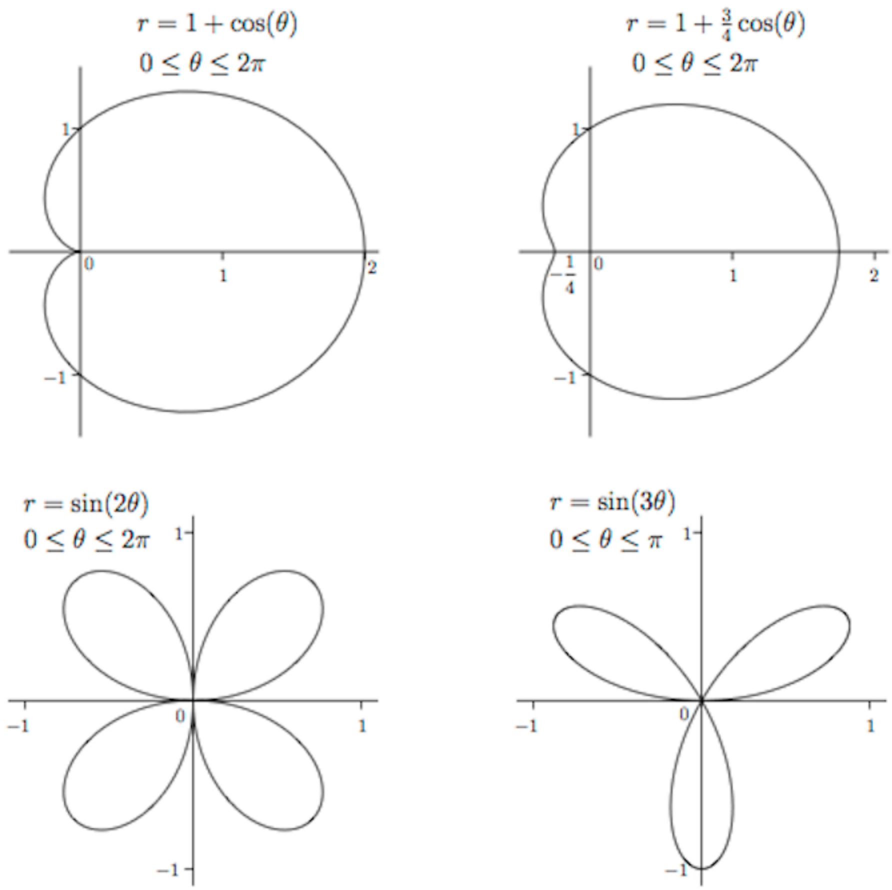
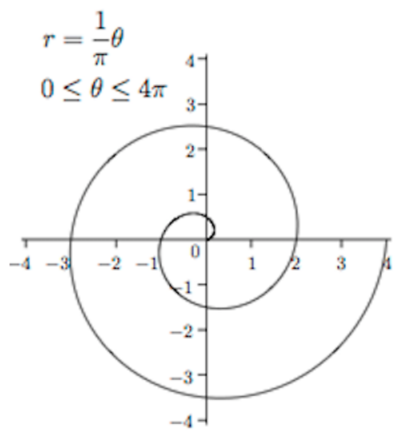
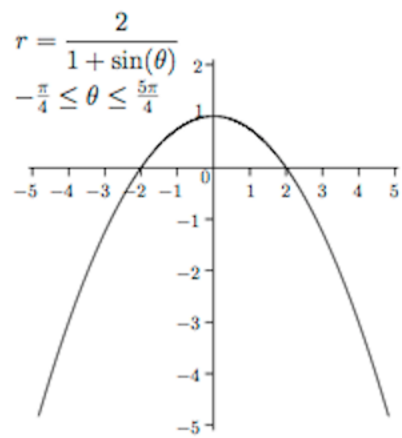
Polar area

Here are some fancy examples of polar curves from *The Calculus Lifesaver*.







Integration to find areas

The idea for (one-dimensional) integration in polar coordinates is that we know r as a function of θ. For example, we had the circle centered at $(0, 3/2)$ given by

$$r = 3\sin\theta$$

We imagine dividing up the circle into little triangles, sectors where

$$\theta \to \theta + \Delta\theta$$

The sector is approximately a triangle with side r and base $r \times \Delta\theta$ (the latter is the length of the arc of the circle on its circumference).

