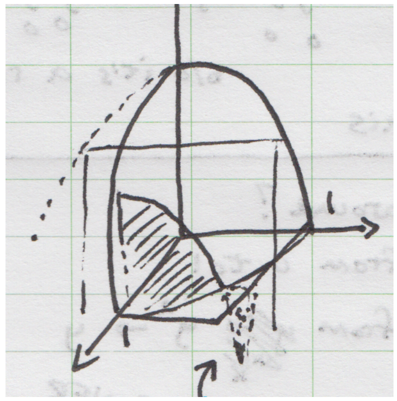
$$\int_{y=0}^{y=1} e^{y^2} y\ dy = \frac{1}{2} e^{y^2}\ \Big|_{y=0}^{y=1} = \frac{1}{2}(e-1)$$

Paraboloid example

In his introduction to double integrals, Prof. Auroux describes the problem of finding the volume under the surface

$$z = 1 - x^2 - y^2$$

Visualizing surfaces can be difficult, but here, just set $x = 0$ or $y = 0$ (separately), then you see that we have a parabola. This solid is a paraboloid, opening downward, with its apex at $(0, 0, 1)$.



The first attempt integrates over the square region $[0,1] \times [0,1]$. As he points out, this is a bit misguided, because for part of this region, the paraboloid is below the x, y-axis. (If $x = y$ and $z = 0$, $x = 1/\sqrt{2}$. Nevertheless,

$$\int_0^1 \int_0^1 1 - x^2 - y^2 \; dy \; dx$$

the inner integral is

$$y - x^2 y - \frac{1}{3} y^3 \Big|_0^1 = \frac{2}{3} - x^2$$

and the outer one is

$$\int_0^1 \frac{2}{3} - x^2 \; dx$$

$$= \frac{2}{3} x - \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3}$$

The way to do this problem and actually obtain the volume of the quarter paraboloid is to set up the bounds of integration properly, over the quarter disk. We can still have $x = [0,1]$ in the outer integral, but for the inner one we use $y = [0, \sqrt{1 - x^2}]$. The changed upper bound makes all the difference. Now, we have

$$\int_0^1 \int_0^{\sqrt{1-x^2}} 1 - x^2 - y^2 \ dy \ dx$$

the inner integral is

$$(1 - x^2)y - \frac{1}{3}y^3 \ \Big|_0^{\sqrt{1-x^2}}$$

$$= (1 - x^2)\sqrt{1 - x^2} - \frac{1}{3}(1 - x^2)^{3/2}$$

$$= \frac{2}{3}(1 - x^2)^{3/2}$$

Switch to polar coordinates for the outer integral (see the chapter **Change of variables**):

$$x = \sin\theta$$

$$dx = \cos\theta \ d\theta$$

$$\sqrt{1 - x^2} = \cos\theta$$

we have

$$= \frac{2}{3} \int \cos^3\theta \cos\theta \ d\theta$$

For the moment, just look it up

$$= \frac{2}{3} \ [\ \frac{\cos^3\theta \ \ \sin\theta}{3} + \frac{3}{4}(\frac{\theta}{2} + \frac{1}{2}\sin\theta \ \ \cos\theta) \ ] \ \Big|_0^{\pi/2}$$

At the upper bound, $\cos\pi/2 = 0$ so we get

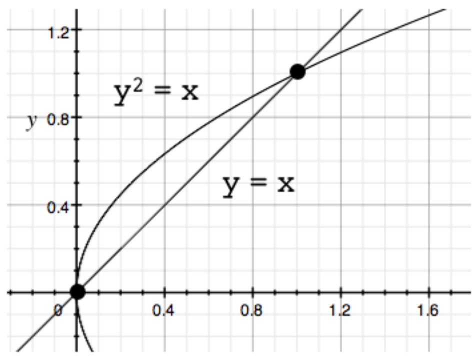
$$\frac{2}{3} \ \frac{3}{4} \ \frac{1}{2} \ \frac{\pi}{2}$$

and at the lower bound, $\sin 0 = 0, \theta = 0$ so we get 0. The result is $\pi/8$.

We will revisit the paraboloid later on.

Two curves

In Auroux's second example, we have the line $y = x$ and the curve $x = y^2$. These two curves cross at $(0,0)$ and $(1,1)$, with the line below the curve between these two endpoints.



If we integrate dx first, the limits will be $x = y^2 \to x = y$, while if we integrate dy first the limits are $y = x \to y = \sqrt{x}$.

Let's do the area function again ($f(x,y) = 1$).

$$\int\int_R 1 \ dA = ?$$

Start with dx first

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \ dx \ dy$$

The inner integral is just

$$\int_{x=y^2}^{x=y} \ dx = x \ \Big|_{x=y^2}^{x=y} = y - y^2$$

so the outer integral is

$$\int_{y=0}^{y=1} y - y^2 \ dy = \frac{1}{2}y^2 - \frac{1}{3}y^3 \ \Big|_{y=0}^{y=1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Doing it the other way

$$\int_{x=0}^{x=1} \int_{y=x}^{y=\sqrt{x}} \, dy \, dx$$

The inner integral is

$$\int_{y=x}^{y=\sqrt{x}} \, dy = y \, \Big|_{y=x}^{y=\sqrt{x}} = \sqrt{x} - x$$

so the outer integral is

$$\int_{x=0}^{x=1} \sqrt{x} - x \, dx = \frac{2}{3}x^{3/2} - \frac{1}{2}x^2 \, \Big|_{x=0}^{x=1} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

Chapter 114

Change of variables

We usually pick a coordinate system with the axes perpendicular, scaled in the same units, with the label x for horizontal and y for vertical. But for some problems, it can be useful to change the coordinate system. There are a number of possible ways to do this, as we'll see.

There are also some basic rules to follow to insure that the areas and other integrals determined in the new coordinate system match up with those in the standard one.

Probably the simplest example is a linear stretching of one dimension, say, the x-axis. Let's think about the problem of determining the area of the rectangle with one corner at $(0, 0)$ and the other corner at $(2, 1)$. Although it seems like overkill, we're going to use single variable calculus to do it. The upper edge is $y = f(x) = 1$.

$$A = \int_{x=0}^{x=2} f(x)\ dx = \int_0^2 1\ dx = x \ \Big|_0^2 = 2$$

It's not strictly necessary to write the 1 for $f(x)$ but I try to do it, to remind myself that we are looking for an area and haven't just forgotten some other $f(x)$.

The answer seems to be correct.

Now, define a new variable u which is exchanged at a rate of 2 u's for every x. If

$$x = 1 \Rightarrow u = 2$$
$$x = 0 \Rightarrow u = 0$$

This means that

$$u = 2x$$

We leave y unchanged.

We take the exact same shape, (with no change in the area), but change the horizontal coordinate system to be defined in terms of u. The point $(2, 1)$ becomes the point $(4, 1)$ in the new coordinate system. So we write

$$A = \int_{u=0}^{u=4} f(u)\ du$$

(This is wrong, but bear with me).

We have adjusted the limits of integration, since before we had the upper limit of $x = 2$, now we have the upper limit of $u = 4$. That part is correct. $f(x)$ is a constant, so everywhere $f(x) = f(u) = 1$ no matter the value of x (or u).

Our mistake is that $du \neq dx$.

$$du = 2\ dx$$
$$\frac{1}{2}du = dx$$

so we substitute

$$A = \int_{x=0}^{x=2} f(x)\ dx = \int_{u=0}^{u=4} f(u)\ \frac{1}{2}du$$

$$= \frac{1}{2} \int_0^4 f(u)\ du$$

$$= \frac{1}{2} \int_0^4 1\ du$$

$$= \frac{1}{2} \times 4 = 2$$

which is correct.

We can do the very same problem in an even more complicated way, using multi-variable calculus. Above, we imagine that what we are doing is slicing the area vertically into many slices with tiny width dx and height $f(x)$ and adding all these together.

We can also imagine that we divide the area up into many little boxes of dA and do the summation this way:

$$A = \iint\limits_R 1\ dA$$

The little boxes of area dA have width dx and height dy.

$$A = \iint\limits_R 1\ dA = \int_{y=0}^{y=1} \int_{x=0}^{x=2} 1\ dx\ dy$$

We evaluate the *inner* integral first, *keeping y constant.*

$$\int_{x=0}^{x=2} 1\ dx = x\ \Big|_0^2 = 2$$

Then do the outer integral:

$$= \int_{y=0}^{y=1} 2\ dy = 2y\ \Big|_0^1 = 2$$

(The real advantage of this is that we can substitute another function for $f(x, y) = 1$—see the Center of Mass chapter). I introduce the two variable method as a way of approaching the next two problems.

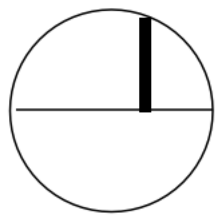
Circle

Consider a circle of radius a centered at the origin.

$$x^2 + y^2 = a^2$$

$$y = \sqrt{a^2 - x^2}$$

This problem is symmetrical so we will only do it one way, integrating over dy first.



Again, we will do the area function, and we will do only the first quadrant

$$\int_{x=0}^{x=a} \int_{y=0}^{y=\sqrt{a^2-x^2}} dy\ dx$$

The inner integral is just

$$\int_{y=0}^{y=\sqrt{a^2-x^2}} dy = \sqrt{a^2 - x^2}$$

so now we have for the outer integral

$$\int_{x=0}^{x=a} \sqrt{a^2 - x^2}\ dx$$

Substitute

$$x = a\cos\theta$$

$$dx = -a\sin\theta\ d\theta$$

For the limits we will have, when

$$x = 0 \Rightarrow \theta = \pi/2$$

$$x = a \Rightarrow \theta = 0$$

Then

$$\int_{x=0}^{x=a} \sqrt{a^2 - x^2}\ dx$$

$$= -\int_{\theta=\pi/2}^{0} \sqrt{a^2 - a^2\cos^2\theta}\ a\sin\theta\ d\theta$$

$$= \int_{0}^{\pi/2} \sqrt{a^2 - a^2\cos^2\theta}\ a\sin\theta\ d\theta$$

$$= a^2 \int_{0}^{\pi/2} \sin^2\theta\ d\theta$$

$$= \frac{a^2}{2}\ [\theta - \sin\theta\cos\theta\ ]\ \Big|_{0}^{\pi/2}$$

$$= \frac{a^2}{2}\frac{\pi}{2} = \frac{\pi}{4}a^2$$

Circle again

Let's try to find the area of a circle of radius a in a different way. In terms of x and y we had previously

$$\iint\limits_R\ dA = \int_{x=0}^{x=a} \int_{y=0}^{y=\sqrt{a^2-x^2}}\ dy\ dx$$

$$= \int_{x=0}^{x=a} \sqrt{a^2 - x^2}\ dx$$

There is an easier way to do this than trig substitution, and that is to change to polar coordinates. A naive attempt is

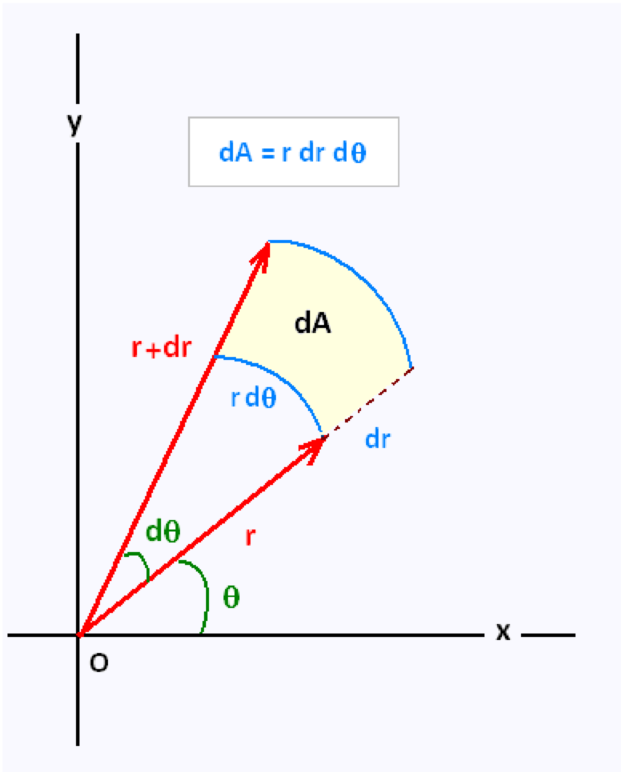
$$\iint\limits_R dA = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} dr\ d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} a\ d\theta = 2\pi a$$

Obviously, this is wrong. What happened is that the area element for a little bit of area dA has an extra factor of r:

$$dA = dx\ dy = r\ dr\ d\theta$$

This is the area element dA in the xy-plane.



Notice that the little piece of the radius dr is a length, but the little piece of angle $d\theta$ is *not a length*. To get the length of the curvy part of the area element dA, we need to multiply the change in angle by the radius. So

$$dA = rd\theta \cdot \ dr$$

(usually written $r\ dr\ d\theta$). To get an area, you must multiply two lengths.

http://www.scientificsentence.net/Equations/CalculusIII/

The length of the curvy segment of arc depends on how far out we are on the radius.

Finishing the problem:

$$\iint_R \ dA = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \ r\ dr\ d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{1}{2}a^2\ d\theta = \pi a^2$$

sphere volume

method: multi-variable calculus

Now, let's try to figure out what the volume elements are for a sphere.

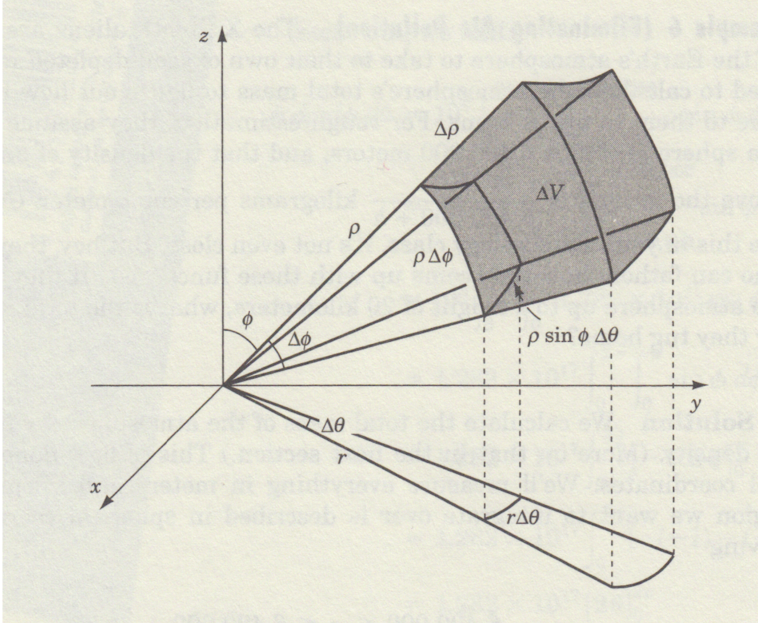


Figure 9.23 Varying each coordinate in spherical coordinates.

The standard way of labeling everything (the parametrization) begins with the polar angle. This is the angle made by the volume element with the positive z-axis.

The mathematicians call this angle ϕ. (The physicists call it θ, but don't get me started).

The other angle is the standard one from polar coordinates, which is the angle with the positive x-axis, or θ. And for the sphere, usually the radius is denoted by ρ rather than r, just to remind us that we are dealing with a sphere.

The two straight sides of the otherwise curvy little box dV lie along two radii, and their length is just $d\rho$.

The two sides which are somewhat vertical here lie on two different great circles, centered on the origin. Imagine traveling around the world on a constant line of longitude. You would travel about $24,901$

miles, the circumference of the earth.

To get the length, multiply $d\phi$ by the radius ρ to obtain $\rho\ d\phi$.

The tricky parts of the volume elements are the sides involving $d\theta$. These also lie on a circle, but it is not a great circle. Instead these are horizontal slices perpendicular to the z-axis.

Imagine traveling around the world on a constant line of latitude, say at 60 degrees north. You would not travel $24,901$ miles but something less than that. The radius and circumference of this circle depend on ϕ.

Looking at the projection in the xy-plane, you should see that the circle has radius r where $r = \rho \sin\phi$ and therefore the length of these guys is

$$r\ d\theta = \rho \sin\phi\ d\theta$$

Putting it all together we get two factors of ρ and one of $\sin\phi$ so:

$$dV = \rho^2 \sin\phi\ d\theta\ d\phi\ d\rho$$

Obtaining the volume element is the hard part.

The triple integral can be done in any order. Most commonly ρ is done first and θ last, with the θ integral independent of the others by symmertry, so it contributes 2π to the final result.

$$dV = \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$$

volume integral

We set up a triple integral

$$V = \iiint dV$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{0}^{R} \rho^2 \sin\phi \ d\rho \ d\phi \ d\theta$$

The integral is easy because the different parts are independent. The only tricky part is the upper bound on ϕ, which is equal to π. Imagine rotating the circle in the xy-plane through the full range of π. We need rotate only by a half-circle to cover the entire volume.

Using independence, we can even rewrite this as

$$= \int_{\theta=0}^{2\pi} d\theta \int_{\phi=0}^{\pi} \sin\phi \ d\phi \int_{0}^{R} \rho^2 \ d\rho$$

We get a factor of 2π from the outside integral and $R^3/3$ from the inside, and the middle is

$$\int_{0}^{\pi} \sin\phi \ d\phi = -\cos\phi \ \Big|_{0}^{\pi} = 2$$

Altogether, that is

$$2\pi \cdot \frac{\rho^3}{3} \cdot 2 = \frac{4}{3}\pi R^3$$

Chapter 115

Parametrization

The topic of parametrization (more rarely called parameterization) is a powerful abstraction.

Probably the simplest example is to describe motion in the plane using a single variable, often called t, because it can stand in for the time.

We can write

$$x = f(t), \quad y = g(t)$$

or more simply

$$x = x(t), \quad y = y(t)$$

or even

$$\mathbf{r} = \langle x(t), y(t) \rangle$$

As the variable t goes between some bounds, the vector $\mathbf{r}$ traces out a curve in the plane. The essential reason that one variable can generate a curve is that the curve is one-dimensional, even though it "lives" in $\mathbb{R}^2$.

Examples include the circle of radius r

$$\mathbf{r} = r\langle \cos t, \sin t \rangle$$

and the ellipse

$$\mathbf{r} = r\langle a\cos t, b\sin t\rangle$$

In $\mathbb{R}^3$ we might make a spiral

$$\mathbf{r} = r\langle a\cos t, b\sin t, t\rangle$$

For a region in the plane, we need two variables. We have often used x and y, and now more and more we use r and θ. For example a region parametrized by r, θ could have area

$$S(r,\theta) = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} \, dS$$

By careful analysis and remembering that "area is length times length and an angle is not a length" we learned that the area element in polar coordinates is

$$dA = dr \;\; \cdot r \; d\theta$$

usually written

$$= r \; dr \; d\theta$$

A more formal statement is that the area elements in terms of x, y $(dA = dx \; dy)$ are related to those in terms of r, θ $(dS = dr \; d\theta)$ by the Jacobian

$$dA = J \; dS$$

where

$$J = | \; x_r y_\theta - x_\theta y_r \; |$$

Since

$$x = r\cos\theta$$

$$x_r = \cos\theta, \quad x_\theta = -r\sin\theta$$

and

$$y = r \sin\theta$$

$$y_r = \sin\theta, \quad y_\theta = r\cos\theta$$

we have

$$J = |\ r\cos^2\theta + r\sin^2\theta\ | = r$$

$$dA = J\ dS = rdS$$

$$dx\ dy = r\ dr\ d\theta$$

Other examples in $\mathbb{R}^2$ include the surface of a cylinder, parametrized by (z, θ) with constant radius r, and the surface of a sphere, parametrized by (ϕ, θ) with constant radius ρ. For the latter, the surface area is

$$S = \int dS = \rho^2 \int_0^{2\pi} \int_0^{\pi} \sin\phi\ d\phi\ d\theta$$

$$= 4\pi\rho^2$$

We can also consider changes of coordinates. A key feature is that we require area elements (with the appropriate Jacobian) and lengths to be invariant.

See the chapters on surface area and change of variables for details.

We can go on to $\mathbb{R}^3$, but I would rather look at examples of surfaces parametrized by two variables.

To get the surface area of a solid whose surface is parametrized by u, v we simply integrate

$$S = \iint f(u, v)\ du\ dv$$

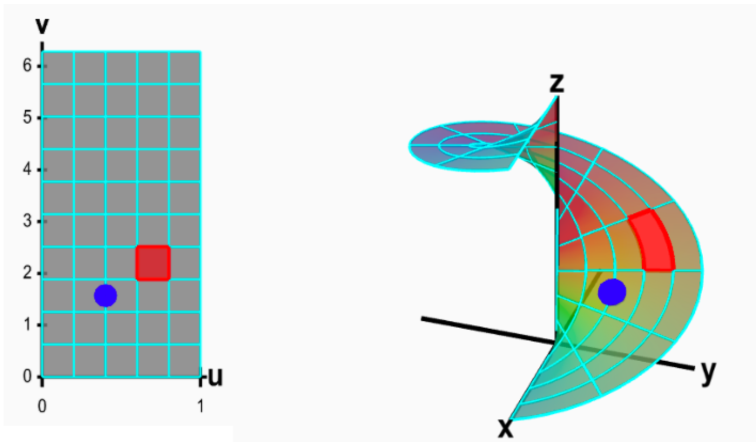
where for the area $f(u, v) = 1$. We can use Fubini's theorem to write

$$\iint\ du\ dv = \iint\ dv\ du$$

(with certain restrictions). And as usual, while the outer integral covers the whole range of that variable, the interval for the inner variable may depend on the outer variable.

Here is an example from the web:

http://mathinsight.org/parametrized_surface_area_examples



We stretch the rectangle on the left to be a helicoid on the right. It can be parametrized as

$$\mathbf{\Phi}(r,t) = (r\cos t, r\sin t, t)$$

where now r is not constant but a variable. The region can be written

$$D = [0,1] \times [0, 2\pi]$$

To get the area we take the partial derivatives:

$$\mathbf{\Phi}_r = \cos t, \sin t, 1$$

$$\mathbf{\Phi}_t = -r\sin t, r\cos t, 0$$

The Jacobian is the absolute value of the cross-product:

$$J = |\ (-r\cos t)\hat{\mathbf{i}} - (-r\sin t)\hat{\mathbf{j}} + (r\cos^2 t + r\sin^2 t)\hat{\mathbf{k}}\ |$$

$$= \sqrt{r^2\cos^2 t + r^2\sin^2 t + r^2}$$
$$= \sqrt{2r^2} = \sqrt{2}\ r$$

The area is then

$$A = \int_0^{2\pi}\int_0^1 J\ dr\ dt$$

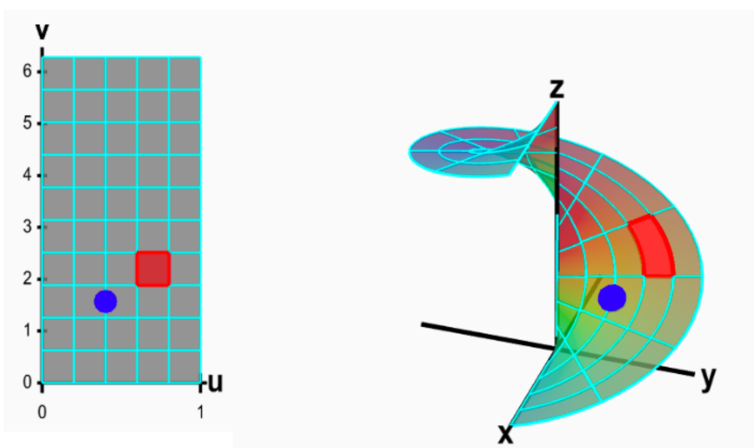
r and t are independent so

$$= 2\pi \int_0^1 \sqrt{2}\ r\ dr$$

$$= 2\pi\sqrt{2}\ \int_0^1 r\ dr$$

$$\pi r^2\sqrt{2}$$

This is the area of an ellipse stretched in the z-direction, Its projection in the plane is a circle. Each little piece tilts up at an angle of 45 degrees.



Seeing this as a tilted ellipse is a really interesting viewpoint.

Jacobian

We've seen that the area element in polar coordinates is

$$r\ dr\ d\theta$$

and the volume element in spherical coordinates is

$$dV = \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$$

We want to develop this in a more systematic way.

Ellipse

Let's think about a standard ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Following the examples in the previous chapter, we might think about trying to compute the area of this ellipse as follows

$$A = \iint\limits_R 1\ dA = \iint\limits_R 1\ dx\ dy$$

The problem is that we do not know how to specify the region for an ellipse, at least, not simply.

However, a little trick can make that problem go away!

We will change both the horizontal and vertical dimensions by constant factors (different for x and y). We compress — well, the opposite of stretching — x by the factor $1/a$ and similarly compress y by the factor $1/b$. We do this by making a change of variables

$$x = au$$

$$dx = a\ du$$

$$y = bv$$
$$dy = b\ dv$$

So
$$dx\ dy = ab\ du\ dv$$

Substituting
$$\frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} = u^2 + v^2 = 1$$

The substitution has converted the ellipse into a circle of radius 1 and area $A = \pi$. and
$$A = \iint\limits_R 1\ dx\ dy = \iint\limits_R ab\ du\ dv$$
$$= ab \iint\limits_R 1\ du\ dv$$

Now, we know the area of the region in u, v coordinates, it is a circle of radius 1 and its area is just equal to π. So finally
$$A = ab \iint\limits_R 1\ du\ dv = \pi ab$$

This is a really simple, beautiful result. The two copies of r in the formula $A = \pi r^2$ become $a \times b$. Both dimensions make equivalent contributions to the area, as we'd expect.

The more formal way to do this is to compute what's called the Jacobian. It gives the ratio between areas determined in two different coordinate systems. We take the partial derivatives of x with respect to u and v, and similarly for y.

$$x_u = a$$
$$x_v = 0$$
$$y_u = 0$$
$$y_v = b$$

The two partials (x_v and y_u) are zero because x does not depend on v and y does not depend on u.

We evaluate the determinant of this matrix:

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

If necessary, we take its absolute value. And that's the factor for converting between the two coordinate systems.

Sometimes the Jacobian is written as

$$J = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}$$

but this doesn't change anything, because $det(A) = det(A^T)$.

To summarize:

$$dx\ dy = J\ du\ dv$$

where J is computed as described.

Circle

The Jacobian is done like this:

$$x = r \cos \theta$$

$$y = r\sin\ \theta$$

We compute

$$x_r = \frac{\partial x}{\partial r} = cos\ \theta$$

$$x_\theta = \frac{\partial x}{\partial \theta} = -r\ sin\ \theta$$

and similarly for y. Then

$$J = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\ \theta & -r\ sin\ \theta \\ \sin\ \theta & r\ cos\ \theta \end{vmatrix} = r(cos^2\ \theta + sin^2\ \theta) = r$$

This is the factor for the ratio of areas under the two systems, and that's why we have $r\ dr\ d\theta$ in the integral. Notice that when we took the partial derivatives, they were partials of x, y with respect to r, θ, and we end up multiplying $dr\ d\theta$ by J.

Chapter 116

Jacobian

General approach to parametrization and the Jacobian

Suppose we wish to determine an area by integration and we're working with the unit square $x = 0 \to x = 1$ and $y = 0 \to y = 1$, sometimes written as $[\ 0, 1\ ] \times [\ 0, 1\ ]$. The area is clearly just 1. Now we want to make a change of variables for some reason, maybe to work on a function that's easier to deal with after the substitution, or because we have some weird bounds in our problem.

$$u = 3x - 2y$$

$$v = x + y$$

We figure out the "exchange rate" for area by tracing out the parallelogram formed by this linear transformation

$$(0, 0) \to (0, 0)$$
$$(1, 0) \to (3, 1)$$
$$(0, 1) \to (-2, 1)$$

If we think of the vectors from $0,0$ to $(3,1)$ as $\langle 3,1\rangle$, and from $0,0$ to $(-2,1)$ as $\langle -2,1\rangle$, the area of the parallelogram formed by these vectors is given by the determinant

$$|\langle 3,1\rangle \times \langle -2,1\rangle| = \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} = 5$$

The other way to do this calculation is (as we've been doing)

$$u_x = 3$$

$$u_y = -2$$

$$v_x = 1$$

$$v_y = 1$$

The Jacobian

$$J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} = 3 - (-2) = 5$$

Or more simply

$$J = u_x v_y - u_y v_x = 3 - (-2) = 5$$

And again, since we took the derivatives with respect to x and y, we multiply $dx\ dy$ by J.

Each element of the area determined in uv units is worth 5 of an element in xy units.

$$5 \iint\limits_R f(x,y)\ dx\ dy = \iint\limits_R f(u,v)\ du\ dv$$

$$\boxed{du\ dv = J\ dx\ dy} \tag{116.1}$$

To say the above one more time in slightly different language, we have

$$u = u(x, y)$$

$$v = v(x, y)$$

If we change x by a little bit Δx and y by a little bit Δy, by how much do u and v change? The linear approximation is

$$\Delta u = u_x \Delta x + u_y \Delta y$$

$$\Delta v = v_x \Delta x + v_y \Delta y$$

So for $\Delta x = 1$, the vector $< 1, 0 >$ becomes

$$< u_x, v_x >$$

and $< 0, 1 >$ becomes

$$< u_y, v_y >$$

and the area of the parallelogram formed by these two vectors (the area in uv-coordinates) is the absolute value of the cross product (think of them as lying in the plane, so there is only one term)

$$< u_x, v_x > \ \times < u_y, v_y >$$

$$J = u_x v_y - u_y v_x$$

So for each unit $dx\ dy$ we get $du\ dv = J\ dx\ dy$ in the uv-coordinate system.

Tilted ellipse

Here is the equation of an ellipse, although that may be hard to recognize at first.

$$x^2 + 4xy + 13y^2 = 16$$

If you remember (or look up) the formula

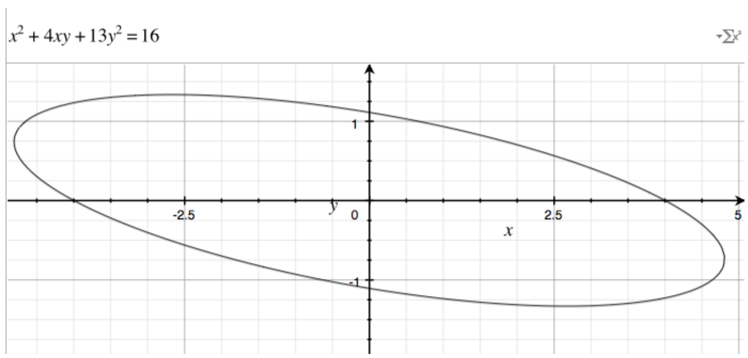
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$$A, B, C = 1, 4, 13$$

The discriminant is

$$B^2 - 4AC = 16 - 52 = -36 < 0$$

Since it's < 0, this is an ellipse. Or we could just get a plotting program like Grapher to plot it



If we knew the angle of the tilt, and there is a method for that, we could rotate to a new coordinate system and just compute the area as πab. However, there is another way which I think is easier. We "complete the square" to remove the term $4xy$ which "mixes" x and y. Since

$$(x + 2y)^2 = x^2 + 4xy + 4y^2$$

Our equation is transformed as follows

$$x^2 + 4xy + 13y^2 = 16$$

$$[\ x^2 + 4xy + 4y^2\ ]\ + 9y^2 = 16$$

$$(x + 2y)^2 + 9y^2 = 16$$

We do a substitution almost like before, but modified:

$$u = x + 2y$$

$$v = 3y$$

so now we have

$$u^2 + v^2 = 16$$

This is a circle of radius 4 and area 16π. Now we just need the Jacobian:

$$u_x = 1$$

$$u_y = 2$$

$$v_x = 0$$

$$v_y = 3$$

$$J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3$$

$$du\ dv = 3\ dx\ dy$$

When we took the partial derivatives, they were partials of u, v with respect to x, y, so we end up multiplying $dx\ dy$ by J.

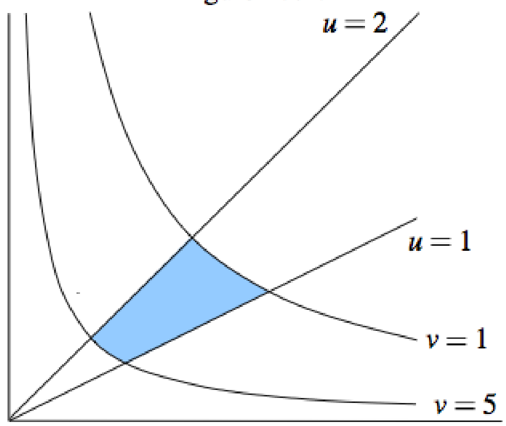
$$\frac{1}{3} du\ dv = dx\ dy$$

We need to multiply the area by this factor to give a final answer of $16\pi/3$.

Varberg example

The next example is from Varberg, 17.16.

Figure 17.17



We have the lines $x = y$ and $x = 2y$ and the curves $xy = 1$ and $xy = 5$. Looks like xy would be a good variable to have so

$$u = \frac{x}{y}$$
$$v = xy$$

These suggestions come from Varberg, not me. :)

The boundaries of the region are just $u = 1 \to u = 2$ and $v = 1 \to v = 5$. Rearranging:

$$x = uy$$
$$x = \frac{v}{y}$$
$$x^2 = xx = uy\,\frac{v}{y} = uv$$

For the Jacobian, it is important to solve for x, y in terms of u, v, and not the other way around, so that we'll have terms containing u and v in the final integral.

$$x = \sqrt{uv}$$
$$y^2 = \frac{x}{u}\frac{v}{x} = \frac{v}{u}$$

$$y = \sqrt{\frac{v}{u}}$$

So

$$x_u = \frac{1}{2}\sqrt{\frac{v}{u}}$$

$$x_v = \frac{1}{2}\sqrt{\frac{u}{v}}$$

$$y_u = -\frac{1}{2u}\sqrt{\frac{v}{u}}$$

$$y_v = \frac{1}{2}\sqrt{\frac{1}{uv}}$$

The Jacobian is then

$$x_u y_v - x_v y_u$$

$$\frac{1}{2}\sqrt{\frac{v}{u}}\ \frac{1}{2}\sqrt{\frac{1}{uv}} + \frac{1}{2}\sqrt{\frac{u}{v}}\ \frac{1}{2u}\sqrt{\frac{v}{u}}$$

$$= \frac{1}{4u} + \frac{1}{4u} = \frac{1}{2u}$$

The area is

$$A = \iint\limits_R 1\ dx\ dy = \iint\limits_R \frac{1}{2u}\ du\ dv$$

$$= \int_{u=1}^{2} \int_{v=1}^{5} \frac{1}{2u}\ dv\ du$$

$$= 2\int_{u=1}^{2} \frac{1}{u}\ du = 2\ln 2$$

spherical coordinates

In a previous chapter, we found the volume element for the sphere as

$$dV = \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$

by analyzing the sides of the volume element. We can do this more formally, using the Jacobian. It is a bit of a mess but we get there in the end.

In 2D we write

$$x = r\cos\theta$$

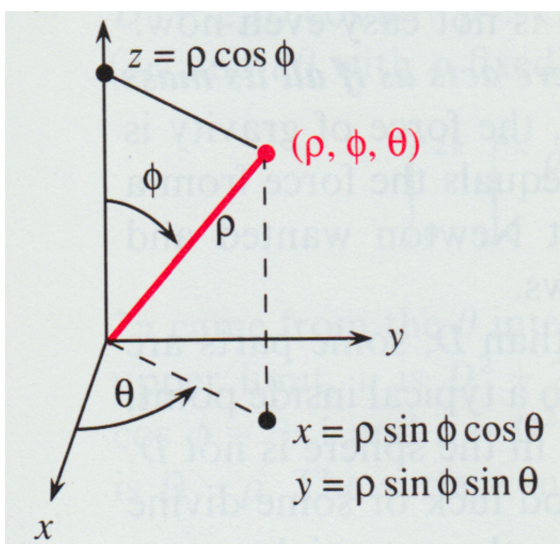
$$y = r\sin\theta$$

In 3D we write

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$

$$z = \rho\cos\phi$$



We need to compute 9 partial derivatives:

$$x_\rho = \sin\phi\cos\theta$$

$$x_\phi = \rho\cos\phi\cos\theta$$

$$x_\theta = -\rho\sin\phi\sin\theta$$

$$y_\rho = \sin\phi\sin\theta$$

$$y_\phi = \rho\cos\phi\sin\theta$$

$$y_\theta = \rho\sin\phi\cos\theta$$

$$z_\rho = \cos\phi$$

$$z_\phi = -\rho\sin\phi$$

$$z_\theta = 0$$

Compute the determinant of this 3×3 matrix:

$$\begin{vmatrix} x_\rho = \sin\phi\cos\theta & x_\phi = \rho\cos\phi\cos\theta & x_\theta = -\rho\sin\phi\sin\theta \\ y_\rho = \sin\phi\sin\theta & y_\phi = \rho\cos\phi\sin\theta & y_\theta = \rho\sin\phi\cos\theta \\ z_\rho = \cos\phi & z_\phi = -\rho\sin\phi & z_\theta = 0 \end{vmatrix}$$

We should use the third row or the third column because $z_\theta = 0$.

We choose the third row because we notice that

$$x_\phi y_\theta = \rho\cos\phi\cos\theta\ \rho\sin\phi\cos\theta = \rho^2\sin\phi\cos\phi\cos^2\theta$$

$$x_\theta y_\phi = -\rho\sin\phi\sin\theta\ \rho\cos\phi\sin\theta = -\rho^2\sin\phi\cos\phi\sin^2\theta$$

So subtracting, we get

$$\rho^2\sin\phi\cos\phi(\cos^2\theta + \sin^2\theta)$$

$$= \rho^2 \sin\phi \cos\phi$$

Multiplying by z_ρ gives

$$= \rho^2 \sin\phi \cos^2\phi$$

which should look vaguely familiar.

For the second term of the determinant we compute

$$x_\rho y_\theta = \sin\phi \cos\theta \ \rho \sin\phi \cos\theta = \rho \sin^2\phi \cos^2\theta$$

$$x_\theta y_\rho = -\rho \sin\phi \sin\theta \ \sin\phi \sin\theta = -\rho \sin^2\phi \sin^2\theta$$

So, subtracting, we get

$$= \rho \sin^2\phi(\cos^2\theta + \sin^2\theta)$$

$$= \rho \sin^2\phi$$

Multiplying by z_ϕ gives

$$-\rho^2 \sin^3\phi$$

Finally, subtract the second term from the first:

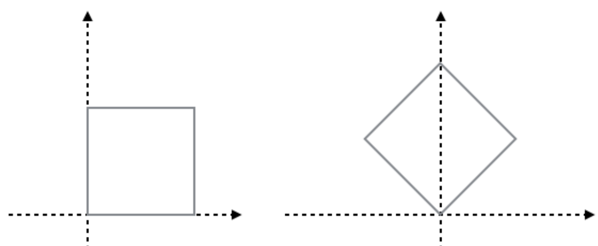
$$\rho^2 \sin\phi \cos^2\phi + \rho^2 \sin^3\phi = \rho^2 \sin\phi$$

Which is exactly what we said before!

Chapter 117

Tilted square

I want to compute the area of the "tilted square" — in this example, rotated by 45 degrees with vertices at $(0,0)$, $(1,1)$, $(2,0)$, and $(-1,1)$.



It is also stretched, note that this is a square with sides of length $\sqrt{2}$ and thus area equal to 2.

Suppose we integrate first over y and then over x. to the right of the origin, the lower bound is the line $y = x$ and the upper bound is $y = 2 - x$.

To the left of the origin, the lower bound is $y = -x$ and the upper bound is $y = 2 + x$. But we can use symmetry and just compute the part for $x \geq 0$, then double it to get the final answer.

As we said, we integrate first over y:

$$A = \int_0^1 \int_{y=x}^{y=2-x} dy\ dx$$

The inner integral is

$$\int_x^{2-x} dy = y \Big|_x^{2-x}$$

$$= 2 - x - x = 2 - 2x = 2(1-x)$$

and the outer integral is then

$$= 2 \int_0^1 1 - x\ dx$$

$$= 2\ [\ x - \frac{x^2}{2}\ ]\ \Big|_0^1 = 2\ \frac{1}{2} = 1$$

For the whole area we have another factor of 2 and the total is thus 2. This matches what we get from basic geometry.

Now, let's compute the average value of y over the region. We need to get what is called the *moment* of y:

$$\int \int y\ dy\ dx$$

We will then divide by the area to get the average value. This is equivalent to the single variable case:

$$\bar{x} = \frac{\int_a^b x\ dx}{b-a}$$

The integral is:

$$\int_0^1 \int_{y=x}^{y=2-x} y\ dy\ dx$$

The inner integral is

$$\frac{y^2}{2} \Big|_x^{2-x} = \frac{1}{2}(4 - 4x + x^2 - x^2) = 2(1 - x)$$

which is the same as before. The outer integral is also the same.

$$2 \int_0^1 1 - x \ dx = 2 \ [\ (x - \frac{x^2}{2}) \Big|_0^1 \ ] = 1$$

And then $\bar{y}$ is equal to this value divided by the area of the region, which is also equal to 1, leaving the answer as just 1. We can see from the symmetry of the problem that this must be correct. Moving along the x-axis (for each value of x), we're looking at a line where for each value of $y = 1+\delta$ above the value $y = 1$ there is another value $y = 1-\delta$ below. So the average value of y is indeed just 1.

For practice, suppose we reverse the order and compute the x integral first.

Because the equation relating the upper bound of x as a function of y changes at $y = 1$ we split the integral into two parts:

$$\int_0^1 \int_{x=0}^{x=y} \ dx \ dy + \int_1^2 \int_{x=0}^{x=2-y} \ dx \ dy$$

The inner integral is just y for the first term and $2 - y$ for the second (but remember the bounds are different). So we obtain

$$= \int_0^1 y \ dy + \int_1^2 (2 - y) \ dy$$

$$= \frac{y^2}{2} \Big|_0^1 + 2y \Big|_1^2 - \frac{y^2}{2} \Big|_1^2$$

$$= \frac{1}{2} + 2 - \frac{3}{2} = 1$$

This matches what we got by computing the y-integral first, as it must.

Now we compute the average value of the function $f(x,y) = y$ over the same region.

$$\int_0^1 \int_{x=0}^{x=y} y \ dx \ dy + \int_1^2 \int_{x=0}^{x=2-y} y \ dx \ dy$$

The inner integrals are

$$xy \ \Big|_0^y = y^2$$

and

$$xy \ \Big|_0^{2-y} = y(2-y)$$

The outer integral is

$$\int_0^1 y^2 \ dy + \int_1^2 2y \ dy - \int_1^2 y^2 \ dy$$

$$= \frac{y^3}{3} \ \Big|_0^1 + y^2 \ \Big|_1^2 - \frac{y^3}{3} \ \Big|_1^2$$

$$= \frac{1}{3} + 3 - \ [\ \frac{8}{3} - \frac{1}{3}\ ] = 1$$

A different way to do the area problem to use a change of variable, which simplifies that problem quite a bit, at the expense of changing the area element.

We would like to tilt the square back to horizontal so that

$$(0,0) \to (0,0)$$

$$(1,1) \to (1,0)$$

$$(0,2) \to (1,1)$$

$$(-1,1) \to (0,1)$$

By guessing I find that the transformation that does this is:

$$u = \frac{1}{2}(x+y)$$

$$v = \frac{1}{2}(y-x)$$

Since this is a linear transformation and it gives the correct answers for the vertices, it works for all interior points as well.

For the area problem, the area of the re-tilted square is equal to 1 in the u, v-coordinate system.

That's a good reason to do this transformation. We get to the correct value for the area in x, y-coordinates by remembering that the area elements are not the same. That is

$$dx\ dy \neq du\ dv$$

The scaling factor is obtained computing the absolute value of the determinant of the Jacobian (a mouthful for sure), but it is just

$$J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}$$

So the area elements are related by

$$du\ dx = J\ dx\ dy$$

The area as computed under the transformation is one-half the area in the standard coordinate system. We obtained 1 as the answer in u, v-coordinates so we multiply by 2 and obtain the answer for x, y-coordinates, which matches what we had before.

We can also compute the average value of y, but first we need an expression for y in terms of u and v. I add the above equations to obtain:

$$y = u + v$$

So the integral we must compute for the average value is

$$\frac{1}{2}\iint y\ dy\ dx = \iint (u+v)\ du\ dv$$

The limits are easy

$$= \int_0^1 \int_0^1 u + v\ du\ dv$$

The inner integral is

$$\int u + v\ du = \frac{u^2}{2} + uv\ \Big|_0^1 = \frac{1}{2} + v$$

The outer integral is then

$$\int_0^1 \frac{1}{2} + v\ dv = \frac{1}{2} + \frac{1}{2} = 1$$

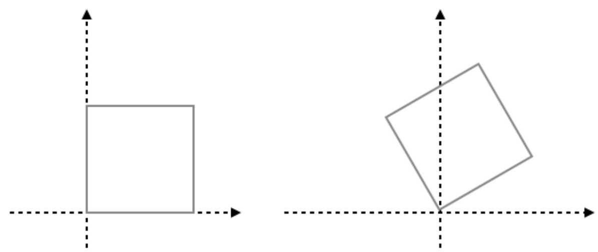
remembering the factor of 1/2, the final answer is 1/2. This, divided by the area is the average value of y in the u, v-coordinate system.

So how does this stack up against the original answer for $\bar{y}$ and what the geometry tells us?

Still working on this, but my feeling about it is that we are looking at y as we slide from $(-1, 1) \to (1, 1)$ in x, y coordinates which is the same as sliding from $(1, 0)$ to $(0, 1)$ in u, v-coordinates.

Since we have $y = 1 \to 0$ linearly with x, the answer is obviously correct.

The real advantage of the transformation approach is that we can tilt to any angle.



Recall that if we turn a vector by an angle ϕ counter-clockwise, the equations are

$$u = x\cos\phi - y\sin\phi$$
$$v = x\sin\phi + y\cos\phi$$

It is easy to get mixed up and do the calculations with the wrong sign. When we *start* with the tilted square and then transform it to the standard orientation we can think of it as either a CW orientation of the points, or a CCW rotation of the coordinates. In any event, the sign of sine in the equations is the opposite of what we have above.

$$u = x\cos\phi + y\sin\phi$$
$$v = -x\sin\phi + y\cos\phi$$

We check this by asking about $\phi = \pi/2$. Then we have

$$u = y$$
$$v = -x$$

The vectors or points:

$$(0,1) \rightarrow (1,0)$$
$$(-1,0) \rightarrow (0,1)$$

which is correct for a clockwise turn of 90 degrees. Since the equations work for unit vectors along the x and y-axes, they will work for any vector or any point.

In contrast to what we did above, this is not a stretching transformation.

$$J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} -\sin\phi & \cos\phi \\ -\cos\phi & -\sin\phi \end{vmatrix} = \sin^2\phi - (-\cos^2\phi) = 1$$

The limits are great: $0 \to 1$. The area of the square is

$$\frac{1}{J}\int_0^1\int_0^1 \ du\ dv = 1$$

To find the moment of y we can go back to our equations

$$u = x\cos\phi + y\sin\phi$$

$$v = -x\sin\phi + y\cos\phi$$

To get, say, y as a function of u and v we can isolate y as follows

$$u\sin\phi = x\sin\phi\cos\phi + y\sin^2\phi$$

$$v\cos\phi = x\sin\phi\cos\phi + y\cos^2\phi$$

Adding:

$$y = u\sin\phi + v\cos\phi$$

or we can remember to just switch the sign of the sines when we switch letters in the original equation:

$$x = u\cos\phi - v\sin\phi$$

$$y = u\sin\phi + v\cos\phi$$

Hence

$$\iint y \; dA = \int \int u \sin\phi + v \cos\phi \; du \; dv$$

where ϕ is a constant.

The inner integral is:

$$\int_0^1 (u \sin\phi + v \cos\phi) \; du = \frac{1}{2} \sin\phi + \cos\phi \; v$$

and the outer integral is

$$\int_0^1 (\frac{1}{2} \sin\phi + \cos\phi \; v) \; dv$$

$$= \frac{1}{2}(\cos\phi + \sin\phi)$$

If we do the integrals for x as well then we will have:

$$\bar{x} = \frac{1}{2}(\cos\phi - \sin\phi)$$

$$\bar{y} = \frac{1}{2}(\cos\phi + \sin\phi)$$

The average value of x and y are these values divided by the area (which was equal to 1).

We can see that we have our signs correct here. If we tilt counter-clockwise, then the average value of x passes through zero and goes negative, which the average value of y will stay positive, reaching a maximum at $\phi = \pi/4$.

Notice that if there is no turning ($\phi = 0$) or if $\phi = \pi/2$, then the average value $\bar{x} = \bar{y} = 1/2$, as expected.

If $\phi = \pi/4$ then $\bar{x} = 0$, as expected, and $\bar{y} = 1/\sqrt{2} \approx 0.7$.

And if $\phi = \pi/2$ then $\bar{x} = -1/2$ and $\bar{y} = 1/2$.

If we differentiate and set the derivative equal to zero, the maximum $\bar{y}$ is:

$$\frac{d}{d\phi}\bar{y} = \frac{1}{2}(-\sin\phi + \cos\phi) = 0$$

$$\sin\phi = \cos\phi$$

$$\phi = \frac{\pi}{4}$$

Part XXVIII

Partial Derivatives

Chapter 118

Del x Del y

In calculus we start with a single variable $y = f(x)$, but the world is often more complicated than that. For example, the Ideal Gas Law is

$$PV = nRT$$

Considering the number of moles of gas as fixed, there are still three variables. Pressure is a function of temperature and volume.

$$P = \frac{1}{V} nRT$$

Or consider:

$$f(x, y) = a^2 - x^2 - y^2$$

where x and y are variables and a is a constant.

We ask "how does f change when x and y change"? We ask the question about one variable at a time, for example about x while holding y constant, and write

$$\frac{\partial f}{\partial x} = -2x$$

Since the y^2 term is taken to be constant when evaluating, its derivative $\partial/\partial x$ is zero.

On the other hand for

$$f(x,y) = xy$$
$$\frac{\partial f}{\partial x} = y$$
$$\frac{\partial f}{\partial y} = x$$

The official definitions use the difference quotient

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}$$
$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, + \Delta y) - f(x,y)}{\Delta y}$$

There is another notation for partial derivatives which I find very useful (Schey, who wrote a wonderful book on multivariable calculus, doesn't like it):

$$f_x = \frac{\partial f}{\partial x}$$

so

$$\nabla f = \langle f_x, f_y \rangle$$

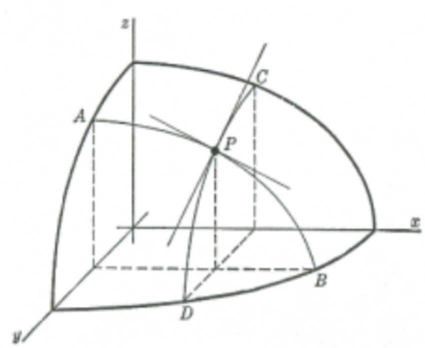


Fig. 1

In the figure, f_x is the slope of the tangent line parallel to the x-axis, and f_y the slope parallel to the y-axis.

gradient

It turns out that the direction in which the slope *increases the fastest* is

$$\langle f_x, f_y \rangle$$

we call this the gradient of f or ∇f.

$$\nabla f = \ \langle f_x, f_y \rangle$$

For example, suppose

$$f(x, y) = xy^2 + y$$

then

$$\nabla f = \ \langle y^2, 2xy + 1 \rangle$$

A widely used "abuse of notation" is to consider ∇ as an operator with this definition in three dimensions:

$$\nabla = \ [\frac{\partial}{\partial x}]\ \hat{\mathbf{i}} + \ [\frac{\partial}{\partial y}]\ \hat{\mathbf{j}} + \ [\frac{\partial}{\partial z}]\ \hat{\mathbf{k}}$$

tangent plane

We can use the partial derivatives to get an equation for the tangent plane to the surface at a point.

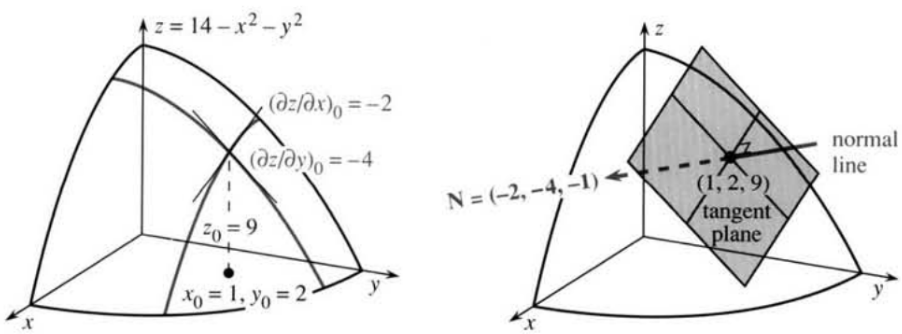


Fig. 13.7 The tangent plane contains the x and y tangent lines, perpendicular to $\mathbf{N}$.

If we know a point (x_0, y_0, z_0) where the derivatives are f_x and f_y, then the equation of the plane tangent to the surface at that point is

$$z - z_0 = f_x(x - x_0) + f_y(y - y_0)$$

$$\Delta z = f_x \Delta x + f_y \Delta y$$

a rearrangement

$$z = z_0 + f_x(x - x_0) + f_y(y - y_0)$$

is called the tangent approximation.

normal vector

Consider this vector which lies in the tangent plane and has constant y:

$$\mathbf{u} = \langle 1, 0, f_x \rangle$$

That is the very definition of slope. We move 1 unit in the x direction and not at all in the y direction and the change in the z direction is f_x. The same for the orthogonal vector

$$\mathbf{v} = \langle 0, 1, f_y \rangle$$

Starting with two vectors in the plane, their cross-product is orthogonal to both, and thus is a normal vector for the plane.

$$\mathbf{u} \times \mathbf{v} = \mathbf{N} = \ \langle -f_x, -f_y, 1 \rangle$$

To get the unit normal, divide by the length

$$\hat{\mathbf{n}} = \frac{\langle -f_x, -f_y, 1 \rangle}{\sqrt{1 + {f_x}^2 + {f_y}^2}}$$

chain rule

Suppose that x, y and z are all functions of some other variable, a parameter t. Then the chain rule says that

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\ \frac{dx}{dt} + \frac{\partial f}{\partial y}\ \frac{dy}{dt} + \frac{\partial f}{\partial z}\ \frac{dz}{dt}$$

We might write the differential

$$df = \frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy + \frac{\partial f}{\partial z}\ dz$$

or even

$$df = f_x\ dx + f_y\ dy + f_z\ dz$$

However, ∂x by itself doesn't mean anything.

Here is an example from Strang: suppose we have a triangle where two of the sides are a and b and the angle between them is θ. Then the area is

$$A = f(a, b, \theta) = \frac{1}{2} ab \cos\theta$$

and

$$f_a = \frac{1}{2} b \sin\theta$$

$$f_b = \frac{1}{2} a \sin\theta$$

$$f_\theta = -\frac{1}{2} ab \cos\theta$$

then

$$df = \frac{1}{2} b \sin\theta\ da + \frac{1}{2} a \sin\theta\ db + \frac{1}{2} ab \cos\theta$$

mixed partial derivatives

In one dimension we get a single second derivative, but with multiple variables there are more derivatives. Here's an example:

$$f(x,y) = xy^2 + e^x + ye^y$$

The first derivatives are

$$f_x = y^2 + e^x$$

$$f_y = 2xy + e^y + ye^y$$

There are four different second derivatives:

$$f_{xx} = e^x$$

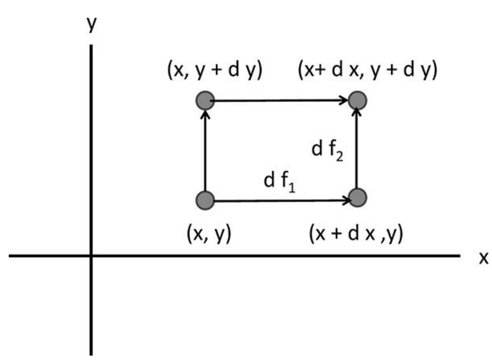
$$f_{yy} = 2x + 2e^y + ye^y$$

$$f_{xy} = 2y$$

$$f_{yx} = 2y$$

The mixed partials are equal. It turns out that always happens. In fact it is a tremendous help in thinking about vector fields as we'll see later.

Shankar has a great explanation of why this is true. If we are making a small change in x and a small change in y, we can imagine doing it by two different tiny paths:



Here we have a view of the plane, but keep in mind the value $f(x, y)$ forms a surface above the plane in $\mathbb{R}^3$.

We can either change x first and then y or first y and then x. We need to get to the same height $z = f(x, y)$ for either path.

Write the differential for the first step from (x, y) to $(x + dx, y)$ as

$$df_1 = f_x \Big|_{(x,y)} dx$$

We take f_x evaluated at (x, y) and multiply by the change dx

But now the value of the function has become $f + df_1$. It is this value which we plug into the formula to calculate the new df_2

$$df_2 = f_y \Big|_{(x+dx,y)} dy$$

We evaluate the derivative with respect to y at the new point $(x+dx, y)$ and that is what we use to multiply by the change dy.

As Shankar says: "because the partial derivative is itself just another function of x and y, we may write to leading order in dx."

$$f_y \Big|_{(x+dx,y)} = \frac{\partial}{\partial y} [\ f(x, y)\ ] \ + \frac{\partial}{\partial y} [\ f_x\ dx]$$

$$= f_y \Big|_{(x,y)} + f_{xy} \Big|_{(x,y)} dx$$

By f_{xy} we understand that the derivative with respect to y is taken second.

So now when we take the step up, from the *new* point we make the change in z as

$$df_2 = f_y \Big|_{(x+dx,y)} dy$$

$$= f_y \Big|_{(x,y)} dy + f_x f_y \Big|_{(x,y)} dx\ dy$$

and combining the two terms

$$df = df_1 + df_2$$

$$= f_x \Big|_{(x,y)} dx + f_y \Big|_{(x,y)} dy + f_{xy} \Big|_{(x,y)} dx\ dy$$

The insight is that if we had gone by the other path we would have the same thing (with the first two terms switched) except the last term would be

$$f_{yx}\ dy\ dx$$

But these must be equal:

$$f_{xy}\ dx\ dy = df = f_{yx}\ dy\ dx$$

Hence

$$f_{xy} = f_{yx}$$

And that's worth remembering.

Chapter 119

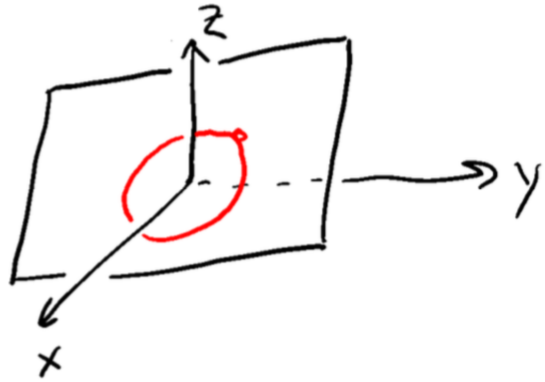
Lagrange

This chapter is about the method of Lagrange multipliers. We have a function $f(x, y)$ and we want to maximize it, but the variables are not independent, e.g. $g(x, y) = c$ where c is a constant. Let's work an example and then come back to the justification afterward.

example

https://gravityandlevity.wordpress.com/2018/07/06/lagrange_multipliers/

> You are living on an inclined plane described by the equation $z = -2x + y$, but you can only move along the circle described by $x^2 + y^2 = 1$. What is the highest point (largest z) that you can reach? What is the lowest point?



example 0

$$f(x, y) = -2x + y$$
$$g(x, y) = x^2 + y^2 = 1$$

So

$$f_x = -2 = \lambda g_x = \lambda(2x)$$
$$f_y = 1 = \lambda g_y = \lambda(2y)$$

Then

$$x = -\frac{1}{\lambda}$$
$$y = \frac{1}{2\lambda}$$

Use the constraint to solve for λ:

$$\frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 1$$
$$\frac{5}{4\lambda^2} = 1$$
$$\lambda = \pm\frac{\sqrt{5}}{2}$$

Solve for x and y. The maximum occurs when

$$x = -\frac{2}{\sqrt{5}}, \qquad y = \frac{1}{\sqrt{5}}$$

$$f(x, y) = \frac{5}{\sqrt{5}} = \sqrt{5}$$

alternative

An alternative approach not using the method would be to solve the constraint to eliminate one variable, like this:

$$y = \sqrt{1 - x^2}$$

so

$$f(x, y) = -2x + \sqrt{1 - x^2}$$

Take the derivative and set it equal to zero

$$f_x = -2 + \frac{1}{2}\ \frac{1}{\sqrt{1 - x^2}}\ (-2x) = 0$$

$$\frac{x}{\sqrt{1 - x^2}} = -2$$

$$x^2 = 4(1 - x^2)$$

$$5x^2 = 4$$

$$x = \pm\frac{2}{\sqrt{5}}$$

This is not so easy in general because the derivative usually gets messy.

idea

According to

https://gravityandlevity.wordpress.com/2018/07/06/lagrange_multipliers/

> The key idea behind the method of Lagrange multipliers is that, instead of trying to reduce the number of variables, you increase the number of variables by adding a set of unknown constants (called Lagrange multipliers). What you get in exchange for increasing the number of variables, however, is a new function (commonly denoted Λ), for which all the variables are independent. With this magic new function you can do the optimization simply by taking the derivative of Λ with respect to each variable one at a time. This function (called the Lagrange function) is:

$$\Lambda(x, y, \lambda) = f(x, y) - \lambda \ g(x, y)$$

where $g(x, y)$ is the constraint rewritten with everything on the left-hand side. In this example:

$$x^2 + y^2 - 1 = 0$$

so

$$\Lambda(x, y, \lambda) = -2x + y - \lambda(x^2 + y^2 - 1)$$

In the function Λ, x is no longer dependent on y, and taking the derivatives is simple.

Set equal to zero, these are:

$$\Lambda_x = -2 - 2\lambda x = 0$$

$$\Lambda_y = 1 - 2\lambda y = 0$$
$$\Lambda_\lambda = -(x^2 + y^2 - 1) = 0$$

The third equation is just the constraint. But the first two equations are just what we had before. Solve for x and y and plug into $g(x, y)$ to obtain λ and it's exactly the same.

summary of the method

Another way to say this is that the method of Lagrange multipliers finds a solution to this equation

$$\nabla f = \lambda \nabla g$$

If there is a constrained max (or min), it will satisfy this equation. In words, we can say that at a point which satisfies the constraint $g(x, y) = c$, and maximizes or minimizes $f(x, y)$, the gradients of the two functions are equal, within some constant λ.

example 1

Suppose

$$f(x, y) = xy$$
$$g(x, y) = x + y = 10$$

This could be the famous maximum area problem where x and y are the sides of a rectangle, the semiperimeter is given, and the objective is to pick x and y to maximize the area. We see a solution for this in Calculus 1, namely, solve g for one of the variables.

$$x = 10 - y$$

and substitute:

$$h(y) = (10 - y)y = 10y - y^2$$

$$h'(y) = 10 - 2y = 0$$

$$h''(y) = -2$$

Since the second derivative is < 0, this is a maximum, and the solution is $y = 5, x = 5$.

Using the Lagrange method we say

$$\nabla f = \lambda \nabla g$$

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

Plugging in, we obtain two equations

$$f_x = y = \lambda g_x = \lambda$$

$$f_y = x = \lambda g_y = \lambda$$

So, clearly $x = y$.

example 2

Corral gives this problem. Find the points on the circle $x^2 + y^2 = 80$ that are either closest to or furthest from the point $P = (1, 2)$.

The circle equation is the constraint. We need to find an equation for distance, which we will then maximize.

$$d = \sqrt{(1 - x)^2 + (2 - y)^2}$$

We can simplify by saying that when d^2 is a min or a max, so is d. So now we have

$$f(x,y) = d^2 = (1-x)^2 + (2-y)^2 = 1 - 2x + x^2 + 4 - 4y + y^2$$

$$f_x = -2 + 2x$$

$$f_y = -4 + 2y$$

$$g_x = 2x$$

$$g_y = 2y$$

So

$$\nabla f = \lambda \nabla g$$

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

Plugging in, we obtain two equations

$$-2 + 2x = \lambda 2x$$

$$-4 + 2y = \lambda 2y$$

Solve for λ and set them equal

$$\frac{-4+2y}{2y} = \frac{-2+2x}{2x}$$

$$\frac{-2+y}{y} = \frac{-1+x}{x}$$

$$-2x + xy = -y + xy$$

$$y = 2x$$

Since $x^2 + y^2 = 80$

$$x^2 + 4x^2 = 80$$

and so $x = \pm 4$ and the solutions are $(4, 8)$, $(-4, -8)$. Here is the diagram from Corral

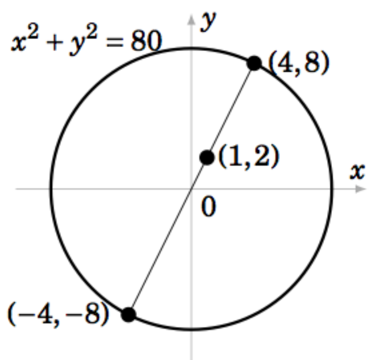


Figure 2.7.1

example 3

Consider the inverted paraboloid

$$z = x^2 + y^2$$

Now ask, which values (x, y) give a maximum (or minimum) value on the surface, that are also on the line $y = 1 + x$?

Try to visualize what we're asking here. We have as our surface a kind of elongated and inverted bowl, with its vertex at the origin, opening up. The vertical plane that cuts through the x, y-plane along the given line traces out a curve in its intersection with the surface.

Since the line is symmetric with respect to the origin and the surface is also, I predict that the answer will be that $x = y$. Let's see.

$$f(x, y) = x^2 + y^2$$

$$\nabla f = \langle\ 2x, 2y\rangle$$

Be sure to re-cast the line equation as a function $g(x, y)$:

$$g(x, y) = y - x = 1$$

$$\nabla g = \langle\ -1, 1\rangle$$

So we have that

$$\lambda\ \nabla f = \nabla g$$

This is two equations:

$$-2x\lambda = 1$$

$$2y\lambda = 1$$

$$-\frac{1}{2x} = \frac{1}{2y}$$

$$-x = y$$

Go back to g to plug in and solve for x:

$$-x - x = 1, \quad x = -\frac{1}{2}$$

and

$$y = -x = - - \frac{1}{2} = \frac{1}{2}$$

The answer is as predicted.

example 4

Here is one from Paul's *Calculus*. Find the dimensions of a rectangular box with maximum volume, subject to the constraint that the surface area is fixed.

$$V = xyz$$

$$A = 2(xy + xz + yz) = \text{constant}$$

Now

$$\nabla V = < f_x, f_y, f_z >$$

we have then three equations plus a fourth (the one for area above).

$$f_x = \lambda g_x = yz = 2\lambda(y + z)$$

$$f_y = \lambda g_y = xz = 2\lambda(x + z)$$

$$f_z = \lambda g_z = xy = 2\lambda(x + y)$$

Paul uses a nice trick to solve this. Take the first two equations, multiply eqn. 1 by x and eqn. 2 by y:

$$xyz = 2\lambda x(y + z)$$

$$xyz = 2\lambda y(x + z)$$

$$xy + xz = xy + yz$$

$$x = y$$

By symmetry, $x = y = z$.

example 5

Auroux gives this problem: on the curve of the hyperbola $xy = 3$, find the point closest to the origin.

The distance to the origin is

$$d = \sqrt{x^2 + y^2}$$

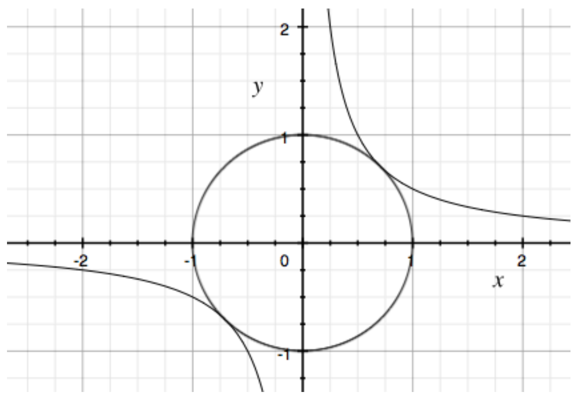
but we can again simplify things a bit because if d^2 is a minimum, then d is a minimum. So the function we need to minimize is

$$f(x, y) = x^2 + y^2$$

subject to the constraint $xy = 3$.

The graph of $f(x, y)$ is a *surface*—a paraboloid with circular cross-section—with apex at $(0, 0, 0)$, and opening up. $x^2 + y^2$ is a *level curve* of f, at the value $f = a$.

In this figure we have the situation as described, except that $xy = 1/2$.



The idea of the method is that, at the maximum, the circle just touches the hyperbola. And at the point of contact, the gradient of the circle function is parallel to the gradient of the hyperbola function, so the two are equal when one is multiplied by a constant that is usually designated λ.

$$\nabla f = \lambda \nabla g$$

We have

$$f(x, y) = x^2 + y^2$$

$$f_x = 2x$$

$$f_y = 2y$$

$$g(x, y) = xy = c$$

$$\nabla f = \lambda \nabla g$$

so the two equations we get are

$$2x - \lambda y = 0$$

$$\lambda x - 2y = 0$$

Auroux uses some linear algebra trickery to solve this. A matrix equation $A\mathbf{v} = \mathbf{0}$ has solutions other than $\mathbf{v} = 0$ only if $\det(\mathrm{A}) = 0$. So that's what we do:

$$A = \begin{bmatrix} 2 & -\lambda \\ \lambda & -2 \end{bmatrix}$$

The determinant is

$$-4 + \lambda^2 = 0$$

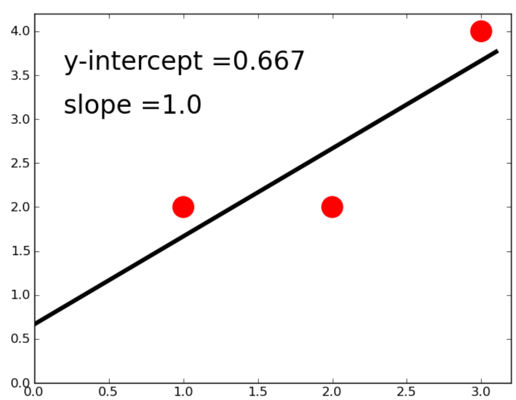
$$\lambda = \pm 2$$

Using $2x - \lambda y = 0$,for $\lambda = 2$, we obtain $2x - 2y = 0$ and $x = y$, while for $\lambda = -2$, we obtain $2x + 2y = 0$ and $x = -y$. Exactly as you would predict from the figure.

Chapter 120

Least squares

We want to decide how to draw the best line through a bunch of data points, In the figure below, given the three data points, we need to find the slope and y-intercept of a line that maximizes the fit according to some idea. The criterion that is used most often is called "least squares." In this method, the line that minimizes the (sum of the squares of the) vertical distance between each point and the line is chosen as the "best fit."



As usual, the equation of the line is in the form y = slope times x + intercept. We will call these C and D:

$$y = C + Dx$$

We need to find C and D. Let's write out the distances between the data points and our (as yet undetermined) line. The points are (1,2), (2,2), and (3,4). The magnitude of the "error" at times 1, 2 and 3 is the difference between the value on the line and the value we actually observed.

$$e1 = C + (1)D - 2$$

$$e2 = C + (2)D - 2$$

$$e3 = C + (3)D - 4$$

The sum of the squared errors is

$$S = (C + D - 2)^2 + (C + 2D - 2)^2 + (C + 3D - 4)^2$$

Note that this is a function of both C and D, so the procedure will be to take the two partial derivatives, and set them both equal to zero.

$$\frac{\partial S}{\partial C} = 2(C + D - 2) + 2(C + 2D - 2) + 2(C + 3D - 4)$$

In the second equation, we will pick up coefficients of D by the chain rule

$$\frac{\partial S}{\partial D} = 2(C + D - 2) + (2)2(C + 2D - 2) + (3)2(C + 3D - 4)$$

Let's work with the first equation.

$$2(C + D - 2) + 2(C + 2D - 2) + 2(C + 3D - 4) = 0$$

$$C + D - 2 + C + 2D - 2 + C + 3D - 4 = 0$$

$$3C + 6D - 8 = 0$$

$$3C + 6D - 8 = 0$$

$$3C = -6D + 8$$

Leave it that way for a second. Now look at equation 2.

$$\frac{\partial S}{\partial D} = 2(C + D - 2) + (2)2(C + 2D - 2) + (3)2(C + 3D - 4)$$

$$2(C + D - 2) + (2)2(C + 2D - 2) + (3)2(C + 3D - 4) = 0$$

$$C + D - 2 + 2C + 4D - 4 + 3C + 9D - 12 = 0$$

$$6C + 14D - 18 = 0$$

Substitute for 3C into equation 2

$$6C + 14D - 18 = 2(-6D + 8) + 14D - 18 = 0$$

$$2D - 2 = 0$$

$$D = 1$$

$$3C = -6D + 8 = 2$$

$$C = \frac{2}{3}$$

If we calculate the actual errors we have

$$e1 = \frac{2}{3} + 1 - 2 = -\frac{1}{3}$$

$$e2 = \frac{2}{3} + 2 - 2 = \frac{2}{3}$$

$$e3 = \frac{2}{3} + 3 - 4 = -\frac{1}{3}$$

Notice that the sum of the errors is equal to 0. This is always true of these solutions.

There are better formulas for a real problem. For example, using μ for the mean

$$slope = A = \frac{\mu(xy) - \mu(x)\mu(y)}{\mu(x^2) - \mu(x)^2}$$

$$intercept = B = \mu(y) - A\ \mu(x)$$

Notice that the line goes through the $\mu(x), \mu(y)$. For our problem, we calculate

$$\mu(x) = 2, \qquad \mu(y) = \frac{8}{3}$$

$$\mu(xy) = \frac{(1*2 + 2*2 + 3*4)}{3} = \frac{18}{3}$$

$$\mu(x)\mu(y) = \frac{16}{3}$$

$$\mu(x^2) = \frac{1+4+9}{3} = \frac{14}{3}$$

$$\mu(x)^2 = 4$$

$$A = \frac{\mu(xy) - \mu(x)\mu(y)}{\mu(x^2) - \mu(x)^2} = \frac{\frac{18}{3} - \frac{16}{3}}{\frac{14}{3} - \frac{12}{3}} = 1$$

$$B = \mu(y) - A\ \mu(x) = \frac{8}{3} - 2 = \frac{2}{3}$$

Just for fun, we can check using R:

```
> x = c(1,2,3)
> y = c(2,2,4)
> fit <- lm(y ~ x)
> fit

Call:
lm(formula = y ~ x)

Coefficients:
```

```
(Intercept)            x
     0.6667       1.0000
```

Here is a derivation of the formula. We write the sum of the squared deviations as

$$D = f(a,b) = \sum [\ y_i - (ax_i + b)\ ]^2$$

Take the partial derivative with respect to each variable and set it equal to 0

$$\frac{\partial D}{\partial a} = \sum [\ y_i - (ax_i + b)\ ](-x_i) = 0$$

$$\frac{\partial D}{\partial b} = \sum [\ y_i - (ax_i + b)\ ](-1) = 0$$

Rearranging the first equation

$$\frac{\partial D}{\partial a} = \sum -x_i y_i + a x_i^2 + b x_i = 0$$

Now the second

$$\frac{\partial D}{\partial b} = \sum\ x_i a + b - y_i = 0$$

To give two equations

$$\sum x_i^2 a + \sum x_i b = \sum x_i y_i$$

$$\sum x_i a + \sum b = \sum y_i$$

This is just a 2 x 2 linear system in a and b. Time for Cramer's Rule! The denominator is

$$N \sum x_i^2 - (\sum x_i)^2$$

So for the slope a and intercept b we get:

$$a = \frac{N \sum x_i y_i - \sum x_i \sum y_i}{N \sum x_i^2 - (\sum x_i)^2}$$

$$b = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{N \sum x_i^2 - (\sum x_i)^2}$$

All terms can be converted to means by dividing by N. Each double sum needs two divisions by N, which makes the leading N's disappear.

$$a = \frac{\mu_{xy} - \mu_x \mu_y}{\mu_{x^2} - (\mu_x)^2}$$

$$b = \frac{\mu_y \mu_{x^2} - \mu_x \mu_{xy}}{\mu_{x^2} - (\mu_x)^2}$$

A little algebra will confirm that

$$b = \mu_y - a\mu_x$$

as we had above. Again, the point (μ_x, μ_y) satisfies the equation of the line for the "best fit." It is on the line.

Part XXIX

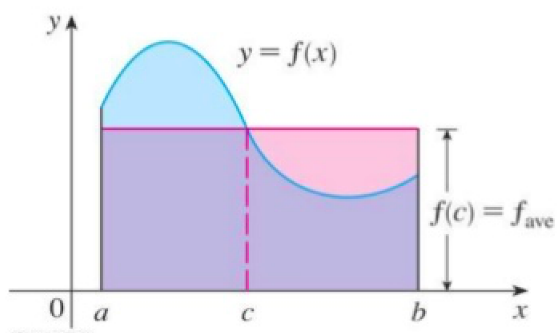
More physics

Chapter 121

Average value

Consider the problem of the *average value* of a function $f(x)$ over some interval $[a, b]$.

One way to think of that would be, by analogy to a list of numbers, to collect the values of $f(x)$ at each point along the graph of the function, and then compute the mean. Of course, there is an infinite number of such points, so that's a bit of a problem.



Suppose we integrate, and find the total area under the curve $y = f(x)$ between $x = a \to x = b$. Take the number calculated from the definite integral, divide by the distance $b - a$, and plot the result as a height h so as to form a rectangle above the x-axis.

Clearly h is a reasonable stand-in for the average value of $f(x)$, since the rectangle formed from $[a,b]\times[0,h]$ gives the same area as we found under the curve by integration. Call this $\bar{f}(x)$ (f bar). Define it to be

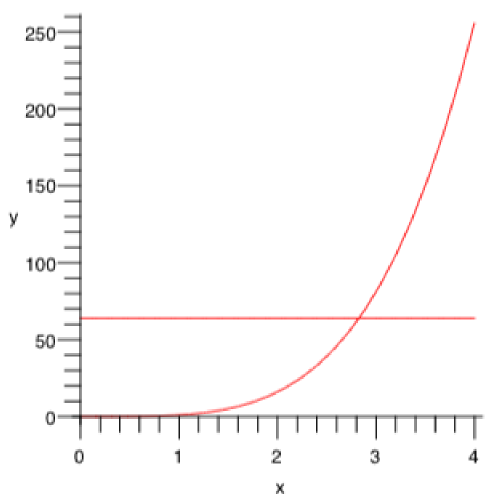
$$\begin{aligned}\bar{f}(x) &= \frac{h}{b-a}\\ &= \frac{\int_a^b f(x)\ dx}{b-a}\end{aligned}$$

The average value of a function over an interval is the integral of the function divided by the length of the interval. To see the analogy with what we'll do later, write

$$\bar{f}(x) = \frac{\int_a^b f(x)\ dx}{\int_a^b\ dx}$$

The denominator is simply the length of the interval.*

example



Which is bigger, the rectangle of height 64 and width 4 or the area under x^4 between 0 and 4?

We integrate $\int x^4$ and obtain $x^5/5$ evaluated between 0 and 4, giving $4^5/5$ or $4^4 \times 0.8$. This is just smaller than the area of the rectangle which is $4^3 \times 4$.

The average value of the function $f(x) = x^4$ over this interval is 0.8×4^3, obtained by dividing the result of the integral by the length of the interval.

mean value theorem

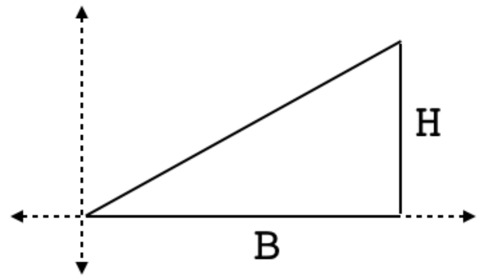
It is worth pointing out that the **mean value theorem for integrals** states if $h = \int f(x)\,dx$ over the interval $[a, b]$, there is at least one value x_0 in the same interval, such that $f(x_0) = h$.

geometric centroid of a triangle

Next, we want to know, what is the position of the average point inside some geometric figure. That point is called the centroid. For example, where is the average point in a triangle? For a physical object with constant density, the centroid would be the **center of mass**.

If we consider a right triangle having its base along the x-axis, and we call that length B, then the height would be H. Let the pointy end be at the origin, and the equation of the line describing the bounding line of the triangle is $y = H/B \cdot x$.

One reason to do the triangle as an example is that we already know the answer. Ceva's theorem tells us that the centroid is 2/3 of the way toward the fat end, for both x and y.



In trying this problem, I first thought, what is the average value of y? Maybe it's

$$\bar{y} \stackrel{?}{=} \int_0^B y \; dx$$

This turns out not to be correct. It gives what we asked for, the average value of y, the distance of the upper edge from the x axis, times x. The result is the area of the triangle.

$$= \int_0^B \frac{H}{B} x \; dx$$
$$= \frac{H}{B} \; \frac{x^2}{2} \; \Big|_0^B$$

That's not what we're looking for.

Surely $\int x \; dx$ is not right either. It will turn out that $\int xy \; dx$ is the answer. This approach is suggested by extending the idea of average value from above into two dimensions. We had

$$\bar{f}(x) = \frac{\int_a^b f(x) \; dx}{\int_a^b \; dx}$$

Substitute

$$\bar{f}(x,y) = \frac{1}{A} \iint f(x)\ dA$$
$$= \frac{\iint f(x)\ dA}{\iint\ dA}$$

where $A = \iint dA$ is the total area.

To get the average of value of the x-coordinate, just do (leaving the area A aside for the moment):

$$\iint f(x)\ dA = \int \int x\ dy\ dx$$

We decide we will do $\int dy$ first and go back to the figure to get y as a function of x:

$$\int_{x=0}^{x=B} \int_{y=0}^{y=Hx/B} x\ dy\ dx$$

The inner integral is simply

$$\int_0^{Hx/B} x\ dy = xy\ \Big|_0^{Hx/B}$$
$$= Hx^2/B$$

and the outer integral is

$$\int_0^B \frac{H}{B} x^2\ dx = \frac{H}{B}\ \frac{x^3}{3}\ \Big|_0^B$$
$$= \frac{HB^2}{3}$$

The last step is to divide by the area $A = HB/2$, obtaining

$$\frac{HB^2}{3} \ \frac{2}{HB} = \frac{2}{3} \ B$$

which is correct.

The question is, can we see anything that would be useful for expressing this idea in terms of one-variable calculus?

The inner integral is $\int x \ dy$ with *constant* x, or just xy.

