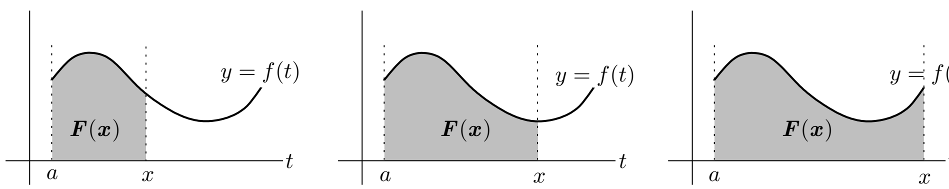
Our second generalization is that the integrand doesn't have to be t^2. It can be any continuous function of t. Let's suppose the integrand is $f(t)$. If a is some constant number, then let's define

$$F(x) = \int_a^x f(t)\,dt.$$

For example, if $a = 0$ and $f(t) = t^2$, you get the original function F from above. In general, for any number x, the value $F(x)$ is the signed area (in square units) between the curve $y = f(t)$, the t-axis, and the vertical lines $t = a$ and $t = x$. Here is an example of what this might look like for three different values of x:



The above pictures are reminiscent of a curtain with fixed left-hand edge, while the right-hand edge slides back and forth. The only unrealistic aspect is that the curtain rod at the top is pretty warped, unless the function f is constant! In any case, note that the function F comes directly from the choice of the integrand $f(t)$ and the number a. By splitting up the integral, you can show that changing the number a just changes the function F by a constant. All these ideas will be very important in the next couple of sections. . . .

17.2 The First Fundamental Theorem

Here's the goal: find

$$\int_a^b f(x)\,dx$$

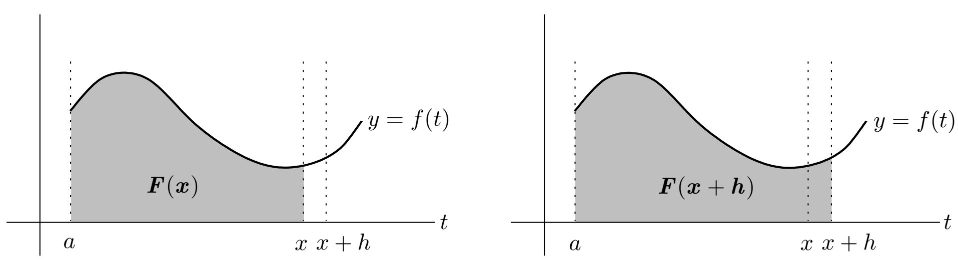
without using Riemann sums. Let's do three things which are not really obvious at all:

1. First, let's change the dummy variable to t and write the above integral as
$$\int_a^b f(t)\,dt.$$
As we saw in the previous section, this doesn't make any difference—the name of the dummy variable doesn't matter.
2. Now, let's replace b by a variable x to get a new function F, defined like this:
$$F(x) = \int_a^x f(t)\,dt.$$
This is exactly the sort of function that we looked at in the previous section. Eventually we're going to want the value of $F(b)$, which is

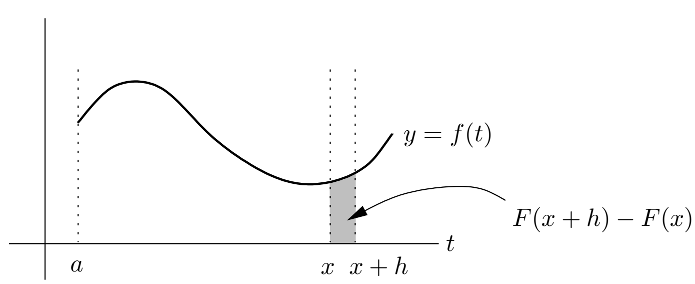
exactly the integral in step 1 above, but first let's see what we can understand about F in general.

3. So we have this new function F. It's like a brand new shiny toy to play with. Since we've spent so much time differentiating functions, let's try differentiating this one with respect to the variable x. That is, we consider
$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt.$$
Understanding the nature of $F'(x)$ will allow us to find $F(x)$ in general. Once we've done that, we can find $F(b)$, which is exactly the integral we want.

The expression
$$\frac{d}{dx}\int_a^x f(t)\,dt$$
might just about be the weirdest thing we've looked at so far in this book. Let's see how to unravel it. Pick your favorite number x and find $F(x)$. Then wobble x a little bit—let's move it to $x + h$, where h is a small number. So now our function value is $F(x + h)$. Here's a picture of the situation:



As you can see, x and $x + h$ are pretty close to each other. The values of $F(x)$ and $F(x+h)$ are pretty close to each other too—they represent the two shaded areas above (respectively). Now, to differentiate F, we have to find
$$\lim_{h\to 0}\frac{F(x+h) - F(x)}{h}.$$
The difference $F(x+h) - F(x)$ is just the difference between the two shaded areas, which is itself just the area of the thin little region (with curved top) between $t = x$ and $t = x + h$:



