Complex functions

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Contents

Part I

Introduction

Chapter 1

Preface

One motivation for learning about complex functions is that the theory is often described as being very beautiful. It also shows how certain more difficult integrals can be solved.

Marsden gives three examples that he says are either very difficult or impossible if we are restricted to just the real numbers:

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\, dx = \frac{\pi}{\sin \alpha\pi}$$

$$\int_0^{2\pi} \frac{1}{a+\sin\theta}\, d\theta = \frac{2\pi}{\sqrt{a^2-1}}$$

Here's another from Nahin.

$$\int_{-\infty}^\infty \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{e}$$

Maybe we can learn to solve these before we're done.

Chapter 2

Arithmetic

Consider the functions

$$x^2 + 1 = 0$$

and

$$x^2 + x + 1 = 0$$

For the first equation, it is easy to see that there is no solution among the real numbers since x^2 is always positive or zero. So adding 1 to x^2 cannot bring the sum back to zero.

Visualizing the same function geometrically, this is just the simple parabola $y = x^2$ shifted up by one unit, moving its vertex from $(0, 0)$ to $(0, 1)$. Plotting shows that the graphs of both the above functions never cross the x-axis—there are no values that lie on the curve and also on the line $y = 0$.

It is often said that complex numbers arose in the context of finding solutions to such polynomials, however, as Nahin writes in his book *An imaginary tale*, this is not really true. We'll explore some of this history in the next chapter.

The ingenious solution to this problem was to invent a new kind of number

$$i = \sqrt{-1}$$
$$i^2 = -1$$

Once we accept that $i = \sqrt{-1}$ then we can factor

$$(x + i)(x - i) = x^2 - i^2$$
$$= x^2 - (-1) = x^2 + 1$$

so $x = \pm\, i$ are both solutions to the equation

$$(x + i)(x - i) = 0$$

For the second one

$$x^2 + x + 1 = 0$$

we can plot it, or we can recall the quadratic formula for solutions to

$$ax^2 + bx + c = 0$$

for real constants a, b and c. The formula is

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

When $4ac > b^2$, then the solutions to the quadratic formula involve the square root of a negative number. Here the formula gives

$$x = \frac{1}{2}(-1 \pm \sqrt{-3})$$

Take the positive root and square it

$$x^2 = \frac{1}{4}(-1 + \sqrt{-3})^2$$

$$= \frac{1}{4}(-2 - 2\sqrt{-3})$$
$$= \frac{1}{2}(-1 - \sqrt{-3})$$

Adding this to $x + 1$ we obtain

$$x^2 + x + 1$$
$$= \frac{1}{2}(-1 - \sqrt{-3}) + \frac{1}{2}(-1 + \sqrt{-3}) + 1$$

the terms with $\sqrt{-3}$ cancel, giving

$$= -\frac{1}{2} - \frac{1}{2} + 1 = 0$$

In fact, now that we have i available, any square root like $\sqrt{-(a^2)}$, where a is a real number, can be factored as $\sqrt{-1}\ \sqrt{a^2} = ia$.

warning

Note that the converse is not necessarily true. Consider

$$i^2 = \sqrt{-1} \cdot \sqrt{-1} \stackrel{?}{=} \sqrt{(-1) \cdot (-1)} = \sqrt{1}$$

Now, $\sqrt{1}$ has two solutions or roots (since -1×-1 and 1×1 are both equal to 1), but we choose the positive root when thinking about $\sqrt{x}$ as a *function*. However, i^2 was defined to be equal to -1, not 1. What's the deal?

The problem is that the equality with a question mark is not valid

$$\sqrt{-1} \cdot \sqrt{-1} \neq \sqrt{(-1) \cdot (-1)}$$

which explains why this "proof" is erroneous.

Expressions that involve the square root of a negative real number, like $\sqrt{-1} = i$ and $\sqrt{-3} = \sqrt{3}\, i$, are called imaginary (or *purely* imaginary).

Numbers that contain both a real and an imaginary part, like $1 + i$, are termed complex numbers, and imaginary numbers are considered to be complex numbers with the real part equal to 0.

The set of complex numbers $\mathbb{C}$ includes the real numbers:

$$\mathbb{R} \subset \mathbb{C}$$

or, as is written, that $\mathbb{R}$ is a subset of $\mathbb{C}$.

We write complex numbers z as combinations like

$$z = a + ib$$

where a and b are both real numbers. a is the real part, and b the imaginary part of the complex number z.

Two useful identities come from factoring $i^2 = -1$:

$$i = -\frac{1}{i}$$

$$-i = \frac{1}{i}$$

It turns out that for much of what is done with complex numbers the fact that i equals $\sqrt{-1}$ is not even relevant.

Instead, we simply think of *ordered pairs* of real numbers (a, b) and the i notation is a bookkeeping device, a marker to remind us that when we multiply two complex numbers

$$(a + ib)(c + id) = ac + iad + ibc + i^2 bd$$

the last term gets a minus sign:

$$ib \cdot id = -bd$$

The result of multiplying $ib \cdot id$ is a real number with the sign flipped, while a real number a times an imaginary number id is equal to iad and

$$(a + ib)(c + id) = ac - bd + i(ad + bc)$$

dual equality

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal

$$z_1 = z_2 \iff a = c \text{ and } b = d$$

if and only if both the real and the imaginary parts of z_1 and z_2 are equal.

matrix form

Another idea to keep track of the same information is in matrix form, namely:

$$z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Such matrices can be added and multiplied in the normal way and give the desired results for complex numbers. Thus:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \times \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{bmatrix} = \begin{bmatrix} u & -v \\ v & u \end{bmatrix}$$

Geometric interpretation

Yet another powerful way to think about complex numbers is to use the complex plane (sometimes called the Argand plane), where points are plotted with the real part along the horizontal axis and the imaginary part along the vertical axis.

This figure is from Brown & Churchill.

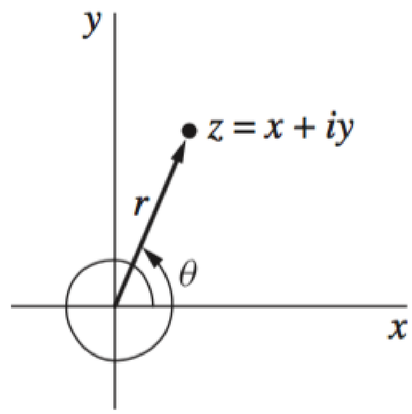


FIGURE 6

Looking at the graph, the distance of any point from the origin is denoted by r, and θ is the angle the ray makes with the positive x-axis in a CCW direction. This should be familiar from standard polar coordinates.

Switching notation to

$$z = x + iy$$

To plot the complex number z we go out x units along the real (horizontal) axis and then up y units along the imaginary (vertical) axis.

The statement that $\mathbb{R} \subset \mathbb{C}$ is equivalent to the observation that the Argand plane contains the horizontal axis. Real numbers have the form $z = x + i \cdot 0 = x$.

More generally, though

$$x = r\cos\theta$$

$$y = r\sin\theta$$

and

$$\begin{aligned} x + iy &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

$$= re^{i\theta}$$

where the last part makes use of Euler's famous equation. r is called the **modulus** and θ is called the **argument** or **phase**.

If you look very carefully at the figure above the argument θ is actually $\theta + 2\pi$.

All multiples $k \cdot 2\pi$ for $k \in 0, \pm 1, \pm 2 \dots$ are valid.

Depending on the calculation one form is often easier to handle.

Addition is simpler with $a + ib$ (the Cartesian format) since

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

while multiplication is more straightforward with the polar format.

Matrices work well for both addition and multiplication.

Here is multiplication in polar coordinates

$$re^{i\theta}\ \rho e^{i\phi} = r\rho\ e^{i(\theta+\phi)}$$

We multiply the distances and add the angles. Here is the square function:

$$(re^{i\theta})^2 = r^2 e^{i2\theta}$$

Multiplication of $z_1 = r_1 e^{i\theta_1}$ by $z_2 = r_2 e^{i\theta_2}$ stretches r_1 (the length of z_1) by the factor r_2 (the length of z_2), and rotates z_1 by adding a phase shift of θ_2 to the original angle θ_1.

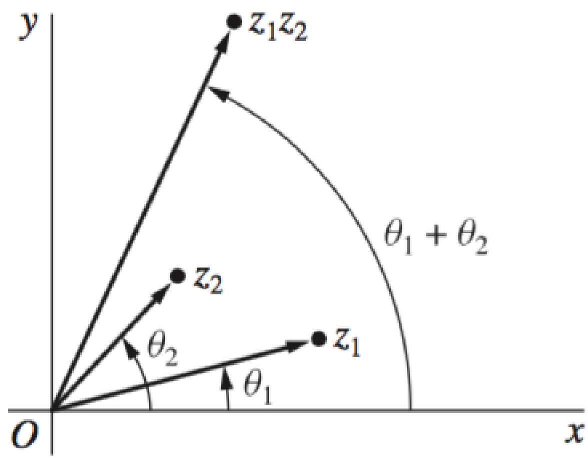


FIGURE 9

The person who originally discovered this representation was Caspar Wessel.

Since the calculations can be tedious, I wrote a Python script to do the calculations for roots and powers.

https://gist.github.com/telliott99/916bc75a73e515968debe48ef418d738

Chapter 3

Conjugate

Consider the complex number:

$$z = x + iy$$

The complex conjugate of z (called $z*$ or $\bar{z}$) is given by:

$$z* = x - iy$$

The real part of $z*$ is the same as the real part of z, while the imaginary part has the sign switched.

length of z

The *length* of z squared is equal to z multiplied by its complex conjugate

$$\begin{gathered} zz* = (x+iy)(x-iy) \\ = x^2 - ixy + ixy - i^2y^2 \\ = x^2 + y^2 \\ = (r\cos\theta)^2 + (r\sin\theta)^2 \\ = r^2 \end{gathered}$$

Again, r is the length of the ray from the origin to z as plotted in the complex plane.

$$r^2 = zz*$$

$$r = \sqrt{zz*}$$

The point corresponding to $z*$ in the complex plane has the same overall distance from the origin and the same x-component as z, but the sign change on y means that $z*$ is reflected across the x-axis from z.

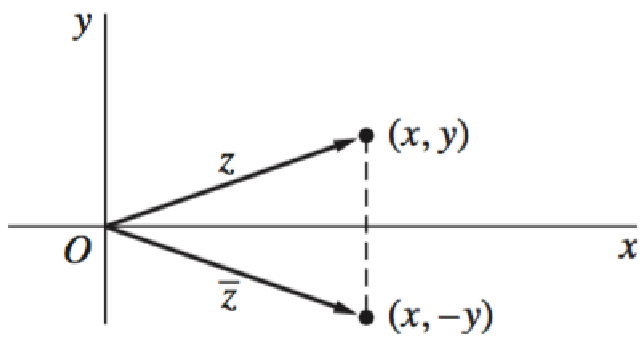


FIGURE 5

In polar coordinates, if $z = re^{i\theta}$ then $z* = re^{i(-\theta)} = re^{-i\theta}$. So

$$zz* = re^{i\theta}\ re^{i-\theta} = r^2e^0 = r^2$$

Multiplication of z by $z*$ makes the product entirely real.

If we consider addition rather than multiplication of the complex conjugate we observe that it also gives an entirely real result:

$$z + z* = x + iy + x - iy = 2x$$

while subtraction gives an entirely imaginary result:

$$z - z* = x + iy - x + iy = i2y$$

conjugate of several values

Another result (that we state without proof) is that if we have an expression involving several complex numbers:

$$w = f(z_1, z_2 \ldots)$$

we can obtain the complex conjugate of the whole thing by substituting the complex conjugate of each component:

$$w* = f(z_1*, z_2 * \ldots)$$

As an example, let us compute the powers of z and $z*$ using the binomial theorem:

$$z = x + iy$$
$$z^2 = x^2 + 2x(iy) + (iy)^2$$
$$z^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$
$$z^4 = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4$$

and the conjugate:

$$z* = x + (-iy)$$
$$(z*)^2 = x^2 + 2x(-iy) + (-iy)^2$$
$$(z*)^3 = x^3 + 3x^2(-iy) + 3x(-iy)^2 + (-iy)^3$$
$$(z*)^4 = x^4 + 4x^3(-iy) + 6x^2(-iy)^2 + 4x(-iy)^3 + (-iy)^4$$

It makes things simpler if we leave the minus signs and the powers of i for the moment.

Now, any even power of i is wholly real. So all we really need to do to form the conjugate is to switch the sign of the odd powers. Since they're odd powers, it makes no difference if we do this inside the parentheses or in front of each term.

So then,

$$\begin{aligned}(z^2)* &= x^2 - 2x(iy) + (iy)^2 \\ &= x^2 + 2x(-iy) + (iy)^2 \\ &= (z*)^2\end{aligned}$$

Furthermore, we can slip an extra minus sign inside any even power without changing the value:

$$\begin{aligned}(z^3)* &= x^3 - 3x^2(iy) + 3x(iy)^2 - (iy)^3 \\ &= x^3 + 3x^2(-iy) + 3x(iy)^2 + (-iy)^3 \\ &= x^3 + 3x^2(-iy) + 3x(-iy)^2 + (-iy)^3 \\ &= (z*)^3\end{aligned}$$

$$\begin{aligned}(z^4)* &= x^4 - 4x^3(iy) + 6x^2(iy)^2 - 4x(iy)^3 + (iy)^4 \\ &= x^4 - 4x^3(iy) + 6x^2(-iy)^2 - 4x(iy)^3 + (-iy)^4 \\ &= (z*)^4\end{aligned}$$

It is clear that this pattern will continue with higher powers.

Part II

Cauchy Riemann

Chapter 4

Difference quotient

This section contains a general discussion of differentiation of complex functions. A complex function takes as input a complex number, and emits as output another complex number. Often w is used for the output:

$$w = f(z)$$

More concretely, a complex number is simply an ordered pair of real numbers, and a complex function is a pair of functions defined on the real numbers:

$$f(z) = f(x + iy) = u(x, y) + i \cdot v(x, y)$$

The functions u and v each take a pair of real numbers and emit one real number. They are connected in f through the fact that u and v have the same input.

Finally, the output of v is multiplied by i. Here is an example:

$$z = x + iy$$

$$z^2 = (x + iy)(x + iy)$$

$$= x^2 - y^2 + 2ixy$$

So

$$u(x, y) = x^2 - y^2$$
$$v(x, y) = 2xy$$

Another one is:

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{z*}{z*}$$
$$= \frac{x - iy}{(x + iy)(x - iy)}$$
$$= \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2}$$

Often a simplified notation is employed:

$$f(z) = u + i \cdot v$$

Since the inputs cover the entire complex plane, we cannot plot graphs as with real functions. Instead, one version of the complex plane is *mapped* by the function into a different version of the complex plane.

Cauchy-Riemann

This chapter gives us a first glimpse of the important Cauchy-Riemann conditions and justifies one of the formulas for calculating the derivative

$$f'(z) = u_x + iv_x$$

As an example of its use, consider the complex exponential

$$f(z) = e^z$$

If we write $z = x + iy$ then

$$f(z) = e^{x+iy}$$
$$= e^x e^{iy}$$

and (from Euler):

$$e^{iy} = \cos y + i \sin y$$

so

$$f(z) = e^x \cos y + i e^x \sin y$$

Using the formula, it can be shown easily that the derivative is the same as the function itself, just as for the case of real numbers.

$$u(x, y) = e^x \cos y$$
$$u_x = e^x \cos y = u$$
$$v(x, y) = e^x \sin y$$
$$v_x = e^x \sin y = v$$

Hence

$$f'(z) = u_x + i v_x = z$$

definition

We define the derivative $f'(z)$ of a complex function $f(z)$ similarly to the derivative of a real function:

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

if the limit exists.

Alternatively, with Δ notation, we might write:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

A crucial difference from real functions is that there are only two directions from which to approach a given real number x, while there is an infinite number of ways of approaching z in the Argand plane. The limit is over *all* possible ways of approaching z.

If the limit exists, the function f is called differentiable and $f'(z)$ is the derivative. Consider

$$f(z) = u(x, y) + iv(x, y)$$

Then

$$f'(z) = \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}$$

fixed y

We tame this beast by looking at two specific paths.

Looking at the special path along the x-axis where $\Delta y = 0$ we obtain

$$f'(z) = \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x}$$

Rearrange the numerator

$$= \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + \frac{iv(x + \Delta x, y) - iv(x, y)}{\Delta x}$$

The first term is

$$u_x = \frac{\partial u}{\partial x}$$

and the second term is

$$iv_x$$

Hence we conclude that

$$f'(z) = u_x + iv_x$$

fixed x

Now look at the special path along the y-axis where $\Delta x = 0$:

$$f'(z) = \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y}$$

Rearrange the numerator

$$= \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + \frac{iv(x, y + \Delta y) - iv(x, y)}{i\Delta y}$$

$$= \frac{1}{i}u_y + v_y$$

Recall that $1/i = -i$

$$f'(z) = v_y - iu_y$$

Putting it together

We require that the limit be the same regardless of the direction of approach to z, so these two expressions for the difference quotient must be equal:

$$f'(z) = u_x + iv_x = -iu_y + v_y$$

Both the real and the imaginary parts must be equal so

$$u_x = v_y$$

$$u_y = -v_x$$

Once differentiability is established, we can use whichever path we want to evaluate the derivative.

As we said at the beginning, in looking at various complex functions we can use this fact:

$$f'(z) = u_x + iv_x$$

One consequence is that

$$\frac{df}{dz} = \frac{\partial f}{\partial x}$$

and since

$$= u_x + iv_x = v_y - iu_y$$

$$= -iu_y + v_y$$

$$= -i(u_y + iv_y)$$

$$= -i\frac{\partial f}{\partial y}$$

We conclude that

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

looking ahead

When we get to integration in a later section we will find that the integral of a complex function is computed as a line integral along a specified curve (often a circle centered either on the origin or on a point z_0).

This curve relates the values of x and y and allows us to parametrize either y in terms of x or more generally, both x and y in terms of a single real variable or parameter t.

When we have a function of such a variable like

$$f(t) = u(t) + iv(t)$$

then the derivative is defined to be

$$f'(t) = u'(t) + iv'(t)$$

where u and v are real-valued functions of a single real variable and so follow the standard rules from introductory calculus. In particular if

$$w(t) = z_0 f(t)$$

then

$$w'(t) = z_0 f'(t)$$

The derivative of a constant times a function is the constant times the derivative of the function.

derivative of z_0 times a function

We can show this by using a little algebra:

$$\begin{aligned}
\frac{d}{dt}\ z_0 f(t) &= \ [\ (x_0 + iy_0)(u + iv)\ ]' \\
&= \ [\ (x_0 u - y_0 v) + i(y_0 u + x_0 v)\ ]' \\
&= (x_0 u - y_0 v)' + i(y_0 u + x_0 v)' \\
&= (x_0 u' - y_0 v') + i(y_0 u' + x_0 v') \\
&= (x_0 + iy_0)(u' + iv') \\
&= z_0 \frac{d}{dt}\ f(t)
\end{aligned}$$

Thus

$$\frac{d}{dt}\ z_0 f(t) = z_0 \frac{d}{dt}\ f(t)$$

which is what we just said.

derivative of exp z_0 t

Another expected result is

$$\frac{d}{dt}\ e^{z_0 t} = z_0 e^{z_0 t}$$

where z_0 is a complex constant and t is a real variable.

To do this one, refer to the definition

$$f'(t) = u'(t) + iv'(t)$$

And now we need to break up the exponential into its real and imaginary parts.

By Euler's equation, we wrote above

$$e^z = e^{x+iy} = e^x \cos y + ie^x \sin y$$

For the exponential of a real variable, but containing a complex constant we have

$$\begin{aligned} e^{z_0 t} &= e^{(x_0+iy_0)t} \\ &= e^{x_0 t}\ e^{iy_0 t} \\ &= e^{x_0 t}\ (\cos y_0 t + i \sin y_0 t) \\ &= e^{x_0 t} \cos y_0 t + ie^{x_0 t}\ \sin y_0 t \end{aligned}$$

Substitution

I find this calculation very confusing. Especially the subscripts. Rather than change letters, we will drop the subscripts on x_0 and y_0 but tell ourselves repeatedly: these are constants. Also, t is a *real* variable.

$$e^{xt} \cos yt + ie^{xt}\ \sin yt$$

Using the definition above we get that the derivative is $u'(t) + iv'(t)$ so the derivative of a sum is the sum of the derivatives.

The first term (u') is (by the product and chain rules):

$$[\ e^{xt} \cos yt\ ]' = xe^{xt} \cos yt - ye^{xt} \sin yt$$

and the second:

$$[\ e^{xt}\ \ \sin yt\ ]' = xe^{xt} \sin yt + ye^{xt} \cos yt$$

Remember that each term in that second one gets an i!

$$i[\ e^{xt}\ \ \sin yt\ ]' = ixe^{xt} \sin yt + iye^{xt} \cos yt$$

Combine the first term from each and factor out the x:

$$x(e^{xt} \cos yt + ie^{xt} \sin yt)$$

Do the same with the second term:

$$y(ie^{xt} \cos yt - e^{xt} \sin yt)$$

the tricky part

$$= iy(e^{xt} \cos yt + ie^{xt} \sin yt)$$

Putting everything together we have just

$$(x + iy)(e^{xt} \cos yt + ie^{xt} \sin yt)$$

Restoring the original naughts, we have just

$$z_0 e^{z_0 t}$$

As promised.

$\square$

Chapter 5

Proofs of CRE

difference quotient

We gave a first proof in the section on differentiation which is repeated more briefly here.

The derivative $f'(z)$ is defined to be the limit of the following difference quotient, if the limit exists.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

where $f(z) = u(x, y) + iv(x, y)$.

The difference quotient is rewritten in terms of u and v as:

$$\frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x) - iv(y)}{\Delta x + i\Delta y}$$

Then we consider two special cases, one where $\Delta y = 0$ and a second where $\Delta x = 0$. The first case yields

$$f'(z) = u_x + iv_x$$

and the second yields

$$f'(z) = -iu_y + v_y$$

The derivative is required to be the same for all directions of approach to the point, so we can equate the two expressions

$$u_x + iv_x = -iu_y + v_y$$

Since both the real and the imaginary parts must be equal, we obtain the CRE:

$$u_x = v_y$$

$$u_y = -v_x$$

□

chain rule

Here is a second approach:

Write:

$$z = x + iy$$

Clearly,

$$\frac{\partial z}{\partial x} = 1, \qquad \frac{\partial z}{\partial y} = i$$

Now,

$$\begin{aligned} w &= f(z) \\ &= u(x, y) + i\ v(x, y) \end{aligned}$$

where u and v are real functions over $\mathbb{R}^2$.

Recalling the chain rule

$$w = u(x, y) + i\ v(x, y)$$

$$\frac{\partial w}{\partial x} = \frac{dw}{dz}\,\frac{\partial z}{\partial x}$$

by the result immediately above (that $\partial z/\partial x = 1$):

$$\frac{\partial w}{\partial x} = \frac{dw}{dz}$$

Similarly

$$\frac{\partial w}{\partial y} = \frac{dw}{dz}\,\frac{\partial z}{\partial y}$$

$$\frac{\partial w}{\partial y} = i\frac{dw}{dz}$$

Hence we can equate the two expressions for dw/dz:

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

Now if we actually compute the partials and plug them in to the last equation, we obtain:

$$u_x + iv_x = -i(u_y + iv_y) = v_y - iu_y$$

Both the real and the imaginary parts must be equal:

$$u_x = v_y$$

$$v_x = -u_y$$

These are (again) the CRE.

□

It is worth taking a breath for a moment and repeating what we just said: the derivative of a differentiable complex function z (what we will call an analytic function) is

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

$$
\begin{aligned}
&= u_x + iv_x \\
&= -i(u_y + iv_y) \\
&= v_y - iu_y
\end{aligned}
$$

Alder

A third, very simple proof is given in Alder:

Suppose $f : C \to C$ is a function, taking $x + iy$ to $u(x, y) + iv(x, y)$, then the derivative is a matrix of partial derivatives:

$$
\begin{matrix} u_x & u_y \\ v_x & v_y \end{matrix}
$$

> the above matrix is the two dimensional version of the slope of the tangent line in dimension one. It gives the linear part (corresponding to the slope) of the affine map which best approximates f at each point.
>
> But at any point $x + iy$, if f is differentiable in the *complex* sense, this must be just a linear complex map, i.e. it multiplies by some complex number. So the matrix must be in our set of complex numbers. In other words, for every value of x it looks like

$$
\begin{matrix} a & -b \\ b & a \end{matrix}
$$

for some real numbers a, b, which change with x.

Of course, this constraint leads directly to the CRE.

□

A very important point is that the CRE and analyticity and differentiability are all related (either a function has all these properties or none of them). For an analytic function, the rules for integration and differentiation are analogous to the real case. For example:

$$\int \frac{1}{3}z^2\ dz = z^3$$

$$\frac{d}{dz}\ \frac{1}{z - z_0} = -\frac{1}{(z - z_0)^2}$$

We will see a lot more of this.

McMahon

Here is yet another proof which I found in McMahon.

https://www.amazon.com/Complex-Variables-Demystified-David-McMahon/dp/007154920X

I include it here because it explains Shankar's statement that by definition an analytic function has no dependence on $z*$. (The $\bar{z}$ notation is used below).

Write

$$z = x + iy, \quad \bar{z} = x - iy$$

so

$$2x = (z + \bar{z})$$

$$x = \frac{1}{2}(z + \bar{z})$$

$$2iy = (z - \bar{z})$$

$$y = \frac{1}{2i}(z - \bar{z}) = -\frac{1}{2}i(z - \bar{z})$$

Take partial derivatives:

$$\frac{\partial x}{\partial z} = \frac{1}{2} = \frac{\partial x}{\partial \bar{z}}$$

and

$$\frac{\partial y}{\partial z} = -\frac{1}{2i} = -\frac{\partial y}{\partial \bar{z}}$$

Then, using the chain rule we write:

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\ [\ \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\ ]$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial x}{\partial \bar{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar{z}}\frac{\partial}{\partial y} = \frac{1}{2}\ [\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\ ]$$

Now apply the two operators (just a matter of a few minus signs):

$$\frac{\partial f}{\partial z} = \frac{1}{2}\ [\ \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\ ]\ [\ u + iv\ ] = \frac{1}{2}\ [\ (u_x + v_y) + i(v_x - u_y)\ ]$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\ [\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\ ]\ [\ u + iv\ ] = \frac{1}{2}\ [\ (u_x - v_y) + i(v_x + u_y)\ ]$$

And now to the point: we *require* the last expression to be zero. $f(z)$ must have no dependence on $\bar{z}$.

As usual, both the real and the imaginary parts must vanish.

$$0 = \frac{1}{2}\ [\ (u_x - v_y) + i(v_x + u_y)\ ]$$

$$u_x = v_y, \quad v_x = -u_y$$

In other words, the CRE apply. And using these conditions, we can rewrite

$$\frac{\partial f}{\partial z} = \frac{1}{2}\ [\ (u_x + v_y) + i(v_x - u_y)\ ]$$

$$= u_x + iv_x$$

□

To put this another way, if we have already established the CRE, we can run this proof backwards to show that for $f(z)$, $\partial f / \partial \bar{z} = 0$.

Sources:

[1] Alder. *An Introduction to Complex Analysis for Engineers.*

http://www.eee.metu.edu.tr/~ccandan/EE202_summer2004/solutions/An%20Introduction%20to%20Complex%20Analysis%20for%20Engineers%20-%20Michael%20Alder.pdf

Chapter 6

Powers

square

Consider

$$f(z) = z^2$$
$$= (x + iy)(x + iy)$$
$$= x^2 - y^2 + i2xy$$
$$u(x, y) = x^2 - y^2$$
$$v(x, y) = 2xy$$

Note that

$$u_x = 2x = v_y$$
$$u_y = -2y = -v_x$$

The Cauchy-Riemann conditions (CRE) hold.

Compute the derivative as follows:

$$\frac{df}{dz} = u_x + iv_x$$
$$= 2x + i2y = 2z$$

or alternatively

$$\frac{df}{dz} = v_y - iu_y$$

$$= 2x - i(-2y) = 2x + i2y = 2z$$

This is the result we would expect to get by simply differentiating f(z) as if it was a real function. For analytic functions this will always be the case.

cube

Let

$$f(z) = z^3 = (x + iy)^3$$

$$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$

$$= x^3 - 3xy^2 + i\ [\ 3x^2y - y^3\ ]$$

So

$$u(x, y) = x^3 - 3xy^2$$

$$v(x, y) = 3x^2y - y^3$$

and

$$u_x = 3x^2 - 3y^2$$

$$v_x = 6xy$$

That means

$$f'(z) = u_x + iv_x$$

$$= 3x^2 - 3y^2 + i6xy$$

$$= 3\ [x^2 + 2x(iy) + (iy)^2\ ]$$

$$= 3z^2$$

We could continue and show that z^n is analytic for any positive integer power of n. Notice the pattern for i:

$$(x+iy)^n = x^n + nx^{n-1}(iy) + (n)(n-1)x^{n-2}(iy)^2 + \dots$$

The progression goes:

$$i^0, i^1, i^2, i^3$$

$$= 1, i, -1, -i$$

and then repeats.

inverse

Using the complex conjugate is a good way to work with the inverse function (or with division by any complex number):

$$\frac{1}{z} = \frac{z*}{zz*} = \frac{x-iy}{x^2+y^2}$$

or in polar notation:

$$\frac{1}{z} = \frac{re^{-i\theta}}{r^2\ e^{i\theta}\ e^{-i\theta}} = \frac{1}{r}\ e^{-i\theta}$$

Let's look at what it means to take the inverse for different z. In every case, the point is reflected across the x-axis (the ray makes an angle $-\theta$ with the x-axis).