We weight the value of x at every point along the base by the number of points with each x, namely, y. The integral is

$$\int xy \ dx$$

and we will need to correct at the end by

$$\int y \ dx$$

which is the area. To restate this:

$$\bar{x} = \frac{\int xy \ dx}{\int y \ dx}$$

The average point in this right triangle lies 2/3 of the base away from the pointy end. By symmetry, the same is true for y. This is **Ceva's Theorem**.

geometric centroid of a half-circle

Consider a half-circle of radius R centered at the origin (just the part above the x-axis). We compute the average values of x and y

$$\bar{x} = \frac{1}{A} \iint x \; dA$$
$$\bar{y} = \frac{1}{A} \iint y \; dA$$

By symmetry, it's clear that $\bar{x} = 0$. Let's compute it anyway. Since we do have the x, we should keep Cartesian coordinates. We integrate with respect to y first:

$$\int_{-R}^{R} \int_{0}^{\sqrt{R^2-x^2}} x \; dy \; dx$$

The inner integral is

$$xy \Big|_{0}^{\sqrt{R^2-x^2}}$$
$$= x\sqrt{R^2 - x^2}$$

The outer integral is

$$\int_{-R}^{R} x\sqrt{R^2 - x^2} \; dx$$
$$= -\frac{1}{3} \; (R^2 - x^2)^{3/2} \; \Big|_{-R}^{R} = 0$$

This is an even function of x so any integral centered about 0 has the same value at both bounds and the value is just zero.

The area is π but zero times $1/\pi$ is still zero.

The average value of y over the circle is calculated as follows.

$$\int_{-R}^{R} \int_{0}^{\sqrt{R^2-x^2}} y \; dy \; dx$$

We can do this in Cartesian or polar coordinates. For the first way the inner integralis

$$\int_0^{\sqrt{R^2-x^2}} y\ dy = \frac{y^2}{2}\ \Big|_0^{\sqrt{R^2-x^2}}$$

$$= \frac{R^2-x^2}{2}$$

The outer integral is

$$\int_{-R}^{R} \frac{R^2-x^2}{2}\ dx$$

$$= \frac{1}{2}\ [\ R^2x - \frac{x^3}{3}\ ]\ \Big|_{-R}^{R}$$

$$= \frac{1}{2}\ [\ (R^3 - \frac{R^3}{3}) - (-R^3 - -\frac{R^3}{3})\ ]$$

It's hard to keep track of the minus signs.

$$= \frac{1}{2}\ [\ (2R^3 - 2\frac{R^3}{3}\ ]$$

$$= \frac{2}{3}R^3$$

If you dare, realize that this

$$\int_{-R}^{R} \frac{R^2-x^2}{2}\ dx$$

is an even function of x, so we can change the lower bound to zero and multiply by 2:

$$= 2\int_0^{R} \frac{R^2-x^2}{2}\ dx$$

The 2 conveniently cancels and we have

$$R^2x - \frac{x^3}{3}\ \Big|_0^{R}$$

which gives the correct answer painlessly.

Perhaps we do want to switch to polar coordinates. I am tempted to write:

$$y = R \sin\theta$$

but that is not correct! I need a variable r

$$y = r \sin\theta$$

So dA becomes:

$$dA = dy\ dx = r\ dr\ d\theta$$

and we have

$$\iint y\ dA = \int_0^{\pi} \int_0^{R} r \sin\theta\ r\ dr\ d\theta$$

$$= \int_0^{\pi} \int_0^{R} \sin\theta\ r^2\ dr\ d\theta$$

The inner integral is

$$\int_0^{R} \sin\theta\ r^2\ dr = \sin\theta \int_0^{R} r^2\ dr$$

$$= \sin\theta\ \frac{R^3}{3}$$

So the outer integral is

$$\frac{R^3}{3} \int_0^{\pi} \sin\theta\ d\theta$$

$$\frac{R^3}{3}\ [\, -\cos\theta \Big|_0^{\pi}\ ] = \frac{2R^3}{3}$$

However, this integral for the numerator is obtained, remember to divide by $(1/2)\pi R^2$ to obtain

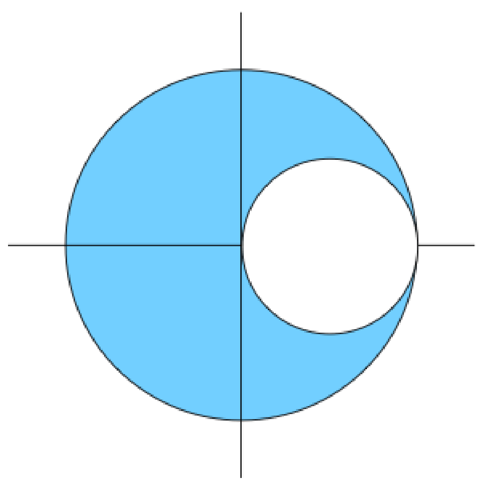
$$\frac{2R^3}{3} \cdot \frac{2}{\pi R^2} = \frac{4R}{3\pi}$$

The geometric centroid of a half-circle is at $(0, 4R/3\pi)$. This is not so obvious. It will become very useful when we consider Pappus Theorem later on.

problem

There is a problem in Varberg that has an easy solution and a hard solution. We've done part of the easy solution here, and the reason I know this is Auroux shows the answer. Varberg does only the hard calculation, which is a little weird.

Figure 17.12



The diagram shows the unit circle at the origin in blue with an inset circle in white that has been removed. The missing circle has radius

$1/2$ and is centered at $1/2, 0$. The problem is to find the average value of the function x over the blue region.

Now, we've just finished showing that the average value of x over the unit circle is zero. But $\bar{x}$ over the small circle will not be zero, and we will obtain minus that value by subtracting from the big circle.

$$\bar{x} = \frac{1}{Area} \iint_R x \ dA$$

Since surely $\bar{x}$ over the small circle is at the origin, $x = 1/2$, multiplied by the area $\pi/4$ and negated because of the subtraction, we predict the answer will be $-\pi/8$.

Let's try to do the problem the way it's done in Varberg.

The idea is to use polar coordinates. It's really hard to see how you could possibly do it in Cartesian coordinates, except in the way that we already used them. How do we describe the x-dependence of the bounds on y? I suppose we could try, but let's go with polar.

We will integrate the top half of the circle, and we'll do it separately for the first and second quadrants.

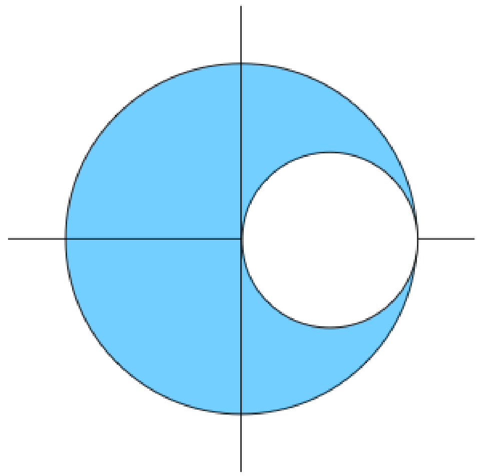
$$\int \int x \ r \ dr \ d\theta$$

$$x = r \cos \theta$$

$$\int \int r \cos \theta \ r \ dr \ d\theta$$

Pretty standard. We will integrate with respect to θ last. So, now our job is to figure out how what the lower limit is on r as θ varies.

Figure 17.12



For the lower limit, when $\theta = 0$, $r = 1$. And when $\theta = \pi/2$, $r = 0$. I don't know how you could be sure it works everywhere if you didn't know the answer, but $\cos\theta$ is correct.

We'll come back to the issue of how this was found later. Let's calculate:

$$\int_0^{\pi/2} \int_{\cos\theta}^1 r\cos\theta \ r \ dr \ d\theta$$

The inner integral is

$$\frac{r^3}{3}\cos\theta \ \Big|_{\cos\theta}^1$$

So the outer integral is then

$$\frac{1}{3}\int_0^{\pi/2} \cos\theta - \cos^4\theta \ d\theta$$

So we will have an outside factor of $1/3$ and $\sin\theta$ between 0 and $\pi/2$, which is just 1, but we have to deal with the fourth power of the cosine. Not fun. We make repeated application of the double-angle formula:

$$\cos^2 s = \frac{1}{2}(1 + \cos 2s)$$

$$\cos^4 s = \frac{1}{4}(1 + \cos 2s)^2$$

$$= \frac{1}{4}(1 + 2\cos 2s + \cos^2 2s)$$

Substitute $t = 2s$

$$\cos^2 2s = \cos^2 t = \frac{1}{2}(1 + \cos 2t)$$

$$= \frac{1}{2}(1 + \cos 4s)$$

Substitute back to the previous version

$$\frac{1}{4}(1 + 2\cos 2s + \cos^2 2s)$$

$$= \frac{1}{4}(1 + 2\cos 2s + \frac{1}{2}(1 + \cos 4s))$$

Rearrange slightly

$$= \frac{3}{8} + \frac{1}{2}\cos 2s + \frac{1}{8}\cos 4s$$

Substitute θ, integrate and get

$$= \frac{3}{8}\theta + \sin 2\theta + \frac{1}{2}\sin 4\theta$$

With limits of $\theta = 0 \to \pi/2$ we have only the first term and that one only at the upper limit

$$= \frac{3}{16}\pi$$

Combine it with the rest of the integral to obtain

$$\frac{1}{3}(1 - \frac{3}{16}\pi)$$

Now, on to the second quadrant.

$$\int_{\pi/2}^{\pi} \int_{0}^{1} r \cos\theta \ r \ dr \ d\theta$$

The inner integral is

$$\frac{1}{3} r^3 \cos\theta \ \Big|_0^1$$

$$= \frac{1}{3}\cos\theta$$

And the outer integral is then

$$\frac{1}{3}\int_{\pi/2}^{\pi} \cos\theta \ d\theta$$

which is just $-1/3$. So finally, we add them up:

$$\frac{1}{3}(1 - \frac{3}{16}\pi) - \frac{1}{3} = -\frac{\pi}{16}$$

Recall that we integrated only the top half of the figure, so multiply by 2 to obtain the same answer that we had much more simply before!

insight

The part I like about this problem is seeing the limits on r. The equation of a circle in polar coordinates is given as

$$r = a\cos\theta + b\sin\theta$$

where $(a/2, b/2)$ is the center of the circle. Since $b = 0$ (we're on the x-axis with $y = 0$), and $a = 1$, the small circle has the equation

$$r = \cos\theta$$

How to check that? Well, we can convert from polar to Cartesian like this. Multiply the equation for the circle by r:

$$r^2 = r\cos\theta$$

Since $x = r\cos\theta$

$$r^2 = x$$

But r^2 is also equal to $x^2 + y^2$ so

$$x = x^2 + y^2$$

$$x^2 - x + y^2 = 0$$

Complete the square:

$$x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4}$$

$$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$$

This is indeed our circle, a circle of radius $1/2$ centered at $O = (1/2, 0)$.

Going back to the polar equation of the small circle $r = \cos\theta$, we can think of this as having θ in sync for the two circles.

As we advance the parameter θ for the large circle, we can find the position of r (where the point is with respect to the origin), at the same value of θ for the small circle. At least, that's how it seems to me.

We have to remember that for an off-center circle (not at the origin) in polar coordinates the r parameter is still with respect to the origin, rather than with respect to the center of the circle.

Chapter 122

Center of mass

As you know, in single variable calculus we can interpret

$$\int_a^b f(x)dx$$

as the area underneath the curve $y = f(x)$ between the lines $x = a$ and $x = b$ (our limits). In multi-variable calculus we compute the double integral over the same region as follows

$$\int_{x=a}^{x=b} \int_{y=0}^{y=f(x)} dy\ dx = \int_{x=a}^{b} y\Big|_0^{f(x)}\ dx = \int_a^b f(x)\ dx$$

To be more general, we'd just say that we compute the double integral over the region R

$$\iint\limits_R dx\ dy$$

with the understanding that we can compute the inner integral with respect to either x or y, whichever is more convenient. Another difference from the single-variable approach is that we can extend this approach by computing

$$\iint\limits_R g(x,y)\ dx\ dy$$

Suppose, for example, that $g(x,y)$ is a function that gives the density of a flat object for each coordinate x, y. In this case we usually use the label $\rho(x,y)$. This integral gives the total mass of the object:

$$M = \iint\limits_R \rho(x,y)\ dx\ dy$$

To find the center of mass we compute

$$M_x = \iint\limits_R \rho(x,y)\ y\ dx\ dy$$

$$M_y = \iint\limits_R \rho(x,y)\ x\ dx\ dy$$

And then finally

$$\bar{x} = \frac{M_x}{M}$$

$$\bar{y} = \frac{M_y}{M}$$

Let's do a simple example. Suppose our region is a rectangle with the origin as one corner and the point $(1,2)$ as the opposite corner. It's just a 2D box of width 1 and height 2. And let's say our density function is $\rho(x,y) = xy$. Then

$$M = \iint\limits_R \rho(x,y)\ dx\ dy = \int_{y=0}^{y=2} \int_{x=0}^{x=1} xy\ dx\ dy$$

The inner integral is

$$\frac{1}{2}x^2y \Big|_0^1 = \frac{1}{2}y$$

and the rest is

$$M = \int_{y=0}^{y=2} \frac{1}{2}y\ dy = \frac{1}{4}y^2 \Big|_0^2 = 1$$

Now

$$M_x = \int_{y=0}^{y=2} \int_{x=0}^{x=1} xy^2\ dx\ dy$$

The inner integral is

$$\frac{1}{2}x^2y^2 \Big|_0^1 = \frac{1}{2}y^2$$

and the rest is

$$M_x = \int_{y=0}^{y=2} \frac{1}{2}y^2\ dy = \frac{1}{6}y^3 \Big|_0^2 = \frac{4}{3}$$

Last, M_y can be done in the same order

$$M_y = \int_{y=0}^{y=2} \int_{x=0}^{x=1} x^2y\ dx\ dy$$

The inner integral is

$$\frac{1}{3}x^3y \Big|_0^1 = \frac{1}{3}y$$

and the rest is

$$M_y = \int_{y=0}^{y=2} \frac{1}{3}y\ dy = \frac{1}{6}y^2 \Big|_0^2 = \frac{2}{3}$$

Thus our center of mass is at the point $2/3, 4/3$. If it had made our lives easier, either integral could be computed with respect to y before x. The answer makes sense. The density increases as we go to the right and up, so the center of mass is offset from the geometric center in the same direction.

Chapter 123

Moment of inertia

Consider a rigid object that rotates about some axis. Suppose it is nailed down at the axis. The motion of a point on the object in the plane perpendicular to the axis of rotation will be a circle.

If the radius r to the point from the axis turns through an angle θ, the arc length moved along the perimeter is

$$s = r\theta$$

We introduce the *angular velocity* ω as the proportion between the clock time t and the angle θ turned since time-zero, in units of radians per second:

$$\theta = \omega t$$

Differentiate both sides with respect to time and the angular velocity can then be seen as the rate of change of the angle

$$\omega = \frac{d\theta}{dt}$$

Alternatively, use this as the definition of ω and integrate to obtain $\omega\Delta t = \Delta\theta$. In Newton's notation we say that $\dot{\theta} = d\theta/dt$ so

$$\omega = \dot{\theta}$$

In the **chapter** on uniform circular motion, we defined the vector $\mathbf{r}$ which goes from the origin to the rotating point. $\mathbf{r}$ is a function of time, namely:

$$\mathbf{r} = r\ \langle \cos \omega t, \sin \omega t \rangle$$

The velocity $\mathbf{v}$ is the time-derivative of $\mathbf{r}$:

$$\mathbf{v} = r\omega\ \langle -\sin \omega t, \cos \omega t \rangle$$

We see that $\mathbf{v} \cdot \mathbf{r} = 0$. The tangential velocity is perpendicular to the vector $\mathbf{r}$ and it points in the direction of the tangent to the circle at the point. Its magnitude is

$$v = r\omega$$

To emphasize its tangential nature we could write v_T.

The acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -r\omega^2\ \langle \cos \omega t, \sin \omega t \rangle$$

We say the acceleration due to circular motion is radial, because it points in the (negative) of the same direction as the radius. Its magnitude is

$$a = r\omega^2 = rv$$

And again, to emphasize its radial nature we could write a_r but we skip this.

We can also view v at the point as the time-derivative of s

$$v = \frac{d}{dt}s = \frac{d}{dt}r\theta = r\frac{d\theta}{dt} = r\dot{\theta}$$

There is only one term from the product rule since $\dot{r} = 0$, the radius doesn't change with time. This matches our previous definitions

$$v = r\omega$$

$$\omega = \dot{\theta}$$

Going back to the tangential velocity we had

$$v = r\dot{\theta} = r\omega$$

and the distance moved is

$$s = r\theta = r\omega t$$

As we said above, for an object turning in a circle at a constant tangential velocity v, there is an acceleration toward to the center of the circle. That acceleration is

$$\begin{aligned} a &= r\omega^2 \\ &= \frac{r^2\omega^2}{r} = \frac{v^2}{r} \end{aligned}$$

If, in addition, the tangential velocity is changing, the tangential acceleration is

$$a_T = \frac{dv_T}{dt} = \frac{d}{dt} r\omega = r\frac{d\omega}{dt}$$

Define the angular acceleration α as

$$\alpha = \frac{d\omega}{dt} = \ddot{\theta}$$

Then

$$a_T = r\alpha$$

To summarize then:

$$\begin{aligned} s &= r\theta \\ v_T &= r\omega = r\dot{\theta} \\ a_T &= r\alpha = r\ddot{\theta} \end{aligned}$$

In rotation problems it is useful to think of the angular velocity ω as an analog of the linear velocity for linear problems. If we do that, then I, the moment of inertia, plays a role analogous to mass.

Kinetic energy:

$$K = \frac{1}{2}mv^2$$

Since $v = r\omega$:

$$K = \frac{1}{2}mr^2\omega^2$$

Define $I = mr^2$, then

$$K = \frac{1}{2}I\omega^2$$

Angular momentum is usually given as the vector product $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p} = m\mathbf{v}$ is the momentum.

If the vectors are at right angles and we just look at the magnitude of the result we have that

$$L = rp$$

so

$$L = rmv = rm\ r\omega = I\omega$$

In the same way that the force is the time-derivative of the momentum, the torque τ is the time derivative of the angular momentum

$$\tau = \frac{d}{dt}L = I\frac{d\omega}{dt} = I\alpha$$

This is the angular version of Newton's second law $F = ma$.

So with this definition

$$I = mr^2$$

we obtain these parallel definitions for angular motion

$$K = \frac{1}{2}I\omega^2$$

$$L = I\omega$$
$$\tau = I\alpha$$

In a sense, the contribution of the radius is shifted from the velocity component (angular velocity) to the mass component (moment of inertia).

The moment of inertia of a collection of discrete masses is the sum of each mass, times the square of the distance to the chosen axis of rotation.

rods

Imagine that we have a uniform (thin) rod, and it's going to rotate around the center of the rod. Choose the center as the origin of coordinates. The ends of the rod are then at $-l/2$ and $+l/2$.

Calculate the mass per unit length M/l, and so in a tiny sliver of the rod of width dx, the mass at that position is $M/l\ dx$ and its moment is $Mx^2/l\ dx$. We add them all up:

$$I = \int_{-l/2}^{l/2} \frac{M}{l} x^2\ dx$$
$$= \frac{M}{l}\ \frac{x^3}{3}\Big|_{-l/2}^{l/2}$$
$$= Ml^2(\frac{1}{24} - -\frac{1}{24}) = \frac{1}{12}Ml^2$$

If we move the axis to the end of the rod, then adjust the coordinate system as well, and we have

$$\int_0^l \frac{M}{l} x^2\ dx$$

$$= \frac{M}{l}\ \frac{x^3}{3}\bigg|_0^l = \frac{1}{3}Ml^2$$

There is a principle called the *parallel axis* theorem which says that

$$I = I_{CM} + md^2$$

The moment around a different axis is equal to the moment around the CM, the center of mass, plus md^2 where d is the distance between the two axes.

The first example was at the CM of the rod, and the distance we moved was $l/2$ so we have

$$I = \frac{1}{12}Ml^2 + M(\frac{l}{2})^2 = Ml^2(\frac{1}{12} + \frac{1}{4}) = \frac{1}{3}Ml^2$$

rings and disks

Imagine the object is a ring, and we're rotating around the center. Think of it as a sum of discrete pieces

$$I = \sum_i m_i r^2$$

But $r = R$ for every piece so we have

$$I = R^2 \sum_i m_i = MR^2$$

For a thin disk, we imagine adding up the contribution for a series of rings with increasing radius. At radius r, the circumference of the ring is $2\pi r$, and the area of the ring is $2\pi r\ dr$. The mass per unit area is $M/\pi R^2$ so the mass of each ring is

$$m = \frac{M}{\pi R^2}\ 2\pi r\ dr = \frac{2M}{R^2} r\ dr$$

Integrate mass times radius squared from $r = 0 \to R$

$$I = \int_0^R r^2 \frac{2M}{R^2} r \; dr$$

$$= \frac{2M}{R^2} \left. \frac{r^4}{4} \right|_0^R = \frac{1}{2} M R^2$$

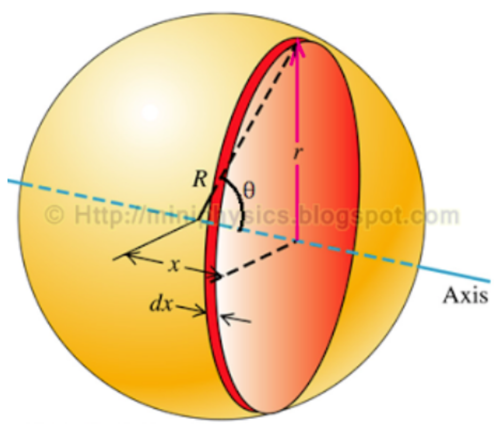
Now, if we were to move the axis of rotation to the edge, we would have

$$I = \frac{MR^2}{2} + MR^2 = \frac{3}{2} M R^2$$

sphere

Start with the sphere. It's analogous to the ring, but harder. I struggled with this one, by not using the appropriate slant height when calculating the area of a slice. I found another approach online:

http://www.miniphysics.com/uy1-calculation-of-moment-of-inertia-of_04.html



and I liked it so much I want to show that approach. We slice the sphere perpendicular to the axis of rotation. For each slice we have a ring of radius r whose moment is

$$dI = r^2\ dm$$

where

$$dm = \frac{M}{A}\ dA = \frac{M}{4\pi R^2}\ dA$$

The trick is to express the area of the ring (parametrize our slices) in terms of an angle θ, shown in the figure ($\theta = 0 \to \pi$, where $\theta = \pi/2$ is perpendicular to the axis). Doing this, we get the correct area for the width of the ring, namely $R\ d\theta$, and a total area for the ring of

$$dA = 2\pi r\ R\ d\theta$$

so

$$dm = \frac{M}{4\pi R^2}\ dA = \frac{M}{4\pi R^2}\ 2\pi r\ R\ d\theta = \frac{M}{2R}\ r\ d\theta$$

and then

$$dI = r^2\ dm = \frac{M}{2R}\ r^3\ d\theta$$

Now, we need to get a relationship between r and θ, but from the diagram it's clear that

$$r = R\sin\theta$$

so we have the integral

$$I = \int dI = \frac{MR^2}{2}\int_0^\pi \sin^3\theta\ d\theta$$

This is pretty easy. We do

$$\sin^3\theta\ d\theta = (1-\cos^2\theta)\sin\theta\ d\theta$$

the integral is just

$$\int \sin^3\theta\ d\theta = -\cos\theta + \frac{\cos^3\theta}{3}\ \Big|_0^\pi = (1-\frac{1}{3}) - (-1+\frac{1}{3}) = \frac{4}{3}$$

and the answer is

$$\frac{2}{3}MR^2$$

solid ball

With the previous result in hand the solid ball is pretty easy. We imagine a series of concentric spheres with increasing radius $r = 0 \to R$. Each sphere has

$$dI = \frac{2}{3} dm\ r^2$$

$$dm = \frac{M}{4/3\pi R^3}\ 4\pi r^2 = \frac{3M}{R^3} r^2$$

so

$$dI = \frac{2M}{R^3} r^4$$

$$I = \int\ dI = \frac{2M}{R^3} \int_0^R r^4$$

$$= \frac{2M}{R^3}\ \frac{R^5}{5} = \frac{2}{5} MR^2$$

parallel axis theorem

Our first proof of the parallel axis theorem is from Fitzpatrick. Choose the origin of coordinates to be at the center of mass of the body.

Orient the z-axis with the axis of rotation. Orient the new axis so that the new moment of inertia lies along the x-axis at $x = d; y = 0$.

Since the center of mass is at the origin, by definition

$$\iiint x\ dx\ dy\ dz = 0$$

with the integral taken over the volume of the body. The same is true for y and z.

The square of the distance of any point in the body from the z-axis is $x^2 + y^2$, so the moment of inertia with respect to the center of mass is

$$I_{CM} = M\frac{\iiint (x^2 + y^2)\ dx\ dy\ dz}{\iiint\ dx\ dy\ dz}$$

The new moment of inertia simply has an additional displacement d with respect to x

$$I' = M\frac{\iiint ((x-d)^2 + y^2)\ dx\ dy\ dz}{\iiint\ dx\ dy\ dz}$$

Expanding and taking the constant d outside the integral

$$= M\frac{\iiint (x^2 + y^2)\ dx\ dy\ dz}{\iiint\ dx\ dy\ dz} - 2dM\frac{\iiint x\ dx\ dy\ dz}{\iiint\ dx\ dy\ dz} + d^2M\frac{\iiint\ dx\ dy\ dz}{\iiint\ dx\ dy\ dz}$$

We see immediately that the middle integral is zero and the first term is I_{CM} so we have

$$I' = I_{CM} + d^2M\frac{\iiint\ dx\ dy\ dz}{\iiint\ dx\ dy\ dz}$$

The third term is just Md^2

$$I' = I_{CM} + Md^2$$

I think this derivation assumes constant density, but it will still work with variable density, just add a function $\delta(x, y, z)$ which never has to be evaluated.

Just in case this fails for some reason that I can't see, let me show you Professor Shankar's version.

We will do this by summation, just pretend we are summing over lots of individual little mass elements with position vectors $\mathbf{r}_i$ from the center of mass.

$$I_{CM} = \sum_i m_i|\mathbf{r}_i|^2 = \sum_i m_i(\mathbf{r}_i \cdot \mathbf{r}_i)$$

Now we move the axis of rotation to a new position with position vector $\mathbf{d}$ from the center of mass. Notice that $\mathbf{d}$ will be the same for every m_i. The new moment is

$$I'_{CM} = \sum_i m_i(\mathbf{r}'_i \cdot \mathbf{r}'_i)$$

where

$$\mathbf{d} + \mathbf{r}'_i = \mathbf{r}_i$$
$$\mathbf{r}'_i = \mathbf{r}_i - \mathbf{d}$$

and so

$$\mathbf{r}'_i \cdot \mathbf{r}'_i = (\mathbf{r}_i - \mathbf{d}) \cdot (\mathbf{r}_i - \mathbf{d})$$
$$= \mathbf{r}_i \cdot \mathbf{r}_i - 2\mathbf{r}_i \cdot \mathbf{d} + \mathbf{d} \cdot \mathbf{d}$$

substitute into the moment calculation

$$I'_{CM} = \sum_i m_i(\mathbf{r}_i \cdot \mathbf{r}_i) + \sum_i m_i|(-2)\mathbf{r}_i \cdot \mathbf{d}| + \sum_i m_i(\mathbf{d} \cdot \mathbf{d})$$

Now, the first term is I_{CM}, and in the third term, since $\mathbf{d}$ does not vary with i, it is just d^2 and we can pull it out of the sum

$$I'_{CM} = I_{CM} + \sum_i m_i(-2)(\mathbf{r}_i \cdot \mathbf{d}) + d^2 \sum_i m_i$$

$$I'_{CM} = I_{CM} + \sum_i m_i(-2)(\mathbf{r}_i \cdot \mathbf{d}) + Md^2$$

We're almost there. We need the middle term to vanish.The coordinates were set up so that $\mathbf{d}$ lies along the x-axis, so $\mathbf{r}_i \cdot \mathbf{d}$ is just the constant d times the x-component of $\mathbf{r}_i$ for each vector.

$$\sum_i m_i(-2)(\mathbf{r}_i \cdot \mathbf{d}) = -2d \sum_i m_i \mathbf{r}_{ix}$$

$\sum_i m_i \mathbf{r}_{ix}$ is the x-component of the center of mass *in this coordinate system*. But we've chosen that point to be the origin. So this term vanishes, leaving

$$I'_{CM} = I_{CM} + Md^2$$

This argument seems to be the same as the first, couched in the language of vectors and using finite sums instead of integrals.

Chapter 124

Volume of the sphere

This chapter is mainly review. We look again at the problem with which we started the book, finding the volume contained inside a sphere.

Officially, the sphere is the set of points at a specified distance R from the origin (i.e., it's hollow), and the volume inside is technically called a "ball". But we'll just use the phrase "volume of the sphere" here.

I will put my favorite method first. It uses single-variable calculus.

Method of spheres

I'm not sure what its official name is, so I just made one up.

We think about how the volume of the sphere depends on r in the interval $r = [0, R]$. An incremental change dr changes the volume by adding a thin shell of volume equal to the surface area of the sphere $4\pi r^2$ times dr. That is

$$dV = 4\pi r^2 \ dr$$

$$V = \int dV = \int_0^R 4\pi r^2 \ dr$$

$$= 4\pi \ \ \frac{1}{3} r^3 \ \Big|_0^R = \frac{4}{3}\pi R^3$$

It's really as simple as that.

Of course, you need to know the formula for the surface area to do it that way. Here's a quick proof. Parametrize the surface using the polar angle ϕ and the radial angle θ. As usual, θ has bounds $[0, 2\pi]$ but ϕ has bounds $[0, \pi]$. (If this is new to you, we will discuss it in detail later in the chapter).

If the radius is R, the surface area element has sides $R \ d\phi$ while the top and bottom have $R \sin \phi \ d\theta$ so we integrate

$$\int_0^{2\pi} \int_0^{\pi} R^2 \sin \phi \ d\phi \ d\theta$$

$$= 2\pi R^2 \int_0^{\pi} \sin \phi \ d\phi$$

$$= 2\pi R^2 \ [\ - \cos \phi \ ] \Big|_0^{\pi} = 4\pi R^2$$

.

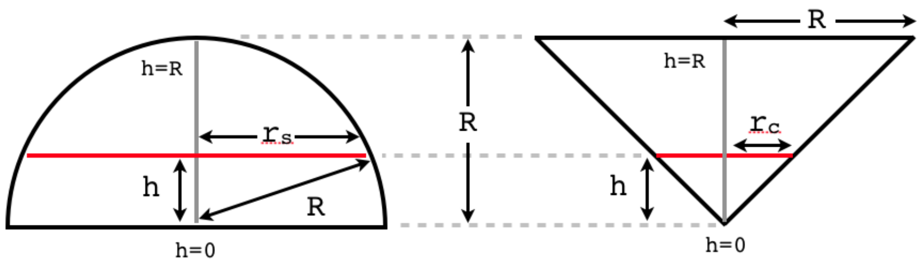
The area element above is the volume element that we'll talk about later in the chapter, just missing the factor dr

Next, we briefly repeat the calculations shown previously, and after that new techniques with double (and triple) integrals will be added to our repertoire.

Archimedes

Archimedes used a method of "slices", which is very close to what we think of as the method of integration.

Consider a hemisphere of radius R oriented with its base in the xy-plane. Position this next to a cone having the same radius R and the same height (also R). Kind of a squat-looking cone.



Now invert the cone, so that its tip is on the plane and the fattest part is farthest away.

What Archimedes found is that for *any* cross-section parallel to the xy-plane, at some height h, the slices have the property that the area of the sphere's cross-section plus the area of the cone's cross-section is a constant. Furthermore, together they equal πR^2, the area of the cross-section of a cylinder with radius.

Pythagoras gives us the radius of the cross-section of the sphere (which is a circle) at height h from the xy-plane as

$$r_s^2 = R^2 - h^2$$

Since the cone has total height and radius both equal to R, and since it's inverted, at any height h, the radius at that height is equal to h

$$r_c^2 = h^2$$

Addition of the areas of the two cross-sections gives

$$\pi r_s^2 + \pi r_c^2 = \pi(R^2 - h^2) + \pi(h^2) = \pi R^2$$

which is equal to the area of the cross-section of a cylinder with radius R.

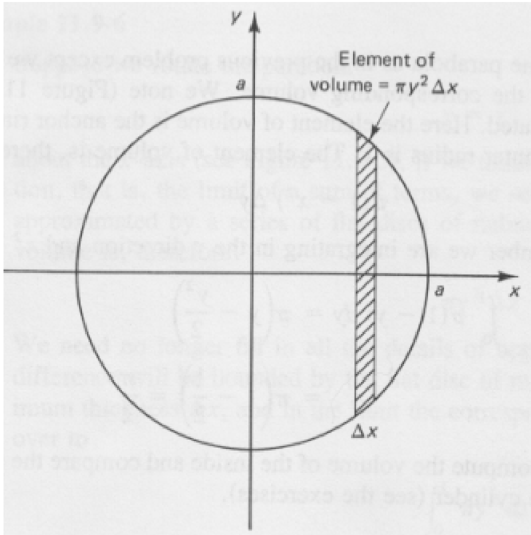
When we add up all the slices from bottom to top, since the cross-sections add to πR^2 at any height, the total of all the cross-sections, which are volumes, add up too.

We conclude that **hemisphere plus cone equals cylinder**.

Since the volumes of the cylinder and cone are πR^3 and $\pi R^3/3$, respectively, the volume of the hemisphere is the difference, and the volume of the sphere is twice that, namely $4/3\pi R^3$.

Method of disks

Consider a function $y = f(x)$, and imagine that we rotate the graph of the function around the x-axis. The rotational symmetry allows us to calculate the volume as a single integral.



The method is to "slice" the volume using slices perpendicular to the x-axis. These cross-sections are circles, because of the rotation. Every slice has volume equal to the area of that slice, πy^2, times the width dx, so the total volume is obtained by adding up all of the slices:

$$V = \int \pi y^2 \; dx$$

Our specific problem is to obtain a sphere, which we get by rotating the top half of the graph of a circle

$$f(x) = +\sqrt{R^2 - x^2}$$

The bounds are $x = -R \to R$. So the integral is just

$$V = \pi \int_{-R}^{R} (R^2 - x^2) \; dx$$

It makes it slightly easier to notice at this point that that $R^2 - x^2$ is an even function of x so

$$\pi \int_{-R}^{R} (R^2 - x^2) \; dx = 2\pi \int_{0}^{R} (R^2 - x^2) \; dx$$

We integrate from $[0, R]$ and then multiply by 2.

$$= 2\pi \ [\ (R^2 x - \frac{x^3}{3}) \Big|_0^R \ ]$$

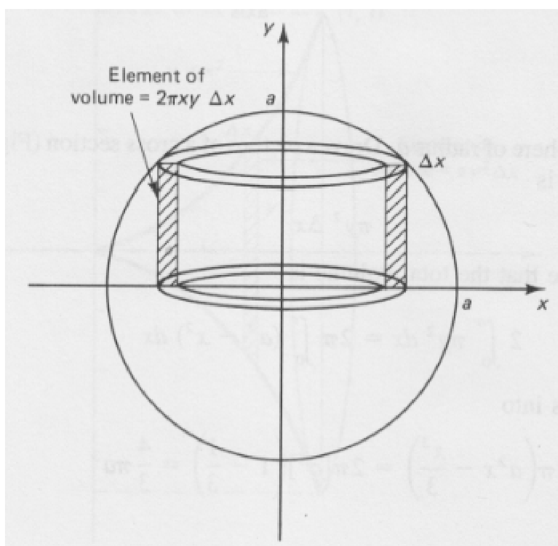
$$= 2\pi \ [\ R^3 - \frac{1}{3} R^3 \ ] \ = \frac{4}{3} \pi R^3$$

Method of shells

The third approach is the method of shells. Here is a picture of what we're doing, from Hamming's Calculus text. The notation is different but the idea is the same.

We'll work with the hemisphere, above the xy-plane.

Let's divide the sphere up into concentric cylinders or shells, and let r vary from $0 \to R$.



The circumference of the shell at each point is

$$C = 2\pi r$$

and the height of each is

$$h = \sqrt{R^2 - r^2}$$

The volume of each very thin cylinder is the product

$$dV = Ch\ dr = 2\pi r \sqrt{R^2 - r^2}\ dr$$

and we want to integrate this over the interval $[0, R]$:

$$V = 2\pi \int_{r=0}^{r=R} r \sqrt{R^2 - r^2}\ dr$$

$$= 2\pi\ [-\frac{1}{3}\pi(R^2 - r^2)^{3/2}\ ]\ \Big|_0^R$$

$$= 2\pi\ [-\frac{1}{3}\pi\ [\ -(R^2)^{3/2}\ ]$$

$$= \frac{2}{3}\pi R^3$$

Multiply by two to obtain the total volume.

Double integral

We now move to multi-variable calculus, and consider the sphere as a function $f(x, y)$ of points in the xy-plane. The graph of this function is plotted on the z-axis, above the points. We consider one octant of the hemisphere, the part where all three of $x, y, z > 0$. The equation is

$$R^2 = x^2 + y^2 + z^2$$

$$z = f(x, y) = \sqrt{R^2 - x^2 - y^2}$$

The "shadow" of the sphere is a quadrant of the circle of radius R (since $z = 0$ in the xy-plane). For each little area element dA inside the shadow, we find the distance up to the graph of the function as $f(x, y)$.

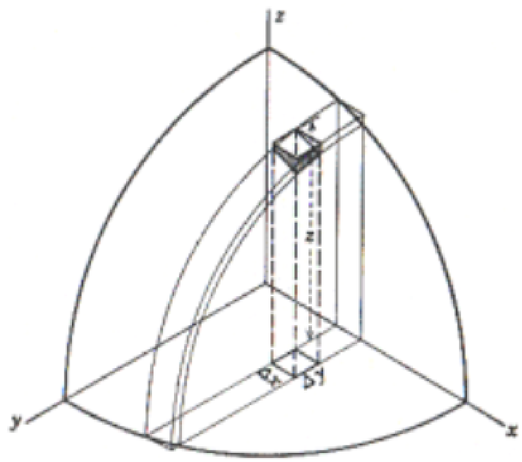


Fig. 10

Suppose we try using Cartesian coordinates, and integrate first with respect to y and then with respect to x. The volume is this double integral

$$V = \int_{x=0}^{x=R} \int_{y=0}^{\sqrt{R^2-x^2}} \sqrt{R^2 - x^2 - y^2}\ dy\ dx$$

This looks rather forbidding, but it won't be too bad. Let us first, for the moment, substitute $a = \sqrt{R^2 - x^2}$ and keep in mind that this is a constant for the inner integral. So that part is now

$$\int_0^a \sqrt{a^2 - y^2}\ dy$$

(We looked at this integral in the section on **cosine squared**).

A little investigation reveals that the right approach to this is a trig substitution, namely

$$y = a \sin\theta$$
$$dy = a \cos\theta \; d\theta$$

We will have

$$\int \sqrt{a^2 - a^2 \sin^2\theta} \; a \cos\theta \; d\theta$$
$$= a^2 \int \sqrt{1 - \sin^2\theta} \cos\theta \; d\theta$$
$$= a^2 \int \cos^2\theta \; d\theta$$

The crucial thing is to change the limits. We need to find the value of θ at the old limits for y, namely

$$y = 0, \quad 0 = a \sin\theta, \quad \theta = 0$$
$$y = a, \quad a = a \sin\theta, \quad \theta = \frac{\pi}{2}$$

So

$$a^2 \int_0^{\pi/2} \cos^2\theta \; d\theta$$

We've done cosine-squared many times by now. We use the double-angle version:

$$= a^2 \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta) \; d\theta$$
$$= a^2 \; \frac{1}{2} \; (\theta + \frac{1}{2} \sin 2\theta) \; \Big|_0^{\pi/2}$$

Almost everything is zero

$$= \frac{a^2}{2}(\frac{\pi}{2} + 0 - 0 - 0)$$

$$= a^2\frac{\pi}{4} = (R^2 - x^2)\frac{\pi}{4}$$

The outer integral is

$$\int_{x=0}^{x=R} (R^2 - x^2)\frac{\pi}{4}\ dx$$

$$= \frac{\pi}{4}(R^2x - \frac{1}{3}x^3)\ \Big|_0^R$$

$$= \frac{\pi}{4}\ \frac{2}{3}R^3 = \frac{1}{6}\pi R^3$$

Recall that this is for one octant, so for the whole sphere we multiply by 8 and obtain:

$$V = \frac{8}{6}\pi R^3 = \frac{4}{3}\pi R^3$$

An easier approach would be to use polar coordinates. We had

$$R^2 = x^2 + y^2 + z^2$$

$$z = f(x,y) = \sqrt{R^2 - x^2 - y^2}$$

We let $x^2 + y^2 = r^2$ and remember that the area element in polar coordinates adds a factor of r (from Jacobian):

$$V = \iint \sqrt{R^2 - r^2}\ r\ dr\ d\theta$$

The integrals are independent. So just do the outer one first

$$= 2\pi \int_0^R \sqrt{R^2 - r^2}\ r\ dr$$

$$= 2\pi \ [\ -\frac{1}{3}(R^2 - r^2)^{3/2} \ ] \Big|_0^R$$

$$= \frac{2}{3}\pi R^3$$

and since we did the volume above the xy-plane, we need to multiply by 2 for the final answer.

Triple integral

For the triple integral, we are integrating the volume element over the entire range of values inside the sphere.

$$V = \iiint dV$$

We can do this integral relatively easily in both cylindrical and spherical coordinates. For completeness, I'm going to try it in Cartesian coordinates as well.

Consider the one-eighth of the sphere that has $x > 0, y > 0, z > 0$.

Let's work from the outside in. We will do x last. The limits on x are $x = 0 \to R$. Simple enough. The shadow of the sphere in the xy-plane is a circle with radius R and so the limits on y are $y = 0 \to \sqrt{R^2 - x^2}$.

And then, naturally enough, the limits on z are $z = 0 \to \sqrt{R^2 - x^2 - y^2}$.

So our integral is

$$\int_{x=0}^{R} \int_{y=0}^{\sqrt{R^2-x^2}} \int_{z=0}^{\sqrt{R^2-x^2-y^2}} dz \ dy \ dx$$

The inner integral is trivial. The middle integral is then

$$\int_{y=0}^{\sqrt{R^2-x^2}} \sqrt{R^2 - x^2 - y^2}\ dy$$

This is exactly the integral we solved above. Setting $a^2 = R^2 - x^2$, the integral is of the form

$$\int \sqrt{a^2 - y^2}\ dy$$

If you look it up, you might find an answer like this:

$$\frac{a^2}{2} \sin^{-1} \frac{y}{a} + \frac{y\sqrt{a^2 - y^2}}{2}$$

which at first sight looks much different than the solution we showed above. There are two reasons why it is so different.

The first is that there are two different but equivalent results for the integral of cosine squared (or sine squared). And second, in this version, rather than stay with θ or t from the trig substitution, we have gone back to the original variable.

Plugging in the upper limit $y = \sqrt{R^2 - x^2}$ we obtain

$$\frac{R^2 - x^2}{2} \sin^{-1} \frac{\sqrt{R^2 - x^2}}{\sqrt{R^2 - x^2}} + \frac{\sqrt{R^2 - x^2}\sqrt{R^2 - x^2 - (R^2 - x^2)}}{2}$$

Luckily, the second term is zero. And since $\pi/2 = \sin^{-1}(1)$, for this part we have just

$$= \frac{R^2 - x^2}{2} \cdot \frac{\pi}{2}$$

At the lower limit ($y = 0$), the first term includes $\sin^{-1}(0)$, which equals zero, and the second term has a factor of y, so the whole thing is just zero.

Finally, we come to the outer integral, which is

$$\frac{\pi}{4} \int_0^R R^2 - x^2 \ dx$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} R^3$$

$$= \frac{\pi}{6} R^3$$

Since there are eight such volumes in the whole sphere, we obtain the familiar answer.

Triple integral in cylindrical coordinates

The equation of the sphere in Cartesian coordinates is:

$$R^2 = x^2 + y^2 + z^2$$

Converting to polar coordinates we have

$$r^2 = x^2 + y^2$$

$$R^2 = z^2 + r^2$$

$$z = \sqrt{R^2 - r^2}$$

Recall that the area element in polar coordinates is $dA = r \ dr \ d\theta$.

If we integrate first with respect to z we have

$$\int \int \int dz\ r\ dr\ d\theta$$

The limits on r and θ are as usual:

$$\int_0^{2\pi} \int_0^R \int dz\ r\ dr\ d\theta$$

In particular, we integrate over the entire shadow of the sphere on the xy-plane. Now, for each value of r and θ we must find the limits on z. From above, we get

$$\int_0^{2\pi} \int_0^R \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} dz\ r\ dr\ d\theta$$

The inner integral is just

$$z \Big|_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} = 2(\sqrt{R^2 - r^2})$$

So the middle integral is then

$$\int_0^R 2(\sqrt{R^2 - r^2})\ r\ dr$$

$$= -\frac{2}{3}(R^2 - r^2)^{3/2} \Big|_0^R = \frac{2}{3}R^3$$

The outer integral is trivial

$$\int_0^{2\pi} \frac{2}{3}R^3\ d\theta = \frac{4}{3}\pi R^3$$

Triple integral in spherical coordinates

Here is the figure from *How to Ace Calculus*

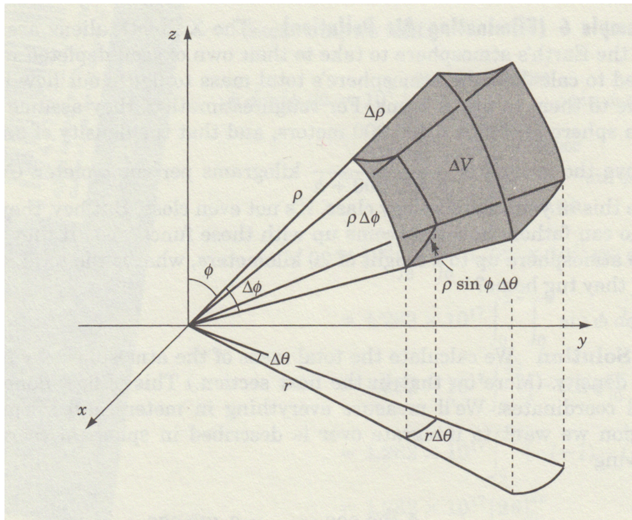


Figure 9.23 Varying each coordinate in spherical coordinates.

In spherical coordinates, we see that the volume element is

$$dV = \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$

And having used that, our work is essentially done. We just set up the integral

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$

The peculiarity of this is that the limits on ρ do not depend on the angles ϕ and θ. So now the inner integral is just

$$= \int_0^{R} \rho^2 \sin\phi \; d\rho = \frac{1}{3} R^3 \sin\phi$$

And the term $R^3/3$ is a constant. So we put that aside and evaluate the middle integral

$$\int_0^{\pi} \sin\phi \; d\phi$$

$$= -\cos\phi \,\Big|_0^{\pi} = -(-2) = 2$$

So in the end we have

$$2\pi \cdot 2 \cdot \frac{1}{3} R^3 = \frac{4}{3}\pi R^3$$

Deriving the formula we used

Up above in the first section of volume integrals (Cartesian coordinates) we used this formula

$$\int \sqrt{a^2 - y^2} \; dy = \frac{a^2}{2} \sin^{-1} \frac{y}{a} + \frac{y\sqrt{a^2 - y^2}}{2}$$

Let me re-write it in a more familiar but equivalent form. We use a simple trig substitution.

$$\frac{x}{a} = \sin t$$

$$x = a \;\sin t$$

$$dx = a \;\cos t \; dt$$

Using Pythagoras

$$\sqrt{a^2 - x^2} = a \cos t$$

Substituting

$$\int \sqrt{a^2 - x^2} \; dx$$

$$= \int a \cos t \; a \cos t \; dt$$

$$= a^2 \int \cos^2 t \ dt$$

An old friend! I'm not going to work this one out from scratch, we've seen it several times. One of the equivalent forms for this integral is

$$\int \cos^2 t \ dt = \frac{1}{2}(t + \sin t \cos t)$$

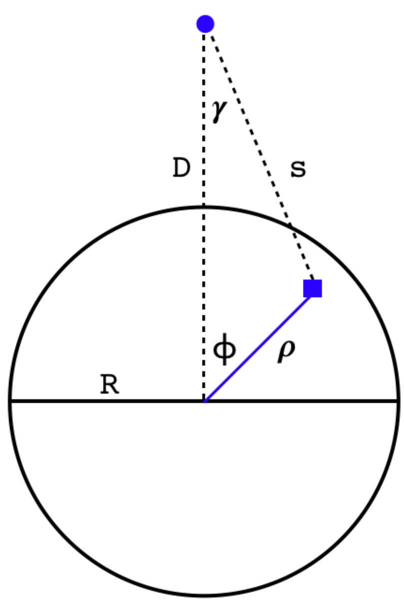
Picking up the outside factor of a^2 and substituting we obtain

$$\begin{aligned} &= a^2 \ \frac{1}{2}(\sin^{-1} \frac{x}{a} + \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a}) \\ &= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x\sqrt{a^2 - x^2}}{2} \end{aligned}$$

Chapter 125

Attraction of the earth

Here we derive what is probably Newton's most famous result, that the mass of a spherical body behaves in terms of gravitational attraction as if it were concentrated as a "point mass" at the very center of the sphere.

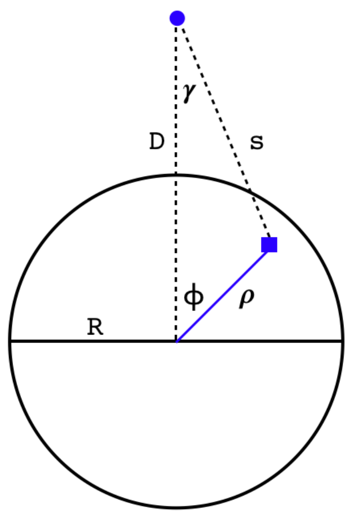


Our notation is shown in the figure. A point mass m (blue circle) lies at a distance D from the sphere's center. Strang and Kline both use

D for this, so I have retained it. The sphere has mass M and radius R.

For the ray from the center to the volume element (or a part of a spherical belt, see later) I have used ρ from Strang, while s is the distance from the volume element to the test mass m.

We are free to choose a coordinate system; it is convenient to orient the z-axis so that it aligns with D; the ray of distance ρ to the little volume element dV (blue square) makes the usual polar angle ϕ as shown. We will also need the angle γ.



There are two different ways of proceeding. One, as indicated by use of dV, is to do a triple integral over the volume. Spherical coordinates are best for this, as always. For this approach, we integrate first over ϕ, then over ρ and finally add 2π from the radial angle θ, which is the outer integral.

An alternative approach is to consider a hollow shell, with ρ fixed. One first proves that the shell acts as if its mass were at the center, then many layers of concentric shells all add up to the total result. The shell approach is not really that much different from the volume integral, which is relatively simple to do in the order given above, once we massage it a bit.

There is another fairly distinct variation for both methods, which I learned from Kline, and that is to use s rather than ϕ as the variable of integration. I had never thought of doing that!

Using s as the variable combined with the method of shells gives an easy extension to a result that we have already used in the book, for the problem with a **tunnel** through the earth. We cover that in the next chapter.

setting up the integral

The inverse square law for gravitation says that the force varies like $1/s^2$ for each small mass element dM contained in a small volume element dV.

The force on the test mass m due to each little piece of mass is

$$dF = \frac{Gm}{s^2}\ dM$$

where $dM = M/V\ dV$ so

$$dF = \frac{GmM}{V}\ \frac{1}{s^2}\ dV$$

Thus, we will have a constant factor of GmM/V multiplying the result. We do not write this any more but remember it for later.

In addition to the fact that the distance s is a variable, the interesting aspect of the integral is that the force, which acts in the direction of s, has a sideways component with respect to D, namely $F \sin\gamma$, that cancels by radial symmetry, and a vertical component $F \cos\gamma$, which does not cancel.

Thus, to get the part of the force that we care about, the part that does not cancel, we have our remembered constant times the summed contributions from each little piece of volume dV:

$$\int_V \frac{1}{s^2} \cos\gamma \ dV$$

$$= \iiint \ [\frac{1}{s^2} \cos\gamma \ ] \ \rho^2 \sin\phi \ d\rho \ d\phi \ d\theta$$

It turns out that these two effects (of $1/s^2$ and $\cos\gamma$) even out so that the value of the integral turns out to be $1/D^2$ times the volume. We talked about the average value of a function **here**. We say that the average value of the function

$$[\frac{1}{s^2} \cos\gamma \ ]$$

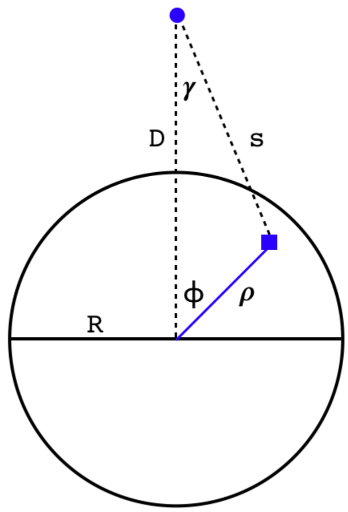
is $1/D^2$.

integral using ϕ

The inner integral is taken with respect to ϕ (holding ρ and θ constant).

$$I = \int \ [\frac{1}{s^2} \cos\gamma \ ] \ \rho^2 \sin\phi \ d\phi$$

We come up against the complication that s and γ are themselves both variables. To make progress, we must express everything in terms of the single variable, ϕ.



The law of cosines comes to the rescue. First, to get s^2 in terms of ϕ

$$s^2 = D^2 + \rho^2 - 2D\rho\cos\phi$$

And then for γ:

$$\rho^2 = D^2 + s^2 - 2Ds\cos\gamma$$

$$\cos\gamma = \frac{D^2 + s^2 - \rho^2}{2Ds}$$

For simplicity, we substitute only for $\cos\gamma$ and not yet for s^2

$$I = \int \frac{1}{s^2} \ [\ \frac{D^2 + s^2 - \rho^2}{2Ds}\ ]\ \rho^2 \sin\phi \ d\phi$$

Even after showing that restraint, I think we can agree that this looks like a bit of a mess.

But now we have an inspiration! Make the substitution:

$$u = s^2$$

$$= D^2 + \rho^2 - 2D\rho\cos\phi$$

The derivative is

$$du = 2D\rho\ \sin\phi\ d\phi$$

$$\rho \ \sin\phi \ d\phi = \frac{1}{2D} \ du$$

With that substitution we generate a factor of $\sin\phi \ d\phi$ in terms of something related to s^2 and can see a path forward:

$$I = \int \frac{1}{s^2} \ [\ \frac{D^2 + s^2 - \rho^2}{2Ds} \ ] \ \rho^2 \sin\phi \ d\phi$$

$$= \int \frac{1}{u} \cdot \frac{u + D^2 - \rho^2}{2D\sqrt{u}} \cdot \rho \cdot \frac{1}{2D} \ du$$

$$= \frac{\rho}{4D^2} \int \frac{u + D^2 - \rho^2}{u^{3/2}} \ du$$

The integral breaks into two parts which are pretty easy

$$= \frac{\rho}{4D^2} \ [\int \frac{1}{\sqrt{u}} \ du + \int \frac{D^2 - \rho^2}{u^{3/2}} \ du \ ]$$

$$= \frac{\rho}{4D^2} \ [\ 2\sqrt{u} - 2 \ \frac{D^2 - \rho^2}{\sqrt{u}} \ ]$$

$$= \frac{\rho}{2D^2} \ [\ \sqrt{u} - \ \frac{D^2 - \rho^2}{\sqrt{u}} \ ]$$

Now, reverse the substitution, but do it gradually. Think about

$$u = s^2 = D^2 + \rho^2 - 2D\rho\cos\phi$$

The bounds on ϕ are the usual $[0, \pi]$ At the upper bound $\cos\phi = -1$ and

$$u = D^2 + \rho^2 + 2D\rho = (D + \rho)^2$$

$$\sqrt{u} = D + \rho$$

So the term in brackets is

$$\sqrt{u} - \ \frac{D^2 - \rho^2}{\sqrt{u}}$$

$$= D + \rho - \frac{(D+\rho)(D-\rho)}{D+\rho} = 2\rho$$

You don't see that kind of simplification every day!

At the lower bound, $\cos\phi = 1$ and

$$u = D^2 + \rho^2 - 2D\rho = (D-\rho)^2$$

$$\sqrt{u} = D - \rho$$

The term in brackets is

$$= D - \rho - \frac{(D+\rho)(D-\rho)}{D-\rho} = -2\rho$$

Since we subtract the value at the lower bound, we get another minus sign and the terms in the brackets add up to 4ρ. The whole integral is

$$I = \frac{\rho}{2D^2} \cdot 4\rho = 2\frac{\rho^2}{D^2}$$

To finish up, integrate next over ρ and obtain

$$\frac{2}{D^2} \int_0^R \rho^2 \ d\rho = \frac{2}{D^2} \ \frac{R^3}{3}$$

Finally, we pick up 2π from the outer integral:

$$\frac{4}{3}\pi R^3 \ \frac{1}{D^2} = \frac{V}{D^2}$$

Recall that we left aside a constant factor of mMG divided by the volume V. Hence the force is:

$$F = \frac{mMG}{V} \ \frac{V}{D^2} = \frac{mMG}{D^2}$$

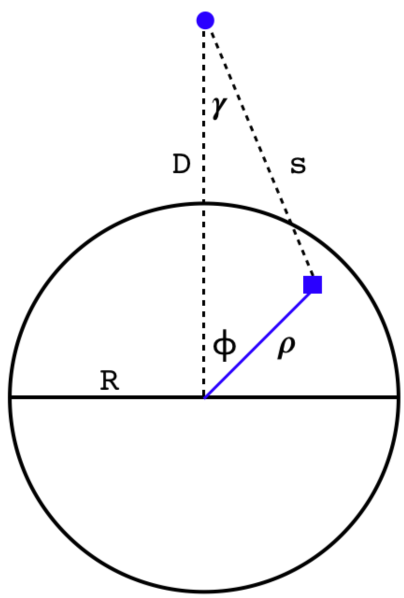
The essence of our result is that

$$\int_V \left[\ \frac{1}{s^2} \cos\gamma \ \right] dV = \frac{V}{D^2}$$

The *average value* of $f(x, y, z)$ is defined to be

$$\frac{\int_V f(x, y, z)\ dV}{\int_V\ dV}$$

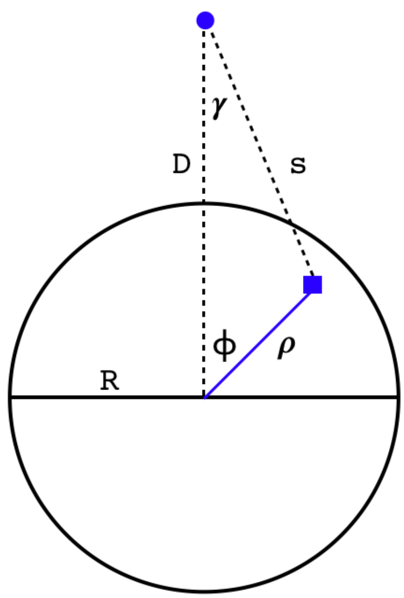
The average value of the function $1/s^2\ \cos\gamma$ taken over the whole sphere is V/D^2 divided by V or just $1/D^2$.



Remarkable.

Chapter 126

Tunnel through the earth



We attack the same problem as we did in the previous chapter, except we set up the integral in a different way, one that allows us to solve a variant problem, that of the hollow sphere.

integral using s

We repeat the analysis with s as the variable. Go back to the integral

$$I = \int [\frac{1}{s^2} \cos\gamma \ ] \ \rho^2 \sin\phi \ d\phi$$

$$= \int \frac{1}{s^2} \ [\ \frac{D^2 + s^2 - \rho^2}{2Ds} \ ] \ \rho^2 \sin\phi \ d\phi$$

We need to get rid of $\sin\phi \ d\phi$, and turn it into something times ds.

The law of cosines for ϕ again:

$$s^2 = D^2 + \rho^2 - 2D\rho\cos\phi$$

Carry out implicit differentiation! (D and ρ are constant).

$$2s \ ds = 2D\rho \sin\phi \ d\phi$$

$$\frac{s}{D} \ ds = \rho \sin\phi \ d\phi$$

So

$$I = \int \frac{1}{s^2} \ [\ \frac{D^2 + s^2 - \rho^2}{2Ds} \ ] \ \rho \ \frac{s}{D} \ ds$$

$$= \frac{\rho}{2D^2} \int \frac{1}{s^2} \ [\ D^2 + s^2 - \rho^2 \ ] \ ds$$

Forget the leading factor for a moment, the integral is

$$\int \ ds + \int \frac{D^2 - \rho^2}{s^2} \ ds$$

$$= s - \frac{D^2 - \rho^2}{s}$$

$$= s - \frac{(D + \rho)(D - \rho)}{s}$$

This might look somewhat familiar.

Now the question is, what are the bounds on s? The minimum value is when D is aligned with ρ and $s = D - \rho$, while the maximum is the other way with $s = D + \rho$. It is important that even though we have substituted s for ϕ as the variable of integration, we are still doing this first integral with ρ constant.

We must evaluate:

$$s - \frac{(D+\rho)(D-\rho)}{s} \Bigg|_{D-\rho}^{D+\rho}$$

This is exactly the same evaluation that we performed for the first integral! The answer has not changed.

So what's the point?

The thing is that now, we can move the test mass m inside a hollow sphere easily. To see how to do that, we set up the problem as a hollow sphere or shell problem. However, the integral will be the one we just solved.

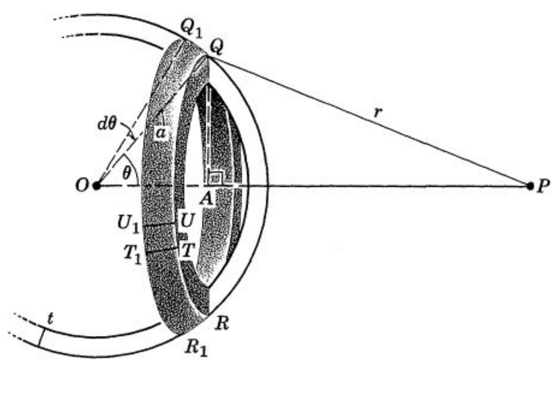
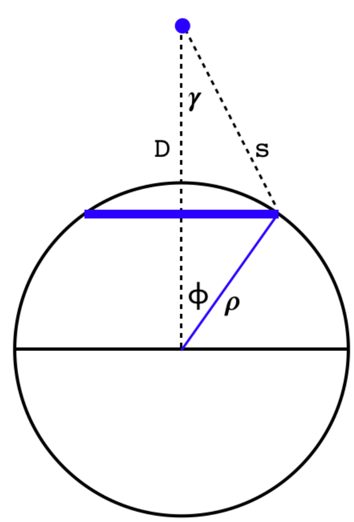


Figure 16-18



We consider a band or belt of the hollow shell, where each piece is at a constant distance from the centerline. In our notation the radius of the belt (a perpendicular dropped from the point Q to point A in the left-hand panel) is $\rho \sin \phi$.

The circumference of the belt is $2\pi\rho \sin \phi$, the width is $\rho \ d\phi$ and so the surface area is

$$A = 2\pi\rho \sin \phi \ \rho \ d\phi$$

(The nice thing here is that $\rho d\phi$ gives the correct width of the belt easily. We saw this approach before in looking at the moment of inertia for a sphere.

Following Kline, rather than calculate the mass, we take t as the thickness of the (very thin) belt and μ as the density. (This difference is not important to the argument).

$dM = \mu \ dV$ so the mass of the belt is

$$dM = \mu t \ 2\pi\rho^2 \sin \phi \ d\phi$$

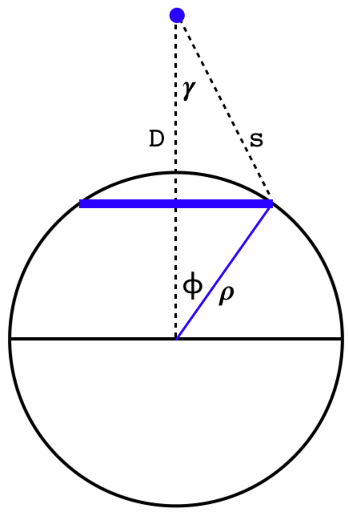
and the total force on the test mass is

$$\frac{Gm\mu t}{s^2} \ 2\pi\rho^2 \sin \phi \ d\phi$$

The last part of the initial setup is that the net force is the component which acts along the central axis. dF is really

$$dF = \frac{Gm\mu t}{s^2} \ 2\pi\rho^2 \sin \phi \ d\phi \cdot \cos \gamma$$

If we were following the wikipedia derivation we would now get everything in terms of ϕ and integrate over $\phi = [0, \pi]$, but as we said above, Kline does something different. He expresses all variables in terms of s instead.



Rather than go through the law of cosines (we would get the same result), we can pretty much just read off the diagram, in the right triangle the side adjacent to ϕ is $\rho \cos \phi$, so the rest of D is the side adjacent to γ, namely $D - \rho \cos \phi$ and

$$\cos \gamma = \frac{D - \rho \cos \phi}{s}$$

This helps, but it's not all that we need. We've exchanged γ for ϕ and s, but we need all s with no ϕ. From the law of cosines (yet again):

$$s^2 = D^2 + \rho^2 - 2D\rho \cos \phi$$

Kline does a bit of clever algebraic manipulation. Rearrange:

$$s^2 - D^2 - \rho^2 = -2D\rho \cos \phi$$

Add $2D^2$ to both sides

$$s^2 + D^2 - \rho^2 = 2D^2 - 2D\rho \cos \phi$$

$$\frac{s^2 + D^2 - \rho^2}{2D} = D - \rho \cos \phi$$

and thus

$$\cos\gamma = \frac{D - \rho\cos\phi}{s} = \frac{s^2 + D^2 - \rho^2}{2Ds}$$

That takes care of γ.

Now rewrite the force as

$$dF = \frac{Gm\mu t}{s^2}\ 2\pi\rho^2\ [\ \frac{s^2 + D^2 - \rho^2}{2Ds}\ ]\ \sin\phi\ d\phi$$

We still have $\sin\phi\ d\phi$. We use this again:

$$s^2 = D^2 + \rho^2 - 2D\rho\cos\phi$$

Carry out implicit differentiation as before:

$$2s\ ds = -2D\rho\ \sin\phi\ d\phi$$

$$\sin\phi\ d\phi = \frac{s}{D\rho}\ ds$$

Back to the force

$$dF = \frac{Gm\mu t}{s^2}\ 2\pi\rho^2\ [\ \frac{s^2 + D^2 - \rho^2}{2Ds}\ ]\ \frac{s}{D\rho}\ ds$$

We are done with the fancy algebra. Just cancel 2, s and ρ top and bottom

$$= \frac{Gm\mu t}{s^2}\ \pi\rho\ [\ \frac{s^2 + D^2 - \rho^2}{D}\ ]\ \frac{1}{D}\ ds$$

Rearrange slightly

$$= \frac{Gm\mu t\ \pi\rho}{D^2}\ [\ \frac{s^2 + D^2 - \rho^2}{s^2}\ ]\ ds$$

This expression is what we have to integrate to get the total force. Forget about the constant leading factor for a moment. The integral is straightforward. It breaks into two parts:

$$\int\ [\ 1 + \frac{D^2 - \rho^2}{s^2}\ ]\ ds$$

$$= s - \frac{(D+\rho)(D-\rho)}{s}$$

And this is exactly the same one we already solved, with the same bounds. The answer is 4ρ. Picking up our leading factor, the force is

$$F = \frac{Gm\mu t \; 4\pi\rho^2}{D^2}$$

Do you see that $4\pi\rho^2 \; \mu t$ is the mass of the shell? Call that mass S. We have then

$$F = \frac{GmS}{D^2}$$

The mass S acts in gravitation as if it were concentrated at the center, a distance D from the test mass m.

To finish the problem, we need to add up an infinite number of infinitely thin shells ranging over the entire radius of the solid ball. For this purpose, we return to

$$F = \frac{Gm\mu t \; 4\pi\rho^2}{D^2}$$

Recall that t is the thickness of a thin shell. In the limit, this thickness is $d\rho$ so we have for the entire ball

$$F = \int_0^R \frac{Gm\mu \; 4\pi\rho^2}{D^2} \; d\rho$$

but this is just

$$= \frac{Gm\mu}{D^2} \; V$$

and $\mu \; V$ is the entire mass of the ball so

$$F = \frac{GmM}{D^2}$$

This is exactly as we had before. It is nice to do things by a different method and get the same answer. However, we will now see another value of this approach. As promised, the test mass moves inside the hollow sphere:

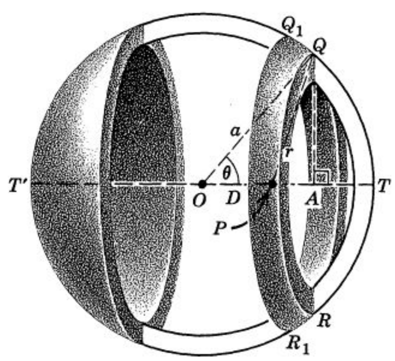


Figure 16-22

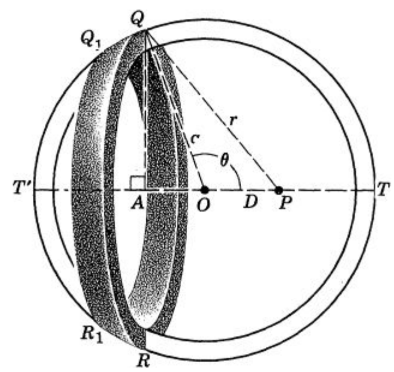
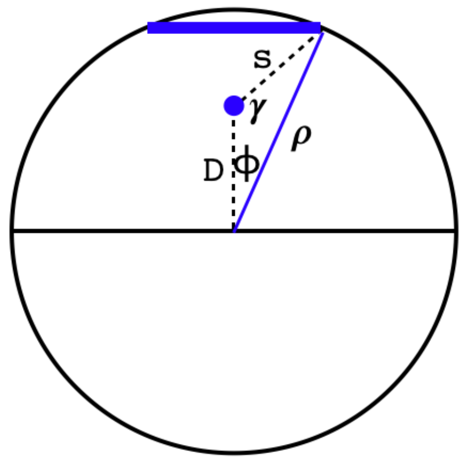


Figure 16-23

One change with the new setup is that ϕ may be obtuse when P is inside. That's OK, the law of cosines is still good even for an obtuse angle, which ϕ has become (θ in Kline's diagram Fig 16-23).

What is different and important is the *bounds* on s.



Now $D < \rho$ (the radius of the shell) and the bounds on s have changed as well.

When s extends to the very top on the same side of the shell where the point P is located, $s + D = \rho$ so

$$s = \rho - D$$

while on the very bottom of the other side $s = \rho + D$.

$$s = [\rho - D, \rho + D]$$

This change of sign at the lower bound makes all the difference.

We evaluate the integral:

$$s - \frac{(D+\rho)(D-\rho)}{s} \Bigg|_{\rho-D}^{\rho+D}$$

At the upper bound, we get

$$(\rho + D) - \frac{(D+\rho)(D-\rho)}{\rho + D} = 2\rho$$

As before. But at the lower bound

$$(\rho - D) - \frac{(D+\rho)(D-\rho)}{\rho - D}$$

flip some signs

$$= (\rho - D) + \frac{(D+\rho)(-D+\rho)}{\rho - D}$$

Canceling top and bottom

$$(\rho - D) + (D + \rho) = 2\rho$$

Subtracting the latter from the former, we end up with zero. There is no net force within the hollow sphere.

Now consider a solid ball, like the earth, and move inside as in the tunnel problem.

Imagine dividing the problem into two parts, break the ball into an outer hollow sphere, and an inner solid ball.

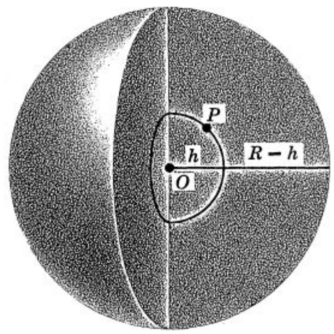


Figure 16-25

The part outside the current location of the object exerts no net force.

The inner part exerts a force proportional to its mass, which is its relative volume times the total mass, over a distance which is smaller than initially. If the distance penetrated is h, the effective mass is

$$\frac{4/3\pi h^3}{4/3\pi R^3}M = \frac{h^3}{R^3}M$$

and the force is exerted over the new distance h so we have

$$F = Gm\ \frac{h^3}{R^3}M\ \frac{1}{h^2} = \frac{GmM}{R^2}\ \frac{h}{R}$$

as we said **here**. A simple answer.

Part XXX

Paths and Surfaces

Chapter 127

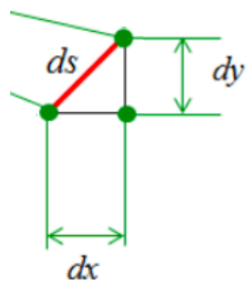
Line integrals

A line integral sums the values taken on by a specified function of x and y in $\mathbb{R}^2$, or x, y, z in $\mathbb{R}^3$, evaluated at a large number positions n along a particular curve, in the limit as $n \to \infty$.

Despite the involvement of x and y (and perhaps z), a line integral is an integral of a single variable. Because the points are all *on the curve*, they can be related. In many cases x and y will be parametric equations (functions of a parameter like t), or we might just express y in terms of x. In any case, the integral will be of a single variable.

A simple application of a line integral is to find the length of a curve. A more sophisticated one yields the work done when moving along a curve, or the flux across a curve, and there are certainly others.

The basic formula can be derived by doing algebra with differentials. Think of a small element of the curve's path, ds, as a right triangle with side lengths dx and dy.



Then Pythagoras tells us that

$$ds^2 = dx^2 + dy^2$$

We "divide and multiply" on the right by dx^2 to give

$$ds^2 = [1 + \frac{dy^2}{dx^2}]\ dx^2$$

then take the square root

$$ds = \sqrt{1 + (\frac{dy}{dx})^2}\ dx$$
$$ds = \sqrt{1 + (y')^2}\ dx$$

Example 0

Here is one where we already know the answer: the arc length along the boundary of the circle of radius R in the first quadrant. Again, the formula is

$$ds = \sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

So we want

$$L = \int ds = \int_0^R \sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

The circle is

$$x^2 + y^2 = R^2$$

By implicit differentiation, we easily obtain

$$2x\ dx + 2y\ dy = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$(\frac{dy}{dx})^2 = \frac{x^2}{y^2}$$

So the integral is

$$= \int_0^R \sqrt{1 + \frac{x^2}{y^2}}\ dx$$

$$= \int_0^R \frac{1}{y}\sqrt{y^2 + x^2}\ dx$$

$$= R\int_0^R \frac{1}{y}\ dx$$

$$= R\int_0^R \frac{1}{\sqrt{R^2 - x^2}}\ dx$$

This can be solved by a trig substitution:

$$x = R\sin t$$

$$dx = R\cos t\ dt$$

$$\sqrt{R^2 - x^2} = R\cos t$$

So we have

$$= R \int \frac{1}{R\cos t} R\cos t \ dt = Rt$$

The slightly tricky part is the limits on t. The lower limit was $x = 0$, so now we need $R\sin t = 0$, so $t = 0$. And the upper limit was $x = R$, so now we need $R\sin t = R$ so $t = \pi/2$. The integral is $\pi R/2$ and the whole circumference is 4 times that or $C = 2\pi R$.

Another, simpler way to do this calculation is to use the parametrized circle $(x = \cos\theta, y = \sin\theta)$. Go back to the original definition of the element of arc ds:

$$ds^2 = dx^2 + dy^2$$

$$L = \int ds = \int \sqrt{dx^2 + dy^2}$$

$$= \int \sqrt{\cos^2\theta + \sin^2\theta} \ d\theta$$

$$= \int \ d\theta$$

Here, we can just go all the way around the circle from $\theta = 0 \Rightarrow 2\pi$. And for a circle of radius a we have $a\cos\theta$ and $a\sin\theta$ which gives a factor of a^2 under the square root, yielding an extra factor of a in the end.

example 1

Consider

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$ds = \sqrt{1 + (\frac{dy}{dx})^2}\ dx$$
$$ds = \sqrt{1 + 4x^2}\ dx$$

The arc length is the integral of ds

$$L = \int \sqrt{1 + 4x^2}\ dx$$
$$= 2 \int \sqrt{(\frac{1}{2})^2 + x^2}\ dx$$

This will be a minor challenge (see trig substitutions). Rather than struggle with it, just set $a = \frac{1}{2}$ and look up the answer in a table of integrals

$$\int \sqrt{a^2 + x^2}\ dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \ln\ |\ x + \sqrt{a^2 + x^2}\ |$$

substitute back for $a = 1/2$

$$\frac{x}{2}\sqrt{\frac{1}{4} + x^2} + \frac{1}{8} \ln\ |\ x + \sqrt{\frac{1}{4} + x^2}\ |$$

Suppose the limits are $x = 1$ and $x = 0$. At the upper limit, we have

$$\frac{1}{2}(\sqrt{1.25}) + \frac{1}{8}\ \ln\ (1 + \sqrt{1.25})$$
$$\sqrt{1.25} \approx 1.118$$
$$\ln(2.118) \approx 0.7505$$
$$(0.5)(1.118) + (0.125)(0.7505) = 0.559 + 0.0938 = 0.6528$$

while at the lower limit the first term is 0 and the second is

$$\frac{1}{8}\ \ln\frac{1}{2} = -(0.125)\ \ \ln 2 = -(0.125)(0.693) = -0.0866$$

Subtracting

$$0.6528 + 0.0866 = 0.7394$$

Remembering the factor of two we get 1.4788

Not exactly pretty, but it works. Check by numerical integration

```
import scipy
f = lambda x: x**2
scipy.integrate.quad(f,0,1)
```

This check gives the expected result 0.33333..

```
g = lambda x: sqrt(1 + 4*x**2)
scipy.integrate.quad(g,0,1)
```

results in 1.47894

example 2

Consider the ellipse

$$y^2 + 2x^2 = 2$$
$$y = \sqrt{2 - 2x^2}$$

By implicit differentiation:

$$2y\ dy + 4x\ dx = 0$$
$$\frac{dy}{dx} = -\frac{2x}{y}$$

Integrate the path element ds:

$$L = \int ds$$

$$= \int \sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

$$= \int \sqrt{1 + \frac{4x^2}{y^2}}\ dx$$

$$= \int \sqrt{\frac{4x^2 + y^2}{y^2}}\ dx$$

$$= \int \sqrt{\frac{2 + 2x^2}{2 - 2x^2}}\ dx = \int \sqrt{\frac{1 + x^2}{1 - x^2}}\ dx$$

This can't be solved in "closed form." The length of an ellipse can only be computed numerically.

Take one-quarter of the perimeter. Let x range from $x = 0$ ($y = \sqrt{2}$) to $x = 1$ ($y = 0$).

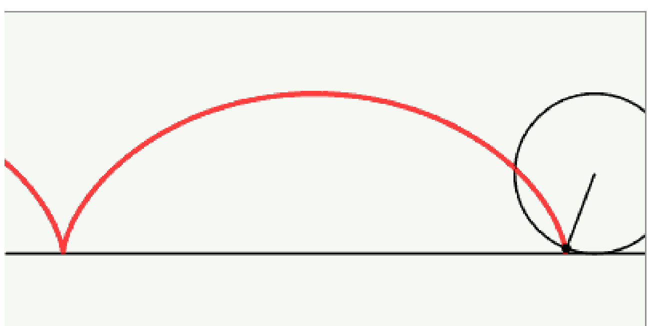
```
>>> from math import sqrt
>>> def f(x):
...     r = (1 + x**2)*1.0/(1 - x**2)
...     return sqrt(r)
...
>>> from scipy.integrate import quad
>>> quad(f,0,1)
(1.9100988945138324, 7.71731567539291e-11)
```

1.91 is the expected result.

Chapter 128

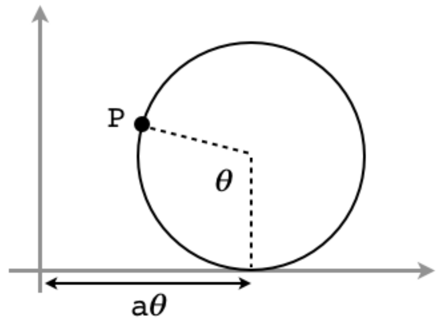
Cycloid

We imagine a bicycle with one tire marked at a particular point on the rim, say with fluorescent paint or a small light. Time starts at $t = 0$ with that point P in contact with the x axis at $(0, 0)$. Then we start rolling the bike. As the tire rotates our fixed point P on the rim traces a curve



We want to find equations that give the position of the point P as a function of time. We will parametrize the curve, yielding parametric equations $x(t)$, $y(t)$.

The second diagram shows the angle through which the wheel has turned as θ, but we will use t for θ here.



The x displacement of the vertical straight down from the center of the tire is just at, where a is the radius of the wheel, it is equal to the arc on the circumference of the wheel from the point which is currently in contact with the ground, going around up to P.

It is reasonably easy to derive the desired parametric equations, using vectors, especially once you know the answer. For x, we have the vector that goes from $(0, 0)$ to the contact point with the ground. As indicated in the figure, that is at.

We need to subtract the distance $a\ sin\ t$ from that. Basically the rationale is that the motion is a standard parametric circle which has been rotated by 90 degrees clockwise and then inverted. The rotation changes cosine to sine, and the inversion the subtraction.

It's easier to see for $t < \pi/2$, but it is true always. Check some other values of t like π or $3\pi/2$ to confirm. This is the usual circular motion.

For y, we have a constant factor of a above the x axis, then the additional displacement is $-a\ cos\ t$. So for $t = 0$ we have the additional displacement is -a (we were on the ground), for $t = \pi/2$ it is zero, and for $t = \pi$ it is plus a for a total of $2a$.

The parametric equations are then

$$x(t) = at - a \sin t$$

$$y(t) = a - a\cos t$$

Taking derivatives:

$$x'(t) = a - a\cos t$$

$$y'(t) = a\sin t$$

The derivation above did a little mental gymnastics with the circle, flipping it and setting $t = 0$ when the point is at the bottom. As an alternative, leave the circle in its usual orientation, with an angle s to the positive x axis.

It can be seen easily that s and t are related by the equation

$$s = 3\pi/2 - t$$

The vector from the center of the circle to the point on the edge is just the standard one for a point on a circle of radius a:

$$a\ \langle \cos s, \sin s \rangle$$

For the x component:

$$\cos s = \cos 3\pi/2 - t$$

$$= \cos 3\pi/2 \cos t + \sin 3\pi/2 \sin t$$

Recall that $\cos 3\pi/2 = 0$ and $\sin 3\pi/2 = -1$ so

$$\cos s = -\sin t$$

And for the y component

$$\sin s = \sin 3\pi/2 - t$$

$$= \sin 3\pi/2 \cos t - \sin t \cos 3\pi/2$$

$$= -\cos t$$

The vector is then

$$a \langle \cos s, \sin s \rangle = a \langle -\sin t, -\cos t \rangle$$

In addition, we have to add another vector, one extending from the origin to the center of the wheel. The y component is constant, it is just a. The x-component is the distance the wheel has traveled from its initial position (the distance between the origin and the point of contact with the x-axis, which is at, shown as $a\theta$ in the figure).

Hence the vector to the point is:

$$a \langle -\sin t, -\cos t \rangle + \langle at, a \rangle$$

$$a \langle t - \sin t, 1 - \cos t \rangle$$

which matches what we had before.

Arc length

We wish to determine the arc length and area under the curve for one complete revolution of the wheel.

