Calculus

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Part I

Fundamentals

Chapter 1

Introduction

This was supposed to be a short book, an exploration of problems like the volume of the cone and sphere, or even just the area of a circle, with some simple physics thrown in. These questions contain within them the heart of calculus: infinities both large and small. I imagine myself looking over Archimedes' shoulder as he explains it to me.

I wrote many of the early chapters originally as short explanations for my son Sean as he studied calculus in high school. It bothers me that so often the good stuff gets left out — the ideas which make you go ... wow. Now, years later, I still find a lot of pleasure in trying to understand what Kepler and Newton did. It took a genius to figure it out the first time, but it is within anyone's grasp to appreciate what they found.

Then I thought, why not include other favorite problems like the area of the ellipse, the "headlight" problem for the parabola, or the reflective property of the ellipse, and the length and area under the cycloid curve (the "light on a bicycle wheel"). These are problems where calculus easily produces answers that can be checked by more elaborate geometric arguments. In fact, this book might as well be titled .. *Best* *of Calculus and Geometry.*

So here we are, with a somewhat longer book.

In the introduction to his book *Calculus*, Morris Kline says

> Anyone who adds to the plethora of introductory calculus texts owes an explanation, if not an apology, to the mathematical community.

I think of this book as akin to ultralight backpacking. We shed weight so as to ascend peaks rapidly, skimming the best of calculus — focusing on geometry and physics, and slinging differentials with abandon. Epsilon is a bit player in the production. Starting with an intuitive notion of adding up many small pieces, we put integrals to work early solving problems.

Going fast allows time to get a view of sophisticated topics, among others, line integrals for work and flux, Newton's proof that a spherical mass acts as a point mass, and integration of a parametrized surface like the torus. Not to mention Kepler's Laws, and a derivation of the Gaussian distribution from first principles.

We do not disdain proof. Proof is central to the enterprise. We prove the Pythagorean Theorem, and the quotient rule for derivatives, as well as Green's Theorem. There is a fun chapter on induction. We prove that π is a constant. In fact, the word "proof" appears nearly 200 times in the text and one of its most interesting features is the natural use of proofs that I have tried to make as simple and easy to follow as possible.

My favorite authors on calculus are Morris Kline, Richard Hamming, and Gil Strang. Sylvanus Thompson's simple book is my favorite first text, and it's even a Project Gutenberg project:

https://www.gutenberg.org/files/33283/33283-pdf.pdf

Having said what I like, briefly, here are some things I don't like.

The rigorous approach to calculus pioneered by Cauchy in the 1820's and exported to American schools by Richard Courant in the 1940's is a bad idea. We must motivate rigorous proof by demonstrating utility first. As Ian Stewart says, "proofs come *after* understanding." Courant's method is the way to teach the subject the second or even third time through.

Thompson:

> You don't forbid the use of a watch to every person who does not know how to make one. You don't object to the musician playing on a violin that he has not himself constructed. You don't teach the rules of syntax to children until they have already become fluent in the use of speech. It would be equally absurd to require general rigid demonstrations to be expounded to beginners in the calculus.

A second thing I dislike is calculus problems that are gratuitously arithmetic. Calculus consists of bright ideas, not complicated ones; if the computation is difficult, it's usually *not* a good problem. Also, a good problem often is one with a physical or practical foundation. Having said that, if a course could integrate elementary programming with calculus, I would be very happy.

Finally, a saying attributed to Manaechmus (speaking to Alexander the Great), "there is no royal road to geometry". Which means, practically, learning mathematics requires that you follow the argument with pencil and paper and work out each step yourself, to your own satisfaction. That is the only way of really learning, and at heart, one of the reasons I wrote this book.

I express my sincere thanks to the authors of my favorite books, which are listed in the references and mentioned at various places in the text. Almost everything in here was appropriated from them, and styled to my taste. I offer my profound thanks also to Eugene Colosimo, S.J. He was, for me, the best of a bunch of very special teachers.

If I stole your figure off the internet, I'm sorry. I intended to redraw it but have not yet found the time.

Update (Feb 2020): I have broken the original book into parts.

Part one is Geometry and topics for Precalculus. It is here:

https://github.com/telliott99/precalculus

Part two is single variable Calculus. It is here:

https://github.com/telliott99/calculus

The original book is here:

https://github.com/telliott99/calculus_book

Chapter 2

Simple slopes

To introduce the two fundamental ideas in calculus, consider two measuring devices used while driving a car. Most good drivers look fairly often at the speedometer, which measures speed or velocity, or how fast you're going.

On the other hand, if someone gives you directions like — go three and a half miles and then turn left (where the old gas station used to be), you need to be watching your odometer.

Distance divided by time is velocity. Velocity times time equals distance. We can think of speed and velocity as the same for now.

Velocity is the *rate of change* of distance with time, it has units like miles per hour or feet per second (15 mph is exactly 22 feet per second; $15 \cdot 5280 = 22 \cdot 3600$).

In calculus we say that

- velocity is the **derivative** of the distance with respect to time
- distance is the **integral** of the velocity with respect to time

We can speak of velocity at a particular time t, as in "our current velocity is 60 miles per hour." But the distance, the integral, must be evaluated between appropriate starting and stopping points for the time.

In our example, you must first look at your odometer *before* you start on that 3.5 mile drive, and subtract the initial from the final value.

time-dependence

Distance equals velocity times time.

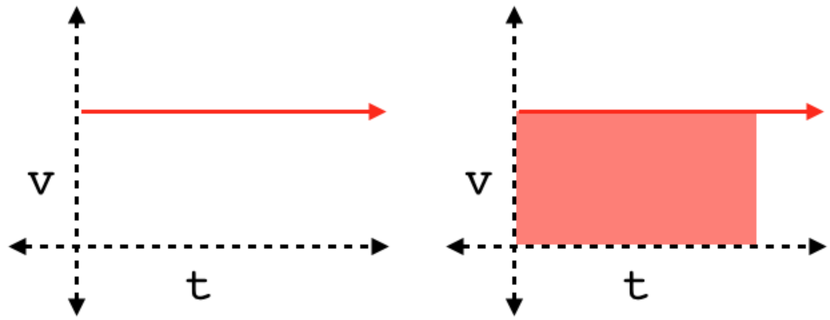
This is easy if the velocity is constant. Travel west on the interstate at exactly 60 miles per hour for 2 hours and your distance will be 120 miles from where you started (provided you don't start in Los Angeles).

It is standard to use s to refer to the distance traveled and v for velocity. If the velocity is constant then:

$$s = vt$$

According to the internet, s is from the Latin "spatium", for "space, room, or distance."

Suppose we plot velocity as a *function of time* with v on the y-axis and t on the x-axis.



Since the velocity is constant, the result is a straight horizontal line.

Furthermore, the distance traveled is the *area under the curve* (and above the x-axis) which is the area of a rectangle with sides v and t and as we said

$$s = vt$$

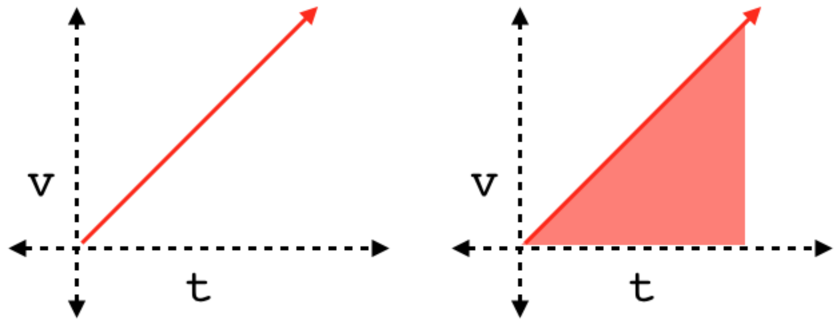
However, for most interesting problems the velocity is not constant.

Imagine maintaining pressure on the gas pedal in the car steadily so that, starting from a stop at zero time, after 1 second your velocity is 10 mph, after 2 seconds it is 20 mph, after 3 seconds, 30 mph. If we continue at the same rate of acceleration, we'll go from 0 to 60 mph in 6 seconds, which is quite a respectable time.

This example has constant acceleration. Here, we say that v is a constant function of time, and write

$$v = at$$

where a is the acceleration.



What about the distance? It turns out that the distance is once again the area under the curve.

Since a isn't zero, v must change with time.

If a is non-zero and constant, then v changes at a constant rate. Starting from 0, the final velocity will be $v = at$, but the distance traveled is no longer the product

$$s = v \times t \stackrel{?}{=}$$

because this v is the final velocity and that is not the correct v to use. For variable velocity, the distance traveled is the *average* velocity times the time. For smooth (constant) acceleration from zero to v, the average velocity is the average of the initial and final velocities:

$$v_{\text{avg}} = \frac{1}{2}\ (v_i + v_f) = \frac{1}{2}\ v$$

So the correct equation is:

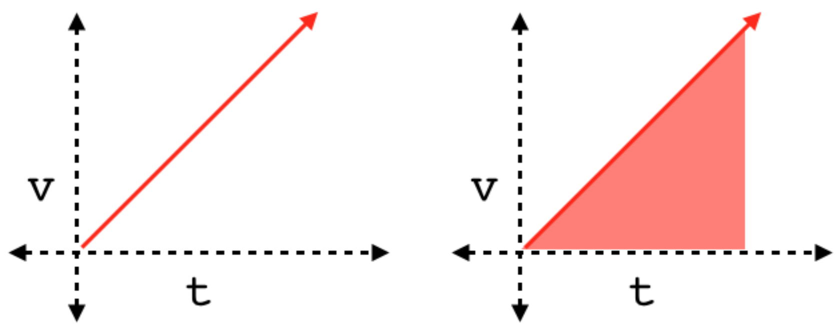
$$s = v_{\text{avg}}\ t = \frac{1}{2}\ v \cdot t$$

and since $v = at$

$$s = \frac{1}{2} a t^2$$

In this case, if we plot velocity as a function of time, we obtain a straight line that extends diagonally up with respect to the x-axis. The distance traveled is the area under the curve, below the line and above the x-axis.

The shape whose area is needed is a triangle. This also accounts for the factor of 1/2.



You probably know that if a mass m is dropped from a tall building like the Tower of Pisa, then the distance it has fallen goes like the square of the time. The equation is:

$$s = \frac{1}{2} g t^2$$

where g is the acceleration due to gravity.

Notice that this is the same equation as we just obtained.

The reason is that g is approximately constant near the surface of the earth, its value is about 10 in units of m/s^2. A fall of four seconds is about 80 meters.

Galileo knew this formula (at least, he knew the t^2 part of it), which he obtained not from experiments at the Tower of Pisa, but by timing the descent of balls down an inclined plane.

initial position and velocity

If you want to be more complete and say that the starting point is not necessarily the origin of the coordinate system, add a constant s_0 to describe the initial distance from the origin and obtain:

$$s = vt + s_0$$

and similarly, a constant v_0 to describe the initial velocity as shown above.

The full equation of motion is

$$s = \frac{1}{2}at^2 + v_0t + s_0$$

We'll say much more about this later.

power rule

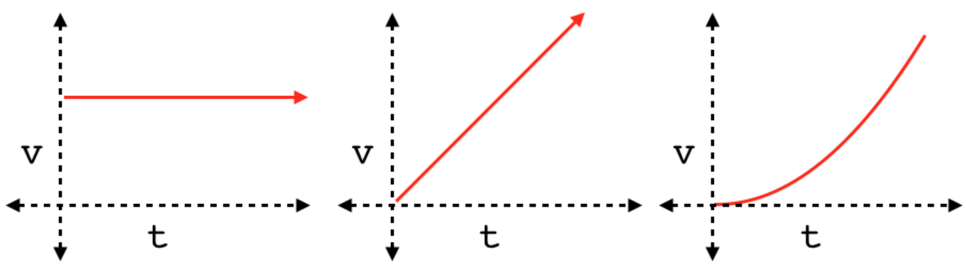
We will introduce the theory of calculus more formally in the next section of the book. For now, we just talk about a simple rule called the power rule.

Switching notation to y and x, suppose that y is a *function* of x and write $y = f(x)$.

Here are three types of dependency (with c as a constant), with three corresponding types of graph.

- $y = c$
- $y = cx$
- $y = cx^2$

These are (respectively) the equations of: (i) a horizontal line, since y is constant, (ii) any other non-vertical line (y is proportional to x), and (iii), a parabola.



Suppose we are at some particular point on the curve, x.

We ask "what happens if we change x a little bit" and use the notation dx to refer to this little bit of x.

What happens to y? y will usually change by a small amount. Call that amount dy.

case 0

We can call this case 0 because we can write it as

$$y = cx^0 = c$$

Of course, in this case

$$y = c$$

y does not actually depend on x at all. The change dy resulting from a change in x, dx, is zero. That is what the curve plotted above tells us (left panel).

$$y = c, \quad dy = 0 \cdot dx$$

The ratio dy/dx is the slope of the curve formed by plotting y against x. We call that slope the *derivative* of the function $f(x)$.

Divide both sides by dx and rewrite the above as:

$$\frac{dy}{dx} = 0$$

This plot is a horizontal line with slope 0.

case 1

Here, y is a linear function of x, the change dy is the change dx multiplied by c:

$$y = cx, \quad dy = c \cdot dx$$

rearranging.

$$\frac{dy}{dx} = c$$

In analytical geometry, we calculate the slope of a line as $\Delta y/\Delta x$.

For a line, the slope is constant and so it doesn't matter which two points with coordinates $(x, y), (x', y')$ we choose for the calculation. The following is true for *any* two points on the line:

$$m = \frac{\Delta y}{\Delta x} = \frac{y - y'}{x - x'}$$

Above we had the example where $v = at$ with constant a. Then $dv/dt = a$.

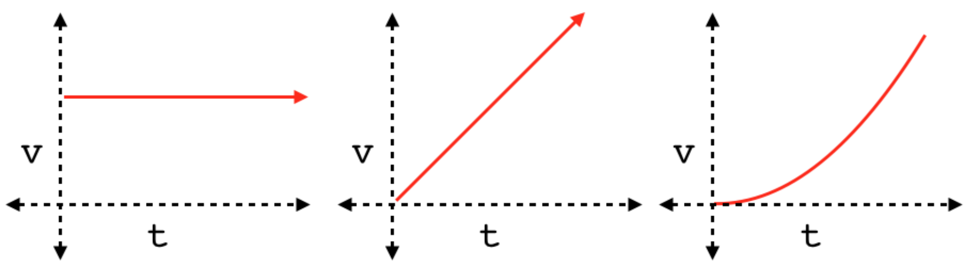
case 2

This case is different.

$$y = cx^2$$

We finally get to using some calculus.

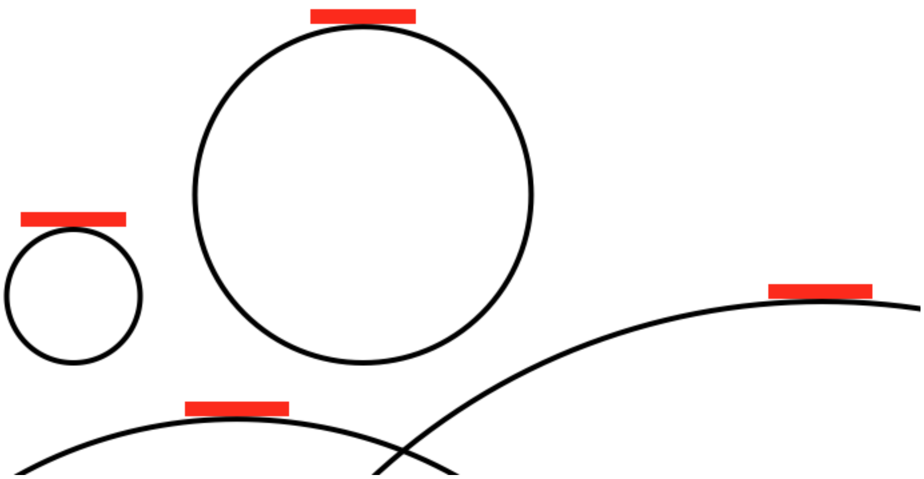
For a parabola, the slope of the curve at a point (the slope of the tangent to the curve $y = cx^2$) depends on the choice of x. The slope is steeper the further out you go in a positive direction on the x-axis (right panel).



It seems impossible to compute the slope of this curve in the standard way, by picking two points (x, y) and (x', y') and then calculating $\Delta y/\Delta x$, because the slope changes as we go out along the curve. You'll get a different answer for each different x.

key idea

The insight is that if x is sufficiently close to x' the slope is approximately constant. It's like saying that the earth is flat *locally*. If you detect any curvature, just zoom in a bit. In the figure below



the distance to the circle from the end of a line of fixed length decreases as we increase the size of the circle.

In calculus, we don't make the curves larger, we make the distance between x and x' smaller and smaller until it is so small, that the circle, or the parabola in the other figure, becomes flat. I have just magnified the figure so we could see it.

The error, the distance between the end of the red line above and the circle, gets smaller and smaller as a fraction of the line's length. Our approximation gets better and better. Just zoom in until the line is a good enough approximation to the shape of the circle, if the curve doesn't look flat enough, zoom in some more.

As we are accelerating in the car, with constantly changing velocity, we can still have a unique velocity at a particular instant in time.

In mathematical language, for a very small change Δx in either direction from x, we get the same slope, *if* Δx is small enough.

If it's not, we can always make it smaller. That's the beauty of the real numbers.

As you accelerate from 0 to 60, there must be at least one moment in time when your velocity is 50.

Or, put still another way, when they built your house they didn't worry about the curvature of the earth.

If r is the radius of the earth in feet, and the house is $a = 50$ feet long, the drop due to curvature is $r - b$ where $b = \sqrt{r^2 - l^2}$

```
r = 21120000
b = 21119999.9999408
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That is about 0.00006 feet over the length of a 50 foot house, about a thousand times less than 1/16 of an inch. It is nearly 6 orders of magnitude, one part in a million.

Since the changes in x and y are so small, we use the new nomenclature: dy and dx.

power rule

To actually calculate slopes for curves (and straight lines), use the power rule.

For a horizontal line with zero slope:

$$y = c$$

$$\frac{dy}{dx} = 0$$

For a line with a slope c:

$$y = cx$$

$$\frac{dy}{dx} = c$$

For the parabola, the rule says that if $y = cx^2$, the slope or derivative is

$$\frac{dy}{dx} = 2cx$$

We've been writing c as the constant, so as not to confuse it with a, the acceleration. In analytic geometry, a parabola is usually written with a constant a, called the shape factor:

$$y = ax^2$$

Then, the slope is $2ax$.

If we had

$$y = ax^2 + bx + c$$

with a, b, c all constant, then the slope would be $2ax + b$.

The above uses our three rules from above, plus one more, that when taking the derivative of a polynomial, the derivative of the whole is simply the summed derivatives for each term.

For the equation of motion under gravity

$$s = \frac{1}{2}at^2 + v_0t + s_0$$

$$v = \frac{ds}{dt} = at + v_0$$

$$\frac{dv}{dt} = a$$

Notice how the 1/2 and the 2 cancel in the second equation.

Continuing to the cubic, if y depends on x^3 like

$$y = cx^3$$

then

$$\frac{dy}{dx} = 3cx^2$$

The general form of the power rule is that if

$$y = x^n$$

then

$$\frac{dy}{dx} = nx^{n-1}$$

The exponent has been reduced by 1 power, and the value of that exponent applied as a factor in front of the expression.

This rule had already been discovered before Newton. It's a toss-up whether Fermat or Cavalieri was first. We will prove this later, but for now we just want to introduce the idea and practice using it.

note

If you already know some calculus you're probably jumping out of your chair while reading this chapter because you've had it pounded into you that dy/dx is not a quotient and believe that you can't simply multiply both sides of the equation by dx.

Well, you can. And I'll explain why as we go along.

Chapter 3

Easy pieces

Integration

Differentiation breaks things up into small pieces dx or dr. Integration adds up many little pieces. The symbol for integration is a relaxed S that stands for summation: $\int$.

As Thompson says

> The word "integral" simply means "the whole." If you think of the duration of time for one hour, you may (if you like) think of it as cut up into 3600 little bits called seconds. The whole of the 3600 little bits added up together make one hour.

We boldly claim that from the point of view of problem-solving, integration is simply the inverse of differentiation.

Mathematicians hate this kind of talk, because it trivializes a profound statement, the fundamental theorem of calculus.

But for practical problem-solving our counter-claim is that this profundity *doesn't matter*. It is also likely to confuse the beginning student, another reason to put it aside for the time being. We'll return to this issue later, when we cover the theory of the subject very lightly.

The sum of a bunch of small pieces dy is equal to the sum of a bunch of small pieces dx times cx, when $dy/dx = cx$ describes how y changes with small changes in x at any particular point.

The key idea is *at any point*. The relationship between dy and dx depends on where you are on the curve. That's why we need integration.

Write

$$dy = f(x)\ dx$$

We want to solve

$$\int dy = \int f(x)\ dx$$

The sum of all the little pieces dy is just y

$$y = \int f(x)\ dx$$

Now, this surely sounds a little vague. But it will turn out that

$$F(x) = \int f(x)\ dx = y$$

exactly when the derivative of $F(x)$ is $f(x)$:

$$\frac{dF}{dx} = F'(x) = f(x)$$

This is the first of two bright ideas we need to solve an equation like $\int f(x)\ dx$. Just find $F(x)$ such that the derivative of $F(x)$ is $f(x)$.

Area of the circle

Let's spend some time analyzing the area of a circle. This provides crucial insight into what integral calculus can do.

Integration is used to compute areas and volumes, and other sums, by adding up many little pieces.

To calculate the area of a circle, we find the pieces we will use with one of three basic strategies: rings, slices of pie, or rectangles of area underneath the function obtained by solving $x^2 + y^2 = R^2$ (using the positive square root). These three approaches are illustrated in the figure above.

rings

In the first approach (left panel), we imagine the area being computed by adding up the individual areas of a series of very thin, concentric rings.

The total area to be computed is that of a circle of a definite, fixed size, and we denote the radius of this circle by capital R, a constant. On the other hand, the series of rings ranges from the origin of the circle to the circumference of the outmost ring. Each one of this progression of rings has a radius, so we use the lowercase r to describe them, with r being a variable—r varies from 0 at the origin to R at the outside of the circle.

Think about an individual ring, for example the outermost ring, which is similar to the circular peel or rind surrounding a thin slice of lemon. We are working with areas here, in two dimensions, so the slice we imagine to be infinitely thin, and we are working with it as a cross-section or ring.

The area of the ring is the length times the width. The length is the circumference, $2\pi R$ for the outermost ring, but in general, for any of the inner rings it is $2\pi r$. The length is multiplied by the width of the slice, which is a small element of radius, dr. The small element of area contributed by an individual ring is dA:

$$dA = 2\pi r \ dr$$

Another way to explain this equation is to ask the question:

how does area change with increasing radius?

If we take a circle and increase its radius by a little bit, how does the area change? The answer is, it changes in proportion to the circumference, $2\pi r$.

Another way to say the same thing is that the derivative is

$$\frac{dA}{dr} = 2\pi r$$

Proceeding from the first equation, the total area is the sum of the areas for the series of rings.

$$A = \int dA = \int_0^R 2\pi r \ dr$$

It's worth emphasizing how this view is different than the examples of integration one usually sees first in a calculus book: these pieces of area are not rectangles but circles. But it poses most clearly the question we are trying to answer, "how does area change as r changes"?

In order to actually determine a value for the area we need two principles. The first is, as we mentioned before, that the solution to

$$\int f(x)\ dx$$

is $F(x)$ if and only if the derivative of $F(x)$ is equal to $f(x)$.

Continuing with our problem

$$\int 2\pi r\ dr = 2\pi \int r\ dr$$

In this step we used a fundamental rule that a constant can come "out from under" the integral sign. That's not surprising. We already know that (at least in the power rule) the derivative of a constant times some function is that constant times the derivative of the function. We will show that is a general rule later.

Now, we need to find a function whose derivative is r.

$$2\pi \int r\ dr$$

We know that function, it is r^2, with an extra factor of 1/2.

$$= 2\pi\ [\ \frac{1}{2}\ r^2\ ]\ = \pi r^2$$

Combining all the coefficients we have $\int 2\pi r\ dr = \pi r^2$ precisely because the derivative of πr^2 is just $2\pi r$.

The second principle we need comes from the Fundamental Theorem of Calculus, which takes account of the bounds on the integral (in this case 0 and R). The bounds are written attached to the integral as

$$\int_0^R$$

and on the expression to be evaluated attached to a vertical bar

$$\Big|_{r=0}^{r=R}$$

like this

$$2\pi \int_{r=0}^{r=R} r\ dr = \pi\ r^2 \Big|_{r=0}^{r=R}$$

We say that the answer is this function, "evaluated between the bounds 0 and R."

The value of such a definite integral is $F(x)$ evaluated at the upper limit minus the value of $F(x)$ evaluated at the lower limit:

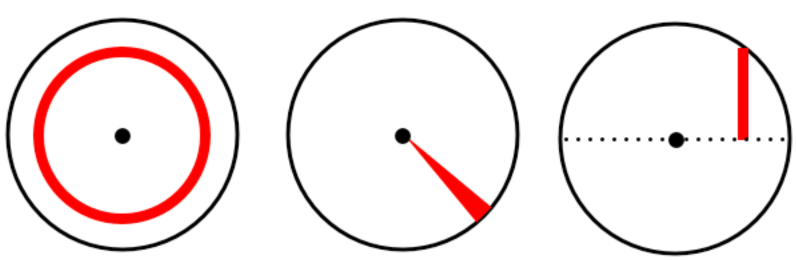
$$= \pi R^2 - \pi(0)^2 = \pi R^2$$

which appears to be correct.

Note in passing that the lower bound doesn't have to be 0, it could be some $\rho < R$. Then we'd have the area of a ring rather than a circle. And another thing, it's not uncommon to leave out the variable from the bounds, and write it like this:

$$2\pi \int_0^R r\ dr$$

wedges



In the second method (middle panel), we need to first find the area of a wedge. For a thin enough slice, this is a triangle, with a familiar formula: one-half the base times the height. The height is R, the radius of the circle.

For the base we need the length of a piece of arc of a circle. Recall that by definition, if we have a unit circle, then the angle of a wedge is equal to the arc it cuts out, and vice-versa, the arc is equal to the angle. (Thus, the total length if we go all the way around the unit circle is 2π).

For a circle with radius R, the length going all the way around is $2\pi R$, and the length of arc for any angle θ is θ times R.

The area we want is built up of a series of wedges that are almost infinitely slender, with angle $d\theta$, so these wedges have bases measuring $R\ d\theta$. The area of each triangular wedge is one-half the height times the base or

$$dA = \frac{1}{2} R\ R\ d\theta$$

For the total area

$$A = \int dA = \int \frac{1}{2} R\ R\ d\theta$$

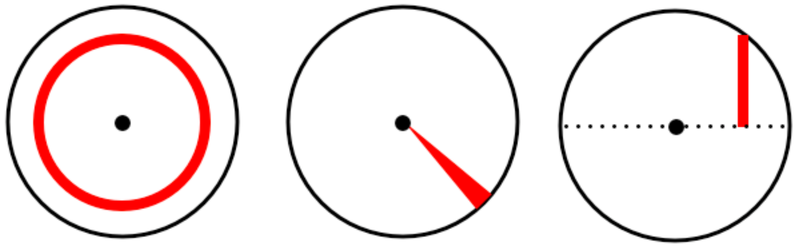
again we see that constants can come outside the integral

$$= \frac{1}{2} R^2 \int_{\theta=0}^{\theta=2\pi}\ d\theta$$

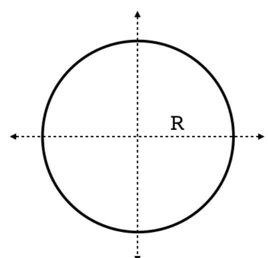
$$= \frac{1}{2} R^2 \theta\ \Big|_{\theta=0}^{\theta=2\pi}$$

$$= \pi R^2$$

area under the curve



The third view (right panel) is the most familiar, but has a somewhat harder calculation. We calculate the area under the positive square root in the equation for a circle (right panel), lying above the x-axis, and then multiply by two to get the whole thing.



$$x^2 + y^2 = R^2$$
$$y = f(x) = \sqrt{R^2 - x^2}$$

To get the area, we need to integrate:

$$\int y \ dx = \int_{-R}^{R} \sqrt{R^2 - x^2} \ dx$$

We will work through this problem **later**, after we review a few more techniques that are useful in doing integration problems.

Of course, the answer will turn out to be just what you'd expect. In fact, this must be so. If we solve the same problem by correctly using two different techniques and get different answers, then at least one of the techniques is wrong.

The area beneath the circle $y = \sqrt{R^2 - x^2}$ and above the x-axis is

$$\frac{1}{2}\pi R^2$$

which is multiplied by 2 to get the area of the whole circle.

Volume of the sphere

We think about how the volume of the sphere depends on r ($r = 0 \rightarrow R$). An incremental change dr changes the volume by adding a thin shell of volume equal to the surface area of the sphere ($4\pi r^2$) times dr. That is

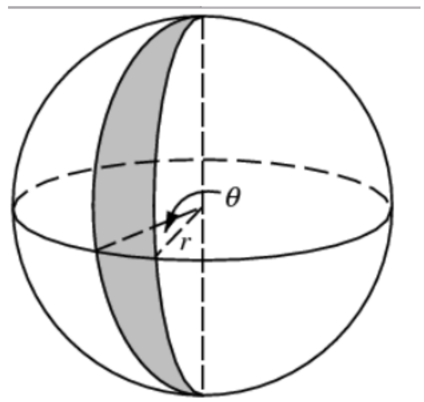
$$dV = 4\pi r^2 \ dr$$

$$V = \int dV = \int_0^R 4\pi r^2 \ dr$$

$$= 4\pi \ \ \frac{1}{3}r^3 \ \Big|_0^R = \frac{4}{3}\pi R^3$$

It's really as simple as that. Of course, you need to know the formula for the surface area to do it that way. Alternatively, if you know the volume of the sphere, taking the derivative is an easy way to get a formula for the surface area.

The image shows a "spherical lune", or segment of the surface of the sphere, as an aid to visualizing the whole surface.



We'll say a lot more about the volume of the sphere **later**.

technical note

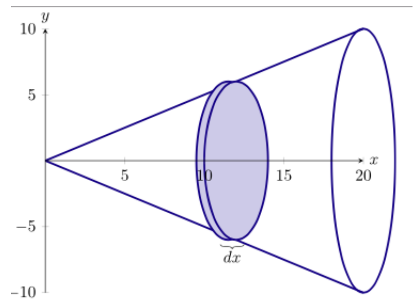
We should point out that this connection between volume and surface area is not true for *every* solid.

As an example, the surface area of a cube of side s is $6s^2$, which would have volume $2s^3$ if the relationship were always correct. In fact, there is something special about the *radial symmetry* of circles and spheres, and their lack of sharp corners and edges.

Here is one more example, to calculate the volume of a cone.

volume of a cone

We lay a cone along the x-axis with its vertex at the origin, opening to the right.



The cone is three-dimensional with the third axis (z) coming up out of the page. The intersection with the xy-plane is a triangle.

Can you see that in the xy-plane y is a linear function of x, i.e. $y = kx$ where k is a constant. The constant k is actually the ratio of the radius R to the height H. That is equal to $\Delta y/\Delta x$.

$$y = \frac{R}{H}x$$

If we slice the cone into thin sections perpendicular to the x-axis, each little piece is a circle with radius y and area πy^2. For a thin enough slice, the volume is that area times the width of the slice:

$$dV = \pi y^2 \ dx$$

Finding the volume of an individual piece is the important part of the calculus argument.

Now we just substitute the value of y in terms of x

$$dV = \pi \ [\frac{R}{H}]^2 x^2 \ dx$$

add up all the little volumes by setting up the integral

$$V = \int dV = \int \pi \ [\frac{R}{H}]^2 x^2 \ dx$$

We apply the basic rule that constant terms can move "out from under" the integral sign:

$$= \pi \ [\frac{R}{H}]^2 \ \int x^2 \ dx$$

This is a corollary of the result that constants are just carried through in taking the derivative.

We recognize that the value x lies in the interval between 0 and H, $[0, H]$, so these are the "bounds" on the integral, which we write as $\int_0^H$:

$$= \pi \ [\frac{R}{H}]^2 \int_0^H \ x^2 \ dx$$

and then just follow the rule for doing a problem like this: $\int x^2 = x^3/3$. So

$$= \pi \ [\frac{R}{H}]^2 \ [\frac{x^3}{3}] \ \Big|_0^H$$

$$= \frac{1}{3}\pi R^2 H$$

This is the answer precisely because the derivative of the result $(x^3/3)$ is equal to the integrand we started with (x^2).

Once again, we obtain the formula of one-third times the area of the base times the height. No matter what the shape of the base is, the area of each slice will be proportional to x^2 and we will end up with a formula involving one-third at the end.

We will see several other methods for obtaining this result.

Note in passing that we can obtain the volume of a frustum (a cone whose top has been cut off) as

$$= \pi \ [\frac{R}{H}]^2 \ [\frac{x^3}{3}] \ \Big|_{h_1}^{h_2}$$

$$= \pi \; [\frac{R}{H}]^2 \; [\frac{{h_2}^3}{3} - \frac{{h_1}^3}{3} \;]$$

The geometers have given us an even more elegant formula (**here**).

Part II

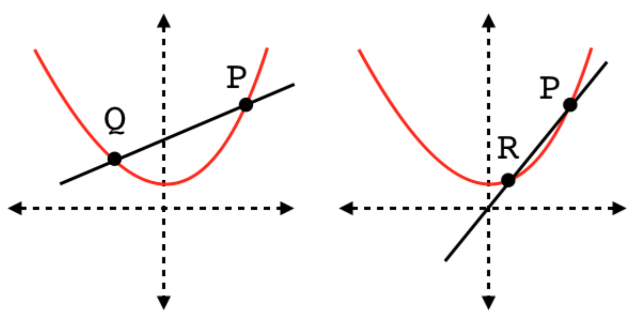
A little theory

Chapter 4

Difference quotients

In this chapter we look at the geometric interpretation of the derivative — which is the traditional way to begin calculus. The general approach was developed by Fermat.

Think for a minute about a curve such as the one shown in the figure, corresponding to some unspecified function $f(x)$, which looks like it's probably a parabola.



At an arbitrary point P on the curve, for some value of x, we plot $y = f(x)$. This is Descartes' genius idea. The point on the graph of $f(x)$ at x has coordinates $P = (x, f(x))$.

Now consider a point Q near P but also on the curve, For the x-

coordinate of Q, a small change is made to x.

We might call that small amount Δx, but many authors use h, a simpler notation, and we will do so as well. The value of the function at $x + h$ is $f(x + h)$ and so Q has coordinates $Q = (x + h, f(x + h))$.

In this example, h is negative, but that makes no difference. We drew it that way so it's easier to see how the approximation to the slope gets better as we go along.

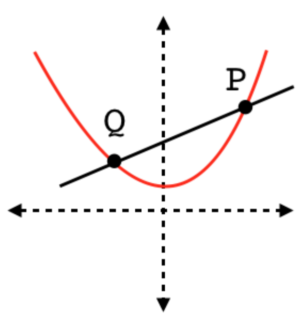
The slope of the (secant) line connecting Q and P is

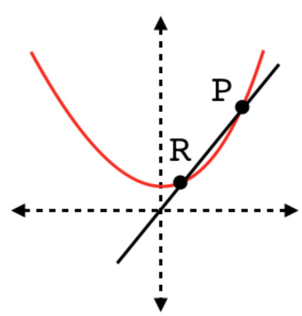
$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h}$$

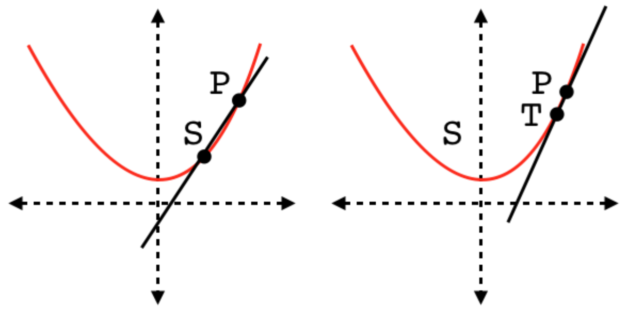
This is a famous quantity, it's called the **difference quotient**.

The goal of differential of calculus is to find the slope of the *tangent* to the curve at the point P. What we have is an expression for the slope of the secant line PQ, which is close but not quite the same thing.

To go from the secant to the tangent, we ask "what happens to this expression as h gets smaller and smaller and approaches zero." The second point where the secant meets the curve comes closer and closer to the first one.







In mathematical language, we say the slope of the tangent is equal to the limit of the difference quotient as h tends to 0:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We'll say a bit more about limits in the next chapter, but for the moment you can think about

$$\lim_{h \to 0}$$

as meaning, "substitute $h = 0$ and see what happens to the expression of interest."

x squared

Let's try a couple of examples and look for a pattern.

$$f(x) = x^2$$

For this function, we write that the difference quotient is

$$\frac{(x+h)^2 - x^2}{h}$$

$$= \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \frac{2xh + h^2}{h}$$

Now divide by the denominator h

$$= 2x + h$$

Finally, to get the slope of the tangent, we evaluate the limit

$$\lim_{h \to 0} 2x + h = 2x$$

In evaluating the limit, we ask: what happens to this expression as h approaches 0. In this case, it cannot actually reach zero, because then our previous step of dividing by h would not be allowed. But we let h become really really small, and take advantage of the property of the limit which says that an expression can have a limit at c even if it can't be evaluated at c itself.

At every point on the curve $y = x^2$, the slope of the tangent line to the curve is $2x$. So the slope at $x = 0$ is 0, and the slope at $x = 2$ is 4, and so on.

This process of computing the difference quotient and then finding the limit as $h \to 0$ is called "taking the derivative." It produces an expression which is called the derivative of y with respect to x, in this case

$$\frac{dy}{dx} = 2x$$

and we can interpret this as the slope of the tangent to the curve of $f(x)$ at the point x.

Another useful shorthand uses the f from $f(x)$. We adopt the convention that the derivative of $f(x)$ can be written $f'(x)$.

$$f'(x) = 2x$$

To be even more succinct we might write y' for $f'(x)$.

If we repeat this exercise with a leading constant a (that is, for $f(x) = ax^2$), we find that every term in the numerator of the difference quotient will contain a, and the final result will be $2ax$. Constants just get carried through.

square root

Now look at the square root:

$$f(x) = \sqrt{x}, \quad (x \geq 0)$$

The difference quotient for this function is

$$\frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Clean up the numerator by multiplying by the conjugate

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} \quad \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{x+h-x}{h\ (\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{h}{h\ (\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

We evaluate the limit

$$\frac{dy}{dx} = \lim_{h \to 0} \ \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

inverse

Consider the inverse function

$$f(x) = 1/x, \quad (x \neq 0)$$

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Clean up the numerator

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h} \quad \frac{(x)(x+h)}{(x)(x+h)}$$

$$= \frac{x - (x+h)}{h\ (x)\ (x+h)}$$

$$= \frac{-h}{h\ (x)\ (x+h)}$$

$$= -\frac{1}{(x)\ (x+h)}$$

We evaluate the limit:

$$\lim_{h \to 0}\ -\frac{1}{(x)\ (x+h)}$$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

There's a pattern here. We will use the notation $f'(x)$ to indicate the slope of the curve $f(x)$ at x

$$f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x$$

$$f(x) = \sqrt{x} = x^{1/2} \quad \Rightarrow \quad f'(x) = \frac{1}{2}x^{-1/2}$$

$$f(x) = \frac{1}{x} = x^{-1} \quad \Rightarrow \quad f'(x) = -\frac{1}{x^2} = -x^{-2}$$

The general formula is

$$f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1}$$

This is easily proved (for integer n) using the binomial expansion for $(x+h)^n$ for integral n ($n \in 1, 2, \dots$). We need only the first three terms:

$$(x+h)^n = x^n + nx^{n-1}h + n\frac{(n-1)}{2}x^{n-2}h^2 + \dots$$

The key point is that the last term shown and all subsequent terms contain powers of h^2 or higher.

After division by h, for each of these terms there will remain one or more terms of h, and in the limit $\lim_{h \to 0}$ these become zero.

$$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + n\frac{(n-1)}{2}\, x^{n-2}h^2 + \dots - x^n}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h + n\frac{(n-1)}{2}\, x^{n-2}h^2 + \dots}{h}$$

$$= \lim_{h \to 0} nx^{n-1} + n\frac{(n-1)}{2}\, x^{n-2}h + \dots$$

$$= nx^{n-1}$$

Another question is what to do with a sum or difference of polynomials, such as

$$f(x) + g(x)$$

If you write out the difference quotient

$$\frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

everything can be exactly as before, just grouping all terms with $f(x)$ and those with $g(x)$ separately.

$$[f(x) + g(x)]' = f'(x) + g'(x)$$

We showed above by computing the difference quotient directly that

$$f(x) = \sqrt{x}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Here is another approach to the same problem. Consider

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

Solve for x as a function of y:

$$x = \sqrt{y}$$

We can do algebra with *differentials* (with some constraints):

$$\frac{dy}{dx}\,\frac{dx}{dy} = 1$$

$$2x\,\frac{dx}{dy} = 1$$

$$\frac{dx}{dy} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}$$

In observing the inverse relationship, remember that x and y are related by the equation $y = x^2$. For example, when $x = 2$, $dy/dx = 2x = 4$.

Using the relationship $f(x)$, when $x = 2$, $y = 4$, and $dx/dy = 1/2\sqrt{y} = 1/2\sqrt{4} = 1/4$, which is indeed the inverse of 4.

In this last section, after solving for x as a function of y, y is the *independent* variable. We can switch back to our usual notation:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

problem

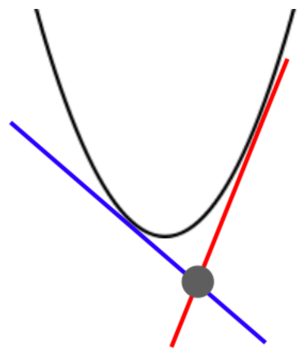
I found the following problems on the web. They are great practice and show what kinds of problems this approach of differentiation can solve. To prove:

Let $(a, f(a))$ and $(b, f(b))$ be two distinct points on the graph of a differentiable function f. Suppose that the tangent lines of f at these two points intersect, and call the point of intersection (c, d). Verifying the following facts is elementary.

1. **If $f(x) = x^2$, then $c = (a + b)/2$, the arithmetic mean of a and b.**
2. **If $f(x) = \sqrt{x}$, then $c = \sqrt{ab}$, the geometric mean of a and b.**
3. **If $f(x) = 1/x$, then $c = 2ab/(a + b)$, the harmonic mean of a and b.**

1

Here is a diagram for the first one:



The claim is that the x-coordinate of the point will be half-way between the x-coordinates for the two points on the parabola. We have:

$$y = f(x) = x^2$$

$$y' = f'(x) = 2x$$

At $x = a$, the slope is $2a$ and the equation of a line through the point (a, a^2) is

$$y - a^2 = 2a(x - a)$$

At $x = b$, the equation is

$$y - b^2 = 2b(x - a)$$

To see where the lines cross, we set the y's to be equal, and solve for x:

$$2a(x - a) + a^2 = 2b(x - b) + b^2$$

$$2ax - a^2 = 2bx - b^2$$

$$2x(a - b) = a^2 - b^2$$

$$= (a + b)(a - b)$$

$$x = \frac{1}{2}(a + b)$$

2

We have:

$$y = f(x) = \sqrt{x}$$
$$y' = f'(x) = \frac{1}{2\sqrt{x}}$$

At $x = a$, the slope is $1/2\sqrt{a}$ and the equation of a line through the point $(a, \sqrt{a})$ is

$$y - \sqrt{a} = \frac{1}{2\sqrt{a}}\ (x - a)$$

At $x = b$, the equation is

$$y - \sqrt{b} = \frac{1}{2\sqrt{b}}\ (x - b)$$

We set the y's to be equal

$$\frac{1}{2\sqrt{a}}\ (x - a) + \sqrt{a} = \frac{1}{2\sqrt{b}}\ (x - b) + \sqrt{b}$$

and solve for x. Multiply by $2\sqrt{a}\sqrt{b}$

$$(x - a)\sqrt{b} + 2a\sqrt{b} = (x - b)\sqrt{a} + 2b\sqrt{a}$$

Multiply through and cancel

$$x\sqrt{b} + a\sqrt{b} = x\sqrt{a} + b\sqrt{a}$$
$$x(\sqrt{b} - \sqrt{a}) = b\sqrt{a} - a\sqrt{b}$$
$$= \sqrt{a}\sqrt{b}(\sqrt{b} - \sqrt{a})$$
$$x = \sqrt{ab}$$

3

We have:

$$y = f(x) = \frac{1}{x}$$

$$y' = f'(x) = -\frac{1}{x^2}$$

At $x = a$, the slope is $-1/a^2$ and the equation of a line through the point $(a, 1/a)$ is

$$y - 1/a = -\frac{1}{a^2}\ (x - a)$$

At $x = b$, the equation is

$$y - 1/b = -\frac{1}{b^2}\ (x - b)$$

We set the y's to be equal

$$-\frac{1}{a^2}\ (x - a) + 1/a = -\frac{1}{b^2}\ (x - b) + 1/b$$

and solve for x:

$$\left(\frac{1}{b^2} - \frac{1}{a^2}\right)x = 2\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$\left(\frac{1}{b} + \frac{1}{a}\right)x = 2$$

$$(a + b)x = 2ab$$

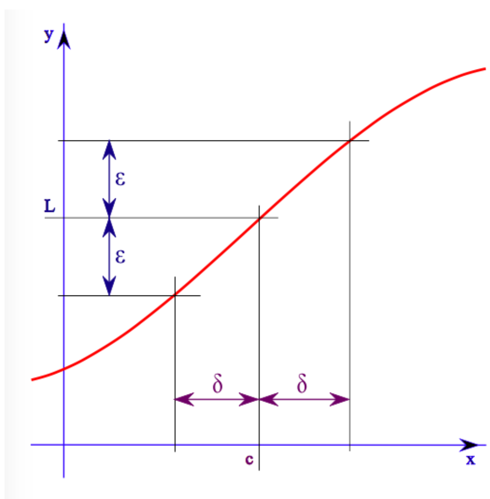
$$x = \frac{2ab}{a + b}$$

Chapter 5

Limit concept

Limit concept

Consider the graph of a function $f(x)$. We might choose a power of x similar to $y = x^2$ or $y = x^3 - x$, which affirmatively has two properties that are of interest here: continuity and differentiability (we'll get to those ideas in a bit). Let's just say $y = f(x)$ is a "good" function. The functions we deal with in this book are all "good."



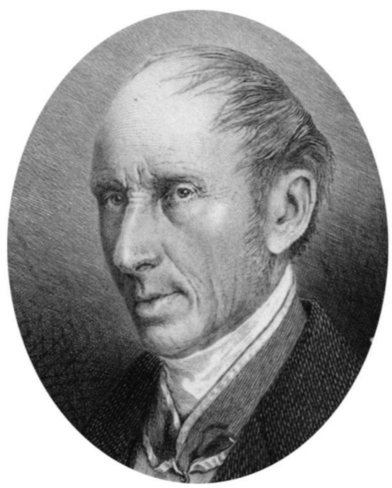
Focus on the neighborhood of a point on the x-axis, $x = c$.

By inspection of the graph, for points near c, the value of f at those points is not too different from L.

(It is also true here that the value of $f(x)$ at c is equal to L. This matters for continuity but not for limits).

We would like to say that the *limit* of $f(x)$ as x *approaches* c is equal to L. The idea is that we can make $f(x)$ as close to L as we please, provided we choose x sufficiently close to c.

> When the values successively attributed to a variable approach indefinitely to a fixed value, in a manner so as to end by differing from it by as little as one wishes, this last is called the limit of all the others. —Cauchy



Modern mathematicians don't like that word "approach", which conjures up movement and the involvement of time.

They also don't like reasoning from what they see in a graph, in part because no graph can show the whole function for the general case. To free ourselves from graphs and pictures, we will use an algebraic method from the formal apparatus of calculus.

There are two equivalent approaches, neighborhoods, and epsilon-delta formalism. Let's look at neighborhoods briefly.

neighborhoods

First, an *interval* between two real numbers a and b ($a < b$) contains every real number $a < x < b$.

$$(a, b) = x \mid a < x < b$$

The "|" means x "such that" the condition $a < x < b$ holds.

A *closed* interval $[a, b]$ includes the endpoints, $a \leq x \leq b$, while an *open* interval (a, b) excludes them. Half-open intervals like $[a, b)$ may be defined, and an interval with $\pm\infty$ as an endpoint is always open on that end, for example: $[a, \infty)$, because infinity *is not a number*.

Any open interval with a point p as its midpoint is called a *neighborhood* of p. Let r be the distance from p to the boundary of a particular neighborhood; r may be large or very very small. We denote a neighborhood of p as $N(p)$. $N(p)$ consists of all those values of x such that

$$|x - p| < r$$

.

which we would write more formally as

$$N(p) = x \mid |x - p| < r$$

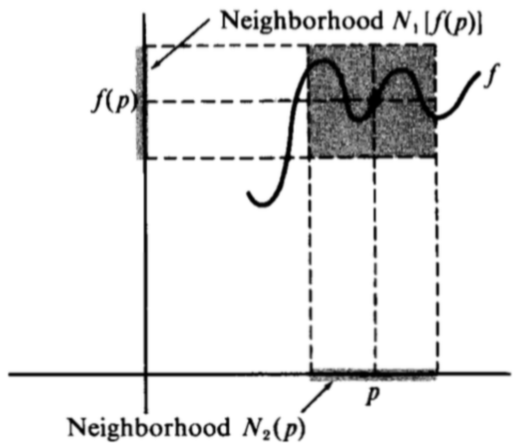
.

To say that the limit $f(x) \to L$ exists, we mean that for every neighborhood $N_1(L)$, no matter how small, there exists some neighborhood $N_2(p)$ such that $f(x)$ is contained within $N_1(L)$, written as

$$f(x) \in N_1(L)$$

whenever $x \in N_2(p)$.

If $N_1(L)$ is very small, then $N_2(p)$ may need to be very small as well, to guarantee that $f(x)$ is contained within N_1. Here is an example where this condition is satisfied.



The idea of a neighborhood is a nice abstraction to hide the apparatus of modern calculus, which we save for the Addendum.

An important fact about limits has to with the case where $x = p$. It is *not* necessary that $f(p) = L$. This relaxed condition is in fact crucial for calculus.

example 1

Limits can be easy or hard, depending on the problem. Here is one found in the previous chapter on difference quotients:

$$\lim_{h \to 0} \frac{2xh + h^2}{h}$$

When you see something like this, what you are supposed to do is reason about what happens as the variable h approaches 0 (gets smaller and smaller). The first step in that is to figure out what would happen if h actually would become zero.

Here, each term has a limit of 0 when h is zero, so we will have $0/0$. The zero on the bottom is trouble, it means that the expression becomes undefined.

However, suppose we first cancel h on top and bottom to obtain

$$\lim_{h \to 0} \frac{2x + h}{1} = \lim_{h \to 0} \ 2x + h$$

Now, the answer is just $2x$. This is valid as long as h approaches zero but is never actually equal to it.

Recall that we can have a limit for $f(x)$ as x approaches c, even if $f(c)$ does not exist.

example 2

Here is another important expression. What is the value of $f(x)$ as h approaches zero?

$$\cos h < f(x) < \frac{1}{\cos h}$$

Since $\cos 0 = 1$, the two outside terms both approach 1 in the limit as h approaches zero. Since $f(x)$ lies between them, it must also approach 1.

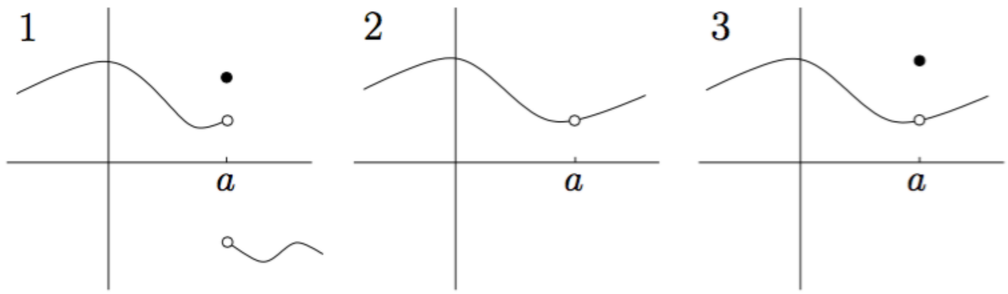
This is called the *squeeze theorem.*

The magical thing is that this is true even if, when $h = 0$, $x = 0/0$. We'll see this when we look at calculus of sine and cosine.

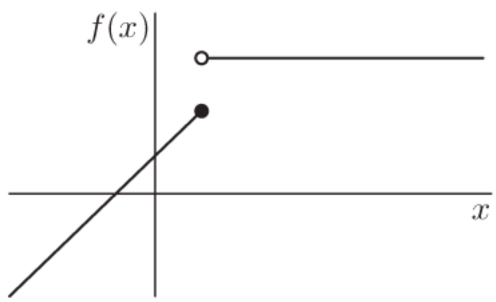
Continuity

Continuity has an intuitive definition: as Euler said, if we can graph a function *without lifting our pencil from the paper*, then the function is continuous.

Here are some graphs showing examples of how continuity can fail.



A filled circle means that the function yields that y-value for the corresponding x-value of the point, while an open circle means it does not. The function may yield some other value, or simply be undefined.



$$f(x) = \begin{cases} x+1 & \text{if } x \leq 1 \\ 3 & \text{if } x > 1 \end{cases}$$

For a function to be continuous at a point $x = c$, we imagine that if we vary x in neighborhood of c, then $f(x)$ should not change in value by too much.

Again, we will call that value L, the limit of $f(x)$ as $x \to c$. For L to exist we require that the two one-sided limits be equal. If we approach

c from the high side ($x > c$) or the low side ($x < c$), the limit must be the same.

Very important: continuity requires, in addition, that $f(c)$ be equal to L.

Differentiability

For a function to be differentiable, we require that the limit

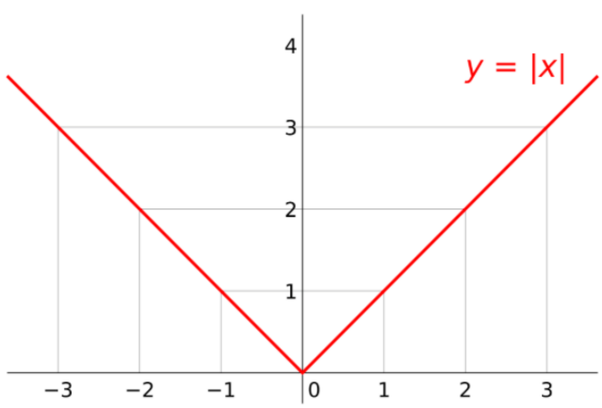
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. An example of a function that is continuous but not differentiable at a particular point is the absolute value function.

example: absolute value

An algebraic definition of the absolute value function is piecewise:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



The function $f(x) = |x|$ is continuous at $x = 0$ because the two one-sided limits exist and are equal to each other. They are also equal to $f(0) = 0$.

However, there is no defined slope at $x = 0$. The difference quotient gives different results for positive Δx (positive slope) than for negative Δx (negative slope).

Without getting too technical

> Note that the graph of the absolute value function is "all in one piece", but has a "sharp point" at the origin. We will not attempt to make these descriptions precise, other than to say that the fact that the graph comes "all in one piece" is a feature of continuity, and that graphs of differentiable functions are "smooth" in that they do not have "sharp points." The unambiguous and demonstrably true statement here is that the absolute value function is continuous at 0 but is not differentiable at 0.

https://oregonstate.edu/instruct/mth251/cq/Stage5/Lesson/diffVsCont.html

practical limits

- Plug in the value and see what happens. No problem here:

$$\lim_{x \to 2} \frac{x+1}{x^2+3} = \frac{3}{7}$$

- Division by zero isn't allowed. But we can factor:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{x-3} = \lim_{x \to 3} x + 3 = 6$$

- Limit at infinity. Convert to a limit at zero:

$$\lim_{x\to\infty} \frac{x^2+3}{3x^2+x+1} = \lim_{1/x\to 0} \frac{1+3/x^2}{3+1/x+1/x^2} = \frac{1}{3}$$

Chapter 6

Higher derivatives

We have defined the derivative of a function $f(x)$ as a limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

It is the limit of the difference quotient as $h \to 0$.

We introduced the power rule to obtain the derivative of integer powers of x. In addition, we said that the derivative of a sum of two or more functions is the sum of the derivatives.

The derivative is just a function itself. Consider a quadratic like

$$y = ax^2 + bx + c$$

The derivative is

$$y' = 2ax + b$$

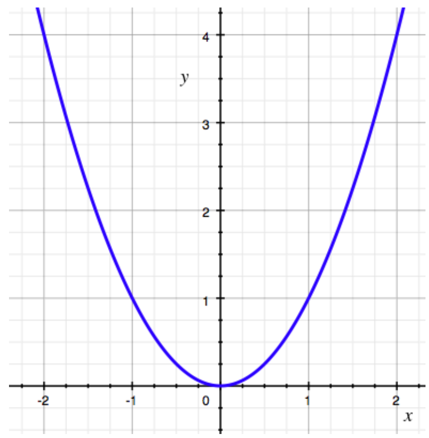
Since the derivative is just a function, why not take the derivative of the derivative, which is called

$$\frac{d^2y}{dx^2}$$

or more compactly, y double prime:

$$y'' = 2a$$

What is the meaning of the second derivative? It gives the slope of the slope, or how the slope is changing with a change in x. For a parabola that opens up, like this one



the second derivative is positive. This means that the slope continues to increase as x increases.

There is nothing to stop us from taking more derivatives. For a quadratic $y''' = 0$, which is not very interesting, but consider the cubic:

$$y = x^3$$
$$y' = 3x^2$$
$$y'' = 6x$$
$$y''' = 6$$

extrema

A very important use of the derivative is to find a maximum or minimum of a function. At such a point the slope is zero because the curve is headed sideways, just for a moment. For the quadratic, the slope is zero at the vertex:

$$y' = 0 = 2ax + b$$

$$x = -\frac{b}{2a}$$

You should recognize this equation from geometry. Without getting into details, it is obtained there by completing the square. Or alternatively, write the equation of a parabola whose vertex is (h, k)

$$(y - k) = a(x - h)^2$$

multiply out

$$y = ax^2 - 2ahx + h^2 + k$$

By comparison with the standard form

$$y = ax^2 + bx + c$$

it's clear that

$$-2ahx = bx$$

so the x-coordinate of the vertex is

$$h = -\frac{b}{2a}$$

The vertex is the maximum or minimum of a quadratic, depending on the sign of a. We can tell the difference by looking at the second derivative again:

$$y'' = 2a$$

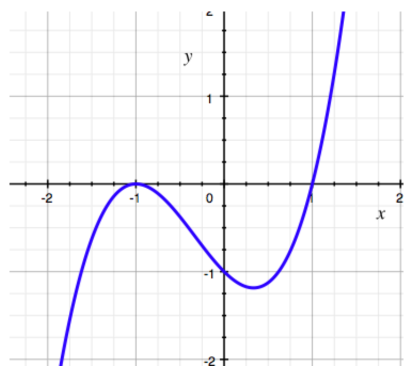
If $a > 0$ (a parabola that opens up), then the second derivative is positive and we have found a minimum value. If $y'' < 0$, we have a maximum.

Consider the cubic

$$\begin{aligned} y &= (x+1)(x+1)(x-1) \\ &= (x^2+2x+1)(x-1) \\ &= x^3+x^2-x-1 \end{aligned}$$

From the first form, we can easily see that the roots are $x = \pm 1$. These are the values where the function crosses the y-axis, that is, where its value is zero.

The graph looks like this:



The first derivative is

$$\begin{aligned} y' &= 3x^2+2x-1 \\ &= (3x-1)(x+1) \end{aligned}$$

This expression is zero when $x = -1$ or $x = 1/3$. That does match the places where the curve is horizontal, as we can see.

The second derivative is

$$y'' = 6x+2$$

For the first value $x = -1$, the second derivative is negative, and this corresponds to a local maximum for the function. For $x = 1/3$, the second derivative is positive, and this is a minimum. We say "local" because there may be more extreme values, as is the case here.

A maximum corresponds to a negative value for the slope of the slope because the slope is first positive, then zero, then negative. Its change with increasing x is negative.

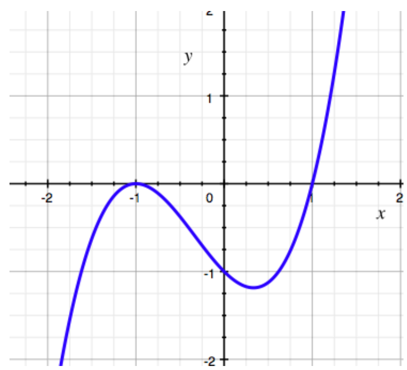
Again, the second derivative of our cubic is

$$y'' = 6x + 2$$

Setting this equal to zero, we obtain

$$x = -\frac{1}{3}$$

This point on the curve is an *inflection point*. It is a point (actually the only point for this curve) where the rate of change of the slope, which is negative to the left $x = -1/3$, changes to positive, and there is an instant where it is zero.



We will see this in connection with the Gaussian or normal curve. It's an interesting fact that the first standard deviation corresponds to the inflection point of the curve. At that point the second derivative of the function is equal to zero.

Rectangular area

Here is a classic problem.

We wish to construct a rectangle with the maximum area *for a fixed perimeter* (without the second statement the area would be infinite). Let's call the sides x and y, and the semi-perimeter S (constant) and so our constraint is that

$$S = x + y$$

The area is then

$$A = xy$$

substitute

$$\begin{aligned} A &= x(S - x) \\ &= Sx - x^2 \end{aligned}$$

Take the first derivative and set it equal to zero:

$$A' = S - 2x = 0$$

$$x = \frac{S}{2} = y$$

A square has the maximum area for a given perimeter constructed with right angles, as expected. We'll see many challenging problems of this type later on.

Chapter 7

Differentials

infinitesimals

We say that the derivative dy/dx is the slope of the tangent to the curve $y = f(x)$ at some particular point (x, y); it is the slope of a line that just touches the curve.

And it is frequently called the slope of the curve at the point (x, y).

We saw a formal definition for the derivative in terms of an expression for Δy as a function of Δx, which is then divided by Δx. We determine what is called the "limit" as Δx approaches zero.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

There appears to be a contradiction here. On one hand, we're saying that Δx must approach zero. On the other hand, it cannot really be zero, because division by zero is not defined. So how close does it have to be to zero?

Are dy and dx small, really small, really really small, or almost zero?

The official answer requires a cumbersome apparatus of limits, and it would say that dy/dx is not a quotient at all, but rather a single entity, the limit of a quotient, as we just said:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

However, our simple answer is that dx and dy are separable entities and they are just as small as they need to be. Take the example we used previously:

$$\frac{dy}{dx} = 3cx^2$$

Put the dx on the right-hand side:

$$dy = 3cx^2 \ dx$$

What this expression says is that for a small change in x which we call dx, we will obtain a small change in y called dy with the given relationship. $3cx^2$ gives the proportionality between dx and dy.

Suppose we evaluate $3cx^2$ at some particular x_0. Then it is a number, it has a fixed value depending on where we are on the curve. So write it as $k = 3cx_0^2$ and then:

$$dy = k \ dx$$

When we write this, we are making a *linear approximation* to the quadratic function. dy is not exactly equal to $k \ dx$, for most situations we are ignoring quadratic and higher terms.

Here, we treat dy and dx as very small but non-zero quantities. If there should ever be a problem because we've chosen Δx too large, just reduce it by some factor ($1/10$, 10^{-6}, 1/googol), whatever is needed,

https://en.wikipedia.org/wiki/Googol

and try again until the problem disappears (it will). Make dx and dy really really small. If that's not small enough, try making them smaller still.

By this trick, we free ourselves from limits. If you want to multiply by dx on both sides of an equality

$$\frac{dy}{dx} = nx^{n-1}$$

$$dx \cdot \frac{dy}{dx} = dy = nx^{n-1}\ dx$$

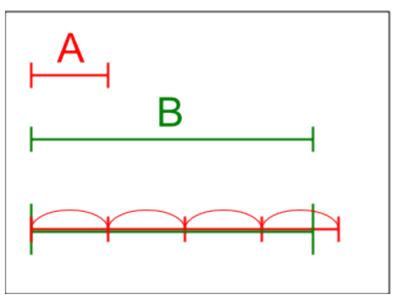
Feel free, go ahead and do it.

fancy

If you want to read more about the derivative as a limit, why dy/dx is not a quotient, and so on, you can look at any standard calculus book. Or start here

https://math.stackexchange.com/questions/21199/is-frac-textrmdy-textrmdx-not-a-ratio

The bottom line is that we want dy and dx to be very small compared to y and x, but one of the properties of the real numbers is that no matter how small small we choose A (or dx), there exists a positive integer n such that $n \cdot A > B$ (or $n \cdot dx > x$). This is called the Archimedean property of the real numbers.



Effectively what limits and neighborhoods do is to say, OK smart guy, you go first. Pick n. Then once you've picked n very large, we can always find dx very very small so that $n \cdot dx$ is still small compared with x. That's the whole trick.

However, in practice none of this is a problem because we view dy and dx as very small. Although often we only care about their ratio, some times we will need to separate them. This is legal, trust me.

Chapter 8

Fundamental theorem of calculus

> Calculus has a long history. Although Newton and Leibniz are credited with the invention of calculus in the late 1600s, almost all the basic results predate them. One of the most important is what is now called the Fundamental Theorem of Calculus (ftc), which relates derivatives to integrals.

https://mathcs.clarku.edu/~ma120/FTC.pdf

The usual way to begin the study of calculus is to think about the slope of the tangent to a curve at a point. If the point is some particular value of x, say $x = a$, then this is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

For any x in the domain of $y = f(x)$, we say that this slope is the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We may call this construct by various names such as the derivative of y with respect to x, dy/dx, or $f'(x)$.

The notation dy/dx is due to Liebnitz, and $f'(x)$ is due to Lagrange. For situations where x is a function of time t, Newton would write $\dot{x}$, $\ddot{x}$ and so on. Newton called his method ”fluxions”, concentrating on derivatives with respect to time.

$$y = f(x)$$
$$\frac{dy}{dx} = f'(x)$$

Evaluation of this limit for $f(x) = x^n$, $f(x) = e^x$, and $f(x) = \sin(x)$ then follows.

Some rules (product rule, chain rule and so on) will be introduced that allow us to calculate derivatives of more complicated functions. We also learn to keep note of various functions and their derivatives because it is essential to be able to ”go backwards.”

The inverse of differentiation is integration. By definition

$$y = f(x) = \int dy$$

Now

$$\frac{dy}{dx} = f'(x)$$
$$dy = f'(x)\ dx$$
$$y = f(x) = \int dy = \int f'(x)\ dx$$

There is an idea in integration which is really profound. We already introduced it in previous chapters by considering a ball or solid sphere in 3D space and the outer surface of the ball (technically, that *is* the sphere, but no matter). As Archimedes showed 2200 years ago, the volume of the sphere is this function of the cube of the radius.

$$V = f(r) = \frac{4}{3}\pi r^3$$

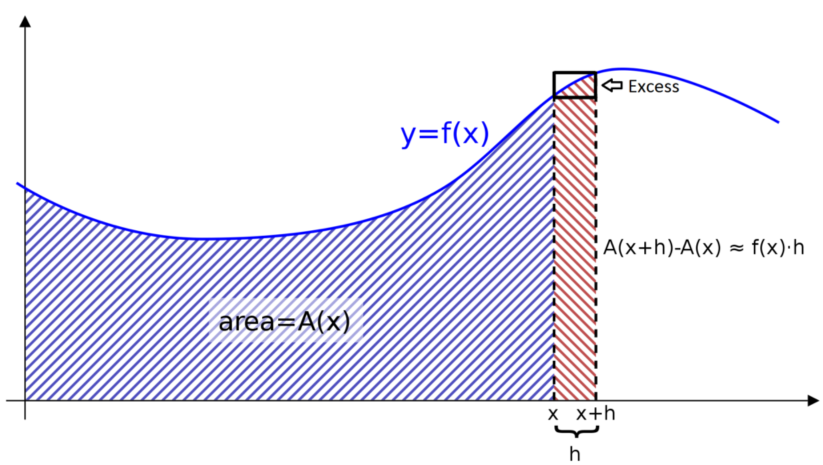
The question was: how does the volume change when the radius changes by a little bit? The big idea is to realize that the answer is exactly the same as the limit we gave above, it is the derivative of V with respect to r.

$$\frac{d}{dr}V = \frac{dV}{dr} = \frac{d}{dr}\frac{4}{3}\pi r^3 = 4\pi r^2$$

It is no accident that this result is the same as the formula for the surface area of the sphere. Increasing the radius r by a little bit dr, the new volume that is added is approximately the surface area $4\pi r^2$ times dr, the volume of the shell at the radius r.

$$dV = 4\pi r^2 \; dr$$

This is a completely general idea. The way it is usually introduced is to consider the graph of a function $f(x)$ in the plane.



We think not just about $f(x)$ itself, but the total area underneath the curve. Area is a function. Its value depends on the bounds. Let us fix the left-hand boundary (say, at $x = a$), but leave the right-hand bound as a variable, call it x.

The big idea is that the total area under the curve (the region in blue) is some as yet unknown function of x, $A = F(x)$.

The way we find $F(x)$ is to ask the question: how does the area $F(x)$ change when x changes by a little bit, say h? If you look at the figure it is clear that the answer is the area of the red rectangle, the added area is just $f(x)$ times h. If h is small enough, the answer is exact.

Recast in mathematical terms

$$\lim_{h \to 0} \ f(x) \ h = \lim_{h \to 0} \ F(x + h) - F(x)$$

We just divide by h

$$\lim_{h \to 0} \ f(x) = \lim_{h \to 0} \ \frac{F(x + h) - F(x)}{h}$$

and recognize that the term on the left does not depend any longer on h so

$$f(x) = \lim_{h \to 0} \ \frac{F(x + h) - F(x)}{h}$$

$f(x)$ is the derivative of $F(x)$:

$$f(x) = F'(x)$$

To find the area function F, we just need to find a function that, when we differentiate it, gives us $f(x)$.

The fundamental theorem of calculus states this principle. Usually, a new variable is introduced to remove any confusion that might arise with respect to x, which in the discussion above, we actually used in two different ways. Write

$$F(x) = \int_a^x f(t)\ dt$$

In the equation above, the real variable is x. t is what's called a dummy variable, since it might replaced with any other symbol without changing anything.

We have two different functions F and f. The value for each will vary with the value of x (since we evaluate at the right-hand bound $t = x$). The value of F depends also on the left-hand bound a. Anyway, having written

$$F(x) = \int_a^x f(t)\ dt$$

The Fundamental Theorem of Calculus (FTC) states:

$$F'(x) = f(x)$$

which is what we figured out before.

The FTC has a second part, which is

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

This gives us the way in which areas (and volumes and so on) are actually calculated. Start with the function $f(x)$. We find $F(x)$, and then just evaluate it at the endpoints a and b. The difference is the area under the curve $f(x)$ between the two bounds a and b.

The proof of this second part is usually done with what are called Riemann sums. We look at those a bit later in the book.

Part III

Three rules for differentiating

Chapter 9

Chain rule

Here's a classic problem leading to the next idea. Temperature in the atmosphere depends on the altitude, decreasing about 3 degrees F for each 1000 feet increase in altitude above sea level.

$$T = T_0 - 3h$$

$$\frac{dT}{dh} = -3$$

(In degrees F per thousand feet).

Suppose we're ascending a mountain road at a rate of 500 feet per minute.

$$\frac{dh}{dt} = 0.5$$

(In thousands of feet per minute).

What is the rate of change of temperature with time? It will turn out that

$$\frac{dT}{dt} = \frac{dT}{dh} \cdot \frac{dh}{dt} = -3 \times 0.5 = -1.5$$

(In degrees F per minute).

chain rule

A compound function $f(g(x))$ means that we apply the function g to the input x, then feed the output of that to the function f. A simple example would be $\sin 2x$.

The chain rule says that if we have a compound function then

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

If we break this down a bit we can write:

$$t = g(x)$$
$$y = f(t) = f(g(x))$$

Then

$$y' = f'(t) \cdot g'(x) = f'(g(x)) \cdot g'(x)$$

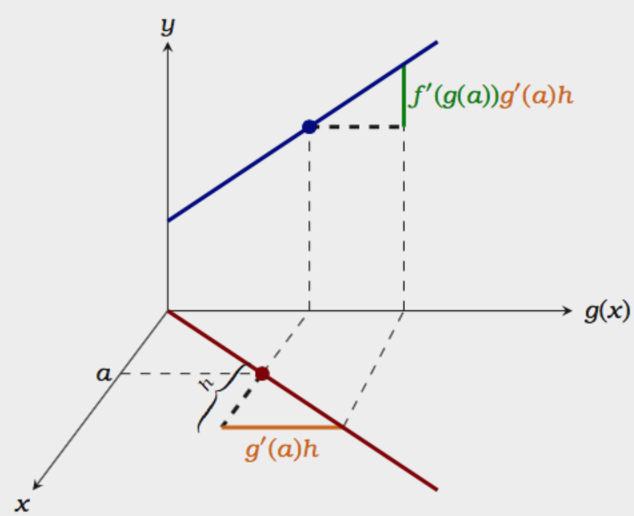


Figure 6.1: A geometric interpretation of the chain rule. Increasing a by a "small amount" h, increases $f(g(a))$ by $f'(g(a))g'(a)h$. Hence,

$$\frac{\Delta y}{\Delta x} \approx \frac{f'(g(a))g'(a)h}{h} = f'(g(a))g'(a).$$

For example

$$\frac{d}{dx}\sqrt{1-x^2} = \frac{1}{2}\ \frac{1}{\sqrt{1-x^2}}\ (-2x)$$
$$= -\frac{x}{\sqrt{1-x^2}}$$

What we've done is to treat $1-x^2$ as one whole expression. The factor of $-2x$ comes from the term $g'(x)$ in the chain rule.

We can do the same problem more slowly and clearly by substituting a new variable

$$t = 1 - x^2$$

Take the derivative:

$$\frac{dt}{dx} = -2x$$

Now rewrite the original $f(x)$ as $f(t)$:

$$y = f(x) = \sqrt{1-x^2}$$
$$y = f(t) = \sqrt{t}$$
$$\frac{dy}{dt} = \frac{1}{2}\ \frac{1}{\sqrt{t}}$$

What we really wanted to know was dy/dx. The chain rule says that

$$\frac{dy}{dx} = \frac{dy}{dt}\ \frac{dt}{dx}$$

(you can think of the factors of dt as canceling, I won't tell anyone).

$$= \frac{1}{2}\ \frac{1}{\sqrt{t}}\ (-2x)$$
$$= -\frac{x}{\sqrt{t}}$$
$$= -\frac{x}{\sqrt{1-x^2}}$$

common roots

There are two common roots that we will run into in this book. The first is the one given above where we might also have a constant a^2 and then

$$\frac{d}{dx}\sqrt{a^2 - x^2} = -x \ \frac{1}{\sqrt{a^2 - x^2}}$$

The power is $1/2$ which gives a factor of $1/2$ from the power rule. The new power is then $1/2 - 1 = -1/2$, and we pick up a factor of $-2x$ from the chain rule. The 2's cancel.

The second is

$$\frac{d}{dx}(a^2 - x^2)^{3/2} = -3x \ \sqrt{a^2 - x^2}$$

The power is $3/2$ which gives a factor of $3/2$ from the power rule. The new power is then $3/2 - 1 = 1/2$, and we pick up a factor of $-2x$ from the chain rule. The 2's cancel.

We'll see both of these again a number of times.

proofs

Proofs of the chain rule and the product rule and its corollaries in the next chapter can be found in almost any standard calculus text.

The proof of the chain rule above basically depends on a property of limits. If

$$\lim_{x \to c} f(x) = L$$

$$\lim_{x \to c} g(x) = M$$

then

$$\lim_{x \to c} f(x) \ g(x) = LM$$

If you'd like to know more, there is an extensive chapter on Limits in the Addendum.

Here's a sketch of a proof, from Kline. Let $y = f(u)$ and $u = f(x)$. Then what is dy/dx?

At some particular point x_0, for a small change Δx

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

In the example at the beginning of the chapter we had degrees F per thousand feet, and thousands of feet per minute. Since the relationships were linear, the relationships were exact, but they will hold for other functions in the limit that Δx becomes very small. This can also be seen clearly in the graphic above.

So now we take the limit of both sides.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

The left-hand side is just dy/dx. The right hand side is the limit of a product which, as we just said, is the product of the two limits. The second one is just du/dx.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{du}{dx}$$

Now, as $\Delta x \to 0$ also, so does $\Delta u \to 0$, otherwise the limit du/dx would not exist. Hence

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = \frac{dy}{du}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

□

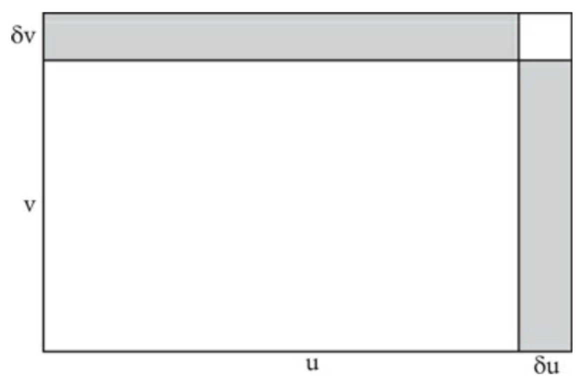
Chapter 10

Product and quotient rules

The product rule says that if we have functions $u(x)$ abbreviated as u, and $v(x)$ as v then

$$(uv)' = u'v + uv'$$

Here is one pictorial explanation:



If we have

$$(u + \delta u)(v + \delta v)$$

The product is

$$uv + u\delta v + v\delta u + \delta u \cdot \delta v$$

The difference quotient loses the first term by subtracting uv. The last term is much smaller than the others because it is $\delta \times \delta$ and can be neglected.

Here is another view:

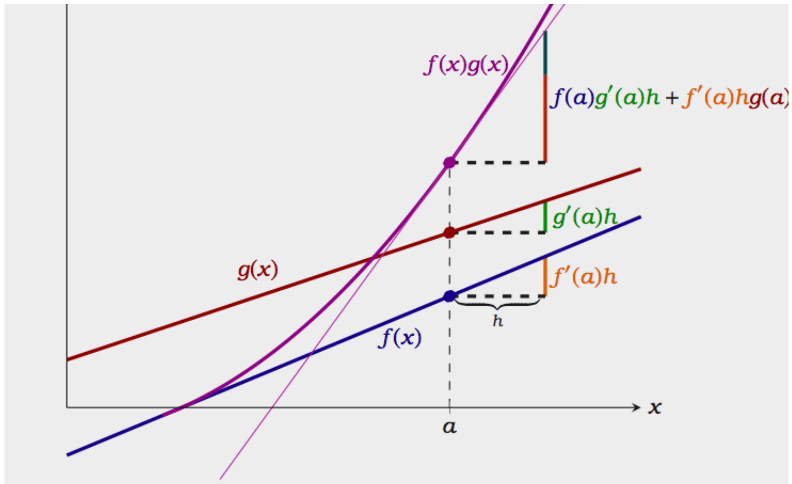


Figure 5.1: A geometric interpretation of the product rule. Since every point on $f(x)g(x)$ is the product of the corresponding points on $f(x)$ and $g(x)$, increasing a by a "small amount" h, increases $f(a)g(a)$ by the sum of $f(a)g'(a)h$ and $f'(a)hg(a)$. Hence,

We can check this in the simplest possible ways

$$(x)' = (1 \cdot x)' = 0 \cdot x + 1 \cdot 1 = 1$$

$$(x^2)' = (x \cdot x)' = 1 \cdot x + x \cdot 1 = 2x$$

$$(x^4)' = (x^3 \cdot x)' = x^3 \cdot 1 + 3x^2 \cdot x = 4x^3$$

or

$$(x^4)' = (x^2 \cdot x^2)' = x^2 \cdot 2x + 2x \cdot x^2 = 4x^3$$

and so on.

It is worth pointing out that one can build the general rule

$$(x^n)' = nx^{n-1}$$

by a chain of induction based on the product rule.

notes

There is a story that Leibnitz got this rule wrong at first — the claim is that he thought the answer was $f'(x)g'(x)$.

However, there isn't much evidence to support this. It seems to be a slander of Leibnitz by an English supporter of Newton, as part of the wars over who invented calculus. The supposition was shown in an unpublished notebook belonging to Liebnitz, as an example of an error.

https://mathoverflow.net/questions/181422/did-leibniz-really-get-the-leibniz-rule-wrong

The other observation is that the rule

$$(uv)' = u'v + uv'$$

was classically recited as "this times the derivative of that, plus that times the derivative of this." I learned it that way originally. But if you look closely at the equation you will see that I write the reverse these days.

The reason is that the version given above makes it easier to remember the quotient rule.

inverse

The product rule says that if we have functions $u(x)$ (abbreviated u), and $v(x)$ then

$$(uv)' = u'v + uv'$$

Remember that if u is not a simple function of x but a compound function like $f(g(x))$, then u' needs to take account of the inside function $g(x)$. By the chain rule

$$u' = f'(g(x)) \cdot g'(x)$$

As an example, when using the power rule might write

$$(\frac{1}{v})' = (v^{-1})' = -v^{-2} = -\frac{1}{v^2}$$

and this is fine so far as it goes.

But suppose v is a compound function, like $v(t)$, the correct result is really

$$(\frac{1}{v(t)})' = -\frac{1}{v^2}\ v'(t)$$

by the chain rule.

quotient rule

Accepting this result, use it and the product rule:

$$(u \cdot \frac{1}{v})' = u'\ \frac{1}{v} + u\ (\frac{1}{v})'$$

$$= u'\ \frac{1}{v} + u(-\frac{1}{v^2})v'$$

$$= \frac{u'v - uv'}{v^2}$$

The is known as the quotient rule.

As mentioned, take care to write the product rule as $u'v + uv'$ so as to get the quotient rule by just flipping the sign of the second term and then dividing by v^2.

You may find yourself wondering whether you have the sign right. Test with these two examples:

$$\frac{d}{dx}\ \frac{x}{1} = \frac{1 \cdot 1 - x \cdot 0}{1^2} = 1$$

$$\frac{d}{dx}\ \frac{1}{x} = \frac{1 \cdot 0 - 1 \cdot 1}{x^2} = \frac{-1}{x^2}$$

That looks correct. If you have the sign wrong this won't work out.

Strang

Gil Strang has many wonderful takes on calculus that I haven't seen in other books. For example:

Use the product rule here:

$$v \cdot \frac{1}{v} = 1$$

Take d/dx of both sides. By the product rule

$$\frac{dv}{dx} \cdot \frac{1}{v} + v \cdot \frac{d}{dx}\ (\frac{1}{v}) = 0$$

rearrange

$$v \cdot \frac{d}{dx}\ (\frac{1}{v}) = -\frac{dv}{dx} \cdot \frac{1}{v}$$

so

$$\frac{d}{dx} \cdot (\frac{1}{v}) = -\frac{1}{v^2} \cdot \frac{dv}{dx}$$

That's what we said!

This result extends the power rule to negative integers:

$$\frac{d}{dx}(x^{-n}) = \frac{d}{dx} \cdot (\frac{1}{x^n})$$

$$\frac{d}{dx} \cdot (\frac{1}{x^n}) = -\frac{1}{(x^n)^2}\ nx^{n-1}$$
$$= -nx^{-n-1}$$

Another one: we establish the power rule, by induction:

$$\frac{d}{dx}(u^{n+1})$$

$$= \frac{d}{dx}(u^n \cdot u)$$

$$= (nu^{n-1} \cdot \frac{du}{dx}) \cdot u + u^n \cdot \frac{du}{dx}$$

$$= (n+1)u^n \cdot \frac{du}{dx}$$

Since we used induction, the result applies only to integer n. Later we will see proofs that extend to rational and even irrational but real numbers n using implicit and logarithmic differentiation.

triple product

You don't see it much, but there is a triple product rule:

$$(uvw)' = u'vw + uv'w + uvw'$$

One might guess this is true by symmetry. Also

$$(uv \cdot 1)' = u'v \cdot (1) + uv' \cdot (1) + uv \cdot (1)'$$

The third term is zero, so we obtain just

$$u'v + uv'$$

Part IV

Standard integrals

Chapter 11

Powers and Polynomials

In this chapter we review briefly the most common integrals. Additional depth is given in a later chapter on **Techniques of integration**.

powers of x

The first **differentiation** that is done in basic calculus is

$$\frac{d}{dx}(x^n) = [x^n]' = nx^{n-1}$$

This answer is correct regardless of whether n is positive or negative, an integer or a rational number, or even irrational.

The proof for positive integers $n \in \{2, 3\}$ is simple. A general proof for positive integers is fairly easy using the binomial theorem.

Implicit differentiation will give a simple proof for all rational exponents. Still later we will have a proof for all real numbers. See **here**. We will finally find a function that yields x^{-1} as its derivative when we talk about the natural logarithm.

Ignoring some subtleties, we can **integrate** by reversing the process.

We write

$$x^n = \int nx^{n-1}\ dx$$

A real math book will tell you that when the formula stands alone like this, we should add a constant of integration: $x^n + C$. But we are mostly using integrals to calculate areas and volumes, which means we determine the value of a definite integral by computing the difference of two expressions which both contain C, so it cancels.

If the factor of n is not given in the problem, we insert it as a divisor up front:

$$\int x^{n-1}\ dx = \frac{1}{n} x^n$$

$$\int x^n\ dx = \frac{x^{n+1}}{n+1}$$

$$\int \sqrt{x}\ dx = \frac{2}{3} x^{3/2}$$

$$\int \frac{1}{\sqrt{x}} = 2\sqrt{x}$$

A common convention for notation is that $\int f(x) = F(x)$. Or, the other way around $f(x)$ is the derivative of $F(x)$, so $f(x) = F'(x)$. Let's make a table.

$f(x)$	$F(x)$
1	x
x	$x^2/2$
x^2	$x^3/3$
x^n	$x^{n+1}/n+1$
$\frac{1}{\sqrt{x}}$	$2\sqrt{x}$
$\sqrt{x}$	$\frac{2}{3}x^{3/2}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
e^x	e^x
$1/x$	$\ln x$
$a\cos ax$	$\sin ax$
$\cos ax$	$\frac{1}{a}\ \sin ax$
ae^{ax}	e^{ax}
e^{ax}	$\frac{1}{a}\ e^{ax}$

We will look at proofs for these trig functions and the exponential and logarithm in some detail in the next few chapters. For now, let's accept them provisionally.

fractional powers

Two additional integrals that we here add come from differentiations as seen in the chapter on the chain rule. With a^2 a constant:

$$\frac{d}{dx}\sqrt{a^2-x^2} = -x\ \frac{1}{\sqrt{a^2-x^2}}$$

$$\frac{d}{dx}(a^2 - x^2)^{3/2} = -3x\ \sqrt{a^2 - x^2}$$

To integrate, we reverse directions. The first one is

$$\int x\ \frac{1}{\sqrt{a^2 - x^2}}\ dx = -\ \sqrt{a^2 - x^2}$$

To check this, do the differentiation. The factor of $1/2$ on the right-hand side from the power and the leading -1 are canceled by the factor of -2 from the chain rule. Those are easy. The important thing is that we have that x under the integral sign.

The second one is

$$\int 2x\ \sqrt{a^2 - x^2}\ dx = -\frac{2}{3}\ (a^2 - x^2)^{3/2}$$

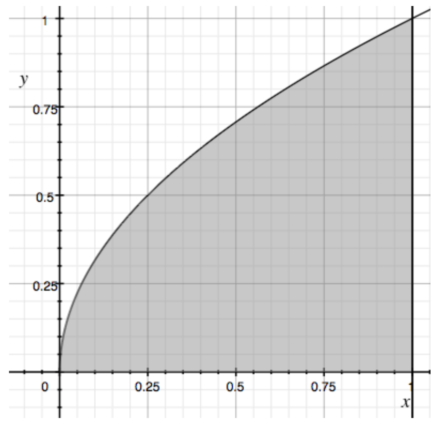
Again, check by differentiating the right-hand side. We get a factor of $3/2$ from the power, which cancels the leading $2/3$, then we get $-2x$ from the chain rule and cancel the minus sign. Factors of 2 and minus signs are easy, but we must have that x on the left, otherwise the answer will be much different.

square root

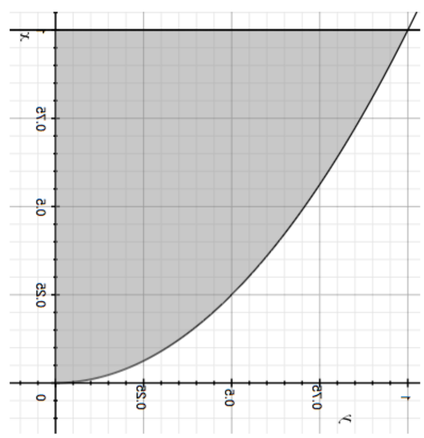
We will show that the integral of the square root gives the correct answer. Suppose we plot

$$y = \sqrt{x}$$

over the interval $[0, 1]$. The area under this curve is the shaded region.



The area seems to be more than 1/2, but how much more? Now, take the very same plot, flip it over and rotate 90 degrees. We obtain something that is more familiar, namely the plot of $y = x^2$. The area under this curve is the white region



Clearly

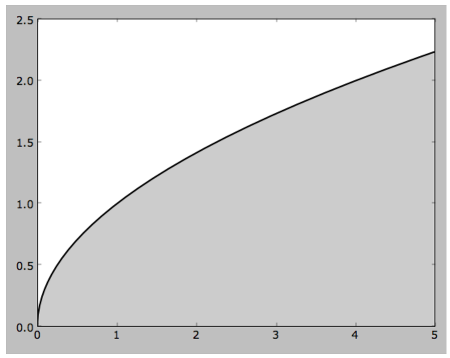
$$\int_0^1 \sqrt{x}\ dx + \int_0^1 x^2\ dx = 1$$

If you look in the list of formulas above (or just calculate) you will find that the first integral is equal to 2/3 and the second one is equal to 1/3.

This result holds for other intervals, not just $[0, 1]$.

Let's integrate to find the area under the curve in the interval $[0, b^2]$ (shaded gray below). We use b^2 for reasons that will be obvious in a minute.

$$A = \int_0^{b^2} \sqrt{x}\ dx = \frac{2}{3}x^{3/2}\ \Big|_0^{b^2} = \frac{2}{3}b^3$$



Now, consider x as a function of y:

$$x = y^2$$

Integrate to find the area between the curve and the y-axis (shaded white above). The only trick is that the bounds are $[0, b]$ because we want to add the two areas together to form a rectangle and the point on the curve at the upper bound will be $x = b^2, y = b$.

$$\int_0^b y^2\ dy = \frac{1}{3}y^3\ \Big|_0^b = \frac{b^3}{3}$$

Adding the results together, we obtain b^3 as the area of a box with width $\Delta x = b^2$ and height $\Delta y = b$, which is correct.

reversing the product rule

Playing around with the product rule can lead to the discovery of other important integrals that we will encounter later. The most important is

$$
\begin{aligned}
(\sin x \cos x)' &= \cos x \cos x - \sin x \sin x \\
&= \cos^2 x - \sin^2 x
\end{aligned}
$$

Recall the most basic trig identity:

$$
\sin^2 x + \cos^2 x = 1
$$

$$
-\sin^2 x = \cos^2 x - 1
$$

so

$$
(\sin x \cos x)' = 2\cos^2 x - 1
$$

integrating

$$
\sin x \cos x = 2 \int \cos^2 x \ dx - x
$$

$$
\int \cos^2 x \ dx = \frac{1}{2}(x + \sin x \cos x)
$$

That is one we'll want to remember. We will use several techniques of integration to derive it from first principles, but this is the easiest way. See **here**.

Here are a few others:

$$
(x \ln x)' = \ln x + \frac{x}{x} = \ln x + 1
$$

$$
\int \ln x \ dx = x \ln x - x
$$

$$
(x^2 \ln x)' = 2x \ln x + x
$$

$$
\int 2x \ln x = x^2 \ln x - \frac{x^2}{2}
$$

$$
(xe^x)' = e^x + xe^x
$$

$$
\int xe^x \ dx = xe^x - e^x
$$

Stirling's approximation

For an application, consider

$$\int \ln x \; dx = x \ln x - x$$

Computing $n!$ gets unwieldy for large n (at least, without computers). It comes up in probability and other places. There is a famous formula for $n!$ called Stirling's approximation which comes in a more precise version

$$n! \approx \sqrt{2\pi n}\ (\frac{n}{e})^n$$

and a less precise one

$$\ln n! \approx n \ln n - n$$

The latter is easily derived:

$$\ln n! = \ln 1 + \ln 2 + \ldots \ln n$$

$$= \sum_{k=1}^{n} \ln k$$

$$\approx \int_1^n \ln x \; dx$$

which we showed is

$$= x \ln x - x \; \Big|_1^n$$

$$= (n \ln n - n) - (1 \ln 1 - 1)$$

$$= n \ln n - n + 1$$

$$\approx n \ln n - n$$

Chapter 12

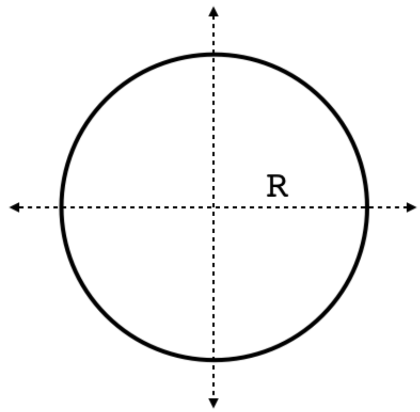
Shells and disks

This chapter is more challenging that the previous ones. But it's worth it, we will show the real power of calculus in finding volumes easily.

volume of the sphere by disks

We already saw how Archimedes found the volume of the sphere. Here we repeat the calculation using calculus of a single variable. This is the first example of what is called a "solid of revolution." We imagine revolving a curve (here, the top half of a circle), around an axis (the x-axis). Revolving the curve forms a surface, and we will consider the volume contained inside that surface.

Draw a circle of radius R, centered at the origin.



Points on the circle obey the standard equation

$$x^2 + y^2 = R^2$$

Rearrange and take the square root to get y as a function of x

$$y = f(x) = \sqrt{R^2 - x^2}$$

For this *function*, we would emphasize that we must choose one root, typically the positive one corresponding to the upper half-circle. The reason is that we need to have each x correspond to a *unique* value of y. If a vertical line cuts through two different curves for a single x, then we don't have a function.

Here, though, it doesn't matter, because are interested in vertical slices through the whole sphere. We will use the area of each slice, with radius y and area πy^2. Squaring, which makes the square root in $y = \sqrt{R^2 - x^2}$ go away, simplifies the integrals in this chapter a lot.

Move from left to right, from $x = -R$ to $x = R$ and add up the areas of all those slices:

$$V = \int_{-R}^{R} \pi y^2 \ dx$$

$$= \pi \int_{-R}^{R} (R^2 - x^2) \ dx$$

$$= \pi(R^2 x - \frac{x^3}{3}) \ \Big|_{-R}^{R}$$

At the upper bound, the term in parentheses is $2/3R^3$, and at the lower bound it is $-2/3R^3$, so subtracting we obtain

$$V = \frac{2}{3}\pi R^3 - (-\frac{2}{3}\pi R^3)$$

$$= \frac{4}{3}\pi R^3$$

as expected.

If you find yourself saying, that's just like what Archimedes did, well, yeah...

Evaluation of the lower bound is confusing (I find it so), so we could note that here the integrand $(R^2 - x^2)$ is an *even* function of x. What that means is

$$f(x) = f(-x)$$

and so

$$\int_{-R}^{R} f(x) \ dx = 2 \int_{0}^{R} f(x) \ dx$$

Take the integral from 0 to R and multiply by two. At this lower bound the value of the integral is zero.

There are several other derivations that use only one-variable calculus. My favorite of all is to integrate the surface area as a function of r, the radius.

We revisit this problem in some detail later, using multi-variable approaches (**here**).

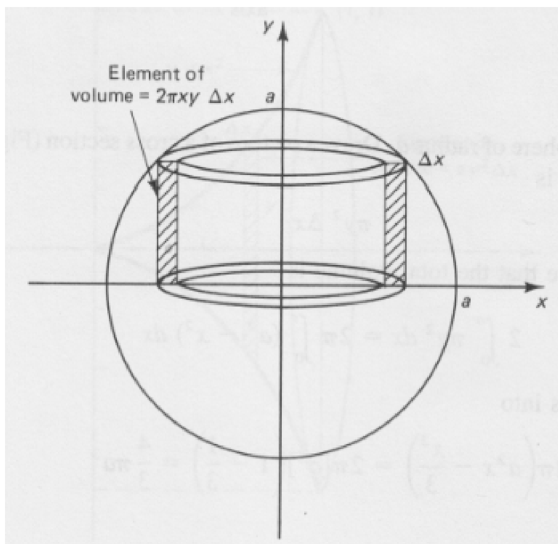
Separately, we will use the theorems of Pappus to find the volume of a "solid of revolution", namely rotation of a half-circle with its base on the x-axis, around that same axis. You can read about it here

http://mathworld.wolfram.com/PappussCentroidTheorem.html

We look at this toward the end of the book (**here**).

volume of the sphere by shells

Here is a picture of what we're doing, from Hamming's *Calculus*. The notation is different but the idea is the same.



We'll work with the hemisphere, above the xy-plane.

Let's divide the sphere up into concentric cylinders or shells, and let r vary from $0 \to R$. The circumference of the shell at each point is

$$C = 2\pi r$$

and the height of each is

$$h = \sqrt{R^2 - r^2}$$

The volume of each very thin cylinder is

$$dV = Ch\ dr = 2\pi r\sqrt{R^2 - r^2}\ dr$$

and we want

$$\int_0^R 2\pi r\sqrt{R^2 - r^2}\ dr$$

$$= -\frac{2}{3}\pi(R^2 - r^2)^{3/2}\Big|_0^R$$

We saw how this works in a previous chapter. To check

$$\frac{d}{dx}\ -\frac{2}{3}\ (R^2 - r^2)^{3/2} = \sqrt{R^2 - r^2}\ (2r)$$

That looks correct. Evaluate the result:

$$= -\frac{2}{3}\pi\ [\ -(R^2)^{3/2}\ ]$$

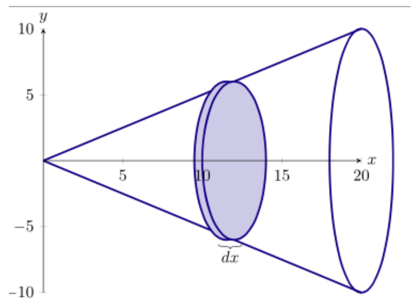
$$= \frac{2}{3}\pi R^3$$

Multiply by two to obtain the total volume.

That integral is one a little more challenging than most that we've had so far. Luckily there are not to many like it, and you will see the same one again and again.

volume of the cone by disks

We did this earlier. It's repeated here for comparison with the other approach.



We have a cone oriented so that it is symmetric about the x-axis, with its vertex at the origin and oriented so that it gets larger as we head to the right.

The equation for the line along the edge of the cone is that the y corresponding to each x is proportional to x with proportionality constant R/H. y is like the radius of the cone and x is like the height.

$$y = \frac{R}{H}x$$

When $x = 0$, $y = 0$, and when $x = H$, $y = R$. Since the formula is linear and it checks at two places, it is correct everywhere.

At each x we slice perpendicular to the x-axis obtaining a circle of radius y and area πy^2. We sum the areas of all these circles

$$V = \int_0^H \pi y^2 \; dx$$

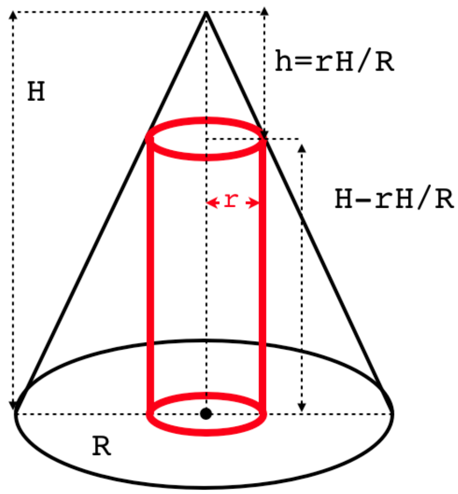
$$= \pi \int_0^H (\frac{R}{H}x)^2 \; dx$$

$$= \pi \; \frac{R^2}{H^2} \; [\frac{x^3}{3} \;] \; \Big|_0^H$$

$$= \frac{1}{3}\pi H R^2$$

shells

There is another way to ”slice” the figure, which is the method of shells.



We think of the volume as constructed from a series of concentric cylinders. Let’s use the same letters we had previously, H for total height and R for base radius. At a height h measured down from the top, the radius r is

$$r = h\frac{R}{H}$$

You can check this by similar triangles, or by calculation at two points, as we did above.

Each cylinder has circumference

$$C = 2\pi r = 2\pi h\frac{R}{H}$$

The height of the cylinder is $H - h$, and the lateral surface area of the shell is

$$\begin{aligned} SA &= C(H-h) \\ &= 2\pi h\frac{R}{H}(H-h) \end{aligned}$$

$$= 2\pi \frac{R}{H}(Hh - h^2)$$

We add up all the shells for $h = 0 \to h = H$

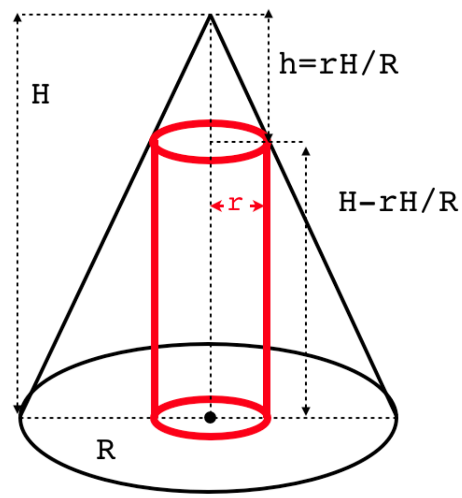
$$V = \int A \; dh = \int_0^H 2\pi \frac{R}{H}(Hh - h^2)$$

$$= 2\pi \frac{R}{H}(\frac{1}{2}Hh^2 - \frac{1}{3}h^3)\Big|_0^H$$

$$= 2\pi \frac{R}{H}(\frac{1}{6}H^3)$$

$$= \frac{1}{3}\pi R^2 H$$

Varying r instead of h



In the previous section we used h as the variable of integration, but we might just as well use r. In that case, r will vary from $r = 0 \to r = R$. At each value, the circumference will be

$$C = 2\pi r$$

and the height of the cylinder will be

$$H - \frac{H}{R}r$$

The volume is the sum of all the little pieces of cylinder volume

$$\begin{aligned}
V &= \int_{r=0}^{r=R} 2\pi r(H - \frac{H}{R}r)\ dr \\
&= 2\pi H \int_{r=0}^{r=R} r - \frac{1}{R}r^2\ dr \\
&= 2\pi H\ (\frac{r^2}{2} - \frac{1}{R}\frac{r^3}{3})\ \Big|_0^R \\
&= 2\pi H(\frac{1}{6}R^2) = \frac{1}{3}\pi R^2 H
\end{aligned}$$

Part V

Sine and cosine

Chapter 13

A famous limit

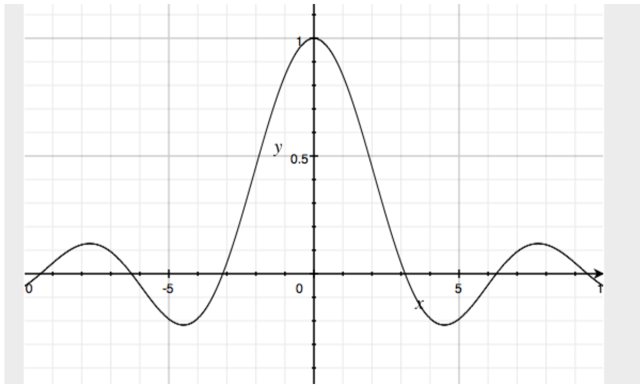
In this unit, we emphasize trigonometric and exponential functions. These are called transcendental functions because they "transcend algebra", meaning that they cannot be expressed as finite polynomials. We will see, however, that they can be expressed as *infinite* polynomials or series.

A famous limit

The fundamental results of calculus with respect to trigonometric functions depend on the value of this limit

$$\lim_{x \to 0} \frac{x}{\sin x}$$

The limit of the ratio of the angle to its sine as the angle gets very small is equal to 1. One way to explore this is to use a plotting application:



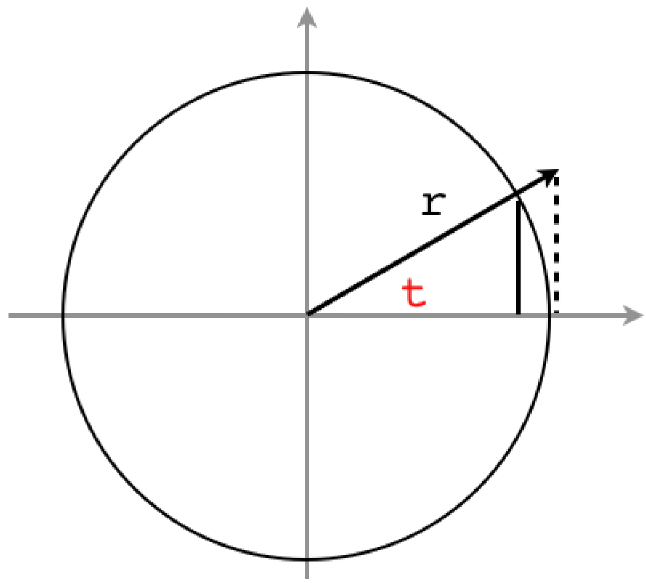
or a calculator such as that embedded in Python