There is no change in length for $r = 1$. But if say

$$z = 1 + i = (1,1) = \sqrt{2}\ e^{i \cdot \pi/4}$$

then the new point has $r = \frac{1}{\sqrt{2}}$ and it is located at

$$\frac{1}{z} = \frac{1}{\sqrt{2}} e^{-i \cdot \pi/4} = (\frac{1}{2}, -\frac{1}{2}) = \frac{1}{2} - i\frac{1}{2}$$

differentiation

We wish to differentiate

$$\begin{aligned} f(z) &= 1/z \\ &= \frac{z*}{zz*} \\ &= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \end{aligned}$$

Let's do the partial derivatives. $u(x, y)$ has x in both the numerator and the denominator.

Recall the quotient rule (using unfamiliar symbols g and h):

$$(g/h)' = (g'h - gh')/h^2$$

which we check by differentiating $x/1$.

- u_x

$$u(x, y) = \frac{x}{x^2+y^2}$$

$$u_x = (x^2 + y^2 - x \cdot 2x) \cdot \frac{1}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

- v_y

$$v(x, y) = -\frac{y}{x^2+y^2}$$

To do v_y just switch x and y in the result above, but remember to then multiply by the leading factor of -1:

$$v_y = (-1)\ \frac{x^2 - y^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

Thus $u_x = v_y$

- u_y

$$u(x,y) = x(x^2+y^2)^{-1}$$
$$u_y = x(-1)\cdot 2y(x^2+y^2)^{-2} = \frac{-2xy}{(x^2+y^2)^2}$$

- v_x

$$v(x,y) = -y(x^2+y^2)^{-1}$$
$$v_x = -y(-1)\cdot 2x(x^2+y^2)^{-2} = \frac{2xy}{(x^2+y^2)^2}$$

So we have that the CRE are satisfied (except at $z=0$) and the derivative is

$$\begin{aligned}\frac{df}{dz} &= u_x + iv_x \\ &= \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2} \\ &= \frac{1}{(x^2+y^2)^2}\ (y^2-x^2+i2xy)\end{aligned}$$

We expect that this should be (in disguise) $-1/z^2$. Let's see:

$$\frac{1}{z} = \frac{z*}{zz*}$$
$$\frac{1}{z^2} = \frac{(z*)^2}{(zz*)^2}$$

The denominator is certainly correct since

$$zz* = x^2+y^2$$

What about the denominator?

$$(z*)^2 = (x - iy)(x - iy) = x^2 - y^2 - i2xy$$

so

$$-(z*)^2 = (-1)(x - iy)(x - iy) = y^2 - x^2 + i2xy$$

Everything checks.

powers: de Moivre's formula

Let n be an integer:

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

Suppose $r = 1$:

$$z^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$$

This is de Moivre's formula.

Suppose $n = 2$, then

$$(\cos\theta + i\sin\theta)^2$$

$$= \cos^2\theta - \sin^2\theta + i\ 2\sin\theta\cos\theta$$

Equating with the right-hand side of de Moivre's formula:

$$\cos^2\theta - \sin^2\theta + i\ 2\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$$

we find that

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

We already know these, they are the double angle formulas.

Suppose $n = 3$, then

$$(\cos\theta + i\sin\theta)^3$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

we find that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

and so on.

We can just check that last one for $\theta = \pi/6$:

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

$$1 = 3(\frac{\sqrt{3}}{2})^2\ \frac{1}{2} - (\frac{1}{2})^3$$

Multiply both sides by 2^3:

$$8 = 3(\sqrt{3})^2 - 1$$

That looks correct.

Chapter 7

Polar CRE

In this chapter we will derive the CRE conditions for polar coordinates and show how to compute the derivative in the same system. The take home lesson is that there is an extra factor for each:

Recall that the CRE in Cartesian coordinates are:

$$u_x = v_y$$

$$v_x = -u_y$$

It turns out that r, θ is similar except for a factor of r, which goes on the partial with respect to r

$$ru_r = v_\theta$$

$$rv_r = -u_\theta$$

Also,

$$f'(z) = e^{-i\theta}\ (u_r + iv_r)$$

We'll also see two derivations for each, a simple one and a more careful but also more complicated one.

A clever way to derive the CRE in polar coordinates is to take advantage of the result that we obtained in Cartesian coordinates.

short and sweet

The function $f(z) = z = x + iy$ is "analytic" and obeys the CRE.

Write the same function in polar coordinates:

$$z = re^{i\theta}$$

and then separate it into a completely real part $u(r, \theta)$ and a completely imaginary part $v(r, \theta)$:

$$z = r(\cos\theta + i\sin\theta)$$

$$= r\cos\theta + ir\sin\theta$$

Observe that

$$u_r = \cos\theta$$

$$v_\theta = r\cos\theta$$

so we deduce that

$$ru_r = v_\theta$$

Similarly

$$v_r = \sin\theta$$

$$u_\theta = -r\sin\theta$$

so

$$-rv_r = u_\theta$$

There are the CRE in polar coordinates. Carrying out the same computation for *any* analytic function would give the same result (with some other expression in the middle, of course).

$$ru_r = v_\theta$$

$$rv_r = -u_\theta$$

Notice the similar format to the Cartesian version, with the addition of a factor of r.

It reminds me of the Jacobian from multi-variable calculus.

polar derivative

This will also turn out to be very similar to the Cartesian version, with an extra factor out front:

$$f'(z) = e^{-i\theta} \ [\ u_r + iv_r \ ]$$

Let's just assume that the derivative is equal to what we would expect, within some unknown factor of k:

$$z' = k(u_r + iv_r)$$

and now we know that for this function the derivative is equal to 1:

$$f(z) = z$$

$$z' = 1 = k(u_r + iv_r)$$

if we write in polar coordinates:

$$z = r\cos\theta + ir\sin\theta$$

Then

$$u_r + iv_r = \cos\theta + i\sin\theta$$

What is the factor that multiplies this expression to give 1? Clearly

$$e^{-i\theta}(\cos\theta + i\sin\theta) = 1$$

So $k = e^{-i\theta}$.

CRE derivation by the chain rule

We know equations to go back and forth between x, y and r, θ so it is not hard to imagine that we can always re-write u and v as

$$z = u\ [\ x(r,\theta), y(r,\theta)\ ] + iv\ [\ x(r,\theta), y(r,\theta)\ ]$$

or more succinctly:

$$z = u(r,\theta) + iv(r,\theta)$$

Now we ask about relations between the partial derivatives. Let us first make a table of them:

$$x = r\cos\theta$$

$$x_r = \cos\theta, \qquad x_\theta = -r\sin\theta$$

$$y = r\sin\theta$$

$$y_r = \sin\theta, \qquad y_\theta = r\cos\theta$$

Clearly, we want expressions involving u_r, v_θ etc. Write:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

or in a convenient shorthand

$$u_r = u_x x_r + u_y y_r = u_x\cos\theta + u_y\sin\theta$$

The other three are

$$u_\theta = u_x x_\theta + u_y y_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta)$$

$$v_r = v_x x_r + v_y y_r = v_x(\cos\theta) + v_y(\sin\theta)$$

$$v_\theta = v_x x_\theta + v_y y_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta)$$

Our key insight is to use the relations given by the CRE in Cartesian coordinates

$$u_x = v_y$$

$$u_y = -v_x$$

Thus, starting with the first expression for partial derivatives above

$$u_r = u_x \cos\theta + u_y \sin\theta$$

Use the CRE in x, y to substitute in terms of v:

$$u_r = v_y \cos\theta + (-v_x) \sin\theta$$

we can see that this is different from the fourth expression above

$$v_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta)$$

only by a factor of r:

$$ru_r = v_\theta$$

which is what needed to prove.

The other one is:

$$u_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta)$$

$$= v_y(-r\sin\theta) - v_x(r\cos\theta)$$

compare with

$$v_r = v_x(\cos\theta) + v_y(\sin\theta)$$

We need a factor of $-r$:

$$u_\theta = -rv_r$$

Like the original Cartesian version but with an extra factor of r on the partials with respect to r.

polar derivative, more carefully

To get the derivative, start with the version that we know for x, y coordinates:

$$f'(z) = u_x + iv_x$$

Our problem is to define $f'(z)$ in terms of u_r and v_r.

Substitute for u_x first. Go back to the two equations involving u_x above

$$u_r = u_x \cos\theta + u_y \sin\theta$$

$$u_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta)$$

Multiply the first by $\cos\theta$

$$u_r \cos\theta = u_x \cos^2\theta + u_y \sin\theta\cos\theta$$

and the second by $-\sin\theta/r$

$$-\frac{u_\theta}{r}\sin\theta = u_x \sin^2\theta - u_y \sin\theta\cos\theta$$

add

$$u_x = u_r \cos\theta - \frac{u_\theta}{r}\sin\theta$$

Substitute $u_\theta/r = -v_r$

$$u_x = u_r \cos\theta + v_r \sin\theta$$

Now get the second and fourth equations, with v_x

$$v_r = v_x \cos\theta + v_y \sin\theta$$

$$v_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta)$$

Multiply the first by $\cos\theta$

$$v_r \cos\theta = v_x \cos^2\theta + v_y \sin\theta\cos\theta$$

and the second by $-\sin\theta/r$:

$$-\frac{v_\theta}{r}\sin\theta = v_x \sin^2\theta - v_y \sin\theta\cos\theta$$

add

$$v_x = v_r\cos\theta - \frac{v_\theta}{r}\sin\theta$$

Subtitute $v_\theta/r = u_r$

$$v_x = v_r\cos\theta - u_r\sin\theta$$

Combine the two results:

$$f'(z) = u_x + iv_x$$

$$= u_r\cos\theta + v_r\sin\theta + i\ [\ v_r\cos\theta - u_r\sin\theta\ ]$$

Group terms with u_r and v_r separately:

$$= u_r(\cos\theta - i\sin\theta) + v_r(\sin\theta + i\cos\theta)$$

Multiply the second term by $1 = -i \cdot i$

$$= u_r(\cos\theta - i\sin\theta) + iv_r(-i\sin\theta + \cos\theta)$$

$$f'(z) = e^{-i\theta}\ [\ u_r + iv_r\ ]$$

And since

$$ru_r = v_\theta$$

$$rv_r = -u_\theta$$

then

$$f'(z) = \frac{1}{r}\ e^{-i\theta}\ [\ v_\theta - iu_\theta\ ]$$

compare this with

$$f'(z) = v_y - iu_y$$

and notice that the factor in front is just $1/z$

example 1

Let's see if we can do an example. Suppose

$$f(z) = \sqrt{z}$$

Written in terms of r, θ we have

$$f(z) = \sqrt{r}e^{i\theta/2}$$

$$= \sqrt{r}\cos\theta/2 + i\sqrt{r}\sin\theta/2$$

Then

$$u_r = \frac{\cos\theta/2}{2\sqrt{r}}$$

$$v_r = \frac{\sin\theta/2}{2\sqrt{r}}$$

and

$$[\sqrt{z}\ ]' = \ [\ \sqrt{r}e^{i\theta/2}\ ]' = e^{-i\theta}\ [\ u_r + iv_r\ ]$$

$$= \frac{1}{2\sqrt{r}}\ [\ e^{-i\theta}(\cos\theta/2 + i\sin\theta/2)\ ]$$

$$= \frac{1}{2\sqrt{r}}e^{-i\theta/2} = \frac{1}{2\sqrt{z}}$$

example 2

Let's try

$$f(z) = \frac{1}{z} = \frac{1}{r}e^{-i\theta}$$

$$= \frac{1}{r}\cos -\theta + i\frac{1}{r}\sin -\theta$$

$$= \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$$

So

$$u_r = -\frac{1}{r^2}\cos\theta$$

$$v_r = \frac{1}{r^2}\sin\theta$$

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$= \frac{1}{r^2}(e^{-i\theta})(-\cos\theta + i\sin\theta)$$

$$= \frac{1}{r^2}(e^{-i\theta})(-\cos -\theta - i\sin -\theta)$$

$$= \frac{1}{r^2}(e^{-i\theta})(-1)(e^{-i\theta})$$

$$= -\frac{1}{r^2 e^{i2\theta}} = -\frac{1}{z^2}$$

Chapter 8

CRE examples

The function

$$f(x, y) = \frac{x^2}{x^2 + y^2}$$

has some problems: first, it is not defined at the origin $(0, 0)$ but also, as we approach the origin along the x-axis and the y-axis we get different limiting values, namely

$$f(x, 0) = \frac{x^2}{x^2} = 1$$

$$f(0, y) = \frac{0}{y^2} = 0$$

Rewriting it in polar coordinates ($x = r\cos\theta, r^2 = x^2 + y^2$):

$$f(r, \theta) = \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$$

Shankar says: the function f is generally a function of *two* complex variables, z and its complex conjugate:

$$z = x + iy$$

$$z* = x - iy$$

which can be written in terms of x and y as

$$x = \frac{z + z*}{2}$$

$$y = \frac{z - z*}{2i}$$

Generally, the value of f depends on both z and $z*$, but we will be very interested in functions which depend only on z and not $z*$. The reason for this is that only such functions have the property that the derivative at a point does not depend on the direction from which we approach that point.

Consider the function:

$$f(x, y) = x^2 - y^2$$

$$= \frac{(z + z*)^2}{4} + \frac{(z - z*)^2}{4}$$

$$= \frac{1}{4} \; [\; z^2 + 2zz * + z *^2 + z^2 - 2zz * + z*^2 \;]$$

$$= \frac{z^2 + z*^2}{2}$$

This function is not a function only of z but of both z and $z*$.

We say that f is an *analytic* function of z if it does not depend on $z*$. Shankar says this means that "x and y enter f *only* in the combination $x + iy$".

The famous Cauchy-Riemann Equations (CRE) are true for $f \iff f$ is an analytic function of z.

For:

$$f(x, y) = u(x, y) + iv(x, y)$$

The CRE conditions are:

$$u_x = v_y$$
$$u_y = -v_x$$

Consider:

$$f(x, y) = x^2 - y^2 + i2xy$$

CRE requires

$$u_x = 2x \stackrel{?}{=} v_y = 2x$$
$$v_x = 2y \stackrel{?}{=} -u_y = 2y$$

The function is analytic. As Shankar says, this is expected because:

$$x^2 - y^2 + 2ixy = (x + iy)(x + iy) = z^2$$

Consider:

$$f(x, y) = \cos y - i \sin y$$

CRE requires:

$$u_x = 0 \stackrel{?}{=} v_y = -\cos y$$
$$v_x = 0 \stackrel{?}{=} -u_y = -\sin y$$

This is "impossible" since there is no y that satisfies both of the conditions. And it's not surprising since

$$y = \frac{z - z*}{2i}$$

Consider:

$$f(x, y) = x^2 + y^2$$

CRE requires:

$$u_x = 2x \stackrel{?}{=} v_y = 2y$$
$$u_y = 0 \stackrel{?}{=} -v_x$$

CRE are only satisfied if $x = y$. Also not surprising since

$$x^2 + y^2 = zz*$$

Consider:

$$f(x, y) = x^2 - y^2$$

CRE requires:

$$u_x = 2x \stackrel{?}{=} v_y = -2y$$

which is true if $x = y$.

$$u_y = 0 \stackrel{?}{=} -v_x = 0$$

But "no importance is given to functions which obey the CRE only at isolated points or on lines."

Consider:

$$f(x, y) = e^x \cos y + ie^x \sin y$$

CRE requires:

$$u_x = e^x \cos y \stackrel{?}{=} v_y = e^x \cos y$$

$$u_y = -e^x \sin y \stackrel{?}{=} -v_x = -\sin y e^x$$

Both are true, so this one does satisfy CRE.

Shankar doesn't mention it here but the last function is special, it is $f(z) = e^z$:

$$e^x \cos y + ie^x \sin y$$

$$= e^x(\cos y + i \sin y)$$

$$= e^x e^{iy}$$

$$= e^{x+iy}$$

$$= e^z$$

and we did this one in the previous section.

For functions of interest, it may often be true that CRE fails at particular points called *singularities*.

Consider:

$$f(x,y) = \frac{1}{z} = \frac{z*}{zz*} = \frac{x - iy}{x^2 + y^2}$$

We need:

$$u_x = \frac{d}{dx}\ \frac{x}{x^2 + y^2} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$v_y = \frac{d}{dy}\ (-\frac{y}{x^2 + y^2}) = -\frac{x^2 - y^2}{(x^2 + y^2)^2} = u_x$$

$$u_y = 0 = v_x$$

But the function blows up at the origin. This described by saying it has a pole at the origin. The function

$$f(z) = \frac{c}{z}$$

where c is a constant, also blows up at the origin. We say that the *residue* of the pole at the origin is c.

Chapter 9

Roots

Consider the square root function $\sqrt{z}$.

For the modulus part, we see that $\sqrt{r} \cdot \sqrt{r}$ is obviously equal to r, and what we need to determine the argument is to find an angle that is one-half of the original one, which leads us to

$$\sqrt{z} = \sqrt{re^{i\theta}}$$

$$= \sqrt{r}\ e^{i(\theta/2)}$$

However, recall from trigonometry that if

$$\theta' = \theta + 2k\pi$$

for integer k, then

$$\sin\theta' = \sin\theta$$

We can even say that θ' is equal to θ since the result for a given r maps to the same point in the plane.

This means that a second solution to the square root problem is

$$\sqrt{z} = \sqrt{r}\ e^{i(\theta/2+\pi)}$$

because, again, $\sqrt{r} \cdot \sqrt{r} = r$ and

$$[\ e^{i(\theta/2+\pi)} \cdot\ e^{i(\theta/2+\pi)}\ ]\ = e^{i(\theta/2+\pi)} = e^{i\theta}$$

example

Consider

$$z = e^{i\pi/3}$$

We don't have to worry about r, since it is equal to 1. One solution to the square root is

$$\sqrt{z} = e^{i\pi/6}$$

The second one is

$$\sqrt{z} = e^{i(\pi/6+\pi)} = e^{i(7\pi/6)}$$

which lies in the third quadrant.

To check this:

$$[\ e^{i(7\pi/6)} \cdot e^{i(7\pi/6)}\ ]\ = e^{i(14\pi/6)} = e^{i\pi/3}$$

For the square root, there is only one additional distinct solution, since one-half of $4\pi + \theta = 2\pi + \theta/2$ which is no different than $\theta/2$.

However, the cube root has 3 solutions and in general the n^{th} root has n solutions.

Consider points on the unit circle with $r = 1$ (so $\sqrt{r} = r$) and suppose

$$\theta = \pi/2$$

so

$$z = e^{i\pi/2}$$

Points with $\theta = \pi/2$ lie directly above the origin on the imaginary axis (there is no real component). This point is one unit from the origin so it is the point $(0 + i \cdot 1) = i$. Thus

$$e^{i\pi/2} = i$$

Note that

$$(e^{i\pi/2})^2 = e^{i(\pi/2+\pi/2)}$$
$$= e^{i\pi} = -1 = i^2$$

We can justify this last step by geometry ($\theta = \pi$), or by using Euler's equation

$$e^{i\theta} = cos\theta + i\sin\theta$$
$$e^{i\pi} = cos\pi + i\sin\pi = -1 + i \cdot 0 = -1$$

square root of i

$\sqrt{e^{i\pi/2}} = \sqrt{i}$ has two possible values. One is

$$\sqrt{e^{i\pi/2}} = (e^{i\pi/2})^{1/2} = e^{i\pi/4}$$

Let's just check. The point is at a distance 1 from the origin and angle $\theta = \pi/4$. We go equal distances along the real and imaginary axes:

$$x = \cos\theta = \frac{1}{\sqrt{2}}$$

$$y = \sin\theta = \frac{1}{\sqrt{2}}$$

So we have that the square is:

$$(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})^2 = \frac{1}{2} - \frac{1}{2} + 2i\frac{1}{2}$$

$$= 0 + i = i$$

the second solution is

$$\sqrt{e^{i\pi/2}} = e^{i \cdot 5/4\pi}$$

which can be plotted as

$$x = \cos\theta = -\frac{1}{\sqrt{2}}$$

$$y = \sin\theta = -\frac{1}{\sqrt{2}}$$

The square is the same except the first term is $(-1/\sqrt{2})^2$, so the result is unchanged. It's a bit counter-intuitive that squaring a number may possibly reduce the phase angle, but you can think of it as modular arithmetic (mod 2π).

In general, if we're working with the complex number

$$re^{i\theta}$$

and we want the nth root, the modulus is just

$$\rho = r^{1/n}$$

And the question always is, what's the angle?

$$\phi = \frac{\theta + 2k\pi}{n}, \quad k = 0, 1, 2 \dots n - 1$$

roots of unity

Let's say we want the cube roots of 1. Obviously, all the roots will have length 1. What about the angles? The starting angle $\theta = 0$, so $\phi = 2k\pi/3$ and

$$\phi_1 = \frac{2\pi}{3}$$

$$\phi_2 = \frac{4\pi}{3}$$
$$\phi_3 = \frac{6\pi}{3} = 0$$

Notice that the first and second roots are complex conjugates because

$$\phi_1 + \phi_2 = \frac{6\pi}{3} = 2\pi = 0$$

Suppose our number is $z = -8i$ and we want the cube roots. Writing the number in polar coordinates:

$$z = 8e^{3\pi/2}$$

All of the roots have the same modulus, 2, since $2^3 = 8$. The are three roots which differ in their arguments. Since $\theta = 3\pi/2$, these are:

$$\phi_1 = \frac{\theta}{3} = \frac{\pi}{2}$$
$$\phi_2 = \frac{\theta + 2\pi}{3} = \frac{\pi}{2} + \frac{2\pi}{3} = \frac{5\pi}{6}$$
$$\phi_3 = \frac{\theta + 4\pi}{3} = \frac{\pi}{2} + \frac{4\pi}{3} = \frac{7\pi}{6}$$

Notice that the second and third roots are complex conjugates.

We take the original angle and multiply by the power that the root corresponds to. Then, divide 2π up into that many pieces, and add k pieces where k runs from 0 to $r - 1$.

When the argument for z is θ_0, a general formula for the angle of the nth root of z is:

$$\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n} \quad k = 0, \pm 1, \pm 2 \ldots$$

We derive this as follows:

$$z = re^{i\theta}$$

$$z^{1/n} = (re^{i(\theta+2k\pi)})^{1/n}$$

Writing only the argument part

$$(e^{i(\theta+2k\pi)})^{1/n} = e^{i(\theta/n+2k\pi/n)}$$

Nahin's puzzle

In one of his books Nahin starts by posing this question: suppose we are given that

$$x + \frac{1}{x} = 1$$

Without computing x, find the value of

$$x^7 + \frac{1}{x^7}$$

Nahin says that if you are the type to just start right in trying to figure this out, then you will like his book.

From its placement in this section, you might just guess the answer. First of all, no real x solves the equation

$$x + \frac{1}{x} = 1$$

as you will see if you use the quadratic formula. So let's change nomenclature and call it z.

(Of course, we were not supposed to *compute* z).

We may guess that z is a complex number with length 1 so that the lengths don't change with powers or roots.

Then, all that happens is that θ changes in such a way that

$$7\theta = \theta = \frac{\theta}{7}$$

To actually compute z, multiply by z, rearrange, and solve:

$$z^2 + 1 = z$$

$$z^2 - z + 1 = 0$$

From the quadratic equation:

$$z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

The square of the length is

$$r^2 = zz*$$
$$= (\frac{1}{2} + i\frac{\sqrt{3}}{2})(\frac{1}{2} - i\frac{\sqrt{3}}{2})$$
$$= \frac{1}{4} + \frac{3}{4} = 1$$

The angle we seek has tangent equal to $1/\sqrt{3}$. You may recognize the sine and cosine of $\pi/3$ as the real and imaginary components of z.

So if

$$z = e^{i\pi/3} = (\frac{1}{2} + i\frac{\sqrt{3}}{2})$$
$$\frac{1}{z} = e^{-i\pi/3} = (\frac{1}{2} - i\frac{\sqrt{3}}{2})$$

then when doing the addition the imaginary parts of z cancel and we have that

$$z + \frac{1}{z} = \frac{1}{2} + \frac{1}{2} = 1$$

The other special attribute of this value for z is that the length is 1 so all powers of r are 1. As for the angle, $\pi/3$ is special in that $7 \times \pi/3 = 2\pi + \pi/3 = \pi/3$. Now it's not strictly true that *the* 7th root of θ is equal to θ (since there are 7 distinct roots). But I hope you can see that there is at least one such root.

Part III

Transcendentals

Chapter 10

Sine and cosine

cosine and sine

Start by recalling Euler's formula for *real* x:

$$e^{ix} = \cos x + i \sin x$$

Substitute $-x$ for x

$$e^{-ix} = \cos -x + i \sin -x$$
$$= \cos x - i \sin x$$

Addition gives:

$$2 \cos x = e^{ix} + e^{-ix}$$
$$\cos x = \frac{1}{2} \left(e^{ix} + e^{-ix} \right)$$

Subtraction:

$$2i \sin x = e^{ix} - e^{-ix}$$
$$\sin x = \frac{1}{2i} \left(e^{ix} - e^{-ix} \right)$$

Our old friends:

$$\cosh x = \frac{1}{2}\ (e^x + e^{-x})$$
$$\sinh x = \frac{1}{2}\ (e^x - e^{-x})$$

complex versions

The complex counterparts of the real trigonometric functions can be explained by saying that Euler's formula is also good for a complex number z (a math book would define them by their power series).

By the same algebra, this gives

$$\cos z = \frac{1}{2}\ (e^{iz} + e^{-iz})$$
$$\sin z = \frac{1}{2i}\ (e^{iz} - e^{-iz})$$

Now we see that the complex sine and cosine have properties just like their real cousins.

We will do the complex hyperbolic functions in the next chapter.

period

The above definition of cosine is

$$\cos z = \frac{1}{2}\ (e^{iz} + e^{-iz})$$

then

$$cos(z + 2\pi) = \frac{1}{2}\ (e^{iz}e^{i2\pi} + e^{-iz}e^{-i2\pi})$$

but

$$e^{i2\pi} = \cos 2\pi + i \sin 2\pi = 1$$

and the same for $e^{-i2\pi}$, so

$$\cos(z + 2\pi) = \cos z$$

The *period* of the complex cosine and sine is 2π, just as for the real function.

derivatives

Take derivatives is straightforward:

$$\sin z = \frac{1}{2i}\ (e^{iz} - e^{-iz})$$

$$\frac{d}{dz}\ \sin z = i \cdot \frac{1}{2i}\ (e^{iz} + e^{-iz}) = \cos z$$

Similarly

$$\cos z = \frac{1}{2}\ (e^{iz} + e^{-iz})$$

$$\frac{d}{dz}\ \cos z = \frac{i}{2}\ (e^{iz} - e^{-iz})$$

$$= -\frac{1}{2i}\ (e^{iz} - e^{-iz}) = -\sin z$$

Also

$$\sin -z = \frac{1}{2i}\ (e^{-iz} - e^{iz}) = -\sin z$$

$$\cos -z = \frac{1}{2}\ (e^{-iz} + e^{iz}) = \cos z$$

separating real and imaginary parts of trig functions

Since

$$\cos z = \frac{1}{2}\ (e^{iz} + e^{-iz})$$

if we let $z = iy$ then

$$\cos iy = \frac{1}{2}\ (e^{i^2 y} + e^{-i^2 y})$$

$$= \frac{1}{2}\ (e^{-y} + e^{y}) = \cosh y$$

Similarly

$$\sin iy = \frac{1}{2i}\ (e^{i^2 y} - e^{-i^2 y})$$

$$= \frac{1}{2i}\ (e^{-y} - e^{y})$$

$$= -\frac{1}{2i}\ (e^{y} - e^{-y})$$

$$= -\frac{1}{i} \sinh y = i \sinh y$$

Hence

$$\cos iy = \cosh y$$

$$\sin iy = i \sinh y$$

So now if we let $z = x + iy$ and use the standard addition formula

$$\cos z = \cos(x + iy)$$

gives

$$\cos z = \cos x \cos iy - \sin x \sin iy$$

Since $\cos iy = \cosh y$ and $\sin iy = i \sinh y$:

$$= \cos x \cosh y - i \sin x \sinh y$$

and what's nice about this is that we have the real and imaginary parts of the complex cosine easily visible.

Similarly

$$\sin z = \sin(x + iy)$$

$$\sin z = \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

These are very similar to the sum of angles results for real numbers. It's just that $z = x + iy$ means the y gets the hyperbolic functions and sinh has a leading factor of i.

using the exponential to get cos z and sin z

We can obtain the same results by working through the formulas using the complex exponential.

