# Complex functions

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# Part I

# Introduction

# Chapter 1

# Preface

One motivation for learning about complex functions is that the theory is often described as being very beautiful. It also shows how certain more difficult integrals can be solved.

Marsden gives three examples that he says are either very difficult or impossible if we are restricted to just the real numbers:

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\, dx = \frac{\pi}{\sin \alpha\pi}$$

$$\int_0^{2\pi} \frac{1}{a+\sin\theta}\, d\theta = \frac{2\pi}{\sqrt{a^2-1}}$$

Here's another from Nahin.

$$\int_{-\infty}^\infty \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{e}$$

Maybe we can learn to solve these before we're done.

# Chapter 2

# Arithmetic

Consider the functions

$$x^2 + 1 = 0$$

and

$$x^2 + x + 1 = 0$$

For the first equation, it is easy to see that there is no solution among the real numbers since $x^2$ is always positive or zero. So adding 1 to $x^2$ cannot bring the sum back to zero.

Visualizing the same function geometrically, this is just the simple parabola $y = x^2$ shifted up by one unit, moving its vertex from $(0, 0)$ to $(0, 1)$. Plotting shows that the graphs of both the above functions never cross the $x$-axis—there are no values that lie on the curve and also on the line $y = 0$.

It is often said that complex numbers arose in the context of finding solutions to such polynomials, however, as Nahin writes in his book *An imaginary tale*, this is not really true. We'll explore some of this history in the next chapter.

The ingenious solution to this problem was to invent a new kind of number

$$i = \sqrt{-1}$$
$$i^2 = -1$$

Once we accept that $i = \sqrt{-1}$ then we can factor

$$(x+i)(x-i) = x^2 - i^2$$
$$= x^2 - (-1) = x^2 + 1$$

so $x = \pm\, i$ are both solutions to the equation

$$(x+i)(x-i) = 0$$

For the second one

$$x^2 + x + 1 = 0$$

we can plot it, or we can recall the quadratic formula for solutions to

$$ax^2 + bx + c = 0$$

for real constants $a$, $b$ and $c$. The formula is

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

When $4ac > b^2$, then the solutions to the quadratic formula involve the square root of a negative number. Here the formula gives

$$x = \frac{1}{2}(-1 \pm \sqrt{-3})$$

Take the positive root and square it

$$x^2 = \frac{1}{4}(-1 + \sqrt{-3})^2$$

$$= \frac{1}{4}(-2 - 2\sqrt{-3})$$
$$= \frac{1}{2}(-1 - \sqrt{-3})$$

Adding this to $x + 1$ we obtain

$$x^2 + x + 1$$
$$= \frac{1}{2}(-1 - \sqrt{-3}) + \frac{1}{2}(-1 + \sqrt{-3}) + 1$$

the terms with $\sqrt{-3}$ cancel, giving

$$= -\frac{1}{2} - \frac{1}{2} + 1 = 0$$

In fact, now that we have $i$ available, any square root like $\sqrt{-(a^2)}$, where $a$ is a real number, can be factored as $\sqrt{-1}\,\sqrt{a^2} = ia$.

**warning**

Note that the converse is not necessarily true. Consider

$$i^2 = \sqrt{-1} \cdot \sqrt{-1} \stackrel{?}{=} \sqrt{(-1) \cdot (-1)} = \sqrt{1}$$

Now, $\sqrt{1}$ has two solutions or roots (since $-1 \times -1$ and $1 \times 1$ are both equal to 1), but we choose the positive root when thinking about $\sqrt{x}$ as a *function*. However, $i^2$ was defined to be equal to $-1$, not 1. What's the deal?

The problem is that the equality with a question mark is not valid

$$\sqrt{-1} \cdot \sqrt{-1} \neq \sqrt{(-1) \cdot (-1)}$$

which explains why this "proof" is erroneous.

Expressions that involve the square root of a negative real number, like $\sqrt{-1} = i$ and $\sqrt{-3} = \sqrt{3}\, i$, are called imaginary (or *purely* imaginary).