```
>>> for i in range(1,100):
...     f = 1.0/i
...     print i, sin(f)/f
...
1 0.841470984808
..
97 0.999982286557
98 0.99998264621
99 0.999982995019
>>>
```

but these are (to be honest) cheating because when they calculate the sine of the angle they use a shortcut based on calculus.

Here is an actual proof that the ratio is equal to 1.

About notation: in the section above we used x as the variable name for an angle. Historically, of course, Greek letters were favored: θ, ϕ and so on. I use these occasionally, but most frequently I use s or both s and t. I prefer them because I feel like θ takes me a moment longer to process each time I see it than t does. Perhaps more important, t is a bit easier to typeset.



Consider the right triangle with radius r (the one that lies entirely inside the circle). Its base is $r\cos t$ and its height is $r\sin t$, so its area is

$$\begin{aligned} A &= \frac{1}{2}\cdot r\cos t\cdot r\sin t \\ &= \frac{1}{2}r^2\sin t\cos t \end{aligned}$$

Consider next the sector of the circle (piece shaped like a slice of pie) containing the same angle, t. Recall that t is the length of the portion of the circumference along this sector (if t is measured in radians). If the circle is not a unit circle, then multiply by the radius.

t is some fraction of the total angular measure of the circle, namely $t/2\pi$, and we multiply by the total area of the circle to get the area of the sector:

$$A = \frac{t}{2\pi}\pi r^2 = \frac{1}{2}r^2 t$$

Finally, consider the right triangle containing the dotted line, whose base has length r. Because it is a similar triangle with the first one, its height (that dotted line) is in the same ratio to r, the base of the triangle, as $\sin t$ is to $\cos t$. Thus, its length is $r \tan t$.

The area of this triangle is

$$\begin{aligned} A &= \frac{1}{2} \cdot r \cdot r \tan t \\ &= \frac{1}{2} r^2 \frac{\sin t}{\cos t} \end{aligned}$$

Since the first triangle is smaller than the sector, and the sector is smaller than the second triangle, *no matter how small* t becomes:

$$\frac{1}{2} r^2 \sin t \cos t < \frac{1}{2} r^2 t < \frac{1}{2} r^2 \frac{\sin t}{\cos t}$$

Now cancel $r^2/2$

$$\sin t \cos t < t < \frac{\sin t}{\cos t}$$

and divide by $\sin t$

$$\cos t < \frac{t}{\sin t} < \frac{1}{\cos t}$$

As $t \to 0$, both $\cos t$ and $1/\cos t$ approach the same limit, 1. Therefore the ratio gets squeezed, and it approaches the same limit as well.

$$\lim_{x \to 0} \frac{x}{\sin x} = 1$$

Since the limit is 1, the inverse approaches the same limit. We have proved the basic limit:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

□

Chapter 14

Sine and cosine

Difference quotient for sine

The limit just obtained allows us to find the derivatives of sine and cosine.

Set up the difference quotient for sine:

$$\frac{\sin(x+h)-\sin x}{h}$$

Using the addition of angles formula:

$$=\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}$$

Group the terms containing $\sin x$ and $\cos x$ separately

$$=\sin x\ \frac{(\cos h-1)}{h}+\cos x\ \frac{\sin h}{h}$$

Evaluating the limit as $h\to 0$, the second term is

$$\cos x\ \lim_{h\to 0}\frac{\sin h}{h}$$

We can pull $\cos x$ out of the limit, because it does not depend on h. By the main result above (our "famous" limit), the limit part is equal to 1, so the whole expression is just equal to $\cos x$.

We will show in just a minute that the first term is zero, which means that we have in the end:

$$\frac{d}{dx} \sin x = \cos x$$

The derivative of the sine is the cosine.

second limit

We massage that left-hand term from above as follows. $\sin x$ can come out because it does not depend on h. We must then evaluate

$$\lim_{h \to 0} \frac{(\cos h - 1)}{h}$$

Now

$$\frac{\cos h - 1}{h} = \frac{\cos h - 1}{h} \cdot \frac{\cos h + 1}{\cos h + 1}$$

The numerator on the right is

$$\begin{aligned}(\cos h - 1)(\cos h + 1) &= \cos^2 h - 1 \\ &= -\sin^2 h\end{aligned}$$

so each term on the right gets one copy of $\sin h$:

$$-\frac{\sin h}{h} \cdot \frac{\sin h}{\cos h + 1}$$

The limit as $h \to 0$ of the first factor is equal to 1 as we saw before, but the second one is $0/2 = 0$, so the whole thing is zero.

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

as promised.

Derivative of the cosine

Set up the difference quotient for cosine:

$$\frac{\cos(x+h) - \cos x}{h}$$

Using addition of angles

$$\frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

Grouping like terms

$$= \cos x \ \frac{(\cos h - 1)}{h} - \sin x \ \frac{\sin h}{h}$$

But we just showed that

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

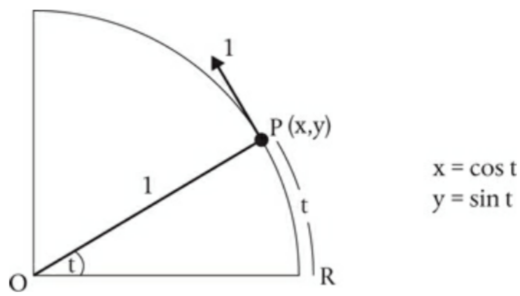
so the first term is zero. By the original limit derived above, the second term is

$$\lim_{h \to 0} \left[\ -\sin x \ \frac{\sin h}{h} \ \right] \ = -\sin x$$

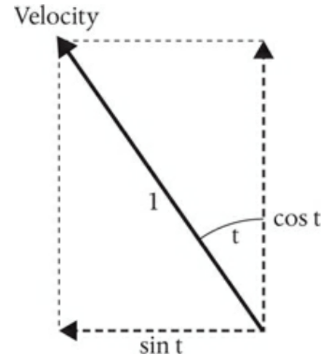
The derivative of the cosine is minus the sine.

$$\frac{d}{dx} \ \cos x = -\sin x$$

Another view



74. Finding the rates of change.



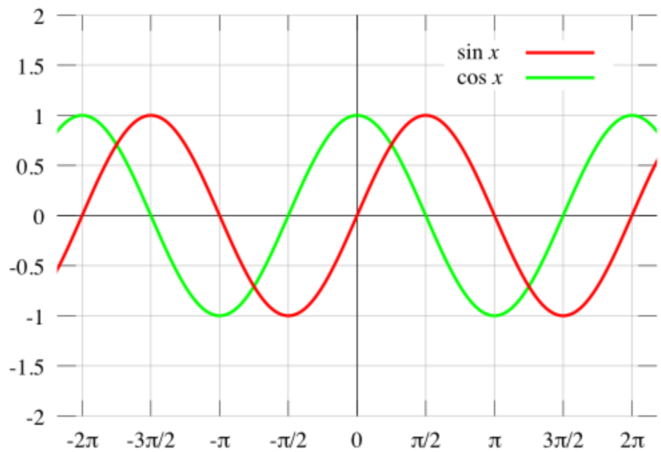
75. The velocity components.

Above are two figures from Acheson:

> One merit of radian measure — together with a unit radius — is that the distance travelled, PR, is not just proportional to the angle POR — it is actually equal to it, and will therefore be t.
>
> So P travels a distance t in time t and therefore goes round and round the circle at unit speed. Its velocity at any moment is therefore 1, directed along the tangent.
>
> And because the tangent is perpendicular to the radius OP, this direction of motion makes an angle t with the y-axis (right panel).
>
> Now, moving with speed 1 in the direction shown is equivalent to moving in the negative x-direction with speed sin t, at the same time as moving in the y-direction with speed cos t.

That is, the x-component, $\cos t$, has a velocity of $-\sin t$, which is its derivative. And the y-component, $\sin t$, has a velocity of $\cos t$.

graphs



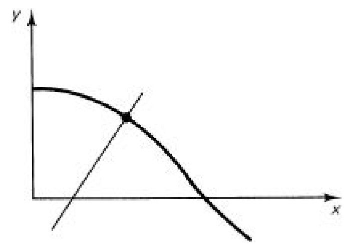
By examining the graphs of sine and cosine, we can see that the results obtained above make sense. The maximum slope of the sine function occurs when $\theta = 0$ because $\cos\theta = \cos 0 = 1$ there, and that matches the plot.

At the top of the arc for sine, when $\theta = \pi/2$, then $\sin\theta = 1$. The curve changes from increasing to decreasing at the very peak, and just for an instant, the slope is zero and the curve is horizontal. The corresponding value for $\cos\theta = \cos\pi/2 = 0$.

Furthermore, the slope of sine is positive when cosine is positive, while the slope of cosine is positive when sine is negative.

example

Here is a problem from Hamming that we can solve using these results.



The curve is $\cos x$ and the question is, if we form the norm (perpendicular) to the curve, does it ever pass through the origin?

The slope of the cosine is $-\sin x$. The slope we seek is perpendicular to that, so its product must yield -1. Thus $m = 1/\sin x$.

We use the point slope formula:

$$\frac{y - y_0}{x - x_0} = \frac{1}{\sin x}$$

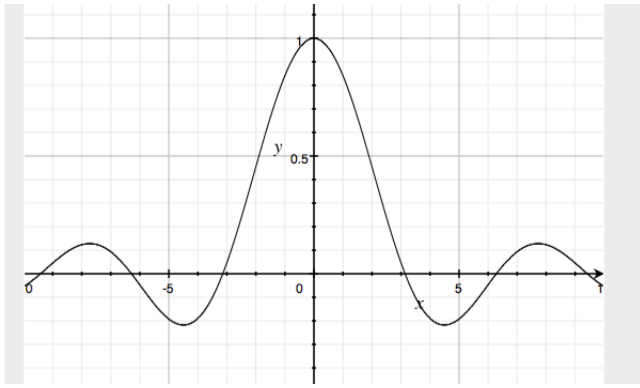
At the point $(x_0, y_0) = (0, 0)$

$$\frac{y}{x} = \frac{1}{\sin x}$$

But $y = \cos x$ so

$$\begin{aligned}
\sin x \cos x &= x \\
\frac{1}{2}\sin 2x &= x \\
\sin 2x &= 2x
\end{aligned}$$

One solution is $(0, 0)$. Are there any more? One way to answer this is to go back to the figure we started with where we plot $\sin x/x$:



No. $\sin x = x$ happens only at $x = 0$. In the figure, $\sin x/x = 1$ happens only at $x = 0$.

For an analytical proof, we observe that the slope of $\sin x$ is equal to $\cos x$ which is equal to 1 at $x = 0$ and for every x after that

$$0 < \cos x < 1, \qquad 0 < x < \frac{\pi}{2}$$

the slope of $\sin x$ is less than 1, while the slope of $y = x$ is equal to 1 everywhere.

The result is that the x rises faster than $\sin x$ everywhere after $x = 0$.

Other trig functions

We use the quotient rule, described **here**.

$$\frac{u'}{v} = \frac{u'v - uv'}{v^2}$$

Check that we've remembered it correctly:

$$[\ \frac{x}{1}\ ]' = \frac{1 \cdot 1 - x \cdot 0}{1} = 1$$

The derivative of the tangent is

$$[\ \frac{\sin x}{\cos x}\ ]' = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

and the secant (inverse cosine):

$$[\ \frac{1}{\cos x}\ ]' = \frac{-(-\sin x)}{\cos^2 x}$$
$$= \sec x \tan x$$

To round out the trig functions we have the cosecant and the cotangent. They aren't seen often, except as part of a strategy of making calculus more difficult than it needs to be.

The cosecant is the inverse of the sine:

$$\frac{d}{dx}\ \frac{1}{\sin\theta} = -\frac{1}{\sin^2\theta}\ \cos\theta$$
$$= -\csc\theta\cot\theta$$

And the cotangent is of course

$$\frac{d}{dx}\ \frac{\cos\theta}{\sin\theta} = \frac{-\sin\theta\sin\theta - \cos\theta\cos\theta}{\sin^2\theta}$$
$$= -\frac{1}{\sin^2\theta} = -\csc^2\theta$$

Notice the similarity to the secant and tangent, with a change of sign as well as substitution of the cotangent and cosecant.

differentiation trick

We used the sum of angles formulas (derived **here**) above. Here are two simple tricks requiring knowledge of the derivatives that help in the finding the sum of angles.

First, if we treat t as a constant and differentiate with respect to s (or vice-versa) we can go between formulas pretty easily:

Start from

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$

Differentiate

$$-\sin(s+t)\ ds = -\sin s \cos t\ ds - \cos s \sin t\ ds$$

$$\sin(s+t) = \sin s \cos t + \cos s \sin t$$

Or start from

$$\sin(s+t) = \sin s \cos t + \cos s \sin t$$

Differentiate

$$\cos(s+t)\ ds = \cos s \cos t\ ds - \sin s \sin t\ ds$$

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$

offsets to cosine

To rearrange the sum of angles formula above, we also need expressions for $\sin\theta$ in terms of cosine with an offset.

I find a simple approach is to take the derivative. We have the offset to sine to find cosine as

$$\cos\theta = \sin(\theta + \frac{\pi}{2})$$

the derivative is

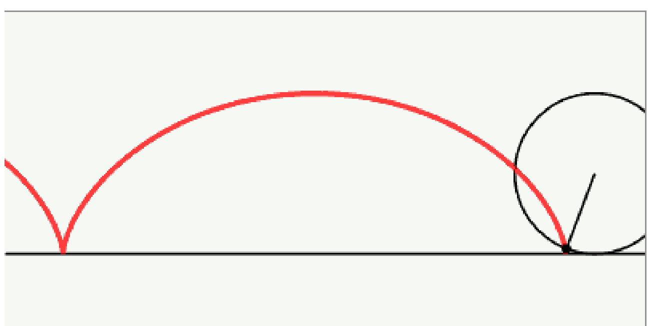
$$-\sin\theta = \cos(\theta + \frac{\pi}{2})$$

$$\sin\theta = -\cos(\theta + \frac{\pi}{2}) = \cos(\theta - \frac{\pi}{2})$$

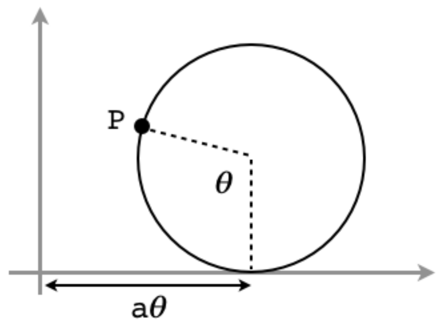
Chapter 15

Cycloid area

We imagine a bicycle with one tire marked at a particular point on the rim, say with fluorescent paint or a small light. Time starts at $t = 0$ with that point P in contact with the x axis at $(0, 0)$. Then we start rolling the bike. As the tire rotates our fixed point P on the rim traces a curve called the cycloid.

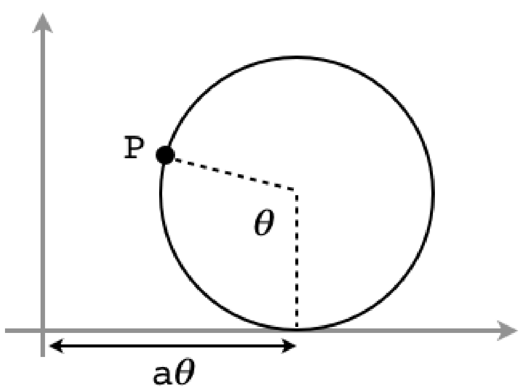


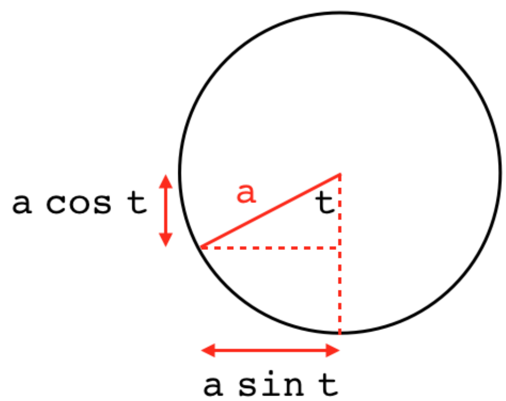
We want to find equations that give the position of the point P as a function of time.



It turns out that there is no simple expression for y in terms of x. However, there is a fairly simple *parametrization* of the curve, yielding parametric equations $x(t)$, $y(t)$ (or θ as labeled in the diagram).

The unusual thing is that we will designate $t = 0$ as being the time when the point P is at the bottom, and then the angle increasing in the clockwise direction, opposite to the usual case. The circle has radius a, for the general case.





It's a bit tricky to think about, but the equations for both y and x as a function of t have two terms. y is easier. The distance of point P above the $x-$axis is the radius of the circle, *minus* the length shown in the right panel as $a \cos t$.

$$y = a - a \cos t$$

The distance of the center of the circle from the $y-$ axis is at, the distance along the arc of the circle from P to the point currently on the bottom. But we must account for the distance of P left or right of the center, which is $-a \sin t$. Thus:

$$x = at - a \sin t$$

Now, going back to the original diagram, the area under the curve is just the integral of y.

That is

$$A = \int y \; dx$$

But we change variables to t. What is dx? $dx = (a - a \cos t) \; dt$.

$$A = \int (a - a \cos t)(a - a \cos t) \; dt$$

$$A = a^2 \int (1 - \cos t)^2 \; dt$$

The bounds on the integral are given by the fact that we want one complete revolution of the circle.

$$A = a^2 \int_0^{2\pi} (1 - \cos t)^2 \; dt$$

That will give a nice simplification, since any simple trigonometric function gives an integral of zero over that range. Let's see

$$(1 - \cos t)^2 = 1 - 2\cos t + \cos^2 t$$

We will need antiderivatives. The first two are easy, but what about $\cos^2$?

Recall the product rule: $(uv)' = u'v + uv'$. If you play around with various products, you will encounter:

$$\frac{d}{dx}\sin x \cos x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

From this we can rearrange and integrate to obtain:

$$2\int \cos^2 x \; dx = x + \sin x \cos x$$

Putting it all together we have:

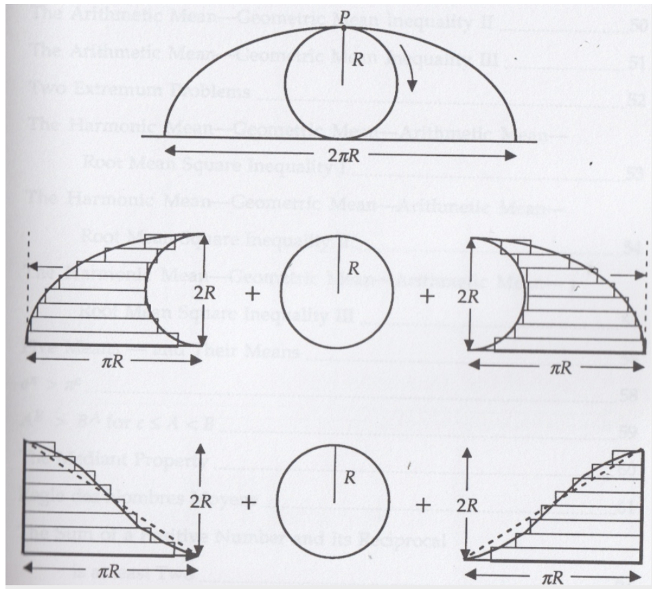
$$A = a^2 \int_0^{2\pi} (1 - \cos t)^2 \; dt$$

$$= a^2 \; [\; t - 2\sin t + \frac{1}{2}(t + \sin t \cos t)\Big|_0^{2\pi} \;]$$

Those terms with sin and cos are just zero over the period 2π. That leaves

$$A = a^2 \; \frac{3}{2}t\Big|_0^{2\pi} = 3\pi a^2$$

A simple, beautiful answer.



In this figure, the radius is R, so the circle is πR^2. We pick up two copies of $(1/2)\pi R \cdot 2R$ which gives us a total of $3\pi R^2$.

Part VI

Exponential

Chapter 16

Exponential and logarithm

Principal and interest

Suppose I put 100 dollars in the bank, and the people at the bank say that after one year, they will give me an additional \$10 at that time. We say that they are paying 10% interest for the year on the principal P of \$100.

However, suppose I bargain with them. I get them to promise to pay me half the interest (5%) at the six-month mark, and the rest after one year. My account will hold \$105 after six months, and the interest due for the second half will be 5% of \$105, which is \$5.25 for a total of \$10.25.

The equation to describe this situation is that if the rate of interest for the year is r and the year is broken up into n periods when interest will be paid, the total amount at the end will be:

$$A = P(1 + \frac{r}{n})^n$$

In the example, we have $r = 0.10$ and $n = 2$ so

$$A = 100(1 + 0.05)^2 = 110.25$$

This is compound interest. If there are additional years t, the exponent will be nt rather than n.

And now we start wondering what happens if the bank pays every month so that $n = 12$ or every day so $n = 365$ or even every second. What happens if the interest is compounded *continuously*?

$$A = \lim_{n\to\infty} P\ [\ (1 + \frac{r}{n})^n\ ]$$

Now it turns out that in the limit as n approaches ∞ these two expressions are equal

$$(1 + \frac{r}{n})^n = \ [\ (1 + \frac{1}{n})^n\ ]^r$$

The same factor r can be either in the numerator of the second term inside or up in the exponent outside.

A quick proof is:

$$\lim_{n\to\infty} (1 + \frac{r}{n})^n$$
$$= \lim_{n\to\infty} (1 + \frac{r}{n})^{(n/r)r}$$

Define $m = n/r$ and so as $n \to \infty$, so does $m \to \infty$ and then we have

$$\lim_{m\to\infty} (1 + \frac{1}{m})^{(m)r}$$

and the r is outside. m is just a dummy variable so we write:

$$\lim_{n\to\infty}\ [\ (1 + \frac{1}{n})^n\ ]^r$$

Therefore, going back to what we were working on, let us bring out the factor r and obtain

$$A = P(1 + \frac{1}{n})^{nr}$$

$$A = P\ [\ (1 + \frac{1}{n})^n\ ]^r$$

Thus, the important question is, what is the value of this expression?

$$A = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

It does not depend on r. It will turn out that this limit is equal to the number e.

$$e = 2.71828\ 18284\ 59045 \ldots$$

First, though, let us review some properties of logarithms.

working with logarithms

The logarithm and exponential functions are inverses. If we have that

$$y = b^x$$

for some $b > 0, b \neq 1$, then we say that

$$x = \log_b\ y$$

Putting them together

$$y = b^{\ \log_b y}$$

The usual bases are 10 (common logarithm, $\log_{10}$, or just log), e (natural logarithm or ln), and 2 (binary logarithm or $\log_2$).

The rules for exponents are simple, if p and q are two numbers and we know the logarithms of p and q to base b

$$p = b^u; \quad q = b^v$$

then their product can be computed as:

$$pq = b^u \cdot b^v = b^{u+v}$$

It helps if we can actually compute b^{u+v}. In the old days there were tables of logarithms, so you just looked up the answer in the table.

The second rule is that:

$$(b^u)^v = b^{uv}$$

And in terms of logarithms we write

$$\log_b(b^u)^v = \log b^{uv} = v \log_b(b^u)$$

For example

$$2^2 = 2 \times 2 = 4$$

$$2^3 = 2 \times 2 \times 2 = 8$$

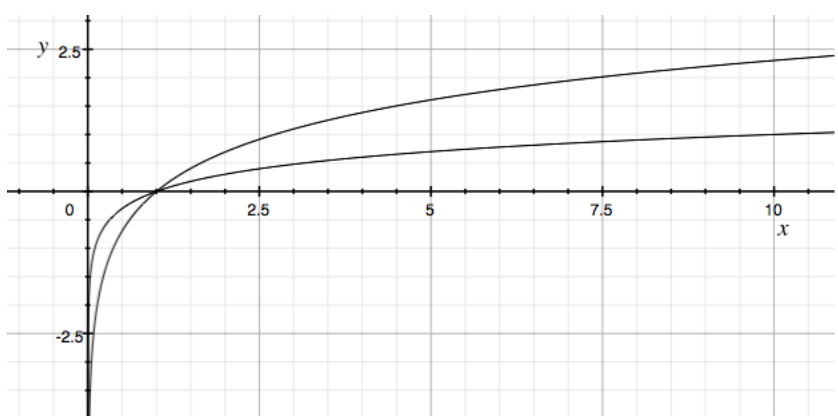
$$4 \times 8 = 2^2 \times 2^3 = 2^{2+3} = 2^5$$

$$= 2 \times 2 \times 2 \times 2 \times 2 = 32$$

and

$$(2^2)^3 = 4^3 = 64 = 2^6 = 2^{2\times 3}$$

Here is a plot of $\log_{10}(x)$ and $\ln x$:



The first function reaches the value 1 when $x = 10$ and the second reaches the value 1 when $x = e$. Both have the value 0 at $x = 1$

because $b^0 = 1$ for any base, so the logarithm to any base of 1 is equal to 0.

It turns out that if we take the logarithm of x (where x is any number > 1) to two *different* bases, the ratio of the logarithms is a constant, independent of the value of x.

change of bases

This is nicely shown by the change of bases formula.

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Start with an expression with b as the base:

$$y = b^x$$

and by the definition of the logarithm

$$x = \log_b y$$

To derive the formula, take the logarithm to the base a on both sides of the first expression:

$$\log_a y = \log_a (b^x)$$

Now, just invoke the second rule on the right-hand side

$$= x \log_a b$$

and substitute for x from the second expression above

$$= \log_b y \log_a b$$

We're basically done.

y can be any value, so replace it by x

$$\log_a x = \log_b x \log_a b$$

Rearranging:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

One way I remember this is that first the logarithms to different bases are connected by some constant k

$$\log_b x = k \log_a x$$

and we substitute for k the inverse of the log to the *same* base as we have in the numerator:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

that is, I remember that we want $\log_a$ something *over* $\log_a$ something on the right.

Alternatively, you might look at the other formula

$$\log_a x = \log_a b \log_b x$$

and imagine the b's canceling in some way.

One other thing we can do is to set $x = a$ in the above formula. We start from

$$\log_b x = \frac{\log_a x}{\log_a b}$$

then with $x = a$

$$\log_b a = \frac{\log_a a}{\log_a b}$$

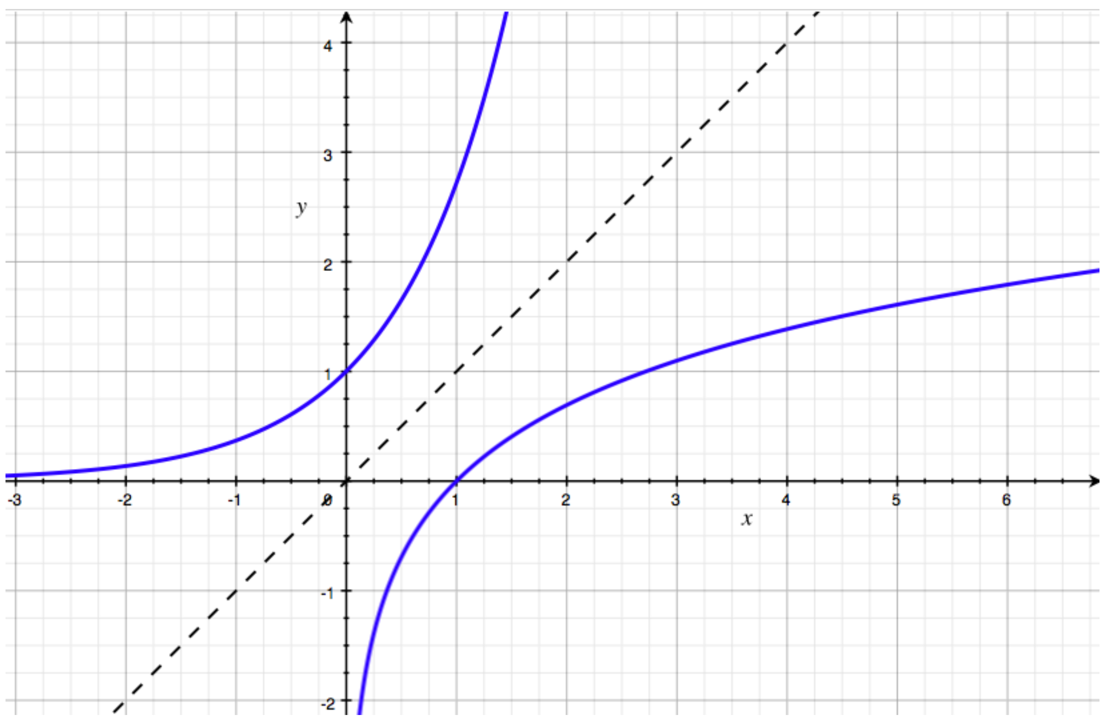
but $\log_a a = 1$ so

$$\log_b a = \frac{1}{\log_a b}$$

And that makes perfect sense. If we multiply by some factor k to convert from the logarithm in base a to base b, we must multiply by the inverse of the same factor to convert back again.

For the figure above of the common log (base 10) and the natural logarithm, $\ln 10 = 2.303$, and that looks about right, when $x = 10$ the first function is 1.0 and the second one is about 2.3.

The logarithm and the exponential are inverse functions, we can see that if we plot them together:



The upper curve is $y = e^x$ and the lower one is $y = \ln x$. As inverse functions, they are symmetric about the line $y = x$.

fractional exponents

The introduction above dealt mainly with integer exponents, but of course you know that the practical use of logarithms depends on fractional values. The simplest way to see how this works is to consider the square root.

$$\sqrt{2} \times \sqrt{2} = 2$$

If we think about what the exponent u to the base 2 would be such that

$$2^u = \sqrt{2}$$

We observe that by the rules for exponents

$$\sqrt{2} \times \sqrt{2} = 2^u \times 2^u = 2^{u+u} = 2^1$$

That is

$$u + u = 1$$

so $u = 1/2$. By the same logic the n^{th} root of b is $b^{1/n}$. And of course

$$(b^2)^{1/2} = b^{2 \times 1/2} = b^1$$

Feynman has a nice description of how logarithms were calculated (see Lectures, volume 1, Chapter 22, Algebra;

http://www.feynmanlectures.caltech.edu/I_22.html)

The basic idea is to take repeated square roots of the base (10), and then combine those to form the required value.

Less than 1

Fractional exponents leads to consideration of $0 < x < 1$. Write

$$x \ \frac{1}{x} = 1$$

Take the logarithm of both sides

$$\log(x \ \frac{1}{x}) = \log 1 = 0$$

$$= \log x + \log \frac{1}{x}$$

Thus

$$\log \frac{1}{x} = -\log x$$

Chapter 17

Half life

A sample of phosphate containing some percentage of the radioactive isotope ^{32}P emits beta particles (energetic electrons of about 0.5 MeV), which are emitted as the nucleus decays into ^{32}S. These can be measured with a Geiger counter, by scintillation counting or by other means (like exposure to X-ray film).

I used a device very similar to this. We would count radioactivity by holding up tubes at some distance like 10 cm, and evaluating how loud the device "screamed". Not quantitative, just trying to find the peak of activity eluted off a chromatographic column.

The measured activity decreases or decays with a half-life of just over 14 days (14.29, to be more precise).

If you want to solve a problem with a given amount of the radioisotope N_o at time-zero ($t = 0$), and you are asked for the amount at time t, what do you do?

If the time is an even multiple of the half-life, it's easy. For one half-life, multiply by $\frac{1}{2}$, for two half-lives multiply by $\frac{1}{4}$, for n half-lives multiply by $(\frac{1}{2})^n$.

If the time is not an even multiple of the half-life, you need the following equation (where T is the half-life and k is a rate constant)

$$N = N_o \, e^{-kt}$$

If you are given the half-life, you will also need that

$$kT = \ln 2 = 0.693$$

I want to show where this equation comes from to motivate our future discussion of the exponential. We will learn two equivalent definitions. The first is that

$$\frac{d}{dx} \, e^x = e^x$$

The second is that

$$\frac{d}{dx} \, \ln(x) = \frac{1}{x}$$

or

$$\int \frac{1}{x} \, dx = \ln(x)$$

In radioactive decay, each atom has a fixed, characteristic probability of disintegrating in the next short time interval dt, although it is impossible to tell in advance *which* nuclei will decay.

The probability varies for different types of radioactive atom (^{3}H, ^{14}C, ^{238}U, etc.), but for each phosphorus atom of this isotope in our sample of ^{32}P it is the same.

As a result, the number of atoms dN that will disintegrate or decay in the short time dt is proportional to N, the number currently present. A fixed fraction of all the atoms will be transformed. We write

$$dN = kN\ dt$$

Slinging differentials, we rearrange and integrate

$$\int \frac{dN}{N} = \int k\ dt$$

The answer is just

$$\ln(N) = kt + C_0$$

Form the exponential on both sides

$$N = Ce^k t$$

$(C = e^{C_0})$. We evaluate the constant C by setting $t = 0$ and find that $C = N_o$ so

$$N = N_o\ e^{kt}$$

Finally, in decay problems it is usual to let k be positive and introduce a minus sign

$$N = N_o\ e^{-kt}$$

As we said

$$kT = \ln 2 = 0.693$$

This is very useful to remember, because frequently we are given a half-life T and asked to compute using the equation with e^{-kt}. It will save time to convert T to k quickly.

The derivation is as follows. By definition, after one half-life has elapsed, when the time $t = T$, $N = N_o/2$.

$$\frac{N_o}{2} = N_o \; e^{-kT}$$

$$\frac{1}{2} = \; e^{-kT}$$

$$2 = \; e^{kT}$$

$$\ln 2 = \; kT$$

Equations for the growth of populations work similarly, with

$$N = N_o \; e^{kt}$$

and again, if the problem gives you an even number of doublings, or generations, just use that. For growth equations it is usual to use another symbol like g for the number of generations, where

$$N = 2N_o \; e^{kg}$$

but the same relation holds between g and k (because of the switched minus sign in e^{kt}).

$$\ln 2 = \; kg$$

guy from Philadelphia

Here is a crime scene example. According to Newton's Law of Cooling

$$T(t) = T_e + (T_0 - T_e)e^{-kt}$$

Suppose Tony Soprano whacks some wise guy from Philly at time-zero, and immediately drags the body into the meat cooler (or better yet, lures him in there first).

The temperature of the stiff at time t is given by the above equation. We'll say very roughly that $T_0 = 37$ (Celsius, formerly Centrigrade) and the environmental temperature $T_e = -3$ so the temperature as a function of time is given by

$$T = 40e^{-kt} - 3$$

$$3 + T = 40e^{-kt}$$

As the crime scene investigator, you need a value for k in order to find the time of death. One way to obtain that is to determine the temperature at two different times. You take the temperature of the body on arrival at the crime scene (an unknown time τ after death) and obtain a value of 27 degrees C. One hour later ($\tau+1$), after photographs have been taken and the scene dusted for fingerprints, you measure it to be 17 C. We have

$$\log\frac{30}{40} = -k\tau$$

$$\log\frac{20}{40} = -k(\tau + 1)$$

Subtracting

$$\begin{aligned} k &= \log\frac{30}{40} - \log\frac{20}{40} \\ &= \log(\frac{30}{40} \cdot \frac{40}{20}) \\ &= \log 1.5 = 0.40 \end{aligned}$$

Since $\log\frac{30}{40} = -0.288$, we have from the first equation that

$$-0.288 = -k\tau = -0.40\tau$$

$$\tau \approx 0.72$$

It was only about 45 minutes after the hit when you first arrived.

Chapter 18

Another famous limit

Consider the function of the variable x defined by the power b^x, where b is some arbitrary constant.

$$f(x) = b^x$$

We will show later that the difference quotient whose limit defines $f'(x)$ can be manipulated to give the following form

$$b^x \lim_{h \to 0} \frac{b^h - 1}{h}$$

For this reason, we are interested in the properties of the limit

$$\lim_{h \to 0} \frac{b^h - 1}{h}$$

First, it is clear that the limit does not depend on x, so it is just a number.

working with the constant

We can easily calculate the value of

$$\lim_{h \to 0} \frac{b^h - 1}{h}$$

Set $h = 0.00001$ and use Python. For the following values of b I get:

- $b = 2$ gives 0.693
- $b = 2.71828 \cdots = e$ gives 1.000
- $b = 10$ gives 2.303

This gives us one *definition* of e. It is the value of b for which this limit is equal to 1.

It will not surprise you to learn that these values correspond to the natural logarithm of the base.

playing with the limit expression

So starting from the definition that e is the value for which

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

We can rearrange the equation as follows:

$$\lim_{h \to 0} e^h = \lim_{h \to 0} 1 + h$$

$$\lim_{h \to 0} e = \lim_{h \to 0} (1 + h)^{1/h}$$

$$e = \lim_{h \to 0} (1 + h)^{1/h}$$

since e is just a constant (we haven't demonstrated that it is legal to manipulate limits in this way, but you can).

We can also substitute $nh = 1$ and so $n = 1/h$ and the expression can be written as

$$e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$$

which is the first expression that we introduced for e.

All three limits are equivalent:

$$e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$$

$$= \lim_{h\to 0}(1 + h)^{1/h}$$

and

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1$$

We can calculate e from this equation as well, but it doesn't converge so fast. For $n = 100000$, using Python, I got $e = 2.718168$, which is only correct to 3 decimal places.

On the other hand, one can use several approaches including the binomial theorem to evaluate

$$e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$$

and show that it is equivalent to

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

which is yet another one of the many definitions for e.

The first three terms are $1 + 1 + 1/2$, which is reasonably close. After six terms, we have $e = 2.718$. The series converges rapidly because the inverse factorials get small very quickly. We calculate e as about 2.71828.

the function

In general we are interested in the *function* e^x and not just the number e.

Start with a general exponential (base b) and consider the problem of finding its slope. We want to see what happens to the exponential function when we vary x by a small amount h.

We want the difference quotient

$$\lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$$

for some unspecified base b. We can rewrite this as

$$\lim_{h \to 0} \frac{b^x(b^h - 1)}{h}$$

The great insight is to note that b^x does not depend on h, so it doesn't change as we go to the limit, and thus we have

$$b^x \lim_{h \to 0} \frac{b^h - 1}{h}$$

the limit is not dependent on x, so it is a constant. Therefore, the slope of the exponential

$$y = b^x$$

for some base b is

$$y' = cb^x$$

The slope is the same as the function, up to a constant.

the exponential function

The number e is such that if $b = e$, then $c = 1$ and we will have:

$$y = e^x$$

$$y' = e^x$$

The function e^x is its own derivative, which leads to all of its amazing properties.

Write

$$e^x = \lim_{n\to\infty} [\ (1 + \frac{1}{n})^n\ ]^x$$

$$e^x = \lim_{n\to\infty} (1 + \frac{1}{n})^{nx}$$

It turns out this is exactly the same as

$$= \lim_{n\to\infty} (1 + \frac{x}{n})^n$$

which seems counterintuitive.

But the result follows from

$$(1 + b)^{mn} = (1 + mb)^n$$

a quick proof

We saw this proof in the previous chapter. We want to show that

$$\lim_{n\to\infty} (1 + \frac{r}{n})^n = \lim_{n\to\infty} [\ (\frac{1}{n})^n\ ]^r$$

Start with

$$\lim_{n\to\infty} (1 + \frac{r}{n})^n = \lim_{n\to\infty} (1 + \frac{r}{n})^{(n/r)r}$$

Define $m = n/r$ and so as $n \to \infty$, so does $m \to \infty$ and then we have

$$\lim_{m\to\infty} (1 + \frac{1}{m})^{(m)r}$$

and the r is outside. m is just a dummy variable so we write:

$$\lim_{n\to\infty} [\ (1 + \frac{1}{n})^n\ ]^r$$

$\square$

proof using the binomial

Write the binomial expansion for both versions. For the first, we have

$$(1+b)^{mn} = 1+(mn)b+\frac{(mn)(mn-1)}{2!}b^2+\frac{(mn)(mn-1)(mn-2)}{3!}b^3+\dots$$

In the limit as $n \to \infty$, the subtracted values of $k = 1, 2, 3\dots$ are negligible and we have

$$= 1 + (mnb) + \frac{1}{2!}(mnb)^2 + \frac{1}{3!}(mnb)^3 + \dots$$

For the second we get

$$(1 + mb)^n = 1 + n(mb) + \frac{n(n-1)}{2}(mb)^2 + \frac{n(n-1)(n-2)}{3!}(mb)^3 + \dots$$

and with the same limit, we get the identical formula.

Going back to

$$e^x = \lim_{n\to\infty} (1 + \frac{x}{n})^n$$

$$= \frac{1}{0!}(\frac{x}{n})^0 + +\frac{n}{1!}(\frac{x}{n})^1 + \frac{n(n-1)}{3!}(\frac{x}{n})^2 + \frac{n(n-1)(n-2)}{3!}(\frac{x}{n})^3 + \cdots$$

The n's cancel and we have

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

This approach assumes that the binomial applies to fractional exponents, which is a bit complicated to show.

$\square$

sketch of a proof using Taylor series

A more sophisticated way to do this is to use Taylor series, together with the fact that the derivative of e^x is equal to e^x.

We can find the value of e^x near $x = 0$ as

$$e^x = \sum_{n=0}^{n=\infty} \frac{f^n(0)}{n!} x^n$$

In particular, all the derivatives are just $e^x = 1$ at $x = 0$ so

$$e^x = \sum_{n=0}^{n=\infty} \frac{x^n}{n!}$$

$$= \frac{x^0}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$$

and if $x = 1$ we have that

$$e^1 = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} \dots$$

The exponential is its own derivative

As we've emphasized, the most important fact about the exponential function $f(x) = e^x$ is that this function is its own derivative.

Above, we derived the famous series for

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots$$

Note that a good approximation for small x is

$$e^x \approx 1 + x$$

If you need more precision, you can add another term:

$$e^x \approx 1 + x + \frac{x^2}{2!}$$

It is easy to see that the derivative of this series is the series itself.

Take $\frac{d}{dx}$ of the last series.

$$\begin{aligned} \frac{d}{dx}\ e^x &= 0 + (1)\frac{x^{1-1}}{1!} + (2)\frac{x^{2-1}}{2!} + (3)\frac{x^{3-1}}{3!} + \cdots \\ \frac{d}{dx}\ e^x &= 0 + \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots \\ &= e^x \end{aligned}$$

Each exponent n that comes down through the power rule, finds an n in $n! = n \times (n-1) \times (n-2) \dots$ to cancel, leaving $n-1$ in the exponent as well as $(n-1)!$.

Part VII

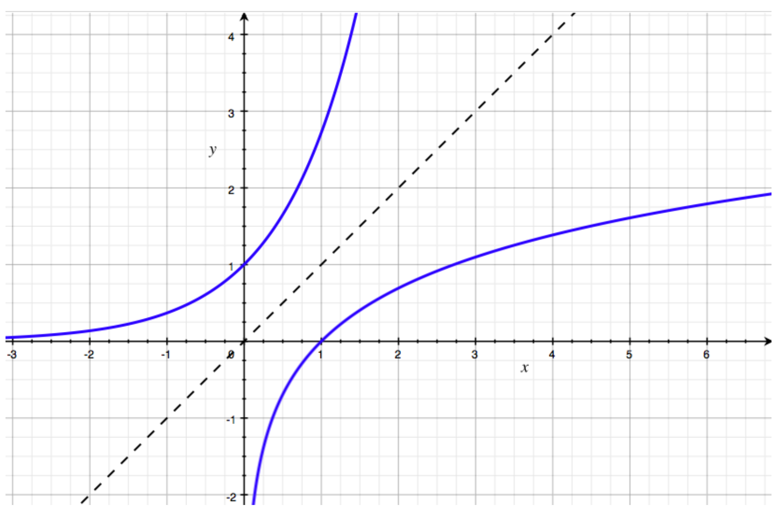
Logarithm

Chapter 19

Exponential to logarithm

The logarithm and the exponential function are inverses of each other. We will prove this assertion here.

It seems reasonable enough if we plot them together. The upper curve is $y = e^x$ and the lower one is $y = \ln x$.



As inverse functions, e^x and $\ln x$ are symmetric about the line $y = x$.

If we consider a particular x value, for example $x = 1$, then the slope of the curve $y = e^x = e$ at the point $(1, e)$ is the inverse of the slope of the curve $y = \ln x$ when $x = e$ (at the point $(e, 1)$).

from exponential to logarithm

The logarithm can be defined as follows:

$$\int \frac{1}{t}\, dt = \ln(t)$$

It is possible to prove this starting from the assumption that e is its own derivative, or that e is the limit we discussed above. Alternatively, we can start from this statement and derive the facts about e. We will do the latter in a later chapter.

Now, differentiate both sides to find the derivative of the logarithm. By the FTC:

$$\frac{d}{dt} \int \frac{1}{t}\, dt = \frac{1}{t} = \frac{d}{dt} \ln(t)$$

This is great because we never did generate x^{-1} by differentiating powers of x.

And now we know how to go back the other way, using the definition that the exponential function is its own derivative to establish the first statement above.

The proof is so simple that if you blink, you'll miss it.

$$y = e^x$$

$$\frac{dy}{dx} = e^x = y$$

Invert

$$\frac{dx}{dy} = \frac{1}{y}$$

$$dx = \frac{1}{y}\, dy$$

Integrate

$$\int dx = x = \int \frac{1}{y}\ dy$$

And what is x? It is $\ln y$!

$$\ln(y) = \int \frac{1}{y}\ dy$$

And since y is just a letter, we can write the same for x, or t

$$\ln x = \int \frac{1}{x}\ dx, \qquad \ln t = \int \frac{1}{t}\ dt$$

One of the prettiest things I've ever seen.

The math purists don't like it, but in general it is OK to do algebra with differentials. One important restriction is that dy/dx (and dx/dy) should not be equal to zero. And a second one is that we just remember we are never passing to the limit, just getting really, really, really close. (As close as you like).

Now that we have this definition, we have another way of estimating the value of e. We add up the areas for little slices under the function $y = 1/x$ until the total reaches 1. The corresponding value of $x = e$.

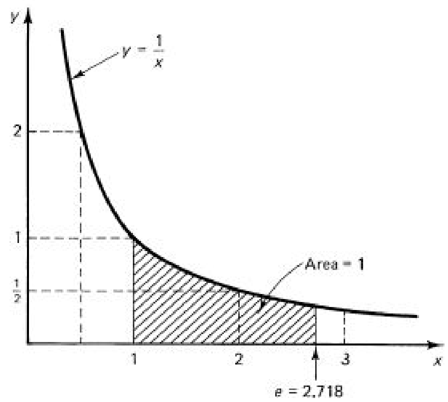


Figure 14.3-1 Estimate of e

(from Hamming).

For example, we might try intervals of 0.1 and do

$$0.1 \cdot \frac{1}{1} + 0.1 \cdot \frac{1}{1.1} + 0.1 \cdot \frac{1}{1.2} + \cdots + 0.1 \cdot \frac{1}{2.7}$$

reverse direction

Above we showed that $(e^x)' = e^x$ implies

$$\ln x = \int \frac{1}{x} \, dx$$

Differentiate both sides

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

We want to go backward now, to show that the derivative of the function $f(x) = e^x$ is itself.

Start with

$$\ln(e^x) = x$$

$$\frac{d}{dx}\ln(e^x) = \frac{d}{dx}x = 1$$

but using the property we just proved and the chain rule, this is also

$$\frac{d}{dx}\ln(e^x) = \frac{1}{e^x}\ \frac{d}{dx}e^x$$

so these two expressions are equal and

$$\frac{1}{e^x}\ \frac{d}{dx}e^x = 1$$

$$\frac{d}{dx}e^x = e^x$$

Magic.

This is really the primordial differential equation.

Strang's view of the series

Gil Strang has a nice introduction to the exponential (for real numbers). He says we want to "construct a function" for $y = e^x$. The first, amazing, property (I) is that the derivative of the function is equal to the function itself.

$$y = y'$$

The second property, a boundary condition, is that at $x = 0$, we want $y = 1$. That's because e is not a variable and we want $e^x = e^0$ to be equal to 1 like every other exponential we know. [A third property that will turn out to be true is $e^{x_1+x_2} = e^{x_1} \cdot e^{x_2}$.]

Using the second condition we write:

$$y(x) = 1$$

Whatever else is true, if there are no terms containing x (because $x = 0$) then $y = 1$. By I we must have that the derivative is equal to the function:

$$y(x) = 1$$
$$y'(x) = 1$$

But now, if we try to evaluate the derivative starting from $y(x)$, where does that 1 come from? It must come from

$$y(x) = 1 + x$$
$$y'(x) = 1$$

Now, though, it's no longer true that $y = y'$ so we fix that

$$y(x) = 1 + x$$
$$y'(x) = 1 + x$$

And where does that x come from in the derivative? It must come from

$$y(x) = 1 + x + \frac{x^2}{2}$$
$$y'(x) = 1 + x$$

And so on ... Following this procedure, we build up the definition

$$\begin{aligned} y(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\ &= \sum_0^\infty \frac{x^n}{n!} \end{aligned}$$

The test for convergence of this infinite series is to find the values of x such that:

$$\lim_{n\to\infty} \frac{|A_{n+1}|}{|A_n|} < 1$$

That is, we need

$$\lim_{n\to\infty} \frac{x}{n+1} < 1$$

But this is true for any x.

Chapter 20

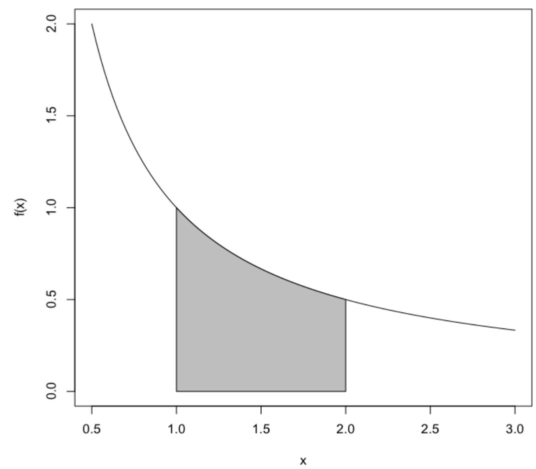
Logarithm to exponential

In more advanced treatments (analysis), what they do is to investigate a particular function which we will call L, and show that the function L has all the properties of the logarithm, *so it is the logarithm.*

We follow that path here, and then after that go backwards from the logarithm to the other properties of e, like its numerical value, and the fact that the derivative of e^x is e^x itself.

This approach comes straight from David Jerrison's lecture in Calculus 1 (MIT online course). We define the logarithm function as

$$L(x) = \int_1^x \frac{dt}{t}$$



For example, the logarithm of 2 is the area under the curve above, $f(x) = 1/x$, between $1 \leq x \leq 2$. Having defined

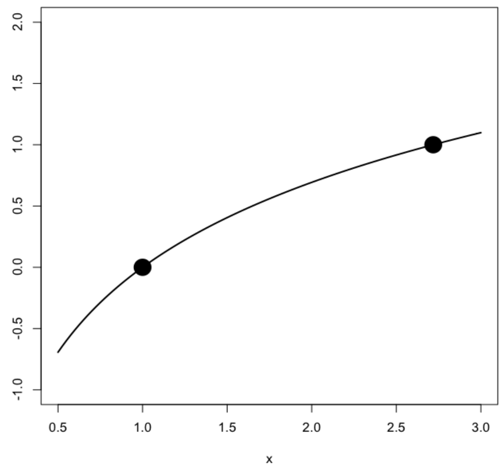
$$L(x) = \int_1^x \frac{dt}{t}$$

By the Fundamental Theorem of Calculus (part II) we have

• Property 1

$$L'(x) = \frac{1}{x}$$

The slope of the logarithm function is always positive ($x > 0$), but is undefined for $x = 0$



- Property 2

$$L(1) = \int_1^1 \frac{dt}{t} = 0$$

This property is by definition. It fits with our use of exponents, where $b^0 = 1$.

- Property 3

$$L''(x) = -\frac{1}{x^2}$$

Although the area under the curve $ln(x)$ is always increasing, so the slope is always positive, the rate of increase of the slope is always decreasing, so the shape is concave down.

- Property 4

$$L(e) = 1$$

This is by definition as well. In extending to exponents it means x is a single-valued function of y, so we can write $y = L(x) \iff e^y = x$.

- Property 5

$$L(ab) = L(a) + L(b)$$

To show that this last statement is true involves showing that

$$\int_1^{ab} \frac{dt}{t} = \int_1^{a} \frac{dt}{t} + \int_a^{ab} \frac{dt}{t}$$

is both true and equivalent to the first statement.

For the arguments a and ab we have

$$L(ab) = \int_1^{ab} \frac{dt}{t}$$

$$L(a) = \int_1^{a} \frac{dt}{t}$$

Both of these are true by definition. The one that takes a little work is

$$L(b) = \int_a^{ab} \frac{dt}{t}$$

We do a substitution. Let $au = t$, then $a\ du = dt$ and

$$L(b) = \int \frac{a\ du}{au} = \int \frac{du}{u}$$

At this point we run into a new idea. When we change the variable we also must change the bounds. At the first one, we have $t = a$ so

$$u = \frac{t}{a} = \frac{a}{a} = 1$$

At the upper bound, we have $t = ab$ so

$$u = \frac{t}{a} = \frac{ab}{a} = b$$

The integral $\int_a^{ab}$ becomes $\int_1^{b}$ and we have

$$L(b) = \int_1^{b} \frac{du}{u}$$

which is again, true by definition.

Thus, the function L has the property that

$$L(ab) = L(a) + L(b)$$

which is one of the two major properties of logarithms.

- Property 6

To see that the second property is also true, start with

$$L(a^r) = \int_1^{a^r} \frac{dt}{t}$$

Substitute $t = u^r$, so $dt = ru^{r-1}du$. The bounds are changed as follows (r can be anything):

$$t = 1, \quad t = u^r \rightarrow u = 1$$
$$t = a^r, \quad t = u^r \rightarrow u = a$$

This gives

$$\begin{aligned} L(a^r) &= \int_{t=1}^{t=a^r} \frac{dt}{t} \\ &= \int_{u=1}^{u=a} \frac{1}{u^r}(ru^{r-1})du \\ &= r\int_1^a \frac{du}{u} = rL(a) \end{aligned}$$

That is

$$L(a^r) = rL(a)$$

As Dunham says (using A for L)

> "these properties of the hyperbolic area—namely $A(ab) = A(a) + A(b)$ and $A(a^r) = rA(a)$—exactly mirror the corresponding properties of logarithms. Clearly something interesting is afoot."

Difference quotient for logarithm

As seen in Hamming, we can also go back to the definition of the logarithm as the inverse of the exponential

$$f(x) = \log_b x$$

write the difference quotient

$$f'(x) = \lim_{h \to 0} \frac{\log_b(x+h) - \log_b x}{h}$$

and then use the properties of logarithms to rearrange it as follows:

$$= \lim_{h \to 0} \frac{\log_b(\frac{x+h}{x})}{h}$$

$$= \lim_{h \to 0} \frac{\log_b(1 + \frac{h}{x})}{h}$$

$$= \lim_{h \to 0} \log_b \ [\ (1 + \frac{h}{x})^{1/h}\ ]$$

$$= \lim_{h \to 0} \frac{1}{x}\ [\ \log_b(1 + \frac{h}{x})^{x/h}\ ]$$

Finally

$$= \frac{1}{x}\ \lim_{h \to 0}\ [\ \log_b(1 + \frac{h}{x})^{x/h}\ ]$$

So it's clear that we will need to evaluate the term for which we are taking the logarithm, in the limit

$$= \lim_{h \to 0}(1 + \frac{h}{x})^{x/h}$$

Let $t = h/x$. Then this becomes

$$= \lim_{t \to 0}(1 + t)^{1/t}$$

which ought to look familar from previous chapters. It is one of the definitions of e. We have then that

$$\frac{d}{dx}\log_b x = \frac{1}{x}\ \log_b e$$

If we use the natural logarithm, then we have

$$\frac{d}{dx}\ln x = \frac{1}{x}\ \ln e = \frac{1}{x}$$

There is another derivation which is essentially identical to this one in videos on Khan Academy.

Chapter 21

Famous limit revisited

e as a limit

Here, we look at a more rigorous proof to demonstrate that the number e is equivalent to this limit:

$$\lim_{n \to \infty} (1 + \frac{1}{n})^n$$

http://aleph0.clarku.edu/~djoyce/ma122/elimit.pdf

So we start with a different definition of e and show that it is equivalent. We begin with this definition of the natural logarithm:

$$\ln x = \int_1^x \frac{1}{t}\ dt$$

We have worked through some consequences above, following David Jerrison's MIT lecture. It can be shown fairly easily that this function has all the properties of the natural logarithm.

In addition to that definition, we need two more properties, first:

$$\ln 1 = \int_1^1 \frac{1}{t}\ dt = 0$$

fairly obvious, since the upper and lower bounds are equal, and then second, our definition of e. It is the number such that

$$\ln e = \int_1^e \frac{1}{t}\ dt = 1$$

So here's the proof. Let t be any number in an interval $[1, 1 + 1/n]$. We're interested in what happens as n gets large. We have that

$$1 \leq t \leq 1 + \frac{1}{n}$$

If we invert each term, then $\leq$ becomes $\geq$, but we will instead rearrange the terms:

$$\frac{1}{1 + \frac{1}{n}} \leq \frac{1}{t} \leq 1$$

The only tricky step is this one: for each of the above, we integrate the variable t between the endpoints 1 and $1 + 1/n$, remembering that n is just a number and so is $1 + 1/n$, so we have

$$\int_1^{1+1/n} \frac{1}{1 + \frac{1}{n}}\ dt \leq \int_1^{1+1/n} \frac{1}{t}\ dt \leq \int_1^{1+1/n} 1\ dt$$

The first integral is a constant times t evaluated between $1 + 1/n$ and 1 which is equal to the constant times $1/n$:

$$[\ \frac{1}{1 + \frac{1}{n}}\ ]\ \frac{1}{n} = \frac{1}{1 + n}$$

The second one is $\ln(1 + 1/n)$ by the definition of the logarithm, and the third is the same integral as the first but without the constant, so we have that:

$$\frac{1}{1 + n} \leq \ln(1 + \frac{1}{n}) \leq \frac{1}{n}$$

From here on, we just rearrange things a bit. Raising e to the power of each term doesn't change the inequality:

$$e^{\frac{1}{1+n}} \leq 1 + \frac{1}{n} \leq e^{1/n}$$

The left-hand inequality

$$e^{\frac{1}{1+n}} \leq 1 + \frac{1}{n}$$

can be raised to the power $(n+1)$ giving:

$$e \leq (1 + \frac{1}{n})^{n+1}$$

then divide by $(1 + \frac{1}{n})$

$$\frac{e}{1 + 1/n} \leq (1 + \frac{1}{n})^n$$

We notice that, in the limit as $n \to \infty$, this becomes

$$e \leq \lim_{n\to\infty} (1 + \frac{1}{n})^n$$

Similarly for the right-hand inequality,

$$1 + \frac{1}{n} \leq e^{1/n}$$

raise to the power n giving:

$$(1 + \frac{1}{n})^n \leq e$$

and in the limit as $n \to \infty$, this becomes

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n \leq e$$

Call the limit L.

The only way that $e \leq L$ and $L \leq e$ can both be true is if e is equal to the limit in question. This is the squeeze theorem. Hence

$$e = \lim_{n\to\infty} (1 + \frac{1}{n})^n$$

Chapter 22

e is irrational

e is irrational

I found a nice proof of the irrationality of e in the calculus text by Courant and Robbins. It is a proof by contradiction. We start by assuming that e is rational.

$$e = \frac{p}{q}, \quad p, q \in \mathbb{N}$$

We make use of the infinite series representation of e

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

From this, it is obvious that $e > 2$. If you're interested, there is a proof that $e < 3$ in the book.

Equating the series representation to the rational fraction p/q:

$$\frac{p}{q} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Multiply both sides by $q!$. For the left-hand side, we have

$$e\ q! = \frac{p}{q}\ q! = p(q-1)!$$

We won't need to do anything more with this, but note that since $e\, q!$ is equal to $p(q-1)!$, we can see that the left-hand side, $e\, q!$, is clearly an integer. Therefore, the right-hand side must also be an integer. This is the series

$$q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{(q-1)!} + \frac{q!}{q!} + \frac{q!}{(q+1)!} + \cdots$$

Now, $q!$ is obviously an integer. And for every integer $k < q$, $k!$ divides $q!$ evenly

$$\frac{q!}{k!} = q \times (q-1) \times (q-2) \cdots \times (q-k+1)$$

In our series

$$q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{(q-1)!} + \frac{q!}{q!} + \frac{q!}{(q+1)!} + \cdots$$

all the terms to the left of $q!/(q-1)!$ are integers, as is $q!/(q-1)! = q$ and $q!/q! = 1$.

So now our concern is with the fractions that follow. We will show that these sum up to something less than 1. We have

$$\frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots$$

Since $q >= 2$

$$\frac{1}{(q+1)} <= \frac{1}{3}$$

$$\frac{1}{(q+1)(q+2)} <= (\frac{1}{3})^2$$

and so on, and the entire remaining series of fractions is less than or equal to

$$\frac{1}{3} + (\frac{1}{3})^2 + (\frac{1}{3})^3 + \cdots$$

This is the geometric series with $r = 1/3$ and first term equal to r, and the sum is known to be

$$\frac{1}{3}(1/(1-\frac{1}{3})) = \frac{1}{2}$$

Since the right-hand side is equal to an integer plus something "less than or equal to $\frac{1}{2}$", it is not an integer, and cannot be equal to the left-hand side, which is equal to an integer. We have reached a contradiction. Therefore e cannot be equal to p/q, for $p, q \in \mathbb{N}$.

Part VIII

Series

Chapter 23

Infinite series

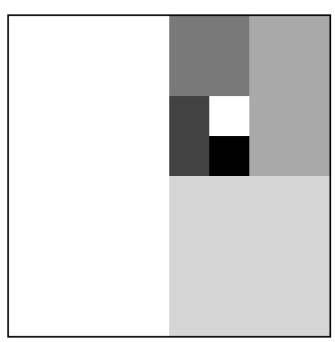
Strang says the most important series of all is the *geometric* series and I think that's right:

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

Probably the most well-known example has $x = 1/2$:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

Although the number of terms in this series is infinite, the sum is finite. Here is a "visual proof" for the sum starting at the second term:



In this figure, we see that $1/2+1/4+1/8+\cdots=1$. So the sum of the series written above is equal to $1+1=2$.

We'd like to determine algebraically or analytically what the sum is. There is a simple approach to this. Let

$$S = 1 + x + x^2 + x^3 + \dots$$

$$Sx = x + x^2 + x^3 + \dots$$

$$S - Sx = 1$$

$$= S(1-x)$$

$$S = \frac{1}{1-x}$$

This seems to work. For the first example, with $x = 1/2$, we obtain 2, as expected (starting the series at 1). Also, multiplying

$$(1-x)(1+x+x^2+x^3+\dots)$$

apparently cancels every term after the first 1.

Unfortunately, there's a problem. Above, we assumed that S is a number. But infinity is *not* a number. For some values of x, the series is not finite.

Consider $x = 1$:

$$1 + x + x^2 + x^3 + \cdots = 1 + 1 + 1 + 1 + \dots$$

And when $x = -1$, the series oscillates:

$$1 - 1 + 1 - 1 + \dots$$

So the answer we obtained before is the value of the sum, but only when that value exists.

The careful way to analyze this says, let us look at sums when we stop the series early, after n terms. The partial sum S_n is

$$S_n = 1 + x + x^2 + x^3 + \cdots + x^n$$

This value, S_n is certain to be finite if x is finite. We do what we did before.

$$(1 - x)S_n = S_n - xS_n$$

On the right-hand side, multiplication by $1-x$ produces a "telescoping sum" so that

$$= 1 + x - x + x^2 - x^2 \cdots + x^n - x^n - x^{n+1}$$

$$= 1 - x^{n+1}$$

Division gives

$$S_n = \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x} - \frac{x^{n+1}}{1 - x}$$

To evaluate the infinite series, we ask what this limit is as n grows larger

$$\lim_{n\to\infty} \frac{x^{n+1}}{1 - x}$$

If

$$x^{n+1} \to 0 \quad \text{as} \quad n \to \infty$$

then the series converges. This only happens when $|x| < 1$ and we call this range of values the "radius of convergence" of the series.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \quad \iff \quad -1 < x < 1$$

For something like

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + \ldots$$

write

$$\sum_{n=0}^{\infty} ax^n = a\sum_{n=0}^{\infty} x^n = \frac{a}{1-x}$$

To compute the sum of a geometric series, use this formula:

$$\frac{\text{first term}}{1 - \text{common ratio}}$$

example

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

This is the classic geometric series. The common ratio is 1/2. Since the ratio is between -1 and 1, the series converges. The first term is 1/2 and the value of the sum is:

$$\frac{1/2}{(1-1/2)} = 1$$

Alternatively

$$\sum_{n=0}^{\infty} \frac{1}{2^n}$$

This is the same as before except the first term is 1 and the sum is

$$\frac{1}{(1-1/2)} = 2$$

Repeating decimals

Consider

$$\sum_{n=1}^{\infty} \frac{1}{10^n}$$

$$= \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots$$

This one is also a geometric series. The sum is obviously

$$= 0.1111\ldots$$
$$= \frac{1}{9}$$

But the formula works here too.

Other interesting repeating decimals are:

$$n = 0.012345679012345679012345679\ldots$$

To see this in a more familiar form, multiply by 10^9

$$1,000,000,000\ n = 12345679.012345679012345679 0\ldots$$

The difference is

$$999,999,999\ n = 12,345,679$$
$$\frac{12345679}{999,999,999} = \frac{1}{81}$$

or even

$$\frac{1}{243} = 0.004115226337448559670781893\ldots$$

The repeat has 25 digits.

http://mathworld.wolfram.com/243.html

Harmonic series

The harmonic series is

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

Obviously, n cannot be equal to 0 so we start at $n = 1$.

The harmonic series diverges. A classic proof is to group the terms:

$$= \frac{1}{1} + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \dots$$

Clearly, we can continue this grouping operation forever. The next group has denominators from $9 \dots 16$, and in general, from $2^{n-1} + 1$ up to 2^n.

The sum for each group is at least $1/2$, so the series is larger term by term than:

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

But this series is clearly divergent, so the harmonic series, whose sum is larger term by term, also diverges.

According to Acheson, this proof dates to 1350 and can be attributed to the French scholar, Nicole Oresme.

A fun fact about this series is that if we consider the related series of inverse primes, that is

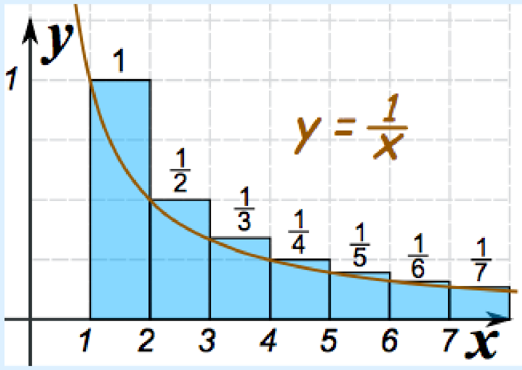
$$\sum_{n=2}^{\infty} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots$$

This sum *also* diverges (see Maor's book on infinity). Maor also says that the Nth partial sum of the harmonic series, the sum of the first N terms, obeys this inequality:

$$\ln N < S_N < \ln N + 1$$

From this, we deduce that the sum of the first one googol (10^{100}) terms of the series, is just a bit more than 230. Even so, the harmonic series diverges.

Here is another proof:



1/x vs harmonic series area

The area under the boxes is equal to the sum of the harmonic series. We will show that the integral, the area under the smooth curve, diverges to ∞. Since the harmonic series is larger than that, it also diverges.

The integral is

$$I = \int_1^\infty \frac{1}{x}\ dx$$

which we evaluate by substituting a finite upper limit

$$I = \int_1^a \frac{1}{x}\ dx$$

$$= \ln x \ \Big|_1^a = \ln a$$

But as $a \to \infty$, the logarithm also tends to ∞, so it diverges.

Related to geometric series

Again, the geometric series is:

$$S = 1 + x + x^2 + x^3 + \dots$$

Differentiate. The right-hand side is

$$1 + 2x + 3x^2 + 4x^3 + \dots$$

and the left-hand side is

$$\frac{d}{dx}\ [\ \frac{1}{1-x}\ ]\ = \frac{1}{(1-x)^2}$$

Try multiplying out

$$\begin{gathered}
(1-x)(1+2x+3x^2+4x^3+\dots) \\
= 1+2x-x+3x^2-2x^2+4x^3-3x^3+\dots) \\
= 1+x+x^2+x^3+\dots \\
= \frac{1}{1-x}
\end{gathered}$$

It checks. And since

$$\begin{gathered}
(\frac{1}{1-x})(\frac{1}{1-x}) = \frac{1}{(1-x)^2} \\
(1+x+x^2+\dots)(1+x+x^2+\dots) \\
= 1+2x+3x^2+4x^3+5x^4+\dots
\end{gathered}$$

Integrate

$$\begin{gathered}
\frac{1}{1-x} = 1+x+x^2+x^3+\dots \\
\int \frac{1}{1-x}\ dx = \int 1+x+x^2+x^3+\dots\ dx \\
-\ln|1-x| = x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\dots
\end{gathered}$$

Let $x = 1/2$

$$-\ln\frac{1}{2} = \ln 2 = \frac{1}{2}+\frac{1}{8}+\frac{1}{24}+\frac{1}{64}+\dots$$

We can calculate $\ln 2$.

This converges fairly slowly. Still 20 terms gives six places correct (0.693147).

Change variables

Start with the geometric series

$$1 + x + x^2 + x^3 + \dots$$

Replace x by $-x^2$:

$$1 - x^2 + x^4 - x^6 + \dots = \frac{1}{1 + x^2}$$

Now, does that right-hand side look familiar? Perhaps not, if you haven't seen the trigonometric functions. Take it on faith that

$$\int \frac{1}{1 + x^2}\, dx = \tan^{-1} x$$

where $\tan^{-1}$ or arc tangent is the *inverse function* to the tangent. That is, if

$$x = \tan\theta$$

then

$$\theta = \tan^{-1} x$$

Integrate the left-hand side:

$$\int 1 - x^2 + x^4 - x^6 + \dots\ dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

Set $x = 1$. The angle with $\tan^{-1}\theta = 1$ is $\theta = \pi/4$. Thus:

$$\tan^{-1} 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

We have found a series that equals π! How cool is that?

This particular series converges extremely slowly. It can be improved in a couple ways. One way is by combining adjacent terms: $1/5 - 1/7 = 2/35; 1/9 - 1/11 = 2/99$ and so on.

Another way is to set $x = 1/\sqrt{3}$, then the angle with that tangent is $\pi/6$ (recall that $\sin \pi/6 = 1/2$ and $\cos \pi/6 = \sqrt{3}/2$).

So

$$\begin{aligned}
\frac{\pi}{6} &= \frac{1}{\sqrt{3}} - \frac{1}{3}\,(\frac{1}{\sqrt{3}})^3 + \frac{1}{5}\,(\frac{1}{\sqrt{3}})^5 - \frac{1}{7}\,(\frac{1}{\sqrt{3}})^7 + \dots \\
&= \frac{1}{\sqrt{3}}(1 - \frac{1}{3}\,(\frac{1}{3}) + \frac{1}{5}\,(\frac{1}{3})^2 + \frac{1}{7}\,(\frac{1}{3})^3 \dots) \\
&= \frac{1}{\sqrt{3}}(1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \frac{1}{729} - \frac{1}{2673})
\end{aligned}$$

```
>>> from math import sqrt
>>> r = 1/sqrt(3)
>>> r * (1 - 1.0/9 + 1.0/45 - 1.0/189 + 1.0/729 - 1.0/2673)
0.5235514642438139
>>>
```

$\pi/6$ is equal to 0.5235987755... So we have only first 4 places correct. This series converges fairly quickly and is actually pretty easy to compute.

There are numerous other series for π:

http://mathworld.wolfram.com/PiFormulas.html

another example

Alcock has an interesting series in *Mathematics Rebooted*

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

It's the harmonic series, except that every term is squared. Does this series converge?

We're tempted to compare it to the geometric series with $r = 1/2$. Unfortunately the terms of the geometric series ultimately become smaller than those of our new series, so that's no help.

This is true even if we pick r smaller, like $r = 1/10$. Eventually, our new series will have larger terms.

$$\text{harmonic: } 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \dots$$

$$\text{new: } 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} + \frac{1}{100} + \dots$$

$$\text{geometric: } 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \dots$$

The trick is to find another series which we can show converges (it's not one I've seen before).

Alcock gives the formula for the sum, and we need to prove this by induction. Leaving off the first term

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$$

As $n \to \infty$, this is equal to 1.

Before we get started, it is fair to ask where this formula comes from. On the one hand, we note that

$$\frac{1}{n(n+1)} > \frac{1}{(n+1)^2}$$

so it will clearly do the job for us, term by term. Alternatively, we could write

$$\frac{1}{n(n-1)} > \frac{1}{n^2}$$

We may have to drop a term or two at the very beginning to line things up, but this is never a problem.

Now, we notice that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

In other words, any sum will be a telescoping sum. So the result is simple, just what we had above with a different symbol.

$$\sum_{n=1}^{n=k} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{k+1}$$

Now for the proof by induction. Clearly this works for the base case. We need to add the next term (for $n+1$) from the left-hand side to the right-hand side and show that it reduces to the correct form.

$$(1 - \frac{1}{n+1}) + \frac{1}{(n+1)(n+2)}$$

$$= 1 - \ [\ \frac{1}{n+1} - \frac{1}{(n+1)(n+2)}\ ]$$

$$= 1 - \ [\ \frac{n+2-1}{(n+1)(n+2)}\ ]$$

$$= 1 - \ [\ \frac{1}{n+2}\ ]$$

which is indeed the sum on the right updated from n to $n+1$. So the solution is correct, the answer is finite, and thus the series does converge.

Then, comparison with our series shows the one which we just proved convergent, is bigger term by term. So our series converges as well.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\dots$$
$$\frac{1}{2\cdot 2}+\frac{1}{3\cdot 3}+\frac{1}{4\cdot 4}+\frac{1}{5\cdot 5}+\dots$$

The second series is clearly smaller than the first, which is equal to 1.

This is a bit of a sneaky example, because the sum (including the first term) is a famous series related to π, found in the reference at the end of the last section:

$$\frac{\pi^2}{6}=\sum_{1}^{\infty}\frac{1}{k^2}$$

The fly and the train

Two trains (or bicycles) are 20 miles apart and headed directly toward each other at a speed of 10 mph. A fly starts from the front of the train on the left, flies at a speed of 15 mph to the tip of the train on the right, then turns around immediately and flies back to the first. This cycle continues until the trains meet, ending everything.

How far does the fly fly?

http://mathworld.wolfram.com/TwoTrainsPuzzle.html

There is a hard way and an easy way to do this problem. The story was that being shown the problem, Johnny Von Neumann did it instantly, and when asked how he did it, said that he used infinite series (this is the hard way).

We will sum an infinite series. It takes a little time to set up, but we know how to solve it. Let's do one round to see what happens.

∘ initial distance between trains: 20 miles

∘ relative speed of fly to second train: 25 miles per hour

∘ time to meet: $20/25 = 0.8$ hour

∘ distance the fly travels: $0.8 \times 15 = 12$ miles

∘ each train travels $0.8 \times 10 = 8$ miles

∘ distance separating the trains after this round: 4 miles

The ratio $4/20 = 1/5$ miles allows us to set up the series:

$$
\begin{gathered}
12 + 12\frac{1}{5} + 12(\frac{1}{5})^2 + \ldots \\
12\ [\ 1 + \frac{1}{5} + (\frac{1}{5})^2 + \ldots\ ] \\
1/1 - r = 5/4
\end{gathered}
$$

Total distance = 15 miles.

The easy way is that the closing speed of the two trains, 20 miles per hour, is the same as the initial separation of 20 miles, so the trains meet in 1 hour. The fly travels 15 miles in one hour.

We should be able to get the series from the train speeds, the fly speed and the initial distance. Let

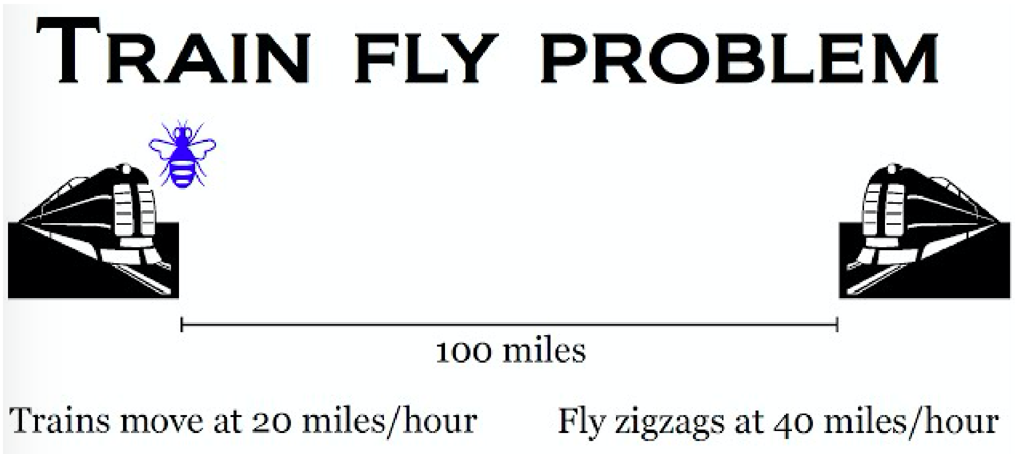
$$d = \text{ initial distance/(train speed + fly speed)} = \frac{20}{15 + 10}$$

The common ratio is

$$r = 1 - d = \frac{1}{5}$$

The initial value a is the fly speed times d.

Here is a nice graphic for a variant that I found on the web.



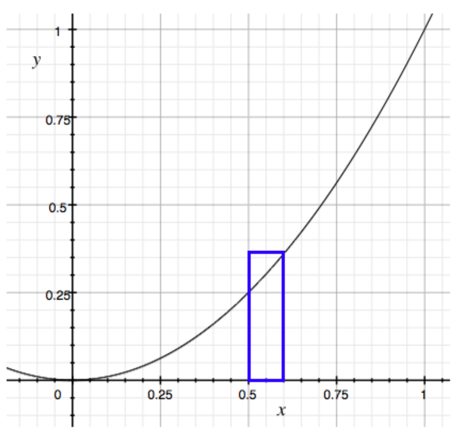
Chapter 24

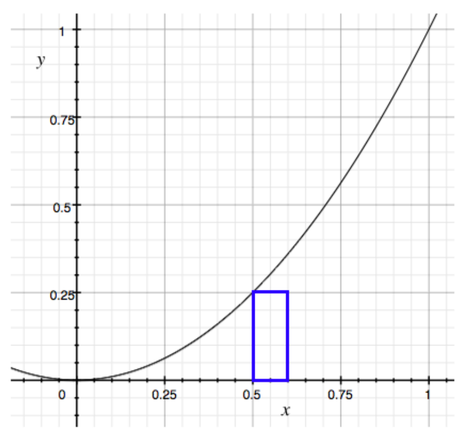
Riemann sums

Riemann sums

We introduced integration as simply the reverse of differentiation. The fundamental theorem of calculus gives us a means of evaluating integrals between two bounds.

Starting with Courant, however, calculus courses have sought a more formal approach. The first (though not the only) method is to compute what are called Riemann sums for the the area bounded under a curve.





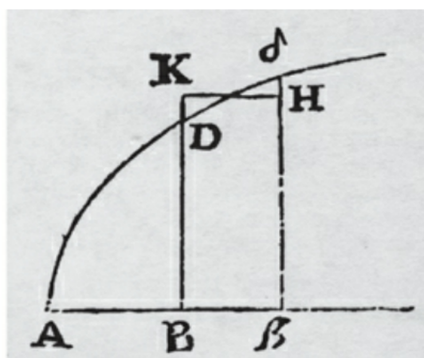
Our first example is to calculate the area under the curve $f(x) = x^2$.

The areas of many small rectangles are added to form the sum.

The key is to set up a calculation or expression for the area in terms of a variable number of rectangles, N. Although any rectangle can only be an approximation to a curved surface, if we use many skinny rectangles the approximation will be very good, and *in the limit* as $N \to \infty$, as we use an infinite number of rectangles, it will be exactly right.

The proof that it *is* right is to show that the sum of a set of rectangles bounding the curve below, and the sum for a second set bounding the curve above, *converge to the same limit* as $N \to \infty$.

Another way to see this is to look at this diagram from Newton's book *De Analysi* (from Acheson's book *The Calculus Story*).



Newton's argument is that if we draw a box as illustrated, there must be one height, one point along the top of the rectangle where the area under the curve but not in the box, to the right, is exactly equal to the area over the curve and in the box, to the left. At that point, the errors exactly cancel.

If this point is always bounded by the y-values of the left and right sides of the box, then as the boxes get smaller and smaller, the balance point will always be included somewhere in the rectangle, so the answer will come out exactly right. (This latter assumption breaks down at maxima and minima, but only for finite boxes. Let's admire the diagram and move on to the actual method).

example

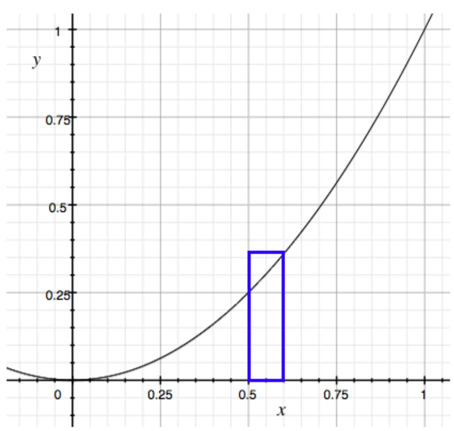
We start with x^2. It is the simplest power curve, and we actually know the answer due to work by Archimedes, called the **quadrature**.

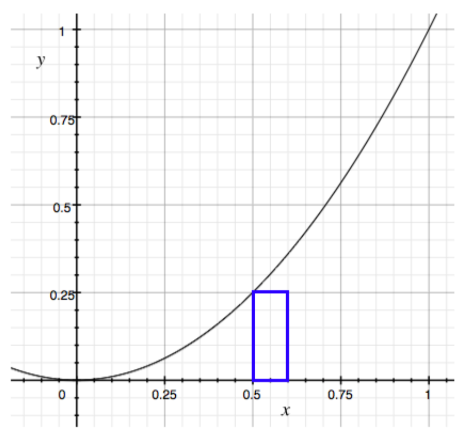
Consider the region bounded by the x-axis, the y-axis, the line $x = 1$, and the curve $y = x^2$.

Partition the region on the x-axis between $x = 0$ and $x = 1$ into N segments. Each segment will contain a tall, thin rectangle that extends from the x-axis up to the curve.

Here is a figure that illustrates the basic idea. The region between 0 and 1 is divided into 10 segments, so the width of each segment is 0.1.

In the left panel, the blue rectangle shown is the sixth segment; its left and right bounds are $x = 0.5$ and $x = 0.6$. The height is $x^2 = 0.6^2 = 0.36$. The right panel is the same, except the height corresponds to the value at the left-hand bound $x^2 = 0.5^2$.





We compute the area (using the first set of rectangles) as

$$\begin{aligned} A &= 0.1(0.1^2 + 0.2^2 + \cdots + 1.0^2) \\ &= 0.1(0.01 + 0.04 + \cdots + 1.0) \\ &= 0.1(3.85) = 0.385 \end{aligned}$$

This is obviously an over-estimate of the area (as it will be for any function that increases over the interval), but the trick is that as the number of rectangles becomes very large, the result will converge to the exact area we want.

area as a limit

We divide the region into N intervals. Each interval has width $1/N$. The x-values of the right hand side of these boxes are

$$\frac{1}{N}, \frac{2}{N} \cdots \frac{N}{N}$$

We can rewrite this as

$$\sum_{k=1}^{N} \frac{k}{N}$$

We will first compute the sums for the set of boxes that is an overestimate, it uses the x-value of the right hand side of each box times the width of each box:

$$A = \sum_{k=1}^{N} f(\frac{k}{N}) \cdot \frac{1}{N}$$

Since the function is x^2 this is

$$A = \sum_{k=1}^{N} (\frac{k}{N})^2 \cdot \frac{1}{N}$$

And since N is a constant, it can be pulled out from the summation:

$$A = \frac{1}{N^3} \sum_{k=1}^{N} k^2$$

Now we need an expression for the sum of the squares of the first N integers. We will show below that the formula is $N \cdot (N+1)/2 \cdot (2N+1)/3$. Distributing the factor of $1/N^3$ over each term, we obtain

$$A = \frac{N}{N} \cdot \frac{N+1}{2N} \cdot \frac{2N+1}{3N}$$

If $N \to \infty$ and we take the limit, the result is the value of the integral

$$\begin{aligned} I &= \lim_{N \to \infty} \frac{N}{N} \cdot \frac{N+1}{2N} \cdot \frac{2N+1}{3N} \\ &= 1 \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} \end{aligned}$$

The reason is that as $N \to \infty$ the difference between N and $N+1$ (or $N-1$ and N becomes negligible compared to the size of N, therefore the ratio $(N+1)/2N$, for example is equal to $1/2$ in the limit. In other words, for, say

$$\lim_{N \to \infty} \frac{1}{2N} \ (N+1) = \lim_{N \to \infty} \frac{1}{2}(1 + \frac{1}{N}) = \frac{1}{2}$$

We must still compute the sums for the set of boxes that is an under-estimate (for finite N), with the x-value used for the function taken from the left-hand side. This is the series

$$\frac{0}{N}, \frac{1}{N} \dots \frac{N-1}{N}$$

If the function is $f(x) = x^2$ the sum is

$$A = \sum_{k=0}^{N-1} (\frac{k}{N})^2 \cdot \frac{1}{N}$$

Since the first term is zero, just remove it and start the index from 1

$$A = \sum_{k=1}^{N-1} (\frac{k}{N})^2 \cdot \frac{1}{N}$$

For the integral, we obtain

$$I = \lim_{N \to \infty} \frac{N-1}{N} \cdot \frac{N}{2N} \cdot \frac{2N-1}{3}$$

We obtain exactly the same result as before. This is really the proof that the method works. In the limit of infinite N, the methods which over-estimate and under-estimate the area converge to the same value, therefore the result is exactly correct.

integer sums

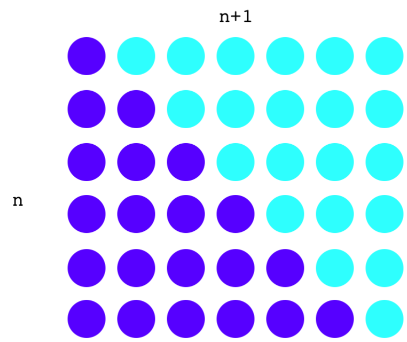
The first series that one usually sees of this type is

$$\sum_{k=1}^{n} k$$

The sum of the integers from $1 \to N$. There is a simple formula

$$\frac{n(n+1)}{2}$$

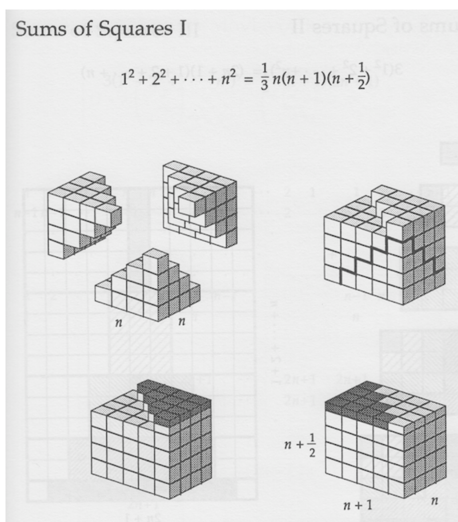
Here is a famous "proof without words":



For Riemann sums, we need a series one level higher. We need the sum of the squares of the first n integers. The answer is what we had before, multiplied by another term

$$\frac{n(n+1)}{2}\,\frac{2n+1}{3}$$

Here is another "proof without words."



For much more detail, see the chapter in the Addendum.

back to the Riemann sum

We plug that expression into the Riemann Sum:

$$= \frac{1}{N^3} \frac{N(N+1)}{2} \, \frac{2N+1}{3}$$

Each of the terms in $N(N+1)(2N+1)$ is grouped with one of the N's in the denominator at the left

$$= \frac{1}{6} \, \frac{N}{N} \, \frac{(N+1)}{N} \, \frac{(2N+1)}{N}$$

In the limit as N gets very large.

$$\lim_{N\to\infty} \frac{N}{N} = 1$$

$$\lim_{N\to\infty} \frac{N+1}{N} = \lim_{N\to\infty} 1 + \frac{1}{N} = 1$$

$$\lim_{N\to\infty} \frac{2N+1}{N} = \lim_{N\to\infty} 2 + \frac{1}{N} = 2$$

Thus, the final answer is 1/3, which agrees with Archimedes.

n cubed

The height of the first interval is $(1/N)^3$ and that of the kth interval is $(k/N)^3$. The total area is:

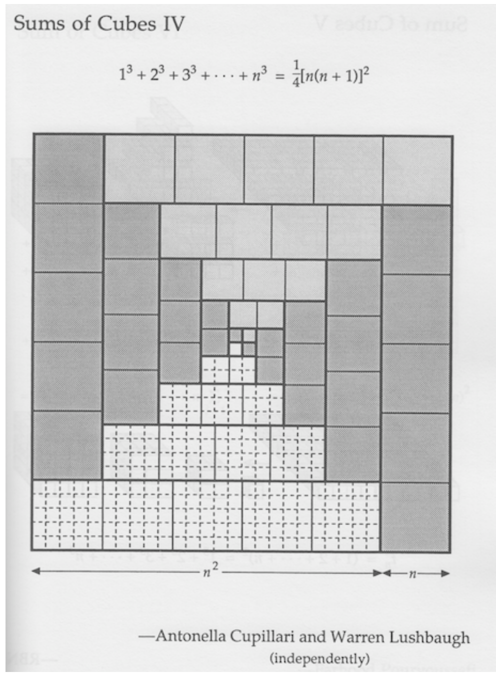
$$\sum_{k=1}^{N} (\frac{k}{N})^3 \times \frac{1}{N}$$

Since N is a constant, it can be pulled out from the summation:

$$\frac{1}{N^4} \sum_{k=1}^{N} k^3$$

So now we need an expression for the sum of the cubes of the first N integers.

Yet another ingenious "proof without words."



$$\sum_{k=1}^{N} k^3 = \frac{1}{4}\ [\ n(n+1)\ ]^2$$

$$= \frac{1}{4}\ N(N+1)N(N+1)$$

As before, each of the four factors of N in the denominator cancels an N or $N+1$ on top and we're left with just $1/4$.

It turns out that if you let the interval be $[0,b]$ or even $[a,b]$, we obtain the expressions you will be used to from integral calculus, namely

$$\int_a^b n^2 = \frac{n^3}{3}\ \Big|_a^b$$

and

$$\int_a^b n^3 = \frac{n^4}{4} \Big|_a^b$$

Chapter 25

Fermat area

In computing Riemann sums, it isn't required that the intervals have the same width, only that the width of the largest goes to zero in the limit.

Courant and John describe a variation on Riemann sums using intervals of unequal (but graduated) width. This "trick" allows them to derive the formula for

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$
$$\int_a^b x^n \, dx = \frac{b^{n+1} - a^{n+1}}{n+1}$$

for all natural numbers n first, and then with some elaborations, for real n except $n = -1$.

This result (for integers) is due to Fermat and was achieved about 1640 (i.e. 25 years before Newton). I found that proof on the web

http://fredrickey.info/hm/CalcNotes/Fermat-Integration.pdf and there is also a good discussion in Maor's *e, the Story of a Number*. We'll look at Fermat's proof here and save the other for the Addendum (**here**). Fermat's version achieves simplicity by using the interval $[0, b]$ with its lower bound at zero.

derivation

Let E be a positive constant less than 1. Divide the region $[0, b]$ into subintervals with boundaries

$$\dots bE^3,\ bE^2,\ bE,\ b$$

How do we get the width of the largest rectangle to decrease to zero? By taking the limit $E \to 1$.

Construct rectangles in the usual way that circumscribe the curve $y = x^n$ and add up their areas. For the ith rectangle, the width is

$$bE^i - bE^{i+1}$$

$(bE^{i+1} < bE^i)$, and the height is

$$(bE^i)^n$$

so the overall sum is

$$S = \sum_{i=0}^{\infty} (bE^i)^n\ (bE^i - bE^{i+1})$$

$$= b^{n+1} \sum_{i=0}^{\infty} (E^i)^n\ (E^i - E^{i+1})$$

$$= b^{n+1} \sum_{i=0}^{\infty} (E^i)^{n+1}\ (1 - E)$$

$$= b^{n+1}\ (1-E)\ \sum_{i=0}^{\infty}(E^{n+1})^i$$

Since E is a positive constant less than 1, E^{n+1} is also. Let $q = E^{n+1}$. The sum becomes

$$\sum_{i=0}^{\infty} q^i = q^0 + q^1 + q^2 + q^3 + \dots$$

Recall that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

for $|x| < 1$.

So, going back to E we have

$$S = b^{n+1}\ (1-E)\ \frac{1}{1-E^{n+1}}$$

and using the same identity again

$$1 - x = \frac{1}{1 + x + x^2 + x^3 + \dots}$$

so

$$S = b^{n+1}\ \frac{1}{(1 + E + E^2 + E^3 + \dots)(1 - E^{n+1})}$$

All the terms in the infinite series starting at E^{n+1} acquire counterparts with a minus sign, hence

$$= b^{n+1}\ \frac{1}{1 + E + E^2 + E^3 + \cdots + E^n}$$

Now take the limit as $E \to 1$. The fraction becomes just

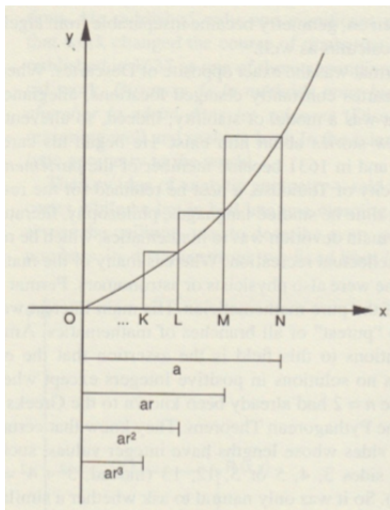
$$\frac{1}{1 + E + E^2 + E^3 + \cdots + E^n} = \frac{1}{n+1}$$

and we have

$$\int_0^b x^n = \frac{b^{n+1}}{n+1}$$

which is what we get when we evaluate x^{n+1} on the interval $[0, b]$ and then divide by $n + 1$.

This diagram from Maor uses slightly different nomenclature — the interval points are of the form $a, ar, ar^2 \ldots$ (moving from right to left).



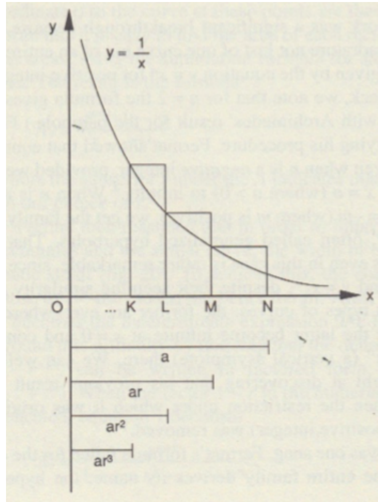
Saint Vincent

It turns out that the above analysis applies for integer $n < -1$,

All except for the most important case, the hyperbola with $n = -1$. The problem that we run into is division by zero.

However, a Belgian Jesuit named Saint Vincent noticed something about the intervals underneath the curve $y = 1/x$. This work was completed about 1631 and published in 1647.

Starting at N and moving backward, let's compute the width (h), height h and area A for each interval.



N:

$$w = a - ar = a(1-r), \quad h = \frac{1}{a}, \quad A = 1 - r$$

M:

$$w = ar - ar^2 = ar(1-r), \quad h = \frac{1}{ar}, \quad A = 1 - r$$

L:

$$w = ar^2 - ar^3 = ar^2(1-r), \quad h = \frac{1}{ar^2}, \quad A = 1 - r$$

Maor:

> This means that as the distance from 0 grows geometrically, the corresponding areas grow in equal increments, that is, arithmetically, and this remains true even when we go to the limit ... But this in term implies that the relationship between area and distance is logarithmic.

The area under the curve 1/x is the logarithm. It took some time to figure out that the base of the logarithm was e.

Chapter 26

Numerical integration

It has been shown that some important integrals cannot be "solved" analytically (i.e. we cannot find $F(x)$). For example, the normal distribution (with mean and standard deviation both equal to 1) is described by this probability density function:

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

To find the expected value or probability that the value lies between bounds a and b we should compute:

$$\int_a^b f(x)\, dx$$

However, there is no function $F(x)$ such that $F'(x) = f(x)$, so we cannot solve the equation in the normal way by computing $F(b) - F(a)$.

To compute the integral, we fall back on Riemann sums. Divide the closed range $[a, b]$ into N rectangles whose individual height is the value $f(x)$ somewhere in the rectangle, and then compute the sum of

$\Delta x \times f(x)$ over the whole interval. A simple approach uses rectangles of constant width (equal to $b - a/N$).

Using Python:

https://gist.github.com/telliott99/5a1190217a130c7ee01dee17ea483f7b

I hope the flow is clear. The function `get_xvalues` generates a list of x-values starting from the middle of the first step past a and continuing to the last step just before b. `integrate` simply computes $f(x)$ for each x-value, sums all of those values, and adjusts for the width of the steps (rectangles).

We integrate $f(x) = x^2$ over the ranges $[0, 1]$ and $[1, 2]$ and obtain the expected results (1/3 and 7/3).

We also integrate the normal probability density function over ranges $[-2, 2]$ to $[-10, 10]$. With standard deviation equal to 1, we obtain the expected result that 95% of the density lies within two standard deviations of the mean. Essentially all of the density lies within four standard deviations of the mean.

And finding that the total area equals 1 confirms that the normalization factor $1/\sqrt{2\pi}$ is correct. That is, the value of the unnormalized integral is equal to $\sqrt{2\pi}$.

Refinements

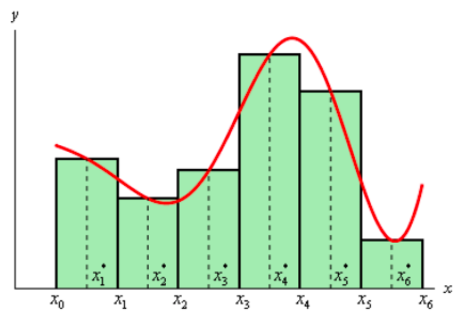
Classically, the major improvement to be made to this algorithm is to make a more accurate estimation of the area for each small rectangle. This is not so important with fast computers. For example, with a million steps rather than 100, I obtain