The area of each little sector is

$$\frac{1}{2}r^2 d\theta$$

example

For this problem:

$$r = 3\sin\theta$$

The total area is then

$$\int_0^{2\pi} \frac{1}{2} r^2 d\theta$$
$$= \int_0^{2\pi} \frac{1}{2}(3\sin\theta)^2 d\theta$$
$$= \frac{9}{2}\int_0^{2\pi} \sin^2\theta d\theta$$

This looks hard but we've done it before. One way is to recall that

$$[\ \sin\theta\cos\theta\ ]' = -\sin^2\theta + \cos^2\theta$$
$$= 1 - 2\sin^2\theta$$

Integrate

$$\int [\ \sin\theta\cos\theta\ ]'\ d\theta = \int (1 - 2\sin^2\theta)\ d\theta$$
$$\sin\theta\cos\theta = \theta - 2\int \sin^2\theta\ d\theta$$

Hence

$$\int \sin^2\theta\ d\theta = \frac{1}{2}(\theta - \sin\theta\cos\theta)$$

So our answer is

$$= (\frac{9}{2})\ \frac{1}{2}(x - \sin\theta\cos\theta)\ \Big|_0^{2\pi}$$

$$= (\frac{9}{2})\ \frac{1}{2}(2\pi)$$
$$= \frac{9\pi}{4}$$

which is correct for a circle with radius 3/2.

example

The second example is from *How to ace the rest of calculus*. We have two circles, both of radius 1. The first one is centered at the origin. We are given the equation of the second in polar coordinates:

$$r = 2\cos\theta$$

Plugging in some values for θ: and calculating r:

$$\theta = 0 \to r = 2$$
$$\theta = \frac{\pi}{6} \to r = \frac{\sqrt{3}}{2}$$
$$\theta = \frac{\pi}{4} \to r = \sqrt{2}$$
$$\theta = \frac{\pi}{3} \to r = 1$$
$$\theta = \frac{\pi}{2} \to r = 0$$

We can also convert to x, y-coordinates. Multiply by r:

$$r^2 = 2r\cos\theta$$

Substituting $r^2 = x^2 + y^2$ and $x = r\cos\theta$:

$$x^2 + y^2 = 2x$$

Complete the square:

$$(x^2 - 2x + 1) + y^2 = 1$$

$$(x-1)^2 + y^2 = 1$$

The second circle is centered at $(1, 0)$. Note that for this circle it is *not* true that $x^2 + y^2 = 1$.

Now, the problem given is to calculate the shaded area in the figure.

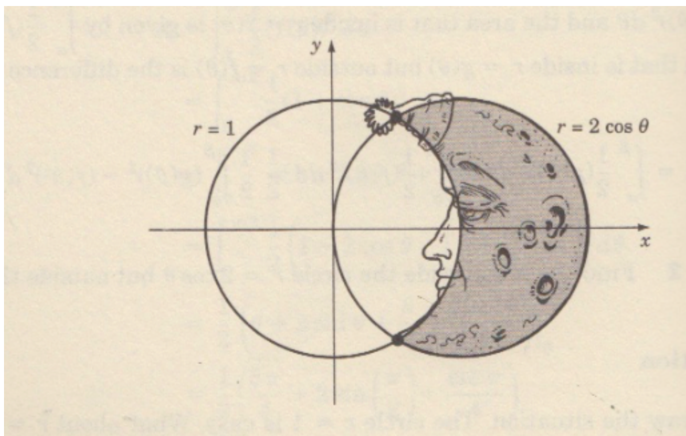


Figure 3.13 Area inside $r = 2\cos\theta$ but outside $r = 1$.

First, we must find the value of θ at the points of intersection between the two circles. We solve the two equations simultaneously:

$$y^2 = 1 - x^2$$

$$(x-1)^2 + y^2 = (x-1)^2 + 1 - x^2 = 1$$

$$-2x + 2 = 1$$

$$2x = 1$$

$$x = \frac{1}{2}$$

$$y = \sqrt{1 - x^2} = \pm\frac{\sqrt{3}}{2}$$

$$\theta = \tan^{-1}\frac{y}{x} = \pm\sqrt{3}$$

Look it up:

$$\theta = \pm\frac{\pi}{3}$$

or notice that we are on the unit circle so $\cos\theta = x = 1/2$, $\theta = \pm\pi/3$. That's the hard way. The easy way is

$$r = 1 = 2\cos\theta$$

$$\theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$$

The area of an arc of the unit circle is the r^2 times one-half the arc length in radians.

$$A = \frac{1}{2}\int r^2 \ d\theta$$

We will subtract the area of the inner arc from that covered by the outer one

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(2\cos\theta)^2 - 1 \ d\theta$$

Recall that

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - 1 + \cos^2\theta$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

so

$$(2\cos\theta)^2 = 4\ \frac{1}{2}(1+\cos 2\theta) = 2(1+\cos 2\theta)$$

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(2\cos\theta)^2 - 1 \ d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 2\cos 2\theta + 1 \ d\theta$$

$$= \frac{1}{2} \left[\sin 2\theta + \theta \right] \Big|_{-\pi/3}^{\pi/3}$$

Since $\sin 2\pi/3 = \sqrt{3}/2$:

$$= \frac{1}{2}(\sqrt{3} + \frac{2\pi}{3}) = \frac{\sqrt{3}}{2} + \frac{\pi}{3}$$