Numbers that contain both a real and an imaginary part, like $1 + i$, are termed complex numbers, and imaginary numbers are considered to be complex numbers with the real part equal to 0.

The set of complex numbers $\mathbb{C}$ includes the real numbers:

$$\mathbb{R} \subset \mathbb{C}$$

or, as is written, that $\mathbb{R}$ is a subset of $\mathbb{C}$.

We write complex numbers $z$ as combinations like

$$z = a + ib$$

where $a$ and $b$ are both real numbers. $a$ is the real part, and $b$ the imaginary part of the complex number $z$.

Two useful identities come from factoring $i^2 = -1$:

$$i = -\frac{1}{i}$$

$$-i = \frac{1}{i}$$

It turns out that for much of what is done with complex numbers the fact that $i$ equals $\sqrt{-1}$ is not even relevant.

Instead, we simply think of *ordered pairs* of real numbers $(a, b)$ and the $i$ notation is a bookkeeping device, a marker to remind us that when we multiply two complex numbers

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd$$

the last term gets a minus sign:

$$ib \cdot id = -bd$$

The result of multiplying $ib \cdot id$ is a real number with the sign flipped, while a real number $a$ times an imaginary number $id$ is equal to $iad$ and

$$(a + ib)(c + id) = ac - bd + i(ad + bc)$$

### dual equality

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal

$$z_1 = z_2 \iff a = c \text{ and } b = d$$

*if and only if* both the real and the imaginary parts of $z_1$ and $z_2$ are equal.

### matrix form

Another idea to keep track of the same information is in matrix form, namely:

$$z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Such matrices can be added and multiplied in the normal way and give the desired results for complex numbers. Thus:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \times \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{bmatrix} = \begin{bmatrix} u & -v \\ v & u \end{bmatrix}$$

### Geometric interpretation

Yet another powerful way to think about complex numbers is to use the complex plane (sometimes called the Argand plane), where points are plotted with the real part along the horizontal axis and the imaginary part along the vertical axis.

This figure is from Brown & Churchill.

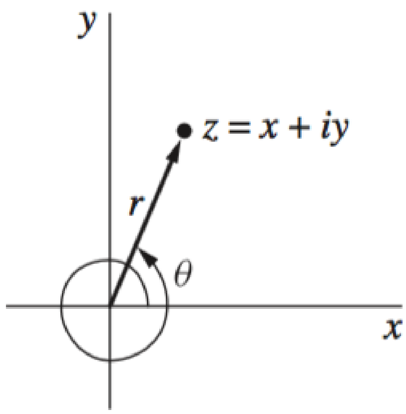



**FIGURE 6**

Looking at the graph, the distance of any point from the origin is denoted by $r$, and $\theta$ is the angle the ray makes with the positive $x$-axis in a CCW direction. This should be familiar from standard polar coordinates.

Switching notation to

$$z = x + iy$$

To plot the complex number $z$ we go out $x$ units along the real (horizontal) axis and then up $y$ units along the imaginary (vertical) axis.

The statement that $\mathbb{R} \subset \mathbb{C}$ is equivalent to the observation that the Argand plane contains the horizontal axis. Real numbers have the form $z = x + i \cdot 0 = x$.

More generally, though

$$x = r\cos\theta$$

$$y = r\sin\theta$$

and

$$\begin{aligned} x + iy &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

$$= re^{i\theta}$$

where the last part makes use of Euler's famous equation. $r$ is called the **modulus** and $\theta$ is called the **argument** or **phase**.

If you look very carefully at the figure above the argument $\theta$ is actually $\theta + 2\pi$.

All multiples $k \cdot 2\pi$ for $k \in 0, \pm 1, \pm 2 \ldots$ are valid.

Depending on the calculation one form is often easier to handle.

Addition is simpler with $a + ib$ (the Cartesian format) since

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

while multiplication is more straightforward with the polar format.