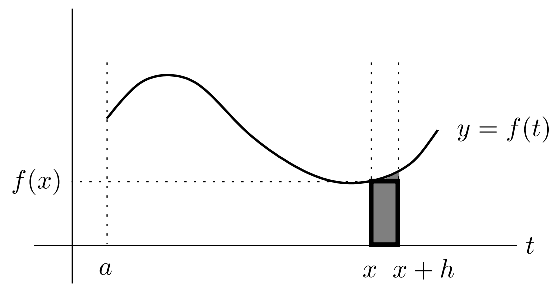
You can see this in symbols by splitting up the integral for $F(x+h)$ at $t=x$, like this:

$$F(x+h)=\int_a^{x+h} f(t)\,dt=\int_a^x f(t)\,dt+\int_x^{x+h} f(t)\,dt=F(x)+\int_x^{x+h} f(t)\,dt.$$

Rearranging, we get

$$F(x+h)-F(x)=\int_x^{x+h} f(t)\,dt,$$

which is exactly the shaded area (in square units) of the thin strip above. Actually, it's not a strip, since the top is curved, but it's **almost** a strip when h is small. The height of the strip at the left-hand side is $f(x)$ units, so we can approximate the thin region by a rectangle with base going from x to $x+h$ and height from 0 to $f(x)$, like this:



The base of the rectangle is h units, and the height is $f(x)$ units, so the area is $hf(x)$ square units. If h is small, then this is a good approximation to the integral we want. That is,

$$F(x+h)-F(x)=\int_x^{x+h} f(t)\,dt\cong hf(x).$$

Dividing by h, we have

$$\frac{F(x+h)-F(x)}{h}\cong f(x).$$

The approximation gets really good when h is really close to 0. It should be true, then, that the approximation is perfect in the limit as $h\to 0$:

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=f(x).$$

As we'll see in Section 17.8 below, the above formula is indeed true; we conclude that

$$F'(x)=f(x).$$

Let's summarize our conclusion as follows:

First Fundamental Theorem of Calculus: for f continuous on $[a,b]$, define a function F by

$$F(x)=\int_a^x f(t)\,dt \qquad \text{for } x \text{ in } [a,b].$$

Then F is differentiable on (a,b) and $F'(x)=f(x)$.

In short, you can write the whole thing as

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x).}$$

So our weird expression simplifies down to $f(x)$!

A common concern with this last formula is that a appears on the left-hand side but not on the right-hand side. This actually makes sense, believe it or not. Suppose that A is some other number in (a, b), and set

$$F(x) = \int_a^x f(t)\,dt \qquad \text{and} \qquad H(x) = \int_A^x f(t)\,dt.$$

Then, as we saw in Section 17.2 above, F and H differ by a constant:

$$F(x) = H(x) + C$$

for some constant C. If we differentiate, the constant goes away and we see that $F'(x) = H'(x)$ for all x in (a, b). So the actual choice of a doesn't affect the derivative. In terms of the curtain, we only care how fast it's being pulled and how high the rail is at the right-hand point. Where it happens to be attached at the fixed left-hand end doesn't affect the **rate** of area being swept out all the way over at the right-hand part of the curtain.

17.2.1 Introduction to antiderivatives

Now, let's pause for breath. We started with some function f of the variable t, as well as some number a; then we constructed a new function F of the variable x. Differentiating F gives us back the original function f, except now we evaluate it at x instead of t. Weird!

OK, weird, but really useful. It actually solves our whole darn problem. Let's see how. Suppose that $f(t) = t^2$ and $a = 0$, so that

$$F(x) = \int_0^x t^2\,dt.$$

The First Fundamental Theorem tells us that $F'(x) = f(x)$. Since $f(t) = t^2$, we have $f(x) = x^2$; this means that $F'(x) = x^2$. In other words, F is a function whose derivative is x^2. We say that F is an *antiderivative* of x^2 (with respect to x). Can you think of any other function whose derivative is x^2? Here are a few:

$$G(x) = \frac{x^3}{3}, \qquad H(x) = \frac{x^3}{3} + 7, \qquad \text{and} \qquad J(x) = \frac{x^3}{3} - 2\pi.$$

In each case, you can check that the derivative is x^2. In fact, any function of x of the form

$$\frac{x^3}{3} + C \qquad \text{for some constant } C$$

is an antiderivative of x^2. Are there any others? The answer is no! We actually saw this in Section 11.3.1 of Chapter 11. If two functions have the same derivative, they differ by a constant. This means that all the antiderivatives of x^2 differ by a constant. Since one of the antiderivatives is $x^3/3$, then any other antiderivative must be $x^3/3+C$, where C is constant. Wait a second—the weird function F above is also an antiderivative of x^2. This means that

$$F(x) = \int_0^x t^2\,dt = \frac{x^3}{3} + C$$

for some constant C. Now all we have to do is find C. We know that

$$F(0) = \int_0^0 t^2\,dt = 0.$$

So we have

$$0 = \frac{0^3}{3} + C.$$

This means that $C = 0$. We now have the formula we've been looking for:

$$\int_0^x t^2\,dt = \frac{x^3}{3}.$$

Finally, we can integrate t^2 from 0 to any number! In particular, if we replace t by 1 and then by 2, we get our well-worn formulas

$$\int_0^1 t^2\,dt = \frac{1^3}{3} = \frac{1}{3} \qquad \text{and} \qquad \int_0^2 t^2\,dt = \frac{2^3}{3} = \frac{8}{3}.$$

