Introduction

Kaplan gives some rules for computing residues, which we explore in this chapter.

RULE I At a simple pole z_0 (that is, a pole of first order),

$$\text{Res}\,[f(z), z_0] = \lim_{z \to z_0} (z - z_0) f(z)$$

example

The example is one we already worked.

$$f(z) = \frac{1}{z^2 + 1} = \frac{1}{(z-i)(z+i)}$$

$$\text{Res}\,[f(z), z = i] = \lim_{z \to i} (z - i) \frac{1}{(z-i)(z+i)}$$

$$= \lim_{z \to i} \frac{1}{z + i} = \frac{1}{2i}$$

review

To summarize the key points about Cauchy 2 and residues, this is the theorem

$$\oint_C \frac{f(z)}{z - z_0}\ dz = 2\pi i\ f(z_0)$$

By definition the residue at a simple pole is defined to be

$$b_1 = \lim_{z \to z_0} (z - z_0)\ f(z)$$

Just think of it as in the limit that $z \to z_0$, the denominator $z - z_0$ on the left is a constant, so we can multiply both sides by $z - z_0$ to obtain the result for the residue.

$$\oint f(z)\ dz = 2\pi i \sum \text{Res}$$

The value of the integral is $2\pi i$ times the sum of all the residues enclosed by the path.

derivative rule

We show here that

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\ dz$$

and there are more formulas for higher derivatives. Therefore

$$2\pi i\ f'(a) = \oint_C \frac{f(z)}{(z-a)^2}\ dz$$

Again, the Cauchy formula is:

$$\oint_C \frac{f(z)}{z - z_0}\ dz = 2\pi i f(z_0)$$

rewrite with a for z_0

$$\oint_C \frac{f(z)}{z - a}\ dz = 2\pi i f(a)$$

We take the partial with respect to a of both sides:

$$\frac{\partial}{\partial a}(\frac{f(z)}{z-a}) = \frac{f(z)}{(z-a)^2}$$

so

$$\oint_C \frac{f(z)}{(z-a)^2}\ dz = 2\pi i f'(a)$$

Thus

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\ dz$$

More generally

$$f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\ dz$$

So

$$\frac{2\pi i}{n!} f^n(a) = \oint_C \frac{f(z)}{(z-a)^{n+1}}\ dz$$

example

$$\oint_C \frac{e^z}{z^3 - z^2 - 5z - 3} = \oint_C \frac{e^z}{(z+1)^2(z-3)}$$

Recall the general formula

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\ dz$$

If the contour includes $z = -1$ but not $z = 3$ then

$$f(z) = \frac{e^z}{(z-3)}$$

so

$$f'(z) = \frac{(z-4)e^z}{(z-3)^2}$$

Hence

$$\oint_C \frac{e^z}{z^3 - z^2 - 5z - 3}\ dz = \oint_C \frac{e^z}{(z+1)^2(z-3)}$$
$$= \oint_C \frac{f(z)}{(z+1)^2}\ dz$$

for $f(z) = e^z/(z-3)$ and

$$= 2\pi i f'(-1) = 2\pi i \; \frac{-5}{e} \; \frac{1}{(-4)^2}$$

$$= \frac{-5\pi i}{8e}$$

Kaplan

Here is rule II from Kaplan. Rule I is in the previous section.

RULE II At a pole of order N $(N = 2, 3, \dots)$,

$$\text{Res}\ [f(z), z_0] = \lim_{z \to z_0} (z - z_0) \frac{g^{(N-1)}(z)}{(N-1)!}$$

where

$$g(z) = (z - z_0)^N f(z)$$

example

$$f(z) = \frac{1}{z(z-2)^2}$$

We have a pole of first order at $z_0 = 0$ and one of second order at $z_0 = 2$. At the first

$$\text{Res}\ (0) = \lim_{z \to 0} \frac{1}{(z-2)^2} = \frac{1}{4}$$

For the other one, remove the factor of $1/(z-2)^2$ and compute the $N - 1$ (first) derivative of what's left

$$\frac{d}{dz} \frac{1}{z} = -\frac{1}{z^2}$$

$$\text{Res}\ (2) = \lim_{z \to 2} -\frac{1}{z^2} = -\frac{1}{4}$$

Don't forget to divide by $(N-1)!$, which is just 1 in this case. The total is just zero.

As a check, let's do this by partial fractions.

$$\frac{1}{z(z-2)^2} = \frac{A}{(z-2)^2} + \frac{B}{z(z-2)} + \frac{C}{z}$$

Hence in putting all terms over a common denominator, for the numerator we have

$$1 = Az + B(z-2) + C(z-2)^2$$

From which we get three equations:

$$-2B + 4C = 1$$

$$Az + Bz - 4Cz = 0$$

$$Cz^2 = 0$$

Hence $C = 0$, so $B = -1/2$ and $A = 1/2$ and we obtain

$$\frac{1}{z(z-2)^2} = \frac{1/2}{(z-2)^2} - \frac{1/2}{z(z-2)}$$

which we check by doing

$$1/2 \cdot z - 1/2 \cdot (z-2) = 1$$

So how to deal with

$$\frac{1/2}{(z-2)^2} - \frac{1/2}{z(z-2)}$$

The first term has a pole of order 2 at $z_0 = 2$. We remove that factor and compute the $N-1$ (first) derivative of what's left, which is just zero.