Work backward from the answer:

$$\cos x \cosh y = \frac{(e^{ix} + e^{-ix})(e^y + e^{-y})}{4}$$

$$= \frac{e^{ix}e^y + e^{ix}e^{-y} + e^{-ix}e^y + e^{-ix}e^{-y}}{4}$$

and then also

$$i \sin x \sinh y = \frac{(e^{ix} - e^{-ix})(e^y - e^{-y})}{4}$$

$$= \frac{(e^{ix}e^y - e^{ix}e^{-y} - e^{-ix}e^y + e^{-ix}e^{-y})}{4}$$

Subtraction gives cancelations:

$$= \frac{e^{ix}e^{-y} + e^{-ix}e^y}{2}$$

And now there's a trick. The exponents of the first product in the numerator add to give

$$ix - y = i(x + iy) = iz$$

the second is

$$-ix + y = -(ix - y) = -iz$$

So we have just

$$\frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \cos z$$

The sine was

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Working with one term at a time, we have

$$\sin x \cosh y = \frac{(e^{ix} - e^{-ix})}{2i} \cdot \frac{(e^y + e^{-y})}{2}$$

$$= \frac{e^{ix}e^y + e^{ix}e^{-y} - e^{-ix}e^y - e^{-ix}e^{-y}}{4i}$$

and

$$i \cos x \sinh y = -\frac{\cos x \sinh y}{i}$$

$$= -\frac{1}{i} \cdot \frac{(e^{ix} + e^{-ix})}{2} \cdot \frac{(e^y - e^{-y})}{2}$$

$$= -\frac{e^{ix}e^y - e^{ix}e^{-y} + e^{-ix}e^y - e^{-ix}e^{-y}}{4i}$$

$$= \frac{-e^{ix}e^y + e^{ix}e^{-y} - e^{-ix}e^y + e^{-ix}e^{-y}}{4i}$$

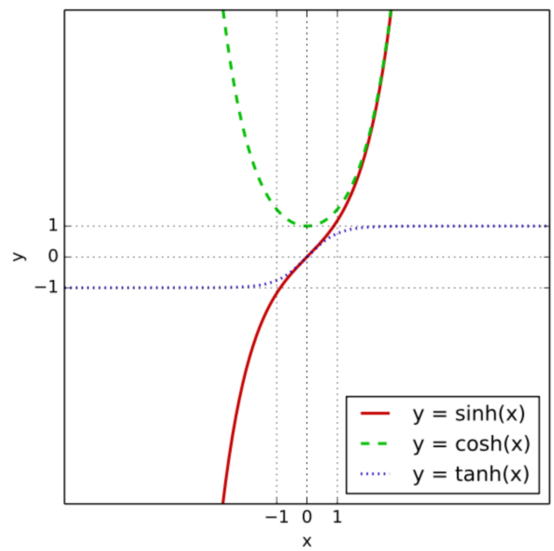
Addition gives cancelations:

$$= \frac{e^{ix}e^{-y} - e^{-ix}e^{-y}}{2i}$$

Recall from above that $ix - y = iz$ and $ix + y = -iz$ so

$$= \frac{e^{iz} - e^{-iz}}{2i}$$
$$= \sin z$$

zeroes



cosh is never zero, while only $\sinh 0 = 0$.

So if we look again at

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

and ask, where is this function equal to zero?

Both parts must vanish. Since cosh is never zero, $\sin x$ must be zero. This happens for $x = 2k\pi$.

The cosine of this x is equal to 1, that means $\sinh y$ must be 0 which only happens for $y = 0$.

So the zeroes of the complex sine function are at $z = 2k\pi + 0i$.

Alternatively, go back to the original definition:

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

which vanishes only for

$$e^{iz} = e^{-iz} = \frac{1}{e^{iz}}$$

$$e^{2iz} = \; [\; e^{iz} \;]^2 \; = 1$$

$$e^{iz} = \pm 1$$

$$e^{i(x+iy)} = \pm 1$$

$$e^{-y} e^{ix} = \pm 1$$

$$e^{-y}(\cos x + i \sin x) = \pm 1$$

The imaginary part must be zero, so $x = 2k\pi$.

The other part must be equal to ± 1, so $y = 0$ and $\cos 2k\pi = 1$, which works.

For the cosine

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

This is equal to zero when

$$e^{iz} = -e^{-iz} = -\frac{1}{e^{iz}}$$

$$e^{2iz} = -1$$

$$e^{i(x+iy)} = \pm i$$

$$e^{-y}(\cos x + i \sin x) = \pm i$$

In this case we need $\cos x = 0$ and then $y = 0$ and $\sin x = 1$ will work. $x = (2k+1)\pi/2$.

Recall that

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

Since cosh is never zero, $\cos x$ must be zero. Then either $\sin x = 0$ or $\sinh y = 0$. Only the latter works for the non-imaginary part, so we have that $y = 0$.

summary

The definition:

$$\cos z = \frac{1}{2}\ (e^{iz} + e^{-iz})$$

$$\sin z = \frac{1}{2i}\ (e^{iz} - e^{-iz})$$

$$\cosh x = \frac{1}{2}\ (e^{x} + e^{-x})$$

$$\sinh x = \frac{1}{2}\ (e^{x} - e^{-x})$$

A pair of identities

$$\cos iy = \cosh y$$

$$\sin iy = i \sinh y$$

By the sum of angles formula, or by manipulating the exponential

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Chapter 11

More trigonometry

analyticity

We proved before that the complex exponential obeys the CRE, which means that it is analytic. There is a theorem that says that if we add two analytic functions together, the result is also analytic. Hence, the trigonometric functions are analytic.

But, just to check this result, let's write them out in terms of u and v and see whether the partial derivatives follow the CRE conditions:

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Taking the derivatives:

$$u(x, y) = \sin x \cosh y$$

$$u_x = \cos x \cosh y$$

$$u_y = \sin x \sinh y$$

and

$$v(x, y) = \cos x \sinh y$$

$$v_x = -\sin x \sinh y$$

$$v_y = \cos x \cosh y$$

So we see that indeed

$$u_x = v_y$$

$$u_y = -v_x$$

The CRE are satisfied and therefore, the complex sine is analytic.

Similarly we have that

$$\cos z = \cos(x + iy)$$

$$= \cos x \cos iy - \sin x \sin iy$$

$$= \cos x \cosh y - i \sin x \sinh y$$

So

$$u(x, y) = \cos x \cosh y$$

$$u_x = -\sin x \cosh y$$

$$u_y = \cos x \sinh y$$

and

$$v(x, y) = -\sin x \sinh y$$

$$v_x = -\cos x \sinh y$$

$$v_y = -\sin x \cosh y$$

So we see that

$$u_x = v_y$$

$$u_y = v_x$$

Thus the complex cosine is also analytic.

We can also prove that:

$$\sin^2 z + \cos^2 z = 1$$

The easy way is

$$\cos^2 z + \sin^2 z = \ [\frac{e^{iz} + e^{-iz}}{2}\ ]^2 + [\frac{e^{iz} - e^{-iz}}{2i}\ ]^2\ ]$$

$$= \frac{e^{2iz} + 2 + e^{-2iz} - e^{2iz} + 2 - e^{-2iz}}{4}$$

$$= 1$$

series

On the other hand, Shankar defines the trig functions and the exponential using series in the same way as the real versions:

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \cdots = \sum_0^\infty (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \cdots = \sum_0^\infty (-1)^n \frac{z^{2n}}{(2n)!}$$

$$\sinh z = \sum_0^\infty \frac{z^{2n+1}}{(2n+1)!}$$

$$\cosh z = \sum_0^\infty \frac{z^{2n}}{(2n)!}$$

and showing that they converge for any z.

complex hyperbolics

The definition is analogous to the real case:

$$\cos z = \frac{1}{2}\ [\ e^z + e^{-z}\ ]$$

$$= \frac{1}{2}\ [\ e^{i(x+iy)} + e^{-i(x+iy)}\ ]$$

$$= \frac{1}{2}\ [\ e^{ix-y} + e^{-ix+y}\ ]$$

Double the top and the bottom

$$= \frac{e^{ix-y} + e^{-ix+y} + e^{ix-y} + e^{-ix+y}}{4}$$

The pattern in the exponents is

$$+- \quad -+ \quad +- \quad -+$$

We reach a new pattern by first switching the order to

$$-+ \quad +- \quad -+ \quad +-$$

$$= \frac{e^{-ix+y} + e^{ix-y} + e^{-ix+y} + e^{ix-y}}{4}$$

then add and subtract terms with $++$ and $--$, like this:

$$= \frac{e^{ix+y} + e^{-ix+y} + e^{ix-y} + e^{-ix-y}}{4} - \frac{e^{ix+y} - e^{-ix+y} - e^{ix-y} + e^{-ix-y}}{4}$$

Now we realize that we can factor the first term as:

$$= \frac{(e^{y} + e^{-y})}{2}\ \frac{(e^{ix} + e^{-ix})}{2}$$

$$= \cosh y \cos x$$

The second term is:

$$= -\frac{(e^{y} - e^{-y})}{2}\ \frac{(e^{ix} - e^{-ix})}{2}$$

$$
\begin{aligned}
&= -i\frac{(e^{y} - e^{-y})}{2}\ \frac{(e^{ix} - e^{-ix})}{2i} \\
&= -i \sinh y \sin x
\end{aligned}
$$

Putting it all together:

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

That required a lot of bookkeeping, and now we have to go back and repeat it all for the sine. And this is basically a repeat of the derivation in the last chapter. It's just nice to see the factoring trick.

Chapter 12

Exponential

Consider first the generic complex number

$$z = x + iy$$

and write

$$\begin{aligned} f(z) &= e^z \\ &= e^{x+iy} \\ &= e^x e^{iy} \end{aligned}$$

We can visualize the complex exponential as having a modulus or length e^x and argument or angle θ of y.

Then, using Euler's formula we can decompose this:

$$e^x e^{iy} = e^x(\cos y + i \sin y)$$

$$= e^x \cos y + i e^x \sin y$$

So if $f(z) = u(x, y) + iv(x, y)$, the real part of e^z is

$$u = e^x \cos y$$

with partial derivatives

$$u_x = e^x \cos y$$

$$u_y = -e^x \sin y$$

and the imaginary part of e^z is

$$v = e^x \sin y$$

with partial derivatives

$$v_x = e^x \sin y$$

$$v_y = e^x \cos y$$

Hence

$$u_x = e^x \cos y = v_y$$

$$u_y = -e^x \sin y = -v_x$$

In other words, these two important conditions hold for the complex exponential:

$$u_x = v_y$$

$$u_y = -v_x$$

These are the famous Cauchy-Riemann equations (CRE) or CR conditions.

When the CRE are satisfied then the function in question is a "good" function — it is one we can do calculus with. It has a derivative.

For this reason, the complex exponential e^z is said to be analytic.

(Which, according to Shankar, we could have predicted, since it depends only on z and not on $z*$).

derivative

We showed before that we can evaluate the derivative along $\Delta y = 0$ as:

$$f'(z) = u_x + iv_x$$

We obtain

$$= e^x \cos y + ie^x \sin y = z$$

The exponential is its own derivative.

This is tremendously important because we want our definitions for complex functions to give the standard results when z has only a real part, i.e. when $y = 0$.

Now, once more we recall Euler's formula (for a real variable θ or x):

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{ix} = \cos x + i\sin x$$

Substitute $-x$ for x:

$$e^{-ix} = \cos -x + i\sin -x$$

$$= \cos x - i\sin x$$

By addition and subtraction we obtain:

$$2\cos x = e^{ix} + e^{-ix}$$

$$\cos x = \frac{1}{2}\ (e^{ix} + e^{-ix})$$

and

$$2i\sin x = e^{ix} - e^{-ix}$$

$$\sin x = \frac{1}{2i}\ (e^{ix} - e^{-ix})$$

we'll see a lot more of this coming up.

alternative derivations

Another proof that the derivative of the complex exponential is as we would hope and expect:

$$\frac{d}{dz}e^z = e^z$$

uses a Taylor series. Shankar says to define e^z in the same way as e^x. For the real series:

$$e^x = \sum_0^\infty \frac{x^n}{n!}$$

which we know converges, since the ratio of successive terms is

$$R = \frac{x^{n+1}}{(n+1)!}\ \frac{n!}{x^n} = \frac{x}{n+1}$$

We ask, for what values of x is the limit

$$\lim_{n\to\infty} R = 0\ ??$$

This is true for all x.

For the complex exponential:

$$e^z = \sum_0^\infty \frac{z^n}{n!}$$

and again we see that

$$\frac{d}{dz}\ e^z = e^z$$

differentiating the series term by term.

Another approach (from McMahon) uses the limit definition:

$$\frac{d}{dz}f(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\frac{d}{dz}e^z = \lim_{\Delta z \to 0} \frac{e^{z+\Delta z} - e^z}{\Delta z}$$

and just as in the real case, we factor out

$$= e^z \lim_{\Delta z \to 0} \frac{e^{\Delta z} - 1}{\Delta z}$$

This limit will turn out to be equal to 1.

Use Euler's formula to get this expression in x and y:

$$\frac{e^{\Delta z} - 1}{\Delta z} = \frac{e^{\Delta x + i\Delta y} - 1}{\Delta x + i\Delta y}$$

$$= \frac{e^{\Delta x}(\cos\Delta y + i\sin\Delta y) - 1}{\Delta x + i\Delta y}$$

$$= \frac{(e^{\Delta x}\cos\Delta y - 1) + ie^{\Delta x}\sin\Delta y}{\Delta x + i\Delta y}$$

The real part of the numerator is

$$\lim_{\Delta x, \Delta y \to 0} e^{\Delta x}\cos\Delta y - 1$$

Both the Δx and the Δy term tend to 1 in the limit, so the entire expression for the real part of the numerator is equal to zero. We are left with

$$\lim_{\Delta x, \Delta y \to 0} \frac{ie^{\Delta x}\sin\Delta y}{\Delta x + i\Delta y}$$

The trick is that we actually set $x = 0$ *first*

$$\lim_{\Delta y \to 0} \frac{ie^0 \sin\Delta y}{0 + i\Delta y} = \lim_{\Delta y \to 0} \frac{\sin\Delta y}{\Delta y} = 1$$

and the last part is the famous limit from calculus.

other properties

The complex exponential

$$e^z = e^x e^{iy}$$

has some properties that are not shared with the real exponential. As we saw before, the angle $\theta + 2\pi = \theta$ (and $2\pi = 0$), so any angle is really a family of angles with different $\theta + 2\pi k$ for integer k.

In particular, $e^z = e^x e^{iy}$ is periodic with a period of $2\pi i$.

Additionally, it is possible for e^z to be negative. Consider that it is possible that

$$e^z = -1$$

as follows. Let

$$z = 0 + i\pi$$

This is a point on the y-axis, a distance π from the origin, and purely imaginary.

Then

$$e^x = e^0 = 1$$

and

$$e^{iy} = e^{i\pi} = -1$$

So

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y) = 1(-1) = -1$$

On the other hand, e^z **cannot be zero**.

$$e^z = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y)$$

For $x \in \mathbb{R}$, $e^x > 0$.

So the only way this could be zero would be if we can find a y such that $\sin y$ and $\cos y$ were both zero. Since there is no such y, we conclude that e^z cannot be equal to zero.

Chapter 13

Logarithm

Nearly everything works for the logarithm of z similarly to the real numbers, except for the issue of multiple phase angles or complex arguments. For example

$$\log(z) = \log(re^{i\theta})$$

$$= \log(r) + \log(e^{i\theta})$$

$$= \ln r + i\theta$$

but we may have any multiple of $k \cdot 2\pi$ added to θ

$$\log(z) = \ln r + i(\theta + k2\pi)$$

We call one particular range of 2π the range for the *principal value* of the function.

Here it is natural to make the range go from $-\pi < \theta < \pi$. The reason is that the negative x-axis consists of negative real numbers, for which the natural logarithm isn't defined, and neither is the complex logarithm.

So we exclude that from the domain of the complex logarithm.

This is called a "branch cut," where we take one particular branch of this multi-valued function.

Here is a derivation.

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$r = |z| = \sqrt{x^2 + y^2}$$

The logarithm of z is w

$$w = \log z \iff e^w = z$$

So what about w? Well, in general, it's a complex number

$$w = s + it$$

so

$$e^w = e^{s+it} = e^s(\cos t + i\sin t)$$

Equating the two we get

$$r(\cos\theta + i\sin\theta) = e^s(\cos t + i\sin t)$$

Hence

$$s = \ln r$$

$$t = \theta$$

$$w = \ln r + i\theta$$

different base

What is

$$i^i = ?$$

The complex logarithm of i is

$$\log i = \ln r + i\theta = \ln 1 + i\frac{\pi}{2} = i\frac{\pi}{2}$$

Write

$$a^z = (e^{\log a})^z$$

$$i^i = (e^{\log i})^i = (e^{i\pi/2})^i = e^{-\pi/2}$$

Not only is i to the *ith* power computable, it is entirely real. It is ≈ 0.2079.

derivative

When we study the Cauchy-Riemann equations we will show that if $f(z)$ is differentiable, then the CRE hold. The converse theorem is also true, that if the CRE hold, then $f(z)$ is differentiable, and its derivative is

$$f'(z) = u_x + iv_x$$

We have that the logarithm function is

$$\log(z) = \ln|z| + i\theta$$

Rewriting in terms of x and y we have that

$$\log(x+iy) = \ln(\sqrt{x^2+y^2}) + i\tan^{-1}(\frac{y}{x})$$

$$\log(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}(\frac{y}{x})$$

So

$$u(x,y) = \frac{1}{2}\ln(x^2+y^2)$$

$$u_x = \frac{1}{2}\,\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$$

$$u_y = \frac{y}{x^2+y^2}$$

and

$$v(x,y) = \tan^{-1}(\frac{y}{x})$$

$$v_x = \frac{1}{1+(y/x)^2}\ y\ (-\frac{1}{x^2}) = \frac{-y}{x^2+y^2}$$

$$v_y = \frac{1}{1+(y/x)^2}\ \frac{1}{x} = \frac{x}{x^2+y^2}$$

We see that CRE are satisfied and that means that the derivative is

$$[\ \log z\ ]' \ = u_x + iv_x$$

$$= \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

$$= \frac{1}{x^2+y^2}(x - iy)$$

$$= \frac{1}{|z|^2}\ z*$$

$$= \frac{1}{zz*}\ z* = \frac{1}{z}$$

The derivative of the complex logarithm is the inverse of z, completely analogous to the real case.

Chapter 14

Summary 1

We have expressions $f(z) = u(x,y) + iv(x,y)$ for all standard complex functions.

Powers can be computed easily

$$z = x + iy$$

$$(x+iy)^2 = x + 2x(iy) + (iy)^2$$
$$(x+iy)^3 = x + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$
$$\dots$$

We worked with the inverse function using the complex conjugate $z* = x - iy$ so

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{z*}{z*}$$

Exponential:

$$e^z = e^{x+iy} = e^x(\cos x + i \sin x)$$

Sine and cosine:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$= \cos x \cosh y - i \sin x \sinh y$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$= \sin x \cosh y + i \cos x \sinh y$$

Logarithm:

$$w = \log z = \ln r + i\theta$$

derivatives

We showed that all of these functions are analytic, with $u_x = v_y$ and $u_y = -v_x$, so therefore, their derivatives can be computed as $f'(z) = u_x + iv_x$.

When this is done, they turn out to be just what you'd want:

$$(z^n)' = nz^{n-1}$$
$$(e^z)' = e^z$$
$$(\sin z)' = \cos z$$
$$(\cos z)' = -\sin z$$
$$(\log z)' = \frac{1}{z}$$

The only ones we haven't done are the roots.

We will not do a general proof, but let's go through the square root. It will remind us of the special features of polar CRE and derivatives.

square root

$$f(z) = \sqrt{z}$$

Use polar notation so $z = re^{i\theta}$ and then

$$f(z) = \sqrt{z} = \sqrt{r}e^{i\theta/2}$$

Using Euler:

$$= \sqrt{r}(\cos\theta/2 + i\sin\theta/2)$$
$$u = \sqrt{r}\cos\theta/2$$
$$v = \sqrt{r}\sin\theta/2$$

The partials are:

$$u_r = \frac{1}{2\sqrt{r}}\cos\theta/2$$
$$u_\theta = \frac{\sqrt{r}}{2}(-\sin\theta/2)$$

and

$$v_r = \frac{1}{2\sqrt{r}}\sin\theta/2$$
$$v_\theta = \frac{\sqrt{r}}{2}\cos\theta/2$$

At first we're worried ($u_r \neq v_\theta$), but then we recall the polar CRE have an extra factor of r:

$$ru_r = v_\theta$$
$$rv_r = -u_\theta$$

So the CRE do obtain, and we can get the derivative.

Next, we recall the second unusual thing about the polar derivative:

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

Leave aside the factor of $e^{-i\theta}$ out front and just combine:

$$
\begin{aligned}
u_r + iv_r &= \frac{1}{2\sqrt{r}} \cos\theta/2 + \frac{1}{2\sqrt{r}} \sin\theta/2 \\
&= \frac{1}{2\sqrt{r}} \; (\cos\theta/2 + i\sin\theta/2) \\
&= \frac{e^{i\theta/2}}{2\sqrt{r}}
\end{aligned}
$$

which appears problematic, but the extra factor gives us just what we need

$$
\begin{aligned}
f'(z) &= e^{-i\theta} \cdot \frac{1}{2\sqrt{r}} \; e^{i\theta/2} \\
&= \frac{1}{2\sqrt{r}} \; e^{-i\theta/2} \\
&= \frac{1}{2\sqrt{r}} \cdot \frac{1}{e^{i\theta/2}} \\
&= \frac{1}{2\sqrt{z}}
\end{aligned}
$$

Just exactly analogous to the real function.

Part IV

Integration

Chapter 15

Integration

In this chapter, we begin working complex integrals. One way to do them relies on the Cartesian definitions:

$$z = x + iy$$

$$dz = dx + i\ dy$$

$$f(x, y) = u(x, y) + i\ v(x, y)$$

$$w = f(z) = u(x, y) + i\ v(x, y)$$

thus

$$\int f(z)\ dz = \int (u + iv)(dx + idy)$$

$$= \int u\ dx - v\ dy + i\ [\ u\ dy + v\ dx\ ]$$

This looks like but really is not a double integral, because x and y are *related*, either as parametric equations in t or θ, or simply because $y = f(x)$.

Introduction

A complex function is a function that produces a complex number as the result. The most general case is that the input is a complex number as well.

Se we could write:

$$w = f(z)$$

where both w and z are complex numbers.

Complex functions are differentiated and integrated in a way that is similar to real functions, but with some differences. We've already seen that the derivatives are like their real cousins, and the integrals will be too.

A main difference is that the integrals are line integrals. We will set up integrals which look like multi-variable integrals, but the two variables are connected because they lie on a path.

Because they are line integrals and most integrals will evaluate to something like e^{it}, going around on a circle in a closed path usually gives a value of zero. The exceptions are functions that contain z^* in some, that are non-analytic.

However, we often but not always restrict our attention to functions that are analytic at most points, paying attention to points in the complex plane where they have poles (or singularities). Since they are not defined at these points, they cannot be analytic there.

getting started

The key is to write the integral as

$$\int f(z)\ dz = \int (u + iv)(dx + idy)$$

Now group the pieces as:

$$= \int u \; dx - v \; dy + iv \; dx + iu \; dy$$

The integral of a complex function is defined as a sum of integrals of two real variables. Just as with line integrals for real functions of x and y, the variables are related by the curve over which we will integrate.

Recall that for the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M \; dx + N \; dy$$

we parametrize the curve to get the integral over a single variable.

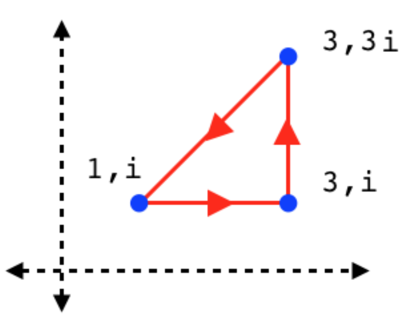
We can view y as a function of x or perhaps, we will be able to parametrize both x and y as functions of t.

example: z

The function is simply $z = x + iy$. The integral is

$$= \int x \; dx - y \; dy + ix \; dy + iy \; dx$$

Now we must get y in terms of x from the curve. Suppose the curve goes from $(1, i)$ to $(3, i)$, then to $(3, 3i)$ and finally back to where we started.



We have three segments. Along the first part, the path is in the positive x direction, with no change in y, $y = 1$, a constant, so $dy = 0$, and the integral is (writing the non-zero parts only):

$$I = \int x\ dx + i \cdot 1\ dx$$

$$= \int_{x=1}^{x=3} x + i\ dx$$

$$= \frac{x^2}{2} + ix\ \Big|_1^3 = 4 + 2i$$

Along the second part, we are moving in the positive y direction with $dx = 0$ and $x = 3$ so

$$= \int_{y=1}^{y=3} -y\ dy + 3i\ dy$$

$$= -\frac{y^2}{2} + 3iy\ \Big|_1^3 = -4 + 6i$$

Along the third path, both dx and dy are non-zero. The parametrization is $y = f(x) = x$. Hence $dy = dx$.

$$= \int x\ dx - x\ dx + ix\ dx + ix\ dx$$

$$= 2i \int x\ dx$$

For the closed path, where we end up back at the starting point, $C3$ should be moving from $(3, 3)$ to $(1, 1)$ so we have

$$2i\ \frac{x^2}{2}\ \Big|_3^1 = 2i(-\frac{8}{2}) = -8i$$

Notice that

$$\int_{C1} + \int_{C2} = 8i = -\int_{C3}$$

If we follow the curve $C3$ from $(3,3)$ to $(1,1)$, the whole thing is just zero. We'll see that this is not a coincidence.

example 2

Suppose the function is

$$f(z) = y - x - i3x^2$$

So $u = (y - x)$ and $v = -3x^2$ and the integral is

$$= \int u\ dx - v\ dy + i\ [\ u\ dy + v\ dx\ ]$$

$$\int (y-x)dx + 3x^2dy + i\ [\ (-3x^2)dx + (y-x)dy\ ]$$

The path goes from the origin to the point $z = 1 + i$ either directly (C) or by first going up vertically ($C1$) and then across (C_2).

For the vertical part (C_1) we have that $x = 0$ and $dx = 0$.

$$\int (y-x)dx + 3x^2dy + i(-3x^2)dx + i(y-x)dy$$

$$I_1 = \int i(y-x)dy = \int iydy$$

It's important to recognize that although we are proceeding from the point $z = 0$ to the point $z = i$, the upper bound on this integral is not i but $y = 1$! Hence

$$I_1 = i\frac{y^2}{2}\ \Big|_0^1 = \frac{i}{2}$$

For the horizontal part (C_2) we have that $y = 1$ and $dy = 0$ so

$$I_2 = \int (y - x)dx + i(-3x^2)dx$$

$$= \int (1 - x)dx + i(-3x^2)dx$$

x goes from 0 to 1

$$= x - \frac{x^2}{2} - ix^3 \Big|_0^1 = \frac{1}{2} - i$$

Therefore the total

$$I = \frac{i}{2} + \frac{1}{2} - i = \frac{1}{2}\,(1 - i)$$

When going directly from the origin to $1 + i$ we have $y = x$ so $dy = dx$ and

$$I = \int (y - x)dx + 3x^2dy + i(-3x^2)dx + i(y - x)dy$$

$$= \int 3x^2dx + i(-3x^2)dx$$

$$= x^3 \ - i\ x^3 \Big|_0^1 = 1 - i$$

And around the closed curve going backward along C:

$$I1 = i\ \frac{1}{2}$$

$$I2 = \frac{1}{2} - i$$

$$I3 = 1 - i$$

That last one is in the direction $0 \rightarrow 1$ so we must subtract it:

$$\oint f(z)\ dz = i\frac{1}{2} + \frac{1}{2} - i - (1-i) = -\frac{1}{2} + i\ \frac{1}{2}$$

This time, even though we returned to our starting point (traversing a *closed* path), the result is not zero.

Notice that

$$f(z) = y - x - i3x^2$$

$$u_x = -1 \neq v_y = 0$$

This function is not analytic.

example: z revisited

Above we wrote:

$$\int z\ dz = \int x\ dx - y\ dy + ix\ dy + iy\ dx$$

And then said: now we must get y in terms of x from the curve.

But what if we don't worry about the curve?

Just write $y = f(x)$ and $dy = f'(x)\ dx$ and see what happens:

$$\int x\ dx - f(x)\ f'(x)\ dx + i\ [\ x\ f'(x)\ dx + f(x)\ dx$$

It helps that we know the answer:

$$\frac{1}{2}z^2 = \frac{1}{2}(x^2 - y^2 + i2xy)$$

$$\int x\ dx = \frac{x^2}{2}$$

So that's the first term. then

$$\int f(x)\ f'(x)\ dx$$

but this is just

$$\frac{1}{2}\ [f(x)]^2 = \frac{1}{2}y^2$$

We're on to something!

For the imaginary part:

$$\int x\ f'(x)\ dx + f(x)\ dx$$

Knowing the answer, we can see that the integrand is the derivative

$$[\ x\ f(x)\ ]'$$

by the product rule. But that's just xy, so this matches.