```
> python numerical_int.py
0.333333333333
```

```
2.33333333333
>
```

for the first two integrations, in about two seconds.

The calculation above uses the midpoint rule, where $f(x)$ is evaluated at the midpoint of the range.



The step size is computed and used to generate a list of values where each rectangle starts, then half the step is added to give the midpoint.

If we think of a and b as the bounds for each small rectangle, then the average of a and b is the midpoint:

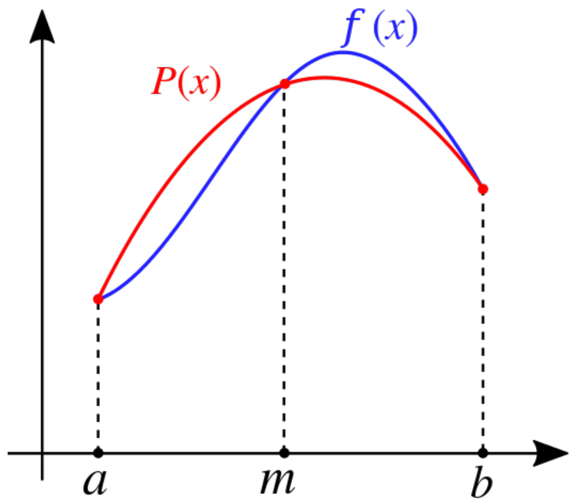
$$m = \frac{a+b}{2}$$

We evaluate $f(m)$, the function at the midpoint, and then multiply by the width:

$$M = f(m) \cdot (b-a)$$

Simpson's rule is a more sophisticated approach that uses:

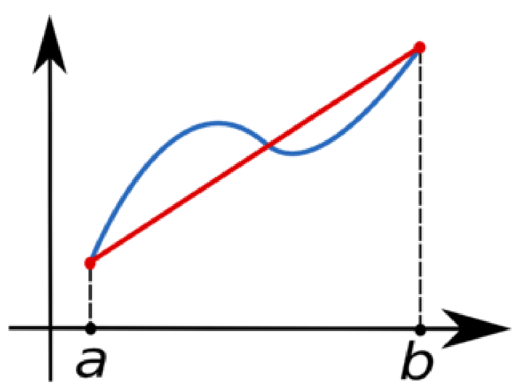
$$\frac{f(a) + 4f(m) + f(b)}{6} \cdot (b-a)$$



We sample once each from a and b, and four times from m, and average those samples.

The trapezoidal rule is

$$\frac{f(a)+f(b)}{2}\cdot(b-a)$$



Simpson's rule is really just a combination of the other two rules, namely, it is equal to $(2M+T)/3$. It weights the value at each endpoint as $1\times$ and then the midpoint as twice the combined values at the endpoints.

Essentially, this fits a parabola to the points a, m, b and then computes the area. It is really Archimedes' result (quadrature) in disguise.

Consider the parabola $y = -x^2 + 1$ between $x = -1 \to 1$ (the points where it crosses the x-axis on its way down). Our task is to find the correct y-value to use as the average height of the function in this region. Inverting the standard result for the area under $y = x^2$, the area under this parabola is $2/3$ of the area above it. The result we seek is $2/3$.

So a quadratic approximation to the area samples four times at the vertex plus once each at $x = \pm 1$. We sample because we understand that the curve is probably not exactly quadratic.

Part IX

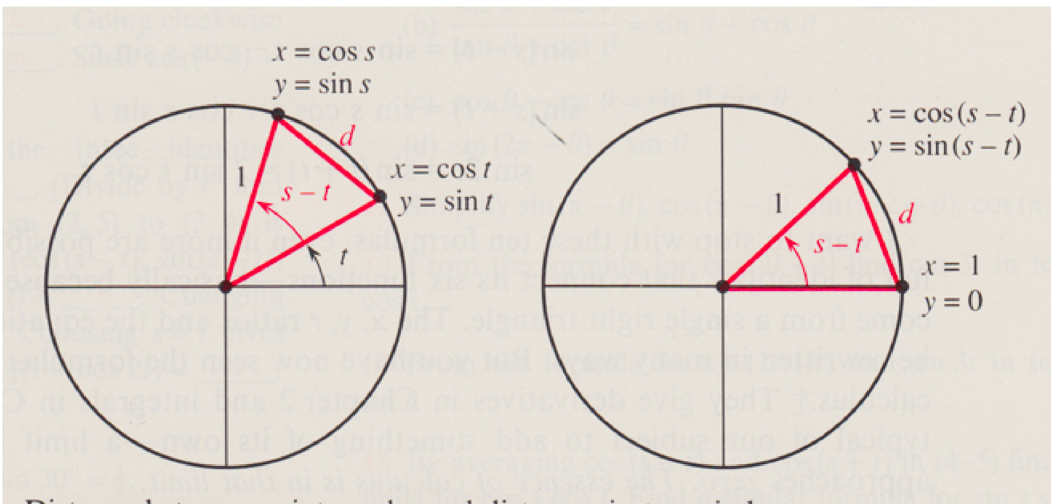
Sum of angles and an application

Chapter 27

Sum of angles again

Strang

For a geometric derivation of the sum of angles formula with minimal setup, I really like this figure from Strang



Distance between points and equal distances in two circles.

We have the same triangle in the two panels, just rotated clockwise on the right.

To compute the distance between two points in the plane we do

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(This is just the Pythagorean theorem in disguise).

We don't actually need to take the square root, let's stick with

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

In the first figure, t is the angle between the lower radius and the x-axis, s is the angle between the upper radius and the x-axis, and as labeled, $s - t$ is the angle between the two radii.

The distance d squared for the left panel is

$$d^2 = (\cos s - \cos t)^2 + (\sin s - \sin t)^2$$

Multiply out:

$$d^2 = \cos^2 s - 2\cos s \cos t + \cos^2 t + \sin^2 s - 2\sin s \sin t + \sin^2 t$$

We have two copies of $\sin^2 + \cos^2$, one for angle s and one for t

$$d^2 = 2 - 2\cos s \cos t - 2\sin s \sin t$$

In the right panel, the two radii have been rotated, preserving the same angle between them.

$$d^2 = (\cos(s-t) - 1)^2 + \sin(s-t)^2$$

(Don't forget the 1).

$$= \cos^2(s-t) - 2\cos(s-t) + 1 + \sin^2(s-t)$$

$$= 2 - 2\cos(s-t)$$

Because the included angle hasn't changed, neither has the distance, so we can equate the two expressions.

$$2 - 2\cos(s - t) = 2 - 2\cos s \cos t - 2\sin s \sin t$$

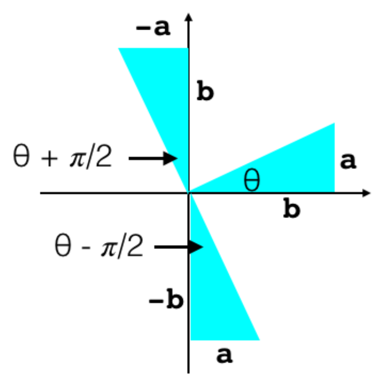
Subtract 2 from both sides, multiply by 1/2, and change sign to give

$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

This is our first formula, for the cosine of the difference of two angles.

getting to sine

Look at the relationships between sine and cosine for angles that are related by addition or subtraction of $\pi/2$.



In the figure, I have simply rotated the same triangle.

From the figure we can easily read off these four identities

$$\sin(\theta + \pi/2) = b = \cos\theta$$

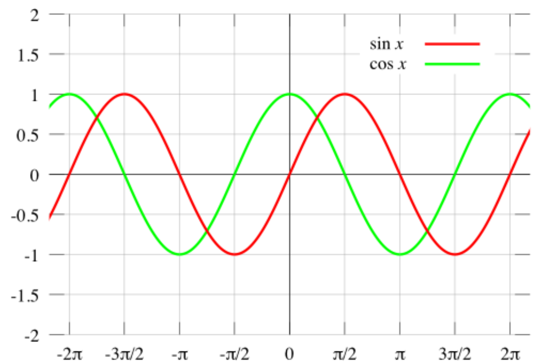
$$\cos(\theta + \pi/2) = -a = -\sin\theta$$

And

$$\sin(\theta - \pi/2) = -b = -\cos\theta$$

$$\cos(\theta - \pi/2) = a = \sin\theta$$

Here is an alternative derivation which proceeds from the graph of sine and cosine versus the angle.



Pick some angle (say $\theta = 0$), then $\cos\theta = 1$. What is the angle for which the sine gives the same result? The sine curve is exactly like the cosine, it is just shifted to the right by a *phase change* of $\pi/2$.

That angle is $\theta + \pi/2$. The phase change is added to the angle:

$$\cos\theta = \sin(\theta + \frac{\pi}{2})$$

Try the same reasoning in reverse.

The cosine curve is exactly like the sine, it is just shifted by a phase change of $-\pi/2$, i.e. to the left. Pick some angle (say $\theta = \pi/2$), then $\sin\theta = 1$. What is the value of the angle for which the cosine gives the same result?

It is $\theta - \pi/2$. The phase change is subtracted from the angle θ:

$$\sin\theta = \cos(\theta - \frac{\pi}{2})$$

In summary, switching sine for cosine gives a valid expression, but there is a difference of *sign* for the phase.

back to our task

So our sum of angles formula (well, really, the difference of angles) was

$$\cos(s-t) = \cos s \cos t + \sin s \sin t$$

Let

$$u = t - \frac{\pi}{2}$$
$$t = u + \frac{\pi}{2}$$

Substitute for t

$$\cos\ [\ s - (u + \frac{\pi}{2})\ ]\ = \cos s \cos(u + \frac{\pi}{2}) + \sin s \sin(u + \frac{\pi}{2})$$

Regroup the left-hand side

$$\cos\ [\ (s-u) - \frac{\pi}{2})\ ]\ = \cos s \cos(u + \frac{\pi}{2}) + \sin s \sin(u + \frac{\pi}{2})$$

Referring to the results we obtained above, cosine something minus $\pi/2$ is the sine of that something:

$$\sin(s-u) = \cos s \cos(u + \frac{\pi}{2}) + \sin s \sin(u + \frac{\pi}{2})$$

Cosine something plus $\pi/2$ is minus the sine:

$$\sin(s-u) = -\cos s \sin u + \sin s \sin(u + \frac{\pi}{2})$$

Sine something plus $\pi/2$ is cosine:

$$\sin(s-u) = -\cos s \sin u + \sin s \cos u$$

Rearrange:

$$\sin(s-u) = \sin s \cos u - \sin u \cos s$$

This is correct, but the path is fraught with error!

For now, memorize. Soon we will see a very simple and effective **aid to memory** due to Euler.

sum of tangents

It is also easy to derive the sum of tangents from the sum of sines and cosines.

$$\tan s + t = \frac{\sin s + t}{\cos s + t}$$
$$= \frac{\sin s \cos t + \cos s \sin t}{\cos s \cos t - \sin s \sin t}$$

Divide by $\cos s \cos t$

$$\tan s + t = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$

$$\tan s - t = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

We will use these for a few problems later in the book.

Chapter 28

Double and half angles

We will find it useful in several problems to be able to compute the sine, cosine and tangent of angle 2θ, knowing the values for θ. These formulas can be rearranged to give the values of $\theta/2$ in terms of θ.

I can't remember these formulas, but derive them from the sum of angles when needed.

cosine

Start with our old friend:

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

Let $s = t$:

$$\cos 2s = \cos^2 s - \sin^2 s$$

Since $\sin^2 s + \cos^2 s = 1$, $-\sin^2 = \cos^2 -1$ so

$$\cos 2s = 2\cos^2 s - 1$$

We can use this formula to compute the value for $2s$ given that for s. To go from 2θ to θ:

$$\cos^2 s = \frac{1}{2}(1 + \cos 2s)$$
$$\cos s = \sqrt{\frac{1}{2}(1 + \cos 2s)}$$

sine

$$\sin s + t = \sin s \cos t + \cos t \sin s$$

Let $s = t$:

$$\sin 2s = 2 \sin s \cos s$$

Put the other way

$$\sin s = \frac{\sin 2s}{2 \cos s}$$

tangent

The formulas for the tangent are easily obtained by substitution. Let us simplify the notation a bit by setting $S = \sin 2t$ and $S' = \sin t$ and similarly for cosine and tangent. From above we have the basic relationships

$$S' = \frac{S}{2C'}$$

and

$$C' = \sqrt{\frac{1}{2}(1 + C)}$$
$$2[C']^2 = 1 + C$$

So the tangent ($T' = \tan s$) is:

$$T' = \frac{S'}{C'} = \frac{S}{2C'} \ \frac{1}{C'} = \frac{S}{2[C']^2}$$

$$= \frac{S}{1+C}$$

That's a fairly remarkable simplification!

Another way to say the same thing:

$$\frac{1}{T'} = \frac{1}{T} + \frac{1}{S}$$

This result can be massaged in various ways. Multiply on the top and bottom of the right-hand side by T

$$T' = \frac{ST}{S+T}$$

Also, since

$$T' = \frac{S}{1+C} = \frac{S'}{C'}$$

$$C' = \frac{S'(1+C)}{S}$$

In going from unprimed (2θ) to prime (θ), it seems that the most straightforward way is to compute

$$C' = \sqrt{\frac{1}{2}(1+C)}$$

$$T' = \frac{S}{1+C}$$

and then the sine last

$$S' = \frac{S}{2C'}$$

check

Let's try checking the results for a known angle

$$2\theta = \pi/3$$

$$S = \sin 2\theta = \frac{\sqrt{3}}{2}, \quad \cos 2\theta = \frac{1}{2}, \quad T = \tan 2\theta = \sqrt{3}$$
$$S' = \sin\theta = \frac{1}{2}, \quad \cos\theta = \frac{\sqrt{3}}{2}, \quad T' = \tan\theta = \frac{1}{\sqrt{3}}$$

Our first equation is

$$T' = \frac{ST}{S+T} = \frac{3/2}{(3/2)\sqrt{3}} = \frac{1}{\sqrt{3}}$$

That looks good. The second one is

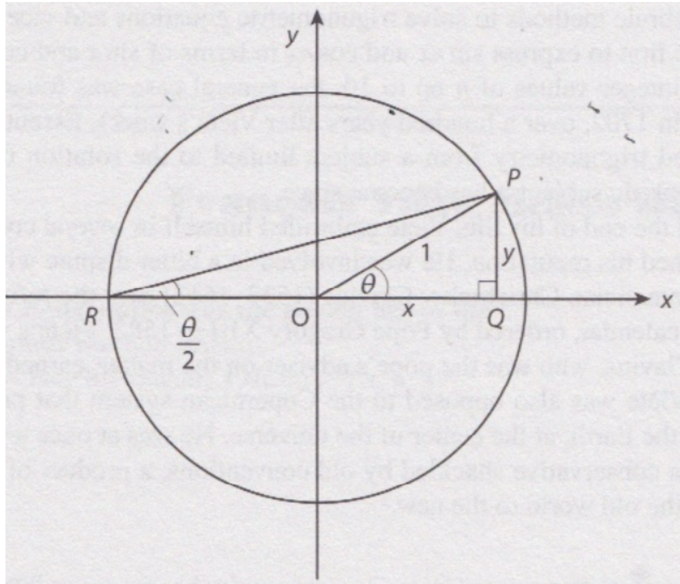
$$S' = \sqrt{\frac{ST'}{2}}$$

$$ST' = \frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}} = \frac{1}{2}$$
$$S' = \sqrt{\frac{1}{2}\ \frac{1}{2}} = \frac{1}{2}$$

These both look correct.

geometric approach

Here are two simple geometric derivations of the half angle formulas for sine and cosine.



For the first, draw an angle θ in a unit circle. We proved the theorem that the angle at the left (on the circle) is equal to $\theta/2$ **here**.

Algebraically, write

$$\cos\frac{\theta}{2} = \frac{1+x}{\sqrt{(1+x)^2+y^2}}$$

$$= \frac{1+x}{\sqrt{1+2x+x^2+y^2}}$$

$$= \frac{1+x}{\sqrt{2+2x}}$$

$$= \sqrt{\frac{1+\cos\theta}{2}}$$

To get the formula for the sine just use the identity

$$\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$$

$$\sin^2\frac{\theta}{2} = 1 - \frac{1+\cos\theta}{2} = \frac{1-\cos\theta}{2}$$

Using the prime notation from above

$$S' = \sqrt{\frac{1-C}{2}}$$

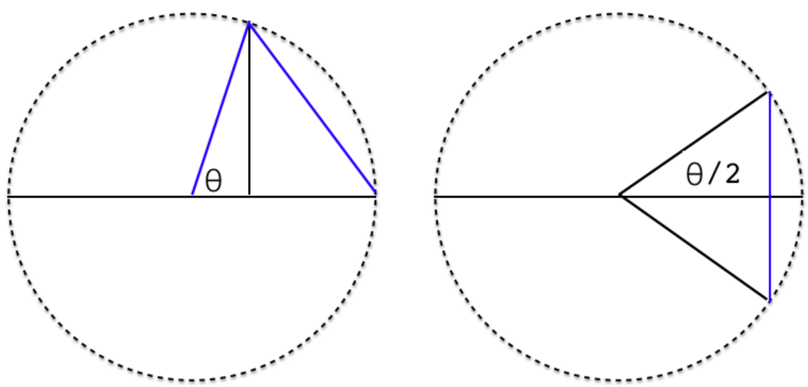
This is easily checked

$$S'^2 + C'^2 = 1$$
$$\frac{1-C}{2} + \frac{1+C}{2} = 1$$

which looks correct.

For the second derivation, again draw an angle θ in a unit circle. As usual, then, the base of the right triangle is $\cos\theta$ and the height is $\sin\theta$ (left panel, below).

Now, also draw the chord of the circle, corresponding to the arc θ.



Notice that the chord is the hypotenuse of a right triangle whose height is $\sin\theta$ and whose base is $1-\cos\theta$. Therefore, the length of the chord is

$$c = \sqrt{(1-\cos\theta)^2 + \sin^2\theta} = \sqrt{2-2\cos\theta}$$

Now, rotate the chord as shown in the right panel. We see that one-half the chord is equal to the sine of $\theta/2$:

$$\sin\theta/2 = \frac{1}{2}\,\sqrt{2-2\cos\theta}$$

$$S' = \frac{1}{\sqrt{2}}\ \sqrt{1-C}$$

and

$$S'^2 + C'^2 = 1$$

$$C'2 = 1 - S'^2 = 1 - \frac{1-C}{2} = \frac{1+C}{2}$$

$$C' = \frac{1}{\sqrt{2}}\ \sqrt{1+C}$$

Part X

Euler

Chapter 29

Euler's equation

According to the historians, Newton came up with a series for e^x about 1669 in his book *De analysi per aequationes numero terminorum infinitas*, although he didn't call e by that name or anything.

Recall that one definition of the exponential is

$$y = e^x = y'$$

We try to *approximate* this function by a series. At $x = 0$, $e^x = e^0 = 1$ so

$$y = 1$$

So far so good. However, the derivative y' is then zero. How do we get a 1 into the derivative? By adding a term of x

$$y = 1 + x$$

Now the derivative matches in its first term

$$y' = 1$$

But we also need

$$y' = y = 1 + x$$

and that x has to come from somewhere in the expression for y. So add $x^2/2$ because its derivative is just x and we obtain

$$y = 1 + x + \frac{x^2}{2}$$

and again we need

$$y' = y = 1 + x + \frac{x^2}{2}$$

so

$$y = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$$

Well, you get the idea. This continues forever.

There are also infinite series for sine and cosine, of course. The have traditionally been attributed to Newton, who described them in the work cited earlier.

According to Stewart (*Significant Figures*), they were known much earlier, discovered by the Indian mathematician, Madhava (approximately 1350-1425 C.E.).

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$

Clearly, the derivative of the first series for sine is equal (term by term) to the series for cosine. And the derivative of the sine is minus the cosine. It all checks.

Actually, we can use what we know about the derivatives of sine and cosine to follow an approach similar to that for the exponential, to come up with these series ourselves.

In particular, we know that the second derivative is minus the function, and the fourth derivative is the function itself.

$$\sin x =$$

The first term can't be 1 because $\sin 0 = 0$ so write

$$\sin x = x + \dots$$

Now, we want $\sin'' x = -\sin x$ so

$$\sin x = x - \frac{x^3}{3!} + \dots$$

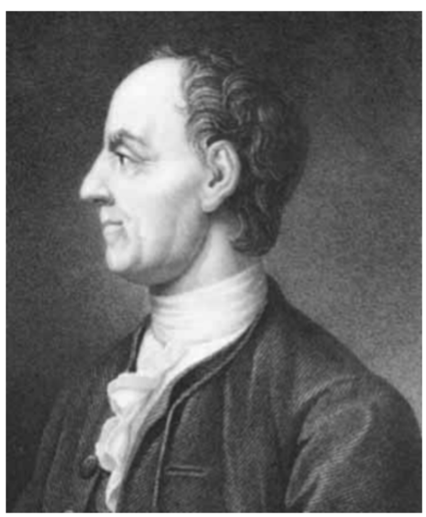
does that. Then $\sin'''' x = \sin x$ so

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

Then just take the derivative of this to get the series for cosine.

noticing a connection

Here is a portrait of Euler where he is not wearing that silly hat.



Euler said, let us consider the complex number

$$z = \cos\theta + i\sin\theta$$

How in the world did he come up with this? Perhaps because he already knew the answer.

My guess is that he looked at the formula

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots$$

and said: it cannot be an accident that the sine and cosine series look so similar, in fact when added together they have exactly the same terms, just with some periodically occuring minus signs.

$$\sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \dots$$

Where would I get alternating plus and minus signs from? Maybe if I substitute ix for x?

$$e^{ix} = \sum_{n=0}^{\infty} \frac{ix^n}{n!} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} \dots$$

Clearly, part of this is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots$$

while the rest is

$$ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} \dots = i\sin x$$

quick and dirty derivation

Let's just see what we can do.

If we assume that calculus is legal with complex numbers (assuming that i is just a number), we can do the following

$$z = \cos\theta + i\sin\theta$$

$$\frac{dz}{d\theta} = -\sin\theta + i\cos\theta$$

(It turns out that it is legal, but the argument requires the fact these functions are all *analytical*, which is beyond our scope here).

And since $i^2 = -1$

$$\frac{dz}{d\theta} = i^2\sin\theta + i\cos\theta$$

$$\frac{dz}{d\theta} = i(i\sin\theta + \cos\theta) = iz$$

Rearrange

$$\frac{1}{z}\ dz = i\ d\theta$$

$$\int \frac{1}{z}\ dz = \int i\ d\theta$$

$$\ln z = i\theta$$

Exponentiate:

$$z = e^{i\theta}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Euler's famous result in a few lines, which he proved more rigorously (but not completely so) by other approaches explored **here**.

addendum

We've really gone fast and loose in this section. One thing that makes series hard is that a series is sometimes valid only for a certain range of x. Consider

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 \ldots$$

The equality is easily verified if you multiply both sides by $1-x$. You get the right-hand side, plus a version of the right hand side containing every term except the first, all with minus signs. It all adds up to 1.

The problem is, the series is obviously not valid for $x = 1$, nor for $x > 1$. Consider $x = 2$. (It gets worse. Consider $x = -1$). In fact, it turns out to be valid only for $|x| < 1$, which is called the radius of convergence.

Luckily, the exponential series and those for sine and cosine *are* valid for all x.

According to Nahin (*An Imaginary Tale*), both Abraham de Moivre and Roger Cotes knew Euler's identity decades before he published it. Which may be a good example of

https://en.wikipedia.org/wiki/Stigler%27s_law_of_eponymy

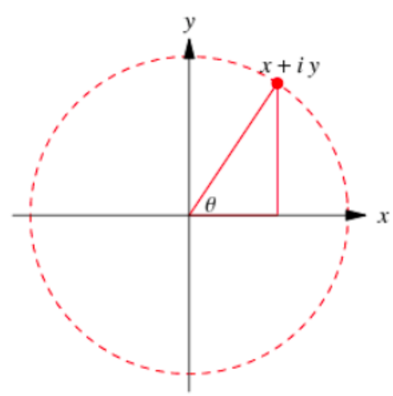
Chapter 30

Euler sum of angles

Euler's equation says that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

One way of thinking about the equation is to view complex numbers as points in the plane. Complex numbers are composed of a real part (say x) and a complex part (iy), where $i = \sqrt{-1}$.



It turns out that $e^{i\theta}$ corresponds to the specification of such a point in radial coordinates in the *Argand* or complex plane.

Later, we will look at a quick derivation of Euler's equation.

One valid proof is to simply plug into the series for the exponential

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

giving

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \dots \\ &= \cos x + i \sin x \end{aligned}$$

which, I realize, we also haven't seen yet. So, take it on faith for the moment.

We can use Euler's equation to get something extremely useful for calculus.

using Euler

Switch notation to s and t

$$e^{is} = \cos s + i \sin s$$

$$e^{it} = \cos t + i \sin t$$

$$e^{i(s+t)} = \cos(s+t) + i \sin(s+t)$$

But

$$e^{i(s+t)} = e^{is} e^{it}$$

$$= (\cos s + i \sin s)\ (\cos t + i \sin t)$$

$$= \cos s \cos t + \cos s\ i \sin t + i \sin s \cos t - \sin s \sin t$$

$$= (\cos s \cos t - \sin s \sin t) + i(\sin s \cos t + \cos s \sin t)$$

We have an equality between two complex numbers, both equal to $e^{i(s+t)}$. For this to be true, both the real and imaginary parts must be equal.

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$
$$\sin(s+t) = \sin s \cos t + \cos s \sin t$$

The addition formulas for sine and cosine.

I find that this is an extremely useful way of remembering how these formulas can be derived.

Chapter 31

deMoivre

The formula says that for integer n

$$[\ \cos x + i \sin x\ ]^n = \cos nx + i \sin nx$$

If we know Euler's formula the derivation is trivial:

$$e^{i\theta} = \cos\theta + i \sin\theta$$

$$(e^{i\theta})^n = \ [\ \cos x + i \sin x\ ]^n$$

$$= e^{i\theta n} = e^{in\theta} = \cos n\theta + i \sin n\theta$$

induction

We can also prove it by induction. Multiply the first formula above by $(\cos x + i \sin x)$. The left-hand side is the form we seek. The right-hand side is

$$(\cos nx + i \sin nx) \cdot (\cos x + i \sin x)$$

$$= \cos nx \cos x - \sin nx \sin x + i(\sin nx \cos x + \cos x \sin nx)$$

Using the sum of angles formulas we obtain

$$= \cos(nx + x) + i \sin(nx + x)$$

$$= \cos(n+1)x + i\sin(n+1)x$$

which completes the inductive step.

The base can be chosen as $n = 1$.

example

Let $n = 3$. Then

$$[\ \cos x + i\sin x\ ]^3 = \cos 3x + i\sin 3x$$
$$= (\cos^2 x - \sin^2 x + 2i(\sin x\cos x)) \cdot (\cos x + i\sin x)$$

Taking the real part of the last expression we have

$$\cos 3x = \cos^3 x - \sin^2 x\cos x - 2\sin^2 x\cos x$$

$$= \cos^3 x - 3\sin^2 x\cos x$$

This can be massaged

$$\cos 3x = \cos x(\cos^2 x - 3\sin^2 x)$$

$$= \cos x(\cos^2 x - 3(1 - \cos^2 x))$$
$$= 4\cos^3 x - 3\cos x$$

standard formula

This agrees with the standard formula:

$$\cos 3x = \cos 2x + x$$

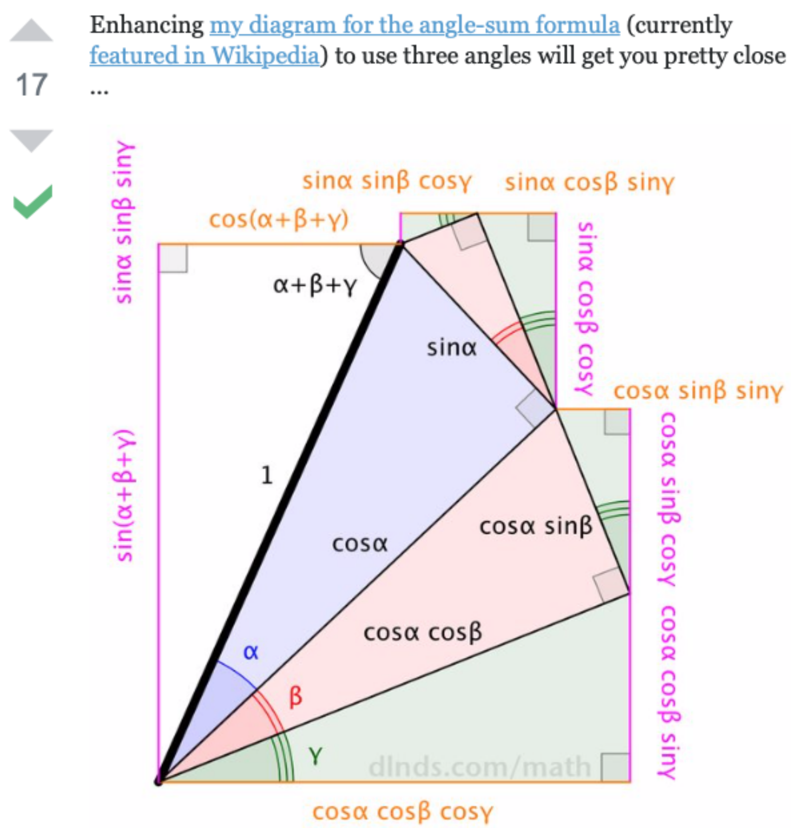
$$= \cos 2x\cos x - \sin 2x\sin x$$
$$= (\cos^2 x - \sin^2 x)\cos x - 2\cos x\sin x\sin x$$

$$= \cos^3 x - \cos x \sin^2 x - 2 \cos x \sin^2 x$$
$$= \cos^3 x - 3 \cos x \sin^2 x$$

From here

https://math.stackexchange.com/questions/852122/picture-intuitive-proof-of-cos3-theta-4-cos3-theta-3-cos-theta

we get a nice geometric derivation.



Note: the formula is not valid for non-integer powers.

https://en.wikipedia.org/wiki/De_Moivre%27s_formula#Roots_of_complex_numbers

Chapter 32

More Euler proofs

additional proofs

Here are sketches of two different derivations of Euler's famous formula, both following Dunham's book about Euler. He starts by simply admiring the formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

If $\theta = \pi$, we have

$$e^{i\pi} = -1 + 0$$

$$e^{i\pi} + 1 = 0$$

(what Feynman called "our jewel").

Dunham says in a video I saw that, if we were going to have a math party, we would invite these five numbers: 0 for arithmetic (additive identity), 1 for multiplication (multiplicative identity), π for geometry, e for calculus, and i for complex functions.

preliminary

Using x is a bit simpler notation, so that's what I'll do here

$$e^{ix} = \cos x + i \sin x$$

Start with the definition of i

$$i = \sqrt{-1}$$

Simple identities that come from it are:

$$i^2 = -1$$

$$-i^2 = 1$$

$$\frac{u}{i} = -iu$$

Having i gives us new factorizations like

$$a^2 + b^2 = (a + bi)(a - bi)$$

since the terms with $\pm$ abi cancel and $-i^2 = 1$. So

$$1 = \cos^2 x + \sin^2 x$$

$$1 = (\cos x + i \sin x)(\cos x - i \sin x)$$

Of course, we could switch sine and cosine here, but this is the convention.

derivation one

Start with the inverse sine function:

$$x = \sin^{-1} y$$

$$y = \sin x$$

$$dy = \cos x \; dx$$

Then we can make a trig substitution and say that the side adjacent to x is $\sqrt{1-y^2}$ and so

$$\cos x = \sqrt{1-y^2}$$

We're interested in the integral

$$\int \frac{1}{\sqrt{1-y^2}} \; dy$$

which is just

$$= \int \frac{1}{\cos x} \; \cos x \; dx = x$$

Now, Euler makes a complex change of variable

$$y = iz$$

$$\frac{1}{1-y^2} = \frac{1}{1+z^2}$$

So

$$x = \int \frac{1}{\sqrt{1-y^2}} \; dy = \int \frac{1}{\sqrt{1+z^2}} \; i \; dz$$

we have converted the integral to having a plus sign under the square root and the answer is

$$= i \ln \, (\sqrt{1+z^2} + z)$$

This follows from a standard trig substitution but it's a bit complicated. It can be checked by differentiating. The derivative is i times

$$\frac{1}{\sqrt{1+z^2}+z} \; (\frac{z}{\sqrt{1+z^2}} + 1)$$

$$= \frac{1}{\sqrt{1+z^2}+z}\ (\frac{z+\sqrt{1+z^2}}{\sqrt{1+z^2}})$$

$$= \frac{1}{\sqrt{1+z^2}}$$

Now, just undo the substitution:

$$z = \frac{y}{i} = \frac{\sin x}{i}$$

$$\sqrt{1+z^2} = \sqrt{1-y^2} = \cos x$$

Hence our previous result

$$x = i \ln\ (\sqrt{1+z^2}+z)$$

is equivalent to

$$x = i \ln\ (\cos x + \frac{\sin x}{i})$$

Recall our two identities involving i. The first one was

$$\frac{u}{i} = -iu$$

So:

$$x = i \ln\ (\cos x + \frac{\sin x}{i})$$

$$= i \ln\ (\cos x - i \sin x)$$

$$ix = -\ln\ (\cos x - i \sin x)$$

$$= \ln\ \frac{1}{(\cos x - i \sin x)}$$

Using the factorization given at the top

$$\frac{1}{\cos u - i \sin u} = \cos u + i \sin u$$

We have that

$$ix = \ln \ \frac{1}{(\cos x - i \sin x)} = \ln \ (\cos x + i \sin x)$$

Exponentiate:

$$e^{ix} = \cos x + i \sin x$$

derivation two

Suppose we try this multiplication:

$$(\cos s + i \sin s)(\cos t + i \sin t)$$

$$= \cos s \cos t + i \sin s \cos t + i \cos s \sin t - \sin s \sin t$$

$$= (\cos s \cos t - \sin s \sin t) + i(\sin s \cos t + \cos s \sin t)$$

$$= \cos(s + t) + i \sin(s + t)$$

set $s = t$ and recall what we started with

$$(\cos s + i \sin s)^2 = \cos 2s + i \sin 2s$$

In fact, Euler showed that this is true for fractional n but I'll assume that part.

$$(\cos s + i \sin s)^n = \cos ns + i \sin ns$$

Now multiply the difference rather than the sum:

$$(\cos s - i \sin s)(\cos t - i \sin t)$$

$$= (\cos s \cos t - \sin s \sin t) - i(\sin s \cos t + \sin t \cos s)$$

$$= \cos(s + t) - i(\sin(s + t))$$

again, with $s = t$

$$(\cos s - i \sin s)^2 = \cos 2s - i \sin 2s$$

$$(\cos s - i\sin s)^n = \cos ns - i\sin ns$$

Restate the two results:

$$(\cos s + i\sin s)^n = \cos ns + i\sin ns$$

$$(\cos s - i\sin s)^n = \cos ns - i\sin ns$$

Add them

$$2\cos ns = (\cos s + i\sin s)^n + (\cos s - i\sin s)^n$$

where the magic happens

Let

$$s = \frac{x}{n}$$

As $n \to \infty$, $s \to 0$, and

$$\sin s \to s$$

(by the famous limit from trigonometry)

$$\cos s \to 1$$

$$\cos x = \cos ns$$

We had

$$2\cos ns = (\cos s + i\sin s)^n + (\cos s - i\sin s)^n$$

which becomes

$$\begin{aligned} 2\cos x &= (1 + is)^n + (1 - is)^n \\ &= (1 + \frac{ix}{n})^n + (1 - \frac{ix}{n})^n \end{aligned}$$

but from the standard limit developed in looking at the exponential

$$e^{ix} = \lim_{n\to\infty} (1 + \frac{ix}{n})^n$$

hence

$$2\cos x = e^{ix} + e^{-ix}$$

We can just integrate this to obtain

$$2i\sin x = e^{ix} - e^{-ix}$$

Or by very similar manipulation to what's in the first part we can also obtain an expression for the sine:

$$2i\sin(ns) = (\cos s + i\sin s)^n - (\cos s - i\sin s)^n$$

which will lead to the same thing

$$2i\sin x = (e^{ix} - e^{-ix})$$

Adding together

$$2(\cos x + i\sin x) = e^{ix} + e^{-ix} + e^{ix} - e^{-ix}$$

$$\cos x + i\sin x = e^{ix}$$

check

Before we stop, we can check the formula. One way is to notice the connection between infinite series expansions for e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots$$

sine:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

and cosine:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots$$

These can almost be added together to give what we seek, except for the problem of the alternating signs. What happens with e^{ix}?

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} \dots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} \dots \end{aligned}$$

The pattern is

$$\sum_{n=0}^{\infty} i^n = 1 + i - 1 - i + 1 \dots$$

And we're there. We just have to recognize that the pattern with e^{ix} has $i \sin x$ so as we said

$$e^{ix} = \cos x + i \sin x$$

Part XI

Serious integration

Chapter 33

Techniques of integration

This chapter surveys some useful techniques for integration. There are four approaches commonly given in introductory calculus:

- u substitution
- trigonometric substitution
- integration by parts (IBP)
- partial fractions

We'll take a quick look at all of these.

Here is an example for substitution.

$$\int \tan t \; dt$$

No doubt, when students first see this, they are mystified. Have I ever seen any $f(x)$ that gives the tangent as its derivative?

Insight comes from writing the two parts of the tangent:

$$\int \frac{\sin t}{\cos t} \; dt$$

We recognize that

$$\frac{d}{dt}\cos t = -\sin t$$

Let

$$u = \cos t$$

$$du = -\sin t \ dt$$

the integral is

$$-\int \frac{1}{u} \ du = -\ln u$$

$$= -\ln \cos t$$

Since the cosine can take on values where the logarithm is not defined (i.e. < 0) the answer is usually given as

$$= -\ln|\cos t|$$

Another classic one is the secant

$$\int \sec x \ dx$$

There is also a trick to this one, multiply top and bottom by $\sec x + \tan x$

$$\int \sec x \ \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

You see that $\sec^2 x$ is the derivative of $\tan x$ and $\sec x \tan x$ is the derivative of $\sec x$ so this is just

$$\int \frac{1}{u} \ du$$

again, namely

$$\int \sec x \ dx = \ln|\sec x + \tan x \ | + C$$

Now compare these two integrals

$$\int x \sqrt{1-x^2}\ dx$$

$$\int \sqrt{1-x^2}\ \ dx$$

The extra value of x makes a big difference in the first one. I look at the x and know we have the derivative of what is inside the square root. So let $u = 1-x^2$ and then $du = -2x\ dx$ and the integral becomes

$$\int (-\frac{1}{2})\ \sqrt{u}\ du = -\frac{1}{3}u^{3/2}$$

$$= -\frac{1}{3}(1-x^2)^{3/2}$$

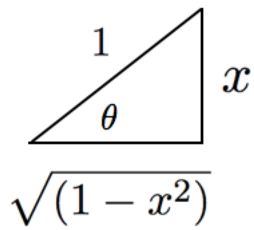
This integral comes up in the problem of finding the average value of x over the unit circle (or half-circle). Since it's an even function of x, the result is zero, if the bounds are centered around zero like as $[-b, b]$.

With practice you will not need to write the substitution. Just say, OK, I know I have the x. Now the integral of the square root is $(1-x^2)^{3/2}$. I need one factor of $2/3$ to neutralize the exponent, and another factor of $-1/2$ for $-x^2$ so I write the answer, and then check by differentiating.

For the second integral, we do not have the derivative of what's inside the square root.

$$\int \sqrt{1-x^2}\ \ dx$$

Nevertheless, this one can be solved using what is called a trig substitution. Consider this figure



We draw a generic right triangle. The figure labels that angle as θ (I often use t just because it's easier to type). Since we have $\sqrt{1-x^2}$, we know we will need x for the side opposite the angle and 1 for the hypotenuse. So let

$$x = \sin\theta$$

$$dx = \cos\theta \; d\theta$$

And from Pythagoras

$$\sqrt{1-x^2} = \cos\theta$$

Take a look at

$$\int \sqrt{1-x^2} \; dx$$

The square root is on the top. If it were on the bottom, the cosines would cancel (we will see that problem later). We get

$$\int \cos^2\theta \; d\theta$$

This is an integral that comes up a lot, and there are several ways to do it. We will solve this **soon**.

For now, the answer is:

$$\int \cos^2\theta \; d\theta = \frac{1}{2} \; [\; \theta + \sin\theta\cos\theta \;]$$

This is easily verified by differentiating:

$$\cos^2\theta = \frac{1}{2}\,[\ 1 - \sin^2\theta + \cos^2\theta\ ]$$
$$= \frac{1}{2}\,[\ \cos^2\theta + \cos^2\theta\ ]$$

Although people always do this problem as described, it's worth pointing out that we could do a different trigonometric substitution. There is no reason not to write

$$x = \cos\theta$$
$$dx = -\sin\theta\ d\theta$$
$$\sqrt{1-x^2} = \sin\theta$$

and so

$$\int \sqrt{1-x^2}\ dx = -\int \sin^2\theta\ d\theta$$

Again, the answer is:

$$\int \sin^2\theta\ d\theta = \frac{1}{2}\,[\ \theta - \sin\theta\cos\theta\ ]$$

Of course this is true since

$$\sin^2\theta + \cos^2\theta = 1$$
$$\int \sin^2\theta\ d\theta + \int \cos^2\theta\ d\theta = \int 1\ d\theta = \theta$$

If you add the two answers given for $\sin^2$ and $\cos^2$ you'll get the same result.

change of variables

u-substitution and trigonometric substitution, or just substitution in general, has one subtle aspect, which is that if you change the variable, you must also change the bounds. Either that, or change back to the original variable at the end. For example, above we had

$$\int x \sqrt{1-x^2}\ dx$$

We let $u = 1 - x^2$ and then $du = -2x\ dx$ and the integral becomes

$$\int (-\frac{1}{2})\ \sqrt{u}\ du = -\frac{1}{3}u^{3/2}$$

Suppose the bounds on x were $[0, 1]$

$$\int_0^1 x \sqrt{1-x^2}\ dx$$

The bounds on the integral in u must be adjusted:

$$x = 0, \qquad u = 1$$

$$x = 1, \qquad u = 0$$

so

$$-\frac{1}{3}u^{3/2}\ \Big|_1^0 = \frac{1}{3}u^{3/2}\ \Big|_0^1 = \frac{1}{3}$$

Alternatively, switch back to x

$$-\frac{1}{3}u^{3/2} = -\frac{1}{3}(1-x^2)^{3/2}\ \Big|_0^1$$

$$= -\frac{1}{3}(0-1) = \frac{1}{3}$$

Our second example above was

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\,[\;\theta + \sin\theta\cos\theta\;]$$

Suppose the bounds on x were $[0,1]$. The first substitution we tried was

$$x = \sin\theta$$

To find the bounds on θ, we must ask, what value of θ gives the correspond value of x? We obtain

$$x = \sin\theta = 0, \qquad \theta = 0$$

$$x = \sin\theta = 1, \qquad \theta = \pi/2$$

So

$$\frac{1}{2}\,[\;\theta + \sin\theta\cos\theta\;]\,\Big|_0^{\pi/2}$$

The second term is 0 at both bounds, and the first is $\pi/2$, which is our answer.

Alternatively, we can switch the variable back to x

$$x = \sin\theta$$

$$\theta = \sin^{-1} x$$

so the answer is

$$\frac{1}{2}\,[\;\theta + \sin\theta\cos\theta\;]\,\Big|_0^{\pi/2} = \frac{1}{2}\,[\;\sin^{-1} x + x\sqrt{1-x^2}\;]\,\Big|_0^{1}$$

And again, the second term is zero at both bounds. The first term is just $\pi/2$, which is the same as we obtained before.

integration by parts (IBP)

Another approach to the integral uses integration by parts.

IBP is a reversal of the product rule. Consider two functions of x, $u(x)$ and $v(x)$. (Drop the (x) notation for the moment).

$$(uv)' = v\ du + u\ dv$$

The idea is that when we integrate this we will have

$$\int (uv)' = uv = \int v\ du + \int u\ dv$$

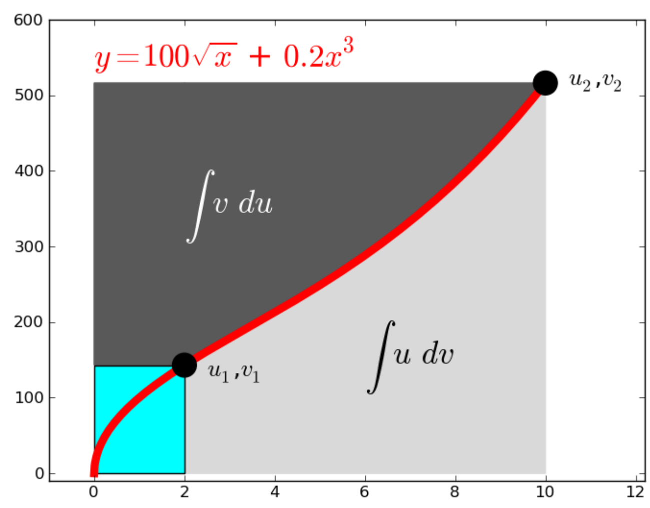
Rearranging, we obtain

$$\int u\ dv = uv - \int v\ du$$

Just to be clear, with an explicit independent variable x this is:

$$\int u \frac{dv}{dx}\ dx = uv - \int v \frac{du}{dx}\ dx$$

Here is a figure from Strang:



examples

As a first example, take

$$\int x\ e^x\ dx$$

If this were

$$\int x\ e^{x^2}\ dx$$

there would be no problem, because upon differentiating x^2 using the chain rule, we will get the x that we see in the example. For the first problem, write

$$u = x$$

$$du = dx$$

$$dv = e^x\ dx$$

this is what we have. Now let's see how this simplifies things

$$v = \int e^x\ dx = e^x$$

Using the formula

$$\int u\ dv = uv - \int v\ du$$

$$uv = xe^x$$

$$\int v\ du = \int e^x\ dx = e^x$$

So

$$\int xe^x\ dx = xe^x - e^x$$

Check

$$\frac{d}{dx}\ [\ xe^x - e^x\ ] = xe^x$$

It is clear that the extra term in the answer is there to eliminate an extra term in the derivative. This pattern is generally true with integration by parts.

The problem we had above, $\cos^2 x$, can be solved by this method. But let's wait and give it its own section.

exponential

Consider the question: what is the average value of x for the negative exponential function over the interval $[0, \infty)$?

Start with

$$\int_0^\infty e^{-kx}\ dx$$

$$= -\frac{1}{k}e^{-kx}\ \Big|_0^\infty$$

Here we are faced with the problem of evaluating a function at the point $x = \infty$ but *infinity is not a number*. The approach is to evaluate the expression at some very large bound b, and then ask what happens if $b \to \infty$. (See **here**).

At the upper bound we get zero, and at the lower bound we are subtracting $-1/k$ so

$$\int_0^\infty e^{-kx}\ dx = \frac{1}{k}$$

Now we want

$$\int_0^\infty x\ e^{-kx}\ dx$$

Let

$$u = x$$

$$du = dx$$

$$dv = e^{-kx}\ dx$$
$$v = -\frac{1}{k}e^{-kx}$$

So IBP says

$$\int u\ dv = uv - \int v\ du$$
$$\int x\ e^{-kx}\ dx = -\frac{x}{k}e^{-kx} + \int \frac{1}{k}\ e^{-kx}\ dx$$
$$= \ [\ -\frac{x}{k}e^{-kx} - \frac{1}{k^2}\ e^{-kx}\ ]\ \Bigg|_0^\infty$$

At the lower limit the first term is zero and the second $-1/k^2$ which we change to $1/k^2$ by subtraction. At the upper limit, the second term is a negative exponential so that goes to zero as $x \to \infty$.

The first term is more complicated.

$$\lim_{x\to\infty}\ xe^{-kx} = ?$$

Rewrite it as a ratio:

$$= \lim_{x\to\infty}\ \frac{x}{e^{kx}}$$

We get infinity for both top and bottom and so invoke **L'Hospital**:

$$= \lim_{x\to\infty}\ \frac{1}{ke^{kx}} = 0$$

The final answer here is

$$\int_0^\infty x\ e^{-kx}\ dx = \frac{1}{k^2}$$

In fact, we can solve this integral for any power x^n in the numerator. Just invoke L'Hospital n times. We will see this elsewhere (**here**).

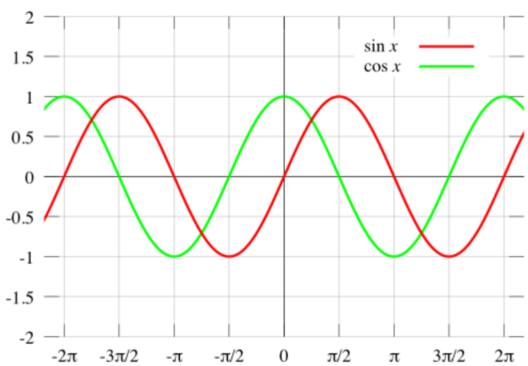
By now, you should have a good handle on the integral as a measure of the area between the curve $f(x)$ and the x-axis.

below the x-axis

One question we haven't dealt with is what happens when the curve dips below the x-axis. Consider the line $y = x - 1$ which crosses the x-axis at $x = 1$ and the y-axis at $y = -1$.

$$\int_0^1 x - 1 \; dx = \frac{x^2}{2} - x \,\bigg|_0^1$$
$$= \frac{1}{2} - 1 = -\frac{1}{2}$$

The absolute value of the area is correct but the sign is negative. When we integrate a function that dips below the x-axis, the result will be negative.



A further demonstration of this is the trigonometric functions $\sin x$ and $\cos x$:

They apparently spend as much of the time below the x-axis as above it. These functions repeat with a period of 2π. A consequence of this is that the integral of sine or cosine over a period of exactly 2π is always equal to zero, regardless of the exact starting point.

$$\int_0^{2\pi} \sin x \ dx = -\cos x \ \Big|_0^{2\pi} = -(-1) - 1 = 0$$

This is true for *any* bounds whose difference is 2π.

area between two curves

A common application of integrals is to give the area between two curves, which may be obtained simply by subtracting one function from another and integrating the difference.

$$A = \int f(x) - g(x) \ dx$$

an easy problem

Consider the two curves $y = \sqrt{x}$ and $y = x^2$. These curves cross at

$$\sqrt{x} = x^2$$

We can see by inspection that this happens at $x = 0$ and $x = 1$.

In this region $(0 \le x \le 1)$ the square root is larger than the square.

Furthermore, we already calculated that the part below $y = x^2$ is $1/3$ of the area of the rectangle with corners $(0, 0)$ and (x, y). The same is true for the area above the square root. From this we can predict what the result will be.

$$A = \int f(x) - g(x) \ dx$$

For this problem

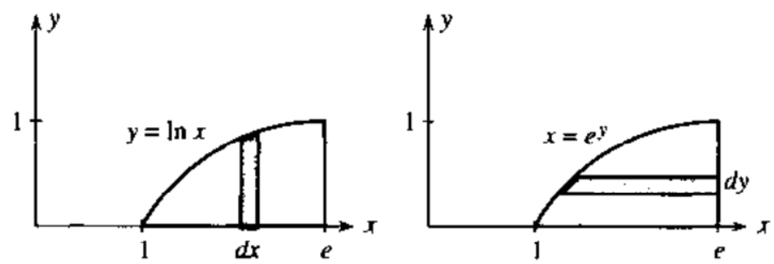
$$A = \int_0^1 \sqrt{x} - x^2 \ dx$$

$$= \frac{2}{3}x^{3/2} - \frac{x^3}{3} \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

a problem done two ways

Consider $y = \ln x$. This can be done two ways, as the figure shows. We can use x as the variable or y.



The standard approach would be x. We want

$$\int_1^e \ln x \ dx$$

We recall fooling around with the derivatives of products to pull this up:

$$= x \ln x - x \Big|_1^e$$

$$= \ [\ e - e\ ] \ - \ [\ 0 - 1\ ] \ = 1$$

It's the other way that we subtract $f(x) - g(x)$:

$$= \int (e - e^y) \ dy$$

$$= ey - e^y \Big|_0^1$$

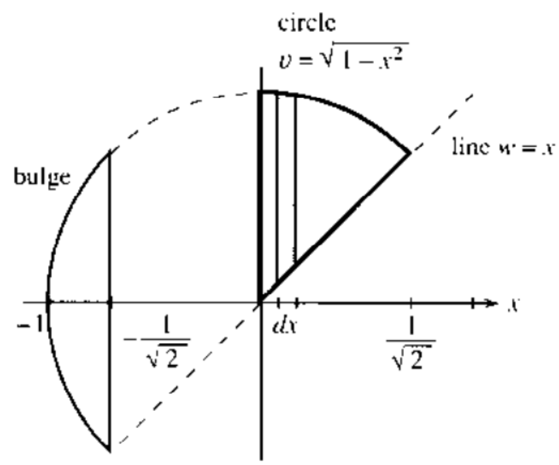
$$= [\ e - e^1\ ] - [\ 0 - 1\ ]$$

$$= 1$$

That looks very smooth but it's got a lot of calculus in it!

a more complicated problem

Consider the two curves $y = x$ and the unit circle $y = \sqrt{1 - x^2}$.



We see that the circle lies above the line for most of its length. However, it's complicated. Part of the figure is below the x-axis, and there is a bulge on the left end.

The key is that we must appreciate the relationship (know which is on top) in order to do integrals of differences between curves properly.

First solve for the points where the two curves cross:

$$y = x = \sqrt{1 - x^2}$$

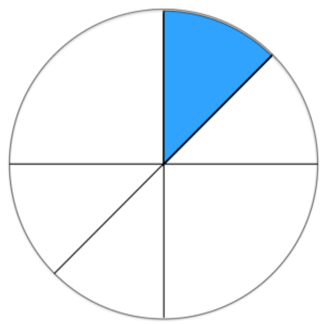
$$2x^2 = 1$$

$$x = \pm\sqrt{1/2} \ = \pm\frac{1}{\sqrt{2}}$$

$$y = \pm\frac{1}{\sqrt{2}}$$

The area of the sector in the first quadrant is pretty easy.

$$A = \int_0^{1/\sqrt{2}} \sqrt{1-x^2} - x \ dx$$



We referred to the solution of $\int \sqrt{1-x^2} \ dx$ previously in this chapter, and we will actually finally solve it in the next. For now, just assume the result:

$$= \frac{1}{2} \ [\ \sin^{-1} x + x\sqrt{1-x^2} \ ] \ - \frac{x^2}{2} \ \Big|_0^{1/\sqrt{2}}$$

At the lower bound everything including $\sin^{-1} x$ is zero, and at the upper bound we have

$$= \frac{1}{2} \ [\ \frac{\pi}{4} + \frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}} \ ] - \frac{1}{4} = \frac{\pi}{8}$$

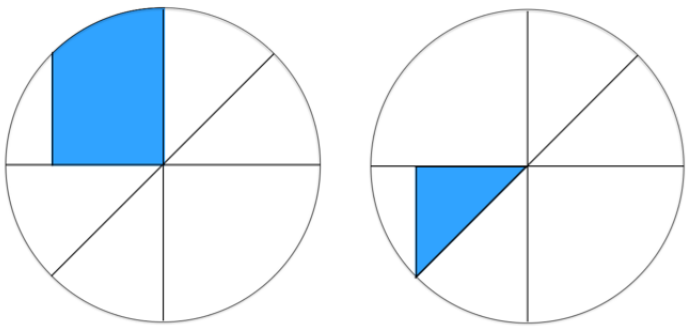
which we confirm from elementary geometry is just a slice of the pie.

For the area to the left of the y-axis we must think a little more. The line $y = x$ dips below the x-axis, so the integral of the second part $g(x)$

$$\int_{-1/\sqrt{2}}^{0} x\ dx$$

is negative.

It's probably less confusing to just do the parts above and below the x-axis separately!



The area above the x-axis is

$$A = \frac{1}{2}\ [\ \sin^{-1} x + x\sqrt{1-x^2}\ ]\ \Big|_{-1/\sqrt{2}}^{0}$$

At the upper bound, we have again zero, and at the lower bound

$$\frac{1}{2}\ [\ -\frac{\pi}{4} + (-\frac{1}{\sqrt{2}})\ \frac{1}{\sqrt{2}}\ ]$$

which must be subtracted, so we change signs and obtain

$$\frac{\pi}{8} + \frac{1}{4}$$

This corresponds to a slice of the pie plus the triangle beneath it, which has two sides of length $1/\sqrt{2}$ and area $1/4$.

For the area below the x-axis:

$$\int_{-1/\sqrt{2}}^{0} x \; dx$$

$$= \frac{x^2}{2} \Big|_{-1/\sqrt{2}}^{0} = -\frac{1}{4}$$

The area below the x-axis is *minus* the result of the integral (the integral yields a negative area).

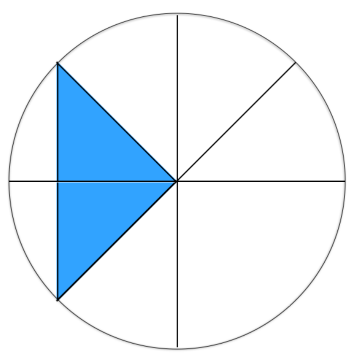
So lose the minus sign

$$A = \frac{1}{4}$$

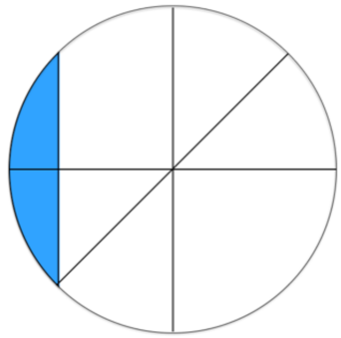
The total for this region (between $-1/\sqrt{2}$ and 0) is then just

$$\frac{\pi}{8} + \frac{1}{4} + \frac{1}{4} = \frac{\pi}{8} + \frac{1}{2}$$

This is effectively a slice of pie plus a triangle of base $2/\sqrt{2}$ and height $1/\sqrt{2}$.



The third part is the bulge to the left of $x = -1/\sqrt{2}$.



We will calculate the area above the x-axis and multiply by two:

$$A = 2 \int_{-1}^{-1/\sqrt{2}} \sqrt{1 - x^2} \ dx$$

$$= 2 \ \frac{1}{2} \ [\ \sin^{-1} x + x\sqrt{1 - x^2} \ ] \ \Big|_{-1}^{-1/\sqrt{2}}$$

The leading factors cancel.

$$= \sin^{-1} x + x\sqrt{1 - x^2} \ ] \ \Big|_{-1}^{-1/\sqrt{2}}$$

At the upper bound we have (as we found before)

$$= -\frac{\pi}{4} - \frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}} = -\frac{\pi}{4} - \frac{1}{2}$$

At the lower bound, $\sin^{-1} x = -\pi/2$ and $x\sqrt{1 - x^2}$ is zero.

We're subtracting, so we have

$$= -\frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{2}$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

Adding all the pieces together

$$A = \frac{\pi}{8} + \frac{\pi}{8} + \frac{1}{2} + \frac{\pi}{4} - \frac{1}{2} = \frac{\pi}{2}$$

which is obviously correct for one-half the unit circle.

Chapter 34

Inverse sine

Inverse sine

This is our first of the inverse trigonometric functions (there are only two really important ones).

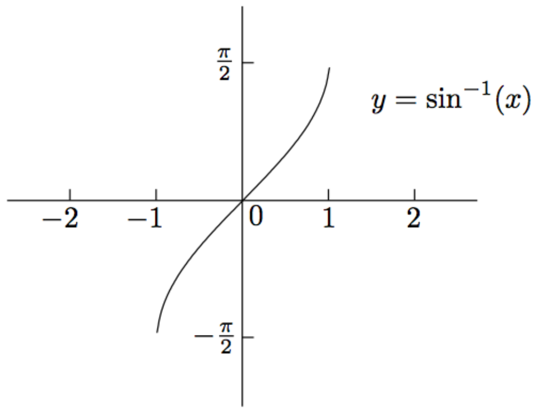
$$y = \sin^{-1} x$$

Read this as y is the arc sine or inverse sine of x.

More usefully, we can say that y is the angle whose sine is x. This means the same thing, we have just solved the first equation for x:

$$x = \sin y$$

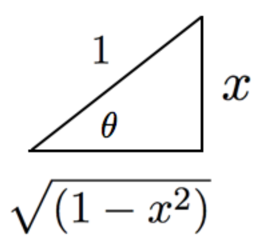
Plot the angle as a function of its sine.



The graph is the same shape as the standard sine curve, but flipped and rotated 90 degrees.

We need to be careful about the interval on which we're working, since if we go too far we will duplicate values and thus, no longer have a *function*. You can see from the plot that the range of y should be $[-\pi/2, \pi/2]$.

We can derive something useful from this with a trig substitution:



Basic trigonometry says that if

$$x = \sin\theta$$

$$\theta = \sin^{-1} x$$

then

$$\cos\theta = \sqrt{1-x^2}$$

so differentiating

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\sin\theta = \cos\theta = \sqrt{1-x^2}$$

Inverting

$$\frac{d\theta}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Furthermore, integrating:

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \int d\theta = \theta$$

$$= \sin^{-1} x$$

and

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

This integral arises in many problems.

Let's switch to using t for θ. Another derivation is to say

$$x = \sin t$$

$$dx = \cos t\ dt$$

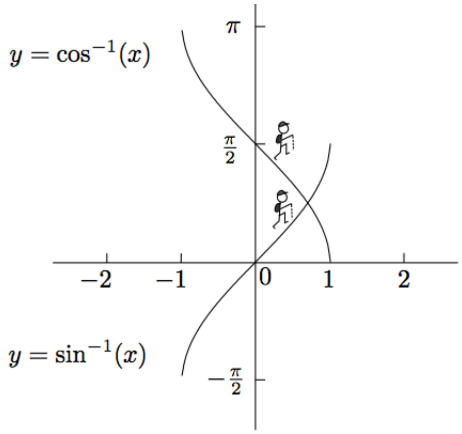
so the integral is

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \int \frac{1}{\cos t}\cos t\ dt = t$$

The complementary angles of a right triangle add up to $\pi/2$, thus

$$\sin^{-1} t + \cos^{-1} t = \pi/2$$

Here's a picture of the two functions together. It is not hard to believe they add up to a constant.



We find the inverse cosine easily by differentiating:

$$\sin^{-1} t + \cos^{-1} t = \pi/2$$

$$\frac{d}{dt}\sin^{-1} t + \frac{d}{dt}\cos^{-1} t = 0$$

$$\frac{d}{dt}\sin^{-1} t = -\frac{d}{dt}\cos^{-1} t$$

where we had

$$\frac{d}{dt}\sin^{-1} t = \frac{1}{\sqrt{1-x^2}}$$

The inverse cosine isn't seen that much because its the very same problem as the inverse sine.

inverse tangent

However, the inverse tangent is important. Let

$$x = \tan t$$

$$t = \tan^{-1} x$$

Differentiating the first equation

$$\frac{dx}{dt} = \sec^2 t$$

If you haven't seen this before, try using the quotient rule on $\sin t / \cos t$. Inverting

$$\frac{dt}{dx} = \cos^2 t$$

Basic trigonometry will show that if

$$x = \tan t$$

then the hypotenuse must be $\sqrt{1 + x^2}$ so

$$\cos t = \frac{1}{\sqrt{1 + x^2}}$$

and then

$$\frac{dt}{dx} = \cos^2 t = \frac{1}{1 + x^2}$$

Integrate

$$\int dt = \int \frac{1}{1 + x^2}\ dx$$

$$t = \int \frac{1}{1 + x^2}\ dx$$

But $t = \tan^{-1} x$ so

$$\tan^{-1} x = \int \frac{1}{1 + x^2}\ dx$$

We will show elsewhere that

$$(1 + x^2) \cdot (1 - x^2 + x^4 - x^6 + \dots) = 1$$

You can check by multiplying it out.

Rearrange and integrate

$$\int \frac{1}{1 + x^2}\ dx = \int 1 - x^2 + x^4 - x^6 + \dots\ dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Combining with what's above

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

The angle whose tangent is 1 is $\pi/4$ so

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

This is the first of a number of series that yield π.

Chapter 35

Cosine squared

This is where we solve some integrals that we put off earlier. We already laid the groundwork by talking about trig substitutions (**here**) and about the inverse sine function, a related topic (**above**).

Let's begin with the integral of $\sqrt{1-x^2}$.

We saw this in the problem of the area of the circle (**here**):

$$\int \sqrt{1-x^2}\ dx$$

We also saw it in the denominator when talking about the inverse sine

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x$$

To solve the first one, we use a trig substitution to give:

$$x = \sin t$$

$$dx = \cos t\ dt$$

$$\sqrt{1-x^2} = \cos t$$

So

$$\int \sqrt{1-x^2}\ dx = \int \cos^2 t\ dt$$

We will finally solve the cosine squared in a second, but just note that the more general version of this problem has a constant, let's call it a:

$$\int \sqrt{a^2-x^2}\ dx$$

To deal with that, factor out an a

$$= a\sqrt{1-(x/a)^2}\ dx$$

Then substitute

$$au = x$$

so

$$a\ du = dx$$

and we obtain

$$= a\sqrt{1-u^2}\ a\ du$$

$$= a^2\sqrt{1-u^2}\ du$$

An equivalent solution is to do it during the trig substitution.

Rather than use 1 for the hypotenuse, use the constant. For the circle we had

$$\int \sqrt{R^2-x^2}\ dx$$

Let

$$x = R\sin t$$

$$dx = R\cos t\ dt$$

$$\sqrt{R^2-x^2} = R\cos t$$

So the integral is

$$R^2 \int \cos^2 t \ dt$$

And if you look back at that problem, I promised that we would come up with a factor of R^2 to make the area come out right. Well there it is. We still need something involving π.

It turns out the answer is

$$\int \cos^2 t \ dt = \frac{1}{2} \ [\ t + \sin t \cos t \ ]$$

We can integrate from $x = [-R, R]$. In that case $t = [-\pi/2, \pi/2]$. Or if we use 0 for the lower bound on x (remembering that we'll need a factor of 2 at the end), we will have $t = 0$ at the lower bound as well. No matter, either way the second term with $\sin t \cos t$ will be zero.

In the first case we'll get $\pi/2$, and in the second case we get $\pi/4$ and then multiply by 2 to get the same thing. Since we only did the top half of the circle we pick up another factor of 2 for that.

Cosine squared

Enough fooling around. We want to find the integral of

$$\int \cos^2 x \ dx$$

which as we've seen is very common in problems using trig substitution and otherwise. The first thing to note is that

$$\int \sin^2 x \ dx$$

is the same problem, because

$$\sin^2 x + \cos^2 x = 1$$

so

$$\int \sin^2 x \; dx + \int \cos^2 x \; dx = \int 1 \; dx = x$$

method 0

I call this method 0 because it's not really methodical, we just guess. If you play around differentiating products of functions (like e^x, $\ln x$, $\sin x$, $\cos x$ and x), you will soon discover that

$$\frac{d}{dx} \; [\; \sin x \cos x \;] \; = \cos^2 x - \sin^2 x$$

which can be manipulated (using $\sin^2 x + \cos^2 x = 1$) to give either

$$\cos^2 x - \sin^2 x = 1 - 2 \sin^2 x$$

or

$$\cos^2 x - \sin^2 x = 2 \cos^2 x - 1$$

Integrating both sides of the $\cos^2$ form we obtain

$$\sin x \cos x = 2 \int \cos^2 x \; dx - x$$

and rearranging:

$$\int \cos^2 x \; dx = \frac{1}{2}(x + \sin x \cos x)$$

method 1

There are two other systematic approaches that can be contrasted. The first, which is arguably the simpler one, is to remember the addition formula for cosine

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

As mentioned earlier, the trick I use to remember these formulas is to work out the consequences for this one:

$$\cos(s-t) = \cos s \cos t + \sin s \sin t$$

This makes perfect sense since if $s = t$ then we get

$$\cos 0 = \cos^2 s + \sin^2 s = 1$$

which we know is correct. So

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$

If $s = t$ then (changing to x)

$$\cos 2x = \cos^2 x - \sin^2 x$$

which we saw above is equal to

$$= 2\cos^2 x - 1$$

The "double angle" formula is then

$$2\cos^2 x = 1 + \cos 2x$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Integrating

$$\int \cos^2 x \ dx = \int \frac{1}{2}(1 + \cos 2x) \ dx$$

$$\frac{1}{2}(x + \frac{1}{2}\sin 2x)$$

We check by differentiating. Leaving the factor of 1/2 out, we obtain for d/dx:

$$1 + \cos 2x$$

which, as we saw above, is equal to $2\cos^2 x$. Remembering the factor of $1/2$, we obtain the expected result.

Comparing our results so far, we have obtained two different answers, namely

$$\int \cos^2 x \; dx = \frac{1}{2}(x + \sin x \cos x)$$

$$\int \cos^2 x \; dx = \frac{1}{2}(x + \frac{1}{2}\sin 2x)$$

which indicates (if there is no mistake), that

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

to see that this is correct, recall the addition formula for sine:

$$\sin(s + t) = \sin s \cos t + \sin t \cos s$$

then if $s = t$

$$\sin 2s = 2\sin s \cos s$$

with a slight rearrangement, this is indeed what we had.

method 2

In the second method, we do a substitution to take advantage of the integration by parts formula

$$\int u \; dv = uv - \int v \; du$$

Let $u = \cos x$, so $du = -\sin x \; dx$, and let $dv = \cos x \; dx$ so $v = \sin x$, so

$$\int \cos^2 x \; dx = \sin x \cos x + \int \sin^2 x \; dx$$

This still seems like not much progress since (as we saw) $\int \sin^2 x \ dx$ is really the same problem as $\int \cos^2 x \ dx$

$$\int \sin^2 x \ dx = \int (1 - \cos^2 x) dx = \int dx - \int \cos^2 x dx$$

but, forging ahead, we combine the two results

$$\int \cos^2 x \ dx = \sin x \cos x + x - \int \cos^2 x dx$$

Rearranging:

$$\int \cos^2 x \ dx = \frac{1}{2} \ [\ \sin x \cos x + x \ ]$$

which is what we had before.

Integration by parts where the result is a related integral can be applied to the general case of $\int \cos^n x \ dx$ with even n, as well as many interesting and more advanced problems. It's worth remembering that it is in our toolbox.

Geometric significance

We ran into the integral of cosine squared in looking at the area of the circle. We'll use a unit circle to make it a bit simpler.

The integral we obtained was

$$\int \cos^2 \theta \ d\theta = \frac{1}{2} \ [\ \theta + \sin \theta \cos \theta \ ]$$

which can be evaluated in two ways. We can figure out the bounds on the substituted variable θ. Since $x = \sin \theta$:

$$x = 0 \Rightarrow \theta = 0$$

$$x = 1 \Rightarrow \theta = \frac{\pi}{2}$$

thus

$$\frac{1}{2}\ [\ \theta + \sin\theta\cos\theta\ ]\ \Big|_0^{\pi/2}$$

and what's nice about this is the second term is zero at both the upper and lower bound, so we end up with $\pi/4$, which is correct for the quarter circle.

Or we can switch back to x. We had

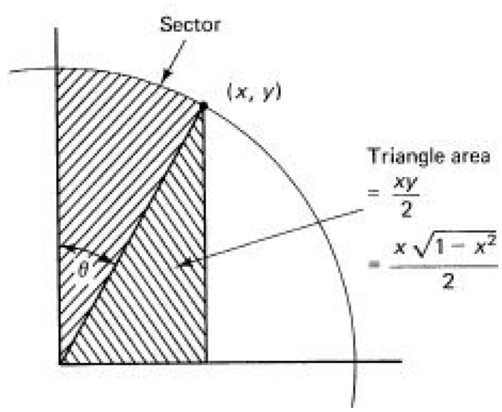
$$\frac{1}{2}\ [\ \theta + \sin\theta\cos\theta\ ]$$

Since $x = \sin\theta$, $\theta = \sin^{-1} x$ (the arc sine of x) so

$$\frac{1}{2}\ [\ \sin^{-1} x + x\sqrt{1 - x^2}\ ]$$

With this approach, we can place the upper bound for this anywhere in $[0, 1]$ without any trouble.

And here is where the geometry is nice. Notice that the area under the curve can be constructed in two parts by drawing the radius to the point (x, y).



There is a sector of the circle and a triangle. These correspond to the two terms of

$$\frac{1}{2} \, [\, \sin^{-1} x + x\sqrt{1-x^2} \,]$$

I find that very illuminating.

alternative parametrization

By the way, there is another approach which gets at the close relationship between $\sin^2\theta$ and $\cos^2\theta$.

For a parametrized unit circle:

$$x = \cos\theta$$

$$dx = -\sin\theta \ d\theta$$

$$y = \sin\theta$$

so

$$\int y \ dx = \int -\sin^2\theta \ d\theta$$

using the formula relating these that we had above

$$= \int \cos^2\theta \ d\theta - \int d\theta$$

I see this and I think, uh oh, we have an extra term here. How will we possibly obtain the same result as before? But it works out!

$$= \frac{1}{2} \, [\, \theta + \sin\theta\cos\theta \,] \ - \theta$$

$$= \frac{1}{2} \, [\, \sin\theta\cos\theta - \theta \,]$$

There is another difference. For the trig substitution we had $x = \sin\theta$, but here we have $x = \cos\theta$. That means

$$x = 0 \Rightarrow \theta = \frac{\pi}{2}$$

$$x = 1 \Rightarrow \theta = 0$$

so when we evaluate the result at these bounds we get

$$= \frac{1}{2} \, [\, \sin\theta\cos\theta - \theta \,] \, \Big|_{\pi/2}^{0}$$

switching bounds means multiplying the expression by -1

$$= \frac{1}{2} \, [\theta - \sin\theta\cos\theta \,] \, \Big|_{0}^{\pi/2}$$

We have switched the sign on the second term $(\sin\theta\cos\theta)$ — compare with the previous answer. But it doesn't matter because it is zero at both of the extreme bounds on the interval. We end up with $\pi/4$, as before.

In 16.4 Hamming turns this argument around. He starts with the picture of the area and works backward to show that $\cos\theta$ is the derivative of $\sin\theta$ without using the limit of $\sin\theta/\theta$, which we worked out **here**.

Chapter 36

Improper integrals

Generally speaking, an integral is improper when one of three conditions holds: (i) the upper bound is $\lim x \to +\infty$, (ii) the lower bound is $\lim x \to -\infty$, or (iii) at one of the bounds, the value of the function is undefined $\to \pm\infty$.

If the function's value becomes undefined in the middle of an interval, first break the integral into pieces.

Then, integrate the function anyway, and if, when we evaluate it at that problematic bound, the result is finite (often 0), then we can use the result.

$$\int_1^\infty \frac{1}{x^2}\ dx$$

$$= -\frac{1}{x}\ \Big|_1^\infty$$

The upper bound is ∞, but the value of the there is zero. So

$$= -(-\frac{1}{1}) = 1$$

On the other hand, the same integral with bounds $[0, 1]$ blows up at the lower bound. That area is infinite.

Here is a second example. Compare:

$$\int_0^1 \frac{1}{x}\ dx$$

$$\int_0^1 \frac{1}{\sqrt{x}}\ dx$$

For both, the value of $f(x)$ becomes infinitely large (the limit does not exist) as $x \to 0$. Nevertheless, the area under the second curve is finite, while that under the first is not.

Informally, the way we roll here is to substitute another bound, like a, which is very small but not zero:

$$\int_a^1 \frac{1}{x}\ dx$$

$$= \ln x \ \Big|_a^1$$

and now we ask, what happens if we plug in 0 for a? The value of the integral "blows up" at the lower bound. $\ln 0$ doesn't exist and the logarithm of a very small number approaches $-\infty$. So this integral can't be evaluated.

If we think of it as a Riemann sum with rectangles of width 1, it is like the harmonic series (with a lower bound of 1)

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

We know this diverges.

On the other hand:

$$\int_a^1 \frac{1}{\sqrt{x}}\ dx$$

$$= 2\sqrt{x}\ \Big|_a^1$$

Now, when we evaluate at the lower bound, with 0 for a, we get 0. Therefore, the value of the integral is just:

$$= 2\sqrt{1} - 0 = 2$$

negative exponential

Here is the negative exponential function:

$$\int_0^\infty e^{-x}\ dx$$

$$= -e^{-x}\ \Big|_0^\infty$$

$$= -(0 - 1) = 1$$

And a variation:

$$\int_0^\infty 2\pi e^{-r^2}\ r\ dr$$

Letting $t = r^2$, this is what we just did:

$$\pi \int_0^\infty -e^{-t}\ dt = \pi$$

The negative exponential often appears with a constant factor, traditionally denoted by λ:

$$\int_0^\infty e^{-\lambda x}\ dx$$

$$= -\frac{1}{\lambda} \, e^{-\lambda x} \, \Big|_0^\infty$$

$$= -\frac{1}{\lambda} \, (0 - 1) = \frac{1}{\lambda}$$

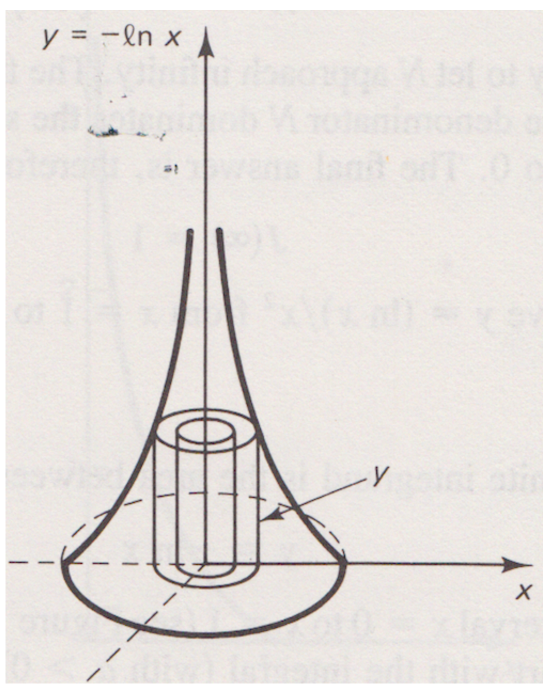
This form of the negative exponential $e^{-\lambda x}$ is a valid (and famous) probability density function if the total value of the integral is equal to 1. We achieve this by "normalizing" it, multiplying by λ:

$$p(x) = \lambda \int e^{-\lambda x} \, dx$$

We'll see all of these again.

log 1/x

The inverse log function provides a fun example of an improper integral with a very simple and finite result. The example (and the figure) are from Hamming.



We consider the function

$$y = -\ln x = \ln \frac{1}{x}$$

The minus sign is used so we are working with positive y, recognizing that the function is not defined at $x = 0$. Rotate the curve around the y axis and find the volume.

To use the method of cylinders, we consider a series of concentric cylindrical surfaces of width dx, ranging from $x = 0$ to $x = 1$. For each value of x, the surface area is the height of the function $h = -\ln x$ times the circumference $2\pi x$ to give a volume for each element of

$$dV = 2\pi x \ y \ dx$$

$$= 2\pi x \ (-\ln x) \ dx$$

We get the total volume by integrating over the interval $[0, 1]$

$$V = \int_0^1 2\pi x \ (-\ln x) \ dx$$

$$= -2\pi \int_0^1 x \ln x \ dx$$

Store the factor of $-\pi$ for second; we will need the 2 sooner. What is

$$\int 2x \ln x \ dx$$

The systematic approach is to use integration by parts, but let's just guess. If we had $F(x) = x^2 \ln x$ then part of the derivative $f(x) = F'(x)$ would be what we want:

$$\begin{aligned}\frac{d}{dx} x^2 \ln x &= 2x \ln x + \frac{x^2}{x} \\ &= 2x \ln x + x\end{aligned}$$

to cancel the extra x, we need another term, namely $-x^2/2$

$$\begin{aligned}F(x) &= x^2 \ln x - \frac{x^2}{2} \\ F'(x) &= 2x \ln x + x - x \\ &= 2x \ln x\end{aligned}$$

So

$$\int 2x \ln x \; dx = x^2 \ln x - \frac{x^2}{2}$$

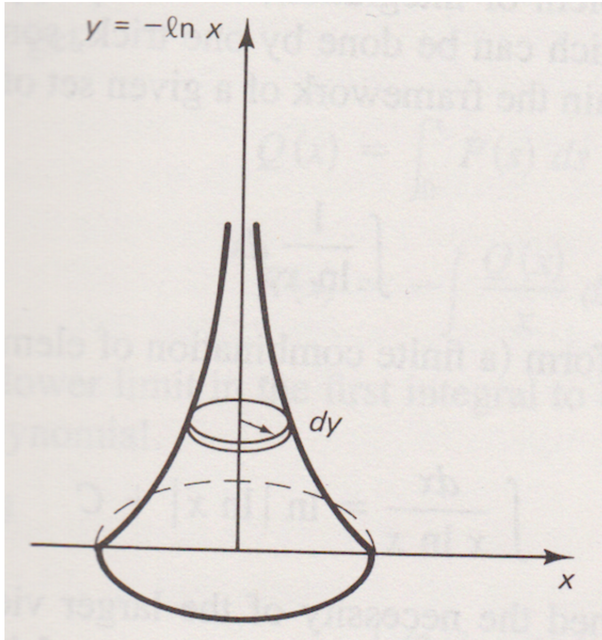
The volume V is equal to $-\pi$ times

$$x^2 \ln x - \frac{x^2}{2} \Big|_0^1$$

At the upper limit, $\ln 1 = 0$ so this is $-1/2$, and the question becomes, what happens to $x^2 \ln x$ as x approaches 0? We guess that x^2 must become small faster than $\ln x$ approaches $-\infty$. So at the lower limit, we will get zero, and the whole thing is

$$V = -\pi \; (-\frac{1}{2}) = \frac{\pi}{2}$$

Let's try by the method of disks and then come back to this limit. We also have a figure for the second approach



Again

$$f(x) = -\ln x$$

As y ranges from 0 to ∞, each disk has a width dy and an area equal to πx^2 so the volume element is

$$dV = \pi x^2 \; dy$$

The next part is really interesting. We follow Hamming. Rather than flip the figure, we can change variables and write

$$y = -\ln x$$

$$dy = -\frac{1}{x} \; dx$$

so the volume element becomes

$$= -\pi x^2 \; \frac{1}{x} \; dx$$

$$= -\pi x \; dx$$

The limits also change. Before we had $y = 0 \to \infty$ and now we have $x = 1 \to 0$ (because $y = -\ln x$), so

$$V = \int_1^0 -\pi x \; dx$$

$$= \pi \int_0^1 x \; dx$$

$$= \frac{\pi}{2} x^2 \Big|_0^1 = \frac{\pi}{2}$$

So it looks like we were right about that limit.

But what is the formal method for evaluating

$$\lim_{x \to 0} x^2 \ln x$$

We convert this to a fraction

$$\lim_{x \to 0} \frac{\ln x}{1/x^2}$$

Since both numerator and the denominator go to ∞ as $x \to 0$, this is an indeterminate form, and we can use L'Hospital's rule (**here**).

We need derivatives of the numerator and the denominator. The numerator gives $1/x$ and the denominator gives $-2/x^3$ so we have

$$\frac{1}{x} \; \frac{1}{-2/x^{-3}} = \frac{x^3}{x}(-\frac{1}{2}) = -\frac{x^2}{2}$$

which in the limit as $x \to 0$, also goes to 0.

Chapter 37

Partial fractions

Strang gives this example:

$$\int \frac{1}{x-2} + \frac{3}{x+2} - \frac{4}{x}\ dx$$
$$= \ln|x-2| + 3\ln|x+2| - 4\ln|x|$$

That seems straightforward enough. Which function would produce that sum?

$$\frac{1}{x-2} + \frac{3}{x+2} - \frac{4}{x}$$
$$= \frac{(x+2)(x) + 3(x-2)(x) - 4(x-2)(x+2)}{(x-2)(x+2)(x)}$$
$$= \frac{x^2 + 2x + 3x^2 - 6x - 4x^2 + 16}{x^3 - 4x}$$
$$= \frac{-4x + 16}{x^3 - 4x}$$

We call this form P/Q, and it's the type of problem we are trying to solve using *partial fractions*. We start by factoring Q (although sometimes, the factors are given).

$$Q = x^3 - 4x = x(x^2 - 4)$$

$$= x(x-2)(x+2)$$

Let's try with a different numerator to see how it works. We write:

$$\frac{P}{Q} = \frac{3x^2 + 8x - 4}{(x-2)(x+2)(x)}$$

$$= \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x}$$

where A,B and C are constants.

We recognize that we can put these three fractions over Q as the common denominator.

These are the partial fractions that add up to P/Q. We need to find the values of A,B and C.

Here are two methods (the first one is slower):

Do what we just said. Put the right-hand side over the common denominator Q:

$$\frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x}$$

$$= \frac{A(x+2)(x) + B(x-2)(x) + C(x-2)(x+2)}{(x-2)(x+2)(x)}$$

Now make the numerators match:

$$3x^2 + 8x - 4 = A(x+2)(x) + B(x-2)(x) + C(x-2)(x+2)$$

$$= Ax^2 + 2Ax + Bx^2 - 2Bx + Cx^2 - 4C$$

We actually have three equations:

$$Ax^2 + Bx^2 + Cx^2 = 3x^2$$

$$2Ax - 2Bx = 8x$$

$$-4C = -4$$

From the last one $C = 1$. From the first one we have:

$$A + B + C = 3$$

$$A + B = 2$$

and then

$$A - B = 4$$

Add them together to get $2A = 6, so A = 3$ and then $B = -1$. We obtain finally

$$\frac{P}{Q} = \frac{3}{x-2} + \frac{-1}{x+2} + \frac{1}{x}$$

second method

The second approach is called the "cover-up method." We have:

$$\frac{3x^2 + 8x - 4}{(x-2)(x+2)(x)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x}$$

Multiply by $(x - 2)$

$$\frac{3x^2 + 8x - 4}{(x+2)(x)} = (\frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x})(x-2)$$

$$= A + \frac{B(x-2)}{x+2} + \frac{C(x-2)}{x}$$

Now evaluate at $x = 2$

$$\frac{3(2)^2 + 8(x) - 4}{(2+2)(2)} = \frac{12 + 16 - 4}{8} = 3 = A$$

Notice that we do not need to actually write

$$A + \frac{B(x-2)}{x+2} + \frac{C(x-2)}{x}$$

Nor, in calculating B, do we need to write

$$\frac{A(x+2)}{x-2} + B + \frac{C(x+2)}{x}$$

since we will pick x to zero out those terms, instead, just substitute $x = -2$ into

$$\frac{3x^2 + 8x - 4}{(x-2)(x)}$$

$$= \frac{3(-2)^2 + 8(-2) - 4}{(-2-2)(-2)}$$

$$B = \frac{12 - 16 - 4}{8} = \frac{-8}{8} = -1$$

For C multiply the left-hand side by x and evaluate at $x = 0$ (to make the A and B terms go away):

$$\frac{3x^2 + 8x - 4}{(x-2)(x+2)} = \frac{-4}{-4} = 1 = C$$

same degree

How about

$$\int \frac{3x^2 + 2x + 7}{x^2 + 1}\, dx$$

To use the method, P must be of a lower degree than Q, but here they both contain multiples of x^2 (degree two). We separate off the term of $3x^2$ by finding another 3:

$$\frac{3x^2 + 2x + 7}{x^2 + 1} = \frac{3x^2 + 3 + 2x + 4}{x^2 + 1}$$

$$= 3 + \frac{2x+4}{x^2+1}$$

Now we just have to solve:

$$\int 3 + \frac{2x}{x^2+1} + \frac{4}{x^2+1}\ dx$$

$$= 3x + \ln(x^2+1) + 4\tan^{-1} x + C$$

repeated factor

$$\frac{2x+3}{(x-1)^2}$$

We have two factors of $x-1$. Solution: use $(x-1)^2$ for one of the fractions:

$$\frac{2x+3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$2x+3 = A(x-1) + B$$

set $x = 1$, then

$$B = 2(1) + 3 = 5$$

and

$$2x + 3 = Ax - A + 5$$

$A = 2$ solves this.

more examples

These few examples are from wikipedia. We would like to simplify

$$\frac{3x+5}{(1-2x)^2}$$

We suppose that this fraction can be decomposed as follows

$$\frac{3x+5}{(1-2x)^2} = \frac{A}{(1-2x)^2} + \frac{B}{(1-2x)}$$

We multiply by the term with B to put everything over a common denominator:

$$\frac{A}{(1-2x)^2} + \frac{B}{(1-2x)}$$

$$= \frac{A}{(1-2x)^2} + \frac{B(1-2x)}{(1-2x)^2}$$

Getting rid of the denominators altogether

$$3x + 5 = A + B(1-2x)$$

Now both the constant terms and the terms in x must be equal:

$$-2Bx = 3x$$

$$B = -\frac{3}{2}$$

$$A + B = 5$$

$$A = \frac{13}{2}$$

And so

$$\frac{3x+5}{(1-2x)^2} = \frac{13/2}{(1-2x)^2} + \frac{-3/2}{(1-2x)}$$

To integrate, we would do this

$$\int \frac{3x+5}{(1-2x)^2}\,dx = \int \frac{13/2}{(1-2x)^2}\,dx + \int \frac{-3/2}{(1-2x)}\,dx$$

$$= \frac{13/4}{(1-2x)} + (3/4)\ln(1-2x)$$

Example 2.

$$f(x) = \frac{1}{x^2 + 2x - 3} = \frac{1}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

$$A(x-1) + B(x+3) = 1$$

$$Ax + Bx = 0$$

$$A = -B$$

$$-B + 3B = 1$$

$$B = \frac{1}{4}$$

$$f(x) = \frac{1}{4}(\frac{-1}{x+3} + \frac{1}{x-1})$$

Chapter 38

Revolution in the air

Surface area

The next topic really is an exciting step forward in calculus. We start looking at surface area using geometric arguments as well as results from calculus of one variable.

I will use S for the surface area. Sometimes for brevity I might write area instead of surface area.

Suppose a function $y = f(x)$ is revolved around the x-axis. Imagine slicing it into disks in the usual way, moving along the x-axis in increments dx.

Now, rather than compute the volume, we want the surface area of the solid. We might try adding up the perimeter of all the disks.

Suppose we start with the simple cone with $R = H$. The cone opens out to the right, with the vertex at the origin.

What we have is the function

$$y = x$$

The circumference at any point x is

$$2\pi y = 2\pi x$$

And the surface area is

$$A = \int x \; dx$$

(this has a subtle error that we will fix).

$$= \int_0^H 2\pi x \; dx$$

$$= \pi x^2 \Big|_0^H$$

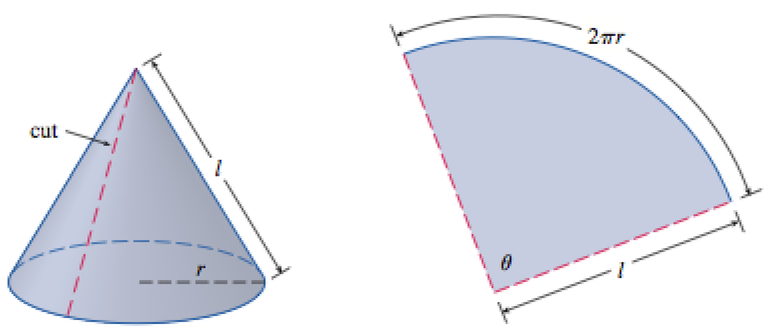
$$= \pi x^2 = \pi H^2$$

and since $H = R$

$$= \pi RH$$

Now, this is obviously not the correct answer.

For the geometry, imagine cutting the surface of a cone directly along the slant and then opening the surface and laying it flat out flat. We end up with a part (a sector) of a circle.



The radius of that circle is the slant height of the cone. The slant is labeled l in the figure (not mine) and the radius is r. Let's use L and R (capital letters for constants) in what follows:

$$L = \sqrt{R^2 + H^2}$$

The total circumference of the circle flat in the plane would be $2\pi L$.

However, the arc length along the sector that we actually used in the previous calculation is the circumference of the base of the cone, which is $2\pi R$.

So the total area of the sector (equivalent to the surface area of the cone) is the total area of the circle, times the ratio of the sector circumference to the total circumference.

$$S = \pi L^2 \, \frac{2\pi R}{2\pi L} = \pi R L$$

The error in our application of calculus to this problem is a factor of L/R, the ratio of the slant height to the radius of the base.

It turns out that what we did wrong was to multiply the circumference at each point by dx. What we should have done is to multiply it by the little increment of slant instead. This is called the path element for the curve ds.

Since

$$L = \sqrt{R^2 + H^2}$$

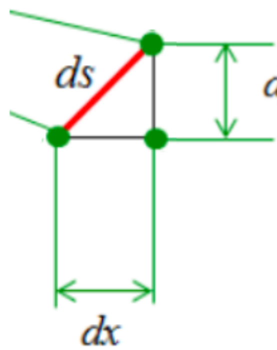
and in this problem

$$R = H$$

$$L = \sqrt{2}R$$

So the answer we obtained was πR^2, whereas the correct answer is πRL, and in this problem $L = \sqrt{2}R$, so the correct answer is $\sqrt{2}\pi R^2$.

path element



For the surface area of a "volume of revolution", instead of dx we need the actual length of the path element along the curve.

From Pythagoras we have

$$ds^2 = dx^2 + dy^2$$

$$= (1 + \frac{dy^2}{dx^2})\ dx^2$$

$$ds = \sqrt{1 + f'(x)^2} \cdot dx$$

We'll use this many times, both for surfaces of volumes of revolution and also for line integrals.

sphere: surface area

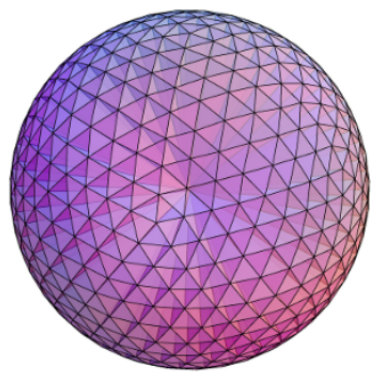
Calculus provides a simple proof for the surface area of a sphere, starting from the formula for the volume of a sphere

$$V = \frac{4}{3}\pi R^3$$

Suppose we take a sphere of radius r. (I use r here because now the radius will be a variable). If we increase the radius by a little bit dr, then how does the volume change? It changes exactly like the surface area! That is

$$dV = S\ dr$$
$$S = \frac{d}{dr}\ V = \frac{d}{dr}\ \frac{4}{3}\pi r^3 = 4\pi r^2$$

Another way to see this is to break up the entire surface area of the sphere into small cones. According to Acheson, this argument is due to Johannes Kepler.



If the number of cones is very large, the base of each one is almost flat. Call the area of the base dS and the height is of course R.

The volume of a single cone is

$$\frac{1}{3}R\ dS$$

If we add up the volumes of all the very thin cones from the entire sphere we will have the volume of the sphere

$$\frac{1}{3}R\ S$$

but we already know this is just $4/3\pi R^3$, so clearly

$$\frac{1}{3}R \cdot S = \frac{4}{3}\pi R^3$$
$$S = 4\pi R^2$$

slices

Another approach is to make a volume of revolution and add up the surface part by the method of slices. This is similar to the volume calculation we did before, except this time, for each slice we need to use the path element ds rather than dx.

Consider a sphere of radius R centered at the origin and make slices perpendicular to the x axis. We have that

$$y = \sqrt{R^2 - x^2}$$

The circumference for each slice is then $2\pi y$.

When we did the volume integral for a sphere in one dimension it was

$$V = \int_{-R}^{R} \pi y^2 \ dx$$

Here we are looking for the surface area, and adding up a bunch of small strips from the perimeter, but the differential is not dx In other words, we can't just do

$$S = \int 2\pi y \ dx$$

For the surface area of a "volume of revolution", instead of dx we need the actual length of the path element along the curve.

$$ds = \sqrt{1 + f'(x)^2} \cdot dx$$

What is the slope of a circle?

$$y = \sqrt{R^2 - x^2}$$

$$\frac{dy}{dx} = f'(x) = -\frac{x}{\sqrt{R^2 - x^2}} = -\frac{x}{y}$$

so

$$ds = \sqrt{1 + f'(x)^2} \cdot dx$$

$$= \sqrt{1 + \frac{x^2}{y^2}}\ dx$$

We want

$$S = \int 2\pi y\ ds$$

$$= \int 2\pi y\ \sqrt{1 + \frac{x^2}{y^2}}\ dx$$

$$= 2\pi \int \sqrt{y^2 + x^2}\ dx$$

$$= 2\pi R \int\ dx$$

$$= 2\pi R\ x\ \Big|_{-R}^{R} = 4\pi R^2$$

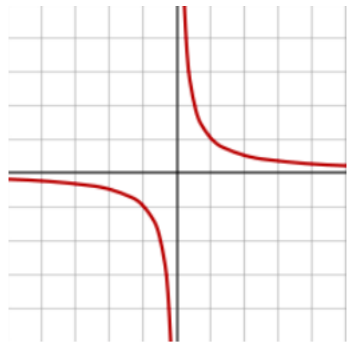
That simplified beautifully. Usually integrals with the path element get messy.

Gabriel's horn

The inverse function is

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$



We consider the curve from $x = 1 \to \infty$.

To get the surface area

$$S = \int 2\pi y \; ds$$

$$= 2\pi \int \frac{1}{x} \; \sqrt{1 + f'(x)^2} \; dx$$

$$= 2\pi \int \frac{1}{x} \; \sqrt{1 + \frac{1}{x^4}} \; dx$$

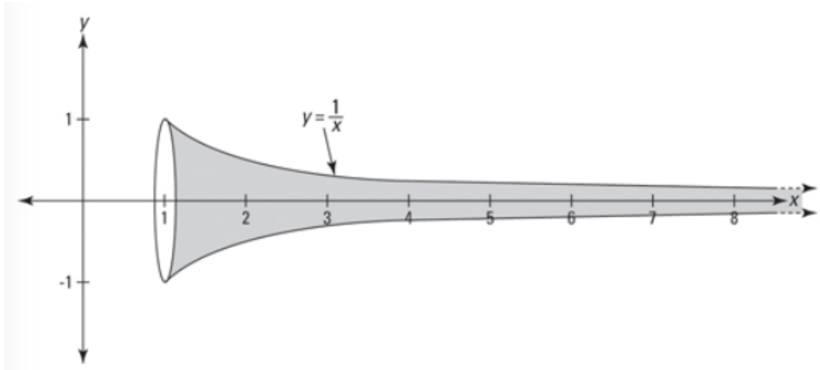
This looks hard! However, we notice that the factor

$$\sqrt{1 + \frac{1}{x^4}} > 1$$

So, if the integral without this factor diverges, the one with it diverges too. And

$$\int_1^\infty \frac{1}{x} \; dx = \ln x \; \Big|_1^\infty$$

certainly diverges at the upper limit.



Why is it so surprising that the surface area of this horn is infinite? It is surprising because the volume is finite.

The volume is

$$\int \pi y^2 \ dx$$

$$= \int \pi \frac{1}{x^2} \ dx$$

$$= \pi \ [-\frac{1}{x} \Big|_1^\infty] = \pi$$

Wait for the inevitable joke: doesn't that blow your mind?

Volumes

A solid of revolution is formed by revolving a curve around a central axis, typically, the x-axis. We can get the volume of the solid by slicing it into disks.

In **Sphere and cone**, we revolved a half-circle to obtain the volume of a sphere. The integral was

$$V = \int_{-R}^{R} \pi y^2 \ dx$$

We also did the cone:

$$V = \pi \int_0^H (\frac{R}{H}x)^2 \ dx$$

However, this approach can be used with any curve or pair of curves.

We also found just above, the volume (and surface area) of Gabriel's horn, using the curve $y = 1/x$.

The volume of the solid formed by rotating the curves $f(x)$ and $g(x)$ around the x-axis on the interval $[a, b]$ ($f(x) < g(x)$ everywhere) is:

$$V = \int_a^b f(x)^2 - g(x)^2 \ dx$$

For $g(x) = 0$ this resolves to the familiar form.

If the curve is given in parametric form (both x and y as a function of t), then

$$V_x = \int_a^b \pi y^2 \ \frac{dx}{dt} \ dt$$
$$V_y = \int_a^b \pi x^2 \ \frac{dy}{dt} \ dt$$

where V_x is revolved around the x-axis, and so on.

The corresponding surface areas are

$$A_x = \int_a^b 2\pi y \ \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \ dt$$
$$A_y = \int_a^b 2\pi x \ \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \ dt$$

Torus

Consider a circle of radius R, displaced upward from the x-axis. The distance from the origin to the center of the circle is a.

The equation of the upper half of this circle is

$$y = \sqrt{R^2 - x^2} + a$$

So

$$y^2 = R^2 - x^2 + a^2 + 2a\sqrt{R^2 - x^2}$$

The equation of the bottom half of the circle is almost identical

$$y = -\sqrt{R^2 - x^2} + a$$

So

$$y^2 = R^2 - x^2 + a^2 - 2a\sqrt{R^2 - x^2}$$

Subtracting the bottom from the top, the first three terms cancel and we have

$$y_{\text{top}}^2 - y_{\text{bottom}}^2 = 4a\sqrt{R^2 - x^2}$$

Don't forget to multiply by π:

$$A = 4\pi a\sqrt{R^2 - x^2}$$

Adding up the area of each slice of the donut

$$V = \int_{-R}^{R} 4\pi a\sqrt{R^2 - x^2}\ dx$$

We need a trig substitution:

$$x = R\sin t$$

$$dx = R\cos t\ dt$$

$$\sqrt{R^2 - x^2} = R\cos t$$

So the integral is $4\pi a$ times

$$R^2 \int \cos^2 t \ dt$$

As always:

$$\int \cos^2 t \ dt = \frac{1}{2} \ [t + \sin t \cos t \ ]$$

And if the bounds are right, that second term will disappear. Let's see.

$$x = [-R, R]$$

$$t = [-\pi/2, \pi/2]$$

The second term does go away, giving π for what's in the brackets, and we obtain finally

$$\begin{aligned} V &= 4\pi a \cdot R^2 \cdot \frac{\pi}{2} \\ &= 2\pi a \cdot \pi R^2 \end{aligned}$$

This is the same as obtained by multiplying the area of the cross-section of the torus by the distance traveled by the center in revolving around the x-axis. The general principle is named after Pappus.

Chapter 39

Powers of sine and cosine

Occasionally, we run into higher powers of the sine and or the cosine.

This chapter explains how to figure these out using integration by parts, but it is complicated enough that if you can, it is probably better to look up the answer in a table of integrals.

If the integrand contains only integral powers like $\sin^m x$ and $\cos^n x$, then there are two cases. If both m and n are even (or only one is present and that power is even), you can use the formula we will develop here. Otherwise, simply separate out one factor of $\sin x$ or $\cos x$ to combine with dx, like this:

$$\begin{aligned}
&\int \sin^5 x \cos^4 x \; dx \\
&= \int \sin^4 x \cos^4 x \sin x \; dx \\
&= \int (1 - \cos^2 x)^2 \cos^4 x \sin x \; dx \\
&= \int (\cos^4 x - 2\cos^6 x + \cos^8 x) \sin x \; dx
\end{aligned}$$

This is just

$$-\int u^4 - 2u^6 + u^8 \ du$$

To handle the case with m and n both even, first use the identity $\sin^2 + \cos^2 = 1$ to convert to all sine or all cosine. Converting the function with the smaller exponent will give the least complicated expression. Now we have an integral of only sine or only cosine raised to an even power. We will use n to represent that power. Let's do cosine first:

$$\int \cos^n x \ dx$$

We use integration by parts:

$$u = \cos^{n-1} x$$

$$du = -(n-1)\cos^{n-2} x \sin x \ dx$$

$$dv = \cos x \ dx$$

$$v = \sin x$$

So $\int \cos^n x \ dx$ is $\int u \ dv$ and this is equal to $uv - \int v \ du$. We have (reversing terms and writing vu):

$$\sin x \cos^{n-1} x + \int \sin x (n-1)\cos^{n-2} x \sin x \ dx$$

The last term is

$$\sin x (n-1)\cos^{n-2} x \sin x$$

$$= (n-1)\cos^{n-2} x \sin^2 x$$

$$= (n-1)\cos^{n-2} x \ (1 - \cos^2 x)$$

$$= (n-1)\cos^{n-2} x - (n-1)\cos^n x$$

The trick is that although we have produced $-(n-1)\cos^n x$ on the right-hand side, this can be moved to the left-hand side, and added to the expression we started with. Placing the above result under the integral and moving all of the $\cos^n x$ terms to the left-hand side, we obtain

$$n \int \cos^n x \; dx = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \; dx$$

We divide by n to obtain the general formula.

$$\boxed{\int \cos^n x \; dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \; dx}$$

A few specific examples:

$$\int \cos^2 x \; dx = \frac{1}{2} \sin x \cos x + \frac{1}{2} \int \cos^0 x \; dx$$

$$= \frac{1}{2}(\sin x \cos x + x)$$

$$\int \cos^4 x \; dx = \frac{1}{4} \sin x \cos^3 x + \frac{3}{4} \int \cos^2 x \; dx$$

$$= \frac{1}{4} \sin x \cos^3 x + \frac{3}{8}(\sin x \cos x + x)$$

$$\int \cos^6 x \; dx = \frac{1}{6} \sin x \cos^5 x + \frac{5}{6} \int \cos^4 x \; dx$$

$$= \frac{1}{6} \sin x \cos^5 x + \frac{5}{24} \sin x \cos^3 x + \frac{5}{16}(\sin x \cos x + x)$$

We will work out the formula for sine below, but notice:

$$\sin^2 x + \cos^2 x = 1$$

$$\int \sin^2 x \ dx + \int \cos^2 x \ dx = \int dx = x$$

Looking at the formula for cosine squared

$$\int \cos^2 x \ dx = \frac{1}{2}(\sin x \cos x + x)$$

it should be clear that we will end up with the same formula for sine squared, but just flip the sign on the term $\sin x \cos x$ to make it go away in the sum. Let's see:

$$\int \sin^n x \ dx$$

We use integration by parts:

$$u = \sin^{n-1} x$$

$$du = (n-1) \sin^{n-2} x \cos x \ dx$$

$$dv = \sin x \ dx$$

$$v = -\cos x$$

For $uv - \int v \ du$ we have:

$$\sin^{n-1} x(-\cos x) - \int (-\cos x)(n-1) \sin^{n-2} x \cos x \ dx$$

Just as before, the last term is

$$\sin^{n-2} x \cos^2 x$$

$$= \sin^{n-2} x(1 - \sin^2 x)$$

$$= \sin^{n-2} x - \sin^n x$$

So the whole thing is:

$$\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^4 x\, dx$$

$$n \int \sin^n x\ dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\ dx$$

$$\boxed{\int \sin^n x\ dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x\ dx}$$

So for $\sin^2 x$:

$$\int \sin^2 x\ dx = -\frac{1}{2} \sin^{n-1} x \cos x + \frac{1}{2} \int\ dx$$

$$\int \sin^2 x\ dx = \frac{1}{2}(-\sin^{n-1} x \cos x + x)$$

As predicted, we have simply switched the sign on the first term.

Part XII

Hyperbolics

Chapter 40

Hyperbolic functions

The hyperbolic functions are defined to be:

$$2\cosh x = e^x + e^{-x}$$

$$2\sinh x = e^x - e^{-x}$$

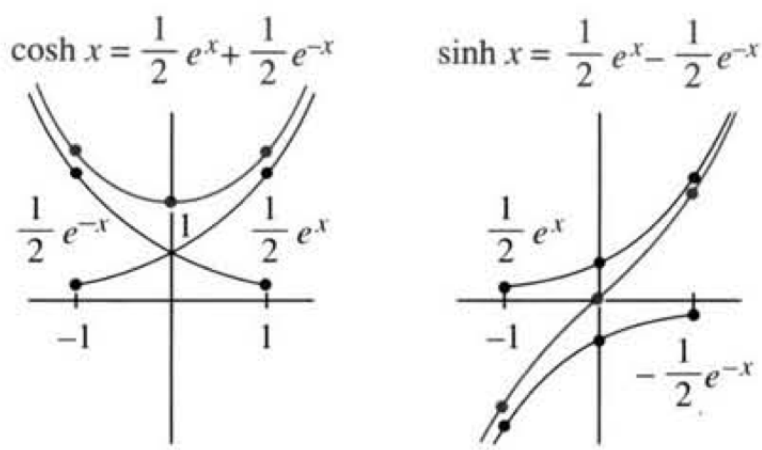


Fig. 6.18 Cosh x and sinh x. The hyperbolic functions combine $\frac{1}{2}e^x$ and $\frac{1}{2}e^{-x}$.

When we work through Euler's formula

$$e^{ix} = \cos x + i\sin x$$

we will find that

$$e^{ix} + e^{-ix} = 2\cos x$$

$$e^{ix} - e^{-ix} = -2\ i\sin x$$

which is in a way, parallel to the hyperbolic definitions.

The difference of squares has a simple value:

$$\cosh^2 t - \sinh^2 t = 1$$

Everything about the hyperbolic sine is reminiscent of the regular trig functions but with a sign change.

A plot of $\sinh t$ on the x-axis and $\cosh t$ on the y-axis yields a hyperbola in the same way the $y^2 - x^2 = 1$ does.

derivatives

$$\frac{d}{dx}2\sinh x = \frac{d}{dx}(e^x - e^{-x}) = e^x + e^{-x} = 2\cosh x$$

$$\frac{d}{dx}2\cosh x = \frac{d}{dx}(e^x + e^{-x}) = e^x - e^{-x} = 2\sinh x$$

Also, note that:

$$2\sinh x + 2\cosh x = 2e^x$$

$$e^x = \sinh x + \cosh x$$

Because of this, and by symmetry, we expect that the series should be

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

The values of the functions at zero are

$$\sinh 0 = 0$$

$$\cosh 0 = 1$$

relativity

The hyperbolic functions come into the mathematics of relativity, where for an observer in a moving reference frame, the following equations hold:

$$x' = \frac{x - vt}{\sqrt{1 - v^2}}$$
$$t' = \frac{t - vx}{\sqrt{1 - v^2}}$$

The quantity s^2 is invariant where

$$s^2 = t^2 - x^2$$

Proof:

$$x'^2 = \frac{x^2 - 2xvt + v^2t^2}{1 - v^2}$$
$$t'^2 = \frac{t^2 - 2xvt + v^2x^2}{1 - v^2}$$
$$t'^2 - x'^2 = \frac{(t^2 - x^2) - v^2(t^2 - x^2)}{1 - v^2}$$
$$= t^2 - x^2$$

The hyperbolic functions come in by defining a parameter θ (the "rapidity")

$$\cosh \theta = \frac{1}{\sqrt{1 - v^2}}$$

Then

$$\sinh^2\theta = \cosh^2\theta - 1 = \frac{1}{1-v^2} - 1 = \frac{v^2}{1-v^2}$$

$$\sinh\theta = \frac{v}{\sqrt{1-v^2}}$$

So we can rewrite

$$x' = \frac{x - vt}{\sqrt{1-v^2}} = x\cosh\theta - t\sinh\theta$$

$$t' = \frac{t - vx}{\sqrt{1-v^2}} = t\cosh\theta - x\sinh\theta$$

$\tanh\theta$

We had

$$\sinh\theta = \frac{v}{\sqrt{1-v^2}}$$

$$\cosh\theta = \frac{1}{\sqrt{1-v^2}}$$

so

$$\tanh\theta = v$$

leading us to explore the properties of the hyperbolic tangent. Going back to the beginning:

$$2\sinh\theta = e^{\theta} - e^{-\theta}$$

$$2\cosh\theta = e^{\theta} + e^{-\theta}$$

$$\tanh\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$$

The derivative is (by the quotient rule):

$$\frac{d}{d\theta}\ \tanh\theta = \frac{\cosh^2\theta - \sinh^2\theta}{\cosh^2\theta}$$

$$= \frac{1}{\cosh^2\theta}$$

Chapter 41

Hyperbolic substitution

I came across this integral:

$$\int \sqrt{x^2 - a^2}\ dx$$

with an unusual substitution method to solve it.

We've had $\sqrt{a^2 - x^2}$ as the integrand before, with the circle. If we manipulate this new expression:

$$y = \sqrt{x^2 - a^2}$$

$$y^2 = x^2 - a^2$$

$$x^2 - y^2 = a^2$$

You can see where the hyperbolic connection comes from.

the answer is

The best way to solve this is to look it up:

$$I = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\ \ln(x + \sqrt{x^2 - a^2})$$

https://www.quora.com/How-do-you-evaluate-the-integral-int-sqrt-x-2-a-2-dx

We shouldn't just accept this of course, but check by differentiating. Make our lives simpler by reserving the factor of $1/2$ from both terms.

We will need

$$[\sqrt{x^2 - a^2}\ ]' = \frac{x}{\sqrt{x^2 - a^2}}$$

The first term is then

$$[x\sqrt{x^2 - a^2}\ ]' = \sqrt{x^2 - a^2} + \frac{x^2}{\sqrt{x^2 - a^2}}$$

$$= \frac{2x^2 - a^2}{\sqrt{x^2 - a^2}}$$

And the second term is a^2 times

$$[\ln(x + \sqrt{x^2 - a^2})\ ]' = \frac{1}{x + \sqrt{x^2 - a^2}}(1 + \frac{x}{\sqrt{x^2 - a^2}})$$

$$= \frac{1}{x + \sqrt{x^2 - a^2}}(\frac{\sqrt{x^2 - a^2} + x}{\sqrt{x^2 - a^2}})$$

$$= \frac{a^2}{\sqrt{x^2 - a^2}}$$

where we have brought back the factor of a^2 in the last step. Subtracting the second term from the first

$$= \frac{2x^2 - 2a^2}{\sqrt{x^2 - a^2}}$$

$$= 2\sqrt{x^2 - a^2}$$

and then finally, recall the factor of $1/2$, which yields the desired result.

hyperbolic substitution

To solve the integral, we use a hyperbolic substitution, which is the interesting part. It will make the integral trivial, however, getting back to the original variable x will be a challenge. Let

$$x = \frac{a}{2}(e^t + e^{-t})$$

If you look again at the hyperbolic functions, you'll see that this is just $x = a \cosh t$. Let's work through the substitution:

$$x^2 = \frac{a^2}{4}\ (e^{2t} + 2 + e^{-2t})$$

$$\sqrt{x^2 - a^2} = \sqrt{\frac{a^2}{4}\ (e^{2t} + 2 + e^{-2t}) - a^2}$$

$$= \sqrt{\frac{a^2}{4}\ (e^{2t} - 2 + e^{-2t})}$$

a bit tricky, but it works.

$$= \frac{a}{2}\ (e^t - e^{-t})$$

$$= a \sinh t$$

So that's going to be a big help. We can get dx by differentiation of $\cosh t$ directly, or work through the exponentials:

$$x = \frac{a}{2}(e^t + e^{-t})$$

$$dx = \frac{a}{2}(e^t - e^{-t})\ dt = a \sinh t\ dt$$

integral

So after the substitution, the integral is

$$\int a^2 \sinh^2 t \ dt$$

which turns out to be easy in terms of the exponentials

$$= \int \frac{a^2}{4} \ (e^t - e^{-t})^2 \ dt$$

$$= \int \frac{a^2}{4} \ (e^{2t} - 2 + e^{-2t}) \ dt$$

$$= \frac{a^2}{4} \ (\frac{e^{2t}}{2} - 2t - \frac{e^{-2t}}{2})$$

That was easy, which is the whole point.

reversing the substitution: term 2

This is the tricky part. We have the integral in terms of t. For a nice value of x we could maybe figure out the new bounds for t but in general we have to go back to x. So

$$x = \frac{a}{2}(e^t + e^{-t})$$

Let $e^t = z$ and $t = \ln z$ so

$$x = \frac{a}{2}(z + \frac{1}{z})$$

$$z^2 - \frac{2x}{a} z + 1 = 0$$

$$z = \frac{(2x/a) \pm \ \sqrt{(2x/a)^2 - 4}}{2}$$

taking only the positive root

$$= \frac{x}{a} + \sqrt{(x/a)^2 - 1}$$

$$= \frac{1}{a}(x + \sqrt{x^2 - a^2})$$

So

$$t = \ln z = \ln|x + \sqrt{x^2 - a^2}| - \ln|a|$$

Recall that the solution to the integral had one term with t that we can now write:

$$-\frac{a^2}{4}\ 2t = -\frac{a^2}{2}\ [\ \ln|x + \sqrt{x^2 - a^2}| - \ln a\ ]$$

The latter part of this gets folded into the constant of integration so

$$= -\frac{a^2}{2}\ \ln|x + \sqrt{x^2 - a^2}| + C$$

which matches the second half of the answer we were given.

reversing the substitution: term 1

The other part is

$$= \frac{a^2}{4}\ (\frac{e^{2t}}{2} - \frac{e^{-2t}}{2})$$

We know the answer so let's work backwards from that

$$x = \frac{a}{2}(e^t + e^{-t})$$

$$\sqrt{x^2 - a^2} = \frac{a}{2}\ (e^t - e^{-t})$$

So

$$x\ \sqrt{x^2 - a^2} = \frac{a^2}{4}(e^t + e^{-t})(e^t - e^{-t})$$

$$= \frac{a^2}{4}(e^{2t} - e^{-2t})$$

Multiply the last expression by 1/2 and we obtain

$$= \frac{a^2}{4}\ (\frac{e^{2t}}{2} - \frac{e^{-2t}}{2})$$

Therefore this expression is equal to

$$\frac{1}{2}x\ \sqrt{x^2 - a^2}$$

which completes the proof.

□

I put this problem in the book because of the interesting substitution, and to give a little more exposure to the hyperbolic functions. It is harder than your typical integral.

I would prefer (with Strang) to just admit that there are a lot of hard integrals that can be solved, and then move on. We have more interesting things to do with our time.

It also helps to remember that for real problems, they usually cannot be solved in "closed form," so learning a bunch of integral manipulations is not so helpful in the real world.

Chapter 42

Hanging chain

The hanging chain, or catenary, is a famous problem solved by Johann Bernoulli about 1700.

https://en.wikipedia.org/wiki/Johann_Bernoulli

solution

This is quite a challenging problem. See

https://gordma.wordpress.com/2014/03/26/we-know-the-lion-by-his-claw/

Bernoulli posed the problem as a public challenge, and said of Newton's anonymous submission: "tanquam ex ungue leonem" ("we know the lion by his claw").

We imagine an ideal hanging cable (like the cable for a suspension bridge, without the vertical cables or bridge deck), although its ends might be at different heights. The cable is ideal: it will not stretch or contract, is perfectly flexible, and has constant mass per unit length.

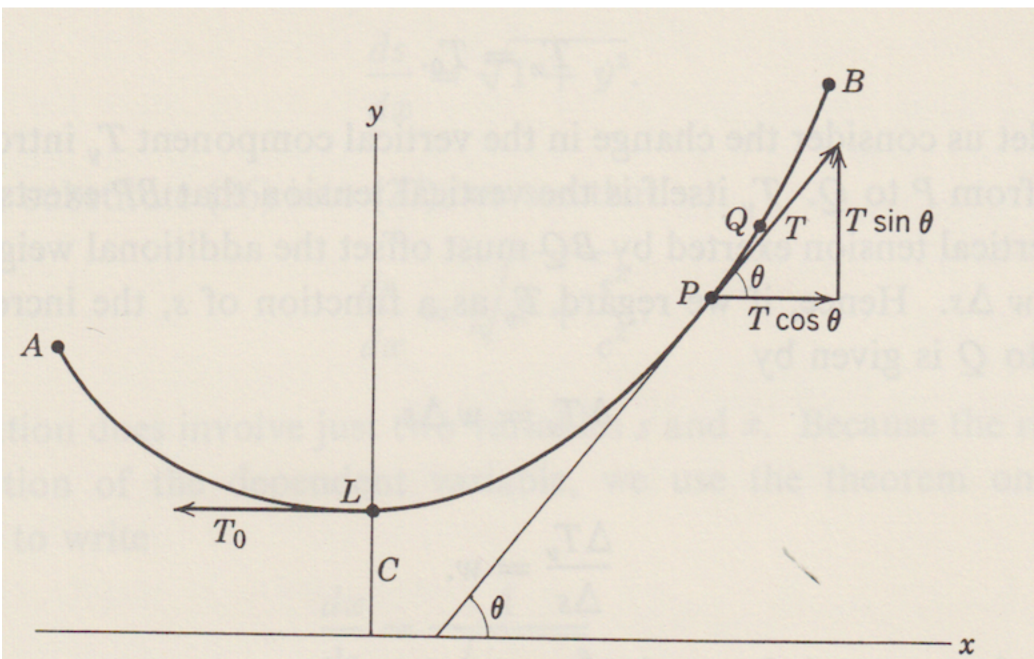


Figure 16-10

Consider a position P on the chain. The part of the chain below P pulls down on it, because of its weight, and the part above pulls up.

This force is called tension, T, and since the cable isn't moving, these forces are equal and opposite for every position on the chain.

The tension is in the direction of the angle θ tangent to the curve. By convention it is positive upward.

We can decompose the tension T into components in the x and y directions:

$$T = T_x + T_y$$

$$T_x = T \cos \theta$$

$$T_y = T \sin \theta$$

The curve traced out by the chain is described by $y = f(x)$, and it has a tangent at each point which is the ratio T_y/T_x.

moving to Q

Now consider moving from P up to the nearby point Q. The weight acting at Q is greater than that at P. If the arc length of L to P is s, we add an additional small length Δs to it in moving from P to Q. The weight of that little segment is the weight per unit length w, $\times \Delta s$.

The key point in the solution to this problem is that the additional force from the added weight changes T_y (because gravity points downward), but does not affect T_x. T_x is the same at every point in the chain, including at the lowest point, which has no weight acting on it. At this point, L, the tension is labeled as T_0, but T_x is equal everywhere.

$$T_x = T_0$$

The change in T_y when moving from P to Q is:

$$\Delta T_y = w \Delta s$$

So

$$\frac{\Delta T_y}{\Delta s} = w$$

and as we pass to the limit:

$$\frac{dT_y}{ds} = w$$

Since w is a constant, this means that

$$T_y = ws + D$$

where D is a constant of integration. But at L, both $s = 0$ and $T_y = 0$, so $D = 0$, and so:

$$T_y = ws$$

This difference in T_y at each point is what is responsible for the change in θ as we trace out the curve of the cable or chain.

Tangent

From above:

$$\frac{T_y}{T_x} = \frac{ws}{T_0}$$

but

$$\frac{T_y}{T_x} = \frac{T \sin \theta}{T \cos \theta}$$

so

$$\frac{ws}{T_0} = \tan\theta$$

And $\tan\theta$ is the slope of the curve so

$$y' = \tan\theta$$

Let $c = T_0/w$ so

$$y' = \frac{ws}{T_0} = \frac{s}{c}$$

Getting to x

The second point of real insight required for this problem is that we now have a differential equation relating y' to s, but we would like to have y (and y') as a function of x.

So finally, we need to relate s to x and y. Recall that s is arc length. We know a formula for that:

$$ds^2 = dx^2 + dy^2$$

and

$$\begin{aligned}\frac{ds}{dx} &= \sqrt{1 + y'^{\,2}} \\ &= \sqrt{1 + s^2/c^2}\end{aligned}$$

Hence

$$dx = \frac{1}{\sqrt{1 + s^2/c^2}}\, ds$$

and our integral is

$$\int \frac{s}{c}\frac{1}{\sqrt{1 + s^2/c^2}}\, ds$$

Integration

We do two substitutions. First, let $u = s/c$ and then $c\ du = ds$ so

$$dx = c\frac{1}{\sqrt{1+u^2}}\ du$$

Second, because we have $\sqrt{1+u^2}$, do a trig substitution with $u/1 = \tan t$. Then $1/\sqrt{1+u^2} = \cos t$, $\sqrt{1+u^2} = \sec t$ and $du = \sec^2 t\ dt$ so

$$\begin{aligned}\int dx &= c\int \cos t\ \sec^2 t\ dt \\ &= c\int \sec t\ dt\end{aligned}$$

Integrate:

$$\begin{aligned}x &= c\ln|\sec t + \tan t| \\ &= c\ln|\sqrt{1+u^2}+u| \\ &= c\ln|\frac{s}{c} + \sqrt{1+s^2/c^2}|\end{aligned}$$

Solve for s

Exponentiate:

$$e^{x/c} = \frac{s}{c} + \sqrt{1+s^2/c^2}$$

Let $z = s/c$. Then

$$\begin{aligned}(e^{x/c} - z)^2 &= 1 + z^2 \\ e^{2x/c} - 2ze^{x/c} + z^2 &= 1 + z^2\end{aligned}$$

$$e^{2x/c} - 2ze^{x/c} = 1$$

$$e^{x/c}(e^{x/c} - 2z) = 1$$

$$e^{x/c} - 2z = e^{-x/c}$$

$$z = \frac{1}{2}(e^{x/c} - e^{-x/c})$$

$$s = c\ \frac{e^{x/c} - e^{-x/c}}{2}$$

Back to y

Finally, recall that

$$y' = \frac{s}{c}$$

So

$$y' = \frac{e^{x/c} - e^{-x/c}}{2}$$

Integrate:

$$y = \frac{1}{2}\ c(e^{x/c} + e^{-x/c})$$

This is the hyperbolic cosine.

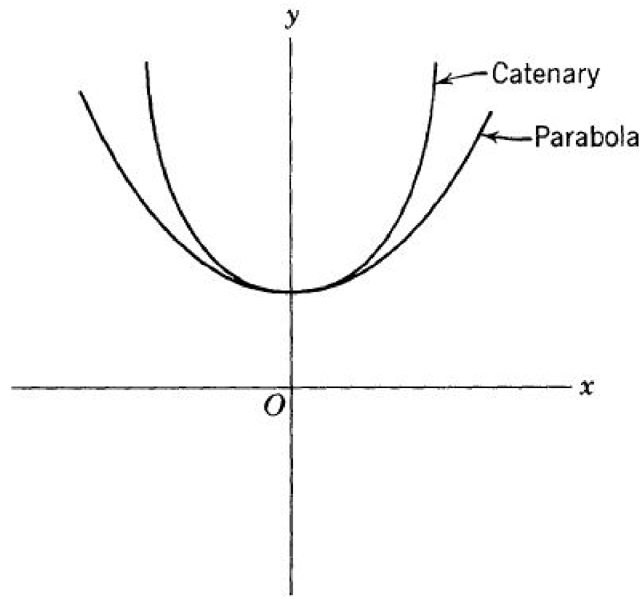
$$y = c\ \cosh \frac{x}{c}$$

And

$$s = c\ \sinh \frac{x}{c}$$

Recall that $c = T_0/w$. There is more to the problem (see Kline, for example), we need to figure out how T_0 depends on the geometry of the problem. But, this seems a good place to stop.

Let's just plot it:



It looks like a parabola but it's steeper.

Part XIII

Fancy differentiation

Chapter 43

Implicit differentiation

In this section we take a look at implicit differentiation, a remarkably powerful technique.

We will prove that the power rule $d/dx\ x^n = nx^{n-1}$ applies not only for integer n, both positive and negative, but also for rational n like $\sqrt{x}$ or $1/x^{3/2}$.

Implicit differentiation

Suppose we have a circle of radius R with equation

$$x^2 + y^2 = R^2$$

$$y = \sqrt{R^2 - x^2}$$

We'd like to find the derivative. It is not too hard by the standard method. By the chain rule

$$y' = \frac{1}{2}\ \frac{1}{\sqrt{R^2 - x^2}}(-2x) = -\frac{x}{y}$$

But there is another way.

$$x^2 + y^2 = R^2$$

Take d/dx of all the terms on both sides. The right-hand side is a constant so

$$2x + \frac{d}{dx}\ y^2 = 0$$

By the chain rule:

$$2x + 2y\ \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

We can also imagine a new variable, call it a parameter t where

$$x = t$$

Think of t like ticks of a clock and x has been scaled to follow t exactly.

Since y is a function of x and x is a function of t, y is also a function of t.

$$y = f(t)$$

Now differentiate with respect to t, using the chain rule:

$$x^2 + y^2 = R^2$$

$$2x\ \frac{dx}{dt} + 2y\ \frac{dy}{dt} = 0$$

It is OK to multiply by dt

$$x\ dx + y\ dy = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

If you find the idea of a parametric equation confusing, read ahead to **this chapter**.

After a while you will not need the parameter or even the chain rule. Just say

$$x^2 + y^2 = R^2$$

$$2x\ dx + 2y\ dy = 0$$

and so on.

Equivalent statements:

$$2x + 2y\ \frac{dy}{dx} = 0$$

$$2x + 2y\ y' = 0$$

derivative of the cosine

To obtain

$$\frac{d}{dx}\ \cos x$$

we originally set up a difference quotient. Here is another way using implicit differentiation and the chain rule.

Start from

$$\sin^2 x + \cos^2 x = 1$$

$$2 \sin x\ (\frac{d}{dx} \sin x) + 2 \cos x\ (\frac{d}{dx} \cos x) = 0$$

Plugging in for the derivative of $\sin x$:

$$2 \sin x \cos x + 2 \cos x (\frac{d}{dx} \cos x) = 0$$

$$\sin x + (\frac{d}{dx} \cos x) = 0$$

Rearranging, we obtain

$$\frac{d}{dx} \cos x = -\sin x$$

power rule

We can use implicit differentiation to prove that the power rule is correct for rational exponents:

$$y = x^{m/n}$$

$$y^n = x^m$$

Differentiate implicitly:

$$ny^{n-1}\ dy = mx^{m-1}\ dx$$

$$\frac{dy}{dx} = \frac{mx^{m-1}}{ny^{n-1}}$$

$$= \frac{m}{n}\ \frac{x^{m-1}}{(x^{m/n})^{n-1}}$$

$$= \frac{m}{n}\ \frac{x^{m-1}}{x^{m-m/n}}$$

Add up the exponents of x:

$$(m-1) - (m - \frac{m}{n}) = \frac{m}{n} - 1$$

The result is the power rule.

$$\frac{d}{dx}\ x^{m/n} = \frac{m}{n}\ x^{m/n-1}$$

Later we will use logarithmic differentiation to show that it applies for *any real number* n. That's quite impressive.

example

Consider the hyperbola

$$xy = c$$

where c is a constant. The standard approach would be

$$y = \frac{c}{x}$$

$$\frac{dy}{dx} = y' = -\frac{c}{x^2}$$

Implicit differentiation:

$$\frac{d}{dx}\ xy = \frac{d}{dx}\ c = 0$$

$$(xy)' = x\ y' + y\ x' = 0$$

$$xy' + y = 0$$

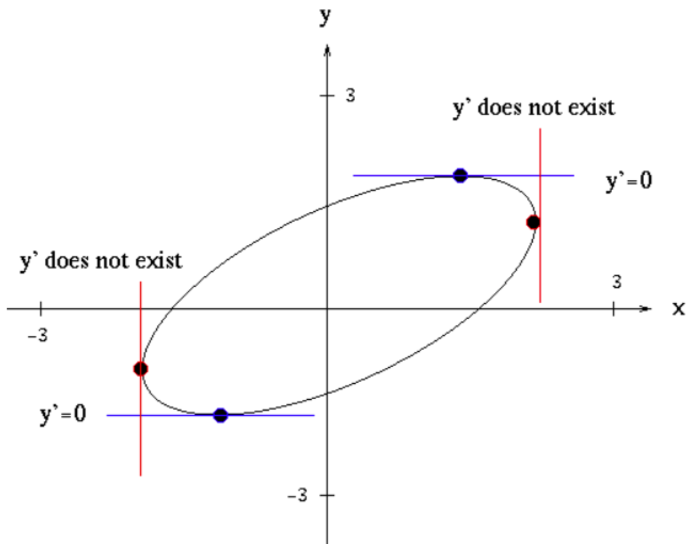
$$y' = -\frac{y}{x}$$

$$= -\frac{c}{x^2}$$

example

Suppose we have the equation of a tilted ellipse:

$$x^2 - xy + y^2 = 3$$



The problem asks us to find the extreme values of x and y on this ellipse. We take derivatives implicitly using the product rule on the second term:

$$x^2 - xy + y^2 = 3$$
$$2x\ dx - (x\ dy + y\ dx) + 2y\ dy = 0$$
$$2x\ dx - y\ dx = -2y\ dy + x\ dy$$
$$\frac{dy}{dx} = \frac{2x - y}{-2y + x}$$

Set y' equal to 0:

$$-\frac{2x - y}{2y - x} = 0$$

The extremes of y occur when $y' = 0$, that is when

$$y = 2x$$

Substituting $y = 2x$ into the original equation we obtain

$$x^2 - 2x^2 + 4x^2 = 3$$

$$x^2 = 1$$

$$x = \pm\, 1$$

Substituting again ($x = \pm\, 1$) into the original equation

$$1 - y + y^2 = 3$$

$$y^2 - y - 2 = 0$$

$$(y-2)(y+1) = 0$$

$$y = 2, -1$$

On the other hand, the maximum values of x occur when y' does not exist:

$$2y = x$$

By symmetry, I claim that this occurs when $y = \pm\, 1$, with analogous solutions for x.

example

Hamming asks us to find the angles at which two curves meet in the first quadrant.

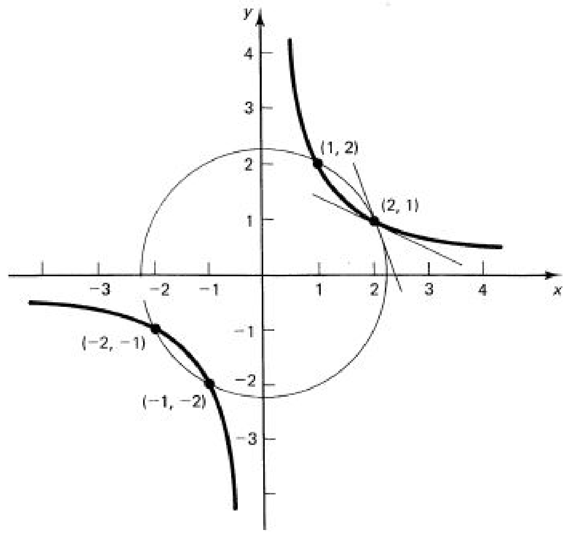


Figure 8.3-1 Angle between a circle and a hyperbola

The first step is to find the points where the curves meet.

$$x^2 + y^2 = 5$$
$$xy = 2$$

so

$$x^2 + (\frac{2}{x})^2 = 5$$
$$x^4 + -5x^2 + 4 = 0$$
$$x^2 = \frac{5 \pm \sqrt{25 - 16}}{2}$$
$$= \frac{5 \pm 3}{2} = 1, 4$$
$$x = 1, 2$$

The corresponding points are $(1, 2)$ and $(2, 1)$.

We have missed two other solutions. This happened when we took the positive square root of x at the last step. However, we are only interested in the first quadrant, so we can ignore those two for now.

Hamming solves this a bit differently, instead, he says to add and subtract $2xy = 4$ from the equation of the circle

$$x^2 + 2xy + y^2 = 9$$

$$(x + y)^2 = 9$$

$$x + y = \pm\, 3$$

and

$$x^2 - 2xy + y^2 = 1$$

$$(x - y)^2 = 1$$

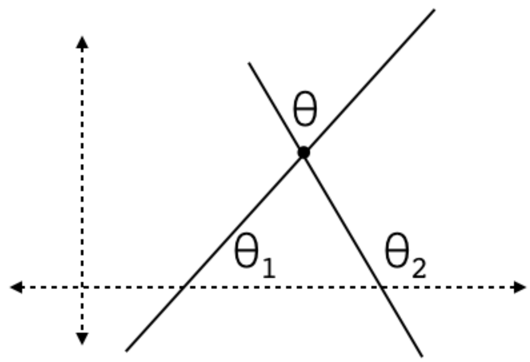
$$x - y = \pm\, 1$$

This leads to the same two answers in the first quadrant: $(2, 1)$ and $(1, 2)$.

We're not done yet. We need the angles.

angle between two lines

Consider two lines with slopes m_1 and m_2 which make angles θ_1 and θ_2 with the positive x-axis. They meet at a point, forming the angle θ, which we take to be the angle through which the first line is rotated counter-clockwise to meet the second.



You should be able to show that $\theta_1 + \theta = \theta_2$.

Look at the triangle formed by the two lines and the x-axis. The (unlabeled) upper angle is equal to θ, by the vertical angles theorem. Therefore $\theta + \theta_1$ plus the (unlabeled) angle at the right side of the triangle add up to 180 degrees. But θ_2 plus this angle is also equal to 180 degrees, by the supplementary angles theorem.

We have then

$$\theta_1 + \theta = \theta_2$$
$$\theta = \theta_2 - \theta_1$$

The slope of a line is equal to the tangent of the angle the line makes with the x-axis.

We'll see the connection in a bit, for now, we just convert the above relationship between angles with what we know in the original problem, slopes, by taking the tangent of both sides:

$$\tan\theta = \tan(\theta_2 - \theta_1)$$

We need the sum (difference) of angles formula for the tangent.

sum of angles, tangent

From the sum of angles for sine and cosine we can write

$$\tan s + t = \frac{\sin s + t}{\cos s + t} = \frac{\sin s \cos t + \cos s \sin t}{\cos s \cos t - \sin s \sin t}$$

Dividing by $\cos s \cos t$ we obtain

$$\tan s + t = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$

Exchanging $-t$ for t changes the sign of the tangent

$$\tan s - t = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

We had

$$\tan \theta = \tan(\theta_2 - \theta_1)$$

so

$$\tan \theta = \tan(\theta_2 - \theta_1) = \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_1 \tan \theta_2}$$

finishing up

We said that the slope of a line is equal to the tangent of the angle the line makes with the x-axis. Therefore, we can substitute slopes for tangents

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

And now we're finally ready to solve the problem we were originally given. We use implicit differentiation to obtain the slopes of the circle and the ellipse, both of which we did earlier.

Suppose we call the hyperbola curve 1

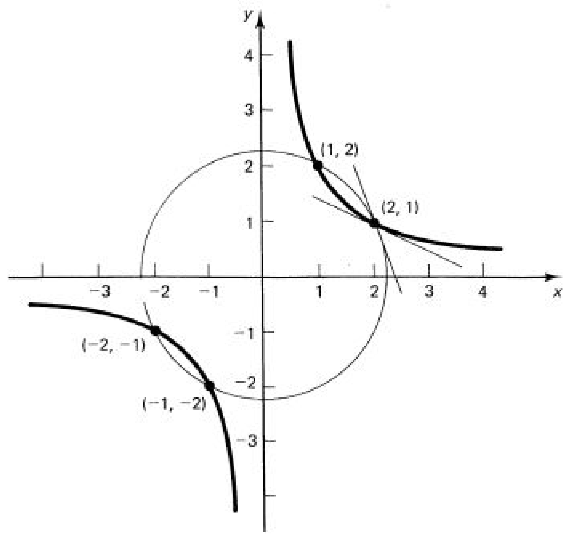


Figure 8.3-1 Angle between a circle and a hyperbola

It has slope (at the point $(2, 1)$):

$$m_1 = y' = -\frac{2}{x^2} = -\frac{1}{2}$$

The circle at the point $(2, 1)$ has slope:

$$m_2 = y' = -\frac{x}{y} = -2$$

This is quite reasonable, if you look at the diagram. So then

$$\tan\theta = \frac{-2 - -1/2}{1+1} = -\frac{3}{4}$$

That's about $-37°$.

The minus sign comes about because we rotate curve 1 (the hyperbola) *clockwise* to meet curve 2 (the circle).

Chapter 44

Logarithmic differentiation

Now that we know the definition

$$\ln x = \int \frac{1}{x}\ dx + C$$

by the fundamental theorem of calculus:

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

We can extend this to a general $f(x)$ using the chain rule

$$\frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)}\ f'(x)$$

Rearrange:

$$\frac{f'(x)}{f(x)} = (\ln(f(x))'$$

or

$$\frac{y'}{y} = \ [\ln y]'$$

Check this with a simple example

$$y = x^2$$

$$y' = 2x$$

$$2x = x^2 \ [\ \frac{d}{dx} \ln x^2 \ ]$$

$$2x = x^2 \ [\ \frac{1}{x^2} \ 2x \ ]$$

Slightly different statement

Write

$$y = f(x)$$

$$\ln y = \ln f(x)$$

Now differentiate implicitly. Imagine that x (and y) are functions of t and differentiate with respect to t. By the chain rule:

$$\frac{1}{y} \ \frac{dy}{dt} = \frac{1}{f(x)} \ f'(x) \ \frac{dx}{dt}$$

Lose the dt:

$$\frac{1}{y} \ dy = \frac{1}{f(x)} \ f'(x) \ dx$$

$$(\ln y)' = \frac{1}{f(x)} \ f'(x)$$

This is a slight variation of what we had above. We can see that it works by a bit of manipulation. We had

$$\frac{1}{y} \ dy = \frac{1}{f(x)} \ f'(x) \ dx$$

$$\frac{1}{y} \ dy = \frac{1}{y} \ y' \ dx$$

$$\frac{1}{y} \ \frac{dy}{dx} = \frac{1}{y} \ y'$$

$$\frac{dy}{dx} = y'$$

But of course.

example

A new problem using this result is to find the derivative of x^x, where the standard methods don't work.

Take the logarithm:

$$\ln(x^x) = x \ln x$$

Now, take the derivative of the logarithm of $f(x)$:

$$1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

Use the rule:

$$y' = y \ \frac{d}{dx} \ln y$$

$$= x^x \ (\ln x + 1)$$

example

In general, any time we have a power of x that is itself a function, we need logarithmic differentiation.

$$y = x^{\cos x}$$

$$\ln y = \cos x \ln x$$

$$\frac{1}{y} \ dy = \ [\ \frac{\cos x}{x} - \sin x \ln x \ ] \ dx$$

$$\frac{dy}{dx} = x^{\cos x} \ [\ \frac{\cos x}{x} - \sin x \ln x \ ]$$

power rule

An important result is to prove the power rule for all real x. We want

$$\frac{d}{dx}x^n$$

where n is not just a positive or negative integer ($\neq -1$), but can be any real number, like π or e or $\sqrt{2}$. Straight-up implicit differentiation is easier for all but the irrational numbers, but it's a subtle thing to talk about real numbers as the limits of rational numbers. We combine logarithmic and implicit differentiation

Write

$$y = x^n$$

$$\ln y = n \ \ln x$$

Imagine both x and y as functions of t with $x = t$, then

$$\frac{d}{dt} \ \ln y = \frac{d}{dt} \ n \ \ln x$$

But n is now *just a constant* so

$$\frac{d}{dt} \ \ln y = n\frac{d}{dt} \ \ln x$$

$$\frac{1}{y} \ \frac{dy}{dt} = n\frac{1}{x} \ \frac{dx}{dt}$$

$$\frac{1}{y} \ dy = n\frac{1}{x} \ dx$$

$$\frac{dy}{dx} = n\frac{y}{x}$$

$$= n\frac{x^n}{x}$$

$$= nx^{n-1}$$

Did you see what just happened! We proved the power rule for real n in a few simple lines. Wow.

example

$$\frac{d}{dx}x^e = e\ x^{e-1}$$
$$\frac{d}{dx}x^\pi = \pi\ x^{\pi-1}$$

example

The following problem is an example of turning calculus into arithmetic, an approach in high school courses which I really dislike. But it's really common to see such problems there so we might as well do one.

Differentiate

$$y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$$

We could use the product, quotient and chain rules for this (and we can be sure it will be messy in the end). An alternative is to use the properties of the logarithm to break the right-hand side into a polynomial. Our formula from above was

$$\frac{y'}{y} = \ [\ln y]'$$

Take the logarithm of both sides for the problem

$$\ln y = \ln(\frac{x^5}{(1-10x)\sqrt{x^2+2}})$$

$$\ln y = \ln(x^5) - \ln(1-10x) - \ln(\sqrt{x^2+2})$$

Now, when we differentiate, it is really implicit differentiation. The three terms are

$$\frac{1}{x^5} 5x^4 \ dx = \frac{5}{x} \ dx$$

$$-\frac{1}{(1-10x)}(-10) \ dx = \frac{10}{(1-10x)} \ dx$$

$$-\frac{1}{\sqrt{x^2+2}} \cdot \frac{1}{2\sqrt{x^2+2}} 2x \ dx = -\frac{x}{x^2+2} \ dx$$

On the left we get (including the dx from the terms on the right-hand side):

$$\frac{1}{y}\frac{dy}{dx} = \frac{y'}{y}$$

$$= \frac{5}{x} + \frac{10}{(1-10x)} - \frac{x}{x^2+2}$$

To finish the problem, we need to multiply through by y

$$y' = \left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)\left(\frac{5}{x} + \frac{10}{(1-10x)} - \frac{x}{x^2+2}\right)$$

which I won't try to simplify.

Chapter 45

L'Hospital

origin

The rule discussed below is attributed to l'Hospital, which is also often written l'Hôpital. The latter spelling is modern French usage, while the one we use (because it is easier to typeset in titles) is English usage. It also happens to be the way l'Hospital spelled his name himself, back in the day.

In any event, this rule was actually discovered by Johann Bernoulli. According to Stewart, these two mathematicians had entered into a business arrangement whereby the Marquis de l'Hospital bought the rights to Bernoulli's mathematical discoveries.

http://www.stewartcalculus.com/data/ESSENTIAL%20CALCULUS%20Early%20Transcendentals/upfiles/projects/ecet_wp_0307_stu.pdf

According to wikipedia,

> In the 17th and 18th centuries, the name was commonly spelled "l'Hospital", and he himself spelled his name that way. However, French spellings have been altered: the silent 's' has been removed and replaced with the circumflex over

the preceding vowel (l'Hôpital). The former spelling is still used in English where there is no circumflex.

limit of a quotient

We are trying to determine the limit of the quotient of two functions

$$\lim_{x\to c} \frac{f(x)}{g(x)}$$

and suppose that we run into trouble because both functions have problems at c. If

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$$

or

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty$$

or

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = -\infty$$

and if

$$\lim_{x\to c} \frac{f'(x)}{g'(x)}$$

exists

Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

Here's a simple example. What is

$$\lim_{x\to 0} \frac{\sin x}{x}$$

We *know* this. it is equal to 1. But $\sin 0 = 0$ so take derivatives

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

If the limit is still of the form 0/0 just repeat the process.

basic proof

If

- f and g are well-behaved (continuously differentiable)
- The first differentiation yields a finite limit as $x \to c$ and
- The form of the quotient is 0/0, with $f(c) = g(c) = 0$

then

$$\lim_{x\to c} \frac{f(x)}{g(x)}$$

$$= \lim_{x\to c} \frac{f(x) - f(c)}{g(x) - g(c)}$$

$$= \lim_{x\to c} \frac{[\ f(x) - f(c)\ ]/(x-c)}{[\ g(x) - g(c)\ ]/(x-c)}$$

$$= \frac{\lim_{x\to c} [\ f(x) - f(c)\ ]/(x-c)}{\lim_{x\to c} [\ g(x) - g(c)\ ]/(x-c)}$$

$$= \frac{f'(x)}{g'(x)}$$

application to the exponential

We will develop a proof that

$$e = \lim_{n\to\infty} (1 + \frac{1}{n})^n$$

using l'Hospital's Rule.

This interesting approach is from the book *Mooculus*. Remember that n is a variable in this section.

The first thing is to rewrite this as an exponential, first for the term in parentheses:

$$1 + \frac{1}{n} = e^{\ln(1+1/n)}$$

and then for the whole thing

$$(1 + \frac{1}{n})^n = (e^{\ln(1+1/n)})^n$$
$$= \ e^{n\ln(1+1/n)}$$

To evaluate the limit, we need to evaluate the limit of the exponent

$$\lim_{n\to\infty} n\ln(1 + \frac{1}{n})$$

At first, it doesn't look like we can use the rule (there is no quotient), but there is a standard trick for these situations. Just rearrange to divide by the inverse

$$= \lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}}$$

Both the top and the bottom limits are easily evaluated to be equal to 0, so we can apply the rule.

We need to evaluate

$$= \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

The derivative of the numerator is (by the chain rule)

$$f'(n) = \frac{-n^{-2}}{1 + 1/n}$$

while the denominator is just

$$g'(x) = -n^{-2}$$

The factor of $-n^{-2}$ cancels from both top and bottom, leaving us with

$$\lim_{n\to\infty} \frac{1}{1 + 1/n} = 1$$

That is:

$$\lim_{n\to\infty} n \ \ln(1 + \frac{1}{n}) = 1$$

Going back to the original problem then

$$\lim_{n\to\infty} \ (1 + \frac{1}{n})^n = \lim_{n\to\infty} \ e^{n \ln(1+1/n)}$$

$$= e^1 = e$$

exponential function

A parallel argument can be used to prove that

$$e^x = \lim_{n\to\infty} \ (1 + \frac{1}{n})^{nx} = \lim_{n\to\infty} (1 + \frac{x}{n})^n$$

with the assistance again of l'Hospital's Rule.

What we need to show is that we can bring x inside the parentheses

$$e^x = \lim_{n\to\infty} (1 + \frac{x}{n})^n$$

Taking logarithms, this is the same as saying that

$$x = \lim_{n\to\infty} n \cdot \ln(1 + \frac{x}{n})$$

As before, we form the quotient

$$\frac{\log(1 + x/n)}{1/n}$$

Use the chain rule to obtain the derivative of the numerator — remember that n is the variable:

$$\frac{1}{1 + x/n} \; (-\frac{x}{n^2})$$

The derivative of the denominator is $-1/n^2$ so we can cancel

$$\frac{f'(x)}{g'(x)} = \frac{1}{1 + x/n} \; (x)$$

We take the limit

$$\lim_{n\to\infty} \frac{1}{1 + x/n} \; (x) = x$$

Part XIV

Applications

Chapter 46

Maximum likelihood

A nice application of logarithmic differentiation is of a set of Bernoulli trials, like a series of coin flips where the coin isn't fair, but instead has a probability p of coming up heads (H) and $1 - p$ of coming up tails (T).

Note that this classic example is actually impossible to achieve. One cannot "weight" a coin to do this, though it is easy with a die (singular of dice).

http://www.stat.columbia.edu/~gelman/research/published/diceRev2.pdf

Now, p is unknown, but we have some data about how the coin performs, and we wish to use the data to estimate p by the method of maximum likelihood. We observe this sequence of trials:

$$HTHHTTTHTHHH$$

Theory says that the probability of observing this sequence of events is dependent on p in the following way:

$$p(1-p)pp(1-p)(1-p)(1-p)p(1-p)ppp = p^7(1-p)^5$$

We call the probability of observing this data, given some underlying probability model p, the likelihood L:

$$L(p) = p^7(1-p)^5$$

(It's called the likelihood, because the total probability is not equal to 1).

In general, since each trial is independent and identically distributed

https://en.wikipedia.org/wiki/Independent_and_identically_distributed_random_variables

we can write that for n trials and k successes we would have

$$L(p) = p^k(1-p)^{n-k}$$

Here, n and k are constants for any particular sequence, but we would like to have the general formula.

To find the maximum for L we differentiate and set that equal to 0.

$$\frac{d}{dp}L = 0$$

However, we note that since $\ln L$ increases and decreases along with L, the value of p that gives a maximum for L also gives a maximum for $\ln L$. So we will take the logarithm of L and set that equal to zero:

$$\frac{d}{dp}\ln L = 0$$

$$\ln L = k \ln p + (n-k)\ln(1-p)$$

Take the derivative d/dp of both sides (we get a minus sign from the chain rule):

$$\frac{d}{dp}\ln L = 0 = \frac{k}{p} - \frac{n-k}{1-p}$$

$$\frac{k}{p} = \frac{n-k}{1-p}$$

Multiply through by $1-p$ and also by $1/k$:

$$\frac{1-p}{p} = \frac{n-k}{k}$$

$$\frac{1}{p} = \frac{n}{k}$$

$$p = \frac{k}{n}$$

As we might have guessed, the maximum likelihood estimate of p is simply the ratio of the observed number of successes to the number of trials: k/n.

Chapter 47

Optimization of area

A general optimization problem expresses some dependence as a function, e.g. $A = f(x)$, where $f(x)$ is moderately complicated. We wish to find the value of x that gives a maximum (or a minimum) for A, perhaps within some limited domain of x, or sometimes over all possible values of x.

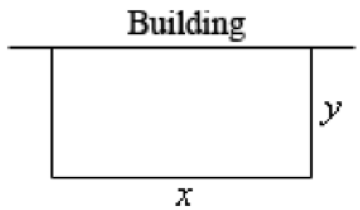
Usually, the first part is to construct the function $f(x)$, using some constraint that is given in the problem statement. Then, the basic method is to find the first derivative A' and set it equal to zero, and solve for x.

There is no shortage of fun (and challenging) problems of this type. We did the rectangular area problem earlier in the chapter on higher derivatives.

In this chapter we do problems involving area or volume. The next chapter has some additional examples.

Three-sided fence

We wish to build a fence, using an existing barn for one of the sides, so we need fencing only on three sides. The total length of available fencing is 500 ft. This is the constraint.



The hard way to solve this is to use the constraint to express y in terms of x:

$$500 = 2y + x$$
$$y = \frac{500 - x}{2}$$

Now write the area as

$$A = xy = 250x - \frac{1}{2}x^2$$

Take the first derivative and set it equal to zero:

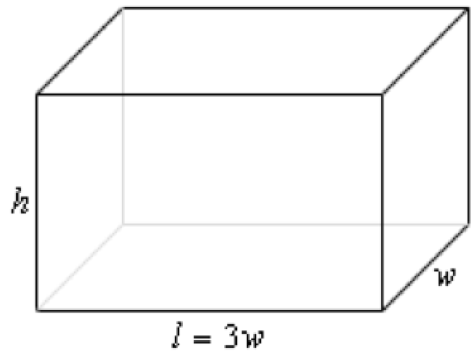
$$A' = 250 - x = 0$$

Clearly, $x = 250$ and $y = 125$.

The easy way (Nahin) is to imagine that we enclose *another* identical rectangular area on the other side of the barn. From the first problem it is known that the two areas together should form a square. Hence, we obtain the half-square as the answer.

Box with an expensive top

We wish to build a box. The box is unusual in that the cost of the top and bottom is more than the sides (10 v. 6 per unit area — let's say it is in square feet but that doesn't really matter).



As noted in the figure, the length of the side is three times the width. We wish to minimize the cost.

The constraint is that the total volume of the box is 50 cubic feet. Using the constraint, we can solve for h in terms of w

$$
\begin{aligned}
50 &= 3wwh \\
h &= \frac{50}{3w^2}
\end{aligned}
$$

The cost C is

$$
\begin{aligned}
C &= 2\ [\ 10 \cdot 3w^2 + 6 \cdot wh + 6 \cdot 3wh\ ] \\
&= 2\ [30w^2 + 24wh\ ] \\
&= 60w^2 + \frac{800}{w}
\end{aligned}
$$

Take the first derivative and set it equal to zero:

$$
C' = 120w - \frac{800}{w^2} = 0
$$

$$
w^3 = \frac{800}{120}
$$

$$w = (\frac{800}{120})^{1/3} = 1.88$$

Box with maximum volume

This problem features a box with a square base and a total surface area of S. We wish to maximize the volume of the box.

The constraint is the surface area (plus the fact of the square base). If b is the base length and h is the height, we have that

$$S = 2b^2 + 4bh$$
$$h = \frac{S - 2b^2}{4b}$$

The volume is

$$V = b^2 h$$
$$= b^2 \ \frac{S - 2b^2}{4b}$$
$$= \frac{S}{4}b - \frac{1}{2}b^3$$

Take the first derivative and set it equal to zero:

$$V' = \frac{S}{4} - \frac{3}{2}b^2 = 0$$
$$S = 6b^2$$
$$b = \sqrt{\frac{S}{6}}$$

We can also find h

$$S = 2b^2 + 4bh$$
$$S = 2\frac{S}{6} + 4h\sqrt{\frac{S}{6}}$$

$$\frac{1}{3}S = 2h\sqrt{\frac{S}{6}}$$
$$h = \frac{1}{\sqrt{6}}\sqrt{S} = b$$

Since $b = h$, what we have is a cube. Not that surprising.

Cylindrical can with maximum volume for its surface area

A cylindrical can is to be formed with a volume of 1.5 cubic liters. What are the dimensions if we wish to minimize the surface area (materials used for construction)?

Suppose that the radius is r and the height is h. The formula for volume tells us that

$$V = 1.5 = \pi r^2 h$$

(Note: the linear dimensions of this volume, and of r, are in tenths of a meter, since a liter is a cubic decimeter—one-tenth of a meter).

The surface area is

$$A = 2\pi r^2 + 2\pi r h$$

Substituting for h

$$A = 2\pi r^2 + 2\pi r(\frac{1.5}{\pi r^2})$$
$$= 2\pi r^2 + \frac{3}{r}$$

Take the first derivative and set equal to zero:

$$A' = 4\pi r - \frac{3}{r^2} = 0$$
$$r^3 = \frac{3}{4\pi}$$

$$r = (\frac{3}{4\pi})^{1/3} = 0.620$$
$$h = \frac{1.5}{\pi r^2} = 1.24$$

Multiply by 10 to get the r and h in centimeters.

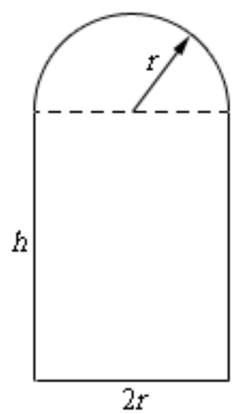
$$r = 6.2 \text{ cm}$$

$$h = 12.4 \text{ cm}$$

It's no accident that $h = 2r$.

Fancy window

We wish to construct this window, with a semi-circular arch added on top of the rectangular region below.



The quantity to maximize is the area of the window. The amount of framing for the window is the constraint, with a total length equal to 12. Strangely, the framing does not include the dotted line. Using the constraint, we can solve for h in terms of r:

$$2h + 2r + \pi r = 12$$

$$h = \frac{12 - (2+\pi)r}{2}$$

The area is

$$\begin{aligned} A &= 2rh + \frac{1}{2}\pi r^2 \\ &= 12r - (2+\pi)r^2 + \frac{1}{2}\pi r^2 \\ &= 12r - 2r^2 - \frac{1}{2}\pi r^2 \end{aligned}$$

We take the first derivative and set it equal to zero:

$$A' = 12 - 4r - \pi r = 0$$

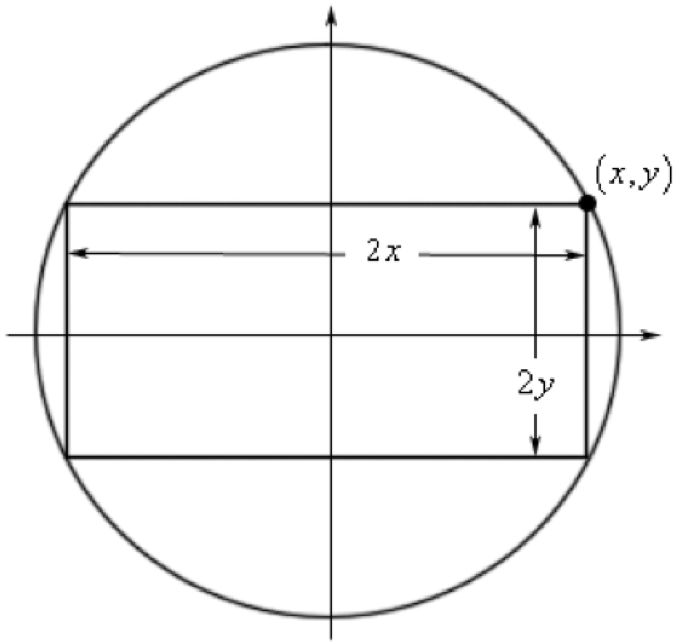
$$r = \frac{12}{4+\pi} = 1.68$$

$$h = \frac{12 - (2+\pi)r}{2} = 1.68$$

That's an interesting result! We should probably revise our drawing, and the client should probably think of a different kind of window.

Rectangle in a circle

Suppose we have a circle of fixed radius R, centered at the origin. Pick a value for x such that $0 \leq x \leq R$. Form the rectangle with all four vertices on the circle. That is, for the y value corresponding to that x, the vertices are $\pm\, x, \pm\, y$.



What this means is that given a particular x

$$y = \sqrt{R^2 - x^2}$$

and therefore the sides of the rectangle are $2x$ and $2\sqrt{R^2 - x^2}$, with area

$$A = xy = 4x\sqrt{R^2 - x^2}$$

We wish to find the value of x which gives the maximum area. We will take the first derivative and set it equal to zero. But, to begin with

$$\frac{d}{dx}\ \sqrt{R^2 - x^2} = -\frac{1}{2}\frac{2x}{\sqrt{R^2 - x^2}}$$

$$= -\frac{x}{\sqrt{R^2 - x^2}}$$

so, using the product rule:

$$A' = (4x)(-\frac{x}{\sqrt{R^2 - x^2}}) + 4(\sqrt{R^2 - x^2}) = 0$$

$$= -\frac{x^2}{\sqrt{R^2 - x^2}} + \sqrt{R^2 - x^2} = 0$$
$$x^2 = R^2 - x^2$$

and since $x^2 + y^2 = R^2$

$$x^2 = x^2 + y^2 - x^2 = y^2$$

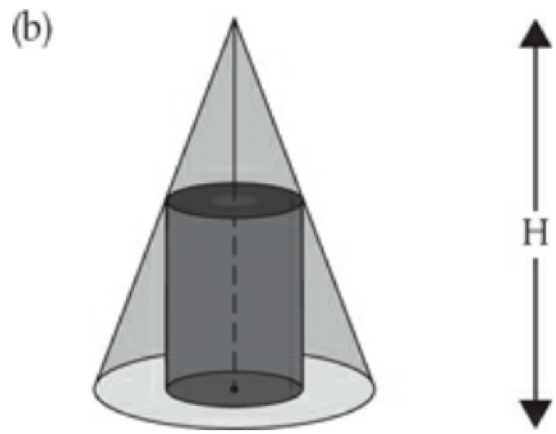
Thus, $x = y$. So the maximum area is for a square. No longer a surprise, I trust.

Mrs. Sidway's problem

This actually a volume problem, but whatever.

In Acheson we find a problem published in a 1714 edition of *The Ladies Diary: or The Woman's Almanack*, posed by a Mrs. Sidway (as a gardening problem, no less).

66. (a) The *Ladies Diary*. (b) Mrs. Sidway's problem.



If a cylinder lies contained within a cone (with its upper edge lying along the surface of the cone), how large should the radius be in order for the cone to have the maximum volume?

I find it easier to think about this problem by first placing the origin at the left of the figure, before moving it to the center for the second part.

The cone has fixed height H and radius R. If we place the origin at the left edge and allow a variable x to range from this origin to the right up to a distance R, you should be able to see first, that for any x

$$\frac{h}{x} = \frac{H}{R}$$

by similar triangles. This is a standard relationship for the cone.

But the radius of the cylinder, measured from the central axis of both cylinder and cone is

$$r = R - x = R - h\frac{R}{H}$$

Rearranging

$$h = \frac{H}{R}(R - r)$$

At this point, we compute the volume as

$$V = Ah = 2\pi r^2 \cdot \frac{H}{R}(R - r)$$

V depends on r and we can take the derivative with respect to r and then set it equal to zero in order to find the maximum.

When we do this, the constants (2π) will carry through into the first derivative and then disappear since we set the result equal to zero. Thus, we can ignore them

$$
\begin{aligned}
V &\approx r^2 \cdot \frac{H}{R}(R - r) \\
&= r^2 \cdot (H - \frac{H}{R}r) \\
&= Hr^2 - \frac{H}{R}r^3
\end{aligned}
$$

The derivative is

$$
\begin{aligned}
0 &= 2Hr - 3\frac{H}{R}r^2 \\
2r &= 3\frac{r^2}{R} \\
\frac{2}{3}R &= r
\end{aligned}
$$

The maximum volume is obtained with r equal to two-thirds the radius of the cone at the bottom.

Chapter 48

Other optimizations

Projectile range

Suppose we fire a cannon where the ball has velocity v at an angle θ with the horizon (straight up would be $\pi/2$ radians). We wish to determine the angle that will give the maximum range. We did this problem in a physics section, but it won't hurt to look at it again here.

The problem has a trick, namely that the distance in the horizontal or x-direction depends on the time (and therefore distance) in the y-direction, since when $y = 0$, the cannonball will fall to earth and not move any more.

If you draw a diagram you will see that

$$v_y = v \sin\theta$$

where v_y is the initial velocity in the y-direction.

The basic equation of motion under gravity is that

$$y = v_y t - \frac{1}{2} g t^2$$

with $g = 32$ so

$$y = v_y t - 16t^2$$

At the point of interest $y = 0$ so

$$0 = v_y t - 16t^2$$
$$v_y t = 16t^2$$

This has two solutions, namely $t = 0$ (not what we are interested in) and

$$v_y = 16t$$
$$t = \frac{v_y}{16}$$
$$= \frac{v \sin\theta}{16}$$

On the other hand, the quantity we are really interested in is the distance in the x-direction. Similarly to v_y,

$$v_x = v \cos\theta$$
$$x = v_x t = v \cos\theta \; \frac{v \sin\theta}{16}$$
$$= \frac{v^2}{16} \; \cos\theta \sin\theta$$

This looks a little strange but all it really says is that the range is a function of the angle θ (and also of the square of the velocity). If you do dimensional analysis at this point you might also be confused unless you remember that the factor of 16 has units of meters per second squared.

We take the first derivative and set it equal to zero:

$$x' = \frac{v^2}{16} \; (\cos^2\theta - \sin^2\theta) = 0$$

$v = 0$ gives a solution to this equation, but it's not the solution we want. We want the solution given by

$$\cos^2\theta - \sin^2\theta = 0$$

The velocity and the gravitational constant have dropped out, which makes sense. It makes intuitive sense that the angle for maximum range (given a velocity), should not depend on that velocity.

The above expression is zero when $\cos\theta = \sin\theta$. If you don't see this you can say:

$$\begin{aligned}
1 - \sin^2\theta - \sin^2\theta &= 0 \\
\sin^2\theta &= \frac{1}{2} \\
\sin\theta &= \frac{1}{\sqrt{2}} \\
\theta &= \frac{\pi}{4}
\end{aligned}$$

An elevation of 45 degrees gives the maximum range.

Closest point to a parabola

Suppose we consider the simple parabola

$$y = x^2$$

Our problem is to find the point(s) (x, y) on the parabola that have the shortest distance to $P = (0, 1)$.

One possibility is that $(0, 0)$ is the minimum. But it will turn out that it is not, and so there will be two such points, which are symmetrical about the $y-$axis. Therefore, we consider only $x \geq 0$.

The distance from any point (x, y) to $P = (0, 1)$ is

$$d = \sqrt{(0 - x)^2 + (1 - y)^2}$$

It is the case that if we minimize d^2, we also minimize d, so let's rewrite the equation as

$$D = (0 - x)^2 + (1 - y)^2$$
$$D = x^2 + 1 - 2y + y^2$$

Now, the constraint is that $y = x^2$ so plugging in we get

$$D = y + 1 - 2y + y^2$$
$$= 1 - y + y^2$$

Take the first derivative (with respect to y) and set it equal to zero:

$$D' = -1 + 2y = 0$$
$$y = \frac{1}{2}$$
$$x = \frac{1}{\sqrt{2}}$$

Check the actual distance:

$$d = \sqrt{(0 - x)^2 + (1 - y)^2}$$
$$= \sqrt{(\frac{1}{\sqrt{2}})^2 + (1 - \frac{1}{2})^2}$$

$$= \sqrt{\frac{1}{2} + \frac{1}{4}}$$
$$= \frac{\sqrt{3}}{2}$$
$$= 0.866$$

$$(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{2})$$

Note that $(1/\sqrt{2}, 1/2)$ is closer to $(0, 1)$ than is $(0, 0)$, as we said.

The slope of the line from $(0, 1)$ to our point $(1/\sqrt{2}, 1/2)$ is

$$\frac{\Delta y}{\Delta x} = \frac{1 - 1/2}{1/\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

The slope of the tangent to the parabola is $2x$, and at $(1/\sqrt{2}, 1/2)$ it is

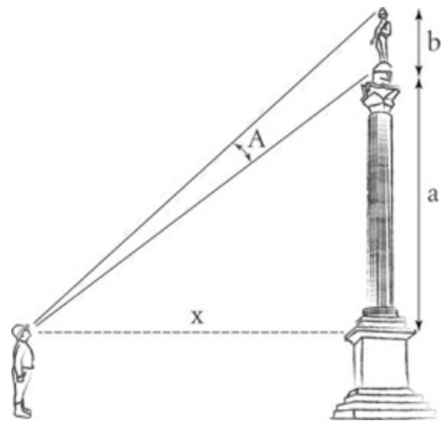
$$m = 2x = 2\frac{1}{\sqrt{2}} = \sqrt{2}$$

Since the product of the slopes is -1, the line corresponding to the minimum distance is perpendicular to the tangent.

The next two problems are about maximizing angles. They are variants with a slight twist.

Nelson's column

Nelson's column is a column with a statue of Nelson on top, naturally. The hero of Trafalgar is honored at London's Trafalgar square. Here is Acheson's sketch:



Somehow we need to maximize the angle A as a function of x, but we are given values that form the tangent of two angles.

However, we know that for θ in the half-open interval $[0, \pi/2)$, as θ increases so does $\tan\theta$. Therefore, if we maximize $\tan\theta$, θ will also be a maximum. This is a standard trick to remember.

Let's use s for the entire angle and t for the lower triangle, then

$$A = s - t$$

and the lengths we are given can be used to express the tangents of s and t.

We derived $\tan s - t$ before, and can do it again:

$$\tan s - t = \frac{\sin s - t}{\cos s - t}$$

$$= \frac{\sin s \cos t - \cos s \sin t}{\cos s \cos t + \sin s \sin t}$$

$$= \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

Plugging in:

$$\tan A = \frac{\frac{a+b}{x} - \frac{a}{x}}{1 + \frac{a(a+b)}{x^2}}$$

The numerator simplifies to b/x, so let's keep the b on top and multiply on the bottom by x

$$\tan A = \frac{b}{x + a(a+b)/x}$$

I got a bit of a mess trying to work with this as it is, so I multiplied by x on top and bottom a second time:

$$\tan A = \frac{bx}{x^2 + a(a+b)}$$

Use the familiar quotient rule to take the derivative and set it equal to zero:

$$0 = \frac{b\ [\ x^2 + a(a+b)\ ] - bx(2x)}{[\ x^2 + a(a+b)\ ]^2}$$

This occurs when the numerator is zero so discard the denominator!

$$0 = b\ [\ x^2 + a(a+b)\ ] - bx(2x)$$

Factor out the b, put the two terms on opposite sides and obtain

$$x^2 + a(a+b) = 2x^2$$

$$x^2 = a(a+b)$$

This is the solution given in Acheson. Furthermore, the statue is fairly small compared to the column's height. If we let $b << a$ then

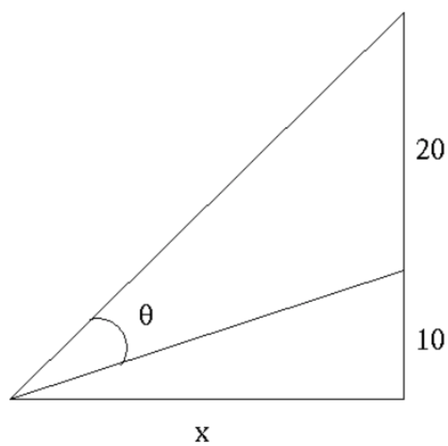
$$x^2 \approx a^2$$

$$x \approx a$$

The appropriate viewing angle is 45 degrees and since the statue is about 169 feet high that would put you in the middle of traffic unless you're quite careful.

movie screen

A movie screen on a wall is 20 feet high and 10 feet above the floor. At what distance x from the front of the room should you position yourself so that the viewing angle θ of the movie screen is as large as possible ?



$$\tan s - t = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

Plugging in the values provided:

$$\tan\theta = \tan s - t$$

$$= \frac{30/x - 10/x}{1 + 300/x^2}$$

$$= 20\ \frac{x}{x^2 + 300}$$

We take the derivative and set it equal to zero. We can ignore the leading factor of 20, obtaining

$$0 = \frac{x^2 + 300 - 2x^2}{(x^2 + 300)^2} = \frac{-x^2 + 300}{(x^2 + 300)^2}$$

This is equal to zero when the numerator is zero, that is, when

$$x = \pm\sqrt{300}$$

Since x is a distance we take the positive square root.

$$x = \sqrt{300} = 10\sqrt{3}$$

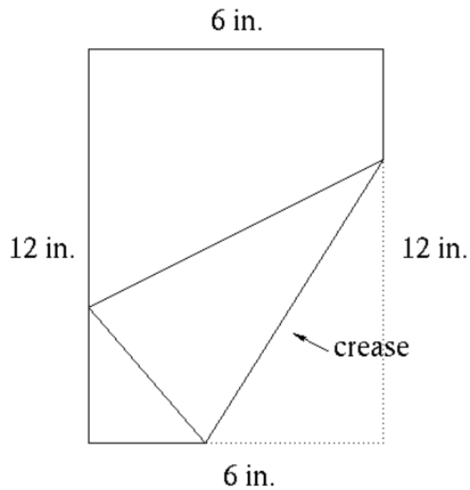
The angles are worth working out. The tangent of the lower angle t is $1/\sqrt{3}$. This is a right triangle with hypotenuse equal to 2 and sine equal to $1/2$. Therefore $t = \pi/6$.

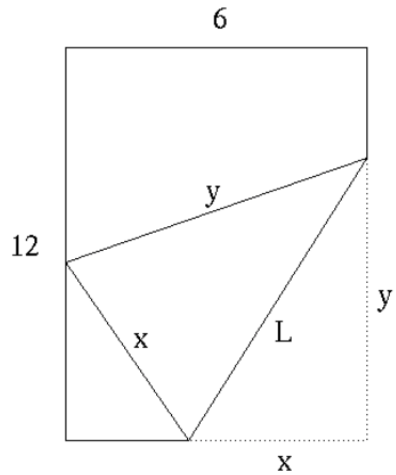
The tangent of the entire angle s is equal to $3/\sqrt{3} = \sqrt{3}$. This is a right triangle with hypotenuse equal to 2 and cosine equal to $1/2$. Therefore $s = \pi/3$.

Therefore the angle to the screen $\theta = s - t$ at the maximum is $\pi/6$ or 30 degrees.

folded paper

Consider a piece of paper with the dimensions 6×12. We pick up the lower right-hand corner and place it against the left side, folding to form a crease.

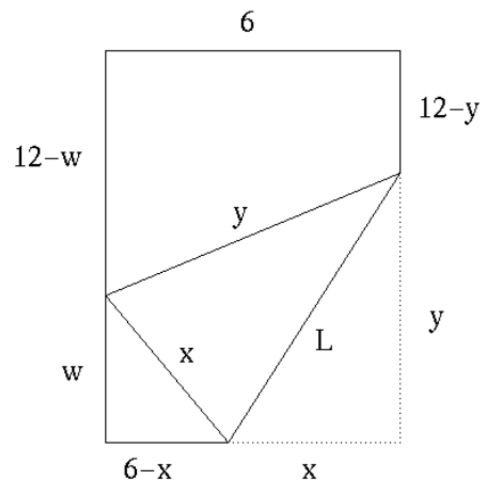




The possible positions on the left-hand side to place that corner range from the bottom up to a distance 6 inches above the bottom. The length of the crease is a variable, and and we wish to find the crease with the minimum length.

The variable distances can be labeled as shown on the right. The length $L^2 = x^2 + y^2$.

I notice that the drawing is not properly scaled (the long dimension is too short). In fact, the angle that the folded part makes on the left-hand side is a right angle. After all, it is a corner of the original sheet, but also we have two congruent triangles, they must both be right triangles.



At this point, I notice that the problem can be re-scaled so the length of the paper is 2 and the width is 1, in order to simplify the arithmetic. I have not re-done the drawings to reflect that yet, but our math will take advantage of it.

The range of the distance x is $[1/2, 1]$, while that for $y = [1, 2]$.

We need to find a relationship between x and y. Start by labeling another variable distance w, as shown above.

The connection that we need can be found by relating x, y and w to the total area of the paper. We have two right triangles with sides x and y and total area xy.

The other triangle has

$$w^2 = x^2 - (1-x)^2$$

$$= 2x - 1$$

$$w = \sqrt{2x-1}$$

and area (we leave w as it is for now).

$$\frac{1}{2}(1-x)w$$

We will need it later, so let's get the derivative of w with respect to x

$$\frac{dw}{dx} = \frac{1}{\sqrt{2x-1}} = \frac{1}{w}$$

Last, we have a rhombus. The average of the two vertical sides is

$$\frac{1}{2}(2 - w + 2 - y) = 2 - \frac{1}{2}(w + y)$$

and since the horizontal side is 1, this is also equal to the area.

area calculation

From the dimensions of the paper, the total area is 2 and this is equal to the three triangles and the rhombus added together

$$2 = xy + \frac{1}{2}(1-x)w + 2 - \frac{1}{2}(w + y)$$

$$4 = 2xy + (1-x)w + 4 - w - y$$

Cancel the 4 and gather terms with y. The left-hand side is

$$y - 2xy = -y(2x - 1) = -yw^2$$

So

$$-yw^2 = (1-x)w - w$$

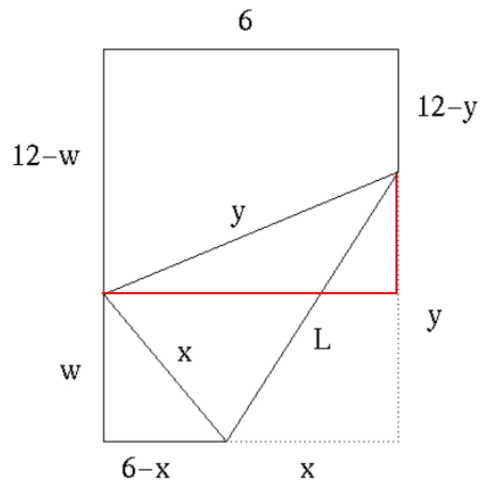
Factor out one w

$$-yw = (1-x) - 1 = -x$$

$$y = \frac{x}{w}$$

That's a nice simplification. We can check that this is correct at one extreme. When $x = w = 1$ the ratio is $1 = y$ and we see that is correct for the fold at 45 degrees. At the other extreme we have $w = 0$ and the ratio is undefined.

And only now do I see that this calculation was unnecessary. Draw the horizontal



Can you see the similar triangles? The angle between w and x is rotated by 90 degrees counter-clockwise to form the angle between the horizontal of length 1 and y, so $w/x = 1/y$, which is exactly what we said.

Now, minimize L

$$L = x^2 + y^2$$
$$= x^2 + \frac{x^2}{w^2}$$

Take the derivative:

$$\frac{dL}{dx} = 2x + \frac{2xw^2 - 2wx^2 \cdot 1/w}{w^4}$$

$$\frac{dL}{dx} = 2x + \frac{2x(2x-1) - 2x^2}{(2x-1)^2}$$

Factor out $2x$ and set equal to zero:

$$0 = 1 + \frac{(2x-1) - x}{(2x-1)^2}$$

$$0 = (2x-1)^2 + x - 1$$
$$4x^2 - 3x = 0$$

Factor out another x

$$4x - 3 = 0$$
$$x = \frac{3}{4}$$

The minimum crease length occurs when x is halfway along its range.

I found the last problem here:

https://www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/maxmindirectory/MaxMin.html

Chapter 49

Related rates

One simple form of related rates problem has two objects moving at right angles from each other, with positions and speeds given in terms of the origin.

For example: "A moves west at x miles per hr, his current position is x_0 miles west of the origin, while B moves south at y miles per hr, and his current position is y_0 miles south of the origin. At what rate are they moving apart?"

For the distance, we use Pythagoras:

$$h^2 = x^2 + y^2$$

All three values are functions of time so

$$2hh' = 2xx' + 2yy'$$

$$h' = \frac{1}{h}(xx' + yy')$$

We will have to calculate h from x_0 and y_0.

Another simple related rates problems involves two quantities with an equation relating the two quantities, e.g. the volume and radius of a sphere, where the sphere is a "balloon being inflated" or something

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 4\pi r^2 \ \frac{dr}{dt}$$

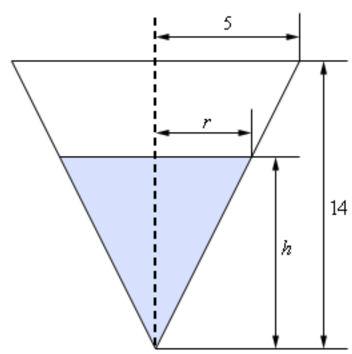
or as usually stated in these problems

$$V' = 4\pi r^2 \ r'$$

If we know V' and r we can calculate r'. Usually, rather than give you r they will give you V, so then

$$r = (\frac{4}{3}V)^{1/3}$$

A related problem :) is where the object is a cone (maybe inverted) and it's filling up with a fluid.



Here, the formula for the volume of a cone is

$$V = \frac{1}{3}\pi r^2 h$$

The problem is that since r and h depend on each other, we can't simply do

$$V' = \frac{1}{3}\pi r^2 h'$$

(this is wrong!)

In this case it's important to realize that the radius r and the height h of the fluid at its current level have the same ratio as the radius R and height H of the container.

$$\frac{r}{h} = \frac{R}{H}$$
$$r = \frac{R}{H}h$$
$$h = \frac{H}{R}r$$

so we can substitute using the relationship between r and h

$$V = \frac{1}{3}\pi r^2 \frac{H}{R} r = \frac{1}{3}\pi \frac{H}{R} r^3$$

Alternatively, we can eliminate r

$$V = \frac{1}{3}\pi \frac{R^2}{H^2} h^3$$

For example, with the figure above, ($R = 5$ and $H = 14$ feet), and given water is draining from the tank at $V' = -2ft^3$ per hour

"At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?"

$$V = \frac{1}{3}\pi\frac{R^2}{H^2}h^3$$
$$V' = \pi\frac{R^2}{H^2}h^2h'$$

We're given V' and h, H and R, so can solve for h'.

The second question is "At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?"

We need r' given h (and R, H, and V')

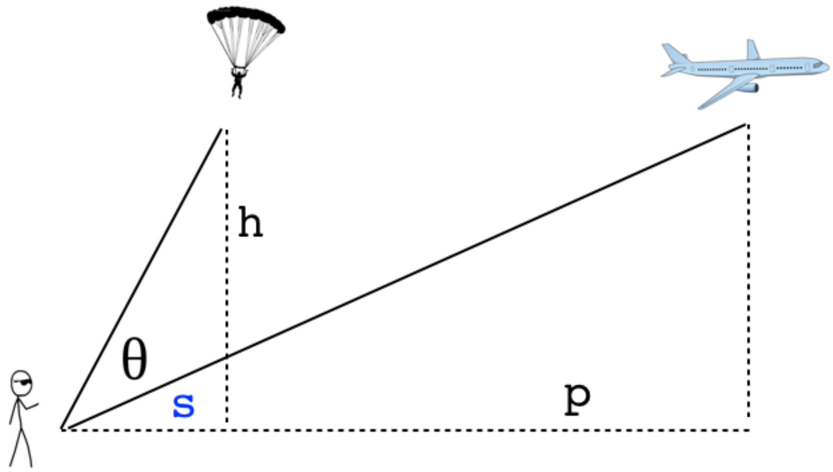
$$V = \frac{1}{3}\pi\frac{H}{R}r^3$$
$$V' = \pi\frac{H}{R}r^2r'$$

$$r = \frac{R}{H}h$$
$$r^2 = (\frac{R}{H})^2h^2$$

Plugging in

$$V' = \pi\frac{R}{H}h^2r'$$

Here is another related rates problem. An airplane and a parachutist are at the same height currently, and in the same direction as you look at them.



The airplane moves away from you at 500 ft/s. The parachutist is floating downward at -10 ft/s and will land 1000 ft away from you. The current value of $h = 2000$ ft. The current value of $p = 8000$ ft. Find $d\theta/dt$.

$$p(t) = p_0 + 500t$$
$$h(t) = h_0 - 10t$$
$$p' = 500$$
$$h' = -10$$

Find equations for the angles involved

$$\tan s = \frac{2000}{p}$$

we use the constant value of 2000 rather than h, which will vary.

$$u = s + \theta$$
$$\tan u = \frac{h}{1000}$$

Take the derivatives. For the airplane

$$\tan s = \frac{2000}{p}$$

$$\frac{d}{dt}\tan s = \sec^2 s\frac{ds}{dt} = \frac{d}{dt}\frac{2000}{p} = -2000\frac{1}{p^2}\frac{dp}{dt}$$

$$\frac{ds}{dt} = -2000\frac{1}{p^2}\frac{dp}{dt}\cos^2 s$$

In the above equation, we know $p = 8000$ and $dp/dt = 500$. We have to find the cosine. If $\tan s = 1/4$ then $\cos s = \sqrt{16/17}$.

$$\frac{ds}{dt} = -2000\ \frac{1}{8000^2}\ 500\ \frac{16}{17}$$

$$\frac{ds}{dt} = -\frac{1}{4}\frac{1}{16}\frac{16}{17} = -\frac{1}{68} = -0.0147$$

For the parachutist

$$\frac{d}{dt}\tan u = \sec^2 u\frac{du}{dt} = \frac{d}{dt}\frac{h}{1000} = \frac{1}{1000}\frac{dh}{dt}$$

$$\frac{du}{dt} = \frac{1}{1000}\frac{dh}{dt}cos^2 u$$

In the above equation, we know $dh/dt = -10$. We have to find the cosine. If $\tan u = 2$ then $\cos u = 1/\sqrt{5}$. So

$$\frac{du}{dt} = 0.001\ (-10)\ \frac{1}{5} = -0.002$$

Since $\theta = u - s$

$$\frac{d\theta}{dt} = \frac{du}{dt} - \frac{ds}{dt} = -0.002 + 0.0147 = 0.0127$$

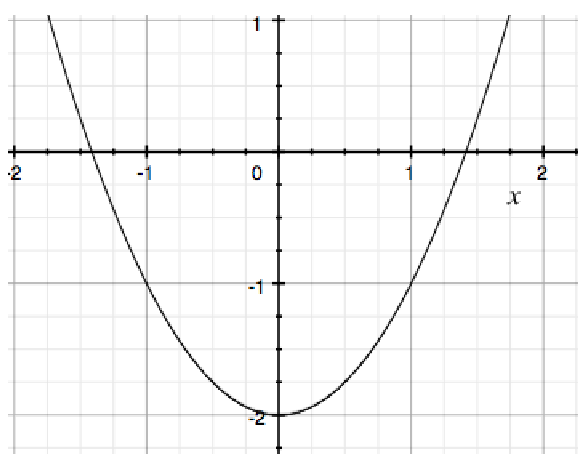
The angle θ between plane and parachutist is *increasing* with time (about 3/4 of a degree per second).

Chapter 50

Newton square root

Here is a method for finding roots of equations quickly, often called Newton's method, or the Newton-Raphson method. As an example, here is a plot of the function

$$f(x) = x^2 - 2$$



which is equal to zero when $x = \pm\sqrt{2}$. That is, we want the roots of the following equation

$$x^2 - 2 = 0$$

and more generally

$$x^2 - N = 0$$

to find the square root of some other number.

Pick a point g (for guess). Then we need to construct the line tangent to the curve at that point, with slope $m = f'(g)$ and ask, where does this line intercept the x-axis?

The slope is $\Delta y / \Delta x$.

$$\frac{f(g) - 0}{g - r} = f'(g)$$

with r being the x-coordinate at the intercept. Rearrange

$$\frac{f(g)}{f'(g)} = g - r$$

$$r = g - \frac{f(g)}{f'(g)}$$

square root problem

For this particular problem, we have

$$f(g) = g^2 - N$$

$$f'(g) = 2g$$

$$r = g - \frac{g^2 - N}{2g} = \frac{1}{2}(g + \frac{N}{g})$$

In other words, r is the average of g and N/g.

Now set $g = r$ and repeat.

This can be encapsulated into the following algorithm:

- Make a guess g and compute N/g
- Average g and N/g to find a new guess

- Repeat until satisfied

The algorithm converges rapidly for most problems.

