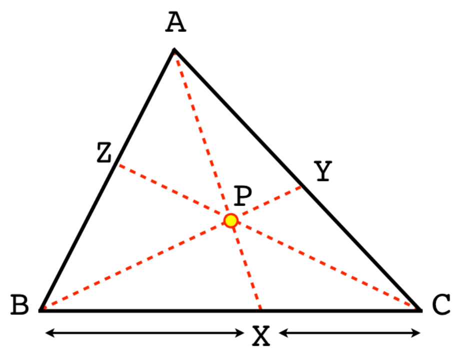
Ceva's Theorem

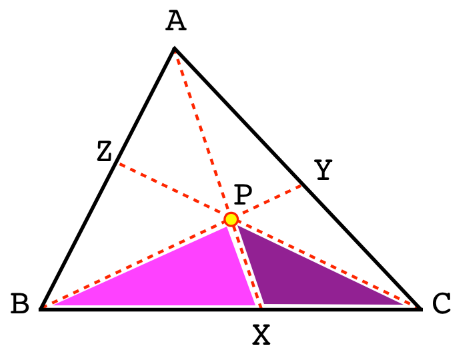
We begin with the triangle shown below, picking a point P to be any point inside the triangle. Now draw line segments from each vertex through P and extend them to the opposing side.



Since P can be anywhere, the ratio can be anything. Let's call it x.

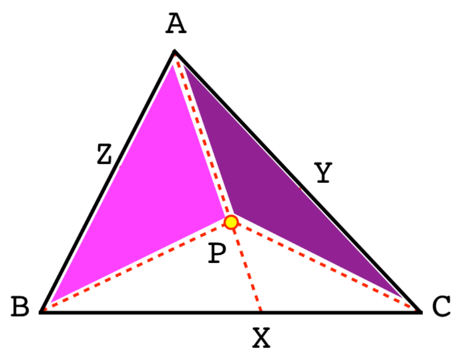
$$BX/XC = x$$

Line AX divides the whole triangle into two parts. We know that the area of $\triangle ABX$ is in the same proportion to the area of $\triangle ACX$ as x, because they share the same height, while x is the ratio of their bases. Now consider the lower pair of triangles $\triangle BPX$ and $\triangle CPX$



These two also have their areas in the ratio x, for the same reason.

The two triangles formed by the difference between these two triangles ($\triangle ABP$ and $\triangle ACP$) are shown here



Again, the same statement is true, and for the same reason. So, altogether, we have that

$$\frac{|ABX|}{|ACX|} = \frac{|BPX|}{|CPX|} = \frac{|ABP|}{|ACP|} = x$$

By the same reasoning, if $y = CY/YA$

$$\frac{|BCP|}{|ABP|} = y$$

and if $z = AZ/ZB$

$$\frac{|ACP|}{|BCP|} = z$$

Then

$$xyz = \frac{|ABP|}{|ACP|}\ \frac{|BCP|}{|ABP|}\ \frac{|ACP|}{|BCP|}$$

But all terms cancel, so

$$xyz = 1$$

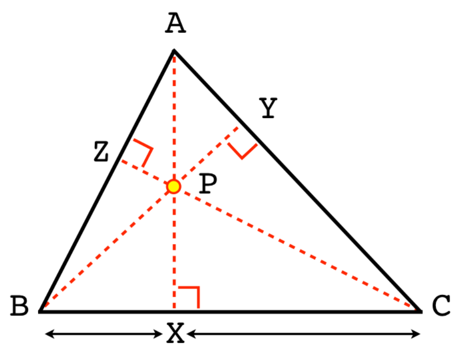
And this is of course true not just for the areas but for the original line segments

$$xyz = \frac{BX}{XC}\ \frac{CY}{YA}\ \frac{AZ}{ZB}$$

This proof also works in reverse,

$$xyz = 1 \iff \text{3 lines cross at point P}$$

So now, for this triangle



if α is the angle at vertex A and so on, then for example,

$$BX = AB\ cos\beta$$

and

$$\frac{BX}{XC} = \frac{AB\ cos\beta}{AC\ cos\gamma}$$
$$\frac{CY}{YA} = \frac{BC\ cos\gamma}{AB\ cos\alpha}$$
$$\frac{AZ}{ZB} = \frac{AC\ cos\alpha}{BC\ cos\beta}$$

When we construct this ratio, all the terms cancel.

$$\frac{ABcos\beta}{AC\ cos\gamma}\ \frac{BC\ cos\gamma}{AB\ cos\alpha}\ \frac{AC\ cos\alpha}{BC\ cos\beta} = 1$$

That means

$$\frac{BX}{XC}\ \frac{CY}{YA}\ \frac{AZ}{ZB} = 1$$

Therefore, the 3 altitudes all cross at a single point. That point is the orthocenter.