We want to use a slightly different version of the usual formula for arc length

$$ds^2 = dx^2 + dy^2$$

$$(\frac{ds}{dt})^2 = (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2$$

$$ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

$$= \sqrt{(a - a\cos\ t)^2 + (a\sin\ t)^2}\ dt$$

This expands to

$$a\sqrt{1 - 2\cos\ t + \cos^2 t + \sin^2 t}\ dt$$

$$= a\sqrt{2 - 2\cos\ t}\ dt$$

The length is

$$L = \int_0^{2\pi} a\sqrt{2 - 2\cos\ t}\ dt$$

$$= a\sqrt{2}\ \int_0^{2\pi} \sqrt{1 - \cos\ t}\ dt$$

double angle

$$\cos(s - t) = \cos s \cos t + \sin s \sin\ t$$

(check: if $s = t$ then $\cos 0 = 1$, which is correct).

So

$$\cos(s + t) = \cos s \cos t - \sin s \sin\ t$$

Let $s = t$ and $u = 2s$, then

$$\cos 2s = \cos u = \cos^2\ (\frac{u}{2}) - \sin^2\ (\frac{u}{2})$$

$$\cos u = 1 - \sin^2\ (\frac{u}{2}) - \sin^2\ (\frac{u}{2})$$

$$2\sin^2\ (\frac{u}{2}) = 1 - \cos u$$

u is just a dummy variable, so we can switch back to t

$$2\sin^2\ (\frac{t}{2}) = 1 - \cos t$$

finishing up

We have that

$$L = a\sqrt{2}\ \int_0^{2\pi} \sqrt{1 - \cos\ t}\ dt$$

And

$$1 - \cos t = 2\sin^2(\frac{t}{2})$$

$$\sqrt{1 - \cos t} = \sqrt{2}\sin(\frac{t}{2})$$

So

$$L = a\sqrt{2}\ \int_0^{2\pi} \sqrt{2}\sin\ (\frac{t}{2})\ dt$$

$$= 2a\ \int_0^{2\pi} \sin\ (\frac{t}{2})\ dt$$

$$= 2a\ (-2)\cos\ (\frac{t}{2})\ \Big|_0^{2\pi}$$

$$= -4a\ (\cos\ \pi - \cos\ 0)$$

$$= -4a\ (-1 - 1) = 8a$$

A simple answer to the problem.

Area under the arc

We want

$$A = \int_{t=0}^{t=2\pi} y\ dx$$

$$= \int_{t=0}^{t=2\pi} (a - a\cos\ t)(a - a\cos\ t)\ dt$$

$$a^2 \int_{t=0}^{t=2\pi} (1 - \cos\ t)(1 - \cos\ t)\ dt$$
$$a^2 \int_{t=0}^{t=2\pi} (1 - 2\cos\ t + \cos^2\ t)\ dt$$
If you don't remember the result for $\int \cos^2 t\ dt$, you can go back to the double angle formula above and convert from $\sin^2$ to $\cos^2$. Otherwise recall it and write:
$$A = a^2(t - 2\sin\ t + \frac{1}{2}t + \frac{1}{4}\sin 2t)\ \Big|_0^{2\pi}$$
$$a^2(2\pi - 0 + \pi + 0 - 0 + 0 - 0 - 0) = 3\pi a^2$$
Also a very simple answer.

Chapter 129

Ellipse area

The area of an ellipse can be computed in several different ways, all interesting. The simplest way is rescaling. In xy-coordinates, the formula is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$(\frac{bx}{a})^2 + y^2 = b^2$$

What this says is that if the x value of each point on the ellipse is re-scaled by a factor of b/a, the result is

$$u = \frac{b}{a}x$$
$$u^2 + y^2 = b^2$$

a circle of radius b and area $A = \pi b^2$. Because the scaling factor is only in the x-direction

$$x = \frac{a}{b}u$$

the area of the ellipse is bigger than the original circle by a factor of a/b

$$A = \pi b^2 \ \frac{a}{b} = \pi ab$$

We might also argue as follows. The area of the ellipse clearly depends on both a and b, so (assuming linearity) we write

$$A = kab$$

where k is an unknown constant. Now, if $a = b$, we obtain

$$A = ka^2$$

but this is just a circle, with known area

$$A = \pi a^2 = ka^2$$

Hence $k = \pi$ and $A = \pi ab$.

single variable calculus

Solve the equation of the ellipse for y

$$y = b\sqrt{1 - \frac{x^2}{a^2}}\ dx$$

We take the positive square root, and integrate from $x = 0 \rightarrow a$, and should obtain 1/4 the area of the ellipse.

$$A = 4b \int \sqrt{1 - \frac{x^2}{a^2}}\ dx$$

The first thing to do is to get rid of the a by substitution. Let $u = x/a$, so $au = x$ and $a\ du = dx$, then

$$A = 4ab \int \sqrt{1 - u^2}\ du$$

The next step is to recognize that $f(x) = \sqrt{1 - u^2}$ is the equation of a circle. Since we are integrating over the first quadrant, the value of the area is just $\pi/4$. The whole thing is π and we pick up the factor ab from outside to give $A = \pi ab$.

If you failed to see this, you can do a trig substitution. If u is the side opposite angle θ, and 1 is the hypotenuse, then

$$\sqrt{1-u^2} = \cos\theta$$

$$u = \sin\theta$$

$$du = \cos\theta \quad d\theta$$

and the integral becomes

$$4ab \int \cos^2\theta \; d\theta$$

Before we do the integration, consider the changing bounds. We originally had $x = 0 \to a$, in changing to u by remembering that

$$au = x$$

we obtain $u = 0 \to 1$. Then, in changing to θ we have

$$u = \sin\theta$$

$$\theta = \sin^{-1} u$$

and we have $\theta = 0 \to 2\pi$. I'm not going to do the integral here, but just give the result

$$\int \cos^2\theta \; d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta)$$

(and there are other ways to write it). But we will take a moment to check that by differentiating

$$\frac{d}{d\theta}\;\frac{1}{2}(\theta + \sin\theta\cos\theta)$$

$$= \frac{1}{2}(1 + \cos^2\theta - \sin^2\theta)$$

$$= \frac{1}{2}(1 + \cos^2\theta + \cos^2\theta - 1) = \cos^2\theta$$

So we need to evaluate

$$4ab\ [\ \frac{1}{2}(\theta + \sin\theta\cos\theta)\ ]\ \Big|_0^{\pi/2}$$

Only one term is non-zero and that is $\theta = \pi/2$ at the upper limit. We obtain

$$A = 4ab\ (\frac{1}{2}\ \frac{\pi}{2}) = \pi ab$$

Green's Theorem

If you know **Green's Theorem**, you can use that.

The statement of the theorem is:

$$\oint_C \mathbf{F} \cdot \mathbf{r} = \iint_R \nabla \times \mathbf{F}\ dA$$

$$\int_C M\ dx + N\ dy = \iint_R (N_x - M_y)\ dx\ dy$$

The theorem equates the line integral around a closed path with an area over the region enclosed by the path (going counter-clockwise).

To start with, if $\mathbf{F}$ is the gradient of some function, we call such a function the potential, and the integral of the work over a closed path is just zero.

Suppose $N_x - M_y = 1$. Then the curl integral is the area of the region. An example would be if $\mathbf{F} = \langle M, N \rangle = \langle -y/2, x/2 \rangle$. Parametrize the ellipse.

$$x = a\cos\theta$$

$$y = b\sin\theta$$

So, for the left hand side we have

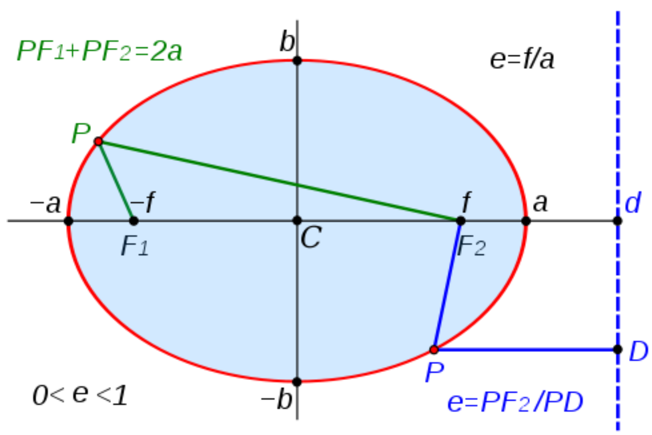
$$\int_C M\ dx + N\ dy = \int_C -\frac{1}{2}y\ dx + \frac{1}{2}x\ dy$$

$$= \int_0^{2\pi} (-\frac{1}{2})(b\sin\theta)\ (-a\sin\theta)\ d\theta\ + (\frac{1}{2})(a\cos\theta)\ (b\cos\theta)\ d\theta$$

$$= \int_0^{2\pi} (\frac{ab}{2}\sin^2\theta + \frac{ab}{2}\cos^2\theta)\ d\theta$$

$$= \frac{ab}{2}\int_0^{2\pi}\ d\theta = \pi ab$$

volume



I want to compute the volume of an ellipsoid. We imagine the solid formed by rotating the ellipse around the x-axis. For each value of x, this solid will have a cross-section whose radius is equal to y, so to get the volume of the ellipse we do

$$V = \int_{-a}^{a} \pi y^2 dx$$

Now,

$$x = a \cos t$$

$$dx = -a \sin t \ dt$$

And we will have to find new limits for the integral. Let's set it up first So

$$V = \pi \int (b^2 \sin^2 t)(-a \sin t) \ dt$$

Previously we had

$$x = -a \to a$$

The lower limit corresponds to $t = \pi$ and the upper limit to $t = 0$.

$$V = \pi a b^2 \int_{\pi}^{0} (\sin^2 t)(-\sin t) \ dt$$

$$= \pi a b^2 \int_{\pi}^{0} (1 - \cos^2 t)(-\sin t) \ dt$$

$$= \pi a b^2 \ [\ \cos t - \frac{1}{3} cos^3 t \ ] \ \Big|_{\pi}^{0}$$

$$= \pi a b^2 \ [\ (1 - \frac{1}{3}) - (-1 + \frac{1}{3}) \ ]$$

$$= \frac{4}{3} \pi a b^2$$

This is quite beautiful. If we consider the three axes in space, for y and z the surface passes through at b, so b counts twice in the volume. If we rotated the other way (around the y axis), we would obtain $\frac{4}{3}\pi a^2 b$.

Chapter 130

Surface integrals

Here we consider some basic facts about surface integrals.

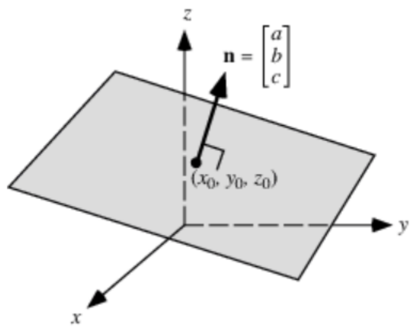
For a point on a surface, we use the tangent plane approximation. Imagine that the surface is tiled, composed of numerous tiny planes. At any point, the surface is locally flat, having uniform slope.

We will need to find the angle θ this plane makes with the horizontal. We use that angle for this calculation:

$$dA = \cos\theta \; dS$$
$$dS = \frac{1}{\cos\theta} \; dA$$

A little piece of area on the surface is made smaller in its projection to the xy-plane by this factor.

Like any plane, we describe a tangent plane by its normal vector, $\mathbf{n}$.



Can you see that $\mathbf{n}$ makes the same angle θ with the z-axis or $\hat{\mathbf{k}}$ unit vector as the surface element makes with the x, y-plane, so that

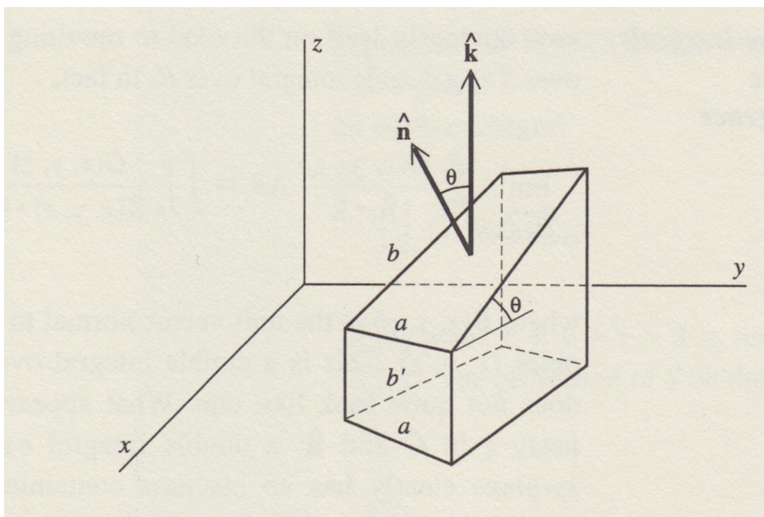


Figure II–14

$$\mathbf{n} \cdot \hat{\mathbf{k}} = |\mathbf{n}||\hat{\mathbf{k}}| \ \cos\theta$$

Since $\hat{\mathbf{k}}$ is a unit vector, $|\hat{\mathbf{k}}| = 1$

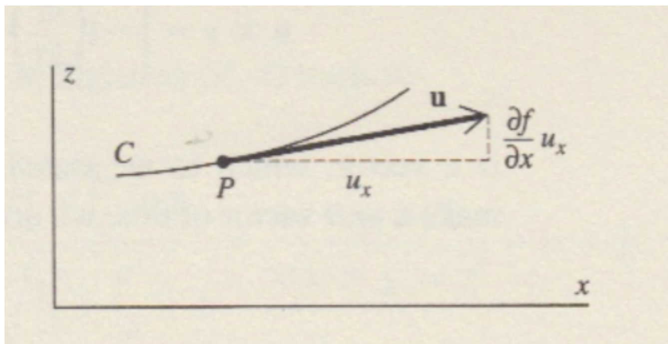
$$\mathbf{n} \cdot \hat{\mathbf{k}} = |\mathbf{n}| \ \cos\theta$$

$$\cos\theta = \frac{\mathbf{n} \cdot \hat{\mathbf{k}}}{|\mathbf{n}|}$$

As always, find $\mathbf{n}$ by first finding two vectors in the plane and then forming the cross-product.

We will use the basis vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$, so think about a cross-section of the plane at the point of interest, parallel to the xz-plane.

The cross-section of the surface is just a line with some slope to it, f_x. For each unit of change in the x or $\hat{\mathbf{i}}$ direction, there is a change of f_x in the $\hat{\mathbf{k}}$ direction and none in the $\hat{\mathbf{j}}$ direction.



So one vector in the plane is

$$\langle 1, 0, f_x \rangle$$

By symmetry, the second is

$$\langle 0, 1, f_y \rangle$$

The cross-product is evaluated by forming the matrix

$$\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{bmatrix}$$

and evaluating its "determinant"

$$\mathbf{n} = \langle -f_x, -f_y, 1 \rangle$$

According to this formulation $\mathbf{n} \cdot \hat{\mathbf{k}}$ is 1!

Returning to what we wrote above

$$\cos\theta = \frac{\mathbf{n} \cdot \hat{\mathbf{k}}}{|\mathbf{n}|} = \frac{1}{|\mathbf{n}|}$$

Our currency exchange rate for area in the plane compared to area on the sloped tangent plane is just

$$dS = \frac{1}{\cos\theta}\ dA$$

$$dS = |\mathbf{n}|\ dA$$

and

$$|\mathbf{n}| = \sqrt{1 + f_x^2 + f_y^2}$$

Hence

$$\iint_S dS = \iint \sqrt{1 + f_x^2 + f_y^2}\ dx\ dy$$

aid to memory

One way to help remembering this formula is by analogy to line integrals, which seem pretty straightforward to set up, though they are often an arithmetic mess.

The small element of the path is ds so by Pythagoras we get

$$ds^2 = dx^2 + dy^2$$

and then a little rearrangement gives:

$$ds^2 = \; [\; 1 + (\frac{dy}{dx})^2 \;] \; dx^2$$

$$ds = \; [\; \sqrt{1 + y'^2} \;] \; dx$$

The area element for surfaces is almost exactly the same:

$$dS = \sqrt{1 + f_x^2 + f_y^2} \; \cdot \; dA$$

So let's just remember the surface area element dS as an extension of ds to 2 dimensions.

plane

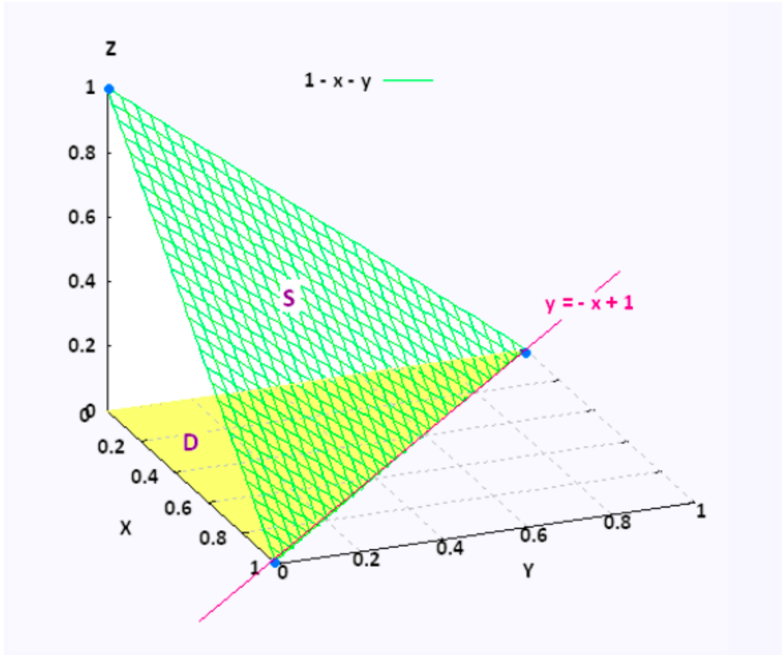
Consider the plane that cuts through each axis at its unit dimension: i.e. $P = (1, 0, 0)$, $Q = (0, 1, 0)$, and $R = (0, 0, 1)$. Purely from symmetry, we could argue that the normal vector should be

$$\mathbf{N} = \langle 1, 1, 1 \rangle$$

By the same argument, the equation of the plane should be

$$x + y + z = c = 1$$

where we evaluate the constant $c = 0$ from looking at the diagram.



Alternatively, obtain two vectors in the plane by subtraction:

$$\mathbf{u} = PQ = \langle 1, -1, 0 \rangle$$

$$\mathbf{v} = PR = \langle 1, 0, -1 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \langle -1, -1, -1 \rangle$$

which is what we had, within sign. So we will say that

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$$

We can get the Jacobian J in two ways. First

$$x + y + z = 1$$

$$z = 1 - x - y$$

$$f_z = -1$$

$$f_y = -1$$

$$J = \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{3}$$

Alternatively

$$|\mathbf{u} \times \mathbf{v}| = |\langle -1, -1, -1 \rangle| = \sqrt{3}$$

Either way, the area element on the surface is

$$dS = J\ dA = \sqrt{3}\ dA$$

The other issue is the bounds for the shadow in the xy-plane. We can see that the slope of the line relating x and y is -1 and the y-intercept is 1 so

$$y = 1 - x$$

Now just set up the double integral

$$\iint_R J\ dA = J \int_0^1 \int_0^{1-x}\ dy\ dx$$

The inner integral is

$$1 - x$$

So the outer integral is

$$\sqrt{3} \int_0^1 1 - x\ dx$$

$$\sqrt{3}\ [\ x - \frac{x^2}{2} \Big|_0^1\ ]\ = \frac{\sqrt{3}}{2}$$

We check this as follows. The lengths of the three sides of the surface we're measuring are each

$$|\mathbf{u}| = |\langle 1, -1, 0 \rangle| = \sqrt{2}$$

The point midway along the base to which we would draw an altitude from R is $(1/2, 1/2, 0)$ so its distance from R is

$$\sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2 + (-1)^2} = \sqrt{\frac{3}{2}}$$

The area of the triangle is

$$\frac{1}{2}\sqrt{2}\,\sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{2}$$

Note: we had the length of the normal vector as $\sqrt{3}$. This is the rate of exchange J between the surface area and the area of the projection in the plane. That projection is just a triangle of area $1/2$, hence the final answer is $\sqrt{3}/2$.

volume

As long as we're here, let's do the volume:

$$V = \iint_R z\ dy\ dx$$

$$= \int_0^1 \int_0^{1-x} (1 - x - y)\ dy\ dx$$

No Jacobian here!

The inner integral is

$$y - xy - \frac{y^2}{2}\ \Big|_0^{1-x}$$

$$= 1 - x - (x - x^2) - \frac{1}{2}(1 - 2x + x^2)$$

$$= \frac{x^2}{2} - x + \frac{1}{2}$$

The outer integral is

$$\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{2} \Big|_0^1 = \frac{1}{6}$$

In solid geometry, we had that

$$V = \frac{1}{3} \text{ base} \times \text{height}$$

which matches.

sphere

Our formula for the surface area element is

$$dS = \sqrt{1 + f_x^2 + f_y^2} \ \cdot \ dA$$

With just this formula we can also calculate the surface area of a sphere. Write

$$x^2 + y^2 + z^2 = a^2$$

where a is the constant radius. Then

$$z = f(x,y) = \sqrt{a^2 - x^2 - y^2}$$

$$f_x = \frac{1}{2} \ \frac{-2x}{\sqrt{a^2 - x^2 - y^2}} = -\frac{x}{z}$$

or by implicit differentiation:

$$z^2 = a^2 - x^2 - y^2$$

$$2z \ dz = -2x \ dx$$

and then we get the same thing as before. Similarly $f_y = -y/z$.

Now use the formula. Plugging in f_x and f_y:

$$dS = \left[\sqrt{(\frac{x}{z})^2 + (\frac{y}{z})^2 + 1} \right] dA$$

$$= \frac{1}{z} \left[\sqrt{(x^2 + y^2 + z^2} \right] dA$$

$$dS = \frac{a}{z} \, dA$$

We want to set up $\int dS$, although it's pretty ugly in x, y. We get

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dy \, dx$$

The inner integral is essentially:

$$\int \frac{1}{\sqrt{c^2 - y^2}} \, dy$$

where $c^2 = a^2 - x^2$. We don't have y so we need to do a trig substitution. This comes out to be

$$\sin^{-1} \frac{y}{c}$$

evaluated between $-c$ and c.

$$\sin^{-1} \frac{y}{c} \bigg|_{-c}^{c} = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$$

The outer integral is then

$$\int_{-a}^{a} \pi a \, dx = 2\pi a^2$$

It is much nicer to switch to r, θ. We do this to check ourselves. Let

$$x^2 + y^2 = r^2$$

$$z^2 = a^2 - x^2 - y^2 = a^2 - r^2$$
$$\iint dS = \frac{a}{z} \ r \ dr \ d\theta$$
$$= \int_0^{2\pi} \int_0^a \frac{a}{\sqrt{a^2 - r^2}} \ r \ dr \ d\theta$$

That extra factor of r from the surface area element (or the Jacobian, if you like), makes everything easy.

The inner integral is essentially $1/\sqrt{u} \ du$
$$\int \frac{1}{\sqrt{a^2 - r^2}} \ r \ dr = -\sqrt{a^2 - r^2}$$

so we obtain
$$a \ [\ -\sqrt{a^2 - r^2} \ ] \ \Big|_0^a = a^2$$

which is zero at the upper bound and $-a^2$ at the lower bound. Subtracting, we obtain a^2, and with the outer integral, the whole thing is $2\pi a^2$.

We are a factor of 2 off from the known result using both methods, and realize that when we wrote
$$z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$$
we were only looking at the upper hemisphere, so we have found the missing factor of 2. The result above is for the hemisphere.

Paraboloid

Our purpose here is to develop two simple examples, the surface areas of the paraboloid and the hemisphere. For the paraboloid, consider one which has its vertex at $z = 1$ and opens down
$$z = 1 - x^2 - y^2$$

When $z = 0$ this is just $x^2 + y^2 = 1$. We find that

$$f_x = -2x$$

$$f_y = -2y$$

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$$

This would be a good time to switch to polar coordinates.

$$x^2 + y^2 = r^2$$

$$\sqrt{(f_x)^2 + (f_y)^2 + 1} = \sqrt{4r^2 + 1}$$

So we have the integral

$$\int \int \sqrt{4r^2 + 1} \;\; r \; dr \; d\theta$$

with the r term coming from the usual source. The inner integral is

$$\frac{1}{12} \; (4r^2 + 1)^{3/2}$$

For the unit circle ($r = 0 \to 1$), and multiplying by 2π for the outer integral, this is

$$2\pi \; \frac{1}{12} \; [\; (5)^{3/2} - 1^{3/2}] = \frac{\pi}{6} \; [\; 5\sqrt{5} - 1 \;]$$

In general, if the limits for the radius are $r = a \to r = b$ we will have

$$\frac{\pi}{6} \; [\; (4b^2 + 1)^{3/2} - (4a^2 + 1)^{3/2} \;]$$

We can check this using the surface of a solid of revolution. Turn the function to have more familiar variables. It is $y = \sqrt{x}$. Rotated around the x-axis, this solid has a cross-section at each point with circumference $2\pi y = 2\pi\sqrt{x}$. The surface area is

$$\int 2\pi\sqrt{x}\ ds$$

The surface area element is

$$ds = \sqrt{1 + f'(x)^2}\ dx$$

(Looks familiar!) So $f'(x)^2 = \frac{1}{4x}$ and we have

$$\int 2\pi\sqrt{x}\ \sqrt{1 + \frac{1}{4}x}\ dx$$

$$2\pi \int \sqrt{x + \frac{1}{4}}\ dx$$

$$\frac{4}{3}\pi\ (x + \frac{1}{4})^{3/2}$$

$$\frac{\pi}{6}\ (4x + 1)^{3/2}$$

evaluated between $x = 0 \to 1$, we obtain

$$\frac{\pi}{6}\ [\ (5)^{3/2} - 1^{3/2}\ ]\ = \frac{\pi}{6}\ [\ 5\sqrt{5} - 1\ ]$$

which matches what we had by the new method.

vector field

To finish up, let's go back to

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\ dS$$

in x, y-coordinates, this is

$$\hat{\mathbf{n}}\ dS = < -f_x, -f_y, 1 > \ \ dx\ dy$$

How do we get this? Above we derived an expression for the surface element dS

$$dS = \ \sqrt{f_x^2 + f_y^2 + a}\ dx\ dy$$

so in R the integral becomes

$$\iint_R \mathbf{F} \cdot \hat{\mathbf{n}}\ \sqrt{1 + f_x^2 + f_y^2}\ dx\ dy$$

Recall from earlier that

$$\hat{\mathbf{n}} = \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}}\ \ < -f_x, -f_y, 1 >$$

The square root cancels, leaving us with

$$\iint_R \mathbf{F} \cdot \ < -f_x, -f_y, 1 > \ \ dx\ dy$$

example

Suppose that

$$\mathbf{F} = z\hat{\mathbf{i}} - y\hat{\mathbf{j}} + x\hat{\mathbf{k}}$$

$$\mathbf{F} = < z, -y, x >$$

and S is the part of the plane $x + 2y + 2z = 2$ in the positive octant, crossing through the points $(2, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$.

Rearrange the equation of the surface

$$f(x, y) = z = 1 - \frac{x}{2} - y$$

$$f_x = -\frac{1}{2}$$
$$f_y = -1$$

We want to compute

$$\iint_R \mathbf{F} \cdot < -f_x, -f_y, 1 > \; dx \; dy$$

Plugging in

$$\mathbf{F} \cdot < -f_x, -f_y, 1 >$$
$$= < z, -y, x > \; \cdot \; < \frac{1}{2}, 1, 1 >$$
$$= \frac{1}{2}(1 - \frac{x}{2} - y) - y + x$$
$$= \frac{1}{2} + \frac{3}{4}x - \frac{3}{2}y$$

So the integral is

$$= \iint_R \frac{1}{2} + \frac{3}{4}x - \frac{3}{2}y \; dx \; dy$$

If we integrate with respect to dy first, using $y = 0 \to y = -x/2 + 1$, the inner integral is

$$= \frac{1}{2}y + \frac{3}{4}xy - \frac{3}{4}y^2 \Big|_{y=0}^{y=-x/2+1}$$

$$= \frac{1}{2}(-\frac{x}{2} + 1) + \frac{3}{4}x(-\frac{x}{2} + 1) - \frac{3}{4}(-\frac{x}{2} + 1)^2$$

Now

$$(-\frac{x}{2} + 1)^2 = \frac{x}{4} - x + 1$$

so we have

$$= \frac{1}{2}(-\frac{x}{2} + 1) + \frac{3}{4}x(-\frac{x}{2} + 1) - \frac{3}{4}(\frac{x}{4} - x + 1)$$

$$= -\frac{x}{4} + \frac{1}{2} - \frac{3}{8}x^2 + \frac{3}{4}x - \frac{3}{16}x + \frac{3}{4}x - \frac{3}{4}$$

$$= -\frac{1}{4} - \frac{x}{4} - \frac{3}{8}x^2 + \frac{3}{4}x - \frac{3}{16}x + \frac{3}{4}x$$

$$= -\frac{1}{4} + \frac{5}{4}x - \frac{3}{16}x - \frac{3}{8}x^2$$

$$= -\frac{1}{4} + \frac{17}{16}x - \frac{3}{8}x^2$$

and then the inner integral is

$$= -\frac{1}{4}x + \frac{17}{32}x^2 - \frac{1}{8}x^3 \Big|_0^1 = \frac{5}{32}$$

The answer given is 1/2 so I suppose there is a mistake somewhere..

example 2

Take the surface of the unit sphere ($z = \sqrt{1 - x^2 - y^2}$ in the positive octant. The region R in the xy-plane is the unit circle in that quadrant.

We find $f_x = -x/z$ and $f_y = -y/z$ as usual and

$$\mathbf{F} = xz\hat{\mathbf{i}} + z^2\hat{\mathbf{k}}$$

$$\mathbf{F} = < xz, 0, z^2 >$$

So

$$\mathbf{F}\cdot \ < -f_x, -f_y, 1 >$$
$$= < xz, 0, z^2 > \ \cdot \ < \frac{x}{z}, \frac{y}{z}, 1 >$$
$$= x^2 + z^2$$
$$= x^2 + 1 - x^2 - y^2$$

We want compute

$$= \iint_R 1 - y^2 \ dx \ dy$$

The first part of this integral is just the area of the quarter circle of radius 1, equal to $\pi/4$. For the second term, we want to integrate over the same region, which suggests the use of polar coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$
$$\iint_R y^2 \ dx \ dy$$
$$= \int_0^{\pi/2} \int_0^1 r^2 \sin^2\theta \ r \ dr \ d\theta$$

If we do $r^3 dr$ first we get a factor of $1/4$ so we have

$$\frac{1}{4}\int_0^{\pi/2} \sin^2\theta \ d\theta$$

$$= \frac{1}{4}\frac{1}{2}(\theta - \frac{1}{2}\sin 2\theta) \ \Big|_0^{\pi/2}$$
$$= \frac{1}{8}(\frac{\pi}{2})$$

Combining with $\pi/4$ (and remembering the minus sign)

$$= \frac{\pi}{4} - \frac{\pi}{16} = \frac{3}{16}\pi$$

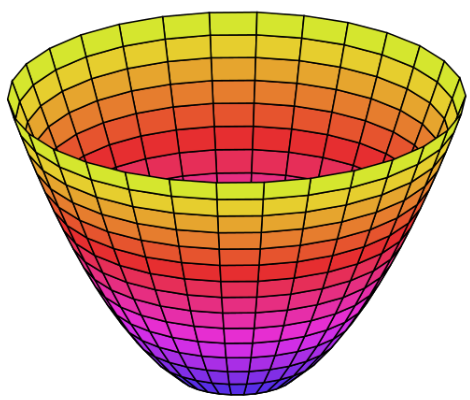
Chapter 131

Paraboloid

volume

A paraboloid is a solid whose vertical cross-sections are parabolas (often oriented along the z-axis).

A paraboloid may open either up, or down.



The cross-sections parallel to the xy-plane are typically circles, though the shape factors for the parabolas in the xz- and yz-planes could be different, leading to an ellipse for the cross-sections.

Consider

$$z = 2 - x^2 - y^2$$

This is a paraboloid that opens down (z gets large and negative when either x or y get large). The vertex is at $z = 2$. When $z = 0$, the cross-section is a circle

$$x^2 + y^2 = 2$$

of radius $r = \sqrt{2}$.

Usually, cylindrical coordinates are good for dealing with this solid. For example, in those coordinates, the surface area element is $dS = r\ dz\ d\theta$.

But let's start with a method from 1-D calculus. Suppose we turn the paraboloid so that its symmetry axis is the x-axis, and its vertex is at the origin. We can model this as being generated by revolution of the graph of $y = \sqrt{x}$ around the x-axis.

The general formulation is that at each point x, we have a circle of radius y and area πy^2. If we consider the width of each slice to be dx, then to get the volume just add all these up

$$V(x) = \int A\ dx = \int \pi x\ dx$$

$$= \pi\ \frac{x^2}{2}\ \Big|_a^b$$

For the unit paraboloid with $a = 0$ and $b = 1$, we get $\pi/2$. And our first paraboloid with $b = 2$ has a volume of

$$V = 2\pi$$

Now, consider the shape factor for the parabola, a. In the standard equation $y = ax^2$, and the larger a is, the faster the parabola grows in the y-direction. But here, we have the parabola "opening" in the $+x$-direction. That is, we have $x = ay^2$ and so $y = \sqrt{x/a}$. Thus

$$V(x) = \int A(x)\ dx = \pi \int \frac{x}{a}\ dx = \frac{\pi}{a} \int x\ dx$$

and we have a factor of $1/a$ for the final volume.

We can ask in another way if these formulas make sense. Consider the parabola $y = x^2$ in the unit square. The area "under" the curve is $1/3$, which is another way of saying that the area "over" the curve and inside the parabola is $2/3$. Compare with the circle, whose area over the unit square is $\pi/4 \approx 3/4$. The circle is a bit "fatter" than the parabola and we expect its volume, when we do the rotation, to be larger in proportion. So this looks reasonable.

another method

Let's find the volume using cylindrical coordinates. I'd also like to generalize the problem. In 1D we orient the vertex at the origin and integrate (usually) from $0 \rightarrow b$. When we turn the volume so that it aligns with the z-axis, we usually place it with the bottom of the desired region in the xy-plane.

Re-write the equation as

$$z = f(x, y) = c - x^2 - y^2$$

where c is the height of the parabola we are measuring.

We are going to use r, θ, z. So we need to find the limits on r. The "shadow" of the paraboloid is a circle in the x, y-plane

$$z = 0 = c - x^2 - y^2$$
$$x^2 + y^2 = r^2 = c$$
$$r = \sqrt{c}$$

The volume in cylindrical coordinates is

$$V = \iiint dV = \iiint r\ dz\ dr\ d\theta$$

What are the bounds on z? Remember, if we integrate first with respect to z then r is *fixed*. For a given r

$$z = c - r^2$$

So, the bounds on z are $z = 0 \to c - r^2$, and the inner integral is just

$$\int_0^{c-r^2} r\ dz = r(c - r^2)$$

This gives us what we would have if we just started by thinking about the double integral of $f(x, y)$ or $f(r, \theta)$ over the region R in the plane.

$$V = \iint f(x, y)\ dx\ dy = \iint f(r, \theta)\ r\ dr\ d\theta$$

The middle integral is

$$\int_0^{\sqrt{c}} cr - r^3\ dr$$

$$= \frac{1}{2}cr^2 - \frac{1}{4}r^4 \Big|_0^{\sqrt{c}}$$

$$= \frac{1}{2}c^2 - \frac{1}{4}c^2 = \frac{1}{4}c^2$$

times 2π from the outer integral. Hence $V = \pi/2 \cdot c^2$.

The way we've set up this problem, the region that we want starts at the xy-plane. So there's not much point in preserving the option of starting evaluation of the middle integral at $a \neq 0$. But if you did decide to do this you could. Just remember to go back and fix the lower bound on z in the inner integral. That would also have to change.

Finally, we could use the method of shells. Imagine a series of concentric cylinders with varying radius.

Each cylinder will have height $c - r^2$ and circumference $2\pi r$ so the lateral surface area is

$$A = 2\pi r(c - r^2)$$

Just integrate from 0 to well, what is the upper bound? When $z = 0$ we have

$$0 = c - r^2$$

$$r = \sqrt{c}$$

Hence the integral is

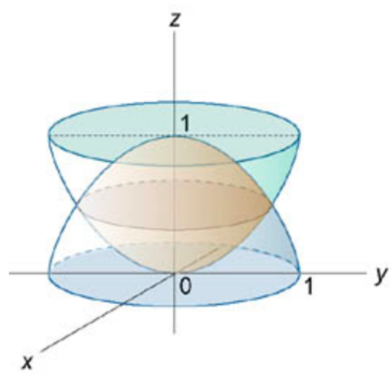
$$V = \int_0^{\sqrt{c}} 2\pi r(c - r^2)\ dr$$

$$= 2\pi(c\frac{r^2}{2} - \frac{r^4}{4}) \Big|_0^{\sqrt{c}}$$

$$= 2\pi(\frac{c^2}{2} - \frac{c^2}{4}) = \frac{\pi}{2}c^2$$

The arithmetic here is exactly the same as for cylindrical coordinates, even though this is an integral of a single variable.

double paraboloid



Just for fun, let's try to do a double paraboloid. In the figure, the paraboloid that opens up is $z = x^2 + y^2$ while the one that opens down is $z = 1 - x^2 - y^2$. To match what we had in the earlier section, I'm going to change the upper one to be $z = 2 - x^2 - y^2$.

We will use cylindrical coordinates to integrate. The key to the problem, as usual, is to find the limits for r and z. First, solve for the intersection of the two surfaces:

$$z = 2 - x^2 - y^2 = x^2 + y^2$$
$$x^2 + y^2 = 1 = r^2$$

So $r = 0 \to 1$. Easy enough. And z ranges from the lower surface to the upper one. Our integral is

$$V = \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} \ dz \ r \ dr \ d\theta$$

The middle integral is

$$\int_0^1 2 - 2r^2 \ r \ dr$$

$$= 2 \ (\frac{r^2}{2} - \frac{r^4}{4}) \ \Big|_0^1$$

$$= 2 \cdot \frac{1}{4} = \frac{1}{2}$$

Multiply by 2π from the outer integral and that gives simply, π. Notice that we have a duplicated version of the first volume, for which we found the answer $\pi/2$. It checks.

paraboloid: surface area

Let's do the surface area paraboloid as well. We can write the equation as

$$z = c - x^2 - y^2$$

This one opens down, and the vertex is at $z = c$.

Suppose the sign of the c term is positive and we want the area above the xy-plane. In the plane, $z = 0$ so

$$x^2 + y^2 = r^2 = c$$

The radius of the circle in the plane is $\sqrt{c}$. That will be the upper bound of the radial integral in polar coordinates.

Recall our formula

$$dS = \ [\ \sqrt{f_x^2 + f_y^2 + 1} \ ] \ dA$$

Here $f_x = -2x$ and $f_y = -2y$ so

$$dS = \left[\sqrt{f_x^2 + f_y^2 + 1} \right] dA$$

$$S = \int dS = \int \sqrt{1 + 4x^2 + 4y^2} \; dA$$

This is, naturally, easier in polar coordinates. $x^2 + y^2 = r^2$ so

$$= \int_0^{2\pi} \int_0^{\sqrt{c}} \sqrt{1 + 4r^2} \; r \; dr \; d\theta$$

The inner integral is

$$= \frac{1}{12} \; (1 + 4r^2)^{3/2} \Big|_0^{\sqrt{c}}$$

$$= \frac{1}{12} \left[\; (1 + 4c)^{3/2} - 1 \; \right]$$

Multiply by 2π to get the whole thing.

A value for c that gives a nice result is $c = 2$, then

$$S = 2\pi \; \frac{1}{12} \left[\; (1 + 4c)^{3/2} - 1 \; \right]$$

$$= 2\pi \; \frac{1}{12} \; (27 - 1) = \frac{26}{6}\pi$$

You can check this by computing this as a surface of revolution. Orient the solid with its axis the same as the x-axis and the vertex at the origin, opening to the right.

We have

$$y = \sqrt{x}$$

$$y' = \frac{1}{2\sqrt{x}}$$

The path element on the surface is

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{1 + y'2}\ dx$$

$$= \sqrt{1 + \frac{1}{4x}}\ dx$$

Multiply the path element by the circumference and integrate to the same bound as we used before:

$$A = \int 2\pi y\ ds$$

$$= 2\pi \int_0^2 \sqrt{x}\ \sqrt{1 + \frac{1}{4x}}\ dx$$

$$= 2\pi \int_0^2 \sqrt{x + \frac{1}{4}}\ dx$$

$$= \frac{4\pi}{3}(x + \frac{1}{4})^{3/2}\ \Big|_0^2$$

$$= \frac{4\pi}{3}(\frac{27}{8} - \frac{1}{8}) = \frac{\pi}{6}\ 26$$

which is the same as what we had before.

Part XXXI

More Geometry

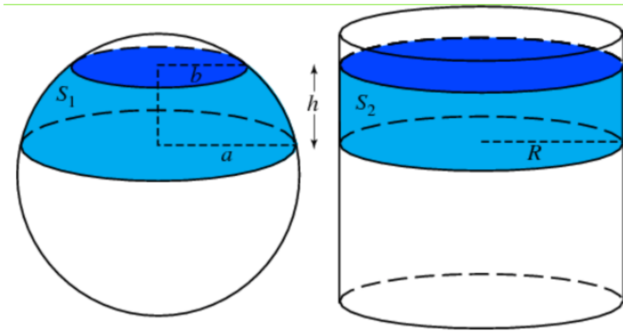
Chapter 132

Hatbox Theorem

A famous result of Archimedes, among many others, is called the hatbox theorem, which states that the surface area of a sphere is equal to the lateral surface area of a cylinder which just encloses it, as we had above. (Lateral area does not count the end pieces).

For a sphere and cylinder of radius R, the cylinder has surface area of the circumference $2\pi R$ times the height $2R$ for a total of $4\pi R^2$.

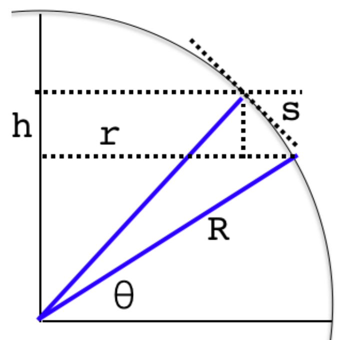
Archimedes showed that this is true not just for the whole, but for any slice or section through the sphere. That's pretty amazing.



Let's sketch a geometric proof briefly.

First, consider a thin strip of surface area extending around the sphere on a great circle (such as the equator). The surface area will be the circumference times the width of the belt, or $2\pi R \times h$.

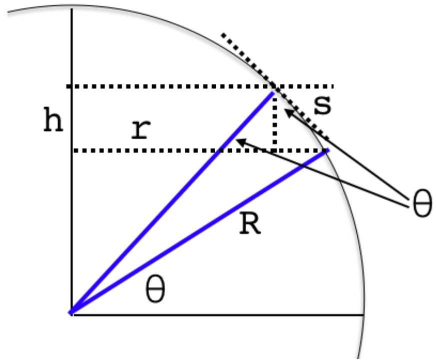
Second, this is true even for a belt that is not at the equator. Consider this figure:



What is the surface area contained between two horizontal cuts of the sphere along the dotted lines? Such a strip is called a "spherical belt". If the slice is very thin, then the circumference at the top and bottom of the slice will be approximately the same, with radius $r = R\cos\theta$.

To get the surface area, we must multiply the circumference $2\pi r$ by the width of the belt. The width is not the height h but s (called the slant height), because of the tilt of the surface.

For a very thin slice, the angle θ won't change much in going from the first blue radius R to the second one. Seeing this, it is then not hard to work out that the angle between s and h in the right triangle containing them both is equal to θ



so

$$\cos\theta = \frac{h}{s}$$

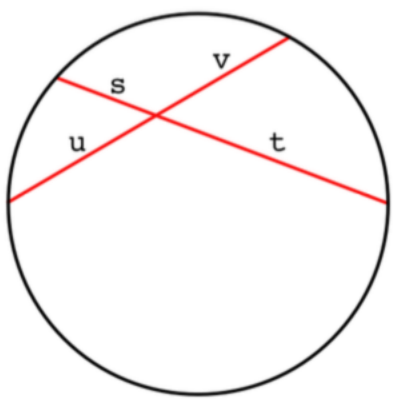
The area is

$$a = 2\pi rs = 2\pi R\cos\theta\frac{h}{\cos\theta} = 2\pi Rh$$

The same as before.

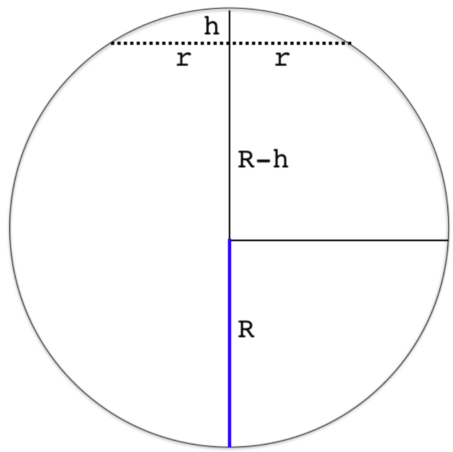
Third, consider the "belt" at the very top of the sphere. This region is more of a "spherical cap", like a contact lens or one of the poles of earth.

We recall a famous result concerning chords of a circle.



$$st = uv$$

This relationship holds for any two chords of the circle. In particular, it holds for the chord consisting of $r + r$ and the chord consisting of $h + (2R - h)$.



By the above theorem

$$r^2 = h(2R - h) = 2Rh - h^2 \approx 2Rh$$

Since h is small, we can neglect a factor of h^2.

The area of the circle is

$$a = \pi r^2 = 2\pi Rh$$

so we have the same rule as above.

Therefore, for *every* belt of height h, the area is $2\pi Rh$.

In summing up the contributions from each belt of width h_i

$$A = \sum a = \sum 2\pi Rh_i = 2\pi R \sum h_i$$

But $\sum h_i$ is just equal to R so we have

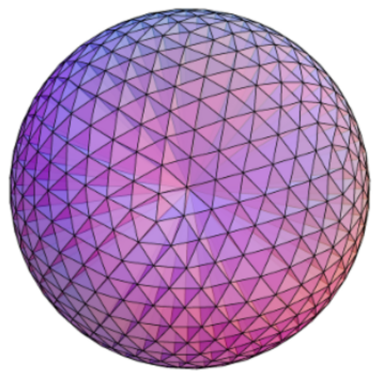
$$A = 2\pi R^2$$

The surface area of a hemisphere of radius R is twice the area of its great circle. Thus, the total area of the sphere is $4\pi R^2$.

geometric proof

Here is a geometric proof that assumes we know the volume of the sphere is

$$V = \frac{4}{3}\,\pi R^3$$



Divide the whole sphere up into triangular prisms. Each one has volume $dV = 1/3\ R\ dA$. So for the whole thing the volume =

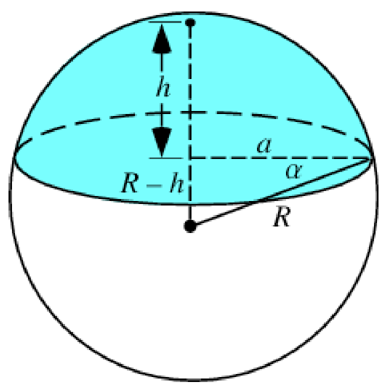
$$\frac{4}{3}\,\pi R^3 = \frac{1}{3} R\ A$$

$$A = 4\pi R^2$$

Chapter 133

Spherical cap

Here is a figure from Wolfram for a spherical cap. We are interested in formulas for the area and volume of the solid obtained by slicing through a sphere, where the height of the cap that is produced is h, and the distance of closest approach to the center of the sphere is $R - h$.



geometry

If we start from the equator, and think about a thin belt going around the sphere, the belt has length equal to the circumference $2\pi R$ and width h, and thus area S:

$$S = 2\pi R h$$

We believe this should be the formula for the surface area of a belt of width h, at least near the equator. In the figure, this width is labeled as $R - h$, because we are more interested in the cap. Thus, for the calculation below, this area will be $2\pi R(R - h)$.

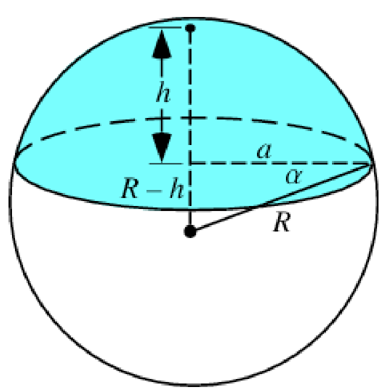
Consider that the total surface area of the hemisphere is $2\pi R^2$ so the area of the cap is the difference

$$S = 2\pi R^2 - 2\pi R(R - h) = 2\pi R h$$

That's a surprising result, that the area of the cap depends only on R and its width (here called h). At least, that is certainly true in the limit as the width of the belt at the equator is very small.

polar cap

Furthermore, if we look in the figure at the right triangle with h and a as the sides, we can draw the hypotenuse of that triangle and call it r (it's not actually labeled in the figure).



It is sometimes called the slant height. We calculate

$$a^2 = R^2 - (R-h)^2 = 2Rh - h^2$$

$$r^2 = a^2 + h^2 = 2Rh - h^2 + h^2 = 2Rh$$

Now think about a very small spherical cap, then it would be almost flat, a circle, and its radius would be r and area

$$S = \pi r^2$$

But $r^2 = 2Rh$, so again we have the same formula for the surface area of a small cap and a belt near the equator!

General case

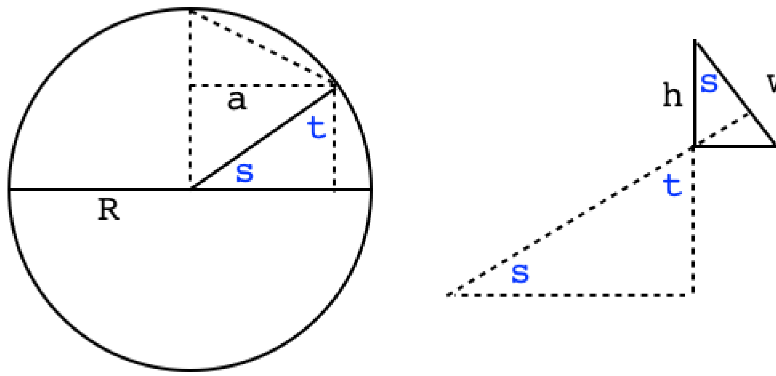
Consider a belt of width h at a position somewhere in the temperate latitudes of the sphere, not close to either the pole or the equator.

We use a thin belt, so that going north toward the pole the surface of the sphere is approximately flat. As before, h is the width of the projection of the belt on the z-axis.

The width w of the belt on the surface is larger than h, because w is not vertical but tilted toward the z-axis. And since the surface is flat, this angle with the z-axis is constant over the width of the belt.

Draw a ray from the center of the sphere to the point where the belt is. The ray makes an angle s with the xy-plane, at the center of the sphere. The radius a at the position of the belt is smaller than R by a factor of $\cos s$.

$$a = R \cos s$$



The tangent to the sphere at this point (namely, w) is perpendicular to the ray. In the right panel, we see a small triangle on the surface of the sphere with sides w and h.

All three of the triangles shown in the right panel are right triangles, with complementary angles s and t (not all of them are labeled). Can you see that the angle between h and w is s?

Therefore, the slant height w of the belt is larger than h by the factor of $\cos s$.

$$h = w \cos s$$

So the true area is

$$2\pi \ a \ w = 2\pi R \cos s \frac{h}{\cos s} = 2\pi R h$$

The cosine of the angle comes in twice, and these factors cancel.

The formula $2\pi R h$ is correct everywhere.

calculus

We imagine rotating a semi-circle around the x-axis to form a volume of revolution. As we've seen in a previous chapter

$$S = \int 2\pi y \sqrt{1 + (\frac{dy}{dx})^2} \ dx$$

The square root comes from the surface area element. This formula looks unwieldy (and often it is). But in this particular case it simplifies dramatically.

If we take a circle as the curve, with formula

$$x^2 + y^2 = R^2$$

$$2x \ dx + 2y \ dy = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

So

$$S = 2\pi \int y \sqrt{1 + \frac{x^2}{y^2}} \ dx$$

$$S = 2\pi \int \sqrt{y^2 + x^2} \ dx$$

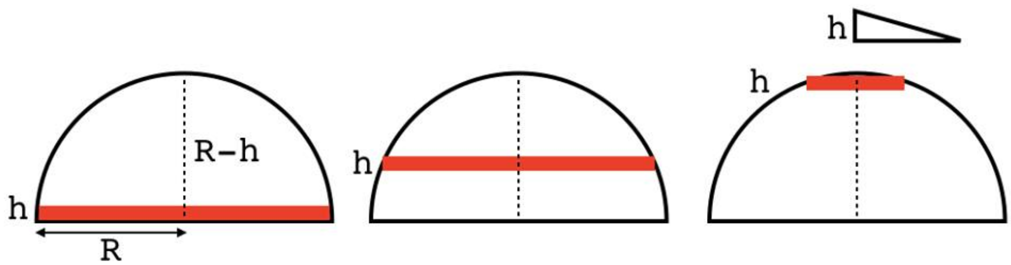
$$S = 2\pi \int R \ dx = 2\pi R x$$

Evaluated between $x = a \to x = b$

$$s = 2\pi R(b - a) = 2\pi R h$$

This makes it very clear that the area does not depend where we are on the sphere. A spherical cap with height h has the same area as a belt of width h wrapped around the equator, or any belt of width h in between the two.

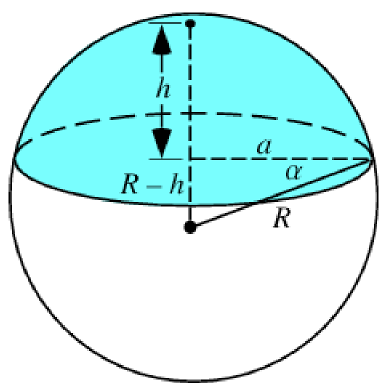
And as we noticed before, the area of a belt or sperical cap, $2\pi Rh$, is equal to the surface area of a cylinder of radius R and height h, the so-called hat-box theorem of Archimedes.



spherical cap: volume

calculus

Here we will derive the formula for the volume of a spherical cap. This is the solid obtained by slicing off a part of a sphere with a plane.



The formula is

$$V_{cap} = \frac{1}{3}\pi h^2(3R - h)$$

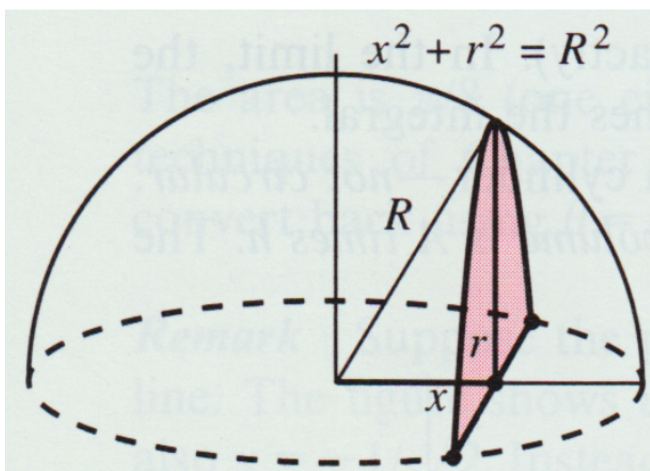
or equivalently

$$V = \pi(Rh^2 - \frac{1}{3}h^3)$$

We can see that this equation makes sense for the extreme case where $h = R$. We get

$$V = \frac{1}{3}\pi R^2(3R - R) = \frac{2}{3}\pi R^3$$

One way to do spherical volumes is by integration of slices as shown in this figure (from Strang)



This approach is basically the same as what I showed when we calculated the volume of the sphere using calculus, before.

In Strang's derivation, at each value of x, the hemisphere (of radius R) has a cross-section that is a half-circle with radius r such that

$$x^2 + r^2 = R^2$$

the area of this hemisphere cross-section is

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(R^2 - x^2)$$

For the whole sphere, each cross-section is a circle with area

$$A = \pi r^2 = \pi(R^2 - x^2)$$

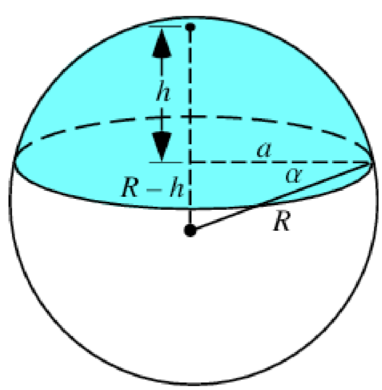
For the volume, we just add up all these slices. To make it simple, take x from $0 \to R$

$$V = \pi \int_0^R (R^2 - x^2)\ dx$$

$$= \pi\ [\ R^2 x - \frac{1}{3}x^3\ ]\ \Big|_0^R = \frac{2}{3}\pi R^3$$

Multiply by a factor of 2 to get the whole thing.

The key insight is, we can get a spherical cap (or a belt), just by changing the lower limit of integration to $x = R - h$!!



We need to evaluate

$$V = \pi\ [\ R^2 x - \frac{1}{3}x^3\ ]\ \Big|_{R-h}^R$$

Leave aside the factor of π, and break the expression into two parts

$$R^2 x \ \Big|_{R-h}^{R} - \frac{1}{3}x^3 \ \Big|_{R-h}^{R}$$

For the left term we get

$$R^3 - R^3 + R^2 h = R^2 h$$

For the right side we get

$$-\frac{1}{3}R^3 + \frac{1}{3}(R-h)^3)$$

$$= -\frac{1}{3}R^3 + \frac{1}{3}R^3 - R^2 h + Rh^2 - \frac{1}{3}h^3$$

Adding left and right terms together, the $R^2 h$ terms cancel, and we have finally

$$V = \pi(Rh^2 - \frac{1}{3}h^3)$$

Factoring out $h^2/3$

$$V = \frac{1}{3}\pi h^2(3R - h)$$

which is the formula we gave at the top.

We can calculate the volume of *any* spherical belt by using the appropriate limits of integration. For example, the belt from $r = 0 \to r = R - h$ has volume

$$V = \pi \ [\ R^2 x - \frac{1}{3}x^3\ ] \ \Big|_{0}^{R-h}$$

Leaving the π aside for now

$$R^2(R-h) - \frac{1}{3}(R-h)^3$$

$$R^3 - R^2h - \frac{1}{3}(R^3 - 3R^2h + 3Rh^2 - h^3)$$

$$\frac{2}{3}R^3 - Rh^2 + \frac{1}{3}h^3$$

With the factor of π

$$V = \pi(\frac{2}{3}R^3 - Rh^2 + \frac{1}{3}h^3)$$

Adding the cap and the belt together:

$$V_{tot} = \pi(Rh^2 - \frac{1}{3}h^3 + \frac{2}{3}R^3 - Rh^2 + \frac{1}{3}h^3)$$

Almost everything cancels

$$V_{tot} = \pi(\frac{2}{3}R^3)$$

The cap and the belt together make up a hemisphere.

Chapter 134

Apple core

45. A hole through a sphere.

Another great problem I want to explore is the shape sometimes called "the cored apple" (Adams et al.) but it is probably more famous as the "napkin ring" problem.

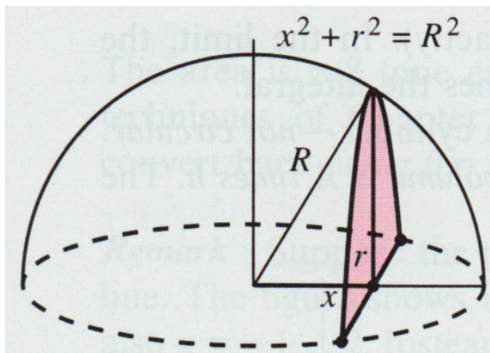
https://en.wikipedia.org/wiki/Napkin_ring_problem

The figure from Acheson, above, is telling. It gives only one length, and that is L, the depth of the hole. The reason is that the volume of the remaining solid is *independent of the size of the sphere*. That's pretty amazing. In fact

$$V = \frac{\pi}{6} L^3$$

fast solution

Recall from the chapter on the spherical cap that we can get the volume of the sphere by integrating slices perpendicular to an axis, say the x-axis. The slices are circles with radius r where



$$r^2 + x^2 = R^2$$

and the area of each slice is

$$A = \pi r^2 = \pi \; [\; R^2 - x^2 \;]$$

The volume is just the integral

$$V = \int_{-R}^{R} \pi \; [\; R^2 - x^2 \;] \; dx$$

Since this is an even function, we can start from 0 and multiply by 2:

$$V = 2 \int_{0}^{R} \pi \; [\; R^2 - x^2 \;] \; dx$$

The integral is easy

$$V = 2\pi \ [\ R^2 x - \frac{x^3}{3} \ ] \ \Big|_0^R$$

What is great about this is that we can get the partial volume by integrating to any height. While we could use $L/2$ for this height, it simplifies the arithmetic to set

$$H = \frac{L}{2}$$

The area of the sphere minus the caps (but including the central cylinder) is

$$V = 2\pi \ [\ R^2 x - \frac{x^3}{3} \ ] \ \Big|_0^H$$
$$= 2\pi \ [\ R^2 H - \frac{1}{3} \ H^3 \ ]$$

All we need now is to find the area of the cylinder.

We are given H. We need the radius r but (referring back to the previous figure) we know that

$$R^2 = H^2 + r^2$$

so the area of the base of the cylinder is

$$A = \pi r^2 = \pi \ [\ R^2 - H^2 \ ]$$

and the volume of the cylinder is

$$V = 2H \cdot \pi \ [\ R^2 - H^2 \ ]$$

Bringing the H inside, we obtain:

$$V = 2\pi \ [\ HR^2 - H^3 \ ]$$

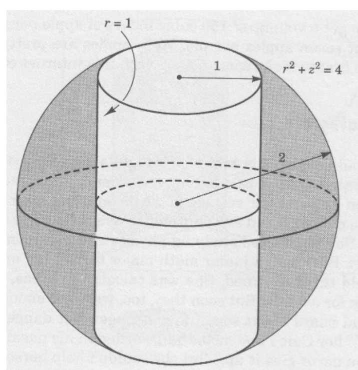
The volume we seek is the volume of the sphere minus its caps, minus the area of the cylinder.

$$V = 2\pi \; [\; R^2 H - \frac{1}{3} \; H^3 - H R^2 + H^3 \;]$$

The magic is that the terms with R^2 cancel so we have

$$\begin{aligned} V &= 2\pi \; [\; -\frac{1}{3} \; H^3 + H^3 \;] \\ &= \frac{4}{3}\pi \; H^3 \\ &= \frac{\pi}{6} \; L^3 \end{aligned}$$

geometry



In Adams we are given the same problem but we don't have L, rather the radius of the sphere and the inset cylinder are given instead.

A sphere of radius 2 has had the central portion consisting of a cylinder plus the two spherical caps removed.

We are given that the cylinder has radius 1. The height L is an unknown.

Recall that

$$R^2 = (\frac{L}{2})^2 + r^2$$

So

$$3 = (\frac{L}{2})^2$$

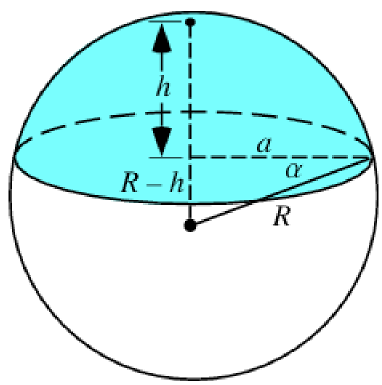
$$L = 2\sqrt{3}$$

Hence, substituting into the formula from the above analysis

$$V = \frac{\pi}{6}\ (2\sqrt{3})^3 = \pi\ \frac{8 \cdot 3 \cdot \sqrt{3}}{6}$$

$$= 4\sqrt{3}\pi$$

Another way to do this problem is to find the volume of each spherical cap. We worked this out in the previous chapter.



We use h here for the height instead of L. If h is the height of the cap, then its volume is

$$V_{cap} = \frac{1}{3}\pi h^2(3R - h)$$

We are given the radius of the core, which is labeled a in the second figure. The equation we get using the Pythagorean theorem is

$$(R - h)^2 + a^2 = R^2$$

$$R^2 - 2Rh + h^2 + a^2 = R^2$$

We know that $R = 2$ and $a = 1$ so this simplifies to

$$h^2 - 4h + 1 = 0$$

Solve this using the quadratic formula to obtain:

$$h = 2 \pm \sqrt{3}$$

Since h cannot be greater than R we take the negative square root:

$$h = 2 - \sqrt{3}$$

$$h^2 = 7 - 4\sqrt{3}$$

We confirm that this value of h solves the quadratic equation.

The volume of one spherical cap is

$$V_{cap} = \frac{1}{3}\pi h^2(3R - h)$$

So

$$\begin{aligned} h^2(3R - h) &= (7 - 4\sqrt{3})(6 - (2 - \sqrt{3})) \\ &= (7 - 4\sqrt{3})(4 + \sqrt{3})) \\ &= 28 - 12 + (-16 + 7)\sqrt{3} \\ &= 16 - 9\sqrt{3} \end{aligned}$$

We have two of these

$$V = 2 \cdot \frac{\pi}{3} \cdot (16 - 9\sqrt{3})$$

The volume of the cylinder is the area of the cross-section

$$\pi \cdot 1^2 = \pi$$

times the height, which is $2(R-h) = 2\sqrt{3}$.

$$V = 2\pi\sqrt{3}$$

We have to subtract both of these from the volume of the whole sphere, which is $32\pi/3$. The desired quantity is

$$V = \frac{32\pi}{3} - 2\pi\sqrt{3} - \frac{2\pi}{3}\ (16 - 9\sqrt{3})$$
$$= \pi\ [\ \frac{32}{3} - 2\sqrt{3} - \frac{2}{3}\ (16 - 9\sqrt{3})\ ]$$
$$= \pi\ [\ -2\sqrt{3} - \frac{2}{3}\ (-9\sqrt{3})\ ]$$
$$= \pi\ [\ 4\sqrt{3})\ ]$$

calculus

There are several more elegant approaches from calculus.

Consider the sphere as a surface above the circle. For the circle we have $x^2 + y^2 = r^2$, where now we are using r as a *variable* that could range from $0 \to R$.

In Cartesian coordinates the circle is

$$x^2 + y^2 + z^2 = R^2$$

Using polar coordinates for x and y we have

$$r^2 + z^2 = R^2$$
$$z = \sqrt{R^2 - r^2} = \sqrt{4 - r^2}$$

We will do a double integral over the the xy-plane, adding up the value of this function for each small area element dA. In polar coordinates, the area element is

$$dA = r\ dr\ d\theta$$

so, for example, the basic area integral is

$$A = \int dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} r\ dr\ d\theta = \int_{\theta=0}^{2\pi} \frac{1}{2} R^2\ d\theta = \pi R^2$$

Now, what we want is to integrate the surface height over the whole area, plugging in from above we get

$$V = \int dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \sqrt{R^2 - r^2}\ r\ dr\ d\theta$$

This integral is not hard to do because we have the derivative of what is under the square root sign. Let $u = R^2 - r^2$. Then

$$du = -2r\ dr$$

$$-\frac{1}{2}\ du = r dr$$

so the inner integral is

$$\int -\frac{1}{2}\sqrt{u}\ du = -\frac{1}{3} u^{3/2}$$

Substituting back, the inner integral (for the whole sphere) is

$$-\frac{1}{3}(R^2 - r^2)^{3/2} \Big|_0^R = \frac{1}{3}(R^2)^{3/2} = \frac{1}{3} R^3$$

When we do the outer integral we pick up an extra factor of 2π, which gives the correct value for the volume of the hemisphere. What is great about this approach is that we don't have to start at $r = 0$.

This simplifies our problem enormously. In the problem, we start from $r = 1$ (and we have $R = 2$). So this gives the volume we seek as

$$V = \int dV = \int_{\theta=0}^{2\pi} \int_{r=1}^{2} \sqrt{4 - r^2} \ r dr d\theta$$

The inner integral is

$$-\frac{1}{3}(4 - r^2)^{3/2} \Big|_1^2 = \frac{1}{3}(3)^{3/2} = \sqrt{3}$$

Now we have

$$V = \int_{\theta=0}^{2\pi} \sqrt{3} \ d\theta$$

which is just $2\pi\sqrt{3}$. Multiply by 2 for the whole apple, we get $4\pi\sqrt{3}$. This matches what we had previously.

rings

Consider a sphere of radius R and a bored hole of radius a. The height of the cylinder (and the part of the sphere that remains) is

$$H^2 = R^2 - a^2$$

Take horizontal cross-sections. For each one, we get a ring (a plane annulus). The area of a cross-section depends on the difference between the outer and the inner rings. The inner ring is constant (a^2) and the outer ring depends on the height above the x, y-plane, h.

$$\text{outer}^2 = R^2 - h^2$$

The difference is

$$R^2 - h^2 - a^2 = H^2 - h^2$$

If we were working before calculus, at this point we would notice that this is the difference between a cylinder of radius H and a cone of variable height h, and use Cavalieri's principle. Does this sound familiar? Look back at Chapter 2 if it does not.

Instead, we let h range from $[0, H]$

$$V = 2 \int_0^H \pi(H^2 - h^2) \ dh$$

The factor of 2 is to include the bottom hemisphere.

$$= 2\pi \ [\ H^2 h - \frac{h^3}{3} \ ] \ \Big|_0^H$$

$$= \frac{4}{3} \pi H^3$$

The result does not depend on R, only H. (However, I don't find this so impressive, since if we specify the problem in terms of the radius a, then R comes in). Also, when $H = R$, we get the correct volume for the whole sphere.

The great advantage of calculus is that we could let the bounds on h be any values in $[0, H]$.

In the previous problem, we had $R = 2$, $a = 1$ and so $H = \sqrt{3}$. We obtain

$$H^3 = 3\sqrt{3}$$

$$\frac{4}{3} \pi H^3 = 4\pi\sqrt{3}$$

which matches our previous result.

cylinders

Or, use the method of cylinders. Let r range over $[a, R]$.

For each r we have a cylinder of circumference $2\pi r$ and height $h = 2\sqrt{R^2 - r^2}$.

The volume is

$$\int_a^R 2\pi r \; 2\sqrt{R^2 - r^2} \; dr$$

$$= 4\pi \; [\; \int_a^R \sqrt{R^2 - r^2} \; r \; dr \;]$$

We already did this integral in a previous section. The result is

$$= -\frac{4}{3}\pi \; [\; (R^2 - r^2)^{3/2} \;]\Big|_a^R$$

$$= \frac{4}{3}\pi \; [\; (R^2 - a^2)^{3/2} \;]$$

$$= \frac{4}{3}\pi \; [\; (H^2)^{3/2} \;]$$

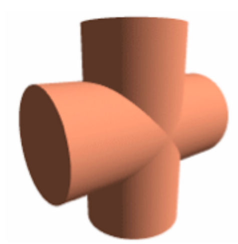
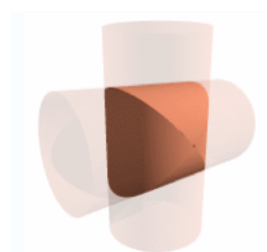
$$= \frac{4}{3}\pi H^3$$

Again.

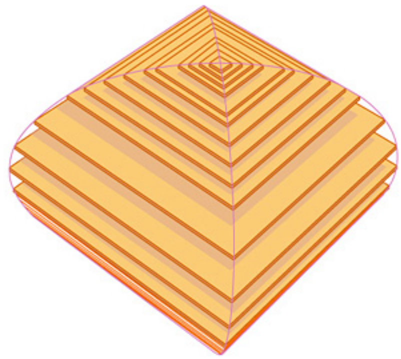
Chapter 135

Stovepipe

I came across an interesting problem in a chapter of Strogatz's *The Joy Of x*. He calls it the stovepipe problem. We want to find the volume of the region formed from the intersection of two cylinders (of equal radius), that meet at right angles.

I really couldn't visualize it, but he told me that the horizontal cross sections of this solid are squares, and gave a picture.



and it makes sense, if you imagine cutting through a potato with a cylindrical bore, then repeating at right angles. At each level, the side is formed by cutting along the edge of the cylinder, hence the opposite sides are parallel. And since the two cuts use the same cylinder, all 4 sides at any particular height are equal, giving a square.

So our problem is to find the length of the side at each value of the height. We deal with the upper half of the solid, for simplicity. If you draw a sketch of the vertical cross-section, at each distance h from the base of the solid, extending up, the remainder of the height down to the base is $R - h$, and the half-length of the side is $s/2 = \sqrt{R^2 - h^2}$.

The volume is obtained by adding up all these slices

$$V = \int_0^R 2\sqrt{R^2 - h^2}\ dh$$

Let's simplify the problem further (for the moment) by dealing with the case where $R = 1$ We need the integral

$$\int \sqrt{1 - h^2}\ dh$$

I certainly didn't know that off the top of my head.

$$\int \sqrt{a^2 - x^2}\ dx = \frac{a^2}{2}\ [\ \sin^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{1 - (\frac{x}{a})^2}\ ]$$

which can be rearranged slightly

$$= \frac{a^2}{2}\ \sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2}$$

for the derivation see **here**.

I'll leave it to you to figure out the whole volume for the general case with radius R.

Chapter 136

Torus

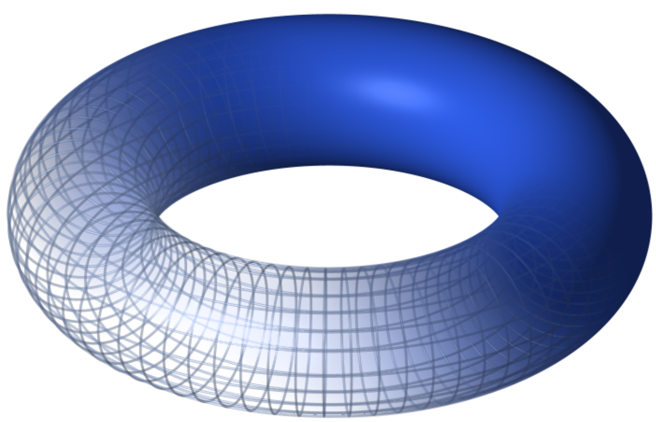
The general theory which allows us to calculate surface areas includes a formula for parametrized surfaces.

For parametrization of a line or a curve we need only one variable.

$$x = \cos t \quad y = \sin t \quad z = ct$$

where t is the parameter and c is a constant, is the parametrization of a helix.

For a surface (like a sphere, or a torus), we will need two parameters, which are usually called u and v.



To get the surface area, we will just calculate

$$V = \int du \, dv$$

over the appropriate range of the two variables. There is an additional constant: we need to figure out the exchange rate for the surface area element dS (composed of sides du and dv) to units of dA (composed of dx and dy). More about that later.

A torus is like a small circle moved around in a larger circle. The small circle is the cross-section of the donut (radius r), while the large circle traces out the path of the donut (radius R). Our parameters will be the angles θ and ϕ. θ describes the position of on the large circle (the position of the center of the small circle) as

$$x_c = R\cos\theta \qquad y_c = R\sin\theta$$

In this approach, the donut's axis of symmetry is the z-axis, and the body of the donut is centered half above the xy-plane and half below it. The second angle, ϕ, describes where we are on the small circle. In particular, the distance above or below the xy-plane will be

$$z = r\sin\phi$$

The only tricky part is the final adjustment to obtain the actual x and y coordinates. For x, when $\phi = 0$, we go out from the origin an additional distance r, while when $\phi = \pi$, we subtract the same distance. Our first attempt is:

$$\begin{aligned} x &= x_c + r\cos\phi \\ &= R\cos\theta + r\cos\phi \end{aligned}$$

While this is correct for $\theta = 0$ it is not correct for other angles. We need an additional factor of $\cos\theta$:

$$x = R\cos\theta + r\cos\phi\cos\theta$$

$$= \cos\theta(R + r\cos\phi)$$

It may not be clear that this works for all angles, but if $\theta = \pi/2$ we add nothing additional to x_c, and that is just what we want. Similarly for y

$$y = y_c + r\cos\phi\sin\theta$$

$$= \sin\theta(R + r\cos\phi)$$

Now, according to the theory, what we need to do is take the partial derivatives of these functions with respect to θ and ϕ. For example:

$$x_\theta = \frac{dx}{d\theta} = \frac{d}{d\theta}\ \cos\theta(R + r\cos\phi)$$

We find that

$$x = \cos\theta(R + r\cos\phi)$$

$$x_\theta = -\sin\theta(R + r\cos\phi)$$

$$x_\phi = -\cos\theta\ r\sin\phi$$

while

$$y = \sin\theta(R + r\cos\phi)$$

$$y_\theta = \cos\theta(R + r\cos\phi)$$

$$y_\phi = -\sin\theta\ r\sin\phi$$

z does not depend on θ so

$$z_\theta = 0$$

$$z_\phi = r\cos\phi$$

The position vector to a point on the surface is

$$\mathbf{r} = \ \langle x, y, z\rangle$$

$$\mathbf{r}_\theta = \ \langle x_\theta, y_\theta, z_\theta\rangle$$

$$\mathbf{r}_\phi = \ \langle x_\phi, y_\phi, z_\phi \rangle$$

The unit for conversion is called the Jacobian. It is the length of the vector $\mathbf{r}_\theta \times \mathbf{r}_\phi$. It's a bit complicated, so let's do the pieces individually. The vector product has three terms

$$\hat{\mathbf{i}} \ y_\theta z_\phi - y_\phi z_\theta$$

$$= \hat{\mathbf{i}} \ \cos\theta(R + r\cos\phi) r\cos\phi - 0$$

$$\hat{\mathbf{j}} \ x_\phi z_\theta - x_\theta z_\phi$$

$$= \hat{\mathbf{j}} \ 0 - \sin\theta(R + r\cos\phi) r\cos\phi$$

The third term is most complicated

$$\hat{\mathbf{k}} \ x_\theta y_\phi - x_\phi y_\theta$$

$$= \hat{\mathbf{k}} \sin\theta(R + r\cos\phi)\sin\theta \ r\sin\phi + \cos\theta \ r\sin\phi\cos\theta(R + r\cos\phi)$$

But notice that we have the same term times $\sin^2\theta$ and $\cos^2\theta$ so this reduces immediately to

$$= \hat{\mathbf{k}}(R + r\cos\phi) \ r\sin\phi$$

The next (and nearly the last) step is to calculate the length of this vector, by squaring each term and adding. Before we do that, notice that all three components contain a factor of $r(R + r\cos\phi)$. We will leave that aside and remember it at the end. The rest of the sum of squares is:

$$\cos^2\theta\cos^2\phi + \sin^2\theta\cos^2\phi + \sin^2\phi$$

$$= \cos^2\phi + \sin^2\phi = 1$$

So the factor we held aside is all that's left

$$|\mathbf{r}_\theta \times \mathbf{r}_\phi| = r(R + r\cos\phi)$$

And the surface integral is just

$$A_S = \int dS = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} |\mathbf{r}_\theta \times \mathbf{r}_\phi| \; d\phi \; d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} r(R + r\cos\phi) \; d\phi \; d\theta$$

Since θ is independent of ϕ and R and r, we get

$$= 2\pi r \int_{\phi=0}^{2\pi} (R + r\cos\phi) \; d\phi$$

But the integral of $\cos\phi$ is just $\sin\phi$, which is zero at the upper and lower bounds on ϕ, hence

$$= 2\pi r \int_{\phi=0}^{2\pi} R \; d\phi$$

$$= 2\pi r \; 2\pi R$$

Which is pretty amazing. We go around the torus along what is called its centroid, traveling a distance $2\pi R$. At each point we have the circumference of the small circle, which is $2\pi r$.

It seems strange that the curvature doesn't make any difference. There is a theorem in geometry (the Theorem of Pappus), with the same result.

Pappus also allows us to calculate the volume as

$$V = \pi r^2 \; 2\pi R$$

Chapter 137

Pappus

Pappus' centroid theorem is actually a pair of theorems about solids of revolution, where a curve C is revolved around a central axis. The two theorems relate to the surface area and volume.

There is a nice article about it at Mathworld:

http://mathworld.wolfram.com/PappussCentroidTheorem.html

Statements

The first theorem states that the surface area SA is the product of the arc length s of the curve C times the distance d traveled by the geometric centroid of C.

The example in the wikipedia article is a torus of minor radius r and major radius R. Then C is the circle of radius r, the centroid of the curve is its center, and this point moves around a circle of radius R the distance $2\pi R$.

The first term is the circumference C of the curve (the small circle) and the total is

$$SA = 2\pi r \ 2\pi R = 4\pi^2 rR$$

One puzzling thing is that we are used to taking account of the slant height in thinking about surface area (though not volume), but the curvature of the surface doesn't seem to be an issue here.

The second theorem depends on the area enclosed between the curve and the axis of rotation.

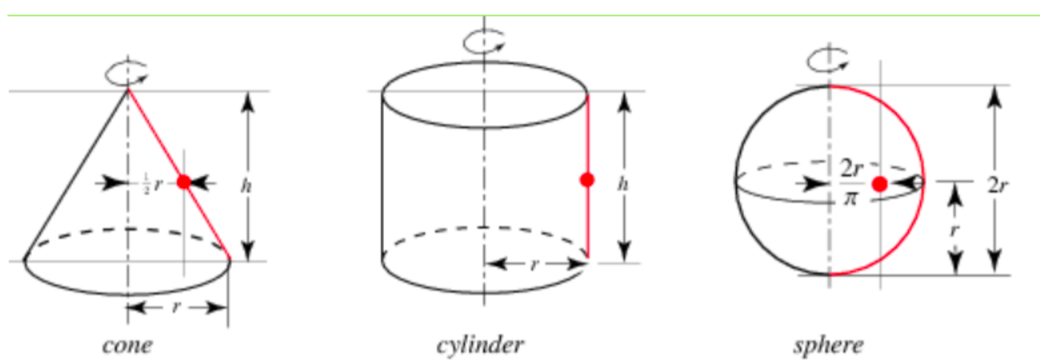
The volume of the solid is the product of this area times the distance d traveled by the geometric centroid of the area (not the curve). For the torus or donut

$$V = \pi r^2 \ 2\pi R = 2\pi^2 Rr^2$$

The main trick here for both theorems is to actually find the geometric centroids. We'll work the two easy cases first: cylinder and cone.

Cylinder surface area

Here is a picture from Wolfram. Our notation is slightly different.



For the cylinder, revolve a parallel line segment around the y-axis. The curve has length H.

The centroid of the parallel line segment (the average distance of each point on the curve from the y-axis) is just the radial distance R, since all the points are the same distance. In addition, the centroid is also halfway along the curve at $H/2$.

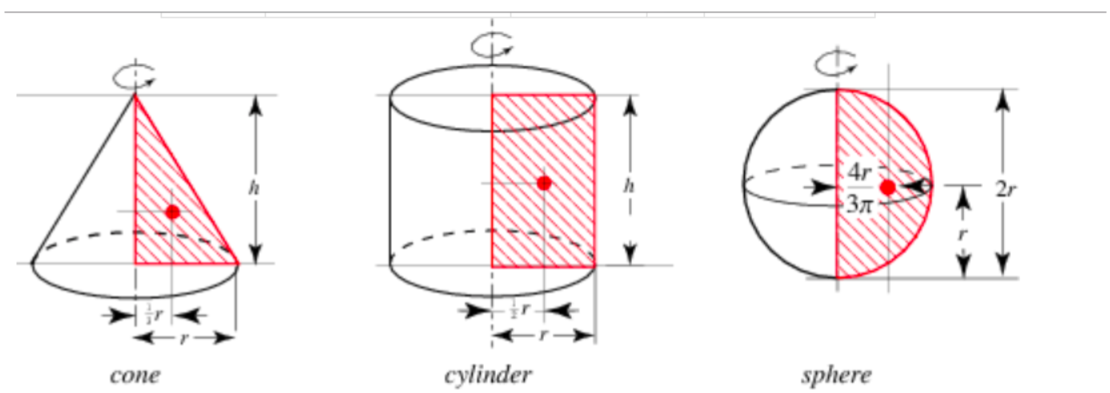
The distance this point travels is $2\pi R$.