This can be made even simpler—we'll do that in the next section. First, I'd like to make one more point. We now have a way of constructing an antiderivative of any continuous function. For example, what is an antiderivative of e^{-x^2}? Just change x to t, pick your favorite number as a left-hand limit of integration (let's say 0 for the moment), and integrate to see that

$$F(x) = \int_0^x e^{-t^2}\,dt \quad \text{is an antiderivative of } e^{-x^2}.$$

The number 0 could be replaced by any number you choose, and the same statement would be true. Of course, you get a different antiderivative for each potential choice of left-hand limit of integration.

17.3 The Second Fundamental Theorem

The example with $f(t) = t^2$ in the previous section points the way to finding $\int_a^b f(t)\,dt$ in general. First, we know that the function F defined by

$$F(x) = \int_a^x f(t)\,dt$$

is an antiderivative of f (with respect to x). We really want to find $F(b)$, since

$$F(b) = \int_a^b f(t)\,dt.$$

We know one more thing:

$$F(a) = \int_a^a f(t)\,dt = 0,$$

because the left-hand and right-hand limits of integration are equal.

Now, suppose we have some other antiderivative of f: let's call it G. Then F and G differ by a constant, so that $G(x) = F(x) + C$. Put $x = a$ and you see that $G(a) = F(a) + C$; since $F(a) = 0$ from above, we have $G(a) = C$. This means that

$$F(x) = G(x) - C = G(x) - G(a).$$

If you replace x by b, you get

$$F(b) = G(b) - G(a).$$

In other words,

$$\int_a^b f(t)\,dt = G(b) - G(a).$$

This is true for **any** antiderivative G. Notice that we've gotten rid of x altogether. So the convention now is to change the dummy variable back to x and also change the letter G to F, arriving at the

Second Fundamental Theorem of Calculus: if f is continuous on $[a, b]$, and F is any antiderivative of f (with respect to x), then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

In practice, the right-hand side is normally written as $F(x)\Big|_a^b$. That is, we set

$$F(x)\Big|_a^b = F(b) - F(a).$$

So, for example, to evaluate

$$\int_1^2 x^2\,dx,$$

start by finding an antiderivative of x^2. We have seen that $x^3/3$ is one antiderivative, so

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2.$$

Now just plug $x = 2$ and $x = 1$ into $x^3/3$, and take the difference:

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2 = \left(\frac{2^3}{3}\right) - \left(\frac{1^3}{3}\right),$$

which works out to be $7/3$. Now, here's another example. Suppose you want to find

$$\int_{\pi/6}^{\pi/2} \cos(x)\,dx.$$

We need an antiderivative of $\cos(x)$. Luckily, we have one at hand: it's $\sin(x)$. After all, the derivative with respect to x of $\sin(x)$ is $\cos(x)$. So, we get

$$\int_{\pi/6}^{\pi/2} \cos(x)\,dx = \sin(x)\Big|_{\pi/6}^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right) = 1 - \frac{1}{2} = \frac{1}{2}.$$

We'll look at more examples of this sort in Section 17.6 below.

17.4 Indefinite Integrals

So far, we've used two different techniques to find definite integrals: limits of Riemann sums (what a pain) and antiderivatives (not so bad). It's quite clear that we're going to have to become pretty adept at finding antiderivatives—in fact, that's going to occupy us for the next couple of chapters after this one. So, we might as well have a shorthand way of expressing antiderivatives without having to write the long word "antiderivative." Inspired by the First Fundamental Theorem, we'll write

$$\int f(x)\,dx$$

to mean "the family of all antiderivatives of f." Bear in mind that any integrable function has infinitely many antiderivatives, but they all differ by a constant. This is what I mean when I say "family." For example,

$$\int x^2\,dx = \frac{x^3}{3} + C$$

for some constant C. This equation literally means that the antiderivatives of x^2 (with respect to x) are precisely the functions $x^3/3 + C$, where C is any constant. It is an error to omit the "$+\,C$" at the end, since that would only give one of the antiderivatives and we need them all.

If you know a derivative, you get an antiderivative for free. In particular:

$$\boxed{\text{if} \quad \frac{d}{dx}F(x) = f(x), \qquad \text{then} \qquad \int f(x)\,dx = F(x) + C.}$$

The above example fits this pattern:

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2, \qquad \text{so} \qquad \int x^2\,dx = \frac{x^3}{3} + C.$$

Similarly, we have

$$\frac{d}{dx}(\sin(x)) = \cos(x), \qquad \text{so} \qquad \int \cos(x)\,dx = \sin(x) + C.$$

One more example for now (there will be many more later!):

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}, \qquad \text{so} \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C.$$

Again, the number C is an arbitrary constant. It's just the nature of things that differentiable functions have only one derivative whereas integrable functions have infinitely many antiderivatives.