Chapter 16

Circular paths

In general, for a parametrized curve, we have

$$\int f[z(t)]\ z'(t)\ dt$$

A particularly important parametrization for complex integrals is the circular path. Such a path can be described as C[origin, radius], as in $C[z_0, r]$, and another frequently used notation is $|z - z_0| = r$.

Usually we are interested in closed circular paths, where z goes around the whole circle and ends up at the start. On such a path, z takes on values with constant r and the only change is in t (or θ).

We can describe this as

$$z = re^{it}$$

with $t \in [0, 2\pi]$.

dz

For this parametrization, we have that

$$dz = ire^{it}\ dt = iz\ dt$$

function: 1

$$\int_C dz = \int_C ire^{it}\ dt = ir\ \frac{e^{it}}{i} = z$$

function: z

$$\int z\ dz = \int re^{i\theta}\ rie^{i\theta} d\theta$$

$$= r^2 i \int e^{i2\theta}\ d\theta$$

$$= r^2 i\ \frac{e^{i2\theta}}{2i} = \frac{1}{2}\ r^2 e^{i2\theta} = \frac{1}{2}\ z^2$$

For any closed path, the starting and ending z are the same, so the value is zero.

Alternatively, the result can be written $\rho e^{i\phi}$, but that exponential is really two trig functions since $e^{i\phi} = \cos\phi + i\sin\phi$, with a period of 2π. Evaluated over any closed path, the result is zero.

For the unit circle, evaluated for the part of the circle between $\theta = 0 \to \pi/2$ we get

$$\frac{1}{2}\ e^{i2\theta}\Big|_0^{\pi/2} = \frac{1}{2}(\ e^{i\pi} - 1)$$

$$= \frac{1}{2}(\cos\pi + i\sin\pi - 1)$$

$$= \frac{1}{2}(-1 - 1) = -1$$

function: z^n**,** $n \in \cdots - 3, -2, 1, 2, 3\ldots$

$$\int z^n\ dz = \int r^n e^{int}\ rie^{it}\ dt$$

$$= ir^{n+1} e^{i(n+1)t}\ dt$$

$$= ir^{n+1} \frac{e^{i(n+1)t}}{i(n+1)}$$
$$= \frac{1}{n+1} \ z^{n+1}$$

Whether we say that z is the same at the beginning and the end, or that we have something like e^{it} with a period of 2π, the result for a full circle will always be zero.

However, the inverse is special. Notice that it is not on the list of exponents.

example: z in terms of (x, y)

Let's repeat the calculation for $\int z \ dz$ in terms of u, v and dx, dy but parametrized as $x = \cos t$ etc.:

$$\int u \ dx - v \ dy + i \ [\ u \ dy + v \ dx \ ]$$

Substitute from $u = x, v = y$:

$$\int x \ dx - y \ dy + i \ [\ x \ dy + y \ dx \ ]$$

Consider a path around the unit circle. Then

$$x = \cos t$$
$$y = \sin t$$
$$dx = -\sin t \ dt$$
$$dy = \cos t \ dt$$

substitute in the integral

$$\int \cos t(-\sin t \ dt) - \sin t \cos t \ dt + i \ [\ \cos^2 t \ dt - \sin^2 t \ dt \ ]$$

$$-2\int \sin t\cos t \ dt + i\int \cos^2 t - \sin^2 t \ dt$$

If this were not a unit circle, each term would have a factor of r, one for the u, v part and one for the differential, so there would be a leading factor of r^2.

Using u-substitution (and reusing the symbol u for the familiar notation), the real part is $\int u \ du = u^2/2$ so

$$I_{re} = -\sin^2 t$$

If you can't remember $\int \cos^2$ why don't we just guess:

$$[\ \sin t\cos t\ ]' = \cos^2 t - \sin^2 t$$

Looks pretty good! So

$$I_{im} = \sin t\cos t$$

Putting it together we get

$$\int z \ dz = -\sin^2 t + i\sin t\cos t$$

Since the trig functions have a period of 2π, any path that is closed has that period, and the value of the integral will be zero.

Otherwise, in say, the first quadrant where $t = 0 \to \pi/2$, at the upper bound we have -1 and at the lower bound we have 0, so the value of the integral is -1, which matches what we got before.

example: $\int z^2$

Let $f(z) = z^2$.

For the path, take the unit circle over the first quadrant from $(1, 0)$ to $(0, 1)$. This is only a part of a circle, so the integral will be non-zero.

There is an easy way to do this, and a hard way.

Let's start by re-checking that this function is analytic, and then do the hard way first.

Write z in terms of x and y:

$$z = x + iy$$

$$z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$$

$$u = (x^2 - y^2)$$

$$v = 2xy$$

$$u_x = 2x = v_y$$

$$u_y = -2y = -v_x$$

The CRE hold.

Also

$$dz = dx + i\ dy$$

So then

$$\int z^2\ dz = \int u\ dx - v\ dy + i\ [\ v\ dx + u\ dy\ ]$$

$$= \int (x^2 - y^2)\ dx - \int 2xy\ dy + i \int 2xy\ dx + i \int (x^2 - y^2)\ dy$$

As before, we must parametrize this using the relationship between x and y along the curve.

$$x = \cos t$$

$$y = \sin t$$

so

$$x^2 - y^2 = \cos^2 t - \sin^2 t = 2\cos^2 t - 1$$

$$2xy = 2\cos t \sin t$$

and

$$dx = -\sin t \ dt$$

$$dy = \cos t \ dt$$

To keep things straight, write the integral in its four separate parts:

$$I_1 = \int (2\cos^2 t - 1) \ (-\sin t \ dt)$$

$$I_2 = -\int 2\cos t \sin t \ (\cos t \ dt)$$

leaving off the i

$$I_3 = \int 2\cos t \sin t(-\sin t \ dt)$$

$$I_4 = \int (2\cos^2 t - 1) \ (\cos t \ dt)$$

I_4 can also be written as $(\cos^2 t - \sin^2 t)(\cos t \ dt)$.

It looks pretty wild, but really, these are fairly easy integrals, since we can use u substitution. Let's make a table for reference:

$$\int \cos^2 t \ (-\sin t \ dt) = \frac{1}{3}\cos^3 t$$

$$\int \sin^2 t \ (\cos t \ dt) = \frac{1}{3}\sin^3 t$$

$$\int \cos^3 t \ dt = \int (1 - \sin^2 t) \ \cos t \ dt = \sin t - \frac{1}{3}\sin^3 t$$

So now we can then just copy the results into the integrals we set up:

$$I_1 = \frac{2}{3}\cos^3 t - \cos t$$

$$I_2 = \frac{2}{3}\cos^3 t$$

The real part is then

$$\frac{4}{3}\cos^3 t - \cos t$$

and

$$I_3 = -\frac{2}{3}\sin^3 t$$

$$I_4 = 2\sin t - \frac{2}{3}\sin^3 t - \sin t$$

The imaginary part is then

$$i\ [\ -\frac{4}{3}\sin^3 t + \sin t\ ]$$

There is symmetry.

At the upper limit, the cosine is zero and the sine is 1 so the imaginary part is only non-zero at the upper limit and we get just $-1/3\ i$.

The real part is only non-zero at the lower limit, that gives 1/3 but it's the lower limit so we subtract, and the answer is

$$-\frac{1}{3} - \frac{1}{3}i = -\frac{1}{3}(-i - 1)$$

easy way

One is to just treat z as if it were a real variable

$$\int z^2\ dz = \frac{1}{3}\ z^3\ \Big|_1^i = \frac{1}{3}(-i - 1)$$

If we go all the way around the unit circle the integral is zero.

example: $1/z$

$$\int_0^{2\pi} \frac{1}{z}\ dz$$

Now we are getting to the core of this subject. While many integrals over a closed path are zero, some are not.

Examining the inverse function, you might want to first confirm that it is analytic by calculating the partial derivatives. We did this already so I'll skip it.

If we are on the unit circle, then

$$z = e^{i\theta}$$

$$dz = ie^{i\theta}d\theta = iz\ d\theta$$

$$\int \frac{dz}{z} = \int \frac{iz}{z}\ d\theta = i\int d\theta = 2\pi i$$

Pretty and pretty easy!

The inverse is an example of a function that *is* analytic, yet the integral around a closed curve that includes the origin is not equal to zero, it is instead equal to $2\pi i$.

If we're centered on the origin but we don't have a unit circle, there will be an r in both the numerator and the denominator, which cancel.

The result is thus independent of the radius of the circle.

example: $\int z^*$

Consider $f(z) = z^*$.

This function is of course *not* analytic, because it involves z^* rather than z, and also because

$$z^* = x - iy$$

so

$$u_x = 1 \neq v_y = -1$$

The CRE do not hold.

Suppose our curve is the circle of radius r centered at the origin, and we proceed between the endpoints $z = -ri \to ri$ (from due south to east to due north). On this half-circle

$$z = re^{i\theta}$$

we have then

$$dz = i\ re^{i\theta}\ d\theta$$

In radial coordinates

$$z^* = re^{-i\theta}$$

so then

$$\int z^*\ dz = \int re^{-i\theta} rie^{i\theta}\ d\theta$$

$$= r^2 i \int_{-\pi/2}^{\pi/2} d\theta = r^2 \pi i$$

Alternatively,

$$zz^* = |z|^2 = r^2$$

$$z^* = \frac{r^2}{z}$$

$$\int z^*\ dz = r^2 \int \frac{1}{z}\ dz$$

Again

$$z = re^{i\theta}$$

$$dz = iz\ d\theta$$

So the integral is just

$$= r^2 \int \frac{1}{z} \ iz \ d\theta$$

$$= r^2 i \int d\theta = r^2 \pi i$$

over the half-circle

inverse in terms of x and y

We can also integrate the inverse function in terms of x and y:

$$\frac{1}{z} = \frac{z*}{zz*} = \frac{x - iy}{x^2 + y^2}$$

So

$$u = \frac{x}{x^2 + y^2}$$

$$v = \frac{-y}{x^2 + y^2}$$

The integral is

$$\int u \ dx - v \ dy + i \ [v \ dx + u \ dy$$

$$= \int \frac{1}{x^2 + y^2} \ [\ x \ dx + y \ dy + i \ (\ -y \ dx + x \ dy) \ ]$$

Suppose we go on a circle of radius R centered on the origin and parametrize in terms of θ. We obtain:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$x^2 + y^2 = r^2$$

and

$$dx = -r \sin\theta \ d\theta$$
$$dy = r \cos\theta \ d\theta$$

We have for the integral

$$\frac{1}{r^2} \int x \ dx + y \ dy + i \ [\ -y \ dx + x \ dy \ ]$$

Each term x, y, dx and dy has a factor of r. So factor that out and cancel what's in front. Then substitute the trig functions:

$$\int \cos\theta \ (-\sin\theta \ d\theta) + \sin\theta \ (\cos\theta \ d\theta) +$$

$$i \ [\ -\sin\theta \ (-\sin\theta \ d\theta) + \cos\theta \ (\cos\theta \ d\theta) \ ]$$

The real part has a big cancelation, leaving:

$$\int i \ [\ -\sin\theta \ (-\sin\theta \ d\theta) + \cos\theta \ (\cos\theta \ d\theta) \ ]$$

and so does the imaginary part, so we have just

$$= \int i \ d\theta = 2\pi i$$

If we integrate the same function around a unit square, we run into problems. First let's do $[0, 0 \times 1, 1]$. We have

$$\int u \ dx - \int v \ dy + i \ [\int v \ dx + \int u \ dy \ ]$$

Along $C1$, $y = 0$ and $dy = 0$ so:

$$\int \frac{x}{x^2 + y^2} \ dx + i \ [\int \frac{-y}{x^2 + y^2} \ dx$$

$$= \int_0^1 \frac{1}{x}\ dx = \ln x\ \Big|_0^1$$

Since $\ln 0$ is not defined, we can't do this.

Logarithms are tricky, no doubt.

If the complex logarithm *Log* z is defined and differentiable along the curve (and it is, along the semicircle from $-i$ to i), we can do this:

$$I = \int_{-i}^{i} \frac{1}{z}\ dz = \text{ Log } z$$

To evaluate this, recall that $z = e^{i\theta}$ recognize that at $z = -i$, $\theta = -\pi/2$, while for $z = i$, $\theta = \pi/2$:

$$= i\theta\ \Big|_{-\pi/2}^{\pi/2} = i\ \frac{\pi}{2} - i\frac{-\pi}{2} = 2i\ \frac{\pi}{2} = \pi i$$

If we're on the unit circle, then we need not worry about the real part of Log $z = \ln r + i\theta$, since $\ln 1$ is equal to zero. Also, if we're centered on the origin, then r is a constant and we get the same number at both bounds, so the difference is zero even if the path is not a full circle.

example: square root

Consider

$$\int \sqrt{z}\ dz$$

along the half-circle of radius 3 starting from the point $z = R$ on the x-axis and proceeding counter-clockwise. We can do this integral even if the "branch" of the square root function that we're using is only defined for $\theta > 0$.

We have that

$$z = re^{i\theta}, \quad \theta = 0 \to \pi$$
$$dz = iz\ d\theta$$
$$= ire^{i\theta}\ d\theta$$

$$\sqrt{z} = \sqrt{r}e^{i\theta/2}$$

so

$$I = \int_0^\pi ir\sqrt{r}e^{i3\theta/2}\ d\theta$$

We need

$$\int e^{i3\theta/2}\ d\theta = \frac{2}{3i}e^{i3\theta/2}\ \Big|_0^\pi$$

easiest to write it out as

$$e^{i3\theta/2}\ \Big|_0^\pi = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} - \cos 0 - i\sin 0$$
$$= 0 + i(-1) - 1 - 0 = -(1+i)$$

Going back to pick up all the factors we left behind:

$$I = -ir\sqrt{r}\ \frac{2}{3i}\ (1+i) = -r\sqrt{r}\ \frac{2}{3}\ (1+i)$$

In the problem, r was actually specified as 3, leading to the cancellation:

$$I = -2\sqrt{3}\ (1+i)$$

We can also do this problem by antiderivatives:

$$\int_r^{-r} \sqrt{z}\ dz = \frac{2}{3}\ z^{3/2}\ \Big|_r^{-r}$$

$$= \frac{2}{3}(r^{3/2}e^{i3\pi/2} - r^{3/2}e^{0})$$
$$= \frac{2}{3}r^{3/2}(e^{i3\pi/2} - 1)$$

and, as we showed above:

$$e^{i3\pi/2} = -i$$

If $r = 3$ we get the same answer as before.

summary

For an analytic function, we can compute the integral by analogy with the real numbers: $\int z\ dz = z^2/2$.

For any closed path, the result is just zero, with some special exceptions.

z^* is special because it's not analytic. $1/z$ is said to be special because it's not defined at $z = 0$, but this begs the question, what about $1/z^2$?

I would rather say that $1/z$ is special because it is z^* in disguise since:

$$\frac{1}{z}\ \frac{z^*}{z^*} = \frac{1}{r^2}\ z^*$$

with r just a constant.

Another reason is that (on the unit circle) we have

$$z = e^{i\theta}$$

so

$$dz = iz\ d\theta$$

and

$$\int \frac{1}{z}\ dz = \int i\ d\theta$$

The z cancels.

For any other power of z we will have some factor of $e^{ki\theta}$ at the end.

Since this is a combination of sine and cosine, it will give zero when integrated over a closed path since $\theta = 0 \to 2\pi$ and the trig functions have a period of 2π no matter where you pick for the starting point (no matter what the value of k is).

Chapter 17

Cauchy theorem

Cauchy's theorem

Cauchy's Integral theorem says that the integral of an analytic function over a closed path is equal to zero:

$$\oint_C f(z)\ dz = 0$$

There is an important restriction: the enclosed region must not contain a singularity.

This result is a consequence of Green's Theorem, which you may remember from multivariable calculus.

Let

$$z = x + iy$$

$$dz = dx + idy$$

$$z = f(x, y) = u(x, y) + iv(x, y)$$

Our integral is

$$= \oint u\ dx - v\ dy + iv\ dx + iu\ dy$$

proof

Green's theorem says that for two real functions of x and y: $M(x,y)$ and $N(x,y)$:

$$\oint_C M dx + N dy = \iint_R (N_x - M_y)\ dx\ dy$$

Back then, M and N were components of a vector field $\mathbf{F}$ and we wrote the shorthand for curl:

$$= \iint_R \nabla \times \mathbf{F}\ dA$$

but the important thing is that the theorem applies to real-valued functions of two real variables $f : \mathbb{R}^2 \to \mathbb{R}^1$, and so it applies to functions like $u(x,y)$ and $v(x,y)$.

Consider the real part of the integral above:

$$I_{Re} = \oint u\ dx - v\ dy$$

Let $M = u$ and let $N = -v$ (notice the minus sign!). Then

$$I_{re} = \oint M\ dx + N\ dy$$

This is equal to a double integral containing:

$$N_x - M_y$$

We have that $N = -v$ so $N_x = -v_x$. And this is equal to u_y by the CRE.

But $M = u$ so $M_y = u_y$. The two terms in the subtraction are equal so the result is zero. Hence, the integral is zero.

For the imaginary part

$$I_{Im} = i \oint v \ dx + u \ dy$$

Let $N = u$ and let $M = v$ (no minus sign). Then

$$I_{re} = \oint M \ dx + N \ dy$$

This is equal to a double integral containing:

$$N_x - M_y$$

But $N_x = u_x$ and $M_y = v_y$ and these terms are equal by the CRE. Therefore this expression is zero.

So the integral for the imaginary part is also zero, and thus the whole thing is zero as well:

$$\oint u \ dx - v \ dy + i \ [\ v \ dx + u \ dy \ ] \ = 0$$

Remember how important it was (for Green's theorem) that the function being integrated be defined everywhere in the region. Well, it's true here as well.

$$\oint_C \frac{1}{z} \ dz \stackrel{?}{=}$$

We've already seen by direct calculation that this integral is *not* zero when the curve C includes the origin, although it zero otherwise.

Path independence

The theorem that says the integral of an analytic function over a closed path (over a region without a singularity), is equal to zero.

$$\oint_C f(z)\ dz = 0$$

This result means, in turn, that the integral of an analytic function between two points z_1 and z_2 is independent of the path taken. Call the two paths $C1$ and $C2$.

Form the closed path by going from z_1 to z_2 over $C1$ and then return to z_1 by going backward over $C2$. The total integral is equal to zero by Cauchy's Theorem.

$$\int_{C1} f(z)\ dz + \int_{-C2} f(z)\ dz = 0$$

But the integral over the path C_2 in the forward direction is just minus the integral over the reverse path $-C2$.

Thus

$$\int_{C1} f(z)\ dz - \int_{C2} f(z)\ dz = 0$$

and then

$$\int_{C1} f(z)\ dz = \int_{C2} f(z)\ dz$$

Chapter 18

Cauchy formula

If we can write an integral in this form:

$$\oint_C \frac{f(z)}{z - z_0}\ dz$$

where $f(z)$ is analytic and defined everywhere in the domain we care about, with this composite function of course not defined at $z = z_0$, the value of the integral is

$$\oint_C \frac{f(z)}{z - z_0}\ dz = 2\pi i\ f(z_0)$$

This is called the Cauchy Integral formula.

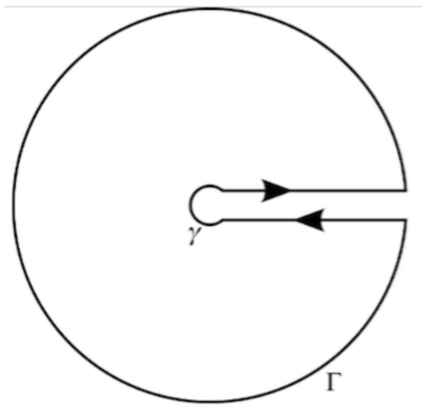
setup

Suppose $f(z)$ is analytic and defined everywhere within some region *except* at a singularity, z_0. For example, suppose we have

$$\frac{f(z)}{z - z_0}$$

We integrate this function around a special closed path in the region of analyticity:

$$\oint \frac{f(z)}{z - z_0}\, dz$$



It's not labeled (I didn't draw the figure) but the singularity z_0 is at the center of the two concentric circles. The "keyhole" excludes z_0 so f is analytic everywhere in the region enclosed by the path.

Cauchy's integral theorem tells us that the total integral is zero.

The straight line segments are identical but traversed in opposite directions, so the net contribution from them is zero.

Therefore, we have that the integral around the outer ring counter-clockwise + the integral around the inner ring clockwise add up to zero.

But reversing the direction of integration on the inner ring (so both paths go in the counter-clockwise direction) changes the sign of the value, hence we have that

$$\oint_{C \text{ outer}} \frac{f(z)}{z - z_0}\, dz - \oint_{C \text{ inner}} \frac{f(z)}{z - z_0}\, dz = 0$$

and

$$\oint_{C \text{ outer}} \frac{f(z)}{z - z_0} \, dz = \oint_{C \text{ inner}} \frac{f(z)}{z - z_0} \, dz$$

We haven't said anything so far about the radius of these rings.

What that means is that the value of the integral around a ring enclosing a singularity is not zero, but its value is *independent of the radius.*

derivation

We can parametrize this path. Each point on one of these curves is given by

$$z = z_0 + re^{i\theta}$$

with $0 \leq \theta \leq 2\pi$ and then

$$z - z_0 = re^{i\theta}$$

This is a circle with the center displaced to z_0.

Since z_0 is a constant

$$dz = ire^{i\theta} d\theta$$

so, substituting for $re^{i\theta}$ above we obtain

$$dz = i(z - z_0) \; d\theta$$

and

$$\oint \frac{f(z)}{z - z_0} \, dz = \oint f(z) \; i \; d\theta$$

$$= i \int_0^{2\pi} f(z) \; d\theta$$

This holds for every circular path enclosing z_0. We may choose r as small as we like, and so we choose it very small ($r \to 0$) so

$$f(z) \to f(z_0) = \text{constant}$$

and in that limit, since it's constant we can bring it out from under the integral sign!

$$i \int_0^{2\pi} f(z)\ d\theta$$

$$= i f(z_0) \int_0^{2\pi} d\theta = 2\pi i\ f(z_0)$$

An alternative (really, equivalent) approach is to say that for a small enough circle

$$\oint \frac{f(z)}{z - z_0}\ dz$$

the value of $f(z)$ approaches $f(z_0)$, a constant, so this becomes

$$= f(z_0) \int \frac{1}{z - z_0}\ dz$$

We know that integral, it is $2\pi i$.

Summarizing:

$$\oint \frac{f(z)}{z - z_0}\ dz = 2\pi i\ f(z_0)$$

What this means is that we can evaluate the integral in question by simply plugging in the value of the function at z_0 and multiplying that by $2\pi i$.

average

Since

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\ dz$$

and z can be parametrized as $z - z_0 = re^{i\theta}$ so that $dz = ire^{i\theta}\ d\theta$:

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \oint \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}\ ire^{i\theta}\ d\theta \\ &= \frac{1}{2\pi} \oint f(z_0 + re^{i\theta})\ d\theta \\ &= \frac{1}{2\pi} \oint f(z)\ d\theta \end{aligned}$$

The value of an analytic function at the center of a circle equals the average (arithmetic mean) of the values on the circumference.

Chapter 19

Cauchy corollary

In this section we follow Beck, so I've used their notation, which is slightly different. In particular, w is a fixed point inside the region.

The Cauchy Integral formula is:

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-w)} \, dz$$

If f is differentiable for all points in some open disk centered at w then f is holomorphic at w. For a holomorphic function f, a specific extension of the Cauchy formula is

$$f'(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-w)^2} \, dz$$

One way this can be obtained is by just differentiating the original formula under the integral sign on the right-hand side.

derivative rule

The Cauchy formula is:

$$\oint_C \frac{f(z)}{z - z_0}\ dz = 2\pi i f(z_0)$$

rewritten with a for z_0

$$\oint_C \frac{f(z)}{z - a}\ dz = 2\pi i f(a)$$

We take the partial with respect to a of both sides:

$$\frac{\partial}{\partial a}(\frac{f(z)}{z - a}) = \frac{f(z)}{(z - a)^2}$$

so

$$\oint_C \frac{f(z)}{(z - a)^2}\ dz = 2\pi i f'(a)$$

Thus

$$f'(a) = \frac{1}{2\pi i}\ \oint_C \frac{f(z)}{(z - a)^2}\ dz$$

More generally

$$f^n(a) = \frac{n!}{2\pi i}\ \oint_C \frac{f(z)}{(z - a)^{n+1}}\ dz$$

So

$$\frac{2\pi i}{n!} f^n(a) = \oint_C \frac{f(z)}{(z - a)^{n+1}}\ dz$$

more carefully

A more formal proof is the following (from Beck).

$$f'(w) = \frac{f(w + \Delta w) - f(w)}{\Delta w}$$

so

$$= \frac{1}{\Delta w}\ [\ \frac{1}{2\pi i}\ \int_\gamma \frac{f(z)}{z-(w+\Delta w)}\ dz - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)}\ dz\ ]$$

$$= \frac{1}{2\pi i}\ \int_\gamma \frac{f(z)}{(z-w-\Delta w)(z-w)}\ dz$$

In putting the two fractions over a common denominator we get a factor of Δw on top which cancels the leading one.

It is now possible to show that the value of this integral approaches what we seek as $\Delta w \to 0$. We will show that the difference of integrals goes to zero as $\Delta w \to 0$.

That difference is

$$= \frac{1}{2\pi i}\int_\gamma (\frac{f(z)}{(z-w-\Delta w)(z-w)} - \frac{f(z)}{(z-w)^2}\ dz$$

$$= \frac{\Delta w}{2\pi i}\int_\gamma \frac{f(z)}{(z-w-\Delta w)(z-w)^2}\ dz$$

We need to show that the integrand is bounded as $\Delta w \to 0$. Then the Δw on top will make the whole thing go to zero.

Let M be the maximum value of the function over the curve. They write:

$$M := \max_{z\in\gamma}|f(z)|$$

Choose $\delta > 0$ such that

$$|z-w| \geq \delta$$

for all z on γ. Then the reverse triangle equality says that

$$|(z-w-\Delta w)(z-w)^2| \geq (|z-w| - |\Delta w|)|z-w|^2$$

$$\geq (\delta - |\Delta w|)\delta^2$$

so

$$\frac{f(z)}{|(z-w-\Delta w)(z-w)^2}| \leq \frac{|f(z)|}{(|z-w|-|\Delta w|)|z-w|^2}$$
$$\leq \frac{M}{(\delta-|\Delta w|)\delta^2}$$

which certainly stays bounded as $\Delta w \to 0$.

This proves the Cauchy Integral formula for f'.

□

The formula for f'' is

$$f''(w) = \frac{1}{\pi i} \int_C \frac{f(z)}{(z-w)^3}\, dz$$

Notice the extra factor of 2

The general rule is:

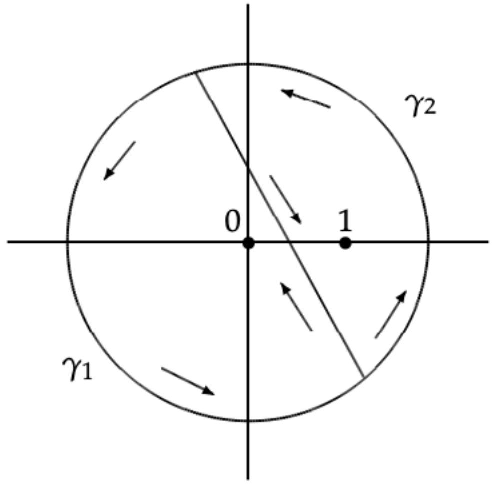
$$f^n(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}}\, dw$$

which can be proved by induction.

example

$$\int \frac{1}{z^2(z-1)}\, dz$$

If the region includes both of the singularities $z = 0$ and $z = 1$ we can split that path into two parts as shown in the figure:



Rewrite the integral as

$$\int \frac{1/(z-1)}{z^2}\ dz + \int \frac{1/z^2}{z-1}\ dz$$

For the first path, γ_1, only the first term is non-zero.

By the corollary

$$2\pi i f'(w) = \int_{\gamma_1} \frac{f(z)}{(z-w)^2}\ dz$$

$w = 0$ and the integral is:

$$= 2\pi i\ [\ \frac{d}{dz}\ \frac{1}{z-1}\bigg|_{z=0} = 2\pi i\ [\ -\frac{1}{(-1)^2} = -2\pi i\ ]$$

For the second path we use

$$2\pi i f(w) = \int_{\gamma_2} \frac{f(z)}{z-w}\ dz$$

with $w = 1$ so

$$= 2\pi i\ [\ \frac{1}{z^2}\bigg|_1\ ] = 2\pi i$$

The total is zero.

partial fractions

We look ahead to a technique we will practice a lot.

The previous problem can also be done by partial fractions. We have

$$\frac{1}{z^2(z-1)}$$

$$= \frac{A}{z^2} + \frac{B}{z(z-1)} + \frac{C}{z-1}$$

so

$$A(z-1) + Bz + Cz^2 = 1$$

$$C = 0$$

$$Az + Bz = 0, \quad A = -B$$

$$A = -1$$

and then

$$-\frac{1}{z^2} + \frac{1}{z(z-1)}$$

which checks.