```
2
1.5
1.4166666666666665
1.4142156862745097
1.4142135623746899
1.414213562373095
```

notes

It is worthwhile to try to make a good first guess. If the method goes wrong (which it can, when equations have bumps or other issues), the problem can be fixed by making a better guess.

You can read more about the method here:

http://www.math.ubc.ca/~anstee/math104/newtonmethod.pdf

For the particular problem that we worked, finding the square root of N, the equation says

$$g' = \frac{1}{2}(g + \frac{N}{g})$$

find your next guess by averaging the current guess g and N/g.

This equation goes back at least as far as Heron of Alexandria (10-70 AD).

Part XV

More series

Chapter 51

Binomial distribution

You are likely familiar with the binomial distribution from algebra. This distribution gives both the coefficients and the powers for each term in an expansion of the form

$$(a+b)^n$$

where n is a positive integer. The first four examples are:

$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

and in general

$$(a+b)^n = c_0a^n + c_1a^{n-1}b + c_2a^{n-2}b^2 + \cdots + c_{n-1}ab^{n-1} + c_nb^n$$

where each of the c_k may be different (though they are inversely symmetric, with $c_0 = c_n$ and $c_1 = c_{n-1} \ldots$).

Ignoring the powers $a^{n-k}b^k$ for the moment, the coefficients are

$$1 \quad 1$$

$$1 \quad 2 \quad 1$$

$$1 \quad 3 \quad 3 \quad 1$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

As you know, this pattern is called Pascal's Triangle. One striking thing about the triangle is that each value which is not on an edge can be formed by adding together the two values that lie directly above it.

The values running down the left and right sides are always 1.

Another approach is that, rather than formatting as a triangle, some people just make vertical columns:

1 1

1 2 1

1 3 3 1

1 4 6 4 1

The first column contains only 1's, the second column is the natural numbers (or positive integers), and the third is the "triangular" numbers, where the difference between successive numbers is the sequence of natural numbers.

We will be very interested in the general case, and therefore we need to set a convention for going across a row n using the index k. Notice that the fourth row $n = 4$ has five terms. It is convenient to let the index run from $k = 0$ to $k = n$.

We can formalize the observation that a value is the sum of the two values above it by saying that the coefficient at position k of row n is the sum of the values at positions k plus position $k-1$ of the preceeding row $n - 1$. For example, for $n = 4$, the $k = 2$ value is 6. 6 is the sum of the values at $k = 2$ and $k = 1$ of the row for $n = 3$. Continuing:

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

The corresponding expansions are:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

The power terms (leaving the coefficients out for the moment) are written in order as decreasing powers of a, starting from n, and increasing powers of b, starting from 0:

$$a^nb^0 + a^{n-1}b^1 + a^{n-2}b^2 + \cdots + a^2b^{n-2} + ab^{n-1} + a^0b^n$$

which is the same as

$$a^n + a^{n-1}b^1 + a^{n-2}b^2 + \cdots + a^2b^{n-2} + ab^{n-1} + b^n$$

understanding the pattern

In multiplying

$$(a+b)(a+b) = a^2 + 2ab + b^2$$

we can think of a generative procedure that goes term-by-term through the second multiplicand, multiplying first by a on the left

$$a \cdot (a+b) = a^2 + ab$$

and then by b on the right

$$(a+b) \cdot b = ab + b^2$$

Finally, add the two results together:

$$a \cdot (a + b) + (a + b) \cdot b$$
$$= a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2$$

We see that the two terms with the same power a^1b^1 come from multiplying $a \cdot (b)$ on the first pass, then adding $(a) \cdot b$ from the second.

This seems quite obvious, but it sets the stage for looking at a more complicated case.

In the expansion for $(a + b)^5$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

the coefficient of the $n = 5$, $k = 2$ term a^3b^2 is 10. We can get that 10 by taking two terms from the expansion for $(a + b)^4$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Namely, the $n = 4$, $k = 2$ term is multiplied by b

$$(4a^3b) \cdot b$$

and the $n = 4$, $k = 3$ term is multiplied by a

$$a \cdot (6a^2b^2)$$

when the two results are added together we obtain

$$(4a^3b) \cdot b + a \cdot (6a^2b^2) = 10a^3b^2$$

For the general case, we see that in writing the expansion for row $n+1$, at the kth term we form the power

$$a^{n+1-k}b^k$$

We have two contributions from the previous row. One comes from multiplying

$$a^{n+1-k}b^{k-1} \cdot b$$

and the second comes from

$$a \cdot a^{n-k}b^k$$

The term that is multiplied by a, namely $a^{n-k}b^k$, is found at the same column, immediately above $a^{n+1-k}b^k$.

The other one, $a^{n-k+1}b^{k-1}$ (the same as $a^{n+1-k}b^{k-1}$), is the term that precedes it. (Recall that powers of a get larger as we go to the left, and powers of b get smaller).

This explains why we obtain the pattern seen in Pascal's triangle as the coefficients for each expansion. As we said before: the coefficient at the position k of row n is the sum of the values at positions k and $k-1$ of row $n-1$.

choose

There is a very useful formula that allows one to find the coefficients of each term in the binomial expansion without actually filling out the triangle. It is called "n choose k".

The official way to write this expression is

$$\binom{n}{k}$$

The formula is:

$$\binom{n}{k} = \frac{n!}{(n-k!)\ k!}$$

This gives the number of different combinations of k objects one can choose from n total objects. For example, suppose you have 5 Bob Marley CD's, and you want to pick 2 of them to take to a party, that is "5 choose 2" and the formula is

$$\frac{5!}{3!\ 2!} = \frac{5 \times 4 \times 3 \times 2}{3 \times 2 \times 2} = \frac{120}{12} = 10$$

This is also the coefficient in row n, position k of Pascal's Triangle (indexing from $k = 0$).

For our example, that is row 5 position 2.

1 5 10 10 5 1

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

which is indeed 10.

Notice a cancelation we can do in our formula. We we rewrite the numerator with $n!$:

$$\frac{n \times (n-1) \cdots \times (n-k+1) \times (n-k)! \cdots}{(n-k)!\ k!}$$

We can cancel the factor of $(n-k)!$ and then have

$$\frac{n(n-1)\cdots(n-k+1)}{k!}$$

In the example:

$$\frac{5 \times 4 \times 3 \times 2}{3 \times 2 \times 2} = \frac{5 \times 4}{2} = \frac{20}{2} = 10$$

It is a good idea to memorize the combinations formula. It comes up repeatedly, in the derivations of calculus, and in probability.

quick derivation of the choose formula

Suppose you have n CD's and you want to pick 2 of them to take with you. There are n choices for the first one—maybe you choose *Natty Dread*. And there are $n-1$ choices for the second one, you pick *Kaya*.

That would give $n \cdot (n-1)$ here, and in the general case $n \cdot (n-1) \cdot \ldots (n-k+1)$. However, the order is not important, you might just as well have picked *Kaya* and then *Natty Dread*. That's why we have to correct for the over-counting. In this particular case we divide by 2, and in the general case divide by k!.

Binomial theorem: proof

A concise statement of the binomial formula is that the general term is of the form

$$\binom{n}{k} a^{n-k} \, b^k$$

and the whole sum is

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} \, b^k$$

where (from the theory of combinations):

$$\binom{n}{k} = \frac{n!}{(n-k)! \; k!}$$

To do an actual calculation, we would first cancel the factor of $(n-k)!$ on top and bottom yielding

$$\binom{n}{k} = \frac{n \times (n-1) \cdots \times (n-k+1)}{k!}$$

Pascal's Lemma

In order to prove the theorem (using induction) we will need the following result:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Here is a simple proof of this result, from the theory of combinations. Imagine that we are considering how many ways there are of forming a committee of k members from a total of n people.

We know that the number of ways of doing this is of course just

$$\binom{n}{k}$$

Now, suppose that among these n people we focus on one particular person, call her Alice. Then there are two types of committees in our collection of combinations: those in which Alice is a member, and those in which she is not.

For all committees of the first type, in addition to Alice, the other $k-1$ members must be drawn from $n-1$ people. The number of ways of doing this is

$$\binom{n-1}{k-1}$$

For the second case, where Alice is not a member, we must recruit all k members from $n-1$ people, since we are leaving Alice out. The number of ways of doing this is

$$\binom{n-1}{k}$$

But putting them together, these must be equal to the total number obtained by the standard analysis, and hence we have that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

This is effectively what we said near the beginning of the introduction to the binomial theorem: in computing the kth coefficient in the nth row, we add the k and $k-1$ values from the preceding row.

This preliminary result is called Pascal's Lemma and it is really the heart of the proof.

In using it below, we will alter its form slightly by substituting $n+1$ for n. Thus:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Induction

If we look at one more term in the expansion for $(a+b)^n$, writing the term preceding the one given above, we have

$$(a+b)^n = \cdots + \binom{n}{k-1}\, a^{n-k+1}\, b^{k-1} + \binom{n}{k}\, a^{n-k}\, b^k + \ldots$$

Notice that the exponent decreases by one for a as we move to the right, while it increases by one for b.

When we form the new general term in the expansion for $(a+b)^{n+1}$, as we said before, we multiply the first term by b and the second one by a, obtaining

$$\cdots + b\binom{n}{k-1}\, a^{n-k+1}\, b^{k-1} + a\binom{n}{k}\, a^{n-k}\, b^k + \ldots$$

$$= \cdots + \binom{n}{k-1}\ a^{n-k+1}\ b^k + \binom{n}{k}\ a^{n-k+1}\ b^k + \dots$$

But these two powers are the same, and since

$$n - k + 1 = (n+1) - k$$

their sum is the general term in the expansion for $(a+b)^{n+1}$

$$= \cdots + C\ a^{(n+1)-k}\ b^k + \dots$$

In adding them, we add their coefficients:

$$\cdots +\ [\ \binom{n}{k-1} + \binom{n}{k}\ ]\ a^{(n+1)-k}\ b^k + \dots$$

Referring back to Pascal's Lemma, we substitute

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

yielding

$$\cdots +\ \binom{n+1}{k} + a^{(n+1)-k}\ b^k + \dots$$

We obtain the general term for the $n+1$ expansion. This completes the inductive part of the proof. It remains to check the binomial formula for a base case like $n = 1$ or $n = 2$, which I invite you to do.

□

Courant's proof

Courant gives a proof by induction which I've transcribed and annotated separately. It duplicates significant parts of what we've already gone through, but there is some new material, which I've added in here.

Pascal's Lemma

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

was "proved" above by an argument about combinations which do or do not include a person named Alice. In Courant's version, the factorials are manipulated to provide a direct mathematical proof.

simplification

As we saw before, in the choose formula, there are terms that cancel. We expand $n!$ partially

$$n! = n \cdot (n-1) \ldots (n-k+1) \cdot (n-k)!$$

The last term is also present in the denominator of our formula

$$C(n,k) = \frac{n!}{k!\ (n-k)!}$$

so we can simplify

$$C(n,k) = \frac{n \cdot (n-1) \ldots (n-k+1)}{k!}$$

We will be interested in coefficients with $k+1$, so let's take a look at n "choose" $k+1$. The original definition would give:

$$C(n,k+1) = \frac{n!}{(k+1)!\ (n-(k+1))!}$$

Remove one set of parentheses

$$= \frac{n!}{(k+1)!\ (n-k-1)!}$$

By the same argument, expand $n!$

$$= \frac{n \cdot (n-1) \dots (n-k+1) \cdot (n-k) \cdot (n-k-1)!}{(k+1)!\ (n-k-1)!}$$

$$= \frac{n \cdot (n-1) \dots (n-k+1) \cdot (n-k)}{(k+1)!}$$

You should convince yourself that the last term in the numerator is correct. It seems somewhat counterintuitive that for $k+1$ the last term is $(n-k)$ rather than say, $(n-k+2)$, as I thought at first.

induction

Let us examine the general statement

$$C(n,k) + C(n,k+1)$$

Rewriting it as the factorial using the simplification we found above

$$\frac{n \cdot (n-1) \dots (n-k+1)}{k!} + \frac{n \cdot (n-1) \dots (n-k+1) \cdot (n-k)}{(k+1)!}$$

We can factor out $(n-k)/(k+1)$ from the second term:

$$= (\frac{n-k}{k+1}) \ \cdot \frac{n \cdot (n-1) \dots \ (n-k+1)}{k!}$$

and we see that the second term is that factor multiplied by the first term.

Hence, the complete sum becomes

$$[\ 1 + (\frac{n-k}{k+1})\ ] \ \cdot \frac{n \cdot (n-1) \dots \ (n-k+1)}{k!}$$

Take the leading factor and put it over a common denominator

$$\frac{(k+1) + (n-k)}{k+1} = \frac{n+1}{k+1}$$

so the expression now becomes

$$= (\frac{n+1}{k+1}) \ \cdot \frac{n \cdot (n-1) \ldots \ (n-k+1)}{k!}$$

$$= \frac{(n+1) \cdot n \cdot (n-1) \ldots \ (n-k+1)}{(k+1)!}$$

rearranging the last term in the numerator slightly

$$= \frac{(n+1) \cdot n \cdot (n-1) \ldots \ [\ (n+1) - k\ ]}{(k+1)!}$$

$$= C(n+1, k+1)$$

This is the correct expression for $n+1$ (because it has $n+1$ in the right places), and it is the correct expression for $k+1$ because it ends with $n+1$ minus k (rather than $k+1$).

recap

We assume that $C(n,k)$ is the correct coefficient for $a^{n-k}b^k$ in the expansion of $(a+b)^n$, and that $C(n,k+1)$ is the correct coefficient for the succeeding term $a^{n-(k+1)}b^{k+1}$ in the same expansion. Multiplication of the first by b and the second by a leads to:

$$[\ C(n,k) + C(n,k+1)\ ]\ a^{n-k}b^{k+1}$$

We need to tweak one exponent slightly when considering this as part of the expansion for $(a+b)^{n+1}$. The n in the exponent for a should be expressed in terms of n referring to the incremented value $n+1$ so it needs to step down one unit, becoming a^{n-k-1} which is equal to $a^{n-(k+1)}$. Thus we have

$$[\ C(n,k) + C(n,k+1)\ ]\ a^{n-(k+1)}b^{k+1}$$

We showed that

$$C(n,k) + C(n,k+1) = (1 + \frac{n-k}{k+1}) \cdot C(n,k)$$

$$= \frac{n+1}{k+1}) \cdot C(n,k)$$

$$= C(n+1,k+1)$$

which is what the binomial theorem gives. This completes the proof by induction. □

probability

The binomial theorem is also used extensively in working with discrete probability.

A simple example is to consider a standard (6-sided), fair die. We ask:

What is the probability that in rolling six such dice (or one die six times), we will observe some distribution of results, like one or more 6's?

There is an easy way to do this. We calculate the probability that no 6 is observed on one roll as 5/6 and on six rolls as

$$(5/6)^6 = 15625/46656 = 0.334898$$

The outcome of at least one 6 is the complement of this so its probability is 0.665102.

The other way to do it is to use the formula that gives the distribution over all possible outcomes:

$$C(n,k)p^k(1-p)^{n-k}$$

In this case, we have

```
C(6,0) = 1
C(6,1) = 6!/(1! 5!) = 6
C(6,2) = 6!/(2! 4!) = 15
C(6,3) = 6!/(3! 3!) = 720/36 = 20
C(6,4) = 15
C(6,5) = 6
C(6,6) = 1

P(0 6's) = 1          (5/6)^6 = 0.334898
P(1 6's) = 6  (1/6)   (5/6)^5 = 0.401878
P(2 6's) = 15 (1/6)^2 (5/6)^4 = 0.200939
P(3 6's) = 20 (1/6)^3 (5/6)^3 = 0.053584
P(4 6's) = 15 (1/6)^4 (5/6)^2 = 0.008038
P(5 6's) = 6  (1/6)^5 (5/6)^1 = 0.000643
P(6 6's) = 1  (1/6)^6         = 0.000021
```

The total is 1.000001, or 1, within the error introduced by truncation.

Suppose instead that we flip a (fair) coin 100 times. What will be the distribution then? Since p = 1 - p = 0.5, the power terms are all the same, namely

$$0.5^{100} = 7.8886e - 31$$

But the factorials are kind of awkward. For example, what is

$$100!/50!50! = ?$$

100! has 158 digits in it. Python can do this calculation, the result is $1.008913e + 29$ so we end up with $P = 0.0796$.

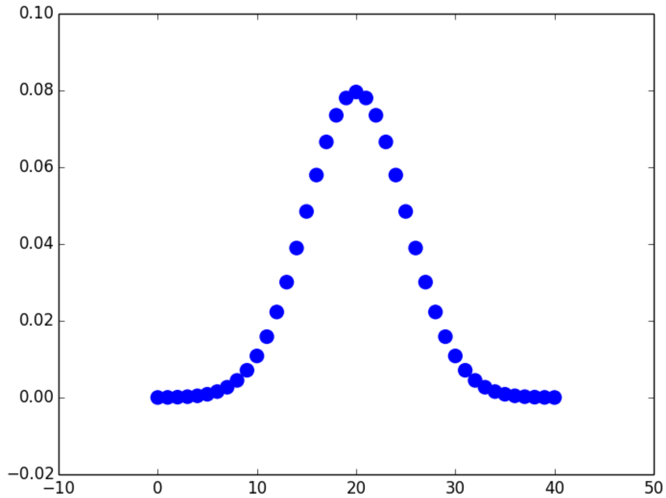
We use Python to compute the factorials in an efficient way, and tabulate the values for the probability from 35-50