The surface area is the product of the arc length H and the distance traveled by the centroid during the revolution

$$SA = H \cdot 2\pi R = 2\pi RH$$

The classical way to obtain a formula for the surface area of the cylinder is to imagine cutting along the length of it, forming a rectangle with width H and length $2\pi R$, which gives the same result.

Cylinder volume



To find the volume of the cylinder, we need to consider the area enclosed by the curve and the y-axis (called a lamina).

For the cylinder, the centroid of the lamina is at $R/2$ and its area is RH. Multiply the area times the distance traveled by the centroid

$$V = RH\ 2\pi \frac{R}{2} = \pi H R^2$$

Cone area

For the cone, we revolve an inclined line segment around the y-axis, with one end lying on the axis and the other at the radius R.

The centroid of this line segment lies at a distance $R/2$ from the y-axis. The distance it travels during the rotation is then πR.

The length of the curve is the slant height s. Multiplying to get the surface area

$$A = \pi R s$$

The classical way to obtain this formula for the surface area of a cone (which we saw in a previous chapter) is to imagine cutting along the slant of the cone to obtain a sector of a circle. The circle has radius s and circumference $2\pi s$ and area πs^2. We take the ratio of the outer perimeter of the sector to the whole circumference, times the whole area

$$\frac{2\pi R}{2\pi s}\ \pi s^2 = \pi R s$$

Cone volume

For the volume of the cone, we need to look at the triangle formed from the inclined line segment. Its area is $Rh/2$. Now, what is its geometric centroid?

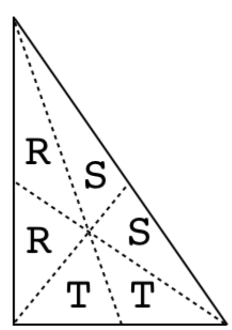
To begin with, I will just use the available result from wikipedia, which is that the centroid of a right triangle is 1/3 of the distance along each side away from the right angle, i.e. $R/3$. The distance it travels during the rotation is then $2\pi R/3$.

The area of the triangle is $(1/2)RH$ and so the volume is

$$V = \frac{1}{2}RH\ 2\pi\frac{R}{3} = \frac{1}{3}\pi R^2 H$$

Centroid of the cone's lamina

For the triangle, we compute the centroid by a geometric argument. The first part of the following holds for any triangle, but I've drawn a right triangle because that's what we've got in the problem (for the cone).



We draw lines from each vertex to the midpoint of the opposite side. The three lines cross at a single point, the centroid. (We looked at the proof of this in Ceva's Theorem). It is easy to see that the areas of the small triangles with the same letter are equal.

For example, both triangles T have the same base (because we drew the median), and the same height. For the same reason

$$R + R + T = S + S + T$$

$$R + R = S + S$$

That is

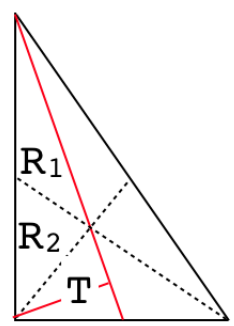
$$R = S$$

As well

$$R = S = T$$

The extension to T follows because the problem is symmetrical.

Now consider the median shown in red in the figure below and the altitude drawn to it.



Both the triangle labeled T and the triangle formed from $R_1 + R_2$ have the same height, namely the altitude drawn in red. But the area of $R_1 + R_2$ together is twice that of T.

Therefore the length of the base of $R_1 + R_2$ (along the median shown in red) must be twice that for the triangle labeled T. The centroid — the point where the lines meet — lies at 2/3 of the distance from the vertex to the opposing side or 1/3 of the way up from the bottom

Since we have a right triangle, then by similar triangles, both the x-coordinate and the y-coordinate of the centroid are at 1/3.

We've seen this result produced by three different arguments in this book. Once when looking at Ceva's Theorem geometrically, once using vectors, once by calculus in the chapter on average value, and now again here. It is not only reassuring to find the same answer each time, but positively required.

There is yet another proof that comes from Euler's construction of the three points on the same line: orthocenter, centroid and circumcenter, and showing that the distance from centroid to orthocenter is twice that from centroid to circumcenter.

Centroids for curves

What we want is the average value of y along the curve (called the "weighted" average $\bar{y}$). Therefore we compute

$$\bar{y} \approx \int y \ ds$$

This result includes a factor of the length of the curve, so we must divide by that at the end (by $\int ds = s$).

$$\bar{y} = \frac{\int y \ ds}{\int ds}$$

(see the chapter on average value).

However, for the cylinder, y was constant so the result is just y. And for the cone, take a shortcut and say "the average value is $R/2$".

For the semicircle, we must actually do the calculation.

Consider the semicircle above the x-axis with equation

$$y = \sqrt{R^2 - x^2}$$

(Note, for the previous discussion we thought about revolving around the y axis. To be consistent with our standard orientation of the semicircle I am switching to revolve around the x-axis here for this calculation).

Each little element of the curve ds is a right triangle with sides dx and dy, and we have seen before that the path element is

$$\sqrt{1+(y')^2}\ dx = ds$$

So our integral for the numerator becomes

$$\bar{y}\ = \int y\ \sqrt{1+(y')^2}\ dx$$

We have

$$y = \sqrt{R^2 - x^2}$$
$$y^2 = R^2 - x^2$$

Using implicit differentiation

$$2y\ dy = -2x\ dx$$
$$y' = -\frac{x}{y}$$
$$(y')^2 = \frac{x^2}{y^2}$$

So the numerator is

$$\int y\ \sqrt{1+\frac{x^2}{y^2}}\ dx$$

We've solved this before. Bring y inside the square root!

$$= \int \sqrt{y^2+x^2}\ dx$$

$$= \int_{-R}^{R} R \; dx = 2R^2$$

The length of the half-circle $s = \pi R$ so

$$\bar{y} = \frac{2R^2}{\pi R} = \frac{2R}{\pi}$$

Sphere surface area

To find the surface area of our solid of revolution (the sphere), multiply the distance traveled by the centroid of the curve, $2\pi\bar{x}$, times the length of the curve, πR

$$A = 2\pi \; \frac{2R}{\pi} \; \pi R = 4\pi R^2$$

Centroids for laminas

The last part is the centroid of the half-circle. Clearly, the average value of x is zero by symmetry. We need to compute the average value of y *over the whole area.*

$$\bar{y} = \frac{\int_R y \; dA}{\int_R dA}$$

We don't actually need to compute the denominator. It is just one-half the area of the circle or one-half πR^2.

We'll do the integral of the numerator both ways, but first by polar coordinates.

We want

$$\int_R y \; dA$$

Here R stands for *region*, but we want to use it below for *radius* so let's suppress it.

$$\int y \ dA$$

To get y in radial coordinates I was tempted to write:

$$y = R \sin \theta$$

but that is not correct! I need a variable r because x and y range over the entire area (this is a double integral, after all).

$$y = r \sin \theta$$

So

$$\iint y \ dA = \int_0^\pi \int_0^R r \sin \theta \ r \ dr \ d\theta$$

$$= \int_0^\pi \int_0^R \sin \theta \ r^2 \ dr \ d\theta$$

(Note the upper bound on θ reflects that the area is the half-circle.

The inner integral is

$$\int_0^R \sin \theta \ r^2 \ dr$$

$$= \sin \theta \ \frac{R^3}{3}$$

So the outer integral is

$$\frac{R^3}{3} \int_0^\pi \sin \theta \ d\theta$$

$$\frac{R^3}{3} \ [-\cos \theta \ \Big|_0^\pi \ ] = \frac{2R^3}{3}$$

We remember to divide by $(1/2)\pi R^2$.

Multiplying by the inverse, we have then:

$$\frac{2R^3}{3} \cdot \frac{2}{\pi R^2} = \frac{4R}{3\pi}$$

Cartesian coordinates

We want

$$\int y \ dA$$

In Cartesian coordinates:

$$= \int_{-R}^{R} \int y \ dy \ dx$$

What are the bounds on y? It ranges from 0 to $\sqrt{R^2 - x^2}$:

$$= \int_{-R}^{R} \int_{0}^{\sqrt{R^2-x^2}} \ y \ dy \ dx$$

Here I was tempted to substitute for y in the integrand, and write

$$= \int_{-R}^{R} \int_{0}^{\sqrt{R^2-x^2}} \ \sqrt{R^2 - x^2} \ dy \ dx$$

but this is *not correct.*

It is actually the same error as when I wanted to write $y = R \sin\theta$ above. The thing is that y as a bound is a constant (for that slice) and the substitution is OK. It is not OK in the integrand, where y is a variable.

The inner integral is just

$$\int y \ dy = \frac{y^2}{2} \ \Big|_{0}^{\sqrt{R^2-x^2}}$$

$$= \frac{R^2 - x^2}{2}$$

The outer integral is

$$\int_{-R}^{R} \frac{R^2 - x^2}{2} \ dx$$

A simplification since both parts are even functions of x

$$= 2 \int_0^R \frac{R^2 - x^2}{2} \ dx$$

$$= \ [\ R^2 x - \frac{x^3}{3}\ ] \ \Big|_0^R$$

$$= R^3 - \frac{R^3}{3} = \frac{2}{3} R^3$$

as before.

Sphere volume

For the sphere, we revolve a half-circle around the x-axis. The centroid for the whole area of the semi-circle was derived above as

$$\bar{y} = \frac{4R}{3\pi}$$

The volume of the solid is the distance traveled by the centroid of the half-circle times the area of the half-circle, $(1/2)\pi R^2$

$$V = 2\pi \ \frac{4R}{3\pi} \ \frac{1}{2} \pi R^2 = \frac{4}{3} \pi R^3$$

alternative

An alternative approach I found on YouTube uses polar coordinates and computes the "center of mass" of a bar in this shape. It's pretty clear from symmetry that the x-coordinate of the center of mass is on the y-axis (at $x = 0$). What we're after is the y-coordinate of the center of mass. By definition

$$y_{cm} = \frac{1}{M} \int y \ dm$$

where dm is a little piece of mass along the curve. We add these all up and divide by the total mass.

For our example, the linear density λ is a constant: $\lambda = M/s$ and $dm = \lambda \ ds$. So we have

$$y_{cm} = \frac{\lambda}{M} \int y \ ds$$

To use polar coordinates, we express y as a function of θ:

$$y = R \sin \theta, \qquad ds = R \ d\theta$$

so we have

$$y_{cm} = \frac{\lambda}{M} \int R \sin \theta \ R \ d\theta$$

$$y_{cm} = \frac{\lambda R^2}{M} \int \sin \theta \ d\theta$$

$$y_{cm} = \frac{\lambda R^2}{M} \ (-\cos \theta) \Big|_{\theta=0}^{\pi}$$

$$y_{cm} = \frac{2\lambda R^2}{M}$$

But $\lambda = M/s$ and $s = \pi R$ so

$$y_{cm} = \frac{M}{\pi R} \frac{2R^2}{M} = \frac{2R}{\pi}$$

which is what we had before.

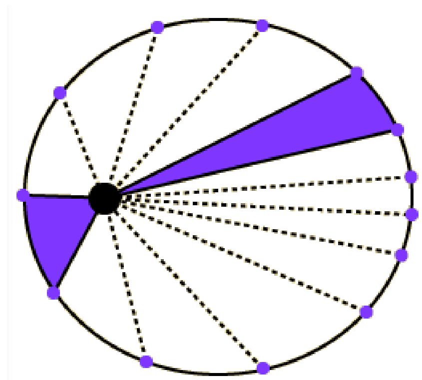
This approach avoids the square roots, but more important it makes it clear why we must divide by the length of the bar. For constant density, the center of mass is equal to the geometric centroid.

Part XXXII

Kepler

Chapter 138

Kepler 2



This is the first of several chapters in which we work through how Kepler's Laws for the orbits of the planets can be derived from Newton's Laws, namely

$$\mathbf{F} = m\mathbf{a}$$

and the inverse square law of gravitation.

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{u}}$$

Kepler's Laws are: first (K1), the orbits of the planets are not circles but ellipses (non-recurrent orbits may be hyperbolic); second (K2), the area or arc "swept out" per unit time is the same no matter where in the orbit the planet is; and third (K3) the period of the orbit is independent of the mass of the planet and its square is proportional to the cube of the length of the semi-major axis of the ellipse.

I also spent some time working on Newton's version of the proof as presented in the *Principia* (see Bressoud's vector calculus book), but he leaves out too many steps. There is also a version "cooked up" by Richard Feynman and discussed in a book called *Feynman's Lost Lecture.*

I never got either of these figured out, but if you want to go this route I recommend starting with Feynman.

For myself, I found that once I cleared up a couple of subtleties, and verified the application of the product rule for differentiation to vector cross-products, it was pretty easy.

circular approximation

The equation of an ellipse in xy-coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a is one-half the "diameter" in the long dimension and b is one-half the length perpendicular to that.

A second way is to give the focal length, the distance of each of the two foci from the center of the ellipse

$$f = \sqrt{a^2 - b^2}$$

Yet another way is to give the *eccentricity*, e, where

$$ea = f$$

Here is a table of planetary eccentricities I found on the web.

Planets: Orbital Properties

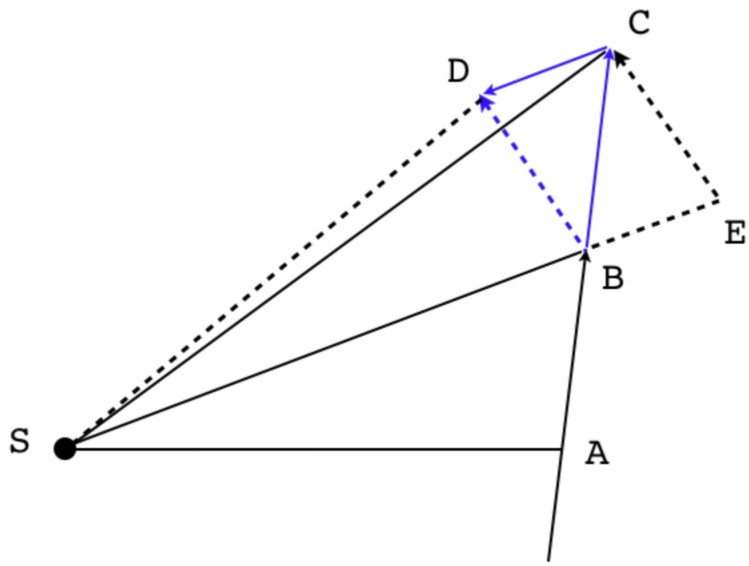
Planet	distance	revolution	eccentricity	inclination
	(A.U.)			(deg)
Mercury	0.387	87.969 d	0.2056	7.005
Venus	0.723	224.701 d	0.0068	3.3947
Earth	1.000	365.256 d	0.0167	0.0000
Mars	1.524	686.98 d	0.0934	1.851
Jupiter	5.203	11.862 y	0.0484	1.305
Saturn	9.537	29.457 y	0.0542	2.484
Uranus	19.191	84.011 y	0.0472	0.770
Neptune	30.069	164.79 y	0.0086	1.769
Pluto	39.482	247.68 y	0.2488	17.142

Mars is the planet that showed Kepler most clearly that the orbits are not circles, but its eccentricity is only 0.09. For Earth this value is only 0.017 which gives a focal length of roughly

$$0.0167 \times 149.6 \times 10^6 \text{ km} \approx 2.5 \times 10^6 \text{ km}$$

which is about 3.5 times the radius of the Sun.

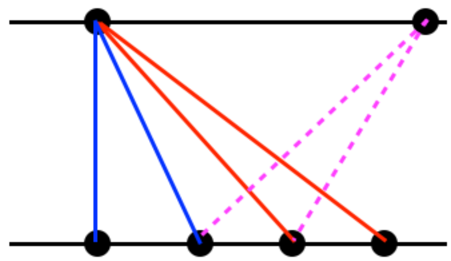
Next, we look at the geometric proof of K2 used by Newton.



We diagram the sun S and a planet at A. Imagine that the force toward the sun is applied discretely. That is, for a small interval Δt, the planet travels from A to B at constant velocity and if undisturbed, would travel to C in the next unit of time.

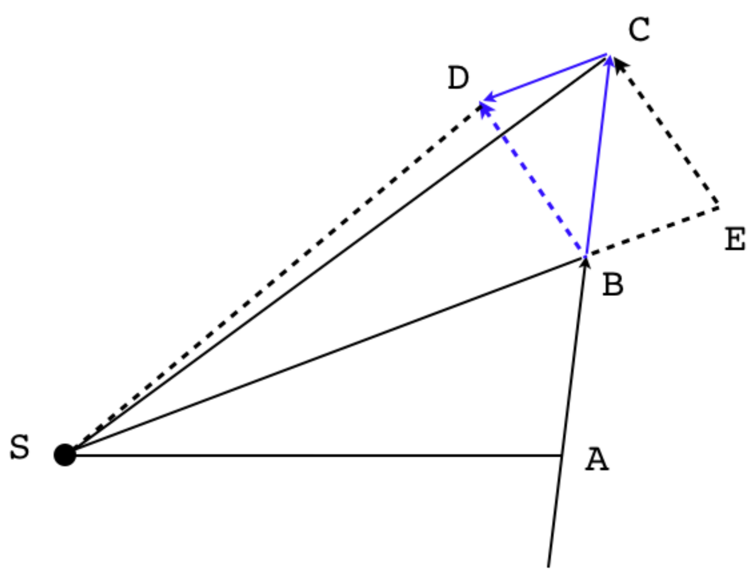
In the absence of a force, the velocity would be constant and so the length of AB is the same as that of BC, and since AB is on the same line as BC, the area of $\triangle ABS$ is equal to the area of $\triangle BCS$.

Proof: draw the vertical line from S to the line containing ABC. The area of either triangle is one-half the length of that altitude times the distance, either AB or BC. The principle is illustrated in the next figure.



Given two parallel lines separated by a distance h, pick two points on one line separated by a distance d and *any* point on the other line. The triangles drawn using those points will all have equal area, namely $(1/2)dh$.

Now, suppose the force is applied at B *toward the sun* along EBS. As a result, the trajectory BC is modified by the change in velocity resulting from application of the force toward the sun. The new path is the additional velocity times Δt. Call the length CD and add it to BC to give the actual trajectory, BD.



CD is parallel to SBE. Therefore, every point on CD has the same altitude to SBE. So any point on CD in a triangle with the same base SB will have the equal area no matter which point is chosen.

The area of $\triangle BDS$ is thus equal to the area of $\triangle BCS$, which was found earlier to be equal to the area of $\triangle ABS$. Since the two triangles from the two intervals of motion have the same area, the area is constant.

Feynman

Richard Feynman gave a famous series of talks at Cornell in 1964 that were videotaped and transcribed into a book. Bill Gates later purchased them and put them on the web, unfortunately with some Microsoft DRM. Still, I have the original book, called *The Character of Physical Law*. This argument is from Chapter 2, *The Relation of Mathematics to Physics*.

It depends on a tiny bit of calculus—specifically, the product rule for differentiation. It also uses the fact that the product rule is valid for vector cross products. (See **here** for a proof).

The rule is that if we have two vectors $\mathbf{a}$ and $\mathbf{b}$ which are functions of time, then

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}$$

In our application the two vectors are the position vector of the planet with respect to the sun, $\mathbf{r}$, and the velocity, which is the time-derivative of that vector.

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$

Or, as the physicists would write it, using Newton's dot notation for the time-derivative:

$$\mathbf{v} = \dot{\mathbf{r}}$$

We are interested in the area of the triangle formed by the vectors $\mathbf{r}$ and $\dot{\mathbf{r}}$ over a small interval of time. The area swept out is constant, as Newton showed, and we will prove again here.

A nice feature of the vector cross-product is that it provides (twice) this area. Namely

$$A = \mathbf{r} \times \dot{\mathbf{r}} = |\mathbf{r}||\dot{\mathbf{r}}| \sin\theta$$

where θ is the angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$, and A is the little bit of area.

Our hypothesis is that A is the same no matter where the planet is in its orbit. Another way to say the same thing is that A doesn't change with time

$$\frac{d}{dt}\ A = \dot{A} = 0$$

Now

$$A = \mathbf{r} \times \dot{\mathbf{r}}$$

and we want to compute $\dot{A}$. Using the product rule it's easy.

$$\dot{A} = \frac{d}{dt}\ (\mathbf{r} \times \dot{\mathbf{r}})$$

$$\dot{A} = \dot{\mathbf{r}} \times \dot{\mathbf{r}}\ +\ \mathbf{r} \times \ddot{\mathbf{r}}$$

As Feynman says: it's just playing with dots.

Let's look at those two terms.

A nice fact about the cross-product is that if the two vectors point in the same direction, then the cross-product is zero. Any vector points in the same direction as itself, so the first term is certainly zero.

$$\dot{\mathbf{r}} \times \dot{\mathbf{r}}\ = 0$$

Next, recall that the second derivative with respect to time of the position is the acceleration vector. According to Newton's second law, the force of gravity points toward the sun, radially.

But of course the position vector also points out radially from the sun. $\mathbf{r}$ and $\ddot{\mathbf{r}}$ are in the same direction (the opposite direction *is* the same direction, multiplied by -1), so the cross-product is again zero.

$$\mathbf{r} \times \ddot{\mathbf{r}} \ = 0$$

So that means the whole thing is zero.

$$\dot{A} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} \ + \ \mathbf{r} \times \ddot{\mathbf{r}} = 0 + 0 = 0$$

We have shown that the time-derivative of the area is zero, so the area is constant, which is Kepler's second law.

Chapter 139

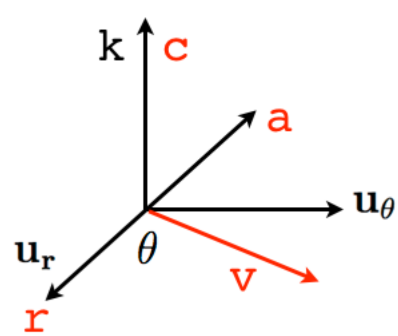
Kepler axes

In this chapter, we introduce some notation and obtain the value of the thing that is constant.

The invariant quantity

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}}$$

(times the mass) is the angular momentum, and the lack of change is the principle of the conservation of angular momentum.



Here is a sketch of the situation. $\mathbf{r}$ is the position vector, extending radially out from the sun to the planet. $\mathbf{u_r}$ is a unit vector in the $\mathbf{r}$ direction, so that

$$\mathbf{r} = r\mathbf{u_r}$$

By the central force hypothesis, the acceleration $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$ is in the $-\mathbf{u_r}$ direction. The source of all our complexity is that the velocity $\mathbf{v} = \dot{\mathbf{r}}$ is not perpendicular to $\mathbf{u_r}$ but makes an angle θ with it.

Earlier we proved that

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{c}$$

is a constant.

Here we give that vector the label $\mathbf{c}$ and a direction. By the properties of the cross product, align $\mathbf{c}$ with $\hat{\mathbf{k}}$.

Since the initial motion is in the xy-plane and so is the acceleration, all the motion takes place in the xy-plane. To put it another way, $\mathbf{c}$ is constant, always in the $\hat{\mathbf{k}}$ direction.

Define $\mathbf{u}_\theta$ as orthogonal to $\mathbf{u_r}$ (and to $\hat{\mathbf{k}}$). $\mathbf{u}_\theta$ is aligned with $\hat{\mathbf{j}}$.

As a result of these definitions:

$$\mathbf{u_r} \times \mathbf{u}_\theta = \hat{\mathbf{k}}$$

$$\hat{\mathbf{k}} \times \mathbf{u_r} = \mathbf{u}_\theta$$

$$\mathbf{u}_\theta \times \hat{\mathbf{k}} = \mathbf{u_r}$$

Reverse the order of any of these pairs, and the result switches sign.

At any given time, $\mathbf{r}$ makes an angle θ with the x-axis, and is at a distance r from the origin, so we write:

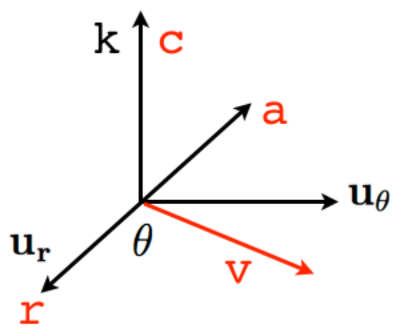
$$\mathbf{r} = \langle r\cos\theta, r\sin\theta, 0\rangle = r\ \mathbf{u_r}$$

$$\mathbf{u_r} = \langle \cos\theta, \sin\theta, 0\rangle$$

$$\mathbf{u}_\theta \perp \mathbf{u_r}$$

$$\mathbf{u}_\theta = \ \langle -\sin\theta, \cos\theta, 0\rangle$$

Verify that the dot-product is zero and that both vectors are unit length.



Now, differentiate $\mathbf{u}_\mathbf{r}$ and $\mathbf{u}_\theta$ (realizing that θ is also a function of time):

$$\frac{d}{dt}\ \mathbf{u}_\mathbf{r} = \dot{\mathbf{u}}_\mathbf{r} = \frac{d\theta}{dt}\ \langle -\sin\theta, \cos\theta, 0\rangle = \ \frac{d\theta}{dt}\ \mathbf{u}_\theta$$

$$\frac{d}{dt}\ \mathbf{u}_\theta = \dot{\mathbf{u}}_\theta = \frac{d\theta}{dt}\ \langle -\cos\theta, -\sin\theta, 0\rangle = -\frac{d\theta}{dt}\ \mathbf{u}_\mathbf{r}$$

Of course, with appropriate choice of units for time t, we can have $\theta = t$, so all of these factors of $d\theta/dt = 1$.

We can also get a parametric expression for the velocity

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}\ (r\mathbf{u}_\mathbf{r}) = \frac{dr}{dt}\mathbf{u}_\mathbf{r} + r\frac{d\theta}{dt}\mathbf{u}_\theta$$

and (with a little more work) we can get the acceleration

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d}{dt}\ (\frac{dr}{dt}\mathbf{u}_\mathbf{r} + r\frac{d\theta}{dt}\mathbf{u}_\theta)$$

$$= \frac{d^2r}{dt^2}\mathbf{u}_\mathbf{r} + \frac{dr}{dt}\dot{\mathbf{u}}_\mathbf{r} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + r\frac{d\theta}{dt}\dot{\mathbf{u}}_\theta$$

We get three terms from differentiating the triple product $r\ d\theta/dt\ \mathbf{u}_\theta$, by a variation on the product rule.

Substitute for the dotted terms from above

$$= \frac{d^2r}{dt^2}\mathbf{u_r} + \frac{dr}{dt}\frac{d\theta}{dt}\ \mathbf{u}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta - r\frac{d\theta}{dt}\frac{d\theta}{dt}\ \mathbf{u_r}$$

Group common terms together

$$= (\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2)\mathbf{u_r} + (2\ \frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2})\mathbf{u}_\theta$$

Now for a nice simplification, look at the factors multiplying $\mathbf{u}_\theta$ and recognize that

$$r\ [\ 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\ ]\ = \frac{d}{dt}(r^2\frac{d\theta}{dt})$$

Therefore, the cofactors for the acceleration in the $\mathbf{u}_\theta$ direction can be re-written as

$$\frac{1}{r}\ [\ \frac{d}{dt}(r^2\frac{d\theta}{dt})\ ]$$

and since if the acceleration is to be only radial (pointed toward the sun), there is no torque (no θ component) and this term must be equal to zero.

$$\frac{1}{r}\ [\ \frac{d}{dt}(r^2\frac{d\theta}{dt})\ ] = 0$$

$$\frac{d}{dt}(r^2\frac{d\theta}{dt}) = 0$$

$$r^2\frac{d\theta}{dt} = h = \text{constant}$$

If we write $d\theta/dt = \omega$, the angular velocity, then $r\omega$ is the speed of the planet, and r times that, times the mass, is the angular momentum. This result is the conservation of angular momentum.

Chapter 140

Varberg

K2

At this point, we have almost all the tools we need to follow the derivation of Kepler's law K2 ("equal areas in equal times") in Varberg *Calculus* (online version, Chapter 14). We just need a bit more discussion of area "swept out" by a planet in a short time.

We revisit the triangle formed by the motion of the planet, and confirm that twice the area of the triangle is equal to

$$h = r^2 \ \frac{d\theta}{dt}$$

The thing that makes it confusing is that whereas Feynman used the velocity for one of the sides of the triangle and showed that

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$$

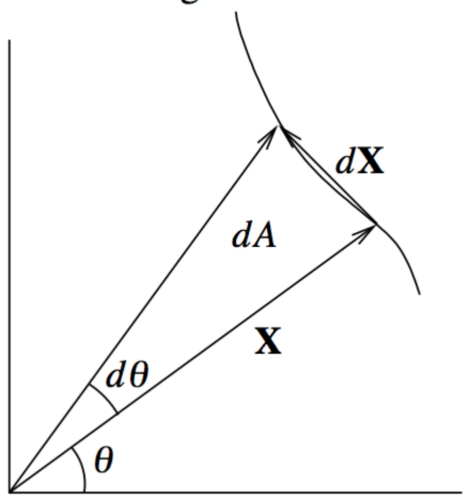
is a constant, Varberg use $d\mathbf{r}$ for their triangle.

It leads to some confusing aspects in the presentation, which I want to work through since I like everything else about their derivation. The bottom line is that before, when we referred to the change in area $\dot{A}$, we were really referring to the change in the change in area $\ddot{A} = 0$.

When Varberg talk about A, they really mean area, the total area since some particular start point. Then $dA/dt = \dot{A}$ is the area swept out in a small unit of time, which is what we've previously called A. So our previous A is the same thing as what Varberg call dA/dt. :)

Here is their diagram

Figure 14.3



They use $\mathbf{X}$ for the position vector, but I will label it as $\mathbf{r}$, following Feynman. This vector "stays always in the xy-plane." Also, I will use $\mathbf{u_r}$ for the vector they call $\mathbf{L}$ and similarly $\mathbf{u}_\theta$ for the vector they call $\mathbf{L}^\perp$.

We are asked to show that

$$2\,\frac{dA}{dt} = \;|\frac{d\mathbf{r}}{dt} \times \mathbf{r}|$$

What I'm going to do is to change the notation a bit and say that in a short time Δt, the area that is swept out is ΔA, corresponding to a length $d\mathbf{r} = \mathbf{v}\Delta t$, and that by the geometry we have

$$2\ \Delta A = |\mathbf{r} \times \mathbf{v}\Delta t|$$

I assert that it is OK to bring Δt out of the cross product (by the rule of scalar multiplication), since it is a scalar quantity, and is a constant at any stage of its future journey to the limit when $\Delta t \to 0$, so I write

$$2\ \Delta A = |\mathbf{r} \times \mathbf{v}|\Delta t$$

Now we have

$$2\ \frac{\Delta A}{\Delta t} = |\mathbf{r} \times \mathbf{v}|$$

and in the limit

$$2\ \frac{dA}{dt} = |\mathbf{r} \times \mathbf{v}|$$

Now this is not quite what we were asked to prove, but recall that

$$\frac{d\mathbf{r}}{dt} \times \mathbf{r} = \dot{\mathbf{r}} \times \mathbf{r} = -\mathbf{r} \times \dot{\mathbf{r}}$$

so the absolute values are the same. Again, the result is that

$$2\ \frac{dA}{dt} = |\dot{\mathbf{r}} \times \mathbf{r}| = |\mathbf{r} \times \dot{\mathbf{r}}| = |\mathbf{h}| = h$$

If you're not completely happy with the argument allowing this step:

$$2\ \Delta A = |\mathbf{r} \times \mathbf{v}\Delta t| = |\mathbf{r} \times \mathbf{v}|\Delta t$$

recall that

$$\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$$

so

$$|\mathbf{r} \times \mathbf{v}\Delta t| = |\mathbf{c}\Delta t| = h\Delta t$$

Varberg *et al.* also give a second argument which which we will go through because it gives us the term

$$r^2 \ \frac{d\theta}{dt} = h$$

By the geometry of the triangle, the area is

$$2 \ dA = r \ rd\theta = r^2 d\theta$$

where r is $|\mathbf{r}|$. And then they say

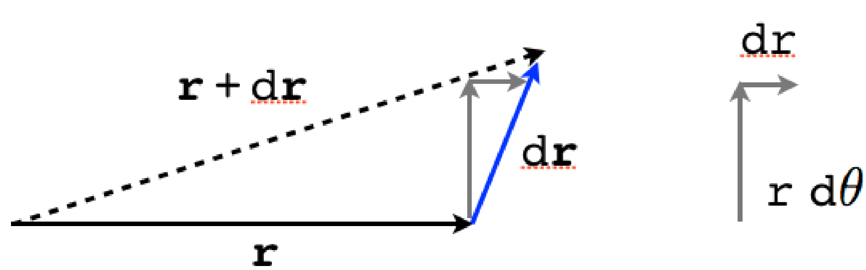
$$2 \ \frac{dA}{dt} = r^2 \ \frac{d\theta}{dt}$$

They do this without comment, but this result assumes that r does not vary with time, although clearly it does (that's the whole point of everything we are doing here). The product rule would give us:

$$\frac{d}{dt} \ r^2 d\theta = 2r \ d\theta \ \frac{dr}{dt} + r^2 \ \frac{d\theta}{dt}$$

This looks a little weird because of the single differential $d\theta$, but what it means is that in the limit, the first term on the right-hand side goes to zero.

Another way of explaining this is to break the area into two parts.



The first part is $(1/2)r$ times $r\ d\theta$, the length of (almost straight) arc perpendicular to $\mathbf{r}$. This is the part we get by assuming that r is constant. And in the limit as $t \to 0$, the resulting $d\theta/dt$ has some value, namely, the angular velocity.

The second part is $(1/2)r\ d\theta$ times dr. This is the part that accounts for the change in r, but it contains two differentials, only one of which gets rescued into some quantity by dt. The other just goes to zero, so the whole thing is zero.

Anyway, let's continue with the argument.

Go back to the right-hand side of what we were asked to prove above

$$2\ \frac{dA}{dt} = |\frac{d\mathbf{r}}{dt} \times \mathbf{r}|$$

and show that it turns into $r^2 d\theta/dt$. Using $\mathbf{u_r}$ for the unit vector in the $\mathbf{r}$ direction, we have

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\mathbf{u_r}) = \frac{dr}{dt}\mathbf{u_r} + r\dot{\mathbf{u}}_\mathbf{r}$$

where the first part is just separating the scalar and unit vector part of $\mathbf{r}$ and the rest is from the product rule. At this point we recall the result that $\dot{\mathbf{u}}_\mathbf{r} = d\theta/dt\ \mathbf{u}_\theta$, so we have

$$= \frac{dr}{dt}\mathbf{u_r} + r\frac{d\theta}{dt}\ \mathbf{u}_\theta$$

So now this is what we need to cross with $\mathbf{r}$, also known as $r\mathbf{u_r}$. We write

$$(\frac{dr}{dt}\mathbf{u_r} + r\frac{d\theta}{dt}\ \mathbf{u}_\theta) \times r\mathbf{u_r}$$

$$= (\frac{dr}{dt}\mathbf{u_r} \times r\mathbf{u_r}) + (r\frac{d\theta}{dt}\ \mathbf{u_\theta} \times r\mathbf{u_r})$$

The first term is zero (the cross-product of $\mathbf{u_r}$ with itself), and because these are unit vectors the absolute value of the second term's vector cross-product is 1

$$|r\frac{d\theta}{dt}\ \mathbf{u_\theta} \times r\mathbf{u_r}| = r^2\frac{d\theta}{dt}\ |\mathbf{u_\theta} \times \mathbf{u_r}| = r^2\frac{d\theta}{dt}$$

So what we've shown is that

$$2\ \frac{dA}{dt} = |\mathbf{r} \times \dot{\mathbf{r}}|$$

and

$$2\ \frac{dA}{dt} = r^2\ \frac{d\theta}{dt}$$

This term ($r^2\ d\theta/dt$) is what Hartig calls c and the other guys call h. As the vector $\mathbf{h}$, it points in the $\hat{\mathbf{k}}$ direction and is the angular momentum but without the mass component.

Chapter 141

Kepler 1

K2 says that orbits "sweep out equal areas in equal times".

Here we start from Newton's force directed toward the sun, the centripetal force, and we will show that the motion stays in a plane, that the orbits are (or can be) ellipses (K1), and also imply K2.

Again, this is Feynman's argument (and notation)

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = 0$$

This is zero because you get two terms from the derivative of the cross-product: one is $\dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$, and the second one is $\mathbf{r} \times \ddot{\mathbf{r}}$, which is zero because these two vectors point in opposite directions by the centripetal force postulate.

Therefore, $\mathbf{r} \times \dot{\mathbf{r}}$ is constant. We will say that

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$$

$$|\mathbf{h}| = h = 2\ \frac{dA}{dt}$$

If $\mathbf{h} = 0$, there is no force, and just straight-line motion. But for $\mathbf{h} \neq 0$, then $\mathbf{r}$ and $\dot{\mathbf{r}}$ are in a plane that doesn't change with time, and $\mathbf{h}$ is the normal vector of that plane.

$$h = |\mathbf{r} \times \dot{\mathbf{r}}| = |\mathbf{r} \times \frac{d\mathbf{r}}{dt}|$$

At this point, Varberg reverse the argument and show that planar motion and K2 imply a centripetal force. But this is just Feynman's dots, which we already went through.

Kepler's First Law K1

Now, we make an additional hypothesis due to Newton, which is that the acceleration is proportional to the inverse square of the distance from the sun (origin), and pointed toward it.

$$\mathbf{a} = \ddot{\mathbf{r}} = -\frac{GM}{r^2}\mathbf{u_r}$$

where (as before) $\mathbf{u_r}$ is the unit vector in the $\mathbf{r}$ direction (i.e. equal to $\mathbf{r}/|\mathbf{r}|$), and GM is a constant.

The first of three main steps in the proof is to take the cross-product with $\hat{\mathbf{k}}$ (as the text says, "this allows us to introduce the area information in vectorial form")

$$\ddot{\mathbf{r}} \times \hat{\mathbf{k}} = -\frac{GM}{r^2}\ \mathbf{u_r} \times \hat{\mathbf{k}} = \frac{GM}{r^2}\ \mathbf{u}_\theta$$

(recall that we "go to the left" for $\mathbf{u}_\theta$).

$$\mathbf{u_r} \times \mathbf{u}_\theta = \hat{\mathbf{k}}$$
$$\hat{\mathbf{k}} \times \mathbf{u_r} = \mathbf{u}_\theta$$
$$\mathbf{u}_\theta \times \hat{\mathbf{k}} = \mathbf{u_r}$$

It should not be surprising that the cross-product brings us back to $\mathbf{u}_\theta$, since we aligned $\mathbf{u_r} \times \mathbf{u}_\theta$ with $\hat{\mathbf{k}}$, but note we're doing the product in reverse order, hence the minus sign.

From our discussion of the unit vectors and parametrization,

$$\frac{d}{dt}\ \mathbf{u_r} = \dot{\mathbf{u}}_\mathbf{r} = \frac{d\theta}{dt}\ \mathbf{u}_\theta = \omega \mathbf{u}_\theta$$

$$\mathbf{u}_\theta = \frac{\dot{\mathbf{u}}_\mathbf{r}}{\omega}$$

and from K2

$$\frac{d\theta}{dt} = \omega = \frac{h}{r^2}$$

Hence

$$\mathbf{u}_\theta = \frac{\dot{\mathbf{u}}_\mathbf{r}}{\omega} = \frac{\dot{\mathbf{u}}_\mathbf{r}}{h/r^2}$$

So the cross-product which we had as

$$\ddot{\mathbf{r}} \times \hat{\mathbf{k}} = \frac{GM}{r^2}\ \mathbf{u}_\theta$$

is equal to

$$= \frac{GM}{r^2}\ \frac{\dot{\mathbf{u}}_\mathbf{r}}{h/r^2}$$

$$= \frac{GM}{h}\ \dot{\mathbf{u}}_\mathbf{r}$$

This is really the key step in our whole adventure.

The second clever thing we do here is to integrate with respect to time

$$\int \ddot{\mathbf{r}} \times \hat{\mathbf{k}} = \int \frac{GM}{h}\ \dot{\mathbf{u}}_\mathbf{r}$$

(remember that GM, h and $\hat{\mathbf{k}}$ are all constant)

$$\dot{\mathbf{r}} \times \hat{\mathbf{k}} = \frac{GM}{h}\ (\mathbf{u_r} + \mathbf{E})$$

where $\mathbf{E}$ is a constant (vector) of integration.

The third step is to form the dot product of both sides with $\mathbf{r}$

$$\mathbf{r} \cdot (\dot{\mathbf{r}} \times \hat{\mathbf{k}}) = \frac{GM}{h}\ \mathbf{r} \cdot (\mathbf{u_r} + \mathbf{E})$$

using a vector identity, the left-hand side is

$$\mathbf{r} \cdot (\dot{\mathbf{r}} \times \hat{\mathbf{k}}) = (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \hat{\mathbf{k}}$$

but

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h} = h\ \hat{\mathbf{k}}$$

so we have

$$h\ \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = h$$

Putting it all together

$$\mathbf{r} \cdot (\ddot{\mathbf{r}} \times \hat{\mathbf{k}}) = h = \frac{GM}{h}\ \mathbf{r} \cdot (\mathbf{u_r} + \mathbf{E})$$

$$\frac{h^2}{GM} = \mathbf{r} \cdot (\mathbf{u_r} + \mathbf{E})$$

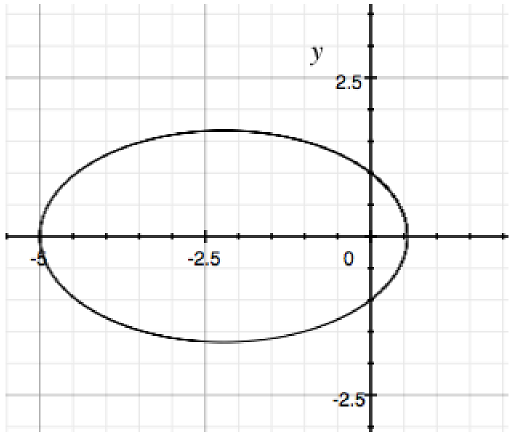
Recall that $\mathbf{u_r}$ is the unit vector in the same direction as $\mathbf{r}$ so that $\mathbf{r} \cdot \mathbf{u_r} = r$.

We can take $\mathbf{E}$ to be in the direction of $\mathbf{r}$ at time-zero so $\mathbf{r} \cdot \mathbf{E}$ is equal to r times e times the cosine of the angle between them at some later time. Since $\mathbf{E}$ is a constant vector of integration, its magnitude e can be anything depending on the initial conditions.

We have then

$$r(1 + e\cos\theta) = \frac{h^2}{GM}$$

These curves in polar coordinates are conic sections. If $e < 1$ the curve is an ellipse.



The curve above is an ellipse with the formula

$$r(1 + 0.8\cos\theta) = 1$$

e is the eccentricity of the ellipse

$$e^2 + \frac{b^2}{a^2} = 1$$

In the figure

$$e^2 = 0.8^2 = 0.64$$
$$\frac{b^2}{a^2} = 1 - 0.64 = 0.36$$
$$\frac{b}{a} = \sqrt{0.36} = 0.6$$

Chapter 142

Kepler 3

In this Chapter we finish up our treatment of Kepler and the orbits of the planets. The first part is Kepler's Third Law (K3).

$$T^2 = \frac{(2\pi)^2}{GM}\ a^3$$

where T is the period, GM is our constant from before, and a is the length of the half-major axis of the ellipse. In other words, the period of an orbit is the 3/2 power of the "radius", technically the semi-major axis of the ellipse.

This formula is pretty easy to derive for a circular orbit (see the chapter on orbital velocity). The problem here is that the velocity is not constant and the orbit is an ellipse. Luckily we have a simple formula for the area of an ellipse.

Start with K2 (Varberg's version, where $\ddot{A} = 0$):

$$2\ \frac{dA}{dt} = h$$

Integrate with respect to time over one revolution obtaining an ellipse with area πab and period T

$$2\pi ab = hT$$

$$T^2 = (\frac{2\pi ab}{h})^2$$

Now, go back to the equation for the orbit

$$r(1 + e\cos\theta) = \frac{h^2}{GM}$$

For simplicity, let $k = h^2/GM$ so

$$r(1 + e\cos\theta) = k$$

$$r = \frac{k}{1 + e\cos\theta}$$

Consider one-half an orbit between $\theta = 0 \to \theta = \pi$. At the start, $\theta = 0$ so

$$r = \frac{k}{(1 + e)}$$

at the end $\theta = \pi$ and

$$r = \frac{k}{(1 - e)}$$

Recall that this r was really $\mathbf{r} \cdot \mathbf{u_r}$. As a signed distance the latter is really minus. So subtracting the former from the latter we get the distance between them is $-a - a = -2a$ and

$$-2a = -\frac{k}{(1 - e)} - \frac{k}{(1 + e)}$$

$$2a = \frac{k}{(1 - e)} + \frac{k}{(1 + e)}$$

$$a = \frac{k}{(1 - e^2)}$$

If you're uncomfortable with this, go back to the chapter on the polar form of the ellipse, we had

$$r(1 + e\cos\theta) = ep = a(1 - e^2)$$

Since $ep = k$, we have the same thing.

For an ellipse

$$1 - e^2 = \frac{a^2 - c^2}{a^2} = \frac{b^2}{a^2}$$

so

$$\frac{b^2}{a^2} = \frac{k}{a}$$
$$b^2 = ka$$

We had

$$T^2 = (\frac{2\pi ab}{h})^2$$

substituting for b^2

$$T^2 = (\frac{2\pi a}{h})^2\ ka$$

and substituting for k

$$= (\frac{2\pi a}{h})^2\ \frac{ah^2}{GM}$$
$$= \frac{(2\pi)^2}{GM}\ a^3$$

which is K3.

GM is the gravitational constant times the mass of the sun. The term due to the angular momentum h has dropped out.

Note that we can get an estimate for GM from observation of the orbits of the planets, and that G can be determined very simply, allowing us to find M, and "weigh the sun".

Summary of K1

unit vectors and velocity

The position vector (from the sun to the planet) is $\mathbf{r}$. Starting from our definition of the unit vector in the $\mathbf{r}$ direction as

$$\mathbf{u_r} = \langle \cos\theta, \sin\theta \rangle$$

where θ is the angle with the positive x-axis, we find $\mathbf{u}_\theta \perp \mathbf{u_r}$

$$\mathbf{u}_\theta = \langle -\sin\theta, \cos\theta \rangle$$

and confirm orthognality

$$\mathbf{u_r} \cdot \mathbf{u}_\theta = 0$$

Remembering that $\theta = \theta(t)$, we easily obtain by the chain rule

$$\dot{\mathbf{u}}_\mathbf{r} = \dot{\theta}\mathbf{u}_\theta$$

$$\dot{\mathbf{u}}_\theta = -\dot{\theta}\mathbf{u_r}$$

r is the magnitude of $\mathbf{r}$

$$\mathbf{r} = r\mathbf{u_r}$$

The velocity $\mathbf{v}$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{u_r} + r\dot{\mathbf{u}}_\mathbf{r} = \dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta$$

We use a vector identity that is easy to prove

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \dot{\mathbf{a}} \times \mathbf{b} + \mathbf{a} \times \dot{\mathbf{b}}$$

to calculate with Feynman's "dots"

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v})$$

$$= \frac{d}{dt}\ (\mathbf{r} \times \dot{\mathbf{r}})$$

$$= \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = 0$$

because any vector's cross-product with itself is zero (including minus itself), which is true for the second term involving the acceleration.

acceleration

An actual expression for the acceleration is just a matter of working through the dots

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d}{dt}\ (\dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta)$$

$$= \ddot{r}\mathbf{u_r} + \dot{r}\dot{\mathbf{u}}_\mathbf{r} + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta$$

substituting for $\dot{\mathbf{u}}_\mathbf{r}$ and $\dot{\mathbf{u}}_\theta$ from above

$$= \ddot{r}\mathbf{u_r} + \dot{r}\dot{\theta}\mathbf{u}_\theta + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta - r\dot{\theta}^2\mathbf{u_r}$$

$$= (\ddot{r} - r\dot{\theta}^2)\mathbf{u_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{u}_\theta$$

Rewrite the coefficient for $\mathbf{u}_\theta$ as

$$\frac{1}{r}(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

angular momentum

We find that the acceleration $\mathbf{a} = \dot{\mathbf{v}}$ has two parts of which the second (in $\mathbf{u}_\theta$)

$$\frac{1}{r}\ \frac{d}{dt}\ r^2\dot{\theta} = 0$$

is zero because $\mathbf{a}$ is all radial. Hence $r^2\dot{\theta} = h$ where h is a constant. Multiplied by the mass m, mh becomes the conserved quantity, angular momentum. It is also twice the area "swept out" and this is the statement of K2.

We get the vector $\mathbf{h}$ by defining the plane of motion as the xy-plane ($\mathbf{u_r} \times \mathbf{u_\theta} = \hat{\mathbf{k}}$) and

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = r\mathbf{u_r} \times (\dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u_\theta})$$

the first term is zero so

$$= r^2\dot{\theta}(\mathbf{u_r} \times \mathbf{u_\theta}) = r^2\dot{\theta}\ \hat{\mathbf{k}}$$

key step

With these preliminary steps we come to the key part of the derivation. I like Varberg's version best. The radial acceleration is

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{u_r}$$

Compute $\mathbf{a} \times \hat{\mathbf{k}}$ (recall that $\mathbf{a}$ is in the $-\mathbf{u_r}$ direction) by recognizing that $-\mathbf{u_r} \times \hat{\mathbf{k}} = \mathbf{u_\theta}$ so

$$\mathbf{a} \times \hat{\mathbf{k}} = \frac{GM}{r^2}\mathbf{u_\theta}$$

but from above $\dot{\mathbf{u}}_\mathbf{r} = \dot{\theta}\mathbf{u_\theta}$ so we have the crucial substitution:

$$\mathbf{a} \times \hat{\mathbf{k}} = \frac{GM}{r^2\dot{\theta}}\dot{\mathbf{u}}_\mathbf{r}$$

$$\mathbf{a} \times \hat{\mathbf{k}} = \frac{GM}{h}\dot{\mathbf{u}}_\mathbf{r}$$

Now we just integrate with respect to time and get

$$\int \mathbf{a} \times \hat{\mathbf{k}} = \int \frac{GM}{h}\dot{\mathbf{u}}_\mathbf{r}$$

$$\mathbf{v} \times \hat{\mathbf{k}} = \frac{GM}{h}\mathbf{u_r} + \mathbf{d}$$

where $\mathbf{d}$ is a constant *vector* of integration. One last trick, we dot with $\mathbf{r}$ and simplify the left-hand side dramatically

$$\mathbf{r} \cdot (\mathbf{v} \times \hat{\mathbf{k}}) = (\mathbf{r} \times \mathbf{v}) \cdot \hat{\mathbf{k}} = \mathbf{h} \cdot \hat{\mathbf{k}} = h$$

So

$$h = \mathbf{r} \cdot (\frac{GM}{h}\mathbf{u_r} + \mathbf{d})$$

$$\frac{h^2}{GM} = \mathbf{r} \cdot (\mathbf{u_r} + \frac{h}{GM}\mathbf{d})$$

Define $k = h^2/GM$ and $e = hd/GM$ and θ as the angle between the constant vector $\mathbf{d}$ and $\mathbf{u_r}$, so finally

$$k = r(1 + e\cos\theta)$$

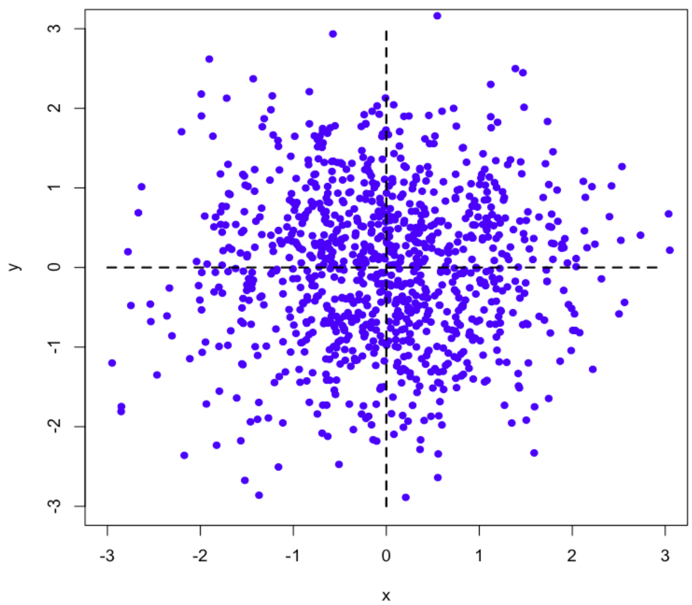
which for $e < 1$ is an ellipse.

Part XXXIII

Distributions

Chapter 143

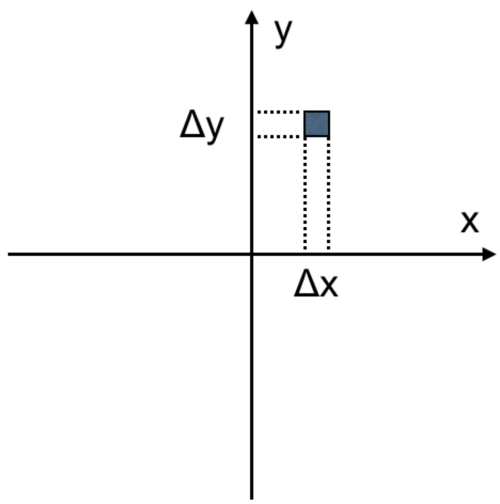
Gaussian derivation



I'm going show a derivation of the Gaussian distribution from first principles. The argument is originally due to Sir John F. W. Herschel.

Imagine that you are throwing darts at the origin of the x,y plane. Under perfect conditions, you would hit the center dead on every time. However, conditions aren't perfect. The wind is gusting, the music is loud, there are other distractions. As a result, small errors creep in and the pattern over time looks like the graphic above.

Now, there is some unknown function for the probability that a dart will land in the interval between x and $x + \Delta x$. Obviously, the probability depends on x, with a maximum at $x = 0$ and then decreasing to zero as x gets large. We designate that function as a probability density function $p(x)$ and evaluate the density over the interval to get the probability that the dart lands in the interval:



$$P = p(x)\Delta x$$

Now we consider a small area of size $\Delta x \Delta y$. If the errors in perpendicular directions are independent, then we expect that we should use the same function p for both x and y and we can get the probability that a dart lands in the small rectangle bounded by x, y and $x + \Delta x$, $y + \Delta y$ as:

$$P = p(x)\Delta x \; p(y)\Delta y$$

In fact, if we assume that the errors do not depend on the orientation of the coordinate system, then the probability is a function only of r, the radial distance from the origin, so we can write

$$P = g(r)\Delta x \ \Delta y$$
$$g(r)\Delta x \ \Delta y = p(x)\Delta x \ p(y)\Delta y$$
$$g(r) = p(x) \ p(y)$$

This assumption of rotational independence leads directly to the answer, as you will see.

Hamming says, since r does not depend on the angle θ, (but x and y do), we can take the partial derivative with respect to θ of $g(r)$ and set it equal to zero, so that:

$$\frac{\partial g(r)}{\partial \theta} = 0 = p(x)\frac{\partial p(y)}{\partial \theta} + p(y)\frac{\partial p(x)}{\partial \theta}$$

What are these derivatives?

$$x = r \ cos\theta$$

$$y = r \ sin\theta$$

$$\frac{\partial p(x)}{\partial \theta} = \frac{\partial p(x)}{\partial x}\frac{\partial x}{\partial \theta}$$
$$\frac{\partial x}{\partial \theta} = -rsin\theta$$
$$\frac{\partial p(x)}{\partial \theta} = p'(x)(-y)$$

$$\frac{\partial p(y)}{\partial \theta} = \frac{\partial p(y)}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial y}{\partial \theta} = rcos\theta$$
$$\frac{\partial p(y)}{\partial \theta} = p'(y)(x)$$

This gives

$$p(x)p'(y)(x) - p(y)p'(x)(y) = 0$$
$$\frac{p'(x)}{p(x)(x)} = \frac{p'(y)}{p(y)(y)} = K$$

What function do we know that has itself as the derivative?

Since

$$p'(x) = Kx\ p(x)$$

Clearly, it is exponential, and an exponential with x^2

$$p(x) = Ae^{Kx^2/2}$$

$$p'(x) = AKx\ e^{Kx^2/2} = Kx\ p(x)$$

Since we assume that large errors are less likely than small ones, $K < 0$, so we can define another constant $V = -1/K$ and

$$p(x) = Ae^{-x^2/2V}$$

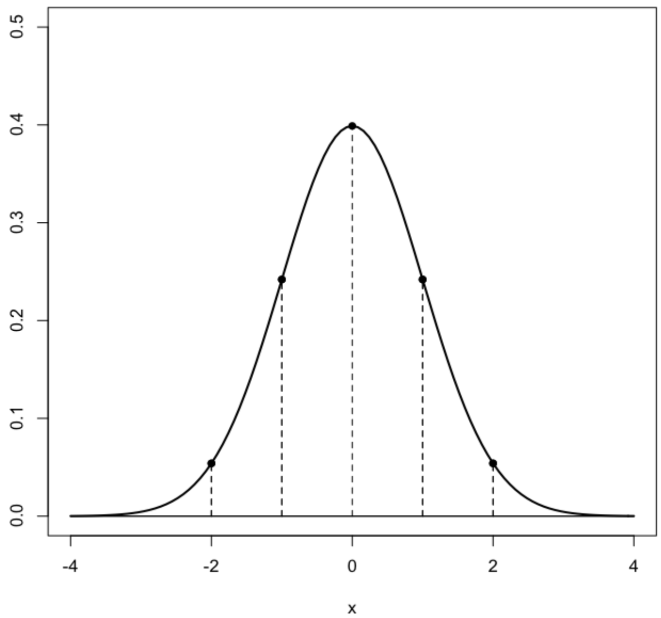
This is the normal distribution with variance V.

It is amazing how far we got with this argument! We assumed: (1) the errors do not depend on the orientation of the coordinate system. (2) errors in perpendicular directions are independent. This means that being too high doesn't alter the probability of being off to the right. (3) large errors are less likely than small errors.

Notice that although we started talking about a probability distribution in two dimensions, the function we end up with is for one dimension.

James Clerk Maxwell used the same argument in three dimensions to derive his expression for the distribution of molecular velocities in a gas.

Derivatives



The normal or Gaussian distribution, plotted above, is usually divided into sections according to $x = \pm n$ standard deviations. It's an interesting fact that the first standard deviation corresponds to the inflection point of the curve. At that point the second derivative of the function is equal to zero.

$$G(x) = \frac{1}{\sigma\sqrt{2\pi}} \ exp \ \{ \ -\frac{1}{2}(\frac{x-\mu}{\sigma})^2 \ \}$$

Let

$$v(x) = -\frac{1}{2}(\frac{x-\mu}{\sigma})^2$$

$$k = \sigma\sqrt{2\pi}$$
$$G(x) = \frac{1}{k}\ e^v$$

$$G'(x) = \frac{1}{k}\ v'e^v$$
$$\frac{dv}{dx} = -\frac{1}{\sigma}(\frac{x-\mu}{\sigma})$$
$$G'(x) = -\frac{1}{k}\ \frac{1}{\sigma}(\frac{x-\mu}{\sigma})e^v$$

$$G''(x) = \frac{1}{k}\ (-\frac{1}{\sigma})(\frac{x-\mu}{\sigma})(-\frac{1}{\sigma})(\frac{x-\mu}{\sigma})e^v + k(\frac{1}{\sigma})e^v$$
$$G''(x) = \frac{1}{k}\ (\frac{1}{\sigma^2})\ [(\frac{x-\mu}{\sigma})^2 - 1]\ e^v$$

We want

$$G''(x) = 0$$

where

$$e^v = exp\ \{\ -\frac{1}{2}(\frac{x-\mu}{\sigma})^2\ \}$$

In the limit as $x \to \pm\infty$, the term above approaches 0, but those are not the solutions we want. So we need

$$(\frac{x-\mu}{\sigma})^2 - 1 = 0$$
$$(x-\mu)^2 = \sigma^2$$
$$x = \mu \pm \sigma$$

The second derivative required some bookkeeping, but was simple in the end.

What about the constant in front? It's there to make the sum of the area under the probability distribution, the cumulative distribution function, equal 1. There is no way to solve the integral. There is a way to compute its value over the intervals $[-\infty, \infty]$ and $[0, \infty]$; alternatively, it can be computed numerically for any interval.

If you do that for e^v as defined above (no leading constant $\frac{1}{k}$), you find that the value is k. So this is a "normalizing constant", to make the whole thing equal to 1.

Chapter 144

Gaussian normalization

Normalizing the normal distribution

The Gaussian or normal distribution is a central tool in probability and statistics. The distribution has the form:

$$p(x) = \int e^{-(x-\mu)^2/2\sigma^2} \, dx$$

where μ is the mean and σ^2 the variance, which determine the placement and shape of the bell. The square root of the variance is σ, the standard deviation. We will assume for what follows that x has been centered so that the mean is equal to zero.

An important result is that there is no way to solve this integral in the usual sense, that is, no "closed form" exists. One simply integrates numerically over an interval $[a, b]$ to obtain the probability that the random variable x lies in the interval $(a \leq x \leq b)$.

Remarkably it *is* possible to derive a value for the integral when the interval is $[-\infty, \infty]$, and even more remarkably, the value is $\sqrt{2\pi} \cdot \sigma$. This leads to writing the "normalized" distribution so that the probability sums to 1 as

$$\frac{1}{\sqrt{2\pi}\cdot\sigma}\int e^{-(x-\mu)^2/2\sigma^2}\ dx$$

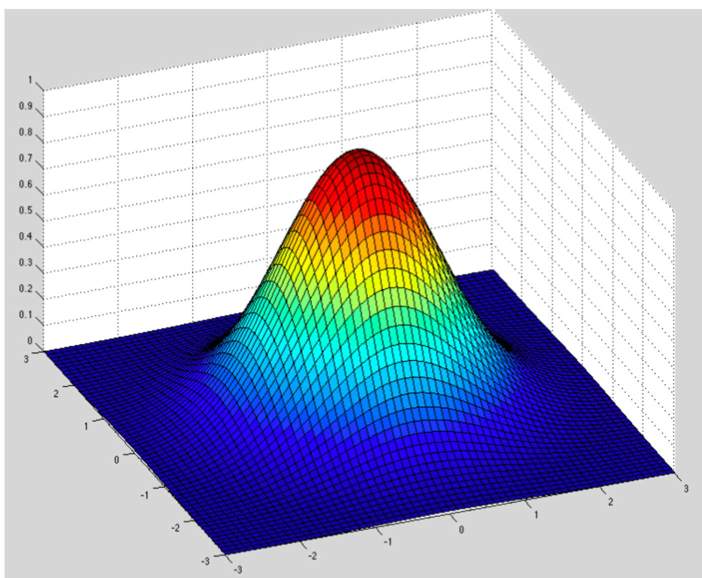
(alternatively, put σ^2 under the square root).

Here is a neat and fairly simple proof that the value of the integral given above is correct. It is based on integrating a volume.

We will show that

$$\int_{-\infty}^{\infty} e^{-kx^2}\ dx = \sqrt{\frac{\pi}{k}}$$

The trick is to imagine the surface that is formed by rotating this function around the z axis in three-dimensions. It looks like this:



The volume is contained under the surface and above the xy-plane. It's a real bell!

The equation of this surface is like two perpendicular copies of the gaussian:

$$z = e^{-(x^2+y^2)/2}$$

For any constant value of y we have the standard normal function for x (with standard deviation equal to 1). If $x \neq 0$ then the height z is reduced by a factor of $e^{-x^2/2}$. The same is true for constant x and variable y.

Any vertical slice is a standard normal, whether it includes the origin or not.

We compute the volume under the surface in two ways. The first way is by horizontal slices perpendicular to the z-axis.

Horizontal slices

$$\int_0^b A(z)\ dz$$

We need an expression for the area of horizontal slices as a function of the height z. We need to find the inverse function for:

$$z = e^{-k(x^2+y^2)}$$

(We use k now rather than 1/2 to make the result more general).

Rearranging:

$$x^2 + y^2 = -\frac{1}{k}\ \ln z$$

The horizontal cross-sections (at $z = c$, with c a constant), are circles of radius r: where

$$x^2 + y^2 = r^2 = -\frac{1}{k}\ \ln z$$

The area, $A = \pi r^2$:

$$A(z) = -\frac{\pi}{k}\ \ln z$$

We will fix the upper bound by inspection. The maximum value of z occurs when $x = y = 0$ so the exponential is equal to 1, otherwise the value is less than 1, so we have that

$$b = e^{-(x^2+y^2)/2} = e^0 = 1$$

This will be the upper bound on z when we calculate the volume.

$$V = -\frac{\pi}{k} \int_0^1 \ln z \; dz$$

Put the leading factor $-\pi/k$ aside for now and consider

$$\int \ln z \; dz = z \ln z - z$$

which is easily verified by differentiating.

We need to evaluate this expression between the bounds ($z = 0 \to 1$). At the upper bound, the first term is zero and the second is equal to -1.

At the lower bound of $z = 0$, the second term is zero.

The problem is the first term, $z \; \ln z$, when $z = 0$. To evaluate this, consider the limit

$$\lim_{z \to 0+} z \; \ln z = \lim_{z \to 0+} \frac{z}{1/\ln z}$$

As $z \to 0+$, $\ln z$ becomes very large. The numerator is just zero and the denominator is $1/-\infty = 0$.

We use L'Hospital's rule. Compute the derivatives:

$$\lim_{z \to 0+} \frac{1}{1/(1/z)} = \lim_{z \to 0+} z = 0$$

As this limit is equal to zero we have zero for the whole expression at the lower bound.

Remembering the extra factor, we have finally $(-\pi/k)(-1) = \pi/k$.

Vertical

The other way is vertical slices. First, label the value of the integral, which we seek, as I:

$$I = \int_{-\infty}^{\infty} e^{-kx^2} \ dx$$

Again, I is what we're looking for. It is **just a number**.

Our function is

$$z = e^{-k(x^2+y^2)}$$

If we take slices perpendicular to the x-axis (with $x =$ constant for any particular slice), the area of each slice is

$$A(y) = \int_{-\infty}^{\infty} e^{-k(x^2+y^2)} \ dy$$

since x is a constant we have

$$= e^{-kx^2} \ \int_{-\infty}^{\infty} e^{-ky^2} \ dy = e^{-kx^2} \ I$$

Now we add up all the little slices to find the volume, which is

$$V = \int_{-\infty}^{\infty} Ie^{-x^2/2} \ dx$$

but I is just a number, so

$$= I \int_{-\infty}^{\infty} e^{-x^2/2} \ dx$$

$$= I^2$$

Alternatively, just say that x and y are independent so that

$$\iint e^{-kx^2} e^{-ky^2} \ dy \ dx$$

$$= \int e^{-kx^2}\ dx \int e^{-ky^2}\ dy = I^2$$

Now we have two different expressions for the same volume, which must then be equal to each other. Thus:

$$\frac{\pi}{k} = I^2$$

$$I = \sqrt{\frac{\pi}{k}}$$

Thinking about the formula that contains a 2 and the variance σ^2 in the denominator of the exponential, $1/k = 2\sigma^2$ so

$$I = \sqrt{2\pi\sigma^2} = \sqrt{2\pi} \cdot \sigma$$

We *divide* by this in order to normalize, to make the value of the whole integral be equal to 1, as it must for a probability distribution.

Change of variables

Taking another look at the bell, we notice that we can change variables and do this faster, not even worrying about the horizontal slices.

$x^2 + y^2 = r^2$ so

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-kx^2} e^{-ky^2}\ dy\ dx = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-kr^2} r\ dr\ d\theta$$

The r is there because of the switch to polar coordinates. (Careful with the bounds on r!). It makes the integral easy

$$\int_0^\infty e^{-kr^2} r\ dr = -\frac{1}{2k} e^{-kr^2}\ \Big|_0^\infty$$

The negative exponential is zero at the upper bound and 1 at the lower one. We pick up a factor of 2π from integrating with respect to θ. So finally we have just

$$2\pi \cdot \frac{1}{2k} = \frac{\pi}{k}$$

which is equal to I^2 as we argued before.

Chapter 145

Combinations

Most people have heard an explanation of permutations somewhere before. Suppose we consider just the two letters a and b. These can be arranged in two orders, or permutations. Either a is first and b second, or the reverse:

$$ab$$
$$ba$$

Either I eat pizza and then I eat ice cream, or I eat ice cream first and then pizza.

Now, how about abx?

One way to reason about this is to say that in starting with a sequence of length two like ab, there are three possible places to add a new letter—in front of or behind ab, or between the two characters. So starting with ab we would have

$$xab \quad axb \quad abx$$

and starting with ba we would have

$$xba \quad bxa \quad bax$$

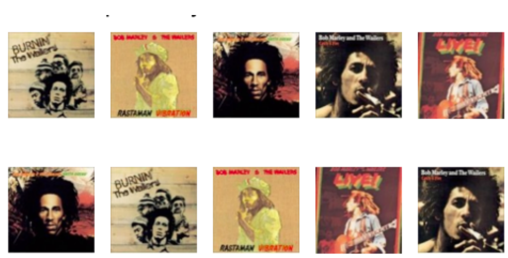
so that's a total of $3 \times 2 = 6$ possibilities.

Another way is to imagine a row of three boxes, where each box will hold a particular letter. We pick one of the three letters for the first position (the first box), one of the two remaining for the second position, and for the third, we have no choice.

In general, for n objects, there are $n \times n - 1 \times \cdots \times 1 = n!$ possible orders.

I have 5 Bob Marley CDs.

There are $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ orders in which I could play my five CDs tonight. I want to hear them all, but I don't want to hear any particular CD more than once. Here are two of the 120 possible orderings.



Now, suppose we consider the problem of picking not all n objects in order, but instead a subset of k objects from that total n. For example, let's pick two CDs from the 5 above. We could pick *Burning* and the Live album, or another group of 2 like *Catch A Fire* and *Natty Dread*.

It's easy to see that the number of possibilities in this problem is 5×4, *as long as the order is important.*

But suppose we don't want to do that. Instead, we want to say that (*Catch A Fire*, *Natty Dread*) is the same as (*Natty Dread*, *Catch A Fire*). We like Bob Marley, but listening to the same album twice in one night is just too much. These are called "combinations."

Let's look at it with letters. Suppose we have $n = 5$ letters *abcde* and we want to pick just $k = 2$. We can pick any of 5 for the first, then any one of the four remaining so there are $5 \times 4 = 20$ possibilities, when considering the ordered pairs.

$$\begin{array}{cccc} ab & ac & ad & ae \\ ba & bc & bd & be \\ ca & cb & cd & ce \\ da & db & dc & de \\ ea & eb & ec & ed \end{array}$$

How many duplicates are there? For every pair like x, y, the reverse ordering y, x is also present.

For any ordering of k items, there are $k!$ permutations (all the permutations of different orderings but containing exactly the same elements). We will have to correct our count by dividing by $k!$.

In general, if we have n total items and choose k of them, we have k terms multiplied together starting with $n \times (n-1) \times \cdots$:

$$n \times (n-1)$$

$$5 \times 4 = 20$$

To get the number of combinations we divide by $k!$, which here equals 2, giving us 10 combinations of $k = 2$ objects from a total collection of $n = 5$ objects.

A second issue is how we compute the number of orders. To get k terms from n total objects, we do

$$n \times (n-1) \cdots \times (n-k+1)$$

It's a bit tricky, but I hope it is clear that for $k = 2$ the last term is $n - 2 + 1 = n - 1$ and for $k = 3$ the last term is $n - 3 + 1 = n - 2$ and so on.

In summary, to count the combinations with k items picked from a set of n total items

$$C = \frac{1}{k!} \;\; n \times (n-1) \cdots \times (n-k+1)$$

That's basically it, but we can simplify by considering one last thing.

$$n! = n \times (n-1) \cdots \times (n-k+1) \times (n-k) \cdots \times 2 \times 1$$

so

$$\begin{aligned} \frac{n!}{(n-k)!} &= \frac{n \times (n-1) \cdots \times (n-k+1)}{1} \times \frac{(n-k) \cdots \times 2 \times 1}{(n-k) \cdots \times 2 \times 1} \\ &= n \times (n-1) \cdots \times (n-k+1) \end{aligned}$$

Therefore, we can rewrite the expression for C above

$$
\begin{aligned}
C &= \frac{1}{k!} \quad n \times (n-1) \cdots \times (n-k+1) \\
&= \frac{n!}{k!\,(n-k)!}
\end{aligned}
$$

This is commonly written

$$
C = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}
$$

Notice the symmetry

$$
\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k}
$$

If I pick two cards from a deck of 52, that is the same problem as picking 50 out of 52, because all I really need to do for the latter is to pick the two to leave behind.

Chapter 146

Exponential distribution

We've seen the exponential function, which applies to compound interest or exponential growth.

Suppose we have an amount or count of "stuff" N (money or rabbits or radioactive phosphorus atoms). It isn't necessary for N to be an integer.

Now, suppose that N changes by an amount ΔN during a short time interval Δt. If the rate of change $\Delta N/\Delta t$ is *proportional* to the amount of stuff present

$$\frac{\Delta N}{\Delta t} = kN$$

Then, in the limit as $\Delta t \to 0$

$$= \frac{dN}{dt} = kN$$

Rearrange

$$\frac{dN}{N} = k\ dt$$

Integrate

$$\log N - \log N_0 = kt$$

$$\log \frac{N}{N_0} = kt$$

Exponentiate:

$$\frac{N}{N_0} = e^{kt}$$

$$N = N_0 e^{kt}$$

The same equation also applies to exponential loss (like radioactive decay). Just add a minus sign in front of the constant of proportionality.

$$N = N_0 e^{-kt}$$

Let's stop for a minute and go back to distributions from standard probability theory: we talk about probability density function $p(x)$ and say that the probability that the random variable x lies in an interval $[a, b]$ is

$$P(a < x < b) = \int_a^b p(x)\ dx$$

Furthermore, the cumulative density function or cdf is usually called $F(x)$ and is

$$F(x) = \int p(x)\ dx$$

$$P(a < x < b) = F(x)\ \Big|_a^b$$

$$P(x < b) = F(x)\ \Big|_0^b = F(b)$$

Suppose we just accept for a moment that the probability density function (pdf) for the negative exponential has this form

$$p(t) = \lambda e^{-\lambda t} \quad (t \geq 0)$$

Use of λ for the constant is traditional. λ is defined to be the the mean number of events per unit time, and the probability that a random event occurs during the interval $[t_1, t_2]$ is the integral

$$\int_{t_1}^{t_2} p(t)\ dt$$

The cumulative distribution function (cdf) corresponding to this pdf is the above integral:

$$\begin{aligned} cdf(x) &= \int \lambda e^{-\lambda t}\ dt \\ &= -e^{-\lambda t} + C \end{aligned}$$

The constant of integration is just 1:

$$cdf(t) = 1 - e^{-\lambda t}$$

It is determined from the boundary conditions. With the lower bound at zero, then as t gets very large the total probability must go to 1.

$$cdf(t) = 1 - e^{-\lambda t}$$

This also works for the other boundary condition, namely $cdf(0) = 0$.

The probability that an event has not yet happened (happens after this time) is 1 minus this or

$$P(x > t) = e^{-\lambda t}$$

We can use this distribution to model the time that passes until a randomly occurring event.

To determine the probability that the event happens after time t we integrate $p(x)$, or just evaluate the cdf at the bounds:

$$\int_t^{\infty} \lambda e^{-\lambda x}\ dx$$

$$= 1 - e^{-\lambda x} \Big|_t^\infty$$

The integral is just the cdf evaluated at the two bounds. To deal with infinity first solve for $\int_t^T$ (with $T > t$ and very large):

$$p(x > t) = 1 - e^{-\lambda x} \Big|_t^T = -e^{-\lambda T} + e^{-\lambda t}$$

The first term becomes zero as $T \to \infty$ so

$$p(x > t) = e^{-\lambda t}$$

The probability that the event will happen after $t = 0$ is equal to 1, as expected.

If λ is the number of events in a unit interval (the mean number of events per unit time), and θ is the mean waiting time until the first event, then

$$\lambda = \frac{1}{\theta}$$

The expected number of events in a short time Δt is just $\lambda \Delta t$.

$\circ$ Suppose the mean number of customers arriving at a bank in a 1-hour interval is 10. The average (waiting) time between customer arrivals is 1/10 of an hour, or 6 minutes. Preferring to measure time in hours, we set

$$\lambda = 10$$

(called a Poisson rate) and use the cdf

$$p(x > t) = e^{-\lambda t} = e^{-(10) \cdot t}$$

The probability of a wait time longer than the mean of 10 minutes is

$$p(x > 1/10) = e^{-\lambda t} = e^{-1}$$

$$= \frac{1}{e} = 0.37$$

and the probability of a wait longer than 30 minutes is $1/e^3 \approx 0.05$.

discrete case: geometric distribution

David Morin:

> Consider a process where we roll a hypothetical 10-sided die once every second. So time is discretized into 1-second intervals.
>
> If the die shows a 1, we'll consider that a success. The other nine numbers represent failure.

We expect to have one success every 10 rolls, and expect to wait 10 seconds for the next success, on the average.

In general, if the probability of success on each trial is p, the average waiting time is $1/p$. With n trials we expect that pn of them will be a success, the average wait time is $n/pn = 1/p$.

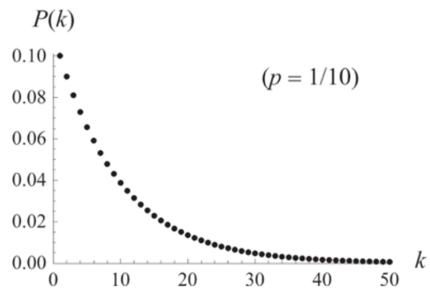
What is the probability that we have to wait through k trials or intervals for a success? We must have failure on the first $k-1$ iterations followed by a single success so

$$P = (1-p)^{k-1}\ p$$

This is called the geometric distribution, because the change in probability when going from one iteration to the next is obtained by multiplying by a constant ratio $1-p$.

> If p = 1/ 10, the distribution is maximum at k = 1 and falls off from that value. Even though k = 10 is the average waiting

time, the probability of the waiting time being exactly k = 10 is only $P(10) = (0.9)^9(0.1) \approx 0.04$.



If p is large (close to 1), the plot of P(k) starts high (at p, which is close to 1) and then falls off quickly, because the factor (1 - p) is close to 0. On the other hand, if p is small (close to 0), the plot of P(k) starts low (at p, which is close to 0) and then falls off slowly, because the factor (1 - p) is close to 1.

derivation of continuous case

Consider a process in which random events occur with an average rate λ.

Chop up that time interval into many small sub-intervals Δt. The probability that an event occurs during a time Δt is (approximately) equal to $\lambda\Delta t$.

(There is a small chance that two or more events will occur in the interval. But if we make Δt small enough, this will be negligible).

The chance that no event occurs is the complement, $1 - \lambda\Delta t$.

Now, we write $q(t)$ for the probability that *no event occurs before* t. Then, the probability that no event occurs before $t + \Delta t$ is the product of these two probabilities, namely

$$q(t + \Delta t) = q(t)(1 - \lambda \Delta t)$$

Rearranging

$$q(t + \Delta t) - q(t) = -\lambda q(t)\Delta t$$
$$\frac{q(t + \Delta t) - q(t)}{\Delta t} = -\lambda q(t)$$

In the limit as $\Delta t \to 0$ the left-hand side becomes the derivative:

$$q'(t) = -\lambda q(t)$$

The negative exponential is a solution to this differential equation:

$$q(t) = e^{-\lambda t} + C$$

Define $F(t)$ as the complement, the probability that an event *does* happen before time t:

$$F(t) = 1 - e^{-\lambda t} - C$$

Evaluation for the initial condition $F(0) = 0$ shows that $C = 0$ so:

$$F(t) = 1 - e^{-\lambda t}$$

This function $F(t)$ is the cumulative density function, or cdf. The pdf is the derivative of the cdf:

$$p(t) = \lambda e^{-\lambda t}$$

memoryless property

For any time t, the probability that an event has not occurred is

$$p(x > t) = e^{-\lambda t}$$

At some later time $t + \Delta t$, the probability is

$$p(x > t + \Delta t) = e^{-\lambda(t+\Delta t)}$$

We ask, what is the ratio

$$\frac{p(x > t + \Delta t)}{p(x > t)} = \frac{e^{-\lambda(t+\Delta t)}}{e^{-\lambda t}}$$

$$= e^{-\lambda \Delta t}$$

This ratio is independent of the time t. The fractional change in probability after a change in time Δt does not depend on where we are on the curve.

Another way of stating this is to say that if we change the time scale of the distribution by choosing some time t as zero-time, and then divide by the probability that an event did happen before that, we obtain the same distribution function back again.

Here is another proof I found on the web.

• A variable X with positive support is *memoryless* if for all $t > 0$ and $s > 0$:

$$P(X > s + t \mid X > t) = P(X > s)$$

Using the definition of conditional probability:

$$P(X > s + t) = P(X > s) \cdot P(X > t)$$

If the two probabilities $P(X > s)$ and $P(X > t)$ are independent, then the distribution is memoryless and then $P(X > s + t)$ is as stated above.

Now, the cdf of the exponential distribution is

$$cdf(x) = 1 - e^{-\lambda x}$$

and the probability that $X > t$ is 1 minus this or

$$p(X > t) = e^{-\lambda t}$$

So we have:

$$\begin{aligned} P(X > s + t) &= P(X > s) \cdot P(X > t) \\ &= e^{-\lambda s} \cdot e^{-\lambda t} \\ &= e^{-\lambda(s+t)} \end{aligned}$$

To put it the other way around, we are looking for a cumulative distribution function with the property that

$$P(X > s + t) = P(X > s) \cdot P(X > t)$$

for *any* s and t so the two probabilities on the right are independent. The exponential has this property since:

$$e^{-\lambda(s+t)} = e^{-\lambda s} \cdot e^{-\lambda t}$$

Expected values: mean and variance

For a continuous variable

$$E(x) = \int x \ f(x) \ dx$$

So here we have

$$E(t) = \int_0^\infty t \ \lambda e^{-\lambda t} \ dt$$

We showed in the Techniques chapter that

$$E(t) = \int_0^\infty t \ \lambda e^{-\lambda t} \ dt = \frac{1}{\lambda}$$

The expected value or mean of the exponential distribution is $1/\lambda$. It is easy to show that the expected value of x^2 is $2/\lambda^2$ and so the variance is

$$V = E[x^2] - (E[x])^2 = \frac{1}{\lambda^2}$$

Chapter 147

Poisson distribution

The Poisson distribution is a special case of the famous binomial distribution for Bernoulli trials. It is an approximation for situations in which the probability of success p on each trial is small, and the number of trials n is large, so that the mean $\lambda = np$ of successes is not too much greater than 1 for the sequence of trials taken as a whole.

