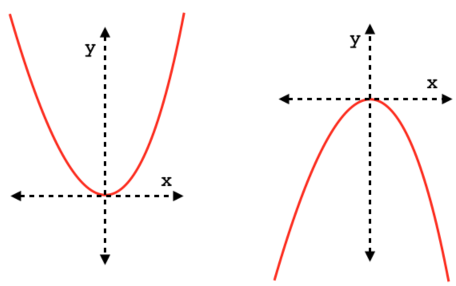
Parabola

direction of opening

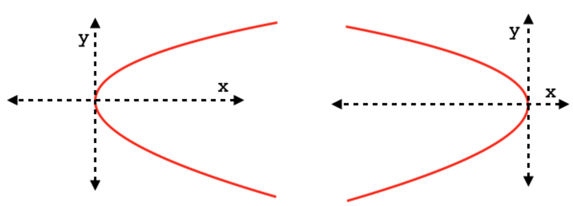
We continue looking at parabolas, the graphs of equations like $y = ax^2$, this time emphasizing the point of view of analytical geometry.

Any simple parabola (with no terms that mix x and y), has its axis of symmetry parallel to either the x- or y-axis.

Below we have $y = ax^2$ with either $a > 0$ (left panel) or $a < 0$ (right panel)



And then we have $x = ay^2$ with either $a > 0$ (left panel) or $a < 0$ (right panel)



In any case, we can just switch$-a$ and a, or x and y as needed, to obtain a figure like the first one, left panel. We focus on parabolas pointing up, with no loss of generality.

Those containing terms like Bxy, we leave aside as a complication to be returned to later.

vertex at different points

Start with a parabola having its vertex at the origin. The equation will be simply $y = ax^2$.

a is called the *shape factor*. It governs how steeply the curve rises (and as we said, by its sign it determines in which direction it opens).

We can see that the vertex is at $(0, 0)$, because (i) $(0, 0)$ is on the curve and (ii) $y \geq 0$, so 0 is the smallest y.

It turns out that any parabola with its vertex at a point other than the origin, can be described by the formula

$$y - k = a(x - h)^2$$

Moving the graph amounts to changing the values of x and y for every point on the curve.

For example, to move the vertex up by two units to $(0, 2)$, add 2 to the value of ax^2 for every x. The result is

$$y = ax^2 + k$$

where k is the amount of vertical shift. This can be rearranged to

$$(y - k) = ax^2$$

The vertex is at $y = 2$ (positive), but the formula says to subtract k from y, which is a bit counter-intuitive.

Changes in x are taken into account *before* squaring. For a parabola whose vertex is on the x-axis, the formula becomes

$$y = a(x - h)^2$$

If you try this for a vertex at $(1, 0)$, and plot the values, you will see that this is correct. For example, x values symmetric on each side of the vertex yield the same y, in the proportion $a(\Delta x)^2$, where $\Delta x = x - h$.

So the general formula for a parabola with its vertex at the point (h, k) is

$$y - k = a(x - h)^2$$

If we work with this a bit, multiplying out:

$$y - k = a(x^2 - 2xh + h^2)$$

$$y = ax^2 - 2ahx + ah^2 + k$$

In this form the cofactors are usually simplified as

$$y = ax^2 + bx + c$$

which we are used to seeing from algebra.

Comparing the two, we see that the cofactors of the x term must be equal:

$$-2ahx = bx$$

$$h = -\frac{b}{2a}$$

and the constant terms must be equal as well

$$c = ah^2 + k$$

$$k = c - ah^2$$

$$= c - \frac{b^2}{4a}$$

We can check this as follows. The first equation is commonly used to find the vertex for a given parabola. The x-value of the vertex is $h = -b/2a$.