We're not done, but what's left is easy, it is just

$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

so ultimately:

$$f(z) = -\frac{1}{z^2} - \frac{1}{z} + \frac{1}{z-1}$$

The path includes both $z = 0$ and $z = 1$. Break it up into the two parts.

The first integral is zero, by Cauchy's theorem when the path does not include $z = 0$, and by our results for $(z - z_0)^{-2}$ when it does.

The second one is $-2\pi i$ for the path including $z = 0$.

For the third, substitute $w = z - 1$, with $dw = dz$. The integral is $2\pi i$ and the total is zero.

Chapter 20

Partial Fractions

In the previous section, we derived Cauchy's formula:

$$\oint_C \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0)$$

One example is

$$\oint_C \frac{1}{z - z_0} \, dz = 2\pi i$$

which is easy to show even without using the new formula.

Example 1

This problem is Beck 4.26. Consider

$$\oint f(z) \, dz = \oint \frac{1}{z^2 + 1} \, dz$$

We see that the denominator is zero when

$$z^2 = -1$$

$$z = \pm \, i$$

Therefore we can factor the denominator as

$$z^2 + 1 = (z + i)(z - i)$$

There are a couple of different ways to handle this. One is to use partial fractions:

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$
$$= \frac{1}{2i} \; [\; \frac{1}{z-i} - \frac{1}{z+i} \;]$$

So the integral is a sum of two integrals:

$$I = \frac{1}{2i} \; [\oint \frac{1}{z-i} \; dz - \oint \frac{1}{z+i} \; dz \;]$$

Suppose the curve is the unit circle centered at i, designated as $C[i, 1]$.

Obviously, this curve contains the singularity $z = i$. The curve goes through the origin, so it does not extend as far as $z = -i$.

Therefore, the second integral is zero (no singularity) and the first is

$$\frac{1}{2i} \; [\; \oint \frac{1}{z-i} \; dz \;] \; = \frac{1}{2i} \; [\; 2\pi i \;]$$

by Cauchy's formula because $f(z_0)$ is 1. Thus the value is just $I = \pi$

According to Beck, as an alternative, rewrite the function as

$$\frac{1}{(z+i)(z-i)} = \frac{(1/z+i)}{z-i}$$

Thus

$$\int \frac{1}{z^2+1} \; dz = \int \frac{(1/z+i)}{z-i} \; dz$$

We have essentially the same thing. The function is

$$\frac{1}{z+i}$$

and when evaluated at i, with result $1/2i$, we obtain

$$\oint \frac{f(z)}{z-z_0}\,dz = 2\pi i\ f(z_0)$$

$$= 2\pi i\ \frac{1}{2i} = \pi$$

Above we have a constant of $1/2i$ which can either be factored out of the integral, or be part of $f(z_0)$. Either way, it's the same result.

partial fractions

We can also do the curve containing both singularities by using the formal apparatus of partial fractions. Write

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

$$= \frac{A}{z+i} + \frac{B}{z-i}$$

We need to determine A and B. When we multiply to put everything over the common denominator (z^2+1) then for the numerators we will have:

$$A(z-i) + B(z+i) = 1$$

All the powers of z must match across the equal sign. Since there is no power of z on the right-hand side, this gives

$$Az + Bz = 0$$

and

$$-Ai + Bi = 1$$

From the first we get that $A = -B$. And so from the second

$$-Ai - Ai = -2Ai = 1$$
$$A = -\frac{1}{2i}$$

Hence the integrand is

$$\frac{1}{z^2+1} == \frac{A}{z+i} + \frac{B}{z-i}$$
$$= -\frac{1}{2i(z+i)} + \frac{1}{2i(z-i)}$$

which matches what we had above:

$$= \frac{1}{2i} \left[\frac{1}{z-i} - \frac{1}{z+i} \right]$$

For the curve including $z = i$ but not $z = -i$ we have that the right-hand integral is 0 by Cauchy's Theorem, and for the left hand side the function is

$$f(z = z_0) = \frac{1}{2i}$$

So the value of the integral is

$$2\pi i f(z_0) = 2\pi i \ \frac{1}{2i} = \pi$$

as before.

The other pole would have

$$f(z = z_0) = -\frac{1}{2i}$$

and the result would be $-\pi$.

A curve enclosing both poles would have for the value of the integral the sum of the two, which is equal to zero.

example

Consider
$$\int_\gamma \frac{z^2}{4 - z^2}\ dz$$
where γ is $|z + 1| = 2$.

Recall that the definition of a circle around z_0 is $|z - z_0| = r$, where r is the radius of the circle. Thus the circle is centered at $z_0 = -1$, which can be checked by looking for values on the real number line that satisfy the equality (yielding -3 and 1).

The denominator of the function can be factored
$$\frac{1}{4 - z^2} = \frac{1}{(2+z)(2-z)}$$

It has zeroes at $z = \pm\, 2$. Only the point $z = -2$ is inside our contour.

So if we split this by partial fractions
$$\frac{1}{(2+z)(2-z)} = \frac{1}{4}\ [\ \frac{1}{2+z} + \frac{1}{2-z}\ ]$$

we can rewrite the integral as
$$I = \int_\gamma \frac{z^2}{4}\ [\ \frac{1}{2+z} + \frac{1}{2-z}\ ]\ dz$$

By Cauchy's Theorem, the second term is zero (no singularity).

The first one is:
$$I = \int_\gamma \frac{z^2}{4}\ (\frac{1}{2+z})\ dz$$
and the value of I is
$$I = 2\pi i f(z_0)$$

where

$$f(z_0) = \frac{z^2}{4} \Bigg|_{z_0=-2} = 1$$

so the integral is just $2\pi i$.

one from wikipedia

Consider

$$g(z) = \frac{z^2}{z^2 + 2z + 2}$$

We want to evaluate the integral:

$$I = \oint g(z)\ dz$$

The denominator

$$z^2 + 2z + 2$$

can be factored.

We plug into the quadratic solution:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 4 \cdot 2}}{2}$$

$$= -1 \pm \ \frac{\sqrt{-4}}{2}$$

$$= -1 \pm i$$

The zeroes of the denominator are z equal to

$$-1 + i\ , \quad -1 - i$$

From this we construct the two factors as

$$z - (-1 + i) = z + 1 - i$$

$$z - (-1 - i) = z + 1 + i$$

We confirm that these two factors multiplied together give back what we started with:

$$\begin{gathered}(z + 1 + i)\ (z + 1 - i) \\ = z^2 + z + iz + z + 1 + i + -iz - i + 1 \\ = z^2 + 2z + 2\end{gathered}$$

So we can factor the denominator and write:

$$\frac{1}{z^2 + 2z + 2} = \frac{A}{z + 1 - i} + \frac{B}{z + 1 + i}$$

Putting these two terms over a common denominator means multiplying the two factors and restoring what we started with.

For the numerator we have

$$A(z + 1 + i) + B(z + 1 - i) = 1$$

$$Az + A + iA + Bz + B - iB = 1$$

Equating terms containing the same power of z gives two simultaneous equations:

$$Az + Bz = 0z$$

and

$$A(1 + i) + B(1 - i) = 1$$

So $A = -B$ and

$$\begin{gathered}A(1 + i) + B(1 - i) = 1 \\ A(1 + i) - A(1 - i) = 1 \\ A2i = 1 \\ A = \frac{1}{2i}, \quad B = -\frac{1}{2i}\end{gathered}$$

The integral is

$$\oint \frac{z^2}{2i} \; [\; \frac{1}{z+(1-i)} - \frac{1}{z+(1+i)} \;] \; dz$$

So we see that we have a sum of integrals of the form

$$\oint \frac{f(z)}{z-z_0}$$

The residues occur at the points

$$z = z_0$$

that is at

$$z = -(1-i) = -1+i$$
$$z = -(1+i) = -1-i$$

If the contour is $|z| = 2$ centered at the origin (the circle of radius 2, then both of the points lie within the contour. ($r^2 = 2$ for both).

We evaluate $2\pi i f(z_0)$ for each

$$f(z) = \frac{z^2}{2i}$$

The first term gives

$$f(z_0) = \frac{(-1+i)^2}{2i} = \frac{1-1-2i}{2i} = -1$$

The second term gives

$$\frac{(-1-i)^2}{2i} = \frac{1+2i-1}{2i} = 1$$

But... this is not quite right. Go back and see that the second term in the integral has a minus sign. Hence the result at this step is -1 for both.

Each of these needs to be multiplied by $2\pi i$ and then summed.

$$I = -4\pi i$$

Chapter 21

Summary 2

A general complex function $f(z)$ takes a complex number z, which is really just an ordered pair (x, y) and feeds that number to a pair of real functions of two real variables, which each output a single real number. So

$$z = x + iy$$
$$f(z) = u(x, y) + v(x, y)$$
$$dz = dx + idy$$

We compute integrals as line integrals along a curve (or contour) by doing

$$\int f(z)\ dz = \int (u + iv)(dx + idy)$$
$$= \int u\ dx - v\ dy + i\ [\ v\ dx + u\ dy\ ]$$

These don't look like it but they are integrals in a single variable, because x and y are related.

There are two kinds of complex functions: *analytic* and otherwise. The analytic functions are *good* functions, they follow the rules we know from basic calculus, and can be differentiated and integrated in analogous forms. We did some examples in x and y like square and triangular paths.

We discovered that integration of analytic functions along a closed path gives a result of zero, except when the function is not defined at some point in the region.

More commonly, integration around a circular contour is of interest, often on a unit circle. In that case, we have a parameter t and the function said to be parametrized.

$$\int_C f(z)\ dz = \int_a^b f\ [\ z\ ]\ z'(t)\ dt$$

For example:

$$z = re^{it}$$

On a unit circle around the origin, r is a constant and

$$dz = r(ie^{it})\ dt = iz\ dt$$

So, for example, the inverse function $1/z$ gives

$$\int \frac{1}{z}\ iz\ dt = i \int dt = 2\pi i$$

Around a closed path, the value of the integral is $2\pi i$.

This occurs despite the fact that $1/z$ obeys the CRE and is analytic. The problem is that it is undefined at the origin and not analytic there.

The factor of $2\pi i$ will come up repeatedly from this point.

Another way to explain this is to say

$$\frac{1}{z} = \frac{z^*}{zz^*}$$

The denominator is $x^2 + y^2 = r^2$ which is constant for any circular path, so we have

$$\int \frac{1}{z}\ dz = k \int z^*\ dz$$

and $z*$ is definitely not analytic since $z^* = x - iy$ and $u_x = 1 \neq v_y = -1$.

We also did some other examples, such as $1/z^2$. On the unit circle

$$z = e^{it}$$

$$dz = iz\ dt$$

so the integral is

$$\int \frac{1}{z^2}\ iz\ dt = \int \frac{1}{z}\ i\ dt$$

$$= i \int e^{-it}\ dt = i\ \frac{1}{-i}\ e^{-it} = -e^{-it}$$

From Euler

$$e^{ix} = \cos x + i \sin x$$

but evaluated around a closed path, this is zero because the sine and cosine have a period of 2π.

Cauchy's integral theorem

Cauchy's first theorem says that:

$$\oint_C f(z)\ dz = 0$$

for an analytic function around a region without any singularity.

We proved this theorem, it follows very simply from Green's theorem.

A corollary of this theorem is that the result of an integral between any two points over two different paths, is equal.

Cauchy's integral formula

With these two results, complex analysis starts to get a bit wild.

If we can write an integral in this form:

$$\oint_C \frac{f(z)}{z - z_0}\ dz$$

where $f(z)$ is analytic and defined everywhere in the domain we care about, with this composite function of course not defined at $z = z_0$.

We parametrize the curve as a circle of radius r around the point z_0

$$z = z_0 + re^{it}$$

z_0 is a constant so

$$dz = rie^{it}\ dt = i(z - z_0)\ dt$$

and then we can simplify the integral as

$$i \oint_C f(z)\ dt$$

It is easy to show that the value of this integral does not depend on the radius of the path, so we let the radius shrink and approach zero.

The magic thing is that $f(z) \to f(z_0)$, but $f(z_0)$ is a *constant.* It can come out from under the integral:

$$= if(z_0) \int_C dt$$

This integral is just 2π, so the whole thing is $2\pi i f(z_0)$ and we can write:

$$\oint_C \frac{f(z)}{z - z_0}\ dz = 2\pi i \cdot f(z_0)$$

This is Cauchy's integral formula.

A simple example is the inverse $1/z$, the numerator is $f(z) = 1$ and $z_0 = 0$ so the result is $2\pi i$ times the value of the function 1 at the origin, which is $2\pi i$ and matches what we got by direct computation.

extension

A specific extension of the Cauchy Integral formula is

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2}\ dz$$

Generally:

$$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\ dz$$

Rearranged:

$$\frac{2\pi i}{n!}\ f^n(z_0) = \int_C \frac{f(z)}{(z - z_0)^{n+1}}\ dz$$

A function f is said to be differentiable at z_0 if the function's domain includes a neighborhood of z_0 and the derivative exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

more

The existence of the derivative at z_0 implies that the function is continuous at that point; however, the converse is not necessarily true.

A function is *analytic* at a point if it has a derivative at that point.

Cauchy's theorem says that the integral around a closed path is zero for a function which is analytic everywhere in a domain.

If such a function is undefined at a limited number of points (e.g. because such values produce zero in the denominator), then those points are called poles or singularities and Cauchy's formula can be used to calculate the value of the integral (called a residue) from the value of the function at those points.

Part V

Series

Chapter 22

Taylor series

This chapter is an introduction to Taylor series for real variables.

Suppose we have a function $f(x)$, but

Shankar:

> "imagine that you don't have access to the whole function. You cannot see the whole thing. You can only zero-in on a tiny region."

around $f(0)$, where you know the value. So the question is, what do we guess the function will do near $f(0)$?

The first approximation is that

$$f(x) \approx f(0)$$

We really can't say anything more. $f(0)$ is the best guess for what the value of the function is (we're talking about continuous and continuously differentiable functions).

Now suppose we know the slope of the function at 0, $f'(0)$. Then, since

$$\Delta y = f'(0)\Delta x = f'(0)(x - 0)$$

we can get a better approximation as the linear approximation:

$$f(x) \approx f(0) + f'(0)\ x + \ldots$$

For most functions, there will be more terms. If f is not a linear function, then the slope won't be constant. So

> "the rate of change itself has a rate of change .. the second derivative."

The term we are going to add is

$$f''(0)\ \frac{x^2}{2}$$

so

$$f(x) \approx f(0) + f'(0)\ x + f''(0)\ \frac{x^2}{2} + \ldots$$

A simple way to see why we have $x^2/2$ is to take derivatives on both sides. The terms like $f'(0)$ and $f''(0)$ are constants, they have been evaluated at $x = 0$. The first derivative is

$$f'(x) \approx f'(0) + f''(0)\ x + \ldots$$

We evaluate at $x = 0$ and the term $f''(0)\ x$ goes away because of the $x = 0$ multiplying the constant $f''(0)$. So we have just

$$f'(x) \approx f'(0)$$

and that matches. Now take the second derivative

$$f''(x) \approx f''(0)$$

and that matches too. We can see a pattern here.

The fourth term is

$$f(x) \approx f(0) + f'(0)\ x + f''(0)\ \frac{x^2}{2!} + f'''(0)\ \frac{x^3}{3!} + \ldots$$

You might not be expecting the factorial which I snuck in there. But if you go back to the exercise above, where we evaluated derivatives, you can see why it works. When we take the first derivative

$$\frac{d}{dx}(f'''(0)\ \frac{x^3}{3!}) = f'''(0)\ \frac{x^2}{2!}$$

the 3 comes down from the power and then turns 3! in the denominator into 2!. The next derivative will bring down the 2. So everything cancels properly.

If you like Σ notation, we can write

$$f(x) = \sum_{n=0}^{\infty} f^n(0)\frac{x^n}{n!}$$

with the understanding that $0! = 1$. The approximation is better the closer x is to 0, and the more terms the better as well.

There is one final wrinkle to this derivation. The series can be modified deal with x near any value a, not just near 0. The modification is

$$f(x) = \sum_{n=0}^{\infty} f^n(a)\frac{(x-a)^n}{n!}$$

This is the Taylor series. The series near $a = 0$ is known as the Maclaurin series.

1/1-x

The first example is

$$f(x) = \frac{1}{1-x}$$

We know the answer to this.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3$$

Proof:

$$1 = (1 - x)(1 + x + x^2 + x^3)$$

Multiplying by 1, the second term x is matched by $-x$ from the first term in the multiplication by $-x$, and so on. The whole thing vanishes, leaving just 1.

We want to evaluate $f(x)$ near 0, let's say, at $x = 0.1$. The correct value of the function is

$$f(x) = \frac{1}{0.9} = 1.11111\ldots$$

Let's try to approximate using the series. We need derivatives

$$f(x) = \frac{1}{1 - x}$$

$$f'(x) = \frac{1}{(1 - x)^2} = (1 - x)^{-2}$$

$$f'(0) = 1$$

so the linear approximation is

$$f(x) \approx 1 + 1x = 1.1$$

For the next term we obtain

$$f''(x) = 2(1 - x)^{-3}$$

The 2 is cancelled by the 2! in the denominator, so this cofactor is 1 and we're left with

$$f''(0)\ \frac{x^2}{2} = x^2 = 0.01$$

And I think we can see where this one is going.

However, you probably remember that this series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

diverges for $|x| \geq 1$, and the Taylor series does too.

The morale of the story is that for some series, there is a radius of convergence and the series is only valid for x within that radius.

binomial

Another very useful series is the binomial.

$$f(x) = (1+x)^n$$

$$f(0) = 1$$

$$f'(0) = n(1+x)^{n-1} = n$$

$$f''(0) = n(n-1)(1+x)^{n-2} = n(n-1)$$

So the series is

$$(1+x)^n \approx 1 + nx + n(n-1)\frac{x^2}{2}$$

We use this one a lot.

A nice application is relativistic energy

$$E = mc^2 f$$

$$f = 1/\sqrt{1 - \frac{v^2}{c^2}}$$

This is, in disguise, a binomial with $n = -1/2$ and $x = -v^2/c^2$ so the expansion is

$$f \approx 1 + nx = 1 + \frac{v^2}{2c^2}$$

so the energy is

$$E \approx mc^2(1 + \frac{v^2}{2c^2})$$

And we see that the second term is just the kinetic energy, $mv^2/2$.

polynomials

The beauty of Taylor Series (despite its complexity) is that it turns any differentiable function into a polynomial. Polynomials are easy to integrate and work with.

The first thing to say about Taylor Series is they give the correct answer for functions that we know. For example, suppose we have

$$f(x) = ax^2 + bx + c = 1$$

We get the derivatives and evaluate them "near" the point $x = 0$.

$$f(x) = ax^2 + bx + c = c$$

$$f'(x) = 2ax + b = b$$

$$f''(x) = 2a$$

The series is then

$$f(x) = c + b(x) + \frac{21}{2!}(x)^2 + \cdots$$

But there are no more terms. That's it. And this is just

$$f(x) = c + bx + ax^2$$

exponential, sine and cosine

Suppose $f(x) = e^x$ and again, we evaluate "near" $x = 0$. We have

$$f(x) = e^x = 1$$

$$f'(x) = e^x = 1$$

$$f''(x) = e^x = 1$$

The series is

$$f(x) = e^x = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots$$

$$f(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Which matches what we already know about e^x. For example, it is obvious that

$$\frac{d}{dx}e^x = e^x$$

Let's try to find something new. Suppose we expand $f(x) = cos\ x$ near $x = 0$

$$f(x) = cos\ x = cos\ 0 = 1$$

$$f'(x) = -sin\ x = -sin\ 0 = 0$$

$$f''(x) = -cos\ x - cos\ 0 = -1$$

$$f'''(x) = sin\ x = sin\ 0 = 0$$

$$f''''(x) = cos\ x = cos\ 0 = 1$$

and this continues in a cycle with period 4. The series is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$f(x) = cos\ x = 1 - \frac{1}{2!}(x-0)^2 + \frac{1}{4!}(x-0)^4 + \cdots$$
$$f(x) = cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Similarly, for $f(x) = sin\ x$ near $x = 0$

$$f(x) = sin\ x = 0$$
$$f'(x) = cos\ x = 1$$
$$f''(x) = -sin\ x = 0$$
$$f'''(x) = -cos\ x = -1$$
$$f''''(x) = sin\ x = 0$$

The series is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$
$$f(x) = sin\ x = x - \frac{1}{3!}(x-0)^3 + \frac{1}{5!}(x-0)^5 + \cdots$$
$$f(x) = sin\ x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

funny series

In Strogatz book (*The Joy of x*), he gives the following series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

and he says that the sum of the series is equal to the natural logarithm of 2:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

with the provision that you have to calculate the sum in the order given.

For example, the second, third and fourth partial sums are:

$$S_2 = \frac{1}{2}; \quad S_3 = \frac{5}{6}; \quad S_4 = \frac{14}{24}; \quad S_5 = \frac{94}{120}$$

with $S_4 = 0.583$ and $S_5 = 0.783$. For any partial sum S_n and the previous sum S_{n-1} the value of the series will be bounded by the two sums.

I thought I would try to show that $\ln 2$ is the correct value for series, by using a Taylor series for the logarithm.

Taylor says we can write a function $f(x)$ (near the value $x = a$) as an infinite sum

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

where f^n means the nth derivative of f and f^0 is just f, and these derivatives are to be evaluated at $x = a$. Near $a = 0$ this simplifies to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n$$

Let's calculate the derivatives of the logarithm:

$$f^0 = \ln x; \quad f^1 = \frac{1}{x} = x^{-1}; \quad f^2 = -x^{-2}; \quad f^3 = 2x^{-3}; \quad f^4 = -3!\ x^{-4}$$

The first thing I notice is that we can't use $a = 0$, since $f^1 = 1/x$ is undefined there. So, let's try $a = 1$. Then (evaluated at $a = 1$)

$$f^0 = \ln x = 0; \quad f^1 = \frac{1}{x} = 1; \quad f^2 = -x^{-2} = -1; \quad f^3 = 2; \quad f^4 = -3!$$

Going back to the definition

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

I get the following series near $a = 1$:

$$\ln x = \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \cdots$$

For the special value $x = 2$, all the terms $(x-1)^n$ go away (which confirms that $a = 1$ is an excellent choice!). We have then

$$\begin{aligned} \ln x &= \frac{0}{0!} + \frac{1}{1!} - \frac{1}{2!} + \frac{2}{3!} - \frac{3!}{4!} + \cdots \\ &= 0 + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \end{aligned}$$

which is what was to be proved.

Chapter 23

Laurent theory

Any function that is analytic inside a disk has a power series (a Taylor series) convergent inside that disk.

But what about a punctured disk?

$$0 < |z - z_0| < R$$

or a donut

$$0 < r < |z - z_0| < R$$

The applicable series is then a Laurent series.

The HELM guys say:

> One of the shortcomings of Taylor series is that the circle of convergence is often only a part of the region in which $f(z)$ is analytic.

https://learn.lboro.ac.uk/archive/olmp/olmp_resources/pages/workbooks_1_50_jan2008/Workbook26/26_6_snglrts_n_resdus.pdf

An example is

$$f(z) = \frac{1}{1 - z}$$

This function is analytic everywhere except at the singularity $z = 1$. The Taylor series expanded around $z = 0$ is

$$1 + z + z^2 + z^3 + \dots$$

which converges to $f(z)$ only inside $|z| = 1$.

> The radius of convergence for a series centered on $z = z_0$ is the distance between z_0 and the nearest singularity.

Boas:

Let $C1$ and $C2$ be two circles with center at z_0. Let $f(z)$ be analytic in the region between the circles. Then $f(z)$ can be expanded in a Laurent series:

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots +$$

$$+\frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots$$

The Taylor part of the series (called the analytic part) usually converges everywhere inside a disk of radius R, while the b part, the principal part, usually converges everywhere outside a disk of radius r, so the combined series is convergent in the annulus, the area between r and R.

Note: if there are several isolated singularities , then there are several annular rings, each with a different Laurent series.

Laurent's Theorem

If $f(z)$ is analytic through a closed annulus D centered at $z = z_0$, then at any point z inside D we can write:

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

$$+ b_1(z - z_0)^{-1} + b_2(z - z_0)^{-2} + \dots$$

where the coefficients are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{1-n}}\, dz$$

(no, they don't match the powers of $(z - z_0)$).

Any polynomial of z is analytic, and quotients of analytic functions are also analytic.

The end result will be that the integral $\int f(z)\, dz$ may be obtained by integrating the right-hand side, where all the terms except one will have an integral equal to zero.

Out of this entire series given above, only one term matters:

$$b_1(z - z_0)^{-1}$$

This is a consequence of Cauchy's Integral theorem.

derivation of Laurent series

We follow

https://www.youtube.com/watch?v=2GC26rJB2L0&list=PLvcbYUQ5t0UFmFXOLwC9E
index=22&t=0s

Fix some particular z in the annulus $0 < r < |z - z_0| < R$.

Choose r_1 and R_1 just inside (or outside, respectively) the boundaries of the donut:

$$0 < r < r_1 < |z - z_0| < R_1 < R$$

Let γ_1 go along (let w take on the values) $|w - z_0| = R_1$ counter-clockwise, with the interior on the left, and let γ_2 go along $|w-z_0| = r_1$, in the opposite direction to γ_1.

Set up a keyhole contour. Then f is analytic in the domain which the line integral encloses, which is required to use Cauchy's formula:

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}\ dw + \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}\ dw$$

The path that links the two circles cancels because we traverse it in opposite directions.

Now rewrite

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)}$$

$$= \frac{1}{w-z_0}\ [\ \frac{1}{1-(z-z_0)/(w-z_0)}\ ]$$

This is a geometric series with initial term

$$\frac{1}{w-z_0}$$

divided by (or multipied by the inverse of) one minus the common ratio

$$\frac{z-z_0}{w-z_0}$$

Since

$$|\frac{z-z_0}{w-z_0}| < 1$$

For the big circle γ_1 we have that this ratio is less than 1 (w runs along R), so the series converges absolutely.

The corresponding series is

$$\sum_{k=0}^{\infty} \frac{(z-z_0)^k}{(w-z_0)^{k+1}}$$

For the small circle, γ_2: the ratio $1/w-z$ is equal to

$$-\frac{1}{z-z_0} \left[\frac{1}{1-(w-z_0)/(z-z_0)} \right]$$

For the small circle γ_2 the flipped ratio is less than 1 since w runs along r, so again the series converges absolutely.

The series is

$$-\sum_{0}^{\infty} \frac{(w-z_0)^j}{(z-z_0)^{j+1}}$$

Rewriting

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_1} f(w) \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{(w-z_0)^{k+1}}\ dw$$
$$-\frac{1}{2\pi i} \int_{\gamma_2} f(w) \sum_{j=0}^{\infty} \frac{(w-z_0)^j}{(z-z_0)^{j+1}}\ dw$$

The sums can come out from under the integral so

$$f(z) = \sum_{k=0}^{\infty} (z-z_0)^k \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(w-z_0)^{k+1}}\ dw$$

$$+\sum_{j=0}^{\infty} z-z_0)^{-j-1} (-\frac{1}{2\pi i}) \int_{\gamma_2} f(w)\ (w-z_0)^j\ dw$$

We can clean up the indices for f_2. Instead of running from $j = 0 \to \infty$ with power $-j-1$, let it be $k = -\infty \to -1$ with power k.

We now have $f_1(z) + f_2(z)$. Now we can write the coefficients for both in the same way:

$$a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}}\, dw$$

and

$$f_1(z) = \sum_0^\infty a_k (z - z_0)^k$$

f_1 is convergent when $|z - z_0| < R$.

$$f_2(z) = \sum_{-\infty}^{-1} a_k (z - z_0)^k$$

f_2 is convergent when $|z - z_0| > r$.

A Laurent series is the combination.

$$\sum_{-\infty}^{\infty 1} a_k (z - z_0)^k$$

It is convergent when both criteria are met.

The principal part of the Laurent series consists of the terms with negative exponents. We can also write that part as

$$\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

Although it seems like we've made things *really* complicated, we haven't really, because when such a series is integrated, the only non-zero term is the $k = -1$ term.

Let's sidestep the problem of determining the coefficients using the formulas given above.

Instead, just say that we seek a series expansion using negative powers of z, and hope to find that it will be valid in the region $|z| > 1$.

Chapter 24

Laurent practice

geometric series

There is a trick to getting a *different* geometric series than $\sum z^n$. Just write

$$\frac{1}{1-z} = (-\frac{1}{z}) \cdot \frac{1}{1-1/z}$$

That's also a geometric series, but in $1/z$. It converges when $|1/z| < 1$ which means $|z| > 1$!!

And it is equal to

$$(-\frac{1}{z}) \cdot (1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} \dots$$

$$-\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} \dots$$

Note that we can with positive z by substituting $-w = z$ and get

$$\frac{1}{1+w} = \frac{1}{w} \; [\; 1 - \frac{1}{w} + \frac{1}{w^2} - \frac{1}{w^3} + \dots \;]$$

$$\frac{1}{1+w} = \frac{1}{w} - \frac{1}{w^2} + \frac{1}{w^3} + \dots$$

go back to z as the variable

$$\frac{1}{1+z} = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \dots$$

Check by multiplying the right-hand side by z and see all the cancellations after the first term.