Recall that the binomial distribution for the number of successes k on n trials, with probability p of success on each trial is:

$$P(k) = \binom{n}{k} p^k \; (1-p)^{n-k}$$

where

$$\binom{n}{k} = \frac{n!}{k! \; (n-k)!}$$

Briefly, the derivation of this formula is that in independent trials with probability of success p and of failure q, for a particular series of successes and failures, say:

S S F S S F F F F S S F F

the probability is

$$P = p\ p\ q\ p\ p\ q\ q\ q\ q\ p\ p\ q\ q = p^6 q^7$$

and more generally, for k successes in n trials

$$P = p^k q^{n-k}$$

Since $q = 1 - p$

$$= p^k (1-p)^{n-k}$$

The factor of

$$\frac{n!}{k!\ (n-k)!}$$

which is ≥ 1, recognizes the occurrence of the combinations, that is, all the permutations of

SSFSSFFFFSSFF

such as

SSSSSSFFFFFFF

that have the *same* number of total successes. Each of these contributes to the probability of k successes in n trials—see

http://en.wikipedia.org/wiki/De_Finetti's_theorem

Simplifying $\binom{n}{k}$

The first step involves the "choose" or combinations term. We are interested in applications where n is large and p is small, and the mean number of successes is roughly near 1. For $k = 2$

$$\begin{aligned} \binom{n}{2} &= \frac{n!}{k!\,(n-k)!} \\ &= \frac{n(n-1)}{k!}\,\frac{(n-2)(n-3)\cdots}{(n-2)(n-3)\cdots} \\ &= \frac{n(n-1)}{k!} \end{aligned}$$

Since n is large

$$\binom{n}{2} = \frac{n(n-1)}{k!} \approx \frac{n^2}{2!}$$

and more generally

$$\binom{n}{k} \approx \frac{n^k}{k!}$$

part 2

The other term we will modify is

$$\begin{aligned} &(1-p)^{n-k} \\ &= \frac{(1-p)^n}{(1-p)^k} \end{aligned}$$

The probability p is quite small, but because n is very large, we cannot just set $1-p$ equal to 1 for the numerator, $(1-p)^n$.

In contrast, if k is a modest size (say, $k < 10$),we *can* reasonably set $1-p = 1$ for the second term, and thus the denominator is equal to 1. The first term contributes much more than the second to the value for this expression.

Finally, we will show that $(1-p)^n \approx e^{-np}$.

Recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \approx 1 + x$$

(for small x)

Since $|p|$ is small, we have

$$e^{-p} \approx 1 - p$$

so

$$(1-p)^n \approx (e^{-p})^n = e^{-np}$$

Alternatively, we note that the binomial expansion (for x near zero) is

$$(1+x)^n \approx 1 + x$$

by a Taylor series. Since both $(1-p)^n$ and e^{-p} are approximately equal to $1-p$ they are approximately equal to each other.

Finishing up

Substituting, we get

$$
\begin{aligned}
P(k) &= \binom{n}{k}\, p^k\, (1-p)^{n-k} \\
&\approx \frac{n^k}{k!}\, p^k\, e^{-np}
\end{aligned}
$$

We introduce a symbol for the mean, $\lambda = np$

$$
\begin{aligned}
P(k) &\approx \frac{n^k}{k!}\, p^k\, e^{-\lambda} \\
&\approx \frac{\lambda^k}{k!}\, e^{-\lambda}
\end{aligned}
$$

This is the Poisson distribution for the number of successes k in n Bernoulli trials where the mean number of successes $\lambda = np$ over the whole series of trials is small.

The Poisson is normalized

The Poisson distribution is normalized, i.e. it sums to 1, and so is a proper probability distribution.

$$
\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\, e^{-\lambda} = e^{-\lambda}\, \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1
$$

since

$$
\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots = e^{\lambda}
$$

Note

I've always used a different set of symbols (from David Freifelder's *Molecular Biology*), although the ones shown above are standard in statistics. In his notation the mean is m and the number of successes is i so the equation is

$$P(i) = \frac{e^{-m}\ m^i}{i!}$$

which I remember as "Emmy!" or more explicitly, "emmii!".

One important propety of the Poisson distribution is that it always simplifies dramatically for $P(0)$, and also for $P(1)$ in the case where the mean is equal to 1.

$$\begin{aligned} P(i) &= \frac{e^{-m}\ m^i}{i!} \\ P(0) &= \frac{e^{-m}\ m^0}{0!} \\ P(0) &= e^{-m} \end{aligned}$$

When the mean is one, $m = 1$

$$\begin{aligned} P(0) &= \frac{1}{e} \\ &\approx 0.368 \end{aligned}$$

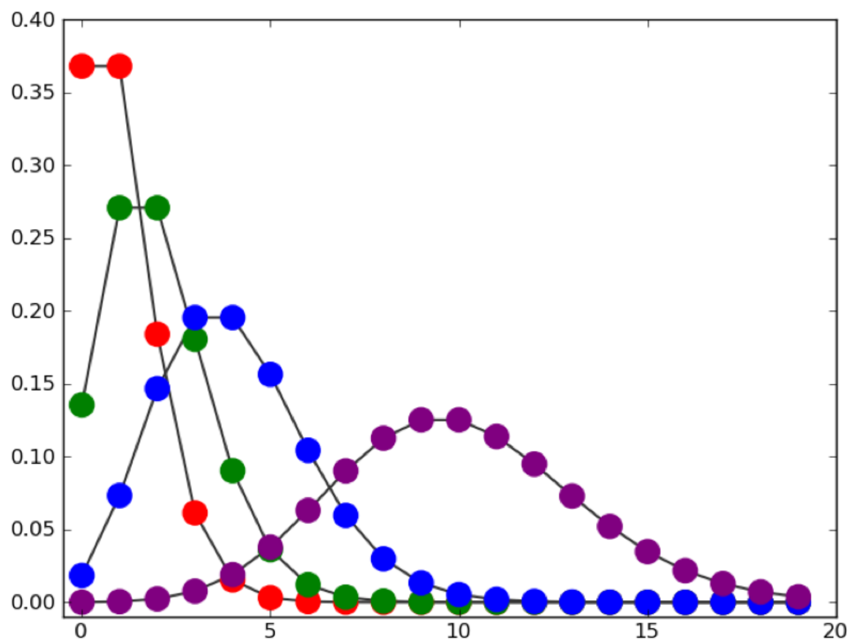
and

$$P(1) = \frac{e^{-1}\ 1^1}{1!} = \frac{1}{e}$$

For example, if we had a box that is divided into one hundred compartments, and if it were possible to throw one hundred marbles into it at random (equal probability for each bin), then at the end of the experiment slightly more than 1/3 of the compartments will still be empty, an equal number will contain one marble, and about 25 percent will contain more than 1. (Grinstead and Snell use the example of V1 rockets aimed at the city of London).

The same holds for a bacterial culture infected with virus at a ratio of one virus per bacterial cell. At an $moi = 1$ ("m.o.i." is short for "multiplicity of infection"), more than one-third of the cells will be uninfected.

Here is a plot of the Poisson distribution for selected values of λ ($m = 1, 2, 5, 10$).



Note the very small values for $P > 10$ when $m < 5$ (red, green, and blue). That's the inverse factorial talking.

Genetics

One application is to bacterial genetics. Consider the Luria-Delbruck experiment, in which mutant bacteria resistant to the action of the bacterial virus T1 were selected. A modern (Darwinian) view appreciates that the mutations which confer virus-resistance pre-exist in the population, having occurred randomly during growth.

An alternative, the Lamarckian view, suggests that the mutations arise *in response* to each bacterium's encounter with the virus.

So if we have an agar plate whose surface contains 10^8 bacterial cells and an excess of virus particles, we can model the Lamarckian view as a process in which each individual cell has an extremely small probability of surviving the phage assault to which it is subjected, but the large number of cells constitutes a large number of trials. We adjust the mean number of successes (phage-resistant colonies per plate) to be near 1. Then, the Poisson approximation should apply.

In particular, if f is the fraction of plates which have no resistant colonies, then in

$$P(i=0) = \frac{e^{-m}\ m^i}{i!}$$

both m^i and $i!$ equal 1 and so

$$P(i=0) = f = e^{-m}$$
$$m = -\ln f$$

For example, if $f = 0.5$, then $m = 0.69$. Now we can calculate $P(i = 1)$, $P(i = 2)$, and the whole distribution. Here is the probability distribution for $m = 1$ and $i = 0 \cdots 5$:

0	0.368
1	0.368
2	0.184
3	0.061
4	0.015
5	0.003

and here is the cumulative distribution:

0	0.368
1	0.736
2	0.92
3	0.981
4	0.996
5	0.999

The probability of observing 6 or more colonies is less than 1/1000.

Crucially, this is *not* what one observes. Instead, trials (plates) containing dozens or even hundreds of colonies are obtained at a frequency of about 1 plate in 10. These are Luria's "jackpots."

In summary, the results are inconsistent with the Lamarckian view, but are easily explained by a model in which mutations occur randomly with respect to each cell division. If a mutation happens to occur early in the growth of a culture, that mutant cell will have a large number of descendants, each of which can form a phage-resistant colony.

Chapter 148

Expected value

David Morin:

> Consider a variable that can take on certain numerical values with certain probabilities. Such a variable is appropriately called a *random variable*

Examples:

∘ A coin is tossed and the outcome is assigned 1 (heads) or 0 (tails).

∘ A coin tossed twice; the outcome is assigned a number based on the total count of heads: 0 (TT), 1 (HT,TH) or 2 (HH).

∘ A fair die is thrown; the outcome is assigned the value facing up.

If we say that X is a random variable that represents the outcomes in the example 3, and the values it can take on are

$$x_1 = 1, x_2 = 2 \dots x_6 = 6$$

The probabilities for the outcomes $x_1, x_2 \dots$ do not have to be equal. In example 2, the probabilities are $p_1 = 0.25$, $p_2 = 0.5$ and $p_3 = 0.25$.

The collection of these probabilities is the *probability distribution* for X.

The *expectation value* or *expected value* of a random variable X is the expected average obtained in a large number of trials of the process (in trials yet to be carried out).

• In a large number of trials each outcome x_i will occur with probability p_i and the expected value is

$$E(X) = \sum_i p_i x_i$$

The expected value $E(X)$ is often written μ_x or even just μ if the meaning is clear. For example 3, the roll of a die, we have

$$\begin{aligned} E = \sum_i p_i x_i &= \frac{1}{6} \cdot 1 + \cdots + \frac{1}{6} \cdot 6 \\ &= \frac{1}{6} \cdot (1 + 2 + 3 + 4 + 5 + 6) \\ &= \frac{21}{3} = 3.5 \end{aligned}$$

Note that here $E(X)$ is not one of the values that X can actually take on.

You might think that the expected value is the value that is most likely to occur. This is *not* the case.

◦ Suppose you flip a coin 4 times ($H = 1, T = 0$). Obviously the expected value is 2. More explicitly, we use the combinations ("choose") formula to write

$$E(X) = \frac{1}{16}\, 0 + \frac{4}{16}\, 1 + \frac{6}{16}\, 2 + \frac{4}{16}\, 3 + \frac{1}{16}\, 4$$

$$= \frac{1}{16}\ (0+4+12+12+4)$$
$$= \frac{32}{16} = 2$$

∘ Flip a coin until heads is obtained. Obviously, the expected value is 1, since we get all zeroes until the last flip, where we get 1.

There is a probability of 1/2 for heads on the first flip. There is a probability of 1/2 that we proceed to the second flip, times $P = 1/2$ that we obtain heads on that one, for a total of $P = 1/4$ for heads on the second flip, and so on.

$$E(X) = \frac{1}{2}\ 1 + \frac{1}{4}(0+1) + \frac{1}{8}(0+0+1) + \cdots = 1$$

∘ On the other hand, if we define X instead as *the total number of times the coin is flipped*, we have

$$E(X) = \sum_{i=1}^{\infty} \frac{1}{2^i}\ i$$
$$E(X) = \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(3) + \frac{1}{16}(4) + \frac{1}{32}(5) + \dots$$
$$= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \dots$$

This is not a series I remembered when working through this initially. There is a trick which I saw in *Grinstead and Snell*, and eventually I remembered another approach which I learned in *Strang*. I'll show both below, but let's work with the series a bit.

The nth term in the series is $n/2^n$. The ratio of successive terms is

$$\frac{n+1}{2^{n+1}}\ \frac{2^n}{n} = \frac{n+1}{2n}$$

which approaches one-half as $n \to \infty$, so we know the series converges (absolutely).

Calculating the sum of the first ten terms:

```
1 0.5
2 1.0
3 1.375
4 1.625
5 1.78125
6 1.875
7 1.9296875
8 1.9609375
9 1.978515625
10 1.98828125
```

With 20 terms the result is 1.999979. It's pretty clear that the result is 2.

For the analytical approach:

$$E(X) = \sum_{i=1}^{\infty} \frac{1}{2^i} \ i = \ ?$$

Consider that the series without the factor of i is just the geometric series with $r = 1/2$ and initial value 1 and its sum is:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

Why not add up many copies of this series, starting each version from the *succeeding index*:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} + \sum_{i=2}^{\infty} \frac{1}{2^i} + \sum_{i=3}^{\infty} \frac{1}{2^i} + \ldots$$

The result will have 2 occurrences of $1/2^2$ and 3 of $1/2^3$ and so on. I claim that:

$$\sum_{i=1}^{\infty} i \ \frac{1}{2^i} = \sum_{i=1}^{\infty} \frac{1}{2^i} + \sum_{i=2}^{\infty} \frac{1}{2^i} + \sum_{i=3}^{\infty} \frac{1}{2^i} + \dots$$

Now, the first term adds up to 1, the second adds up to $1 - 1/2 = 1/2$ (since the first term is missing), the third to $1 - 3/4 = 1/4$ (since the first two terms are missing) and so on giving

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$$

In other words

$$\sum_{i=1}^{\infty} i \ \frac{1}{2^i} = \sum_{i=0}^{\infty} \ \frac{1}{2^i}$$

which is rather surprising.

For another approach to the analysis, start with this equality

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

The formal proof looks at the partial sums as their number increases, but simply multiplying shows that:

$$(1-x)(1 + x + x^2 + x^3 + \dots) \stackrel{?}{=}$$

Multiplying by 1 just gives us the same series. Multiplying by -x gives:

$$-x - x^2 - x^3 + \dots$$

It's clear that every term after the first will cancel, so the sum is just 1, as required.

Now, the trick is to differentiate:

$$\frac{d}{dx}\ 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{d}{dx}\ \frac{1}{1-x}$$

The left-hand side is

$$1 + 2x + 3x^2 + 4x^3 + \ldots$$

This is *almost* our series (with $x = 1/2$):

$$\sum_{i=1}^{\infty} i\ \frac{1}{2^i} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots$$

To obtain the exact series we must subtract the following (again, with $x = 1/2$)

$$1 + x + x^2 + x^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$

Differentiating the other part we obtain

$$\frac{d}{dx}\ \frac{1}{1-x} = \frac{1}{(1-x)^2}$$

(canceling the minus sign from the power and the one from the chain rule).

But as we said, $x = 1/2$ so this part of the sum is equal to

$$\frac{1}{(1-1/2)^2} = 4$$

we have to subtract the other part, which was equal to 2, yielding the final answer.

addition and multiplication of expectation

• Addition rule for expected value (mean).

$$E(X+Y) = E(X) + E(Y)$$

Proof:

In calculating $E(X+Y)$ we look at each value x_i and associated probability p_i, and also at each value y_j and its associated probability q_j.

The joint probability for both x_i and y_j being observed is $p_i q_j$ and the value of $X+Y$ is $x_i + y_j$ so the expected value is

$$E(X+Y) = \sum_i \sum_j p_i q_j (x_i + y_j)$$

Now, breaking up the outer sum, we have for each x_i

$$\sum_j p_i q_j (x_i + y_j) = \sum_j p_i q_j x_i + \sum_j p_i q_j y_j$$

Both x_i and p_i are constants for each inner sum.

The first term on the right-hand side is

$$\sum_j p_i q_j x_i = p_i x_i \sum_j q_j = p_i x_i$$

while the second term is

$$\sum_j p_i q_j y_j = p_i \sum_j q_j y_j = p_i E(Y)$$

Going back to the outer sum, we have then

$$E(X+Y) = \sum_i p_i x_i + \sum_i p_i E(Y)$$

$$= \sum_i p_i x_i + E(Y) \sum_i p_i$$
$$= E(X) + E(Y)$$

□

This does *not* depend on X and Y being independent.

○ X is a fair die and Y a tossed coin. We know that

$$E(X) = 3.5$$

$$E(Y) = 0.5$$

All cases have

$$P = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$$

Enumerating the cases for $X + Y$

$$X = 0, \quad Y = 1 \ldots \quad X = 0, \quad Y = 6$$

$$X = 1, \quad Y = 1 \ldots \quad X = 1, \quad Y = 6$$

The term $x_i + y_i$ summed over all of these cases is

$$(0 + 1) + (0 + 2) + \cdots + (0 + 2) + (1 + 1) + (1 + 2) + \cdots + (1 + 6)$$

$$= 21 + 6 + 21 = 48$$

We must divide by 12 giving 4.0.

• The general case for addition (with multiplication by a constant) is

$$E(aX + bY + c) = aE(X) + bE(Y) + c$$

for constants a, b and c.

• If we have n random variables all associated with the same probability distribution, then

$$E(X_1 + X_2 + \ldots X_n) = E(X_1) + E(X_2) + \ldots E(X_n)$$

$$= nE(X)$$

∘ Consider the permutations of *abc*. There are six:

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba$$

Now, define the value of each permutation as the number of letters which lie in the same positions as they do in *abc*. Namely

$$abc = 3,\ acb = 1,\ bac = 1,\ bca = 0,\ cab = 0,\ cba = 1$$

Let X be a random variable which takes on this value when each of the permutations (called the number of "fixed points") is obtained randomly.

X			Y
a	b	c	3
a	c	b	1
b	a	c	1
b	c	a	0
c	a	b	0
c	b	a	1

Table 6.3: Number of fixed points.

What is the expected value of X?

$$E(X) = \frac{1}{6}\left(3 + 1 + 1 + 0 + 0 + 1\right) = 1$$

Now, we can use the addition rule to generalize this to the case where we have the set $\{1, 2, 3 \dots n\}$, and cannot simply enumerate the cases. *Grinstead and Snell*:

Let Z denote the random permutation. For each i, $1 \leq i \leq n$, let X_i equal 1 if Z fixes i, and 0 otherwise. So if we let F denote the number of fixed points in Z, then

$$F = X_1 + X_2 + \cdots + X_n$$

By the rule:

$$E(F) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

But it is easy to see that for each i

$$E(X_i) = \frac{1}{n}$$

so

$$E(F) = 1$$

- Multiplication rule for expected value (mean).

$$E(X \cdot Y) = E(X) \cdot E(Y)$$

provided X and Y are *independent*.

Proof:

$$E(X) \cdot E(Y) = \sum_i \sum_j p_i q_j (x_i \cdot y_j)$$

And again, for each x_i the inner sum has constant p_i and constant x_i so we can write

$$\sum_j p_i q_j (x_i \cdot y_j) = p_i x_i \sum_j q_j (y_j) = p_i x_i E(Y)$$

Then, provided that q_j, the probability distribution for Y, does not depend on which x we are considering, the various $E(Y)$ are all the same and the outer sum is

$$\sum_i p_i x_i E(Y)$$

and since $E(Y)$ is constant

$$\sum_i p_i x_i E(Y) = E(Y) \sum_i p_i x_i = E(X) \cdot E(Y)$$

□

Note that this equation is not valid:

$$E(X) \cdot E(X) \stackrel{?}{=} E(X^2)$$

since the probability distribution for X *does* depend on X

∘ Consider two dice. If we roll them independently then:

$$E(X+Y) = \frac{1}{36}(1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + 5 \cdot 6 \ldots$$
$$+ 6 \cdot 7 + 5 \cdot 8 + 4 \cdot 9 + 3 \cdot 10 + 2 \cdot 11 + 1 \cdot 12)$$
$$= \frac{1}{36}(2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12)$$
$$= \frac{1}{36}\ 252 = 7.0$$
$$= E(X) + E(Y)$$

∘ But if we roll one die and then simply turn the other one so as to have the *next* number up (e.g. 1 is paired with 2, 2 with 3, and so on, wrapping around so that 6 is paired with 1), we obtain

$$E(X+Y) = \frac{1}{6}\ (3 + 5 + 7 + 9 + 11 + 1)$$
$$\frac{36}{6} \neq 7$$

∘ And if we roll one die and then simply turn the other one so as to have the *same* number up, we obtain

$$E(X+Y) = \frac{1}{6}\ (2 + 4 + 6 + 8 + 10 + 12)$$

$$\frac{42}{6} = 7$$

For dependent variables $E(X+Y)$ depends on the probability distributions of the variables.

Variance

• Variance is defined as:

$$\text{Var}(X) = E[(X-\mu)^2]$$

where $\mu = E(X)$. So

$$\text{Var}(X) = \sum_i p_i (X_i - \mu)^2$$

◦ For throwing a fair die, we found that $E(X) = \mu = 3.5$. The variance is:

$$\begin{aligned} \text{Var}(X) &= \frac{1}{6}\ (2.5^2 + 1.5^2 + 0.5^2 + 0.5^2 + 1.5^2 + 2.5^2) \\ &= \frac{1}{6}\ (2)(8.75) = 2.916 \end{aligned}$$

◦ For a coin flip, we found that $E(x) = \mu = 0.5$

$$\text{Var}(X) = \frac{1}{2}\ [\ (0-0.5)^2 + (1-0.5)^2\ ]\ = 0.25$$

◦ For a biased coin that gives heads with frequency p, we found that

$$E(X) = p \cdot 1 + (1-p) \cdot 0 = p$$

so

$$\text{Var}(X) = p(1-p)^2 + (1-p)(0-p)^2$$

$$= p(1-p)\ [\ (1-p) + p\ ]$$
$$= p(1-p) = pq$$

• Multiplication by a constant gives a factor of the constant squared in the variance.

$$\text{Var}(aX) = a^2\ \text{Var}(X)$$

Proof:

$$E(aX) = aE(X)$$

so

$$\text{Var}(aX) = \sum_i p_i(ax_i - a\mu)^2$$
$$= a^2 \sum_i p_i(x_i - \mu)^2$$
$$= a^2\ \text{Var}(X)$$

□

○ Suppose each dice roll's value is multiplied by a factor of 2. Then

$$E(X) = \frac{1}{6}\ (2 + 4 + 6 + 8 + 10 + 12) = 7$$

$$E\ [\ (X-\mu)^2\ ]\ = \frac{1}{6}\ [\ (-5)^2 + (-3)^2 + (-1)^2 + 1^2 + 3^2 + 5^2)\ ]$$
$$= \frac{1}{6}(70) = 11.66 \approx 2^2 \cdot 2.92$$

• Variances add. This result is central to probability and statistics and it's the main point of this write-up.

$$\text{Var}(X) + \text{Var}(Y) = \text{Var}(X+Y)$$

Proof:

$$\mathrm{Var}(X+Y) = E\ [\ ((X+Y) - (\mu_{x+y}))^2\ ]$$

Since expected values add, we can write

$$\mu_{x+y} = E(X+Y) = E(X) + E(Y) = \mu_x + \mu_y$$

so we can write

$$\mathrm{Var}(X+Y) = E\ [\ ((X+Y) - (\mu_x + \mu_y))^2\ ]$$

Rearrangement gives

$$= E\ [\ ((X - \mu_x) + (Y - \mu_y))^2\ ]$$

Now, we just multiply out

$$= E\ [\ (X-\mu_x)^2 + 2(X-\mu_x)(Y-\mu_y) + (Y-\mu_y)^2\ ]$$

By the addition rule for expected values, we can break this up into three pieces:

$$= E\ [\ (X-\mu_x)^2\ ]\ + E\ [\ 2(X-\mu_x)(Y-\mu_y)\ ]\ + E\ [\ (Y-\mu_y)^2\ ]$$

We will show that the middle term is zero. What remains is the required result.

$$= E\ [\ (X-\mu_x)^2\ ]\ + E\ [\ (Y-\mu_y)^2\ ]$$

$$= \mathrm{Var}(X) + \mathrm{Var}(Y)$$

To show that the middle term is zero, leave aside the factor of 2 (using the multiplication rule) and we have:

$$E\ [\ (X-\mu_x)(Y-\mu_y)\ ]$$

By the multiplication rule for expected values (allowed since we said that X and Y are independent)

$$= E(X-\mu_x) \cdot E(Y-\mu_y)$$

But

$$E(X - \mu_x) = E(X) - \mu_x$$
$$= \mu_x - \mu_x = 0$$

so finally

$$E\ [\ (X - \mu_x)(Y - \mu_y)\ ] = 0$$

□

In fact, if X and Y are not independent, this term is the *covariance*.

• Variances add but they don't multiply.

• Repeated sampling from the same distribution

If all of the X_i are *independent and identically distributed* (i.i.d.) variables, then

$$= \text{Var}(X_1 + X_2 + \cdots + X_n) = n\text{Var}(X)$$

∘ For a biased coin that gives heads with frequency p, we found that

$$E(X) = p \cdot 1 + (1 - p) \cdot 0 = p$$

and

$$\text{Var}(X) = pq$$

For n repeated trials with the same coin, the variance will be npq.

• An alternative form of the definition for variance can be derived pretty easily. Start with the original form

$$\text{Var}(X) = E\ [(X - \mu)^2\ ]$$

Multiply out:

$$= E\ [X^2 - 2X\mu + \mu^2\ ]$$

Use the addition rule for expected values:

$$= E(X^2) - E(2X\mu) + \mu^2 \;]$$

But by the multiplication rule:

$$E(2X\mu) = 2\mu E(X) = 2\mu^2$$

Hence

$$\begin{aligned} E\;[(X-\mu)^2\;] &= E(X^2) - (E(x))^2 \\ &= E(X^2) - \mu^2 \end{aligned}$$

□

∘ Recall that the variance for rolls of a standard die is 2.9166. The expected value is

$$E(X) = \mu = 3.5$$

$$\mu^2 = 12.25$$

And

$$\begin{aligned} E(X^2) &= \frac{1}{6}\;(1+4+9+16+25+36) \\ &= \frac{91}{6} = 15.1666 \end{aligned}$$

And indeed

$$2.9166 = 15.1666 - 12.25$$

The second version is usually preferred for calculation since it only does the subtraction once.

$$\mathrm{Var}(X) = E(X^2) - \mu^2$$

Standard deviation

The standard deviation is defined to be the square root of the variance.

• Since $\mathrm{Var}(aX) = a^2\ \mathrm{Var}(X)$:

$$\mathrm{SD}(aX) = a \cdot \mathrm{SD}(X)$$

• The addition rule for variance can be written as

$$\sigma_X^2 + \sigma_Y^2 = \sigma_{X+Y}^2$$

and it can be generalized as:

$$\sigma_{X1}{}^2 + \sigma_{X2}{}^2 + \cdots = \sigma_{X_1+X_2+\ldots}{}^2$$

$$\sqrt{\sigma_{X1}{}^2 + \sigma_{X2}{}^2 + \ldots} = \sigma_{X_1+X_2+\ldots}$$

• If all of the X_i are *independent and identically distributed* random variables, then this result becomes

$$\sigma_{X_1+X_2+\ldots} = \sqrt{n\sigma_X{}^2} = \sqrt{n}\sigma_X$$

◦ Recall that the variance for a process with odds of success p is

$$\mathrm{Var}\ (X) = np(1-p) = npq$$

The standard deviation is $\sqrt{npq}$. For a fair coin,

$$\sqrt{pq} = \frac{1}{2}$$

The standard deviation of the number of Heads in $n = 100$ fair coin flips is 5 and in $n = 10,000$ flips it is 50.

Suppose you roll a die $10,000$ times. How many of those rolls should yield a 6?

This is equivalent to a process with $p = 0.16$. According to the formulas we've seen

$$E(X) = p = 0.16$$

for one roll and

$$E(X) = np = 1667$$

for all the rolls roll together. The standard deviation is:

$$SD(X) = \sqrt{npq} = 37$$

So three standard deviations is 111 and you are *extremely* unlikely to observe a result outside the range 1667 ± 111. (There are 3 chances in 1000 of this happening).

- Recall the alternative formula for the variance

$$\text{Var}\ (X) = E(X^2) - \mu^2 = \sigma^2$$

Thus

$$E(X^2) = \sigma^2 + \mu^2$$

and if the mean is zero then

$$E(X^2) = \sigma^2$$

Standard error

The standard error of the mean relates to the variance or standard deviation for a series of values taken from some unknown probability distribution.

Suppose we let $\bar{x}$ equal to the mean of a particular draw (say, n values). We now ask, what is

$$\mathrm{Var}(\bar{X}) = \ ?$$

By definition

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

so

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}(\frac{X_1 + X_2 + \cdots + X_n}{n})$$

Recall that multiplication by a constant gives a factor of the constant squared in the variance.

$$= \frac{1}{n^2}\mathrm{Var}(X_1 + X_2 + \cdots + X_n)$$

$$= \frac{1}{n^2}\mathrm{Var}(nX)$$

$$= \frac{1}{n}\mathrm{Var}(X)$$

So the standard deviation of $\bar{x}$ is equal to the standard deviation of X divided by the square root of n.

If we draw repeatedly and calculate $\bar{X}$ as a random variable, it will have a standard deviation that is smaller than the standard deviation of X itself by a factor of $\sqrt{n}$.

• A major result from this discussion is that variances are additive. I found this article on the web

http://apcentral.collegeboard.com/apc/members/courses/teachers_corner/50250.html

which calls this result the Pythagorean Theorem of statistics:

$$SD^2(X \pm Y) = SD^2(X) + SD^2(Y)$$

> Just as the Pythagorean theorem applies only to right triangles, this relationship applies only to independent random variables.

What's nice about this formulation is that it makes it easy to remember that the relationship is true only for *independent* random variables. The article has a great counter-example:

> Consider a survey in which we ask people two questions: During the last 24 hours, how many hours were you asleep? And how many hours were you awake?
>
> There will be some mean number of sleeping hours for the group, with some standard deviation. There will also be a mean and standard deviation of waking hours. But now let's sum the two answers for each person. What's the standard deviation of this sum? It's 0, because that sum is 24 hours for everyone – a constant. Clearly variances did not add here.

Part XXXIV

Vector Calculus 2D

Chapter 149

Vector fields

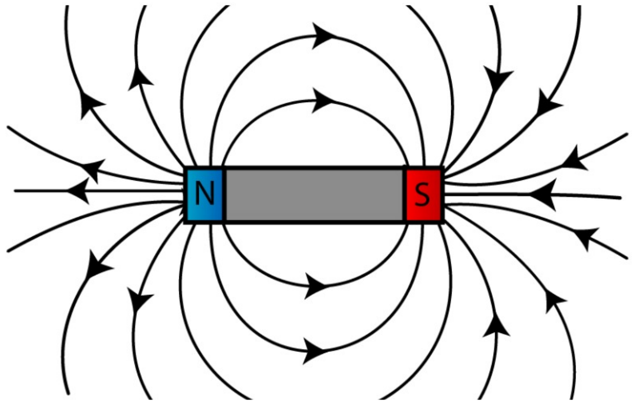
visualization

An electric (or gravitational) field at a point can be defined as the force that would be felt by a unit charge (or mass) placed at that point. So for example

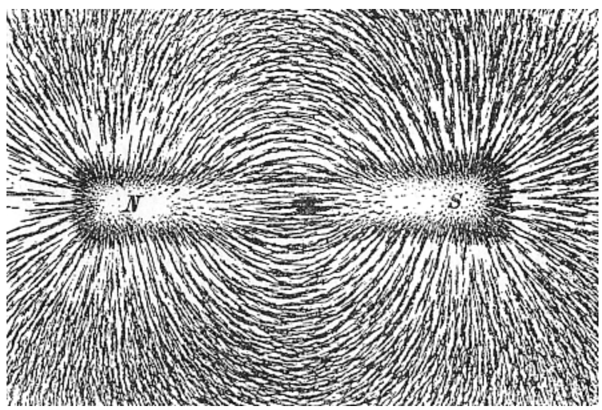
$$\mathbf{F} = q\mathbf{E}$$

Fields, like forces are vectors. That means, to every point in space there corresponds a vector with the magnitude and direction of the field or the force at that point.

Fields are often visualized by drawing lots of little arrows. For example, here is a representation of the magnetic field surrounding a magnet:



A classic physical demonstration of this field is obtained by overlaying the magnet with a sheet of paper and then sprinkling iron filings (small thin pieces) on top.



The filings align with the field, and tend to line up head-to-tail. The field also exists between the lines of filings, but this looks something like the vector field one draws with arrows.

vector fields

In two dimensions, vector field assigns to every point (x, y) in a region R in the plane a vector $\mathbf{F}(x, y)$ with two components

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{i}$$

Each of the component functions M and N takes an ordered pair of real numbers and outputs a real number. As a shorthand, we could write:

$$\mathbf{F} = \langle M, N \rangle$$

In three dimensions we might have $\mathbf{F} = \langle P, Q, R \rangle$. We could also write the components of $\mathbf{F}$ explicitly as some functions of x and y, e.g.

$$\mathbf{R} = \langle x, y \rangle$$

This is the field of radial vectors, all of which point in the direction opposite to the origin, and have length $r = \sqrt{x^2 + r^2}$. We could also normalize to get $\mathbf{R}/r$ or even $\mathbf{R}/r^2$.

The spin field is $\mathbf{S} = \langle -y, x \rangle$. These can be normalized or divided by r^2 as for $\mathbf{R}$.

potential

A *gradient field* is the gradient of some function f which is called the potential. Such fields are conservative, as we'll see later on. Recall that the gradient is

$$\nabla f = \langle f_x, f_y \rangle$$

where by f_x I mean the x-derivative of f. The radial fields are all gradient fields. Consider

$$f(x, y) = \frac{1}{2}(x^2 + y^2)$$

Clearly

$$\nabla f = \langle x, y \rangle = \mathbf{R}$$

The gradient is everywhere perpendicular to the level curves $f(x, y) = c$.

Some fields are gradient fields, and some are not.

What is the potential function for

$$\frac{\mathbf{R}}{r} = \frac{1}{\sqrt{x^2 + y^2}} \ \langle x, y \rangle$$

Well, what function f has as its x-derivative $x/\sqrt{x^2 + y^2}$?

$$\frac{\partial}{\partial x} \ \sqrt{x^2 + y^2} = \, ?$$

That looks right.

Chapter 150

Line integral for work

Here we extend the use of line integrals to calculate work moving along a curve in a field. Review the **previous section** on line integrals if you need to.

Suppose we have x and y as functions of a parameter t. Also we may have a vector field $\mathbf{F}$ where

$$\mathbf{F} = \langle\ M, N\ \rangle$$

or

$$\mathbf{F} = \langle\ P, Q, R\ \rangle$$

It is usual to use M, N in two dimensions and P, Q, R in three.

We are interested in the integral along the curve (for the work done by $\mathbf{F}$):

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F \cdot \hat{\mathbf{T}} ds$$
$$= \int_C P\ dx + Q\ dy + R\ dz$$

This last part seems like a magic trick. We'll see how to justify it in the next section. The crucial insight for evaluation is parametrization of the curve.

example

Suppose

$$\mathbf{F} = \langle\ x, y, z\ \rangle$$

and we have equations for $x(t), y(t), z(t)$, say

$$x = t, \quad y = t, \quad z = 2t^2$$

Now,

$$\begin{aligned} \mathbf{r}(t) &= \ \langle x(t), y(t), z(t)\ \rangle \\ \frac{d\mathbf{r}}{dt} &= \ \langle\ \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\ \rangle \\ &= \ \langle\ 1, 1, 4t\ \rangle \end{aligned}$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\ dt$$

$$= \int_C \langle\ t, t, 2t^2\ \rangle\ \cdot\ \langle\ 1, 1, 4t\ \rangle\ dt$$

$$= \int_C (2t + 8t^3) dt = t^2 + 2t^4$$

Evaluate from say, $t = 0$ to $t = 1$

$$t^2 + 2t^4 = 3$$

It doesn't seem complicated at all, once we have the parametric equations.

theory

The basic line integral is something like this one for work

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

We have a curve C made up of lots of little pieces $d\mathbf{r}$. For each piece, we compute the dot product with the force $\mathbf{F}$, multiplying by the component of the force that is in the same direction as we're headed.

It makes great sense symbolically, but how to compute it? Again

$$\frac{d\mathbf{r}}{dt} = \ \langle \ \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \ \rangle$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \ \langle \ \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \ \rangle \ dt$$

If $\mathbf{F}$ has components

$$\mathbf{F} = \ \langle \ P, Q, R \ \rangle$$

then this becomes

$$\int_C \langle \ P, Q, R \ \rangle \ \cdot \ \langle \ \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \ \rangle \ dt$$

We could even write this

$$\int_C P \ dx + Q \ dy + R \ dz$$

This is a useful mnemonic to remember, but keep in mind that this is a single integral, we can't just do dx and dy separately. We need a single variable and t, the parameter for the curve C, comes to the rescue. (Either that or express y and z as functions of x. We must get all these in terms of t. Then it's OK.

Auroux uses the vector $\hat{\mathbf{T}}$.

$$d\mathbf{r} = ds\ \hat{\mathbf{T}}$$

$\hat{\mathbf{T}}$ is the unit vector in the (tangential) direction of $d\mathbf{r}$, the direction the point (or particle or test mass or test charge) is moving at the moment ($d\mathbf{r}/dt$ is the velocity). ds is the magnitude of $d\mathbf{r}$.

Then

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \frac{ds}{dt}\ \hat{\mathbf{T}}$$

and

$$\frac{ds}{dt}\ \hat{\mathbf{T}} = \frac{dx}{dt}\ \hat{\mathbf{i}} + \frac{dy}{dt}\ \hat{\mathbf{j}} + \frac{dz}{dt}\ \hat{\mathbf{k}}$$

so

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \mathbf{F} \cdot \mathbf{v}\ dt \\
&= \int_C \mathbf{F} \cdot\ \frac{ds}{dt}\ \hat{\mathbf{T}}\ dt \\
&= \int_C \mathbf{F} \cdot\ [\ \frac{dx}{dt}\ \hat{\mathbf{i}} + \frac{dy}{dt}\ \hat{\mathbf{j}} + \frac{dz}{dt}\ \hat{\mathbf{k}}\ ]\ dt \\
&= \int_C \mathbf{F} \cdot\ [\ dx\ \hat{\mathbf{i}} + dy\ \hat{\mathbf{j}} + dz\ \hat{\mathbf{k}}\ ] \\
&= \int_C P\ dx + Q\ dy + R\ dz
\end{aligned}$$

example

It may be necessary to break the curve up into pieces. Suppose we're in $\mathbb{R}^2$

$$\mathbf{F} = \langle x, y \rangle$$

Let C be the unit square. **This relationship provides the parametrization**.

For the first leg we have x ranging from $0 \to 1$ and $y = 0$. So parametrize x using t by setting $x = t$ and let $t = 0 \to 1$.

The integral is

$$= \int_C M\ dx + N\ dy$$

$$= \int_C x\ dx + y\ dy$$

$$= \int_0^1 x\ dx = \frac{1}{2}$$

We could parametrize to $x = t$ here but we don't need to.

We can't just look at the field $\mathbf{F} = \langle x, y \rangle$ and substitute the parameter, unless we know the curve.

In a similar way, on the second leg (up to $(1,1)$), $x = 1$ and y ranges from $0 \to 1$, but notice that even though $x \neq 0$, dx is zero.

$$= \int_C x\ dx + y\ dy$$

$$= \int_0^1 y\ dy = \frac{1}{2}$$

We have exactly the same integral.

For the third leg $y = 1$, $dy = 0$ and $x = 1 \to 0$ so

$$= \int_C x\ dx + y\ dy$$

$$= \int_1^0 x\ dx = -\frac{1}{2}$$

For the third leg $x = 0$, $dx = 0$ and $y = 1 \to 0$ so

$$= \int_C x\ dx + y\ dy$$

$$= \int_1^0 y\ dy = -\frac{1}{2}$$

The total work is $1/2 + 1/2 - 1/2 - 1/2 = 0$.

That's interesting, why is the total work zero? It turns out to be because our force $\mathbf{F} = < x, y >$ is the gradient of a potential function.

$$\mathbf{F} = \nabla \mathbf{f}$$

where

$$\nabla = \ \langle \ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

Can we guess what function? Sure!

$$f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$$

That gives the correct values for the components of $\mathbf{F}$

$$\mathbf{F} = \nabla \mathbf{f} = \nabla(\frac{1}{2}x^2 + \frac{1}{2}y^2)$$

$$= \ \langle f_x, f_y \rangle \ = \ \langle x, y \rangle$$

and since

$$\hat{\mathbf{T}}\ ds = (dx\ \hat{\mathbf{i}} + dy\ \hat{\mathbf{j}})$$

Then, at least in the case where this gradient condition holds, we have

$$\int_C \mathbf{F} \cdot \hat{\mathbf{T}}\ ds = \int \ \langle f_x, f_y \rangle \ \cdot (dx\ \hat{\mathbf{i}} + dy\ \hat{\mathbf{j}})$$

written with the ”del” notation

$$= \int (\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}) \cdot (dx\ \hat{\mathbf{i}} + dy\ \hat{\mathbf{j}})$$

$$= \int \frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy$$

$$= \int df = f(\mathbf{r}_2) - f(\mathbf{r}_1)$$

another example

Suppose $\mathbf{F}$ is $\langle y, x\rangle$ and we want

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\ dx + N\ dy$$

$$= \int_C y\ dx + x\ dy$$

C is a sector of the unit circle between $0 <= \theta <= \pi/4$. We break the curve up into 3 parts.

- C_1 is from $(0, 0)$ to $(0, 1)$.
- C_2 is from $(0, 1)$ to $(1/\sqrt{2}, 1/\sqrt{2})$.
- C_3 is from $(1/\sqrt{2}, 1/\sqrt{2})$ to $(0, 0)$.

For C_1, as before, notice that $y = 0$, and $dy = 0$ so

$$\int_C y\ dx + x\ dy = 0$$

Also, notice that $\mathbf{F}$ is $\langle 0, x\rangle$, so $\mathbf{F} \perp d\mathbf{r}$ and then of course $\mathbf{F} \cdot d\mathbf{r} = 0$.

For C_2 from $(0, 1)$ to $(1, 0)$ Here, we’re on the unit circle. It’s natural to change variables:

$$x = \cos\theta$$
$$dx = -\sin\theta \; d\theta$$
$$y = \sin\theta$$
$$dy = \cos\theta \; d\theta$$

$$\int_C y \; dx + x \; dy$$
$$= \int_C -\sin^2\theta \; d\theta + \cos^2\theta \; d\theta$$
$$= \int_C \cos \; 2\theta \; d\theta$$
$$= \frac{1}{2}\sin \; 2\theta \; \Big|_0^{\pi/4} = \frac{1}{2}$$

For C_3 from $(1/\sqrt{2}, 1/\sqrt{2})$ back to $(0, 0)$, both x and y will change along the path, and we do need to parametrize in some fashion. One way is

$$x = t \quad y = t$$

another way is

$$x = y$$

If we just use $x = y$ and $dx = dy$ then

$$\int_C y \; dx + x \; dy$$
$$2\int_C x \; dx$$

$$= x^2 \Big|_{1/\sqrt{2}}^{0} = -\frac{1}{2}$$

So, once again, the total integral is 0.

And the reason is that $\mathbf{F}$ is (again) the gradient (∇) of a potential function.

By guessing, we obtain this formula for the potential:

$$f(x,y) = xy$$
$$F = \nabla f = \ \langle f_x, f_y \rangle \ = \ \langle y, x \rangle$$

The fundamental theorem of calculus for line integrals:

$$\int_C \nabla f \cdot d\mathbf{r} = f(P1) - f(P2)$$

The example is a closed curve (P1 = P2), so of course it's just 0.

But we can also do each part separately using the method. We get $f(x,y) = (1/\sqrt{2} \times 1/\sqrt{2}) = 1/2$ along C_2 (starting from 0 at C_1), and of course, minus that along C_3, back to $(0,0)$.

In the case where $\mathbf{F}$ is the gradient (∇) of a potential function

$$\mathbf{F} \cdot d\mathbf{r} = (f_x \ \hat{\mathbf{i}} + f_y \ \hat{\mathbf{j}}) \cdot (dx \ \hat{\mathbf{i}} + dy \ \hat{\mathbf{j}})$$

The property that is required for $\mathbf{F}$ to be the gradient of a potential function f:

$$\mathbf{F} = \nabla f = \langle f_x, f_y \rangle$$

is that the mixed second derivatives must be equal. That is, if $\mathbf{F} = \langle M, N \rangle$, then

$$M_y = N_x$$

is both necessary and sufficient for the force to be the gradient of a potential, for the force to be conservative, and for us to be able to evaluate the integral for work by just subtracting the value of the potential at the two endpoints.

In the language of **vector calculus**, this requirement is the same thing as saying that the "curl" of the field must be zero, where curl $F = N_x - M_y$.

Shankar:

> We are beginning to see why certain integrals do not depend on the path. Here is an analogy. Forget about integrals. Imagine I am on some hilly terrain. I start at one point, and I walk to another point. At every portion of my walk, I keep track of my change in altitude, with uphill as positive and downhill as negative. That is like my dU. I add them all up. The total height change will be the difference in the heights of the end points.
>
> Now, you start with me but go on a different path. You wander all over the place but finally stop where I stopped. If you kept track of how long you walked, it won't be the same as my walk. But if you also kept track of how many feet you climbed at each step and added them all up, you would get the same answer I got. I repeat: if what you were keeping track of was the height change in a function, then the sum of all the height changes will simply be the height at the end minus the height at the beginning, independent of the path.

.

There is finally, the issue of sign. In the case of gravitation, the force points toward the earth (or whatever body we are talking about). If we define positive as up, then the force is negative but the potential function is positive — potential energy increases as we go away from the earth. The upshot of this is that in physics the force and the potential have this relationship:

$$\mathbf{F} = -\nabla U = -\langle U_x, U_y \rangle$$

Chapter 151

Introduction to Green's Theorem

flux and curl

We now undertake an exploration of the fundamental theorems of vector calculus.

First, though, a word about flux and curl. Flux seems pretty clear: it is the flow out of a region, across a line in $\mathbb{R}^2$ or across a surface in $\mathbb{R}^3$.

Curl, on the other hand, measures the rotation of a vector field (its absolute value is twice the angular momentum).

As an example, if a two dimensional field $\mathbf{F}$ has components $\langle M, N \rangle$, then

$$\text{curl } \mathbf{F} = N_x - M_y$$

Note: we write M, N for the components of a field in $\mathbb{R}^2$ and P, Q, R for the components of a field in $\mathbb{R}^3$

As we'll see, the result of curl $\mathbf{F}$ is a vector, in this case it points out of the plane. When we calculate with it in 2D we are implicitly forming the dot product with $\hat{\mathbf{k}}$ and then ignoring the third dimension.

right-hand rule

Our convention is that we go on a curve around a region with that region R on our left, and then the curl points up. This is a result of what is called the "right hand rule."

If the value of curl $\mathbf{F}$ is zero, then the work done going around a closed curve is also zero, alternatively if the curl is non-zero, work is done. Think of swimming in a whirlpool.

Against the flow it is hard going, whereas with the flow, it's easy.

In our theorems, the curl will be associated with the line integral for work. The work done in moving along a curve C is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Del: ∇

In three dimensions, we introduce the "del" operator

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

∇ is used in several ways. The first is to indicate the gradient or "grad" of a scalar function $f(x, y, z)$. The result is a vector field:

$$\mathbf{F} = \nabla f = \langle f_x, f_y, f_z \rangle$$

∇ is also used to indicate the *divergence* of a vector field $\mathbf{F}$.

$$\nabla \cdot \mathbf{F}$$

$$= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \ \cdot \ \langle P, Q, R \rangle$$
$$= \langle P_x, Q_y, R_z \rangle$$

The third use is for the curl of **F**, which is written as

$$\nabla \times \mathbf{F}$$

and defined

$$\nabla \times \mathbf{F} = \quad \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

which is basically impossible to remember except by using this convenient device

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

We think of forming the "determinant" of this "matrix."

In two dimensions $R = 0$ and also P_z and Q_z are both zero so

$$\nabla \times \mathbf{F} = \langle 0, 0, Q_x - P_y \rangle$$

or substituting the symbols usually used for the field components in $\mathbb{R}^2$, we have

$$\nabla \times \mathbf{F} = N_x - M_y$$

This is the equation we saw above.

A basic fact in vector calculus is that **if the field is the gradient of a potential function, the curl is zero**.

Let the function be f and the field $\mathbf{F} = \nabla f$, then $M = f_x$ and $N = f_y$. Then the curl is $f_{yx} - f_{xy}$, but for any such function

$$f_{xy} = f_{yx}$$

Equivalent formulations of this are that

- the work integral is independent of the path
- the integral around any closed path is zero.

Calculations

Let's look again at

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

This can be written as

$$= \int_C M\ dx + N\ dy$$

(It can also be written as

$$= \int_C \mathbf{F} \cdot ds\hat{\mathbf{T}}$$

where $\hat{\mathbf{T}}$ is the tangential vector in the direction of motion, in the same direction as the velocity $\mathbf{v}$).

One way to see $\int_C M\ dx + N\ dy$ is to say

$$d\mathbf{r} = \frac{d}{dt}\mathbf{r}\ dt = \ \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle \ \ dt = \ \langle dx, dy \rangle$$

so when we do the dot product with $\mathbf{F}$, we get what is written above.

There is another important integral to be explained below (the one for flux):

$$\int_C \mathbf{F} \cdot \hat{\mathbf{n}}\ ds = \int_C -N\ dx + M\ dy$$

Notice the similar form with the first one. It is also helpful to remember that the unit tangential vector $\hat{\mathbf{T}}$ and the unit normal vector $\hat{\mathbf{n}}$ are orthogonal.

Although these equations look something like a double integral, they are *not*.

We will have a parametrization of the curve in terms of t (or x or θ), and a single integral like $\int_C f(t)\ dt$.

Green—work

We start with two theorems in the plane (typically the xy-plane). These are called Green's Theorem for work, and Green's Theorem for flux.

Green's Theorem for work states that for a closed path

$$\oint \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F}\ dA$$

One sticky point I had here is that the curl produces a vector, yet the formula is usually given as above. That's because this is a special case of Stokes theorem where the term on the right is really

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{k}}\ dA$$

which (since $\nabla \times \mathbf{F}$ is parallel to $\hat{\mathbf{k}}$) gives what we have above.

Alternatively (and best to remember for computation):

$$\int_C M\ dx + N\ dy = \iint_R (N_x - M_y)\ dx\ dy$$

The work done along a closed path around R is equal to the double integral over R of the curl of $\mathbf{F}$.

Remember the whirlpool.

Green—flux

The theorem for work can be reformulated to give a new result, about flux or flow across the curve.

Flux is flow across a curve, or in $\mathbb{R}^3$, through a surface.

Green's Theorem for flux in the plane states that for a closed path C over a region R

$$\int_C \mathbf{F} \cdot \hat{\mathbf{n}}\ ds = \iint_R\ \nabla \cdot \mathbf{F}\ dA$$

where the expression on the left is a line integral, and the quantity on the right is the integral of the divergence of $\mathbf{F}$, symbolized with the "del" operator.

Alternatively

$$\int_C M\ dy - N\ dx = \iint_R\ (M_x + N_y)\ dx\ dy$$

$$\nabla \cdot \mathbf{F}$$

if $\mathbf{F} = \quad \langle M, N \rangle$

$$\nabla \cdot \mathbf{F} = M_x + N_y$$

The divergence of a vector field is a scalar quantity. It measures the net production (or disappearance) of the "substance" that flows in a vector field. If there are no sources or sinks in a region, the divergence of $\mathbf{F}$ will be zero.

Restating the theorem:

$$\oint \mathbf{F} \cdot \hat{\mathbf{n}}\ dS = \iint_R \nabla \cdot \mathbf{F}\ dA$$

Breaking this down, on the left hand side=, $\hat{\mathbf{n}}$ is the unit vector *orthogonal* to $\hat{\mathbf{T}}$. Since $\hat{\mathbf{n}}$ and $\mathbf{n}$ are orthogonal to $\hat{\mathbf{T}}$ and $d\mathbf{r}$, the dot product with $\langle dx, dy \rangle$ must equal zero. Hence, we should have

$$\hat{\mathbf{n}}\ ds = \quad \langle \frac{dy}{dt}, -\frac{dx}{dt} \rangle\ dt = \quad \langle dy, -dx \rangle$$

We put the minus sign on the dx term because of the right-hand rule.

Another way to think about this is that we rotate by 90° counter-clockwise

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{vmatrix} dy \\ -dx \end{vmatrix}$$

so when we compute $\mathbf{F} \cdot \quad \langle dy, -dx \rangle$ we get $\int_C M\ dy - N\ dx$.

Putting it all together, we have

$$\int_C \mathbf{F} \cdot \hat{\mathbf{n}}\ ds = \iint_R \nabla \cdot \mathbf{F}\ dA$$

$$\int_C M\ dy - N\ dx = \iint_R (M_x + N_y)\ dx\ dy$$

Here, the expression on the right *is* a double integral.

This is Green's Theorem for flux. Both theorems are statements about $\mathbb{R}^2$. We will see analogous statements for $\mathbb{R}^3$ later.

examples

State the theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{r} = \iint_R \nabla \times \mathbf{F} \; dA$$

$$\int_C M \; dx + N \; dy = \iint_R (N_x - M_y) \; dx \; dy$$

To start with, if $\mathbf{F}$ is the gradient of some function, we call such a function the potential, and the integral of the work over a closed path is just zero.

$$\mathbf{F} = < y, x >$$

$$f = xy$$

Suppose we take the line integral of $\mathbf{F} \cdot \mathbf{r}$ over the unit square $(0, 0)$ to $(0, 1)$, etc.

$$\oint_C y \; dx + x \; dy =$$

I get $0 + 1 - 1 + 0 = 0$.

A sign change can make all the difference.

$$\mathbf{F} = < -y, x >$$

$$N_x - M_y = 1 - -1 = 2 \neq 0$$

A common field with zero curl in 3D is

$$\mathbf{F} = < yz, xz, xy >$$

$$\nabla \times \mathbf{F} = < R_y - Q_z, P_z - R_x, Q_x - P_y >$$
$$= x - x, y - y, z - z > = \mathbf{0}$$

Auroux

Suppose
$$M = ye^{-x}, \quad N = \frac{1}{2}x^2 - e^{-x}$$
and our curve is a shifted unit circle, centered at $(2,0)$, so $x = 2 + cos\theta$ and $y = sin\theta$.

This is difficult to parametrize for the line integral, because of e^{-x}. The line integral does not look like fun, and the region is no help, but

Instead we can do
$$\iint_R (N_x - M_y)\ dA$$
$$M = ye^{-x}, \quad M_y = e^{-x}$$
$$N = \frac{1}{2}x^2 - e^{-x}, \quad N_x = x + e^{-x}$$
$$\iint_R x\ dA$$
Recall that
$$\bar{x} = \frac{\iint x\ dA}{\iint dA} = (1/Area) \iint x dA$$
By symmetry, $\bar{x} = 2$ so
$$\iint_R x\ dA = 2\pi$$

Alternatively, we could calculate this directly
$$x = 2 + \cos\theta$$

So

$$\iint_R x\ dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{r=1}(2+\cos\theta)\ r\ dr\ d\theta$$

inner

$$(2+\cos\theta)\frac{r^2}{2}\ \Big|_0^1 = 1+\frac{1}{2}\cos\theta$$

outer

$$\int_{\theta=0}^{2\pi}(1+\frac{1}{2}\cos\theta)\ d\theta = (\theta+\frac{1}{2}\sin\theta)\ \Big|_0^{2\pi} = 2\pi$$

which checks.

Paul

Given

$$\mathbf{F} = < xy, x^2y^3 >$$

The curl is

$$N_x - M_y = 2xy^3 - x$$

If the region is the triangle $(0,0) \rightarrow (1,0) \rightarrow (1,2) \rightarrow (0,0)$ then

$$\int_0^1\int_0^{2x} 2xy^3 - x\ dy\ dx$$

inner

$$= \frac{1}{2}xy^4 - xy\ \Big|_0^{2x} = 8x^5 - 2x^2$$

outer

$$\int_0^1 8x^5 - 2x^2\ dx = \frac{8}{6}x^6 - \frac{2}{3}x^3\Big|_0^1 = \frac{8}{6} - \frac{2}{3} = \frac{2}{3}$$

Try the line integral to check it.

ellipse

Of course, my favorite example is the area of the ellipse. Suppose $N_x - M_y = 1$. Then the curl integral is the area of the region. If the components of $\mathbf{F}$ are $N = x/2$ and $M = -y/2$, this condition holds. Parametrize the ellipse.

$$x = a\cos\theta$$

$$y = b\sin\theta$$

So, for the left hand side we have

$$\int_C M\ dx + N\ dy = \int_C -\frac{1}{2}y\ dx + \frac{1}{2}x\ dy$$

$$= \int_0^{2\pi} (-\frac{1}{2})(b\sin\theta)\ (-a\sin\theta)\ d\theta\ + (\frac{1}{2})(a\cos\theta)\ (b\cos\theta)\ d\theta$$

$$= \frac{1}{2}ab \int_0^{2\pi} \sin^2\theta\ d\theta + \int_0^{2\pi} \cos^2\theta\ d\theta$$

$$= \frac{1}{2}ab \int_0^{2\pi}\ d\theta = \pi ab$$

You may wonder why we chose $\mathbf{F} = \langle -y/2, x/2\rangle$ since there are many other values that would work. The reason is that the integral is particularly easy. Let's try one other choice:

$$\mathbf{F} = \langle 0, x\rangle$$

We use the same parametrization from above. The left hand side is:

$$\int_C M\ dx + N\ dy = \int_C 0\ dx + x\ dy$$

$$= \int_0^{2\pi} a\cos\theta\ b\cos\theta\ d\theta$$

$$= ab \int_0^{2\pi} \cos^2\theta \; d\theta$$

$$= \frac{1}{2} ab \; (\theta + \sin\theta\cos\theta)| \Big|_0^{2\pi})$$

$$= \pi ab$$

Chapter 152

Funky field

I ran into what looks like a funky field this morning in a Calc 3 book. Consider:

$$f(x, y) = \theta$$

where θ is, as usual, the angle corresponding to that formed by the ray reaching out to the point in polar coordinates.

$$\theta = \tan^{-1} \frac{y}{x}, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Now compute

$$\mathbf{F} = \nabla f = \ \langle \ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \ \rangle$$

To pre-compute a bit, the derivative of $\tan^{-1} t$ is $1/(1+t^2)$, so our derivatives will both have a factor of

$$\frac{1}{1 + y^2/x^2} = \frac{x^2}{x^2 + y^2}$$

The derivatives are then what we have above times the derivative of the argument to $\tan^{-1}$:

$$\frac{\partial}{\partial x} \ \tan^{-1} \frac{y}{x} = \frac{x^2}{x^2 + y^2} \cdot (\frac{-y}{x^2}) = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial}{\partial y} \ \tan^{-1} \frac{y}{x} = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\mathbf{F} = \frac{1}{x^2+y^2} \ \langle \ -y, x \ \rangle$$

line integral

Now, since $\mathbf{F}$ is the gradient of a potential function you might expect that the line integral around a closed path will be zero.

If we are just presented with $\mathbf{F}$, not knowing anything else, one test to do is to compute the mixed partials and see if they are equal. That is, we should compute:

$$M_y = \frac{d}{dy} \ \frac{-y}{x^2+y^2} = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$N_x = \frac{d}{dx} \ \frac{x}{x^2+y^2} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

The theorem is that *if* the mixed partials are not equal, then the field is not conservative. The converse is also true, but has an additional requirement, as we will see. Let's compute the line integral.

$$\oint M \ dx + N \ dy$$

That factor of $x^2 + y^2$ makes me think polar coordinates should be the way to go (not to mention that we started with $f(x, y) = \theta$). Let's take the unit circle centered at the origin as the path, going counter-clockwise.

$$x = \cos\theta$$

$$dx = -\sin\theta \ d\theta$$

$$y = \sin\theta$$

$$dy = \cos\theta \ d\theta$$

$$x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$$

$$M = \frac{-y}{x^2+y^2} = -\sin\theta$$

$$N = \frac{x}{x^2+y^2} = \cos\theta$$

So

$$\oint M \ dx + N \ dy = \oint \sin^2\theta \ d\theta + \cos^2\theta \ d\theta$$

$$= \oint d\theta = \theta \ \Big|_0^{2\pi} = 2\pi$$

Indeed the integral around this closed path is *not* zero and the field is not conservative. Why not?

The additional requirement that we skirted around above is that the field must be defined everywhere in the region inside the closed curve. But neither component of $\mathbf{F}$ is defined at the origin. Somehow, that deficiency makes it blow up and give a non-zero integral on this path.

It is also worth pointing out that if the path does not include the origin, then the field *is* conservative and the integral will be zero.

path not including the origin

Suppose we go around the displaced unit square starting from $(1, 0)$. The region is $[1, 2] \times [0, 1]$.

$C1$: $y = 0$, $dy = 0$

$$\int_1^2 M \ dx = \int_1^2 \frac{-y}{x^2+y^2} \ dx = \int_1^2 0 \ dx = 0$$

$C2$: $x = 2$, $dx = 0$

$$\int_0^1 N\ dy = \int_0^1 \frac{x}{x^2+y^2}\ dy = \int_0^1 \frac{2}{4+y^2}\ dy$$

Leave aside the factor of 2 for a moment.

The basic integral is $1/a + y^2$, which we deal with by substitution: $\sqrt{a}u = y$, $au^2 = y^2$ and $\sqrt{a}\ du = dy$ so

$$\int \frac{1}{a+y^2}\ dy = \sqrt{a} \int \frac{1}{a+au^2}\ du = \frac{1}{\sqrt{a}} \int \frac{1}{1+u^2}\ du$$

$$= \frac{1}{\sqrt{a}}\ \tan^{-1} u = \frac{1}{\sqrt{a}}\ \tan^{-1} \frac{y}{\sqrt{a}}$$

We have the same bounds, because we have switched back to y as the variable. Recall the factor of 2 and that $a = 4$ so we have

$$= 2 \cdot \frac{1}{2} \tan^{-1} \frac{y}{2}\ \Big|_0^1 = \tan^{-1} \frac{1}{2}$$

Let's leave it in this form for now.

$C3$: $y = 1$, $dy = 0$

$$\int_2^1 M\ dx = \int_2^1 \frac{-y}{x^2+y^2}\ dx = \int_2^1 \frac{-1}{1+x^2}\ dx$$

$$= \int_1^2 \frac{1}{1+x^2}\ dx = \tan^{-1} 2 - \tan^{-1} 1$$

$C4$: $x = 1$, $dx = 0$

$$\int_1^0 N\ dy = \int_1^0 \frac{x}{x^2+y^2}\ dy = \int_1^0 \frac{1}{1^2+y^2}\ dy$$

$$= \tan^{-1} 0 - \tan^{-1} 1$$

So, all together we have:

$$\tan^{-1}\frac{1}{2} + \tan^{-1}2 - 2\tan^{-1}1$$

Now, the angles whose tangents are equal to $1/2$ and 2 (≈ 63 degrees) are not exactly nice, but they are complementary angles and so the sum is equal to $\pi/2$, which *is* nice. The angle whose tangent is equal to 1 is, of course, $\pi/4$. So

$$\frac{\pi}{2} - 2\,\frac{\pi}{4} = 0$$

Everything cancels and the final result is just 0. This field is conservative and the line integral around a closed path is equal to zero, as long as that path does not enclose the origin.

I suppose we could think about a circle of radius 1 displaced from the origin to $(2, 0)$. But I ran into trouble trying to calculate that one.

It is also worth pointing out that conservation fails if we even just touch the origin on the curve. Consider the part of the unit circle in the first quadrant. The integral from $x = 0 \to 1$ is zero since $y = 0$ and $dy = 0$ and so is the other part coming back along the y-axis. But the integral over the arc is $\pi/2$. So the total is not zero.

And now that I think about it, suppose we complete the closed curve, not by going back to zero, but by following some other path. We must have zero for the whole path, and that means that the integral along *any* path from the positive y-axis to the positive x-axis excluding the origin is $-\pi/2$.

Chapter 153

Derivation of Green's Theorem

The curl of $\mathbf{F}$ is defined to be

$$\nabla \times \mathbf{F} = N_x - M_y$$

for $\mathbf{F} = \langle M, N \rangle$.

Curl measures how far the field is from being conservative. Green's Theorem is

$$\oint_C \mathbf{F} \cdot \mathbf{dr} = \iint_R \nabla \times \mathbf{F} \, dA$$

$$\oint_C M \; dx + N \; dy = \iint_R (N_x - M_y) \, dA$$

where $\oint$ is an integral over a *closed path*, traveling in the ccw direction.

The left-hand side "lives on the curve," whereas the right-hand side "lives over the whole region."

Derivation of Green's Theorem for Work

1

We will first show that

$$\oint_C M\ dx = \iint_R -M_y\ dA$$

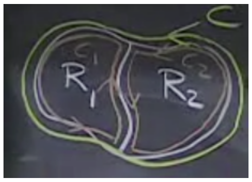
we are computing the special case where N = 0, there is only an x-component in the vector field. But by symmetry

$$\oint_C N\ dy = \iint_R N_x\ dA$$

and the sum is equivalent to the theorem.

2

Next, any complex curve (with some exceptions) can be decomposed into a set of regions, we do the integrals for each one, and the boundary curves between regions cancel.

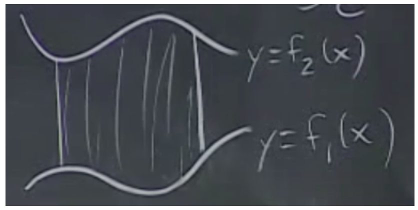


3

So then finally, to prove:

$$\oint_C M\ dx = \iint_R -M_y\ dA$$

We work on the line integral. For a vertically simple region, we have a total of four curves going around. The upper and lower curves are some functions $y = f_1(x)$ and $y = f_2(x)$. We bound these with vertical lines.



For the vertical segments C_2 and C_4 we have $dx = 0$, the other two are

$$\oint_C M\ dx = \oint_{C1} M\ dx + \oint_{C3} M\ dx$$

$$= \int_a^b M(x, f_1(x))\ dx - \int_a^b M(x, f_2(x))\ dx$$

where f_1 is the lower curve and f_2 the upper one. At each point along the curve, we have $y = f(x)$, so we can evaluate what $M(x, y)$ is at that point and then integrate with respect to x. Notice that we have switched the bounds on the second integral, and added a minus sign.

Now look at the right-hand side in the theorem, the integral over the region

$$\iint_R -M_y\ dA = \iint_R -M_y\ dy\ dx$$

and

$$M_y = \frac{\partial M}{\partial y}$$

so

$$I = -\int_{x=a}^{x=b} \int_{y=f_1(x)}^{y=f_2(x)} \frac{\partial M}{\partial y}\ dy\ dx$$

but

$$\frac{\partial M}{\partial y}\ dy = M$$

so the inner integral is just

$$\int_{y=f_1(x)}^{y=f_2(x)} \frac{\partial M}{\partial y}\ dy = M(x, f_2(x)) - M(x, f_1(x))$$

and (remembering the minus sign) the outer integral is

$$-\int_a^b \left[\ M(x, f_2(x)) - M(x, f_1(x))\ \right]\ dx$$

but that is the same as what we had above (taking account of the signs).

Derivation of Green's Theorem for Flux

Green's Theorem for Flux has the same mathematical content as the work theorem, just substituting symbols to look at it in a different light.

The work theorem starts with

$$\oint_C \mathbf{F} \cdot \mathbf{dr} = \int_C \mathbf{F} \cdot \hat{\mathbf{T}}\ ds$$

while the flux theorem starts with

$$\int_C \mathbf{F} \cdot \hat{\mathbf{n}}\ ds$$

where

$$d\mathbf{r} \cdot \hat{\mathbf{n}} = 0$$

This formulation gave, for the work theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\ dx + N\ dy = \iint_R (N_x - M_y)\ dA$$

For the flux theorem the differentials have the same magnitude but they switch places and signs so that they will be orthogonal and thereby satisfy the condition $d\mathbf{r} \cdot \hat{\mathbf{n}} = 0$:

$$\int_C \mathbf{F} \cdot \hat{\mathbf{n}}\ ds = \langle M, N \rangle \cdot \langle dy, -dx \rangle = \int_C M\ dy - N\ dx$$

Rewrite with dx first as usual.

$$\int_C -N\ dx + M\ dy$$

Now apply Green's work theorem.

$$\int_C -N\ dx + M\ dy = \iint_R (M_x - (-N_y)\ dA = \iint_R (M_x + N_y)\ dA$$

As Strang says: "playing with letters has proved a new theorem!...The components M and N can be chosen freely and named freely."

The left-hand side is

$$\oint_C -N\ dx + M\ dy = \oint \mathbf{F} \cdot \hat{\mathbf{n}}\ ds$$

(recall that $\hat{\mathbf{n}}\ ds = \ \langle dy, -dx \rangle$).

while the right-hand side is

$$\iint_R (M_x + N_y)\ dA = \iint_R \nabla \cdot \mathbf{F}\ dA$$

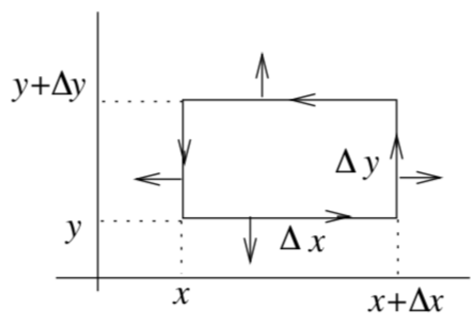
meaning of the divergence

Green's theorem for flux says that

$$\int_C \mathbf{F} \cdot \hat{\mathbf{n}} \; ds = \oint_C -N \; dx + M \; dy$$
$$= \iint_R \nabla \cdot \mathbf{F} \; dA = \iint_R M \; dx + N \; dy$$

The total flux or flow across the curve bounding the region R (heading to the outside) is equal to the *divergence* of $\mathbf{F}$. We obtain this result by a direct analysis, as follows.

We analyze the flow out of a small rectangle. If $\mathbf{F}$ is continuously differentiable, then div $\mathbf{F}$ is is a continuous function, which is therefore approximately constant if the rectangle is small enough. The double integral is approximated by a product, since the integrand is approximately constant.



The flux across the top is the part of $\mathbf{F}(x, y + \Delta y)$ in the $\hat{\mathbf{j}}$ direction times the length of the side:

$$\mathbf{F}(x, y + \Delta y) \cdot \hat{\mathbf{j}} \; \Delta x = N(x, y + \Delta y) \; \Delta x$$

On the bottom we have a minus sign from the dot product and

$$\mathbf{F}(x, y) \cdot \hat{\mathbf{j}} \; \Delta x = -N(x, y) \; \Delta x$$

Adding these two together

$$N(x, y + \Delta y)\ \Delta x - N(x, y)\ \Delta x \approx (\frac{\partial N}{\partial y}\Delta y)\Delta x$$

An equivalent argument gives the flux across the sides as

$$\approx (\frac{\partial M}{\partial x}\Delta x)\Delta y$$

So all four together yield

$$(\frac{\partial N}{\partial y}\Delta y)\Delta x + (\frac{\partial M}{\partial x}\Delta x)\Delta y = (M_x + N_y)\Delta x\ \Delta y$$

which, in the limit as the rectangle becomes very small, is the divergence. If we tile the region with tiny rectangles and sum them up, we obtain

$$\iint_R (M_x + N_y)\ dx\ dy$$

> Continuing our search for a physical meaning for the divergence, if the total flux over the sides of the small rectangle is positive, this means there is a net flow out of the rectangle. According to conservation of matter, the only way this can happen is if there is a source adding fluid directly to the rectangle. If the flow is taking place in a shallow tank of uniform depth, such a source can be visualized as someone standing over the tank, pouring fluid directly into the rectangle. Similarly, a net flow into the rectangle implies there is a sink withdrawing fluid from the rectangle. It is best to think of such a sink as a negative source. The net rate (positive or negative) at which fluid is added directly to the rectangle from above may be called the source rate for the rectangle.

$$\begin{aligned} \text{total flux across } C &= \text{source rate for } R \\ \oint_C M\,dy - N\,dx &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA \end{aligned}$$

http://math.mit.edu/~jorloff/suppnotes/suppnotes02/v4.pdf

Part XXXV

Vector Calculus 3D

Chapter 154

Triple integrals

For double integrals, we have a function $f(x, y)$, although for a simple area this function might be 1, and we think of the value of the function at each point (x, y) over a region R as a height in the z-dimension. So for the double integral what we visualize is a surface in three dimensions, and we are computing the volume underneath the surface over the region R that is its shadow of this in the xy-plane.

We do that by taking slices through this volume that are perpendicular to one of the axes, say the x-axis, and each of this series of thin slices has a volume which is an area times a thickness that is dx. The area calculation is a standard 1D integral of the slice across the range of y. The biggest trick is to remember that the limits for y in the inner integral usually depend on which slice we are talking about (i.e. the current value of x). The bounds of R, the limits for y, often depend on where we are along the x-axis. The outer integral always goes from the smallest to largest values of R in that dimension.

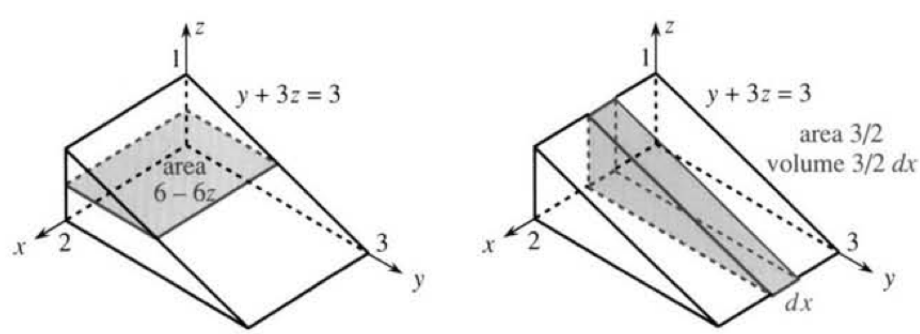
For a triple integral, we can no longer visualize the thing we are computing as a volume, unless you are able to visualize it as a volume in four dimensions. Instead we have two things: a function which associates with a value with every point in 3D space inside some solid, and in conjunction a description of this solid or volume which is the 3D analog of the region R that was the shadow of our surface for a double integral.

Remember that in double integrals it was the shape of the region that determined the bounds on the integrals. It will be similar for triple integrals, but harder to visualize.

Strang starts this topic with a box, and you can't get simpler than that. Suppose it is $[0, 2] \times [0, 3] \times [0, 1]$.

$$
\begin{aligned}
V = \iiint dV &= \int_{z=0}^{1} \int_{y=0}^{3} \int_{x=0}^{2} dx\ dy\ dz \\
&= \int_{z=0}^{1} \int_{y=0}^{3} x\Big|_0^2\ dy\ dz \\
&= 2 \int_{z=0}^{1} y\Big|_0^3\ dz \\
&= 6 \int_{z=0}^{1} dz = 6
\end{aligned}
$$

What about half a box? Call it a doorstop or *prism*. We can slice the object in various ways.



For the approach in the left panel, we have slices of area added up as we go along the z-axis. The question is, what are the bounds if the outer integral is of dz with $z = 0 \to z = 1$. If we do dx as the middle one, notice that x does not depend on z so the bounds are $x = 0 \to x = 2$. The only variable bounds are on y. y depends on z, and the equation for the outer bound is $y = 3 - 3z$. So this integral is

$$\int_{z=0}^{1} \int_{x=0}^{2} \int_{y=0}^{3-3z} dy\ dx\ dz$$

$$\int_{z=0}^{1} \int_{x=0}^{2} 3 - 3z\ dx\ dz$$

$$\int_{z=0}^{1} (3-3z)x \Big|_{0}^{2}\ dz$$

$$\int_{z=0}^{1} 6 - 6z\ dz = 6z - 3z^2 \Big|_{0}^{1} = 3$$

For the approach in the right panel, our slices of area are parallel to the yz-plane. We have the same equation ($y = 3 - 3z$), but solve this time for z.

$$\int_{x=0}^{2} \int_{y=0}^{3} \int_{z=0}^{1-\frac{1}{3}y} dz\ dy\ dx$$

$$\int_{x=0}^{2} \int_{y=0}^{3} 1 - \frac{1}{3}y\ dy\ dx$$

$$\int_{x=0}^{2} y - \frac{1}{6}y^2 \Big|_{0}^{3}\ dx$$

$$\int_{x=0}^{2} \frac{3}{2}\ dx = 3$$

cone

The hard part about the cone is simply coming up with the right volume element for cylindrical coordinates

$$V = \iiint dz\ r\ dr\ d\theta$$

which can be seen if we remember that a small change in $\Delta\theta$ gives a small change in length on the surface of $r\Delta\theta$. And of course we can check it for the cylinder with radius $r = 2$ and height $h = 2$

$$V = \int_0^{2\pi} \int_0^{r} \int_0^{2} dz\ r\ dr\ d\theta$$

$$= 2 \int_0^{2\pi} \int_0^{r} r\ dr\ d\theta$$

$$= 4\pi\ (\frac{1}{2}r^2 \Big|_0^2) = 8\pi$$

$$V = \iiint dz\ r\ dr\ d\theta$$

For a cone with height $h = 1$ and radius $r = 1$, if we fix r, what are the bounds on z? z ranges from $0 \to 1 - r$.

$$V = \int_0^{2\pi} \int_0^{1} \int_0^{1-r} dz\ r\ dr\ d\theta$$

$$= \int_0^{2\pi} \int_0^{1} (1 - r)\ r\ dr\ d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}r^2 - \frac{1}{3}r^3 \Big|_0^1\ d\theta$$

$$2\pi\ \frac{1}{6} = \frac{\pi}{3}$$

For a general cone, an important thing is that the ratio of R/H is fixed so that at any height h

$$r = \frac{R}{H}h$$

$$h = \frac{H}{R}r$$

So again, given fixed r (and θ), what are the bounds on z? They are $0 \to H - \frac{H}{R}r$.

$$V = \int_0^{2\pi} \int_0^R \int_0^{H-\frac{H}{R}r} dz \ r \ dr \ d\theta$$

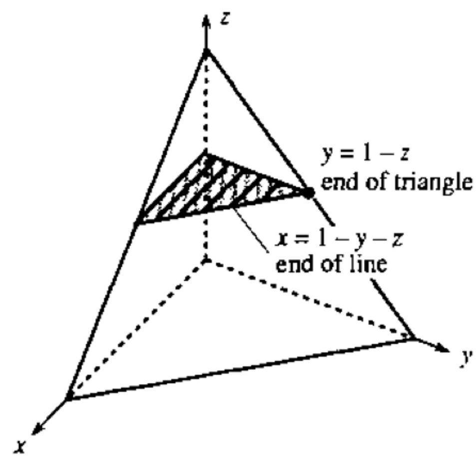
$$V = \int_0^{2\pi} \int_0^R (H - \frac{H}{R}r) \ r \ dr \ d\theta$$

$$V = \int_0^{2\pi} \frac{1}{2}HR^2 - \frac{1}{3}\frac{H}{R}R^3 \ d\theta$$

$$V = \int_0^{2\pi} \frac{1}{6}HR^2 \ d\theta$$

$$V = \frac{1}{3}\pi \ HR^2$$

tetrahedron



We have a tetrahedron with one vertex at the origin and the other 3 on the axes, 1 unit away from the origin. Suppose we slice horizontally, adding up slices as we move along $z = 0 \to 1$. As the figure shows, the lines that we will sum up to give each slice of area are in the x direction. So our integral looks like

$$\iiint dx\ dy\ dz$$

We need to find the bounds for x and y. We do y first. In the yz-plane, the dependence of y on z is $y + z = 1$ or $y = 1 - z$.

$$V = \int_{z=0}^{1} \int_{y=0}^{1-z} \int dx\ dy\ dz$$

What about x? Notice that the equation of the plane is

$$x + y + z = 1$$

So $x = 1 - y - z$ and

$$V = \int_{z=0}^{1} \int_{y=0}^{1-z} \int_{x=0}^{1-y-z} dx\ dy\ dz$$

$$= \int_{z=0}^{1} \int_{y=0}^{1-z} 1 - y - z \; dy \; dz$$

$$= \int_{z=0}^{1} y - \frac{1}{2}y^2 - yz \Big|_{0}^{1-z} \; dz$$

$$= \int_{z=0}^{1} 1 - z - \frac{1}{2}(1-z)^2 - (1-z)z \; dz$$

$$= \int_{z=0}^{1} 1 - z - (1-z)z - \frac{1}{2}(1-z)^2 \; dz$$

$$= \int_{z=0}^{1} (1-z)^2 - \frac{1}{2}(1-z)^2 \; dz$$

$$= \int_{z=0}^{1} \frac{1}{2}(1-z)^2 \; dz$$

$$= -\frac{1}{2}\frac{1}{3}(1-z)^3 \Big|_{0}^{1} = \frac{1}{6}$$

(Faster), notice that for each slice $x = y$ so we could do

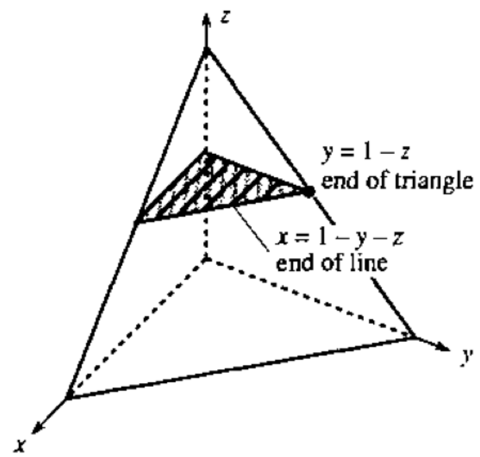
$$V = \int_{z=0}^{1} \int_{y=0}^{1-z} \int_{x=0}^{y} \; dx \; dy \; dz$$

$$V = \int_{z=0}^{1} \int_{y=0}^{1-z} y \; dy \; dz$$

$$V = \int_{z=0}^{1} \frac{1}{2}(1-z)^2 \; dz$$

and so on.

Looking again at the figure



The area of the base is $\frac{1}{2}$ (half the unit square), the height is 1, and there is the extra factor of $\frac{1}{3}$ for the volume of a pyramid.