Then the y-coordinate (k) can obtained by plugging into the given equation:

$$k = a(-\frac{b}{2a})^2 + b(-\frac{b}{2a}) + c$$
$$k - c = \frac{b^2}{4a} - \frac{b^2}{2a}$$
$$= -\frac{b^2}{4a}$$

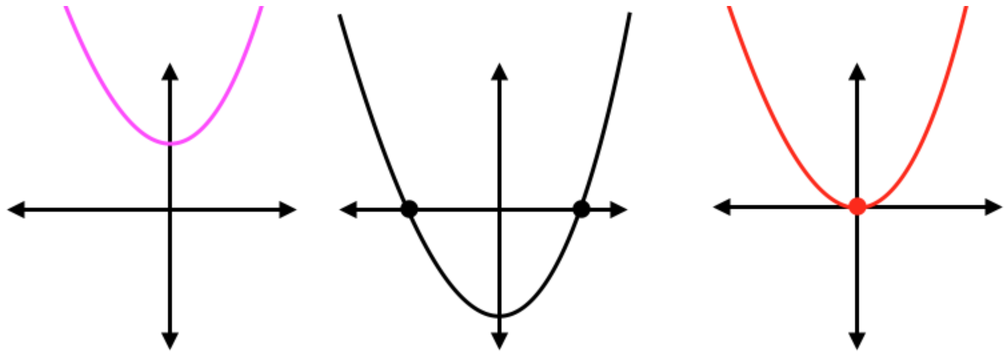
which matches what we had above.

roots

Probably the most common thing we're asked to do with a quadratic equation like this is to find the roots. These are the values of x for which $y = 0$ is a solution. They are the points where the graph of the curve crosses the x-axis.

By trying different possibilities it becomes clear that it is possible to have 0, 1 or 2 roots.

In the figure below, the black curve has two roots, the red curve has one. The latter's equation is $y = x^2$, the former is $y = x^2 - 1$ and then we can see that $x^2 = 1$ has two real solutions $x = \pm 1$.



On the left, the magenta curve does not cross the y-axis. Its equation is $y = x^2 + 1$, and there are no (real) solutions, no values of x that solve the equation when $y = 0$.

$$0 = x^2 + 1$$

$$x^2 = -1$$

To find the roots of

$$ax^2 + bx + c = 0$$

We can guess solutions by trying to factor into a form like:

$$(x - s)(x - t) = 0$$

The case of a single root occurs when $s = t$ so we have $a(x - s)^2 = 0$. A common example of that is a parabola with its vertex at the origin, so $s = 0$ and $y = ax^2$ (right panel, above).

Roots do not have to be integers (or even rational). An arguably more productive and certainly more general approach to finding them is the process of *completing the square*.

completing the square

Suppose we have

$$x^2 + 4x + 2 = 0$$

Here, it is clear that there will not be integer roots. No two integers add to get 4 and multiply to give 2.

The idea is to think about how we could convert $x^2 + 4x$ into a perfect square.

$$(x+2)^2 = x^2 + 4x + 4$$

Since we already have 2 we do this:

$$x^2 + 4x + 2 = 0$$

$$x^2 + 4x + 4 = 2$$

$$(x+2)^2 = 2$$

$$(x+2) = \pm\sqrt{2}$$

$$x = -2 \pm \sqrt{2}$$

Check:

$$x^2 + 4x + 4 = 0$$

$$(-2 \pm \sqrt{2})^2 + 4(-2 \pm \sqrt{2}) + 2 = 0$$

Positive square root:

$$4 - 4\sqrt{2} + 2 - 8 + 4\sqrt{2} + 2 = 0$$

Negative square root:

$$4 + 4\sqrt{2} + 2 - 8 - 4\sqrt{2} + 2 = 0$$

They both check.

quadratic equation

There is a general form for what we just did.

Suppose we have

$$y = ax^2 + bx + c$$

When $y = 0$:

$$ax^2 + bx + c = 0$$

First, multiply through by $1/a$ and place the constant term on the right-hand side:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

We want to change the left-hand side

$$x^2 + \frac{b}{a}x$$

into a perfect square, like

$$(x + \frac{b}{2a})^2$$

This is the first bright idea. We chose $b/2a$ for the last equation, because there will be two copies when computing the square

$$(x + \frac{b}{2a})^2 = x^2 + 2 \cdot \frac{b}{2a}x + \frac{b^4}{4a^2}$$

$$= x^2 + \frac{b}{a}x + \frac{b^4}{4a^2}$$

The second insight is that on the original left-hand side

$$x^2 + \frac{b}{a}x$$

that third term is missing, but *we can fix it*. To maintain equality, simply add the same thing on both sides:

$$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$$

So the left-hand side is now a perfect square. Also, we put the right-hand side over a common denominator

$$(x + \frac{b}{2a})^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$$

Taking the square root

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Now, we just do a bit of rearranging.