For the second term, we have two simple poles at $z_0 = 0$ and $z_0 = 2$. The residues are

$$\text{Res }(0) = \lim_{z \to 0} z \cdot \frac{1}{z(z-2)} = -\frac{1}{2}$$

$$\text{Res }(2) = \lim_{z \to 2} (z-2) \cdot \frac{1}{z(z-2)} = \frac{1}{2}$$

which adds up to zero.

example

$$f(z) = \frac{1}{z^4 + z^3 - 2z^2}$$

where C is the circle $|z| = 3$ with positive orientation.

The denominator can be factored as

$$z^2(z^2 + z - 2) = z^2(z+2)(z-1)$$

so

$$f(z) = \frac{1}{z^2(z+2)(z-1)}$$

There is a pole of order 2 at the origin and simple poles at 1 and -2. All of these lie within the contour $|z| = 3$.

$$\text{Res }(1) = \lim_{z \to 1} (z-1)\ f(z) = \lim_{z \to 1} \frac{1}{z^2(z+2)} = \frac{1}{3}$$

$$\text{Res }(-2) = \lim_{z \to -2} (z+2)\ f(z) = \lim_{z \to -2} \frac{1}{z^2(z-1)} = -\frac{1}{12}$$

For the double pole, we remove the factor of $1/z^2$, take the first derivative of what's left

$$\frac{d}{dz}\ \frac{1}{z^2 + z - 2} = \frac{(-1)(2z+1)}{(z^2+z-2)^2}$$

Evaluate at zero and obtain.

$$\text{Res } (0) = -\frac{1}{4}$$

The total of the residues is

$$\frac{1}{3} - \frac{1}{12} - \frac{1}{4} = 0$$

As Mathews and Howell say:

> The value 0 for the integral is not an obvious answer, and all of the preceding calculations are required to find it.

example

$$f(z) = \frac{1 + e^z}{z^2} + \frac{2}{z}$$

We can break this up into its two component parts. For the first term, the pole is of order $m = 2$ at $z_0 = 0$. We remove the z^2 term and take the $m - 1 = 1$ derivative

$$(1 + e^z)' = e^z$$

Remember to divide by $(m - 1)!$, leaving e^z which is evaluated at the pole giving a residue

$$\text{Res } (0) = e^0 = 1$$

The other term is just 2 times the standard

$$\oint \frac{1}{z}\, dz = 2\pi i$$

Here $I = 4\pi i$ and the residue is 2. Alternatively just use

$$I = 2\pi i f(z_0) = 4\pi i$$

where $f = 2$.

The total of the residues is 3 and the value of the integral is $6\pi i$.

example

$$f(z) = \frac{e^z}{z(z-1)^2}$$

We have a pole of first order at $z = 0$ and one of second order at $z = 1$. At the first

$$\text{Res }[f(z), z = 0] = \lim_{z \to 0} \frac{e^z}{(z-1)^2} = 1$$

For the other one, remove the factor of $1/(z-1)^2$ and compute the $N-1$ (first) derivative of what's left

$$\text{Res }[f(z), z = 1] = \lim_{z \to 1} \left[\ \frac{e^z}{z} \ \right]'$$

$$= \frac{e^z z - e^z}{z^2} \Bigg|_1 = 0$$

Hence

$$\oint f(z) \ dz = 2\pi i \ [\sum \text{Res} \ ] \ = 2\pi i$$

example

$$f(z) = \frac{1}{z(z-2)^4}$$

We have a pole of first order at $z = 0$ and one of fourth order at $z = 2$. At the first

$$\text{Res }[f(z), z = 0] = \lim_{z \to 0} z \ \frac{1}{z(z-2)^4}$$

$$= \lim_{z \to 0} \ \frac{1}{(z-2)^4} = \frac{1}{(-2)^4} = \frac{1}{16}$$

For the other pole recall that

$$\frac{2\pi i}{n!} f^n(a) = \oint_C \frac{f(z)}{(z-a)^{n+1}} \ dz$$

We remove the factor of $1/(z-2)^4$ leaving $f(z) = 1/z$ and then compute the $N-1$ (third) derivative of what's left

$$\text{Res}\,[f(z), z = 2] = \frac{1}{n!}\,\lim_{z\to 2}\,[\,\frac{1}{z}\,]'''$$

$$f(z) = z^{-1}$$

$$f'(z) = -z^{-2}$$

$$f''(z) = 2z^{-3}$$

$$f'''(z) = -6z^{-4}$$

$$\lim_{z\to 2}\,[\,\frac{1}{z}\,]''' = -\frac{6}{16}$$

Don't forget to divide by $(N-1)!$, which is $3! = 6$ in this case. That leaves

$$\text{Res}\,[f(z), z = 2] = -\frac{1}{16}$$

The total of the residues is just zero.

This problem is from Brown and Churchill (p. 234), which they work by doing Laurent series. They get a different answer, namely $-\pi i/8$.

The reason is that they integrate over the contour $0 < |z-2| < 2$, which includes the second pole, but not the first. Multiplying by $2\pi i$ gives their result.