Matrices work well for both addition and multiplication.

Here is multiplication in polar coordinates

$$re^{i\theta}\ \rho e^{i\phi} = r\rho\ e^{i(\theta+\phi)}$$

We multiply the distances and add the angles. Here is the square function:

$$(re^{i\theta})^2 = r^2 e^{i2\theta}$$

Multiplication of $z_1 = r_1 e^{i\theta_1}$ by $z_2 = r_2 e^{i\theta_2}$ stretches $r_1$ (the length of $z_1$) by the factor $r_2$ (the length of $z_2$), and rotates $z_1$ by adding a phase shift of $\theta_2$ to the original angle $\theta_1$.

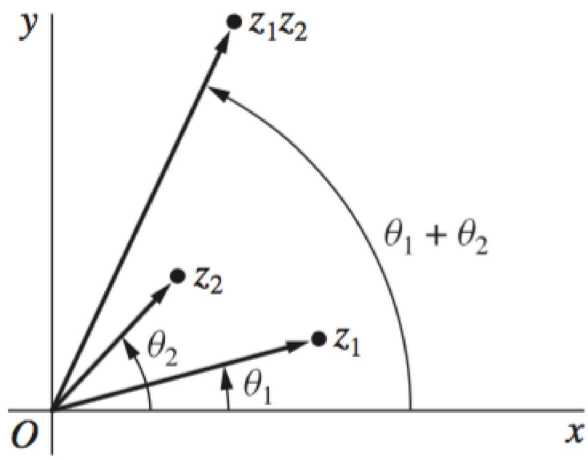


**FIGURE 9**

The person who originally discovered this representation was Caspar Wessel.

Since the calculations can be tedious, I wrote a Python script to do the calculations for roots and powers.

https://gist.github.com/telliott99/916bc75a73e515968debe48ef418d738

# Chapter 3

# Conjugate

Consider the complex number:

$$z = x + iy$$

The complex conjugate of $z$ (called $z*$ or $\bar{z}$) is given by:

$$z* = x - iy$$

The real part of $z*$ is the same as the real part of $z$, while the imaginary part has the sign switched.

### length of $z$

The *length* of $z$ squared is equal to $z$ multiplied by its complex conjugate

$$\begin{aligned} zz* &= (x + iy)(x - iy) \\ &= x^2 - ixy + ixy - i^2y^2 \\ &= x^2 + y^2 \\ &= (r\cos\theta)^2 + (r\sin\theta)^2 \\ &= r^2 \end{aligned}$$

Again, $r$ is the length of the ray from the origin to $z$ as plotted in the complex plane.

$$r^2 = zz*$$

$$r = \sqrt{zz*}$$

The point corresponding to $z*$ in the complex plane has the same overall distance from the origin and the same $x$-component as $z$, but the sign change on $y$ means that $z*$ is reflected across the $x$-axis from $z$.

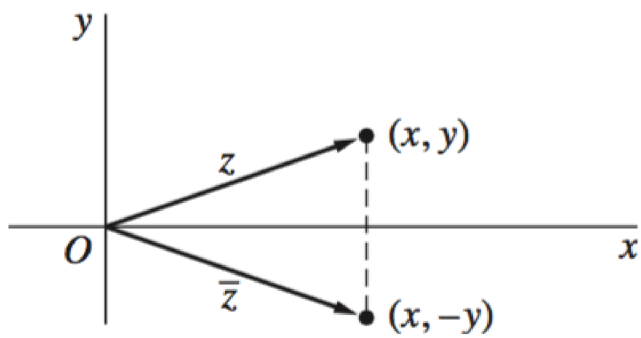



**FIGURE 5**

In polar coordinates, if $z = re^{i\theta}$ then $z* = re^{i(-\theta)} = re^{-i\theta}$. So

$$zz* = re^{i\theta}\ re^{i-\theta} = r^2e^0 = r^2$$

Multiplication of $z$ by $z*$ makes the product entirely real.