A more laborious calculation is that the length of each edge is $\sqrt{2}$, the length of the altitude of any face is

$$h = \sqrt{2}\frac{\sqrt{3}}{2}$$

The area of the face is then

$$\frac{1}{2}\ base \times height = \frac{1}{2}\sqrt{2}\sqrt{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

To get the correct volume, the coordinates of the point in the plane closest to the origin must be

$$P = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

(by symmetry, and because $x + y + z = 1$), and the distance from the origin is then

$$d = \sqrt{(\frac{1}{3})^2 + (\frac{1}{3})^2 + (\frac{1}{3})^2} = \frac{1}{\sqrt{3}}$$

and the volume is

$$V = \frac{1}{3}\ area \times height = \frac{1}{3}\ \frac{\sqrt{3}}{2}\ \frac{1}{\sqrt{3}} = \frac{1}{6}$$

as we expected.

center of mass

Naturally, we do not need such a fancy method to find the volumes of regular solids, nor even for an ellipsoid. However, if we have some other quantity that varies with position like density $= \rho(x, y, z)$, then we just integrate with that too. For example, let's find the average height of z, called $\bar{z}$. (The centroid is at $\bar{x}, \bar{y}, \bar{z}$).

$$\begin{aligned}
\iiint z dV &= \int_{z=0}^{1} \int_{y=0}^{1-z} \int_{x=0}^{x=y} z\ dx\ dy\ dz \\
&= \int_{z=0}^{1} \frac{1}{2} z(1-z)^2\ dz \\
&= \int_{z=0}^{1} \frac{1}{2} z - z^2 + \frac{1}{2} z^3\ dz \\
&= \frac{1}{4} z^2 - \frac{1}{3} z^3 + \frac{1}{8} z^4 \Big|_0^1 = \frac{1}{24} \\
\bar{z} &= \frac{\iiint z\ dV}{\iiint dV} = \frac{1}{24} / \frac{1}{6} = \frac{1}{4}
\end{aligned}$$

Auroux

His first example considers the shape constructed from two surfaces, one above $z = x^2 + y^2$ and the second underneath $z = 4 - x^2 - y^2$. These are both paraboloid surfaces, and they have a plane of symmetry at $z = 2$. We consider the function 1, that is, we want to compute the volume of this solid.

(Incidentally, we know the answer to this problem already. If we turn the paraboloid on its side, and put the vertex at the origin, then we will have the curve $y = \sqrt{x}$. Calculate its volume as a solid of revolution as $\int \pi y^2 \ dx = \pi \int x \ dx = \pi/2 \ x^2$. We obtain 2π with the limits $x = 0 \to x = 2$, this is for each symmetric half of the solid, or 4π total.

So we want to set up the triple integral

$$\iiint dV$$

and the question is, what are the bounds? The solid has vertical mirror image symmetry, and the place where the two functions are equal is

$$4 - x^2 - y^2 = x^2 + y^2$$

$$2 = x^2 + y^2$$

The easiest way to do this is to integrate first with respect to z, since we know that z goes from $x^2 - y^2 \to 4 - x^2 + y^2$. Imagine that we have a dipstick that goes through the engine, it only gets wet in the middle, and the bounds of the wet part are as above. So then the question is, for each value of z that is contained in our solid, what are x and y?

The trick here is that the other bounds are for the circle $x^2 + y^2 = 2$, this is the largest cross-section of our solid. So Auroux sets up this integral

$$\int_{x=-\sqrt{2}}^{x=\sqrt{2}} \int_{y=-\sqrt{2-x^2}}^{y=\sqrt{2-x^2}} \int_{z=x^2+y^2}^{z=4-x^2-y^2} dz \ dy \ dx$$

Notice that these xy-bounds go over the entire region, that is the shadow of the solid in the xy-plane. This is a little counterintuitive to me, given how careful we were in double integrals about the bounds start and stop, but it's because that is all taken care of by the z term. We integrate over the whole disk and in the inside we let z go between the appropriate bounds.

He goes on to set up the integral in cylindrical coordinates. The inner integral is the same, but the outer integrals are changed to polar coordinates (and in particular, $x^2 + y^2 = r^2$

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{2}} \int_{z=r^2}^{z=4-r^2} dz\ r\ dr\ d\theta$$

The inner integral is:

$$\int_{z=r^2}^{z=4-r^2} dz$$

$$= 4 - 2r^2$$

The middle integral is then

$$\int_{r=0}^{r=\sqrt{2}} (4 - 2r^2)\ r\ dr$$

$$= 2r^2 - \frac{1}{2}r^4 \Big|_0^{\sqrt{2}} = 4 - 2 = 2$$

And the outer integral is just

$$\int_{\theta=0}^{\theta=2\pi} 2\ d\theta = 4\pi$$

Chapter 155

Surface normal

n dS six ways

A characteristic feature of vector calculus for physics is that we seem to do everything twice, first in $\mathbb{R}^2$ and then again, in $\mathbb{R}^3$.

My experience has been that $\mathbb{R}^2$ is easy and $\mathbb{R}^3$ is hard, even though already in $\mathbb{R}^2$ we have the ideas of multivariable calculus, line integrals, and even the 2D versions of curl and divergence.

One difference is that we have not just Cartesian but cylindrical and spherical coordinates in $\mathbb{R}^3$. Another even bigger difference is that for some problems, we will substitute surface integrals for line integrals. For example, we will often want to compute

$$\mathbf{F} \cdot \hat{\mathbf{n}} \ dS$$

for some surface.

The second term above ($\hat{\mathbf{n}} \ dS$) is the surface area element as a vector in the direction normal (perpendicular) to the surface, at that point. For problem solving, it is crucial to have a solid understanding of how to derive $\hat{\mathbf{n}}\ dS$ for various geometries or types of coordinates, as well as what it signifies.

In general, one can always find two vectors parallel to the surface

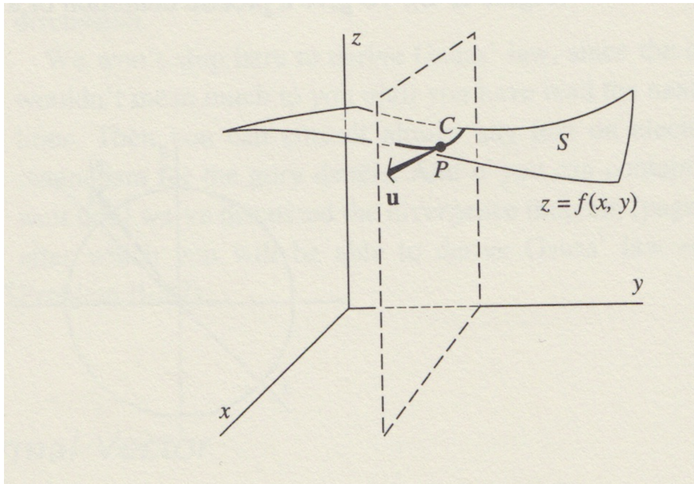


Figure II–4

namely, $\langle 1, 0, f_x \rangle$ and $\langle 0, 1, f_y \rangle$, and then form their cross-product. But some geometries have symmetry that allow us to avoid that.

plane

In the xy-plane things are really easy. The normal vector points straight up. It is $\hat{\mathbf{k}}$ or

$$\hat{\mathbf{n}} = \langle 0, 0, 1 \rangle$$

Similarly, the surface element is just dA, so we have

$$\hat{\mathbf{n}}\ dS = \langle 0, 0, 1 \rangle\ dx\ dy$$

cylinder

On the surface of a cylinder oriented along the z axis, the normal vector points straight out radially (no z component). If we look down from the top at the circular cross-section, the normal vector points out with components

$$\mathbf{n} = \langle x, y, 0 \rangle$$

The length of $\mathbf{n}$ is a, the radius of the cylinder, so

$$\hat{\mathbf{n}} = \frac{1}{a} \ \langle x, y, 0 \rangle$$

The vertical side of the surface area element is just Δz, while in the horizontal direction it is r or a times $\Delta\theta$, thus

$$\Delta S = a \ \Delta\theta \ \Delta z$$

$$dS = a \ d\theta \ dz$$

$$\hat{\mathbf{n}} \ dS = \ \langle x, y, 0 \rangle \ d\theta \ dz$$

Although it seems weird to mix x, y with θ, the idea is to keep things like this until we do the dot product with the field as will usually happen.

sphere

On the surface of a sphere, the normal vector again points straight out. It is

$$\mathbf{n} = \langle x, y, z \rangle$$

as we've seen before. If the sphere has radius a, then the length of $\mathbf{n}$ is a, and

$$\hat{\mathbf{n}} = \frac{1}{a} \ \langle x, y, z \rangle$$

The surface of the sphere is parametrized by just ϕ and θ (no r since it is fixed with $r = a$).

Looking down at the horizontal circular cross-section for a given ϕ, the radius of that circle is $a \sin\phi$, so the horizontal component of the surface area element is $a \sin\phi \ \Delta\theta$. The vertical component is a great circle (radius $r = a$), so its length is just $a \ \Delta\phi$.

$$\Delta S = a^2 \ \sin\phi \ \Delta\phi \ \Delta\theta$$
$$dS = a^2 \ \sin\phi \ d\phi \ d\theta$$

Let's just check:

$$\begin{aligned}\int dS &= \int_0^{2\pi} \int_0^{\pi} a^2 \ \sin\phi \ d\phi \ d\theta \\ &= 2\pi a^2 \int_0^{\pi} \ \sin\phi \ d\phi \\ &= 2\pi a^2 \ [-\cos\phi \ \Big|_0^{\pi} \ ] \\ &= 4\pi a^2\end{aligned}$$

and

$$\hat{\mathbf{n}} \ dS = a \ \langle x, y, z \rangle \ \sin\phi \ d\phi \ d\theta$$

Very quickly, what we have here is $a^2 \ \sin\phi \ d\phi \ d\theta$ times the unit normal, which is $\langle x, y, z \rangle / a$, and that accounts for the single power of a in the result.

graph of a function

If our surface is the graph of a function $g(x, y)$ then we can just remember the formula

$$\hat{\mathbf{n}}\ dS = \ \langle -f_x, -f_y, 1\rangle\ dx\ dy$$

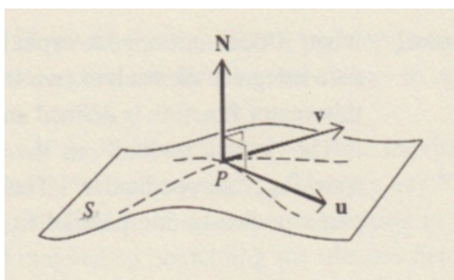
If we forget and need to work it out, the deal is that we get a linear approximation to the plane surface using f_x and f_y so that

$$\mathbf{u} = \langle 1, 0, f_x\rangle\ dx$$

$$\mathbf{v} = \langle 0, 1, f_y\rangle\ dy$$

The cross-product $\mathbf{u} \times \mathbf{v}$ gives

$$\mathbf{n} = \ \langle -f_x, -f_y, 1\rangle$$



Note that the part of $\mathbf{n}$ in the $\hat{\mathbf{k}}$ direction is just 1.

As usual, the length of $\mathbf{n}$ is

$$|\mathbf{n}| = \sqrt{f_x^2 + f_y^2 + 1}$$

but we won't need this because it will cancel.

Just write

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \ \frac{\langle -f_x, -f_y, 1\rangle}{n}$$

dS is larger than its shadow in the xy-plane by exactly this same factor

$$dS \;\; \cos\theta = dA$$

where

$$\begin{aligned} \cos\theta &= \frac{\mathbf{n}\cdot\hat{\mathbf{k}}}{|\mathbf{n}|\ |\mathbf{k}|} \\ &= \frac{1}{|\mathbf{n}|\ |\mathbf{k}|} \\ &= \frac{1}{|\mathbf{n}|} \end{aligned}$$

Hence

$$\begin{aligned} \hat{\mathbf{n}}\ dS = \hat{\mathbf{n}}\ \frac{1}{\cos\theta}\ dA &= \ \frac{\langle -f_x, -f_y, 1\rangle}{n}\ n\ dA \\ &= \ \langle -f_x, -f_y, 1\rangle\ dA \\ &= \ \langle -f_x, -f_y, 1\rangle\ dx\ dy \end{aligned}$$

Depending on whether $\mathbf{n}$ points up or down we may change the sign.

parametrization (parameterization)

More generally, we may have only a parametrization of the surface (it's not an explicit function $f(x,y)$).

$$S = \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

Then

$$\hat{\mathbf{n}}\ dS = |\mathbf{r_u} \times \mathbf{r_v}|\ du\ dv$$

normal vector

Auroux has a last example, in which we only know a normal vector $\mathbf{N}$ to the surface S. Examples include a plane

$$ax + by + cz = d$$

$$\mathbf{N} = \ \langle a, b, c >$$

or S is given by

$$g(x, y, z) = 0$$

$$\mathbf{N} = \nabla g$$

Then

$$dS = \frac{1}{\cos\theta}\ dA = \frac{|\mathbf{N}|}{\mathbf{N} \cdot \hat{\mathbf{k}}}\ dA$$

$$\hat{\mathbf{n}}\ dS = \frac{|\mathbf{N}|\ \hat{\mathbf{n}}}{\mathbf{N} \cdot \hat{\mathbf{k}}}\ dA$$

As Auroux says, what happens if I take the unit normal $\mathbf{n}$, and I multiply it by the length of my other normal $|\mathbf{N}|$? It's just $\mathbf{N}$.

$$\hat{\mathbf{n}}\ dS = \frac{\mathbf{N}}{\mathbf{N} \cdot \hat{\mathbf{k}}}\ dA$$

And again, this is "within sign", depending on how $\hat{\mathbf{n}}$ is oriented.

Chapter 156

The Divergence

In the two previous chapters we worked out Green's Theorem, which relates line integrals in the plane to double integrals over the included region. Each version of Green's Theorem has a short-hand vector expression, together with another differential version in which the meaning is more explicit.

The theorem for work is

$$\oint \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \; dA$$

alternatively

$$\int_C M \; dx + N \; dy = \iint_R N_x - M_y \; dx \; dy$$

The work done along a closed path C around a region R is equal to the double integral over R of the curl of $\mathbf{F}$. If $\mathbf{F}$ is a whirlpool then work will be done, if the force is conservative there is no net circulation and the total work will be zero.

The theorem for flux is

$$\oint \mathbf{F} \cdot \hat{\mathbf{n}} \; ds = \iint_R \nabla \cdot \mathbf{F} \; dA$$

Recall that $d\mathbf{r} = \hat{\mathbf{T}}\ ds$ and $\hat{\mathbf{T}}$ is orthogonal to $\hat{\mathbf{n}}$ so

$$\int_C M\ dy - N\ dx = \iint_R M_x + N_y\ dx\ dy$$

I think that using $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ is not technically correct in $\mathbb{R}^2$, (one should rather say "div F" and "curl F"), since ∇ really lives in three dimensions. As discussed previously, for the curl what we are effectively doing is taking a cross product (with the field zero in the z direction), then using the dot product with $\hat{\mathbf{k}}$.

For the divergence, it may be better to move now to three dimensions for discussion of the analogous theorem, sometimes ascribed to Gauss, although there are other claimants.

Divergence

In $\mathbb{R}^3$ we write:

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\ dS = \iiint_V\ \nabla \cdot \mathbf{F}\ dV$$

$$= \iiint_V\ (P_x + Q_y + R_z)\ dV$$

As a simple example, consider the unit sphere and a radial field $\mathbf{F} = \langle x, y, z \rangle$. In this case, we can do the surface integral because of the radial symmetry.

Clearly the force and the surface normal are parallel everywhere, because, for any x, y, z on the surface of the sphere, the vector to that point is $\langle x, y, z \rangle$ and so is the force at that point.

Therefore, everywhere on the surface, $\mathbf{F}$ is in the same direction as the normal, so

$$\mathbf{F} \cdot \hat{\mathbf{n}} = F$$

We have F times the surface area of the unit sphere.

Everywhere on the unit sphere $F = 1$. We can see this easily for three points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

In general we have $z^2 = 1 - x^2 - y^2$ so

$$F = |\mathbf{F}| = \sqrt{\mathbf{F} \cdot \mathbf{F}}$$

$$= \sqrt{x^2 + y^2 + +z^2}$$

$$= \sqrt{x^2 + y^2 + 1 - x^2 - y^2} = 1$$

We multiply by the surface area of the unit sphere and obtain 4π.

To check this using the right-hand side of the divergence equation, notice that

$$\nabla \cdot \mathbf{F} = P_x + Q_y + R_z = 3$$

Using the theorem, the result (the flux out of the sphere) is simply 3 times the volume of the unit sphere, or finally, 4π. The result is the same.

Other coordinates

The divergence was given as

$$\iiint_V \nabla \cdot \mathbf{F} \; dV$$

The "del" operator is

$$\nabla = < \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >$$

The divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F}$$

if $\mathbf{F} = < P, Q, R >$

$$\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$$

The divergence of a vector field is a scalar quantity.

Although the expression above is often given as the *definition* of divergence, Schey makes a big deal out of saying that he prefers another definition

$$div\ \mathbf{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\ dS$$

This can be used to derive expressions for the divergence in cylindrical and spherical coordinates, where it has a more complicated form. In cylindrical coordinates:

$$div\ \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(F_\theta) + \frac{\partial}{\partial z}(F_z)$$

(Here F_z is not a partial derivative but just the component of $\mathbf{F}$ in the z-direction, and so on).

Similarly, in spherical coordinates the divergence is

$$div\ \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\ \sin\phi}\frac{\partial}{\partial \phi}(\sin\phi F_\phi) + \frac{1}{r\ \sin\phi}\frac{\partial}{\partial \theta}(F_\theta)$$

There are other people who say this makes much more sense, but exploring the consequences seems beyond what I want to do here.

http://math.oregonstate.edu/BridgeBook/book/guide/thm

cylindrical example

Let's look at a few examples to explore divergence in cylindrical coordinates. Suppose we have a cylinder oriented along the z-axis with its length equal to 1 and its radius $r = 2$.

Further, we have a field with some divergence, like $\mathbf{F} = \langle x, y, 0\rangle$.

This field is written in x, y, z coordinates, when we switch to cylindrical coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

We want to check the divergence theorem by computing both sides of

$$\iint_S \mathbf{F} \cdot \mathbf{n}\ dS = \iiint_V\ \nabla \cdot \mathbf{F}\ dV$$

The double integral is an integral over a closed surface, in this case, the cylinder with top and bottom included.

When we rewrite $\mathbf{F}$ in cylindrical coordinates, we will have

$$\mathbf{F} = \langle r, \theta, z\rangle$$

The given field is independent of θ and z and and since

$$r = \sqrt{x^2 + y^2}$$

$$\mathbf{F} = \langle r, 0, 0\rangle$$

Using the definition of divergence from above, we have

$$div\ \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(F_\theta) + \frac{\partial}{\partial z}(F_z)$$

since $\mathbf{F}$ has no z or θ component and the r component is $F_r = r$ (this is *not* a derivative), we have

$$div\ \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(r^2) = 2$$

So the triple integral uses the cylindrical volume element and is just

$$\int_0^{2\pi}\int_0^1\int_0^2 2\ r\ dr\ dz\ d\theta$$

$$= \int_0^{2\pi}\int_0^1 [\ r^2 \Big|_0^2\ ]\ dz\ d\theta = 8\pi$$

Notice that the value of the volume integral scales linearly with z and like r^2.

Now for the surface integral. In standard form, on the sides, the cylinder has

$$\hat{\mathbf{n}}\ dS = \mathbf{r}\ d\theta\ dz = \langle x, y, 0\rangle\ d\theta\ dz$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ dS = \iint_S \langle x, y, 0\rangle \cdot \langle x, y, 0\rangle\ d\theta\ dz$$

$$\iint_S x^2 + y^2\ d\theta\ dz$$

$$\int_0^1\int_0^{2\pi} r^2\ d\theta\ dz = 2\pi r^2 = 8\pi$$

Don't forget the top and bottom of the cylinder. However, since the force is radial, the flux $\mathbf{F}\cdot\hat{\mathbf{n}} = 0$ everywhere on these two surfaces, so the total is still just 8π.

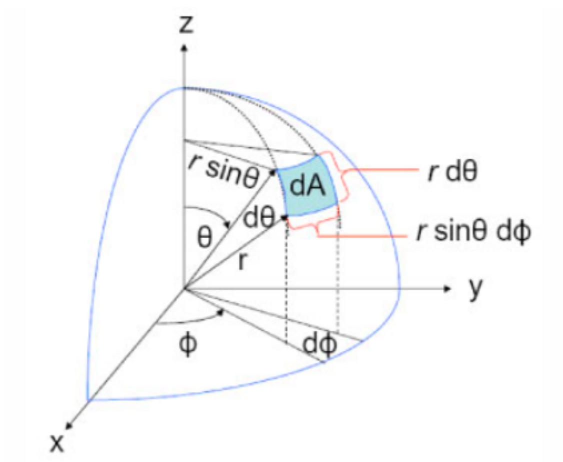
spherical example

Verify the divergence theorem for a *hemisphere* of radius a with $\mathbf{F} = \langle x, y, z \rangle$.

First, restate the theorem

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \iiint_D \nabla \cdot \mathbf{F}$$

In spherical coordinates, the surface area element is



a "box" with sides $r \; d\theta$ and $r \sin\phi \; d\phi$. Here r is given as a so

$$\hat{\mathbf{n}} \; dS = a^2 \; \sin\phi \; d\phi \; d\theta$$

The field was given in x, y, z-coordinates, $\mathbf{F} = \langle x, y, z \rangle$. This is just $\mathbf{r}$. Since $\mathbf{F}$ is orthogonal to the surface at every point the contribution of $\mathbf{F}$ to the dot product is $|\mathbf{F}| = a$, so we have

$$\mathbf{F} \cdot \hat{\mathbf{n}} \; dS = a \cdot a^2 \; \sin\phi \; d\phi \; d\theta$$

and

$$= \iint_S a^3 \; \sin\phi \; d\phi \; d\theta$$

This is a hemisphere so the bounds on ϕ are half the usual

$$= a^3 \int \left[-\cos\phi \,\Big|_0^{\pi/2} \right] d\theta$$

$$= a^3 \int d\theta = 2\pi a^3$$

Don't forget the bottom surface. In this problem, there is a component of the field in the z direction

$$\mathbf{F} \cdot \hat{\mathbf{n}} \; dS = \langle x, y, z \rangle \cdot \langle 0, 0, -1 \rangle \; dx \; dy = -z \; dx \; dy$$

however, the value of the field on the xy-plane is $z = 0$ so there is no flux.

For the divergence,

$$\nabla \cdot \mathbf{F} = 1 + 1 + 1 = 3$$

which is pretty easy!. Now, integrate

$$\iiint_D 3 \; dV$$

Well, the volume is $\frac{2}{3}\pi a^3$ so we obtain $2\pi a^3$, which matches.

OSU example

A problem from OSU asks us to verify the divergence theorem for

$$\mathbf{F} = \langle y, x, z \rangle$$

where the region is

$$0 \leq z \leq 16 - x^2 - y^2$$

The graph of $z = 16 - x^2 - y^2$ is a paraboloid which opens downward and has its vertex at $z = 16$. When $z = 0$ we have a circle of radius $r = 4$.

Recall that

$$\hat{\mathbf{n}}\ dS = \langle -f_x, -f_y, 1\rangle\ dA$$

so for this paraboloid surface we have

$$z = f(x, y) = 16 - x^2 - y^2$$

$$\hat{\mathbf{n}}\ dS = \ \langle 2x, 2y, 1\rangle\ dA$$

This corresponds to $\hat{\mathbf{n}}$ pointing out of the surface. Then

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = \iint_R 4xy + z\ dA$$

$$= \int_{-4}^{4}\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} 4xy + 16 - x^2 - y^2\ dx\ dy$$

xy-coordinates are not a good way to do this problem. Convert to polar coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$dA = r\ dr\ d\theta$$

$$\iint_R (4r^2\sin\theta\cos\theta + 16 - r^2)\ r\ dr\ d\theta$$

The region of integration is the disk of radius $r = 4$

$$\int_0^{2\pi}\int_0^4 (4r^2\sin\theta\cos\theta + 16 - r^2)\ r\ dr\ d\theta$$

The inner integral is

$$\int_0^4 4r^3\sin\theta\cos\theta + 16r - r^3\ dr$$

$$r^4 \sin\theta\cos\theta + 8r^2 - \frac{1}{4}r^4 \Big|_0^4$$
$$= 256\sin\theta\cos\theta + 128 - 64$$
$$= 256\sin\theta\cos\theta + 64$$

The outer integral is

$$\int_0^{2\pi} 64 + 256\sin\theta\cos\theta \; d\theta$$

$$= 128\pi + 256\sin^2\theta \Big|_0^{2\pi}$$
$$= 128\pi$$

There is another part of our solid. That is the disk in the xy-plane. For this disk, the unit normal (pointing out) is just $\langle 0, 0, -1\rangle$.

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}} \; dS = -\iint_R z \; dA$$

but remember that we're on the xy-plane so $z = 0$ and the whole integral is 0.

We're not done yet! We still have to compute

$$\iiint_R \nabla\cdot\mathbf{F}$$

$$= \iiint_R P_x + Q_y + R_z \; dV$$

since $\mathbf{F} = \langle y, x, z\rangle$ this is just equal to 3. So we need

$$3\iiint_R dV$$

If we convert to cylindrical coordinates, we will integrate over the disk of radius $r = 4$. What is the upper bound on z?

$$z = 16 - x^2 - y^2 = 16 - r^2$$

So we have

$$\int_0^{2\pi} \int_0^4 \int_0^{16-r^2} dz\ r\ dr\ d\theta$$

The inner integral is just $16 - r^2$. The middle integral is

$$\int_0^4 16r - r^3\ dr$$

$$= 8r^2 - \frac{1}{4}r^4 \Big|_0^4$$

$$= 128 - 64 = 64$$

Finally, we pick up 2π from the outer integral for a final result of 128π, which matches what we had above.

Or, we could have just said that the solid is a hemisphere of radius 4 so the volume is a standard formula. Again, we need

$$3 \iiint_R dV$$

so

$$3V = 3\ \frac{1}{2}\ \frac{4}{3}\pi 4^3$$

$16^2 = 256$ and one-half of that is 128, times π.

Conceptually, going from 2 to 3 dimensions doesn't seem like a big deal. But the problems are considerably more challenging.

Chapter 157

Stokes

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \; dS$$

Stokes' theorem applies to a curve in space (it does not have to lie in a plane). The theorem says that the work going around a closed curve is equal to the integral over *any surface* with that curve as its boundary, of the component of the curl of $\mathbf{F}$ normal to the surface.

An alternative form (using the fact that $\mathbf{dr} = \langle dx, dy, dz \rangle$, and computing the curl $\nabla \times \mathbf{F}$) is

$$\int_C M \; dx + N \; dy + P \; dz = \iint_R \; \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle \quad \cdot \hat{\mathbf{n}} \; dS$$

When Stokes' theorem is applied to a region in the xy-plane, $\hat{\mathbf{n}} = \hat{\mathbf{k}}$, so only the third term of the curl is non-zero, and $dS = dA$. On the left-hand side $dz = 0$, so this just reduces to

$$\int_C M \; dx + N \; dy = \iint_R \; (N_x - M_y) \; dA$$

which is Green's theorem for work.

example

State the theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\ dS$$

By the usual reasoning, since $d\mathbf{r} = < dx, dy, dz >$, the left-hand side is

$$P\ dx + Q\ dy + R\ dz$$

Now, suppose we have

$$\mathbf{F} = < z, x, y >$$

and C is the unit circle in the xy-plane, then

$$P\ dx + Q\ dy + R\ dz = \oint_C z\ dx + x\ dy + y\ dz = \oint_C x\ dy$$

Parameterize

$$C = \begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

we have

$$\oint_C x\ dy = \int_0^{2\pi} \cos t\ \ \cos t\ dt$$

$$= \frac{1}{2}(t + \frac{1}{2} \sin t)\ \Big|_0^{2\pi} = \pi$$

For the surface, we can use anything that passes through C, let's use the paraboloid for fun.

$$z = 1 - x^2 - y^2$$

We need the curl of $\mathbf{F} = < z, x, y >$

$$\nabla \times \mathbf{F} = < 1, 1, 1 >$$

We need

$$\hat{\mathbf{n}}\ dS = < -f_x, -f_y, 1 >\ dx\ dy = < 2x, 2y, 1 >\ dx\ dy$$

so

$$\iint_R (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\ dS = \iint_R\ 2x + 2y + 1\ dx\ dy$$

Again, C is the unit circle in the xy-plane. To save effort, we can notice that

$$\int x\ dx = \overline{x}$$

What is the *average* value of x over the unit circle? It is just equal to 0. The same thing is true for the second integrand (reverse the order of integration). So we have just

$$\iint_R\ 1\ dx\ dy = \pi$$

which matches what we had above.

Suppose we hadn't seen this. We could just do

$$\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\ x\ dy\ dx$$

$$= \int_{x=-1}^{1} 2\sqrt{1-x^2}\ x\ dy\ dx$$

$$= -\frac{2}{3}\ (1-x^2)^{3/2}\ \Big|_{-1}^{1}$$

At both bounds, $1 - x^2 = 0$, so the whole thing is 0.

Stokes Theorem is:

$$\oint_C \mathbf{F} \cdot \mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\ dS$$

Problem 1

Given

$$\mathbf{F} = \langle yz, xz, xy \rangle$$

Show that the integral

$$\oint_C yz\ dx + xz\ dy + xy\ dz = 0$$

over *any* closed curve C.

One way to do this is to guess the potential function for which $\mathbf{F} = \nabla f$.

$$f(x, y, z) = xyz$$

fulfills this criterion. Since this is true, the curl of $\mathbf{F}$ must be zero. By Stokes theorem, the integral is zero for any closed curve C.

A second approach is to actually calculate the curl

$$\nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$= \langle x - x, y - y, z - z \rangle = \langle 0, 0, 0 \rangle$$

and the dot product with *any* $\hat{\mathbf{n}}$ is zero.

Problem 2

Evaluate

$$\oint_C (y + 2z)dx + (x + 2z)dy + (x + 2y)dz$$

where C is the intersection of the unit sphere $x^2 + y^2 + z^2 = 1$ with the plane $x + 2y + 2z = 0$. This looks fairly hard at first. How to parameterize this curve? But we start by calculating

$$\nabla \times \mathbf{F} = \langle 2 - 2, 2 - 1, 1 - 1 \rangle = \langle 0, 1, 0 \rangle$$

What is $\hat{\mathbf{n}}\ dS$? Our surface is a part of the plane. Notice that $(0,0,0)$ is a solution of the equation for the plane, so it goes through the origin. Therefore, the intersection is a circle of radius 1. The plane has normal vector $\mathbf{n} = \langle 1,2,2 \rangle$ and *unit normal* $\hat{\mathbf{n}} = 1/3\ \mathbf{n}$ so

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} = \frac{2}{3}$$

Thus we have

$$\iint_R (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\ dS = \iint_R \frac{2}{3}\ dS$$

which is just two-thirds the area of the unit circle, or $4/3\pi$.

Problem 3

Evaluate

$$\oint_C y^3\ dx - x^3\ dy + z^3\ dz$$

where C is the intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $x + y + z = b$.

The normal vector to the plane is $\mathbf{n} = \langle 1,1,1 \rangle$. We could certainly parametrize the curve in terms of the angle θ going around the cylinder. z would move from a minimum at $\theta = \pi/4$ to a maximum on the other side of the circle.

Let's try the curl first.

$$\mathbf{F} = \langle y^3, -x^3, z^3 \rangle$$

$$\begin{aligned} \nabla \times \mathbf{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \langle 0, 0, -3x^2 - 3y^2 \rangle \end{aligned}$$

Using the equation of the surface $z = b - x - y$, we get that $f_x = -1 = f_y$ so

$$\hat{\mathbf{n}}\ dS = \langle -f_x, -f_y, 1\rangle\ dx\ dy$$

and

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\ dS = -3x^2 - 3y^2\ dx\ dy$$

$$\begin{aligned}
\iint_R (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\ dS &= \iint_R -3x^2 - 3y^2\ dx\ dy \\
&= -3 \iint_R x^2 + y^2\ dx\ dy
\end{aligned}$$

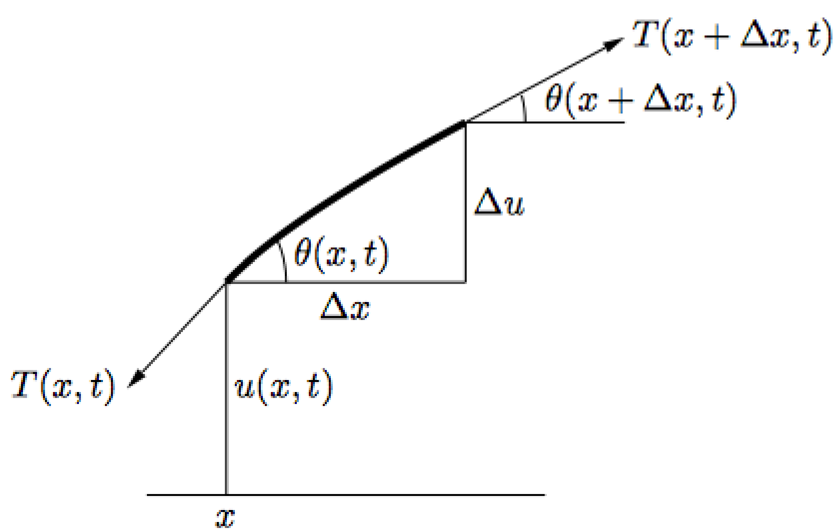
We need to integrate this over a circle of radius a, so switch to polar coordinates

$$\begin{aligned}
&= -3 \int \int r^2\ r\ dr\ d\theta \\
&= -3\ \int \frac{1}{4} a^4\ d\theta \\
&= -\frac{3}{2}\pi a^4
\end{aligned}$$

Chapter 158

Wave equation

This chapter contains a derivation of the wave equation. We consider a violin string pinned down at the ends and then plucked. Here is a short segment of the string (the notation doesn't match exactly what I'm going to use, but it's a place to start).



Here, x is not a variable but just a label for a position on the string.

We start to solve this problem by an approximation, saying that the tension T (the force in the direction shown by the arrows), has the *same magnitude* at both ends of the short interval shown as Δx in the figure.

What differs between the two ends of the interval and provides a net force is the difference in the angle θ at the two positions x and $x+\Delta x$. That force is

$$T \sin \theta_{x+\Delta x} - T \sin \theta_x$$

which, by Newton's Law, is equal to ma. For this small segment of the string

$$T \sin \theta_{x+\Delta x} - T \sin \theta_x = dm \ a$$

where dm is the mass of this small segment. You might be tempted to write $\ddot{x}$ (d^2x/dt^2) for a here, but as we said, in this problem x is just a label for a position on the string.

The value which changes is the displacement, which we will call ψ. Furthermore, if you think about it, it is clear that the displacement ψ is a function of both time and the horizontal coordinate x, so we need the partial derivative

$$T \sin \theta_{x+\Delta x} - T \sin \theta_x = dm \ \frac{\partial^2 \psi}{\partial t^2}$$

Now, dm is the mass of this small segment, which is equal to the mass per unit length times dx.

$$T \sin \theta_{x+\Delta x} - T \sin \theta_x = \mu \ dx \ \frac{\partial^2 \psi}{\partial t^2}$$

On the left hand side we are going to apply the small angle approximation. Recall that

$$\sin\theta \approx \theta$$

(where the next term in the series for $\sin\theta$ is $-\theta^3/3!$). Since $\cos\theta \approx 1$ then

$$\theta \approx \sin\theta \approx \tan\theta$$

If you look back at the figure you will see that according to the labels there

$$\frac{\Delta u}{\Delta x} = \tan\theta$$

Now, u is what we are calling ψ and in the limit as this is really a partial derivative

$$\frac{\partial\psi}{\partial x} = \tan\theta \approx \sin\theta$$

$$T(\left.\frac{\partial\psi}{\partial x}\right|_{x+dx} - \left.\frac{\partial\psi}{\partial x}\right|_{x}) = \mu\ dx\ \frac{\partial^2\psi}{\partial t^2}$$

Now, divide both sides by T and by dx and let $dx \to 0$ and we get

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\mu}{T}\ \frac{\partial^2\psi}{\partial t^2}$$

This is the wave equation, but we will re-write it as

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\ \frac{\partial^2\psi}{\partial t^2}$$

$$v = \sqrt{T/\mu}$$

It will turn out that v is the velocity of the wave.

We just guess the solution

$$\psi(x,t) = A\cos kx + \omega t$$

where k is called the *wave number*.

$$\frac{\partial^2}{\partial x^2}\ \psi(x,t) = -k^2\ \psi(x,t)$$
$$\frac{\partial^2}{\partial t^2}\ \psi(x,t) = -\omega^2\ \psi(x,t)$$

So

$$-k^2 = -\frac{\omega^2}{v^2}$$
$$k = \pm\frac{\omega}{v}$$
$$\pm kv = \omega$$
$$\psi(x,t) = A\cos kx - \omega t$$

At time zero, this function has a maximum at $x = 0$. Wait a time dt, then the maximum is when $k\ dx - \omega\ dt = 0$.

$$\frac{dx}{dt} = \frac{\omega}{k}$$

Substituting $\omega = \pm kv$

$$\frac{dx}{dt} = \pm v$$

and

$$\psi(x,t) = A\cos kx \pm kvt = A\cos k(x \pm vt)$$

Clearly, the crest of the wave is moving at the velocity v.

$$\psi(x,t) = A\cos k(x - vt)$$

describes a wave moving to the right, and the opposite choice of sign means a wave moving to the left.

Note that *any* function $f(x - vt)$ satisfies the wave equation, even

$$Ae^{-k^2(x-vt)^2}$$

If $kx = 2\pi$ the wave repeats and by definition

$$k\lambda = 2\pi$$

$$k = \frac{2\pi}{\lambda}$$

$$v = \frac{\omega}{k} = \frac{\omega\lambda}{2\pi}$$

since $\omega = 2\pi f$

$$v = f\lambda$$

The wavelength times the frequency is equal to the velocity.

Chapter 159

Maxwell's equations

This short write-up is a brief introduction to Maxwell's Equations. I want to show both the "differential" and "integral" forms, and to do that we need to start with a review of the Divergence Theorem and Stokes' Theorem. It's easiest to understand those by looking first at the 2D case using Green's Theorem.

Green and Stokes

Green's Theorem applies to curves and fields that "live in the plane." It comes in two versions. The first one says that the work done going around a closed curve C embedded in a vector field $\mathbf{F}$ is equal to the "curl" of the same field over the region enclosed by the curve. (The curl is only defined in 3D, but here its single component is in the z-direction since the vector field has no z-component).

$$\oint \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \; dA$$

The physical meaning of this is pretty easy to understand. If the field has some circulation (non-zero curl), then the work done will be non-zero as well. With no circulation, there can be no (net) work.

In terms of computation, if $\mathbf{F} = \langle M, N \rangle$, then the left-hand side is

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int \langle M, N \rangle \cdot \langle dx, dy \rangle$$

To actually evaluate this integral, we will need to parametrize the curve and express both x and y in terms of a single variable like x, or perhaps, t.

For this simple example in the plane, the curl of $\mathbf{F}$ has its only component in the z-direction, and its magnitude is $N_x - M_y$, so the double integral over the region is

$$\iint_R \nabla \times \mathbf{F} \; dA = \iint (N_x - M_y) \; dx \; dy$$

This is a real double integral.

For a conservative force, the force is the gradient of some scalar function called the potential ($\mathbf{F} = \nabla\phi = \langle \phi_x, \phi_y \rangle$), so the curl ($\nabla \times \mathbf{F} = N_x - M_y = \phi_{yx} - \phi_{xy}$) is zero, since the mixed second derivatives of a function $\phi(x, y)$ must be equal.

In three dimensions, this becomes Stokes' Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \; dS$$

Stokes theorem applies to a curve in space, which does not have to lie in a plane. The theorem says that the work going around the curve is equal to the integral of the component of the curl of $\mathbf{F}$ normal to the surface, over *any surface* with that curve as its boundary.

Green and Divergence

Green's Theorem for flux in the plane is

$$\oint_C \mathbf{F} \cdot \mathbf{n}\ ds = \iint_R \nabla \cdot \mathbf{F}\ dA$$

Here, we imagine that $\mathbf{F}$ could be a velocity field of some kind, like that describing the flow of a fluid. What the theorem says is that if there is net flow across the boundary, there has to be a source or a sink inside, depending on the sign of the flow.

As before, the left-hand integral needs to be cast in terms of a single variable through parametrization of the curve. The right-hand side is (for $\mathbf{F} = \langle M, N \rangle$)

$$\iint_R \nabla \cdot \mathbf{F}\ dA = \iint (M_x + N_y)\ dx\ dy$$

In three dimensions, the divergence theorem is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\ dS = \iiint_V \nabla \cdot \mathbf{F}\ dV = \iiint_V (P_x + Q_y + R_z)\ dx\ dy\ dz$$

(for $\mathbf{F} = \langle P, Q, R \rangle$).

Gauss's Law

The first of Maxwell's Equations is Gauss's Law, which says that for any surface enclosing a set of charges, the total flux of the electric field through the surface is related to the total charge inside

$$\Phi_E = \frac{Q}{\epsilon_0} = \iint_S \mathbf{E} \cdot d\mathbf{A}$$

So, if we knew the field (magnitude and direction) at every point on the surface, then we could calculate the charge. However, the usual situation is that we know the charge, and for a limited number of special cases we argue from symmetry that the field is everywhere perpendicular to the surface. For example, consider a charged sphere and a Gaussian surface surrounding that sphere at a radius of R from the center of the sphere. By symmetry, the field is radial. We write

$$\frac{Q}{\epsilon_0} = \iint_S \mathbf{E} \cdot d\mathbf{A}$$

$$= \iint_S E \ dA$$

We can do the last step because of the radial field. Then

$$= E \iint_S \ dA = E \ 4\pi R^2$$

Therefore,

$$E = \frac{1}{4\pi\epsilon_0} \ \frac{Q}{R^2}$$

which is easily transformed to Coulomb's Law when we multiply by the value of a test charge.

All of Maxwell's equations have both an integral form like

$$\frac{Q}{\epsilon_0} = \iint_S \mathbf{E} \cdot d\mathbf{A}$$

as well as a differential form, which in this case is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

The integral form integrates over some volume, the differential form refers to a small region of space and describes what the field is doing there.

The way to show the equivalence of these two forms is to go back to the divergence theorem (substituting **E** for **F**)

$$\iint_S \mathbf{E} \cdot \mathbf{n}\, dS = \iiint_V \nabla \cdot \mathbf{E}\, dV$$

Since $\mathbf{n}\, dS$ is really the same as $d\mathbf{A}$, the left-hand side is just Q/ϵ_0 by the first version of the law, which means that

$$\frac{Q}{\epsilon_0} = \iiint_V \nabla \cdot \mathbf{E}\, dV$$

And then the trick is to say that the charge is the integral of the charge density ρ over the volume

$$\frac{1}{\epsilon_0} \iiint_V \rho\, dV = \iiint_V \nabla \cdot \mathbf{E}\, dV$$

Since the integrals are equal, so are the integrands!

$$\frac{1}{\epsilon_0}\rho = \nabla \cdot \mathbf{E}$$

Gauss's Law for magnetism

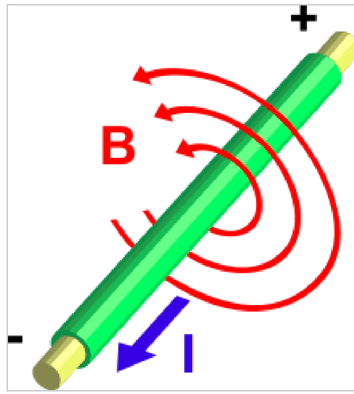
Since there is no such thing as a "magnetic charge" or magnetic monopole

$$\iint_S \mathbf{B} \cdot d\mathbf{A} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

Ampere's Law

Ampere discovered that there is a magnetic field surrounding a wire in which a current is moving



The law says that the circulation of the magnetic field surrounding a current is

$$\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 \sum_i I_i = \mu_0 I$$

Go back to Stokes' Theorem, and this becomes

$$\mu_0 I = \iint_S (\nabla \times \mathbf{B}) \cdot \hat{\mathbf{n}} \; dS$$

Recall the trick we used before. We can view the current as the integral over the whole surface of a current density $\mathbf{j}$, so we obtain

$$\iint_S \mu_0 \; \mathbf{j} \cdot \hat{\mathbf{n}} \; dS = \iint_S (\nabla \times \mathbf{B}) \cdot \hat{\mathbf{n}} \; dS$$

but if the integrals are equal, then so are the integrands. Thus

$$\mu_0 \; \mathbf{j} = \nabla \times \mathbf{B}$$

Now, we're not supposed to know this yet, but later on we will find out that

$$\frac{1}{c^2} = \epsilon_0 \; \mu_0$$

So we can rewrite the previous result as

$$\nabla \times \mathbf{B} = \frac{1}{c^2\ \epsilon_0}\,\mathbf{j}$$

or

$$c^2\ \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0}$$

This equation acquires another term due to Maxwell and the "displacement current"

$$c^2\ \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t}$$

Displacement current

Consider a circuit containing a battery, a switch and a capacitor. Throw the switch and the capacitor will charge. Current flows in the circuit except across the capacitor. According to Ampere's Law, at any instant this relation holds

$$\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

Stokes theorem says the surface through which we measure the current can be displaced from its boundary, where we integrate the magnetic field. Draw the boundary around the wire, but put the surface between the plates of the capacitor. The current across the surface is zero, but there is still a magnetic field.

For a capacitor

$$\mathbf{E} = \frac{Q}{\epsilon_0 A}$$

Outside the capacitor the electric field is zero. So

$$\Phi_E = \iint_S \mathbf{E} \cdot d\mathbf{A} = \mathbf{E}A$$

across the capacitor only.

Thus

$$\Phi_E = \mathbf{E}A = \frac{Q}{\epsilon_0}$$

In this situation, the flux is time-dependent:

$$\frac{d}{dt}\Phi_E = \frac{1}{\epsilon_0}\ \frac{dQ}{dt} = \frac{1}{\epsilon_0}\ I$$

Therefore we substitute $I + \epsilon_0\ d\Phi_E/dt$ for I in Ampere's Law:

$$\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 I + \mu_0\epsilon_0\frac{d\Phi_E}{dt}$$

Faraday's Law

Faraday's Law describes the following experiment.

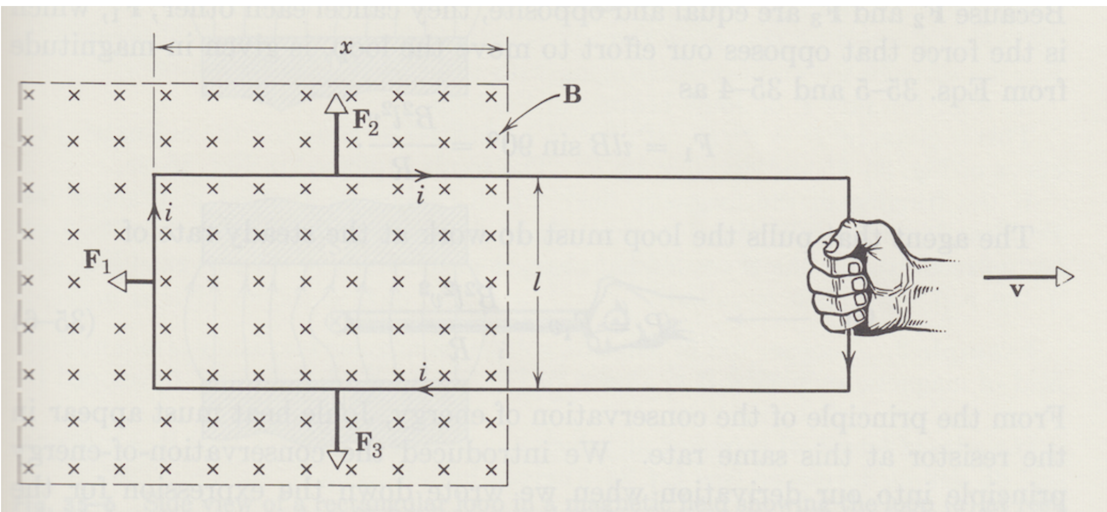


Fig. 35–5 A rectangular loop is pulled out of a magnetic field with velocity v.

A uniform magnetic field is somehow produced that has a boundary. The crosses show that the field points into the page. A loop of wire is pulled through the edge of the field, and the movement causes a current to flow in the loop.

If a current is flowing there must be an EMF ($\mathcal{E}$). The force is

$$\mathcal{E} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$$

As the drawing indicates, the force points perpendicular to the wire. This means that the forces on the top and bottom cancel. It is the unbalanced force on the left-hand side of the loop that makes the current flow. Work is applied to move the loop, this energy input appears as heat in the wire.

One way to figure out which way the current will flow is to remember that the induced current will itself cause a magnetic field. This field will be such as to *counteract the existing field* $\mathbf{B}$. In this loop, the current will flow clockwise, as indicated by the little arrows. The field due to the loop points out of the page.

$$\mathcal{E} = \oint \mathbf{E} + (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

The minus sign is due to Lenz.

Another way to get current to flow would be to vary the magnetic field. The flux resolves into two components:

$$-\frac{d\Phi}{dt} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} + \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$$

By subtraction

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

through Stokes' theorem we obtain

$$\iint_S \nabla \times \mathbf{E} \cdot d\mathbf{A} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

so the same trick as before gives

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t})$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

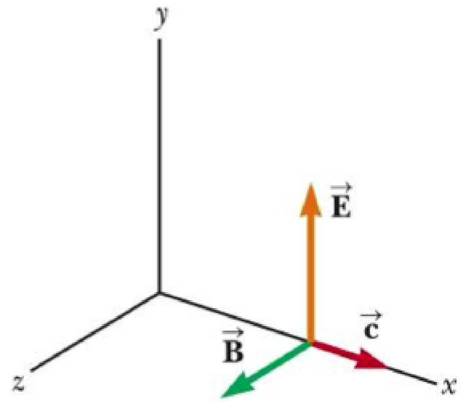
In free space there is no charge and no current so we have just

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \ \frac{\partial \mathbf{E}}{\partial t}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

wave equation

Now, suppose there is an electric field and a magnetic field at right angles to each other.



The electric field $\mathbf{E}$ is all in the y-direction, while the magnetic field $\mathbf{B}$ is all in the z-direction. Both fields are functions of x and t, and we are only concerned with what happens close to the x-axis. We compute the curl of both fields

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \mathbf{E}(x,t) & 0 \end{vmatrix} = |\frac{\partial \mathbf{E}}{\partial x}| \ \hat{\mathbf{k}}$$

Thus by one of our fundamental equations

$$\nabla \times \mathbf{E} = \frac{\partial \mathbf{E}}{\partial x} = -\frac{\partial \mathbf{B}}{\partial t}$$

Similarly

$$\nabla \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \mathbf{B}(x,t) \end{vmatrix} = -|\frac{\partial \mathbf{B}}{\partial x}| \ \hat{\mathbf{k}}$$

Thus, by another of our fundamental equations

$$\frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{\mu_0 \epsilon_0} \ \nabla \times \mathbf{B} = \frac{1}{\mu_0 \epsilon_0} \ \frac{\partial \mathbf{B}}{\partial x}$$

Take the x-derivative of the first result

$$\frac{\partial \mathbf{E}}{\partial x} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} = \frac{\partial}{\partial x}\,(-\frac{\partial \mathbf{B}}{\partial t}) = \frac{\partial}{\partial t}\,(-\frac{\partial \mathbf{B}}{\partial x})$$

$$= -\frac{\partial}{\partial t}\,(\mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t})$$

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} = \mu_0\epsilon_0\,\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

But we know this equation. It is the wave equation.

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} = \frac{1}{c^2}\,\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$c^2 = \frac{1}{\mu_0\epsilon_0}$$

calculation

$$\mu_0 = 4\pi \times 10^{-7}$$

$$\epsilon_0 = 8.8542 \times 10^{-12}$$

$$\frac{1}{\mu_0\epsilon_0} = \frac{1}{111.265 \times 10^{-19}}$$

since $1/111.265 = 0.00898$

$$c^2 = 8.99 \times 10^{16}$$

$$c = 2.99 \times 10^{8}$$

What about the units?

ϵ_0 is in

$$\frac{\text{Farads}}{m} = \frac{s^4 \cdot A^2}{m^3 \cdot \text{ kg}}$$

μ_0 is in

$$\frac{N}{A^{-2}} = \frac{\text{kg } \cdot m}{s^2 \cdot A^2}$$

The product is thus s^2/m^2. Invert and take the square root and we have meters per second, a velocity.

The speed of light in a vacuum was known to be $2.99 \times 10^8 m/s$

Chapter 160

Conservation of energy

The law of conservation of energy is pretty simple in one dimension. If kinetic energy (T) is

$$\frac{1}{2}mv^2$$

then

$$\frac{dK}{dt} = mv\frac{dv}{dt} = mva = Fv = F\frac{dx}{dt}$$

We just cancel the dt and integrate:

$$K_2 - K_1 = \int_{x_1}^{x_2} F\ dx$$

$$= U_1 - U_2$$

where $U = -\int F\ dx$. So

$$U_1 + K_1 = U_2 + K_2$$

To introduce the problem in two dimensions, I found a fun problem involving the position vector and it uses dot notation for the time-derivative, so I put it here because we've done a lot of that in the chapters on Kepler.

The problem comes from Marsden and Tromba's *Vector Calculus.*

First, we just state Newton's second law

$$\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{r}} = m\frac{d^2}{dt^2}\ \mathbf{r}$$

and then make the statement that the force is minus the gradient of the gravitational potential V, which we've seen before

$$\mathbf{F} = -\nabla V(\mathbf{r})$$

The total energy in the system is the sum of the kinetic and potential energy:

$$E = \frac{1}{2}m\ |\dot{\mathbf{r}}|^2 + V$$

And the problem given is to compute d/dt of the energy. According to the book, it's a "simple calculation."

Start with the first term, the kinetic energy. Here is a trick to allow us to work with $|\dot{\mathbf{r}}|^2$.

Recall that

$$\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = |\dot{\mathbf{r}}|^2$$

and we looked at the time-derivative of the dot product of the velocity with itself earlier:

$$\frac{d}{dt}\ \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = 2\ \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}$$

calling them "Feynman's dots". This used the product rule.

So then the time-derivative of the kinetic energy is

$$\frac{d}{dt}\ \frac{1}{2}m\ |\dot{\mathbf{r}}|^2 = m\ \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}$$

$$= \dot{\mathbf{r}} \cdot\ [\ m\ \ddot{\mathbf{r}}\ ]$$

$$= \dot{\mathbf{r}} \cdot \ [-\nabla V(\mathbf{r})\ ]$$

To justify the last step, go back to the first two statements about the force, and equate them

$$m\mathbf{a} = m\ddot{\mathbf{r}} = m\frac{d^2}{dt^2}\ \mathbf{r} = -\nabla V(\mathbf{r})$$

So now we need to evaluate this dot product.

$$\dot{\mathbf{r}} \cdot \ [-\nabla V(\mathbf{r})\ ]$$

V is a function of x, y, z that yields a real number. Its gradient is

$$\nabla V = \ \langle\ \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\ \rangle$$

while

$$\dot{\mathbf{r}} = \ \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$$

so the dot product is

$$\nabla V \cdot \dot{\mathbf{r}} = \nabla V = \ \langle\ \frac{\partial V}{\partial x}\ \frac{dx}{dt}, \frac{\partial V}{\partial y}\ \frac{dy}{dt}, \frac{\partial V}{\partial z}\ \frac{dz}{dt}\ \rangle$$

$$= \frac{d}{dt}V$$

We need to go back and pick up the minus sign we dropped from

$$\dot{\mathbf{r}} \cdot \ [-\nabla V(\mathbf{r})\ ]\ = -\frac{d}{dt}V$$

What was this? We were calculating the time-derivative of the kinetic energy T.

$$\frac{dT}{dt} = -\frac{dV}{dt}$$

And what was the problem? It was to calculate the time-derivative of the total energy:

$$E = T + V$$

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dV}{dt} = -\frac{dV}{dt} + \frac{dV}{dt} = 0$$

And now we see it. The total energy does not change with time. This is the conservation law for energy.

Part XXXVI

Advanced Topics

Chapter 161

Differentiation under the integral sign

In *Surely you're joking, Mr. Feynman*, Richard Feynman says the following:

> I had learned to do integrals by various methods shown in a book that my high school physics teacher Mr. Bader had given me. [It] showed how to differentiate parameters under the integral sign — it's a certain operation. It turns out that's not taught very much in the universities; they don't emphasize it. But I caught on how to use that method, and I used that one damn tool again and again. [If] guys at MIT or Princeton had trouble doing a certain integral, [then] I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools on it before giving the problem to me.

Introduction

Suppose $f(x,t)$ is a function of x and t (defined) on the rectangle R, where $R = [a,b] \times [c,d]$. That is, the bounds of the rectangle are $a \le x \le b$ and $c \le t \le d$.

Suppose also that $\partial f/\partial t$ is continuous on R.

Then, **Leibnitz's Rule** says that

$$\frac{d}{dt} \int_a^b f(x,t)\ dx = \int_a^b \frac{\partial}{\partial t} f(x,t)\ dx$$

If we write

$$g(x,t) = \frac{\partial}{\partial t} f(x,t)$$

Then

$$\frac{d}{dt} \int_a^b f(x,t)\ dx = \int_a^b g(x,t)\ dx$$

We'll look at some uses of this rule in the sections below.

improper integral

Let's start by recalling how to evaluate the improper integral

$$\int_0^\infty e^{-x}\ dx$$

This is accomplished by substituting a finite upper bound b for ∞ and doing the integral, then seeing what happens in the limit as $b \to \infty$.

$$\int_0^b e^{-x}\ dx = -e^{-x}\ \Big|_0^b = -\frac{1}{e^x}\ \Big|_0^b = \frac{1}{e^0} - \frac{1}{e^b}$$

In the limit, the second term becomes zero so the integral is just equal to 1:

$$\int_0^\infty e^{-x}\ dx = 1$$

Now, suppose we have

$$\int_0^\infty e^{-tx}\ dx$$

where t does not depend on x. We know that

$$\int e^{ax}\ dx = \frac{1}{a}\ e^{ax}$$

(easily verified by differentiating), and thus

$$\int_0^b e^{-tx}\ dx = -\frac{1}{t}\ [\ e^{-tx}\ \Big|_0^b\ ]$$

$$= -\frac{1}{t}\ [\ \frac{1}{e^{tx}}\ \Big|_0^b\ ]$$

$$= \frac{1}{t}\ [\ \frac{1}{e^0} - \frac{1}{e^{bt}}\ ]$$

In the limit as $b \to \infty$, the term in brackets becomes $[\ 1 - 0\ ]\ = 1$ so the integral is just $1/t$:

$$\int_0^\infty e^{-tx}\ dx = \frac{1}{t}$$

factorial

Now, apply Liebnitz's rule!

$$\frac{d}{dt}\ \int_0^\infty e^{-tx}\ dx = \int_0^\infty \frac{\partial}{\partial t}\ e^{-tx}\ dx$$

Substitute the result from above on the left-hand side:

$$\frac{d}{dt}\,\frac{1}{t} = -\frac{1}{t^2} = -t^{-2}$$

Differentiate *with respect to* t on the right-hand side:

$$-t^{-2} = \int_0^\infty -x\ e^{-tx}\ dx$$

Rearranging:

$$\int_0^\infty xe^{-tx}\ dx = \frac{1}{t^2}$$

This can be continued indefinitely:

$$\int_0^\infty x^2e^{-tx}\ dx = \frac{2}{t^3}$$

Here we have simply repeated the differentiation with respect to t on both sides, while canceling the minus signs.

And again:

$$\int_0^\infty x^3e^{-tx}\ dx = \frac{2\cdot 3}{t^4}$$

$$\int_0^\infty x^ne^{-tx}\ dx = \frac{n!}{t^{n+1}}$$

Finally, just let $t = 1$:

$$\int_0^\infty x^ne^{-x}\ dx = n!$$

This is Euler's factorial integral. It is often written with t switched in for x and x for n:

$$\int_0^\infty t^xe^{-t}\ dt = x!$$

and the Gamma (Γ) function is defined

$$\Gamma(x+1) = x! = \int_0^\infty t^x e^{-t}\ dt$$

equivalently

$$\Gamma(n) = (n-1)! = \int_0^\infty t^{n-1} e^{-t}\ dt$$

integration by parts

Euler's factorial integral can also be derived using integration by parts starting from

$$\int_0^\infty e^{-x}\ dx = 1$$

which is where we started, above. Now, suppose we have

$$\int_0^\infty x^n e^{-x}\ dx$$

with $n = 1$

$$\int_0^\infty x e^{-x}\ dx$$

let

$$u = x, \quad du = dx$$

$$dv = e^{-x}\ dx, \quad v = -e^{-x}$$

so the integral is

$$= -xe^{-x}\Big|_0^\infty - \int_0^\infty -e^{-x}\ dx$$

The second term is our previous result. At the lower bound, the first term is zero. So we must evaluate:

$$\lim_{x\to\infty} -xe^{-x}$$

$$= \lim_{x \to \infty} \frac{-x}{e^x} = \frac{-\infty}{\infty}$$

Use L'Hospital's Rule and differentiate:

$$= \lim_{x \to \infty} \frac{-1}{e^x} = \frac{-1}{\infty} = 0$$

Therefore, the entire first term is zero and the result of the integral is just 1.

$$\int_0^\infty xe^{-x}\ dx = 1$$

The general case is:

$$\int_0^\infty x^n e^{-x}\ dx$$

As before, we use integration by parts:

$$u = x^n, \quad du = nx^{n-1}\ dx$$

$$dv = e^{-x}\ dx, \quad v = -e^{-x}$$

So the integral is:

$$= -x^n e^{-x} + n \int x^{n-1} e^{-x}\ dx$$

The first term is equal to zero by the analysis we did above. We need to apply L'Hopital's rule repeatedly (n times), and eventually find that the result is zero at the upper bound as well as the lower bound.

The second term is n times the original integral, but with $n-1$ substituted for n. Recalling that we had $f(0) = f(1) = 1$, we find that:

$$\int_0^\infty x^n e^{-x}\ dx = n!$$

proof

The proof of Leibnitz's Rule is pretty straightforward, if we assume that the order of integration in an iterated integral doesn't matter (Fubini's Theorem):

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} g(x,y)\ dx\ dy = \int_{x=a}^{x=b} \int_{y=c}^{y=d} g(x,y)\ dy\ dx$$

(see wikipedia for examples of when it does matter).

I've written the equation in terms of x and y because that's how it is in the original:

http://math.hawaii.edu/~rharron/teaching/MAT203/LeibnizRule.pdf

Suppose we choose for our function the partial derivative of another function $f(x,y)$

$$g(x,y) = \frac{\partial f}{\partial y}$$

and that we take the derivative with respect to y of both sides:

$$\frac{d}{dy} \int_c^d \int_a^b \frac{\partial f}{\partial y}\ dx\ dy = \frac{d}{dy} \int_a^b \int_c^d \frac{\partial f}{\partial y}\ dy\ dx$$

Using one version of the fundamental theorem of calculus, namely:

$$\frac{d}{dt} \int_a^t f(x)\ dx = f(t)$$

the left-hand side of our equality

$$\frac{d}{dy} \int_c^d \int_a^b \frac{\partial f}{\partial y}\ dx\ dy$$

becomes:

$$= \int_a^b \frac{\partial f}{\partial y} \ dx$$

Using the other version of the FTC

$$\int_a^b F'(x) \ dx = F(b) - F(a)$$

the right-hand side

$$\frac{d}{dy} \ \int_{x=a}^{x=b} \ \int_{y=c}^{y=d} g(x,y) \ dy \ dx$$

becomes:

$$= \frac{d}{dy} \ \int_a^b \ [\ f(x,y) - f(x,c) \ ] \ dx$$

but $f(x,c)$ is not a function of y so the partial derivative is just zero for the second term and we have:

$$= \frac{d}{dy} \ \int_a^b \ f(x,y) \ dx$$

Putting the two sides together:

$$\int_a^b \frac{\partial f}{\partial y} \ dx = \frac{d}{dy} \ \int_a^b \ f(x,y) \ dx$$

which is what we wanted to prove.

I am a little shaky on both these steps but I include them based on the source.

Gaussian

Shankar uses as an example a series of functions related to the Gaussian

$$I_0(a) = \int_0^\infty e^{-ax^2} \ dx$$

$$I_1(a) = \int_0^\infty e^{-ax^2} \ x \ dx$$

and so on. The notation $I(a)$ indicates that we will eventually view these guys as a function of the parameter a.

The first one is (one-half) the Gaussian integral and we will just assume the answer here:

$$I_0(a) = \int_0^\infty e^{-ax^2} \ dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} = \frac{1}{2} \ \sqrt{\pi} a^{-1/2}$$

The second is straightforward to evaluate since we have the derivative of what's in the exponent:

$$I_1(a) = \int_0^\infty e^{-ax^2} \ x \ dx = -\frac{1}{2a} \ [e^{-ax^2} \ ] \ \Big|_0^\infty$$

$$= -\frac{1}{2a} \ (-1) = \frac{1}{2a}$$

After that (with higher powers of x) it's not so easy.

What we're going to do is start from $I_0(a)$ and differentiate with respect to a, using the rule from above. We will have:

$$\int_0^\infty \frac{\partial}{\partial a} \ e^{-ax^2} \ dx = \frac{d}{da} \ \frac{1}{2} \ \sqrt{\pi} a^{-1/2}$$

$$\int_0^\infty \ e^{-ax^2} \ (-x^2) \ dx = -\frac{1}{4} \ \sqrt{\pi} a^{-3/2}$$

$$I_2(a) = \int_0^\infty \ e^{-ax^2} \ x^2 \ dx = \frac{1}{4} \ \sqrt{\pi} a^{-3/2}$$

$$I_4(a) = \int_0^\infty \ e^{-ax^2} \ x^4 \ dx = \frac{3}{8} \ \sqrt{\pi} a^{-5/2}$$

and so on. For the odd functions do this:

$$I_1(a) = \int_0^\infty e^{-ax^2}\ x\ dx = \frac{1}{2a}$$

$$\int_0^\infty \frac{\partial}{\partial a}\ e^{-ax^2}\ x\ dx = \frac{d}{da}\ \frac{1}{2a}$$

$$-\int_0^\infty\ e^{-ax^2}\ x^3\ dx = -\frac{1}{2a^2}$$

$$I_3(a) = \int_0^\infty\ e^{-ax^2}\ x^3\ dx = \frac{1}{2a^2}$$

Thus we can solve the whole series of integrals of the form:

$$I_n(a) = \int_0^\infty e^{-ax^2}\ x^n\ dx$$

inverse tangent

Another one is the integral which yields the inverse tangent:

$$\int_0^\infty \frac{1}{1+x^2}\ dx$$

$$= \tan^{-1} x\ \Big|_0^\infty = \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

It's easy to solve this with a trig substitution:

$$x = \tan\theta$$

$$dx = \sec^2\theta\ d\theta$$

$$\frac{1}{1+x^2} = \cos^2\theta$$

The integral is just

$$\int d\theta = \theta = \tan^{-1} x$$

Suppose instead that we have

$$\int_0^\infty \frac{1}{a^2 + x^2}\ dx$$

One way to solve this is to scale x:

$$x = au, \quad dx = a\ du$$

Then

$$\begin{aligned}
\int_0^\infty \frac{1}{a^2 + x^2}\ dx &= a \int_0^\infty \frac{1}{a^2 + a^2 u^2}\ du \\
&= a \int_0^\infty \frac{1}{a^2}\ \frac{1}{1 + u^2}\ du \\
&= \frac{1}{a} \int_0^\infty \frac{1}{1 + u^2}\ du \\
&= \frac{1}{a} \tan^{-1} u \ \Big|_0^\infty = \frac{\pi}{2a}
\end{aligned}$$

So, start with that one and differentiate:

$$\int_0^\infty \frac{\partial}{\partial a}\ \frac{1}{a^2 + x^2}\ dx = \frac{d}{da}\ \frac{\pi}{2a}$$

$$-2a \int_0^\infty \frac{1}{(a^2 + x^2)^2}\ dx = -\frac{\pi}{2a^2}$$

$$\int_0^\infty \frac{1}{(a^2 + x^2)^2}\ dx = \frac{\pi}{4a^3}$$

Chapter 162

Generating functions

generating functions

$f(x)$ is a generating function for the sequence $a0$, $a1$, $a2$, ... if

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

A generating function can sometimes be analyzed to prove something more (like a formula for the coefficients), but at the very least, it gives a way to generate them.

example

The expected value for a function $f(x)$ of a continuous random variable is defined to be

$$\int f(x)\ p(x)\ dx$$

For the exponential distribution $p(x) = \lambda e^{-\lambda x}$ we obtained

$$E[x] = \int_0^{\infty} x\ \lambda e^{-\lambda x}\ dx$$

$$E[x] = \frac{1}{\lambda}$$

using integration by parts (IBP).

(A note on nomenclature: normally we would use t as the variable when talking about the exponential distribution, since it's used for problems involving events happening in time. But we're going to use t for something else near the end, so just stick with x here).

By two applications of IBP we obtained

$$E[x^2] = \int_0^\infty x^2 \ \lambda e^{-\lambda x} \ dx$$

$$E[x^2] = \frac{2}{\lambda^2}$$

Using these results we calculated the variance as

$$\text{Var } (x) = E[(x-\mu)^2]$$

$$= E[x^2] - (E[x])^2$$

$$= E[x^2] - \mu^2$$

$$= \frac{1}{\lambda^2}$$

generating function for exponential distribution

A somewhat easier way is to do the same calculation is to use what is called the moment-generating function. Here the variable t is not for time but is instead a dummy variable.

$$\phi(t) = \frac{\lambda}{\lambda - t}, \quad t < \lambda$$

where the expected value of x is the first derivative of $\phi(t)$ with respect to t, evaluated at $t = 0$:

$$E[x] = \frac{d}{dt}\ \phi(t) = \lambda\ \frac{1}{(\lambda - t)^2}\ \Big|_{t=0} = \frac{1}{\lambda}$$

Similarly

$$E[x^2] = \frac{d^2}{dt^2}\ \phi(t) = 2\lambda\ \frac{1}{(\lambda - t)^3}\ \Big|_{t=0} = \frac{2}{\lambda^2}$$

and so on. This is certainly easier. We obtain the third and fourth moments by the :same method

$$E[x^3] = \frac{d^3}{dt^3}\ \phi(t) = 6\lambda\ \frac{1}{(\lambda - t)^4}\ \Big|_{t=0} = \frac{6}{\lambda^3}$$

Of course, the question is then: where does $\phi(t)$ come from?

derivation

According to

https://www.statlect.com/fundamentals-of-probability/moment-generating-function

With some restrictions (notably $t < \lambda$):

$$E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx}\ p(x)\ dx$$

Recall that for the exponential distribution, the pdf is

$$p(x) = \lambda e^{-\lambda x}$$

so we have

$$\int_{-\infty}^{\infty} e^{tx} \lambda e^{-\lambda x}\ dx$$

The probability is defined to be $p(x) = 0$ for $x < 0$ so the bounds change to

$$\int_0^\infty e^{tx} \lambda e^{-\lambda x} \ dx$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x} \ dx$$

which we note is only finite for $t < \lambda$. The integral is pretty easy but the bounds make it improper:

$$= \frac{\lambda}{t - \lambda} \ e^{(t-\lambda)x} \ \Big|_0^\infty$$

Now, since $t < \lambda$ and so $t - \lambda < 0$), this is a negative exponential e^{-kx}.

Therefore, the upper bound is just zero. The exponential in the lower bound becomes 1 and subtraction of the rest gives:

$$\frac{\lambda}{\lambda - t}$$

So that is the immediate derivation, which doesn't say much about *why* it works, of course.

More generally, we can write

$$E[e^{tX}] = \int e^{tx} \ p(x) \ dx$$

We expand the exponential as a power series:

$$= \int (1 + tx + \frac{t^2x^2}{2} + \dots) \ p(x) \ dx$$

$$= \int \ p(x) \ dx + \int tx \ p(x) \ dx + \int t^2x^2 \ p(x) \ dx + \dots$$

$$= 1 + tE(x) + t^2E(x^2) + \dots$$

So we can see why, to get the kth moment, we take the kth derivative with respect to t and then evaluate with $t = 0$ to knock out all the higher terms.

And *that's* why it works.

Another approach

A moderately advanced (but simplifying) technique that can be used to solve this same problem is called differentiating under the integral sign.

Before we go through that, just review the following improper integral

$$\int_0^\infty e^{-cx} \ dx$$

for some constant c. We solve the integral with a real positive number b as the upper bound

$$\int_0^b e^{-cx} \ dx = -\frac{1}{t} e^{-cx} \Big|_0^b = -\frac{1}{c} \ [\ e^{-cb} - 1 \ ]$$

In the limit as $b \to \infty$, the negative exponential first term goes to zero, so we have simply

$$\int_0^\infty e^{-cx} \ dx = \frac{1}{c}$$

Liebnitz's rule says that you can differentiate on either side of the integral sign

$$\frac{d}{dt} \int_0^\infty e^{-tx} \ dx = \int_0^\infty \frac{\partial}{\partial t} \ e^{-tx} \ dx$$

Use the result from above that this integral is just $1/t$ so the left-hand side is:

$$\frac{d}{dt} \ \frac{1}{t} = -\frac{1}{t^2} = -t^{-2}$$

Differentiate *with respect to t* on the right-hand side:

$$-t^{-2} = \int_0^\infty -x\ e^{-tx}\ dx$$

Rearranging:

$$\int_0^\infty xe^{-tx}\ dx = t^{-2}$$

This can be continued indefinitely:

$$\int_0^\infty x^2 e^{-tx}\ dx = 2t^{-3}$$

Here we have simply repeated the differentiation with respect to t on both sides, while canceling the minus signs.

There is more in the write-up on the technique.

Solving our problem

$$E[x] = \int_0^\infty x\ \lambda e^{-\lambda x}\ dx$$

we get λ times what we have above, namely

$$\lambda\ \int_0^\infty xe^{-\lambda x}\ dx = \lambda\ \lambda^{-2}$$

Thus, cancelling one factor of λ on top and bottom we obtain:

$$E[x] = \frac{1}{\lambda}$$

Similarly

$$E[x^2] = \lambda\ \int_0^\infty x^2 e^{-\lambda x}\ dx$$

$$= \lambda\ 2\lambda^{-3} = \frac{2}{\lambda^2}$$

Chapter 163

Gamma and beta functions

In the chapter on improper integrals, we studied the negative exponential

$$\int_0^\infty e^{-x}\ dx$$

which we formally evaluate at a very large number a and then ask about $a \to \infty$, but in practice just write

$$= -e^{-x}\ \Big|_0^\infty$$

$$= -\ [\ 0 - 1\ ] = 1$$

Euler factorial

Here is a variant

$$\int_0^\infty xe^{-x}\ dx$$

We can do this by guessing. Since

$$[xe^{-x}]' = e^{-x} - xe^{-x}$$

Rearrange

$$xe^{-x} = -[xe^{-x}]' + e^{-x}$$

Integrate both sides and write the bounds

$$\int_0^\infty xe^{-x}\ dx = -xe^{-x}\ \Big|_0^\infty + \int_0^\infty e^{-x}\ dx$$

For the right-hand side, the second term is just 1 by the above result

$$\int_0^\infty xe^{-x}\ dx = -xe^{-x}\ \Big|_0^\infty + 1$$

evaluate a ratio

So what is

$$xe^{-x}\ \Big|_0^\infty$$

Which term wins the race? Write this as a ratio

$$\frac{x}{e^x}$$

At the upper bound we have ∞/∞ so L'Hospital says take the derivative and re-evaluate:

$$\frac{1}{e^x}$$

We have just e^{-x} as $x \to \infty$ which equals zero. At the lower bound

$$xe^{-x} = 0 \cdot 1 = 0$$

Therefore

$$xe^{-x}\ \Big|_0^\infty = 0$$

We proceed to derive a result we will use below.

$$x^2 e^{-x} \Big|_0^\infty = 0$$

Write the ratio

$$\frac{x^2}{e^x}$$

we have ∞/∞ so take the derivative, and then take it again, reaching

$$\frac{2}{e^x}$$

We have $2e^{-x}$ as $x \to \infty$, which equals zero. At the lower bound

$$x^2 e^{-x} = 0^2 \cdot 1 = 0$$

since both bounds are zero, the whole thing is zero.

$$x^2 e^{-x} \Big|_0^\infty = 0$$

Note carefully that $x^n e^{-x}$ is equal to 0 for any value of n at both the upper bound $x \to \infty$ and at $x = 0$. Do you see why?

Finishing the problem from the previous section:

$$\int_0^\infty x e^{-x} \ dx = -x e^{-x} \Big|_0^\infty + 1 = 1$$

n = 2

The next integer is

$$\int_0^\infty x^2 e^{-x} \ dx$$

We do this by IBP. Let

$$u = x^2, \qquad du = 2x\ dx$$

$$dv = e^{-x}\ dx, \qquad v = -e^{-x}$$

$$\int u\ dv = uv - \int v\ du$$

$$\int_0^\infty x^2 e^{-x}\ dx = -x^2 e^{-x}\ \Big|_0^\infty - 2\int_0^\infty -e^{-x}\ x\ dx$$

The first term on the right-hand side is zero, as before.

$$\int_0^\infty x^2 e^{-x}\ dx = 2\int_0^\infty e^{-x}\ x\ dx = 2$$

higher values of n

$$\int_0^\infty x^n e^{-x}\ dx$$

Use IBP

$$u = x^n, \qquad du = nx^{n-1}\ dx$$

$$dv = e^{-x}\ dx, \qquad v = -e^{-x}$$

$$\int u\ dv = uv - \int v\ du$$

$$\int_0^\infty x^n e^{-x}\ dx = -x^n e^{-x}\ \Big|_0^\infty + n\int_0^\infty x^{n-1} e^{-x}\ dx$$

but that first term on the right-hand side is zero so

$$\int_0^\infty x^n e^{-x}\ dx = n\int_0^\infty x^{n-1} e^{-x}\ dx$$

For $n = 3$

$$\int_0^\infty x^3 e^{-x}\ dx = 3\int_0^\infty x^2 e^{-x}\ dx = 3 \cdot 2$$

For $n = 4$

$$\int_0^\infty x^4 e^{-x}\ dx = 4\int_0^\infty x^3 e^{-x}\ dx = 4 \cdot 3 \cdot 2 = 4!$$

The general result is

$$\int_0^\infty x^n e^{-x}\ dx = n\int_0^\infty x^{n-1} e^{-x}\ dx = n!$$

Gamma function

The generalized function is usually written with t as the variable and x as the power

$$\int_0^\infty t^x e^{-t}\ dt = x!$$

The gamma function (Γ) is defined to be

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\ dt = x!$$

The reason for this shift (using $x+1$ in the function definition) is given here:

http://www.sosmath.com/calculus/improper/gamma/gamma.html

It has to do with the domain of x for which the function is defined. The original domain is $(-1, \infty)$, but after the shift it is $(0, \infty)$.

properties

As a result of the definition in the previous section

$$\Gamma(x+1) = x\Gamma(x) = x!$$

so

$$\Gamma(x) = (x-1)!$$

$$\Gamma(1) = \int_0^\infty t^0 e^{-t}\ dt = 1$$

So far we have only considered $\Gamma(x)$ for $x \in \mathbb{N}$, but in fact the formula works for fractional $x = p/q$ which means that we can write $\Gamma(1/2)$ and actually evaluate it.

$$\Gamma(\frac{1}{2}) = \int_0^\infty x^{-1/2} e^{-x}\ dx$$

$$= \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\ dx$$

Substitute $x = t^2$, then $dx = 2t\ dt$ and

$$\int_0^\infty \frac{e^{-t^2}}{t} 2t\ dt$$

$$= 2\int_0^\infty e^{-t^2}\ dt$$

We solved this integral in the chapter on the normal distribution:

$$\int_{-\infty}^\infty e^{-kx^2}\ dx = \sqrt{\frac{\pi}{k}}$$

Here, $k = 1$ so

$$\int_{-\infty}^\infty e^{-x^2}\ dx = \sqrt{\pi}$$

This is an even function so

$$\int_0^\infty e^{-x^2}\ dx = \frac{\sqrt{\pi}}{2}$$

which means the value of the integral given above is just $\sqrt{\pi}$.

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

Some people like to write things like $(-1/2)!$ at this point, and while others prefer to say that the Gamma function gives the same result as the factorial for the positive integers, but also is also defined for rational numbers.

Since

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(\frac{3}{2}) = \frac{1}{2}\cdot\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(\frac{5}{2}) = \frac{3}{2}\cdot\Gamma(\frac{3}{2}) = \frac{3}{2}\cdot\frac{\sqrt{\pi}}{2}$$

Here's a quote from

http://mhtlab.uwaterloo.ca/courses/me755/web_chap1.pdf

> The gamma function is one of the most widely used special functions encountered in advanced mathematics because it appears in almost every integral or series representation of other advanced mathematical functions.

historical aside

In *An Imaginary Tale*, Nahin says that Wallis "knew" the particular result given above, namely $\Gamma(1/2) = \sqrt{\pi}$, although of course the notation Γ was only introduced later.

Like many people in the years just before Newton, he was investigating the area under certain curves. One family of curves Wallis looked at is

$$f(x) = (x - x^2)^n$$

He found that the value of the integral is given by

$$\int_0^1 (x - x^2)^n \, dx = \frac{(n!)^2}{(2n+1)!}$$

I used the program described in the chapter on Numerical Integration

https://gist.github.com/telliott99/5a1190217a130c7ee01dee17ea483f7b

to evaluate the integral for $n = 2, 3, 4$ and obtained:

$$n = 2, \quad 0.033333333625$$

$$n = 3, \quad 0.00714285699705$$

$$n = 4, \quad 0.00158730158727$$

Very similar values are obtained by evaluating

$$\frac{(n!)^2}{(2n+1)!}$$

e.g with $n = 4$

$$\frac{24^2}{362880} = 0.00158730$$

Here is the connection to 1/2. A circle of unit *diameter* placed with its center at $(1/2, 0)$ has the equation:

$$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$$

$$y^2 = \frac{1}{4} - x^2 + x - \frac{1}{4} = x - x^2$$

$$y = \sqrt{x - x^2} = (x - x^2)^{1/2}$$

The area above the x-axis is

$$\int_0^1 \sqrt{x - x^2}\ dx$$

We know the area under this curve from the geometry of the circle, it is just one-half of $\pi/4 = \pi/8$.

Plugging the power of $n = 1/2$ into the factorial formula:

$$\frac{(n!)^2}{(2n+1)!}$$

$$\frac{((1/2)!)^2}{2} = \frac{\pi}{8}$$

$$((1/2)!)^2 = \frac{\pi}{4}$$

$$(1/2)! = \frac{\sqrt{\pi}}{2}$$

How about that!

Beta function

The Gamma function is related to the Beta function, which has wide application in probability.

The beta distribution is

$$f(x) = x^{p-1}(1 - x)^{q-1}$$

The beta function is defined as follows:

$$B(p, q) = \int_0^1 x^{p-1}(1 - x)^{q-1}\ dx, \quad p > 0, q > 0$$

You may recognize this expression as being related to the binomial distribution. The value obtained from the integral can be used to normalize the distribution so that it is a proper pdf.

Here is a great explanation:

http://varianceexplained.org/statistics/beta_distribution_and_baseball

The example is batting averages in baseball.

> Given our batting average problem, which can be represented with a binomial distribution (a series of successes and failures), the best way to represent these prior expectations (what we in statistics just call a prior) is with the beta distribution ...
>
> We expect that the player's season-long batting average will be most likely around .27, but that it could reasonably range from .21 to .35. This can be represented with a beta distribution with parameters $\alpha = 81$ and $\beta = 219$

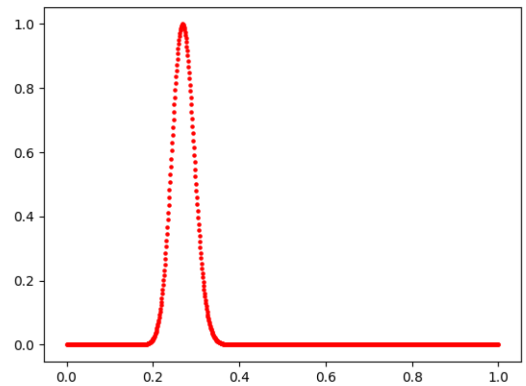
We let

$$f(x) = x^{p-1}(1-x)^{q-1}$$

with $p = 81$ and $q = 219$.