```
35 0.000863855665742
36 0.00155973939648
37 0.00269792760472
38 0.00447287997624
39 0.00711073226993
40 0.0108438667116
41 0.0158690732365
42 0.0222922695466
43 0.0300686426442
44 0.0389525597891
45 0.0484742966264
46 0.0579583981403
47 0.066590499991
48 0.0735270104067
49 0.0780286641051
50 0.0795892373872
```

The distribution is symmetrical about the midpoint.

Adding up values gives the result that 99.8% of the total probability lies between 35-65 inclusive.

Here is a plot of the values from 30-70.



Look familiar?

Chapter 52

Taylor series

Suppose we have a function $f(x)$, but

Shankar:

> "imagine that you don't have access to the whole function. You cannot see the whole thing. You can only zero-in on a tiny region."

around $f(0)$, where you know the value. So the question is, what do we guess the function will do near $f(0)$?

The first approximation is that

$$f(x) \approx f(0)$$

We really can't say anything more. $f(0)$ is the best guess for what the value of the function is (we're talking about continuous and continuously differentiable functions).

Now suppose we know the slope of the function at 0, $f'(0)$. Then, since

$$\Delta y = f'(0)\Delta x = f'(0)(x - 0)$$

we can get a better approximation as the linear approximation:

$$f(x) \approx f(0) + f'(0)\ x + \dots$$

For most functions, there will be more terms. If f is not a linear function, then the slope won't be constant. So

> "the rate of change itself has a rate of change .. the second derivative."

The term we are going to add is

$$f''(0)\ \frac{x^2}{2}$$

so

$$f(x) \approx f(0) + f'(0)\ x + f''(0)\ \frac{x^2}{2} + \dots$$

A simple way to see why we have $x^2/2$ is to take derivatives on both sides. The terms like $f'(0)$ and $f''(0)$ are constants, they have been evaluated at $x = 0$. The first derivative is

$$f'(x) \approx f'(0) + f''(0)\ x + \dots$$

We evaluate at $x = 0$ and the term $f''(0)\ x$ goes away because of the $x = 0$ multiplying the constant $f''(0)$. So we have just

$$f'(x) \approx f'(0)$$

and that matches. Now take the second derivative

$$f''(x) \approx f''(0)$$

and that matches too. We can see a pattern here.

The fourth term is

$$f(x) \approx f(0) + f'(0)\ x + f''(0)\ \frac{x^2}{2!} + f'''(0)\ \frac{x^3}{3!} + \dots$$

You might not be expecting the factorial which I snuck in there. But if you go back to the exercise above, where we evaluated derivatives, you can see why it works. When we take the first derivative

$$\frac{d}{dx}(f'''(0)\ \frac{x^3}{3!}) = f'''(0)\ \frac{x^2}{2!}$$

the 3 comes down from the power and then turns 3! in the denominator into 2!. The next derivative will bring down the 2. So everything cancels properly.

If you like Σ notation, we can write

$$f(x) = \sum_{n=0}^{\infty} f^n(0)\frac{x^n}{n!}$$

with the understanding that $0! = 1$. The approximation is better the closer x is to 0, and the more terms the better as well.

There is one final wrinkle to this derivation. The series can be modified deal with x near any value a, not just near 0. The modification is

$$f(x) = \sum_{n=0}^{\infty} f^n(a)\frac{(x-a)^n}{n!}$$

This is the Taylor series. The series near $a = 0$ is known as the Maclaurin series.

1/1-x

The first example is

$$f(x) = \frac{1}{1-x}$$

We know the answer to this.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3$$

Proof:

$$1 = (1-x)(1 + x + x^2 + x^3)$$

Multiplying by 1, the second term x is matched by $-x$ from the first term in the multiplication by $-x$, and so on. The whole thing vanishes, leaving just 1.

We want to evaluate $f(x)$ near 0, let's say, at $x = 0.1$. The correct value of the function is

$$f(x) = \frac{1}{0.9} = 1.11111\ldots$$

Let's try to approximate using the series. We need derivatives

$$f(x) = \frac{1}{1-x}$$

$$f'(x) = \frac{1}{(1-x)^2} = (1-x)^{-2}$$

$$f'(0) = 1$$

so the linear approximation is

$$f(x) \approx 1 + 1x = 1.1$$

For the next term we obtain

$$f''(x) = 2(1-x)^{-3}$$

The 2 is cancelled by the 2! in the denominator, so this cofactor is 1 and we're left with

$$f''(0)\ \frac{x^2}{2} = x^2 = 0.01$$

And I think we can see where this one is going.

However, you probably remember that this series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

diverges for $|x| \geq 1$, and the Taylor series does too.

The morale of the story is that for some series, there is a radius of convergence and the series is only valid for x within that radius.

binomial

Another very useful series is the binomial.

$$f(x) = (1+x)^n$$

$$f(0) = 1$$

$$f'(0) = n(1+x)^{n-1} = n$$

$$f''(0) = n(n-1)(1+x)^{n-2} = n(n-1)$$

So the series is

$$(1+x)^n \approx 1 + nx + n(n-1)\frac{x^2}{2}$$

We use this one a lot.

A nice application is relativistic energy

$$E = mc^2 f$$

$$f = 1/\sqrt{1 - \frac{v^2}{c^2}}$$

This is, in disguise, a binomial with $n = -1/2$ and $x = -v^2/c^2$ so the expansion is

$$f \approx 1 + nx = 1 + \frac{v^2}{2c^2}$$

so the energy is

$$E \approx mc^2(1 + \frac{v^2}{2c^2})$$

And we see that the second term is just the kinetic energy, $mv^2/2$.

polynomials

The beauty of Taylor Series (despite its complexity) is that it turns any differentiable function into a polynomial. Polynomials are easy to integrate and work with.

The first thing to say about Taylor Series is they give the correct answer for functions that we know. For example, suppose we have

$$f(x) = ax^2 + bx + c = 1$$

We get the derivatives and evaluate them "near" the point $x = 0$.

$$f(x) = ax^2 + bx + c = c$$

$$f'(x) = 2ax + b = b$$
$$f''(x) = 2a$$

The series is then

$$f(x) = c + b(x) + \frac{21}{2!}(x)^2 + \cdots$$

But there are no more terms. That's it. And this is just

$$f(x) = c + bx + ax^2$$

exponential, sine and cosine

Suppose $f(x) = e^x$ and again, we evaluate "near" $x = 0$. We have

$$f(x) = e^x = 1$$

$$f'(x) = e^x = 1$$
$$f''(x) = e^x = 1$$

The series is

$$f(x) = e^x = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots$$

$$f(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Which matches what we already know about e^x. For example, it is obvious that

$$\frac{d}{dx}e^x = e^x$$

Let's try to find something new. Suppose we expand $f(x) = cos\ x$ near $x = 0$

$$f(x) = cos\ x = cos\ 0 = 1$$

$$f'(x) = -sin\ x = -sin\ 0 = 0$$

$$f''(x) = -cos\ x - cos\ 0 = -1$$

$$f'''(x) = sin\ x = sin\ 0 = 0$$

$$f''''(x) = cos\ x = cos\ 0 = 1$$

and this continues in a cycle with period 4. The series is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$f(x) = cos\ x = 1 - \frac{1}{2!}(x-0)^2 + \frac{1}{4!}(x-0)^4 + \cdots$$

$$f(x) = cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Similarly, for $f(x) = sin\ x$ near $x = 0$

$$f(x) = sin\ x = 0$$

$$f'(x) = cos\ x = 1$$
$$f''(x) = -sin\ x = 0$$
$$f'''(x) = -cos\ x = -1$$
$$f''''(x) = sin\ x = 0$$

The series is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$f(x) = sin\ x = x - \frac{1}{3!}(x-0)^3 + \frac{1}{5!}(x-0)^5 + \cdots$$

$$f(x) = sin\ x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

funny series

In Strogatz book (*The Joy of x*), he gives the following series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

and he says that the sum of the series is equal to the natural logarithm of 2:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

with the provision that you have to calculate the sum in the order given.

For example, the second, third and fourth partial sums are:

$$S_2 = \frac{1}{2}; \quad S_3 = \frac{5}{6}; \quad S_4 = \frac{14}{24}; \quad S_5 = \frac{94}{120}$$

with $S_4 = 0.583$ and $S_5 = 0.783$. For any partial sum S_n and the previous sum S_{n-1} the value of the series will be bounded by the two sums.

I thought I would try to show that $\ln 2$ is the correct value for series, by using a Taylor series for the logarithm.

Taylor says we can write a function $f(x)$ (near the value $x = a$) as an infinite sum

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

where f^n means the nth derivative of f and f^0 is just f, and these derivatives are to be evaluated at $x = a$. Near $a = 0$ this simplifies to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n$$

Let's calculate the derivatives of the logarithm:

$$f^0 = \ln x; \quad f^1 = \frac{1}{x} = x^{-1}; \quad f^2 = -x^{-2}; \quad f^3 = 2x^{-3}; \quad f^4 = -3!\, x^{-4}$$

The first thing I notice is that we can't use $a = 0$, since $f^1 = 1/x$ is undefined there. So, let's try $a = 1$. Then (evaluated at $a = 1$)

$$f^0 = \ln x = 0; \quad f^1 = \frac{1}{x} = 1; \quad f^2 = -x^{-2} = -1; \quad f^3 = 2; \quad f^4 = -3!$$

Going back to the definition

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

I get the following series near $a = 1$:

$$\ln x = \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \cdots$$

For the special value $x = 2$, all the terms $(x-1)^n$ go away (which confirms that $a = 1$ is an excellent choice!). We have then

$$\ln x = \frac{0}{0!} + \frac{1}{1!} - \frac{1}{2!} + \frac{2}{3!} - \frac{3!}{4!} + \cdots$$

$$= 0 + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

which is what was to be proved.

Chapter 53

Series convergence

Tests for convergence

Series like the Taylor series can be very helpful in approximating a function. Here we review some common tests to see whether the sum of an infinite series converges to a finite limit, or instead diverges.

Start with the geometric series

$$\sum_{k=0}^{\infty} x^k$$

Suppose we compute the sum of a number of terms n

$$s_n = x^0 + x^1 + x^2 + \cdots + x^n$$

Since this is a finite series, the sum exists so

$$xs_n = x^1 + x^2 + \cdots + x^{n+1}$$

Then

$$s_n - xs_n = (1-x)s_n = 1 - x^{n+1}$$
$$s_n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}, \quad x \neq 1$$

Clearly, if $x > 1$ then $x^{n+1} \to \infty$ as n gets large, and this is true even if $x = 1$. If $x < -1$ then the second term alternates in sign and its absolute value gets very large as n gets large.

Only for $|x| < 1$, as $n \to \infty$, the second term vanishes and we have

$$s_n = \frac{1}{1-x}$$

For example,

$$\sum_{k=0}^{\infty} (\frac{1}{2})^k = \frac{1}{1-1/2} = 2$$

$$\sum_{k=0}^{\infty} (\frac{1}{3})^k = \frac{1}{1-1/3} = \frac{3}{2}$$

and so on.

A second famous series is the harmonic series

$$\sum_{k=0}^{\infty} \frac{1}{k^p}$$

especially with $p = 1$

$$\sum_{k=0}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

This series diverges.

One proof is the following. Assume that the series converges. Then its sum has a limit which we can call L.

$$L = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \dots$$

$$> \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} \dots$$
$$= L$$

a contradiction. Therefore, the harmonic series diverges.

Let us look at some tests of convergence.

Divergence test

The first test requires that the limit of the individual terms a_k must tend to zero

$$\lim_{k \to \infty} a_k = 0$$

if not, then the sum diverges. For example,

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots$$

This clearly diverges, since

$$\lim_{k \to \infty} a_k = 1$$

Or

$$\lim_{k \to \infty} \frac{k}{2k+1}, \quad k \in \{1, 2, \dots\}$$
$$= \lim_{k \to \infty} \frac{1}{2 + 1/k} = \frac{1}{2} \neq 0$$

Another example is the harmonic series

$$\lim_{k \to \infty} \frac{1}{k} = 0, \quad k \in \{1, 2, \dots\}$$

Despite passing this limit test, the harmonic series diverges. Thus, a pass is necessary but not sufficient.

Integral test

The integral test says that a (well-behaved) function $f(x)$

$$\int_1^\infty f(x)\ dx$$

converges $\iff$

$$\sum_{k=1}^{\infty} f(k)$$

also converges. The function must be continuous and integrable, etc.

Let's apply this test to the harmonic series

$$\sum_{k=0}^{\infty} \frac{1}{k}$$

We have

$$\int_1^\infty \frac{1}{x}\ dx = \ln|x|\ \Big|_1^\infty$$

but the upper bound has the limit

$$\lim_{k\to\infty} \ln|k| = \infty$$

In general, for

$$\sum_{k=0}^{\infty} \frac{1}{k^p}$$

if $p > 1$, the sum converges, but not otherwise:

$$\int_1^\infty x^{-p}\ dx = \frac{1}{1-p} x^{1-p}\ \Big|_1^\infty$$

For $p > 1$

$$\lim_{x\to\infty} x^{1-p} = 0$$

On the other hand

$$\int_1^\infty \frac{1}{n^2}\,dn = -\frac{1}{n}\bigg|_1^\infty = 0 - -1 = 1$$

so the $\sum 1/n^2$ converges.

Comparison test

If we compare a series and a convergent series and the test series is smaller term-by-term, then it also converges. Similarly, if a series is larger than a divergent series when compared term-by-term, it also diverges. Any finite number of terms from the beginning of a series may be disregarded before starting the comparison.

Since

$$\sum_{k=0}^{\infty} \frac{1}{k^2}$$

converges, so does

$$\sum_{k=0}^{\infty} \frac{1}{k^2+10}$$

And since

$$\sum_{k=0}^{\infty} \frac{1}{k}$$

diverges, so does

$$\sum_{k=0}^{\infty} \frac{1}{\ln|k+1|}$$

since for $k > 2$

$$\ln|k+1| < k$$

so

$$\frac{1}{\ln|k+1|} > \frac{1}{k}$$

Ratio test

Consider

$$\sum_{k=0}^{\infty} a_k, \quad a_k > 0$$

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = L$$

$$\begin{cases} L < 1 & : \text{converges} \\ L > 1 & : \text{diverges} \\ L = 1 & : \text{inconclusive} \end{cases}$$

As an example

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

Check

$$\lim_{k \to \infty} \frac{1/(k+1)!}{1/k!} = \frac{1}{k+1} < 0$$

This one is also easily checked by the comparison test since

$$k! > k^2, \quad k > 3$$

Since $1/k^2$ converges, so does $1/k!$.

Or

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

$$\lim_{k \to \infty} \frac{1/n+1}{1/n} = \frac{n}{n+1} = 1$$

so the test is inconclusive.

a bit more

In the previous chapter, we looked at Taylor series and showed that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

Actually, this series is really interesting from the point of view of convergence. An infinite series is convergent if all of the terms add up to something finite, like $\ln 2$.

In this case, the terms alternate sign, which makes me wonder. We have the positive terms

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \dots$$

The two terms

$$\frac{1}{5} + \frac{1}{7} = \frac{12}{35} = 0.343 > \frac{1}{3}$$

The next four terms

$$\frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15}$$
$$= \frac{715}{6435} + \frac{585}{6435} + \frac{495}{6435} + \frac{429}{6435} = \frac{2224}{6435}$$
$$= 0.346 > \frac{1}{3}$$

Notice that the decimal sum is increasing.

This is sorta like the harmonic series. It diverges to $+\infty$.

The negative terms are

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \cdots$$
$$= (-\frac{1}{2}) \cdot (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

which *is* the harmonic series. It diverges to $-\infty$.

So two series which separately diverge to $+\infty$ and $-\infty$ add to something finite!

Another amusing rearrangement of the $\ln 2$ series (Acheson) is:

$$
\begin{aligned}
&1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{7}+\frac{1}{8}+\cdots\\
&=(1-\frac{1}{2})-\frac{1}{4}+(\frac{1}{3}-\frac{1}{6})-\frac{1}{8}+(\frac{1}{5}-\frac{1}{10})-\frac{1}{12}+\dots\\
&=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\dots\\
&=\frac{1}{2}(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots)
\end{aligned}
$$

How can the series be equal to one-half of itself?

These groupings of terms with alternating signs are illegal. They do not yield correct values for the sum. However, an adjacent grouping does work, namely:

$$
\begin{aligned}
&(1-\frac{1}{2})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{5}-\frac{1}{6})+(\frac{1}{7}-\frac{1}{8})+\cdots\\
&=\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\frac{1}{56}+\dots
\end{aligned}
$$

which, as you can see, converges rather slowly.

Chapter 54

Newton binomial

standard binomial

As you know, the binomial expansion for the first few positive integers n is

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^n = \sum_{k=0}^{k=n} c_k \ a^{n-k} b^k$$

The coefficients c_k are given by Pascal's triangle or by computing

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

For

$$k = 0, \qquad \frac{n!}{0!(n-0)!} = 1$$

$$k = 1, \qquad \frac{n!}{1!(n-1)!} = n$$

$$k = 2, \qquad \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2!}$$

$$k = 3, \qquad \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{3!}$$

This can also be written as

$$= \frac{n}{1} \cdot \frac{(n-1)}{2} \cdot \frac{(n-2)}{3}$$

Let us just consider binomials of the form $a = 1$, so then substitute x for b

$$(1+x)^n = \sum_{k=0}^{k=n} c_k \; x^k$$

The expansion becomes ($x^0 = 1$)

$$1 + n \cdot x + n \cdot \frac{(n-1)}{2} \cdot x^2 + \frac{n}{1} \cdot \frac{(n-1)}{2} \cdot \frac{(n-2)}{3} \cdot x^3 + \dots$$

This series terminates when $n = k$, and the coefficients are symmetric about $k = n/2$. If n is even, the two middle terms of the sequence are equal.

Now, the natural question is, what will happen if we substitute r for n, where is r is a rational exponent, or may even be negative?

Newton

Newton wrote the binomial expansion in this way

$$(P + PQ)^{m/n} = P^{m/n} + \frac{(m)}{(n)} P^{m/n} Q + \frac{(m)}{(n)} \frac{(m-n)}{(2n)} P^{m/n} Q^2 + \cdots$$

$$+\frac{(m)}{(n)}\frac{(m-n)}{(2n)}\frac{(m-2n)}{(3n)}P^{m/n}Q^3+\cdots$$

This looks a little strange to modern eyes, but it's actually the same as the standard binomial.

Notice that we can factor out $P^{m/n}$ so

$$(1+Q)^{m/n}=1+\frac{(m)}{(n)}Q+\frac{(m)}{(n)}\frac{(m-n)}{(2n)}Q^2+\frac{(m)}{(n)}\frac{(m-n)}{(2n)}\frac{(m-2n)}{(3n)}Q^3+\cdots$$

Then just bring n up into the numerator

$$(1+Q)^{m/n}=1+\frac{m/n}{1}Q+\frac{(m/n)}{1}\frac{(m/n-1)}{2}Q^2+$$

$$+\frac{(m/n)}{1}\frac{(m/n-1)}{2}\frac{(m/n-2)}{3}Q^3+\cdots$$

Substitute r for m/n

$$(1+Q)^r=1+rQ+r\frac{(r-1)}{2}Q^2+r\frac{(r-1)}{2}\frac{(r-2)}{3}Q^3+\cdots$$

This is the binomial with $x=Q$.

One difference is that Newton used it for negative integers and even for a fractional power. A key change is that in these cases the series becomes infinite.

usage

We can use this to compute roots. Suppose $Q=1$ and $r=1/3$, so we are looking for the cube root of 2. The terms of the series are $1+1/3+\ldots$

$$\frac{1}{3}\cdot\frac{-2/3}{2}=-\frac{1}{3^2}$$

$$-\frac{1}{3^2}\cdot\frac{-5/3}{3}=\frac{5}{3^4}$$
$$\frac{5}{3^4}\cdot\frac{-8/3}{4}=-\frac{10}{3^5}$$
$$-\frac{10}{3^5}\cdot\frac{-11/3}{5}=\frac{22}{3^6}$$
$$\frac{22}{3^6}\cdot\frac{-14/3}{6}=\frac{154}{3^8}$$
$$\vdots$$

$$= 1+\frac{1}{3}-\frac{1}{9}+\frac{5}{81}-\frac{10}{243}+\frac{22}{729}-\frac{154}{6561}\dots$$

which seems pretty close (but not that close) to 1.259921.

Another use is to obtain a series for $1/(1+x)$.

$$(1+Q)^r = 1 + rQ + r\frac{(r-1)}{2}Q^2 + r\frac{(r-1)}{2}\frac{(r-2)}{3}Q^3 + \cdots$$
$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots$$

Newton checked this by multiplying:

$$1 = (1+x)(1 - x + x^2 - x^3 + x^4 + \cdots)$$

And if you know that the area under this curve is the logarithm, you can integrate the series for $1/(1+x)$ to obtain

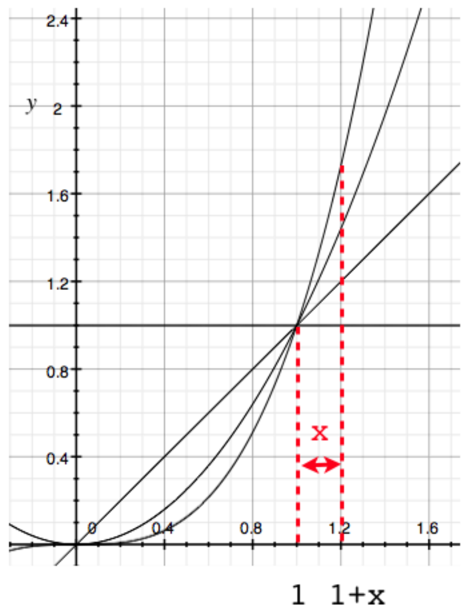
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

The logarithms obtained by this method can be easily verified.

derivation

So far as I know Newton did not provide a proof of his version of the binomial theorem. I found a discussion of how he came to it here:

http://www.quadrivium.info/MathInt/Notes/NewtonBinomial.pdf



Following Wallis, Newton studied expressions for the area under various curves.

powers of x

Above: $y = x^n, n = \{0, 1, 2\}$. These areas were apparently known prior to Newton, and I am not sure exactly how this was done, but I will use results obtained by integration.

It's perhaps a little confusing, but we use x here for two purposes. First, the curves are $f(x) = x^n$. Second, we seek the area under each of these curves between the endpoints 1 and $1 + x$. So for each n, we will compute the integral of x^n and evaluate that between the limits 1 and $1+x$.

- For $y = x^0 = 1$, the area under the curve is just $x \cdot 1 = x$.
- For $y = x$, the area under the curve is

$$\frac{x^2}{2}\Bigg|_1^{1+x} = \frac{(1+x)^2-1}{2} = \frac{2x+x^2}{2} = x + \frac{x^2}{2}$$

- For $y = x^2$, the area under the curve is

$$\frac{x^3}{3}\Bigg|_1^{1+x} = \frac{(1+x)^3-1}{3} = \frac{3x+3x^2+x^3}{3}$$

$$= x + x^2 + \frac{x^3}{3}$$

- For $y = x^3$, the area under the curve is

$$\frac{x^4}{4}\Bigg|_1^{1+x} = \frac{(1+x)^4-1}{4} = \frac{4x+6x^2+4x^3+x^4}{4}$$

$$= x + \frac{3}{2}x^2 + x^3 + \frac{x^4}{4}$$

which we can write as

$$= x + \frac{3}{2}x^2 + \frac{3}{3}x^3 + \frac{x^4}{4}$$

- For $y = x^4$, the area under the curve is

$$\frac{x^5}{5}\Bigg|_1^{1+x} = \frac{(1+x)^5-1}{5}$$

$$= \frac{5x+10x^2+10x^3+5x^4+x^5}{5}$$

$$= x + 2x^2 + 2x^3 + x^4 + \frac{x^5}{5}$$

which we can write as

$$x + \frac{4}{2}x^2 + \frac{6}{3}x^3 + \frac{6}{4}x^4 + \frac{1}{5}x^5$$

∘ For $y = x^5$, we would find

$$x + \frac{5}{2}x^2 + \frac{10}{3}x^3 + \frac{10}{4}x^4 + \frac{5}{5}x^5 + \frac{1}{6}x^6$$

If we look carefully at what we've obtained, we see that there is a sum of terms like x^p/p times a cofactor which goes like Pascal's triangle or a standard binomial expansion (and indeed, that's where it came from). Newton organized the cofactors into a table.

table

p	0	1	2	3	4	5
$x/1$	1	1	1	1	1	1
$x^2/2$	0	1	2	3	4	5
$x^3/3$	0	0	1	3	6	10
$x^4/4$	0	0	0	1	4	10
$x^5/5$	0	0	0	0	1	5
$x^6/6$	0	0	0	0	0	1
$x^7/7$	0	0	0	0	0	0

That is, we have for x^2 that the area is

$$(1)x + (2)\frac{1}{2}x^2 + (1)\frac{1}{3}x^3 = x + x^2 + \frac{1}{3}x^3$$

Newton noticed (as did Pascal) that the pattern of coefficients can be generated by addition. For example, the 6 in the column for $n = 4$

is generated by adding together the entry to its immediate left (3), plus the entry above that (also 3). Thus, having the first row (all 1), and the column under $n = 0$, one can generate the rest of the table mechanically.

Now, Newton says, what happens if we add an additional column for $n = -1$, and we make a rule that the entry in the first row must be 1, because all the other columns have this value.

p	-1	0	1	2	3	4	5
$x/1$	1	1	1	1	1	1	1
$x^2/2$	.	0	1	2	3	4	5
$x^3/3$	.	0	0	1	3	6	10
$x^4/4$	.	0	0	0	1	4	10
$x^5/5$	.	0	0	0	0	1	5
$x^6/6$	.	0	0	0	0	0	1
$x^7/7$	.	0	0	0	0	0	0

How do we fill in the missing entries? By using the addition rule! The first missing value must be a -1, so that it plus the 1 above add together to give the 0 to its right.

p	-1	0	1	2	3	4	5
$x/1$	1	1	1	1	1	1	1
$x^2/2$	-1	0	1	2	3	4	5
$x^3/3$	.	0	0	1	3	6	10
$x^4/4$	.	0	0	0	1	4	10
$x^5/5$	.	0	0	0	0	1	5
$x^6/6$	.	0	0	0	0	0	1
$x^7/7$	.	0	0	0	0	0	0

He filled out the rest of the column for $n = 0$ using this idea.

p	-1	0	1	2	3	4	5
$x/1$	1	1	1	1	1	1	1
$x^2/2$	-1	0	1	2	3	4	5
$x^3/3$	1	0	0	1	3	6	10
$x^4/4$	-1	0	0	0	1	4	10
$x^5/5$	1	0	0	0	0	1	5
$x^6/6$	-1	0	0	0	0	0	1
$x^7/7$	1	0	0	0	0	0	0

This gives the series for $1/(1 + x)$ that we have above.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \dots$$

One can check that it's correct by multiplying out.

$$1 = 1 - x + x - x^2 + x^2 - x^3 + x^3 - x^4 + x^4 + \cdots = 1$$

Using this idea, one can fill in the table for the negative integers. In particular, the column for -2 is

p	-2	-1	0	1	2	3	4	5
$x/1$	1	1	1	1	1	1	1	1
$x^2/2$	-2	-1	0	1	2	3	4	5
$x^3/3$	3	1	0	0	1	3	6	10
$x^4/4$	-4	-1	0	0	0	1	4	10
$x^5/5$	5	1	0	0	0	0	1	5
$x^6/6$	-6	-1	0	0	0	0	0	1
$x^7/7$	7	1	0	0	0	0	0	0

so the series is

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

Multiply by $1+x$. We do this by first multiplying by x

$$x - 2x^2 + 3x^3 - 4x^4 + \dots$$

and then adding to the series itself to obtain

$$1 - x + x^2 - x^3 + \dots$$

This confirms that

$$\frac{1}{(1+x)} = (1+x)(1 - 2x + 3x^2 - 4x^3 + \dots)$$

rational powers

What about fractional powers?

p	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$x/1$	1	1	1	1	1	1	1	1	1
$x^2/2$	-1	.	0	.	1	.	2	.	3
$x^3/3$	1	.	0	.	0	.	1	.	3
$x^4/4$	-1	.	0	.	0	.	0	.	1
$x^5/5$	1	.	0	.	0	.	0	.	0
$x^6/6$	-1	.	0	.	0	.	0	.	0

After a lot of work, Newton comes to two simple ideas.

First, the addition rule remains in place, for entries that are separated by a whole unit. The missing entries in the table above depend not on the entries to the immediate left, but one more column over.

Given just one of these missing entries, the entire row can be filled in. Let's suppose that 1/2 is the entry between 0 and 1. We use the addition rule to fill in the rest of that row:

p	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$x/1$	1	1	1	1	1	1	1	1	1
$x^2/2$	-1	$-1/2$	0	$1/2$	1	$3/2$	2	$5/2$	3
$x^3/3$	1	.	0	.	0	.	1	.	3
$x^4/4$	-1	.	0	.	0	.	0	.	1
$x^5/5$	1	.	0	.	0	.	0	.	0

So then the question is, where does the first entry of $-1/2$ or $1/2$ come from in the row for x^2?

Newton came up with the following pattern, which is just a generalization of the rule: add up the entry to the left plus the entry above it.

$$\begin{array}{cccccc}
a & a & a & a & a \\
b & a+b & 2a+b & 3a+b & 4a+b \\
c & b+c & a+2b+c & 3a+3b+c & 6a+4b+c \\
d & c+d & b+2c+d & a+3b+3c+d & 4a+6b+4c+d \\
.. & & & &
\end{array}$$

However, this won't work for the fractional tables, because $a = 1$ and the increment between whole values in the second row is now $2a = 2$, whereas it should be 1.

To solve this problem, we use the same pattern, but *decouple* the rows from each other by changing the names of the variables.

$$\begin{array}{cccccc}
a & a & a & a & a \\
b & c+b & 2c+b & 3c+b & 4c+b \\
d & e+d & f+2e+d & 3f+3e+d & 6f+4e+d \\
g & h+g & i+2g+h & k+3i+3h+g & 4k+6i+4h+g \\
.. & & & &
\end{array}$$

Consider the first pattern

$$b \quad c+b \quad 2c+b \quad 3c+b \quad 4c+b$$

applied to the row for x^2 where we have, starting with the column 0:

$$0 \quad ? \quad 1 \quad ? \quad 2$$

We can write two equations in two unknowns, namely

$$b = 0$$

$$2c + b = 1$$

so $c = 1/2$. Now, fill in the fractional columns as

$$c + b = 1/2$$

$$3c + b = 3/2$$

For the second pattern

$$d \quad e+d \quad\quad f+2e+d \quad 3f+3e+d \quad 6f+4e+d$$

applied to the third row (x^3) where we have starting with the column 0

$$0 \quad ? \quad 0 \quad ? \quad 1$$

We conclude that

$$d = 0$$

$$f + 2e + d = 0$$

$$6f + 4e + d = 1$$

Solving for $e = -f/2$ and substituting in the last equation:

$$6f - 2f = 1$$

We have $f = 1/4$ and $e = -1/8$ and then fill in the fractional columns as

$$e + d = -1/8$$

$$3f + 3e + d = 3/4 - 3/8 = 3/8$$

This allows us to fill in the rest of the third row.

p	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$x/1$	1	1	1	1	1	1	1	1	1
$x^2/2$	-1	$-1/2$	0	$1/2$	1	$3/2$	2	$5/2$	3
$x^3/3$	1	$3/8$	0	$-1/8$	0	$3/8$	1	$15/8$	3
$x^4/4$	-1	.	0	.	0	.	0	.	1
$x^5/5$	1	.	0	.	0	.	0	.	0

At this point, we can extend the second row to the left indefinitely ($-3/2, -2\ldots$), use the pattern to find a single entry in any given row, then use the addition rule to generate all the other entries in that row.

We will skip the solution and just fill in one of the columns, for the $1/2$ power

p	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$x/1$	1	1	1	1	1	1	1	1	1
$x^2/2$	-1	$-1/2$	0	$1/2$	1	$3/2$	2	$5/2$	3
$x^3/3$	1	$3/8$	0	$-1/8$	0	$3/8$	1	$15/8$	3
$x^4/4$	-1	.	0	$3/48$	0	.	0	.	1
$x^5/5$	1	.	0	$-15/384$	0	.	0	.	0

Here is the final table:

Table 9

p / term	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$\frac{x}{1}$	1	1	1	1	1	1	1	1	1	1	1
$\frac{-x^3}{3}$	-1	$\frac{-1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$\frac{x^5}{5}$	1	$\frac{3}{8}$	0	$\frac{-1}{8}$	0	$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6
$\frac{-x^7}{7}$	-1	$\frac{-5}{16}$	0	$\frac{3}{48}$	0	$\frac{-1}{16}$	0	$\frac{5}{16}$	1	$\frac{35}{16}$	4
$\frac{x^9}{9}$	1	$\frac{35}{128}$	0	$\frac{-15}{384}$	0	$\frac{3}{128}$	0	$\frac{-5}{128}$	0	$\frac{35}{128}$	1
$\frac{-x^{11}}{11}$	-1	$\frac{-63}{256}$	0	$\frac{105}{3840}$	0	$\frac{-3}{256}$	0	$\frac{3}{256}$	0	$\frac{-7}{256}$	0
$\frac{x^{13}}{13}$	1	$\frac{231}{1024}$	0	$\frac{-945}{46080}$	0	$\frac{7}{1024}$	0	$\frac{-5}{1024}$	0	$\frac{7}{1024}$	0

Looking at these values, Newton came up with his version of the binomial, which can generate them, and is where we started.

Pi

Computation of π

The series generated under the 1/2 power is

$$x - \frac{1}{4}\,\frac{x^2}{2} + \frac{3}{16}\,\frac{x^3}{3} - \frac{15}{96}\,\frac{x^4}{4} + \cdots$$

if $x = \sqrt{1-u^2}$, this is

$$x - \frac{1}{4}\,\frac{x^2}{2} + \frac{3}{16}\,\frac{x^3}{3} - \frac{15}{96}\,\frac{x^4}{4} + \cdots$$

and it's the area under the quarter-circle ($\pi/4$), when $x = 1$.

$$\frac{\pi}{4} = 1 - \frac{1}{4}\ \frac{1}{2} + \frac{3}{16}\ \frac{1}{3} - \frac{15}{96}\ \frac{1}{4}$$
$$= 1 - \frac{1}{8} + \frac{3}{48} - \frac{15}{384} + \cdots$$

which is a reasonable series for $\pi/4$.

Taylor series

It's worth mentioning how we would get this series by a modern approach. Write the general Taylor series

$$\sum f^n(x-a)\frac{(x-a)^n}{n!}$$

near $a = 0$

$$\sum f^n(x)\frac{x^n}{n!}$$

The function is

$$f(x) = \frac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$$

The derivatives are

$$f'(x) = -\frac{1}{2}(1+x)^{-3/2} = -\frac{1}{2}$$
$$f''(x) = \frac{3}{4}(1+x)^{-5/2} = \frac{3}{4}$$
$$f'''(x) = -\frac{15}{8}(1+x)^{-7/2} = -\frac{15}{8}$$
$$f''''(x) = \frac{105}{16}(1+x)^{9/2} = \frac{105}{16}$$

The series is then

$$1-\frac{1}{2}x+\frac{3}{4}\frac{x^2}{2}-\frac{15}{8}\;\frac{x^3}{3!}+\frac{105}{16}\;\frac{x^4}{4!}+\dots$$

$$=1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\frac{35}{128}x^4+\dots$$

which matches the table, above.

Part XVI

Practice problems

Chapter 55

Integration problems

This chapter and the next contain a number of problems which you may find challenging and instructive. We have reached the end of our discussion of the theory of calculus at this point, though we will see many more applications later.

Definite integrals

$$\int_0^{\pi/2} \cos x \ dx$$

$$\int_0^{\pi/2} \cos\theta \ d\theta$$

$$\int_{-\pi/2}^{\pi/2} \sin x \ dx$$

$$\int_0^1 x^2 \ dx$$

$$\int_0^1 \sqrt{x} \ dx$$

$$\int_1^3 \frac{1}{x}\ dx$$

$$\int_1^2 3x^2 + 2x + 1\ dx$$

$$\int_0^8 x^{2/3}\ dx$$

$$\int_0^{\pi/4} \cos 2t\ dt$$

$$\int_1^3 \frac{1}{x^2}\ dx$$

$$\int_0^3 \frac{2t}{1+t^2}\ dt\ ; \quad \text{hint: watch the bounds}$$

$$= \int_{u=1}^{u=10} \frac{1}{u}\ du$$

$$= \ln u \Big|_{u=1}^{u=10} = \ln 10 - \ln 1 = \ln 10$$

$$\int_0^1 (3x-2)^3\ dx$$

$$\int_0^1 xe^{-x^2}\ dx$$

$$\int_{-\pi/4}^{\pi/4} \cos 2x\ dx$$

$$\int_{-1}^{1} xe^x \ dx$$

$$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}} \ dx$$

$$\int_0^{e-1} \ln(x+1) \ dx; \quad \text{hint: what is } \frac{d}{dx} x \ln x \ ?$$

$$\int_{\pi/3}^{\pi/2} \tan \frac{\theta}{2} \ \sec^2 \frac{\theta}{2} \ d\theta$$

$$\int_{\pi/2}^{x} \cos t \ dt$$

$$\int_0^{\ln 2} e^{2x} \ dx$$

$$\int_0^2 (x^3 + k) \ dx = 10 \ ; \quad \text{find } k$$

$$\int_0^{\ln 2} e^{2x} \ dx$$

$$\int_1^e \frac{\ln t}{t} \ dt$$

$$\int_0^1 xe^{x^2+1} \ dx$$

Indefinite integrals

$$\int \tan x \ dx$$

$$\int \ln x \ dx$$

$$\int \sec^2 x \ dx$$

$$\int \csc^2 \theta \ d\theta$$

$$\int \tan \theta \sec \theta \ d\theta$$

$$\int (\sqrt{x} + \frac{1}{x^3}) \ dx$$

$$\int \frac{3x^2 + x - 1}{x^2} \ dx$$

$$\int \frac{1}{u - 3} \ du$$

$$\int \cos^2(2x) \ \sin 2x \ dx$$

$$\int \frac{x}{\sqrt{3 - 4x^2}} \ dx$$

$$\int \frac{1}{\sqrt{9 - x^2}} \ dy$$

$$\int \frac{x}{(2 - x^2)^3} \ dx$$

$$\int \frac{e^x}{1 - 2e^x} \ dx$$

$$\int e^{x + e^x} \ dx$$

$$\int (x^3 - \sin 2x)\ dx$$

$$\int \frac{e^{3x}}{e^x}\ dx$$

$$\int \frac{z}{1-4z^2}\ dz$$

$$\int \frac{5}{1+x^2}\ dx$$

$$\int \frac{\cos x}{\sin^2 x}\ dx$$

$$\int \tan^4 t\ \ \sec^2 t\ dt$$

$$\int e^x \cos(e^x)\ dx$$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\ dx$$

$$\int \frac{x+1}{x^2+1}\ dx$$

$$\int \frac{x}{x+a}\ dx\ ;\quad \text{hint: } = \int \frac{x+a-a}{x+a}\ dx$$

$$\int a^u\ du\ ;\quad a = const$$

$$\int e^{4-\ln x}\ dx$$

$$\int x\ \sqrt{x+2}\ dx\ ;\quad \text{hint: } u = x+2$$

$$\int \frac{x}{\sqrt{x+3}}\ dx \ ; \quad \text{hint: } u = x+3$$

$$\int \frac{1+x}{\sqrt{x}}\ dx$$

$$\int \frac{1}{x^2+2x+5}\ dx$$

$$\int \frac{x^2+3}{x-1}\ dx \ : \quad \text{hint: make top divisible by } x-1$$

$$\int \frac{\ln x}{3x}\ dx$$

$$\int \frac{e^x}{e^x+1}\ dx$$

$$\int \frac{1+x}{\sqrt{x}}\ dx$$

$$\int \sin\theta\cos\theta\ d\theta$$

for the last, give both versions of the answer and show they are equal

$$\int (\sin x + \cos x)^2\ dx$$

$$\int (1+\tan x)^2\ dx$$

$$\int \frac{\cos^2 x}{1+\sin x}\ dx$$

$$\int \frac{\sin x}{1+\sin x}\ dx$$

$$\int \sin^3\ dx$$

$$\int \sec^2 x \ \sqrt{5 + \tan x} \ dx$$

$$\int \cos x \ e^{1+\sin x} \ dx$$

$$\int e^x \cos(e^x) \ dx$$

$$\int x \sin x \ dx$$

$$\int e^x \sin x \ dx$$

$$\int \frac{\sin x + \cos x}{e^{-x} + \sin x} \ dx \ ; \quad \text{hint: multiply by } e^x/e^x$$

$$\int \frac{2^{\ln x}}{x} \ dx$$

$$\int \frac{1}{x \ln x} \ dx$$

$$\int \frac{\ln \sqrt{x}}{x} \ dx$$

For absolute value problems, recall that

$$|x| = \begin{cases} x & : x \geq 0 \\ -x & : x < 0 \end{cases}$$

The method is to find the place where the expression inside the absolute value symbols is equal to zero, then integrate piecewise, substituting as shown above.

$$\int_0^2 |t-1|\ dt$$

Since $t-1=0$ when $t=1$ this is

$$\int_0^1 -(t-1)\ dt + \int_1^2 (t-1)\ dt$$
$$= (-\frac{1}{2}t^2+t)\ \Big|_0^1\ + (\frac{1}{2}t^2-t)\ \Big|_1^2$$
$$= (-\frac{1}{2}+1-0+0)+(2-2-\frac{1}{2}+1) = 1$$

Tricky to evaluate.

FTC

There is a perverse desire to make sure you understand the FTC (part 1).

If $F(x)$ is "nice" and

$$F(x) = \int_a^x f(t)\ dt$$

then..

$$F'(x) = \frac{d}{dx}\ F(x) = \frac{d}{dx}\ \int_a^x f(t)\ dt = f(x)$$

Problems: for each $G(x)$ below, find $G'(x)$

$$G(x) = \int_1^x 2t\ dt$$
$$G(x) = \int_0^x (2t^2+\sqrt{t})\ dt$$

$$G(x) = \int_0^x \tan t \ dt$$

$$G(x) = \int_{x^2}^x \frac{t^2}{1+t^2} \ dt$$

The last problem needs first to be manipulated into a (sum of) integrals between a constant (0) on the lower bound and x above, and then the one with x^2 must take account of the fact that if $t = -x^2$ then $dt = -2x \ dx$.

$$G(x) = -\int_0^{-x^2} \frac{t^2}{1+t^2} \ dt + \int_0^x \frac{t^2}{1+t^2} \ dt$$

$$G'(x) = -\frac{x^4}{1+x^4} \ (-2x) + \frac{x^2}{1+x^2}$$

hard one

Here is a problem involving the actual integral we had above. I didn't know how to solve it completely, but I found the answer on the web and can work backward and see that it's correct. Call it a challenge. It looks simple enough:

$$\int \frac{\sqrt{x}}{1-x} \ dx$$

substitute

$$u = \sqrt{x}, \quad u^2 = x, \quad 2u \ du = dx$$

we obtain

$$\int \frac{u}{1-u^2} \ 2u \ du$$

$$2 \int \frac{u^2}{1-u^2} \ du$$

If this were x in the numerator rather than x^2, it would be simple. Still, it looks like it ought to be easy, somehow. The answer is here:

(http://integrals.wolfram.com/index.jsp)

Let's change to x:

$$\int \frac{x^2}{1-x^2}\ dx$$

The first part of the answer is a useful trick for many problems. If the numerator is the same as the denominator, within a constant, then:

$$= -\int \frac{1-x^2-1}{1-x^2}\ dx$$
$$= -\int 1\ dx - \int \frac{1}{1-x^2}\ dx$$

Now the real trick is that the second part can be re-worked because it is a difference of squares

$$(1-x)(1+x) = 1-x^2$$
$$\int \frac{1}{1-x^2}\ dx = \frac{1}{2}\int (\frac{1}{1+x} + \frac{1}{1-x})\ dx$$

If we put the two terms on the right over the common denominator $1-x^2$, then for the numerator we have $1-x+1+x=2$. !! So the whole integral is

$$\int \frac{x^2}{1-x^2}\ dx$$
$$= -x - \frac{1}{2}\ [\ (\ln(1+x) - \ln(1-x))\ ]\ + C$$

$$= -x - \frac{1}{2} \ \ln \frac{(1+x)}{(1-x)} + C$$

I'll leave it to you to work out the answer to the original problem with $\sqrt{x}$.

another hard one

$$\int \frac{\sqrt{x^2+1}}{x} \ dx$$

Substitution: let $x = \tan t$. So opp $= x$, adj $= 1$, hyp $= \sqrt{1+x^2}$.

$$x = \tan t$$
$$dx = \sec^2 t \ dt$$
$$\sqrt{1+x^2} = \sec t$$

So the integral is

$$\int \frac{\sec t}{\tan t} \ \sec^2 t \ dt$$
$$= \int \frac{\sec t}{\tan t}(1+\tan^2 t) \ dt$$

The first term is

$$\int \frac{1}{\sin t} \ dt$$

and the second is

$$\int \sec t \tan t \ dt$$

$$= \int \frac{\sin t}{\cos^2 t}\ dt$$

The second part is easy $(1/\cos t)$. But the first requires more work. Let

$$u = \cos t$$
$$du = -\sin t\ dt$$

We rewrite the integral as

$$\int \frac{\sin t}{\sin^2 t}\ dt$$

$$= -\int \frac{1}{1-u^2}\ du$$
$$= -\frac{1}{2}\int \frac{1}{1+u} + \frac{1}{1-u}\ du$$
$$= -\frac{1}{2}(\ln(1+u) - \ln(1-u))$$

So, in terms of t we have (combining)

$$\frac{1}{\cos t} - \frac{1}{2}(\ln(1+\cos t) - \ln(1-\cos t))$$

In order to substitute back to x, we recall that

$$\frac{1}{\sqrt{1+x^2}} = \cos t$$

and I think we'll just leave it right there. Well, in the original problem we had a definite integral with limits $\sqrt{15}$ and $\sqrt{3}$, so that $\cos t =$

1/4 at the high end and $\cos t = 1/2$ at the low end which makes it considerably easier to evaluate.

$$
\begin{aligned}
&= 4 - \frac{1}{2}(\ln 5/4 - \ln 1/2) - 2 + \frac{1}{2}(\ln 3/2 - \ln 1/2) \\
&= 2 - \frac{1}{2}(\ln 5/2 + \ln 3)
\end{aligned}
$$

Chapter 56

Table of integrals

I thought we could work on the derivations of some of the integrals shown in standard tables.

Here is what I have so far. Each one of the following will come in three versions [A: $(x^2 + 1)$, B: $(x^2 - 1)$, C: $(1 - x^2)$]. Let's group them as follows:

$$\int \frac{1}{\sqrt{x^2+1}}\,dx \tag{56.1}$$

$$\int \sqrt{x^2+1}\,dx \tag{56.2}$$

$$\int \frac{1}{x^2+1}\,dx \tag{56.3}$$

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx \tag{56.4}$$

$$\int \frac{1}{x\sqrt{x^2+1}}\,dx \tag{56.5}$$

$$\int \frac{1}{x^2\sqrt{x^2+1}}\,dx \tag{56.6}$$

In every case, we can also deal with a similar integral containing x^2+a^2 (i.e. a^2 substituted for 1). This is done either by factoring the a part out or by setting up a trig substitution a little differently. For the most part, we'll just keep it simple, but we'll look at the effect of having a^2 in place of 1 for a few examples as shown below.

Let's see how far we get.

1A $\int 1/\sqrt{x^2+1}\ dx$

This one is fairly simple, a "trig substitution". Since we have $\sqrt{x^2+1}$, this suggests a right triangle with two sides x and 1, and hypotenuse $\sqrt{x^2+1}$. We will have a right triangle with angle t, and then imagine that we have opposite and adjacent sides:

$$\text{opp} = x, \quad \text{adj} = 1$$

and hypotenuse

$$\text{hyp} = \sqrt{x^2+1}$$

and, in terms of the trig functions, we obtain:

$$x = \tan t, \quad dx = \sec^2 t\ dt, \quad 1/\sqrt{x^2+1} = \cos t$$

This gives:

$$= \int \cos t \sec^2 t\ dt$$
$$= \int \sec t\ dt$$

$$= \ln | \ \sec t + \tan t \ | + c$$

Substitute back with x and finally we have, in summary:

$$\int \frac{1}{\sqrt{x^2+1}}\ dx = \ln | \ x + \sqrt{x^2+1} \ | + C$$

check

$$\begin{aligned} &\frac{d}{dx} \ln | \ x + \sqrt{x^2+1} \ | \\ &= \frac{1}{x+\sqrt{x^2+1}}(1 + \frac{x}{\sqrt{x^2+1}}) \\ &= \frac{1}{x+\sqrt{x^2+1}}(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}) \\ &= \frac{1}{\sqrt{x^2+1}} \end{aligned}$$

dealing with a^2

Let's think about how we'd have to change things for $x^2 + a^2$. One approach is to change the trig functions. Change the side with length 1 to have length a, then with hypotenuse $\sqrt{x^2+a^2}$, we have

$$\frac{x}{a} = \tan t, \quad dx = a \sec^2 t \ dt, \quad a/\sqrt{x^2+a^2} = \cos t$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\ dx = \int \frac{1}{a} \cos t \ a \sec^2 t \ dt = \int \sec t \ dt$$

We have the same integral and obtain almost the same answer as before

$$= \ln | \ \sec t + \tan t \ | + c$$
$$= \ln | \ \frac{1}{a}(\sqrt{x^2 + a^2} + x) \ | \ + C$$

check

We should check this result.

$$\frac{d}{dx} \ln \ | \ \frac{1}{a}(\sqrt{x^2 + a^2} + x) \ |$$

$$= \frac{a}{x + \sqrt{x^2 + a^2}} \cdot \frac{1}{a}(\frac{x}{\sqrt{x^2 + a^2}} + 1)$$

$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \frac{x + \sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}} = \frac{1}{\sqrt{x^2 + a^2}}$$

factoring method

A second approach is to factor out the a, obtaining

$$\int \frac{1}{\sqrt{x^2 + a^2}} \ dx = \frac{1}{a} \int \frac{1}{\sqrt{(x/a)^2 + 1}} \ dx$$

I think of this as multiplying ”on the top and on the bottom” by $1/a$, with the factor out in front being the multiplication on the top. Now, substitute

$$u = x/a, \quad a \ du = dx$$

$$\frac{1}{a}\int \frac{1}{\sqrt{(x/a)^2+1}}\ dx = \int \frac{1}{\sqrt{u^2+1}}\ du$$

$$= \ln |\ \sqrt{u^2+1} + u\ | + C$$
$$= \ln\ |\ \sqrt{(x/a)^2+1} + x/a\ | + C$$
$$= \ln\ |\ \frac{1}{a}\sqrt{x^2+a^2} + \frac{1}{a}x\ | + C$$
$$= \ln |\ \frac{1}{a}(\sqrt{x^2+a^2} + x)\ |\ + C$$

as we had before.

inverse hyperbolic sine

There is actually *another* version of the answer to this problem. It involves the hyperbolic sine, which is defined as follows

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

The answer is

$$\int \frac{1}{\sqrt{x^2+a^2}}\ dx = \sinh^{-1}\frac{x}{a}$$

Proof. Represent the value of the integral as z, then

$$z = \sinh^{-1}\frac{x}{a}$$

$$\frac{x}{a} = \sinh z$$
$$\frac{x}{a} = \frac{1}{2}(e^z - e^{-z})$$

Solve for z. Let $u = e^z$ so we have

$$\frac{2x}{a} = u - \frac{1}{u}$$
$$au^2 - 2xu - a = 0$$

The roots are

$$u = \frac{2x \pm \sqrt{4x^2 + 4a^2}}{2a}$$
$$u = \frac{x \pm \sqrt{x^2 + a^2}}{a}$$

For real z, $e^z > 0$ so we need the positive root

$$u = \frac{x + \sqrt{x^2 + a^2}}{a}$$

Go back to z

$$e^z = \frac{x + \sqrt{x^2 + a^2}}{a}$$
$$z = \ln|x + \sqrt{x^2 + a^2}| - \ln a$$

1B $\int 1/\sqrt{x^2-1}\ dx$

This one seems a bit easier. Again, do a trig substitution.

$$\text{hyp} = x, \quad \text{adj} = 1, \quad \text{opp} = \sqrt{x^2-1}$$

Now, $x = \sec t$, $dx = \sec t \tan t\ dt$, and $\sqrt{x^2-1} = \tan t$

$$= \int \frac{1}{\tan t} \sec t \tan t\ dt$$
$$= \int \sec t\ dt$$

The same as for # 1A except for the minus sign under the square root.

$$= \ln|\sec t + \tan t| + c = \ln|x + \sqrt{x^2-1}| + C$$

check

$$\frac{1}{x+\sqrt{x^2-1}}\ (1 + \frac{x}{\sqrt{x^2-1}})$$
$$= \frac{1}{x+\sqrt{x^2-1}}\ (\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}) = \frac{1}{\sqrt{x^2-1}}$$

dealing with a^2

$$\int \frac{1}{\sqrt{x^2-a^2}}\ dx$$

Our substitution is now

$$\text{hyp} = x, \quad \text{adj} = a, \quad \text{opp} = \sqrt{x^2-a^2}$$

with $x = a \sec t$, $dx = a \sec t \tan t\ dt$, and $\sqrt{x^2 - a^2} = a \tan t$. Then the integral becomes

$$\int \frac{1}{a \tan t}\ a \sec t \tan t\ dt$$

The integral comes out just as before: $\ln \mid \sec t + \tan t \mid$, but the substitution back to x gives an extra factor of $1/a$ for both terms:

$$= \ln \mid \frac{x}{a} + \frac{1}{a}\sqrt{x^2 - a^2} \mid$$

These two results (A and B) are usually combined and written

$$\int \frac{1}{\sqrt{x^2 \pm a^2}}\ dx = \ln \mid \frac{1}{a}\sqrt{x^2 \pm a^2} + \frac{1}{a}x \mid\ + C$$

1C $\int 1/\sqrt{1 - x^2}\ dx$

This one is just $\sin^{-1} x$. We get that by

$$\text{opp} = x, \quad \text{hyp} = 1, \quad \text{adj} = \sqrt{1 - x^2}$$

So

$$x = \sin t$$
$$t = \sin^{-1} x$$
$$\frac{dx}{dt} = \cos t = \sqrt{1 - x^2}$$
$$\frac{dt}{dx} = \frac{1}{\cos t} = \frac{1}{\sqrt{1 - x^2}}$$

The answer is

$$\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x$$

dealing with a^2

It's easy to see that we will have

$$\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}$$

2A $\int \sqrt{x^2+1}\ dx$

Multiply top and bottom by what's on the top

$$= \int \frac{x^2+1}{\sqrt{x^2+1}}\ dx$$

$$= \int \frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}\ dx$$

The term on the right is # 1A from above. The answer is:

$$\ln|\sqrt{x^2+1}+x| + C$$

Now for the term on the left. That is actually # 4A. The answer (still with an integral in it) is:

$$\int \frac{x^2}{\sqrt{x^2+1}}\ dx = x\sqrt{x^2+1} - \int \sqrt{x^2+1}\ dx + C$$

We've come full circle. Appearing in the answer is minus the same integral that we started with. It's OK. We assemble everything, grouping the two identical terms on the left, divide by 2, and obtain:

$$\int \sqrt{x^2+1}\; dx = \frac{1}{2}\; x\sqrt{x^2+1} + \frac{1}{2}\ln |\; \sqrt{x^2+1} + x\; | + C$$

check

Leave the factor of 1/2 aside for the moment. The derivative of the first term is

$$\sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}}$$

We checked the second term above in # 1A, it is just

$$\frac{1}{\sqrt{x^2+1}}$$

So now we have

$$\begin{aligned} &\sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} \\ &= \frac{x^2+1}{\sqrt{x^2+1}} + \frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} \\ &= \frac{2(x^2+1)}{\sqrt{x^2+1}} \end{aligned}$$

Remember the factor of 1/2, and simplify

$$= \sqrt{x^2+1}$$

Trig substitution

This one can also be done by a trig substitution: $x/a = \tan t$. Then $\sqrt{x^2+a^2}/a = \sec t$, and $dx = \sec^2 t\ dt$ so we have

$$\int \sec^3 t\ dt = \frac{1}{2}(\sec t \tan t + \ln|(\sec t + \tan t)| + C$$

We pick up some factors of a in substituting back to x

$$= \frac{1}{2}(\frac{\sqrt{x^2+a^2}}{a}\ \frac{x}{a} + \ln|\frac{1}{a}(\sqrt{x^2+a^2}+x)| + C$$

2B $\int \sqrt{x^2-1}\ dx$

Use a trig substitution.

$$\text{hyp} = x, \quad \text{adj} = 1, \quad \text{opp} = \sqrt{x^2-1}$$
$$x = \sec t, \quad dx = \sec t \tan t\ dt, \quad \sqrt{x^2-1} = \tan t$$

We have:

$$\int \sqrt{x^2-1}\ dx = \int \sec t \tan^2 t\ dt$$
$$= \int \sec t + \sec^3 t\ dt$$

We've done $\sec^3$ elsewhere:

$$= \ln|\sec t + \tan t\ | + \frac{1}{2}\sec t \tan t + \frac{1}{2}\ln|\sec t + \tan t\ |C$$

$$= \frac{3}{2} \ln |x + \sqrt{x^2 - 1}| + \frac{1}{2} x \sqrt{x^2 - 1} + C$$

check

Do it term by term. The derivative of the first term is

$$= \frac{3/2}{x + \sqrt{x^2 - 1}} \cdot (1 + \frac{x}{\sqrt{x^2 - 1}})$$

$$= \frac{3/2}{x + \sqrt{x^2 - 1}} \cdot (\frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}}) = \frac{3/2}{\sqrt{x^2 - 1}}$$

The second part is

$$= \frac{1}{2}(\sqrt{x^2 - 1} + \frac{x^2}{\sqrt{x^2 - 1}})$$

$$= \frac{1}{2}(\frac{x^2 - 1}{\sqrt{x^2 - 1}} + \frac{x^2}{\sqrt{x^2 - 1}})$$

Putting it all together we have

$$= \frac{3/2 + x^2 - 1/2}{\sqrt{x^2 - 1}} = \sqrt{x^2 - 1}$$

2C $\int \sqrt{1 - x^2}\ dx$

Multiply top and bottom by $\sqrt{1 - x^2}$

$$= \int \frac{1 - x^2}{\sqrt{1 - x^2}}$$

$$= \int \frac{1}{\sqrt{1-x^2}}\ dx - \frac{x^2}{\sqrt{1-x^2}}\ dx$$

The first term is just $\sin^{-1} x$ and the second one is # 4C. Substituting the answer from there:

$$\int \sqrt{1-x^2}\ dx = \sin^{-1} x + x\sqrt{1-x^2} - \int \sqrt{1-x^2}\ dx$$

Two copies of our integral, so:

$$2\int \sqrt{1-x^2}\ dx = \sin^{-1} x + x\sqrt{1-x^2}$$

$$\int \sqrt{1-x^2}\ dx = \frac{1}{2}\sin^{-1} x + \frac{x}{2}\sqrt{1-x^2}$$

check

We differentiate term by term. Remember the factor of 1/2. The first term is just

$$\frac{1}{\sqrt{1-x^2}}$$

while the derivative of $x\sqrt{1-x^2}$ is

$$\frac{-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}$$

$$= \frac{-x^2+1-x^2}{\sqrt{1-x^2}}$$

Adding the terms together we obtain

$$= \frac{1}{\sqrt{1-x^2}} + \frac{-x^2+1-x^2}{\sqrt{1-x^2}}$$

Recall the factor of 1/2

$$= \frac{1-x^2}{\sqrt{1-x^2}} = \sqrt{1-x^2}$$

3A $\int 1/(x^2+1)\ dx$

This one is just $\tan^{-1} x$. We derive this as follows

$$\text{opp} = x, \quad \text{adj} = 1$$

then the hypotenuse is

$$\text{hyp} = \sqrt{x^2+1}$$

We have

$$x = \tan t$$
$$t = \tan^{-1} x$$
$$\frac{dx}{dt} = \sec^2 t = 1 + x^2$$
$$\frac{dt}{dx} = (\tan^{-1} x)' = \frac{1}{1+x^2}$$

Summarizing:

$$\int \frac{1}{x^2+1}\ dx = \tan^{-1} x + C$$

If we started with

$$\int \frac{1}{x^2+a^2}\ dx = \frac{1}{a^2}\int \frac{1}{(x/a)^2+1}\ dx$$

then we pick up a factor of a from the substitution ($u = x/a$, $a\ du = dx$) and end up with

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

3B $\int 1/(x^2-1)\ dx$

Ingenious trick:

$$\begin{aligned}
&\int \frac{1}{x^2-1}\ dx\\
&= \int \frac{1}{(x+1)(x-1)}\ dx\\
&= -\frac{1}{2}\int \frac{1}{x+1} - \frac{1}{x-1}\ dx\\
&= \frac{1}{2}\int \frac{1}{x-1} - \frac{1}{x+1}\ dx\\
&= \frac{1}{2}(\ln\ |\ x-1\ |\ - \ln\ |\ x+1\ |\ ) + C
\end{aligned}$$

check

Leave aside the factor of 1/2.

$$\frac{d}{dx} \ \ln \ | \ x-1 \ | \ \ -\ln \ | \ x+1 \ | \ \ |$$
$$= \frac{1}{x-1} - \frac{1}{x+1}$$
$$= \frac{x+1-x+1}{x^2-1}$$

Recall the factor of 1/2, and we're done.

3C $\int 1/(1-x^2) \ dx$

Ingenious trick, again:

$$= \frac{1}{2} \int \frac{1}{1-x} + \frac{1}{1+x} \ dx$$
$$= \frac{1}{2} - \ln|1-x| + \ln|1+x| + C$$
$$= \frac{1}{2} \ln| \ \frac{1+x}{1-x} \ \ |$$

check

Leave aside the factor of 1/2.

$$\frac{d}{dx} \ \ln| \ 1+x \ \ -\ln| \ 1-x \ |$$
$$= \frac{1}{1+x} + \frac{1}{1-x}$$

$$= \frac{1 - x + 1 + x}{1 - x^2}$$

recall the factor of 1/2, and we're done.

4A $\int x^2/\sqrt{x^2+1}\ dx$

We'll try integration by parts (IBP). Let $u = x$ and $du = dx$, then

$$dv = \frac{x}{\sqrt{x^2+1}}\ dx$$
$$v = \sqrt{x^2+1}$$

We have then

$$= x\sqrt{x^2+1} - \int \sqrt{x^2+1}\ dx$$

and the right-hand term is # 1A.

Recall:

$$\int \sqrt{x^2+1}\ dx = \frac{1}{2}\ [\ x\sqrt{x^2+1} + \ln|\sqrt{1+x^2} + x| + C$$

So ???

$$\int \frac{x^2}{\sqrt{x^2+1}}\ dx = \frac{1}{2}\ [\ x\sqrt{x^2+1} + \ln|\sqrt{1+x^2} + x| + C$$

4B $\int x^2/\sqrt{x^2-1}\ dx$

Use IBP. Let $u = x$, $du = dx$,

$$dv = \frac{x}{\sqrt{x^2-1}}\ dx$$
$$v = \sqrt{x^2-1}$$

So the integral is

$$= \frac{3}{2}x\sqrt{x^2-1} - \frac{3}{2}\ln|x + \sqrt{x^2-1}| + C$$

4C $\int x^2/\sqrt{1-x^2}\ dx$

Use IBP. Let $u = x$, $du = dx$,

$$dv = \frac{x}{\sqrt{1-x^2}}\ dx$$
$$v = -\sqrt{1-x^2}$$

So the integral is

$$= -x\sqrt{1-x^2} + \int \sqrt{1-x^2}\ dx$$

which is # 1C. Kind of circular!

$$= -x\sqrt{1-x^2} + \sin^{-1}x + \frac{x}{2}\sqrt{1-x^2}$$

$$= \sin^{-1}x - \frac{x}{2}\sqrt{1-x^2}$$

5A $\int 1/x\sqrt{x^2+1}\ dx$

$$\text{opp} = x, \quad \text{adj} = a, \quad \text{hyp} = \sqrt{x^2+a^2}$$

$$x = a\tan t$$
$$dx = a\sec^2 t\ dt$$
$$\frac{a}{\sqrt{x^2+a^2}} = \cos t$$

So

$$\int \frac{1}{x\sqrt{x^2+a^2}}\ dx$$
$$= \int \frac{1}{a\tan t}\ \cos t\ a\sec^2 t\ dt$$
$$= \int \frac{1}{\sin t}\ dt$$

It's easy to forget, but we've seen this one. It is just

$$\int \csc t\ dt$$
$$= -\ln\ |\ \csc t + \cot t\ |$$
$$= -\ln\ |\ \csc t + \cot t\ |$$
$$= -\ln\ |\ \frac{\sqrt{x^2+a^2}}{x} + \frac{a}{x}\ |$$

check1

Remember the factor of -1. The first part of the derivative is just

$$\frac{1}{\frac{\sqrt{x^2+a^2}}{x} + \frac{a}{x}}$$

Multiply top and bottom by x

$$\frac{x}{\sqrt{x^2+a^2}+a}$$

Now we need

$$\frac{d}{dx}\ \frac{\sqrt{x^2+a^2}}{x} + \frac{a}{x}$$

The first term is

$$= \left(\frac{x^2}{\sqrt{x^2+a^2}} - \sqrt{x^2+a^2}\right)\frac{1}{x^2}$$

Combined with the second term we have

$$\begin{aligned} &= \left(\frac{x^2}{\sqrt{x^2+a^2}} - \sqrt{x^2+a^2}\right)\frac{1}{x^2} - \frac{a}{x^2} \\ &= \left(\frac{x^2-x^2-a^2}{\sqrt{x^2+a^2}}\right)\frac{1}{x^2} - \frac{a}{x^2} \\ &= \frac{-a^2}{x^2\sqrt{x^2+a^2}} - \frac{a}{x^2} \end{aligned}$$

Finally, multiply by what we got from the logarithm:

$$= \frac{x}{\sqrt{x^2+a^2}+a} \; [\; \frac{-a^2}{x^2\sqrt{x^2+a^2}} - \frac{a}{x^2} \;]$$

$$= \frac{1}{\sqrt{x^2+a^2}+a} \; [\; \frac{-a^2}{x\sqrt{x^2+a^2}} - \frac{a}{x} \;]$$

Factor out $-a$ and recall the -1 so it's just a to remember

$$= \frac{1}{\sqrt{x^2+a^2}+a} \; [\; \frac{a}{x\sqrt{x^2+a^2}} + \frac{1}{x} \;]$$

I don't know what to do with that factor of $\sqrt{x^2+a^2}+a$.

check2

Remember the factor of -1. The derivative of the rest is weird looking for sure. Do it in the variable t! It's just

$$\csc t = \frac{\sqrt{x^2+a^2}}{x}$$

times the derivative of $\csc t$. What is that? It is (using x):

$$(\frac{x^2}{\sqrt{x^2+a^2}} - \sqrt{x^2+a^2})\frac{1}{x^2}$$

For this part I get (by the usual trick)

$$\frac{1}{x^2}\frac{a^2}{\sqrt{x^2+a^2}}$$

needs more work!

5B $\int 1/x\sqrt{x^2-1}\ dx$

This one is the third important inverse trig function $\sec^{-1} x$.

$$\int \frac{1}{x\sqrt{x^2-1}}\ dx = \sec^{-1} x + C$$

How do we get this?

$$\text{hyp} = x, \quad \text{adj} = 1, \quad \text{opp} = \sqrt{x^2-1}$$

We have

$$x = \sec t$$
$$t = \sec^{-1} x$$

$$\frac{dx}{dt} = \sec t \tan t = x\sqrt{x^2-1}$$
$$\frac{dt}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

5C $\int 1/x\sqrt{1-x^2}\ dx$

Recall

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{(1+x)(1-x)}$$
$$= \frac{1}{2}(\frac{1}{1+x} + \frac{1}{1-x})$$

What now?

6A $\int 1/x^2\sqrt{x^2+1}\ dx$

$$\int \frac{1}{x^2\sqrt{x^2+1}}\ dx$$

I struggled with this so, finally, I looked it up in Strang. It turned out I had the right answer, my check by differentiation had a mistake.

$$\int \frac{1}{x^2\sqrt{x^2 \pm a^2}}\ dx = \mp\frac{\sqrt{x^2+a^2}}{a^2x} + C$$

And it's easy enough to check when you know it's right. Use the first case (x^2+a^2). Save the factor of $-1/a^2$ for later.

Recall the quotient rule $(u'v - uv'/v^2)$. The derivative is

$$[\ \frac{x^2}{\sqrt{x^2+a^2}} - \sqrt{x^2+a^2}\ ]\ \frac{1}{x^2}$$
$$[\ \frac{x^2-x^2-a^2}{\sqrt{x^2-a^2}}\ ]\ \frac{1}{x^2}$$

Do the subtraction, cancel using the factor of $-1/a^2$ and we're done. So now, try to derive it (first case) from the integral.

$$\text{opp} = x, \quad \text{adj} = a, \quad \text{hyp} = \sqrt{x^2+a^2}$$

$$x = a\tan t$$
$$dx = a\sec^2 t\ dt$$
$$\frac{a^2}{x^2} = \frac{1}{\tan^2 t} = \frac{\cos^2 t}{\sin^2 t}$$

$$\frac{1}{\sqrt{x^2+a^2}} = \cos t$$

Substituting:

$$\begin{aligned}
&= \int \frac{\cos^2 t}{a^2 \sin^2 t} \cos t \frac{a}{\cos^2 t}\ dt \\
&= \frac{1}{a} \int \cot t \csc t\ dt \\
&= -\frac{\csc t}{a} + c \\
&= -\frac{\sqrt{x^2+1}}{ax}
\end{aligned}$$

6B $\int 1/x^2\sqrt{x^2-1}\ dx$

See # 6A.

6C $\int 1/x^2\sqrt{a^2-x^2}\ dx$

$$\text{opp} = x, \quad \text{adj} = \sqrt{a^2-x^2}, \quad \text{hyp} = a$$

$$\begin{aligned}
x &= a \sin t \\
dx &= a \cos t\ dt \\
\frac{a}{\sqrt{a^2-x^2}} &= \sec t
\end{aligned}$$

$$\int \frac{1}{x^2\sqrt{a^2-x^2}}\ dx$$

$$= \int \frac{1}{a^2 \sin^2 t} \ \frac{1}{a} \sec t \ a \cos t \ dt$$

$$= \frac{1}{a^2} \int \frac{1}{\sin^2 t} \ dt$$

The derivative of $\tan t$ is $\sec^2 t \ dt$, so the derivative of $\cot t$ is $-\csc^2 t \ dt$, and our integral is then $-\cot t$ so

$$= -\frac{1}{a^2} \frac{\sqrt{a^2 - x^2}}{x} + C$$

check

$$\frac{d}{dx} \ \ -\frac{1}{a^2} \frac{\sqrt{a^2 - x^2}}{x}$$

Save the factor of $-1/a^2$. Recall the quotient rule (above). We have

$$(\frac{-x^2}{\sqrt{a^2 - x^2}} - \sqrt{a^2 - x^2}) \ \frac{1}{x^2}$$

$$(\frac{-x^2 - a^2 + x^2}{\sqrt{a^2 - x^2}}) \ \frac{1}{x^2}$$

Do the cancellation in the numerator, recall the factor of $-1/a^2$, and we're done.

Part XVII

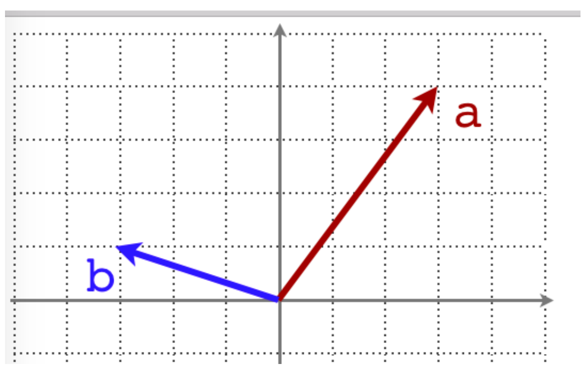
Vectors

Chapter 57

Vector dot product

In this chapter, we look at a few useful properties and operations of vectors in two- and three-dimensional space. I assume that you have already encountered vectors before, so this is not totally new.

From a geometrical point of view, a vector is a mathematical object that has both magnitude and direction. For example, in the standard 2D-coordinate system, the (maroon) vector $\langle\ 3, 4\ \rangle$ goes out from the origin three units in the x-direction and four units in the y-direction.



Vectors are written in bold type:

$$\mathbf{a} = \langle\ 3, 4\ \rangle$$

$$\mathbf{b} = \langle\ -3, 1\ \rangle$$

A vector has one property of a line, slope, but the fixed magnitude means that a vector does not extend to infinity as a line does. The squared length of a vector can be computed as the sum of the squares of its components, according to Pythagoras.

$$(\text{length } \mathbf{a})^2 = |\mathbf{a}|^2 = 3^2 + 4^2$$

By convention, we allow vectors to move about in space. We mean that two vectors of the same length, and pointing in the same direction are considered to be the same object, regardless of where they are located in space. (Some physics problems don't allow this, but in math it's the usual case).

So if we have the vector $\mathbf{v} = \langle\ 1, 1\ \rangle$ starting at the origin $(0, 0)$ and ending at the point $(1, 1)$, and compare it to a second vector $\mathbf{u}$ that starts from $(2, 0)$ and ends at $(3, 1)$, those are considered to be the same vector.

As you might guess, the vector that connects two points (x_1, y_1) and (x_2, y_2) is

$$\mathbf{p} = \ \langle\ x_2 - x_1, y_2 - y_1\rangle$$

If we do the subtraction in reverse we have

$$\mathbf{q} = \ \langle\ x_1 - x_2, y_1 - y_2\rangle$$

$$\mathbf{p} = -\mathbf{q}$$

Vectors add by adding their components:

$$\mathbf{a} = \langle\ 3, 4\ \rangle$$

$$\mathbf{b} = \langle\ -3, 1\ \rangle$$

$$\mathbf{a} + \mathbf{b} = \langle\ 0, 5\ \rangle$$

Subtraction works the same way.

From a linear algebra point of view, a vector is simply an ordered collection of numbers

$$\mathbf{u} = \langle\ u_1, u_2, \cdots\ u_n \rangle$$

where n could be very large, even infinite.

However, a lot of work is done in two or three dimensions (officially $\mathbb{R}^2$ and $\mathbb{R}^3$), and the principles developed there carry over nicely into n-dimensional space. So let's start by thinking about a two-dimensional vector

$$\mathbf{u} = \langle\ u_1, u_2\ \rangle$$

As I've said, the vector $\mathbf{u}$ can be thought of as an arrow that goes from the origin to the point (u_1, u_2). It has both length and direction, with the length given by

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2}$$

and its direction is

$$\frac{u_2}{u_1} = \tan\ \theta, \qquad \theta = \tan^{-1}\ \frac{u_2}{u_1}$$

where θ is the angle the vector makes (rotating counter-clockwise) from the positive x-axis.

Any vector can be converted into a *unit vector*, a vector of length one, by dividing by its length. For example if $\mathbf{u} = \langle\ 1, 2\ \rangle$ then

$$\hat{\mathbf{u}} = \frac{1}{|\mathbf{u}|}\ \mathbf{u} = \langle\ \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$$

$\hat{\mathbf{u}}$ is a unit vector pointing in the same direction as $\mathbf{u}$.

The line through the origin with slope $m = u_2/u_1$ and equation

$$y = mx$$

can be thought of as the extension of vector $\mathbf{u}$ obtained by multiplying some t times $\mathbf{u}$ for all $t \in \mathbb{R}$. We have stretched the vector to infinity, and beyond!

The standard unit vectors point in the direction of the x, y and z axes.

$$\hat{\mathbf{i}} = \langle\ 1, 0, 0\ \rangle$$

$$\hat{\mathbf{j}} = \langle\ 0, 1, 0\ \rangle$$

$$\hat{\mathbf{k}} = \langle\ 0, 0, 1\ \rangle$$

We can write the vector using these unit vectors as

$$\mathbf{a} = \langle\ 3, 4\ \rangle = 3 \cdot \hat{\mathbf{i}} + 4 \cdot \hat{\mathbf{j}}$$

Dot product

We now introduce a procedure for multiplying two vectors, the *dot product*, and derive the relationship between the dot product of two vectors and the angle between them. Suppose we have two vectors

$$\mathbf{a} = \langle a_1, a_2 \rangle$$

$$\mathbf{b} = \langle b_1, b_2 \rangle$$

Geometrically, we might think of these as being one vector extending from the origin in the x, y-plane to the point (a_1, a_2), and the other vector extending from the origin to (b_1, b_2). The dot product is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

We can extend this to a pair of vectors in n-dimensional space

$$\mathbf{a} = \langle a_1, a_2, \ldots a_n \rangle$$

$$\mathbf{b} = \langle b_1, b_2, \ldots b_n \rangle$$

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=0}^{n} a_i b_i$$

The two vectors being multiplied (whose dot product is computed) must have the same dimension, the same n. Also, the result of the multiplication—the dot product—is a number. This is in contrast to another form of vector multiplication (the cross-product) which yields a vector as the result.

notation

The dot ($\cdot$) in the dot product may also used to set apart two multiplicands in scalar multiplication, to increase clarity. So, you ask, how can we tell what is meant? Well, consider

$$v \cdot \frac{1}{v}$$

$$\mathbf{a} \cdot \mathbf{b}$$

It's a dot product if the two objects are vectors, otherwise it's multiplication.

Some properties

The dot product obeys the usual rules: it is associative, commutative and distributive.

The commutative property of the dot product:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

follows from the same property for multiplication of real numbers, since

$$\mathbf{a} \cdot \mathbf{b} = \sum_n a_n b_n$$

$$= \sum_n b_n a_n = \mathbf{b} \cdot \mathbf{a}$$

For the distributive property, suppose

$$\mathbf{b} = \mathbf{c} + \mathbf{d}$$

Then

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d}$$

You can easily verify this by computing each term of the respective products.

$$\mathbf{b} = \ \langle b_1, b_2 \rangle \ = \mathbf{c} + \mathbf{d} = \ \langle c_1 + d_1, c_2 + d_2 \rangle$$

$$\mathbf{a} \cdot \mathbf{b} = a_1(c_1 + d_1) + a_2(c_2 + d_2)$$

$$= a_1 c_1 + a_1 d_1 + a_2 c_2 + a_2 d_2$$

$$= a_1 c_1 + a_2 c_2 + a_1 d_1 + a_2 d_2$$

$$= \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d}$$

Another example that we will need below is

$$(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b}$$

by the commutative property

$$= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\ \mathbf{a} \cdot \mathbf{b}$$

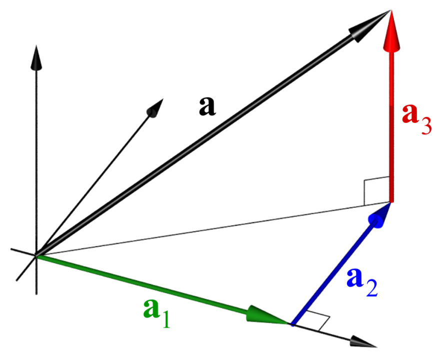
Length of a vector

As we said, the length of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$, designated $|\mathbf{a}|$, is computed by a straightforward application of the Pythagorean Theorem:

$$|\mathbf{a}|^2 = a_1^2 + a_2^2$$

We leave the result as the square for simplicity.

This is easily extended to more dimensions by sequential application of the same method.



In $\mathbb{R}^3$:

$$|\mathbf{a}|^2 = a_1^2 + a_2^2 + a_3^2$$

In $\mathbb{R}^n$:

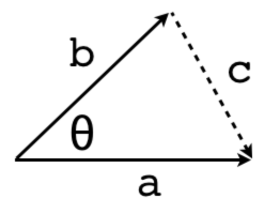
$$|\mathbf{a}|^2 = a_1^2 + a_2^2 + \cdots + a_n^2$$

Notice that

$$|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$$

Relation to θ

Now we are ready for the main idea. Suppose we draw two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^2$ with their tails at the same point. Designate the angle between them as θ and the vector representing the side opposite as $\mathbf{c}$.



The orientation of **c** doesn't matter for the argument that follows. As shown

$$\mathbf{b} + \mathbf{c} = \mathbf{a}$$

$$\mathbf{c} = \mathbf{a} - \mathbf{b}$$

Compute the dot product of **c** with itself

$$\mathbf{c} \cdot \mathbf{c} = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$$

Recalling the result from above, this is

$$\mathbf{c} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\ \mathbf{a} \cdot \mathbf{b}$$

Since

$$|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$$

and so on, we have that

$$\mathbf{c} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\ \mathbf{a} \cdot \mathbf{b}$$

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\ \mathbf{a} \cdot \mathbf{b}$$

Does this remind you of the **law of cosines**?

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Comparing the two equations, we see that

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \cos\theta$$

This relationship is extremely useful because it allows us to compute the cosine of the included angle via the dot product.

Even more important, two vectors which are perpendicular will have $\cos\theta = 0$, so their dot product is zero. Two vectors in pointed in the same direction have $\cos\theta = 1$ so it's just the product of the magnitudes.

This result extends to vectors in $\mathbb{R}^n$. Proof: choose a coordinate system where the two vectors lie in the same plane. Then apply the standard method.

For example, suppose I have the vector

$$\mathbf{u} = \langle p, q \rangle$$

Find a vector $\mathbf{v}$ perpendicular to $\mathbf{u}$.

$$\mathbf{v} = \langle q, -p \rangle$$

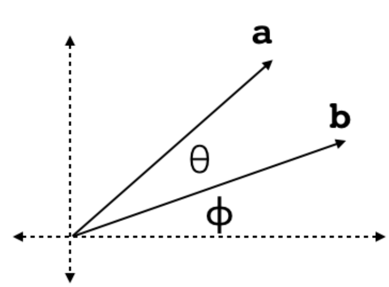
$\mathbf{v}$ is perpendicular to $\mathbf{u}$ because

$$\mathbf{u} \cdot \mathbf{v} = pq + q(-p) = 0$$

How to find a vector in $\mathbb{R}^5$ perpendicular to $\langle 1, 1, 1, 1, 0 \rangle$? Any vector of the form $\langle 0, 0, 0, 0, k \rangle$ will do, where k is some real number.

Alternate derivation

Here is another approach which doesn't depend on knowing the law of cosines, but uses the addition rule for cosine instead.



Vector **a** forms an angle θ with vector **b**. **b** forms an angle ϕ with the x-axis, so the angle between **a** and the x-axis is $\theta + \phi$.

Find the dot product using components. If $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$ then

$$a_x = a\cos(\theta + \phi)$$
$$b_x = b\cos\phi$$
$$a_y = a\sin(\theta + \phi)$$
$$b_y = b\sin\phi$$

So

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= a_x b_x + a_y b_y \\ &= ab\ [\ \cos(\theta+\phi)\cos\phi + \sin(\theta+\phi)\sin\phi\ ]\end{aligned}$$

Using the rule

$$\cos s - t = \cos s\cos t + \sin s\sin t$$

the part in parentheses is

$$\begin{aligned}&\cos(\theta+\phi)\cos\phi + \sin(\theta+\phi)\sin\phi \\ &= \cos(\theta+\phi-\phi) = \cos\theta\end{aligned}$$

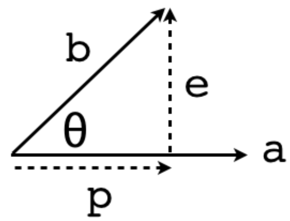
Another important property is that the value of the dot product is *independent* of the coordinate system chosen, because rotation or translation cannot change the lengths of the vectors nor the angle between them.

Projection

If $|\mathbf{a}| = 1$ we say that **a** is a *unit vector*. In that case

$$\mathbf{b}\cdot\mathbf{a} = |\mathbf{b}|\cos\theta$$

Looking at the figure, $|\mathbf{b}|\cos\theta$ is the length of the *projection* of **b** on **a**. (Recall that the dot product is a scalar—a number—and not a vector).



The result, $\mathbf{b} \cdot \mathbf{a} = |\mathbf{b}| \cos\theta$, is the length of the part of $\mathbf{b}$ that extends in the same direction as $\mathbf{a}$. The corresponding vector is

$$\mathbf{p} = (\mathbf{b} \cdot \mathbf{a})\ \mathbf{a}$$

The other component of $\mathbf{b}$ is the part that is perpendicular to $\mathbf{p}$

$$\mathbf{p} + \mathbf{e} = \mathbf{b}$$

We compute $\mathbf{e}$ as the difference $\mathbf{b} - \mathbf{p}$. $\mathbf{e}$ is the part of $\mathbf{b}$ that is perpendicular to the projection. As a final note, the formula given here is a simplification for the situation in which $\mathbf{a}$ is a unit vector. If not, the complete formula is:

$$\mathbf{p} = \frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\ \mathbf{a}$$

Vectors allow simple proofs for some geometric theorems such as Ceva's theorem and the law of cosines.

example

Here is a problem from Nahin:

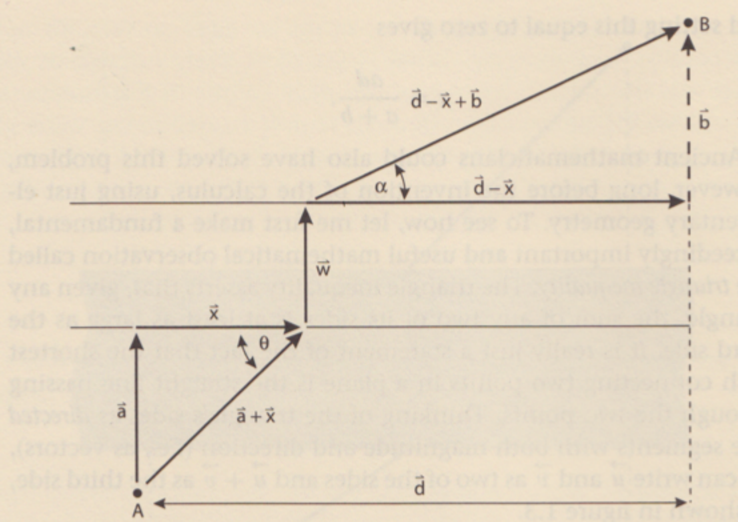


FIGURE 1.4. Bridge geometry in vector notation.

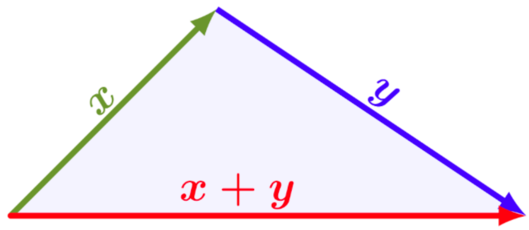
Two towns are on opposite sides of a river at points A and B. It is desired to choose the site of a bridge so as to minimize the distance between the two towns when traveling over the bridge. The problem can be set up algebraically and solved by differential calculus. However, the vector approach is more fun, and allows us to introduce the important *triangle inequality*.

Vectors are shown in the figure: $\mathbf{a}$ is the perpendicular distance from A to the river, and similarly for $\mathbf{b}$. $\mathbf{x}$ determines the placement of the bridge. If the horizontal distance between A and B is $\mathbf{d}$, then $\mathbf{d}-\mathbf{x}$ is the horizontal distance between B and the bridge. The distance across the bridge is $\mathbf{w}$, which cannot be changed. Its length will just be added onto our shortest path.

We want to choose $\mathbf{x}$ so that the path from A to B is the shortest. The path from A to the bridge is $\mathbf{a}+\mathbf{x}$, that from the bridge to B is $\mathbf{b}+\mathbf{d}-\mathbf{x}$ so all together we have (taking the lengths of the vectors)

$$L = |\mathbf{a}+\mathbf{x}| + |\mathbf{b}+\mathbf{d}-\mathbf{x}|$$

The triangle inequality says that the lengths of two sides of a triangle add to be larger than or equal to the length of the third side.



$$|\mathbf{x}| + |\mathbf{y}| \geq |\mathbf{x} + \mathbf{y}|$$

The rule is that the minimal value for the sum $|\mathbf{x}| + |\mathbf{y}|$ occurs when they point in the same direction.

In our problem, the minimum length occurs when $\mathbf{a} + \mathbf{x}$ and $\mathbf{b} + \mathbf{d} - \mathbf{x}$ point in the same direction. In other words, when $\theta = \alpha$.

Then, by similar triangles,

$$\frac{x}{a} = \frac{d-x}{b}$$

$$bx = ad - ax$$

$$x = \frac{ad}{a+b}$$

example

Here is one from Strang.

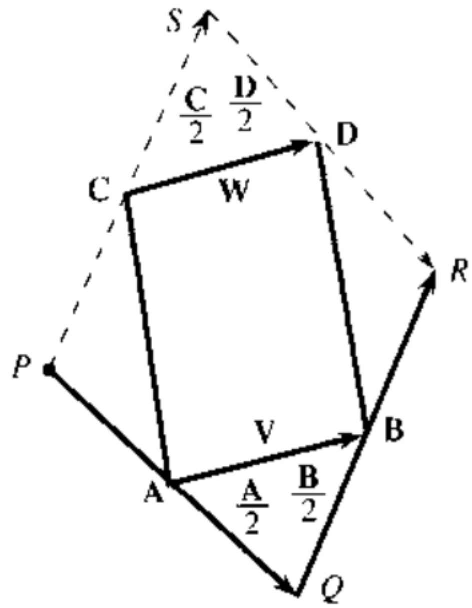


Fig. 11.4 Four midpoints

Consider *any* four-sided figure in space, such as $PQRS$ in the figure. (Note: $|\mathbf{A}| \neq |\mathbf{B}|$, and so on, and S is not co-planar with P, Q, R. I claim that the midpoints of the sides form a parallelogram $ABCD$.

We will prove that $\mathbf{V} = \mathbf{W}$.

The figure makes it almost obvious.

$$\mathbf{V} = \frac{\mathbf{A}}{2} + \frac{\mathbf{B}}{2}$$

$$\mathbf{W} = \frac{\mathbf{C}}{2} + \frac{\mathbf{D}}{2}$$

The segment from P to R can be covered in two ways

$$\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$$

Divide both sides by 2 and obtain

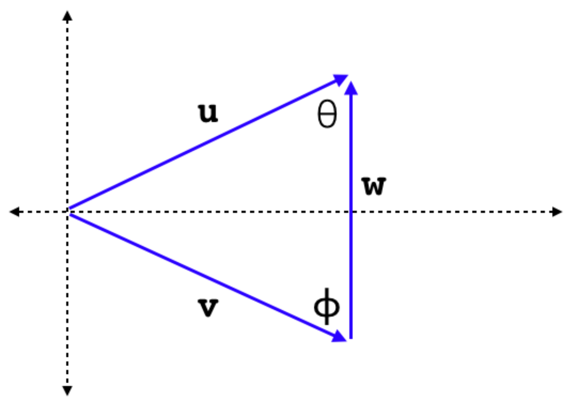
$$\frac{\mathbf{A}}{2} + \frac{\mathbf{B}}{2} = \frac{\mathbf{C}}{2} + \frac{\mathbf{D}}{2}$$

$$\mathbf{V} = \mathbf{W}$$

□

example

And here is one from Euclid:



We are given a triangle with two sides the same length (isosceles). Without loss of generality, draw the triangle with its vertex at the origin and the midpoint of the third side on the x-axis.

To prove: $\theta = \phi$.

Let

$$\mathbf{u} = \langle a, b \rangle$$

$$\mathbf{v} = \langle a, -b \rangle$$

$$\mathbf{w} = \langle 0, 2b \rangle$$

We compute the dot products so that the angle between the vectors is acute and the dot product is > 0.

$$\mathbf{u} \cdot \mathbf{w} = 2b^2$$

$$= |\mathbf{u}||\mathbf{w}| \cos\theta = \sqrt{a^2 + b^2}\ 2b \cos\theta$$

$$\cos\theta = \frac{b}{\sqrt{a^2 + b^2}}$$

which is also obvious from the figure. We didn't need vectors for this.

$$(-\mathbf{w}) \cdot \mathbf{v} = 2b^2$$

$$= \sqrt{a^2 + b^2}\ 2b \cos\phi$$

$$\cos\phi = \frac{b}{\sqrt{a^2 + b^2}}$$

We obtain the same result for $\cos\phi$ as for $\cos\theta$ and then finally

$$\theta = \phi$$

Chapter 58

Vector cross product

Suppose we have two ordinary vectors $\mathbf{u}$ and $\mathbf{v}$. These must be in $\mathbb{R}3$ because the cross-product is only defined for vectors in $\mathbb{R}3$.

Their respective lengths are u and v.

We write the cross-product as

$$\mathbf{u} \times \mathbf{v} = \mathbf{w}$$

The simplest definition is that the magnitude of $\mathbf{w}$ is

$$w = uv \ \sin\theta$$

The symmetry with the dot product is obvious. Also

$$|\mathbf{u} \times \mathbf{v}|^2 + |\mathbf{u} \cdot \mathbf{v}|^2 = (uv)^2$$

The direction is defined by saying that $\mathbf{w}$ is orthogonal to the plane which contains both $\mathbf{u}$ and $\mathbf{v}$, and its sign is given by the right-hand rule. Curl the fingers of your right hand around in the direction from $\mathbf{u}$ to $\mathbf{v}$. Your thumb points in the same direction as $\mathbf{w}$.

The term $\sin\theta$ means that the cross-product of any vector with itself is zero.

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

To make the notation simpler, we define

$$\mathbf{u} = \langle p, q, r \rangle$$

$$\mathbf{v} = \langle x, y, z \rangle$$

and in order to compute the cross product, we form what looks like a really weird matrix

$$\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ p & q & r \\ x & y & z \end{bmatrix}$$

and write its "determinant"

$$\mathbf{u} \times \mathbf{v} = (qz - ry)\ \hat{\mathbf{i}} + (rx - pz)\ \hat{\mathbf{j}} + (py - qx)\ \hat{\mathbf{k}}$$

We can show that the resulting vector is orthogonal to the two starting vectors, $\mathbf{u}$ and $\mathbf{v}$. Test that by forming the dot product with $\mathbf{u}$.

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = p(qz - ry) + q(rx - pz) + r(py - qx)$$

The first and fourth terms cancel, the second and fifth terms cancel, and the third and sixth terms also cancel.

So $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$, and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ as well.

In fact, a very common use for the cross-product is to find the normal vector to a plane in vector calculus.

As an aside, we could have skipped this calculation. The following rule holds for vectors:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

(we will explore triple products below). So

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{u}) \cdot \mathbf{v} = 0$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = -\mathbf{v} \cdot (\mathbf{v} \times \mathbf{u}) = -(\mathbf{v} \times \mathbf{v}) \cdot \mathbf{u}) = 0$$

About the angle

How to show that

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \ \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

According to wikipedia, this is the *definition* of the cross-product, and from this one can derive the expression that we got by setting up our matrix and computing its "determinant." So that is what we are going to do.

I am going to go back to the notation we had before, rather than use subscripts like a_x, etc.

$$\mathbf{u} = \langle p, q, r \rangle$$
$$\mathbf{v} = \langle x, y, z \rangle$$

We proceed from the "determinant" definition of the cross product and show that the length of that vector squared plus the square of the dot product is equal to u^2v^2. By the argument we made above, the magnitude of the cross product is then equal to $uv \sin\theta$.

$$\mathbf{u} \times \mathbf{v} = (qz - ry)\hat{\mathbf{i}} + (rx - pz)\hat{\mathbf{j}} + (py - qx)\hat{\mathbf{k}}$$

$$|\mathbf{u} \times \mathbf{v}|^2 = (qz - ry)^2 + (rx - pz)^2 + (py - qx)^2$$

$$= (qz)^2 - 2qryz + (ry)^2 + (rx)^2 - 2prxz + (pz)^2 + (py)^2 - 2pqxy + (qx)^2$$

$$\mathbf{u} \cdot \mathbf{v} = px + qy + rz$$

$$(\mathbf{u} \cdot \mathbf{v})^2 = (px)^2 + (qy)^2 + (rz)^2 + 2pqxy + 2prxz + 2qryz$$

When we add these together, all the terms with cofactor 2 cancel so that leaves

$$|\mathbf{u} \times \mathbf{v}|^2 + (\mathbf{u} \cdot \mathbf{v})^2$$

$$= (qz)^2 + (ry)^2 + (rx)^2 + (pz)^2 + (py)^2 + (qx)^2 + (px)^2 + (qy)^2 + (rz)^2$$

rearranging terms

$$= (px)^2 + (py)^2 + (pz)^2 + (qx)^2 + (qy)^2 + (qz)^2 + (rx)^2 + (ry)^2 + (rz)^2$$

$$= (p^2 + q^2 + r^2)(x^2 + y^2 + z^2)$$

$$= |\mathbf{u}|^2|\mathbf{v}|^2$$

That was tedious, but it we made it.

All of these properties of the cross-product are connected.

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$$

$$\begin{aligned}\mathbf{a}\times\mathbf{b} = &\ \langle qu - rt, rs - pu, pt - qs\rangle\\ |\mathbf{a}\times\mathbf{b}| &= |\mathbf{a}||\mathbf{b}|\sin\theta\\ |\mathbf{a}\times\mathbf{b}|^2 + (\mathbf{a}\cdot\mathbf{b})^2 &= |\mathbf{a}|^2|\mathbf{b}|^2\end{aligned}$$

Triple products

Suppose we have

$$\begin{aligned}\mathbf{a} = &\ \langle p, q, r\rangle\\ \mathbf{b} = &\ \langle s, t, u\rangle\\ \mathbf{c} = &\ \langle x, y, z\rangle\end{aligned}$$

And

$$\begin{aligned}\mathbf{a}\times\mathbf{b} = &\ \langle qu - rt, rs - pu, pt - qs\rangle\\ \mathbf{b}\times\mathbf{c} = &\ \langle tz - uy, ux - sz, sy - tx\rangle\\ \mathbf{a}\times\mathbf{c} = &\ \langle qz - ry, rx - pz, py - qx\rangle\end{aligned}$$

Algebraically

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= p(tz - uy) + q(ux - sz) + r(sy - tx)\\ \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) &= s(ry - qz) + t(pz - rx) + u(qx - py)\\ \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) &= x(qu - rt) + y(rs - pu) + z(pt - qs)\end{aligned}$$

So

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

The way to remember this is that these are all the same cyclic permutation.

A much simpler proof is to remember that the cross-product $\mathbf{a}\times\mathbf{b}$ is the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$ and the *scalar* triple product is the signed volume of the parallelipiped formed by the three vectors. Signed meaning that $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = -\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a})$ so the area may come out negative, if we order $\mathbf{a}$ and $\mathbf{b}$ differently.

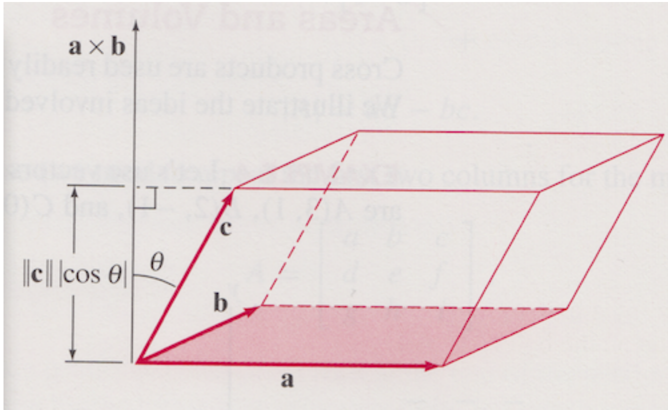


Figure 1.58 The parallelepiped determined by **a**, **b**, and **c**.

Recall that the direction of $\mathbf{a} \times \mathbf{b}$ is perpendicular to both vectors. If we are careful to write the cross-product in the correct order using the right-hand rule, the result of the dot product will always be positive, with the projection of $\mathbf{c}$ onto the cross-product equal to the height of the solid. In particular, for this arrangement, we must write $\mathbf{a} \times \mathbf{b}$, $\mathbf{b} \times \mathbf{c}$, or $\mathbf{c} \times \mathbf{a}$.

It doesn't matter which two vectors we choose as the base of our solid, the volume must come out the same.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

Chapter 59

Point and plane

Construct a plane containing 3 points

Consider three points

$P = (1, 0, 0)$, $Q = (0, 1, 0)$, and $R = (0, 0, 1)$.

Find two vectors in the plane by subtracting the second and third from the first.

$$\begin{aligned} \mathbf{u} &= (1, 0, 0) - (0, 1, 0) \\ &= \langle 1, -1, 0 \rangle \\ \mathbf{v} &= (1, 0, 0) - (0, 0, 1) \\ &= \langle 1, 0, -1 \rangle \end{aligned}$$

Obtain the normal vector by computing the cross product

$$\mathbf{N} = \mathbf{u} \times \mathbf{v} \Rightarrow \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 1\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 1\hat{\mathbf{k}} = \langle 1, 1, 1 \rangle$$

One equation of the plane is then

$$\mathbf{N} \cdot \mathbf{w} = 0$$

for any vector **w** in the plane.

Consider a fixed point in the plane (x_0, y_0, z_0). Then any other point in the plane (x, y, z) yields a vector from the fixed point which, dotted with **N**, yields 0

$$\langle x - x_0, y - y_0, z - z_0 \rangle \ \cdot \ \langle 1, 1, 1 \rangle \ = 0$$

$$x - x_0 + y - y_0 + z - z_0 = 0$$

$$x + y + z = x_0 + y_0 + z_0 = d$$

Plugging in any one of the points yields

$$x + y + z = d = 1$$

find the closest point in the plane

Consider any point in space, e.g. $P = (3, 4, 6)$.

Find the point Q on the plane which is closest to P, the point we arrive at by subtracting some fraction of N from P.

We have a point and a vector

$$Q = P - t\mathbf{N}$$

$$Q = (3, 4, 6) \ - \ t\langle 1, 1, 1 \rangle$$

Since Q is in the plane, its components x, y, z satisfy $x + y + z = 1$!

So

$$(3 - t) + (4 - t) + (6 - t) = 1$$

$$13 - 3t = 1$$

$$t = 4$$

$$Q = (-1, 0, 2)$$

Check that Q is in the plane

$$-1+0+2=1$$

and $P-Q$ is parallel to $\mathbf{N}$

$$P-Q= \ \langle 4,4,4\rangle$$

that is definitely a multiple of $\mathbf{N}$.

point where a vector crosses the plane

Where does the vector $\mathbf{w}$ that goes from the origin to point $P=(3,4,6)$ hit the plane? Call that point R. Again we have a point and a vector

$$R=(0,0,0)+t\mathbf{w}=(0,0,0)+t\ \langle 3,4,6\rangle$$

And again, since R is in the plane, its components x,y,z satisfy $x+y+z=1$. So

$$3t+4t+6t=1$$
$$t=\frac{1}{13}$$
$$R=(\frac{3}{13},\frac{4}{13},\frac{6}{13})$$

Notice that the vector $Q-R$ is in the plane, as it should be

$$(Q-R)\cdot\mathbf{N}=((-1,0,2)-(\frac{3}{13},\frac{4}{13},\frac{6}{13}))\cdot\ \langle 1,1,1\rangle$$
$$=\ \langle\frac{-16}{13},\frac{-4}{13},\frac{20}{13}>\ \cdot\langle 1,1,1\rangle\ =0$$

And, adding the horizontal and vertical components together

$$Q-R+P-Q=P-R=(3,4,6)-(\frac{3}{13},\frac{4}{13},\frac{6}{13})$$
$$=(\frac{36}{13},\frac{48}{13},\frac{72}{13})$$

the result is parallel to $\mathbf{w}$.

Lines in space

One way of specifying a line in 3D-space is as the intersection of two planes. Another way is by giving a vector and a point in space. Let's look at these in turn. Suppose we have the following two planes:

$$x + y - z = 7$$

$$2x - 3y + z = 3$$

Since the x, y, z terms are not related by a multiplicative constant, the planes are not parallel, so they will meet in a line, and the solutions consist of all the points on the line. Let's find one solution, at $x = 0$. Then

$$y - z = 7$$

$$-3y + z = 3$$

Adding

$$-2y = 10$$

$$y = -5$$

$$z = y - 7 = -12$$

Our solution $P_0 = (0, -5, -12)$. Now find a second solution, at $z = -3$

$$x + y = 4$$

$$2x - 3y = 6$$

Solving

$$x = 4 - y$$

$$2(4 - y) - 3y = 6$$

$$8 - 5y = 6$$

$$y = \frac{2}{5}$$

$$x = \frac{18}{5}$$

The second point is $P_1 = (18/5, 2/5, -3)$. Now we have two points on the line. Its equation is

$$L = P_0 + t(P_1 - P_0)$$

$$L = (0, -5, -12) + t(\frac{18}{5}, \frac{27}{5}, 9)$$

We can re-scale the vector that multiplies t to have integer components (or length 1, or whatever we wish). Why not multiply by 5/9?

$$L = (0, -5, -12) + t(2, 3, 5)$$

There is another way to do this problem that might be a little easier. Consider that the equation of the first plane gives its normal vector n_1 as

$$n_1 = < 1, 1, -1 >$$

Similarly the normal vector to the second plane is n_2

$$n_2 = < 2, -3, 1 >$$

Now, the vector that is parallel to the line of intersection is orthogonal to both n_1 and n_2 (Do you see why?) So we compute the cross-product:

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= -2i - 3j - 5k$$

Multiplying by -1 gives what we obtained above.

Chapter 60

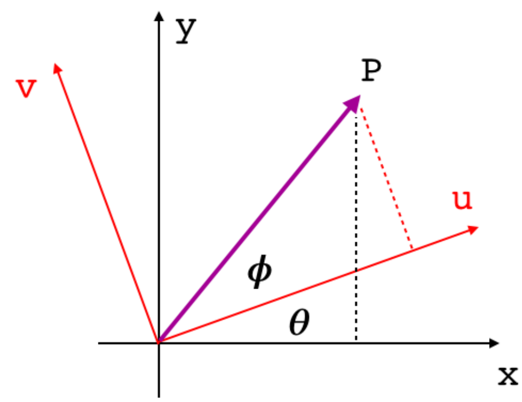
Geometric rotation

Our goal here is to find the equations for rotation of coordinates. We want to be as simple as we can, so that we can (at least try to) remember how the derivation works.

In this chapter we look at geometric approaches. Vector methods are simpler, in my opinion, but they require knowing the idea of matrix multiplication. The geometry can be confusing to set up, but once that's done you can just read the answer off the diagram.

Stewart

I found a nice method in Stewart that depends on knowing the sum of angles formulas, which I urged you to memorize.



Here, we have the standard xy-coordinates in black. The rotated uv-coordinate system is in red, with a rotation angle θ in the counter-clockwise direction.

The ray to the point P has length r. Notice that the coordinates of point P in the u, v system are naturally expressed in terms of ϕ:

$$u = r\cos\phi$$

$$v = r\sin\phi$$

while x and y are naturally expressed in terms of the combined angle $\theta + \phi$.

$$x = r\cos(\theta + \phi)$$

Now, use the sum formula for cosine:

$$x = r\cos\theta\cos\phi - r\sin\theta\sin\phi$$

But we have from above that

$$u = r\cos\phi$$

$$v = r\sin\phi$$

So

$$x = u\cos\theta - v\sin\theta$$

It's as easy as that.

In the same way:

$$
\begin{aligned}
y &= r\sin(\theta+\phi) \\
&= r\sin\theta\cos\phi + r\cos\theta\sin\phi \\
&= u\sin\theta + v\cos\theta
\end{aligned}
$$

Solve for u and v

To convert these formulas to functions $u = f(x, y)$ and $v = f(x, y)$, there is a hard way and an easy way. We do the hard way first:

$$x = u\cos\theta - v\sin\theta$$

so

$$x\cos\theta = u\cos^2\theta - v\sin\theta\cos\theta$$

and

$$y = u\sin\theta + v\cos\theta$$

so

$$y\sin\theta = u\sin^2\theta + v\sin\theta\cos\theta$$

adding:

$$x\cos\theta + y\sin\theta = u$$

similarly:

$$x\sin\theta = u\sin\theta\cos\theta - v\sin^2\theta$$

$$y\cos\theta = u\sin\theta\cos\theta + v\cos^2\theta$$

add minus the first to the second:

$$-x\sin\theta + y\cos\theta = v$$

It *is* easier when you know where you're going. In summary:

$$u = x\cos\theta + y\sin\theta$$

$$v = -x \sin\theta + y \cos\theta$$

and the original pair:

$$x = u \cos\theta - v \sin\theta$$

$$y = u \sin\theta + v \cos\theta$$

The difference is the sign of the sine.

The easy way is to switch x, y for u, v and at the same time, substitute $-\theta$ for θ. What this amounts to is relabeling our diagram with x, y being the rotated axes, and then rotating in the opposite direction (cw rather than ccw).

$$x = u \cos\theta - v \sin\theta$$

switch

$$u = x \cos -\theta - y \sin -\theta$$

$$= x \cos\theta + y \sin\theta$$

(Recall that $\cos -x = \cos x$ and $\sin -x = -\sin x$).

For y

$$y = u \sin\theta + v \cos\theta$$

switch

$$v = x \sin -\theta + y \cos -\theta$$

$$v = -x \sin\theta + y \cos\theta$$

A small test

Rotation of coordinates counter-clockwise (x, y to u, v) gives $-\sin$ in the formula for the vertical component v), whereas clockwise rotation (u, v to x, y) gives $-\sin$ in the formula for the horizontal component x.

One way to see that this is correct is to substitute from the u, v formulas into the x, y ones:

$$x = (x\cos\theta + y\sin\theta)\cos\theta - (-x\sin\theta + y\cos\theta)\sin\theta$$

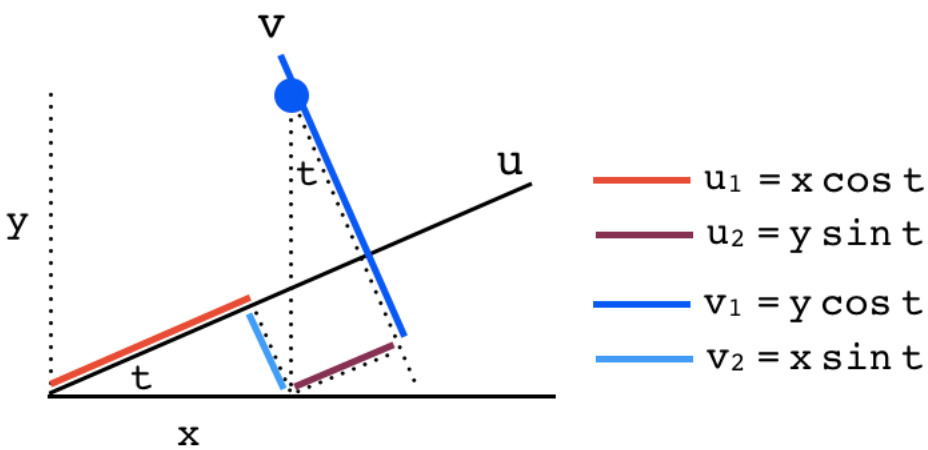
Can you see that if this is multiplied out, we will get $x(\cos^2\theta + \sin^2\theta) = x$ and the terms with y will just cancel?

A similar thing happens with the other one:

$$y = (x\cos\theta + y\sin\theta)\sin\theta + (-x\sin\theta + y\cos\theta)\cos\theta = y$$

standard derivation

Here is a second derivation with a sort of minimalist diagram of rotation



We draw the horizontal x-axis and the rotated u-axis. The angle between them is t. We plot our point and then draw perpendiculars to both axes. To finish the set-up, we draw perpendiculars from the point $(x, 0)$ as shown.

Once the drawing is rendered, we are almost done. You will know that you've done it right if you have both x and y as the *hypotenuse of a right triangle.* Now we just work our way through

$$u_1 = x \cos t$$

$$u_2 = y \sin t$$

So

$$u = x \cos t + y \sin t$$

(All the triangles in the diagram are similar, with small angle t. Can you prove it?)

$$v_1 = y \cos t$$

$$v_2 = x \sin t$$

$$v = -x \sin t + y \cos t$$

Shankar derivation

It seems that the more you talk about rotation, the less clear things become. Nevertheless, I will show one more from Shankar's book, and then work through a calculation.

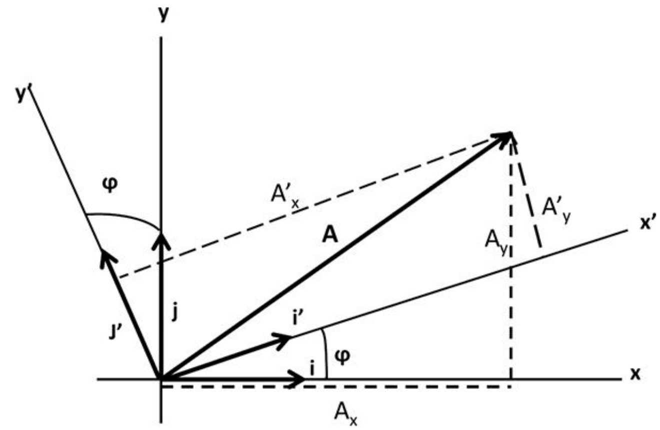


Figure 2.4 The same vector **A** is written as $\mathbf{i}A_x + \mathbf{j}A_y$ in one frame and as $\mathbf{i}'A'_x + \mathbf{j}'A'_y$ in the other. The dotted lines indicate the components in the two frames.

Write $\hat{\mathbf{i}}'$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ as follows.

To reach the end of $\hat{\mathbf{i}}'$ we go out in the $\hat{\mathbf{i}}$ direction. How far? Call it x. In terms of the new unit vector $\hat{\mathbf{i}}'$ we have

$$\frac{x}{|\hat{\mathbf{i}}'|} = \cos\phi$$

But $|\hat{\mathbf{i}}'| = 1$ so

$$x = \cos\phi$$

x is shorter than $\hat{\mathbf{i}}$ by the factor of $\cos\phi$.

We also need to go up in the $\hat{\mathbf{j}}$ direction. Use the other triangle with $\hat{\mathbf{j}}$ rotated to $\hat{\mathbf{j}}'$ (which is similar to the first triangle).

$$\frac{y}{|\hat{\mathbf{j}}'|} = \sin\phi$$

But $|\hat{\mathbf{j}}'| = 1$ so

$$y = \sin\phi$$

y is shorter than $\hat{\mathbf{j}}$ by the factor of $\sin\phi$.

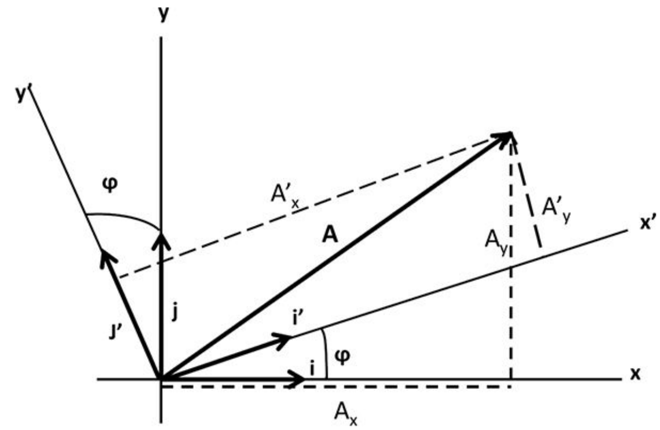


Figure 2.4 The same vector $\mathbf{A}$ is written as $\mathbf{i}A_x + \mathbf{j}A_y$ in one frame and as $\mathbf{i}'A_x' + \mathbf{j}'A_y'$ in the other. The dotted lines indicate the components in the two frames.

So to construct $\hat{\mathbf{i}}'$ we go out in the $\hat{\mathbf{i}}$ direction a distance of $\cos\phi$ and up in the $\hat{\mathbf{j}}$ direction a distance of $\sin\phi$:

$$\hat{\mathbf{i}}' = \hat{\mathbf{i}}\cos\phi + \hat{\mathbf{j}}\sin\phi$$

A similar process will yield

$$\hat{\mathbf{j}}' = \hat{\mathbf{j}}\cos\phi - \hat{\mathbf{i}}\sin\phi$$

Now write the vector $\mathbf{A}$ in two ways:

$$\mathbf{A} = A_x'\,\hat{\mathbf{i}}' + A_y'\,\hat{\mathbf{j}}'$$

$$= (\hat{\mathbf{i}}\cos\phi + \hat{\mathbf{j}}\sin\phi)A_x' + (\hat{\mathbf{j}}\cos\phi - \hat{\mathbf{i}}\sin\phi)A_y'$$

$$= \hat{\mathbf{i}}(\cos\phi A_x' - \sin\phi A_y') + \hat{\mathbf{j}}(\sin\phi A_x' + \cos\phi A_y')$$

or in the usual way

$$\mathbf{A} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}}$$

So

$$A_x = \cos\phi\ A_x' - \sin\phi\ A_y'$$

$$A_y = \sin\phi\ A'_x + \cos\phi\ A'_y$$

Finally compute the length of **A** (squared):

$$A_x^2 = \cos^2\phi\ {A'_x}^2 + \sin^2\phi\ {A'_y}^2 - 2\cos\phi\ A'_x\ \sin\phi\ A'_y$$

$$A_y^2 = \sin^2\phi\ {A'_x}^2 + \cos^2\phi\ {A'_y}^2 + 2\sin\phi\ A'_x\ \cos\phi\ A'_y$$

Add together and notice the cancellation of the mixed A'_x and A'_y terms:

$$A_x^2 + A_y^2 = \cos^2\phi\ {A'_x}^2 + \sin^2\phi\ {A'_y}^2 + \sin^2\phi\ {A'_x}^2 + \cos^2\phi\ {A'_y}^2$$

$$= {A'_x}^2 + \sin^2\phi\ {A'_y}^2 + \cos^2\phi\ {A'_y}^2$$

$$= {A'_x}^2 + {A'_y}^2$$

As we should expect, the length of a vector such as **A** is *invariant*, it does not depend on the choice of coordinate system.

Chapter 61

Vector rotation

If you know something about matrix multiplication, just remember this result:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If not, read on.

Our goal here is to find the equations for rotation of coordinates. We want to be as simple as we can, so that we can remember how the derivation works. We will need a couple of preliminary facts, however.

- a procedure to compute the dot product of two vectors
- two vectors are $\perp$ (orthogonal) $\iff$ the dot product is zero
- the projection of a vector $\mathbf{a}$ onto *any* other **unit** vector $\hat{\mathbf{e}}$ is just the dot product $\mathbf{a} \cdot \hat{\mathbf{e}}$