If we consider addition rather than multiplication of the complex conjugate we observe that it also gives an entirely real result:

$$z + z* = x + iy + x - iy = 2x$$

while subtraction gives an entirely imaginary result:

$$z - z* = x + iy - x + iy = i2y$$

**conjugate of several values**

Another result (that we state without proof) is that if we have an expression involving several complex numbers:

$$w = f(z_1, z_2 \ldots)$$

we can obtain the complex conjugate of the whole thing by substituting the complex conjugate of each component:

$$w* = f(z_1*, z_2 * \ldots)$$

As an example, let us compute the powers of $z$ and $z*$ using the binomial theorem:

$$\begin{aligned} z &= x + iy \\ z^2 &= x^2 + 2x(iy) + (iy)^2 \\ z^3 &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 \\ z^4 &= x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4 \end{aligned}$$

and the conjugate:

$$\begin{aligned} z* &= x + (-iy) \\ (z*)^2 &= x^2 + 2x(-iy) + (-iy)^2 \\ (z*)^3 &= x^3 + 3x^2(-iy) + 3x(-iy)^2 + (-iy)^3 \\ (z*)^4 &= x^4 + 4x^3(-iy) + 6x^2(-iy)^2 + 4x(-iy)^3 + (-iy)^4 \end{aligned}$$

It makes things simpler if we leave the minus signs and the powers of $i$ for the moment.

Now, any even power of $i$ is wholly real. So all we really need to do to form the conjugate is to switch the sign of the odd powers. Since they're odd powers, it makes no difference if we do this inside the parentheses or in front of each term.

So then,

$$\begin{aligned}(z^2)* &= x^2 - 2x(iy) + (iy)^2 \\ &= x^2 + 2x(-iy) + (iy)^2 \\ &= (z*)^2\end{aligned}$$

Furthermore, we can slip an extra minus sign inside any even power without changing the value:

$$\begin{aligned}(z^3)* &= x^3 - 3x^2(iy) + 3x(iy)^2 - (iy)^3 \\ &= x^3 + 3x^2(-iy) + 3x(iy)^2 + (-iy)^3 \\ &= x^3 + 3x^2(-iy) + 3x(-iy)^2 + (-iy)^3 \\ &= (z*)^3\end{aligned}$$

$$\begin{aligned}(z^4)* &= x^4 - 4x^3(iy) + 6x^2(iy)^2 - 4x(iy)^3 + (iy)^4 \\ &= x^4 - 4x^3(iy) + 6x^2(-iy)^2 - 4x(iy)^3 + (-iy)^4 \\ &= (z*)^4\end{aligned}$$

It is clear that this pattern will continue with higher powers.

# Chapter 4

# Definitions

We will often want to write an expression for the distance between two points in the complex plane. Suppose $z$ and $w$ are those two points, with $z = x + iy$ and $w = s + it$.

Subtract:

$$z - w = (x - s) + i(y - t)$$

If we take the modulus of this as

$$|z - w| = \sqrt{(x - s)^2 + (y - t)^2}$$

This is the distance between $z$ and $w$ by the Pythagorean theorem. It is the modulus of the complex number $z - w$.

### neighborhood

One very useful concept is that of the neighborhood. A neighborhood is an open disk of radius $r$ (or $\epsilon$) around a point $z_0$. The points in the disk can be defined as

$$z : |z - z_0| < \epsilon$$

all the points $z$ such that the distance between them is less than $r$.

This is an open disk. It does not include the points on the boundary, defined by $z : |z - z_0| = \epsilon$.

**punctured disk**

A punctured disk or deleted neighborhood is all the points in the neighborhood of $z_0$ except $z_0$ itself:

$$z : 0 < |z - z_0| < \epsilon$$

Sometimes, the missing points are more than just one, but lie in a smaller disk around $z_0$. The region is called an annulus:

$$z : r < |z - z_0| < R$$

**boundaries**

There is a funny kind of language used when talking about sets of points that are either inside, on the boundary of, or outside a set of points. $w$ is an *interior* point of $\mathbf{S}$ if $w \in \mathbf{S}$ and there exists a neighborhood of $w$ that includes no boundary points of $\mathbf{S}$. (This can happen because there is no closest number to a number, whether real or rational).