We can also deal with

$$\frac{1}{a-z} = \frac{1}{a} \cdot \frac{1}{1 - z/a}$$

and then we have a series as $1/a \cdot \sum (z/a)^n$.

Boas example

$$f(z) = \frac{12}{z(2-z)(1+z)}$$

This function has three isolated singularities (at $z = 0, 2, -1$). And expanded around $z_0 = 0$, there will be three regions in which we have *different* series: namely $0 < |z| < 1$, $1 < |z| < 2$ and $|z| > 2$. There will be three series all together.

Start by using partial fractions to obtain:

$$= \frac{4}{z} \cdot (\frac{1}{2-z} + \frac{1}{1+z})$$

Start with the inner punctured disk. We want convergence for the region $|z| < 1$. Since it's less than, we use standard geometric series.

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 \dots$$

$$\frac{1}{2-z} = \frac{1}{2} \cdot \frac{1}{1 - z/2} = \frac{1}{2} \left[\ 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} \dots \right]$$

Add them together

$$= \frac{3}{2} - \frac{3z}{4} + \frac{9z^2}{8} - \frac{15z^3}{16} \dots$$

and multiply by $4/z$ to obtain:

$$= \frac{6}{z} - 3 + \frac{9z}{2} - \frac{15z^2}{4} \dots$$

This is the Laurent series valid in the innermost region.

For the outer region, manipulate each fraction:

$$\frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+1/z}$$

$$\frac{1}{2-z} = -\frac{1}{z} \cdot \frac{1}{1-2/z}$$

We do this so that the geometric series will be convergent for $|z| > 2$ (the ratio is $2/z$ in the second one).

Again, geometric series. Write them separately:

$$\frac{1}{z} \left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} + \dots \right]$$

$$-\frac{1}{z} \left[1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \frac{16}{z^4} \dots \right]$$

Move the minus sign inside:

$$\frac{1}{z} \left[-1 - \frac{2}{z} - \frac{4}{z^2} - \frac{8}{z^3} - \frac{16}{z^4} \dots \right]$$

Add

$$\frac{1}{z} [-\frac{3}{z} - \frac{3}{z^2} - \frac{9}{z^3} - \frac{15}{z^4} + \dots$$

Recall the leading factor of $4/z$, and get another factor of $-3/z$ giving what it has in the book:

$$-\frac{12}{z^3}\ [\ 1+\frac{1}{z}+\frac{3}{z^2}+\frac{5}{z^3}+\ldots\ ]$$

The last part is the annulus in the middle. For this we want convergence for $|z|>1$ and for $|z|<2$. Hence we want

$$\frac{1}{1+z}=\frac{1}{z}\cdot\frac{1}{1+1/z}$$
$$=\frac{1}{z}\ [\ 1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}+\ldots\ ]$$

and

$$\frac{1}{2-z}=\frac{1}{2}\cdot\frac{1}{1-z/2}=\frac{1}{2}\ [\ 1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}\ldots\ ]$$

I need a factor of $1/z$ on the latter (and move the factor of 2 inside:

$$\frac{1}{z}\ [\ \frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\frac{z^4}{16}\ldots\ ]$$

We can add them and remember the leading factor of $4/z$ so it's

$$\frac{4}{z^2}\ [\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\frac{z^4}{16}\cdots+1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}+\ldots\ ]$$

I try to verify one power, pick z^{-1}

$$\frac{6}{z}+\frac{2}{z}$$

and that, unfortunately is not a match. But I know we're close, and you can see the general idea.

example

Consider:

$$f(z) = \frac{z}{(z-1)(z-3)}$$

Suppose we want the series around $0 \leq |z-1| \leq 2$, also written as $C[1,2]$, a circle of radius 2 around the point $z_0 = 1$.

(I like the notation C[origin, radius] since there is no need to change the sign on the second term of $z - z_0$ to get z_0.)

One way to do this is to make a substitution $x = z - 1$, so $z = x + 1$ and we have

$$= \frac{x+1}{(x)(x-2)}$$

Factor out the $1/x$

$$= \frac{1}{x} \ (\frac{x+1}{x-2})$$

(this is easy to restore at the end by multiplying by $1/x$).

And then our goal is to get something like $1/1-r$.

That means we want $x-2$ on top

$$= \frac{1}{x} \ (\frac{x-2+3}{x-2})$$

$$= \frac{1}{x} \ (1 + \frac{3}{x-2})$$

$$= \frac{1}{x} \ (1 - \frac{3}{2-x})$$

$$= \frac{1}{x} \ (1 - \frac{3/2}{1-x/2})$$

$$= \frac{1}{x} \ (1 - \frac{3}{2} \cdot \frac{1}{1-x/2})$$

and now the

$$\frac{1}{1-x/2}$$

can be expanded because that's a geometric series $\sum(x/2)^n$

so

$$\frac{1}{1-x/2} = 1 + \frac{x}{2} + (\frac{x}{2})^2 + \ldots$$

which gives

$$= \frac{1}{x}\ [1 - \frac{3}{2} \cdot (1 + \frac{x}{2} + (\frac{x}{2})^2 + \ldots)\ ]$$

Now, multiplying through by $1/x$ gives

$$-\frac{1}{2x} + \ldots$$

and *nothing else matters.* Reverse the change of variable:

$$= -\frac{1}{2(z-1)} + \ldots$$

which we will integrate as

$$\oint \frac{-1/2}{z-1}\ dz$$

over $C[1,2]$.

Recall the formula for residues:

$$b_1 = \lim_{z \to z_0} (z - z_0)\ f(z)$$

So

$$\text{Res }(1) = \lim_{z \to 1} (z-1)\ \frac{-1/2}{z-1} = -\frac{1}{2}$$

Multiply by $2\pi i$ to obtain $I = -\pi i$.

As a check on this go back to

$$f(z) = \frac{z}{(z-1)(z-3)}$$

$$\begin{aligned}\text{Res } (1) &= \lim_{z\to 1}(z-1)\ \frac{z}{(z-1)(z-3)} \\ &= \lim_{z\to 1}\ \frac{z}{z-3} = -\frac{1}{2}\end{aligned}$$

and we just bypassed the manipulation.

example

These examples can get complicated. Here is one from

http://zimmer.csufresno.edu/~doreendl/128.13f/handouts/Lseriesex.pdf

$$f(z) = \frac{1}{(z-2)(z-1)}$$

This function has poles at $z = 1$ and $z = 2$. If we are asked to write expansions around $z_0 = 0$, then we have three regions of interest and three different expansions.

The first region is the circle of radius 1: $|z| < 1$, the second is $1 < |z| < 2$ and then finally $|z| > 2$.

region 1

Use partial fractions to write:

$$\frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$$

Considering the second term, we bring the minus sign inside

$$= \frac{1}{z-2} + \frac{1}{1-z}$$

We have a geometric series for the second term, and for the other one

$$\frac{1}{z-2} = -\frac{1}{2-z}$$

Our goal is to convert this into something like the geometric series. Factor out the 2 on the bottom like so

$$= -\frac{1}{2} \; [\; \frac{1}{1-z/2} \;]$$

We can do a formal substitution or recognize that this is the geometric series

$$= -\frac{1}{2} \; [\; \sum_{n=0}^{\infty} (z/2)^n \;]$$

We can rewrite this slightly by combining 2^n on the bottom with the 2 out front:

$$= \sum_{n=0}^{\infty} \; [\; \frac{-1}{2^{n+1}} \;] \; z^n$$

which converges for $0 < |z/2| < 1 \Rightarrow 0 < |z| < 2$.

Our series is the sum of these two series, which can be combined as

$$\sum_{n=0}^{\infty} \; [\; 1 - \frac{1}{2^{n+1}} \;] \; z^n$$

region 2

This is the annulus $1 < |z| < 2$. Thus

$$|\frac{1}{z}| < 1 \quad \text{and} \quad |\frac{z}{2}| < 1$$

What they do is to work on the right-hand term of

$$\frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$$

and, as we saw in the previous section transform it into something containing $1/z$, which will be valid in the region $|z| > 1$.

So let's do it:

$$\frac{1}{1-z} = -\frac{1}{z-1}$$

$$= -\frac{1}{z} \cdot \frac{1}{1-1/z}$$

leaving aside the leading factor this is

$$= 1 + \frac{1}{z} + \frac{1}{z^2} \dots$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

add back that factor

$$-\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n}$$

The left-hand term is exactly what we had before:

$$\sum_{n=0}^{\infty} [\ \frac{-1}{2^{n+1}}\ ]\ z^n$$

so we combine them

$$\sum_{n=0}^{\infty} [\ \frac{-1}{2^{n+1}}\ ]\ z^n - \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n}$$

and then just bring that z in the second term inside

$$\sum_{n=0}^{\infty} [\ \frac{-1}{2^{n+1}}\ ]\ z^n - \ \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

or change the index

$$= \sum_{n=0}^{\infty} [\ \frac{-1}{2^{n+1}}\ ]\ z^n - \ \sum_{n=1}^{\infty} \frac{1}{z^{n}}$$

region 3

We do the $1/z$ trick with both terms

$$\frac{1}{z-2} - \frac{1}{z-1}$$

Start with the first one:

$$\frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-2/z}$$

The series is

$$\frac{1}{z}\ \cdot\ \sum_{n=0}^{\infty} [\ \frac{2}{z}\ ]^n$$

$$= \sum_{n=0}^{\infty}\ \frac{2^n}{z^{n+1}}$$

The second term is (leaving off the factor of -1)

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-1/z}$$

The series is

$$\frac{1}{z}\ \cdot\ \sum_{n=0}^{\infty} [\ \frac{1}{z}\ ]^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

Combining the two results and bringing back the factor we get

$$\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

$$= \sum_{n=0}^{\infty} (2^n - 1) \frac{1}{z^{n+1}}$$

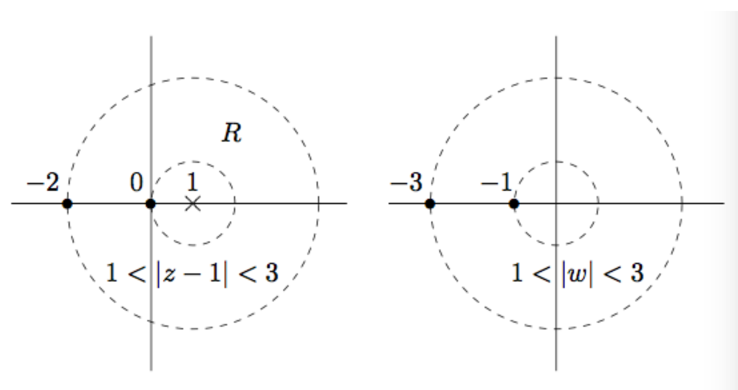
adjust the index

$$= \sum_{n=1}^{\infty} (2^{n-1} - 1) \frac{1}{z^n}$$

example

$$f(z) = \frac{1}{z(z+2)}$$

Suppose the region of interest is an annulus centered on $z = 1$ with $1 < |z-1| < 3$.



The first thing to do is make a substitution that translates the region so that it becomes centered on the origin: $w = z - 1$. Then the function becomes

$$\frac{1}{(w+1)(w+3)}$$

The next thing is to write partial fractions. For the numerator we get

$$A(w+3)+B(w+1)=1$$

$$A=-B=\frac{1}{2}$$

Hence

$$\frac{1}{2}\cdot\ [\ \frac{1}{w+1}-\frac{1}{w+3}\ ]$$

The third step is to convert each of these fractions into something like $1/1-x$.

$$\frac{1}{w+1}=\frac{1}{1-(-w)}$$

$$\frac{1}{w+3}=\frac{1}{3}\cdot\frac{1}{1-(-w/3)}$$

And then the fourth step is to write the series, recalling that we want different forms depending on whether we are in a circle or an annulus.

$$\frac{1}{1-(-w)}$$

$$=\sum_{n=0}^{\infty}(-w)^n=\sum_{n=0}^{\infty}(-1)^n(w)^n,\quad |w|<1$$

$$=-\sum_{n=1}^{\infty}\frac{1}{(-w)^n}=-\sum_{n=1}^{\infty}\frac{(-1)^n}{w^n},\quad |w|>1$$

We pick the second form because our region is $1<|z-1|<3$

A similar thing can be done for the other term. We show only the first series since we are inside the circle.

$$\frac{1}{3} \cdot \frac{1}{1-(-w/3)}$$

$$= \frac{1}{3} \cdot \sum_{n=0}^{\infty} (-1)^n (\frac{w}{3})^n, \quad |w| < 3$$

$$= \sum_{n=0}^{\infty} (-1)^n \ \frac{1}{3^{n+1}} \ w^n, \quad |w| < 3$$

Add the two series together (remembering the minus sign on the second term)

$$- \sum_{n=1}^{\infty} \frac{(-1)^n}{w^n} - \sum_{n=0}^{\infty} (-1)^n \ \frac{1}{3^{n+1}} \ w^n$$

and then picking up the leading factor from

$$\frac{1}{2} \cdot \ [\ \frac{1}{w+1} - \frac{1}{w+3} \ ]$$

so

$$\frac{1}{2} \ [\ - \sum_{n=1}^{\infty} \frac{(-1)^n}{w^n} - \sum_{n=0}^{\infty} (-1)^n \ \frac{1}{3^{n+1}} \ w^n \ ]$$

The last step is to reverse the substitution: $w = z - 1$ and bring the minus sign out front

$$f(z) = -\frac{1}{2} \ [\ \sum_{n=1}^{\infty} \frac{(-1)^n}{(z-1)^n} \sum_{n=0}^{\infty} (-1)^n \ \frac{1}{3^{n+1}} \ (z-1)^n \ ]$$

I don't know if I could ever learn to do this well, but at least the explanations make sense.

Now, if we were to integrate $f(z)$, we would have only one term that gives a non-zero result, namely the first term with $n = 1$

$$-\frac{1}{2}(-1)\frac{1}{z-1}$$

The residue is the cofactor of that term:

$$\text{Res }(1) = \lim_{z \to 1} \frac{1}{2} = \frac{1}{2}$$

Multiply by $2\pi i$ to obtain πi.

simpler view

On the other hand, I would just write the partial fraction:

$$\int \frac{1}{2} \left[\frac{1}{z} - \frac{1}{z+2} \right] dz$$

The curve $C[1,3]$ includes the singularity at $z = 0$, but $z = -2$ is on the boundary, not inside the region. The curve $C[1,1]$ includes neither point.

So for the second curve the integral is zero and for the first one only the $\int 1/z \ dz$ matters. It's an old friend, with value $2\pi i$, the value of the integral is thus πi.

example

Here are two examples from Brown and Churchill.

$$\int \frac{1}{z(z-2)^4} \ dz$$

with singularities at $z = 0$ and $z = 2$.

Use the $1/z$ part to get a geometric series:

$$\frac{1}{z(z-2)^4} = \frac{1}{(z-2)^4} \cdot \frac{1}{(2+z-2)}$$

$$= \frac{1}{(z-2)^4} \cdot \frac{1}{2} \cdot \frac{1}{(1-(-(z-2))/2)}$$

The third term gives the geometric series with common ratio $(z-2)/2$. Those $z-2$ terms will cancel the leading factor. The only term that matters is the cube, which gives:

$$\frac{1}{(z-2)} \cdot \frac{1}{2} \cdot \frac{1}{(-2)^3}$$

We have that $b_1 = -1/16$ so $I = -\pi i/8$.

The second one is:

$$\int e^{1/z^2}\ dz$$

We use the standard series for e^z

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

substituting $1/z^2$

$$1 + \frac{1}{z^2} + \frac{1}{2!\ z^4} + \frac{1}{3!\ z^6} + \dots$$

Since there's no z with a power $n = -1$, the value of the integral is just zero.

Part VI

Residue Theory

Chapter 25

Simple poles

Boas **definitions**:

Consider the Laurent series for $f(z)$ inside some C centered on z_0. Let z_0 be either a regular point or an isolated singular point and there are no other singular points inside C. Then:

∘ If all the b coefficients are zero, $f(z)$ is analytic at $z = z_0$ and we call z_0 a *regular point*.

∘ If $b_n \neq 0$ but all the b's after b_n are zero, then $f(z)$ is said to have a *pole of order* n at $z = z_0$. If $n = 1$ it is called a *simple pole*.

∘ If there are an infinite number of b's different from zero, then $f(z)$ has an *essential singularity* at $z = z_0$.

∘ The coefficient b_1 of $1/(z - z_0)$ is called the *residue* of $f(z)$ at $z = z_0$.

example

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

There are no b's, e^z is analytic, and the residue at $z = z_0$ is 0.

$$\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!z} + \frac{1}{3!} \dots$$

The pole is of order 3, the residue at $z = z_0$ is $1/2!$.

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

There is an essential singularity at $z = z_0$, and the residue at $z = z_0$ is 1.

The residue theorem

Let z_0 be an isolated singular point of $f(z)$, and expand the Laurent series about $z = z_0$ for f. We want to find $\oint f(z)\ dz$. By Cauchy's integral theorem, the integral of the analytical part of the Laurent series is zero (non-negative exponents for $z - z_0$).

It is easy to show that the negative exponents of power -2 and higher also give rise to zero when integrated, because they retain some power of $e^{i\theta}$ which multiplies everything by zero for a closed path.

The exception is the $(z - z_0)$ term, where stuff cancels:

$$\oint \frac{b_1}{z - z_0}\ dz = b_1 \int_0^{2\pi} \frac{rie^{i\theta}}{re^{i\theta}}\ d\theta = 2\pi i\ b_1$$

b_1 is called the residue of $f(z)$ at the singular point inside C. If there is more than one isolated singularities, the value of the integral is $2\pi i$ times the sum of the residues.

The trick of course, is to know what b_1 is equal to.

If we have the Laurent series, then b_1 is the coefficient of the $\frac{1}{z-z_0}$ term.

example

$$f(z) = \frac{e^z}{(z-1)}$$

$$= \frac{e}{(z-1)} \cdot e^{(z-1)} = \frac{e}{(z-1)} \; [\; 1 + (z-1) + \frac{(z-1)^2}{2!} + \ldots$$

$$= e(\frac{1}{(z-1)} + 1 + \frac{(z-1)}{2!} + \ldots$$

Then

$$\oint f(z)\; dz = 2\pi i \cdot e$$

The residue is the coefficient of $1/(z-1)$.

simple pole

If $f(z)$ has a simple pole at $z = z_0$ we find the residue by a simple consequence of Cauchy's integral formula:

$$\int \frac{f(z)}{z - z_0}\; dz = 2\pi i \cdot f(z_0)$$

in the limit that $z \to z_0$ it can come out from the integral sign:

$$\int f(z)\; dz = \lim_{z \to z_0} (z - z_0) \cdot 2\pi i \cdot f(z_0)$$

Stated in terms of the residue:

$$R = \lim_{z \to z_0} (z - z_0) f(z_0)$$

If there is more than one such point

$$\oint f(z)\; dz = 2\pi i \; \sum \; \text{Res}$$

The value of the integral is $2\pi i$ times the sum of all the residues enclosed by the path.

There is no longer a factor of $1/z - z_0$ in the integral, just $f(z)$.

examples

$$f(z) = \frac{z}{(2z+1)(5-z)}$$

The poles are at $z_0 = -1/2$ and $z_0 = 5$.

First multiply the function by $z - z_0$ That gives

$$R(-1/2) = \frac{z}{2(5-z)}\bigg|_{z=-1/2} = \frac{-1/2}{2(11/2)} = -\frac{1}{22}$$

And

$$R(5) = -\frac{z}{2z+1}\bigg|_5 = -\frac{5}{11}$$

For

$$f(z) = \frac{\cos z}{z}$$

The pole is at $z_0 = 0$ so multiply by z:

$$R(0) = \cos z\bigg|_0 = 1$$

residues

We repeat a problem, using residues.

$$\oint f(z)\ dz = \oint \frac{1}{z^2+1}\ dz$$

The formula is:

$$b_1 = \lim_{z \to z_0} (z - z_0)\ f(z)$$

and

$$\oint f(z)\ dz = 2\pi i \sum \text{Res}$$

Evaluate the formula. Our path includes i but not $-i$.

$$b_1 = \lim_{z \to i} (z - i)\ \frac{1}{(z+i)(z-i)}$$

$$= \lim_{z \to i}\ \frac{1}{z+i} = \frac{1}{2i}$$

And by the second equation:

$$I = \pi$$

as before. Seems a bit easier!

If the unit circle had been centered at $-i$, rewrite the function as

$$f(z) = \frac{1/z - i}{z+i}$$

The value of the function is

$$\left.\frac{1}{z-i}\right|_{-i} = -\frac{1}{2i}$$

and the integral is then $-\pi$.

A contour that includes both singularities integrates to zero.

example

In this section we get more practice.

$$\oint_C \frac{e^z}{z^2 - 2z - 3} \; dz$$

The denominator can be factored

$$z^2 - 2z - 3 = (z+1)(z-3)$$

If the disk is $|z| \leq 2$ then it includes only $z_0 = -1$ and the formula is

$$b_1 = \lim_{z \to z_0} (z - z_0) \; f(z)$$

so

$$\begin{aligned} b_1 &= \lim_{z \to -1} (z+1) \; \frac{e^z}{(z+1)(z-3)} \\ &= \lim_{z \to -1} \; \frac{e^z}{(z-3)} \\ &= \frac{e^{-1}}{-1-3} = -\frac{1}{4e} \end{aligned}$$

and

$$\begin{aligned} I &= 2\pi i \; b_1 = 2\pi i \; (-\frac{1}{4e}) \\ &= -\frac{\pi i}{2e} \end{aligned}$$

example

$$\int \frac{5z - 2}{z(z-1)} \; dz$$

There are two simple poles at $z_0 = 0$ and $z_0 = 1$ and the residues are

$$\text{Res } (0) = \lim_{z \to 0} (z - 0) \; \frac{5z-2}{z(z-1)}$$

$$= \lim_{z \to 0} \ \frac{5z-2}{(z-1)}$$

$$= \frac{5 \cdot 0 - 2}{0-1} = 2$$

$$\text{Res } (1) = \lim_{z \to 1} (z-1) \ \frac{5z-2}{z(z-1)}$$

$$= \lim_{z \to 1} \ \frac{5z-2}{z}$$

$$= \frac{5 \cdot 1 - 2}{1} = 3$$

Hence the total of all the residues is 5 and $I = 10\pi i$.

example

Consider

$$\int \frac{1}{z^4 - 1} \ dz$$

We can factor the denominator as

$$z^4 - 1 = (z^2 - 1)(z^2 + 1)$$

$$= (z+1)(z-1)(z+i)(z-i)$$

There are four poles, and each will have a residue.

$$b_1 = \lim_{z \to z_0} (z - z_0) \ f(z)$$

$$\text{Res}(1) = \lim_{z \to 1} (z-1) \ \frac{1}{(z+1)(z-1)(z+i)(z-i)}$$

$$= \lim_{z \to 1} \ \frac{1}{(z+1)(z+i)(z-i)}$$

$$= \frac{1}{2(1+1)} = \frac{1}{4}$$

$$\text{Res}(i) = \lim_{z\to i}(z-i)\ \frac{1}{(z+1)(z-1)(z+i)(z-i)}$$

$$\text{Res}(i) = \lim_{z\to i}\ \frac{1}{(z+1)(z-1)(z+i)}$$

$$= \lim_{z\to i}\ \frac{1}{(z^2-1)(z+i)}$$

$$= \frac{1}{(-2)(2i)} = -\frac{1}{4i} = \frac{i}{4}$$

example

$$\oint_C \frac{e^z}{z^3 - z^2 - 5z - 3} = \oint_C \frac{e^z}{(z+1)^2(z-3)}$$

Recall the general formula

$$f'(a) = \frac{1}{2\pi i}\ \oint_C \frac{f(z)}{(z-a)^2}\ dz$$

If the contour includes $z = -1$ but not $z = 3$ then

$$f(z) = \frac{e^z}{(z-3)}$$

so

$$f'(z) = \frac{(z-4)e^z}{(z-3)^2}$$

Hence

$$\oint_C \frac{e^z}{z^3 - z^2 - 5z - 3}\ dz = \oint_C \frac{e^z}{(z+1)^2(z-3)}$$

$$= \oint_C \frac{f(z)}{(z+1)^2}\ dz$$

for $f(z) = e^z/(z-3)$ and

$$= 2\pi i f'(-1) = 2\pi i\ \frac{-5}{e}\ \frac{1}{(-4)^2}$$

$$= \frac{-5\pi i}{8e}$$

Removable singularities

If the residue turns out to be equal to zero, that is called a removable singularity.

$$I = \int_C \frac{\sin \pi z}{z^2 - 1}\ dz$$

The denominator can be factored into

$$z^2 - 1 = (z+1)(z-1)$$

Suppose C includes only $z = 1$, then

$$\text{Res}(1) = \lim_{z \to 1} (z-1)\ \frac{\sin \pi z}{(z+1)(z-1)}$$

$$= \lim_{z \to 1}\ \frac{\sin \pi z}{(z+1)} = \frac{\sin \pi}{2} = 0$$

Here's a trick:

$$f(z) = z^2 \sin \frac{1}{z}$$

Compute Res(0)

$$\sin \frac{1}{z} = \frac{1}{z} - \frac{1}{3!}\ \frac{1}{z^3} + \frac{1}{5!}\ \frac{1}{z^5} \dots$$

$$z^2 \sin \frac{1}{z} = z - \frac{1}{3!}\ \frac{1}{z} + \frac{1}{5!}\ \frac{1}{z^3} \dots$$

The only non-zero integral term is

$$-\frac{1}{3!}\ \frac{1}{z}$$

and the residue there is

$$\lim_{z \to 0} (z - 0)(-\frac{1}{3!}\ \frac{1}{z})$$

$$= -\frac{1}{3!} = -\frac{1}{6}$$

which we could have just read off the cofactor in the Laurent series.

Chapter 26

Kaplan

Here is Rule II from Kaplan. Rule I is in the previous chapter.

At a pole of order N $(N = 2, 3, \dots)$,

$$\text{Res } [f(z), z_0] = \lim_{z \to z_0} (z - z_0) \frac{g^{(N-1)}(z)}{(N-1)!}$$

where

$$g(z) = (z - z_0)^N f(z)$$

example

$$f(z) = \frac{1}{z(z-2)^2}$$

We have a pole of first order at $z_0 = 0$ and one of second order at $z_0 = 2$. At the first

$$\text{Res } (0) = \lim_{z \to 0} \frac{1}{(z-2)^2} = \frac{1}{4}$$

For the other one, remove the factor of $1/(z-2)^2$ and compute the

$N-1$ (first) derivative of what's left

$$\frac{d}{dz}\frac{1}{z} = -\frac{1}{z^2}$$

$$\text{Res }(2) = \lim_{z\to 2} -\frac{1}{z^2} = -\frac{1}{4}$$

Don't forget to divide by $(N-1)!$, which is just 1 in this case. The total is just zero.

As a check, let's do this by partial fractions.

$$\frac{1}{z(z-2)^2} = \frac{A}{(z-2)^2} + \frac{B}{z(z-2)} + \frac{C}{z}$$

Hence in putting all terms over a common denominator, for the numerator we have

$$1 = Az + B(z-2) + C(z-2)^2$$

From which we get three equations:

$$-2B + 4C = 1$$

$$Az + Bz - 4Cz = 0$$

$$Cz^2 = 0$$

Hence $C = 0$, so $B = -1/2$ and $A = 1/2$ and we obtain

$$\frac{1}{z(z-2)^2} = \frac{1/2}{(z-2)^2} - \frac{1/2}{z(z-2)}$$

which we check by doing

$$1/2 \cdot z - 1/2 \cdot (z-2) = 1$$

So how to deal with

$$\frac{1/2}{(z-2)^2} - \frac{1/2}{z(z-2)}$$

The first term has a pole of order 2 at $z_0 = 2$. We remove that factor and compute the $N-1$ (first) derivative of what's left, which is just zero.

For the second term, we have two simple poles at $z_0 = 0$ and $z_0 = 2$. The residues are

$$\text{Res }(0) = \lim_{z \to 0} z \cdot \frac{1}{z(z-2)} = -\frac{1}{2}$$

$$\text{Res }(2) = \lim_{z \to 2} (z-2) \cdot \frac{1}{z(z-2)} = \frac{1}{2}$$

which adds up to zero.

example

$$f(z) = \frac{1}{z^4 + z^3 - 2z^2}$$

where C is the circle $|z| = 3$ with positive orientation.