- a rotated set of coordinates is a set of orthogonal unit vectors (in whatever direction we choose)

Recall that the dot product is a number computed from the components of any two vectors $\mathbf{a}$ and $\mathbf{b}$ by the following procedure:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Some examples:

$$\langle 3, 2 \rangle \cdot \langle 2, 5 \rangle = 3 \times 2 + 2 \times 5 = 16$$

$$\langle 1, 0 \rangle \cdot \langle 0, 1 \rangle = 0$$

$$\langle \cos\theta, \sin\theta \rangle \cdot \langle -\sin\theta, \cos\theta \rangle = 0$$

As we might expect, the unit vector along the x-axis, usually called $\hat{\mathbf{i}}$, and the unit vector along the y-axis, $\hat{\mathbf{j}}$, are perpendicular to each other (second example, above).

Similarly, for any angle θ, the given vectors $\langle \cos\theta, \sin\theta \rangle$ and $\langle -\sin\theta, \cos\theta \rangle$ are perpendicular. These two examples should suggest to you a general method for finding a second vector orthogonal to one you are given.

The length of a vector $\mathbf{a}$ is represented as $|\mathbf{a}|$, or even just a, and the length squared is

$$a^2 = \mathbf{a} \cdot \mathbf{a}$$

With respect to the projection, an example should also make that clearer. Suppose we are working with two-dimensional vectors and we decide that our new x-axis should be in the direction of the vector $3, 4$. The first thing to do is to re-scale this to be a unit vector. The length squared is

$$\langle 3, 4 \rangle \cdot \langle 3, 4 \rangle = 9 + 16 = 25$$

Hence the length is 5 and our new unit vector $\hat{\mathbf{u}}$ is

$$\hat{\mathbf{u}} = \langle 3/5, 4/5 \rangle$$

We also need a unit vector $\hat{\mathbf{v}}$ such that $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = 0$. We obtain

$$\hat{\mathbf{v}} = \langle -4/5, 3/5 \rangle$$

or

$$\hat{\mathbf{v}} = \langle 4/5, -3/5 \rangle$$

These two vectors are the same vector, just pointing in opposite directions (which is in the same direction, for the purpose of vectors).

Then for *any* vector $\mathbf{a}$, we can compute the same vector in a set of rotated coordinates based on $\mathbf{u}$ and $\mathbf{v}$ as

$$a_u = \mathbf{a} \cdot \hat{\mathbf{u}}$$

$$a_v = \mathbf{a} \cdot \hat{\mathbf{v}}$$

derivation

All we have to do is to think about rotation of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ through an angle θ counter-clockwise.

Start with $\hat{\mathbf{i}}$. The new vector we seek is still a unit vector, but rotated so that it forms an angle θ with the positive x-axis.

The new vector has both $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components. Projection onto $\hat{\mathbf{i}}$ gives a vector with unit length times $\cos\theta$ or just $\cos\theta$, and similarly, the projection onto $\hat{\mathbf{j}}$ gives a length $\sin\theta$. Clearly the squared length is $\cos^2\theta + \sin^2\theta$, so this is a unit vector.

In vector notation we would say that

$$\langle 1, 0 \rangle \ \ \Rightarrow \ \ \langle \cos\theta, \sin\theta \rangle$$

In matrix language the two vectors are related in this way:

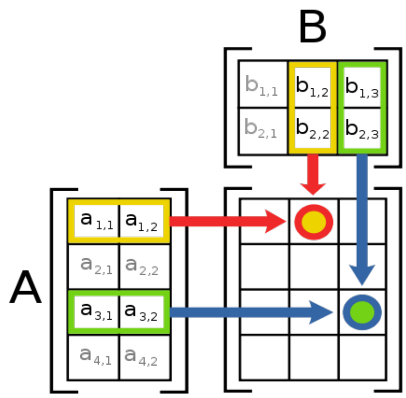
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \cos\ \theta \\ \sin\ \theta \end{bmatrix}$$

where $\Rightarrow$ refers to rotation.

So the question is, what matrix will multiply $\hat{\mathbf{i}}$ to give this result?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos & \theta \\ \sin & \theta \end{bmatrix}$$

Recall that matrix multiplication works like this



For a matrix times a vector, B would have only a single column.

So going back to this:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos & \theta \\ \sin & \theta \end{bmatrix}$$

I hope it's pretty clear that $a = \cos\theta$ and $c = \sin\theta$:

$$\begin{bmatrix} \cos\theta & b \\ \sin\theta & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos & \theta \\ \sin & \theta \end{bmatrix}$$

Can you see why this is true?

On the other hand, rotation of the unit $\hat{\mathbf{j}}$ vector by θ should give

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin & \theta \\ \cos & \theta \end{bmatrix}$$

The minus sign comes because the new unit vector is now sticking out into the second quadrant.

Again, it should be clear that

$$\begin{bmatrix} a & -\sin\ \theta \\ c & \cos\ \theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\ \theta \\ \cos\ \theta \end{bmatrix}$$

Now, just put them together:

$$R_{ccw} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

In particular, a rotation of 90° ccw goes like this

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\hat{\mathbf{i}}$ is rotated to become $\hat{\mathbf{j}}$.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$\hat{\mathbf{j}}$ is rotated to become $-\hat{\mathbf{i}}$.

I claim that since the matrix we found works for both of the unit vectors it will work for any vector, since any vector can be written as a linear combination of the unit vectors

$$\mathbf{a} = a_1\ \hat{\mathbf{i}} + a_2\ \hat{\mathbf{j}}$$

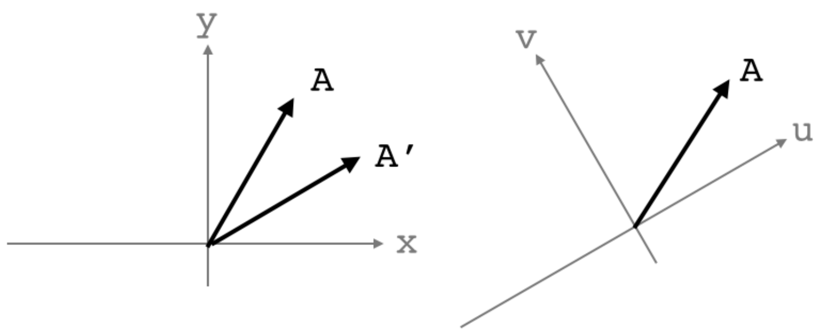
The inverse of the matrix we derived would be used for clockwise rotation and it is just

$$R_{cw} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

You can verify this by remembering the rule for 2×2 or by multiplication

$$R_{cw}\ R_{ccw} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Don't be confused when someone talks about rotation of the coordinate system. Here, the coordinate system stayed fixed but we rotated the vector counter-clockwise. We achieve the same thing (and use the same equation) for a *clockwise* rotation of the coordinate system through an angle θ.



If you really want to rotate the coordinate system counter-clockwise, rotate the vector clockwise.

consequence

One other neat thing comes out of this when we ask about rotation by an angle $s + t$. We can write two equivalent expressions, one by substituting $\theta = s + t$, and the other by doing two sequential applications of the matrix. That is:

$$\begin{bmatrix} \cos(s+t) & -\sin(s+t) \\ \sin(s+t) & \cos(s+t) \end{bmatrix} = \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

Look at the term on the upper-left, $\cos(s+t)$. Sound familiar? Carry out the matrix multiplication on the right for that element

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$

We have derived the cosine addition formula. Similarly, the bottom-left term is for the sine

$$\sin(s+t) = \sin s \cos t + \cos s \sin t$$

yet another way

We can look at this in still a different way. Write

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In this representation, the vector $\langle x, y \rangle$ is a linear combination of the unit vectors $\hat{\mathbf{i}} = \langle 1, 0 \rangle$ and $\hat{\mathbf{j}} = \langle 0, 1 \rangle$.

To rotate the point, we just want to use a different set of unit vectors. The new unit vectors (for the rotated axes) are $\langle \cos\theta, \sin\theta \rangle$ and $\langle -\sin\theta, \cos\theta \rangle$.

If you compute their lengths, it is clear that they are, in fact, unit vectors.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = x \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + y \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Written as a matrix multiplication, this is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Chapter 62

Parametric equations

Suppose we are working in $\mathbb{R}^2$. Consider two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. The vector $\mathbf{v}$ which goes in the direction PQ can be obtained as

$$\mathbf{v} = \langle\, x_2 - x_1, y_2 - y_1 \rangle$$

Written like that as just $\mathbf{v}$ by itself, the default would be that we consider that vector to lie with its tail at the origin. Multiplying by a real number t will extend the vector to any point lying on the line with slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

that goes through the origin.

If $t < 0$ then the vector goes backward.

To start from some other position, add the vector which extends to that point. The position vector is commonly called $\mathbf{r}$. So the complete specification of a line in $\mathbb{R}^2$ would be something like:

$$l = \mathbf{r} + t\mathbf{v}$$

This is called the *parametric* representation of the line.

As a second example, consider a circle of radius R with its center at the origin. We can write the equation of this circle in several ways. In Cartesian coordinates:

$$x^2 + y^2 = R^2$$

In polar coordinates:

$$r = R$$

or parametrically

$$x = R\cos\theta$$

$$y = R\sin\theta$$

where θ takes on values in the interval $[0, 2\pi]$.

With the appropriate units for t, we can write

$$x = R\cos t$$

$$y = R\sin t$$

where t is often understood to be the time, although it doesn't have to be. (Alternatively, we might use the time and an angular velocity ω in the combination as ωt).

We can write something equivalent using only one variable, for the position vector

$$\mathbf{r}(t) = \langle\ R\cos t, R\sin t\rangle = R\ \langle\cos t, \sin t\rangle$$

In three dimensions, we just add another component to $\mathbf{r}$ for z. Perhaps

$$\mathbf{r}(t) = \ \langle\cos t, \sin t, t\rangle$$

which traces out a spiral whose shadow in the xy-plane is the unit circle.

We can do calculus with vector functions of this type: both differentiating and integrating.

The rule is simple: the dimensions are independent. We differentiate each one separately. For example:

$$\mathbf{r} = \ \langle \cos t, \sin t \rangle$$

$$\frac{d}{dt}\ \mathbf{r} = \ \langle -\sin t, \cos t \rangle$$

but

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \mathbf{v}$$

the time-derivative of position is velocity.

So, we observe that, for motion on the unit circle, the velocity is perpendicular to the position vector because

$$\mathbf{r} \cdot \mathbf{v} = \langle\ \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t \rangle = 0$$

By the same logic the acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{r}} = \ \langle -\cos t, -\sin t \rangle$$

The acceleration vector **a** points in the same direction as the position vector **r**, with opposite sign.

Observe that the magnitude of the velocity vector is

$$|\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

This magnitude of **v** is unchanging in time. So on the circle, there is acceleration even though the speed is constant

tangent and normal

We observe that the velocity vector at a point is tangent to the curve. So, if we need a unit tangent vector

$$\hat{\mathbf{T}} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

All of these are functions of t. We could have written:

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

But we resist the urge to do that.

The normal vector $\hat{\mathbf{n}}$ to the curve is perpendicular to $\hat{\mathbf{T}}$. We construct $\hat{\mathbf{n}}$ in the usual way, by taking the x and y components of $\hat{\mathbf{T}}$ and writing:

$$\hat{\mathbf{n}} = \langle -y, x \rangle$$

or $\langle y, -x \rangle$. In either case, the dot product with $\mathbf{v}$ will be zero.

For an ellipse, we can parametrize like this:

$$\mathbf{r} = \langle a \cos t, b \sin t \rangle$$

$$\mathbf{v} = \langle -a \sin t, b \cos t \rangle$$

Now, it is no longer true that the tangent and the position vector are orthogonal. The ellipse is more interesting than the circle in this respect, as we will see.

I would just mention that for a surface, we need two variables in the parametrization. For example, to parametrize the surface of the sphere, we might use the polar angle ϕ and the radial angle θ. Any position on the globe can be specified with its longitude and latitude. We'll see a lot more about this as well.

time-derivative of products

As we mentioned above, to take the derivative with respect to the parameter (such as time), we just go through each component of a vector

$$\mathbf{r} = \langle x, y, z \rangle$$

$$\frac{d\mathbf{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$$

The question arises, what about products? Is there a product rule for vectors? Here's an example

$$\frac{d}{dt}\ \mathbf{r} \cdot \mathbf{v}$$

It turns out that there is.

$$\frac{d}{dt}\ \mathbf{r} \cdot \mathbf{v} = \frac{d\mathbf{r}}{dt} \cdot \mathbf{v} + \mathbf{r} \cdot \frac{d\mathbf{v}}{dt}$$

The reason is that the individual components of the dot product are simple functions of t, and our rule is to differentiate one component at a time.

Let's use Newton's dot notation for d/dt:

$$\mathbf{r} = \langle x, y, z \rangle$$

$$\dot{\mathbf{r}} = \mathbf{v} = \langle \dot{x}, \dot{y}, \dot{z} \rangle$$

$$\mathbf{r} \cdot \dot{\mathbf{r}} = x\dot{x} + y\dot{y} + z\dot{z}$$

The derivative is

$$\frac{d}{dt}\mathbf{r} \cdot \dot{\mathbf{r}} = \frac{d}{dt}(x\dot{x} + y\dot{y} + z\dot{z})$$

$$= \dot{x}\dot{x} + x\ddot{x} + \dot{y}\dot{y} + y\ddot{y} + \dot{z}\dot{z} + z\ddot{z}$$

$$= \dot{x}\dot{x} + \dot{y}\dot{y} + \dot{z}\dot{z} + x\ddot{x} + y\ddot{y} + z\ddot{z}$$

$$= \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \mathbf{r} \cdot \ddot{\mathbf{r}}$$

The same is true of the cross-product. The torque is $\mathbf{F} \times \mathbf{r}$ Let's take the derivative:

$$\frac{d}{dt}\ [\ \mathbf{F} \times \mathbf{r}\ ]\ =$$

Let's write $\mathbf{F} = \langle M, N, P \rangle$, then the cross-product gives a vector with components

$$[\ Nz - Py\ ]\ \hat{\mathbf{i}} + \ [\ Px - Mz\ ]\ \hat{\mathbf{j}} + \ [\ My - Nx\ ]\ \hat{\mathbf{k}}$$

where both M, N, P and x, y, z are *functions* of time.

The time derivative is obtained by the product rule. Again, I will use dots, and here we separate the components onto different lines:

$$[\ \dot{N}z + N\dot{z} - \dot{P}y - P\dot{y}\ ]\ \hat{\mathbf{i}} +$$
$$+\ [\ \dot{P}x + P\dot{x} - \dot{M}z - M\dot{z}\ ]\ \hat{\mathbf{j}} +$$
$$+\ [\ \dot{M}y + M\dot{y} - \dot{N}x - N\dot{x}\ ]\ \hat{\mathbf{k}}$$

but this is just two different cross-products added together. The first one is

$$[\ \dot{N}z - \dot{P}y\ ]\ \hat{\mathbf{i}} + \ [\ \dot{P}x - \dot{M}z\ ]\ \hat{\mathbf{j}} + \ [\ \dot{M}y - \dot{N}x\ ]\ \hat{\mathbf{k}}$$
$$= \dot{\mathbf{F}} \times \mathbf{r}$$

and the second is:

$$[\ N\dot{z} - P\dot{y}\ ]\ \hat{\mathbf{i}} + \ [\ P\dot{x} - M\dot{z}\ ]\ \hat{\mathbf{j}} + \ [\ M\dot{y} - N\dot{x}\ ]\ \hat{\mathbf{k}}$$
$$= \mathbf{F} \times \dot{\mathbf{r}}$$

Putting it all together

$$\frac{d}{dt}\ [\ \mathbf{F} \times \mathbf{r}\ ]\ = \dot{\mathbf{F}} \times \mathbf{r} + \mathbf{F} \times \dot{\mathbf{r}}$$

The product rule for differentiation holds for both the dot product and the cross-product.

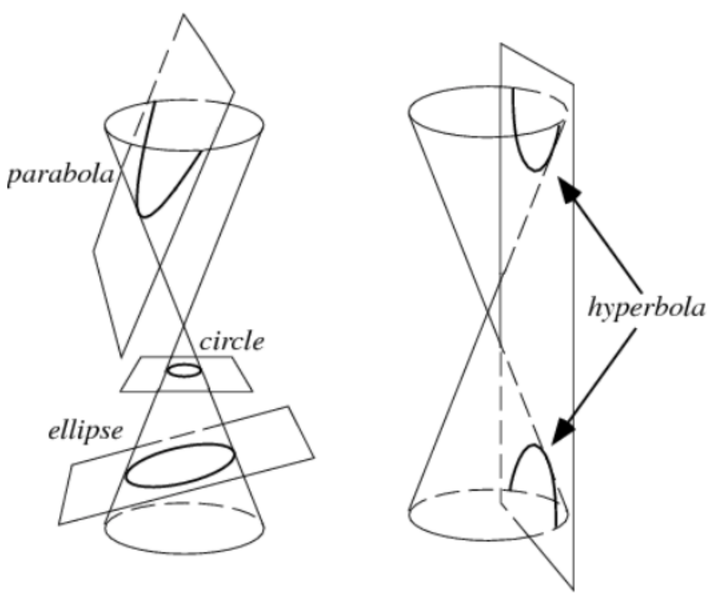
Part XVIII

Conic Sections

Chapter 63

Circle

We now consider what are called quadratic forms, as distinguished from linear equations (i.e., for lines). The quadratics contain a squared term (or a term that mixes x and y).



The simplest example is the equation for a unit circle centered at the origin:

$$x^2 + y^2 = 1$$

Pythagoras tells us that for a point (x, y), the square of the distance from the origin is $x^2 + y^2$. This equation describes all the points whose distance from the origin is equal to $\sqrt{1} = 1$. But all the points equidistant to a point form a circle. We generalize

$$x^2 + y^2 = r^2$$

It is clear that when $y = 0$, $x = \pm\, r$. r is the radius of the circle.

Now, what happens if we displace the unit circle from the origin so its center is at $(1, 0)$? What this amounts to is adding 1 to the x value of every point. If we solve for x

$$x = \sqrt{1 - y^2}$$

and then add 1

$$\begin{aligned} x &= \sqrt{1 - y^2} + 1 \\ (x-1)^2 &= 1 - y^2 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

Or, more generally

$$(x-h)^2 + (y-k)^2 = r^2$$

where the origin of the circle is at (h, k).

Multiplying out:

$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$$

$$x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2)$$

Comparing to the most general form for a quadratic

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

We see that

$$A = 1, \quad B = 1, \quad C = 0$$

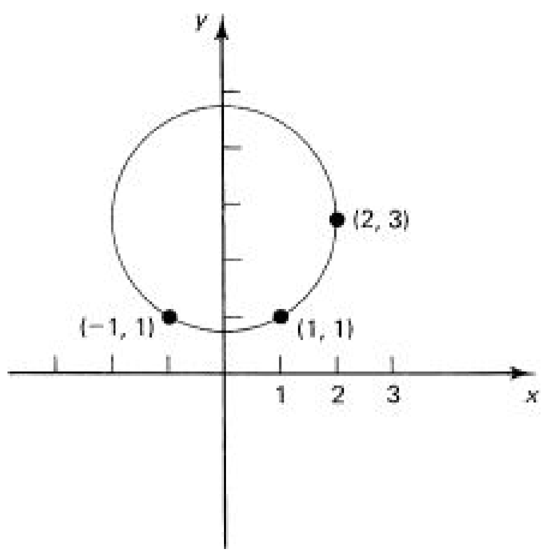
and in fact, this is true for all circles. (If $A = B \neq 1$, just divide all the terms by A).

Moreover

$$D = -2h, \quad E = -2k, \quad F = h^2 + k^2 - r^2$$

This equation can help us solve the following problem from Hamming: find the equation of the circle that passes through the following three points:

$$(-1, 1), (1, 1), (2, 3)$$



We write

$$x^2 + y^2 + Dx + Ey + F = 0$$

From the values of x and y at each of the three points we get

$$1 + 1 - D + E + F = 0$$

$$1 + 1 + D + E + F = 0$$

$$4 + 9 + 2D + 3E + F = 0$$

Three equations in three unknowns. We can do that.

Adding the first two equations together:

$$4 + 2(E + F) = 0$$

so $E + F = -2$.

Subtracting the first two equations (or substituting the result for $E+F$) tells us that $D = 0$.

Adding (-3) times the second equation to the third gives:

$$1 + 6 - D - 2F = 0$$

$$7 - 2F = 0$$

$F = 7/2$, and since $E + F = -2$, $E = -11/2$.

So the solution is

$$x^2 + y^2 - \frac{11}{2}y + \frac{7}{2} = 0$$

You can check that it works for all three points:

$$(-1, 1), (1, 1), (2, 3)$$

The first two are easy, while the third gives

$$4 + 9 - \frac{11}{2}3 + \frac{7}{2} = 0$$

$$8 + 18 - 33 + 7 = 0$$

which looks correct.

completing the square

We can improve this by completing the square. We see that

$$y^2 - \frac{11}{2}y + (\frac{11}{4})^2 = (y - \frac{11}{4})^2$$

We must add that back to the right-hand side of the original to obtain:

$$x^2 + (y - \frac{11}{4})^2 = (\frac{11}{4})^2 - \frac{7}{2}$$

The center is at $(0, 11/4)$. The radius doesn't come out cleanly but r^2 is

$$\frac{121}{16} - \frac{56}{16} = \frac{65}{16}$$

so r is slightly more than 2.

Or recall that we had:

$$D = -2h, \quad E = -2k, \quad F = h^2 + k^2 - r^2$$

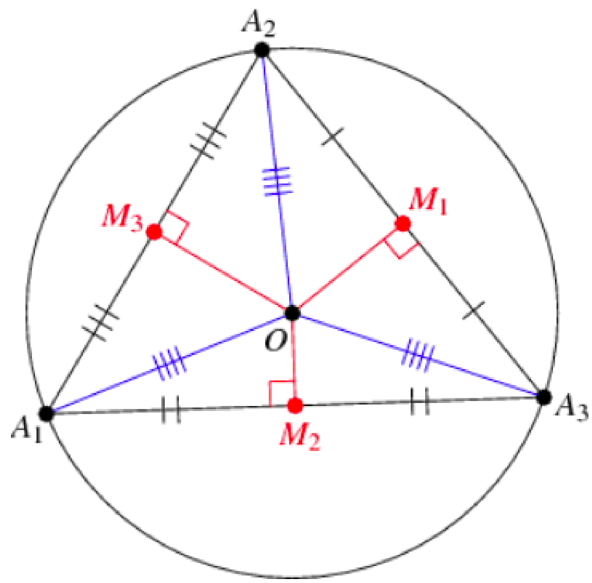
From this, we have that $h = 0$ and $k = -E/2 = 11/4$, and the radius is more complicated, as we said.

plane geometry

We can check our work by solving the problem using a technique from plane geometry. Again, we want the circle passing through three points:

$$(-1, 1), (1, 1), (2, 3)$$

Take two of the points to be placed on a circle and construct the line segment joining them (a chord of the circle). Find the midpoint of the chord and erect a perpendicular bisector through the midpoint. Now, every point lying on the bisector is equidistant from the two starting points. Proof: draw the two triangles including that point, the two starting points and the midpoint of the bisector. The two triangles are congruent. Here is the general picture.

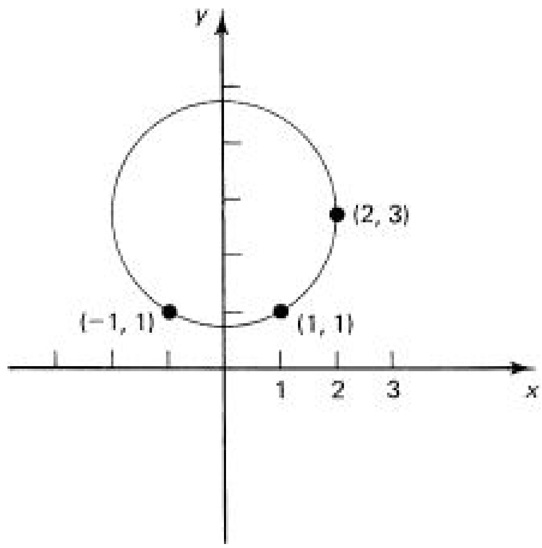


It's a bit trickier to prove that *every* point that is equidistant from the two points lies on the bisector. We assume that.

Since every point that is equidistant from the two points lies on the bisector, the radius of the circle lies on the bisector.

Then, erect a perpendicular bisector of a chord joining another pair chosen from the three points. This new bisector and the first one meet at the center of the circle.

In our case two points $(-1,1),(1,1)$ are symmetric about the y-axis. Therefore it is clear that the perpendicular bisector for these two points is the y-axis.



For the second bisector, form the vector between $(1, 1)$ and $(2, 3)$ as $\mathbf{v} = \langle 1, 2 \rangle$. The midpoint is at $(1, 1) + \mathbf{v}/2 = (3/2, 2)$.

The slope of the bisector is the negative inverse of the slope for the chord which is $-1/2$ so the equation of the bisector is

$$y - y_0 = -\frac{1}{2}(x - x_0)$$

Plugging in the point that we know, we obtain

$$y - 2 = -\frac{1}{2}(x - 3/2)$$

We want to solve for y when $x = 0$, crossing the first bisector, the y-axis

$$y - 2 = -\frac{1}{2}(-3/2)$$
$$y = \frac{11}{4}$$

So the center is at $(0, 11/4)$, which matches what we had before. We compute the distance to one of the points $(1, 1)$ as

$$d = \sqrt{1^2 + (11/4 - 1)^2} = \sqrt{1 + 49/16}$$

which also matches our previous result.

quadratics

The technique of completing the square comes from the standard equation

$$(x + p)^2 = x^2 + 2px + p^2$$

We run into problems where we have the $2px$ but not the p^2. For example

$$x^2 + y^2 + Dx + F = 0$$

Focus on

$$x^2 + Dx$$

we want to turn this into

$$(x + \text{something})^2$$

if D is like $2p$ we need to add something like p^2:

$$x + Dx + \frac{D^2}{4}$$

$$= (x + \frac{D}{2})^2$$

Since we added $D^2/4$ on the left, we must also add it on the right. We obtain

$$(x + \frac{D}{2})^2 + y^2 + F = \frac{D^2}{4}$$

You don't believe me? Multiply it out

$$x^2 + Dx + \frac{D^2}{4} + y^2 + F = \frac{D^2}{4}$$

To form $(x + D/2)^2$ on the left-hand side, we added $D^2/4$ (what we needed) to both sides.

Again, the general equation for a quadratic is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

In starting to work with one of these, the first thing to do is to see if there is a term which "mixes" x and y, that is, whether there is some term like Bxy. If there is, we might think about rotating the curve so that it is in a standard orientation.

We'll talk about how to do that **here**, in the context of the ellipse. However, the approach is general.

Let us assume we've done that, we relabel the new A, C etc. and assume here that $B = 0$.

Once in standard orientation, the next thing we might do is to translate the quadratic so that it is centered on the origin. We do that by completing the square for both x and y. We did some of that in this chapter, and we'll talk more about it **here** in the context of the parabola. Once again, however, the approach is general.

Cases:

- Both A and C present, and $F < 0$. If
  - A and C are both > 0: it's an ellipse.
  - A and C are of opposite signs: it's a hyperbola.
- A, C and F are all negative: it's imaginary.

• Only one squared term is present, but we still have the other variable
$Ax^2 + Ey + F = 0$: it's a hyperbola.

Not every quadratic equation gives a conic. Some are "degenerate". For example, having done all the right manipulations, we might end up with something like

$$A'(x-h)^2 + B'(y-k)^2 = 0$$

which has only $x = h$ and $y = k$ as a solution. It's a point.

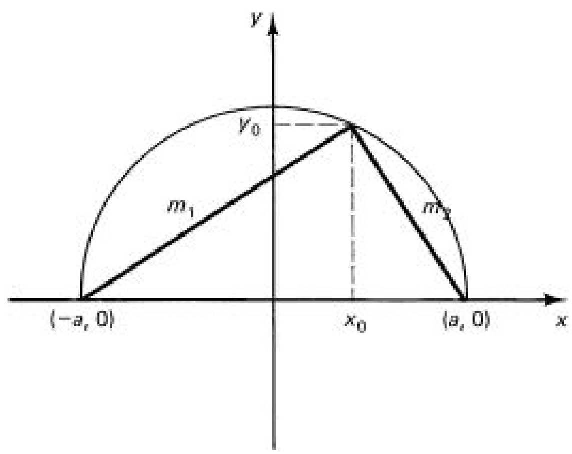


Figure 6.2-3 Angle in a semicircle

Here is another problem from Hamming. We need to prove that the angle above is a right angle. Suppose the equation of the circle is

$$x^2 + y^2 = a^2$$

The point on the circle is (x_0, y_0).

Our first solution uses slopes and points. The line from $(-a, 0)$ to

(x_0, y_0) has slope

$$m_1 = \frac{y_0}{x_0 + a}$$

The line from $(a, 0)$ to (x_0, y_0) has slope

$$m_2 = \frac{y_0}{a - x_0}$$

Two lines meet at a right angle if the product of their slopes is equal to -1.

$$m_1 m_2 = \frac{y_0}{x_0 + a} \ \frac{y_0}{a - x_0}$$

$$= \frac{y_0^2}{a^2 - x_0^2} = \frac{y_0^2}{x_0^2 + y_0^2 - x_0^2} = -1$$

This was not pretty, it's just good exercise.

And here is a proof using vectors and the dot product. Consider the semicircle centered on the origin with radius a, so the ends of the diameter are at $(x = \pm\, a, 0)$.

Form the vectors from those ends to an arbitrary point (x, y) on the perimeter:

$$\mathbf{u} = \ \langle x + a, y \rangle$$

$$\mathbf{v} = \ \langle x - a, y \rangle$$

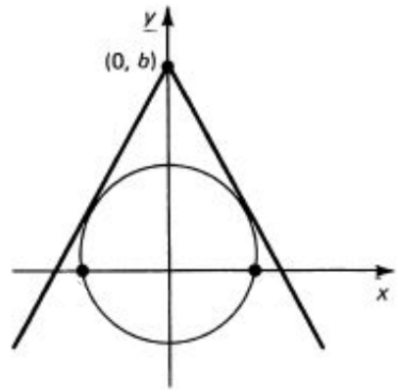
Notice that

$$\mathbf{u} \cdot \mathbf{v} = (x + a)(x - a) + y^2$$

$$= x^2 - a^2 + y^2 = 0$$

because $x^2 + y^2 = a^2$ for any point on the circle.

As our last example, consider the problem of finding the equation of a line tangent to a circle that goes through some arbitrary point b.



We take the circle to have radius a and be centered at the origin. We take the point b to be on the y-axis. The equation of the line on the right side is

$$\frac{y-y_0}{x-x_0} = m = \frac{y-b}{x}$$

$$y = mx + b$$

(well, of course).

For the point or points where the line intersects the circle we also have

$$y = \sqrt{a^2 - x^2}$$

$$\sqrt{a^2 - x^2} = mx + b$$

$$a^2 - x^2 = m^2x^2 + 2bmx + b^2$$

$$(m^2+1)x^2 + 2bmx + b^2 - a^2 = 0$$

From the quadratic equation:

$$x = \frac{-2bm \pm \sqrt{4b^2m^2 - 4(m^2+1)(b^2-a^2)}}{2(m^2+1)}$$

We are looking for the case where there is a single solution so the discriminant under the square root must be equal to zero:

$$4b^2m^2 = 4(m^2+1)(b^2-a^2)$$

$$
\begin{aligned}
m^2b^2 &= m^2b^2 - m^2a^2 + b^2 - a^2 \\
0 &= -m^2a^2 + b^2 - a^2 \\
m &= \pm\frac{\sqrt{b^2 - a^2}}{a}
\end{aligned}
$$

This makes sense since if $a = b$ the single tangent should be horizontal with zero slope. Notice that if $a^2 > b^2$ there is no real solution. This corresponds to having b inside the circle.

Chapter 64

Curvature

Curvature of curves

We would like to have a way to describe the curvature of a curve at a point. The way this is done is to fit a circle to the curve, to construct a circle that has the same first and second derivatives as the curve at the point of interest. (This section is taken from Hamming).

The general equation of a circle is

$$(x-h)^2 + (y-k)^2 = r^2$$

Implicit differentiation gives

$$2(x-h) + 2(y-k)\ y' = 0$$

$$(x-h) + (y-k)\ y' = 0$$

Differentiate again (using the product rule)

$$1 + (y')^2 + (y-k)\ y'' = 0$$

The known values in these equations are the point (x, y), the slope y' and the second derivative y'', taken from the curve we want to analyze.

The unknowns are the parameters of the circle that we're trying to find: h, k, and r.

Now, solve the last equation for

$$y - k = -\frac{1 + (y')^2}{y''}$$

and solve the second equation for

$$x - h = -(y - k)\ y'$$

From these two equations we can find h and k, the center of the circle.

Substituting for $x - h$ into the general equation of the circle we obtain

$$(y - k)^2\ (y')^2 + (y - k)^2 = r^2$$
$$(y - k)^2\ [\ (y')^2 + 1\ ] = r^2$$

Now substitute for $y - k$

$$[\ \frac{1 + (y')^2}{y''}]^2\ [\ (y')^2 + 1\ ] = r^2$$
$$\frac{[\ 1 + (y')^2\ ]^3}{(y'')^2} = r^2$$

Take the square root to find the radius.

$$r = \frac{[\ 1 + (y')^2\ ]^{3/2}}{y''}$$

If the original curve had been a straight line or the second derivative zero at that point, we would face the problem of dividision by zero.

For this reason, it makes sense to define the *curvature* as the inverse of the radius of the fitted circle. κ is used for the curvature:

$$\kappa = \frac{1}{r} = \frac{y''}{[\ 1 + (y')^2\ ]^{3/2}}$$

We write this as the absolute value:

$$\kappa = \frac{1}{r} = |\frac{y''}{[\ 1 + (y')^2\ ]^{3/2}}|$$

because r is always a positive quantity.

So, for example, any straight line (which has $y'' = 0$), has zero curvature.

test with a known circle

Try using a circle centered at $(0, 0)$, as the curve to be fitted. We should get back $h = 0, k = 0, r = r$.

$$x^2 + y^2 = r^2$$
$$2x + 2y\ y' = 0$$
$$y' = -\frac{x}{y}$$

The second derivative is

$$y'' = -\ [\frac{y - y'x}{y^2}\ ]$$

Combined with the previous result

$$y'' = -\ [\frac{y - (-x/y) \cdot x}{y^2}\ ]$$

$$= -\frac{y^2 + x^2}{y^3} = -\frac{r^2}{y^3}$$

Calculate the three values:

$$\kappa = |\frac{y''}{[\ 1 + (y')^2\ ]^{3/2}}|$$
$$= \frac{r^2}{y^3} \cdot \frac{1}{(1 + x^2/y^2)^{3/2}}$$
$$= r^2 \cdot \frac{1}{[\ y^2(1 + x^2/y^2)\ ]^{3/2}}$$
$$= r^2 \cdot \frac{1}{[r^2\ ]^{3/2}}$$
$$= \frac{1}{r}$$

That's exactly what we want. $\kappa = 1/r$ corresponds to a circle of radius r. To find the center of the circle:

$$y - k = -\frac{1 + (y')^2}{y''}$$
$$= \frac{1 + x^2/y^2}{r^2/y^3}$$
$$= \frac{y^2 + x^2}{r^2/y}$$
$$= y$$

For x

$$x - h = -(y - k)\ y'$$
$$= -(y - k)(-\frac{x}{y})$$

Since we found that $k = 0$

$$x - h = -y(-\frac{x}{y}) = x$$

Everything looks correct. Having carried out all of this preparation, let's do a real problem.

parabola

Consider a simple parabola

$$y = x^2 \qquad y' = 2x \qquad y'' = 2$$

The radius is

$$r = |\frac{[\ 1 + (y')^2\ ]^{3/2}}{y''}|$$
$$= \frac{[\ 1 + 4x^2\ ]^{3/2}}{2}$$

The y-coordinate of the origin of the circle, k, is

$$k = y + \frac{1 + (y')^2}{y''}$$

$$= y + \frac{1 + 4x^2}{2}$$

and h is

$$h = x + (y - k)\ 2x$$

If we decide to fit the circle to the parabola at the point $(0, 0)$ we have

$$r = \frac{1}{2} \qquad k = \frac{1}{2} \qquad h = 0$$

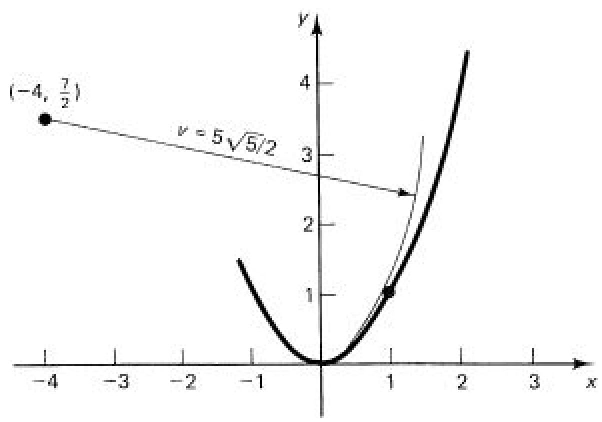
Perhaps more interesting, for the point $(1,1)$ we have

$$r = \frac{5^{3/2}}{2}$$

$$\begin{aligned} k &= y + \frac{1+4x^2}{2} \\ &= 1 + \frac{5}{2} = \frac{7}{2} \end{aligned}$$

$$\begin{aligned} h &= x + (y-k)\ 2x \\ &= 1 + (1 - 7/2)2 = -4 \end{aligned}$$

Which seems to be correct.



Find the slope of the line from the center of the circle to $(1,1)$:

$$\begin{aligned} m &= \frac{1-k}{1-h} \\ &= \frac{1-7/2}{1-(-4)} = \frac{-5/2}{5} = -\frac{1}{2} \end{aligned}$$

Recall that slope of the parabola at $(1, 1)$ is $y' = 2x = 2$. The slope of the line perpendicular to the tangent to the curve at the point $(1, 1)$ is the negative of the reciprocal, which is just what we obtained.

The squared distance between the center and the point should be equal to the radius squared from above:

$$(1 - 7/2)^2 + (1 - (-4))^2$$

$$= \frac{1}{4} \cdot 5^2 + 5^2 = \frac{5}{4} \cdot 5^2 = \frac{5^3}{4}$$

Taking the square root, we obtain what we had above:

$$r = \frac{5^{3/2}}{2}$$

Notice that the point $(1, 1)$ is 5 units horizontally across from the center and $5/2$ units down. If we were to translate the whole thing to the origin and then find the slope of the tangent to the circle at that point it would be

$$-\frac{x}{y} = -\frac{5}{5/2} = -2$$

which is the same as the slope of the parabola at that point, within sign.

The second derivative is

$$y'' = -\frac{r^2}{y^3}$$

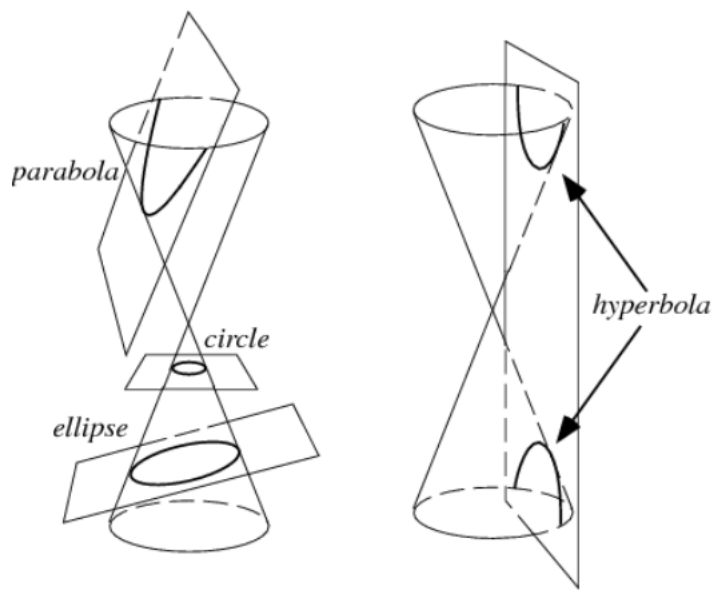
$$= -\frac{5^3/4}{(5/2)^3} = -2$$

which is also the same, within sign. The difference in sign comes about because we have not adjusted the equation for the circle. This point is in the fourth quadrant for the version moved to the origin.

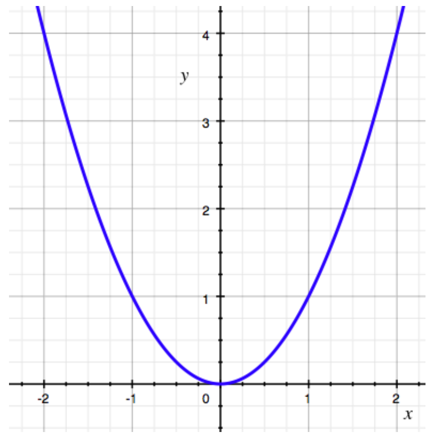
Chapter 65

Parabola

You probably know that the *conic sections* are the parabola, circle and ellipse, and hyperbola.



After the circle, perhaps the most familiar of these is the parabola. Consider a parabola in standard orientation, opening up and with its axis of symmetry pointed straight up and down.



A parabola of this orientation whose vertex is located at the origin and with axis of symmetry $x = 0$ is described by the very simple equation

$$y = ax^2$$

For the one shown above, a is equal to 1, giving $y = x^2$. For example, the points $(1, 1)$ and $(2, 4)$ lie on the curve.

polynomials and quadratics

A **polynomial** is a sum containing various powers of an independent variable, which is usually given as x. For example:

$$y = c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$$

The powers must be positive integers or zero: $n \in \{0, 1, 2 \dots\}$.

Each power x^n is multiplied by its corresponding constant c_n. The original equation might contain multiple coefficients for a given power x^n that are then combined to form the constant.

Each c_n is some real number. It is usual in examples for these constants to be integers, but this is by no means a requirement.

A **quadratic** is a polynomial that contains a term with x^2 but no higher powers of x:

$$y = c_2x^2 + c_1x + c_0$$

This is usually written as

$$y = ax^2 + bx + c$$

where a, b, and c are constants.

A quadratic may or may not contain lower powers of x. That is, either or both of b and c might be equal to zero. All of these are quadratics:

$$y = ax^2$$

$$y = ax^2 + bx$$

$$y = ax^2 + c$$

In general, the roles the constants a and c play in the graph of a quadratic are fairly obvious, while that of b is more subtle.

Changing the value of c shifts the graph up or down by the amount added. Comparing

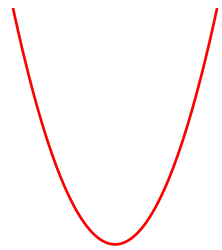
$$y = ax^2$$

$$y = ax^2 + c$$

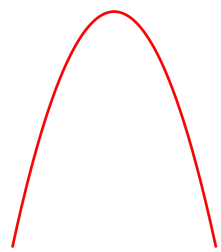
the second graph will be identical to the first, simply shifted up by c.

shape factor

a is called the **shape factor**. If a is positive, then the two "arms" of the parabola open up, and the **vertex** is the minimum value of the graph of the function.



If a is negative, then the graph opens down, and the vertex is the maximum.



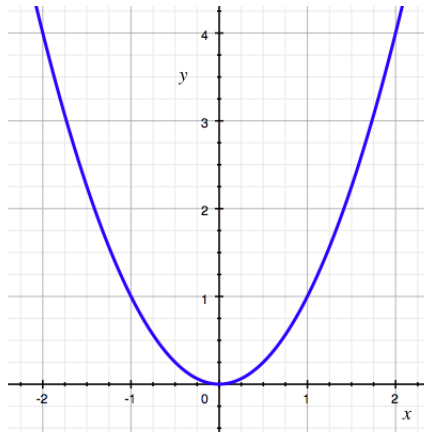
Any quadratic is symmetrical about the vertical axis that goes through its minimum (or maximum) point, the vertex.

For the rest of this discussion, we will only consider $a > 0$. If $a < 0$, the whole graph is flipped upside down. In that case, everywhere I say "minimum", we have a maximum instead.

Suppose

$$y = ax^2$$

x^2 is always greater than or equal to zero ($x^2 \geq 0$), therefore so is the value of this function, $y \geq 0$. The minimum value is the vertex at $(x = 0, y = 0)$.



The graph is symmetrical about the y-axis. This is also evident because $y = ax^2$ is an even function: $f(x) = f(-x)$.

Consider the values of y corresponding to $x = \{1, 2, 3, 4\}$. These are $y = \{a, 4a, 9a, 16a\}$. They increase like the square of x, but *linearly* with a.

If we plot $y = x^2$ and compare it to $y = 4x^2$, every y value in the second plot can be taken from the first one, just multiplied by 4. a stretches the plot linearly in the y-direction.

A substitution of variables $v = y/a$ turns $y = ax^2$ into $v = x^2$.

vertex

Every parabola with the same value of a has exactly the same shape. For the same a, they may differ in the position of the vertex, depending on the other constants, b and c. The graph of a parabola depends only on the shape factor and the position of the vertex.

The coordinates at the vertex are usually given as constants (h, k). Suppose the vertex of the parabola is at (h, k) with $a = 1$. Then the equation of the parabola is

$$(y - k) = (x - h)^2$$

It may seem counterintuitive that we subtract the value of the variable at the vertex, but this is consistent across all of the conic sections.

Rearranging

$$y = (x - h)^2 + k$$

we see that $y - k$ is like adding k to the constant c, it moves the graph up the page.

Expand

$$(y - k) = a(x - h)^2$$

$$y = ax^2 - 2ahx + h^2 + k$$

and compare with the canonical representation

$$y = ax^2 + bx + c$$

The coefficients of corresponding powers must be equal so

$$-2ah = b$$

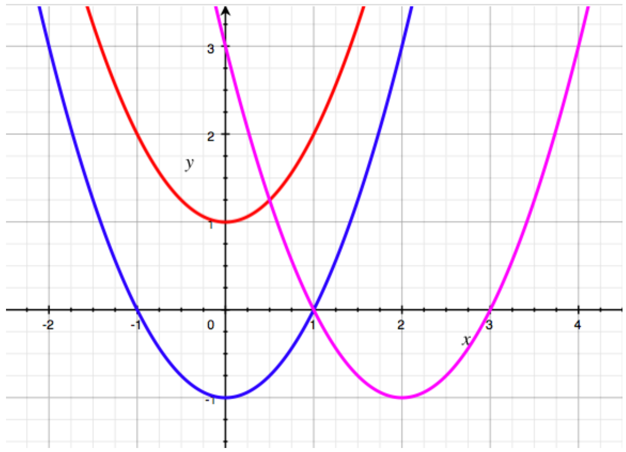
$$h = -\frac{b}{2a}$$

The x-value of the vertex is $x = -b/2a$. Also

$$h^2 + k = c$$

The y-value of the vertex is

$$k = c - h^2 = c - b^2/4a$$



Here are three plots, all with shape factor $a = 1$, which differ only in the position of the vertex: $y = x^2 - 1$ (blue), $y = x^2 + 1$ (red), and $y = (x - 2)^2 - 1$ (magenta).

The two with the vertex below the x-axis have $c = -1$, while the one with the vertex above the x-axis has $c = 1$.

Notice that the magenta one has $h = 2$. To write it in canonical form, expand

$$\begin{aligned} y &= (x - h)^2 - 2 \\ &= x^2 - 2xh + h^2 - 2 \\ &= x^2 - 4x + 2 \end{aligned}$$

Even though $c > 0$ for this one, the vertex is below the x-axis. That's because $b = -2h$ also affects the placement of the vertex. We see that effect here as $c = h^2 - 2$.

roots

The roots of a quadratic are those values of x for which the corresponding value of y is zero.

For example, if $x = 0$ then

$$y = x^2 = 0^2 = 0$$

$x = 0$ is the only value that yields this result, since if $x \neq 0$, then $y = x^2 > 0$.

Suppose we shift the graph of the equation *down*, by subtracting 1, i.e. letting $c = -1$:

$$y = x^2 - 1$$

From examining the plot of this function you will observe that there are two points where the graph of the function crosses the x-axis (where $y = 0$). We can guess them from the plot as $x = \pm 1$, and confirm that result directly by substituting into the equation and checking that we get $y = 0$.

$$y = (1)^2 - 1 = 0$$
$$y = (-1)^2 - 1 = 0$$

Another way to get this answer is to factor the original equation.

$$y = (x + 1)(x - 1)$$

Now it is obvious that either $x = \pm 1$ gives $y = 0$ as the result.

If a quadratic can be factored to the form

$$y = (x - r_1)(x - r_2)$$

then r_1 and r_2 are the roots, because if $x = r_1$ or $x = r_2$, then $y = 0$.

On the other hand, suppose we shift the graph *up*

$$y = x^2 + 1$$

Since x^2 is always positive or zero, there is no value of x which gives $y = 0$. We say that such an equation has no (real) roots. An equivalent statement or observation is that its graph does not cross the x-axis.

Again, if a quadratic can be factored to the form

$$y = (x - r_1)(x - r_2)$$

then

$$y = x^2 - (r_1 + r_2)x + r_1 r_2$$

In the canonical representation

$$ax^2 + bx + c$$

c is the product of the roots, and b is the negative of the sum.

For the example

$$y = x^2 - 1$$

the sum of the roots is 0 and their product is -1. From the first fact:

$$r_2 = -r_1$$

substituting into the second.

$$r_1^2 = 1$$

If you've had practice factoring quadratics with integer roots, you should be very familiar with this fact: c is the product of the roots, and b is the negative of the sum.

three types

In summary, we can classify parabolas into three types.

The first one has a graph that does not cross the x-axis. It has no real roots.

The second type has a graph that does cross the x-axis and has two distinct real roots of the form

$$y = (x - r_1)(x - r_2)$$

The third one has repeated roots

$$y = (x - r)(x - r)$$

This type has a single value of x that yields $y = 0$. This happens when the graph of the parabola just touches the x-axis — the vertex is on the x-axis.

The example given above $y = x^2$ is a special case of this type where $r = 0$.

quadratic formula

The quadratic formula gives the roots of any quadratic. In the case where there are no real roots, the results from the quadratic formula are complex numbers of the form $p \pm q\sqrt{-1}$, (where p and q are real numbers).

The formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You should memorize this.

Examples:

$y = x^2$

$$x = \frac{-(0) \pm \sqrt{0^2 - 4(1)(0)}}{2(1)} = 0$$

$y = x^2 - 1$

$$x = \frac{-(0) \pm \sqrt{0^2 - 4(1)(-1)}}{2(1)} = \pm\frac{\sqrt{4}}{2} = \pm 1$$

$y = x^2 + 1$

$$x = \frac{-(0) \pm \sqrt{0^2 - 4(1)(1)}}{2(1)} = \pm \frac{\sqrt{-4}}{2} = ??$$

The result of the last calculation is a pair of complex numbers. A complex number is a number of the form $p + q\sqrt{-1}$, often written $p + iq$.

We may write the expression under the square root as

$$D = b^2 - 4ac$$

where D stands for discriminant. If $D < 0$ then the result is a pair of complex numbers and we say there are no real roots. If $D = 0$ then there is a single (repeated) root.

Although the existence of complex roots means the graph does not cross the x-axis and there is no x such that $f(x) = 0$, nevertheless these roots do have physical meaning.

The two complex roots are related, they are called complex conjugates.

Label them as $z = p \pm q\sqrt{-1}$ and plug them into the factored quadratic

$$y = (x - z_1)(x - z_2)$$

$$\begin{aligned} y = & \ [\ x - (p + q\sqrt{-1})\ ]\ [\ x - (p - q\sqrt{-1})\ ] \\ & = (x - p - q\sqrt{-1})(x - p + q\sqrt{-1}) \end{aligned}$$

Multiplying this out, the square roots will (always) disappear:

$$y = x^2 - px + xq\sqrt{-1} - px + p^2 - pq\sqrt{-1} - xq\sqrt{-1} + pq\sqrt{-1} + q^2$$

$$= x^2 - 2px + p^2 + q^2$$

$$= (x - p)^2 + q^2$$

The meaning of this is the following: even with complex roots, the factored form gives a real result when multiplied out. The real part of $z = p \pm q\sqrt{-1}$ is p, and this is the value of x at the minimum.

q^2 is the displacement of the vertex up from the x-axis at the minimum.

more about b

Consider the basic equation

$$y = ax^2 + bx + c$$
$$\frac{y-c}{a} = x^2 + \frac{b}{a}x$$

By judicious manipulation we can make the last term $b/a \cdot x$ go away (this is *always* true). The general procedure is called completing the square. Write

$$x^2 + \frac{b}{a}x + \_\_\_ = (x + \_\_\_)^2$$

We seek two values to substitute for the spaces $\_\_\_$.

Now, of course, the $\_\_\_$ on the left-hand side is the square of the second one, on the right.

But there is another constraint. Namely, the second $\_\_\_$ is related to the cofactor of x on the left-hand side.

Recall that

$$x^2 + 2mx + m^2 = (x + m)^2$$

Compare that with

$$x^2 + \frac{b}{a}x + \_\_\_ = (x + \_\_\_)^2$$

Can you see that b/a must be equal to $2m$ and so $m = b/2a$? We need two copies of the second term in the binomial expansion (m), to put as the cofactor of x in the term $2mx$. Since the standard form in the last expression has b/a equivalent to $2m$, we get $b/2a$ equivalent to m.

If that's not clear, just verify that the following works. Write:

$$x^2 + \frac{b}{a}x + \_\_\_ = (x + \frac{b}{2a})^2$$

$$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = (x + \frac{b}{2a})^2$$

Now that we know what is needed to complete the square, go back to our problem.

$$\frac{y - c}{a} = x^2 + \frac{b}{a}x$$

To make the perfect square, we add $(b/2a)^2$ to the right-hand side, and to maintain the equality add the same thing to the left-hand side:

$$\frac{y - c}{a} + (\frac{b}{2a})^2 = x^2 + \frac{b}{a}x + (\frac{b}{2a})^2$$

As we saw above, the right-hand side is also $(x+b/2a)^2$ so we can write

$$\frac{y - c}{a} + (\frac{b}{2a})^2 = (x + \frac{b}{2a})^2$$

Finally, multiply through by a and rearrange slightly

$$y = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

We can get several things from this.

First, the minimum value of y occurs when the squared term is equal to zero, that is when

$$x + \frac{b}{2a} = 0$$

$$x = -\frac{b}{2a}$$

Therefore, the vertex of the parabola is at this value of x. We found this earlier by writing $(y - k) = a(x - h)^2$ and multiplying out.

When $x = -b/2a$, the corresponding value of y is

$$y = a(-\frac{b}{2a} + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

$$= c - \frac{b^2}{4a}$$

This also matches what we had before.

Second and more generally, the roots occur when

$$y = 0 = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

$$a(x + \frac{b}{2a})^2 = \frac{b^2}{4a} - c$$

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which is the quadratic formula.

Third, any quadratic can be rewritten as

$$y = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

$$y = a(x - h)^2 + k$$

translation

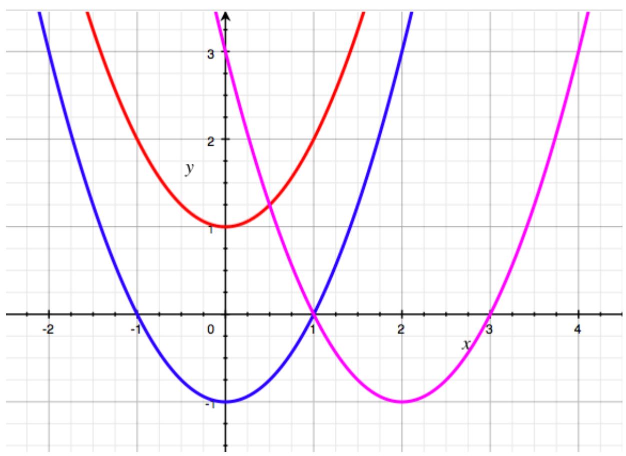
The basic shape depends only on a.

b (combined with $2a$) moves the vertex right or left from the y-axis.

a, b, c all together combine to move it up and down from the x-axis.

If you play around with a plotting application and change b you will find that the shape stays the same, but both the x and the y-values of the vertex will change as b changes.

Here are the three plots again.



$y = x^2 - 1$ (blue), $y = x^2 + 1$ (red), and $y = (x-2)^2 - 1$ (magenta).

The two with the vertex on the y-axis have $h = 0$, the other has $h = 2$.

The plots with real roots have $k = -1$ and $(y - k) = y + 1$, the other has $k = 1$.

For example, the lower right-hand plot is

$$(y + 1) = (x - 2)^2 = x^2 - 4x + 4$$

$$y = x^2 - 4x + 3$$

This can be factored easily:

$$y = (x-1)(x-3)$$

The roots are at $x = 1, x = 3$, which checks.

rotation

You might ask, what about rotation? For example, if we rotate $y = x^2$ by 45 degrees clockwise, what would be the equation to describe it? The short answer is that such equations do exist, and they have terms like xy or uv in them. They are not polynomials of the type we've been describing.

As an example to rotate through 45°, replace x and y by u and v with

$$x = u\cos\theta - v\sin\theta = \frac{u}{\sqrt{2}} - \frac{v}{\sqrt{2}}$$
$$x = ku - kv$$

$$y = u\sin\theta + v\cos\theta = \frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}$$
$$y = ku + kv$$

Substitute for x and y in the standard equation:

$$y = ax^2 + bx + c$$

$$ku + kv = a(ku - kv)^2 + b(ku - kv) + c$$
$$u + v = ak(u^2 - 2uv + v^2) + b(u - v) + \frac{c}{k}$$

Notice the term $-2akuv$ that mixes u and v.

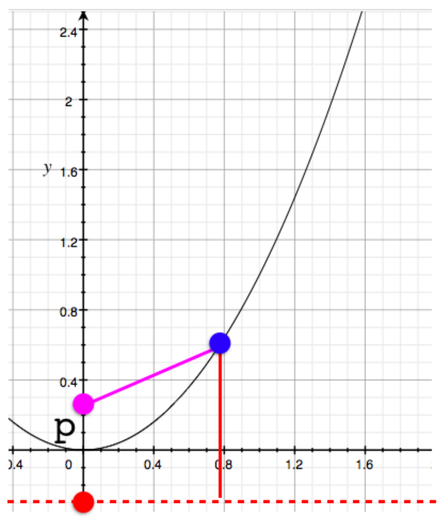
This is a more advanced topic than we can deal with here.

focus and directrix

There is a geometric definition of the parabola. Based on what we said above, without loss of generality, we can translate any parabola to the origin of coordinates, with equation $y = ax^2$.

Now, pick a point on the y-axis a distance p up from the origin, colored magenta in the figure. This point is called the focus.

Then draw a line parallel to the x-axis which intersects the y-axis the same distance p below the origin. This line is called the directrix. It is colored red and is dashed.



The parabola consists of all those points whose distance to the focus is equal to the vertical distance to the directrix.

Pick an arbitrary point on the parabola (in blue), with coordinates (x, ax^2). The squared distance to the focus (magenta point) is

$$\Delta y^2 + \Delta x^2 = (ax^2 - p)^2 + x^2$$

while the squared distance to the directrix (red line) is $(ax^2 + p)^2$.

For the correct choice of p these distances are to be equal:

$$(ax^2 - p)^2 + x^2 = (ax^2 + p)^2$$

$$a^2x^4 - 2apx^2 + p^2 + x^2 = a^2x^4 + 2apx^2 + p^2$$

Canceling two terms on each side

$$-2apx^2 + x^2 = +2apx^2$$

Divide by x^2

$$-2ap + 1 = 2ap$$

$$4ap = 1$$

$$p = \frac{1}{4a}$$

The shape factor a determines the distance of the focus from the origin, which is p, and from the directrix, which is $2p$.

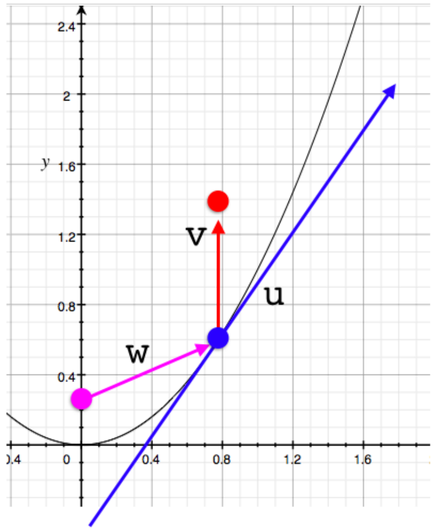
Chapter 66

Headlight problem

The reflective property of the parabola asserts that if a light ray emitted from the focus bounces off any point of the parabola, it then travels off in the vertical direction.

Snell's law for reflection says that the angle of incidence and reflection to the inside surface of the parabola must be equal. It is curious that this law has Snell's name on it, since the fact was known to Euclid, and Heron had a proof of it. The proof depends on the assumption that light travels the shortest path between two points.

In any case, applying that law to this problem, the angle of incidence is the angle of the magenta vector $\mathbf{w}$ with the tangent vector $\mathbf{u}$. This is equal to the angle of reflection, the angle of the tangent $\mathbf{u}$ with the vertical vector $\mathbf{v}$.



We assert that there exists a point on the y-axis (the focus, colored magenta), with the property that when we draw a vector to any point on the parabola, the angle that this vector makes with the tangent to the parabola is equal to the angle the tangent makes with the vertical.

Let the distance of this point from the origin be p. Then

$$\mathbf{w} = \langle x, ax^2 - p \rangle$$

The tangent has slope $2ax$ so

$$\mathbf{u} = \langle 1, 2ax \rangle$$

Scale the vertical to be a unit vector

$$\mathbf{v} = \langle 0, 1 \rangle$$

By the definition of the dot product, the cosine of the angle between $\mathbf{w}$ and $\mathbf{u}$ is

$$\frac{\mathbf{w} \cdot \mathbf{u}}{u\ w}$$

By the equal angle constraint, this is equal to the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$

$$\frac{\mathbf{u} \cdot \mathbf{v}}{u\ v} = \frac{\mathbf{w} \cdot \mathbf{u}}{u\ w}$$

Since $v = 1$ we have

$$w\ (\mathbf{u} \cdot \mathbf{v}) = \mathbf{w} \cdot \mathbf{u}$$

That's the important logic of the solution.

Now it's just algebra: The length of $\mathbf{w}$ is

$$w = \sqrt{x^2 + (ax^2 - p)^2}$$

while

$$\mathbf{u} \cdot \mathbf{v} = 2ax$$

$$\mathbf{w} \cdot \mathbf{u} = x + 2ax(ax^2 - p)$$

So

$$w\ (\mathbf{u} \cdot \mathbf{v}) = \mathbf{w} \cdot \mathbf{u}$$

$$\sqrt{x^2 + (ax^2 - p)^2}\ (2ax) = x + 2ax(ax^2 - p)$$

Divide by $2ax$:

$$\sqrt{x^2 + (ax^2 - p)^2} = \frac{1}{2a} + (ax^2 - p)$$

Square both sides

$$x^2 + (ax^2 - p)^2 = \frac{1}{(2a)^2} + \frac{1}{a}(ax^2 - p) + (ax^2 - p)^2$$

A nice cancelation:

$$x^2 = \frac{1}{(2a)^2} + \frac{1}{a}(ax^2 - p)$$

We can also cancel the x^2:

$$0 = \frac{1}{(2a)^2} + \frac{1}{a}(-p)$$

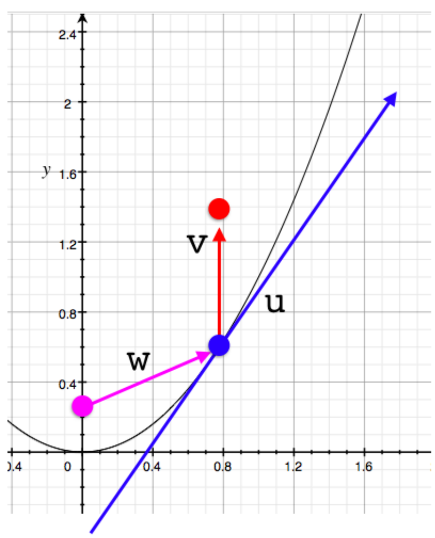
and finally cancel an a:

$$0 = \frac{1}{4a} - p$$

$$p = \frac{1}{4a}$$

The point $(0, 1/4a)$ is, as we saw before, the focus of the parabola.

Since p is independent of x, this property holds for every point on the parabola.



An alternative, more geometric approach is to note that the angle the vector **u** makes with the vertical at (x, ax^2) is equal to the angle **u** makes with the y-axis (just off the image to the bottom).

This angle is equal to the angle between **w** and **u** if and only if the triangle is isosceles, that is, if length of the vector **w** is equal to the distance between $(0, p)$ and the intersection of **u** with the y-axis.

We start by exploring the properties of a line through the point (x, ax^2) with slope equal to $2ax$.

From this point on, the point on the parabola is *fixed*. We want to write an equation for a line with the same slope as the parabola at this point, the same slope as the vector $\mathbf{u}$.

We will be re-using x as a variable. To reduce confusion, label the fixed value at the point as $\hat{x}$, so then $\hat{y} = a\hat{x}^2$, and the slope is $2a\hat{x}$.

The point-slope formula for the line is

$$2a\hat{x} = \frac{\Delta y}{\Delta x} = \frac{y - \hat{y}}{x - \hat{x}} = \frac{y - a\hat{x}^2}{x - \hat{x}}$$

The intersection with the y-axis occurs at $y = 0$ so there

$$\begin{aligned} 2a\hat{x} &= \frac{-a\hat{x}^2}{x - \hat{x}} \\ 2 &= \frac{-\hat{x}}{x - \hat{x}} \\ 2x - 2\hat{x} &= -\hat{x} \\ x &= \frac{\hat{x}}{2} \end{aligned}$$

The intersection of $\mathbf{u}$ with the x-axis is at $\hat{x}/2$.

For the intersection with the y-axis, $x = 0$ and then

$$\begin{aligned} 2a\hat{x} &= \frac{y - a\hat{x}^2}{-\hat{x}} \\ -2a\hat{x}^2 &= y - a\hat{x}^2 \\ y &= -a\hat{x}^2 \end{aligned}$$

What we've discovered is that the point of intersection is the same distance below the x-axis as our point on the parabola $(\hat{x}, a\hat{x}^2)$ is above it.

We could have used congruent triangles proceeding from the discovery above that the intersection of with the x-axis is at $\hat{x}/2$.

Our goal is to show that the triangle is isosceles:

$$a\hat{x}^2 + p = w$$

$$a\hat{x}^2 + p = \sqrt{\hat{x}^2 + (a\hat{x}^2 - p)^2}$$
$$(a\hat{x}^2 + p)^2 = \hat{x}^2 + (a\hat{x}^2 - p)^2$$

Continuing

$$a^2\hat{x}^4 + 2ap\hat{x}^2 + p^2 = \hat{x}^2 + a^2\hat{x}^4 - 2ap\hat{x}^2 + p^2$$

Does this look familiar?

Cancel two terms

$$2ap\hat{x}^2 = \hat{x}^2 - 2ap\hat{x}^2$$
$$4ap\hat{x}^2 = \hat{x}^2$$
$$4ap = 1$$
$$p = \frac{1}{4a}$$

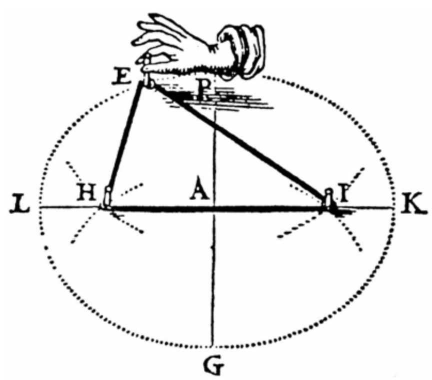
And we already proved this is true, if the magenta point we start from is the focus.

Hence the lengths are equal, the triangle is isosceles, and the corresponding angles are equal. The point we've been using is just the focus.

Chapter 67

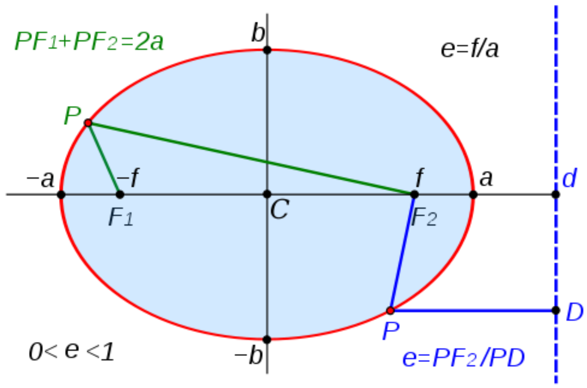
Ellipse

construction



Learning how to draw an ellipse using two pins and a circular piece of string holding a pencil is an early adventure in mathematics. The ellipse is the set of all points whose combined distance to the two pins (foci) is the same.

The drawing is reproduced from a 17th century book in Acheson (see the References).



The pin positions with respect to the origin or center are called the foci, lying at the points shown in the figure as $(\pm f, 0)$.

We will use the notation c: the focus in the first quadrant is at the point $(c, 0)$.

The lengths of the axes (called semi-major and semi-minor) are usually labeled a and b.

Consider the situation when the pencil is at the point $P = (0, a)$. The distance to the left focus is $c + a$, so the length L of the string is twice that

$$L = 2(c + a)$$

The combined distance from each point on the ellipse to the two foci is the length of the string minus the distance between the two foci

$$L - 2c = 2(c + a) - 2c = 2a$$

standard equation

We learn in algebra that the equation for an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We will derive this equation below.

The relation of a and c to b can be seen from the point $Q = (0, b)$ (see previous figure) where the combined distance to the two foci is just

$$QF_1 + QF_2$$

From what we said above the distance is $2a$, but Pythagoras also gives us

$$QF_1 + QF_2 = 2a = 2\sqrt{b^2 + c^2}$$

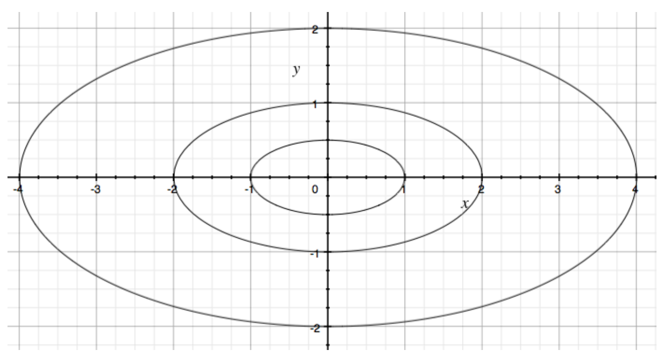
so

$$b^2 + c^2 = a^2$$

$$c^2 = a^2 - b^2$$

Given a and b one can then find c easily.

Here are three ellipses drawn with the same center.



The difference is an adjustment in the value on the right-hand side of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$$

where $r = \{1/2, 1, 2\}$. This is equivalent to scaling both a and b by the same factor of r

$$\frac{x^2}{(ra)^2} + \frac{y^2}{(rb)^2} = 1 = (\frac{x/a}{r})^2 + (\frac{y/b}{r})^2$$

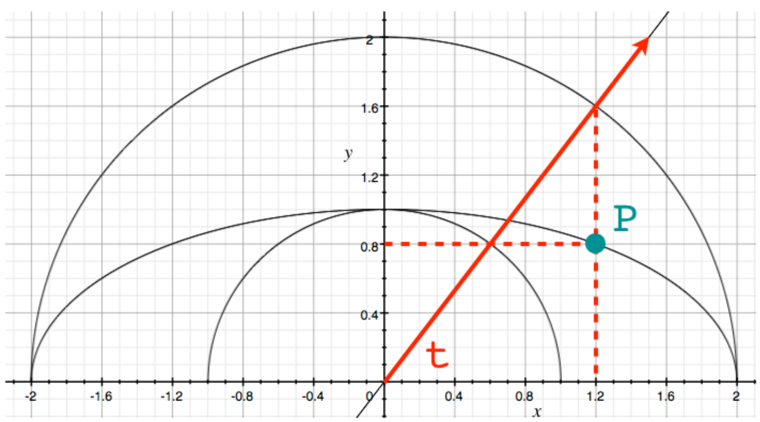
When $r = 2$ we need to make the string a bit less than twice as long, because the length c is also involved:

$$\frac{L_2}{L_1} = \frac{ra + c}{a + c}$$

parametrization

An alternative view is the one below, which shows (black curves) the upper half of two circles of radius $r = 1$ and $r = 2$ and an ellipse whose equation is

$$\frac{x^2}{2^2} + \frac{y^2}{1} = 1$$



Here $a = 2$ and $b = 1$.

The standard parametrization of the ellipse is

$$x = a \cos t$$

$$y = b \sin t$$

which I had trouble visualizing, until I drew the picture. The thing is that the parameter t is *not* the angle that a ray to P makes with the x-axis, as it is for the circle. Instead, to find the x value of P

corresponding to t, we extend the ray with angle t to the larger circle, with radius a, where we read off the x-value as

$$x = a\cos t$$

We go back to find the intersection of the same ray with the small circle to get

$$y = b\sin t$$

The algebraic way to do this is to show that the parametrization is equivalent to the original formulation

$$x^2 = a^2\cos^2 t$$

$$y^2 = b^2\sin^2 t$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$$

as expected.

Derivation of the equation of the ellipse

Although it is a bit tedious, it's a reasonable exercise to derive the equation of the ellipse from the geometric constraint. Recall that a is the length of the semi-major axis and c the distance to each of the foci from the origin.

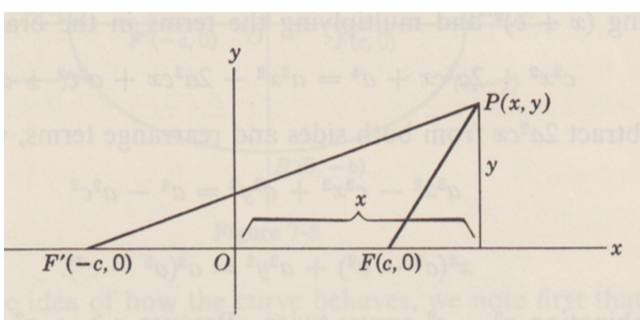


Figure 7-4

For any point x, y on the ellipse, the distance to the focus in the first quadrant is

$$\sqrt{(x-c)^2 + y^2}$$

and combined distances to both foci are equal to $2a$ so

$$2a = \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2}$$

Now we just do some algebra. Pick one square root and rearrange

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

Square both sides

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

Cancel y^2

$$(x-c)^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2$$

But

$$(x+c)^2 - (x-c)^2 = 4xc$$

so

$$0 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 4xc$$
$$a^2 + xc = a\sqrt{(x+c)^2 + y^2}$$

Square again

$$a^4 + 2a^2xc + x^2c^2 = a^2(x^2 + 2xc + c^2 + y^2)$$

$$a^4 + 2a^2xc + x^2c^2 = a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2$$

Cancel $2a^2xc$

$$a^4 + x^2c^2 = a^2x^2 + a^2c^2 + a^2y^2$$

Gather terms

$$a^4 - a^2c^2 = a^2x^2 - x^2c^2 + a^2y^2$$

$$a^2(a^2 - c^2) = x^2(a^2 - c^2) + a^2y^2$$

Recall that $b^2 = a^2 - c^2$

$$b^2a^2 = b^2x^2 + a^2y^2$$

Divide by a^2b^2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

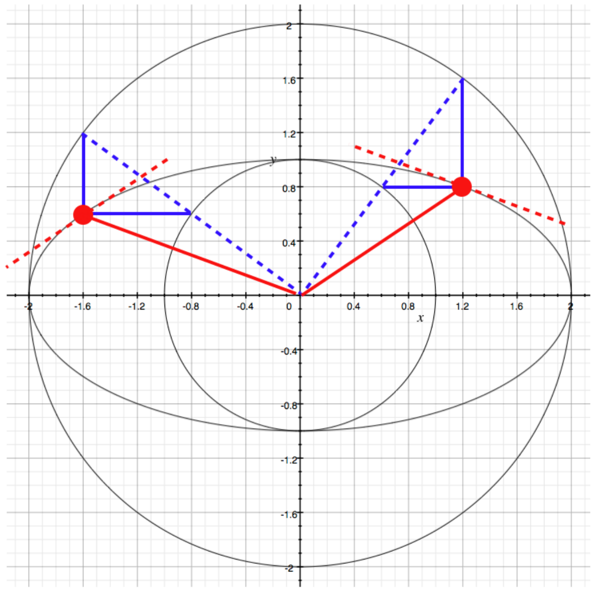
rotation

Let's return to the diagram of the ellipse with two bounding circles of radius a and radius b. There is a new diagram below. Consider the coordinates of the point $P = (x, y)$ (the red dot in the first quadrant) as functions of the angle t. As we said, t is *not* the angle of a ray from the origin to P.

Draw a ray (blue dotted line) from the origin makes an angle t with the x-axis. As before, extend the ray to the outer circle. The radius is a, the angle is t, and

$$a \cos t = x$$

This is the parametrization of the ellipse introduced previously.



The ray drawn with angle t has the same x-intercept with the outer circle as our point P on the ellipse. Similarly, the intercept of the ray with the inner circle has the same y-value as the point P on the ellipse.

We estimate the point $P = (1.2, 0.8) = (6/5, 4/5)$. Using our algebraic equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Recall that $a = 2$ and $b = 1$ so

$$x^2 + 4y^2 = 4$$

Plugging in for x^2 and y^2 we get

$$\frac{36}{25} + 4\left(\frac{16}{25}\right) = \frac{100}{25} = 4$$

as expected. Reading off the intercepts for the ray with angle t (dotted blue line) with the outer circle, we have the point $(1.2, 1.6)$ at a distance

2 from the origin. Thus,

$$\frac{1.2}{2} = 0.6 = \cos t$$

$$t \approx 0.927 \text{ rad} \approx 53^\circ$$

Looking again at the figure, we want to consider what happens for the angle $u = t + \pi/2$. This is the dotted blue ray in the second quadrant.

We might calculate the values of sine and cosine for u, but notice that if we view u as a vector, its *dot product* with t must be equal to zero. The coordinates of the intercept of the rotated vector with the outer circle are $(-1.6, 1.2)$, so the cosine of the angle u is

$$\cos u = -0.8$$

$$u \approx 2.498 = t + \frac{\pi}{2} \text{ rad} \approx 143^\circ$$

We confirm that

$$2.498 - 0.927 = 1.57 = \frac{\pi}{2}$$

The coordinates of the point on the ellipse are $(-1.6, 0.6)$, which we check against the formula

$$x^2 + 4y^2 = 4$$

$$(1.6)^2 + 4(0.6)^2 = 2.56 + 4(0.36) = 4$$

(no clean fractions this for this one).

tangent

Finally, and this is really the crucial result:

the vector to the point, call it Q, on the ellipse (red dot in the second quadrant) is the *tangent to the ellipse* for the point P in the first quadrant.

How did this happen? Recall what we did. We had

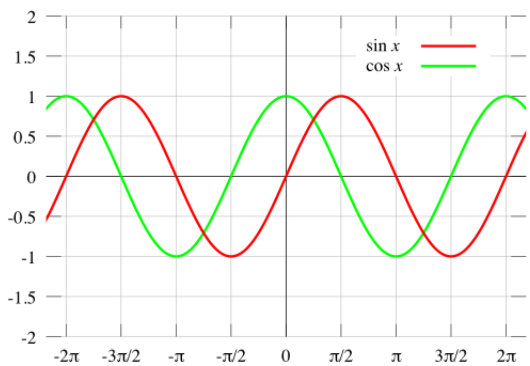
$$x = a\cos t$$

$$y = b\sin t$$

The rotated point $Q = (x', y')$ is

$$x' = a\cos(t + \frac{\pi}{2})$$

$$y' = b\sin(t + \frac{\pi}{2})$$



Sine is like cosine, but shifted to the right by $\pi/2$

$$\cos\theta = \sin(\theta + \frac{\pi}{2})$$

$$\sin\theta = -\cos(\theta + \frac{\pi}{2})$$

So

$$x' = a\cos(t + \frac{\pi}{2}) = -a\sin t$$

$$y' = b\sin(t + \frac{\pi}{2}) = b\cos t$$

Let's look at the position vector, which can be written $\mathbf{r}(t)$, since it's a function of the angle t or the time, but we will just use $\mathbf{r}$. It has components x and y.

$$\mathbf{r} = \langle x, y \rangle = \langle a\cos t, b\sin t \rangle$$

Now, the tangent to the ellipse is precisely the direction in which a particle at (x, y) is currently moving on the ellipse. The tangent vector points in the same direction as the velocity vector, but $\mathbf{v}$ is just the time-derivative of the position vector.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$= \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle$$

$$= \langle -a\sin t, b\cos t \rangle$$

$$= \langle x', y' \rangle$$

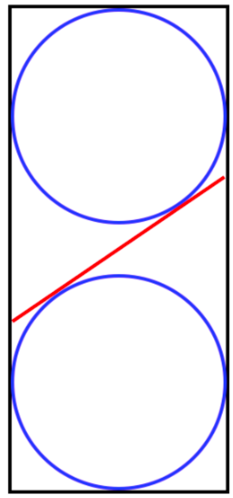
These two methods — using the time-derivative as the tangent, and rotation of t by $\pi/2$ — generate the same vector. And that's the point. :)

Starbird

Here is a neat approach to the ellipse that I saw in one of Michael Starbird's lectures.

Imagine a glass cylinder, shown here in cross-section and colored black. The cylinder has been sliced through at an angle by a plane, and we suppose a flat piece of glass in the shape of an ellipse is glued between the two halves.

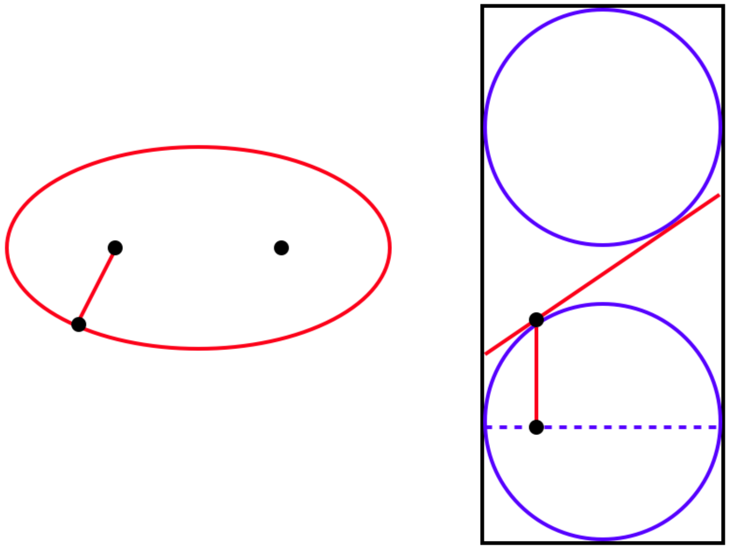
The elongated region in red (formed at the plane of the cut) is the ellipse, and the cylinder is oriented so that at each horizontal position going across the page, the two points on the ellipse are at the same vertical position. We see the plane of the cut edge-on.



Two spheres that fit snugly inside the cylinder lie above and below the ellipse, just touching it. The planar surface of the ellipse is tangent to the spheres, touching each one at a single point.

We claim that the points where the spheres touch the ellipse are the foci of the ellipse.

By the nature of the construction, the two spheres just fit inside the cylinder. That means the intersection where the spheres touch the cylinder is a circle, the lower one is shown with a dotted blue line in the next figure.

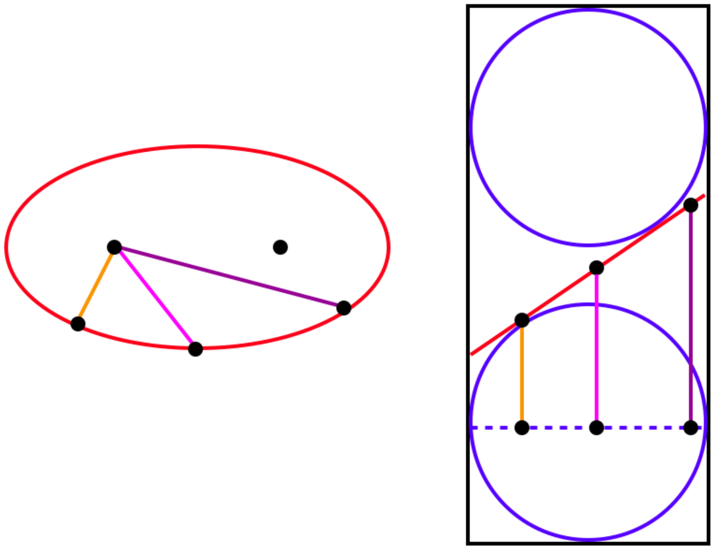


Now consider any point on the ellipse. On the left, we see one point on the ellipse together with two interior points we claim are foci, with a line drawn from our point to one of the foci.

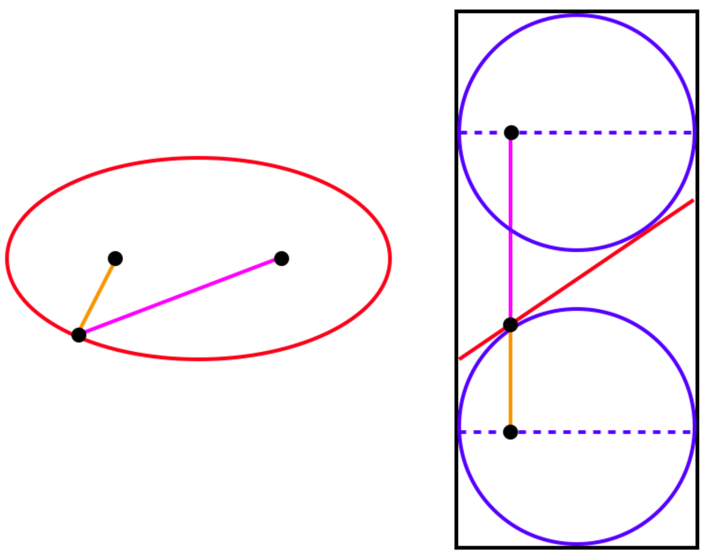
We said that this point is the point where the ellipse touches the lower sphere. We conclude that the line we've drawn from the edge of the ellipse to the focus is a tangent to the sphere.

A second tangent of interest is the perpendicular dropped vertically down the surface of the cylinder, shown in the right panel. Since they are both tangents, this line is the same length as line to the focus.

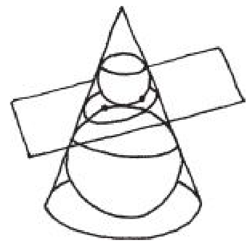
But the construction, and this equality, holds for any point on the ellipse, as shown in the next figure.



Finally, this is true for both spheres (below). The sum of the perpendicular tangents for any point is a constant.



Thus, the points where the spheres touch the ellipse are its foci, because the sum of the distances to any point on the ellipse, which is equal to the sum of the vertical tangents, is a constant.



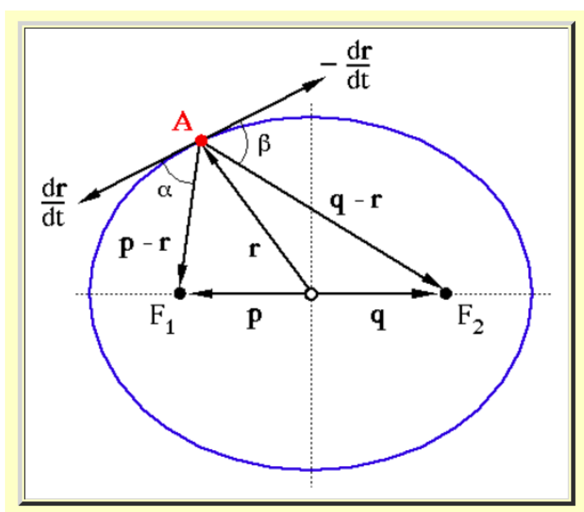
According to Lockhart, the same argument can be used to prove that the cross sections of a cone are ellipses (which seems strange at first since we've been demonstrating that the cross-sections of cylinders are also ellipses).

Chapter 68

Ellipse reflected rays

In any ellipse, the segments from the foci to any point on the ellipse make equal angles with the tangent. This means that light rays emitted from one focus and striking anywhere on the ellipse will pass through the other focus upon reflection. It is the principle behind "whispering galleries."

Here is a vector proof. A simple geometric proof follows.



http://163.178.103.176/Fisiologia/renal/objetivo_1/Medical_

Lithotripsy.htm

We have seen previously that

$$\mathbf{r} = \langle x, y \rangle = \langle a\cos t, b\sin t \rangle$$
$$\mathbf{v} = \dot{\mathbf{r}} = \langle -a\sin t, b\cos t \rangle$$

$\mathbf{v}$ points in the same direction as the tangent.

Now construct a vector from the origin to the focus F as

$$\mathbf{q} = \langle c, 0 \rangle$$

and the vector to F' is $\mathbf{p}$.

The vector corresponding to PF going toward the focus is

$$\mathbf{q} - \mathbf{r}$$

(since $\mathbf{r} + PF$ gets to the same place as $\mathbf{q}$).

and the one corresponding to PF' is

$$\mathbf{p} - \mathbf{r}$$

By the standard definition of an ellipse, the sum of the lengths is a constant

$$|\mathbf{q} - \mathbf{r}| + |\mathbf{p} - \mathbf{r}| = 2a, \quad \text{constant}$$

Lemma about a time-derivative

$\mathbf{r}$ is a vector function of time. We need to establish a property of the time-derivative of such a function. As an example use an arbitrary vector function of time, $\mathbf{w}$.

$$\frac{d}{dt}|\mathbf{w}| = \frac{d}{dt}\sqrt{\mathbf{w} \cdot \mathbf{w}}$$

by the chain rule:

$$= \frac{1}{2\sqrt{\mathbf{w} \cdot \mathbf{w}}} \; \frac{d}{dt}(\mathbf{w} \cdot \mathbf{w})$$

by the product rule

$$= \frac{1}{2\sqrt{\mathbf{w} \cdot \mathbf{w}}} \; (\frac{d\mathbf{w}}{dt} \cdot \mathbf{w} + \mathbf{w} \cdot \frac{d\mathbf{w}}{dt})$$

$$= \frac{1}{|\mathbf{w}|} \; (\frac{d\mathbf{w}}{dt} \cdot \mathbf{w})$$

Hence we have

$$\frac{d}{dt}|\mathbf{w}| = \frac{d\mathbf{w}}{dt} \cdot \frac{\mathbf{w}}{|\mathbf{w}|}$$

The rate of change of the magnitude of **w** is equal to a part of the rate of change of **w** itself, namely that part which points in the same direction as **w** itself. Effectively, what we've done is to decompose a differential change in **w** with time into two parts, one parallel to **w** and one perpendicular to it. The latter does not contribute to a change in the length.

back to the proof

We had

$$|\mathbf{q} - \mathbf{r}| + |\mathbf{p} - \mathbf{r}| = 2a$$

where $2a$ is a constant, so the time-derivative of the left-hand side is also zero.

$$\frac{d}{dt} \; (|\mathbf{q} - \mathbf{r}| + |\mathbf{p} - \mathbf{r}|) = 0$$

Now using the lemma

$$\frac{d}{dt}|\mathbf{q} - \mathbf{r}| = \frac{d}{dt}(\mathbf{q} - \mathbf{r}) \cdot \frac{\mathbf{q} - \mathbf{r}}{|\mathbf{q} - \mathbf{r}|}$$

and since $\mathbf{q}$ is constant:

$$= -\frac{d\mathbf{r}}{dt} \cdot \frac{\mathbf{q} - \mathbf{r}}{|\mathbf{q} - \mathbf{r}|}$$

So for the whole thing we have (rearranging terms):

$$\frac{d\mathbf{r}}{dt} \cdot \frac{\mathbf{q} - \mathbf{r}}{|\mathbf{q} - \mathbf{r}|} = -\frac{d\mathbf{r}}{dt} \cdot \frac{\mathbf{p} - \mathbf{r}}{|\mathbf{p} - \mathbf{r}|}$$

Take a look at that!

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \mathbf{v}$$

so we actually have

$$\mathbf{v} \cdot \frac{\mathbf{q} - \mathbf{r}}{|\mathbf{q} - \mathbf{r}|} = -\mathbf{v} \cdot \frac{\mathbf{p} - \mathbf{r}}{|\mathbf{p} - \mathbf{r}|}$$

The terms dotted with $\mathbf{v}$ are *unit vectors* from the point to the two foci.

Using the definition of the dot product, for a unit vector $\hat{\mathbf{b}}$

$$\mathbf{a} \cdot \hat{\mathbf{b}} = |\mathbf{a}||\mathbf{b}| \cos\theta = |\mathbf{a}| \cos\theta$$

this

$$\mathbf{v} \cdot \frac{\mathbf{q} - \mathbf{r}}{|\mathbf{q} - \mathbf{r}|} = -\mathbf{v} \cdot \frac{\mathbf{p} - \mathbf{r}}{|\mathbf{p} - \mathbf{r}|}$$

is the same as

$$|\mathbf{v}| \cos\alpha = |-\mathbf{v}| \cos\beta$$

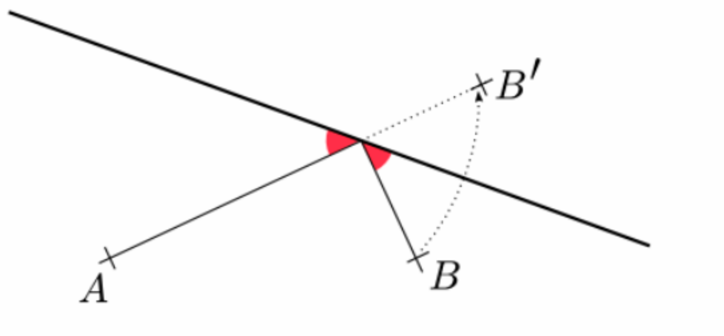
so

$$\alpha = \beta$$

□

geometry

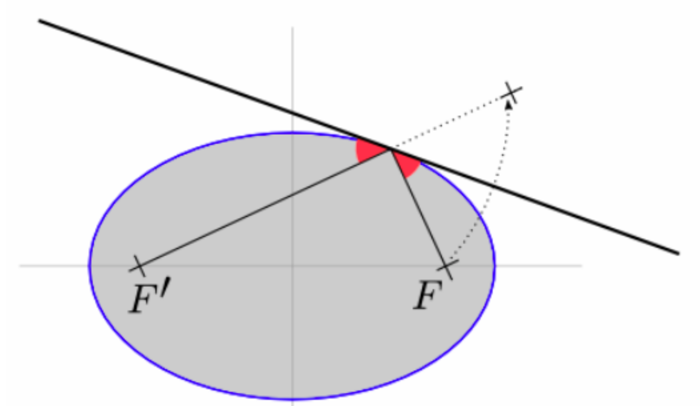
The geometric proof is even simpler. In another **chapter**, we considered the problem of the "shortest path."



The problem is to go from A to the line and then back to B by the shortest path. The clever solution is to place B' on the other side of the line at the same distance away. By definition (see Euclid) the shortest path A to B' is a straight line.

We can use vertical angles (or supplementary angles twice) and then similar triangles to prove that the two angles colored red are equal.

Now consider an enhanced diagram of the same situation:



We draw the tangent to the ellipse. By definition, the tangent has only a single point on the curve. This point lies at a distance $2a$ from the combined foci. All other points on the line are farther away from the two foci than the point of intersection. (You would have to make the string bigger to draw the ellipse that goes through any of those points).

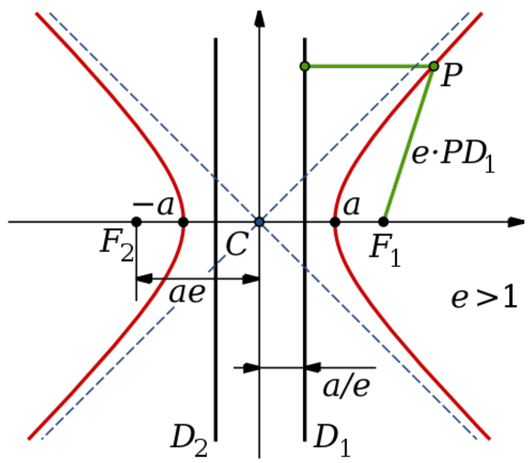
Therefore, the path shown is the shortest path from F' to the tangent and then to F. But we know that for the shortest path the angles colored red are equal.

http://math.stackexchange.com/questions/1063977/how-to-geometrically-prove-the-focal-property-of-ellipse

Chapter 69

Hyperbola

Here is a hyperbola as shown in the wikipedia article on the subject.



Hyperbolas of this type (that open "east-west") have equations of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Rearranging

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2}$$

so the minimum value of x occurs when $y = 0$ and $x = a$.

The *conjugate* hyperbola of this one is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

or equivalently

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

opens ”north-south.”

And, although I will wait to deal with this complication, we have to mention another very common hyperbola

$$xy = c$$

where it must be true that $x \neq 0$ and $y \neq 0$.

Another feature of hyperbolas is the asymptote, the straight line which is approached when $x, y >> a, b$. In the case of the first example

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

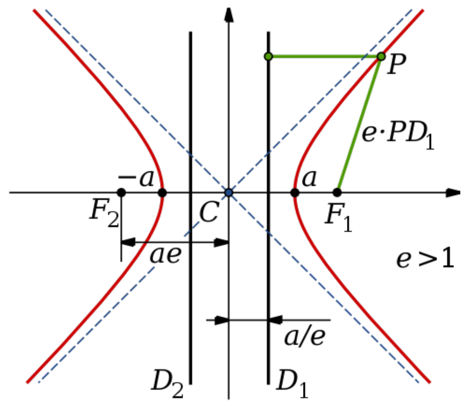
$$y^2 = \frac{b^2}{a^2}x^2 - \frac{1}{a^2}$$

but for large x and y this approaches

$$y^2 = \frac{b^2}{a^2}x^2$$

$$y = \pm\frac{b}{a}x$$

As the diagram suggests:



The following diagram gives geometric meaning to the b coefficient which really derives from the slope of the asymptotic line. We go vertically up from $x = a$ to the asymptote and then go left to the $y-$axis, that intercept is b.

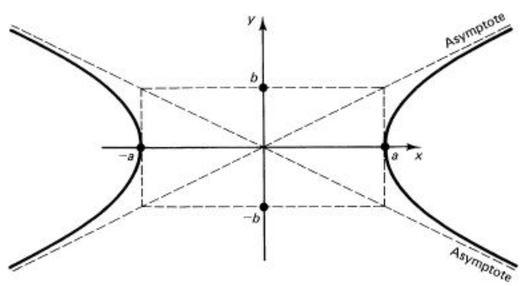


Figure 6.6-1 Hyperbola

geometry

Kline gives the following string and pencil construction for the hyperbola.

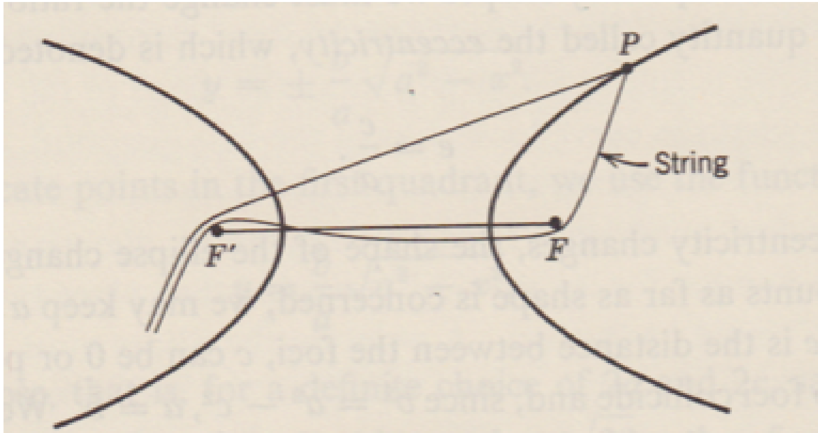


Figure 7-9

Pick two foci F and F' and loop a long piece of string around them, holding it tight. Then place the pencil at some point P on a line between the two foci, at a fixed position in the upper loop.

Now let the string slowly slip up past F' in both directions, increasing the length of PF and PF' by the same amount for each small slip. What this amounts to is that the difference $PF - PF'$ is constant.

If we place the origin halfway between F and F' then

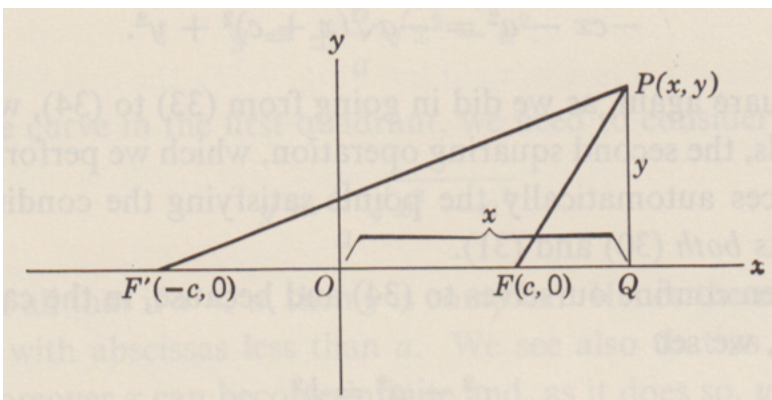


Figure 7-10

$$PF = \sqrt{(x-c)^2 + y^2}$$

$$PF' = \sqrt{(x+c)^2 + y^2}$$

and the difference $PF' - PF$ is

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}$$

and if the constant distance

$$PF' - PF = 2a$$

then

$$\sqrt{((x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

Now we repeat the approach we took for the ellipse:

$$\sqrt{((x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$$

Square

$$(x+c)^2 + y^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x-c)^2 + y^2$$

Cancel y^2

$$(x+c)^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x-c)^2$$

Since

$$(x+c)^2 - (x-c)^2 = 4cx$$

we have

$$4cx = 4a^2 + 4a\sqrt{(x+c)^2 + y^2}$$
$$cx - a^2 = a\sqrt{(x+c)^2 + y^2}$$
$$c^2x^2 - 2ca^2x + a^4 = a^2(x+c)^2 + a^2y^2$$
$$c^2x^2 - 2ca^2x + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2$$
$$(c^2 - a^2)x^2 - a^2y^2 = (c^2 - a^2)a^2$$

Define b^2 slightly differently here

$$b^2 = c^2 - a^2$$

so

$$b^2x^2 - a^2y^2 = b^2a^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

which looks familiar.

Chapter 70

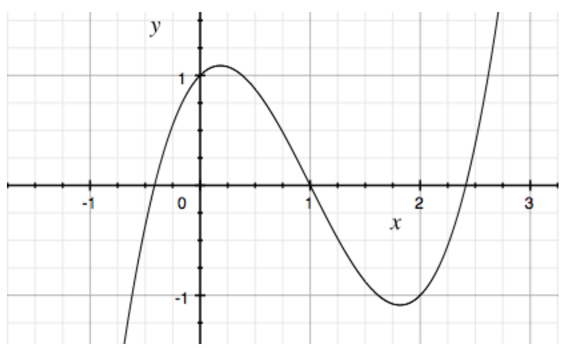
Cubics

Every cubic polynomial equation has at least one term containing x^3 but lacks any higher powers of x such as x^4.

The general equation is

$$y = ax^3 + bx^2 + cx + d$$

and a typical graph $(x^3 - 3x^2 + x + 1)$ looks something like this:



There is an axis of symmetry, here at $x = 1$, and the left half is the negative reflection of the right half about the value of $y = f(1) = 0$.

roots

By the *roots* of the equation, we mean those values of x giving $y = 0$, that is, we are solving

$$ax^3 + bx^2 + cx + d = 0$$

In this case we can always multiply through by $1/a$ so the term x^3 has a coefficient of 1, and if we do that then the coefficients are often renamed as:

$$x^3 + ax^2 + bx + c = 0$$

The cubic is an odd function, so the sign of x carries through in x^3. Since the x^3 term dominates the value of the function for extreme values of x, when $x << 0$, y is large and negative, while for $x >> 0$, y is large and positive. This is clearly seen for the above plot.

As a result, the graph of the function must cross the x-axis at least once, and thus every cubic has at least one real root, where $f(x) = 0$.

From this we conclude that every cubic can be factored into

$$(x - r)(x^2 + sx + t)$$

where $x = r$ is the guaranteed real root, although it isn't always the case that r is an integer, of course.

This expression is equal to zero either when $x = r$ or when the quadratic term is zero.

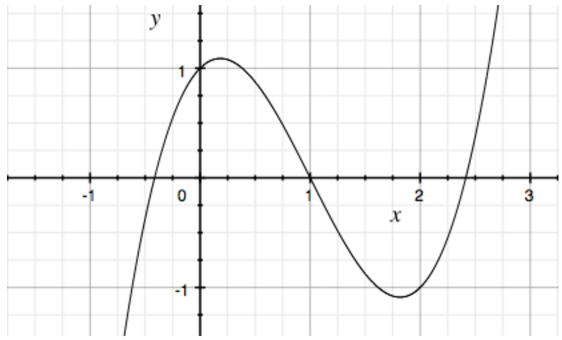
The roots of a quadratic $x^2 + sx + t$ are given by the familiar

$$\frac{-s \pm \sqrt{s^2 - 4t}}{2}$$

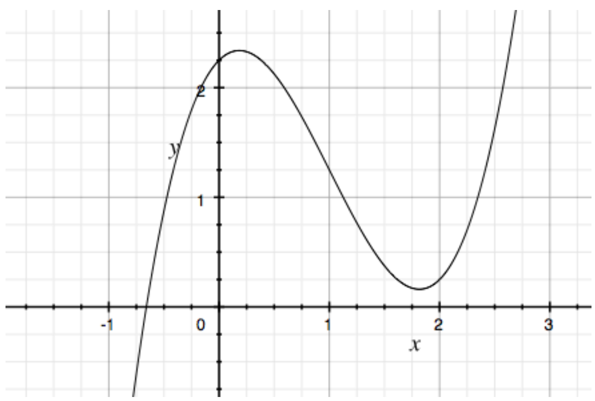
We know that quadratics have either two real roots or none depending on the value of the discriminant under the square root. We consider the case of repeated roots (when the discriminant is zero) as *two* roots.

Therefore, every cubic has either one real root or three of them.

Graphically, we can easily see the general truth of this statement.



In this example from before, near $x = 1.8$, no matter how far the graph goes below the x-axis before it turns the second time, we can add that much to the constant c. The result will be that the graph just touches the x-axis at this point. A tiny bit more and it will not cross at all.



The above graph is the same equation but with 1.25 added to c: $(x^3 - 3x^2 + x + 2.25)$.

factoring

Every cubic has at least one real root and as a consequence, the solution where $y = 0$ can be written as

$$(x - r)(x^2 + sx + t) = 0$$

Multiplying out

$$= x^3 + (s - r)x^2 + (t - rs)x - rt$$

Thus $a = s - r$ and $c = -rt$.

Suppose we know, or have guessed r. We can find s and t by comparison with the original equation.

As an example, consider:

$$(x - 1)(x + 1)(x + 2) = 0$$

$$= (x - 1)(x^2 + 3x + 2)$$

$$= x^3 + 2x^2 + 5x - 2$$

We plot this, guess that $x = 1$ is a root, check and find that $x = 1$ solves the equation. Thus $r = 1$ and since

$$c = -2 = -rt$$

then $t = 2$. Also

$$a = 2 = s - r = s - 1$$

and $s = 3$.

Alternatively, there is a formalism called synthetic division for deriving s and t. I have a simple version of this I like better than the complete formal approach. Consider

$$x^3 - 5x^2 - 2x + 24 = 0$$

Suppose we are given that $x = -2$ is a solution, which is easily checked. Now write:

$$\begin{gathered} x^3 - 5x^2 - 2x + 24 \\ = (x+2)(x^2 + \_\_\_\ x + \_\_\_) = 0 \end{gathered}$$

The cofactor of x^2, the first term on the right, is clearly just 1, so that we get the desired x^3 in the product.

Then we see that, multiplying by 2 we get $2 \times x^2 = 2x^2$, where the desired result is $-5x^2$. We need another $-7x^2$. Therefore the cofactor of x on the right must be -7 so that $-7x^2 + 2x^2 = -5x^2$:

$$(x+2)(x^2 - 7x + \_\_\_) = 0$$

Then we see that, multiplying by 2 we have $2 \times -7x = -14x$, where the desired result is $-2x$. We need another $12x$. Therefore, the constant term on the right must be 12 so that $-14x + 12x = -2x$.

$$(x+2)(x^2 - 7x + 12) = 0$$

And finally, we check the whole thing, multiplying 2×12 to give the desired constant, 24. This must work out if we've done the rest correctly and $x = -2$ is really a solution.

Inspired guessing can help. Consider

$$x^3 - 4x^2 - 9x + 36 = 0$$

You may notice that $a \times b = c$. This tells us that $1 - 4$ is a factor of $-9 + 36$. We guess that $x = 4$ is a solution:

$$(x-4)(x^2 + \_\_\_\ x + \_\_\_) = 0$$

We don't need anything more than $-4x^2$ so the cofactor of x on the right is 0 and write

$$(x-4)(x^2 + \_\_\_) = 0$$

For the constant, we guess -9 so as to get $-9x$ and then check $-4 \times -9 = 36$.

$$(x-4)(x^2-9) = 0$$

Finally, we can factor the second term

$$(x-4)(x+3)(x-3) = 0$$

relating roots to cofactors

Suppose a cubic has three distinct real roots, meaning there are real numbers p, q, r such that

$$(x-p)(x-q)(x-r) = 0$$

Multiplying out we would obtain for the constant term: $-pqr$.

$$(x^2 - qx - px + pq)(x-r) = 0$$
$$x^3 - qx^2 - px^2 + pqx - rx^2 + qrx + prx - pqr = 0$$
$$x^3 - (p+q+r)x^2 + (pq+qr+pr)x - pqr = 0$$

So

$$d = -pqr$$

Furthermore

$$a = -(p+q+r)$$
$$b = pq + qr + pr$$

If there are three real roots, they multiply to give the constant term.

$$(x-1)(x+2)(x+1)$$

$$= (x^2 + x - 2)(x + 1)$$
$$= x^3 + 2x^2 - x - 2$$

$1 \times -2 \times -1 = -2.$

Actually, a related statement is true even if two of the roots are not real. In principle, we can factor out the single real root, leaving a quadratic.

We said that the roots of a quadratic $x^2 + sx + t$ are given by

$$\frac{-s \pm \sqrt{s^2 - 4t}}{2}$$

If the second and third roots are imaginary they are so because what is under the square root is negative, so the above can be written as

$$z = u \pm iv$$

These consist of two complex numbers, which are complex conjugates. Their product is a real number:

$$(u + vi)(u - vi) = u^2 + v^2$$

When we plug the three roots into $(x - p)(x - q)(x - r)$ we get

$$(x - u + vi)(x - u - vi)(x - r)$$
$$= x^2 - ux - vix - ux + u^2 + uvi + vix - uvi + v^2)(x - r)$$

The imaginary terms with i cancel.

$$= (x^2 - 2ux + u^2 + v^2)(x - r)$$
$$= x^3 - 2ux^2 + u^2x + v^2x - rx^2 + 2urx - u^2r - v^2r$$
$$= x^3 - (2u + r)x^2 + (u^2 + 2ur + v^2)x - r(u^2 + v^2)$$

The result is all real coefficients.

$$d = -r(u^2 + v^2)$$

and d is the product of the three roots.

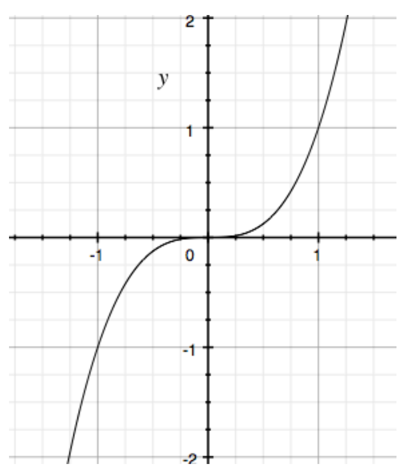
extreme points

The points where the graph turns around can be found by taking the derivative and setting it equal to zero.

$$y = ax^3 + bx^2 + cx + d$$
$$\frac{dy}{dx} = 3ax^2 + 2bx + c = 0$$
$$x = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a}$$

Not all cubics have a downward sloping segment. This happens when the quadratic for the slope has no real roots, i.e. when the square root term is less than or equal to zero. A simple example of this is when $b = 0$, such as
$$y = x^3$$

This obviously has 3 equal real roots, all zero.



The slope is $2x^2$, which is never negative and equal to zero only at $x = 0$.

repeated roots

A cubic can have repeated roots. Suppose

$$(x-p)^2(x-q) = 0$$

$$= (x^2 - 2px + p^2)(x-q)$$

$$= x^3 - qx^2 - 2px^2 + 2pqx + p^2x - p^2q$$

$$= x^3 - (q+2p)x^2 + (2pq+p^2)x - p^2q$$

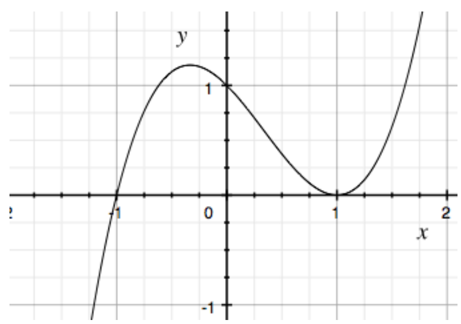
And, as before, the product of the roots is $-d$.

Graphically, two repeated roots means that one of the extreme points is also a root.

$$(x+1)(x-1)(x-1)$$

$$= (x+1)(x^2 - 2x + 1)$$

$$= x^3 - x^2 - x + 1$$



The slope is zero when

$$3x^2 - 2(q+2p)x + (2pq+p^2) = 0$$

translation

Continuing with $y = x^3$, when we add or subtract a value from y the plot is shifted up or down, similarly, changes to x shift the same curve to the left or right.

For example:

$$(x-1)^3 = x^3 - 3x^2 + 3x - 1$$

What this means is that the cofactors a and b may be non-zero and the shape still be the same as $y = x^3$. (The fact that a, b, c conform to the cubic expansion is a tipoff, however). The following section describes what is also essentially a horizontal translation.

depressed cubic

Tartaglia discovered that the quadratic term can be removed from a cubic

$$x^3 + ax^2 + bx + c$$

by an inspired substitution, $x = u - a/3$. Actually, I find the arithmetic a bit confusing, so I will further substitute $v = a/3$ and so $x = u - v$.

$$(u-v)^3 + a(u-v)^2 + b(u-v) + c$$

Now, expand each power of $u - v$ in order.

The cubic binomial $(u-v)^3$ has cofactors of 3 for the inner terms

$$(u-v)^3 = (u-v)(u^2 - 2uv + v^2)$$
$$= u^3 - 2u^2v + uv^2 - u^2v + 2uv^2 - v^3$$
$$= u^3 - 3u^2v + 3uv^2 - v^3$$

Switch the order so that the power of u is last in each term

$$= u^3 - 3vu^2 + 3v^2u - v^3$$

The quadratic is

$$a(u-v)^2 = a\ [\ u^2 - 2uv + v^2\ ]$$
$$= au^2 - 2avu + av^2$$

The linear term is just $bu - bv$.

Finally, collecting all the terms and grouping them by powers of u

$$= u^3\ [\ -3v + a\ ]\ u^2 +\ [\ 3v^2 - 2av + b\ ]\ u +\ [\ -v^3 + av^2 - bv + c\ ]$$

The bright idea is that the cofactor of u^2

$$-3v + a$$

is equal to zero by the terms of the substitution ($v = a/3$).

That leaves:

$$= u^3 +\ [\ 3v^2 - 2av + b\ ]\ u +\ [\ -v^3 + av^2 - bv + c\ ]$$

If we write

$$m = 3v^2 - 2av + b$$
$$n = -v^3 + av^2 - bv + c$$

then the cubic is

$$u^3 + mu + n = 0$$

We can reverse the second substitution ($v = a/3$). We have one less term in each formula, which is simplified a bit, but this also makes the formulas more awkward.

$$m = 3\frac{a^2}{3^2} - 2a\frac{a}{3} = \frac{a^2}{3} - 2\frac{a^2}{3} = -\frac{a^2}{3} + b$$

and

$$\begin{aligned} n &= -\frac{a^3}{3^3} + a\frac{a^2}{3^2} - \frac{ab}{3} + c \\ &= 2\frac{a^3}{3^3} - \frac{ab}{3} + c \end{aligned}$$

Here's an example. Consider:

$$x^3 + 3x^2 - x + 1 = 0$$

$$a = 3, \quad b = -1, \quad c = 1$$

So

$$m = -\frac{a^2}{3} + b = -\frac{a^2}{3} - 1 = -4$$

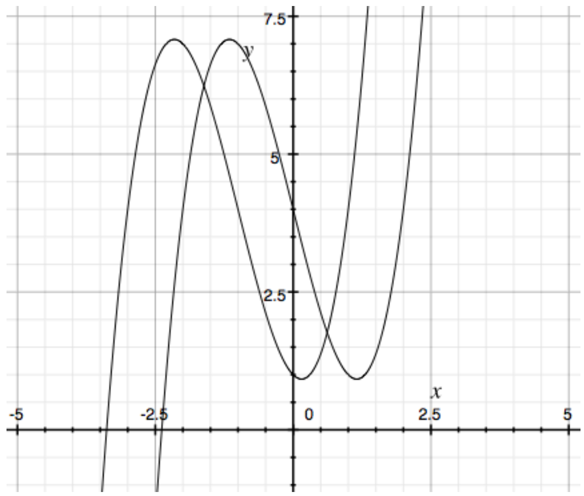
The constant n is

$$n = \frac{2a^3}{3^3} - \frac{ab}{3} + c$$
$$2 + 1 + 1 = 4$$

So the transformed version is

$$u^3 - 4u + 4$$

And this is indeed the same curve, simply displaced to the right by 1 unit, as the substitution $x = u - a/3$ or $x = u - 1$ implies.



The real root is also the same (taking into account the translation).

An example from Nahin is

$$x^3 - 15x^2 + 81x - 175 = 0$$

The coefficient of u is

$$(-\frac{a^2}{3} + b)$$

$$= -\frac{(-15)^2}{3} + 81 = -75 + 81 = 6$$

The constant is

$$\frac{2a^3}{3^3} - \frac{ab}{3} + c$$

$$= \frac{2(-15)^3}{3^3} - \frac{(-15)81}{3} - 175$$

$$= -150 + 405 - 175 = -20$$

Hence we have

$$u^3 + 6u - 20 = 0$$

By trial and error, we find that $u = 2$ is a solution.

Reversing the substitution, $3v = a$. This is the cofactor of x^2, the a from the original equation! So $v = a/3 = -15/3 = -5$.

Thus $x = u - v = 2 - (-5) = 7$. Check by substitution:

$$(7)^3 - 15(7^2) + 81(7) - 175 = 0$$

Factor out 7

$$7^2 - 15(7) + 81 - 25 = 0$$

$$49 - 105 + 81 - 25 = 0$$

which checks.

The resulting equation (lacking a quadratic term), is called a *depressed* cubic.

$$u^3 + mu + n = 0$$

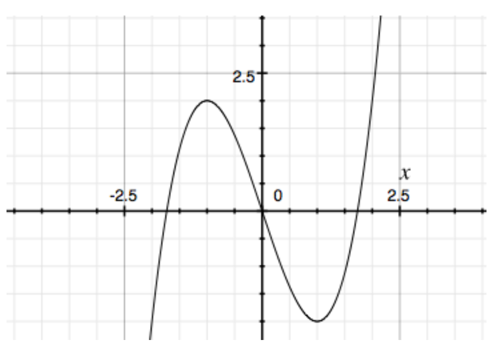
The result is kind of amazing. For any cubic containing ax^2, we can obtain the same curve without any quadratic term.

form of the curve

The constant simply displaces y by some value. The coefficient of x^3 stretches the curve.

From consideration of the depressed quadratic, you can see that the essential form is conferred by the cofactor of x in, say

$$y = x^3 - 3x$$



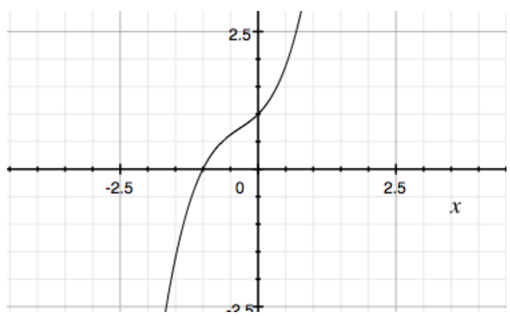
The larger the value of b, the bigger the deviations before the curve turns back. It's curious that the extreme points are exactly $y = 2$ here. That's because

$$\frac{dy}{dx} = 3x^2 - 3 = x^2 - 1$$

Hence they occur at $x = \pm 1$, where $y = \pm 2$.

In an expression like $y = x^3 + ax^2 + bx + c$, increasing a makes the central displacement more pronounced, while increasing b makes it less pronounced. Interestingly, having all three constants equal to 1 makes it go away altogether.

$$y = x^3 + x^2 + x + 1$$



$x = -1$ has solution $y = 0$, and that's the single real root because

$$x^3 + x^2 + x + 1 = (x + 1)(x^2 + 1)$$

and we know $x^2 + 1$ has $i = \pm\sqrt{-1}$ as its solution.

Also, we note for this example

$$y = x^3 + x^2 + x + 1$$

Get the slope as the derivative and set it equal to zero:

$$y' = 3x^2 + 2x + 1 = 0$$

for which the roots are

$$x = \frac{-2 \pm \sqrt{4 - 12}}{6}$$

The discriminant is negative, so there is no x that gives a slope of zero.

The minimum value of y' is $y'' = 0$

$$y'' = 6x + 2 = 0$$

$$x = -\frac{1}{3}$$

$$y' = \frac{3}{9} - \frac{2}{3} + 1 = \frac{2}{3}$$

Solving the depressed cubic

Which brings us finally to Cardano, and the solution of the cubic.

Dunham has a picture of the geometrical division of a cube that Cardano visualized,

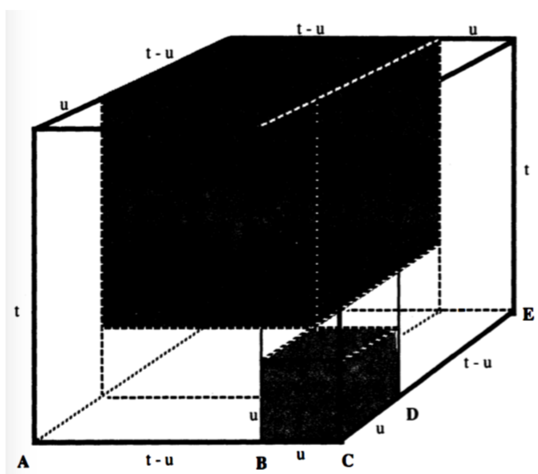


FIGURE 6.1

However, with modern notation, we can get there pretty simply from algebra.

$$(t - u)^3 = t^3 - 3t^2u + 3tu^2 - u^3$$

$$(t-u)^3 + 3t^2u - 3tu^2 = t^3 - u^3$$
$$(t-u)^3 + 3ut(t-u) = t^3 - u^3$$

Now let $x = t - u$

$$x^3 + 3tux = t^3 - u^3$$

Substitute $m = 3tu$ and $n = t^3 - u^3$:

$$x^3 + mx = n$$

This is a depressed cubic.

$$x^3 + mx - n = 0$$

The idea is to start with a depressed cubic we want to solve, and use that to get values for m and n.

If we can then determine values for t and u, $x = t - u$ will be the solution that we seek.

We have the two conditions: $m = 3tu$ and $n = t^3 - u^3$. Solve the first for u

$$u = \frac{m}{3t}$$

and substitute into the second:

$$n = t^3 - \frac{m^3}{(3t)^3}$$

Multiply both sides by t^3

$$nt^3 = t^6 - \frac{m^3}{27}$$

Looks like it's getting more complicated.

But this is a quadratic equation in disguise!

$$t^6 - nt^3 - \frac{m^3}{27} = 0$$

By the quadratic formula:

$$t^3 = \frac{n \pm \sqrt{n^2 + 4m^3/27}}{2}$$

Take the positive square root

$$= \frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}$$

and take its cube root:

$$t = \ [\ \frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}\ ]^{1/3}$$

Since $u^3 = t^3 - n$, just subtract n from the expression for t^3 before taking the cube root.

$$u = \ [\ -\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}\ ]^{1/3}$$

Then

$$x = t - u$$

$$= [\ \frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}\ ]^{1/3} - \ [\ -\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}\ ]^{1/3}$$

We can write this more simply by pre-computing

$$r = \frac{n}{2}, \qquad s = \frac{m^3}{27}$$

Then

$$x = \ [\ r + \sqrt{r^2 + s}\ ]^{1/3} - \ [\ -r + \sqrt{r^2 + s}\ ]^{1/3}$$

Here is Cardano (recall we are solving $x^3 + mx - n = 0$):

Cube one-third the coefficient of x; add to it the square of one-half the constant of the equation; and take the square root of the whole. You will duplicate [repeat] this, and to one of the two you add one-half the number you have already squared and from the other you subtract one-half the same . . . Then, subtracting the cube root of the first from the cube root of the second, the remainder which is left is the value of x.

example

How about Cardano's example:

$$x^3 + 6x - 20 = 0$$

Clearly, 2 is a solution to this. It is the only real root.

We have $m = 6$ and $n = 20$, so $n/2 = 10$.

$m = 6$ so

$$\frac{m^3}{27} = \frac{6^3}{27} = \frac{(2 \cdot 3)^3}{3^3} = 2^3 = 8$$

and then

$$x = \ [\ 10 + \sqrt{100 + 8}\ ]^{1/3} - \ [\ -10 + \sqrt{100 + 8}\ ]^{1/3}$$

These two terms are

$$(10 + \sqrt{108})^{1/3} = 2.732$$

$$(-10 + \sqrt{108})^{1/3} = 0.732$$

The difference is indeed very close to 2.

example 2

Another famous example is

$$x^3 - 15x - 4 = 0$$

Guessing, we obtain $x = 4$ as one root.

Now, to factor out $(x - 4)$:

$$x^3 - 15x - 4 = (x - 4)(\_\_\_\ x^2 + \_\_\_\ x + \_\_\_)$$

$$x^3 - 15x - 4 = (x - 4)(x^2 + \_\_\_\ x + \_\_\_)$$

$$x^3 - 15x - 4 = (x - 4)(x^2 + 4x + \_\_\_)$$

$$x^3 - 15x - 4 = (x - 4)(x^2 + 4x + 1)$$

The last multiplication to give the constant works, which provides a check on the whole thing.

We solve the quadratic as

$$x = \frac{-4 \pm \sqrt{16 - 4}}{2} = -2 \pm \sqrt{4 - 1} = -2 \pm \sqrt{3}$$

Check the positive root:

$$(-2 + \sqrt{3})^2 + 4(-2 + \sqrt{3}) + 1$$

$$= 4 - 4\sqrt{3} + 3 - 8 + 4\sqrt{3} + 1$$

$$= 0$$

So we have three real roots. Notice that

$$4 + (-2 + \sqrt{3}) + (-2 - \sqrt{3}) = 0$$

The sum of the roots is zero.

Now use Cardano's solution to solve

$$x^3 - 15x - 4$$

First

$$r = \frac{n}{2} = -2$$

$$s = \frac{m^3}{27} = \frac{-15^3}{27} = -125$$

$$\begin{aligned} x = \ & [\ r + \sqrt{r^2 + s}\ ]^{1/3} - \ [\ -r + \sqrt{r^2 + s}\ ]^{1/3} \\ = \ & [\ -2 + \sqrt{4 + -125}\ ]^{1/3} - \ [\ 2 + \sqrt{4 + -125}\ ]^{1/3} \\ = \ & [\ -2 + \sqrt{-121}\ ]^{1/3} - \ [\ 2 + \sqrt{-121}\ ]^{1/3} \\ = \ & [\ -2 + \sqrt{-121}\ ]^{1/3} + \ [\ -2 - \sqrt{-121}\ ]^{1/3} \end{aligned}$$

That seems strange at first. We have three real roots, but Cardano's solution gives an expression which is the sum of two imaginary numbers.

The resolution is that the two numbers here are complex conjugates. What we have is

$$[\ z\ ]^{1/3} + \ [\ z*\ ]^{1/3}$$

where

$$z = -2 + 11i$$

If we write this in polar form

$$z = re^{i\theta}$$

$$z* = re^{-i\theta}$$

so

$$z^{1/3} = r^{1/3}e^{i\theta/3}$$

$$z*^{1/3} = r^{1/3} e^{i(-\theta/3)}$$

The sum is

$$r^{1/3} \; [\; e^{i\theta/3} + e^{i(-\theta/3)} \;]$$

The term in the brackets is the sum of a complex number and its complex conjugate, $w + w*$, which is completely real, so the whole thing is completely real.

$$e^{i\theta/3} + e^{i(-\theta/3)} = 2 \cos \; (\theta/3)$$

To actually do the calculation

$$z = -2 + 11i$$

$$zz* = (-2 + 11i)(-2 - 11i) = -4 + 121$$

$$r = \sqrt{zz*} = \sqrt{117}$$

$$r^{1/3} = 2.211$$

For the angle

$$\theta = \tan^{-1} -\frac{11}{2} = -1.391$$

$$\theta/3 = -0.46$$

The term in the brackets is

$$2 \cos \; (\theta/3) = 2 \cos \; (-0.46) = 1.788$$

The whole thing is

$$r^{1/3} \; [\; e^{i\theta/3} + e^{i(-\theta/3)} \;] = 2.211(1.788) \approx 4$$

A much simpler method is to notice that

$$(2 + \sqrt{-1})^3 = (2 + \sqrt{-1})(4 + 4\sqrt{-1} - 1)$$

$$= (2 + \sqrt{-1})(3 + 4\sqrt{-1})$$
$$= 6 + 3\sqrt{-1} + 8\sqrt{-1} - 4$$
$$= 2 + 11\sqrt{-1} = 2 + \sqrt{-121}$$

The same result is obtained with $-2 + \sqrt{-1})^3$.

Hence

$$x = \ [\ -2 + \sqrt{-121}\ ]^{1/3} - \ [\ 2 + \sqrt{-121}\ ]^{1/3}$$
$$= 2 + 2 = 4$$

example 3

$$x^3 + 3x - 2 = 0$$

It might be simplest to try $m = 3$ and $n = 2$, so $n/2 = 1$.

$m = 3$ so

$$\frac{m^3}{27} = 1$$

Then

$$x = \ [\ 1 + \sqrt{2}\ ]^{1/3} - \ [\ -1 + \sqrt{2}\ ]^{1/3}$$

These two terms are

$$= 1.3415 - 0.745 = 0.596$$

This doesn't quite match the plot, however (which is closer to 0.7).

The arithmetic is tiresome, so write a script to do it.

script