A boundary point is one for which each neighborhood contains points both $\in \mathbf{S}$ and not $\in \mathbf{S}$.

An exterior point of a set is a point for which there exists a neighborhood with no points in $\mathbf{S}$.

A set is open if it does not contain any of its boundary point, closed if it contains all of its boundary points. The punctured disk is neither open nor closed.

An open set is connected if each pair of points can be joined by a polygonal line. An open set that is connected is called a *domain.* Any neighborhood is a domain.

A set **S** is *bounded* if every point of the set lies within a circle $|z| = R$.

A point is an *accumulation* point of **S** if each deleted neighborhood of the point contains at least one point of **S**, A closed set contains all of its accumulation points. A set is closed if an only if it includes all of its accumulation points. [more work needed]

### curves and paths

Typically we have paths (often designated $\gamma$) which are parametrized curves $\gamma(t)$ for $a \leq t \leq b$. The curve generates points $x, y$ for each $t$.

The value of an integral over a curve does not depend on the particular parametrization.

The length of the path is just $\int_a^b |\gamma'(t)\ dt$.

We often designate closed circular paths in the complex plane as $C[w, r]$ where $r$ is the radius and $w$ is the center of the curve $z : |z - w| = r$.

### Limit

The distance also shows up in limits. The limit

$$\lim_{z \to z_0} f(z) = L$$

is defined by saying that given $\epsilon > 0$ and the statement $|f(z) - L < \epsilon|$, then one can find $\delta > 0$ such that $|z - z_0| < \delta$ implies that conclusion.

We can also use the language of neighborhoods: the limit exists if for any neighborhood defined for $f(x) - L$ in terms of $\epsilon$ we can guarantee that if $|z - z_0|$ is in the neighborhood defined by radius $\delta$, the conclusion is true.

Let a function $f$ be defined at all points $z$ in some deleted neighborhood of $z_0$, then the statement

$$\lim_{z \to z_0} f(z) = w_0$$

means that the point $w = f(z)$ can be made arbitrarily close to $w_0$ if we choose the point $z$ close enough to $z_0$ (though distinct from it).

Formally, for each positive number $\epsilon$, there exists a positive number $\delta$ such that

$$|z - z_0| < \delta \quad \Rightarrow |f(z) - w_0| < \epsilon$$

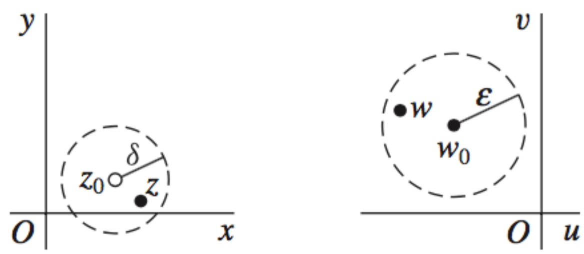



**FIGURE 23**

If the limit of a function exists at a point, it is unique.

## Continuity

A function $f$ is continuous as a point $z_0$ if all three conditions hold:

$$\lim_{z \to z_0} f(z) = f(z_0)$$

which of course requires

$$f(z_0) \text{ exists}$$

$$\lim_{z \to z_0} f(z) \text{ exists}$$

## Differentiable

A function $f$ is said to be differentiable at if the function's domain includes a neighborhood of $z_0$ and the derivative exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

The existence of the derivative at $z_0$ implies that the function is continuous at that point; however, the converse is not true.

## Analytic

A function is analytic at a point if it has a derivative at that point.

## Entire

An entire function is a function that is analytic at each point in the entire finite plane.