The denominator can be factored as

$$z^2(z^2 + z - 2) = z^2(z+2)(z-1)$$

so

$$f(z) = \frac{1}{z^2(z+2)(z-1)}$$

There is a pole of order 2 at the origin and simple poles at 1 and -2. All of these lie within the contour $|z| = 3$.

$$\text{Res }(1) = \lim_{z \to 1} (z-1)\ f(z) = \lim_{z \to 1} \frac{1}{z^2(z+2)} = \frac{1}{3}$$

$$\text{Res }(-2) = \lim_{z \to -2} (z+2)\ f(z) = \lim_{z \to -2} \frac{1}{z^2(z-1)} = -\frac{1}{12}$$

For the double pole, we remove the factor of $1/z^2$, take the first derivative of what's left

$$\frac{d}{dz}\,\frac{1}{z^2+z-2} = \frac{(-1)(2z+1)}{(z^2+z-2)^2}$$

Evaluate at zero and obtain.

$$\text{Res }(0) = -\frac{1}{4}$$

The total of the residues is

$$\frac{1}{3} - \frac{1}{12} - \frac{1}{4} = 0$$

As Mathews and Howell say:

> The value 0 for the integral is not an obvious answer, and all of the preceding calculations are required to find it.

example

$$f(z) = \frac{1+e^z}{z^2} + \frac{2}{z}$$

We can break this up into its two component parts. For the first term, the pole is of order $m = 2$ at $z_0 = 0$. We remove the z^2 term and take the $m - 1 = 1$ derivative

$$(1+e^z)' = e^z$$

Remember to divide by $(m-1)!$, leaving e^z which is evaluated at the pole giving a residue

$$\text{Res }(0) = e^0 = 1$$

The other term is just 2 times the standard

$$\oint \frac{1}{z}\,dz = 2\pi i$$

Here $I = 4\pi i$ and the residue is 2. Alternatively just use

$$I = 2\pi i f(z_0) = 4\pi i$$

where $f = 2$.

The total of the residues is 3 and the value of the integral is $6\pi i$.

example

$$f(z) = \frac{e^z}{z(z-1)^2}$$

We have a pole of first order at $z = 0$ and one of second order at $z = 1$. At the first

$$\text{Res } [f(z), z = 0] = \lim_{z \to 0} \frac{e^z}{(z-1)^2} = 1$$

For the other one, remove the factor of $1/(z-1)^2$ and compute the $N - 1$ (first) derivative of what's left

$$\text{Res } [f(z), z = 1] = \lim_{z \to 1} \left[\ \frac{e^z}{z} \ \right]'$$

$$= \frac{e^z z - e^z}{z^2} \Bigg|_1 = 0$$

Hence

$$\oint f(z) \ dz = 2\pi i \ [\ \sum \text{Res} \ \ ] \ = 2\pi i$$

example

$$f(z) = \frac{1}{z(z-2)^4}$$

We have a pole of first order at $z = 0$ and one of fourth order at $z = 2$. At the first

$$\text{Res } [f(z), z = 0] = \lim_{z \to 0} z \ \frac{1}{z(z-2)^4}$$

$$= \lim_{z\to 0} \frac{1}{(z-2)^4} = \frac{1}{(-2)^4} = \frac{1}{16}$$

For the other pole recall that

$$\frac{2\pi i}{n!} f^n(a) = \oint_C \frac{f(z)}{(z-a)^{n+1}}\ dz$$

We remove the factor of $1/(z-2)^4$ leaving $f(z) = 1/z$ and then compute the $N-1$ (third) derivative of what's left

$$\text{Res}\ [f(z), z = 2] = \frac{1}{n!} \lim_{z\to 2}\ [\ \frac{1}{z}\ ]'''$$

$$f(z) = z^{-1}$$

$$f'(z) = -z^{-2}$$

$$f''(z) = 2z^{-3}$$

$$f'''(z) = -6z^{-4}$$

$$\lim_{z\to 2}\ [\ \frac{1}{z}\ ]''' = -\frac{6}{16}$$

Don't forget to divide by $(N-1)!$, which is $3! = 6$ in this case. That leaves

$$\text{Res}\ [f(z), z = 2] = -\frac{1}{16}$$

The total of the residues is just zero.

This problem is from Brown and Churchill (p. 234), which they work by doing Laurent series. They get a different answer, namely $-\pi i/8$.

The reason is that they integrate over the contour $0 < |z-2| < 2$, which includes the second pole, but not the first. Multiplying by $2\pi i$ gives their result.

Chapter 27

Quotients

RULE III If $A(z)$ and $B(z)$ are analytic in a neighborhood of z_0, $A(z_0) \neq 0$, and $B(z)$ has a zero at z_0 of order 1, then

$$f(z) = \frac{A(z)}{B(z)}$$

has a pole of first order at z_0 and

$$\text{Res } [f(z), z_0] = \frac{A(z_0)}{B'(z_0)}$$

example

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$$

$$B = z^2 + 1, \quad B' = 2z$$

$$\text{Res } [f(z), z = i] = \frac{1}{2i}$$

$$\text{Res } [f(z), z = -i] = -\frac{1}{2i}$$

example

$$f(z) = \frac{ze^z}{z^2 - 1}$$

Both top and bottom are analytic. The poles of $B(z)$ are at $\pm$ 1. $A(z) \neq 0$ at those points. We have

$$\frac{A(z)}{B'(z)} = \frac{ze^z}{2z}$$

$$\left.\frac{ze^z}{2z}\right|_{z=1} = \frac{e}{2}$$

$$\left.\frac{ze^z}{2z}\right|_{z=-1} = e^{-1}$$

RULE IV If $A(z)$ and $B(z)$ are analytic in a neighborhood of z_0, $A(z_0) \neq 0$, and $B(z)$ has a zero at z_0 of order 2, then

$$\text{Res}\ [f(z), z_0] = \frac{6A'B'' - 2AB'''}{3B''^2}$$

example

$$f(z) = \frac{e^z}{z(z-1)^2}$$

$$B = z(z^2 - 2z + 1) = z^3 - 2z^2 + z$$

$$B' = 3z^2 - 4z + 1$$

$$B'' = 6z - 4$$

$$B''' = 6$$

So

$$\frac{6A'B'' - 2AB'''}{3B''^2} = \frac{6(e^z)(6z-4) - 2e^z(6)}{3(6z-4)^2}$$

Evaluate at $z = 1$:

$$\frac{6e(2) - 2e(6)}{12} = 0$$

So only the pole at $z = 0$ contributes.

Chapter 28

Summary 3

Cauchy's residue formula was

$$\oint \frac{f(z)}{z - z_0}\; dz = 2\pi i \cdot f(z_0)$$

Two corollaries:

$$\oint \frac{f(z)}{(z - w)^2}\; dz = 2\pi i \cdot f'(z_0)$$

generally

$$\oint \frac{f(z)}{(z - z_0)^{n+1}}\; dz = \frac{1}{n!}\; 2\pi i \cdot f^n(z_0)$$

The residue of $f(z)$ at a singularity z_0 is the coefficient of the $(z-z_0)^{-1}$ term in the Laurent series for $f(z)$, if we can write it.

It is also given by

$$\begin{aligned} R &= \lim_{z \to z_0} (z - z_0) \cdot f(z_0) \\ &= \frac{1}{2\pi i} \oint f(z)\; dz \end{aligned}$$

Part VII

More

Chapter 29

Real Integrals

inverse tangent

Note the complex function

$$\oint_C^{\infty} \frac{1}{1+z^2}\ dz$$

We solved this one previously. Basically we factor the integrand as

$$-\frac{1}{2i}\ [\ \frac{1}{z+i} - \frac{1}{z-i}\ ]$$

We will integrate over a path that includes only the pole at $z = i$, so only the second term contributes, and by Cauchy 2 the value is

$$I = 2\pi i f(z_0) = 2\pi i \cdot (-\frac{1}{2i}) \cdot (-1) = \pi$$

Now we do the same integral on the real axis, with different limits:

$$\int_0^{\infty} \frac{1}{1+x^2}\ dx$$

We know the answer to this one, it is

$$\tan^{-1} x \,\Big|_0^\infty = \frac{\pi}{2}$$

Since $f(x)$ is an even function, the integral over $-\infty \to \infty = \pi$.

We feel there ought to be a connection between the two results: real and complex.

Suppose we draw a different curve (contour) extending on its base from $-\infty \to \infty$: the real axis. That integral is $\int f(z)\ dz$ but y and dy are both zero so it becomes just $\int f(x)\ dx$ with the result shown.

How to complete the contour? Imagine a semicircle in the upper half-plane with $R \to \infty$. That is, parametrize

$$\gamma(\theta) = Re^{i\theta}, \quad \theta \in [0, \pi]$$
$$\gamma\ '(\theta) = iRe^{i\theta}$$

The integral is

$$\int_0^\pi \frac{1}{1 + R^2 e^{i2\theta}}\ iRe^{i\theta}\ d\theta$$
$$= i \int_0^\pi \frac{1}{1/Re^{i\theta} + Re^{i\theta}}\ \ d\theta$$

Now what?

We'll try to be more formal later, but just for now, it's clear that as $R \to \infty$, this integrand goes to 0. So we have that the total integral for the complex case is equal to the integral for the real part plus this extra half-circle which is zero.

What this means is that if we had not know the result for the real integral, we could deduce it from the fact that the whole complex integral has value equal to π, and the part over this complex half-circle is zero.

Gaussian

application of Cauchy 1

The function we'll be working with is one we introduced before:

$$u(x,y) = e^{-x^2} e^{y^2} \cos 2xy$$

$$v(x,y) = e^{-x^2} e^{y^2} (-\sin 2xy)$$

Everything will simplify pretty quickly. Divide the path into its four parts and compute each separately: Over $C1$, $y = 0$ and $dy = 0$ so we have:

$$\int_{C1} = \int u \ dx = \int_0^a e^{-x^2} e^0 \cos 0 \ dx = \int_0^a e^{-x^2} \ dx$$

C2 ($x = a$, $dx = 0$):

$$\int_{C2} = -\int_0^b e^{-a^2} e^{y^2} (-\sin 2ay) \ dy$$

C3 ($y = a$, $dy = 0$):

$$\int_{C3} = \int_a^0 e^{-x^2} e^{b^2} (\cos 2bx) \ dx$$

C4 ($x = 0$, $dx = 0$):

$$\int_{C4} = \int_b^0 e^{y^2} (-\sin 0) \ dy = 0$$

So all together:

$$\int_0^a e^{-x^2} \ dx - \int_0^b e^{-a^2} e^{y^2} (-\sin 2ay) \ dy + \int_a^0 e^{-x^2} e^{b^2} \cos 2bx \ dx = 0$$

$$\int_0^a e^{-x^2} \ dx = e^{-a^2} \int_0^b e^{y^2} (-\sin 2ay) \ dy + e^{b^2} \int_0^a e^{-x^2} \cos 2bx \ dx$$

Let $a \to \infty$. Then

$$e^{-a^2} \to 0$$

so the first term on the right side goes to zero and we have:

$$\int_0^\infty e^{-x^2}\ dx = e^{b^2} \int_0^\infty e^{-x^2} \cos 2bx\ dx$$

But we know the value of the left-hand side, it is

$$\int_0^\infty e^{-x^2}\ dx = \frac{\sqrt{\pi}}{2}$$

so

$$\int_0^\infty e^{-x^2} \cos 2bx\ dx = \frac{\sqrt{\pi}}{2}\ e^{-b^2}$$

The Gaussian that we know, is a special case of this general form.

example

We will develop a proof that

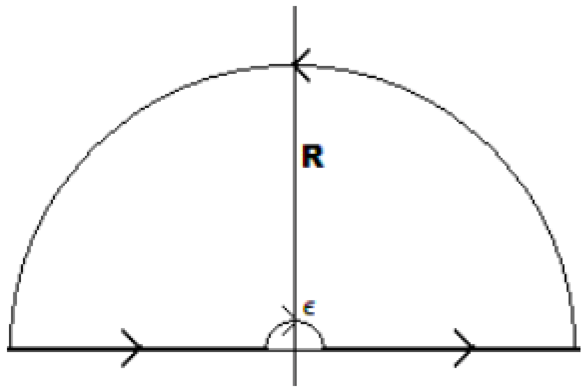
$$\int_{-\infty}^\infty \frac{\sin x}{x}\ dx = \frac{\pi}{2}$$

Start by considering the function

$$f(z) = \frac{e^{iz}}{z}$$

Obviously there is a pole at the origin.

Now we consider a specially designed contour that avoids the origin by going around it clockwise in a semi-circle of radius ϵ.



Since the integral avoids the pole, its value is zero.

small semi-circle

Consider the pieces next, starting with the small semi-circle. In the limit as $\epsilon \to 0$

$$f(z) = \frac{e^{iz}}{z}$$

$$\begin{aligned}\text{Res }(0) &= \lim_{z\to 0} z\ \frac{e^{iz}}{z} \\ &= \lim_{z\to 0} e^{iz} \\ &= 1\end{aligned}$$

We multiply by πi for the half-circular path, and put in a minus sign since we are going counter-clockwise.

$$I = -\pi i$$

As a check write

$$e^{iz} = \cos z + i \sin z$$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots + iz - i\frac{z^3}{3!}$$

Multiply by $1/z$ to obtain

$$= \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} \cdots + i - i\frac{z^3}{3!}$$

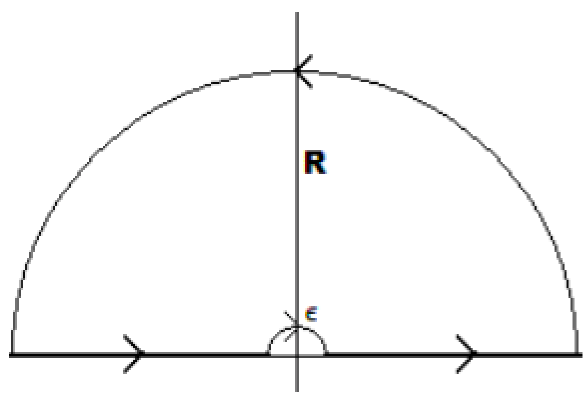
In the limit $z \to 0$,

$$= \frac{1}{z} + i$$

Or just look at the integral term by term of the series. The only non-zero term is

$$\int_{\pi}^{0} \frac{1}{z}\ dz = -\pi i$$

large semi-circle



For the large semi-circle at radius R we have

$$z = Re^{i\theta}$$

$$\int \frac{e^{iz}}{z}\ dz$$

The absolute value of the denominator is

$$|z| = |Re^{i\theta}| = R$$

The numerator is

$$e^{iz} = e^{iRe^{i\theta}}$$

$$= e^{R(i\cos\theta - \sin\theta)}$$

$$= e^{-R\sin\theta}\ e^{Ri\cos\theta}$$

Go back to the fundamental definition of length for a complex number: $|w|^2 = ww*$. The square of the length of the numerator is

$$|e^{-R\sin\theta}|^2\ [\ e^{Ri\cos\theta} e^{R(-i)\cos\theta}\ ]$$

$$= |e^{-R\sin\theta}|^2$$

So the absolute value of the numerator is just

$$e^{-R\sin\theta}$$

and the absolute value of

$$|\frac{e^{iz}}{z}| = \frac{e^{-R\sin\theta}}{R}$$

now

$$dz = Re^{i\theta}\ d\theta$$

$$|dz| = |Rd\theta|$$

so

$$\int |\frac{e^{iz}}{z}\ dz| = \int \frac{e^{-R\sin\theta}}{R}\ Rd\theta$$

$$= \int e^{-R\sin\theta}\ d\theta$$

but in the limit as $R \to \infty$, the integrand goes to zero.

The last two segments lie on the real axis. The first

$$\int_{x=-R}^{-\epsilon} \frac{e^{ix}}{x}\ dx$$

(since y and dy are both zero. Now, tsubstitute $-x$ for x

$$\int_{-x=-R}^{-\epsilon} \frac{e^{-ix}}{-x}\ d-x$$

$$= \int_{x=R}^{\epsilon} \frac{e^{-ix}}{x}\ dx$$

the second one is just

$$\int_{x=\epsilon}^{R} \frac{e^{ix}}{x}\ dx$$

so then in the limit as $\epsilon \to 0$ and $R \to \infty$ this is

$$\int_0^\infty \frac{e^{ix}}{x} + \frac{e^{-ix}}{x}\ dx$$

$$= \int_0^\infty 2i\ \frac{\sin x}{x}\ dx$$

Adding together all four pieces and equating them to the first result for the whole contour (zero), we have

$$0 = -\pi i + 0 + 2i \int_0^\infty \frac{\sin x}{x}\ dx$$

Rearranging

$$\pi i = 2i \int_0^\infty \frac{\sin x}{x}\ dx$$

$$\int_0^\infty \frac{\sin x}{x}\ dx = \frac{\pi}{2}$$

Complicated example using Cauchy 2

Consider a semicircle of radius R lying in the first two quadrants with its diameter on the real x-axis, and a function $f(z)$. We wish to evaluate:

$$= \oint_C \frac{e^{iaz}}{b^2 + z^2}\ dz$$

where a and b are positive constants.

It is apparent that $f(z)$ has singularities at $z = \pm ib$, where $b < R$. In particular, we are interested in what happens as $R \to \infty$.

Along the x-axis, we have $y = 0$ and $dy = 0$, so $dz = dx$ and

$$\int_{C1} = \int_{-R}^{R} \frac{e^{iax}}{b^2 + x^2}\ dx$$

Along the semi-circular arc, we have $r = R$ and $\theta = 0 \to \pi$ and

$$z = Re^{i\theta}$$

$$dz = iRe^{i\theta}\ d\theta$$

$$\int_{C2} = \int_{0}^{\pi} \frac{e^{ia(Re^{i\theta})}}{b^2 + R^2 e^{i2\theta}}\ iRe^{i\theta}\ d\theta$$

Thus

$$\oint_C z\ dz = \oint_C \frac{e^{iaz}}{b^2 + z^2}\ dz$$

$$= \int_{-R}^{R} \frac{e^{iax}}{b^2 + x^2}\ dx + \int_{0}^{\pi} \frac{e^{ia(Re^{i\theta})}}{b^2 + R^2 e^{i2\theta}}\ iRe^{i\theta}\ d\theta$$

Rewriting the integrand for the integral on the left-hand side:

$$\frac{e^{iaz}}{b^2 + z^2} = \frac{e^{iaz}}{(z + ib)(z - ib)} = \frac{e^{iaz}}{i2b}\left(\frac{1}{z - ib} - \frac{1}{z + ib}\right)$$

Having factored out $1/i2b$, rewrite the integral as

$$\frac{1}{i2b}\left(\oint \frac{e^{iaz}}{z - ib}\, dz - \oint \frac{e^{iaz}}{z + ib}\, dz\right) = \int_{-R}^{R} \frac{e^{iax}}{b^2 + x^2}\, dx + \int_{0}^{\pi} \frac{e^{ia(Re^{i\theta})}}{b^2 + R^2 e^{i2\theta}}\, iRe^{i\theta}\, d\theta$$

That's quite a mouthful!

The second term on the left-hand side has a singularity at $z = -ib$, which is *outside* the region (actually, below it) and hence by Cauchy 1 that integral is zero.

So now

$$\frac{1}{i2b} \oint \frac{e^{iaz}}{z - ib} \, dz = \int_{-R}^{R} \frac{e^{iax}}{b^2 + x^2} \, dx + \int_0^{\pi} \frac{e^{ia(Re^{i\theta})}}{b^2 + R^2 e^{i2\theta}} \, iRe^{i\theta} \, d\theta$$

Looking at the other contour integral

$$\frac{1}{i2b} \oint \frac{e^{iaz}}{z - ib} \, dz$$

we have a singularity at $z = z_0 = ib$, which, as R becomes large, is inside the semicircular region and thus by Cauchy 2 the integral is equal to $2\pi i \ f(z_0)$ where

$$f(z_0) = e^{iaz_0} = e^{iaib} = e^{-ab}$$

and so we have

$$\begin{aligned} \frac{1}{i2b} \oint \frac{e^{iaz}}{z - ib} \, dz &= \frac{1}{i2b} 2\pi i e^{-ab}, \quad R > b \\ &= \frac{\pi}{b} e^{-ab} \end{aligned}$$

Putting it all together

$$\frac{\pi}{b} e^{-ab} = \int_{-R}^{R} \frac{e^{iax}}{b^2 + x^2} \, dx + \int_0^{\pi} \frac{e^{ia(Re^{i\theta})}}{b^2 + R^2 e^{i2\theta}} \, iRe^{i\theta} \, d\theta$$

Nahin shows that the second integral on the right-hand side vanishes as $R \to \infty$. The reason is that we have R^2 in the denominator and only R in the numerator.

So

$$\frac{\pi}{b}e^{-ab} = \int_{-\infty}^{\infty} \frac{e^{iax}}{b^2 + x^2}\ dx$$

$$\frac{\pi}{b}e^{-ab} = \int_{-\infty}^{\infty} \frac{\cos(ax)}{b^2 + x^2}\ dx + i\int_{-\infty}^{\infty} \frac{\sin(ax)}{b^2 + x^2}\ dx$$

The imaginary part of the left-hand side is zero, so by the equality we must have that

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{b^2 + x^2}\ dx = 0$$

"which is no surprise since the integrand is an odd function of x". But the other result (from the real part) is:

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{b^2 + x^2}\ dx = \frac{\pi}{b}e^{-ab}$$

In the special case $a = b = 1$ we obtain

$$\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2}\ dx = \frac{\pi}{e} = 1.15572735$$

which is not only an integral we didn't know how to do before, but a remarkable fraction as the result.

Chapter 30

Trig Integrals

Karkhar gives this problem

$$\int_0^{2\pi} \frac{1}{2 + \cos\theta} \; d\theta$$

Before we start I'd just point out that the equation for an ellipse in polar coordinates (with one focus at the origin) is

$$r = \frac{b^2}{a - c\cos\theta}$$

If we neglect the minus sign (which just flips the orientation along the x-axis), let $a = 2$ and $c = 1$ and

$$b^2 = a^2 - c^2 = 3$$

rewrite

$$\int_0^{2\pi} \frac{3}{2 + \cos\theta} \; d\theta$$

What this looks like to me is the integral of $r \; d\theta$ around an ellipse with $a = 2$ and $b = \sqrt{3}$. This would be $r d\theta$ added up over the perimeter of that ellipse, i.e. the area.

Go back to the given problem. Let

$$z = e^{i\theta}$$

$$dz = iz\ d\theta$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

Use this result, but go back to z:

$$\int_{|z|=1} \frac{1}{2 + (1/2)(z + 1/z)}\ \frac{1}{iz}\ dz$$

$$= \frac{1}{i} \int_{|z|=1} \frac{1}{2z + (1/2)(z^2 + 1)}\ dz$$

$$= \frac{2}{i} \int_{|z|=1} \frac{1}{z^2 + 4z + 1}\ dz$$

The roots of the denominator are $-2 \pm \sqrt{3}$. One of these roots $(-2 + \sqrt{3})$ lies within our contour, which is just the unit circle.

Carry out partial fractions:

$$\frac{1}{z^2 + 4z + 1} = \frac{A}{z - (-2 + \sqrt{3})} + \frac{B}{z - (-2 - \sqrt{3})}$$

$$Az + A2 + A\sqrt{3} + Bz + B2 - B\sqrt{3} = 1$$

Hence $A = -B$ and

$$A2 + A\sqrt{3} - A2 + A\sqrt{3} = 1$$

$$2A\sqrt{3} = 1$$

$$A = \frac{1}{2\sqrt{3}}$$

The term we want is the one with $z_0 = (-2+\sqrt{3})$ and that has coefficient A. Hence the value is

$$2\pi i \ (\frac{1}{2\sqrt{3}}) = \frac{\pi i}{\sqrt{3}}$$

Pick up the leading factor of $2/i$ and obtain $2\pi/\sqrt{3}$.

Going back to the argument about the ellipse at the beginning, multiplied by 3 gives $2\sqrt{3}\pi$. This is exactly the area of an ellipse with $a = 2$ and $b = \sqrt{3}$.

residues

The zeros are at

$$-2 \pm \ \sqrt{3}$$

If we call these two values a_1 and a_2 with

$$a_1 = -2 + \sqrt{3}$$

$$a_2 = -2 - \sqrt{3}$$

then

$$f(z) = \frac{1}{(z-a_1)(z-a_2)}$$

We see that only $a_1 = -2+\sqrt{3}$ is within the contour over which we're integrating, so using the formula for residues

$$\text{Res } (a_1) = \lim_{z \to a_1} (z-a_1) \ \frac{1}{(z-a_1)(z-a_2)}$$

$$= \lim_{z \to a_1} \frac{1}{z-a_2}$$

$$= \frac{1}{a_1 - a_2}$$

Now

$$
\begin{aligned}
a_1 - a_2 &= (-2 + \sqrt{3}) - (-2 - \sqrt{3}) \\
&= 2\sqrt{3}
\end{aligned}
$$

so

$$\frac{1}{a_1 - a_2} = \frac{1}{2\sqrt{3}}$$

To get the value of the whole integral we have to pick up the leading factor of $2/i$, giving

$$\frac{2}{i} \cdot \frac{1}{2\sqrt{3}}$$

we also need to multiply the result by $2\pi i$

$$I = 2\pi i \cdot \frac{2}{i} \cdot \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

As we said before, if we get back to the argument about the ellipse at the beginning, multiplying by 3 gives $2\sqrt{3}\pi$, which is exactly the area of an ellipse with $a = 2$ and $b = \sqrt{3}$.

Chapter 31

Conformal mapping

conformal mapping

At this point let's just remind ourselves how the visual representation of a complex function differs from the more familiar case of a function $f : \mathbb{R}^1 \to \mathbb{R}^1$.

In the real case, the first dimension is the independent variable x and the second is $y = f(x)$ and the derivative is the *slope* of the curve produced by plotting pairs of $x, f(x)$.

In the complex case, our numbers z are points in the complex plane. They are *mapped* to other complex numbers in a different complex plane, which is often called w, where $w = f(z) = u(x, y) + iv(x, y)$.

The derivative does not have any notion of slope. Our requirement for differentiability included the constraint that the derivative of the function at a point z_0 must be the same no matter from what direction we approach that point. This leads to the CRE and the study of only analytic functions. Many such functions may have isolated points at which they are not defined, and that will still be OK.

In the figure, is shown the mapping corresponding to the complex function $w = f(z) = z^2$.

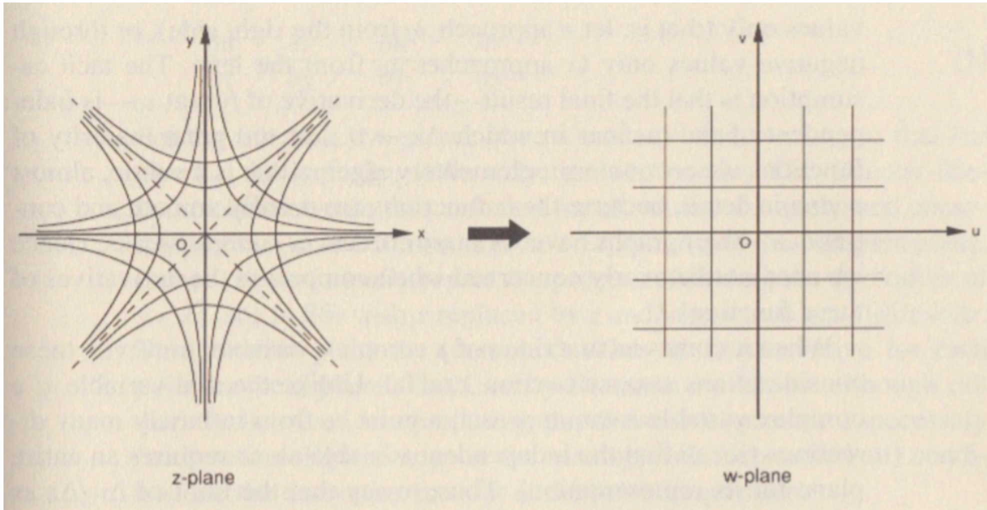


FIG. 71. Mapping by the complex function $w = z^2$.

$$z = x + iy$$
$$z^2 = x^2 - y^2 + i2xy$$

The functions $u = x^2 - y^2$ and $v = 2xy$ are both hyperbolas. Consider what happens in the case where $u = c$ where c is a constant. The values of x and y that satisfy this constrain lie on hyperbolas in the z-plane. For example, the points on the curve $xy = 1/2$ correspond to the points on the curve $v = 1$, which is a straight vertical line in the w plane.

In this sense, the hyperbolic curves in the z-plane shown in the figure are mapped into rectangular grid in the w-plane. An important note here is that the angles where these curves meet are the same in both the z-plane and the w-plane. In both cases the lines meet at right angles.

According to wolfram

http://mathworld.wolfram.com/ConformalMapping.html

A conformal mapping, also called a **conformal map**, conformal transformation, angle-preserving transformation, or biholomorphic map, is a transformation that preserves local angles. An analytic function is conformal at any point where it has a nonzero derivative.

looking ahead

As motivation to do the work that is coming, consider these statements from the summary article in wikipedia:

One of the central tools in complex analysis is the line integral. The line integral around a closed path of a function that is holomorphic everywhere inside the area bounded by the closed path is always zero, which is what the Cauchy integral theorem states. The values of such a holomorphic function inside a disk can be computed by a path integral on the disk's boundary, as shown in (Cauchy's integral formula).

Path integrals in the complex plane are often used to determine complicated real integrals, and here the theory of residues among others is applicable (see methods of contour integration). A "pole" (or isolated singularity) of a function is a point where the function's value becomes unbounded, or "blows up". If a function has such a pole, then one can compute the function's residue there, which can be used to compute path integrals involving the function; this is the content of the powerful residue theorem.

harmonics

Boas says this about analytic functions (that satisfy the CRE).

"If $f(z) = u = iv$ is analytic in a region, then u and v satisfy Laplace's equation, that is, u and v are harmonic functions..."