```
# Cardano's method for solving
# x^3 + mx = n
```

```
import sys
from math import sqrt

def simplify(a,b,c):
    f = 1.0*a/3
    g = a*f   # a^2/3
    m = -g + b
    h = f*g   # a^3/3^2
    j = h/3   # a^3/3^3
    return m, (-j + h - (a*b*1.0)/3 + c)

def cardano(m,n):
    c = (m**3)/27
    h = n/2.0
    r = 0.3333333333

    rad1 = ( h + sqrt(h**2 + c))
    rad2 = (-h + sqrt(h**2 + c))
    return round(rad1**r - rad2**r, 5)
```

```
> python
..
>>> from cubics import *
>>> simplify(3,-1,1)
(-4.0, 4.0)
>>> cardano(6,20)
2.0
>>> cardano(3,2)
0.59607
```

We get Cardano's result, and confirm the calculation for the second example. This suggests that the error lies in the plotting program.

practical solving

A practical approach to real problems involves first plotting the function so as to know whether there is one real root or three, and get an idea of their values.

For example

plotter.py

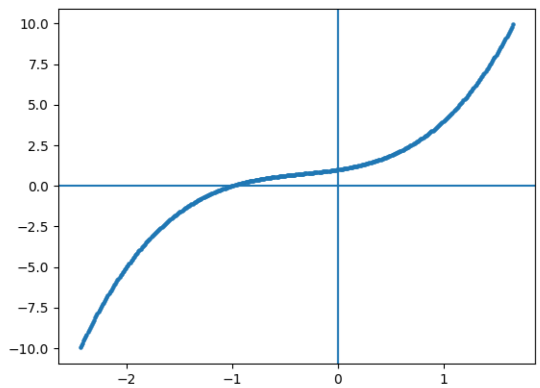
```
from matplotlib import pyplot as plt
import numpy as np

def plot(X,Y):
    plt.scatter(X,Y,s=5)
    plt.axhline()
    plt.axvline()
    #plt.axes().set_aspect('equal')
    plt.savefig('x.png')

def cubic(a,b,c,d):
    def f(x):
        return a*x**3 + b*x**2 + c*x + d
    L = np.linspace(-10,10,2000)
    X = list()
    Y = list()
    for x in L:
        y = f(x)
        if y < -10 or y > 10:
            continue
        X.append(x)
```

```
        Y.append(y)
    plot(X,Y)

cubic(1,1,1,1)
```



An actual solver might look something like this:

guess.py

```
import numpy as np

a,b,c = 1, 1, 1

def f(x):
    return x**3 + a*x**2 + b*x + c

def getX(x1,x2):
    N = 1000
    return np.linspace(x1,x2,N)

# assumes we go from f(x) < 0 to f(x) > 0
def guess(x1, x2):
```

```
    std_order = f(x1) < f(x2)

    print 'guess'
    print 'x1 = ', str(x1)
    print 'x2 = ', str(x2)
    print 'y1 = ', str(f(x1))
    print 'y2 = ', str(f(x2))
    print

    X = getX(x1,x2)
    if not std_order:
        X.reverse()
    assert f(X[0]) < 0 and f(X[-1]) > 0

    for i,x1 in enumerate(X):
        x2 = X[i+1]
        # must happen
        if f(x2) > 0:
            if not std_order:
                return x2, x1
            return x1, x2

def close(r):
    e = 1e-12
    return not (r > e or r < -e)

x1 = -2
x2 =  0
i = 0

while i < 100:
```

```
    print i+1
    x1, x2 = guess(x1, x2)
    if close(x2 - x1):
        break
    i += 1
```

output

```
> python guess.py
1
guess
x1 =  -2
x2 =  0
y1 =  -5
y2 =  1

2
guess
x1 =  -1.001001001
x2 =  -0.998998998999
y1 =  -0.00200400701102
y2 =  0.001999998999

3
guess
x1 =  -1.000001002
x2 =  -0.999998997997
y1 =  -2.00400801575e-06
y2 =  2.00400399997e-06

4
guess
```

```
x1 =  -1.000000001
x2 =  -0.999999998997
y1 =  -2.00601180111e-09
y2 =  2.00601202316e-09

5
guess
x1 =  -1.0
x2 =  -0.999999999999
y1 =  -2.00772731773e-12
y2 =  2.00817140694e-12