I like to think of this equation in two pieces. The first part is the discriminant:

$$D = b^2 - 4ac$$

So we can rewrite the equation as

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

which is the quadratic equation. Written out in full it is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula always works to find the roots of an equation, if they exist.

The quantity under the square root is called the discriminant. If $D < 0$ then $\sqrt{D}$ does not exist in the real numbers and there is no x such that

$y = 0$. That corresponds to the case where the parabola does not cross the x-axis.

If $D = 0$ then there is a single root, and the graph just touches the x-axis.

check the answer

We assert that when x takes on those two values, it is a solution for

$$ax^2 + bx = -c$$

Take the positive root:

$$x = \frac{1}{2a}\ [-b + \sqrt{D}\ ]$$

Compute ax^2

$$\begin{aligned} ax^2 &= \frac{1}{4a}\ [b^2 - 2b\sqrt{D} + D\ ] \\ &= \frac{1}{4a}\ [2b^2 - 2b\sqrt{D} - 4ac\ ] \\ &= \frac{1}{2a}\ [b^2 - b\sqrt{D} - 2ac\ ] \end{aligned}$$

and then bx

$$bx = \frac{1}{2a}\ [-b^2 + b\sqrt{D}\ ]$$

All the terms of bx cancel terms in ax^2 when we add $ax^2 + bx$. What's left is $-2ac$ in the numerator so then

$$\frac{1}{2a}\ (-2ac) = -c$$

If we had started with the negative root, the square roots in both ax^2 and bx would each change sign, but they would still be opposite signs and hence, cancel.

simpler method

The quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is not particularly complicated, and even less so if we first divide by a. (This changes the shape of the parabola but doesn't change the roots).

Nevertheless, students have trouble with it. There is an easier way. The germ of the idea is contained in what we just said.

We recognize that any quadratic can be rewritten as

$$(x - s)(x - t) = 0$$

where s and t are the roots we seek when solving

$$x^2 - (s + t)x + st = 0$$

Compared with the standard form (with a divided into b and c) we have that

$$b = -(s + t)$$

$$st = c$$

And of course, these two relationships are what students exploit when guessing the numbers s and t. The problem with that approach is that it so rarely yields integer values, especially in real situations.

However, there is other structure in the problem. Any parabola has mirror image symmetry about an axis that goes through the vertex. As a result, the values $x = s$ and $x = t$, which lie on a horizontal, are equally distant from the value x takes on at the vertex.

Let us call that middle value m for median. Equidistant means that

$$m = \frac{s + t}{2}$$

m is closely related to b:

$$2m = -b$$
$$m = -\frac{b}{2}$$

The second bright idea here is that, because of the same symmetry, s and t are equidistant from m. Let us call that distance d, so then

$$m - d = s, \qquad m + d = t$$

$$st = m^2 - d^2$$

and $st = c$ which gives

$$d^2 = m^2 - c$$

Using this method it is a four-step process, but each step is simple:

- Divide by a, giving x^2, then redefine b/a as b and c/a as c.
- Compute $m = -b/2$.
- Compute $d = \sqrt{m^2 - c}$.
- Compute s,t as $m \pm d$.

The roots are symmetric because of the $\pm$ symbol. They are called *conjugates*, and they have the property that added, the square roots cancel, and when multiplied together, what's under the square root comes into the product.

So even if D, or $m^2 - c = d^2$ is negative and its square root does not exist in the real numbers, nevertheless:

$$\begin{aligned}(x - s)(x - t) &= (x - (m - d)) \cdot (x - (m + d)) \\ &= ((x - m) + d) \cdot ((x - m) - d) \\ &= (x - m)^2 - d^2\end{aligned}$$

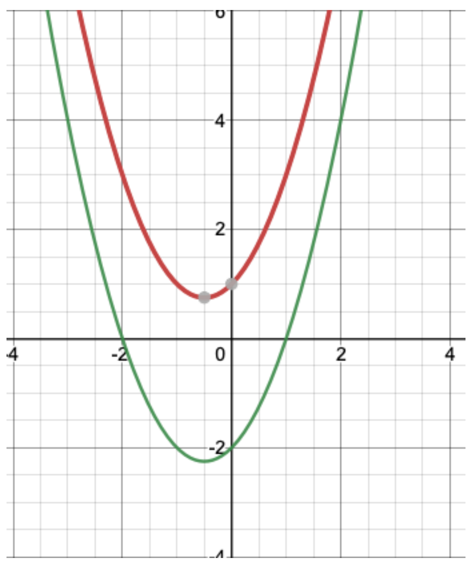
which we may recognize from the steps of the algorithm.

