Basics of the hyperbolic functions

Recalling Euler's formula:

$$e^{ix} = \cos x + i \sin x$$

we obtain a similar formula with $-ix$:

$$e^{-ix} = \cos(-x) + i\ \sin(-x) = \cos x - i\ \sin x$$

Adding and subtracting

$$e^{ix} + e^{-ix} = 2 \cos x$$

$$e^{ix} - e^{-ix} = -2\ i \sin x$$

The hyperbolic functions are defined similarly, but without i:

$$2 \cosh x = e^{x} + e^{-x}$$

$$2 \sinh x = e^{x} - e^{-x}$$

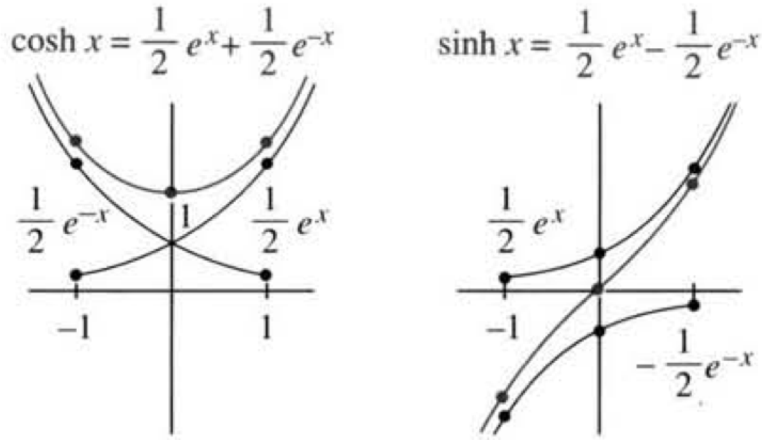

Fig. 6.18 Cosh x and sinh x. The hyperbolic functions combine $\frac{1}{2}e^x$ and $\frac{1}{2}e^{-x}$.

The difference of squares has a simple value:

$$\cosh^2 t - \sinh^2 t = 1$$

Everything about the hyperbolic sine is reminiscent of the regular trig functions but with a sign change.

A plot of $\sinh t$ on the x-axis and $\cosh t$ on the y-axis yields a hyperbola in the same way the $y^2 - x^2 = 1$ does.

derivatives

$$\frac{d}{dx}2\sinh x = \frac{d}{dx}(e^x - e^{-x}) = e^x + e^{-x} = 2\cosh x$$
$$\frac{d}{dx}2\cosh x = \frac{d}{dx}(e^x + e^{-x}) = e^x - e^{-x} = 2\sinh x$$

Also, note that:

$$2\sinh x + 2\cosh x = 2e^x$$
$$e^x = \sinh x + \cosh x$$

Because of this, and by symmetry, we expect that the Taylor series expansions should be

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

The values of the functions at zero are

$$\sinh 0 = 0$$

$$\cosh 0 = 1$$

So, for example, the expansion for cosh is

$$\sinh x = \sum_{n=0}^{\infty} \frac{f^n(0)\ x^n}{n!}$$

$$= \frac{0 \cdot 1}{0!} + \frac{1 \cdot x}{1!} + \frac{0 \cdot x^2}{2!} + \frac{1 \cdot x^3}{3!} + \dots$$

and so on.

relativity

The hyperbolic functions come into the mathematics of relativity, where for an observer in a moving reference frame, the following equations hold:

$$x' = \frac{x - vt}{\sqrt{1 - v^2}}$$

$$t' = \frac{t - vx}{\sqrt{1 - v^2}}$$

The quantity s^2 is invariant where

$$s^2 = t^2 - x^2$$

Proof:

$$x'^2 = \frac{x^2 - 2xvt + v^2t^2}{1 - v^2}$$

$$t'^2 = \frac{t^2 - 2xvt + v^2x^2}{1 - v^2}$$

$$t'^2 - x'^2 = \frac{t^2 - x^2 + v^2x^2 - v^2t^2}{1 - v^2}$$

$$= \frac{t^2 - x^2 + v^2(x^2 - t^2)}{1 - v^2}$$

$$= \frac{t^2 - x^2 - v^2(t^2 - x^2)}{1 - v^2} = t^2 - x^2$$

The hyperbolic functions come in by defining a parameter θ (the "rapidity")

$$\cosh\theta = \frac{1}{\sqrt{1 - v^2}}$$

Then

$$\sinh^2\theta = \cosh^2\theta - 1 = \frac{1}{1 - v^2} - 1 = \frac{v^2}{1 - v^2}$$

$$\sinh\theta = \frac{v}{\sqrt{1 - v^2}}$$

So we can rewrite

$$x' = \frac{x - vt}{\sqrt{1 - v^2}} = x\cosh\theta - t\sinh\theta$$

$$t' = \frac{t - vx}{\sqrt{1 - v^2}} = t\cosh\theta - x\sinh\theta$$

And our identity from above is

$$t'^2 - x'^2 = (t^2\cosh^2\theta - 2xt\sinh\theta\cosh\theta + x^2\sinh^2\theta)$$

$$-(x^2\cosh^2\theta - 2xt\sinh\theta\cosh\theta + t^2\sinh^2\theta)$$

the terms starting with $2xt$ cancel and we have

$$\begin{aligned}
t'^2 - x'^2 &= (t^2\cosh^2\theta + x^2\sinh^2\theta - x^2\cosh^2\theta - t^2\sinh^2\theta) \\
&= t^2(\cosh^2\theta - \sinh^2\theta) - x^2(\cosh^2\theta - \sinh^2\theta) \\
&= t^2 - x^2
\end{aligned}$$

$\tanh\theta$

We had

$$\begin{aligned}
\sinh\theta &= \frac{v}{\sqrt{1-v^2}} \\
\cosh\theta &= \frac{1}{\sqrt{1-v^2}}
\end{aligned}$$

so

$$\tanh\theta = v$$

leading us to explore the properties of the hyperbolic tangent. Going back to the beginning:

$$\begin{aligned}
2\sinh\theta &= e^\theta - e^{-\theta} \\
2\cosh\theta &= e^\theta + e^{-\theta} \\
\tanh\theta &= \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}
\end{aligned}$$

The derivative is (by the quotient rule):

$$\begin{aligned}
\frac{d}{d\theta}\ \tanh\theta &= \frac{\cosh^2\theta - \sinh^2\theta}{\cosh^2\theta} \\
&= \frac{1}{\cosh^2\theta}
\end{aligned}$$

Shankar has a problem involving two angles

$$\begin{aligned}
2\sinh(\theta+\phi) &= e^{\theta+\phi} - e^{-\theta-\phi} \\
&= e^\theta e^\phi - e^{-\theta}e^{-\phi}
\end{aligned}$$