>
```

It's pretty clear that $x = -1$ is the real root.

$$x^3 + x^2 + x + 1 = (x + 1)(x^2 + 1)$$

The product of the two other roots is $x^2 + 1$, that is, they are $\pm\, i$.

Recall that $d = -pqr$ so

$$d = -\ [\ (-1) \times i \times -i\ ] = -\ [\ (-1) \times 1\ ] = 1$$

Part XIX

Gravitation

Chapter 71

Falling bodies

The simplest kind of motion is movement in a single dimension. To analyze it, the first thing we need to do is to pick an origin for our coordinate system, the place where x or $y = 0$. For a gravity problem the only dimension is up and down, and usually it's most convenient to pick the ground as the origin. It is conventional to use y as the variable.

Next, we need to decide when to start the clock, to say when is $t = 0$.

Such problems often have constant acceleration due to gravity, with $\mathbf{a} = -9.8$ m/sec^2. For simplicity we can take $\mathbf{a} = -10$ m/sec^2. The minus sign indicates that our coordinate system assigns positive numbers to positions further above us than ground-level, while the acceleration due to gravity points toward the earth. In 1D, we need not worry about vectors and direction, we just have to remember the convention about sign.

Velocity is the derivative of position with respect to time.

$$v = \frac{dy}{dt}$$

and acceleration is the second derivative

$$a = \frac{d^2y}{dt^2} = \frac{dv}{dt}$$

The two pieces of information with which we usually start the analysis are the initial position y_0 and the initial velocity v_0. We seek an equation that will tell us the current position, $y(t)$, given these two values plus the acceleration. Rather than do a formal integration, we make a a guess based on the relationship to the second derivative above:

$$y(t) \approx at^2 + C$$

We need to adjust the guess to get rid of the 2 that will come down when we take the first derivative.

$$y(t) = \frac{1}{2}at^2 + C$$

We also realize that the constant C can include terms of two types. First, anything like t times something will go away when we take the second derivative, so we should write

$$y(t) = \frac{1}{2}at^2 + C_1t + C_2$$

(where C_1 and C_2 are now different constants of integration). If we take the first derivative we have

$$v(t) = \frac{dy}{dt} = at + C_1$$

and we recognize that C_1 is just the initial velocity

$$v(0) = v_0 = 0 + C_1$$

so we have

$$y(t) = \frac{1}{2}at^2 + v_0t + C_2$$

Finally, we realize that, for $t = 0$ we have

$$y(0) = C_2 = y_0$$

so, finally

$$y(t) = \frac{1}{2}at^2 + v_0 t + y_0$$

which should be very familiar. With this equation in hand, knowing a, v_0 and t_0, we have only two variables, y and t. Given t it is easy to solve for y. Similarly, we can solve for v given t using

$$v(t) = at + v_0$$

It can be a little awkward to find t corresponding to a given y, but the last equation provides a nice trick for this. Solve for t (simplify the notation by writing v for $v(t)$):

$$v - v_0 = at$$

$$t = \frac{v - v_0}{a}$$

Now, given v (for example $v = 0$ at the top of the curve) we can find t. We can also plug this into the other equation and get something simple and useful. Rather than write $y(t)$ just write y so

$$y = \frac{1}{2}at^2 + v_0 t + y_0$$

$$2(y - y_0) = at^2 + 2v_0 t$$

$$2(y - y_0) = a\ (\frac{v - v_0}{a})^2 + 2v_0\ (\frac{v - v_0}{a})$$

$$2a(y - y_0) = (v - v_0)^2 + 2v_0(v - v_0)$$

$$= v^2 - 2vv_0 + v_0^2 + 2v_0 v - 2v_0^2$$

$$= v^2 - v_0^2$$

So finally,

$$2a(y - y_0) = v^2 - v_0^2$$
$$v^2 = v_0^2 + 2a(y - y_0)$$

Rather than find the time first and plug into the standard equation to get the velocity, we can go directly between velocity and position or vice versa.

Here are our equations re-written for gravity problems:

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$v - v_0 = -gt$$

$$v^2 = v_0^2 - 2g(y - y_0)$$

general examples

One simple question is: what is the time to fall from a given height h.

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

The initial velocity is zero, so

$$y - y_0 = -\frac{1}{2}gt^2$$

$$= 0 - h = -h$$

$$t = \sqrt{\frac{2h}{g}}$$

And given this time, the terminal velocity is:

$$v = v_0 - gt = 0 - g\sqrt{\frac{2h}{g}} = -\sqrt{2gh}$$

The second simple situation is to fire a projectile up with initial velocity v_0. Then we ask, what is the maximum height. This is one way

$$v^2 = v_0^2 - 2g(y - y_0)$$

At the maximum height $v = 0$:

$$0 = v_0^2 - 2gh$$

$$h = \frac{v_0^2}{2g}$$

Another way is to first get the time:

$$v - v_0 = -gt$$

$$-v_0 = -gt$$

$$t = \frac{v_0}{g}$$

Now the height is

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$$

$$h = -\frac{1}{2}g(\frac{v_0}{g})^2 + v_0\frac{u}{g}$$

$$h = \frac{v_0^2}{2g}$$

The object returns to earth when the height is equal to zero

$$0 = -\frac{1}{2}gt^2 + v_0 t$$

$$u = \frac{1}{2}gt$$

$$t = \frac{2v_0}{g}$$

One-half the time is spent going up, and the other half coming down. It's worth pointing out that the change in potential energy at height h is equal to

$$U = mgh = mg\frac{v_0^2}{2g} = \frac{1}{2}mv_0^2$$

which is equal to the kinetic energy at launch.

clever derivation

Shankar offers this derivation in his first Physics lecture. Write

$$\frac{dv}{dt} = a$$

Multiply both sides by v

$$v\frac{dv}{dt} = av$$

The first key step is to recognize that the left-hand side is equal (by the chain rule) to

$$v\frac{dv}{dt} = \frac{d}{dt}(\frac{v^2}{2})$$

So rewrite what we had including this and on the right-hand side use the definition $v = dx/dt$

$$\frac{d}{dt}(\frac{v^2}{2}) = a\ \frac{dx}{dt}$$

The second key is to recognize that we can get rid of dt and just think about this as an equality between differentials

$$d(\frac{v^2}{2}) = a\ dx$$

Now integrate

$$\int d(\frac{v^2}{2}) = \int a\ dx$$

for the constants of integration use the initial values

$$\frac{v^2}{2} - \frac{v_0^2}{2} = a(x - x_0)$$

That should look familiar:

$$v^2 = v_0^2 + 2a(x - x_0)$$

numerical examples

Suppose you are on the roof of a building of height $y_0 = 15$ m and throw a rock upward with velocity $v_0 = 10$ m/s. We find the maximum height as the position where $v = 0$. From the second equation

$$t = \frac{v - v_0}{a} = \frac{0 - 10}{-10} = 1 \text{ s}$$

$$y = \frac{1}{2}at^2 + v_0 t + y_0 = (-5)t^2 + (10)t + 15 = 20 \text{ m}$$

How fast is it going when it hits the ground? From the third equation

$$v^2 = v_0^2 + 2a(y - y_0) = (10)^2 + 2(-10)(-15) = 400$$

$$v = \sqrt{v^2} = \sqrt{400} = \pm\ 20 \text{ m/s}$$

There are two solutions, the one with negative value corresponds to the rock hitting the ground at the end of the throw. The positive velocity is the same rock being thrown from the ground upward with velocity 20 m/s. This will also hit the ground with velocity -20 m/s.

To find the time when the rock hits the ground, from the first equation

$$y(t) = \frac{1}{2}at^2 + v_0 t + y_0 = 0 = (-5)t^2 + 10t + 15$$

$$t^2 - 2t - 3 = 0 = (t - 3)(t + 1) = 0$$

So either $t = 3$ or $t = -1$ seconds. The first solution is the one we thought we wanted, the second corresponds to the positive velocity situation we had above. For a throw from the ground up, the trajectory is symmetric, two seconds going up and two seconds coming back down again. And reusing the second equation, for the part coming down $v - v_0 = at$, so $v = at = -20$ m/s.

example

A ball is thrown so that it goes upward with a velocity of 16 m/s. If $g = 32\ ft/s^2$, what is the position of the ball at time t?

We have the distance equation

$$h = h_0 + v_0 t - \frac{1}{2} g t^2$$

We set $h_0 = 0$, $v_0 = 16$ and $g = 32$

$$h = 16t - 16t^2$$

We wish to know when $h = 0$

$$0 = 16t(1 - t)$$

$t = 0$ is a solution, which is obviously correct. The ball starts with $h = 0$ at $t = 0$. The other solution is $t = 1$. The ball returns to $h = 0$ at $t = 1$.

Notice also that

$$v = v_0 - gt = 16 - 32t$$

so when $v = 0$

$$0 = v_0 - gt = 16 - 32t$$

$$16 = 32t$$

and $t = 1/2$. The trajectory of this ball is a parabola. It reaches its vertex when the upward velocity is zero ($t = 1/2s$). It returns to the earth in a time equal to that which was needed for its ascent.

example

Find t if a ball is dropped from a height = 392 feet, for $h_0 = 392$ and $v_0 = 0$ The distance equation is

$$h = h_0 + v_0 t - \frac{1}{2}gt^2$$

We have $h_0 = 392$ and $v_0 = 0$

$$0 = 392 - \frac{1}{2}gt^2$$

$$784 = 16t^2$$

$$\frac{784}{16} = 49 = t^2$$

$$t = 7$$

maximum range

Here is a problem in 2D. A ball is thrown making an angle θ with respect to the horizontal. What value of θ will give the maximum horizontal distance?

$$x(t) = v_x t$$

$$y(t) = v_y t - \frac{1}{2}gt^2$$

$$v_x = v\cos\theta$$

$$v_y = v\sin\theta$$

We find the time t when $y = 0$ and the ball has come back down to earth. We can remove one factor of t from each term on the right (we lose a possible solution but it's the one we already know, $y = 0$ at $t = 0$).

$$y(t) = 0 = v_y t - \frac{1}{2}gt^2$$

$$0 = v_y - \frac{1}{2}gt$$

$$t = \frac{2}{g}v_y$$

Substitute for t in the equation for $x(t)$ above

$$x(t) = v_x t = v_x \frac{2}{g} v_y$$

converting it to $x(\theta)$

$$x(\theta) = v\cos\theta\ (\frac{2}{g})\ v\sin\theta$$

$$= \ \frac{2v^2}{g}\sin\theta\cos\theta$$

Remembering the sum of angles formula ($\sin 2s = 2\sin s\cos s$):

$$= \ \frac{v^2}{g}\sin 2\theta$$

This is a maximum (for fixed v) when $\sin 2\theta$ is a maximum (equal to 1, so $\theta = \pi/4$. Alternatively

$$x(\theta) = \frac{2v^2}{g}\sin\theta\cos\theta$$

$$\frac{dx}{d\theta} = (\frac{2v^2}{g})\ [\ \cos^2\theta - \sin^2\theta\ ]$$

Set the first derivative equal to zero. Eliminate the constants in front:

$$0 = -\sin^2\theta + \cos^2\theta$$

$$\sin\theta = \cos\theta$$

$$\theta = \tan^{-1} 1 = \frac{\pi}{4} = 45^\circ$$

Chapter 72

Escape from the earth

In this chapter, we will calculate the energy required to move a body which is initially at some distance r_1 from the center of the earth (say, R, the earth's radius), to a position at some other distance r_2.

Eventually, we will consider problems related to the orbits of planets around the sun. But let's start with idealized circular orbits and consider escape velocity which does not require vector calculus to solve.

gravitational potential

First, suppose you stand on the earth's surface and throw a ball straight up into the air with a velocity v. It reaches some maximum height where its vertical velocity is zero. It has traded kinetic energy for potential energy. Be sure to move out of its way as it comes down.

Potential energy due to gravity increases with height above the earth.

Suppose we climb the steps up to the top of the leaning Tower of Pisa and then drop a marble over the edge. At the start the marble has zero velocity and at the end it has some velocity which we can calculate, neglecting air resistance.

We have the basic equation of motion with acceleration

$$y = \frac{1}{2}at^2 + v_0 t + y_0$$

and its time-derivative

$$v = \frac{dy}{dt} = at + v_0$$

We can obtain a formula that does not involve the time

$$v^2 - v_0^2 = 2a(y - y_0)$$

Do this by starting with equation 1 and rearranging:

$$2(y - y_0) = at^2 + 2v_0 t$$

then solve equation 2 for $t = (v - v_0)/a$ and substitute

$$\begin{aligned} 2(y - y_0) &= a\frac{(v - v_0)^2}{a^2} + 2v_0\frac{v - v_0}{a} \\ 2a(y - y_0) &= (v - v_0)^2 + 2v_0(v - v_0) \\ &= v^2 - v_0^2 \end{aligned}$$

For gravity

$$v^2 - v_0^2 = -2g(y - y_0)$$

We write the sign of the acceleration as negative, that is, $-g$, where it's understood that $g > 0$.

The usual choice is to have the coordinate system point up. If the ball is over my head, then $|v| < |v_0|$ (both going up and coming down), so the left-hand side is negative, which matches $y > y_0$ only if the sign on the acceleration is minus.

Later, when we calculate work, in some cases, the relevant force will be the one which we have applied to oppose gravity, and that force points up.

Call the distance $y - y_0 = h$. Then

$$v^2 - v_0^2 = -2gh$$

At the top, $v = 0$

$$v_0^2 = 2gh$$

This gives

$$\frac{1}{2}mv_0^2 = mgh$$

Which you probably recognize. At a height h above the ground there is a potential energy difference of mgh. After dropping through a height h, all this potential energy is converted to kinetic energy $mv^2/2$.

$$v^2 = 2gh$$

$$v = \sqrt{2gh}$$

escape velocity

Now, imagine an object starting from rest on the surface of the earth and then giving it enough velocity in an idealized trajectory (simply the vertical direction) so that it can move far enough away to be free from gravity altogether. In this problem the force decreases as the object moves away.

A simple approach is to use the principle of conservation of energy: we will impart enough kinetic energy so that the increase in potential energy after the motion is just balanced. The question is how to compute the potential energy.

Later on, in vector calculus we will use this expression for work

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

We must use an integral because the force is a function of r, and we use the dot product because the force and the changing position vector are not always aligned.

The work done over the course of the motion (force times distance) is equal to the energy added to the object. The vector equation says that only the component of the force in the same direction as the motion contributes.

Luckily, we don't need to use a vector equation for this problem. Just place the center of the earth at the origin, and treat all of the mass of the earth as being at that point. (We will develop Newton's proof of this later, see **here**).

Then, consider the force and motion as occurring only in one direction. While x or y could be used for the variable, the conventional choice is r. All of the (scalar) force and motion is in the r direction.

In the vector approach, we learn that the force $\mathbf{F}$ is the gradient of a scalar function (called the potential energy)

$$\mathbf{F} = -\nabla U$$

In 1D this just amounts to doing the scalar integral.

The usual derivation is that the change in potential energy in going from configuration a to b, is minus the work done in going from a to b where

$$W_{ab} = \int_a^b F(r) \ dr$$

and then

$$\Delta U = U_b - U_a = -W_{ab}$$

For the first example above, we have $F = -mg$ and so

$$W = \int_0^h -mg\ dy = -mgh$$

(Gravity does work on an object when it falls in the gravitational field, it does negative work on the object as it rises). Then

$$\Delta U = mgh$$

For the second case, where the change is large enough that the force is not constant, we write $F(r)$

$$F(r) = -\frac{GmM}{r^2}$$

$$\begin{aligned} W_{ab} &= \int_a^b -\frac{GmM}{r^2}\ dr \\ &= \frac{GmM}{r}\ \Big|_a^b \\ &= GmM\ [\frac{1}{r_b} - \frac{1}{r_a}\ ] \end{aligned}$$

then

$$\begin{aligned} \Delta U = U_b - U_a &= -W_{ab} \\ &= GmM\ [\frac{1}{r_a} - \frac{1}{r_b}\ ] \end{aligned}$$

We pick a convenient reference point, namely $b \to \infty$. The upper bound of ∞ makes this an *improper* integral. We say "oh we're not really going to infinity, just really far away, and then wonder, what would happen if we did go to infinity."

We also agree that this configuration is defined to have zero potential energy. Then

$$U_b - U_a = GmM\ [\frac{1}{r_a} - \frac{1}{r_b}\ ]$$

$$-U_a = \frac{GmM}{r_a}$$

$$U_a = -\frac{GmM}{r_a}$$

If we're thinking about the potential due to gravity, the force points toward the earth. The potential energy is larger the further away from earth you go, and is largest at infinity, where by definition it is equal to zero. Thus, all other potential energies are negative. To be on earth is, in that sense, to be trapped, since it requires an input of energy to lift you to some larger r and less negative U.

This is, in magnitude, equal to the work we would have to do pitting some force against gravity to lift a mass m to completely escape from the earth's gravity. The larger the starting radius the less work there is to do. Another way to state this is that the extra potential energy at some radius $r_1 > r_2$ is exactly equal to the work required to move between these points.

How much kinetic energy do we need to impart? Using the standard form for K, we obtain this expression, valid for the earth's surface:

$$\frac{1}{2}mv^2 = \frac{GM}{R}m$$

$$\frac{1}{2}v^2 = \frac{GM}{R}$$

$$v = \sqrt{\frac{2GM}{R}}$$

The velocity needed is independent of the object's mass (though of course it will take more energy to get a bigger object from zero up to a certain velocity). This velocity is called the escape velocity.

If we recall that the acceleration due to gravity at the surface of the earth is

$$g = \frac{GM}{R^2}$$

substituting

$$v = \sqrt{2gR}$$

We need to watch the units.

If we use English (Imperial) units, $a = 32$ feet per sec^2. $R = 3959$ miles, and in feet that is $3959 \cdot 5280$. Hence $v = \sqrt{2 \cdot 32 \cdot 3959 \cdot 5280} \approx 36500$ feet per second.

15 miles per hour is exactly equal to 22 feet per second.

$$15\ \frac{\text{miles}}{\text{hour}} \cdot 5280\ \frac{\text{feet}}{\text{mile}} \cdot \frac{1}{3600}\ \frac{\text{hour}}{\text{second}} = 22\ \frac{\text{feet}}{\text{second}}$$

The velocity works out to about $24,900$ miles per hour. That's a lot. A point on the equator of the earth rotates $24,900$ miles a day.

In MKS units, $a = 9.8$ meters per sec^2 and the earth's radius is 6371 km so

$$\begin{aligned} v &= \sqrt{2aR} \\ &= \sqrt{2 \cdot 0.0098 \cdot 6371} \\ &\approx 11.2\ \frac{\text{km}}{\text{sec}} \end{aligned}$$

The energy required to achieve a stable orbit around the earth has two parts: a radial part that gives the needed potential energy and a tangential part that gives the necessary orbital velocity. In the next chapter we will look at orbital velocities.

Kline

Morris Kline uses a different approach. We start with the acceleration due to gravity whose absolute value is

$$a = \frac{GM}{r^2}$$

You might think we should write this with a minus sign because a points down, while r increases going up. However, as we said before, this is not correct. The reason is that the force that must be exerted to escape from gravity points *up*.

Write this explicitly as the second derivative of position with respect to time

$$a = \frac{d^2r}{dt^2} = \frac{GM}{r^2}$$

Now, he says, we'd like to integrate both sides (with respect to t, naturally). That would give us $v(t)$ on the left. But what is written on the right is not explicitly a function of time. What to do?

Use the chain rule!

Manipulating the left-hand side:

$$\frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$$

Clever, eh? Hence

$$v\frac{dv}{dr} = \frac{GM}{r^2}$$

Moving dr to the right-hand side and integrating, we obtain

$$\frac{v^2}{2} = -\frac{GM}{r} + C$$

You may recognize a connection between kinetic and potential energy here, we are just missing the mass m.

We can evaluate the constant C by recognizing that we want $v = 0$ for some particular r that we're going to choose. It may be $r \to \infty$ as we had above, or it might be some other r.

What we could do is just to deal with the velocity and the radius at the two points, subtract, and then the constant goes away.

$$\frac{v_2^2}{2} - \frac{v_1^2}{2} = -\ [\ \frac{GM}{r_2} - \frac{GM}{r_1}\ ]$$

Let's call the radius where $v = 0$, r_1. There

$$0 = \frac{GM}{r_1} + C$$

$$C = -\frac{GM}{r_1}$$

$$\frac{v^2}{2} = \frac{GM}{r} - \frac{GM}{r_1}$$

Now ask what happens in the limit as $r_1 \to \infty$, and the r of interest to us is the radius of the earth R

$$\lim_{r_1 \to \infty} \frac{GM}{r_1} = 0$$

$$\frac{v^2}{2} = \frac{GM}{R}$$

and this is the same result as we had before.

Kline II

Kline also uses a different argument to achieve the same goal.

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}$$

Multiply both sides by

$$\frac{dr}{dt} \cdot \frac{d^2r}{dt^2} = -\frac{GM}{r^2} \cdot \frac{dr}{dt}$$

The derivative is just a function. Pretend we don't know what it is, call it u, and substitute the left-hand side *only*.

$$u = \frac{dr}{dt}$$

$$u \ \frac{du}{dt} = -\frac{GM}{r^2} \cdot \frac{dr}{dt}$$

Now, multiply by dt and integrate. Or if you prefer, reverse the chain rule and integrate both sides with respect to t.

$$\frac{1}{2}u^2 = \frac{GM}{r} + C$$

Recall that u is really just v

$$v = \frac{dr}{dt} = u$$

$$\frac{1}{2}v^2 = \frac{GM}{r} + C$$

A bit of sleight of hand.

Chapter 73

Uniform circular motion

Here we want to think about some problems related to the orbits of planets. The first topic is uniform circular motion. In this analysis we will idealize the orbits of the planets as circles.

The eccentricity of an ellipse is defined so that

$$ea = f$$

where $f = \sqrt{a^2 - b^2}$ is the distance from the center to either focus. If $e = 0$ then the curve is a circle rather than an ellipse.

According to wikipedia, the eccentricity of orbit of the objects known to Kepler (6 planets plus the moon) are

Mercury	0.024
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.054
Moon	0.055

These vary with time.

Looking at Earth, the distance from the sun is very nearly 150×10^6 km ($147,098,074$ at perihelion and $152,097,701$ km at aphelion, with a mean value of 149.6×10^6 km).

That is plus or minus 2.5×10^6 km or about 200 earth diameters.

Besides the eccentricity of the orbit, there is also the complication that the earth does not "revolve around the sun" but instead both earth and sun revolve around their common center of mass.

However the mass of the sun is about 1.989×10^{30} kg, while that of the earth is only about 5.972×10^{24} kg. The ratio is about 3.3×10^5.

So the center of mass is displaced from the center of the sun by about

$$\frac{5.972 \times 10^{24}}{1.989 \times 10^{30}} \, 150 \times 10^6 \approx 450$$

That's 450 km. Since the sun's radius is about $695,000$ km, the center of mass is very close to the center of the sun.

basic equations

We should be quite familiar with polar coordinates, we write

$$x = R\cos\theta$$

$$y = R\sin\theta$$

(I'm going to use R for the radius since it's a constant in this analysis. Also, we will also have $\mathbf{r}(t)$, the position vector).

We are talking about objects that change position with time, so we need to introduce time here somehow. We will say that

$$\theta = \omega t$$

Our clock ticks in seconds, and ω (in units of radians per second) tells how θ is calibrated with respect to the clock.

So now we can write the components of

$$\mathbf{r}(t) = \langle\ x(t), y(t)\ \rangle$$

$$= R\ \langle \cos\theta, \sin\theta \rangle$$

$$= R\ \langle \cos\omega t, \sin\omega t \rangle$$

We have the magnitude R times a unit vector.

And then we just differentiate to find the velocity and acceleration

$$\mathbf{r}(t) = R\ \langle \cos\omega t, \sin\omega t \rangle$$

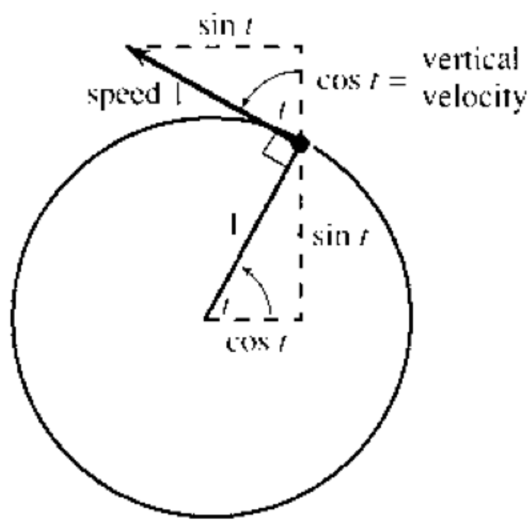
$$\mathbf{v}(t) = \omega R\ \langle -\sin\omega t, \cos\omega t \rangle$$

$$\mathbf{a}(t) = -\omega^2 R\ \langle \cos\omega t, \sin\omega t \rangle$$

$$= -\omega^2 R\ \mathbf{r}(t)$$

Notice that $\mathbf{v}(t)$ is orthogonal to $\mathbf{r}(t)$ (compute the dot product to see this)

$$\mathbf{v}(t) \cdot \mathbf{r}(t) = 0$$



and the acceleration is on exactly the same line as $\mathbf{r}$ but points inward, toward the sun or origin of the system.

$$\mathbf{a}(t) = -\omega^2 R\ \mathbf{r}(t)$$

Perhaps if we adjusted our clock to have the appropriate units of time, we wouldn't need ω, but it gives the magnitude of the velocity.

$$v = |\mathbf{v}|$$

$$= |\omega R\ \langle -\sin\omega t, \cos\omega t\rangle|$$

$$= \sqrt{\omega^2 R^2(\sin^2\omega t + \cos^2\omega t)}$$

$$= \sqrt{\omega^2 R^2(\sin^2\theta + \cos^2\theta)}$$

$$= \omega R$$

and

$$a = |\mathbf{a}| = \omega^2 R$$

We combine these to get an important identity

$$a = \omega^2 R = (\frac{v}{R})^2 R = \frac{v^2}{R}$$

$$aR = v^2$$

$$v = \sqrt{aR}$$

We can apply the formula to find the orbital velocity for an object going around the earth. The acceleration due to gravity is

$$a = \frac{GM}{R^2}$$

and then the orbital velocity is

$$v = \sqrt{aR} = \sqrt{\frac{GM}{R}}$$

So called low-earth orbits start at a height of about 160 km. We add that to the earth's radius (6371 km)

$$v = \sqrt{aR} = \sqrt{0.0098 \cdot 6531} = 8.00$$

in kilometers per second.

Compare this to the radial velocity at earth's equator which is about

$$40075/(24 \cdot 3600) = 4.63$$

in km per second. There's a reason why rockets are launched from Florida.

Of course, the radial velocity depends on latitude. One must multiply by the cosine of the latitude. The radial velocity at the north pole is zero.

In addition to the energy from the orbital velocity, we also need to add the potential energy to get to a particular orbit. We showed previously that

$$V = -GM/R$$

$$\Delta V = -GM(\frac{1}{R_2} - \frac{1}{R_1}) = GM(\frac{1}{R_1} - \frac{1}{R_2})$$

We can use these to look at some other orbits. Geostationary orbit 42,000 km Moon orbit avg = 385,000 km

The variation in satellite orbits is pretty extreme.

energy

We did this calculation in the previous chapter.

The force due to gravity is

$$\mathbf{F} = -\frac{GmM}{r^2}\hat{\mathbf{r}}$$

The potential is a function, which when we take

$$-\frac{d}{dr}U = \mathbf{F}$$

$$U = -\frac{GmM}{r} + C$$

Define

$$U_\infty = 0 \to C = 0$$

The total energy is

$$E = K + U = \frac{1}{2}mv^2 - \frac{GmM}{r}$$

if we want an object to just reach $r = \infty$ with zero energy, starting from $r = R$, then

$$0 = \frac{1}{2}mv^2 - \frac{GmM}{R}$$

$$v^2 = \frac{2GM}{R}$$

This is the escape velocity.

If the force is as given above, the magnitude of the acceleration is

$$a = \frac{GM}{r^2}$$

but for uniform circular motion we had

$$a = \frac{v^2}{r} = \frac{GM}{r^2}$$

$$v^2 = \frac{GM}{r}$$

or at the surface of the earth

$$v^2 = \frac{GM}{R}$$

This is the orbital velocity.

So, near the earth's surface, the escape velocity is approximately $\sqrt{2}$ times the orbital velocity.

Finally, we know that the velocity is distance divided by time, so for one full revolution it is

$$v = \frac{2\pi R}{T}$$

but

$$v^2 = \frac{GM}{R} = (\frac{2\pi R}{T})^2$$

Rearranging

$$T^2 = \frac{(2\pi)^2}{GM} R^3$$

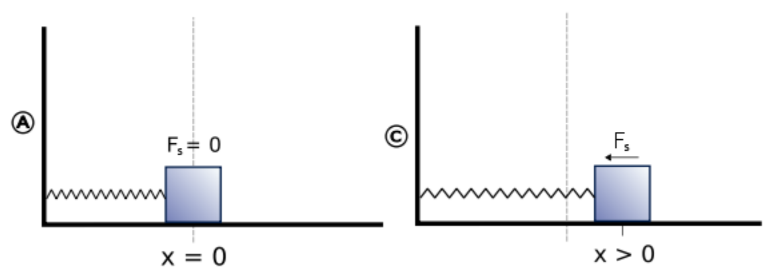
This is Kepler's Third Law: the square of the period is proportional to the cube of the orbit's radius.

The derivation is easy if we assume the orbits are circles, which is pretty close to being true. It is also true for an elliptical orbit. That is a bit harder calculation, which we will get to later.

Chapter 74

Harmonic oscillator

Here we look at the classical equations for oscillation, using as one of the sources the Physics lectures by Prof. Shankar. The simplest example is called the mass and spring system.



In the picture, the left panel is the equilibrium position, while the right view is the mass displaced to the right by some distance x. We pulled it there and have just released it. The force due to the spring is $F = -kx$. Not shown is the picture when the mass goes to the left of the equilibrium position and compresses the spring. To begin with, we will use a frictionless table. Without friction, the mass will oscillate back and forth forever.

We don't need vectors for this since we are working in one dimension. By experiment, we find that the force is proportional to the displacement and directed toward the equilibrium position.

$$F = -kx$$

Newton's Law says that $F = ma$ so

$$F = ma = m\ddot{x} = -kx$$

$$\ddot{x} + \frac{k}{m}x = 0$$

This equation is easy to solve. We need something whose second derivative is minus itself, within some constant. A good choice is

$$x(t) = A\cos\omega t$$

We choose $\cos t$ rather than $\sin t$ because we want $x(0) \neq 0$.

$$\ddot{x}(t) = -\omega^2 A\cos\omega t$$

so that

$$[\ -\omega^2 + \frac{k}{m}\ ]\ A\cos\omega t = 0$$

A corresponds to $x(0)$, which we want to be non-zero. This equation is zero (for all t) only when

$$\frac{k}{m} = \omega^2$$

$$\omega = \sqrt{k/m}$$

Observe that A can be anything we want. It corresponds to the maximum amplitude of the oscillation, determined by how far we pull out the mass to start things off.

In contrast, ω is determined by the characteristics of the spring, and is inversely proportional to the mass. ω is also related to the frequency of oscillation. If T is the time period for one complete oscillation, then $\omega T = 2\pi$, or if f is the frequency of oscillation, then $\omega = 2\pi f$.

phase

Finally, this solution assumes that $v_0 = 0$

$$\dot{x}(t) = -\omega A \sin \omega t$$

$$\dot{x}(0) = -\omega A \sin \omega (0) = 0$$

That is more restrictive than necessary. We can deal with this in various ways, one is to add a term containing the sine to $x(t)$

$$x(t) = A \cos \omega t + B \sin \omega t$$

$$\dot{x}(t) = -\omega A \sin \omega t + B \cos \omega t$$

$$\dot{x}(0) = B = v_0$$

Another way is to add a phase term to the angle, which doesn't change the derivatives

$$x(t) = A \cos \omega t + \phi$$

$$\dot{x}(t) = -\omega A \sin \omega t + \phi$$

$$\dot{x}(0) = v_0 = -\omega A \sin \phi$$

It is interesting to see why these amount to the same thing. Recall

$$\cos \omega t + \phi = \cos \omega t \cos \phi - \sin \omega t \sin \phi$$

But ϕ is a constant so if $-B = \sin \phi$ and $A = \cos \phi$

$$\cos \omega t + \phi = A \cos \omega t + B \sin \omega t$$

However, for now we will agree to set our clock to zero when the mass has just been released and has zero velocity, at the maximum amplitude of the oscillation.

Conservation of Energy

At this point, Prof. Shankar does this calculation

$$x(t) = A\cos\omega t$$

$$v = \dot{x} = -\omega A \sin\omega t$$

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$= \frac{1}{2}m\omega^2 A^2 \sin^2\omega t + \frac{1}{2}kA^2\cos^2\omega t$$

But $\omega^2 = k/m$ so

$$= \frac{1}{2}kA^2 \sin^2\omega t + \frac{1}{2}kA^2\cos^2\omega t$$

$$E = \frac{1}{2}kA^2$$

Not only is this independent of time, but we can write

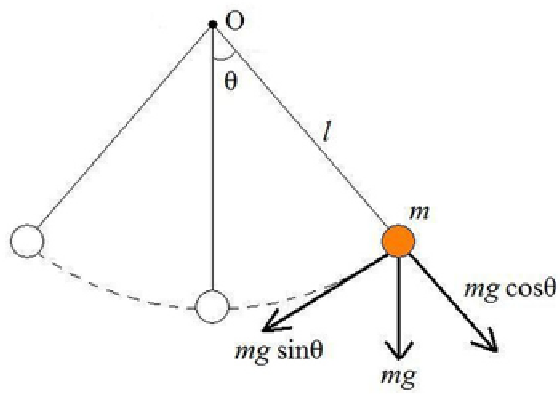
$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Given A and x, we can find v, and so on.

Chapter 75

Pendulum

A useful extension of the harmonic oscillator is to the problem of the pendulum.



The *torque* on the mass is computed from the component of the gravitational force perpendicular to the rod, which is $-mg\sin\theta$

The vector components are drawn a bit too long in the figure, the total force should be the hypotenuse of a right triangle with sides $F\sin\theta$ and $F\cos\theta$.

$$\tau = -mgL\sin\theta$$

We apply the small angle approximation $\sin x \approx x$ and obtain:

$$\tau = -mgL\theta$$

Torque is related to angular momentum the way force is related to momentum

$$\tau = I\ddot{\theta}$$

so

$$I\ddot{\theta} + mgL\theta = 0$$

This is exactly the equation we solved above. In particular

$$\omega = \sqrt{\frac{mgL}{I}} = \sqrt{\frac{mgL}{mL^2}} = \sqrt{\frac{g}{L}}$$

The period T times the angular frequency is 2π

$$T\omega = 2\pi$$

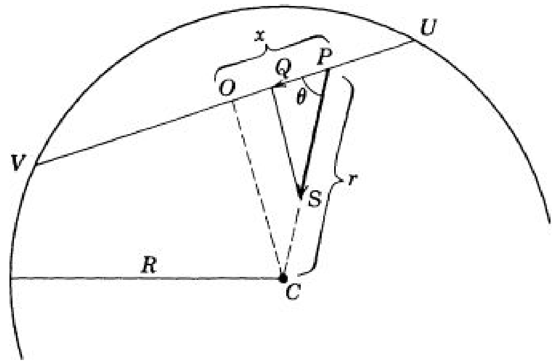
$$T = 2\pi\sqrt{\frac{L}{g}}$$

The period is independent of the mass.

Tunneling through the earth

Kline (Fig 10-18) has a fun problem.

Imagine that a tunnel has been bored straight through the earth between U and V in the figure and we consider the motion of a particle that is free to move through the tunnel — maybe a railroad car on some kind of track.



The car will move under the force of gravity. Kline does not derive it here, but the force inside the earth is variable depending on the position in the tunnel, partly due to the change in r, and partly due to the fact that the mass outside the current radius r does not generate any net force. Rather than the standard

$$F = \frac{GmM}{r^2}$$

the formula is

$$F = \frac{GmM}{R^3} r$$

Kline says, let $k = GM/R^3$ so

$$F = kmr$$

We choose the coordinate system as shown, with the origin at the midpoint of the tunnel. The effect of the force is to move the car along the tunnel, so we need to calculate the part of the force that points in that direction. It is simply $F \cos\theta$, where $\cos\theta = x/r$. Somewhat miraculously, with a nice cancelation, the force becomes

$$F = kmx$$

We take x positive to the right, while the force is in the negative x-direction so switch signs

$$F = -kmx$$

Apply Newton's second law, obtaining a differential equation

$$F = -kmx = m\ddot{x}$$

$$-kx = \ddot{x}$$

We know the solution to this one. The second derivative of the function is minus the function. A general solution is

$$x(t) = A \sin t + B \cos t$$

A factor of $\sqrt{k}$ is needed inside the trig functions:

$$x(t) = A \sin(\sqrt{k} \cdot t) + B \cos(\sqrt{k} \cdot t)$$

Check by differentiating. Now we use two initial conditions, one is that $v_0 = 0$.

$$v_0 = \dot{x}(0) = A\sqrt{k} \cos(\sqrt{k} \cdot 0) - B\sqrt{k} \sin(\sqrt{k} \cdot 0)$$

At $t = 0$, $\sin t = 0$, so we need the first term to be zero, with $A = 0$ and the equation reduces to

$$x(t) = B \cos(\sqrt{k} \cdot t)$$

The second is that $x(0) = x_0$ so $B = x_0$ and

$$x(t) = x_0 \cos(\sqrt{k} \cdot t)$$

We have a periodic or oscillatory motion. The period is

$$T\sqrt{k} = 2\pi$$

$$T = \frac{2\pi}{\sqrt{k}}$$

We defined

$$k = GM/R^3$$

but recall that at the surface of the earth

$$32m = \frac{mMG}{R^2}$$

$$32 = \frac{MG}{R^2}$$

so

$$k = \frac{GM}{R^3} = \frac{32}{R}$$

and

$$T = 2\pi\sqrt{\frac{R}{32}}$$

With the radius measured in feet

$$R = 3959 \cdot 5280 = 20903520$$

$$\frac{1}{\sqrt{k}} = 808.2$$

$T = 5077$ seconds or about 85 minutes.

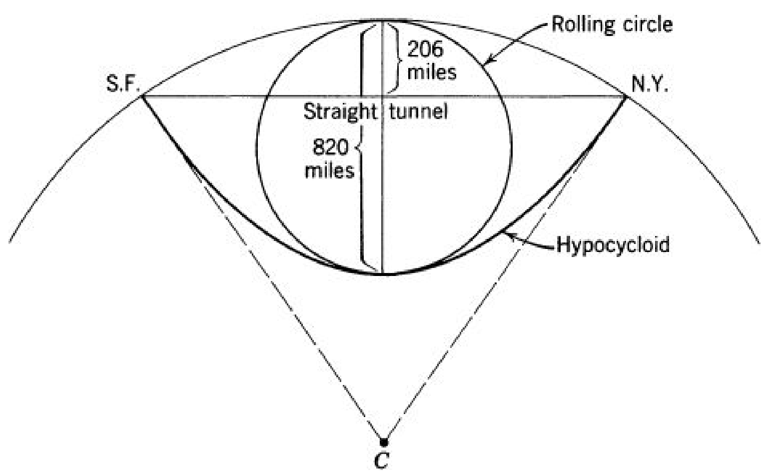
It is very interesting that this result does not depend on the location or the length of the tunnel.

The fastest time occurs for a hypocycloidal path.

Kline:

> The following data give some idea of what can be gained by using a hypocycloidal path. Suppose that a straight tunnel

is dug between New York and San Francisco. Along the surface of the earth the two cities are about 2,575 miles apart, but the tunnel would be 2,530 miles long. At the midpoint the tunnel would be 206 miles below the surface; the maximum velocity acquired by the object, which would be at the midpoint, would be 1.57 mi/ sec; and a one-way journey, according to (reference), would take about 42 minutes. The hypocycloidal path would be 2940 miles long. At the midpoint the tunnel would be 820 miles below the surface; the velocity of the object at that point would be 3 mi/ sec; and the one-way journey would take 26 minutes.



We can calculate the velocity for the straight line path.

$$\dot{x}(t) = -x_0\sqrt{k}\sin(\sqrt{k} \cdot t)$$

which has a maximum when the sine term is equal to 1 , at one-quarter and three-quarters of the period.

$$|v_{\max}| = x_0\sqrt{k} = 1265/808 = 1.57 \text{ mi/sec}$$

which matches the text.

We can check the calculation another way, by using the conservation of energy. We have that the potential energy lost is mgh (it shouldn't matter that g changes as we go inside the earth). This is equal to the kinetic energy gained:

$$\frac{1}{2}mv^2 = mgh$$

$$v = \sqrt{2 \cdot 32 \cdot 206 \cdot 5280} = 8343 \text{ ft/sec} = 1.58 \text{ mi/sec}$$

Another angle is to ask how things would differ on the moon? Given:

mass is 1/81; radius is 3/11; gravity is 1/6.

k goes like M/R^3 or $1/81 \times (11/3)^3 = 11^3/3^7$

$\sqrt{k}$ goes like

$$\frac{11\sqrt{11}}{3^3\sqrt{3}} = \frac{11\sqrt{11}}{27\sqrt{3}} = 0.78$$

Since T is proportional to the inverse it's about 1.28 times longer. Not as much different as you might expect.

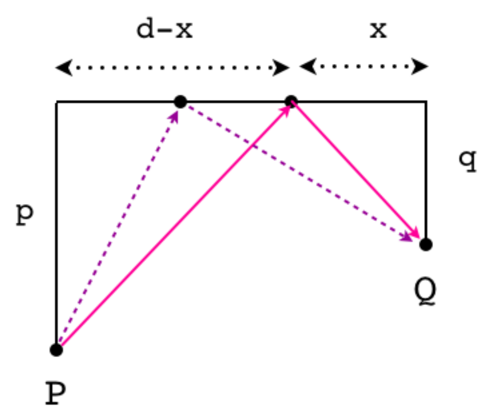
Part XX

Time

Chapter 76

Shortest path

As shown in the figure below, we have two points P and Q, which might be, say, the origin and destination of a journey that must also go to the river (horizontal line at the top). The point where we reach the river can be adjusted, and we seek the path that has the shortest overall distance.



There is a hard way to do this problem, and an easy way. The hard way involves a tiny bit of calculus (to do the minimization). I'll show that one first.

Depending on where we put the point on the river, we have a horizontal distance x to that point from Q, and a horizontal distance $d-x$ to the same point from P.

The path consists of two parts, from P to the river, and from the river back to Q. That distance is

$$\sqrt{p^2 + (d-x)^2} + \sqrt{q^2 + x^2}$$

We take the derivative with respect to x. It's a little tricky.

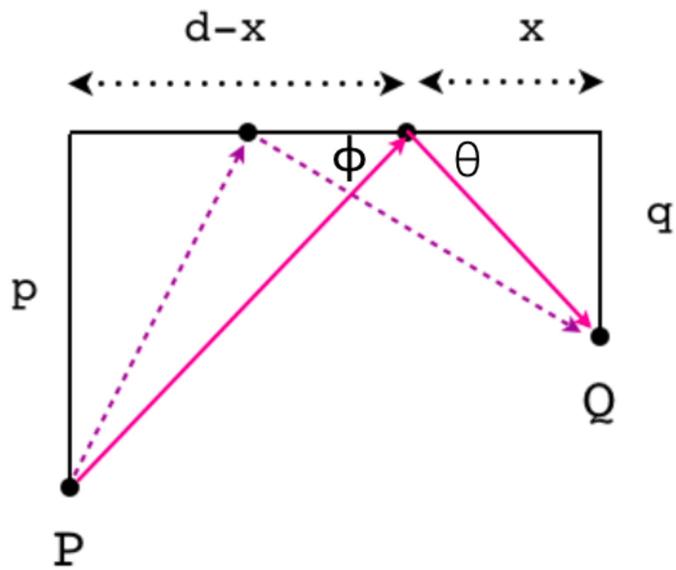
For the first term, we get the denominator from applying the power rule to the square root (plus a factor of 1/2), then we apply the chain rule to $(d-x)^2$ and get $2(d-x)$ in the numerator, and finally a factor of -1 from $-x$. The twos cancel. The second term is similar.

Set the result equal to 0:

$$-\frac{d-x}{\sqrt{p^2 + (d-x)^2}} + \frac{x}{\sqrt{q^2 + x^2}} = 0$$

$$\frac{x}{\sqrt{q^2 + x^2}} = \frac{(d-x)}{\sqrt{p^2 + (d-x)^2}}$$

It's just algebra from this point. But notice that we don't need it. Both of these ratios correspond to the cosine of an angle.



$$\frac{x}{\sqrt{q^2 + x^2}} = \cos\theta$$

$$\frac{(d-x)}{\sqrt{p^2 + (d-x)^2}} = \cos\phi$$

Because of this, the angles are equal as well.

Here's the algebra:

$$\frac{x^2}{q^2 + x^2} = \frac{(d-x)^2}{p^2 + (d-x)^2}$$

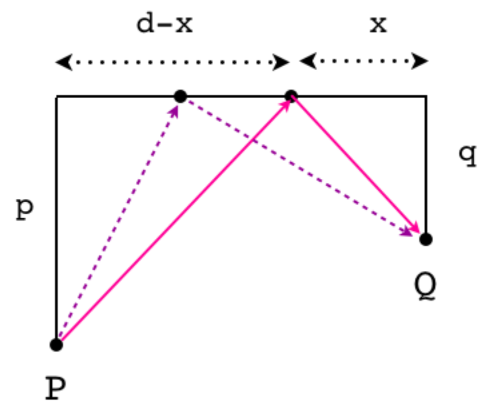
invert and simplify

$$\frac{q^2}{x^2} + 1 = \frac{p^2}{(d-x)^2} + 1$$

cancel +1 and take square roots

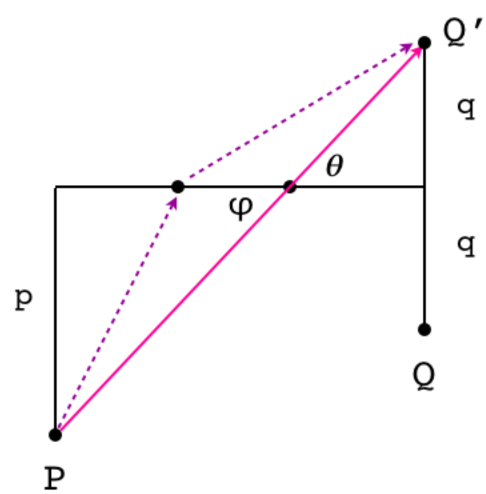
$$\frac{q}{x} = \frac{p}{(d-x)}$$

The result is that the two triangles should be similar (since their sides are in proportion and they are both right triangles).



Another way to say it is that the angle where we come from P to the river, and the angle by which we leave the river to Q should be equal.

Now for the easy way.



We draw a vertical up the same distance q to point Q'—the mirror image of Q. Minimizing the distance to Q' is the same problem because it's exactly the same distance.

What's the shortest distance between two points? A straight line from P to Q'. With a straight line then the two angles ϕ and θ are equal and the similarity of the triangles follows immmediately.

This is a famous result in physics. It's true for light rays, that when you shine a light from P at a mirror, the light rays arriving at Q come by the shortest path. The law about the angles being equal is called the "law of reflection" and it was known to Euclid.

Pool players who know nothing about about Euclid know this result, and they use it all the time in making bank shots.

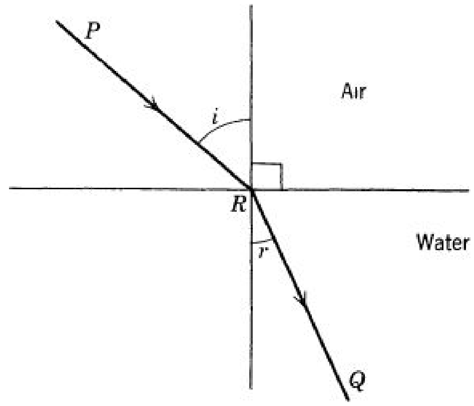
But this is only the beginning. The general principle is called "least action."

Snell's Law

A variation on the previous problem yields Snell's Law for refraction.

Consider the problem of a light ray passing from P to Q, where P is in air, and Q is in a medium like water, with a higher refractive index and lower speed of light.

Figure 8-9



The physical principle is that light takes the path of shortest time. We need to find the R that makes this true.

Suppose the total horizontal distance between P and Q is d. Let x be the horizontal distance from P to R, then $d - x$ is the horizontal distance from R to Q. The vertical distances are fixed, let's call them p (for PR) and q (for QR).

The time taken is the distance divided by the speed. Let the speed of light in air be u and the speed of light in water be v. There are two segments of the trip:

$$t_1 = \frac{\sqrt{x^2 + p^2}}{u}$$

$$t_2 = \frac{\sqrt{(d - x)^2 + q^2}}{v}$$

$$t = t_1 + t_2 = \frac{\sqrt{x^2 + p^2}}{u} + \frac{\sqrt{(d - x)^2 + q^2}}{v}$$

We have time t as a function of x and we take the first derivative and set it equal to zero:

$$\frac{x}{u\sqrt{x^2+p^2}} + \frac{-(d-x)}{v\sqrt{(d-x)^2+q^2}} = 0$$

$$\frac{x}{u\sqrt{x^2+p^2}} = \frac{(d-x)}{v\sqrt{(d-x)^2+q^2}}$$

Rather than fool with the square roots, notice that

$$\sqrt{x^2+p^2} = PR$$

and

$$\sqrt{(d-x)^2+q^2} = RQ$$

so

$$\frac{x}{u\sqrt{x^2+p^2}} = \frac{(d-x)}{v\sqrt{(d-x)^2+q^2}}$$

becomes

$$\frac{x}{u\ PR} = \frac{(d-x)}{v\ RQ}$$

Furthermore $x/PR = \sin\theta_i$ and $(d-x)/RQ = \sin\theta_r$ so

$$\frac{\sin\theta_i}{u} = \frac{\sin\theta_r}{v}$$

The sines of the angles for each side of the barrier are in the same ratio as the velocities in the respective medium.

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{u}{v}$$

$$\sin\theta_r = \frac{v}{u}\ \sin\theta_i$$

Since the speed of light in air is higher than in water $u > v$, $v/u < 1$ which means that $\sin\theta_r < \sin\theta_i$ and thus $\theta_r < \theta_i$.

We can also use the refractive index n which is proportional to the reciprocal of the speed.

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_r}{n_i}$$

Chapter 77

Across the river

The problem we want to solve is as follows. There is a (very smooth) river that flows with a speed u, while you and your twin brother each swim with a speed v. You have used your trigonometry skills to mark out a position which is up-river from the start point a distance equal to the width of the river, d.

Your brother wants to race (across and back versus up and down), and he's willing to let you pick which way you want to swim. Which should you choose?

To solve this problem, first recall that the time to travel any distance d at a constant speed v is

$$t = d/v$$

Also, the time to swim up-river (and back) depends on your relative speed with respect to the land, going up-river it is $v - u$ and coming down-river it is $v + u$. The total time for the first trip is

$$t_1 = \frac{d}{v-u} + \frac{d}{v+u} = d(\frac{v+u+v-u}{v^2-u^2}) = \frac{2dv}{v^2-u^2}$$

For the second trip, across and back, you need to swim upstream at some angle θ in order that the current will sweep you back down to a point on the line that goes directly across the river. The vectors make a little right triangle with hypotenuse v and far side u. The effective speed is $\sqrt{v^2 - u^2}$. So, just as for the up and back trip, there is a requirement that $v > u$. You should probably calculate θ before leaving shore, $\theta = \sin^{-1}(u/v)$.

Having calculated the speed, the time is just

$$t_2 = \frac{2d}{\sqrt{v^2-u^2}}$$

The ratio of the two times is

$$\begin{aligned}\frac{t_1}{t_2} &= \frac{2dv}{v^2-u^2}\frac{\sqrt{v^2-u^2}}{2d} \\ &= \frac{v}{\sqrt{v^2-u^2}}\end{aligned}$$

Let's check this result quickly. Suppose $v = 5, u = 3$ (in m/s^2). Suppose we go up-river 7200 m at 2 m/s (taking 1 hr) and then back down-river 7200 m at 8 m/s for 0.25 hr or 5/4 hr total.

Pythagoras tells us that we go across the river at 4 m/s. If the one-way distance is 7200 m, then we do each half in $1800s = 0.5 + 0.5 = 1.0$ hr. Going across is faster.

Which is a bit surprising. The average speed up- and down-river is 5, yet we are slower than going across with a constant speed of 4. That's because we already spent a whole hour at 2 m/s going up-river, we still have to come back!

There is a very famous experiment which depends on this difference.

http://en.wikipedia.org/wiki/Michelson-Morley_experiment

Chapter 78

Relativity and time

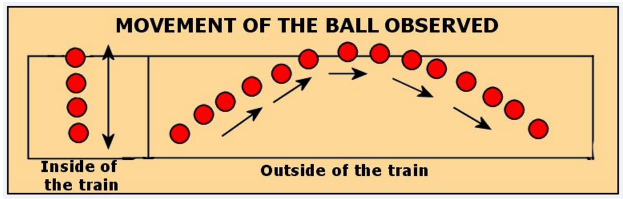
relative motion

Imagine that you are seated in a train looking out the window. Another train is just next to you, sliding past on the adjacent track, moving to your right.

All you can see out the window is the other train and what is happening inside. This is an idealized world (what Einstein called a Gedanken or thought experiment) so trains do not make noise or lurch from side to side as they move.

Galileo's classical **relativity** says that there is no physical test you can do from inside your train to decide which train is moving. It might be that the other train is moving or it could be that your train is moving. You can't tell.

vertical and horizontal independence



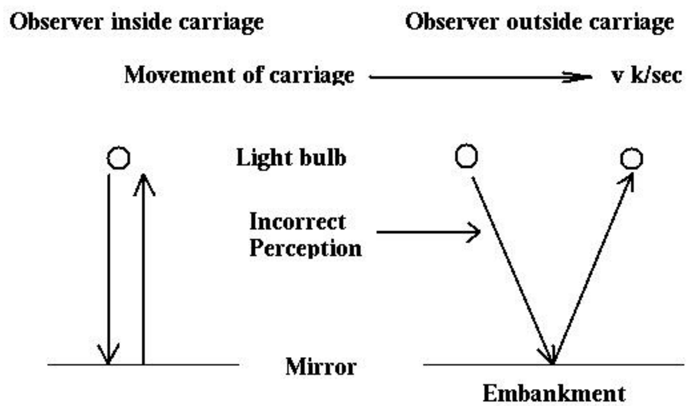
Suppose there is a guy sitting in the other train and he throws a ball straight up in the air and then it comes down. His view of things is shown on the left, and your view of things is on the right.

You see a horizontal movement that he does not see, because of the relative velocity of the trains.

However, movement in the vertical direction is independent of the horizontal direction and vice-versa, so the time each of you measures for the flight of the ball is the same, governed by its initial velocity upward and the downward force of gravity. So is the vertical velocity of the ball at each point, and your calculations for the acceleration due to gravity will be the same.

man with a flashlight

Now suppose instead that the man in the other train is holding a flashlight overhead, pointed at a mirror on the floor. He turns it on very briefly, emitting a light ray that travels (from his perspective) vertically.



He observes the time when the light returns back from the mirror. To make things simpler for our calculations, let us just consider the path from the flashlight to the floor. The light traveled down a distance e in time s.

He can calculate the *speed of light* as distance divided by time

$$c = \frac{e}{s}$$

your view

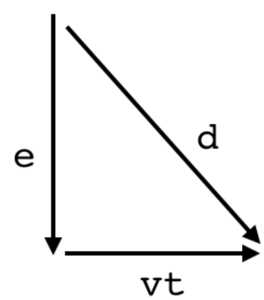
Suppose also that the light starts on its journey at precisely the same time as the man, the flashlight and that spot on the floor are all directly opposite you, in your train.

Considered from your perspective, the light ray will cover a longer distance, because of the relative velocity between you and the other train.

What you see is that the man turned on the flashlight while he was exactly opposite, but by the time the light ray hits the floor, the floor has moved to the right. So the light travels to the floor by an angled path with distance d, hitting the point directly below the other guy when that point is some distance off to your right (it is similar to the example with the thrown ball).

Of course, you must be going very fast for this to be obvious, because the speed of light is so large ($3 \cdot 10^8$ m/sec or about $186,300$ mph).

Suppose that by your watch, the time taken for this to happen is t. e is the vertical component of d, and the horizontal component is the relative speed of the two trains v multiplied by your own time t. Speed times time equals distance.



You can also calculate the speed of light. You saw it move a distance d in a time t so

$$c = \frac{d}{t}$$

Einstein's principle of relativity

One of Einstein's fundamental principles of relativity is that *the speed of light is the same for all observers.*

We must both obtain the *same* velocity for light:

$$c = \frac{e}{s} = \frac{d}{t}$$

Since the distances are clearly not equal ($d \neq e$), neither can the times be equal: $s \neq t$.

That is a basic paradox of relativity. There is no longer such a thing as absolute time. And if two observers who are in relative motion cannot agree on the time, they will not be able to agree on whether two events are simultaneous or not.

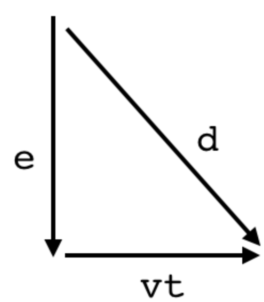
calculation

A little algebra remains.

We can rearrange what's above above slightly

$$e = cs, \quad d = ct$$

Since d is the hypotenuse of a right triangle with sides e and vt



Pythagoras tells us that

$$d^2 = e^2 + (vt)^2$$

Substitute $d = ct$ and $e = cs$:

$$(ct)^2 = (cs)^2 + (vt)^2$$

Isolate the terms containing t:

$$(ct)^2 - (vt)^2 = (cs)^2$$

Factor out t^2

$$[\ c^2 - v^2\ ]\ t^2 = c^2 s^2$$

Divide by c^2

$$[\ \frac{c^2 - v^2}{c^2}\ ]\ t^2 = s^2$$

$$[\ 1 - \frac{v^2}{c^2}\ ]\ t^2 = s^2$$

$$\sqrt{1 - \frac{v^2}{c^2}} \cdot t = s$$

Define

$$\gamma = \sqrt{1 - \frac{v^2}{c^2}}$$

so

$$\gamma\ t = s$$

The factor γ shows up often in relativity.

Look again

$$\gamma = \sqrt{1 - \frac{v^2}{c^2}}$$

If $v = 0$ then $\gamma = 1$, so nothing really changes if the other train is not moving.

As v gets larger, v^2/c^2 is always positive so we have the square root of something smaller than 1, and thus $\gamma < 1$.

Since

$$\gamma\ t = s$$

and $\gamma < 1$, it must be that $s < t$.

The moving observer's value for time, s, is smaller than what you measure, t.

If v gets larger, γ gets smaller.

If v were to become as large as c, then $\gamma \to 0$, and then, time would stand still.

Part XXI

Electric fields

Chapter 79

Field of an infinite wire

These are two classic problems: find the electric field $\mathbf{E}$ for an infinite wire charged with density λ or an infinite sheet charged with density σ (positive charge). Let's start with the wire.

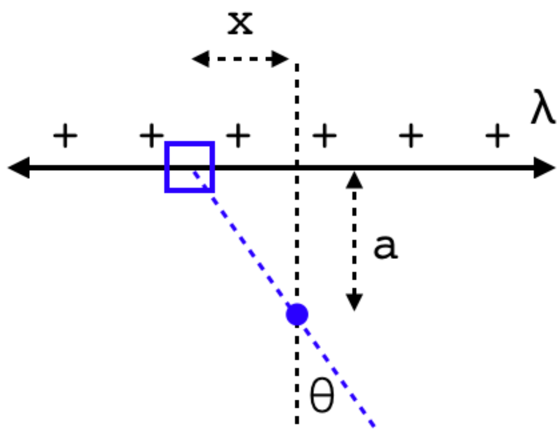
The force between two charges is given by Coulomb's Law:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{\mathbf{e}}_r$$

while the field is the force on a unit test charge due to a given charge.

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{e}}_r$$

infinite wire



Consider a point at a distance a from the wire. Call the nearest position on the wire $x = 0$. The force from a small element at position x points radially away from the element.

Break that force into two components, the component horizontal in the figure will be canceled by a component in the other direction from its counterpart at $-x$. The vertical component is $F \cos\theta$.

Coulomb says that

$$E = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}$$

If the small element has width dx and the density is λ, the total charge in that element is λdx. The distance from the element to the position we are evaluating is $\sqrt{x^2 + a^2}$. So the small part of the electric field is

$$dE = \frac{\lambda}{4\pi\epsilon_0}\frac{dx}{(a^2 + x^2)}$$

There is a further factor of $\cos\theta$ to account for the fact that only the vertical part of the force is not canceled. In terms of a and x:

$$\cos\theta = \frac{a}{\sqrt{a^2 + x^2}}$$

$$dE = \frac{\lambda a}{4\pi\epsilon_0} \frac{dx}{(a^2+x^2)^{3/2}}$$

To integrate this, we need a trig substitution. Because of the $a^2 + x^2$, choose the tangent. (Also, Shankar says, we will want to integrate to ∞ as a limit, so neither sine nor cosine will do).

$$x/a = \tan\theta$$

$$x = a\tan\theta$$

$$dx = a\sec^2\theta \; d\theta$$

We already have

$$a/\sqrt{a^2+x^2} = \cos\theta$$

$$\frac{1}{(a^2+x^2)^{3/2}} = \frac{\cos^3\theta}{a^3}$$

Thus

$$dE = \frac{\lambda a}{4\pi\epsilon_0} a\sec^2\theta \; \frac{\cos^3\theta}{a^3} \; d\theta$$

$$dE = \frac{\lambda}{4\pi\epsilon_0 a}\cos\theta \; d\theta$$

$$E = \int \frac{\lambda}{4\pi\epsilon_0 a}\cos\theta \; d\theta$$

$$= \frac{\lambda}{4\pi\epsilon_0 a} \; \sin\theta$$

The limits require care. The original limits on x were (for an infinite wire) $x = -\infty \to \infty$. Now $x = \tan\theta$, so $\theta = -\pi/2 \to \pi/2$. Evaluating $\sin\theta$ between those limits we get $1 - -1 = 2$.

$$E = \frac{\lambda}{2\pi\epsilon_0 a}$$

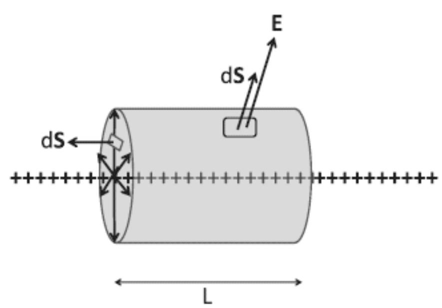
The field falls off like $1/a$ rather than $1/a^2$. We visualize the electric field lines as being perpendicular to the wire. If we take a cross-section in the perpendicular plane, the field lines spread like 1 over the distance. It is a one-dimensional spreading.

We can get the same result using Gauss's Law. We haven't introduced it properly yet, but just state it here:

$$\int_S \mathbf{E} \cdot \mathbf{r} = \frac{Q}{\epsilon_0}$$

The flux of the electric field across a closed surface is equal to the charge enclosed Q times the constant $1/\epsilon_0$, which is about 9×10^9.

Surround a section of wire of length L with a cylindrical Gaussian surface.



The amount of charge enclosed is λL. Gauss says

$$\int_S \mathbf{E} \cdot \mathbf{r} = \frac{Q}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$$

By symmetry, the field is radial, and so the dot product with the surface elements on the ends of the cylinder is zero. The dot product with the surface elements on the curved part is just $E\ dS$ so the whole integral is

$$\int_S \mathbf{E} \cdot \mathbf{dS} = E \int dS = E\ 2\pi a L$$

$$E\ 2\pi a L = \frac{\lambda L}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi a \epsilon_0}$$

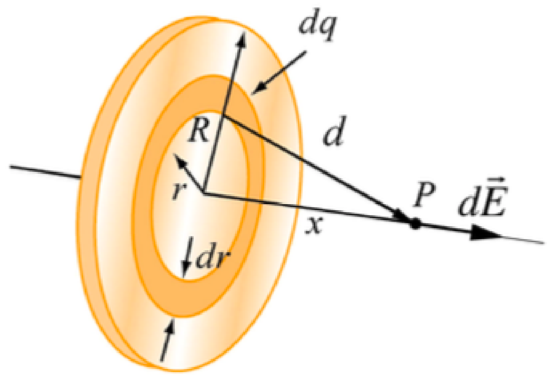
and that matches what we had before.

Chapter 80

Field of an infinite sheet

Here we evaluate the field at a point a distance a away from an infinite sheet. The charge density is σ.

We start by thinking about a ring with radius r centered on the position that is closest to the point where we're conducting the evaluation. The normal vector passes through the center of the ring and also through our point. The distance between them is x



The distance between points on the ring and the point where the field is being evaluated is d.

The force is directed from each little segment on the ring toward the point, but the part of the force that is not perpendicular is canceled in each case, by an opposite component coming from the piece of the ring on the other side. So again, we will have a factor of $\cos\theta$.

The relevant distance squared for Coulomb's Law is

$$d^2 = r^2 + x^2$$

the factor of $\cos\theta$ is

$$\cos\theta = \frac{x}{d} = \frac{x}{\sqrt{r^2 + x^2}}$$

So Coulomb says:

$$dE = \frac{dq}{4\pi\epsilon_0}\frac{x}{(r^2 + x^2)^{3/2}}$$

The ring has width dr and length $2\pi r$, so it has area $2\pi r\ dr$ and charge $2\pi r\sigma\ dr$. We have

$$dE = \frac{2\pi r\sigma}{4\pi\epsilon_0}\frac{x}{(r^2 + x^2)^{3/2}}\ dr$$

$$dE = \frac{x\sigma}{2\epsilon_0}\frac{r}{(r^2 + x^2)^{3/2}}\ dr$$

The denominator is similar to what we had in the first problem, but now we have $r\ dr$ up top.

The integral is

$$E = \int dE = \int \frac{x\sigma}{2\epsilon_0}\frac{r}{(r^2 + x^2)^{3/2}}\ dr$$

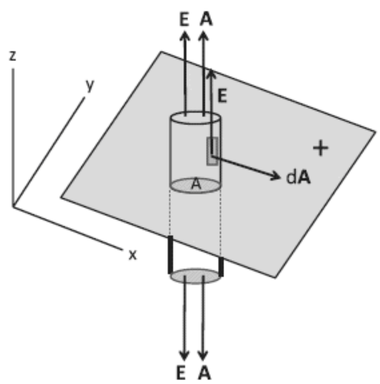
$$= \frac{x\sigma}{2\epsilon_0}(-\frac{1}{\sqrt{r^2 + x^2}})$$

We evaluate between $r = 0 \to \infty$. The term in parentheses becomes $- - 1/\sqrt{x^2}$, so the answer is finally

$$E = \frac{\sigma}{2\epsilon_0}$$

Remarkably, the field is independent of x. If we take a cross-section in the perpendicular plane, the field lines do not spread out.

As before, we can also use Gauss's Law.



The Gaussian surface is a cylinder, as shown. The enclosed charge is σA. Gauss says

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \frac{Q}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

Only the end caps of the cylinder intercept any field lines (so the dot product on the curved parts is zero), and for these the field and the normal vector point in the same direction so

$$\int_S \mathbf{E} \cdot \mathbf{dS} = E \int dS$$

We must be careful here. There are *two* end caps

$$\int dS = 2A$$

$$= 2A\ E = \frac{\sigma A}{\epsilon_0}$$
$$E = \frac{\sigma}{2\epsilon_0}$$

sphere

The problem of the sphere comes up in gravitation as well as electrostatics. We will sidestep the determination of how a sphere or a solid ball behaves. (See the chapter on the shell theorem, due to Newton). We will instead use Gauss's Law, which says the total electric flux through a surface enclosing a charge Q is

$$\Phi_E = \frac{Q}{\epsilon_0}$$

The flux is defined to be the dot product of the electric field with the surface element, integrated over the entire surface.

$$\Phi_E = \iint_S \mathbf{E} \cdot d\mathbf{A}$$

If we have a sphere, it is radially symmetric. Therefore, we expect that the electric field will be perpendicular to the surface of the sphere, and to any (imaginary) sphere that we might draw around the sphere at a radius r from the center. This means that the dot product is just EdA, so

$$\Phi_E = \iint_S EdA = E \iint_S dA = 4\pi r^2 E$$
$$\frac{Q}{\epsilon_0} = 4\pi r^2 E$$
$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\ \hat{\mathbf{r}}$$

$$\mathbf{F} = q\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\ \hat{\mathbf{r}}$$

Chapter 81

Field of a dipole

Consider a dipole with charge $+q$ on one end, at $(a, 0)$, and charge $-q$ on the other end $(-a, 0)$. We want to calculate the field due to the dipole at various points in the xy-plane. That gives us everything, since every other plane formed by rotation of this one around the x-axis has the same field.

This material comes from volume II of Shankar's Physics. The second part, using the potential, has the calculus. The first part, using the field, is there for contrast and because he uses various tricks to approximate and simplify the answer.

The relevant equation is a variation on Coulomb's Law:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_o} \cdot \frac{1}{r^2} \cdot \hat{\mathbf{e_r}}$$

starting with the field, E

For a general point (x, y) the vector from the positive pole to the point is

$$(x - a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

the squared distance is

$$(x-a)^2 + y^2$$

the unit vector is

$$\frac{(x-a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x-a)^2 + y^2)^{1/2}}$$

so

$$\mathbf{E}_+ = \frac{q}{4\pi\epsilon_o} \cdot \frac{(x-a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x-a)^2 + y^2)^{3/2}}$$

For the other pole, we get

$$\mathbf{E}_- = -\frac{q}{4\pi\epsilon_o} \cdot \frac{(x+a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x+a)^2 + y^2)^{3/2}}$$

The total field is the sum of these two.

$$\mathbf{E} = \frac{q}{4\pi\epsilon_o} \cdot \ [\frac{(x-a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x-a)^2 + y^2)^{3/2}} - \frac{(x+a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x+a)^2 + y^2)^{3/2}} \ ]$$

special solutions

On the $x-$axis, $y = 0$ so

$$\mathbf{E} = \frac{q}{4\pi\epsilon_o} \cdot \ [\frac{(x-a)\hat{\mathbf{i}}}{(x-a)^3} - \frac{(x+a)\hat{\mathbf{i}}}{(x+a)^3} \ ]$$

$$= \frac{q}{4\pi\epsilon_o} \cdot \ [\frac{(x+a)^2 - (x-a)^2}{(x-a)^2 \ (x+a)^2} \ ] \ \hat{\mathbf{i}}$$

$$= \frac{q}{4\pi\epsilon_o} \cdot \ [\frac{4ax}{(x^2 - a^2)^2} \ ] \ \hat{\mathbf{i}}$$

Define the **dipole moment** as

$$\mathbf{p} = 2aq \ \hat{\mathbf{i}}$$

so

$$\mathbf{E} = \frac{\mathbf{p}}{4\pi\epsilon_o} \cdot \frac{2x}{(x^2 - a^2)^2}$$

For $x >> a$ (and since $y = 0$, $r = x$):

$$\mathbf{E} = \frac{1}{2\pi\epsilon_o} \cdot \frac{\mathbf{p}}{r^3}$$

On the $y-$axis, $x = 0$ so

$$\mathbf{E} = \frac{q}{4\pi\epsilon_o} \cdot \ [\frac{(-a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((-a)^2 + y^2)^{3/2}} - \frac{a\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((a)^2 + y^2)^{3/2}} \ ]$$

$$= \frac{q}{4\pi\epsilon_o} \cdot \ [\frac{2(-a\hat{\mathbf{i}})}{(a^2 + y^2)^{3/2}} \ ]$$

$$= -\frac{1}{4\pi\epsilon_o} \cdot \ \frac{\mathbf{p}}{(a^2 + y^2)^{3/2}}$$

For $y >> a$

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_o} \cdot \ \frac{\mathbf{p}}{r^3}$$

Compare with the x-axis:

$$\mathbf{E} = \frac{1}{2\pi\epsilon_o} \cdot \frac{\mathbf{p}}{r^3}$$

In both cases, the field falls like $1/r^3$, but numerically it is twice as large along the x-axis for a given r. Also, along the x-axis the field points to the right, while along the y-axis it points to the left.

general solution

Shankar does some slick thinking to simplify the general case. We have a complicated beast:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_o} \cdot \ [\ \frac{(x-a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x-a)^2 + y^2)^{3/2}} - \frac{(x+a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x+a)^2 + y^2)^{3/2}}\ ]$$

Shankar says "$\mathbf{E}$ vanishes when $a = 0$. The net field is non-zero only becauase $a \neq 0$, and the non-zero part will start out as the first power of a in a Taylor series expansion. To keep the dimension of the field the same, the extra a must really be a/r", and we saw that in the simple cases above because we had an extra $\mathbf{p}/r$ and $\mathbf{p}$ is proportional to a.

Then: (the equation) has two parts, each with a numerator divided by the denominator, or the numerator times the inverse denominator. We can get the single power of a from either term, and the a_0 term from the other. If we get a^1 from the numerator we may set $a = 0$ in the denominator and vice-versa.

Consider

$$\mathbf{E}_+ = \frac{q}{4\pi\epsilon_o} \cdot \frac{(x-a)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x-a)^2 + y^2)^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_o} \cdot \frac{\mathbf{r} - a\hat{\mathbf{i}}}{((x-a)^2 + y^2)^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_o} \cdot \ [\ \frac{\mathbf{r}}{((x-a)^2 + y^2)^{3/2}} - \frac{a\hat{\mathbf{i}}}{((x-a)^2 + y^2)^{3/2}}\ ]$$

Expand

$$= \frac{q}{4\pi\epsilon_o} \cdot \ [\ \frac{\mathbf{r}}{((x^2 - 2ax + a^2 + y^2)^{3/2}} - \frac{a\hat{\mathbf{i}}}{((x^2 - 2ax + a^2 + y^2)^{3/2}}\ ]$$

$$\approx \frac{q}{4\pi\epsilon_o} \cdot \ [\ \frac{\mathbf{r}}{((x^2 - 2ax + y^2)^{3/2}} - \frac{a\hat{\mathbf{i}}}{((x^2 + y^2)^{3/2}}\ ]$$

In the first term, we keep $\mathbf{r}$ on top (which is a^0), so we keep a^1 but not a^2 on the bottom. In the second term, we keep a^1 on top, so we don't need to keep any terms involving a on the bottom. Now substitute $r^2 = x^2 + y^2$:

$$\mathbf{E}_+ \approx \frac{q}{4\pi\epsilon_o} \cdot \ [\ \frac{\mathbf{r}}{(r^2 - 2ax)^{3/2}} - \frac{a\hat{\mathbf{i}}}{r^3}\ ]$$

In dealing with $\mathbf{E}_-$, we change the sign on both q and a

$$\mathbf{E}_- \approx -\frac{q}{4\pi\epsilon_o} \cdot \ [\ \frac{\mathbf{r}}{(r^2 + 2ax)^{3/2}} + \frac{a\hat{\mathbf{i}}}{r^3}\ ]$$

so the total field due to the dipole is

$$\mathbf{E} \approx \frac{q}{4\pi\epsilon_o} \cdot \ [\ \frac{\mathbf{r}}{(r^2 - 2ax)^{3/2}} - \frac{\mathbf{r}}{(r^2 + 2ax)^{3/2}} - 2\frac{a\hat{\mathbf{i}}}{r^3}\ ]$$

Then, bringing out $1/r^3$

$$\approx \frac{q}{4\pi\epsilon_o r^3} \cdot \ [\ \frac{\mathbf{r}}{(1 - 2ax/r^2)^{3/2}} - \frac{\mathbf{r}}{(1 + 2ax/r^2)^{3/2}} - 2a\hat{\mathbf{i}}\ ]$$

A last simplification comes from $(1+z)^n = 1 + nz + \ldots$

$$(1 - 2ax/r^2)^{-3/2} = 1 - \frac{-3}{2}\ \frac{2ax}{r^2}$$

$$= 1 + \frac{3ax}{r^2}$$

Hence

$$\mathbf{E} \approx \frac{q}{4\pi\epsilon_o r^3} \cdot \ [\ -2a\hat{\mathbf{i}} + \mathbf{r}(1 + \frac{3ax}{r^2}) - \mathbf{r}(1 - \frac{3ax}{r^2})\ ]$$

$$\mathbf{E} \approx \frac{q}{4\pi\epsilon_o r^3} \cdot \ [\ -2a\hat{\mathbf{i}} + 3\mathbf{r}\frac{2ax}{r^2}\ ]$$

$$\approx \frac{1}{4\pi\epsilon_0 r^3}\ [\ \ -\mathbf{p} + 3\mathbf{r}\ \frac{(\mathbf{p}\cdot\mathbf{r})}{r^2}\ ]$$

since $\mathbf{p}\cdot\mathbf{r} = 2axq$.

starting with the potential, V

The potential is the integral of the field dotted with $\mathbf{r}$, going to some agreed-upon end point, like ∞, with zero potential. The basic equation is

$$V = \frac{q}{4\pi\epsilon_0 r}$$

In this case we add the signed contributions from each pole

$$V = \frac{q}{4\pi\epsilon_0}\ [\frac{1}{r_+} - \frac{1}{r_-}\ ]$$

That's it for the potential.

We will simplify by moving things to the numerator like so

$$V = \frac{q}{4\pi\epsilon_0}\ [\frac{r_- - r_+}{r_+ r_-}\ ]$$

But first, extend this by defining the position vector $\mathbf{r}$ relative to the center of the dipole so that in terms of vectors

$$\mathbf{r}_+ = \mathbf{r} - a\hat{\mathbf{i}}$$

$$\mathbf{r}_- = \mathbf{r} + a\hat{\mathbf{i}}$$

and

$$r_+ = \sqrt{\mathbf{r}_+\cdot\mathbf{r}_+}$$

$$= \sqrt{r^2 + a^2 - 2\mathbf{r}\cdot a\hat{\mathbf{i}}}$$

Approximate (for $r >> a$) by setting $a^2 = 0$ and pulling out r:

$$r_+ = r\sqrt{1 - 2\mathbf{r}\cdot a\hat{\mathbf{i}}/r^2}$$

$$r_- = r\sqrt{1 + 2\mathbf{r} \cdot a\hat{\mathbf{i}}/r^2}$$

Simplify using the binomial expansion

$$r_+ \approx r(1 - \mathbf{r} \cdot a\hat{\mathbf{i}}/r^2)$$

$$r_- \approx r(1 + \mathbf{r} \cdot a\hat{\mathbf{i}}/r^2)$$

That numerator will be

$$r_- - r_+ = r(1 + \mathbf{r} \cdot a\hat{\mathbf{i}}/r^2) - r(1 - \mathbf{r} \cdot a\hat{\mathbf{i}}/r^2)$$

$$= 2\mathbf{r} \cdot a\hat{\mathbf{i}}/r$$

So now, going back to

$$V = \frac{q}{4\pi\epsilon_0}\ [\frac{1}{r_+} - \frac{1}{r_-}\ ]$$

in the brackets, in the numerator we will have what we got in the line before and in the denominator we have

$$r_+r_- = r\sqrt{1 + 2\mathbf{r} \cdot a\hat{\mathbf{i}}/r^2} \cdot r\sqrt{1 - 2\mathbf{r} \cdot a\hat{\mathbf{i}}/r^2}$$

Approximating the square roots as just equal to 1 (since $a << r$), that gives

$$\approx r^2$$

which leaves us with

$$V = \frac{q}{4\pi\epsilon_0}\ [\frac{2\mathbf{r} \cdot a\hat{\mathbf{i}}/r}{r^2}\ ]$$

The dipole moment is

$$\mathbf{p} = 2aq\hat{\mathbf{i}}$$

so

$$V = \frac{1}{4\pi\epsilon_0} \, [\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \,]$$

In terms of $\mathbf{r} = \langle x, y \rangle$

$$\mathbf{p}\cdot\mathbf{r} = 2aq\hat{\mathbf{i}}\cdot\mathbf{r} = px$$

and $r^2 = x^2 + y^2$ so

$$V = \frac{p}{4\pi\epsilon_0} \, [\frac{x}{(x^2+y^2)^{3/2}} \,]$$

compute the field

$$\mathbf{E} = -\nabla V$$

$$E_x = -\frac{p}{4\pi\epsilon_0} \, \frac{\partial}{\partial x} \, [\, \frac{x}{(x^2+y^2)^{3/2}} \,]$$

The partial derivative is $(u'v - uv')/v^2$ which is

$$= \frac{(x^2+y^2)^{3/2} - x \cdot 3/2(x^2+y^2)^{1/2} \cdot 2x}{[(x^2+y^2)^{3/2}]^2}$$

$$= \frac{1}{(x^2+y^2)^{3/2}} - \frac{3x^2}{(x^2+y^2)^{5/2}}$$

$$= \frac{1}{r^3} - \frac{3}{r^3} \, \frac{x^2}{r^2}$$

so

$$V = \frac{p}{4\pi\epsilon_0 r^3} \, (3\cos^2\theta - 1)$$

Similarly

$$E_y = -\frac{p}{4\pi\epsilon_0} \, \frac{\partial}{\partial y} \, [\, \frac{x}{(x^2+y^2)^{3/2}} \,]$$

The partial is

$$(-\frac{3}{2})x \ \frac{1}{(x^2+y^2)^{5/2}} \ 2y = \frac{-3xy}{r^5}$$

$$= -\frac{1}{r^3} \ 3\sin\theta\cos\theta$$

Combining these results, the field is

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3}(3\cos^2\theta\hat{\mathbf{i}} - \hat{\mathbf{i}} + 3\sin\theta\cos\theta\hat{\mathbf{j}}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3}(3p\cos^2\theta\hat{\mathbf{i}} - p\hat{\mathbf{i}} + 3p\sin\theta\cos\theta\hat{\mathbf{j}}$$

Since

$$\mathbf{p} = p\hat{\mathbf{i}}$$

The middle term is $-\mathbf{p}$.

The unit vector is

$$\hat{\mathbf{e}}_r = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$$

so the other two terms give

$$3p\cos\theta(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})$$

$$= 3p\cos\theta \ \hat{\mathbf{e}}_r$$

$$= (3\mathbf{p}\cdot\hat{\mathbf{e}}_r) \ \hat{\mathbf{e}}_r - \mathbf{p}$$

tacking on the part out front we have

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3} \ [\ (3\mathbf{p}\cdot\hat{\mathbf{e}}_r) \ \hat{\mathbf{e}}_r - \mathbf{p} \ ]$$

which matches what we had above at the end of the section on the field

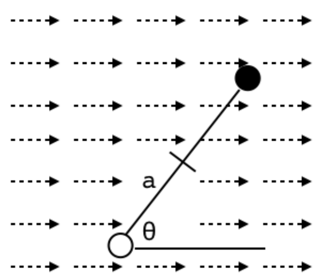
$$\approx \frac{1}{4\pi\epsilon_0 r^3}\ [\ \ -\mathbf{p} + 3\mathbf{r}\ \frac{(\mathbf{p}\cdot\mathbf{r})}{r^2}\ ]$$

allowing for the fact that $\mathbf{r}/r = \hat{\mathbf{e}}_r$.

Chapter 82

Dipole in a field

Consider a dipole with a total separation $2a$ between the charges, $+q$ and $-q$. The dipole lies in a uniform electric field E (say it is horizontal) at an angle to the field lines of θ.



The forces on the two charges from the field cancel.

However, there are also two torques, which both tend to align the dipole with the field. For the charge $+q$, there is a force Eq. The torque is that component of the force perpendicular to the axis of the dipole, $F \sin \theta$, times the distance from the axis of rotation at the point where the force is applied, which is equal to a.

$$\tau = aEq\sin\theta$$

Since there are two charges, the torque is double:

$$\tau = 2aEq\sin\theta$$

Even though the second charge is negative, the force on it tends to rotate the dipole in the same direction as the force on the positive charge.

$2aq$ is equal to the dipole moment, p so

$$\tau = pE\sin\theta$$

but this is

$$\tau = \mathbf{p} \times \mathbf{E}$$

The order of the two terms in the cross-product is a matter of definition: $\tau = \mathbf{r} \times \mathbf{F}$. This order makes the resulting vector point into the page.

There is a potential energy when the dipole lies at an angle to the field. For work done by a force we would write

$$U = -W = \int \mathbf{F} \cdot d\mathbf{r}$$

The equivalent formulation for torque is

$$\int \tau \cdot d\theta = \int_{\theta_0}^{\theta} pE\sin\theta$$

$$= -pE\cos\theta \,\Big|_{\theta_0}^{\theta}$$

If we choose θ_0 corresponding to zero potential energy

$$U = pE\cos\theta = \mathbf{p} \cdot \mathbf{E}$$

When θ is 90 degrees and the dipole is perpendicular to the field, the potential energy is pE.

Chapter 83

Capacitor

A capacitor consists of two charged objects: parallel plates, or concentric sphere and shell. The plates are separated by a dielectric (insulator). A capacitor stores energy in the form of the electric field between the plates.

The potential difference is dependent on the amount of charge that is present, and physical characteristics like the size and geometry of the plates and the distance between them.

For the example of two (infinite) parallel plates, we verified elsewhere the formula for the electric field

$$E = \frac{\sigma}{\epsilon_0}$$

This result is easy to see if we visualize an electric field with lines of flux. By symmetry, there is nowhere to go for a line that leaves the positive plate but directly toward the negative plate. There is no space for the lines to spread out with distance, hence, no dependence on the separation.

For a finite capacitor, if we ignore edge effects, the same basic result holds.

The voltage difference does depend on the distance a separating the plates.

$$V = Ea = \frac{\sigma a}{\epsilon_0}$$

Now since $\sigma = Q/A$

$$V = \frac{Q}{A}\,\frac{\sigma a}{\epsilon_0}$$

We consolidate all the terms except the charge into a factor $C = \epsilon_0 A/a\sigma$ so that gives

$$V = Ea = \frac{Q}{C}$$

Capacitance C relates voltage to charge

$$CV = Q$$

The higher the capacitance, the smaller the voltage for a given amount of stored charge. The permittivity ϵ_0 changes to ϵ for other materials, where $\epsilon = k\epsilon_0$. For example, for paper, $k = 3.5$ or so.

The capacitance is defined to be the ratio of the electric charge on each conductor to the potential difference. The unit is the farad, which is equal to one coulomb per volt. Typical values might be microfarads μF.

A very useful property of capacitors is that they block direct current, yet allow alternating current to pass. Also, when combined in appropriate circuits, they can be used to tune a circuit to a particular resonant frequency (e.g. in a radio).

energy

If we consider a capacitor with a charge Q on it, work must be done to bring a small amount of positive charge dQ to the positively charged plate. Work is equal to force times distance.

The energy stored in a capacitor can be calculated as the work done in moving a small amount of charge "from" one plate "to" another.

$$dW = E \ dq \ a = V \ dq$$

$$W = \int dW = \int V \ dq = \int \frac{Q}{C} \ dq$$

Hence, in building up a charge Q on a capacitor the work done is just

$$W = \frac{Q^2}{2C} = \frac{(CV)^2}{2C} = \frac{1}{2}CV^2$$

The integral here takes account of the fact that the voltage at the time any small charge dq is transferred depends on how much charge is currently on the plates.

Since current does not actually flow across the capacitor, for each electron that leaves the positive plate, one must join the negative plate.

discharge circuit

Consider a resistor R and a capacitor C, and a switch. The capacitor is initially charged. Close the switch and what happens? Use the standard rules and go around the circuit looking at the voltage drops across the components. We get

$$\frac{Q}{C} - IR = 0$$

Define the current as $I = -dq/dt$, and then

$$\frac{Q}{C} = IR = -\frac{dq}{dt}R$$

$$\frac{dq}{Q} = -\frac{1}{RC}dt$$
$$Q = Q_0 e^{-t/RC}$$

The charge decays exponentially with time with characteristics governed by RC. In particular, if $Q = Q_0/2$ then

$$\ln 2 = \frac{T_{1/2}}{RC}$$

The current is the time derivative of the charge

$$I = -\frac{dq}{dt}$$
$$= \frac{1}{RC}Q_0 e^{-t/RC}$$
$$= \frac{1}{RC}I_0$$

In exactly the same way as for the charge

$$\frac{dI}{dt} == -\frac{1}{RC}I_0$$
$$RC\ \frac{dI}{dt} + I_0 = 0$$

When you hook up a voltage and a capacitor, there is an initial current, but as the capacitor plates acquire charge the current dies away exponentially.

energy

On the other hand,onsider a circuit containing a battery (or EMF $\mathcal{E}$), a resistor R and a capacitor C, and a switch. The capacitor is initially uncharged. Close the switch and what happens? Use the standard rules and go around the circuit looking at the voltage

series and parallel

Everyone has probably seen the equations for resistors.

$$V = IR$$

Put two resistors in series, and each one must carry the full current, but the voltage drop is distributed, part of it over each resistor separately. Hence:

$$\begin{aligned} V_{tot} &= V_1 + V_2 = IR_1 + IR_2 \\ &= I(R_1 + R_2) = IR_e \end{aligned}$$

where R_e is the equivalent resistance for the two resistors together.

$$R_1 + R_2 = R_e$$

In contrast, if the resistors are in parallel, they have the same voltage drop and the current is distributed.

$$I = \frac{V}{R}$$

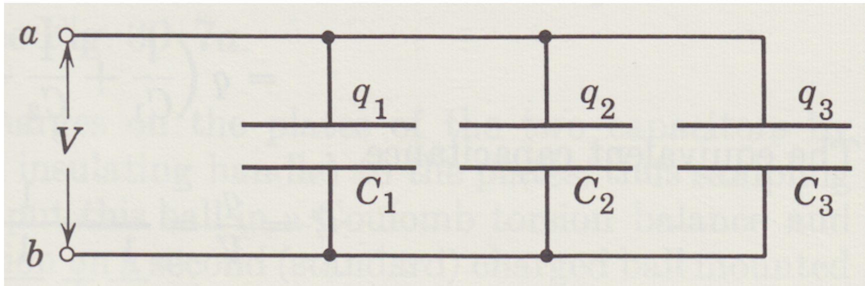
$$I = I_1 + I_2 = \frac{V}{R_e}$$

Thus

$$\begin{aligned} \frac{V}{R_e} &= \frac{V}{R_1} + \frac{V}{R_2} \\ \frac{1}{R_e} &= \frac{1}{R_1} + \frac{1}{R_2} \end{aligned}$$

Resistances add in series, and their reciprocals add, in parallel.

Capacitors are just the opposite. Capacitance is additive in parallel, but the reciprocals add in series.



In parallel, both components have the same voltage, but the charge is additive.

$$V = \frac{Q_1}{C_1}$$

$$V = \frac{Q_2}{C_1}$$

$$Q_{tot} = C_e V = C_1 V + C_2 V$$

Thus

$$C_e = C_1 + C_2$$

In series, the charge is the same and the voltages add

$$V_{tot} = \frac{Q}{C_1} + \frac{Q}{C_2} = \frac{Q}{C_e}$$

$$\frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{C_e}$$

AC circuit

If we put a capacitor into an AC circuit, then the charge is

$$q(t) = CVe^{j\omega t}$$

where as before C is a number that describes the capacity of the capacitor (with units of coulombs/volt), and i is renamed to be j because the electrical engineers use i and I for current. Anyway the point is that the current across such a device is

$$i(t) = \frac{dq}{dt} = j\omega CVe^{j\omega t} = j\omega q(t)$$

To solve this differential equation, we need to find a function where

$$i = \frac{dq}{dt} = j\omega q$$

If the voltage goes like the cosine, then this is a problem.

What we are going to do is to write the voltage as

$$V(t) = V_o e^{j\omega t}$$

where $j = \sqrt{-1}$. Then

$$q(t) = CV_o e^{j\omega t}$$

so the current is

$$i(t) = j\omega CV_o e^{j\omega t}$$

so the equivalent of the resistance for the capacitor is

$$X_C = \frac{1}{j\omega C}$$

$$= -\frac{j}{\omega C}$$

What does this mean? It means that when the voltage is at the peak of its cycle, the current through the capacitor is 90 degrees out of phase with it.

Chapter 84

Work, energy and potential

To keep things simple, in this chapter we will use only one dimension. The beauty of orthogonal directions (or *basis vectors* for space) is independence: the net force or displacement or whatever we care about is the sum of what happens in x, plus what happens in y, and the same for z.

The second nice feature is that when two vectors are not parallel, say the force and the distance, then we take the component which is in the same direction using the dot product: $\mathbf{F} \cdot d\mathbf{r} = F \cos\theta \times r$. If we adopt the convention that this has already been done implicitly, then we can just write force times distance or Fd.

The only thing left is to deal with the fact that in some cases the force might not be constant over distance (or time or whatever pieces we're adding up), so we set up the integral $\int F dx$ and add up for each piece the value of F for the corresponding position x_i.

work-energy

Previously, we **derived** this relationship for motion with acceleration a:

$$v^2 - v_0^2 = 2a(x - x_0)$$

starting from the basic equation of motion

$$x = \frac{1}{2}at^2 + v_0t + x_0$$

and the fact that velocity is the time-derivative of position

$$v = \frac{dx}{dt} = 2a + v_0$$

Newton's second law says that $a = F/m$ so

$$v^2 - v_0^2 = 2a(x - x_0)$$

$$v^2 - v_0^2 = 2\frac{F}{m}(x - x_0)$$

Rearranging

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = F(x - x_0)$$

The term $\frac{1}{2}mv^2$ is called the kinetic energy. The change in kinetic energy is equal to the force, times the distance over which it operates.

$$K - K_0 = \Delta K = Fd$$

If the force is not constant then

$$\Delta K = \int F\ dx$$

We make another definition: Fd is the work done by the force

$$W = Fd = \Delta K$$

Suppose we take our marble back up to the leaning Tower of Pisa and drop it, then the work done by the force of gravity on the marble during its fall through height h is $W = Fd = mgh$, which is equal to the kinetic energy acquired during the fall. If the initial velocity is zero then

$$\frac{1}{2}mv^2 = mgh$$

$$v = \sqrt{2gh}$$

Work has a sign. If the work done is in the direction that the force acts, it's positive.

potential energy

If we consider the marble as it is balanced on the wall at the top of the tower (or for that matter while it is still in my pocket), we say that it has more potential energy than it does before I carry it up, or after its arrival back down at the bottom.

The change in potential energy is exactly equal to the change in kinetic energy.

There is a funny business about the sign. By definition:

$$\Delta K = K - K_0 = \int_{x_0}^{x} F \ dx$$

Suppose the function G is the integral of $F \ dx$ then

$$\Delta K = G(x) - G(x_0) = G - G_0$$

so

$$K - G = K_0 - G_0$$

We don't like those minus signs.

The solution is to define $U = -G$, where U is called the potential energy. Then

$$K + U = K_0 + U_0$$

This is the law of conservation of energy: the kinetic energy K plus the potential energy U is constant.

Going back to G we see that

$$G' = \frac{d}{dx} G = F$$

so

$$-\frac{d}{dx} U = F$$

Minus the derivative of the poential energy is equal to the force.

The vector form of this relationship is

$$-\nabla U = \mathbf{F}$$

visualization

The relationships derived above are quite easy to visualize. In practice there can be confusion about the sign, but the basic idea is simple.

Consider this 550 foot tall "mountain" on the island of Iwo Jima. (It has a special importance for me. The fact that I am here (or was) is due to something that *didn't* happen there but could have, in Feb-Mar 1945).

I found a topographical map on the web

The lines connect positions with equal height. They connection positions of equal potential, and are what Auroux calls level curves.

The lines shown on the map are roughly circular (at least near the bottom), and the smaller the circle the greater the height.

The potential energy U increases as the height increases. The gradient of U, ∇U points perpendicularly to the level curves.

∇U points in the direction of increasing height. If you happen to stumble and fall, however, the direction that gravity pushes you is downhill, and it will be exactly opposite to the direction in which ∇U points. Hence $\mathbf{F} = -\nabla U$.

There are, as we said issues about sign. For example, when work is done by a force that is increasing the potential energy, that force acts in the opposite direction from gravity or from the electric field. Also, in electricity, we must worry about the sign of the charge that is responding to the field.

mass and spring

In the mass and spring system, we find experimentally that

$$F = -kx$$

using the definitions above we find that

$$U = -\int F dx = \frac{1}{2}kx^2$$

and conservation of energy says that

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \text{const}$$

If the mass is pulled to the right for an initial displacement A and then let go, at any time afterward (idealizing with no friction)

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2$$

$$\frac{k}{m}(A^2 - x^2) = v^2$$

$$v = \sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}$$

We **solved** the differential equation of this system

$$F = -kx = ma = m\ddot{x}$$

$$\ddot{x} + \frac{k}{m}x = 0$$

One solution is

$$x(t) = A\cos\omega t$$

$$\ddot{x}(t) = -\omega^2 x(t) = -\frac{k}{m}x(t)$$

so

$$\omega^2 = \frac{k}{m}$$

$$\omega = \pm \frac{k}{m}$$

ω is the angular frequency (it is called that because one can view the oscillation as the projection of circular motion onto one dimension).

Going back to the equation for the velocity

$$v = \pm\, \omega\sqrt{A^2 - x^2}$$

If we want the velocity for $x = 0$ it is just $v = \pm\, A\omega$. It is plus or minus depending on the direction of travel.

If the time t is equal to one period $t = T$, then the product ωT is equal to 2π so $\omega = 2\pi/T$. The units are radians per second. One can also write $\omega = 2\pi f$ where f is the frequency.

potential

The last topic for this chapter is potential. Potential is not the same as potential energy. It is rather, that work done by a force, gravitational or electrical or whatever, *on a unit charge or mass.*

Let's start with the force itself. As our example, take the electric force.

The force exerted at a point on a unit test charge is called the electric field $\mathbf{E}$. Let's agree to work in one dimension so that $\mathbf{F} = F$ and $\mathbf{E} = E$ and so on. Then

$$F = qE$$

Now recall from above that the work $W = Fd$, work is the product of force times distance.

Potential is the work with the charge separated out, it is

$$W = qEd = qV$$

$$V = Ed$$

Potential is also called *voltage* and has units of volts.

Electric field strength is defined in terms of newtons per coulomb, which now makes sense. One volt is defined as one joule per coulomb.

Power is the time-derivative of the work

$$P = \frac{d}{dt}W = \frac{d}{dt}\ qEd$$

So, for a constant electric field, since current $i = dq/dt$

$$P = iEd = Vi$$

and if the resistor follows Ohm's Law ($i = V/R$) then

$$P = Vi = i^2R$$

example

A light bulb has a resistance of $R = 240\Omega$. If the circuit is $120V$, then the current will be $i = V/R = 0.5A$, one-half an amp, and the power will be

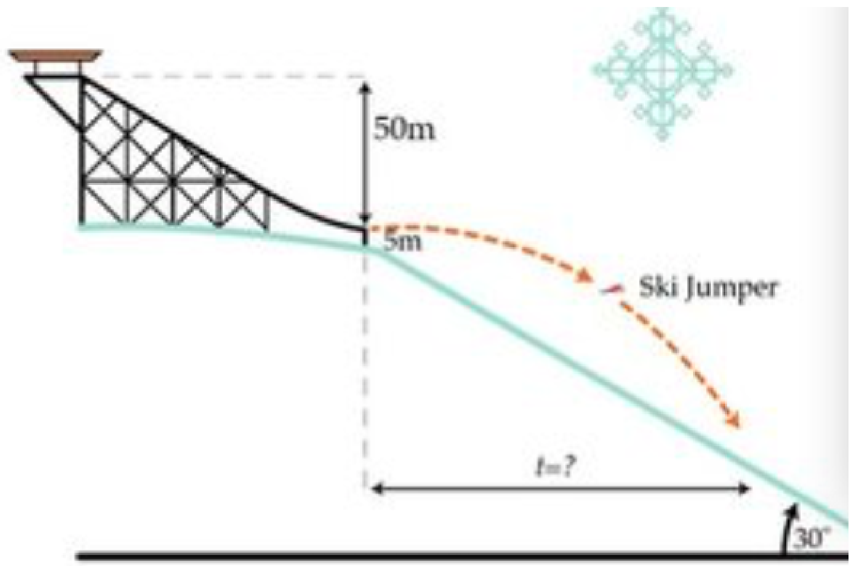
$$P = i^2R = 60W$$

60 watts.

In my shop I have a tool with a $1 - 1/2$ hp (horsepower) motor. 1 hp $= 746W$. The current it draws is $P/V = 1119W/120V = 9.3A$. That should be OK for a standard 15 amp circuit.

ski jumper

This problem doesn't have any calculus in it but it's still fun.



A ski jumper goes down a ramp of 50 meters height which is the same angle as the slope, 30 degrees. At the very end of the ramp all velocity is converted into horizontal motion by a small lip. We may not neglect friction: the coefficient of kinetic friction $\mu = 0.05$.

Given some horizontal velocity v, the equations of motion (taking y positive downward) are:

$$x = vt$$
$$y = \frac{1}{2}gt^2 - 5$$

(We take the origin of coordinates to be 5 m below the release point.) Solve for the time when

$$\frac{y}{x} = \tan\theta = \frac{1}{\sqrt{3}}$$

If we can ignore the extra 5 feet, we would have

$$\tan\theta = \frac{g}{2v}t$$

$$t = \frac{2v}{g\sqrt{3}}$$

$$x = v^2\frac{2}{g\sqrt{3}} = \frac{v^2}{g\cos\theta}$$

If we cannot neglect it we have

$$\tan\theta = \frac{g}{2v}t - \frac{5}{vt}$$

$$\frac{1}{2}gt^2 - v\tan\theta\ t - 5 = 0$$

Given v we can solve for t and then compute vt.

first part

The force of gravity is mg, reduced by the factor of $\cos\theta$. The force which determines friction is that pointed perpendicular into the slope, $mg\sin\theta$. Friction opposes gravity along the ramp, giving a net force of

$$mg\cos\theta - \mu mg\sin\theta$$

The work done is the force times the distance, the length of the ramp, which is $h\sin\theta$. We have

$$W = mgh(\cos\theta - \mu\sin\theta)\sin\theta$$

This is equal to the kinetic energy at take-off so

$$\frac{1}{2}\,mv^2 = mgh(\cos\theta - \mu\sin\theta)\sin\theta$$

$$v^2 = 2gh(\cos\theta - \mu\sin\theta)\sin\theta$$

so finally

$$\begin{aligned} x &= \frac{1}{g\cos\theta}\;2gh(\cos\theta - \mu\sin\theta)\sin\theta \\ &= 2h\;(1 - \mu\tan\theta)\sin\theta \\ &= 100\;(1 - \frac{0.05}{\sqrt{3}})\frac{1}{2} = 50 - \frac{2.5}{\sqrt{3}} \end{aligned}$$

Part XXII

Theta and r

Chapter 85

Polar coordinates

In polar coordinates points are plotted in terms of distance from the origin, r and the angle θ that this ray makes with the positive x-axis. Converting from x, y to r, θ is pretty easy:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

To go the other way, use Pythagoras to write

$$x^2 + y^2 = r^2$$

$$\theta = \tan^{-1}(\frac{y}{x}), \quad x \neq 0$$

In polar coordinates, as in Cartesian (xy) coordinates, the equation of a circle depends on whether it is at the origin or not. If it is at the origin, then something like

$$r = 3$$

defines the graph. But if it's away from the origin, then the equations are of the form:

$$r = a\cos\theta + b\sin\theta$$

Let's do a derivation and then look at examples. We can manipulate these equations to go back to Cartesian coordinates.

$$r = 2h\cos\theta + 2k\sin\theta$$

The reason for choosing these particular coefficients will become clear shortly. Substitute x and y.

$$r = 2h \cdot \frac{x}{r} + 2k \cdot \frac{y}{r}$$

so

$$r^2 = 2hx + 2ky$$

$$x^2 + y^2 = 2hx + 2ky$$

$$[\ x^2 - 2hx\ ]\ +\ [\ y^2 - 2ky\ ]\ = 0$$

complete both squares

$$[\ x^2 - 2hx + h^2\ ]\ +\ [\ y^2 - 2ky + k^2\ ]\ = h^2 + k^2$$

$$(x-h)^2 + (y-k)^2 = h^2 + k^2$$

That is, for an equation of the form

$$r = 2h\cos\theta + 2k\sin\theta$$

the origin is at (h, k) and the radius is

$$r = \sqrt{h^2 + k^2}$$

If the equation contains only $\sin\theta$ then compute k equal to one-half the coefficient of $\sin\theta$, with the origin at $(0, k)$ and radius $r = k$.

Similarly, if the equation contains only $\cos\theta$ then compute h equal to one-half the coefficient of $\cos\theta$, with the origin at $(h, 0)$ and radius $r = h$.

examples

For example

$$r = 3\sin\theta$$

is a circle centered at $(0, 3/2)$, with a radius of $3/2$ (it passes through the origin and the point $(x = 0, y = 3)$. All such circles (with just one of $\sin\theta$ or $\cos\theta$) have this property.

And

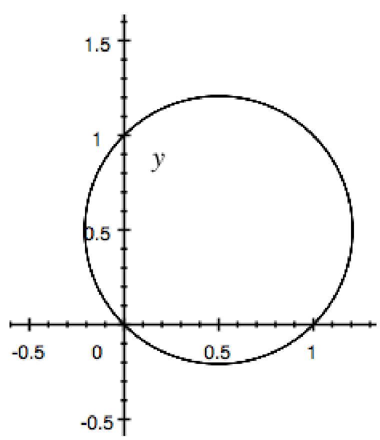
$$r = \sin\theta + \cos\theta$$

is a circle centered at $(1/2, 1/2)$ with a radius squared:

$$r^2 = h^2 + k^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}$$

$$r = \frac{1}{\sqrt{2}}$$

We see that all circles of this form pass through the origin.



other conic sections

Parabolas look like this:

$$r = \frac{1}{1 \pm a\sin\theta}$$
$$r = \frac{1}{1 \pm a\cos\theta}$$

The ones with $\sin\theta$ open up and down, the others open left and right. If the sign of the a term is negative, the parabola opens up or to the right.

The general formulas are

$$r = \frac{ep}{1 \pm e\sin\theta} = p\,\frac{1}{1/e + \sin\theta}$$
$$r = \frac{ep}{1 \pm e\cos\theta} = p\,\frac{1}{1/e + \cos\theta}$$

where e is the eccentricity ($e = 1$ for a parabola). If $e < 1$ then we have an **ellipse**.

parabola

Consider

$$r = \frac{2}{1 + \sin\theta}$$

Plot it to see. Or convert to Cartesian coordinates:

$$r + r\sin\theta = 2$$
$$\frac{y}{r} = \sin\theta$$
$$r + y = 2$$
$$r^2 = (2 - y)^2 = x^2 + y^2$$
$$4 - 4y + y^2 = x^2 + y^2$$
$$y - 1 = -\frac{1}{4}x^2$$

ellipse

If we measure r and θ from a focus, then

$$x = c + r\cos\theta$$

$$y = r\sin\theta$$

one can derive a formula for the ellipse in terms of r, θ

$$r = \frac{ep}{1 \pm e\cos\theta}$$

(We talk more about this **here**). For example if

$$r = \frac{1}{1 + e\cos\theta}$$

with $0 < e < 1$, we will get an ellipse.

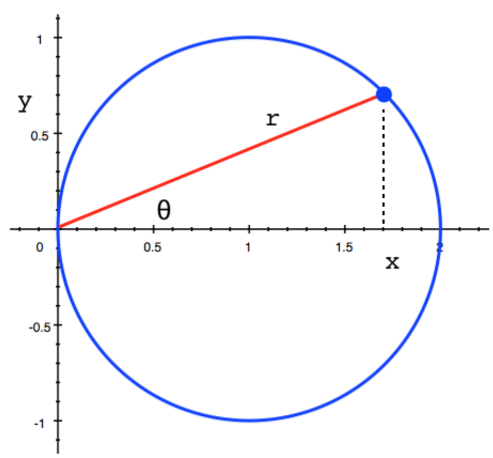
Chapter 86

Polar conics

circle

A very simple circle in polar coordinates is $r = a$. There is no θ-dependence when the circle has its center at the origin.

For a circle of radius a centered at $(a, 0)$ then



$$a^2 = (x - a)^2 + y^2$$
$$x^2 - 2ax + y^2 = 0$$

Always, $x = r\cos\theta$ and $y = r\sin\theta$ so

$$r^2(\sin^2\theta + \cos^2\theta) - 2ar\cos\theta = 0$$
$$r^2 - 2ar\cos\theta = 0$$
$$r = 2a\cos\theta$$

If the center of the circle is on the y-axis the equation is similar but with $\sin\theta$. A more general equation is

$$r = 2h\cos\theta + 2k\sin\theta$$

which is a circle that touches the origin, and has its center at (h, k).

The most general equation is with the circle anywhere in the plane. If we remember to specify the center at (s, ϕ) in *radial* coordinates, then the law of cosines easily yields

$$r^2 + s^2 - 2rs\cos(\theta - \phi) = a^2$$

reverse

Start from

$$r = 2h\cos\theta + 2k\sin\theta$$

Always, $x = r\cos\theta$ and $y = r\sin\theta$ so

$$r = 2h\frac{x}{r} + 2k\frac{y}{r}$$
$$r^2 = 2hx + 2ky$$
$$x^2 + y^2 = 2hx + 2ky$$

Easily rearrange and complete the square:

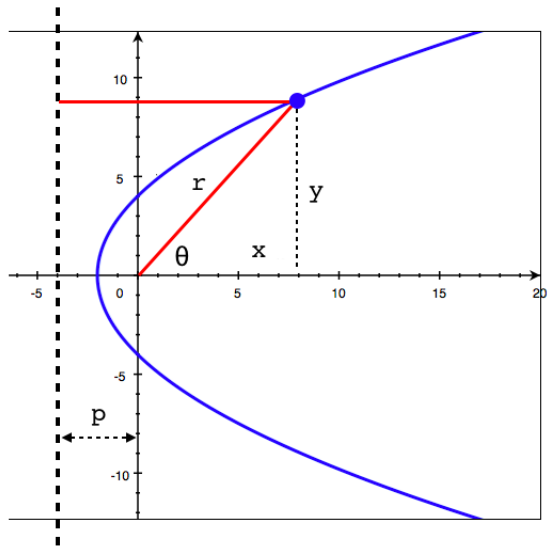
$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = h^2 + k^2$$
$$(x - h)^2 + (y - k)^2 = h^2 + k^2$$

For a circle touching the origin, $h^2 + k^2 = a^2$

$$(x - h)^2 + (y - k)^2 = a^2$$

parabola

To derive the equation for a parabola in polar coordinates it is convenient to rotate from the standard orientation by 90 degrees CW. In this way θ will have its usual relationship with the x-axis.



The origin of coordinates is placed at the focus, the distance to the vertex for this parabola is 2, and the distance from the origin to the directrix is $p = 4$.

Note that in Cartesian coordinates, this parabola will be of the form is $x = ay^2$ because of the rotation.

The distance from the focus to a general point (x, y) is just r. The distance from the directrix to the point is $p+x$. The geometric constraint gives simply:

$$r = p + x$$

We make the standard substitution $x = r\cos\theta$.

$$r = p + r\cos\theta$$

Some rearrangement gives the standard equation

$$r = \frac{p}{1 - \cos\theta}$$

For a vertically oriented parabola we would have $\sin\theta$ instead.

reverse

To go back to Cartesian coordinates, reverse the substitution for x:

$$r = \frac{p}{1 - x/r}$$

$$r - x = p$$

$$r^2 = (x + p)^2$$

Use $r^2 = x^2 + y^2$:

$$x^2 + y^2 = x^2 + 2px + p^2$$

$$y^2 = 2px + p^2$$

$$\frac{1}{2p}y^2 = x + \frac{p}{2}$$

This looks unusual. However, the equation that was actually plotted was $r = 4/(1 + \cos\theta)$ ($p = 4$).

Note: here we have used p as the distance from the focus to the directrix, which is twice the distance to the vertex. If we call the latter distance c, the $p = 2c$. Previously we showed that $4ac = 1$, so $a = 1/4c$. Thus we obtain $a = 1/8$:

$$\frac{1}{8}y^2 = x + 2$$

This shape factor matches the plot (four units above the axis (at $x = 0$ is two units to the right of the vertex) and the vertex is at $(-2, 0)$. a is unusually small, the reason is so the parabola will open quickly, giving room to put all the labels in the diagram.

ellipse

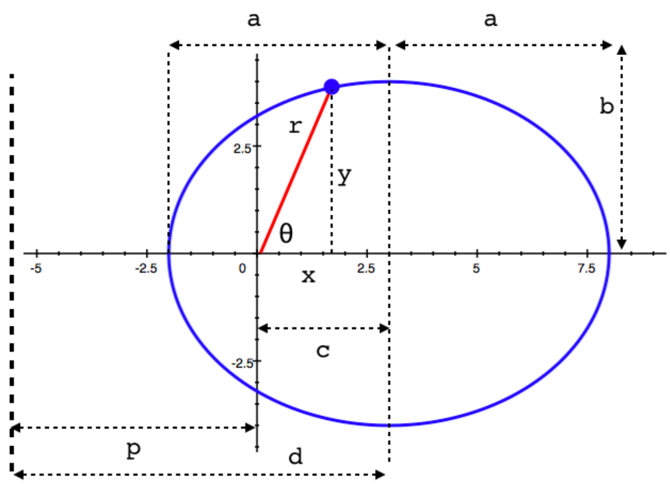
From the geometry of the ellipse, with the center at the origin, it is fairly easy to show that

$$a^2 = b^2 + c^2$$

and derive the equation of the ellipse in Cartesian coordinates:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

To derive the equation for an ellipse in polar coordinates it is convenient to shift the origin of coordinates to be the left focus of the ellipse at $(-c, 0)$.



The ellipse plotted here has $a = 5, b = 4$ and so $c = \sqrt{a^2 - b^2} = 3$. It has been shifted so the focus at $(-3, 0)$ is the origin of coordinates.

The eccentricity e is defined by the geometric constraint (next), which can be shown to be equivalent to $c/a = 0.6$.

Let p be the distance from the focus to the directrix, and let d be the distance from the directrix to the center of the ellipse.

The ellipse can be defined by its **geometric constraint**.

This says that for any point on the ellipse, the ratio of the distance from the focus (and here, the origin) to the point (that is, r), when divided by the distance from the point to the directrix, $x + p$, is equal to a constant, which we will call the eccentricity e.

$$\frac{r}{x+p} = \frac{r}{p + r\cos\theta} = e$$

$$r = e(p + r\cos\theta)$$

We simply rearrange to isolate r

$$r(1 - e\cos\theta) = ep$$

$$r = \frac{ep}{1 - e\cos\theta}$$

reverse

Going back is more complicated for the ellipse. Reverse the substitution $x/r = \cos\theta$.

$$r(1 - ex/r) = ep$$

$$r - ex = ep$$

$$r = ex + ep$$

There's a *magic* substitution that we will justify below:

$$ep = a(1 - e^2)$$

Using that, we have

$$r = ex + a(1 - e^2)$$

Use $r^2 = x^2 + y^2$:

$$x^2 + y^2 = e^2x^2 + 2exa(1 - e^2) + a^2(1 - e^2)^2$$

Combine cofactors for x^2, obtaining $(1-e^2)$ and then divide through by $(1-e^2)$:

$$x^2 + \frac{y^2}{1-e^2} = 2exa + a^2(1-e^2)$$

Complete the square for x by adding $(ea)^2$ to both sides

$$x^2 - 2exa + (ea)^2 + \frac{y^2}{1-e^2} = a^2(1-e^2) + (ea)^2$$

$$(x-ea)^2 + \frac{y^2}{1-e^2} = a^2(1-e^2) + (ea)^2$$

We asserted that $ea = c$. Simplify the right-hand side at the same time:

$$(x-c)^2 + \frac{y^2}{1-e^2} = a^2$$

This is great, because we need to shift the origin of coordinates back to the center of the ellipse by exactly this amount.

Unfortunately, I have not discovered any way to make that derivation simpler.

solve for $1-e^2$

To deal with $1-e^2$, recall that the basic geometry says

$$a^2 - c^2 = b^2$$

$$1 - (\frac{c}{a})^2 = \frac{b^2}{a^2}$$

Since $c = ea$

$$1 - e^2 = \frac{b^2}{a^2}$$

so what we had simplifies as the inverse of that times y^2

$$(x-c)^2 + \frac{a^2}{b^2}\ y^2 = a^2$$

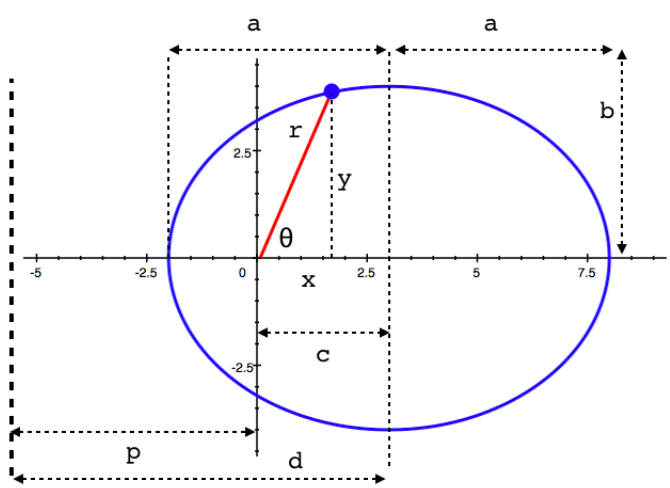
$$\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

Exactly what we want. Furthermore, we can view this derivation in reverse as a proof of $c = ea$ for an ellipse with this equation in Cartesian coordinates, shifted out to focus $(-c, 0)$.

Now let's explain the substitution:

$$ep = a(1 - e^2)$$

It is easiest to start by finding an expression for d, then getting e and p.



solve for d and e

Applying the geometric constraint to the extreme left end:

$$\frac{a-c}{d-a} = e$$

At the very top of the ellipse the distance to the focus is $\sqrt{b^2 + c^2}$ but this is also just a, which means

$$\frac{a}{d} = e = \frac{a-c}{d-a}$$

so

$$ad - a^2 = ad - cd$$

thus

$$d = \frac{a^2}{c}$$

One can also obtain this result by equating the ratios for the left and right ends of the ellipse.

Notice that the ratio a/d obeys the geometric constraint:

$$\frac{a}{d} = e = \frac{ac}{a^2} = \frac{c}{a}$$

We have proved $ae = c$, using only the geometry.

A longer, but pretty, proof is to start from the ratio for the extreme right end:

$$e = \frac{a+c}{d+a}$$

substitute $d = a^2/c$

$$= \frac{a+c}{a^2/c + a}$$

Multiply top and bottom by $1/a$

$$e = \frac{1 + c/a}{a/c + 1}$$

Put top and bottom over common denominators

$$e = \frac{(a+c)/a}{(a+c)/c} = \frac{c}{a}$$

solve for p

$$p = d - c = \frac{a^2}{c} - c$$

$$= \frac{a^2 - c^2}{c} = \frac{b^2}{c}$$

finally

$$\begin{aligned} pe &= \frac{b^2}{c} \cdot \frac{c}{a} = \frac{b^2}{a} \\ &= \frac{a^2 - c^2}{a} \\ &= \frac{a^2 - e^2a^2}{a} \\ &= a(1 - e^2) \end{aligned}$$

which is the special substitution we used.

summary of the summary

The circle (touching the origin), parabola (rotated to the right), and the ellipse are, in order:

$$r = 2h\cos\theta + 2k\sin\theta$$

$$r = \frac{p}{1 - \cos\theta}$$

$$r = \frac{ep}{1 - e\cos\theta}$$

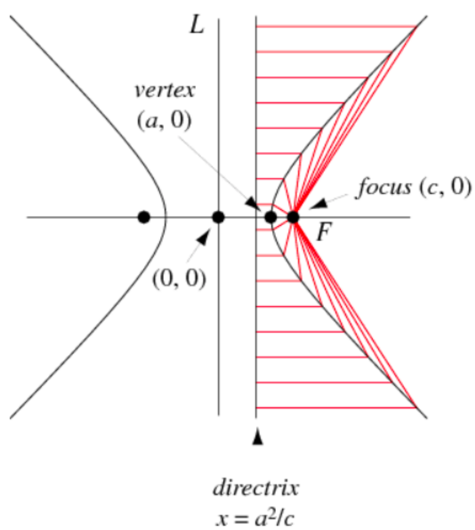
In the last case, note that $0 < e < 1$, so the parabola is the same but with $e = 1$.

Chapter 87

Polar hyperbola

We can also use a focus-directrix approach to the definition of the hyperbola. It will turn out to give the same formula as that of the ellipse and the parabola, but there are a few twists and turns along the way.

http://mathworld.wolfram.com/Hyperbola.html



The equation will turn out to be similar to that for the ellipse and parabola

$$r = \frac{ep}{1 - e\cos\theta}$$

except that $e > 1$, as we will see.

First, though, recall the equation that we derived before, namely

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

meaning of a and b

This gives a hyperbola of the type graphed above, opening "east-west". As we used in the derivation, c is the distance from the origin to each focus. Although the value of x is never 0, the value of y can be, and at those two points we have

$$x^2 = a^2$$

$$x = \pm a$$

So a is the horizontal distance to points on the curve along the x-axis.

As for b, we rearrange the basic equation

$$b^2x^2 - a^2y^2 = a^2b^2$$

asymptotes

Associated with the hyperbola is a pair of lines called asymptotes. Their equation is

$$b^2x^2 - a^2y^2 = 0$$

Factoring

$$(bx + ay)(bx - ay) =$$

which has solutions when

$$y = \pm\frac{b}{a}x$$

For a hyperbola in standard orientation like this, these lines go through the origin.

eccentricity

Notice that, unlike the ellipse, $a < c$. We define the eccentricity with the same equation as for the ellipse

$$ea = c$$

but now realize that for a hyperbola $e > 1$.

Recall that we defined

$$b^2 = c^2 - a^2, \qquad (c > a)$$

Hence

$$b^2 = (ea)^2 - a^2 = a^2(e^2 - 1)$$

whereas before for the ellipse we had

$$b^2 = a^2(1 - e^2)$$

directrix

For the directrix we will assume the answer, and then show that it leads to the desired properties. From the diagram above we read that on the directrix

$$x = \frac{a^2}{c}$$

That is, the distance d from the y-axis is equal to

$$d = \frac{a^2}{c}$$

and since $c = ea$

$$d = \frac{a}{e}$$

which is just what we had before with the ellipse:

$$ea = c$$

$$ed = a$$

(except that now with the hyperbola $e > 1$ and $c > a > d$ whereas before with the ellipse $e < 1$ and $d > a > c$).

On the x-axis the distance from the focus to the curve is $c - a$ and from the curve to the directrix is $a - d$. We consider the ratio of the two distances

$$\frac{c-a}{a-d} = \frac{ea-a}{a-a/e}$$

$$= \frac{a(e-1)}{a(1-1/e)}$$

$$= \frac{(e-1)}{(1-1/e)}$$

Multiply by e on top and bottom:

$$= e\frac{(e-1)}{(e-1)}$$

$$= e$$

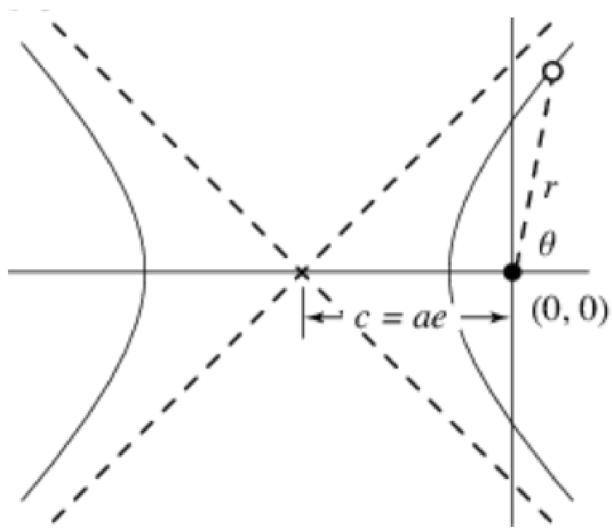
definition of p

As before, in a similar way to the directrix $d = a^2/c$, we define the focal parameter p as

$$p = \frac{b^2}{c}$$

$$= \frac{c^2 - a^2}{c}$$
$$= \frac{e^2a^2 - a^2}{c}$$
$$= \frac{a^2(e^2 - 1)}{c}$$
$$= \frac{a(e^2 - 1)}{e}$$

polar coordinates



The geometric constraint is

$$\frac{PF}{PD} = e$$

where PF is just r and the problem is to evaluate the length of PD.

$$PD = r\cos\theta + (c - d)$$

Hence we have that

$$e = \frac{r}{r\cos\theta + (c - d)}$$

$$er\cos\theta + e(c-d) = r$$
$$r(e\cos\theta - 1) = -e(c-d)$$
$$r = \frac{e(c-d)}{1-e\cos\theta}$$

The numerator

$$e(c-d) = e(ea - \frac{a}{e})$$
$$a(e^2-1)$$
$$= ep$$

So finally we obtain:

$$r = \frac{ep}{1-e\cos\theta}$$

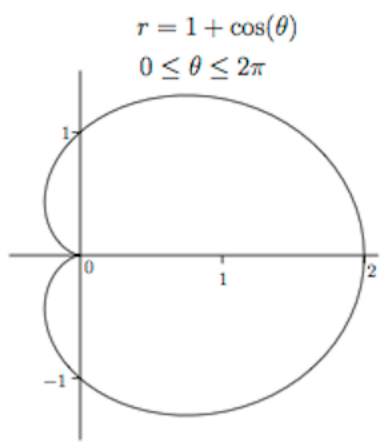
as the equation of a standard hyperbola in polar coordinates.

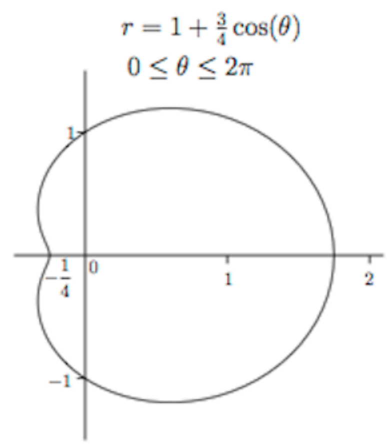
We recovered the same equation for all three conic sections: parabola, ellipse and hyperbola. The only difference is the value of e. Here $e > 1$, for the parabola $e = 1$ and for the ellipse $e < 1$.

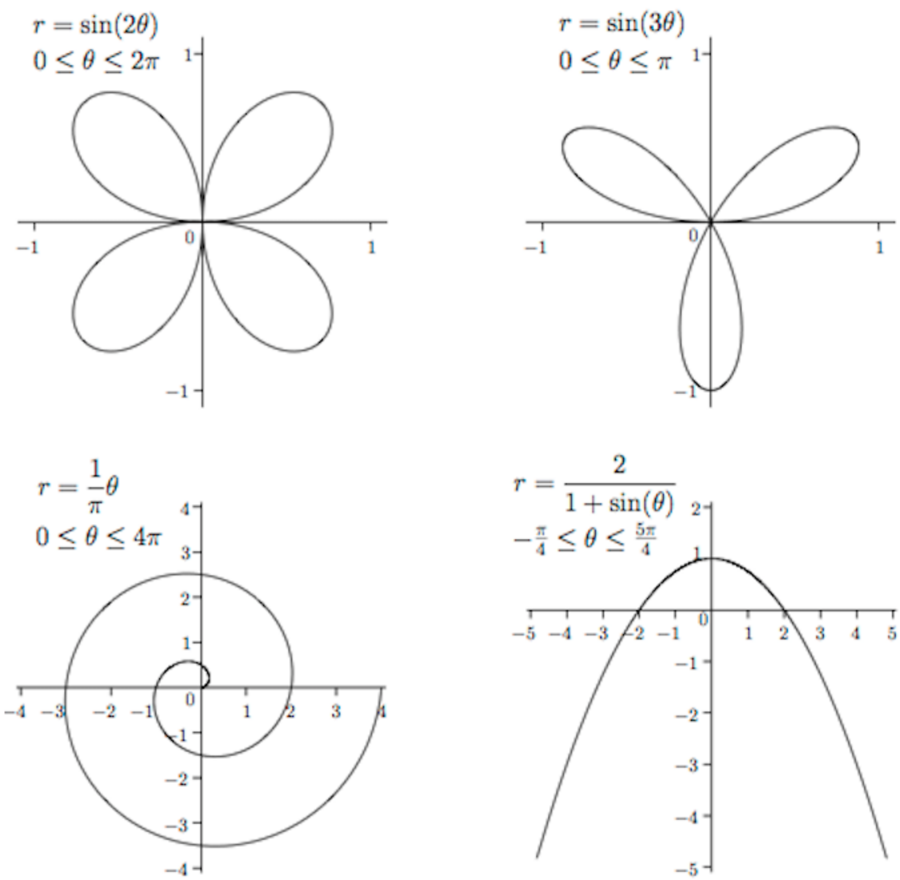
Chapter 88

Polar area

Here are some fancy examples of polar curves from *The Calculus Lifesaver*.







Integration to find areas

The idea for (one-dimensional) integration in polar coordinates is that we know r as a function of θ. For example, we had the circle centered at $(0, 3/2)$ given by

$$r = 3\sin\theta$$

We imagine dividing up the circle into little triangles, sectors where

$$\theta \to \theta + \Delta\theta$$

The sector is approximately a triangle with side r and base $r \times \Delta\theta$ (the latter is the length of the arc of the circle on its circumference).

The area of each little sector is

$$\frac{1}{2} r^2 d\theta$$

example

For this problem:

$$r = 3\sin\theta$$

The total area is then

$$\int_0^{2\pi} \frac{1}{2} r^2 d\theta$$
$$= \int_0^{2\pi} \frac{1}{2} (3\sin\theta)^2 d\theta$$
$$= \frac{9}{2} \int_0^{2\pi} \sin^2\theta d\theta$$

This looks hard but we've done it before. One way is to recall that

$$[\ \sin\theta\cos\theta\ ]' = -\sin^2\theta + \cos^2\theta$$
$$= 1 - 2\sin^2\theta$$

Integrate

$$\int [\ \sin\theta\cos\theta\ ]'\ d\theta = \int (1 - 2\sin^2\theta)\ d\theta$$
$$\sin\theta\cos\theta = \theta - 2\int \sin^2\theta\ d\theta$$

Hence

$$\int \sin^2\theta \ d\theta = \frac{1}{2}(\theta - \sin\theta\cos\theta)$$

So our answer is

$$= (\frac{9}{2}) \ \frac{1}{2}(x - \sin\theta\cos\theta) \ \Big|_0^{2\pi}$$

$$= (\frac{9}{2}) \ \frac{1}{2}(2\pi)$$

$$= \frac{9\pi}{4}$$

which is correct for a circle with radius 3/2.

example

The second example is from *How to ace the rest of calculus.* We have two circles, both of radius 1. The first one is centered at the origin. We are given the equation of the second in polar coordinates:

$$r = 2\cos\theta$$

Plugging in some values for θ: and calculating r:

$$\theta = 0 \rightarrow r = 2$$

$$\theta = \frac{\pi}{6} \rightarrow r = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{4} \rightarrow r = \sqrt{2}$$

$$\theta = \frac{\pi}{3} \rightarrow r = 1$$

$$\theta = \frac{\pi}{2} \rightarrow r = 0$$

We can also convert to x, y-coordinates. Multiply by r:

$$r^2 = 2r\cos\theta$$

Substituting $r^2 = x^2 + y^2$ and $x = r\cos\theta$:

$$x^2 + y^2 = 2x$$

Complete the square:

$$(x^2 - 2x + 1) + y^2 = 1$$

$$(x-1)^2 + y^2 = 1$$

The second circle is centered at $(1, 0)$. Note that for this circle it is *not* true that $x^2 + y^2 = 1$.

Now, the problem given is to calculate the shaded area in the figure.

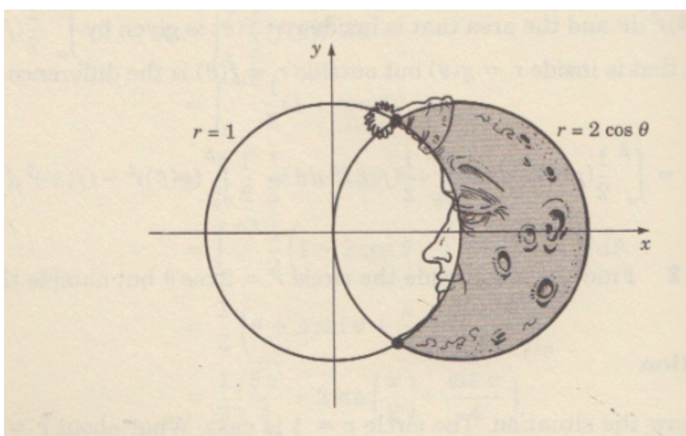


Figure 3.13 Area inside $r = 2\cos\theta$ but outside $r = 1$.

First, we must find the value of θ at the points of intersection between the two circles. We solve the two equations simultaneously:

$$y^2 = 1 - x^2$$

$$(x-1)^2 + y^2 = (x-1)^2 + 1 - x^2 = 1$$

$$-2x + 2 = 1$$
$$2x = 1$$
$$x = \frac{1}{2}$$
$$y = \sqrt{1 - x^2} = \pm\frac{\sqrt{3}}{2}$$
$$\theta = \tan^{-1}\frac{y}{x} = \pm\sqrt{3}$$

Look it up:

$$\theta = \pm\frac{\pi}{3}$$

or notice that we are on the unit circle so $\cos\theta = x = 1/2$, $\theta = \pm\pi/3$. That's the hard way. The easy way is

$$r = 1 = 2\cos\theta$$
$$\theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$$

The area of an arc of the unit circle is the r^2 times one-half the arc length in radians.

$$A = \frac{1}{2}\int r^2\ d\theta$$

We will subtract the area of the inner arc from that covered by the outer one

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(2\cos\theta)^2 - 1\ d\theta$$

Recall that

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - 1 + \cos^2\theta$$
$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

so

$$(2\cos\theta)^2 = 4\ \frac{1}{2}(1+\cos 2\theta) = 2(1+\cos 2\theta)$$

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta)^2 - 1\ d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 2\cos 2\theta + 1\ d\theta$$

$$= \frac{1}{2}\ [\ \sin 2\theta + \theta\ ]\ \Big|_{-\pi/3}^{\pi/3}$$

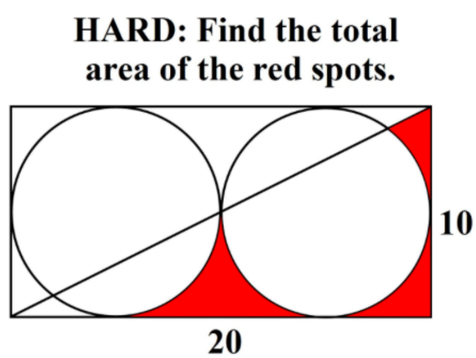
Since $\sin 2\pi/3 = \sqrt{3}/2$:

$$= \frac{1}{2}(\sqrt{3} + \frac{2\pi}{3}) = \frac{\sqrt{3}}{2} + \frac{\pi}{3}$$

Chapter 89

Circular segment

I found a hard geometry problem on the web:



We know it's hard because it says so!

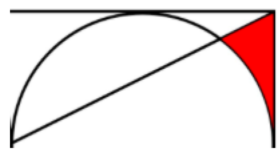
I liked it particularly because there are at least four different ways to calculate the answer using basic geometry and trigonometry, plus standard integration as well as polar integration. I got the same answer each time, so I have increased confidence in the result.

It is included here because it's a fun problem, and one of the methods was polar integration to find the area.

As a first step, we observe that the problem has been made artificially complicated by using these particular values for the side lengths. If both dimensions are scaled down by a factor of 5, then we obtain a rectangle with side lengths 4 and 2. The two circles become unit circles. We must just remember to re-scale to obtain the final answer, multiplying areas by 25.

The area of any arched corner segment is pretty easy, since 4 of them put together are equal to the difference between the area of a square with side length 2 and a unit circle: $4 - \pi$, so each one is $1 - \pi/4$.

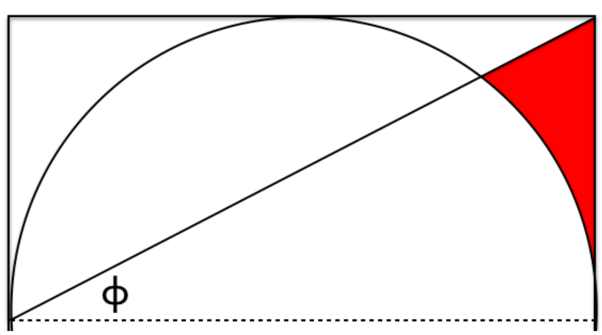
The real difficulty is the upper right-hand corner.



One of the arches is divided into two pieces, and we are supposed to only count the red part of the arch.

The basic right triangle that we see repeated in these images has side lengths in the ratio 1 : 2. Its area is just 1, and the smaller angle is

$$\phi = \tan^{-1} 0.5 \approx 0.4636 \text{ rad } \approx 26.565 \text{ deg}$$



That's not a nice round number, but OK.

My first thought was to calculate the area cut off by the chord of a circle, called a "circular segment". Then we could calculate the white part of the divided arch:

$$\text{triangle} \; - \; \text{segment} \; - \; \text{arch}$$

$$1 - \; \text{segment} \; - (1 - \pi/4)$$

and subtract that from one whole arch to get the red part.

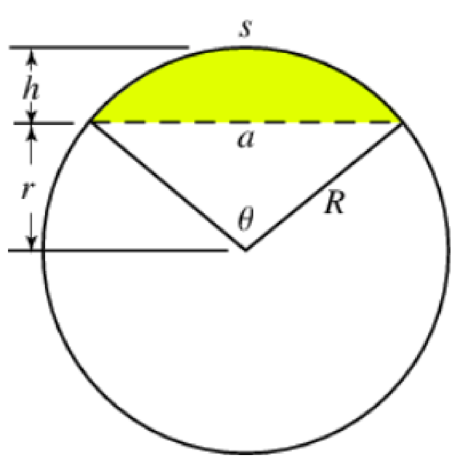
$$(1 - \pi/4) - \; [\; 1 - \; \text{segment} \; - (1 - \pi/4) \;]$$

$$1 - \frac{\pi}{2} + \; \text{segment}$$

We will use this result at the end for our final answer.

There is an easier way, which we find by exploring this direction just a little further.

A circular segment is like a polar cap, but in two dimensions.



http://mathworld.wolfram.com/CircularSegment.html

We carefully distinguish between the circular segment, in yellow, and the circular sector, which is the area of that slice of the circular pie swept out by the angle θ.

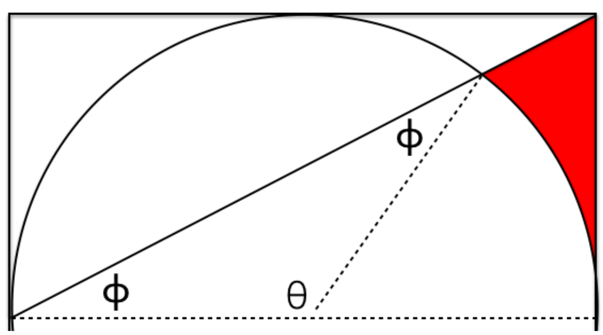
The area of the circular sector with central angle θ is the fraction of the total circular angle, times the area of a unit circle. The result is just half the angle.

$$\frac{\theta}{2\pi} \cdot \pi = \frac{\theta}{2}$$

For the actual calculation of a circular segment, we would need not only the angle θ, but also r and a, which we would need to derive from θ by applying the Pythagorean theorem and/or trigonometry. It can be done! However, we see a better way.

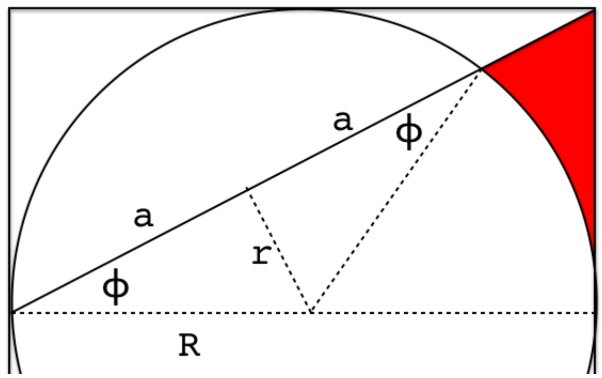
isosceles triangles

The first key idea to draw θ on our diagram, and realize that θ is the apex angle of an isosceles triangle. The two sides are both radii of our circle, and so are equal to each other! Therefore $\theta = \pi - 2\phi$.



So now we just calculate the circular sector swept out by θ and figure out the area of the isosceles triangle, subtract to find the spherical segment and go on from there. The beauty of this is we do not need to know the formulas for circular segments..

Let's continue by finding the lengths and areas of parts of the central isosceles triangle with angles ϕ-ϕ-θ.



The triangle area is easy because one-half of it is a triangle similar to the original. This means that $r/a = \tan\phi = 0.5$ so $a/r = 2$ and

$$a = 2r$$

Furthermore

$$r = R\sin\phi$$

$$a = R\cos\phi$$

The area of the entire isosceles triangle is two of these smaller ones:

$$A = ar = R^2 \sin\phi\cos\phi$$

We can also do it solely in terms of r

$$A = 2r^2$$

$$= 2R^2 \sin^2\phi$$

For this unit circle

$$A = 2\sin^2\phi$$

This works because

$$\cos\phi = 2\sin\phi$$

so

$$\sin\phi\cos\phi = 2\sin^2\phi$$

Because of the square we get an exact answer:

$$\sin\phi = \frac{1}{\sqrt{5}}$$

$$\sin^2\phi = \frac{1}{5}$$

$$A = 2\sin^2\phi = 0.4$$

Later, we will have a use for h:

$$h = R - r = R(1 - \sin\phi)$$

$$= 1 - \sin\phi$$

We could now move on to consideration of the circular sector with angle θ. But wait!

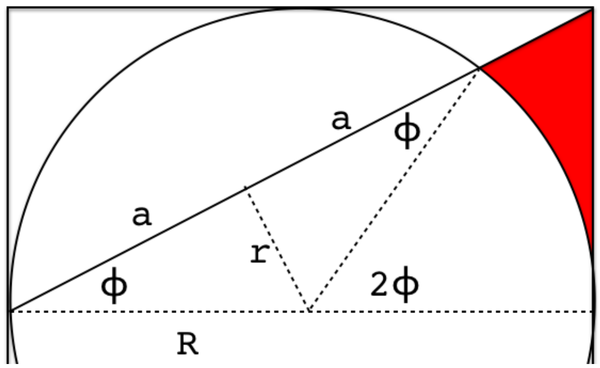
the other triangle

Notice at this point a different circular sector, just to the right of the isosceles triangle. Several different familiar theorems gives the angle as 2ϕ.

◦ $\theta + \phi + \phi = \pi$ but at the same time θ plus the unknown angle also equals π. Thus, the angle is 2ϕ.

◦ ϕ and the unknown angle both sweep out the same arc on the circle, but the unknown angle is at the origin of the circle, and ϕ is on the perimeter. A famous theorem says the the unknown angle is twice ϕ. The same argument in reverse is a *proof* of the theorem.

○ Draw a horizontal line (not shown) intersecting the angled line and the perimeter of the circle, at the upper-right, just above and to the right of the letter ϕ. The angle between the new horizontal and the angled line is ϕ, by another famous theorem (interior angles ...), and thus the unknown angle is twice ϕ, by that same famous theorem.



The area of the sector is, by the same calculation we did before, the ratio of the angle to 2π times the area of the unit circle.

$$\frac{2\phi}{2\pi}\,\pi = \phi$$

calculation 1: ignore the circular segment

So far we have that the area of the isosceles triangle is $2\sin^2\phi = \sin\phi\cos\phi$.

The circular sector with angle 2ϕ has area ϕ.

The part of the large lower triangle that is not in red is

$$\text{circular sector} \ + \ \text{isosceles triangle}$$

$$\phi + 2\sin^2\phi$$

Then what we want is to subtract that from the large triangle:

$$A = 1 - \phi - 2\sin^2\phi$$

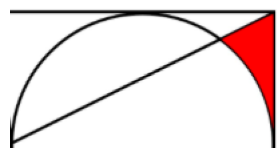
$$= 1 - \approx 0.46 - 0.4$$

I get ≈ 0.14, which seems reasonable.

Let's try to remember that: the red part of the arch is $1 - \phi - 2\sin^2\phi$.

calculation two: polar area

We can also use calculus, namely to compute a polar area. Look again at



What if we could get the area of the lower triangle minus the red part?

This is a pretty easy integral in polar coordinates. Since the angle is usually given as θ, for the moment we relabel ϕ as θ. We also reuse the variable a.

Set up a circle of radius a with its left edge at the origin, the equation of that circle in polar coordinates is

$$r = 2a\cos\theta$$

For example, radius $a = 1$ gives

$$\text{at } \theta = 0, \quad r = 2$$

$$\text{at } \theta = \frac{\pi}{4}, \quad r = \sqrt{2}$$

$$\text{at } \theta = \frac{\pi}{2}, \quad r = 0$$

The arc of the upper semi-circle that we're seeing is $\theta = 0 \to \frac{\pi}{2}$

$$\text{at } \theta = \phi, \quad r = 2\cos\phi$$

Since $\cos\phi = 2/\sqrt{5}$, $r = 4/\sqrt{5}$.

You don't believe me that this is correct? We have four equations. Always:

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Do not cancel the $r = 1$ yet.

We also have the equation of this circle. The general equation is

$$r = 2h\cos\theta + 2k\sin\theta$$

where (h, k) is the origin of the circle. Here, the origin is at $h = 1$ so

$$r = 2\cos\theta$$

Now comes the magic. Substitute for $\cos\theta$:

$$r = 2\frac{x}{r}$$

$$r^2 = 2x$$

but

$$r^2 = x^2 + y^2$$

We obtain

$$x^2 + y^2 = 2x$$

$$x^2 - 2x + y^2 = 0$$

Complete the square and add the same term on the right

$$(x-1)^2 + y^2 = 1$$

This is indeed a unit circle with origin at $(1, 0)$.

polar area

The area is made up of many small wedges with angle θ, and $r = f(\theta) = 2\cos\theta$. The wedges are approximately triangles with area $1/2 \cdot r \cdot r\ d\theta$. So the total area is

$$A = \int \frac{1}{2}\ [\ f(\theta)\ ]^2\ d\theta$$

$$= \frac{1}{2}\ 4 \int \cos^2\theta\ d\theta$$

$$= 2\ [\ \frac{1}{2}(\theta + \sin\theta\cos\theta)\ ]\ \Big|_0^\phi$$

At the lower bound, this is zero so

$$A = \phi + \sin\phi\cos\phi$$

For this ϕ we have that:

$$A = \phi + 2\sin^2\phi$$

This should be equal to the area of the circular sector of angle 2ϕ plus the area of the isosceles triangle, and if you look back, you'll see that we have a match.

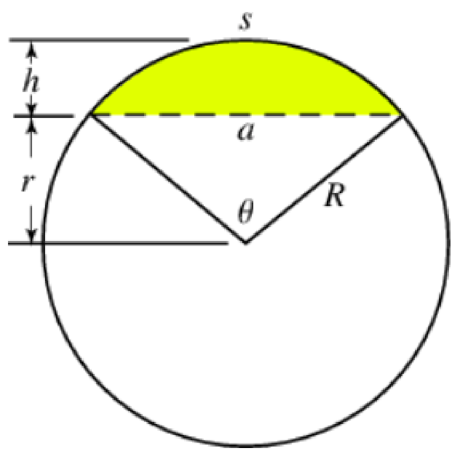
To get the red arch, we must subtract this from the triangle, which has unit area.

$$A = 1 - \phi - 2\sin^2\phi$$

That matches what we had before. To finish up, we do this two more ways, both requiring the area of the circular segment.

calculation 3: standard calculus

We find the circular segment by standard integral calculus:



Recall that the area of (the upper half of) a circle drawn in standard orientation (x horizontal) is

$$\int \sqrt{R^2 - x^2} \ dx$$

The bounds we need are $R - h$ to R.

Here we have rotated a quarter-turn and are integrating vertically, so we'll use y as the variable. The area of the half-circle to the right of the y-axis is

$$\int_{R-h}^{R} \sqrt{R^2 - y^2} \ dy$$

This is a unit circle so

$$A = \int_{1-h}^{1} \sqrt{1 - y^2} \ dy$$

The answer (done many times by now)

$$\frac{1}{2} \ [\ \sin^{-1} y + y\sqrt{1 - y^2} \ ] \ \Big|_{1-h}^{1}$$

At the upper bound we get $\pi/4$. For the lower bound we need h

$$h = 1 - \sin\phi$$

so that bound is just $\sin\phi$. The first term in parentheses is

$$\sin^{-1}(\sin\phi)$$

The angle whose sine is $\sin\phi$ is just ϕ!

The second term is

$$(\sin\phi)\sqrt{1-(\sin\phi)^2}$$
$$= \sin\phi\cos\phi$$

Altogether, we have

$$\frac{\pi}{4} - \frac{1}{2}(\phi + \sin\phi\cos\phi)$$

$$= \frac{\pi}{4} - \frac{\phi}{2} - \sin^2\phi$$

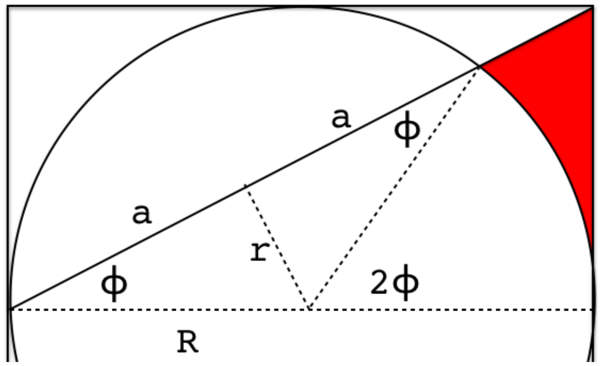
This is (almost) the circular segment.

It is off by a factor of 2. The reason is that the integral only gives the area of that part of the circle that is above the x-axis. We must multiply the provisional answer by 2.

$$= \frac{\pi}{2} - \phi - 2\sin^2\phi$$

Wait for the calculation of the red part of the arch, starting from the circular segment.

calculation four: geometry



We have that the area of the isosceles triangle is exactly

$$\sin\phi\cos\phi = 2\sin^2\phi = 0.4$$

The area of the circular sector with angle θ is

$$\frac{\theta}{2\pi}\,\pi = \frac{\theta}{2}$$

in terms of angle ϕ:

$$= \frac{1}{2}(\pi - 2\phi)$$
$$= \frac{\pi}{2} - \phi$$

The area of the circular segment is the area of the circular sector minus that of the isosceles triangle:

$$\frac{\pi}{2} - \phi - 2\sin^2\phi$$

This matches what we had for the third calculation, by integration of the equation of the circle.

the red part of the arch

At the very beginning we calculated the red part of the arch as:

$$1 - \frac{\pi}{2} + \text{ segment}$$

Now we have the circular segment with angle θ (in terms of ϕ) as

$$\frac{\pi}{2} - \phi - 2\sin^2\phi$$

So the answer for the red part of the arch is

$$1 - \frac{\pi}{2} + \frac{\pi}{2} - \phi - 2\sin^2\phi$$

$$1 - \phi - 2\sin^2\phi$$

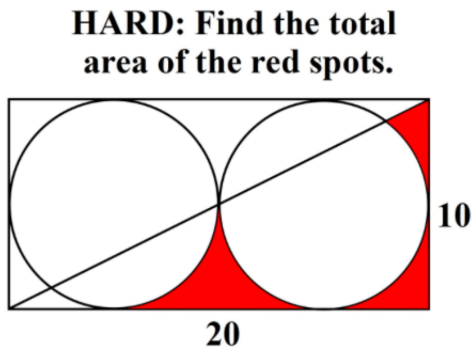
This matches parts one and two. All four methods give the same answer, which is quite a relief.

finishing up

To meet the problem statement, we must add this fractional arch to 3 complete copies, and then multiply the whole thing by the square of the scaling factor (because this is an area):

$$25\ [1 - \phi - 2\sin^2\phi\ \ + 3(1 - \pi/4)\ ]$$

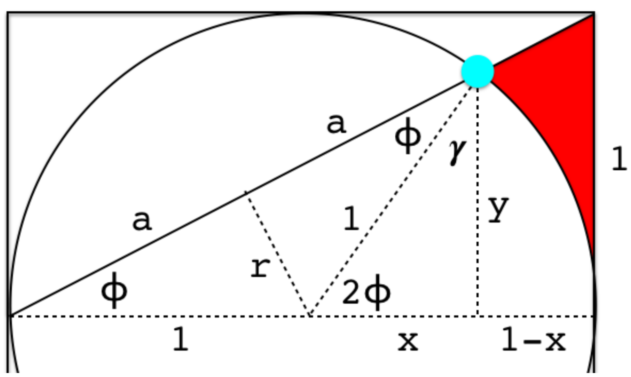
I'm too tired to calculate it.



A fifth way

Later on, I thought of another approach, also pretty simple. Recall the point described in words way back in the section on **the other triangle**.

Here, that point is shown in cyan:



From the diagram we can see that

$$\frac{y}{2a} = \sin\phi$$

$$y = 2a\sin\phi$$

To get x and y in terms of only ϕ we can proceed as follows. A basic result we obtained before was that

$$a = \cos\phi = 2\sin\phi$$

so

$$\begin{aligned} y &= 2a\sin\phi \\ &= 4\sin^2\phi \\ &= \cos^2\phi \\ &= (\frac{2}{\sqrt{5}})^2 = \frac{4}{5} \end{aligned}$$

x looks more complicated but since $x^2+y^2 = 1$, we can at least calculate $x = 3/5$.

Notice that

$$y = \sin 2\phi$$

and by the double-angle formula

$$y = 2\sin\phi\cos\phi = \cos^2\phi$$

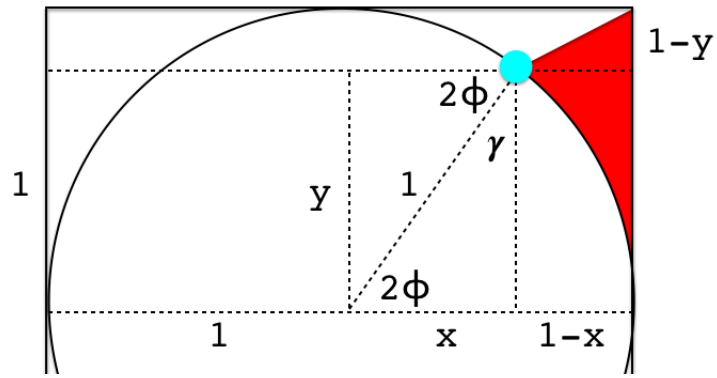
as before.

So for x we have

$$\begin{aligned} x &= \cos 2\phi \\ &= \cos^2\phi - \sin^2\phi \\ &= 2\cos^2\phi - 1 = 2y - 1 \end{aligned}$$

which is confirmed by doing the arithmetic.

Redraw the figure slightly



Notice that part of the red arch is the area to the right of the circle, and the rest is a triangle.

Let $x = f(y) = \sqrt{1 - y^2}$ be the function and calculate the relevant area (the part of the circle in the first quadrant lying below the horizontal line at y) as

$$A = \int_0^y \sqrt{1 - y^2}\ dy$$
$$= \frac{1}{2}\ [\sin^{-1} y + y\ \sqrt{1 - y^2}\ ]\ \Big|_0^y$$

At the lower bound everything is zero and at the upper bound

$$A = \frac{1}{2}(\sin^{-1} y + y\ \sqrt{1 - y^2})$$

From the figure, we can see that $2\phi = \sin^{-1} y$ so

$$A = \frac{1}{2}(2\phi + xy)$$
$$= \phi + \frac{xy}{2}$$

Leave this as it is for the moment.

The area of the red arch below the horizontal line is y minus this.

$$A = y - \phi - \frac{xy}{2}$$

Finally we must add the area of the triangle above. That area is

$$\frac{1}{2}(1-x)(1-y)$$

$$= \frac{1}{2}(1-x-y+xy)$$

Putting it all together

$$A = y - \phi - \frac{xy}{2} + \frac{1}{2}(1-x-y+xy)$$

The xy terms cancel.

$$A = y - \phi + \frac{1}{2}(1-x-y)$$

Recall that $x = 2y - 1$

$$A = y - \phi + \frac{1}{2}(1-2y+1-y)$$

$$= 1 - \phi - \frac{1}{2}y$$

And $y = 4\sin^2\phi$ so

$$A = 1 - \phi - 2\sin^2\phi$$

This matches what we had before.

We obtained the same area five ways (well, at least four ways, two are variants of each other).

Part XXIII

Analysis

Chapter 90

Triangle inequality

Introduction

Here we prove the theorem known as the **triangle inequality**, which is used in proving numerous other theorems in analysis. It involves the concept of absolute value.

Absolute value

The absolute value function is typeset as $|x|$ and defined as follows:

$$f(x) = |x| = \begin{cases} 0, & x = 0 \\ x, & x > 0 \\ -x, & x < 0 \end{cases}$$

The definition can be simplified by combining the first two cases into one statement: $|x| = x$ for $x \geq 0$. But in what follows we will usually consider the three cases separately.

theorem

We will prove that

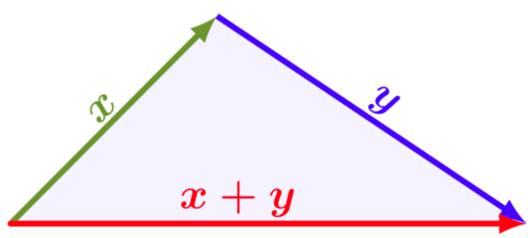
$$|x + y| \leq |x| + |y|$$

This inequality holds regardless of the signs of x and y.

Aside on geometry

The triangle inequality is named after the version from geometry.

When considering the lengths of the sides of a triangle (or the distances between three points in the plane), the lengths of any two sides added together are greater than the length of the third side. Equality is found for the limiting case when the triangle collapses.



Euclid teaches us that the shortest distance between any two points is a straight line (in this figure from the web, $\mathbf{x} + \mathbf{y}$ refers to the *vector sum* of $\mathbf{x}$ and $\mathbf{y}$).

For the real number line, consider any set of three points. The largest distance is equal to the other two added together

$$d(x, y) = d(x, z) + d(z, y)$$

Any other single distance is less than the sum of the remaining two.

For points in R^1 the absolute value gives the distance between them. For vectors in R^2 the absolute value gives the length (which is also the distance between vertices).

One way to prove the theorem

$$|x + y| \leq |x| + |y|$$

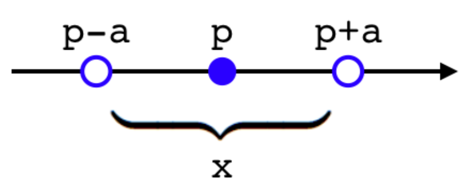
is to go through cases. It turns out that if x and y are both positive or both negative, the above relationship is an equality, equivalent to what we said above about the largest distance is equal to the other two added together.

It is only when the terms have opposite signs that the result is $<$.

For a formal approach (especially helpful for the case of different signs), we follow Apostol. We consider a preliminary theorem first, which is another result frequently used in analysis.

Interval with midpoint

Consider an open interval whose endpoints are equidistant from a central point p, where the distance to the boundary is a (as a *distance*, certainly $a > 0$), and x is contained somewhere in the open interval $(p - a, p + a)$



We can translate this picture to algebra as

$$p - a < x < p + a$$

Adding $-p$ to each term

$$-a < x - p < a$$

The preliminary theorem states that the above is true if and only if

$$|x - p| < a$$

We should read the last statement as saying that the distance between x and p is less than a.

Taking the statements above

$$|x - p| \leq a \iff -a \leq x - p \leq a$$

and rewriting $x - p$ as just x we obtain:

$$|x| \leq a \iff -a \leq x \leq a$$

This is the statement we will prove.

preliminary preliminary theorem

There is actually another theorem that comes before the preliminary theorem.

$$-|x| \leq x \leq |x|$$

proof

We just do this one by cases.

∘ If $x \geq 0$ then $x = |x|$ and $-|x| < 0$ so $x \geq -|x|$.

∘ If $x < 0$, then $|x| = -x$ and $x = -|x|$ and also $|x| \geq 0$ so $x < |x|$.

preliminary theorem

$$|x| \leq a \iff -a \leq x \leq a$$

proof

Let a be a constant real number $a > 0$ and the starting proposition is

$$|x| \leq a$$

Add the expression $-a - |x|$ to both sides to obtain

$$-a \leq -|x|$$

From the preliminary preliminary theorem we obtain

$$-|x| \leq x \leq |x|$$

We combine all three parts:

$$-a \leq -|x| \leq x \leq |x| \leq a$$

which simplifies to

$$-a \leq x \leq a$$

This completes the forward proof.

To prove the converse, we are assuming that

$$x \leq a$$

and

$$-a \leq x$$

Now if $x \geq 0$ we have $|x| = x$ so

$$|x| = x \leq a$$

On the other hand for $x < 0$, we start with the first part

$$-a \leq x$$

Add $a - x$ to both sides

$$-x \leq a$$

and since $x < 0$, $|x| = -x$ and we have

$$|x| = -x \leq a$$

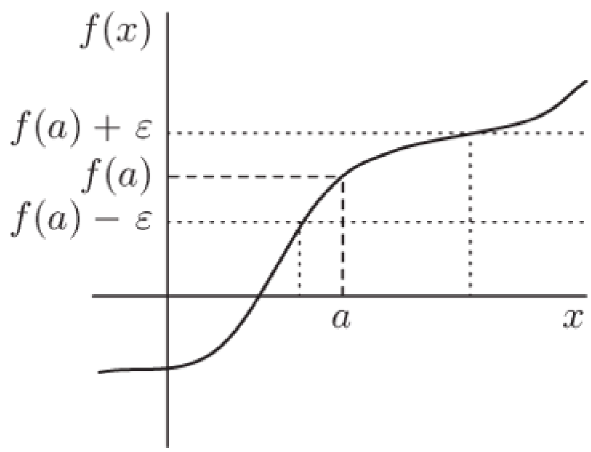
In both cases we have

$$|x| \leq a$$

and this completes the proof.

□

continuity example



As motivation for this whole topic, in the definition of **continuity** of a function $f(x)$ at a point a, we play the epsilon-delta game. Choose an *arbitrary* positive real number $\epsilon > 0$.

Then we say, f is continuous at a if and only if we can find δ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

If the distance (or difference) between x and a is less than δ, then it follows that the difference $|f(x) - f(a)|$ is less than ϵ.

The last statement is exactly the same as

$$-\epsilon < f(x) - f(a) < \epsilon$$

The advantage of this formulation is that there are no absolute value signs so we can rearrange the above to give:

$$f(a) - \epsilon < f(x) < f(a) + \epsilon$$

We will see multiple examples of this in real analysis.

alternative proof (by examining cases)

The statement

$$-a < x - p < a \Rightarrow |x - p| < a$$

may be proved informally by going exhaustively through the cases, as follows.

(1) If $x = p$ then $x - p = 0 < a$ and $0 > -a$;

(2) If $x > p$, then $x - p > 0$ and so

$$|x - p| = x - p$$

Then

$$x - p < a$$

implies

$$|x - p| = x - p < a$$

and since the absolute value is positive it is certainly $> -a$.

(3) If $x < p$ then $x - p < 0$ so

$$|x - p| = -(x - p) = p - x$$

Then the first part of the assumption is

$$-a < x - p$$

Add $a + p - x$ to both sides

$$p - x < a$$

which implies

$$|x - p| = p - x < a$$

and again the absolute value is positive so it is certainly $> -a$.

This proves the theorem.

□

Triangle inequality

Let us change the notation of the previous theorem so we can reuse the variable x in the next part:

$$|\phi| \le a \iff -a \le \phi \le a$$

Now start by considering

$$-|x| \le x \le |x|$$

Now add two such inequalities

$$-\ [\ |x| + |y|\ ]\ \le x + y \le |x| + |y|$$

Recall the theorem we just proved with its new notation and focus on the right-hand side

$$-a \leq \phi \leq a$$

Equate a with $|x| + |y|$ and ϕ with $x + y$.

Then the expression that we had above by adding two inequalities

$$-\ [\ |x| + |y|\ ]\ \leq x + y \leq |x| + |y|$$

can be seen as equivalent to the right-hand side from the theorem.

$$-a \leq \phi \leq a$$

Therefore, the left-hand side must also be true, namely

$$|\phi| \leq a$$

$$|x + y| \leq |x| + |y|$$

This proves the triangle inequality.

□

alternative proof (by examining cases)

Let $m > n > 0$. If $n > m$, just switch x and y. We deal with the case $m = n$ and x, y of different signs in the last part.

• x and y with the same sign

○ $x = m, y = n$

$$|x + y| = |m + n| = m + n$$

$$m + n = |m| + |n| = |x| + |y|$$

○ $x = -m, y = -n$

$$|x + y| = | - m - n| = | - (m + n)| = m + n$$

$$m + n = |-m| + |-n| = |x| + |y|$$

• one of x, y is equal to zero

∘ $x = m, y = 0$

$$|x + y| = |m + 0| = |m| = m$$

$$m = m + 0 = |m| + |0| = |x| + |y|$$

∘ $x = -m, y = 0$

$$|x + y| = |-m + 0| = |-m| = m$$

$$m = m + 0 = |-m| + |0| = |x| + |y|$$

• one of x, y is less than zero

Here we see the inequality. Let $p = m - n$ (we assume $m > n$ so $p > 0$).

$$p = m - n$$

$$p < p + 2n$$

$$p < m + n$$

∘ $x = -m, y = n$

$$|x + y| = |-m + n| = |-p| = p < m + n$$

$$m + n = |-m| + |n| = |x| + |y|$$

so

$$|x + y| < |x| + |y|$$

∘ $x = m, y = -n$

$$|x + y| = |m - n| = |p| = p < m + n$$

$$m + n = |m| + |-n| = |x| + |y|$$

so

$$|x+y| < |x| + |y|$$

∘ Finally, suppose that $m = n$. Then $x = m, y = -m$.

$$|x+y| = |m + -m| = 0 < |m| + |-m| = |x| + |y|$$

□

We next consider a related theorem and some corollaries:

$$|x-y| \geq |x| + |y|$$

$$|xy| = |x|\ |y|$$

$$|x-y| = |y-x|$$

$$|x-y| \leq |x-z| + |z-y|$$

Additional theorems

The first one is trivial: the triangle inequality applies no matter if y is negative. So if we think of $y > 0$ and then add a minus sign to it:

$$|x-y| \geq |x| + |y|$$

multiplication of absolute values

$$|xy| = |x|\ |y|$$

Obviously true for equal signs or one of x, y equal to zero. For different signs $(m, n > 0; \quad x = -m, y = n)$:

$$|xy| = |(-m)n| = |-mn| = mn = |-m|\ |n| = |x|\ |y|$$

□

theorem: change of sign

$$|x - y| = |y - x|$$

easy proof

Regardless of whether $x \geq 0$ or $x < 0$, by the definition of absolute value

$$|x| = |-x|$$

Substitute $x - y$ for x

$$|x - y| = |-(x - y)| = |y - x|$$

first proof

Suppose that $m, n > 0$ and $x = m, y = -n$.

○ $m > n$ so $m - n = p > 0$

$$|x - y| = |m - n| = |p| = p$$

$$|y - x| = |n - m| = |-p| = p$$

so

$$|x - y| = |y - x|$$

Having $n > m$ does not change the argument

○ $n > m$ so $m - n = -p < 0$

$$|x - y| = |m - n| = |-p| = p$$

$$|y - x| = |n - m| = |p| = p$$

theorem: third variable

$$|x - z| \leq |x - y| + |z - y|$$

proof

Start with the triangle inequality

$$|x+y| \leq |x| + |y|$$

Suppose that what is meant is $x, y > 0$. It doesn't matter since the triangle theorem holds regardless of the signs of the two terms.

Replace x by $x - y$ and y by $y - z$

$$|x - y + y - z| \leq |x - y| + |y - z|$$

$$|x - z| \leq |x - y| + |z - y|$$

We can substitute letters if you prefer something different

$$|x - y| \leq |x - z| + |y - z|$$

We can switch the order for any term by the first corollary above. So

$$|x - y| \leq |x - z| + |y - z|$$

goes to

$$|x - y| \leq |x - z| + |z - y|$$

theorem: subtraction

$$|x - y| \geq |x| - |y|$$

proof

$$|x+y| \leq |x| + |y|$$

Replace x by $x - y$

$$|x| \leq |x - y| + |y|$$

$$|x - y| \geq |x| - |y|$$

Summary

$$|x+y| \leq |x| + |y|$$
$$|xy| = |x|\ |y|$$
$$|x-z| \leq |x-y| + |y-z|$$
$$|x-y| \geq |x| - |y|$$

Extension

theorem

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

proof

By induction. Let x be equal to the sum of all the terms before the last one.

$$x = x_1 + x_2 + \cdots + x_{n-1}$$

We may assume that

$$|x| \leq |x_1| + |x_2| + \cdots + |x_{n-1}|$$

By the triangle inequality

$$|x + x_n| \leq |x| + |x_n|$$

But using the assumption we can substitute for $|x|$ and write

$$|x + x_n| \leq |x_1| + |x_2| + \cdots + |x_{n-1}| + |x_n|$$

So

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

□

theorem

$$| \int_0^1 f(x)\ dx \ | \leq \int_0^1 |f(x)|\ dx$$

We won't prove this one, but the analogy with the previous result is pretty clear.

Chapter 91

Archimedean property

This simple idea can be stated in a variety of equivalent forms.

The simplest is that the real numbers are not bounded above in $\mathbb{N}$. No matter how large a real number x that we take, we can always find an integer that is larger.

Formal statements of the theorem all start like this: for any (arbitrary) real number x

$$\forall\ x \in \mathbb{R}$$

we can find a natural number n such that

$$\exists\ n \in \mathbb{N} \mid$$

- $n > x$

proof

By contradiction. Suppose that no integer n exceeds x. Then x is an upper bound for $\mathbb{N}$.

By the completeness axiom (more about this ahead), $\mathbb{N}$ must have a real least upper bound or supremum. Let β be this number.

$\beta - 1$ is not a bound for $\mathbb{N}$ (because β is the least upper bound). So there must be a positive integer $n_0 > \beta - 1$. But then $n_0 + 1$ is $\in \mathbb{N}$ but also $n_0 + 1 > \beta$, so β is not an upper bound for $\mathbb{N}$.

$\square$

An equivalent statement is that for any real a, however small, and any real x, however large, we can find

- $na > x$.

In the immortal words of somebody-or-other: if we have a bathtub full of water and a teaspoon, we can empty the bathtub (given enough time).

If you prefer a small real number. For any real number ϵ, however small, $(\forall\ \epsilon \in \mathbb{R})$

- $\frac{1}{n} < \epsilon$

Beck says:

> Theorem 7.6 (the Archimedean property) essentially says that **infinity is not part of the real numbers**... The Archimedean Property underlies the construction of an infinite decimal expansion for any real number, while the Monotone Sequence Property shows that any such infinite decimal expansion actually converges to a real number.

For example, when we are writing the decimal expansion of $\sqrt{2}$, we must stop somewhere. The Archimedean Property says that regardless of your specification of the difference between the "true value" $\sqrt{2}$ and the value of the truncated expansion, we can find a rational number $\frac{1}{n} < \epsilon$.

The axiom of completeness guarantees that this sequence converges, and we define $\sqrt{2}$ as the limit of the convergent sequence. (Coming below).

Apostol and Stewart

Apostol goes through this development:

• The set $\mathbf{P}$ of positive integers is *unbounded above*. The proof is to assume that P is bounded above. Then there is a largest element n of $\mathbf{P}$ which is less than the bound.

But by definition $n + 1$ is $\in \mathbf{P}$.

• For every real x there exists a positive integer n such that $n > x$. Proof: if this were not so, then x would be an upper bound for $\mathbf{P}$.

Now, simply replace x with y/x:

• For every real y/x there exists a positive integer n such that $n > y/x$. Thus $nx > y$.

Apostol:

> Geometrically it means that any line segment, no matter how long, may be covered by a finite number of line segments of a given positive length, no matter how small. In other words, a small ruler used often enough can measure arbitrarily large distances. Archimedes realized that this was a fundamental property of the straight line and stated it explicitly as one of the axioms of geometry.

Stewart's definition is:

Given a real number $\epsilon > 0$, there exists a positive integer n such that

$$\frac{1}{10^n} < \epsilon$$

This is certainly compatible with the other definitions. If n is an integer than so is 10^n. So ϵ is Apostol's (small) positive length and if we can choose N so that $N\epsilon$ is as large as we please, we can certainly choose it so that $N\epsilon > 1$.

I interpret this as follows: in distinguishing two real numbers a and b (say, by trying to find another number that lies between them), if $a - b = \epsilon$ is the distance between them, we can always find

$$\frac{1}{10^n} < \epsilon$$

and so always find another number (either real or rational) that lies between a and b.

examples

○ $(1, \frac{1}{2}, \frac{1}{3}, \dots)$ converges to 0.

proof

You tell me how close you want the values to get to zero, say, within ϵ of zero. Given ϵ, the Archimedean property guarantees we can find $1/N < \epsilon$. Then for all $n > N$ we have

$$\frac{1}{n} < \frac{1}{N} < \epsilon$$

○ The sequence $(1, 2, 3, \dots)$ is not bounded. Any unbounded sequence fails to converge.

○ The sequence

$$1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \dots$$

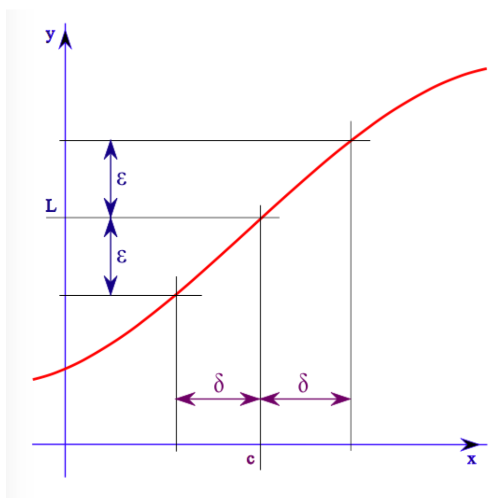
is the sequence of partial sums of the harmonic series, which does not converge.

Chapter 92

Limits and continuity

Limits

Consider the graph of a function $f(x)$. We might choose a power of x similar to $y = x^2$ or $y = x^3 - x$, which affirmatively has two properties that are of interest here: continuity and differentiability.

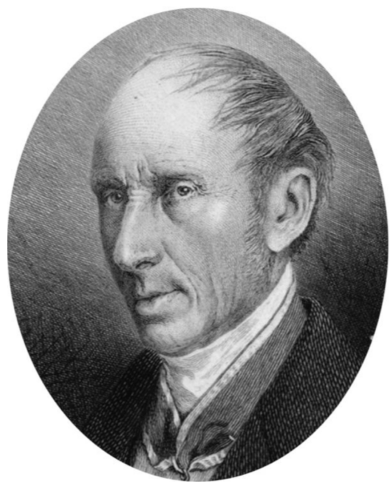


We focus on the neighborhood of a point on the x-axis, $x = c$.

By inspection of the graph we see that the value of $f(x)$ at c is equal to L, and furthermore, for points near c, the value of f at those points is not too different from L.

We would like to say that the *limit* of $f(x)$ as x *approaches* c is equal to L. The idea is that we can make $f(x)$ as close to L as we please, provided we choose x sufficiently close to c.

> When the values successively attributed to a variable approach indefinitely to a fixed value, in a manner so as to end by differing from it by as little as one wishes, this last is called the limit of all the others. —Cauchy



Modern mathematicians don't like that word "approach", which conjures up movement and the involvement of time, and they don't like reasoning from what they see in a graph, in part because no graph can show the whole function for the general case. Instead we will use an algebraic method from the formal apparatus of calculus.

There are two equivalent approaches, neighborhoods, and epsilon-delta formalism. Let's look at neighborhoods briefly first.

neighborhoods

First, an *interval* between two real numbers a and b ($a < b$) contains every real number $a < x < b$.

$$(a, b) = x \mid a < x < b$$

The "$|$" means x "such that" the condition $a < x < b$ holds.

A *closed* interval $[a, b]$ includes the endpoints ($a \le x \le b$), while an *open* interval (a, b) excludes them. Half-open intervals like $[a, b)$ may be defined, and an interval with $\pm\ \infty$ as an endpoint is always open on that end, for example: $[a, \infty)$.

Any open interval with a point p as its midpoint is called a *neighborhood* of p. The distance r from p to the boundary of a particular neighborhood may be large or very very small. We denote a neighborhood of p as $N(p)$.

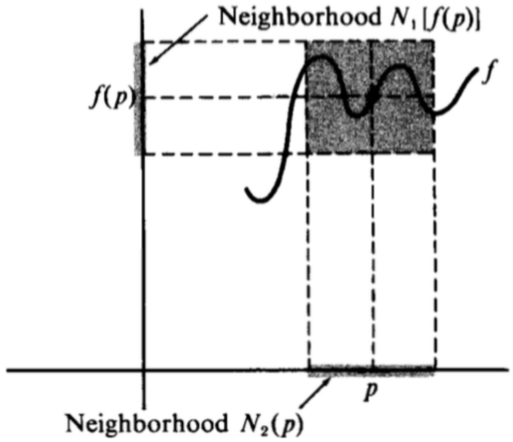
$$N(p) = x \ \text{ such that } \ |x - p| < r$$

.

To say that the limit $f(x) \to L$ exists, we mean that for every neighborhood $N_1(L)$, there exists some neighborhood $N_2(p)$ such that $f(x) \in N_1(L)$ whenever $x \in N_2(p)$.

For a limit, we exclude the point $x = p$. It is not necessary that $f(p) = L$.

The idea of a neighborhood is a nice abstraction to hide the apparatus of modern calculus, which we look at next.



epsilon-delta game

The formal method uses numbers called ϵ and δ and is originally due to Bolzano.

We say that, *if* for all points x within a specified distance δ of c, we find that $f(x)$ lies within a specified distance ϵ from L, *then* the limit is L.

To do this we must choose ϵ first. That's why I call it a game. Why don't you go first? Choose ϵ, which provides a constraint on how close to L you want the value of $f(x)$ to be: you require that $|f(x) - L| < \epsilon$. The *distance* from $f(x)$ to L must be less than ϵ.

Now that I know your ϵ, I must try to find a suitable δ. If I can, then you get another chance, and will presumably choose a smaller ϵ.

If I can show that it is possible to find a δ to guarantee that your constraint is satisfied for *all* values of $\epsilon > 0$ *no matter how small*, then I win and the limit exists. If not, it doesn't.

Here is the formal definition:

$$\forall\ \epsilon > 0, \exists\ \delta > 0 \mid \forall\ x$$

For all (arbitrary) ϵ, there exists $\delta > 0$ such that for all x satisfying

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

We describe the limit defined above by saying that

$$\lim_{x \to c} f(x) = L$$

The limit as x tends to, or approaches, c is equal to L.

Important points about limits:

• We do not require that $f(c) = L$.

The function $f(x)$ may or *may not* have the value L at $x = c$ and the limit can still exist and be equal to L. Suppose we have $f(x) = x$, whose graph is the line $y = x$, except that we decide to define $f(0) = 1$, leaving a hole in our line $y = x$ at the point where $x = 0$. The limit of $f(x)$ at $x = 0$ is equal to 0, despite the fact that $f(0) = 1$.

Alternatively, suppose that we only allow values of x in the open interval (a, b), and the limit $x \to a+$ (from the right) does exist. Since we have restricted the domain of f to values $x > a$ the limit $x \to a-$ certainly does not exist, and in fact the left-hand endpoint a is not in the domain of f.

We say that such a limit $(x \to a+)$ is a *one-sided* limit. If the two one-sided limits do not agree at a particular value of c, then the (two-sided) limit does not exist.

• Limits must be unique.

proof

If

$$\lim_{x \to c} f(x) = L$$

$$\lim_{x \to c} f(x) = M$$

Then $M = L$.

The proof is by contradiction. Suppose $L \neq M$. Let

$$\epsilon = \frac{|L - M|}{10}$$

There is an N_1 such that if $n > N_1$, then $|a_n - L| < \epsilon$.

There is an N_2 such that if $n > N_2$, then $|a_n - M| < \epsilon$.

Let $N = \max(N_1, N_2)$.

If $n > N$ then $|a_n - L| < \epsilon$ and $|a_n - M| < \epsilon$.

By the triangle inequality:

$$|L - M| \leq |a_n - L| + |M - a_n|$$

But also

$$|a_n - L| + |M - a_n| < \frac{2}{10}|L - M|$$

so

$$|L - M| \leq \frac{2}{10}|L - M|$$

which is impossible.

We have reached a contradiction. Therefore, $L = M$.

$\square$

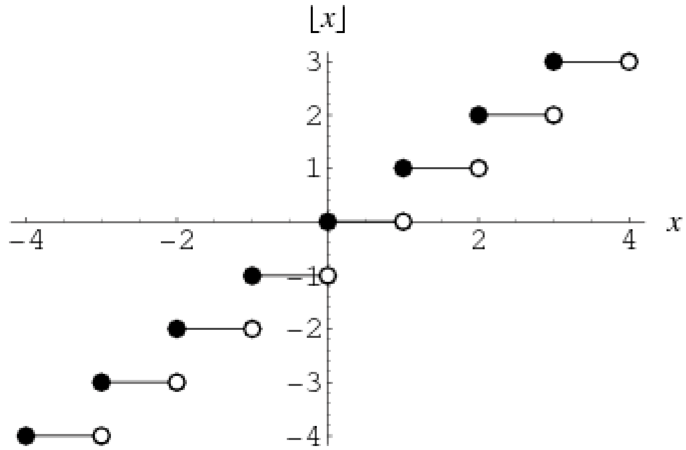
- We allow the existence of a limit as x approaches ∞

$$\lim_{x \to \infty} f(x) = L$$

To define this limit, play the epsilon-delta game (typically, using c instead of δ) and say that if, when $x > c$, $|f(x) - L| < \epsilon$, the limit "at" ∞, or as x *tends to* ∞, exists and has the value L.

example: floor

Consider the "floor" function, which is defined on the real numbers and has the value of the largest integer less than or equal to x.



The floor function has one-sided limits (from the right) at integral values of x, but the limit at $x = 2$, for example, does not exist, because those two one-sided limits are not the same.

example: inverse

Consider the function $f(x) = 1/x$. This function is undefined at $x = 0$ since division by zero is not defined. As x gets close to zero from the right, $1/x$ takes on larger and larger positive values.

Some people will say that limits can have infinite values. In the case of $f(x) = 1/x$, informally, we accept that the limit as $x \to 0+$ exists and has the value ∞. Speaking more formally, we might say that the function "diverges" or "grows without bound".

In any case since for $f(x) = 1/x$

$$\lim_{x \to 0+} \neq \lim_{x \to 0-}$$

so the limit as $x \to 0$ does not exist (abbreviated D.N.E.).

example: sine of 1/x

The trigonometric functions sine and cosine are, of course, periodic. For any value of θ

$$\sin\theta = \sin(\theta \pm 2\pi)$$

The maximum values of the sine function (for $\theta > 0$) occur at

$$\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2} \ldots$$

The corresponding maximum values of $\theta = 1/x$ occur at

$$x = \frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \frac{2}{13\pi} \ldots$$

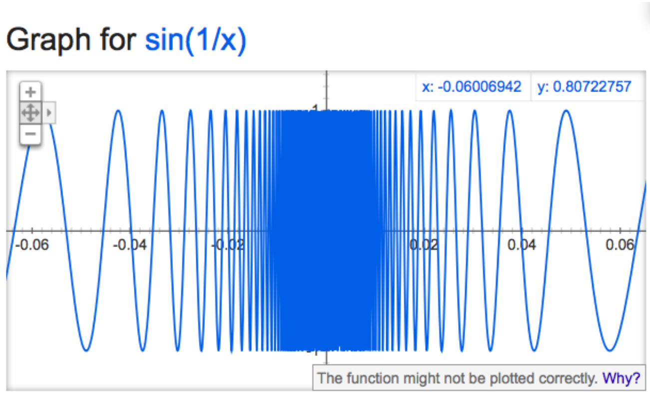
The corresponding decimal values are approximately

$$x = 0.6366, 0.1273, 0.0707, 0.0490$$

As $\pi/2 + 2k\pi$ gets larger, the corresponding values for the inverse get smaller, and more closely spaced together.

Now, $1/x$ grows without bound as $x \to 0$. This means that there is an infinite number of places where the value of the function $\sin(1/x)$ is equal to 1 and indeed, takes on all possible values in its range $[-1, 1]$, and this occurs more and more rapidly as $x \to 0$.

In short, the value oscillates and does so more extremely the closer $x \to 0$.



The limit as $x \Rightarrow 0$ D.N.E.

Calculating limits

The limit of a function $f(x)$ at a point a is written

$$\lim_{x \to a} f(x) = L$$

The formal definition is:

$$\forall \ \epsilon > 0, \exists \ \delta > 0 \mid \forall \ x,$$

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

You tell me the ϵ you require with $|f(x) - L| < \epsilon$, and I will try to find the right δ.

For a typical function, it's a good guess that $L = f(a)$.

$$|f(x) - f(a)| < \epsilon$$

which we can write without the absolute value bars (see Triangle write-up):

$$-\epsilon < f(x) - f(a) < \epsilon$$

example 1

Suppose $f(x) = 3x$ and we're interested in the point $a = 5$. Then set $L = f(a) = 15$.

$$-\epsilon < f(x) - f(a) < \epsilon$$

$$-\epsilon < 3x - 15 < \epsilon$$

$$-\frac{\epsilon}{3} < x - 5 < \frac{\epsilon}{3}$$

If we set $\delta = \epsilon/3$ we'll be good. And in general for a function $f(x) = cx$ with c a constant, at the point a, we can use

$$|x| - a < \frac{\epsilon}{c}$$

example 2

Suppose $f(x) = x^2$ and we're interested in the point $a = 2$. Then set $L = f(a) = a^2 = 4$.

$$-\epsilon < f(x) - f(a) < \epsilon$$

$$-\epsilon < x^2 - a^2 < \epsilon$$

or

$$|x^2 - a^2| < \epsilon$$

Now we argue as follows:

$$|x^2 - a^2| = |(x - a)(x + a)|$$

$$= |x - a|\ |x + a|$$

(The last step follows from $|xy| = |x||y|$ which is true even if $xy < 0$).

To get started suppose we require that *at least*

$$|x - a| < 1$$

$$-1 < x - a < 1$$
$$a - 1 < x < a + 1$$
$$2a - 1 < x + a < 2a + 1$$
$$|x + a| < 2a + 1$$

Then going back to

$$|x - a|\ |x + a| < \epsilon$$
$$|x - a|\ |x + a| < |x - a|\ (2a + 1) < \epsilon$$

and

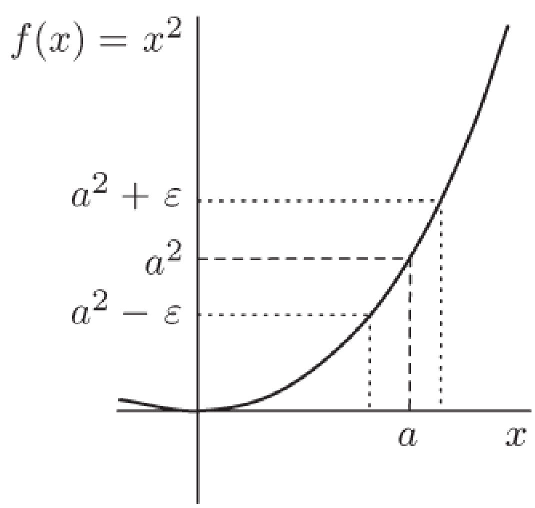
$$|x - a| < \frac{\epsilon}{(2a + 1)}$$

Remembering the first condition we set:

$$|x - a| < \min(\frac{\epsilon}{(2a + 1)}, 1) = \delta$$

And what we notice is that for $f(x) = x^2$, at least for some a (and depending on the value of ϵ that is chosen), the value of δ required depends on a.

That should not be too surprising.



The same ϵ will require a smaller δ the farther out we go on the curve.

example 3

Now consider the inverse function $f(x) = 1/x$. Suppose we're interested in the point $a = 3$ where we expect the limit to be $L = 1/3$. For this to be true we must guarantee that

$$|\frac{1}{x} - \frac{1}{3}| < \epsilon$$

for arbitrary ϵ.

Factor

$$|\frac{1}{x} - \frac{1}{3}| = |\frac{3-x}{3x}| = \frac{1}{3}\ \frac{1}{|x|}\ |3-x|$$

We showed in the write-up on the triangle inequality that $|a - x| = |x - a|$ so

$$= \frac{1}{3}\ \frac{1}{|x|}\ |x-3|$$

Here, we need to make sure that $|x|$ is not too *small*, so $1/|x|$ is not too large.

First require that $|x - 3| < 1$. Then

$$-1 < x - 3 < 1$$

$$2 < x < 4$$
$$\frac{1}{4} < \frac{1}{x} < \frac{1}{2}$$

This means that $1/x > 0$ so

$$\frac{1}{|x|} = \frac{1}{x} < \frac{1}{2}$$

We now have
$$|\frac{1}{x} - \frac{1}{3}| = \frac{1}{3}\ \frac{1}{|x|}\ |3 - x|$$
provided $|x - 3| < 1$ and also with this condition $1/|x| < 1/2$ so
$$|\frac{1}{x} - \frac{1}{3}| < \frac{1}{6}\ |x - 3|$$
Hence if $\delta = |x - 3| < 6\epsilon$, the above expression is $< \epsilon$ and we're done. Officially we need:
$$|x - 3| < \min\ (6\epsilon, 1)$$

Combining Limits

Assume that
$$\lim_{x \to c} f(x) = L$$
$$\lim_{x \to c} g(x) = M$$

We want to show that
$$\lim_{x \to c} f(x) + g(x) = L + M$$

The limit of the sum is the sum of the limits.

Let $\epsilon > 0$ be arbitrary.

Then the existence of the limits means that
$$\forall\ \epsilon, \exists\ \delta_1 > 0 \mid \forall\ x,\ 0 < |x - c| < \delta_1 \to |f(x) - L| < \epsilon/2$$

and
$$\forall\ \epsilon, \exists\ \delta_2 > 0 \mid \forall\ x,\ 0 < |x - c| < \delta_2 \to |g(x) - M| < \epsilon/2$$

Let
$$\delta = \ \min\ (\delta_1, \delta_2)$$

Now for $|x - c| < \delta$:

$$|f(x) - L + g(x) - M| < \epsilon$$

But by the triangle inequality the left-hand side is

$$|f(x) - L| + |g(x) - M| \leq |f(x) - L + g(x) - M|$$

so

$$|f(x) - L| + |g(x) - M| < \epsilon$$

which proves the theorem.

proof of the product rule for limits

Assume that

$$\lim_{x \to c} f(x) = L$$
$$\lim_{x \to c} g(x) = M$$

We want to show that

$$\lim_{x \to c} f(x) \cdot g(x) = LM$$

The limit of the product is the product of the limits.

We need to show that

$$f(x) \cdot g(x) - LM$$

is small.

Subtract $Lg(x)$ and add it back

$$f(x) \cdot g(x) - LM = f(x) \cdot g(x) - Lg(x) + Lg(x) - LM$$
$$= (f(x) - L)g(x) + L(g(x) - M)$$

Take the absolute value on both sides

$$|f(x) \cdot g(x) - LM| = |(f(x) - L) \cdot g(x) + L \cdot (g(x) - M)|$$

Use the triangle inequality to split up the sum:

$$\leq |(f(x) - L) \cdot g(x)| + |L \cdot (g(x) - M)|$$

This can be further massaged to

$$= |f(x) - L| \cdot |g(x)| + |L| \cdot |g(x) - M|$$

Write the whole thing:

$$|f(x) \cdot g(x) - LM| \leq |(f(x) - L)||g(x)| + |L||(g(x) - M)|$$

Now, play the epsilon-delta game: you pick ϵ and then I concentrate on a region so close to c that

$$|f(x) - L| < \epsilon$$

and

$$|g(x) - M| < \epsilon$$

If your epsilon is too large it would mess things up (why?), so in that case I will pick $|g(x) - M| = 1$.

Then I have

$$|f(x) - L| < \epsilon$$

$$|g(x) - M| < \epsilon$$

$$|g(x)| < |M| + 1$$

Go back to the equation we obtained above

$$|f(x) \cdot g(x) - LM| \leq |(f(x) - L)||g(x)| + |L||(g(x) - M)|$$

substitute on the right-hand side

$$
\begin{aligned}
&|(f(x) - L)||g(x)| + |L||(g(x) - M)| \\
&\leq \epsilon\ (|M| + 1) + |L|\ \epsilon \\
&\leq \epsilon\ (|M| + |L| + 1)
\end{aligned}
$$

That is:

$$|f(x) \cdot g(x) - LM| \leq \epsilon\ (|M| + |L| + 1)$$

as Adrian Banner says in *Calculus Lifesaver*:

> That's almost what I want! I was supposed to get ϵ on the right-hand side, but I got an extra factor of $|M|+|L|+1$. This is no problem—you just have to allow me to make my move again, but this time I'll make sure that $|f(x) - L|$ is no more than $\epsilon/2(|M| + |L| + 1)$, and similarly for $|g(x) - M|$. Then when I replay all the steps, ϵ will be replaced by $\epsilon/(|M| + |L| + 1)$, and at the very last step, the factor $|M| + |L| + 1$ will cancel out and we'll just get our ϵ. So we have proved the result.

More formal proof of the product rule

Suppose that

$$\lim_{x \to c} f(x) = L$$

$$\lim_{x \to c} g(x) = M$$

To prove:

$$\lim_{x \to c} f(x) \cdot g(x) = LM$$

proof

Let $\epsilon > 0$. By the definition of limits we can find three numbers δ_1, δ_2 and δ_3 such that

if $0 < |x - c| < \delta_1$:

$$|f(x) - L| < \frac{\epsilon}{2(1 + |M|)}$$

if $0 < |x - c| < \delta_2$:

$$|g(x) - M| < \frac{\epsilon}{2(1 + |L|)}$$

and third, if $0 < |x - c| < \delta_3$:

$$|g(x) - M| < 1$$

Now write

$$|g(x)| = |g(x) - M + M|$$

use the triangle inequality

$$|g(x)| \leq |g(x) - M| + |M|$$

Then according to (3), if if $0 < |x - c| < \delta_3$:

$$|g(x) \leq 1 + |M|$$

Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.

Then if $0 < |x - c| < \delta$:

$$|f(x) \cdot g(x) - LM| = |f(x) \cdot g(x) - L \cdot g(x) + L \cdot g(x) - LM|$$

by the triangle inequality (again)

$$\leq |f(x) \cdot g(x) - L \cdot g(x)| + |L \cdot g(x) - LM|$$

The next step is to factor (see below):

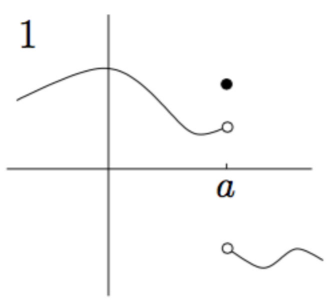
$$
\begin{aligned}
&\leq |g(x)||f(x) - L| + |L||g(x) - M| \\
&< (1 + |M|)\,\frac{\epsilon}{2(1 + |M|)} + (1 + |L|)\,\frac{\epsilon}{2(1 + |L|)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
|f(x) \cdot g(x) - LM| &< \epsilon
\end{aligned}
$$

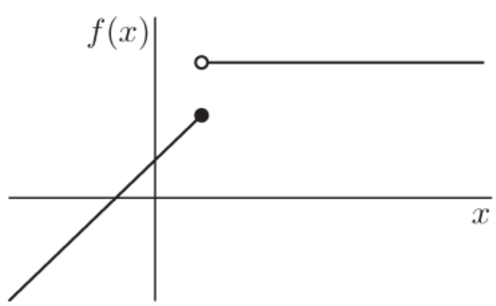
which completes the proof.

Continuity

Continuity has an intuitive definition: if we can graph a function *without lifting our pencil from the paper*, then the function is continuous.

Here are some graphs showing examples of how continuity can fail.





$$f(x) = \begin{cases} x + 1 & \text{if } x \leq 1 \\ 3 & \text{if } x > 1 \end{cases}$$

For a function to be continuous at a point $x = c$, we imagine that if we vary x in neighborhood of c, then $f(x)$ should not change in value by too much.

Again, we will call that value L, the limit of $f(x)$ as $x \to c$. For L to exist we require that the two one-sided limits be equal.

In addition, it must also be true that $f(c) = L$.

fancy definitions

If we had not previously developed the concept of a limit, we might proceed as follows: a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $c \in \mathbb{R}$ if and only if

$$\forall \, \epsilon > 0 \; \exists \, \delta > 0 \;\text{ such that, if } |x - c| < \delta, \text{ then } |f(x) - f(c)| < \epsilon$$

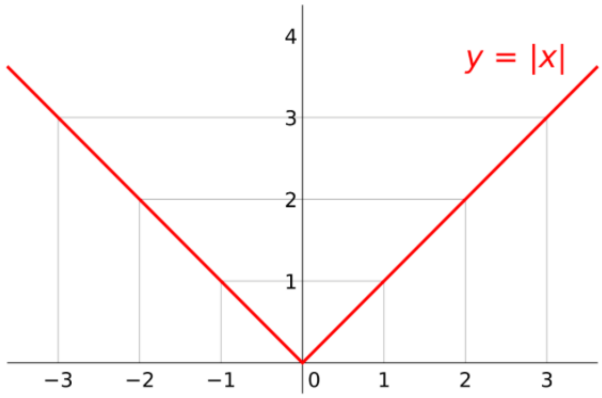
Above we talked about functions, here is a definition that involves sequences.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $L \in \mathbb{R}$. We say that f is continuous at L, if, whenever (a_n) is a sequence that converges to L, then the sequence (b_n) defined by $f(a_n)$ also converges and its limit is equal to $f(L)$. We say that f is continuous if it is continuous at all $L \in \mathbb{R}$.

example: absolute value

An algebraic definition of the absolute value function is piecewise:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



The function $f(x) = |x|$ is continuous at $x = 0$ because the two one-sided limits exist and are equal to each other. They are also equal to $f(0) = 0$.

example: constant

Suppose $f(x) = a$ for some real number a. Then no matter what ϵ is chosen and no matter what real number c is chosen

$$f(x) = f(c) = a$$

so

$$|f(x) - f(c)| < \epsilon$$

example: x

Suppose $f(x) = x$.

$$\lim_{x \to c} f(x) = c = f(c)$$

so f is continuous at c.

Or choose $\delta = \epsilon$. Then, if $|x - c| < \delta$

$$c - \delta < x < c + \delta$$

$$f(x) = x$$
$$f(x) - c < \delta = \epsilon$$
$$|f(x) - c| < \epsilon$$

so f is continuous at c

example: constant factor

Suppose $f(x) = cx$ ($c \in \mathbb{R}$). Use the $\epsilon - \delta$ game to prove that f is continuous.

Proof: the function "stretches" x by a factor of c. Hence δ will also be stretched. Set $\delta = \epsilon/c$. Then, if $|x - a| < \delta$ we have

$$|f(x) - f(a)| = |cx - ca| = c|x - a| < c\delta = c\epsilon/c = \epsilon$$

Hence f is continuous at every $a \in \mathbb{R}$.

example: product rule

How to prove that $f(x) = x^2$ is continuous? One way is to try adjusting δ based on the value of a (e.g. min $(|\sqrt{a^2 + \epsilon} \pm a|)$), but a better way is to invoke the product rule.

If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both continuous at $a \in \mathbb{R}$, then fg is continuous at a.

First, prove $f(x) = x$ is continuous. Then define $f(x) = g(x) = x$. So $x^2 = f(x)g(x)$ in continuous.

By induction then, all powers $f(x) = x^n$ are continuous.

proof of the product rule for continuity

Let f and g be functions defined on an open subset of $\mathbb{R}$. We have that f and g are both continuous at c which means that

$$\lim_{x \to c} f(x) = L$$

$$\lim_{x \to c} g(x) = M$$

Then

$$\lim_{x \to c} f(x) \cdot g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = LM$$

To obtain this result we have used the product rule for limits.

example: inverse

Consider the function $f(x) = 1/x$. Above we pointed out that this function is undefined at $x = 0$ since division by zero is not defined. But there is nothing to stop us from defining the function piecewise, like so:

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

As x gets close to zero from the right, $1/x$ continues to take on larger and larger positive values, but then dives to 0 at $x = 0$ and then further dives toward $-\infty$ as we pass to the left of zero.

Trick question: give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is not continuous at zero. If you said the inverse, that is not correct. The reason is that the inverse is not $f : \mathbb{R} \to \mathbb{R}$ because *it is not defined at zero.*

On the other hand, $f(x) = \sin(x)$ *is* $f : \mathbb{R} \to \mathbb{R}$ even though the values actually output by the function are $f(x) \in [-1, 1]$. That interval is the *image* of the function, any codomain so that the image is contained in the codomain will do.

Chapter 93

FTC proof

David Joyce has proofs of the two statements of the FTC

http://aleph0.clarku.edu/~ma120/FTCproof.pdf

which we will follow here.

The two statements are often called FTC-1 and FTC-2. Citing historical precedent, Joyce calls the second one the FTC and the first its inverse, or FTC^{-1}.

FTC

The FTC is what we use when we evaluate definite integrals. If F is an antiderivative of f, then:

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

We will require that f be *continuous* on $[a, b]$. Strictly speaking, this isn't necessary, but it makes the proof simpler. For a function with a finite number of discontinuities, one can just chop up the integral into its component pieces.

For the inverse statement (FTC^{-1}), we require again that f be continuous on $[a, b]$ and F be the accumulation function defined by

$$F(x) = \int_a^x f(t)\ dt$$

Then the theorem is that F is differentiable on $[a, b]$ and its derivative is f. That is

$$F'(x) = f(x) \quad \text{for } x \in [a, b]$$

This is usually written

$$\frac{d}{dx} \int_a^x f(t)\ dt = f(x)$$

We have adopted the "dummy" variable t to avoid confusion.

proof of the inverse FTC

We start with the inverse theorem. First of all, since f is continuous, it is integrable, so we know that the integral

$$F(x) = \int_a^x f(t)\ dt$$

actually exists.

We need to show that $F'(x) = f(x)$.

We go back to the definition of the derivative

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} [\ F(x+h) - F(x)\ ]$$

$$= \lim_{h \to 0} \frac{1}{h} [\ \int_a^{x+h} f(t)\ dt) - \int_a^x f(t)\ dt\ ]$$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\ dt$$

We will show that this limit equals $f(x)$ We will only prove the case where $h > 0$. The other proof is similar but has minus signs in various places.

On the interval $[x, x+h]$, $f(t)$ has a minimum value m and a maximum value M (by the extreme value theorem). So

$$m \leq f(t) \leq M$$

for every $x \in [a, b]$, and when we integrate each term of the inequality we get

$$\int_x^{x+h} m\ dt \leq \int_x^{x+h} f(t)\ dt \leq \int_x^{x+h} M\ dt$$

Since m and M are constants and $\int dt = h$ between these limits:

$$hm \leq \int_x^{x+h} f(t)\ dt \leq hM$$

dividing through by h

$$m \leq \frac{1}{h} \int_x^{x+h} f(t)\ dt \leq M$$

Now, as $h \to 0$, all values of f on the interval $[x, x + h]$ approach the same value, and in particular, $m \to f(x)$ and $M \to f(x)$. Being squeezed between them

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) = f(x)$$

□

proof of the FTC

Let

$$G(x) = \int_a^x F'(t)\ dt$$

Take derivatives on both sides

$$G'(x) = \frac{d}{dx}\ \int_a^x F'(t)\ dt$$

so

$$G'(x) = F'(x)$$

by the theorem we just proved.

Therefore $G(x)$ and $F(x)$ differ at most by a constant

$$G(x) = F(x) + C$$

for $x \in [a, b]$.

In particular, at $x = a$ we have

$$G(a) = F(a) + C$$

but

$$G(a) = \int_a^a F'(t)\ dt = 0$$

Hence

$$F(a) = -C$$

At $x = b$ we have

$$G(b) = F(b) + C$$

but $C = -F(a)$ so

$$G(b) = F(b) - F(a)$$

By the original definition of G

$$G(b) = \int_a^b F'(t)\ dt$$

Hence

$$\int_a^b F'(t)\ dt = F(b) - F(a)$$

$\square$

Chapter 94

Courant Riemann

intervals of unequal width

Courant and John describe a variation on Riemann sums using intervals of unequal (but graduated) width. This "trick" allows them to derive the formula for

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\int_a^b x^n \, dx = \frac{b^{n+1} - a^{n+1}}{n+1}$$

for all natural numbers n first, and then with some elaborations, for real n except $n = -1$.

We subdivide the interval $[a, b]$ by points with spacing that increases by a factor of q at each step

$$a, \; aq, \; aq^2, \; \ldots \; aq^{n-1}, \; aq^n$$

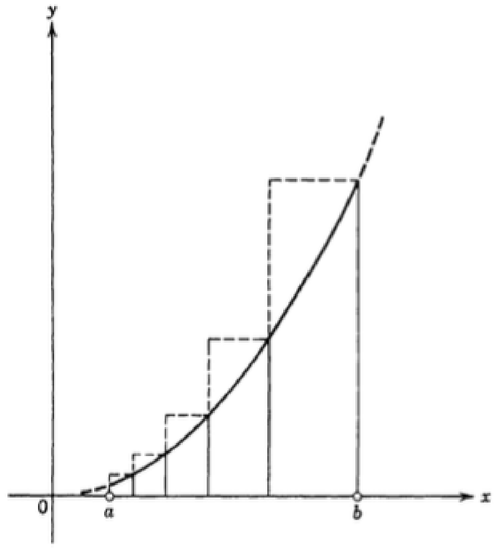


Figure 2.13 Area under a parabolic arc by geometric subdivision.

At the final, nth step we have

$$aq^n = b$$

Solving for the common ratio q we have

$$q = (b/a)^{1/n}$$

The points of division are

$$x_i = aq^i$$

The width of the ith rectangle is

$$\Delta x_i = aq^i - aq^{i-1} = aq^i \ (1 - \frac{1}{q})$$

$$= aq^i \ [\ \frac{q-1}{q} \ ]$$

The widest rectangle is the last one

$$\Delta x_n = aq^n \ [\ \frac{q-1}{q} \ ]$$

$$= b \; [\; \frac{q-1}{q} \;]$$

In the usual way, we will let the number of rectangles $n \to \infty$.

At the same time, since

$$q = (b/a)^{1/n}$$

then $q \to 1$.

So then $\Delta x_n \to 0$, and so do all the other rectangles, which are smaller.

The function of interest is to raise x to the positive integer power p. For each rectangle, the area is

$$A_i = x_i^p \; \Delta x_i$$

$$= (aq^i)^p \; aq^i \; [\; \frac{q-1}{q} \;]$$

$$= a^{p+1} \; (q^i)^{p+1} \; [\; \frac{q-1}{q} \;]$$

$$= a^{p+1} \; (q^{p+1})^i \; [\; \frac{q-1}{q} \;]$$

For the integral, we need to add all these up (from $i = 1$ to $i = n$):

$$I = \sum_{i=1}^{n} a^{p+1} \; (q^{p+1})^i \; [\; \frac{q-1}{q} \;]$$

We can take out values that don't depend on i from the summation:

$$I = a^{p+1} \; [\; \frac{q-1}{q} \;] \; \sum_{i=1}^{n} (q^{p+1})^i$$

Recall that for a geometric series with common ratio r the nth sum (starting from $i = 0$) is

$$S_n = 1 + r + r^2 \cdots + r^n = \sum_{i=0}^{n} r^i$$

$$= \frac{1 - r^n}{1 - r} = \frac{r^n - 1}{r - 1}$$

Substituting q for r:

$$S_n = \frac{q^n - 1}{q - 1}$$

For the expression above

$$\sum_{i=1}^{n} (q^{p+1})^i$$

we factor out one q^{p+1} so as to start from $i = 0$

$$= q^{p+1} \sum_{i=0}^{n} (q^{p+1})^i$$

and then the common ratio is q^{p+1} and the sum is

$$\sum_{i=0}^{n} (q^{p+1})^i = \frac{(q^{p+1})^n - 1}{q^{p+1} - 1} = \frac{q^{n(p+1)} - 1}{q^{p+1} - 1}$$

The whole sum or integral I that we seek is

$$I = a^{p+1} \left[\frac{q-1}{q} \right] q^{p+1} \frac{q^{n(p+1)} - 1}{q^{p+1} - 1}$$

$$= a^{p+1} (q-1) q^p \frac{q^{n(p+1)} - 1}{q^{p+1} - 1}$$

$$= a^{p+1} (q-1) q^p \frac{(b/a)^{p+1} - 1}{q^{p+1} - 1}$$

Since

$$a^{p+1} \left[(b/a)^{p+1} - 1 \right] = b^{p+1} - a^{p+1}$$

we obtain

$$I = \left[b^{p+1} - a^{p+1} \right] q^p \frac{q - 1}{q^{p+1} - 1}$$

Referring to the sum for a geometric progression again, we have from above

$$S_n = \frac{q^n - 1}{q - 1}$$

So (for $q \neq 1$) and $n = p + 1$, the inverse of that is what we have for the right-hand term

$$\frac{q - 1}{q^{p+1} - 1} = \frac{1}{S_{p+1}}$$

where

$$S_{p+1} = 1 + q + q^2 + \cdots + q^{p+1}$$

Substituting

$$I = \;[\; b^{p+1} - a^{p+1} \;]\; q^p \; \frac{1}{1 + q + q^2 + \cdots + q^{p+1}}$$

As we saw near the beginning, as $n \to \infty$, $q \to 1$, and so do all the powers of q so the term

$$q^p = 1$$

and also

$$1 + q + q^2 + \cdots + q^{p+1} = p + 1$$

so the fraction is just equal to $1/(p + 1)$ and we have finally:

$$I = \;[\; b^{p+1} - a^{p+1} \;]\; \frac{1}{p + 1}$$

which is what we sought to prove.

If you look carefully at this proof, you'll see that it follows Fermat exactly, but is more general by including a as the lower bound.

Part XXIV

Addendum

Chapter 95

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