Advanced techniques for Integrals

Shankar has this example for the volume of a sphere of radius R. We compute the volume for the part of the hemisphere that is in the first quadrant. The surface is $f(x,y) = \sqrt{R^2 - x^2 - y^2}$ so the integral is

$$\int_0^R \int_0^{\sqrt{R^2-y^2}} \sqrt{R^2 - x^2 - y^2}\ dx\ dy$$

There are several ways to approach this but what he does is an unusual trig substitution where the hypotenuse is $\sqrt{R^2 - y^2}$. The substitution is

$$x = \sqrt{R^2 - y^2} \sin\theta$$
$$dx = \sqrt{R^2 - y^2} \cos\theta\ d\theta$$

(This integral is over the values of x with y *fixed*, it may be treated as a constant.) The integrand is

$$\sqrt{R^2 - x^2 - y^2} = \sqrt{R^2 - y^2} \cos\theta$$

When $x = 0 \rightarrow \theta = 0$, and when $x = \sqrt{R^2 - y^2} \rightarrow \theta = \pi/2$ so the inner integral becomes

$$\int_0^{\pi/2} \sqrt{R^2 - y^2} \cos\theta \sqrt{R^2 - y^2} \cos\theta\ d\theta$$

$$R^2 - y^2 \int_0^{\pi/2} \cos^2\theta\ d\theta$$

We've done this one many times. One form of the answer is

$$\frac{1}{2}(\theta - \sin\theta\cos\theta)$$

For any interval involving $n\pi/2, n \in \{0, 1, 2 \dots\}$, the second term disappears. So the result for $\theta = 0 \to \pi/2$ is just $\pi/4$.

The outer integral becomes then

$$\frac{\pi}{4}\int_0^R (R^2 - y^2)\ dy$$

$$= \frac{\pi}{4}(R^2 y - \frac{y^3}{3})\Big|_0^R = \frac{\pi}{4}\ \frac{2}{3}R^3$$

But there are eight such regions, so the total volume is

$$8\ \frac{\pi}{4}\ \frac{2}{3}R^3 = \frac{4}{3}\pi R^3$$

The point of this example is that use of $\sqrt{R^2 - y^2}$ as the hypotenuse is useful and allowed because y is constant, and it simplifies the bounds as well as the integral itself.

Gaussian integral

We've worked this out elsewhere, but let's take a look at the following, called the Gaussian integral:

$$\int_0^\infty e^{-ax^2}\ dx$$

Shankar calls this $I_0(a)$, that is, it is a function of a:

$$I_0(a) = \int_0^\infty e^{-ax^2}\ dx$$

and it is $I_0(a)$ because it is the zeroth version of a family of functions

$$I_n(a) = \int_0^\infty x^n e^{-ax^2}\ dx$$

For the version

$$I_0(a) = \int_0^\infty e^{-ax^2}\ dx$$

$$2I_0(a) = \int_{-\infty}^\infty e^{-ax^2}\ dx$$

there is a special trick to convert this to polar coordinates

$$4I_0(a)^2 = \int_{-\infty}^\infty e^{-ax^2}\ dx \int_{-\infty}^\infty e^{-ay^2}\ dy$$

Since the bounds of the x and y integrals are independent, as are the integrals themselves, we can turn this into the double integral

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-ax^2} e^{-ay^2}\ dx\ dy$$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-a(x^2+y^2)}\ dx\ dy$$

which can be converted to polar coordinates as

$$= \int_0^{2\pi} \int_{-\infty}^\infty e^{-ar^2}\ r\ dr\ d\theta$$

$$= 2\pi \int_0^\infty e^{-ar^2}\ r\ dr$$

$$= 2\pi\ (-\frac{1}{2a})\ (e^{-ar^2}\ \Big|_0^\infty)$$

At the upper limit we get 0, and at the lower limit we get

$$e^{-\infty} - e^0 = 0 - 1 = -1$$

so the minus signs and the factors of two cancel and we have

$$4I_0(a)^2 = \frac{\pi}{a}$$

$$I_0(a) = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

For the rest of the family, the first one is easy

$$I_1(a) = \int_0^\infty xe^{-ax^2}\ dx$$

$$= -\frac{1}{2a}(e^{-ax^2}\ \Big|_0^\infty) = \frac{1}{2a}$$

We get an approach to the second one in the following way. Start with

$$I_0(a) = \int_0^\infty e^{-ax^2}\ dx$$

Differentiate with respect to a!

$$\frac{d}{da}\int_0^\infty e^{-ax^2}\ dx = \int_0^\infty \frac{\partial}{\partial a}e^{-ax^2}\ dx$$

$$= \int_0^\infty -x^2e^{-ax^2}\ dx$$

For the left-hand side, we have

$$\frac{d}{da}\ \frac{1}{2}\sqrt{\frac{\pi}{a}} = -\frac{\sqrt{\pi}}{4}\frac{1}{a\sqrt{a}} = \frac{1}{4a}\sqrt{\frac{\pi}{a}}$$

So finally

$$\int_0^\infty x^2e^{-ax^2}\ dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}}$$