In this final form, we have only d^2.

example

Suppose we take an ordinary quadratic like $x^2 + x - 2$ which has real roots, and make them not real by adding 3:

$$x^2 + x + 1$$



It should be apparent that the first equation can be factored

$$x^2 + x - 2 = (x + 2)(x - 1)$$

which explains the roots at $x = -2, x = 1$.

Working with the other equation, $x^2 + x + 1$, we compute

$$m = -\frac{b}{2} = -\frac{1}{2}$$

and

$$(m + d)(m - d) = m^2 - d^2 = c$$

$$d^2 = m^2 - c = \frac{1}{4} - 1 = -\frac{3}{4}$$
$$d = \frac{\sqrt{-3}}{2}$$

The roots are

$$x = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}$$

So we can rewrite the equation as

$$(x + \frac{1}{2} - \frac{\sqrt{-3}}{2})(x + \frac{1}{2} + \frac{\sqrt{-3}}{2})$$

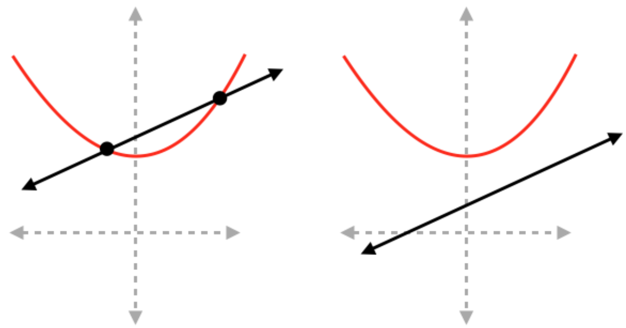
which becomes a difference of squares:

$$= (x + \frac{1}{2})^2 - (\frac{\sqrt{-3}}{2})^2$$
$$= x^2 + x + \frac{1}{4} + \frac{3}{4}$$
$$= x^2 + x + 1$$

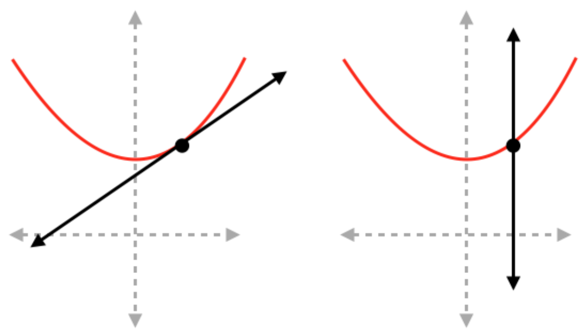
The roots go away as we said, and in this one, even the fractions disappear.

tangent lines

Consider a parabola and a line on the same graph. There are four possibilities for the intersection of the line and the parabola. First and second: two points (left panel), and no points (right panel).

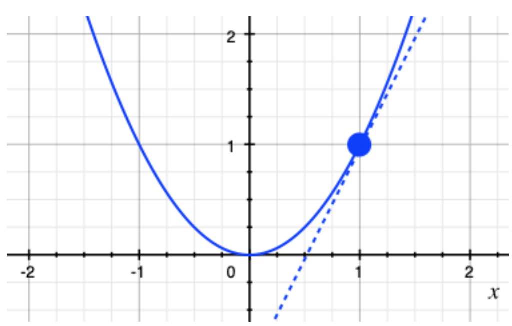


The other two possibilities both have a single point of intersection. The tangent line at a point (left panel), and a vertical line (right panel).



We are particularly interested to know the equations of tangent lines to the parabola, which includes their slopes. Consider the simplest example: $y = x^2$.

The point $(1, 1)$ is on the curve, because $(x = 1, y = 1)$ satisfies the equation $y = x^2$.



Suppose we know that the slope of the tangent to the curve at the point $(1, 1)$ is equal to 2.