Laplace's equation is:

$$\nabla^2 f = 0$$

Consider the function

$$u(x, y) = x^2 - y^2$$
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
$$= 2 - 2 = 0$$

To find the function $v(x, y)$ such that $z = u + iv$ is analytic, use the CRE:

$$u = x^2 - y^2$$
$$v_y = u_x = 2x$$
$$v_x = -u_y = 2y$$

So it looks like $2xy$ will work. In particular

$$z = x^2 - y^2 + i2xy + \text{constant}$$

But of course

$$x^2 - y^2 + i2xy = (x + iy)^2 = z^2$$

which does not depend on $z*$.

To explore why this is true, take the second derivatives of the CRE:

$$u_x = v_y$$
$$u_y = -v_x$$

$$u_{xx} = v_{yx}$$

$$u_{xy} = v_{yy}$$
$$u_{yx} = -v_{xx}$$
$$u_{yy} = -v_{xy}$$

But the mixed partials must be equal so

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy}$$
$$u_{xx} + u_{yy} = 0$$
$$v_{xx} = -u_{yx} = -u_{xy} = -v_{yy}$$
$$v_{xx} + v_{yy} = 0$$

Part VIII

Addendum

Chapter 32

Extra

In what follows $z = re^{is}$ or $z = x + iy$, as convenient.

We also use $w = \rho e^{it}$ or $w = u + iv$.

Properties of the conjugate: P1

$$(z + w)^* = x - iy + u - iv = z^* + w^*$$

P2

$$(zw^*)^* = (re^{is}\rho e^{-it})^* = (r\rho e^{i(s-t)})^* = r\rho e^{i(t-s)} = z^*w$$

P3

$$z + z* = x + iy + x - iy = 2x = 2Re\ z$$

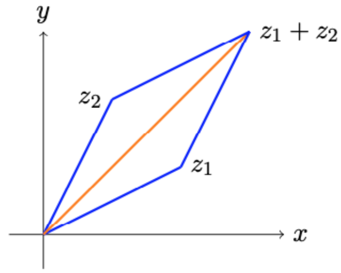
P4

$$|z| = |z^*|$$

Length of a product:

$$|z||w| = |re^{is}|\ |\rho e^{-it}| = r\ \rho = |zw|$$

Triangle Inequality



Triangle inequality: $|z_1| + |z_2| \geq |z_1 + z_2|$

$$|z+w|^2 = (z+w)(z+w)^*$$

$z+w$ is a complex number. The square of the length of a complex number is equal to the number multiplied by its modulus.

$$= (z+w)(z^* + w^*)$$

P1 above.

$$= zz^* + zw^* + z^*w + ww^*$$

Distributivity of multiplication.

$$= zz^* + zw^* + (zw^*)^* + ww^*$$

P2 above.

$$= |z|^2 + 2Re(zw*) + |w|^2$$

From the definition of the conjugate and P3.

Now we transition to the inequality

$$\leq |z|^2 + 2|zw^*| + |w|^2$$

since the real part of a complex number is less than or equal to its length (only equal if it is purely real).

$$= |z|^2 + 2|z||w^*| + |w|^2$$

See the length of a product, above.

$$= |z|^2 + 2|z||w| + |w|^2$$

P4 above.

$$= (|z| + |w|)^2$$

from basic multiplication.

The inequality follows from taking the square root of the first and last expressions:

$$|z + w| \leq |z| + |w|$$

□

reverse triangle equality

Surprisingly tricky...

$$|z_2 - z_1| \geq |\ |z_2| - |z_1|\ |$$

https://math.stackexchange.com/questions/127372/reverse-triangle-inequality-proof

By the standard triangle inequality:

$$|x| + |y - x| \geq |x + y - x| = |y|$$

By subtraction

$$|y - x| \geq |y| - |x|$$

By the same logic or just substituting symbols

$$|y| + |x - y| \geq |y + x - y| = |x|$$

$$|x - y| \geq |x| - |y|$$

Let

$$t = |y - x|$$

But $|a| = |-a|$ so also

$$t = |x - y|$$

Let $u = |y| - |x|$ We have

$$t \geq u$$

$$t \geq -u$$

Since $|a| = |-a|$:

$$|t| \geq |u|$$

Then

$$|y - x| \geq |\ |y| - |x|\ |$$

□

Chapter 33

Partial fractions real

Strang gives this example:

$$\int \frac{1}{x-2} + \frac{3}{x+2} - \frac{4}{x}\ dx$$
$$= \ln|x-2| + 3\ln|x+2| - 4\ln|x|$$

That seems straightforward enough. Which function would produce that sum?

$$\frac{1}{x-2} + \frac{3}{x+2} - \frac{4}{x}$$
$$= \frac{(x+2)(x) + 3(x-2)(x) - 4(x-2)(x+2)}{(x-2)(x+2)(x)}$$
$$= \frac{x^2 + 2x + 3x^2 - 6x - 4x^2 + 16}{x^3 - 4x}$$
$$= \frac{-4x + 16}{x^3 - 4x}$$

We call this form P/Q, and it's the type of problem we are trying to solve using *partial fractions*. We start by factoring Q (although sometimes, the factors are given).

$$Q = x^3 - 4x = x(x^2 - 4)$$

$$= x(x-2)(x+2)$$

Let's try with a different numerator to see how it works. We write:

$$\frac{P}{Q} = \frac{3x^2 + 8x - 4}{(x-2)(x+2)(x)}$$

$$= \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x}$$

where A,B and C are constants.

We recognize that we can put these three fractions over Q as the common denominator.

These are the partial fractions that add up to P/Q. We need to find the values of A,B and C.

Here are two methods (the first one is slower):

Do what we just said. Put the right-hand side over the common denominator Q:

$$\frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x}$$

$$= \frac{A(x+2)(x) + B(x-2)(x) + C(x-2)(x+2)}{(x-2)(x+2)(x)}$$

Now make the numerators match:

$$3x^2 + 8x - 4 = A(x+2)(x) + B(x-2)(x) + C(x-2)(x+2)$$

$$= Ax^2 + 2Ax + Bx^2 - 2Bx + Cx^2 - 4C$$

We actually have three equations:

$$Ax^2 + Bx^2 + Cx^2 = 3x^2$$

$$2Ax - 2Bx = 8x$$

$$-4C = -4$$

From the last one $C = 1$. From the first one we have:

$$A + B + C = 3$$

$$A + B = 2$$

and then

$$A - B = 4$$

Add them together to get $2A = 6, so A = 3$ and then $B = -1$. We obtain finally

$$\frac{P}{Q} = \frac{3}{x-2} + \frac{-1}{x+2} + \frac{1}{x}$$

second method

The second approach is called the "cover-up method." We have:

$$\frac{3x^2 + 8x - 4}{(x-2)(x+2)(x)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x}$$

Multiply by $(x-2)$

$$\frac{3x^2 + 8x - 4}{(x+2)(x)} = (\frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x})(x-2)$$

$$= A + \frac{B(x-2)}{x+2} + \frac{C(x-2)}{x}$$

Now evaluate at $x = 2$

$$\frac{3(2)^2 + 8(x) - 4}{(2+2)(2)} = \frac{12 + 16 - 4}{8} = 3 = A$$

Notice that we do not need to actually write

$$A + \frac{B(x-2)}{x+2} + \frac{C(x-2)}{x}$$

Nor, in calculating B, do we need to write

$$\frac{A(x+2)}{x-2} + B + \frac{C(x+2)}{x}$$

since we will pick x to zero out those terms, instead, just substitute $x = -2$ into

$$\frac{3x^2 + 8x - 4}{(x-2)(x)}$$
$$= \frac{3(-2)^2 + 8(-2) - 4}{(-2-2)(-2)}$$
$$B = \frac{12 - 16 - 4}{8} = \frac{-8}{8} = -1$$

For C multiply the left-hand side by x and evaluate at $x = 0$ (to make the A and B terms go away):

$$\frac{3x^2 + 8x - 4}{(x-2)(x+2)} = \frac{-4}{-4} = 1 = C$$

same degree

How about

$$\int \frac{3x^2 + 2x + 7}{x^2 + 1}\, dx$$

To use the method, P must be of a lower degree than Q, but here they both contain multiples of x^2 (degree two). We separate off the term of $3x^2$ by finding another 3:

$$\frac{3x^2 + 2x + 7}{x^2 + 1} = \frac{3x^2 + 3 + 2x + 4}{x^2 + 1}$$

$$= 3 + \frac{2x+4}{x^2+1}$$

Now we just have to solve:

$$\int 3 + \frac{2x}{x^2+1} + \frac{4}{x^2+1}\ dx$$

$$= 3x + \ln(x^2+1) + 4\tan^{-1} x + C$$

repeated factor

$$\frac{2x+3}{(x-1)^2}$$

We have two factors of $x-1$. Solution: use $(x-1)^2$ for one of the fractions:

$$\frac{2x+3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$2x+3 = A(x-1) + B$$

set $x = 1$, then

$$B = 2(1) + 3 = 5$$

and

$$2x + 3 = Ax - A + 5$$

$A = 2$ solves this.

more examples

These few examples are from wikipedia. We would like to simplify

$$\frac{3x+5}{(1-2x)^2}$$

We suppose that this fraction can be decomposed as follows

$$\frac{3x+5}{(1-2x)^2} = \frac{A}{(1-2x)^2} + \frac{B}{(1-2x)}$$

We multiply by the term with B to put everything over a common denominator:

$$\begin{aligned} &\frac{A}{(1-2x)^2} + \frac{B}{(1-2x)} \\ &= \frac{A}{(1-2x)^2} + \frac{B(1-2x)}{(1-2x)^2} \end{aligned}$$

Getting rid of the denominators altogether

$$3x+5 = A + B(1-2x)$$

Now both the constant terms and the terms in x must be equal:

$$-2Bx = 3x$$

$$B = -\frac{3}{2}$$

$$A + B = 5$$

$$A = \frac{13}{2}$$

And so

$$\frac{3x+5}{(1-2x)^2} = \frac{13/2}{(1-2x)^2} + \frac{-3/2}{(1-2x)}$$

To integrate, we would do this

$$\int \frac{3x+5}{(1-2x)^2}\ dx = \int \frac{13/2}{(1-2x)^2}\ dx + \int \frac{-3/2}{(1-2x)}\ dx$$

$$= \frac{13/4}{(1-2x)} + (3/4)\ln(1-2x)$$

Example 2.

$$f(x) = \frac{1}{x^2 + 2x - 3} = \frac{1}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

$$A(x-1) + B(x+3) = 1$$

$$Ax + Bx = 0$$

$$A = -B$$

$$-B + 3B = 1$$

$$B = \frac{1}{4}$$

$$f(x) = \frac{1}{4}(\frac{-1}{x+3} + \frac{1}{x-1})$$

Chapter 34

Taylor series

This chapter is an introduction to Taylor series for real variables.

Suppose we have a function $f(x)$, but

Shankar:

> "imagine that you don't have access to the whole function. You cannot see the whole thing. You can only zero-in on a tiny region."

around $f(0)$, where you know the value. So the question is, what do we guess the function will do near $f(0)$?

The first approximation is that

$$f(x) \approx f(0)$$

We really can't say anything more. $f(0)$ is the best guess for what the value of the function is (we're talking about continuous and continuously differentiable functions).

Now suppose we know the slope of the function at 0, $f'(0)$. Then, since

$$\Delta y = f'(0)\Delta x = f'(0)(x - 0)$$

we can get a better approximation as the linear approximation:

$$f(x) \approx f(0) + f'(0)\ x + \ldots$$

For most functions, there will be more terms. If f is not a linear function, then the slope won't be constant. So

> "the rate of change itself has a rate of change .. the second derivative."

The term we are going to add is

$$f''(0)\ \frac{x^2}{2}$$

so

$$f(x) \approx f(0) + f'(0)\ x + f''(0)\ \frac{x^2}{2} + \ldots$$

A simple way to see why we have $x^2/2$ is to take derivatives on both sides. The terms like $f'(0)$ and $f''(0)$ are constants, they have been evaluated at $x = 0$. The first derivative is

$$f'(x) \approx f'(0) + f''(0)\ x + \ldots$$

We evaluate at $x = 0$ and the term $f''(0)\ x$ goes away because of the $x = 0$ multiplying the constant $f''(0)$. So we have just

$$f'(x) \approx f'(0)$$

and that matches. Now take the second derivative

$$f''(x) \approx f''(0)$$

and that matches too. We can see a pattern here.

The fourth term is

$$f(x) \approx f(0) + f'(0)\ x + f''(0)\ \frac{x^2}{2!} + f'''(0)\ \frac{x^3}{3!} + \ldots$$

You might not be expecting the factorial which I snuck in there. But if you go back to the exercise above, where we evaluated derivatives, you can see why it works. When we take the first derivative

$$\frac{d}{dx}(f'''(0)\ \frac{x^3}{3!}) = f'''(0)\ \frac{x^2}{2!}$$

the 3 comes down from the power and then turns 3! in the denominator into 2!. The next derivative will bring down the 2. So everything cancels properly.

If you like Σ notation, we can write

$$f(x) = \sum_{n=0}^{\infty} f^n(0) \frac{x^n}{n!}$$

with the understanding that $0! = 1$. The approximation is better the closer x is to 0, and the more terms the better as well.

There is one final wrinkle to this derivation. The series can be modified deal with x near any value a, not just near 0. The modification is

$$f(x) = \sum_{n=0}^{\infty} f^n(a) \frac{(x-a)^n}{n!}$$

This is the Taylor series. The series near $a = 0$ is known as the Maclaurin series.

1/1-x

The first example is

$$f(x) = \frac{1}{1-x}$$

We know the answer to this.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3$$

Proof:

$$1 = (1-x)(1+x+x^2+x^3)$$

Multiplying by 1, the second term x is matched by $-x$ from the first term in the multiplication by $-x$, and so on. The whole thing vanishes, leaving just 1.

We want to evaluate $f(x)$ near 0, let's say, at $x = 0.1$. The correct value of the function is

$$f(x) = \frac{1}{0.9} = 1.11111\ldots$$

Let's try to approximate using the series. We need derivatives

$$f(x) = \frac{1}{1-x}$$

$$f'(x) = \frac{1}{(1-x)^2} = (1-x)^{-2}$$

$$f'(0) = 1$$

so the linear approximation is

$$f(x) \approx 1 + 1x = 1.1$$

For the next term we obtain

$$f''(x) = 2(1-x)^{-3}$$

The 2 is cancelled by the 2! in the denominator, so this cofactor is 1 and we're left with

$$f''(0)\ \frac{x^2}{2} = x^2 = 0.01$$

And I think we can see where this one is going.

However, you probably remember that this series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

diverges for $|x| \geq 1$, and the Taylor series does too.

The morale of the story is that for some series, there is a radius of convergence and the series is only valid for x within that radius.

binomial

Another very useful series is the binomial.

$$f(x) = (1+x)^n$$

$$f(0) = 1$$
$$f'(0) = n(1+x)^{n-1} = n$$
$$f''(0) = n(n-1)(1+x)^{n-2} = n(n-1)$$

So the series is

$$(1+x)^n \approx 1 + nx + n(n-1)\frac{x^2}{2}$$

We use this one a lot.

A nice application is relativistic energy

$$E = mc^2 f$$

$$f = 1/\sqrt{1 - \frac{v^2}{c^2}}$$

This is, in disguise, a binomial with $n = -1/2$ and $x = -v^2/c^2$ so the expansion is

$$f \approx 1 + nx = 1 + \frac{v^2}{2c^2}$$

so the energy is

$$E \approx mc^2(1 + \frac{v^2}{2c^2})$$

And we see that the second term is just the kinetic energy, $mv^2/2$.

polynomials

The beauty of Taylor Series (despite its complexity) is that it turns any differentiable function into a polynomial. Polynomials are easy to integrate and work with.

The first thing to say about Taylor Series is they give the correct answer for functions that we know. For example, suppose we have

$$f(x) = ax^2 + bx + c = 1$$

We get the derivatives and evaluate them "near" the point $x = 0$.

$$f(x) = ax^2 + bx + c = c$$

$$f'(x) = 2ax + b = b$$

$$f''(x) = 2a$$

The series is then

$$f(x) = c + b(x) + \frac{21}{2!}(x)^2 + \cdots$$

But there are no more terms. That's it. And this is just

$$f(x) = c + bx + ax^2$$

exponential, sine and cosine

Suppose $f(x) = e^x$ and again, we evaluate "near" $x = 0$. We have

$$f(x) = e^x = 1$$

$$f'(x) = e^x = 1$$

$$f''(x) = e^x = 1$$

The series is

$$f(x) = e^x = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots$$

$$f(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Which matches what we already know about e^x. For example, it is obvious that

$$\frac{d}{dx}e^x = e^x$$

Let's try to find something new. Suppose we expand $f(x) = cos\ x$ near $x = 0$

$$f(x) = cos\ x = cos\ 0 = 1$$

$$f'(x) = -sin\ x = -sin\ 0 = 0$$

$$f''(x) = -cos\ x - cos\ 0 = -1$$

$$f'''(x) = sin\ x = sin\ 0 = 0$$

$$f''''(x) = cos\ x = cos\ 0 = 1$$

and this continues in a cycle with period 4. The series is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$f(x) = cos\ x = 1 - \frac{1}{2!}(x-0)^2 + \frac{1}{4!}(x-0)^4 + \cdots$$

$$f(x) = cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Similarly, for $f(x) = sin\ x$ near $x = 0$

$$f(x) = sin\ x = 0$$

$$f'(x) = cos\ x = 1$$

$$f''(x) = -sin\ x = 0$$

$$f'''(x) = -cos\ x = -1$$

$$f''''(x) = sin\ x = 0$$

The series is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$f(x) = sin\ x = x - \frac{1}{3!}(x-0)^3 + \frac{1}{5!}(x-0)^5 + \cdots$$

$$f(x) = sin\ x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

funny series

In Strogatz book (*The Joy of x*), he gives the following series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

and he says that the sum of the series is equal to the natural logarithm of 2:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

with the provision that you have to calculate the sum in the order given.

For example, the second, third and fourth partial sums are:

$$S_2 = \frac{1}{2}; \quad S_3 = \frac{5}{6}; \quad S_4 = \frac{14}{24}; \quad S_5 = \frac{94}{120}$$

with $S_4 = 0.583$ and $S_5 = 0.783$. For any partial sum S_n and the previous sum S_{n-1} the value of the series will be bounded by the two sums.

I thought I would try to show that $\ln 2$ is the correct value for series, by using a Taylor series for the logarithm.

Taylor says we can write a function $f(x)$ (near the value $x = a$) as an infinite sum

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

where f^n means the nth derivative of f and f^0 is just f, and these derivatives are to be evaluated at $x = a$. Near $a = 0$ this simplifies to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n$$

Let's calculate the derivatives of the logarithm:

$$f^0 = \ln x; \quad f^1 = \frac{1}{x} = x^{-1}; \quad f^2 = -x^{-2}; \quad f^3 = 2x^{-3}; \quad f^4 = -3!\, x^{-4}$$

The first thing I notice is that we can't use $a = 0$, since $f^1 = 1/x$ is undefined there. So, let's try $a = 1$. Then (evaluated at $a = 1$)

$$f^0 = \ln x = 0; \quad f^1 = \frac{1}{x} = 1; \quad f^2 = -x^{-2} = -1; \quad f^3 = 2; \quad f^4 = -3!$$

Going back to the definition

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

I get the following series near $a = 1$:

$$\ln x = \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \cdots$$

For the special value $x = 2$, all the terms $(x-1)^n$ go away (which confirms that $a = 1$ is an excellent choice!). We have then

$$\begin{aligned}\ln x &= \frac{0}{0!} + \frac{1}{1!} - \frac{1}{2!} + \frac{2}{3!} - \frac{3!}{4!} + \cdots \\ &= 0 + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\end{aligned}$$

which is what was to be proved.

Chapter 35

Cubics

quadratic

Some of the earliest examples of problems where the square root of a negative number arises involve a right triangle of a specified area and perimeter.

Nahin says, suppose a right triangle has area 7 and perimeter 12. Find the two sides.

Label the sides as a and b.

We can get some idea of where this problem is headed by supposing that the triangle is also isosceles with $a = b$. Then

$$\frac{1}{2}ab = 7$$

$$ab = 14$$

$$a^2 = 14$$

so $a = \sqrt{14}$, and the perimeter is

$$p = a + b + \sqrt{a^2 + b^2}$$

$$= 2\sqrt{14} + \sqrt{14+14} = 12.77$$

The perimeter we are given is smaller than that

However, an isosceles right triangle has the smallest possible perimeter for a given area (the largest area for a given perimeter), hence there is no such pair a, b. The problem as posed has no solution.

Proof:

Let k be a constant and x and $k-x$ be the given sides. The area is

$$A = \frac{1}{2}x(k-x)$$

$$= -\frac{1}{2}x^2 + \frac{k}{2}x$$

The extreme point is

$$\frac{dA}{dx} = 0 = -x + \frac{k}{2}$$

$$x = \frac{k}{2}$$

The second derivative is $-1 < 0$, which shows that this is a minimum. We can also see the same thing from the negative cofactor of x^2 in the equation.

$$A = -\frac{1}{2}x^2 + \frac{k}{2}x$$

Doing the algebra of the original problem anyway, we solve two simultaneous equations

$$a \cdot b = 14$$

$$p = a + b + \sqrt{a^2+b^2} = 12$$

Isolate and then remove the square root in the second one

$$a^2 + b^2 = (12 - a - b)^2$$

$$= 12^2 - 12a - 12b - 12a + a^2 + ab - 12b + ab + b^2$$

Collect terms and cancel a^2 and b^2

$$0 = 12^2 - 24a - 24b + 2ab$$

$$0 = 72 - 12a - 12b + ab$$

Substituting from the first equation given above

$$0 = 72 - 12a - 12 \cdot \frac{14}{a} + 14$$

$$0 = 36 - 6a - 6 \cdot \frac{14}{a} + 7$$

$$-6a^2 + 43a - 84 = 0$$

$$6a^2 - 43a + 84 = 0$$

To solve this, use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

However, $b^2 = 43^2 = 1849$ is less than $4ac = 4 \cdot 6 \cdot 84 = 2016$. We end up with

$$x = \frac{43 \pm \sqrt{-167}}{12}$$

The classic answer at this point is just to say, these values do not exist. The graph would be a parabola opening up ($a > 0$) whose vertex lies above the x-axis.

But suppose these two *complex* roots do have meaning.

They are, of course, complex conjugates: $p + iq$ and $p - iq$. If they are substituted into the factored form of the quadratic:

$$y = [\ x - (p + iq)\ ]\ [\ x - (p - iq)\ ]$$

$$
\begin{aligned}
y = \; & [\; x - p - iq) \;] \, [\; x - p + iq) \;] \\
= x^2 - px + iqx - px + p^2 - ipq - iqx + ipq + q^2 \\
= x^2 - 2px + p^2 + q^2 \\
y = (x-p)^2 + q^2
\end{aligned}
$$

p is the value of x at the vertex, corresponding to the minimum value of y, which is equal at that point to q^2.

Recall that the slope is

$$y' = 2ax + b$$

at the minimum, it equals zero so

$$0 = 2ax + b$$

$$x = -\frac{b}{2a}$$

This value of x makes the factored form equal to zero. It is also the first term in

$$-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

cubics

In general, people just ignored problems with negative square roots — sometimes explicitly — until Cardano came to cubic polynomials.

Briefly, he discovered that any cubic like

$$x^3 + ax^2 + bx + c = 0$$

can be converted to a *depressed* cubic of the form

$$x^3 + px + q = 0$$

Any cubic has either one real root and two complex ones, or else three real roots.

We need to look at Cardano's formula to solve the depressed cubic. He was actually solving a problem like

$$x^3 + mx = n$$

(with m and n both positive), we re-write this as

$$x^3 + mx - n = 0$$

Define

$$r = \frac{n}{2}, \qquad s = \frac{m^3}{27}$$

Then Cardano showed that a real root (the only one or one of the three) is

$$x = \ [\ r + \sqrt{r^2 + s}\ ]^{1/3} + \ [\ r - \sqrt{r^2 + s}\ ]^{1/3}$$

Clearly, depending on the values of m and n, and thus r and s, $r^2 + s$ may be a negative number.

Cardano could not just ignore this issue, because the formula works to give a real result. He struggled with this.

Even today, it is hard to see the resolution because of the cube root.

Write what is in the brackets as a generalized complex number, in polar format

$$x = \ [\ re^{i\theta}\ ]^{1/3} + \ [\ re^{-i\theta}\ ]^{1/3}$$

Recall that complex multiplication goes like so:

$$r_1 e^{i\theta}\ r_2 e^{i\phi} = r_1 r_2 e^{i(\theta+\phi)}$$

So a cubic is

$$re^{i\theta} \cdot re^{i\theta} \cdot re^{i\theta} = r^3 e^{i3\theta}$$

With a change of variable, complex eponentiation is as follows:

$$[\ re^{i\theta}\ ]^{1/3} = r^{1/3}\ e^{i\theta/3}$$

$$[\ re^{-i\theta}\ ]^{1/3} = r^{1/3}\ e^{-i\theta/3}$$

The cube roots of complex conjugates are also complex conjugates!

When added together, the imaginary parts cancel, leaving an entirely real result.

$$x = r^{1/3}\ (e^{i\theta/3} + e^{-i\theta/3})$$

The term in brackets is clearly a sum $z + z*$, which is real, with the value twice the real component of the complex number.

example

Let's figure out an example arithmetically. The math is a little messy but we'll try to get through it. One of the problems studied by Cardano is

$$x^3 = 15x + 4$$

All terms are positive, which is typical for the time. We try the solution $x = 4$ and find it works out.

The Tartaglia formula gives

$$r = 4/2 = 2$$

$$s = (-15)^3/27 = -125$$

so we have that

$$x = \ [\ r + \sqrt{r^2 + s}\ ]^{1/3} + \ [\ r - \sqrt{r^2 + s}\ ]^{1/3}$$

$$x = \ [\ 2 + \sqrt{-121}\ ]^{1/3} + \ [\ 2 - \sqrt{-121}\ ]^{1/3}$$

This is easy to solve if one happens to know that

$$(2 \pm \sqrt{-1})^3 = 2 \pm \sqrt{-121}$$

Hence

$$x = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4$$

Let's try to calculate this:

$$(2 + \sqrt{-1})^3 = 2 + \sqrt{-121}$$

Usually, we would think that the polar format would make for easier calculation. However, let's go forward using the Cartesian format

$$\begin{aligned}(2+i)^3 &= (3+4i)(2+i)\\ &= 6-4+11i\\ &= 2+11i\end{aligned}$$

Pretty easy.

To use the polar format, let's compute the cube root:

$$(2+11i)^{1/3} = ?$$

We need the polar form of $2 + 11i$. We obtain

$$r = \sqrt{2^2 + 11^2} = \sqrt{125}$$

$$\theta = \tan^{-1} 11/2 = 1.391$$

Then

$$r' = r^{1/3} = \sqrt{5}$$

$$\theta' = \theta/3 = 0.46346$$

To convert back to Cartesian coordinates:

$$x = r \cdot \cos\theta = \sqrt{5} \cdot 0.8944 = 2.0$$

$$y = r \cdot \sin\theta = \sqrt{5} \cdot 0.4472 = 1.0$$

The result is $2 + i$, as expected.

example

Here is another problem from Nahin showing that the real component of a complex solution may have application in the real world.

> Imagine that a man is running at his top speed of v feet per second, to catch a bus that is stopped at a traffic light. When he is still a distance of d feet from the bus, the light changes and the bus starts to move away from the running man with a constant acceleration of a feet per second per second. When will the man catch the bus?

Let the origin of coordinates be the traffic light and x_m and x_b be the positions of the man and the bus. At $t = 0$, $x_b = 0$ and $x_m = -d$. For an arbitrary time t

$$x_b = \frac{1}{2}at^2$$
$$x_m = -d + vt$$

If the man is to catch the bus at $t = T$, the positions are the same

$$x_m(T) = x_b(T)$$

$$-d + vT = \frac{1}{2}aT^2$$

This is a quadratic

$$\frac{1}{2}aT^2 - vT + d = 0$$

In general, the solution for T may be complex, if

$$v^2 - 2ad < 0$$

Rearranging

$$d > v^2/2a$$

For such values there is no catching the bus.

Nahin rearranges the equation to give

$$T^2 - 2\frac{v}{a}T + 2\frac{d}{a} = 0$$

The quadratic formula gives

$$T = \frac{2v/a \pm \sqrt{4v^2/a^2 - 8d/a}}{2}$$
$$= \frac{v}{a} \pm \sqrt{v^2/a^2 - 2d/a}$$

Even for a complex result, the real part is

$$T = \frac{v}{a}$$

But notice: the separation between the man and the bus is

$$s = x_b - x_m$$
$$= \frac{1}{2}at^2 + d - vt$$

At what time is the man closest to the bus? That occurs when

$$\frac{ds}{dt} = at - v = 0$$
$$t = \frac{v}{a}$$

This is the real part of the result above.

If the man does catch the bus ($\sqrt{v^2/a^2 - 2d/a}$ is real), it worth thinking about the two solutions to the quadratic. Which is the correct one and what is the meaning of the second?

Chapter 36

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