Here is a Python script to plot the distribution:

https://gist.github.com/telliott99/74bb7a47adb692b60fcbf00dd9829901



The mean of the beta distribution is simply $p/p+q = 81/(81+219) = 0.27$. The magnitudes of p and q determine the width of the peak, chosen here to approximate $[0.21, 0.35]$. Nearly all the density for historical batting averages lies in this range.

The beta function we defined is the integral of this pdf over the range $[0, 1]$. We'll see that this integral can be expressed in terms of the Gamma function, and this makes it easy to calculate.

$$B(p,q) = \frac{\Gamma(p)\ \Gamma(q)}{\Gamma(p+q)}$$

derivation

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\ dx, \quad p > 0, q > 0$$

If we make a trigonometric substitution $x = \sin^2\theta$, we have

$$B(p,q) = \int (\sin^2\theta)^{p-1}\ (\cos^2\theta)^{q-1}\ 2\sin\theta\cos\theta\ d\theta$$

$$= 2 \int (\sin\theta)^{2p-1}\ (\cos\theta)^{2q-1}\ d\theta$$

At the lower bound $x = 0, \theta = 0$, and at the upper bound $x = 1, \theta = \pi/2$ so

$$B(p,q) = 2 \int_0^{\pi/2} (\sin\theta)^{2p-1}\ (\cos\theta)^{2q-1}\ d\theta$$

Relationship to the Gamma function

Recall that

$$\Gamma(p) = \int_0^{\infty} t^{p-1} e^{-t}\ dt$$

Substitute $t = x^2$, $dt = 2x\ dx$ so

$$\Gamma(p) = 2 \int_0^{\infty} x^{2(p-1)} x\ e^{-x^2}\ dx$$

$$= 2 \int_0^{\infty} x^{2p-1}\ e^{-x^2}\ dx$$

x and y are both *dummy* variables in what follows so we can write

$$\Gamma(q) = 2 \int_0^{\infty} y^{2q-1}\ e^{-y^2}\ dy$$

Multiplying

$$\Gamma(p)\ \Gamma(q) = 4 \int_0^{\infty} x^{2p-1}\ e^{-x^2}\ dx\ \int_0^{\infty} y^{2q-1}\ e^{-y^2}\ dy$$

$$= 4 \int_0^{\infty} \int_0^{\infty} x^{2p-1} y^{2q-1}\ e^{-x^2}\ e^{-y^2}\ dx\ dy$$

x and y are independent, so we can think of this as a double integral in the plane and switch to polar coordinates with $e^{-x^2} \cdot e^{-y^2} = e^{-(x^2+y^2)} = e^{-r^2}$

$$\Gamma(p)\ \Gamma(q) = 4 \int_0^{\pi/2} \int_0^{\infty} (r \sin\theta)^{2p-1} (r \cos\theta)^{2q-1}\ e^{-r^2}\ r\ dr\ d\theta$$

The bounds on θ are the first quadrant of the plane, the same quadrant where the first double integral was taken.

The r part and the θ part are independent so we have two components (split the factor of 4):

$$I_r = 2 \int_0^{\infty} r^{2p+2q-1} e^{-r^2}\ dr$$

compare with this expression from above:

$$\Gamma(p) = 2 \int_0^{\infty} x^{2p-1}\ e^{-x^2}\ dx$$

We see that

$$I_r = \Gamma(p+q)$$

For the θ part of the integral

$$I_\theta = 2 \int_0^{\pi/2} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1}\ d\theta$$

compare with the definition of the beta function

$$B(p,q) = 2 \int_0^{\pi/2} (\sin\theta)^{2p-1}\ (\cos\theta)^{2q-1}\ d\theta$$

We see that these are identical

$$I_\theta = B(p,q)$$

Combining the results:

$$\Gamma(p)\ \Gamma(q) = I_r\ I_\theta = \Gamma(p+q)\ B(p,q)$$

and so

$$B(p,q) = \frac{\Gamma(p)\ \Gamma(q)}{\Gamma(p+q)}$$

$$= \int_0^1 x^{p-1}(1-x)^{q-1}\ dx, \quad p > 0, q > 0$$

application

If you want to see an application to Bayesian probability look at this great post:

https://alexanderetz.com/2015/07/25/understanding-bayes-updating-priors-via-the-likelihood/

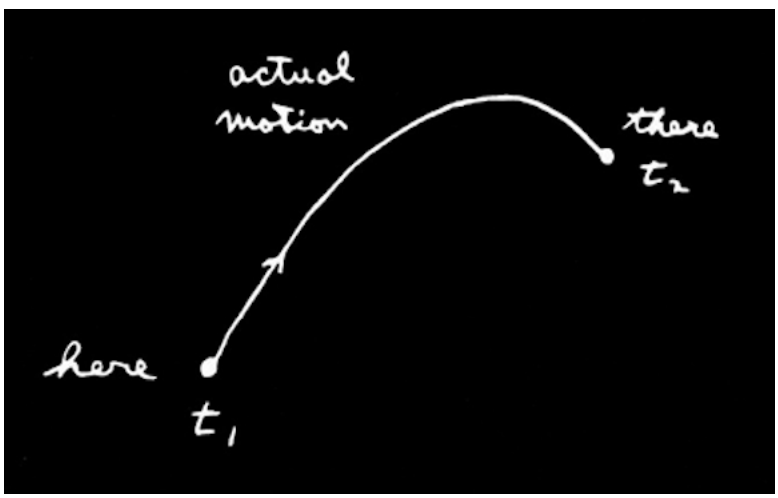
The basic idea in Bayes is that we have some current information about the process of interest (called the prior), and that we update probabilities of whatever we're trying to estimate based on new data (called the likelihood). In the example, we're trying to estimate the probability of success of a 3-point shot in basketball.

Mathematically, the update happens by multiplying the likelihood times the prior. These are two integrals. The multiplication is easy to do if the two integrals are of the same type (called conjugate distributions), for example, if they are both beta functions. That is natural when the data is of this type, from a series of Bernoulli trials.

Chapter 164

Least action

In the second volume of his *Lectures on Physics*, Feynman has a great discussion of the principle he calls "least" or "stationary" action. I want to follow his argument here.



The basic principle is that the path followed by an object is such that it minimizes the difference between the kinetic and potential energy. This turns out to be equivalent to Newton's Laws.

In mathematical language, the principle is that this integral

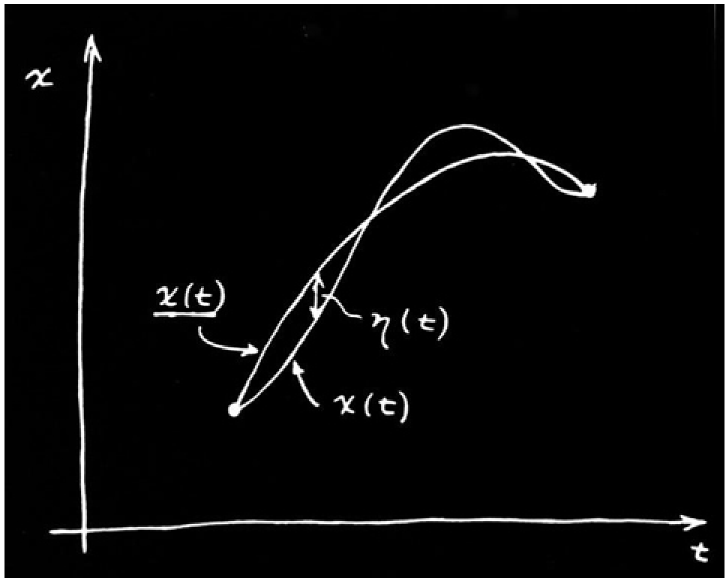
$$\text{Action} = S = \int_{t_1}^{t_2} [\ \frac{1}{2}m(\frac{dx}{dt})^2 - mgx\ ]\ dt$$

well, he says it's not a "minimum" but it is a critical point, where the linear approximation of how it changes for a little change in the path is zero. There are only second order terms in the correction if we shift the path a little bit.

This reminds us of an ordinary minimization problem, but it is much more complicated. The area uses what is called the calculus of variations. Another example of a problem in this area is to find which curve of a given length encloses the most area. You know that the answer is a circle, but you will need the calculus of variations to prove it.

Here is what we are going to do. We will label the path with the least action as $\underline{x}$. We will make a small perturbation to $\underline{x}$ such that

$$x(t) = \underline{x}(t) + \eta(t)$$
$$\frac{dx}{dt} = \frac{d\underline{x}}{dt} + \frac{d\eta}{dt}$$



We write

$$S = \int_{t_1}^{t_2} [\ \frac{m}{2}(\frac{d\underline{x}}{dt} + \frac{d\eta}{dt})^2 - V(\underline{x} + \eta)\ ]\ dt$$

$$(\frac{d\underline{x}}{dt} + \frac{d\eta}{dt})^2 = (\frac{d\underline{x}}{dt})^2 + 2\frac{d\underline{x}}{dt}\frac{d\eta}{dt} + (\frac{d\eta}{dt})^2$$

We will put any term involving η^2 into "a little box called 'second and higher order'" and not worry about it.

So the kinetic energy is

$$\frac{m}{2}(\frac{d\underline{x}}{dt})^2 + m\frac{d\underline{x}}{dt}\frac{d\eta}{dt}$$

We also have the potential energy

$$V(\underline{x} + \eta)$$

We write a standard Taylor Series expansion (η is small)

$$V(\underline{x}+\eta) = V(x) + \eta V'(\underline{x}) + \frac{\eta^2}{2}V''(\underline{x}) + \ldots$$

again neglecting second-order terms we have then

$$V(\underline{x}+\eta) = V(x) + \eta V'(\underline{x})$$

so

$$S = \int_{t_1}^{t_2} \left[\ \frac{m}{2}(\frac{d\underline{x}}{dt})^2 + m\frac{d\underline{x}}{dt}\frac{d\eta}{dt} - V(x) + \eta V'(\underline{x})\ \right]\ dt$$

We want to concentrate on the difference between this S and $\underline{S}$, the path of least action. Let's call that difference δS

$$\delta S = \int_{t_1}^{t_2} \left[\ m\frac{d\underline{x}}{dt}\frac{d\eta}{dt} - \eta V'(\underline{x})\ \right]\ dt$$

Now, we are going to use integration by parts to make the derivative of η disappear.

$$\begin{aligned}
\frac{d}{dt}(\eta f) &= \eta\frac{df}{dt} + f\frac{d\eta}{dt} \\
f\frac{d\eta}{dt} &= \frac{d}{dt}(\eta f) - \eta\frac{df}{dt} \\
\int f\frac{d\eta}{dt} &= \int \frac{d}{dt}(\eta f) - \int \eta\frac{df}{dt} \\
\int f\frac{d\eta}{dt} &= \eta f - \int \eta\frac{df}{dt}
\end{aligned}$$

$$\int f \frac{d\eta}{dt}\ dt = \eta f - \int \eta \frac{df}{dt}\ dt$$

In our problem

$$f = m\frac{d\underline{x}}{dt}$$

We had

$$\delta S = \int_{t_1}^{t_2} [\ m\frac{d\underline{x}}{dt}\frac{d\eta}{dt} - \eta V'(\underline{x})\ ]\ dt$$

and now we have

$$\delta S = m\frac{d\underline{x}}{dt}\eta(t)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}(\ m\frac{d\underline{x}}{dt})\ \eta(t)\ dt - \int_{t_1}^{t_2} V'(\underline{x})\eta(t)\ dt$$

Feynman says: "Now comes something which always happensthe integrated part disappears." The reason is that we must start and finish all the paths at the same place. This means that η is zero at both t_1 and t_2, and so the first term goes away. We factor out the $\eta(t)$ in the rest of it to obtain

$$\delta S = \int_{t_1}^{t_2} [\ -\frac{d}{dt}(\ m\frac{d\underline{x}}{dt}) - V'(\underline{x})\ ]\ \eta(t)\ dt$$

$$\delta S = \int_{t_1}^{t_2} [\ -m\frac{d^2\underline{x}}{dt^2} - V'(\underline{x})\ ]\ \eta(t)\ dt$$

Our principle of least action says that $\delta S = 0$ for *any* η. So that means that what multiplies η must be equal to zero.

$$-m\frac{d^2\underline{x}}{dt^2} - V'(\underline{x})$$

$$-V'(\underline{x}) = m\frac{d^2\underline{x}}{dt^2}$$
$$-V'(\underline{x}) = ma$$

and the force is the derivative of the potential with respect to position (with a minus sign), thus

$$F = ma$$

There is a lot more in the lecture, but I will leave it at this. Feynman closes this section by saying

> One remark: I did not prove it was a minimum - maybe it's a maximum. In fact, it doesn't really have to be a minimum. It is quite analogous to what we found for the principle of least time which we discussed in optics. There also, we said at first it was least time. It turned out, however, that there were situations in which it wasn't the least time. The fundamental principle was that for any first-order variation away from the optical path, the change in time was zero; it is the same story. What we really mean by least is that the first-order change in the value of S, when you change the path, is zero. It is not necessarily a minimum.

Part XXXVII

Addendum

Chapter 165

Value of pi revisited

As discussed in a previous **chapter**, Archimedes used paired inscribed and circumscribed polygons to develop an iterative procedure that can be used to calculate the value of π *to any desired accuracy.* Although the method is beautiful, his argument is unwieldy in detail, so we used modern trigonometry to achieve the same result more economically.

There are, in addition, two other sets of formulas that also reach this end, one based on perimeters, and the other on areas. These formulas are intriguing because they are simple, and it is not surprising that they are connected.

For example, consider a circle of unit *diameter*, so that π is equal to the perimeter. If p and P are the inside and outside perimeters for polygons whose sectors have central angle θ, and the same symbols are used with primes for angle $\theta/2$, then:

$$P' = 2\frac{pP}{p+P}$$

$$p' = \sqrt{pP'}$$

The corresponding formulas for inside (a) and outside (A) areas are (for a circle of unit radius)

$$A' = 2\frac{a'A}{a' + A}$$

$$a' = \sqrt{aA}$$

Notice that these two similar sets of formulas are subtly different. For example, to go from p and P to the primed version, we start with the first formula, while for area we must start with the square root. Part of our purpose in this chapter is to show that this works. (I must confess, I still do not have a simple explanation for *why* it is true).

inspiration

Originally, I was thinking about trying to implement Archimedes actual method for calculating π. However, the details of the approach are pretty painful. Instead, I worked through the problem using trigonometry.

It's striking that the formulas for the inside and outside perimeters are so simple, namely $n \sin \theta$ and $n \tan \theta$. The rest just follows from the half-angle formulas.

The web page which originally got me started with the harmonic and geometric mean formulas has been preserved by the wayback machine:

https://web.archive.org/web/20171024182015/http://personal.bgsu.edu/~carother/pi/Pi3d.html

On the very same day that I was revising the previous chapter to better integrate these two approaches, I came across another page which gives a "proof without words" of Gregory's Theorem (that is our subject).

https://divisbyzero.com/2018/09/28/proof-without-word-gregorys-theorem/

It gives these two formulas:

$$I_{2n} = \sqrt{I_n C_n}$$

$$C_{2n} = \frac{2}{1/I_{2n} + 1/C_n}$$

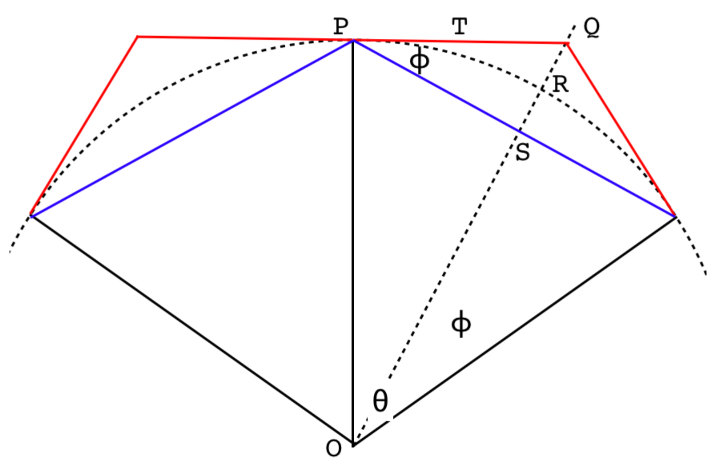
I found this notation a bit awkward, so I substituted the versions given above:

$$a' = \sqrt{aA}$$

$$A' = 2\frac{a'A}{a' + A}$$

Here, we mainly follow the development from that page and its "proof without words". One difference is that we will start with the geometry and work backward to the formulas. Let's deal with the perimeter first and then do the area.

basic setup



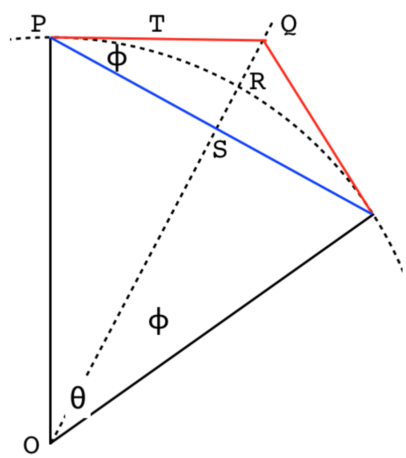
Draw a circle centered at O (only an arc of the circle is shown).

Points on the circle are chosen such that the arc length is an integral fraction of the whole. Equivalently, set $n\theta = 2\pi$.

Two adjacent sectors are shown in the figure above. The two polygons might be drawn so that the vertices of the internal and external figures are on the same ray, with parallel sides. However, the construction shown is more convenient.

The precise scale does not matter to the argument (nor the value of n). If it should turn out that the arc length as drawn is not exactly right, increase or decrease the radius of the circle and then fit it to the figure, keeping two points on the perimeter, and adjust O to be at the center of the adjusted circle.

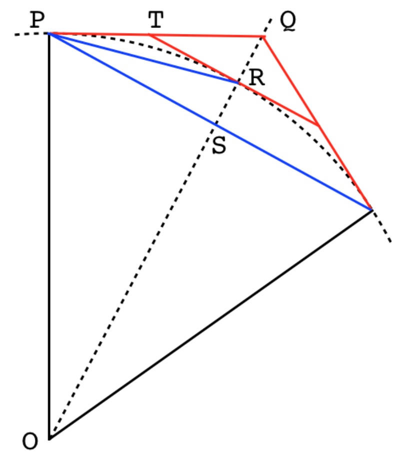
Two red lines comprise this sector's external perimeter P, while a single blue line is the inscribed perimeter p. The lines of the external perimeter are both tangent to the circle, and the whole figure is symmetric in each sector, with one blue and two red lines.



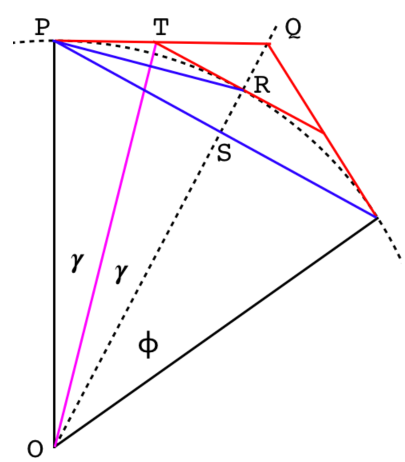
$\angle PSR$ is a right angle. Proof: we simply appeal to symmetry, or point out the congruent triangles. Since $\phi = \theta/2$, we have SAS.

Next, draw the perimeters p' and P' for the polygon with $2n$ sides and sector angle $\phi = \theta/2$.

It is convenient to rotate the internal perimeter by $\theta/2$ with respect to the external one, a bit to the left when we draw p' and a bit to the right for P'. Both p' and P' touch the circle at R.



A central relationship we use below is that $\triangle PRT$ is isosceles. For a proof, draw OT and appeal to symmetry.

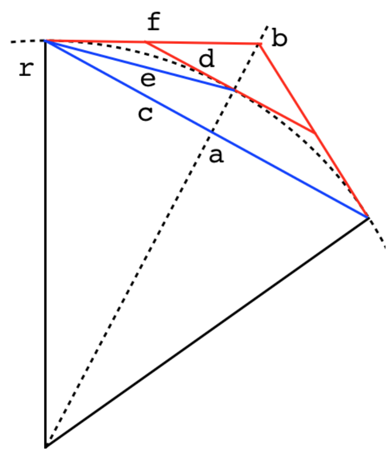


Or note that OT bisects $\angle POQ$ so $\triangle POT \cong \triangle ROT$ by SAS.

A consequence is that PR bisects $\angle QPS$. This can also be proved by an argument based on the sum of internal angles for an n-gon,

It looks as if the segment of the vertical that extends beyond the radius might be equal to that part below down to what looks like the "strut" of a kite. However, this is not true. We will show what this ratio is equal to in just a bit.

Rather than use the vertices as points of reference, we will label the line segments.

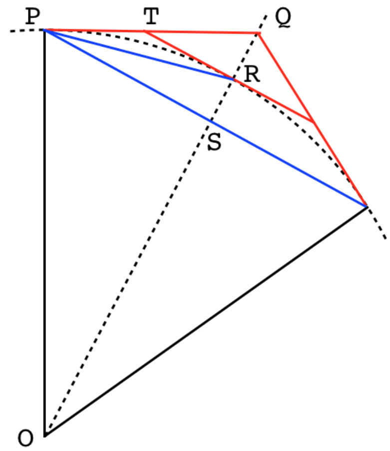


Just to be clear: a is the part of the radius extended to point S above, while b extends to Q. c and d are the lengths of the indicated lines *in the half-sector*, not all the way across, and f is the entire length of PQ.

We're ready to proceed.

basic geometry: perimeters

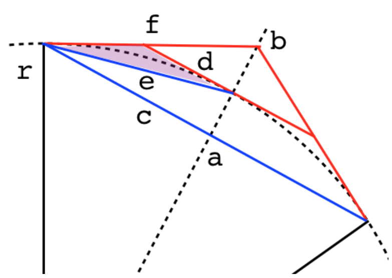
As we said, the key observation is that $\triangle PRT$ is isosceles.



Because of that, and since $\angle SPR = \angle PRT$ by the alternate interior angles theorem, $\angle SPR = \angle TPR$.

Therefore the cosines are also equal, namely:

$$\frac{c}{e} = \frac{e/2}{d}$$



(To see the midpoint of e, drop an altitude in the isosceles triangle, shown in purple).

Therefore:

$$2dc = e^2$$

Now, c is the entirety of p in this half-sector. But d is only one-half of P'.

Hence $2d \cdot c$ is equal to pP', and since $e = p'$, we have that

$$pP' = [p']^2$$

which was our second rule for the perimeters.

The first rule was

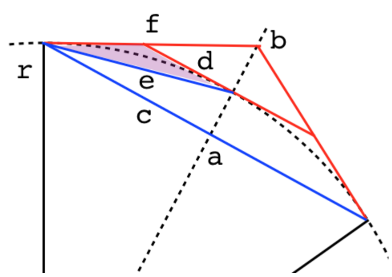
$$P' = 2\frac{pP}{p+P}$$

In geometric terms, we must show that

$$2d = 2\frac{cf}{c+f}$$

$$cd + df = cf$$

Taking another look at the diagram:



The small triangle with base d ($\triangle QRT$ above) has slanted side $f - d$ (subtracting d because, again, $\triangle PRT$ is isosceles). By similar triangles, we have

$$\frac{d}{f-d} = \frac{c}{f}$$

$$df = cf - cd$$

$$cd + df = cf$$

But this is what we needed to prove.

$\square$

basic geometry: areas

The area formulas for inside (a) and outside (A) polygons are those for a circle of unit radius (so that π is the area):

$$A' = 2\frac{a'A}{a' + A}$$

$$a' = \sqrt{aA}$$

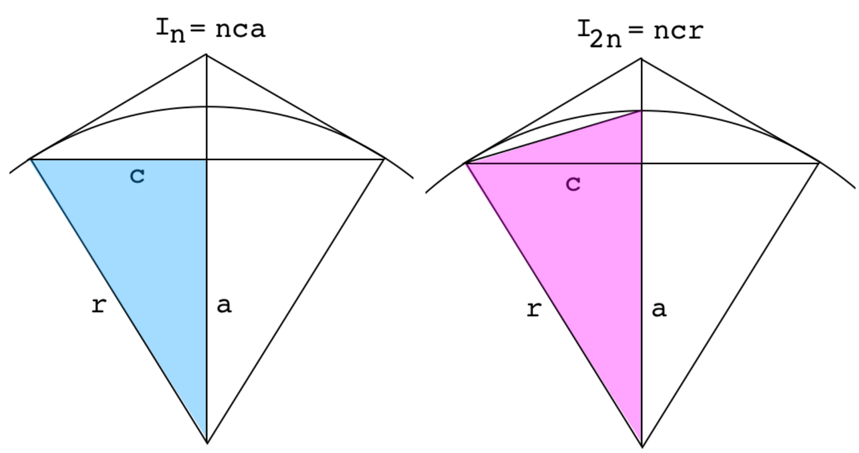
However, having reached this point, we need another symbol for area, because a is currently the line segment corresponding to p/n. Let's use I and C for the inside and outside areas, to match the source.

We will also adopt their n and $2n$ notation, It's a bit clumsy but that will make it easier to match things up.

$$C_{2n} = 2 \cdot \frac{I_{2n}C_n}{I_{2n} + C_n}$$

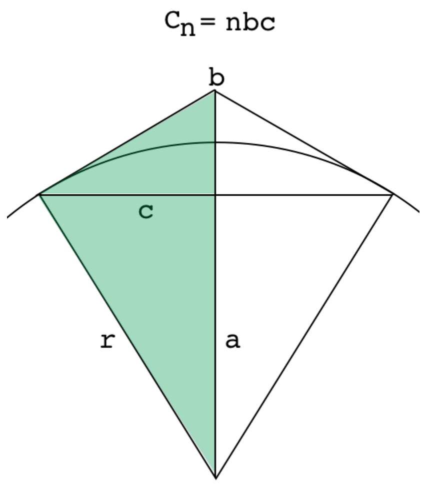
$$I_{2n} = \sqrt{I_n C_n}$$

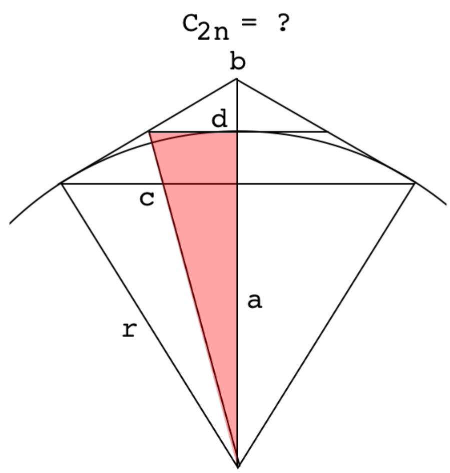
The first two areas are I_n and I_{2n}



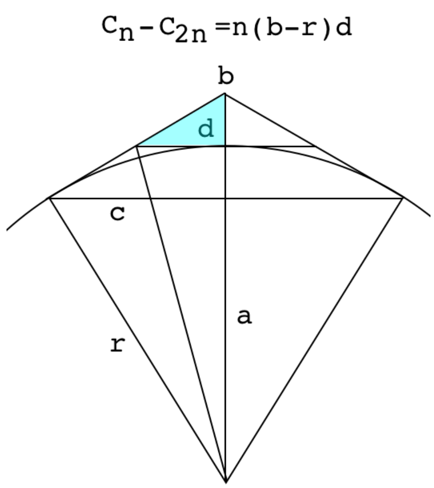
We compute these areas for the whole sector of angle θ, so there are two congruent triangles with base a (or base r) and height c. Multiply by n if you like to get the entire polygon, but every expression will have a factor of n, and we'll be looking at ratios, so we can just not worry about it.

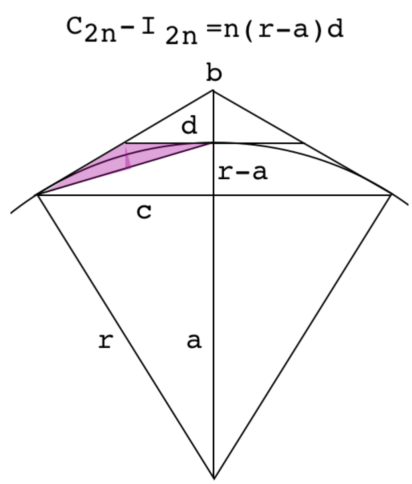
The third easy one is C_n:





We write the last one (C_{2n}) as two different differences.





Let's gather all these expressions in one place, forming ratios:

$$\frac{I_{2n}}{I_n} = \frac{ncr}{nca} = \frac{r}{a}$$

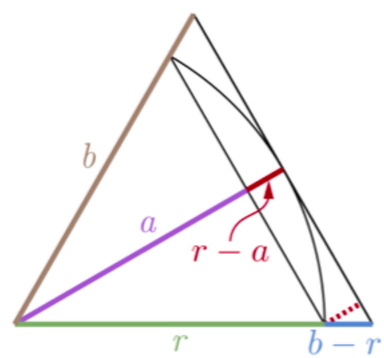
$$\frac{C_n}{I_{2n}} = \frac{ncb}{ncr} = \frac{b}{r}$$

$$\frac{C_n - C_{2n}}{C_{2n} - I_{2n}} = \frac{n(b-r)d}{n(r-a)d} = \frac{b-r}{r-a}$$

We will prove that these three ratios are all equal to each other.

We will have used the geometry to prove what the source calls their Lemmas, and those can be used in turn to prove the original Gregory formulas.

But the proof is easy:



It's just a matter of similar triangles:

$$\frac{r}{a} = \frac{b}{r} = \frac{b-r}{r-a}$$

That's the "without words" part.

For that very last part, you can work out the dimensions of the tiny similar triangle, or you can say:

$$\frac{r}{a} = \frac{b}{r}$$

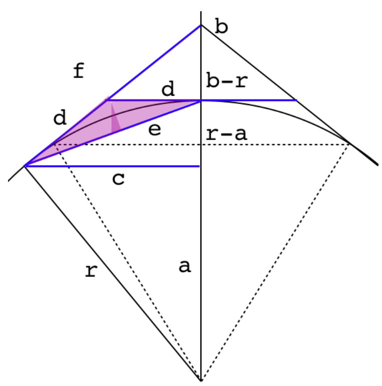
$$\frac{r}{a} - \frac{a}{a} = \frac{b}{r} - \frac{r}{r}$$

$$\frac{r-a}{a} = \frac{b-r}{r}$$

which is easily rearranged to give the desired result.

□

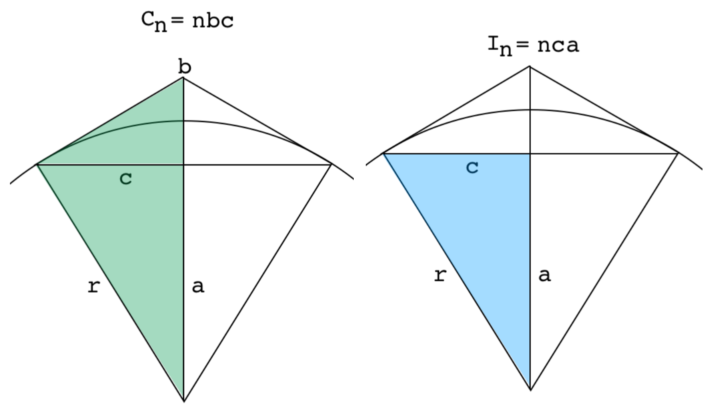
This can also be proved using the **angle bisector theorem**.



The side labeled e bisects the angle formed by the two sides labeled c and f. Therefore

$$\frac{b-r}{f} = \frac{r-a}{c} \quad \Rightarrow \quad \frac{b-r}{r-a} = \frac{f}{c}$$

But f and c are two sides of a triangle which is similar to the colored portions below:



Therefore

$$\frac{b}{r} = \frac{r}{a} = \frac{f}{c} = \frac{b-r}{r-a}$$

As we said.

algebra

Moving on to the geometric mean formula is not hard. From above we have that

$$\frac{I_{2n}}{I_n} = \frac{C_n}{I_{2n}}$$

$$[I_{2n}]^2 = I_n C_n$$

Translated back into the A, a area notation

$$a' = \sqrt{aA}$$

This is just what we wanted to show.

For the other formula, what we have is:

$$\frac{C_n - C_{2n}}{C_{2n} - I_{2n}} = \frac{C_n}{I_{2n}}$$

$$I_{2n}(C_n - C_{2n}) = C_n(C_{2n} - I_{2n})$$

$$2I_{2n}C_n = C_nC_{2n} + I_{2n}C_{2n}$$
$$= C_{2n}(C_n + I_{2n})$$

So

$$C_{2n} = 2 \cdot \frac{I_{2n}C_n}{C_n + I_{2n}}$$
$$C_{2n} = 2 \cdot \frac{1}{1/I_{2n} + 1/C_n}$$

And we're done. In our preferred notation

$$A' = 2 \cdot \frac{1}{1/a' + 1/A}$$

historical note

The area-based formulas given above are due to James Gregory.

https://divisbyzero.com/2018/09/28/proof-without-word-gregorys-theorem/

As an aside, the Fundamental Theorem of Calculus (FTC) is usually thought about (taught and learned) using the language of functions, and ascribed mainly to Leibnitz, with some credit to the two Isaacs, Newton and his university lecturer, Barrow.

https://arxiv.org/abs/1111.6145

Amazingly enough, Gregory published a geometric (Euclidean) proof of the FTC in 1668! That predates Liebnitz (1693) by more than 25 years. This is motivation to give considerable credit to individuals other than Newton and Liebnitz (e.g. Fermat, Pascal, Wallis, Gregory, etc.) in the invention of the calculus.

test

I wrote a simple test of the area formulas using Python.

The script is here:

https://gist.github.com/telliott99/5269b48672cdaeca95c9c9c9d163321d

It gives this output:

```
> python script.py
    4 2.0000000000 4.0000000000
    8 2.8284271247 3.3137084990
   16 3.0614674589 3.1825978781
   32 3.1214451523 3.1517249074
   64 3.1365484905 3.1441183852
  128 3.1403311570 3.1422236299
  256 3.1412772509 3.1417503692
  512 3.1415138011 3.1416320807
 1024 3.1415729404 3.1416025103
 2048 3.1415877253 3.1415951177
 4096 3.1415914215 3.1415932696
 8192 3.1415923456 3.1415928076
16384 3.1415925766 3.1415926921
32768 3.1415926343 3.1415926632
65536 3.1415926488 3.1415926560
>
```

The digits of the output appear to be identical or nearly so. The only difference is that in this script I computed 2^n to give the number of sides. In the previous chapter, we just print n.

details

That's very curious. The first four lines of output from the perimeter version:

```
  2 2.8284271247  4.0000000000
 3 3.0614674589  3.3137084990
 4 3.1214451523  3.1825978781
 5 3.1365484905  3.1517249074
```

and the first five from the area version:

```
    4 2.0000000000 4.0000000000
    8 2.8284271247 3.3137084990
   16 3.0614674589 3.1825978781
   32 3.1214451523 3.1517249074
   64 3.1365484905 3.1441183852
```

It's pretty clear that we are doing the same calculation. It's just that the first column is shifted up by one row.

To confirm that, the perimeter calculation is:

initialization:

$$p = 2\sqrt{2} \qquad P = 4$$

recurrence:

$$P' = \frac{2pP}{p + P} \qquad p' = \sqrt{pP'}$$

The area version is:

initialization:

$$a = 2 \qquad A = 4$$

recurrence:

$$a' = \sqrt{aA} \qquad A' = \frac{2a'A}{a' + A}$$

They give identical results: $A = P$, at each round, but a matches p', or to put it the other way around, p' is retarded by one cycle compared to a'.

Let's try one round of calculation by hand:

$$p = 2\sqrt{2} \qquad P = 4$$

$$P' = \frac{2pP}{p+P} = \frac{2 \cdot 2\sqrt{2} \cdot 4}{2\sqrt{2}+4} = \frac{2 \cdot 2\sqrt{2} \cdot 4}{2\sqrt{2}(1+\sqrt{2})} = \frac{8}{1+\sqrt{2}} = 3.31371$$

$$p' = \sqrt{pP'} = \sqrt{2\sqrt{2} \cdot \frac{8}{1+\sqrt{2}}} = 4\sqrt{\frac{1}{1+1/\sqrt{2}}} = 3.06147$$

The area calculation:

$$a' = \sqrt{aA} = \sqrt{2 \cdot 4} = \sqrt{8} = 2.828427$$

$$A' = \frac{2a'A}{a'+A} = \frac{2 \cdot \sqrt{8} \cdot 4}{\sqrt{8}+4} = \frac{8}{1+\sqrt{2}}$$

A' is the same as P'.

The next round for a' is

$$a' = \sqrt{aA} = \sqrt{\sqrt{8} \cdot \frac{8}{1+\sqrt{2}}} = 4\sqrt{\frac{1}{1+1/\sqrt{2}}}$$

Perhaps someday I'll have a deeper understanding, Undoubtedly, there is a series here.

Chapter 166

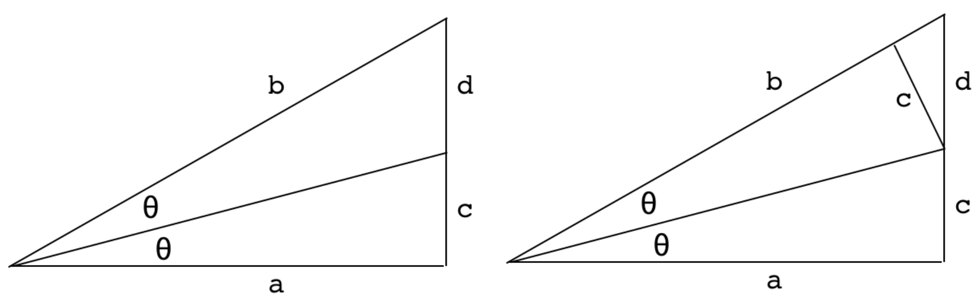
Archimedes and pi

We're going to follow a page that in turn follows Archimedes argument for the approximation of π.

https://itech.fgcu.edu/faculty/clindsey/mhf4404/archimedes/archimedes.html

Before we start, let's review some ideas related to angle bisection. Recall that if we have an angle bisector in a right triangle (left panel), the theorem says that

$$\frac{a}{c} = \frac{b}{d}$$



The proof (which we showed in an earlier chapter) involves drawing the altitude of the top triangle, forming two congruent triangles and a smaller one which is easily shown to be similar to the original triangle (right panel). By similar triangles, we have

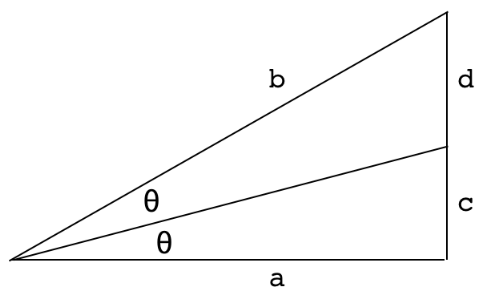
$$\frac{d}{c} = \frac{b}{a}$$

which can be rearranged to give the desired statement. A corollary follows:

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a+b}{b} = \frac{c+d}{d}$$

$$\frac{a+b}{c+d} = \frac{b}{d} = \frac{a}{c}$$

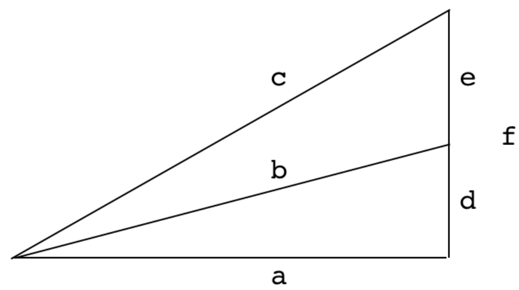


This doesn't seem obvious to me, in fact it seems counter-intuitive. Nonetheless, we will use it extensively for what follows.

overview

There are three steps which we will repeat (eventually, four times).

Let's relabel the figure now:



• Obtain ratios for the cosecant and cotangent (c/f and a/f). In the first round, these are just $\sqrt{3}$ and 2.

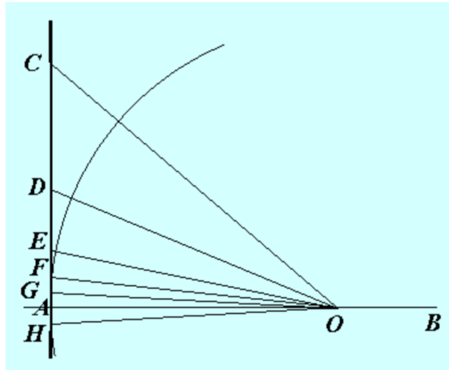
• Add them together, obtaining $(c+a)/f$ and observe that this ratio is also equal to a/d, by the corollary of the angle bisector theorem above. This gives us the cotangent of the half-angle.

• Obtain the cosecant of the half-angle, b/d, by using the Pythagorean theorem. Namely:

$$a^2 + d^2 = b^2$$
$$\sqrt{\frac{a^2}{d^2} + 1} = \frac{b}{d}$$

Part A, round 1

Draw a circle with radius OA and tangent AC, and let $\angle AOC$ be one-third of a right angle.



Note: the figure appears to have been compressed in width. The angle bisectors don't look right and the original angle looks more like 45 than 30. We'll use it anyway.

In what follows we list the claim first:

- $OA : AC > 265 : 153$

Followed by the proof:

Since the triangle is a 30-60-90 triangle, $OA = \sqrt{3}$ and $AC = 1$, so the ratio is just $\sqrt{3}$. 265/153 is a (very good) approximation, just slightly smaller than the true value.

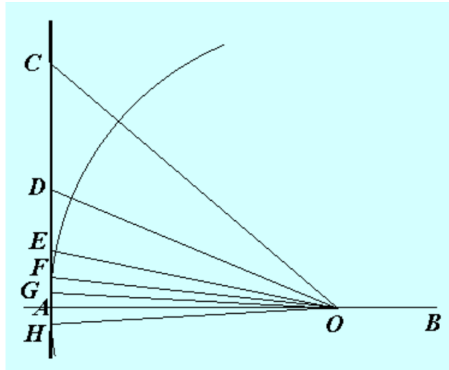
- $OC : AC = 306 : 153$

The cosecant = 2. The denominator has been chosen to match the previous ratio.

Now draw the angle bisector OD.

- $CO : OA = CD : DA$

This is just the angle bisector theorem.



- $(CO + OA) : CA = OA : AD$

This ratio is equal to that from the bisector theorem by the corollary. Start with

$$CO : OA = CD : DA$$
$$\frac{CO + OA}{OA} = \frac{CD + DA}{DA} = \frac{CA}{DA}$$

This crucial step gives us the cotangent of the half-angle formed by the angle bisector OD.

- $OA : AD > 571 : 153$

We just add numerators for the first two ratios above, leaving the result over the common denominator.

- $OD : AD > 591\ 1/8 : 153$

Finally, we want $OD : AD$. By the Pythagorean Theorem

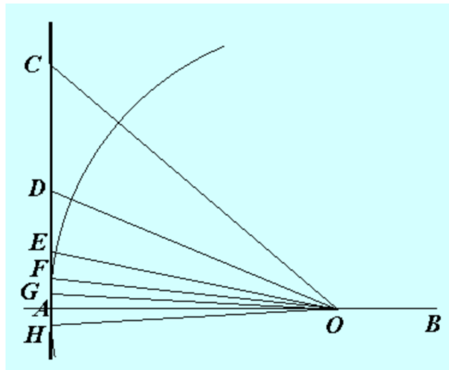
$$OD^2 = OA^2 + AD^2$$
$$(OD : AD)^2 = (OA : AD)^2 + 1$$

Now do $571^2 = 326041; 153^2 = 23409$ so the sum of numerators is 349450 and the square root is 591 1/6.

Archimedes approximates the result as 591 1/8 : 153. Remember that we are looking for a lower bound, so smaller is OK.

Part A, round 2

Now draw the angle bisector OE.



• From above, we have that $OA : AD > 571 : 153$ and $OD : AD > 591\ 1/8 : 153$.

• $OA : AE > 1162\ 1/8 : 153$

This calculation invokes the angle bisector corollary again. Rather than repeat the derivation, just add the inputs:

$$591\ 1/8 : 153 + 571 : 153$$

which adds to give the result above, 1162 1/8 : 153.

• $OE : AE > 1172\ 1/8 : 153$

Use the Pythagorean theorem to write:

$$OE^2 = AE^2 + OA^2$$

$$\frac{OE^2}{AE^2} = \frac{OA^2}{AE^2} + 1$$

We have $(1162\ 1/8)^2$ and $153^2 = 23409$.

Write

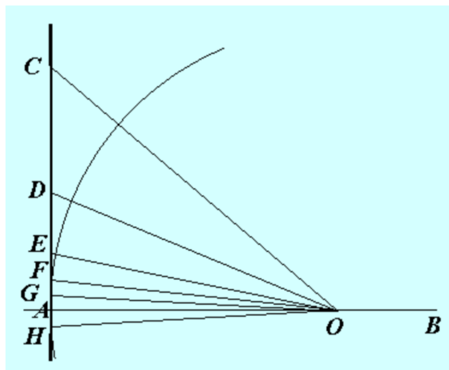
$$1162^2 + 1162/4 + 153^2 = 1350244 + 290\ 1/2 + 23409$$

$$= 1373943\ 1/2 + 1/64 = 1373943\ 33/64$$

The square root is 1172 1/8.

Part A, round 3

Now draw the angle bisector OF.



• From above, we have that $OA : AE > 1162\ 1/8 : 153$ and $OE : AE > 1172\ 1/8 : 153$.

• $OA : AF > 2334\ 1/4 : 153$

This calculation invokes the angle bisector corollary again.

$$\frac{OA}{FA} = \frac{OE}{EA} + \frac{OA}{EA}$$
$$1162\ 1/8 : 153 + 1172\ 1/8 : 153$$

which adds to give the result above.

• $OF : FA > 2339\ 1/4 : 153$

Use the Pythagorean theorem to write:

$$\frac{OF^2}{FA^2} = \frac{OA^2}{FA^2} + 1$$

We have $(2334\ 1/4)^2$ and $153^2 = 23409$.

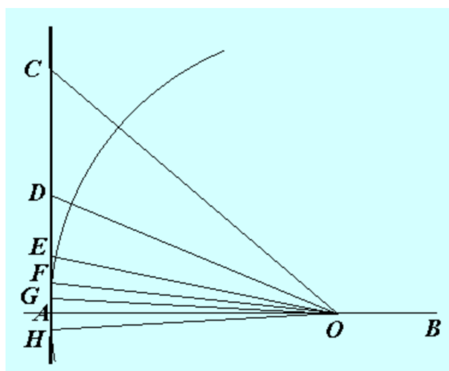
Write

$$\begin{aligned} 2334^2 + 2334/2 + 153^2 &= 5447556 + 1167 + 23409 \\ &= 5472132 + 1/16 \end{aligned}$$

The square root is 2339 1/4.

Part A, round 4

Now draw the angle bisector OG.



• From above, we have that $OA : FA > 2334\ 1/4 : 153$ and $OF : FA > 23391/4 : 153$

Add

• $OA : AG > 4673\ 1/2 : 153$

We're almost done. The original distance AC was 1/12 the perimeter of a circumscribed polygon, so we would multiply by 12 to get the ratio to the radius, but we want the ratio to the diameter so that gives a factor of 2 on the bottom for a total factor of 6.

There is an additional factor for the four "halvings" of $2^4 = 16$. Hence we obtain

$$153 \times 96 = 14688$$

and then invert to get the ratio of the circumference to the diameter:

$$\frac{14688}{4673\ 1/2} = 3 + \frac{668\ 1/2}{4673\ 1/2}$$

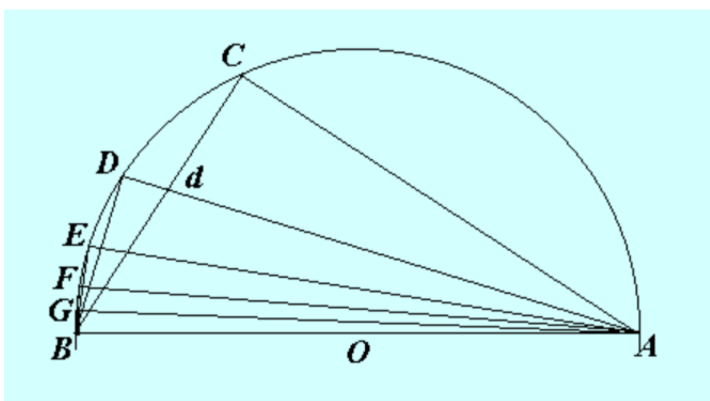
The fraction is just less than 1/7.

$1/7 = 0.142857$, while $668\ 1/2/4673\ 1/2 = 0.14304$.

We conclude that $\pi < 3\ 1/7$.

Part B

For Part B we use this diagram for an inscribed polygon.



As before $\triangle ABC$ is a $30 - 60 - 90$ right triangle.

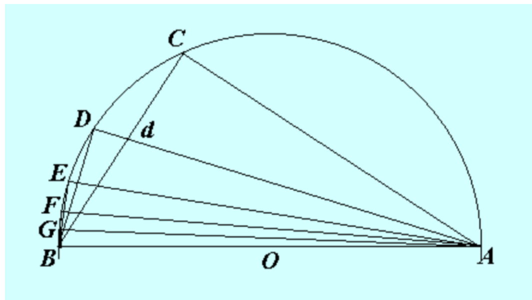
- $AC : BC < 1351 : 780$.

This ratio is an approximation for $\sqrt{3}$. It is an even better approximation than the previous one, and also, crucially, it is just slightly *more* than the true value, whereas 265/153 was slightly less.

Part B, round 1

Let AD bisect the angle, and then join BD.

- $\angle BAD = \angle dAC = \angle dBD$.



The first statement just restates the construction as an angle bisector. The second follows from the fact that the two angles have vertices on the circle and cut off the same arc.

As a consequence, $\triangle dBD \sim \triangle dAC$.

- $AD : DB < 2911 : 780$

Start with the similar triangles above and write three ratios of long side (not hypotenuse) to short side

$$AD : BD = BD : Dd = AC : Cd$$

Note: the source has $AB : Bd$ but this seems to be an error. That is a ratio of two hypotenuses and so is not equal to the others. As a result, I was unable to follow this part of the proof:

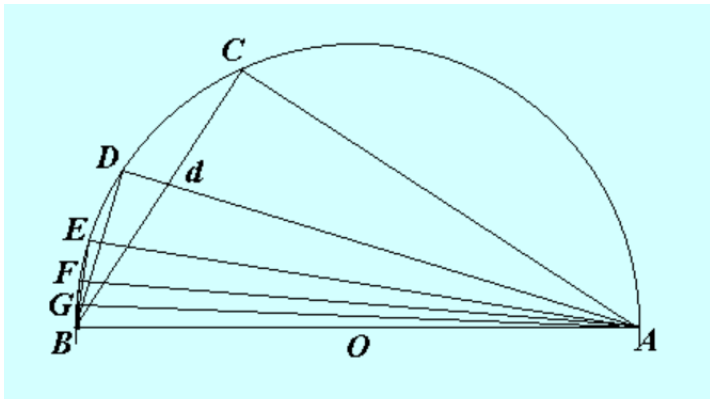
$$\begin{aligned} AD : BD = BD : Dd &= AB : Bd \\ &= (AB + AC) : (Bd + Cd) \\ &= (AB + AC) : BC \end{aligned}$$

or $(BA + AC) : BC = AD : DB.$

However, I was able to prove the last statement

$$(AB + AC) : BC = AD : DB$$

The proof is as follows.



We have that $\triangle ABC$ is a right triangle and that AD and thus Ad is the angle bisector for $\angle BAC$. Therefore, we have by our favorite theorem that

$$(AB + AC) : BC = AC : Cd$$

We also have that $\triangle ABD$ is a right triangle and by virtue of the angle bisector construction, $\triangle ABD$ is similar to $\triangle ACd$. Therefore:

$$AC : Cd = AD : DB$$

These two lines combine to give the desired result.

The ratio $AD : DB$ is what we need going forward, and we get that from the other part: $(AB + AC) : BC$. We have that the cotangent $AC : BC < 1351 : 780$, and $AB : BC$ is the cosecant whose value is 2 so we multiply $780 \times 2 = 1560$, and then add 1351 to get the numerator of the result listed above.

- $AB : BD < 3013\ 3/4 : 780$

From the Pythagorean theorem: $AD^2 + BD^2 = AB^2$ so

$$AB^2 : BD^2 = AD^2 : BD^2 + 1$$

$AD : DB < 2911 : 780$ So we obtain $2911^2 = 8473921$ and $BD^2 = 608400$ so we have that

$$AB^2 : BD^2 = 9082321 : 608400$$

$$AB : BD < 3013\ 3/4 : 780$$

Part B, round 1 summary

Let's summarize what we did in round 1. We started with the cosecant and the cotangent for $\triangle ABC$, namely $AB : BC$ and $: AC : BC$.

We used the relationship $(AB + AC) : BC = AD : DB$ to obtain the cotangent of the bisected angle, and then we used the Pythagorean theorem in this form

$$AB^2 : BD^2 = AD^2 : BD^2 + 1$$

to get the cosecant from the cotangent. Thus we have

- $AD : DB < 2911 : 780$, the cotangent.
- $AB : BD < 3013\ 3/4 : 780$, the cosecant.

Part B, round 2

Now, let AE bisect the angle, and then join BE.

Rather than go through the geometry again, let's just substitute letters. First the cotangent

$$(AB + AD) : BD = AE : EB$$

Then the cosecant.

$$AB^2 : BE^2 = AE^2 : BE^2 + 1$$

For the first part we have 2911 : 780 + 3013 3/4 : 780 = 5924 3/4 : 780.

We reduce the denominator to 240. This amounts to dividing by 3 1/4. 5924 3/4 divided by 3 1/4 is exactly equal to 1823.

- $AE : EB = 1823 : 240$, the cotangent.

For the second part we have the previous number squared and added to 1 and then take the square root. $1823^2 = 3323329; 240^2 = 57600$; so we have 3380929 and the square root is $< 1838\ 3/4$, but the source gives the fraction as a bit larger 1838 9/11.

- $AB : BE = 1838\ 9/11 : 240$, the cosecant.

Part B, round 3

Now, let AF bisect the angle, and then join BF. Substitute letters (carefully, looking at the diagram). First the cotangent

$$(AB + AE) : BE = AF : FB$$

Then the cosecant.

$$AB^2 : BF^2 = AF^2 : BF^2 + 1$$

For the first part we have $1838\ 9/11 : 240 + AE : EB = 1823 : 240 = 3661\ 9/11 : 240$.

We reduce the denominator, this time to 66. This amounts to multiplication by 11/40. So the numerator is multiplied by the same factor giving

- $AF : FB = 1007 : 66$, the cotangent.

For the second part we have the previous number squared and added to 1 and then take the square root.$1007^2 = 1014049; 66^2 = 4356$, so we have 1018405 and the square root is 1009 1/6.

- $AB : FB = 1009\ 1/6 : 66$, the cosecant.

Part B, round 4

Finally, let AG bisect the angle, and then join BG. Substitute letters. First the cotangent

$$(AB + AF) : BF = AG : GB$$

Then the cosecant.

$$AB^2 : BG^2 = AG^2 : BG^2 + 1$$

For the first part we have

- $AG : GB = 2016\ 1/6 : 66$, the cotangent.

Now do $2016^2 = 4064256; 66^2 = 4356$ so that's 4068612 and the square root is 2017 1/12 while the source gives

- $AB : GB < 2017\ 1/4 : 66$, the cosecant.

Almost done. The side BG is a side of an inscribed regular polygon of 96 sides. We multiply $66 \times 96 = 6336$ and compute the ratio of the inverse.

I am not sure how Archimedes came up with it, but it is easy to verify that the ratio which is less than π is greater than:

$$\frac{6336}{2017\ 1/4} > 3\ 10/71$$

We combine parts A and B to make our final statement that

$$3\ 10/71 < \pi < 3\ 1/7$$

Chapter 167

Square root of 3

Archimedes uses two approximations for $\sqrt{3}$: 265/153 and 1350/780. (Recall that $\sqrt{3}$ is irrational so it doesn't have an exact decimal representation).

I wrote a script to search for these and other close approximations.

```
> python approx_sqrt3.py
     0       0       0       1       1
     1       1       2       2       1
     3       5       2       6       9
     4       6      12       7       1
    11      19       2      20      37
    15      25      50      26       1
    41      71       2      72     141
    56      96     192      97       1
   153     265       2     266     529
   209     361     722     362       1
   571     989       2     990    1977
   780    1350    2700    1351       1
  2131    3691       2    3692    7381
```

```
  2911   5041  10082   5042      1
  7953  13775      2  13776  27549
 10864  18816  37632  18817      1
>
```

The first column has the denominator i (we search from 1 to 50000).

The algorithm is really brute force. For each possible i, we search all the integers larger than i until we find one j such that $3 \cdot i^2 < j^2$. In other words, j is the smallest integer such that j/i is larger than $\sqrt{3}$.

Having the closest j (and $j-1$) for each i, we test whether

$$j^2 - 3 \cdot i^2 < 5$$

$$3 \cdot i^2 - (j-1)^2 < 5$$

If eiher is true, we print all the values e.g.

```
   153    265      2    266    529
```

In the third column we find repeatedly, 2.

What this means is that the square of the value in column 2, plus 2, is exactly three times the square of the value in column 1. For example:

```
153^2 = 23409
265^2 = 70225
3 x 23409 = 70227
```

Since $265^2/(153^2+2) = 3$, 265/153 is just barely less than $\sqrt{3}$.

The error is $2/23409 \approx 8 \times 10^{-5}$.

In column 5 we see the number 1 repeated.

This is the difference between 3 times the square of the value in column 4 and the square of the value in column 1. For example:

```
780 1350  2700 1351      1

780^2 = 608400
1351^2 = 1825201
3 x  608400 = 1825200
```

So 1351/780 is just barely greater than $\sqrt{3}$, the error is $1/608400 \approx 1.6 \times 10^{-6}$.

Continued fractions

There is another way to find such numbers. A continued fraction is an expression like:

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}$$

This particular continued fraction is equal to the famous number ϕ.

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}$$

But notice, the second term on the right-hand side is $1/\phi$ so we can write

$$\phi = 1 + \frac{1}{\phi}$$

$$\phi^2 = \phi + 1$$

For more on ϕ see **here**.

Square roots can be represented as continued fractions. We look first at the slightly easier case of $\sqrt{2}$, before tackling $\sqrt{3}$.

$$(\sqrt{2}-1)(\sqrt{2}+1) = 1$$

Rearrange to get a substitution we will use again

$$\sqrt{2}-1 = \frac{1}{\sqrt{2}+1}$$

At the same time, add one and subtract one on the bottom right:

$$\sqrt{2}-1 = \frac{1}{2+\sqrt{2}-1}$$

substitute

$$= \frac{1}{2+\frac{1}{\sqrt{2}+1}}$$

Add one and subtract one again and then substitute again

$$= \frac{1}{2+\frac{1}{2+\sqrt{2}-1}} = \frac{1}{2+\frac{1}{2+\frac{1}{\sqrt{2}+1}}}$$

Clearly, this goes on forever.

$$\sqrt{2} = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}$$

The numerators are all 1, so this is a *simple* continued fraction for $\sqrt{2}$.

The continued fraction representation of $\sqrt{2}$ is $[1:2]$, meaning that there is an initial 1 followed by repeated 2's.

This fraction goes on forever (since $\sqrt{2}$ is irrational). To turn this into an approximate decimal representation of $\sqrt{2}$, ignore the $\ldots$. Then the last fraction is $5/2$. Invert and add, repeatedly:

```
2 + 1/2 = 5/2
2 + 2/5 = 12/5
2 + 5/12 = 29/12
2 + 12/29 = 71/29
2 + 29/71 = 171/71
2 + 71/171 = 413/171
```

To terminate we need to use that initial 1:

```
1 + 171/413 = 584/413 = 1.414043
```

To six places, $\sqrt{2} = 1.414213$. We have only three places, but can easily get more.

square root of 3

The continued fraction representation of $\sqrt{3}$ is $[1, 1, 2, 1, 2, \ldots]$, which can be shortened to $[1 : (1, 2)]$.

Here is a derivation:

$$
\begin{aligned}
(\sqrt{3} - 1)(\sqrt{3} + 1) &= 2 \\
\sqrt{3} - 1 &= \frac{2}{\sqrt{3} + 1} \\
\frac{\sqrt{3} - 1}{2} &= \frac{1}{\sqrt{3} + 1}
\end{aligned}
$$

both of which we will use again. However, going further, add and subtract on the bottom right

$$\sqrt{3}-1=\frac{2}{\sqrt{3}+1}=\frac{2}{2+\sqrt{3}-1}$$

Divide top and bottom by 2

$$=\frac{1}{1+\frac{\sqrt{3}-1}{2}}$$

and substitute giving

$$=\frac{1}{1+\frac{1}{\sqrt{3}+1}}$$

That's the end of step 1.

Now, for the second step, we focus on that last fraction

$$\frac{1}{\sqrt{3}+1}=\frac{1}{2+\sqrt{3}-1}=\frac{1}{2+\frac{2}{\sqrt{3}+1}}$$

Then for step three, we focus again on the last fraction, which is what we worked with in the first part.

$$\frac{2}{\sqrt{3}+1}=\frac{1}{1+\frac{1}{\sqrt{3}+1}}$$

So now both terms repeat:

$$\sqrt{3}-1=\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\dots}}}}$$

which is $[1 : (1, 2)]$, as we said.

We can get approximations for $\sqrt{3}$ similar to what we did for $\sqrt{2}$. Unlike previously, here there are two possibilities. We start with either one of

$$1 + \frac{1}{2 + \ldots}$$
$$2 + \frac{1}{1 + \ldots}$$

and proceed by ignoring the dots.

The first gives

```
1 + 1/2 = 3/2
2 + 2/3 = 8/3
1 + 3/8 = 11/8
2 + 8/11 = 30/11
1 + 11/30 = 41/30
2 + 30/41 = 112/41
1 + 41/112 = 153/112

1 + 112/153 = 265/153 = 1.732026
```

The actual value is $\sqrt{3} = 1.732051$, to six places. We have four.

The second gives

```
2 + 1 = 3
1 + 1/3 = 4/3
2 + 3/4 = 11/4
1 + 4/11 = 15/11
2 + 11/15 = 41/15
1 + 15/41 = 56/41
2 + 41/56 = 153/56
1 + 56/153 = 209/153
```

```
2 + 153/209 = 571/209
1 + 209/571 = 780/571

1 + 571/780 = 1351/780 = 1.732051
```

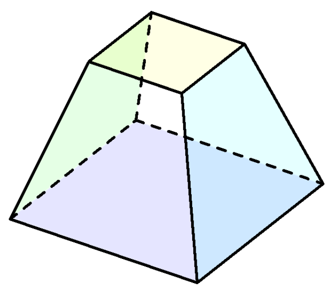
The actual value is $\sqrt{3} = 1.732051$, to six places. We have all six.

It is believed that this is how Archimedes came up with those approximations. (He doesn't say).

Chapter 168

Frustum

A frustum is the (bottom) part of a larger pyramid or cone that remains after the original solid is cut by a horizontal plane and the upper, small pyramid or cone removed.



If we call the dimensions of the larger pyramid H (height) and B (base), then its volume is

$$V = \frac{1}{3}HB^2$$

Similarly, if we call the dimensions of the pyramid that has been removed h and b its volume is

$$V = \frac{1}{3}hb^2$$

The volume of the frustum is just the difference

$$V = \frac{1}{3}(HB^2 - hb^2)$$

If we're dealing with a cone rather than a pyramid, then replace B^2 with R^2 and b^2 with r^2 and multiply the whole thing by π.

alternative formula

However, there is another formula for the volume of the frustum, which is perhaps more interesting.

If we call the altitude or height of the frustum a, where $a = H - h$, this formula is

$$V = \frac{1}{3}a(B^2 + Bb + b^2)$$

$$V = \frac{1}{3}(H - h)(B^2 + Bb + b^2)$$

We'd like to derive this. The key insight here is that by similar triangles

$$\frac{b}{h} = \frac{B}{H}, \qquad h = \frac{b}{B}H$$

while

$$\begin{aligned} a &= H - h \\ &= H - \frac{b}{B}H \\ &= H(1 - \frac{b}{B}) \\ &= H(\frac{B - b}{B}) \end{aligned}$$

reverse proof

The proof proceeds easily in the reverse direction. Start with the answer:

$$V = \frac{1}{3} a(B^2 + Bb + b^2)$$

Substitute for a

$$V = \frac{1}{3}\ H(1 - \frac{b}{B})(B^2 + Bb + b^2)$$

Part of this simplifies dramatically

$$(1 - b/B)(B^2 + Bb + b^2)$$

$$= B^2 + Bb + b^2 - bB - b^2 - \frac{b^3}{B}$$

$$= B^2 - \frac{b^3}{B}$$

Hence we have that

$$V = \frac{1}{3}\ H(B^2 - -\frac{b^3}{B})$$

Multiplying out, the first term is $1/3\ HB^2$, as desired.

The second is

$$-\frac{1}{3}\ \frac{H}{B} b^3$$

Recall that $h = bH/B$ so this is just $-1/3\ hb^2$, and we're done.

$\square$

forward proof

How would you proceed if you didn't know the answer? The formula we can easily work out is for the frustrum as the difference in volumes of two pyramids:

$$V = \frac{1}{3}\ (HB^2 - hb^2)$$

and it's not much of a stretch to substitute for $h = Hb/B$

$$V = \frac{1}{3}\ (HB^2 - \frac{H}{B}b^3)$$
$$= \frac{H}{3}\ (B^2 - \frac{b}{B}b^2)$$

and then since

$$a = H(1 - \frac{b}{B})$$

you notice the connection to the previous expression and imagine trying to factor out

$$(1 - \frac{b}{B})(p + q + r) = (B^2 - \frac{b}{B}b^2)$$

where p, q and r will need to be determined. Obviously, we will need two sets of cancellations.

We need a term of B^2

$$(1 - \frac{b}{B})(B^2 + q + r) = (B^2 - \frac{b}{B}b^2)$$

but from $(b/B) \cdot B^2$ we get a term $-bB$ that needs a corresponding term bB from somewhere.

Similarly we must have a term b^2 to generate $b/B \cdot b^2$ so

$$(1 - \frac{b}{B})(B^2 + q + b^2) = (B^2 - \frac{b}{B}b^2)$$

but then from the 1 we get a term b^2 that needs a corresponding $-b^2$.

Then the inspiration: bB gives us both of these things.

$$(1 - \frac{b}{B})(B^2 + bB + b^2) = (B^2 - \frac{b}{B}b^2)$$

I'm not saying it was easy!

Now we just pick up the $(1/3)H$

$$V = \frac{1}{3}H(1 - \frac{b}{B})(B^2 + bB + b^2)$$

and recall that

$$a = H(1 - \frac{b}{B})$$

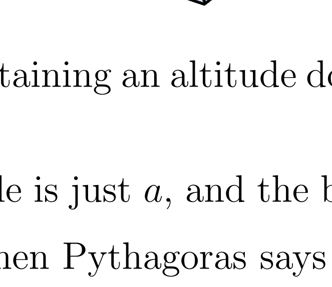
so

$$V = \frac{1}{3}a(B^2 + bB + b^2)$$

slant height

The slant height is the length of a cone or frustum along its outside edge. In the case of a cone, we can obtain it from the height and 1/2 the length of the base using the Pythagorean theorem.

For a frustum



consider the triangle containing an altitude down from the outside edge on the top.

The height of the triangle is just a, and the base has length $(B - b)/2$.

If the slant height is c then Pythagoras says that

$$c^2 = a^2 + (\frac{B - b}{2})^2$$

$$a = \sqrt{c^2 - (\frac{B-b}{2})^2}$$

Chapter 169

Pythagorean triples

From the chapter on the Pythagorean theorem:

Pythagorean triples

The triples which are not multiples of another triple are called *primitive*. For every integer m, n, with $m > n$, a Pythagorean triple is given by Euclid's formula:

$$a = m^2 - n^2 \qquad b = 2mn \qquad c = m^2 + n^2$$

We must choose m and n of opposite parity (one even and one odd). Otherwise, a, b and c will all be even and the triple won't be primitive.

It is easy to see why this works:

$$a^2 + b^2 = (m^2 - n^2)^2 + (2mn)^2$$

$$= m^4 - 2m^2n^2 + n^4 + 4m^2n^2$$

$$= m^4 + 2m^2n^2 + n^4$$

$$= (m^2 + n^2)^2 = c^2$$

all primitive triples

Maor gives a proof that *all primitive triples* can be found using Euclid's formula.

Consider

$$a^2 + b^2 = c^2$$

For a primitive triple, a and b should be of opposite parity, with one even and one odd. For if not:

Suppose a and b both even. Then $a = 2m$ and $b = 2n$ and

$$a^2 + b^2 = 4(m^2 + n^2) = c^2$$

Thus c is even and the triple is not primitive.

Otherwise, suppose a and b both odd. Then $a = 2j + 1$ and $b = 2k + 1$ and

$$a^2 + b^2 = 4j^2 + 4j + 1 + 4k^2 + 4k + 1 = c^2$$

Hence c^2 is even but not divisible by 4.

On the other hand, if a and b are odd, so are a^2 and b^2, thus their sum is even. But if c^2 is even, then so is c, thus

$$(c)^2 = (2i)^2$$

and c^2 must be divisible by 4. We have reached a contradiction.

Therefore only one of a and b is even. Let $a = 2t$.

$$(2t)^2 + b^2 = c^2$$

$$(2t)^2 = c^2 - b^2$$

$$(2t)^2 = (c + b) \cdot (c - b)$$

$$t^2 = \frac{c + b}{2} \cdot \frac{c - b}{2}$$

Since a^2 is even and b^2 is odd, c^2 is odd, so c is also odd.

Therefore the sum $c+b$ and difference $c-b$ are both even. This means that the terms on the right-hand side in the last expression above are integers.

b and c are also both relatively prime. If they were not, then the sum and difference would share this factor and the same factor would also be shared by a, contrary to the assumption that this is a primitive triple.

The left-hand side

$$t^2 = \frac{c+b}{2} \cdot \frac{c-b}{2}$$

is a perfect square, so the right-hand side is also.

On the right, since the two terms are relatively prime, each term must itself be a perfect square, otherwise their product would not be a perfect square.

So we can write:

$$u^2 = \frac{c+b}{2}$$
$$v^2 = \frac{c-b}{2}$$

Adding the two equations we obtain

$$u^2 + v^2 = c^2$$

Since c and c^2 are both odd, this shows that u and v have *opposite* parity. Subtracting

$$u^2 - v^2 = b^2$$

Finally

$$t^2 = u^2 v^2$$

$$t = uv$$

$$a = 2t = 2uv$$

There is a small table of triples in this discussion of Euclid X:29 by Joyce:

https://mathcs.clarku.edu/~djoyce/elements/bookX/propX29.html

	1	3	5	7	9	11	13
3	3 : 4 : 5						
5	5 : 12 : 13	15 : 8 : 17					
7	7 : 24 : 25	21 : 20 : 29	35 : 12 : 37				
9	9 : 40 : 41	27 : 36 : 45	45 : 28 : 53	63 : 16 : 65			
11	11 : 60 : 61	33 : 56 : 65	55 : 48 : 73	77 : 36 : 85	99 : 20 : 101		
13	13 : 84 : 85	39 : 80 : 89	65 : 72 : 97	91 : 60 : 109	117 : 44 : 125	143 : 24 : 145	
15	15 : 112 : 113	45 : 108 : 117	75 : 100 : 125	105 : 88 : 137	135 : 72 : 153	165 : 52 : 173	195 : 28 : 197

code

Here is a Python script to generate triples by exhaustive search:

https://gist.github.com/telliott99/b543f41d84155bc9171df68b6350e256

And here is one that implements Euclid's formula:

https://gist.github.com/telliott99/144c1a7e90740eb1614ca8ceb5bdeed9

Here is some output (m, n, a, b, c) from the second script, sorted on m and n:

```
> python triples2.py
  1   2   3   4   5
  1   4   8  15  17
  1   6  12  35  37
```

```
  1   8  16  63  65
  1  10  20  99 101
  1  12  24 143 145
  1  14  28 195 197
  2   3   5  12  13
  2   5  20  21  29
  2   7  28  45  53
  2   9  36  77  85
  2  11  44 117 125
  2  13  52 165 173
  3   4   7  24  25
  3   8  48  55  73
  3  10  60  91 109
  3  14  84 187 205
  4   5   9  40  41
  4   7  33  56  65
  4   9  65  72  97
  4  11  88 105 137
  4  13 104 153 185  not in list
...
```

To test triples for being primitive, we look for a greatest common divisor of a and b equal to 1. All such triples have m and n of opposite parity.

The last entry says "not in list" because the limit set for the exhaustive search was exceeded. With a larger search, this triple would be found (or it can just be confirmed by direct computation).

When sorted on a, b, c all triples found by exhaustive search appear to also be found by Euclid's formula, but this wasn't tested explicitly. That could be done easily, as an exercise.

There are some interesting patterns in lists of triples. Here are two:

3	4	5
5	12	13
7	24	25
9	40	41
11	60	61
13	84	85
15	112	113
17	144	145
19	180	181
21	220	221
23	264	265
25	312	313
27	364	365

The first entry doesn't fit the pattern. But starting with $5, 12, 13$, for every step $\Delta a = 2$, we get Δb increasing in steps of 4, with $c = b + 1$.

Below, starting with $8, 15, 17$, for every step $\Delta a = 4$, we get Δb increasing in steps of 8, with $c = b + 2$.

8	15	17
12	35	37
16	63	65
20	99	101
24	143	145
28	195	197

Chapter 170

Courant Riemann

intervals of unequal width

Courant and John describe a variation on Riemann sums using intervals of unequal (but graduated) width. This "trick" allows them to derive the formula for

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\int_a^b x^n \, dx = \frac{b^{n+1} - a^{n+1}}{n+1}$$

for all natural numbers n first, and then with some elaborations, for real n except $n = -1$.

We subdivide the interval $[a, b]$ by points with spacing that increases by a factor of q at each step

$$a, \; aq, \; aq^2, \; \ldots \; aq^{n-1}, \; aq^n$$

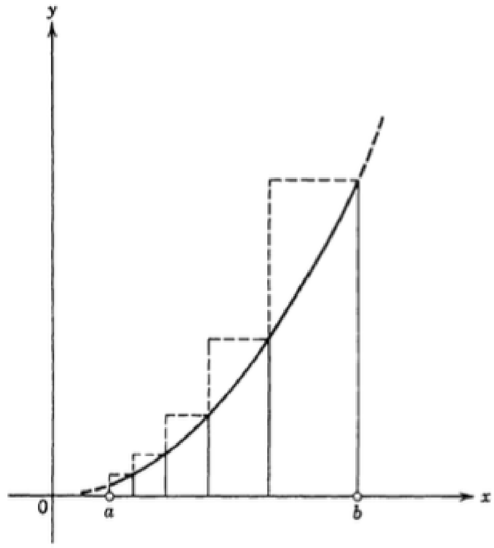


Figure 2.13 Area under a parabolic arc by geometric subdivision.

At the final, nth step we have

$$aq^n = b$$

Solving for the common ratio q we have

$$q = (b/a)^{1/n}$$

The points of division are

$$x_i = aq^i$$

The width of the ith rectangle is

$$\Delta x_i = aq^i - aq^{i-1} = aq^i\ (1 - \frac{1}{q})$$

$$= aq^i\ [\ \frac{q-1}{q}\ ]$$

The widest rectangle is the last one

$$\Delta x_n = aq^n\ [\ \frac{q-1}{q}\ ]$$

$$= b \ [\ \frac{q-1}{q} \ ]$$

In the usual way, we will let the number of rectangles $n \to \infty$.

At the same time, since

$$q = (b/a)^{1/n}$$

then $q \to 1$.

So then $\Delta x_n \to 0$, and so do all the other rectangles, which are smaller.

The function of interest is to raise x to the positive integer power p. For each rectangle, the area is

$$A_i = x_i^p \ \Delta x_i$$

$$= (aq^i)^p \ aq^i \ [\ \frac{q-1}{q} \ ]$$

$$= a^{p+1} \ (q^i)^{p+1} \ [\ \frac{q-1}{q} \ ]$$

$$= a^{p+1} \ (q^{p+1})^i \ [\ \frac{q-1}{q} \ ]$$

For the integral, we need to add all these up (from $i = 1$ to $i = n$):

$$I = \sum_{i=1}^{n} a^{p+1} \ (q^{p+1})^i \ [\ \frac{q-1}{q} \ ]$$

We can take out values that don't depend on i from the summation:

$$I = a^{p+1} \ [\ \frac{q-1}{q} \ ] \ \sum_{i=1}^{n} (q^{p+1})^i$$

Recall that for a geometric series with common ratio r the nth sum (starting from $i = 0$) is

$$S_n = 1 + r + r^2 \cdots + r^n = \sum_{i=0}^{n} r^i$$

$$= \frac{1-r^n}{1-r} = \frac{r^n-1}{r-1}$$

Substituting q for r:

$$S_n = \frac{q^n-1}{q-1}$$

For the expression above

$$\sum_{i=1}^{n} (q^{p+1})^i$$

we factor out one q^{p+1} so as to start from $i = 0$

$$= q^{p+1} \ \sum_{i=0}^{n} (q^{p+1})^i$$

and then the common ratio is q^{p+1} and the sum is

$$\sum_{i=0}^{n} (q^{p+1})^i = \frac{(q^{p+1})^n - 1}{q^{p+1}-1} = \frac{q^{n(p+1)}-1}{q^{p+1}-1}$$

The whole sum or integral I that we seek is

$$I = a^{p+1} \ [\ \frac{q-1}{q} \ ] \ q^{p+1} \ \frac{q^{n(p+1)}-1}{q^{p+1}-1}$$

$$= a^{p+1} \ (q-1) \ q^p \ \frac{q^{n(p+1)}-1}{q^{p+1}-1}$$

$$= a^{p+1} \ (q-1) \ q^p \ \frac{(b/a)^{p+1}-1}{q^{p+1}-1}$$

Since

$$a^{p+1} \ [\ (b/a)^{p+1} - 1 \ ] \ = b^{p+1} - a^{p+1}$$

we obtain

$$I = \ [\ b^{p+1} - a^{p+1} \ ] \ q^p \ \frac{q-1}{q^{p+1}-1}$$

Referring to the sum for a geometric progression again, we have from above

$$S_n = \frac{q^n - 1}{q - 1}$$

So (for $q \neq 1$) and $n = p + 1$, the inverse of that is what we have for the right-hand term

$$\frac{q - 1}{q^{p+1} - 1} = \frac{1}{S_{p+1}}$$

where

$$S_{p+1} = 1 + q + q^2 + \cdots + q^{p+1}$$

Substituting

$$I = \left[\ b^{p+1} - a^{p+1}\ \right] q^p \ \frac{1}{1 + q + q^2 + \cdots + q^{p+1}}$$

As we saw near the beginning, as $n \to \infty$, $q \to 1$, and so do all the powers of q so the term

$$q^p = 1$$

and also

$$1 + q + q^2 + \cdots + q^{p+1} = p + 1$$

so the fraction is just equal to $1/(p + 1)$ and we have finally:

$$I = \left[\ b^{p+1} - a^{p+1}\ \right] \frac{1}{p + 1}$$

which is what we sought to prove.

If you look carefully at this proof, you'll see that it follows Fermat exactly, but is more general by including a as the lower bound.

Chapter 171

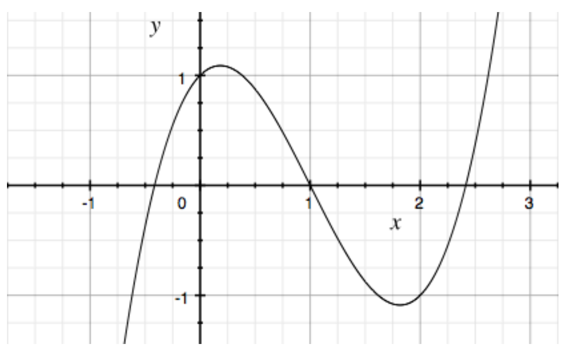
Cubics

Every cubic polynomial equation has at least one term containing x^3 but lacks any higher powers of x such as x^4.

The general equation is

$$y = ax^3 + bx^2 + cx + d$$

and a typical graph $(x^3 - 3x^2 + x + 1)$ looks something like this:



There is an axis of symmetry, here at $x = 1$, and the left half is the negative reflection of the right half about the value of $y = f(1) = 0$.

roots

By the *roots* of the equation, we mean those values of x giving $y = 0$, that is, we are solving

$$ax^3 + bx^2 + cx + d = 0$$

In this case we can always multiply through by $1/a$ so the term x^3 has a coefficient of 1, and if we do that then the coefficients are often renamed as:

$$x^3 + ax^2 + bx + c = 0$$

The cubic is an odd function, so the sign of x carries through in x^3. Since the x^3 term dominates the value of the function for extreme values of x, when $x << 0$, y is large and negative, while for $x >> 0$, y is large and positive. This is clearly seen for the above plot.

As a result, the graph of the function must cross the x-axis at least once, and thus every cubic has at least one real root, where $f(x) = 0$.

From this we conclude that every cubic can be factored into

$$(x - r)(x^2 + sx + t)$$

where $x = r$ is the guaranteed real root, although it isn't always the case that r is an integer, of course.

This expression is equal to zero either when $x = r$ or when the quadratic term is zero.

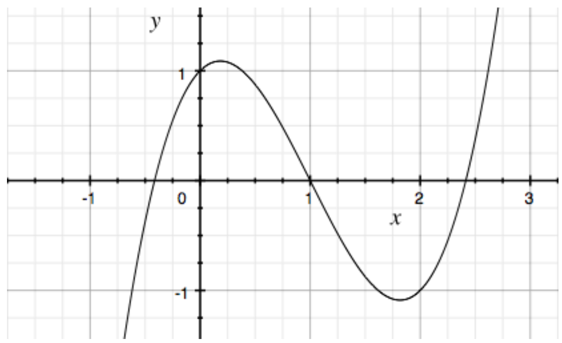
The roots of a quadratic $x^2 + sx + t$ are given by the familiar

$$\frac{-s \pm \sqrt{s^2 - 4t}}{2}$$

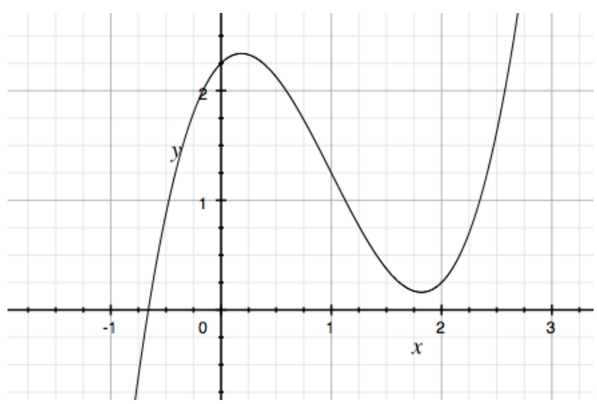
We know that quadratics have either two real roots or none depending on the value of the discriminant under the square root. We consider the case of repeated roots (when the discriminant is zero) as *two* roots.