The equation of the tangent line is

$$y - y' = m(x - x')$$

Plugging in for $(x', y') = (1, 1)$:

$$y - 1 = 2(x - 1)$$

$$y = 2x - 1$$

Now suppose that we knew only the parabola and this slope, but we did not know the point where the tangent meets the curve, and so do not know the y-intercept.

We have the equation of a line:

$$y = 2x + y_0$$

We seek points which are simultaneously on the line and the curve. They must satisfy both equations.

Since this is a tangent line, we seek the value for which this expression has only a single solution. The tangent ”touches” the curve at a single point.

Substitute for y from the equation for the curve:

$$x^2 = 2x + y_0$$

$$x^2 - 2x - y_0 = 0$$

Use the quadratic formula to set up an expression for x:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There is a single solution when the part under the square root (the discriminant) is equal to zero.

$$b^2 - 4ac = 0$$
$$b^2 = 4ac$$
$$(-2)^2 = 4(-y_0)$$
$$y_0 = -1$$

Therefore, the equation of the tangent line is $y = 2x - 1$, which matches what we had before.

$y = 2x + y_0$ is a *family* of lines. For $y_0 = -1$, there is a single solution for x to be both on the line and the parabola. For $y_0 < -1$, there are no solutions, while for $y_0 > -1$ there are two solutions, because the line actually traces out a chord or secant of the parabola, passing through the curve at two points.

The general solution is as follows:

$$y = ax^2 + bx + c$$

$$y = mx + y_0$$

The point(s) of intersection are given by (x, y) satisfying both equations:

$$ax^2 + bx + c = y = mx + y_0$$

Then

$$ax^2 + (b - m)x + c - y_0 = 0$$

From the quadratic equation the solutions are

$$x = \frac{(m - b) \pm \sqrt{(b - m)^2 + 4a(c - y_0)}}{2a}$$

For the case of the tangent line, there is a single solution, which happens when the discriminant is zero and then

$$x = \frac{m-b}{2a}$$
$$m = 2ax + b$$

As we've been saying. The slope of the tangent to the parabola at a point (x, ax^2) is equal to $2ax$, but there is another term b when the equation of the parabola is $ax^2 + bx + c$.

We can find the equation of the line by finding y_0, the value of y when $x = 0$.

$$m = 2ax = \frac{y - y_0}{x - 0}$$
$$2ax^2 = ax^2 - y_0$$
$$y_0 = -ax^2$$

Note: do not make the mistake of writing the equation of the line now as

$$y = mx + y_0 = 2ax \cdot x - ax^2$$

This is wrong because m and y_0 were determined for a particular point on the parabola, but in the equation of the line $y = mx + y_0$, *that* x is any x. Going back to the prime notation we should write:

$$y = mx + y_0 = 2ax' \cdot x - ax'^2$$

where x is a variable and x' is a constant.

Kline 4-23

The slope of the parabola has some simple interesting properties.

For example, pick two points (x, y) and (x', y') on such that the line joining them goes through the focus.

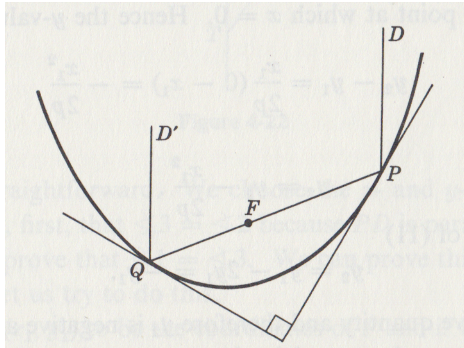


Figure 4-23

The tangents through these two points form a right angle, as shown in the figure.

Proof.

Recall that when the vertical DP is extended downward, the angle that it makes with the tangent, ϕ, is equal to the angle DP itself makes with the tangent.

Furthermore, FP makes the same angle with the tangent, by the "headlight property". The angle of incidence is equal to the angle of reflection.

The situation is this:

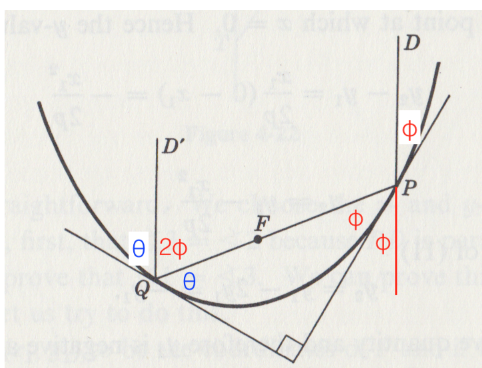


Figure 4-23

Since DP is parallel to $D'P$, by alternate interior angles the angle

$D'QF$ has measure 2ϕ. Using the headlight property again we have that $\theta = \theta$.

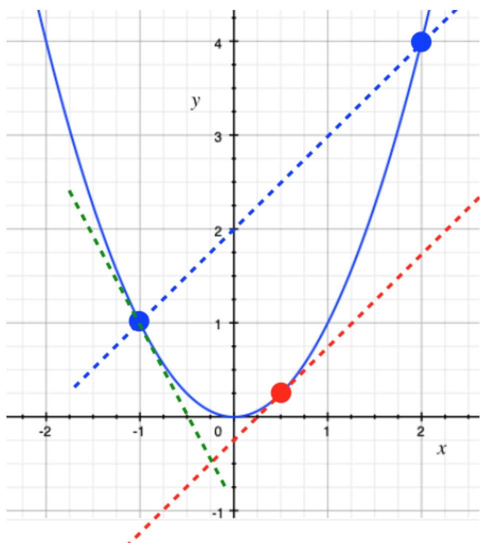
Since $2\phi + 2\theta$ is 180 degrees, θ and ϕ are complementary, so the angle where the tangents meet is a right angle.

$\square$

further comment

Pick any two points (x, y) and (x', y') on our standard parabola.

The slope of the line that connects those two points is equal to the slope of the parabola at the point whose x-value is halfway in between.



For the first part:

$$m = \frac{y' - y}{x' - x}$$
$$= \frac{ax'^2 - ax^2}{x' - x}$$
$$= a \left[\frac{x'^2 - x^2}{x' - x} \right]$$

$$= a(x' + x)$$

For the midpoint

$$x_m = \frac{1}{2}(x' + x)$$

and the slope is

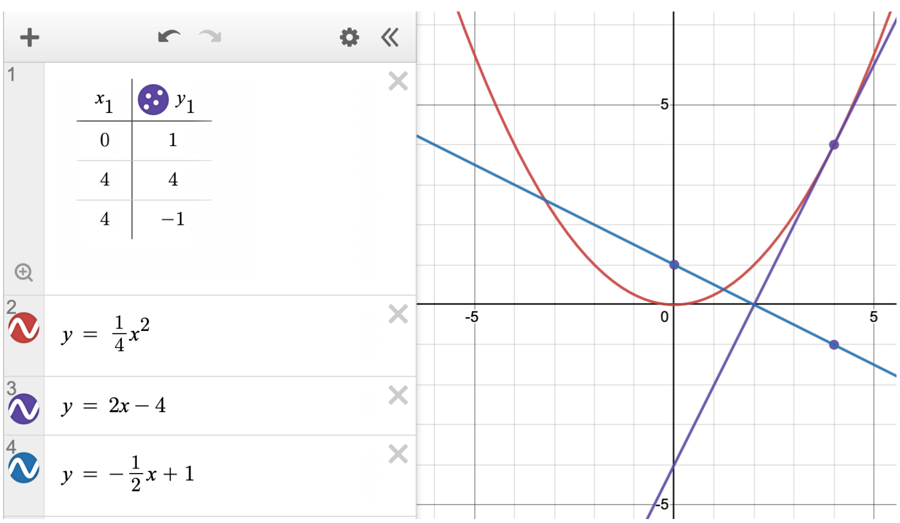
$$2a \cdot \frac{1}{2}(x' + x)$$
$$= a(x' + x)$$

A similar result is that if we pick any two points (x, y) and (x', y'), and draw their slopes, the point where the two slope lines meet has its x-value exactly halfway in between x and x'.

perpendicular bisector

Roberval's idea was that the tangent should split the difference between the lines to the focus and to the directrix.

Another way to phrase that is to say that the tangent to (x, ax^2) is the perpendicular bisector of a line connecting the focus and the point on the directrix with the same x.



But the perpendicular bisector crosses the x-axis at $x/2$. So then the slope is just

$$\frac{ax^2}{x/2} = 2ax$$