Therefore, every cubic has either one real root or three of them.

Graphically, we can easily see the general truth of this statement.



In this example from before, near $x = 1.8$, no matter how far the graph goes below the x-axis before it turns the second time, we can add that much to the constant c. The result will be that the graph just touches the x-axis at this point. A tiny bit more and it will not cross at all.



The above graph is the same equation but with 1.25 added to c: $(x^3 - 3x^2 + x + 2.25)$.

factoring

Every cubic has at least one real root and as a consequence, the solution where $y = 0$ can be written as

$$(x - r)(x^2 + sx + t) = 0$$

Multiplying out

$$= x^3 + (s - r)x^2 + (t - rs)x - rt$$

Thus $a = s - r$ and $c = -rt$.

Suppose we know, or have guessed r. We can find s and t by comparison with the original equation.

As an example, consider:

$$(x - 1)(x + 1)(x + 2) = 0$$

$$= (x - 1)(x^2 + 3x + 2)$$

$$= x^3 + 2x^2 + 5x - 2$$

We plot this, guess that $x = 1$ is a root, check and find that $x = 1$ solves the equation. Thus $r = 1$ and since

$$c = -2 = -rt$$

then $t = 2$. Also

$$a = 2 = s - r = s - 1$$

and $s = 3$.

Alternatively, there is a formalism called synthetic division for deriving s and t. I have a simple version of this I like better than the complete formal approach. Consider

$$x^3 - 5x^2 - 2x + 24 = 0$$

Suppose we are given that $x = -2$ is a solution, which is easily checked. Now write:

$$x^3 - 5x^2 - 2x + 24$$
$$= (x+2)(x^2 + \_\_\_\ x + \_\_\_) = 0$$

The cofactor of x^2, the first term on the right, is clearly just 1, so that we get the desired x^3 in the product.

Then we see that, multiplying by 2 we get $2 \times x^2 = 2x^2$, where the desired result is $-5x^2$. We need another $-7x^2$. Therefore the cofactor of x on the right must be -7 so that $-7x^2 + 2x^2 = -5x^2$:

$$(x+2)(x^2 - 7x + \_\_\_) = 0$$

Then we see that, multiplying by 2 we have $2 \times -7x = -14x$, where the desired result is $-2x$. We need another $12x$. Therefore, the constant term on the right must be 12 so that $-14x + 12x = -2x$.

$$(x+2)(x^2 - 7x + 12) = 0$$

And finally, we check the whole thing, multiplying 2×12 to give the desired constant, 24. This must work out if we've done the rest correctly and $x = -2$ is really a solution.

Inspired guessing can help. Consider

$$x^3 - 4x^2 - 9x + 36 = 0$$

You may notice that $a \times b = c$. This tells us that $1 - 4$ is a factor of $-9 + 36$. We guess that $x = 4$ is a solution:

$$(x-4)(x^2 + \_\_\_\ x + \_\_\_) = 0$$

We don't need anything more than $-4x^2$ so the cofactor of x on the right is 0 and write

$$(x-4)(x^2 + \_\_\_) = 0$$

For the constant, we guess -9 so as to get $-9x$ and then check $-4 \times -9 = 36$.

$$(x-4)(x^2-9) = 0$$

Finally, we can factor the second term

$$(x-4)(x+3)(x-3) = 0$$

relating roots to cofactors

Suppose a cubic has three distinct real roots, meaning there are real numbers p, q, r such that

$$(x-p)(x-q)(x-r) = 0$$

Multiplying out we would obtain for the constant term: $-pqr$.

$$(x^2 - qx - px + pq)(x-r) = 0$$

$$x^3 - qx^2 - px^2 + pqx - rx^2 + qrx + prx - pqr = 0$$

$$x^3 - (p+q+r)x^2 + (pq+qr+pr)x - pqr = 0$$

So

$$d = -pqr$$

Furthermore

$$a = -(p+q+r)$$

$$b = pq + qr + pr$$

If there are three real roots, they multiply to give the constant term.

$$(x-1)(x+2)(x+1)$$

$$= (x^2+x-2)(x+1)$$

$$= x^3+2x^2-x-2$$

$1 \times -2 \times -1 = -2.$

Actually, a related statement is true even if two of the roots are not real. In principle, we can factor out the single real root, leaving a quadratic.

We said that the roots of a quadratic x^2+sx+t are given by

$$\frac{-s \pm \sqrt{s^2-4t}}{2}$$

If the second and third roots are imaginary they are so because what is under the square root is negative, so the above can be written as

$$z = u \pm iv$$

These consist of two complex numbers, which are complex conjugates. Their product is a real number:

$$(u+vi)(u-vi) = u^2+v^2$$

When we plug the three roots into $(x-p)(x-q)(x-r)$ we get

$$(x-u+vi)(x-u-vi)(x-r)$$

$$= x^2-ux-vix-ux+u^2+uvi+vix-uvi+v^2)(x-r)$$

The imaginary terms with i cancel.

$$= (x^2-2ux+u^2+v^2)(x-r)$$

$$= x^3 - 2ux^2 + u^2x + v^2x - rx^2 + 2urx - u^2r - v^2r$$
$$= x^3 - (2u + r)x^2 + (u^2 + 2ur + v^2)x - r(u^2 + v^2)$$

The result is all real coefficients.

$$d = -r(u^2 + v^2)$$

and d is the product of the three roots.

extreme points

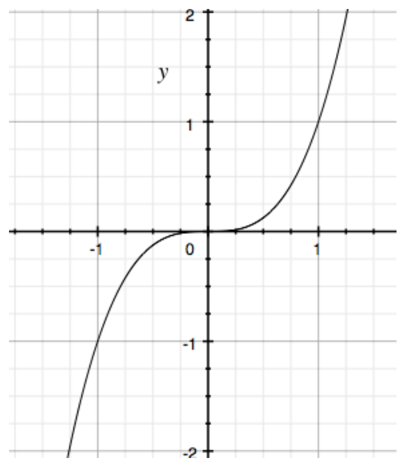
The points where the graph turns around can be found by taking the derivative and setting it equal to zero.

$$y = ax^3 + bx^2 + cx + d$$
$$\frac{dy}{dx} = 3ax^2 + 2bx + c = 0$$
$$x = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a}$$

Not all cubics have a downward sloping segment. This happens when the quadratic for the slope has no real roots, i.e. when the square root term is less than or equal to zero. A simple example of this is when $b = 0$, such as

$$y = x^3$$

This obviously has 3 equal real roots, all zero.



The slope is $2x^2$, which is never negative and equal to zero only at $x = 0$.

repeated roots

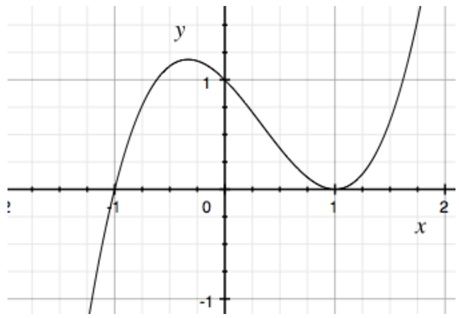
A cubic can have repeated roots. Suppose

$$(x-p)^2(x-q) = 0$$
$$= (x^2 - 2px + p^2)(x-q)$$
$$= x^3 - qx^2 - 2px^2 + 2pqx + p^2x - p^2q$$
$$= x^3 - (q+2p)x^2 + (2pq + p^2)x - p^2q$$

And, as before, the product of the roots is $-d$.

Graphically, two repeated roots means that one of the extreme points is also a root.

$$(x+1)(x-1)(x-1)$$
$$= (x+1)(x^2 - 2x + 1)$$
$$= x^3 - x^2 - x + 1$$



The slope is zero when

$$3x^2 - 2(q + 2p)x + (2pq + p^2) = 0$$

translation

Continuing with $y = x^3$, when we add or subtract a value from y the plot is shifted up or down, similarly, changes to x shift the same curve to the left or right.

For example:

$$(x - 1)^3 = x^3 - 3x^2 + 3x - 1$$

What this means is that the cofactors a and b may be non-zero and the shape still be the same as $y = x^3$. (The fact that a, b, c conform to the cubic expansion is a tipoff, however). The following section describes what is also essentially a horizontal translation.

depressed cubic

Tartaglia discovered that the quadratic term can be removed from a cubic

$$x^3 + ax^2 + bx + c$$

by an inspired substitution, $x = u - a/3$. Actually, I find the arithmetic a bit confusing, so I will further substitute $v = a/3$ and so $x = u - v$.

$$(u - v)^3 + a(u - v)^2 + b(u - v) + c$$

Now, expand each power of $u - v$ in order.

The cubic binomial $(u - v)^3$ has cofactors of 3 for the inner terms

$$(u - v)^3 = (u - v)(u^2 - 2uv + v^2)$$
$$= u^3 - 2u^2v + uv^2 - u^2v + 2uv^2 - v^3$$
$$= u^3 - 3u^2v + 3uv^2 - v^3$$

Switch the order so that the power of u is last in each term

$$= u^3 - 3vu^2 + 3v^2u - v^3$$

The quadratic is

$$a(u - v)^2 = a\ [\ u^2 - 2uv + v^2\ ]$$
$$= au^2 - 2avu + av^2$$

The linear term is just $bu - bv$.

Finally, collecting all the terms and grouping them by powers of u

$$= u^3\ [\ -3v + a\ ]\ u^2 +\ [\ 3v^2 - 2av + b\ ]\ u +\ [\ -v^3 + av^2 - bv + c\ ]$$

The bright idea is that the cofactor of u^2

$$-3v + a$$

is equal to zero by the terms of the substitution ($v = a/3$).

That leaves:

$$= u^3 + \ [\ 3v^2 - 2av + b\ ]\ u + \ [\ -v^3 + av^2 - bv + c\ ]$$

If we write

$$m = 3v^2 - 2av + b$$

$$n = -v^3 + av^2 - bv + c$$

then the cubic is

$$u^3 + mu + n = 0$$

We can reverse the second substitution ($v = a/3$). We have one less term in each formula, which is simplified a bit, but this also makes the formulas more awkward.

$$m = 3\frac{a^2}{3^2} - 2a\frac{a}{3} = \frac{a^2}{3} - 2\frac{a^2}{3} = -\frac{a^2}{3} + b$$

and

$$n = -\frac{a^3}{3^3} + a\frac{a^2}{3^2} - \frac{ab}{3} + c$$

$$= 2\frac{a^3}{3^3} - \frac{ab}{3} + c$$

Here's an example. Consider:

$$x^3 + 3x^2 - x + 1 = 0$$

$$a = 3, \qquad b = -1, \qquad c = 1$$

So

$$m = -\frac{a^2}{3} + b = -\frac{a^2}{3} - 1 = -4$$

The constant n is

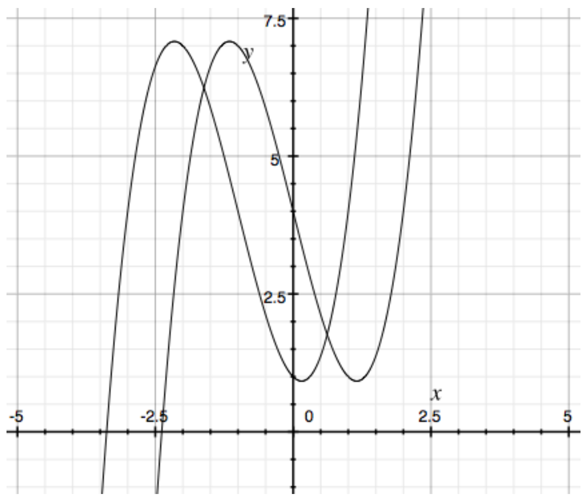
$$n = \frac{2a^3}{3^3} - \frac{ab}{3} + c$$

$$2 + 1 + 1 = 4$$

So the transformed version is

$$u^3 - 4u + 4$$

And this is indeed the same curve, simply displaced to the right by 1 unit, as the substitution $x = u - a/3$ or $x = u - 1$ implies.



The real root is also the same (taking into account the translation).

An example from Nahin is

$$x^3 - 15x^2 + 81x - 175 = 0$$

The coefficient of u is

$$(-\frac{a^2}{3} + b)$$

$$= -\frac{(-15)^2}{3} + 81 = -75 + 81 = 6$$

The constant is

$$\frac{2a^3}{3^3} - \frac{ab}{3} + c$$

$$= \frac{2(-15)^3}{3^3} - \frac{(-15)81}{3} - 175$$

$$= -150 + 405 - 175 = -20$$

Hence we have

$$u^3 + 6u - 20 = 0$$

By trial and error, we find that $u = 2$ is a solution.

Reversing the substitution, $3v = a$. This is the cofactor of x^2, the a from the original equation! So $v = a/3 = -15/3 = -5$.

Thus $x = u - v = 2 - (-5) = 7$. Check by substitution:

$$(7)^3 - 15(7^2) + 81(7) - 175 = 0$$

Factor out 7

$$7^2 - 15(7) + 81 - 25 = 0$$

$$49 - 105 + 81 - 25 = 0$$

which checks.

The resulting equation (lacking a quadratic term), is called a *depressed* cubic.

$$u^3 + mu + n = 0$$

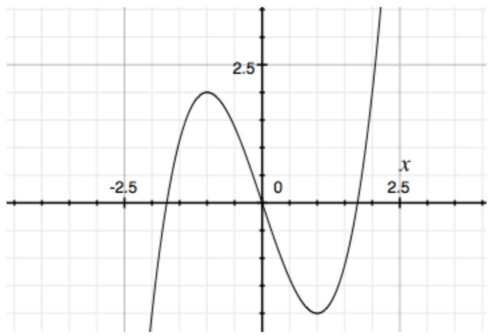
The result is kind of amazing. For any cubic containing ax^2, we can obtain the same curve without any quadratic term.

form of the curve

The constant simply displaces y by some value. The coefficient of x^3 stretches the curve.

From consideration of the depressed quadratic, you can see that the essential form is conferred by the cofactor of x in, say

$$y = x^3 - 3x$$



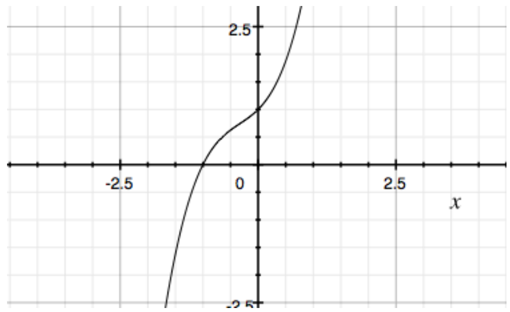
The larger the value of b, the bigger the deviations before the curve turns back. It's curious that the extreme points are exactly $y = 2$ here. That's because

$$\frac{dy}{dx} = 3x^2 - 3 = x^2 - 1$$

Hence they occur at $x = \pm 1$, where $y = \pm 2$.

In an expression like $y = x^3 + ax^2 + bx + c$, increasing a makes the central displacement more pronounced, while increasing b makes it less pronounced. Interestingly, having all three constants equal to 1 makes it go away altogether.

$$y = x^3 + x^2 + x + 1$$



$x = -1$ has solution $y = 0$, and that's the single real root because

$$x^3 + x^2 + x + 1 = (x + 1)(x^2 + 1)$$

and we know $x^2 + 1$ has $i = \pm\sqrt{-1}$ as its solution.

Also, we note for this example

$$y = x^3 + x^2 + x + 1$$

Get the slope as the derivative and set it equal to zero:

$$y' = 3x^2 + 2x + 1 = 0$$

for which the roots are

$$x = \frac{-2 \pm \sqrt{4 - 12}}{6}$$

The discriminant is negative, so there is no x that gives a slope of zero.

The minimum value of y' is $y'' = 0$

$$y'' = 6x + 2 = 0$$

$$x = -\frac{1}{3}$$

$$y' = \frac{3}{9} - \frac{2}{3} + 1 = \frac{2}{3}$$

Solving the depressed cubic

Which brings us finally to Cardano, and the solution of the cubic.

Dunham has a picture of the geometrical division of a cube that Cardano visualized,

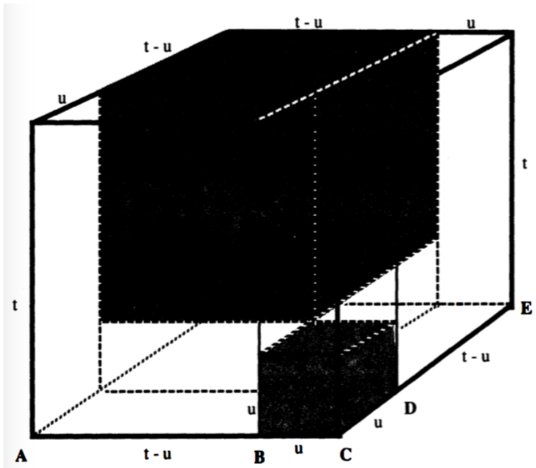


FIGURE 6.1

However, with modern notation, we can get there pretty simply from algebra.

$$(t-u)^3 = t^3 - 3t^2u + 3tu^2 - u^3$$
$$(t-u)^3 + 3t^2u - 3tu^2 = t^3 - u^3$$
$$(t-u)^3 + 3ut(t-u) = t^3 - u^3$$

Now let $x = t - u$

$$x^3 + 3tux = t^3 - u^3$$

Substitute $m = 3tu$ and $n = t^3 - u^3$:

$$x^3 + mx = n$$

This is a depressed cubic.

$$x^3 + mx - n = 0$$

The idea is to start with a depressed cubic we want to solve, and use that to get values for m and n.

If we can then determine values for t and u, $x = t - u$ will be the solution that we seek.

We have the two conditions: $m = 3tu$ and $n = t^3 - u^3$. Solve the first for u

$$u = \frac{m}{3t}$$

and substitute into the second:

$$n = t^3 - \frac{m^3}{(3t)^3}$$

Multiply both sides by t^3

$$nt^3 = t^6 - \frac{m^3}{27}$$

Looks like it's getting more complicated.

But this is a quadratic equation in disguise!

$$t^6 - nt^3 - \frac{m^3}{27} = 0$$

By the quadratic formula:

$$t^3 = \frac{n \pm \sqrt{n^2 + 4m^3/27}}{2}$$

Take the positive square root

$$= \frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}$$

and take its cube root:

$$t = \ [\ \frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}\ ]^{1/3}$$

Since $u^3 = t^3 - n$, just subtract n from the expression for t^3 before taking the cube root.

$$u = \ [\ -\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}\ ]^{1/3}$$

Then

$$x = t - u$$
$$= [\ \frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}\ ]^{1/3} - \ [\ -\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}\ ]^{1/3}$$

We can write this more simply by pre-computing

$$r = \frac{n}{2}, \qquad s = \frac{m^3}{27}$$

Then

$$x = \ [\ r + \sqrt{r^2 + s}\ ]^{1/3} - \ [\ -r + \sqrt{r^2 + s}\ ]^{1/3}$$

Here is Cardano (recall we are solving $x^3 + mx - n = 0$):

> Cube one-third the coefficient of x; add to it the square of one-half the constant of the equation; and take the square root of the whole. You will duplicate [repeat] this, and to one of the two you add one-half the number you have already squared and from the other you subtract one-half the same . . . Then, subtracting the cube root of the first from the cube root of the second, the remainder which is left is the value of x.

example

How about Cardano's example:

$$x^3 + 6x - 20 = 0$$

Clearly, 2 is a solution to this. It is the only real root.

We have $m = 6$ and $n = 20$, so $n/2 = 10$.

$m = 6$ so

$$\frac{m^3}{27} = \frac{6^3}{27} = \frac{(2 \cdot 3)^3}{3^3} = 2^3 = 8$$

and then

$$x = \ [\ 10 + \sqrt{100 + 8}\ ]^{1/3} - \ [\ -10 + \sqrt{100 + 8}\ ]^{1/3}$$

These two terms are

$$(10 + \sqrt{108})^{1/3} = 2.732$$

$$(-10 + \sqrt{108})^{1/3} = 0.732$$

The difference is indeed very close to 2.

example 2

Another famous example is

$$x^3 - 15x - 4 = 0$$

Guessing, we obtain $x = 4$ as one root.

Now, to factor out $(x - 4)$:

$$x^3 - 15x - 4 = (x - 4)(\_\_\_\ x^2 + \_\_\_\ x + \_\_\_)$$

$$x^3 - 15x - 4 = (x - 4)(x^2 + \_\_\_\ x + \_\_\_)$$

$$x^3 - 15x - 4 = (x - 4)(x^2 + 4x + \_\_\_)$$

$$x^3 - 15x - 4 = (x - 4)(x^2 + 4x + 1)$$

The last multiplication to give the constant works, which provides a check on the whole thing.

We solve the quadratic as

$$x = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{4-1} = -2 \pm \sqrt{3}$$

Check the positive root:

$$\begin{aligned}
&(-2+\sqrt{3})^2 + 4(-2+\sqrt{3}) + 1 \\
&= 4 - 4\sqrt{3} + 3 - 8 + 4\sqrt{3} + 1 \\
&= 0
\end{aligned}$$

So we have three real roots. Notice that

$$4 + (-2+\sqrt{3}) + (-2-\sqrt{3}) = 0$$

The sum of the roots is zero.

Now use Cardano's solution to solve

$$x^3 - 15x - 4$$

First

$$r = \frac{n}{2} = -2$$

$$s = \frac{m^3}{27} = \frac{-15^3}{27} = -125$$

$$\begin{aligned}
x = &\ [\ r + \sqrt{r^2+s}\ ]^{1/3} - \ [\ -r + \sqrt{r^2+s}\ ]^{1/3} \\
= &\ [\ -2 + \sqrt{4 + -125}\ ]^{1/3} - \ [\ 2 + \sqrt{4 + -125}\ ]^{1/3} \\
= &\ [\ -2 + \sqrt{-121}\ ]^{1/3} - \ [\ 2 + \sqrt{-121}\ ]^{1/3} \\
= &\ [\ -2 + \sqrt{-121}\ ]^{1/3} + \ [\ -2 - \sqrt{-121}\ ]^{1/3}
\end{aligned}$$

That seems strange at first. We have three real roots, but Cardano's solution gives an expression which is the sum of two imaginary numbers.

The resolution is that the two numbers here are complex conjugates. What we have is

$$[\ z\ ]^{1/3} + \ [\ z* \ ]^{1/3}$$

where

$$z = -2 + 11i$$

If we write this in polar form

$$z = re^{i\theta}$$

$$z* = re^{-i\theta}$$

so

$$z^{1/3} = r^{1/3}e^{i\theta/3}$$

$$z*^{1/3} = r^{1/3}e^{i(-\theta/3)}$$

The sum is

$$r^{1/3}\ [\ e^{i\theta/3} + e^{i(-\theta/3)}\ ]$$

The term in the brackets is the sum of a complex number and its complex conjugate, $w + w*$, which is completely real, so the whole thing is completely real.

$$e^{i\theta/3} + e^{i(-\theta/3)} = 2\cos\ (\theta/3)$$

To actually do the calculation

$$z = -2 + 11i$$

$$zz* = (-2 + 11i)(-2 - 11i) = -4 + 121$$

$$r = \sqrt{zz*} = \sqrt{117}$$

$$r^{1/3} = 2.211$$

For the angle

$$\theta = \tan^{-1} -\frac{11}{2} = -1.391$$

$$\theta/3 = -0.46$$

The term in the brackets is

$$2\cos\,(\theta/3) = 2\cos\,(-0.46) = 1.788$$

The whole thing is

$$r^{1/3}\ [\ e^{i\theta/3} + e^{i(-\theta/3)}\ ] = 2.211(1.788) \approx 4$$

A much simpler method is to notice that

$$(2+\sqrt{-1})^3 = (2+\sqrt{-1})(4+4\sqrt{-1}-1)$$

$$= (2+\sqrt{-1})(3+4\sqrt{-1})$$

$$= 6+3\sqrt{-1}+8\sqrt{-1}-4$$

$$= 2+11\sqrt{-1} = 2+\sqrt{-121}$$

The same result is obtained with $-2+\sqrt{-1})^3$.

Hence

$$x = \ [\ -2+\sqrt{-121}\ ]^{1/3} - \ [\ 2+\sqrt{-121}\ ]^{1/3}$$

$$= 2+2 = 4$$

example 3

$$x^3 + 3x - 2 = 0$$

It might be simplest to try $m = 3$ and $n = 2$, so $n/2 = 1$.

$m = 3$ so

$$\frac{m^3}{27} = 1$$

Then

$$x = \ [\ 1 + \sqrt{2}\ ]^{1/3} - \ [\ -1 + \sqrt{2}\ ]^{1/3}$$

These two terms are

$$= 1.3415 - 0.745 = 0.596$$

This doesn't quite match the plot, however (which is closer to 0.7).

The arithmetic is tiresome, so write a script to do it.

script

```
# Cardano's method for solving
# x^3 + mx = n

import sys
from math import sqrt

def simplify(a,b,c):
    f = 1.0*a/3
    g = a*f   # a^2/3
    m = -g + b
    h = f*g   # a^3/3^2
    j = h/3   # a^3/3^3
```

```
    return m, (-j + h - (a*b*1.0)/3 + c)

def cardano(m,n):
    c = (m**3)/27
    h = n/2.0
    r = 0.3333333333

    rad1 = ( h + sqrt(h**2 + c))
    rad2 = (-h + sqrt(h**2 + c))
    return round(rad1**r - rad2**r, 5)
```

```
> python
..
>>> from cubics import *
>>> simplify(3,-1,1)
(-4.0, 4.0)
>>> cardano(6,20)
2.0
>>> cardano(3,2)
0.59607
```

We get Cardano's result, and confirm the calculation for the second example. This suggests that the error lies in the plotting program.

practical solving

A practical approach to real problems involves first plotting the function so as to know whether there is one real root or three, and get an idea of their values.

For example

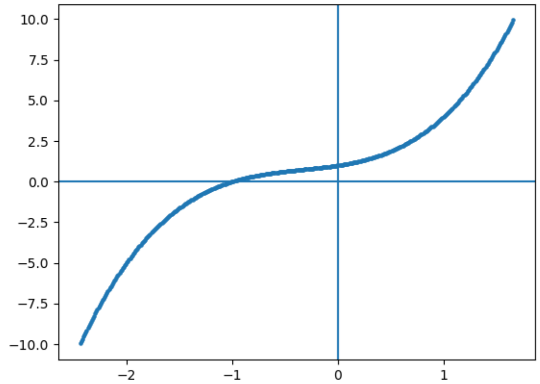
plotter.py

```
from matplotlib import pyplot as plt
import numpy as np

def plot(X,Y):
    plt.scatter(X,Y,s=5)
    plt.axhline()
    plt.axvline()
    #plt.axes().set_aspect('equal')
    plt.savefig('x.png')

def cubic(a,b,c,d):
    def f(x):
        return a*x**3 + b*x**2 + c*x + d
    L = np.linspace(-10,10,2000)
    X = list()
    Y = list()
    for x in L:
        y = f(x)
        if y < -10 or y > 10:
            continue
        X.append(x)
        Y.append(y)
    plot(X,Y)

cubic(1,1,1,1)
```



An actual solver might look something like this:

guess.py

```
import numpy as np

a,b,c = 1, 1, 1

def f(x):
    return x**3 + a*x**2 + b*x + c

def getX(x1,x2):
    N = 1000
    return np.linspace(x1,x2,N)

# assumes we go from f(x) < 0 to f(x) > 0
def guess(x1, x2):
    std_order = f(x1) < f(x2)

    print 'guess'
    print 'x1 = ', str(x1)
    print 'x2 = ', str(x2)
```

```
    print 'y1 = ', str(f(x1))
    print 'y2 = ', str(f(x2))
    print

    X = getX(x1,x2)
    if not std_order:
        X.reverse()
    assert f(X[0]) < 0 and f(X[-1]) > 0

    for i,x1 in enumerate(X):
        x2 = X[i+1]
        # must happen
        if f(x2) > 0:
            if not std_order:
                return x2, x1
            return x1, x2

def close(r):
    e = 1e-12
    return not (r > e or r < -e)

x1 = -2
x2 =  0
i = 0

while i < 100:
    print i+1
    x1, x2 = guess(x1, x2)
    if close(x2 - x1):
        break
    i += 1
```

output

```
> python guess.py
1
guess
x1 =  -2
x2 =  0
y1 =  -5
y2 =  1

2
guess
x1 =  -1.001001001
x2 =  -0.998998998999
y1 =  -0.00200400701102
y2 =  0.001999998999

3
guess
x1 =  -1.000001002
x2 =  -0.999998997997
y1 =  -2.00400801575e-06
y2 =  2.00400399997e-06

4
guess
x1 =  -1.000000001
x2 =  -0.999999998997
y1 =  -2.00601180111e-09
y2 =  2.00601202316e-09
```

```
5
guess
x1 =  -1.0
x2 =  -0.999999999999
y1 =  -2.00772731773e-12
y2 =  2.00817140694e-12

>
```

It's pretty clear that $x = -1$ is the real root.

$$x^3 + x^2 + x + 1 = (x+1)(x^2+1)$$

The product of the two other roots is $x^2 + 1$, that is, they are $\pm\, i$.

Recall that $d = -pqr$ so

$$d = -\ [\ (-1) \times i \times -i\ ] = -\ [\ (-1) \times 1\ ] = 1$$

Chapter 172

Real numbers

Integers

The *natural* or counting numbers which everyone learns very early in life are $1, 2, 3$ and so on.

One can get hung up on the question of whether the natural numbers would exist without the problem of counting four sheep or all ten of our fingers. We will not worry about that.

Mathematicians refer to the *set* of natural numbers and give this set a special symbol, $\mathbb{N}$. We write

$$\mathbb{N} = \{1, 2, 3 \dots\}$$

Sometimes people say that $0 \in \mathbb{N}$ (0 is a part of the set), but most do not, and we will follow the definition given above. To construct the set $\mathbb{N}$, start with the smallest element, 1. Then add successive elements by forming $a_{n+1} = a_n + 1$.

The dots mean that this sequence of numbers continue forever.

theorem

$\mathbb{N}$ is an infinite set. There is no largest number in $\mathbb{N}$, no largest $n \in \mathbb{N}$ (no largest n *in* $\mathbb{N}$).

Proof: the proof is by contradiction. Suppose $\mathbb{N}$ does have a largest number, a. Well, what about $a + 1$? This is clearly also a member of the set.

A second important property of $\mathbb{N}$, as mentioned, is that there is a least number in the set. If pairwise comparisons are carried out, a single element, the number 1, has the property that $1 \leq n$ for all numbers in $n \in \mathbb{N}$.

Since we can also find the least member of the set excluding 1, usually written $\mathbb{N} \setminus 1$, we can order every number in $\mathbb{N}$.

This property is called the **well-ordered** property.

The set $\mathbb{Z}$ contains all the members of $\mathbb{N}$ plus their negatives, as well as the special number 0, often called the additive identity.

$$\mathbb{Z} = \{\cdots - 2, -1, 0, 1, 2, \ldots\}$$

$\mathbb{Z}$ stands for the German word *Zahlen*, Number. The set $\mathbb{Z}$ are usually referred to as the integers.

$\mathbb{Z}$ is also an infinite set and also has the well-ordered property. To show this simply order all numbers $p > 0$ with respect to zero using $<$, and all the numbers $n < 0$ using $>$.

Rationals

The set $\mathbb{Q}$ (for quotient) are the rational numbers. The name is derived from ratio. Here is one definition, from Courant and Robbins.

$$\mathbb{Q} = \{\frac{p}{q}\} \text{ for } p \in \mathbb{Z}, q \in \mathbb{N}$$

Notice that we have defined $q > 0$. If p/q is to be less than zero, then it is enough that one of p or q be less than zero. However, most authors don't make a big deal out of this, and going forward will just say $p, q \in \mathbb{Z}, q \neq 0$.

q must not be zero because division by zero is not defined. We *could* choose to allow division by zero, but would quickly run into logical contradictions.

decimal representation

Every rational number can be represented as a decimal, using the method called long division.

Consider 1/2

$$2\overline{)1.000}$$

We say that 2 does not *go into* 1, so we have the first part of our result as 0, followed by a decimal point. But 2 does go into 10 exactly 5 times, giving 0.5. The remainder is zero and so the division process terminates.

Consider 1/8.

$$8\overline{)1.000}$$

◦ 8 goes into 10 once, leaving 2 as remainder

◦ 8 goes into 20 twice, leaving 4.

◦ 8 goes into 40 exactly 5 times with no remainder.

The result is 0.125.

The other possibility is that in going through the process a remainder comes up that has been seen previously. If this happens then the sequence will repeat forever.

If we don't terminate with zero, then this must eventually happen, because there are only as many as q possible remainders.

Thus, for example

$$1/7 = 0.142857142857\ldots$$

which contains 142857, repeating.

decimals to fractions

Conversely, every repeating decimal can be represented as a rational number. For example

$1 \times r = 0.142857142857\ldots$

$1000000 \times r = 142857.142857\ldots$

$999999 \times r = 142857$

$$r = \frac{142857}{999999} = \frac{1}{7}$$

(since 142857 goes into 999999 exactly 7 times).

You can do this trick with

$r = 0.333\ldots$

$10 \times r = 3.33\ldots$

$9 \times r = 3$

$$r = \frac{3}{9} = \frac{1}{3}$$

or even $r = 0.4999\ldots$

$10 * r = 4.999\ldots$

$9 * r = 4.5$

$$r = \frac{4.5}{9} = \frac{1}{2}$$

and

$r = 0.9999\ldots$

$10 * r = 9.999\ldots$

$9 * r = 9$

$$r = \frac{9}{9} = 1$$

This is one of the subtleties of numbers. In what sense can we say that

$$0.5 = 0.4999\ldots$$

$$1 = 0.9999\ldots$$

Most everyone is OK with the example $1/3 = 0.3333\ldots$ but some may be uneasy with the other two.

Ultimately, we justify the result as defined by evaluation of a limit.

Consider 0.9999. For the number of places in the result $n \in \mathbb{N}$, then as $n \to \infty$ the number being shown approaches 1 as its limit. We'll come back to this after introducing the real numbers.

ordering

For two rational numbers a and b there are only three cases: either $a = b$, $a < b$ or $b < a$.

$$\frac{p}{q} < \frac{s}{t} \iff pt < qs$$

p/q is less than s/t if and only if $pt < qs$. Ordering of the integers guarantees ordering of the rational numbers.

Note: we used the property that if

$$a < b$$

then for $c > 0$

$$ca < cb$$

intervals

We denote the numbers greater than u and less than v as lying in the interval (u, v). With parentheses symbols, the interval described is *open*, it does not include the boundary values.

To describe a *closed* interval, write $[u, v]$. This interval includes all the values in the first one, plus it also includes u and v.

Because of the density property described below, an interval such as

$$I = [0, 1]$$

contains an *infinite* quantity of rational numbers.

density

Consider the set of all points

$$x = \frac{p}{10^n}$$

for all natural numbers n and integers p.

It is clear that simply by increasing the value of n, we can construct a set of equally spaced rational numbers as tightly clustered as we wish.

The rational numbers are said to be *dense* on the number line.

theorem

• Between *any* two rational numbers it is always possible to find another rational number.

We describe this situation by writing that

$$\forall\ u, v \in \mathbb{Q}\ \exists\ w \in \mathbb{Q} \mid w \in (u, v)$$

For every open interval whose bounds are rational numbers, there exists another rational number between u and v.

Suppose we have the numbers $p/q, s/t \in \mathbb{Q}$ and that $p/q < s/t$. The average of the two is

$$r = \frac{1}{2}\ [\ \frac{p}{q} + \frac{s}{t}\ ]\ = \frac{pt + sq}{2qt}$$

Thus, r is rational and is equal to the average of p/q and s/t, so it lies between them.

More formally, as r is defined above, it has the property

$$\frac{p}{q} < r < \frac{s}{t}$$

Proof: substitute p/q for s/t in the equation of the average, defining r above. The resulting number is smaller than r, because the substituted value is smaller than s/t

$$\frac{1}{2}\left[\ \frac{p}{q} + \frac{p}{q}\ \right] \ < \frac{1}{2}\left[\ \frac{p}{q} + \frac{s}{t}\ \right] \ = r$$

but the left-hand side is equal to p/q. Now do the same thing with s/t.

$$r = \frac{1}{2}\left[\ \frac{p}{q} + \frac{s}{t}\ \right] \ < \frac{1}{2}\left[\ \frac{s}{t} + \frac{s}{t}\ \right]$$

Reals

unit square

It seemed at first that a perfectly consistent system of mathematics could be built up out of only the rational numbers. However, it was discovered that some "numbers" cannot be represented by a rational number p/q with integer p and q.

Probably the simplest example is to consider a unit square and its diagonal. The unit square has sides of length 1 and Pythagoras tells us that the length of the diagonal is $\sqrt{1^2 + 1^2} = \sqrt{2}$.

Clearly, this is a valid physical length (just draw the diagonal, then transfer it to the number line).

Yet as you probably know, we cannot solve the equation $(p/q)^2 = 2$ for integer p and q (that is, rational number p/q).

And generally, many numbers (defined as solutions to equations like $r^2 = 2$ or $r^2 = 3$) cannot be represented by a rational number p/q with integer p and q.

Despite the density of the rational numbers — recall the infinite number of rational numbers in the interval $[0, 1]$, nevertheless there are values that cannot be represented as rational numbers. This situation is described by saying that the rational number line has "holes" in it.

no holes

We introduce the concept of the real numbers $\mathbb{R}$ to include all the numbers we know so far: the natural numbers, the integers, the rational numbers $p/q \in \mathbb{Q}$, *plus* the irrational numbers like $\sqrt{2}$ and also π and e. In this section, when we talk about real numbers we are really exploring the properties of those real numbers which are not rational, the set $-\mathbb{Q}$.

The discovery that one cannot find integer p and q such that

$$(\frac{p}{q})^2 = 2$$

is due to the Pythagorean school and was most unwelcome since it screwed up their cherished theory of the universe.

The standard proof is by contradiction. We suppose that p and q exist with this property. Crucially, if p and q have a common factor we can certainly remove it by division, so that we must start with p/q already in lowest terms.

An elegant way to state this is to use the theorem which says that each number has a unique prime *factorization.* For example:

$$12 = 2 \times 2 \times 3$$

$$256 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

$$4841 = 47 \times 103$$

So for any p and q if they have a common factor it is certain that we can find it, and remove it by division.

Another important preliminary result is that the square of an even number is an even number.

Proof: every (positive) even number can be expressed as $n = 2k,\ k \in \mathbb{N}$, and then

$$n^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2 = 2m$$

is even. On the other hand, every (positive) odd number has an odd square because it has the form $n = 2k + 1,\ k \in \mathbb{N}$ so

$$n^2 = (2k+1)^2 = 2\ [\ 2 \cdot (k^2 + k)\ ]\ + 1 = 2m + 1$$

and so is odd.

Returning to

$$(\frac{p}{q})^2 = \frac{p^2}{q^2} = 2$$

A simple rearrangement

$$p^2 = 2q^2$$

shows that p^2 must be even, and from the above argument, p must be even. Rewrite $p = 2p'$. Then

$$\frac{(2p')^2}{q^2} = 2$$

$$4p'^2 = 2q^2$$

$$2p'^2 = q^2$$

Hence q^2 is even and so is q, which contradicts the assumption of p and q as having no common factors.

This proves that integer p and q with the property $p/q = \sqrt{2}$ cannot be found.

proof 2

This is such an important result that we discuss another proof of it, following Stewart and Tall's *The Foundations of Mathematics*. It uses the prime factorization theorem from above.

Euclid's lemma says that if a prime p divides the product of two integers a and b, then p must divide at least one of a and b.

It's a bit convoluted to prove so we will assume that part.

From Euclid's lemma we obtain the Fundamental Theorem of arithmetic, also called the unique factorization theorem. This says that every integer larger than 1 is either prime or is a unique product of prime factors.

Suppose $s > 1$ is the product of prime factors in two different ways:

$$p = p_1 \cdot p_2 \dots$$

$$q = q_1 \cdot q_2 \dots$$

By Euclid's lemma, p_1 must divide one of q_1 etc. But these are all prime. Therefore p_1 equals one of the q_i, say q_1. Thus

$$\frac{p}{p_1} = p_2 \dots$$

$$\frac{p}{q_1} = q_2 \dots$$

This can be done for each of the factors p_i. Therefore the factorization is unique.

Unique prime factorization can be used in turn to prove that $\sqrt{2}$ is irrational. Since every integer p is a product $p_1 p_2 \dots$, and every rational number p/q is

$$\frac{p_1 \cdot p_2 \dots}{q_1 \cdot q_2 \dots}$$

where no p_i equals any q_i.

Every rational number squared is

$$\frac{p_1^2 \cdot p_2^2 \dots}{q_1^2 \cdot q_2^2 \dots}$$

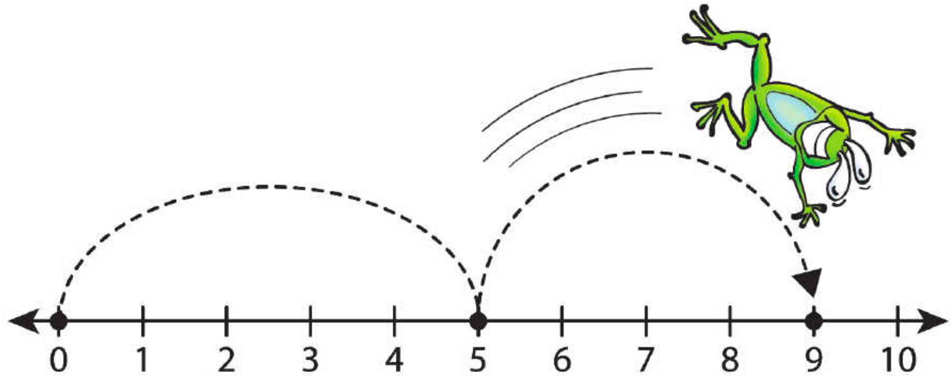
But 2 has only itself as a factor and that factor only occurs once. There is no rational number with this property.

density

number line

A simple tool to visualize all of the real numbers is the familiar number line. Here is the number line with numbers marked from $\mathbb{N}$, but obviously we could also draw one for $\mathbb{Z}$ or $\mathbb{Q}$.

We explore the application of the number line to $\mathbb{R}$ as we proceed.



We might simply assume that to every point on the number line there corresponds a rational or irrational number, and that this total collection obeys the same laws of arithmetic as the rational numbers do.

As mentioned above, the need for the real numbers is indicated by empty "holes" in the number line corresponding to the irrational numbers like $\sqrt{2}$.

A problem that arises is how to specify an irrational number non-geometrically and other than as the solution to an equation such as $r^2 = 2$. In all cases we write particular real numbers as *approximations*. For example, the square root of 2 lies between 1 and 2 because

$$1^2 = 1 < 2$$

$$2^2 = 4 > 2$$

Implying that $\sqrt{2} < 2$. At the second place:

$$1.4^2 = 1.96 < 2$$

$$1.5^2 = 2.25 > 2$$

Implying that $\sqrt{2} < 1.5$. At the third:

$$1.41^2 = 1.9881 < 2$$

$$1.42^2 = 2.0164 > 2$$

Implying that $\sqrt{2} < 1.42$. and at the seventh place

$$1.414213^2 = 1.9999984093689998.. < 2$$

$$1.414214^2 = 2.0000012377960004 > 2$$

and so on.

We can never write down the decimal value of $\sqrt{2}$ exactly, but only approximate it to greater and greater precision. The decimal value goes on forever.

Because any repeating decimal can be written as a fraction, we know that the sequence cannot repeat.

The real number $\sqrt{2}$ is defined to be the limit of this sequence

$1.4, 1.41, 1.414, \ldots 1.414214 \ldots$

as the number of terms $n \to \infty$.

In a similar way, the number e can be viewed as

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n$$

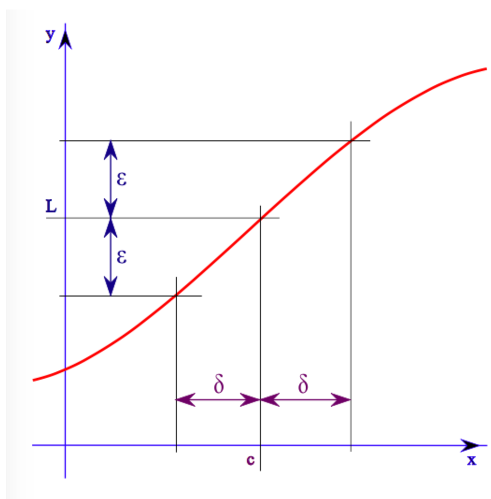
And the number π can be viewed as the limit of the method of exhaustion applied to the area of a unit circle.

Chapter 173

Limits and continuity

Limits

Consider the graph of a function $f(x)$. We might choose a power of x similar to $y = x^2$ or $y = x^3 - x$, which affirmatively has two properties that are of interest here: continuity and differentiability.

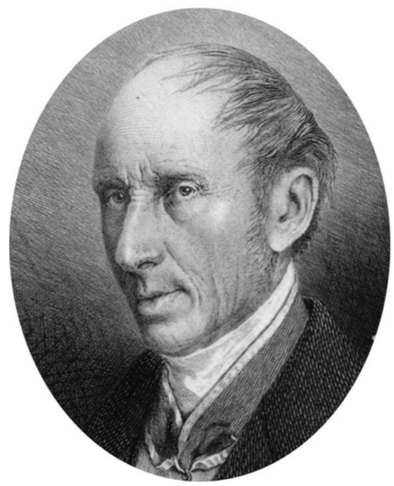


We focus on the neighborhood of a point on the x-axis, $x = c$.

By inspection of the graph we see that the value of $f(x)$ at c is equal to L, and furthermore, for points near c, the value of f at those points is not too different from L.

We would like to say that the *limit* of $f(x)$ as x *approaches* c is equal to L. The idea is that we can make $f(x)$ as close to L as we please, provided we choose x sufficiently close to c.

> When the values successively attributed to a variable approach indefinitely to a fixed value, in a manner so as to end by differing from it by as little as one wishes, this last is called the limit of all the others. —Cauchy



Modern mathematicians don't like that word "approach", which conjures up movement and the involvement of time, and they don't like reasoning from what they see in a graph, in part because no graph can show the whole function for the general case. Instead we will use an algebraic method from the formal apparatus of calculus.

There are two equivalent approaches, neighborhoods, and epsilon-delta formalism. Let's look at neighborhoods briefly first.

neighborhoods

First, an *interval* between two real numbers a and b $(a < b)$ contains every real number $a < x < b$.

$$(a, b) = x \mid a < x < b$$

The "|" means x "such that" the condition $a < x < b$ holds.

A *closed* interval $[a, b]$ includes the endpoints $(a \leq x \leq b)$, while an *open* interval (a, b) excludes them. Half-open intervals like $[a, b)$ may be defined, and an interval with $\pm\ \infty$ as an endpoint is always open on that end, for example: $[a, \infty)$.

Any open interval with a point p as its midpoint is called a *neighborhood* of p. The distance r from p to the boundary of a particular neighborhood may be large or very very small. We denote a neighborhood of p as $N(p)$.

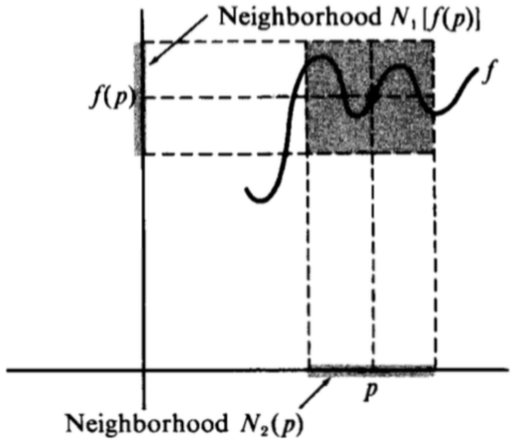
$$N(p) = x \quad \text{such that} \quad |x - p| < r$$

.

To say that the limit $f(x) \to L$ exists, we mean that for every neighborhood $N_1(L)$, there exists some neighborhood $N_2(p)$ such that $f(x) \in N_1(L)$ whenever $x \in N_2(p)$.

For a limit, we exclude the point $x = p$. It is not necessary that $f(p) = L$.

The idea of a neighborhood is a nice abstraction to hide the apparatus of modern calculus, which we look at next.



epsilon-delta game

The formal method uses numbers called ϵ and δ and is originally due to Bolzano.

We say that, *if* for all points x within a specified distance δ of c, we find that $f(x)$ lies within a specified distance ϵ from L, *then* the limit is L.

To do this we must choose ϵ first. That's why I call it a game. Why don't you go first? Choose ϵ, which provides a constraint on how close to L you want the value of $f(x)$ to be: you require that $|f(x) - L| < \epsilon$. The *distance* from $f(x)$ to L must be less than ϵ.

Now that I know your ϵ, I must try to find a suitable δ. If I can, then you get another chance, and will presumably choose a smaller ϵ.

If I can show that it is possible to find a δ to guarantee that your constraint is satisfied for *all* values of $\epsilon > 0$ *no matter how small*, then I win and the limit exists. If not, it doesn't.

Here is the formal definition:

$$\forall\, \epsilon > 0, \exists\, \delta > 0 \mid \forall\, x$$

For all (arbitrary) ϵ, there exists $\delta > 0$ such that for all x satisfying

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

We describe the limit defined above by saying that

$$\lim_{x \to c} f(x) = L$$

The limit as x tends to, or approaches, c is equal to L.

Important points about limits:

- We do not require that $f(c) = L$.

The function $f(x)$ may or *may not* have the value L at $x = c$ and the limit can still exist and be equal to L. Suppose we have $f(x) = x$, whose graph is the line $y = x$, except that we decide to define $f(0) = 1$, leaving a hole in our line $y = x$ at the point where $x = 0$. The limit of $f(x)$ at $x = 0$ is equal to 0, despite the fact that $f(0) = 1$.

Alternatively, suppose that we only allow values of x in the open interval (a, b), and the limit $x \to a+$ (from the right) does exist. Since we have restricted the domain of f to values $x > a$ the limit $x \to a-$ certainly does not exist, and in fact the left-hand endpoint a is not in the domain of f.

We say that such a limit ($x \to a+$) is a *one-sided* limit. If the two one-sided limits do not agree at a particular value of c, then the (two-sided) limit does not exist.

- Limits must be unique.

proof

If

$$\lim_{x\to c} f(x) = L$$
$$\lim_{x\to c} f(x) = M$$

Then $M = L$.

The proof is by contradiction. Suppose $L \neq M$. Let

$$\epsilon = \frac{|L-M|}{10}$$

There is an N_1 such that if $n > N_1$, then $|a_n - L| < \epsilon$.

There is an N_2 such that if $n > N_2$, then $|a_n - M| < \epsilon$.

Let $N = \max(N_1, N_2)$.

If $n > N$ then $|a_n - L| < \epsilon$ and $|a_n - M| < \epsilon$.

By the triangle inequality:

$$|L - M| \leq |a_n - L| + |M - a_n|$$

But also

$$|a_n - L| + |M - a_n| < \frac{2}{10}|L - M|$$

so

$$|L - M| \leq \frac{2}{10}|L - M|$$

which is impossible.

We have reached a contradiction. Therefore, $L = M$.

$\square$

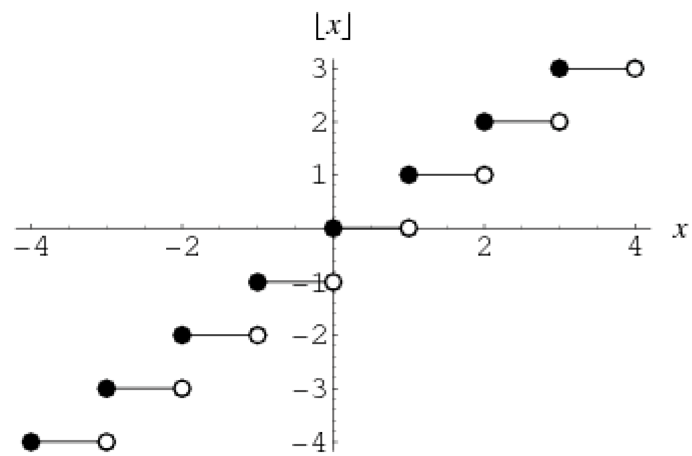
• We allow the existence of a limit as x approaches ∞

$$\lim_{x\to\infty} f(x) = L$$

To define this limit, play the epsilon-delta game (typically, using c instead of δ) and say that if, when $x > c$, $|f(x) - L| < \epsilon$, the limit "at" ∞, or as x *tends to* ∞, exists and has the value L.

example: floor

Consider the "floor" function, which is defined on the real numbers and has the value of the largest integer less than or equal to x.



The floor function has one-sided limits (from the right) at integral values of x, but the limit at $x = 2$, for example, does not exist, because those two one-sided limits are not the same.

example: inverse

Consider the function $f(x) = 1/x$. This function is undefined at $x = 0$ since division by zero is not defined. As x gets close to zero from the right, $1/x$ takes on larger and larger positive values.

Some people will say that limits can have infinite values. In the case of $f(x) = 1/x$, informally, we accept that the limit as $x \to 0+$ exists and has the value ∞. Speaking more formally, we might say that the function "diverges" or "grows without bound".

In any case since for $f(x) = 1/x$

$$\lim_{x \to 0+} \neq \lim_{x \to 0-}$$

so the limit as $x \to 0$ does not exist (abbreviated D.N.E.).

example: sine of 1/x

The trigonometric functions sine and cosine are, of course, periodic. For any value of θ

$$\sin\theta = \sin(\theta \pm 2\pi)$$

The maximum values of the sine function (for $\theta > 0$) occur at

$$\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2} \dots$$

The corresponding maximum values of $\theta = 1/x$ occur at

$$x = \frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \frac{2}{13\pi} \dots$$

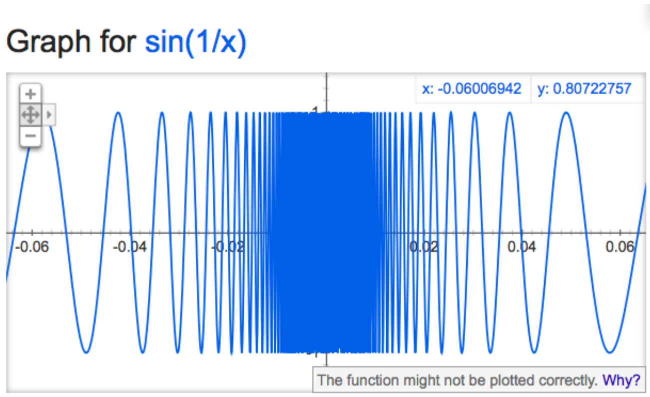
The corresponding decimal values are approximately

$$x = 0.6366, 0.1273, 0.0707, 0.0490$$

As $\pi/2 + 2k\pi$ gets larger, the corresponding values for the inverse get smaller, and more closely spaced together.

Now, $1/x$ grows without bound as $x \to 0$. This means that there is an infinite number of places where the value of the function $\sin(1/x)$ is equal to 1 and indeed, takes on all possible values in its range $[-1, 1]$, and this occurs more and more rapidly as $x \to 0$.

In short, the value oscillates and does so more extremely the closer $x \to 0$.



The limit as $x \Rightarrow 0$ D.N.E.

Calculating limits

The limit of a function $f(x)$ at a point a is written

$$\lim_{x \to a} f(x) = L$$

The formal definition is:

$$\forall\ \epsilon > 0, \exists\ \delta > 0 \mid \forall\ x,$$

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

You tell me the ϵ you require with $|f(x) - L| < \epsilon$, and I will try to find the right δ.

For a typical function, it's a good guess that $L = f(a)$.

$$|f(x) - f(a)| < \epsilon$$

which we can write without the absolute value bars (see Triangle write-up):

$$-\epsilon < f(x) - f(a) < \epsilon$$

example 1

Suppose $f(x) = 3x$ and we're interested in the point $a = 5$. Then set $L = f(a) = 15$.

$$-\epsilon < f(x) - f(a) < \epsilon$$

$$-\epsilon < 3x - 15 < \epsilon$$

$$-\frac{\epsilon}{3} < x - 5 < \frac{\epsilon}{3}$$

If we set $\delta = \epsilon/3$ we'll be good. And in general for a function $f(x) = cx$ with c a constant, at the point a, we can use

$$|x| - a < \frac{\epsilon}{c}$$

example 2

Suppose $f(x) = x^2$ and we're interested in the point $a = 2$. Then set $L = f(a) = a^2 = 4$.

$$-\epsilon < f(x) - f(a) < \epsilon$$

$$-\epsilon < x^2 - a^2 < \epsilon$$

or

$$|x^2 - a^2| < \epsilon$$

Now we argue as follows:

$$|x^2 - a^2| = |(x - a)(x + a)|$$

$$= |x-a|\ |x+a|$$

(The last step follows from $|xy| = |x||y|$ which is true even if $xy < 0$).

To get started suppose we require that *at least*

$$|x-a| < 1$$

$$-1 < x-a < 1$$

$$a-1 < x < a+1$$

$$2a-1 < x+a < 2a+1$$

$$|x+a| < 2a+1$$

Then going back to

$$|x-a|\ |x+a| < \epsilon$$

$$|x-a|\ |x+a| < |x-a|\ (2a+1) < \epsilon$$

and

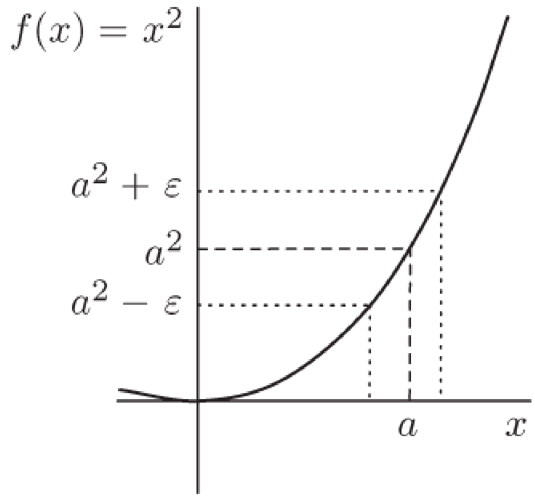
$$|x-a| < \frac{\epsilon}{(2a+1)}$$

Remembering the first condition we set:

$$|x-a| < \min(\frac{\epsilon}{(2a+1)}, 1) = \delta$$

And what we notice is that for $f(x) = x^2$, at least for some a (and depending on the value of ϵ that is chosen), the value of δ required depends on a.

That should not be too surprising.



The same ϵ will require a smaller δ the farther out we go on the curve.

example 3

Now consider the inverse function $f(x) = 1/x$. Suppose we're interested in the point $a = 3$ where we expect the limit to be $L = 1/3$. For this to be true we must guarantee that

$$|\frac{1}{x} - \frac{1}{3}| < \epsilon$$

for arbitrary ϵ.

Factor

$$|\frac{1}{x} - \frac{1}{3}| = |\frac{3-x}{3x}| = \frac{1}{3}\ \frac{1}{|x|}\ |3-x|$$

We showed in the write-up on the triangle inequality that $|a - x| = |x - a|$ so

$$= \frac{1}{3}\ \frac{1}{|x|}\ |x-3|$$

Here, we need to make sure that $|x|$ is not too *small*, so $1/|x|$ is not too large.

First require that $|x-3| < 1$. Then

$$-1 < x - 3 < 1$$

$$2 < x < 4$$
$$\frac{1}{4} < \frac{1}{x} < \frac{1}{2}$$

This means that $1/x > 0$ so

$$\frac{1}{|x|} = \frac{1}{x} < \frac{1}{2}$$

We now have

$$|\frac{1}{x} - \frac{1}{3}| = \frac{1}{3}\ \frac{1}{|x|}\ |3 - x|$$

provided $|x-3| < 1$ and also with this condition $1/|x| < 1/2$ so

$$|\frac{1}{x} - \frac{1}{3}| < \frac{1}{6}\ |x - 3|$$

Hence if $\delta = |x-3| < 6\epsilon$, the above expression is $< \epsilon$ and we're done. Officially we need:

$$|x - 3| < \min\ (6\epsilon, 1)$$

Combining Limits

Assume that

$$\lim_{x\to c} f(x) = L$$
$$\lim_{x\to c} g(x) = M$$

We want to show that

$$\lim_{x \to c} f(x) + g(x) = L + M$$

The limit of the sum is the sum of the limits.

Let $\epsilon > 0$ be arbitrary.

Then the existence of the limits means that

$$\forall\ \epsilon, \exists\ \delta_1 > 0 \mid \forall\ x,\ 0 < |x - c| < \delta_1 \to |f(x) - L| < \epsilon/2$$

and

$$\forall\ \epsilon, \exists\ \delta_2 > 0 \mid \forall\ x,\ 0 < |x - c| < \delta_2 \to |g(x) - M| < \epsilon/2$$

Let

$$\delta = \ \min\ (\delta_1, \delta_2)$$

Now for $|x - c| < \delta$:

$$|f(x) - L + g(x) - M| < \epsilon$$

But by the triangle inequality the left-hand side is

$$|f(x) - L| + |g(x) - M| \leq |f(x) - L + g(x) - M|$$

so

$$|f(x) - L| + |g(x) - M| < \epsilon$$

which proves the theorem.

proof of the product rule for limits

Assume that

$$\lim_{x \to c} f(x) = L$$

$$\lim_{x \to c} g(x) = M$$

We want to show that

$$\lim_{x \to c} f(x) \cdot g(x) = LM$$

The limit of the product is the product of the limits.

We need to show that

$$f(x) \cdot g(x) - LM$$

is small.

Subtract $Lg(x)$ and add it back

$$\begin{aligned} f(x) \cdot g(x) - LM &= f(x) \cdot g(x) - Lg(x) + Lg(x) - LM \\ &= (f(x) - L)g(x) + L(g(x) - M) \end{aligned}$$

Take the absolute value on both sides

$$|f(x) \cdot g(x) - LM| = |(f(x) - L) \cdot g(x) + L \cdot (g(x) - M)|$$

Use the triangle inequality to split up the sum:

$$\leq |(f(x) - L) \cdot g(x)| + |L \cdot (g(x) - M)|$$

This can be further massaged to

$$= |f(x) - L| \cdot |g(x)| + |L| \cdot |g(x) - M|$$

Write the whole thing:

$$|f(x) \cdot g(x) - LM| \leq |(f(x) - L)||g(x)| + |L||(g(x) - M)|$$

Now, play the epsilon-delta game: you pick ϵ and then I concentrate on a region so close to c that

$$|f(x) - L| < \epsilon$$

and

$$|g(x) - M| < \epsilon$$

If your epsilon is too large it would mess things up (why?), so in that case I will pick $|g(x) - M| = 1$.

Then I have

$$|f(x) - L| < \epsilon$$
$$|g(x) - M| < \epsilon$$
$$|g(x)| < |M| + 1$$

Go back to the equation we obtained above

$$|f(x) \cdot g(x) - LM| \leq |(f(x) - L)||g(x)| + |L||(g(x) - M)|$$

substitute on the right-hand side

$$\begin{aligned} &|(f(x) - L)||g(x)| + |L||(g(x) - M)| \\ &\leq \epsilon\ (|M| + 1) + |L|\ \epsilon \\ &\leq \epsilon\ (|M| + |L| + 1) \end{aligned}$$

That is:

$$|f(x) \cdot g(x) - LM| \leq \epsilon\ (|M| + |L| + 1)$$

as Adrian Banner says in *Calculus Lifesaver*:

> That's almost what I want! I was supposed to get ϵ on the right-hand side, but I got an extra factor of $|M|+|L|+1$. This is no problem—you just have to allow me to make my move again, but this time I'll make sure that $|f(x) - L|$ is no more than $\epsilon/2(|M| + |L| + 1)$, and similarly for $|g(x) - M|$. Then when I replay all the steps, ϵ will be replaced by $\epsilon/(|M| + |L| + 1)$, and at the very last step, the factor $|M| + |L| + 1$ will cancel out and we'll just get our ϵ. So we have proved the result.

More formal proof of the product rule

Suppose that

$$\lim_{x \to c} f(x) = L$$

$$\lim_{x \to c} g(x) = M$$

To prove:

$$\lim_{x \to c} f(x) \cdot g(x) = LM$$

proof

Let $\epsilon > 0$. By the definition of limits we can find three numbers δ_1, δ_2 and δ_3 such that

if $0 < |x - c| < \delta_1$:

$$|f(x) - L| < \frac{\epsilon}{2(1 + |M|)}$$

if $0 < |x - c| < \delta_2$:

$$|g(x) - M| < \frac{\epsilon}{2(1 + |L|)}$$

and third, if $0 < |x - c| < \delta_3$:

$$|g(x) - M| < 1$$

Now write

$$|g(x)| = |g(x) - M + M|$$

use the triangle inequality

$$|g(x)| \leq |g(x) - M| + |M|$$

Then according to (3), if if $0 < |x - c| < \delta_3$:

$$|g(x) \le 1 + |M|$$

Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.

Then if $0 < |x - c| < \delta$:

$$|f(x) \cdot g(x) - LM| = |f(x) \cdot g(x) - L \cdot g(x) + L \cdot g(x) - LM|$$

by the triangle inequality (again)

$$\le |f(x) \cdot g(x) - L \cdot g(x)| + |L \cdot g(x) - LM|$$

The next step is to factor (see below):

$$\le |g(x)||f(x) - L| + |L||g(x) - M|$$

$$< (1 + |M|)\ \frac{\epsilon}{2(1 + |M|)} + (1 + |L|)\ \frac{\epsilon}{2(1 + |L|)}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

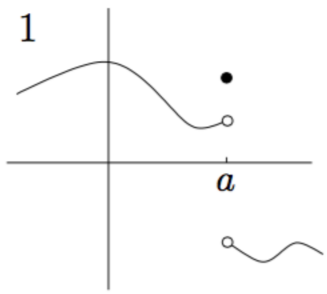
$$|f(x) \cdot g(x) - LM| < \epsilon$$

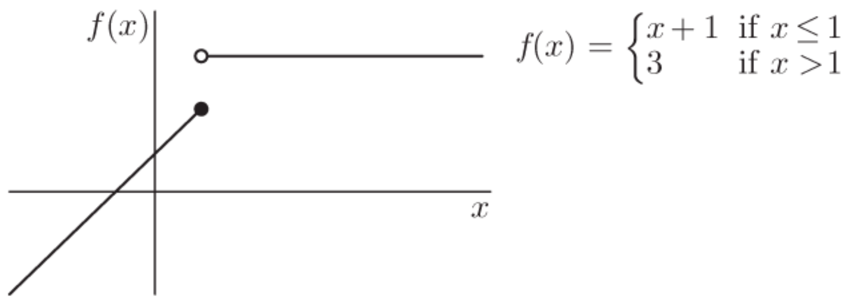
which completes the proof.

Continuity

Continuity has an intuitive definition: if we can graph a function *without lifting our pencil from the paper*, then the function is continuous.

Here are some graphs showing examples of how continuity can fail.





For a function to be continuous at a point $x = c$, we imagine that if we vary x in neighborhood of c, then $f(x)$ should not change in value by too much.

Again, we will call that value L, the limit of $f(x)$ as $x \to c$. For L to exist we require that the two one-sided limits be equal.

In addition, it must also be true that $f(c) = L$.

fancy definitions

If we had not previously developed the concept of a limit, we might proceed as follows: a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $c \in \mathbb{R}$ if and only if

$$\forall\ \epsilon > 0\ \exists\ \delta > 0 \ \text{ such that, if } |x - c| < \delta, \text{ then } |f(x) - f(c)| < \epsilon$$

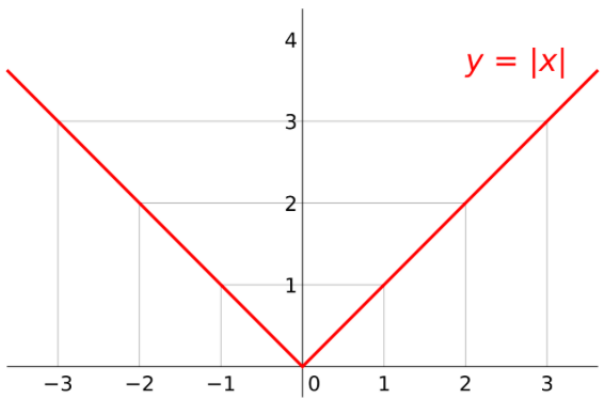
Above we talked about functions, here is a definition that involves sequences.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $L \in \mathbb{R}$. We say that f is continuous at L, if, whenever (a_n) is a sequence that converges to L, then the sequence (b_n) defined by $f(a_n)$ also converges and its limit is equal to $f(L)$. We say that f is continuous if it is continuous at all $L \in \mathbb{R}$.

example: absolute value

An algebraic definition of the absolute value function is piecewise:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



The function $f(x) = |x|$ is continuous at $x = 0$ because the two one-sided limits exist and are equal to each other. They are also equal to $f(0) = 0$.

example: constant

Suppose $f(x) = a$ for some real number a. Then no matter what ϵ is chosen and no matter what real number c is chosen

$$f(x) = f(c) = a$$

so

$$|f(x) - f(c)| < \epsilon$$

example: x

Suppose $f(x) = x$.

$$\lim_{x \to c} f(x) = c = f(c)$$

so f is continuous at c.

Or choose $\delta = \epsilon$. Then, if $|x - c| < \delta$

$$c - \delta < x < c + \delta$$

$$f(x) = x$$

$$f(x) - c < \delta = \epsilon$$

$$|f(x) - c| < \epsilon$$

so f is continuous at c

example: constant factor

Suppose $f(x) = cx$ $(c \in \mathbb{R})$. Use the $\epsilon - \delta$ game to prove that f is continuous.

Proof: the function "stretches" x by a factor of c. Hence δ will also be stretched. Set $\delta = \epsilon/c$. Then, if $|x-a| < \delta$ we have

$$|f(x) - f(a)| = |cx - ca| = c|x-a| < c\delta = c\epsilon/c = \epsilon$$

Hence f is continuous at every $a \in \mathbb{R}$.

example: product rule

How to prove that $f(x) = x^2$ is continuous? One way is to try adjusting δ based on the value of a (e.g. min $(|\sqrt{a^2+\epsilon} \pm a|)$), but a better way is to invoke the product rule.

If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both continuous at $a \in \mathbb{R}$, then fg is continuous at a.

First, prove $f(x) = x$ is continuous. Then define $f(x) = g(x) = x$. So $x^2 = f(x)g(x)$ in continuous.

By induction then, all powers $f(x) = x^n$ are continuous.

proof of the product rule for continuity

Let f and g be functions defined on an open subset of $\mathbb{R}$. We have that f and g are both continuous at c which means that

$$\lim_{x\to c} f(x) = L$$

$$\lim_{x\to c} g(x) = M$$

Then

$$\lim_{x\to c} f(x) \cdot g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = LM$$

To obtain this result we have used the product rule for limits.

example: inverse

Consider the function $f(x) = 1/x$. Above we pointed out that this function is undefined at $x = 0$ since division by zero is not defined. But there is nothing to stop us from defining the function piecewise, like so:

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

As x gets close to zero from the right, $1/x$ continues to take on larger and larger positive values, but then dives to 0 at $x = 0$ and then further dives toward $-\infty$ as we pass to the left of zero.

Trick question: give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is not continuous at zero. If you said the inverse, that is not correct. The reason is that the inverse is not $f : \mathbb{R} \to \mathbb{R}$ because *it is not defined at zero.*

On the other hand, $f(x) = \sin(x)$ *is* $f : \mathbb{R} \to \mathbb{R}$ even though the values actually output by the function are $f(x) \in [-1, 1]$. That interval is the *image* of the function, any codomain so that the image is contained in the codomain will do.

Chapter 174

FTC proof

David Joyce has proofs of the two statements of the FTC

http://aleph0.clarku.edu/~ma120/FTCproof.pdf

which we will follow here.

The two statements are often called FTC-1 and FTC-2. Citing historical precedent, Joyce calls the second one the FTC and the first its inverse, or FTC^{-1}.

FTC

The FTC is what we use when we evaluate definite integrals. If F is an antiderivative of f, then:

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

We will require that f be *continuous* on $[a, b]$. Strictly speaking, this isn't necessary, but it makes the proof simpler. For a function with a finite number of discontinuities, one can just chop up the integral into its component pieces.

For the inverse statement (FTC^{-1}), we require again that f be continuous on $[a, b]$ and F be the accumulation function defined by

$$F(x) = \int_a^x f(t)\ dt$$

Then the theorem is that F is differentiable on $[a, b]$ and its derivative is f. That is

$$F'(x) = f(x) \quad \text{for } x \in [a, b]$$

This is usually written

$$\frac{d}{dx} \int_a^x f(t)\ dt = f(x)$$

We have adopted the "dummy" variable t to avoid confusion.

proof of the inverse FTC

We start with the inverse theorem. First of all, since f is continuous, it is integrable, so we know that the integral

$$F(x) = \int_a^x f(t)\ dt$$

actually exists.

We need to show that $F'(x) = f(x)$.

We go back to the definition of the derivative

$$F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\ F(x + h) - F(x)\ \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\int_a^{x+h} f(t)\ dt) - \int_a^x f(t)\ dt \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\ dt$$

We will show that this limit equals $f(x)$ We will only prove the case where $h > 0$. The other proof is similar but has minus signs in various places.

On the interval $[x, x+h]$, $f(t)$ has a minimum value m and a maximum value M (by the extreme value theorem). So

$$m \leq f(t) \leq M$$

for every $x \in [a, b]$, and when we integrate each term of the inequality we get

$$\int_x^{x+h} m\ dt \leq \int_x^{x+h} f(t)\ dt \leq \int_x^{x+h} M\ dt$$

Since m and M are constants and $\int dt = h$ between these limits:

$$hm \leq \int_x^{x+h} f(t)\ dt \leq hM$$

dividing through by h

$$m \leq \frac{1}{h} \int_x^{x+h} f(t)\ dt \leq M$$

Now, as $h \to 0$, all values of f on the interval $[x, x+h]$ approach the same value, and in particular, $m \to f(x)$ and $M \to f(x)$. Being squeezed between them

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) = f(x)$$

□

proof of the FTC

Let
$$G(x) = \int_a^x F'(t)\ dt$$
Take derivatives on both sides
$$G'(x) = \frac{d}{dx}\ \int_a^x F'(t)\ dt$$
so
$$G'(x) = F'(x)$$
by the theorem we just proved.

Therefore $G(x)$ and $F(x)$ differ at most by a constant

$$G(x) = F(x) + C$$

for $x \in [a, b]$.

In particular, at $x = a$ we have

$$G(a) = F(a) + C$$

but
$$G(a) = \int_a^a F'(t)\ dt = 0$$
Hence
$$F(a) = -C$$
At $x = b$ we have
$$G(b) = F(b) + C$$
but $C = -F(a)$ so
$$G(b) = F(b) - F(a)$$

By the original definition of G

$$G(b) = \int_a^b F'(t)\ dt$$

Hence

$$\int_a^b F'(t)\ dt = F(b) - F(a)$$

□

Chapter 175

More Euler proofs

additional proofs

Here are sketches of two different derivations of Euler's famous formula, both following Dunham's book about Euler. He starts by simply admiring the formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

If $\theta = \pi$, we have

$$e^{i\pi} = -1 + 0$$

$$e^{i\pi} + 1 = 0$$

(what Feynman called "our jewel").

Dunham says in a video I saw that, if we were going to have a math party, we would invite these five numbers: 0 for arithmetic (additive identity), 1 for multiplication (multiplicative identity), π for geometry, e for calculus, and i for complex functions.

preliminary

Using x is a bit simpler notation, so that's what I'll do here

$$e^{ix} = \cos x + i \sin x$$

Start with the definition of i

$$i = \sqrt{-1}$$

Simple identities that come from it are:

$$i^2 = -1$$

$$-i^2 = 1$$

$$\frac{u}{i} = -iu$$

Having i gives us new factorizations like

$$a^2 + b^2 = (a + bi)(a - bi)$$

since the terms with $\pm\ abi$ cancel and $-i^2 = 1$. So

$$1 = \cos^2 x + \sin^2 x$$

$$1 = (\cos x + i \sin x)(\cos x - i \sin x)$$

Of course, we could switch sine and cosine here, but this is the convention.

derivation one

Start with the inverse sine function:

$$x = \sin^{-1} y$$

$$y = \sin x$$
$$dy = \cos x \ dx$$

Then we can make a trig substitution and say that the side adjacent to x is $\sqrt{1-y^2}$ and so

$$\cos x = \sqrt{1-y^2}$$

We're interested in the integral

$$\int \frac{1}{\sqrt{1-y^2}} \ dy$$

which is just

$$= \int \frac{1}{\cos x} \ \cos x \ dx = x$$

Now, Euler makes a complex change of variable

$$y = iz$$
$$\frac{1}{1-y^2} = \frac{1}{1+z^2}$$

So

$$x = \int \frac{1}{\sqrt{1-y^2}} \ dy = \int \frac{1}{\sqrt{1+z^2}} \ i \ dz$$

we have converted the integral to having a plus sign under the square root and the answer is

$$= i \ln \left(\sqrt{1+z^2} + z\right)$$

This follows from a standard trig substitution but it's a bit complicated. It can be checked by differentiating. The derivative is i times

$$\frac{1}{\sqrt{1+z^2}+z} \ \left(\frac{z}{\sqrt{1+z^2}} + 1\right)$$

$$= \frac{1}{\sqrt{1+z^2}+z} \ (\frac{z+\sqrt{1+z^2}}{\sqrt{1+z^2}})$$

$$= \frac{1}{\sqrt{1+z^2}}$$

Now, just undo the substitution:

$$z = \frac{y}{i} = \frac{\sin x}{i}$$

$$\sqrt{1+z^2} = \sqrt{1-y^2} = \cos x$$

Hence our previous result

$$x = i \ln\ (\sqrt{1+z^2}+z)$$

is equivalent to

$$x = i \ln\ (\cos x + \frac{\sin x}{i})$$

Recall our two identities involving i. The first one was

$$\frac{u}{i} = -iu$$

So:

$$x = i \ln\ (\cos x + \frac{\sin x}{i})$$

$$= i \ln\ (\cos x - i \sin x)$$

$$ix = -\ln\ (\cos x - i \sin x)$$

$$= \ln\ \frac{1}{(\cos x - i \sin x)}$$

Using the factorization given at the top

$$\frac{1}{\cos u - i \sin u} = \cos u + i \sin u$$

We have that

$$ix = \ln\ \frac{1}{(\cos x - i\sin x)} = \ln\ (\cos x + i\sin x)$$

Exponentiate:

$$e^{ix} = \cos x + i\sin x$$

derivation two

Suppose we try this multiplication:

$$(\cos s + i\sin s)(\cos t + i\sin t)$$

$$= \cos s\cos t + i\sin s\cos t + i\cos s\sin t - \sin s\sin t$$
$$= (\cos s\cos t - \sin s\sin t) + i(\sin s\cos t + \cos s\sin t)$$
$$= \cos(s+t) + i\sin(s+t)$$

set $s = t$ and recall what we started with

$$(\cos s + i\sin s)^2 = \cos 2s + i\sin 2s$$

In fact, Euler showed that this is true for fractional n but I'll assume that part.

$$(\cos s + i\sin s)^n = \cos ns + i\sin ns$$

Now multiply the difference rather than the sum:

$$(\cos s - i\sin s)(\cos t - i\sin t)$$

$$= (\cos s\cos t - \sin s\sin t) - i(\sin s\cos t + \sin t\cos s)$$
$$= \cos(s+t) - i(\sin(s+t))$$

again, with $s = t$

$$(\cos s - i \sin s)^2 = \cos 2s - i \sin 2s$$

$$(\cos s - i \sin s)^n = \cos ns - i \sin ns$$

Restate the two results:

$$(\cos s + i \sin s)^n = \cos ns + i \sin ns$$

$$(\cos s - i \sin s)^n = \cos ns - i \sin ns$$

Add them

$$2 \cos ns = (\cos s + i \sin s)^n + (\cos s - i \sin s)^n$$

where the magic happens

Let

$$s = \frac{x}{n}$$

As $n \to \infty$, $s \to 0$, and

$$\sin s \to s$$

(by the famous limit from trigonometry)

$$\cos s \to 1$$

$$\cos x = \cos ns$$

We had

$$2 \cos ns = (\cos s + i \sin s)^n + (\cos s - i \sin s)^n$$

which becomes

$$2 \cos x = (1 + is)^n + (1 - is)^n$$

$$= (1 + \frac{ix}{n})^n + (1 - \frac{ix}{n})^n$$

but from the standard limit developed in looking at the exponential

$$e^{ix} = \lim_{n\to\infty}(1 + \frac{ix}{n})^n$$

hence

$$2\cos x = e^{ix} + e^{-ix}$$

We can just integrate this to obtain

$$2i\sin x = e^{ix} - e^{-ix}$$

Or by very similar manipulation to what's in the first part we can also obtain an expression for the sine:

$$2i\sin(ns) = (\cos s + i\sin s)^n - (\cos s - i\sin s)^n$$

which will lead to the same thing

$$2i\sin x = (e^{ix} - e^{-ix})$$

Adding together

$$2(\cos x + i\sin x) = e^{ix} + e^{-ix} + e^{ix} - e^{-ix}$$

$$\cos x + i\sin x = e^{ix}$$

check

Before we stop, we can check the formula. One way is to notice the connection between infinite series expansions for e^x:

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\dots$$

sine:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

and cosine:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots$$

These can almost be added together to give what we seek, except for the problem of the alternating signs. What happens with e^{ix}?

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} \dots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} \dots \end{aligned}$$

The pattern is

$$\sum_{n=0}^{\infty} i^n = 1 + i - 1 - i + 1 \dots$$

And we're there. We just have to recognize that the pattern with e^{ix} has $i \sin x$ so as we said

$$e^{ix} = \cos x + i \sin x$$

Chapter 176

References

- Auroux. *Multivariable Calculus*

https://ocw.mit.edu/courses/mathematics/18-02-multivariable-calculus-fall-2007/

- Acheson. *The Calculus Story.*
- Adams, Thompson, Hass. *How to Ace the Rest of Calculus.*
- Alcock. *Mathematics Rebooted.*
- Banner. *The Calculus Lifesaver.*
- Colley. *Vector Calculus.*
- Corral. *Vector Calculus.*
- Courant, Robbins, Stewart. *What is Mathematics?.*
- Dunham. *Euler: The Master of Us All.*
- Dunham. *Journey Through Genius.*
- Dunham. *The Mathematical Universe.*
- Feynman. *The Character of Physical Law.*

• Fowler, Snapp. *Mooculus.*

• Goodstein, Goodstein. *Feynman's Lost Lecture.*

• Hamming. *Methods of Mathematics Applied to Calculus, Probability, and Statistics.*

• Kline. *Calculus.*

• Lockhart. *Measurement.*

• Maor. *e, the Story of a Number.*

• Maor. *To Infinity and Beyond.*

• Marsden, Tromba. *Vector Calculus.*

• Morin. *Probability.*

• Nahin. *When least is best.*

• Savov. *No bullshit guide to math and physics.*

http://www.minireference.com

• Schey. *Div, grad, curl*

• Shankar. *Fundamentals of Physics, vol I and II.*

• Spivak. *The Hitchhiker's Guide to Calculus.*

• Stewart. *Significant Figures.*

• Strang. *Calculus.*

• Strogatz. *The Joy of x.*

• Thompson. *Calculus Made Easy.*

• Varberg, Purcell, Rigdon. *Calculus with Differential Equations.*