## Singular point

A point $z_0$ is called a singular point of a function $f$ if $f$ fails to be analytic at $z_0$ but is analytic at some point in every neighborhood of $z_0$ .

A singular point $z_0$ is said to be isolated if, in addition, there is a deleted neighborhood of $z_0$ throughout which $f$ is analytic.

## Pole

An isolated singular point is called a pole. For example

$$\frac{b_1}{z - z_0}$$

has a pole at $z_0$, since it is undefined there. A pole of order $m$ would be

$$\frac{b_1}{(z-z_0)^m}$$

**Holomorphic and meromorphic**

Holomorphic is used as a synonym for analytic. A function $f$ is said to be meromorphic in a domain $D$ if it is analytic throughout $D$ except for poles.

**Limit of a sequence**

An infinite sequence of complex numbers $z_1 \ldots z_n$ has a limit $L$ if, for each positive number $\epsilon$, there exists a positive integer $n_0$ such that $|z_n - L| < \epsilon$ whenever $n > n_0$.

**Cauchy sequence**

# Part II

# Cauchy Riemann

# Chapter 5

# Difference quotient

This section contains a general discussion of differentiation of complex functions. A complex function takes as input a complex number, and emits as output another complex number. Often $w$ is used for the output:

$$w = f(z)$$

More concretely, a complex number is simply an ordered pair of real numbers, and a complex function is a pair of functions defined on the real numbers:

$$f(z) = f(x + iy) = u(x, y) + i \cdot v(x, y)$$

The functions $u$ and $v$ each take a pair of real numbers and emit one real number. They are connected in $f$ through the fact that $u$ and $v$ have the same input.

Finally, the output of $v$ is multiplied by $i$. Here is an example:

$$z = x + iy$$

$$z^2 = (x + iy)(x + iy)$$

$$= x^2 - y^2 + 2ixy$$

So

$$u(x, y) = x^2 - y^2$$
$$v(x, y) = 2xy$$

Another one is:

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{z*}{z*}$$
$$= \frac{x - iy}{(x + iy)(x - iy)}$$
$$= \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}$$

Often a simplified notation is employed:

$$f(z) = u + i \cdot v$$

Since the inputs cover the entire complex plane, we cannot plot graphs as with real functions. Instead, one version of the complex plane is *mapped* by the function into a different version of the complex plane.

### Cauchy-Riemann

This chapter gives us a first glimpse of the important Cauchy-Riemann conditions and justifies one of the formulas for calculating the derivative

$$f'(z) = u_x + iv_x$$

As an example of its use, consider the complex exponential

$$f(z) = e^z$$

If we write $z = x + iy$ then

$$\begin{aligned} f(z) &= e^{x+iy} \\ &= e^x e^{iy} \end{aligned}$$

and (from Euler):

$$e^{iy} = \cos y + i \sin y$$

so

$$f(z) = e^x \cos y + ie^x \sin y$$

Using the formula, it can be shown easily that the derivative is the same as the function itself, just as for the case of real numbers.

$$u(x, y) = e^x \cos y$$

$$u_x = e^x \cos y = u$$

$$v(x, y) = e^x \sin y$$

$$v_x = e^x \sin y = v$$

Hence

$$f'(z) = u_x + iv_x = z$$

**definition**

We define the derivative $f'(z)$ of a complex function $f(z)$ similarly to the derivative of a real function:

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

if the limit exists.

Alternatively, with $\Delta$ notation, we might write:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

A crucial difference from real functions is that there are only two directions from which to approach a given real number $x$, while there is an infinite number of ways of approaching $z$ in the Argand plane. The limit is over *all* possible ways of approaching $z$.

If the limit exists, the function $f$ is called differentiable and $f'(z)$ is the derivative. Consider

$$f(z) = u(x, y) + iv(x, y)$$

Then

$$f'(z) = \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}$$

**fixed y**

We tame this beast by looking at two specific paths.

Looking at the special path along the $x$-axis where $\Delta y = 0$ we obtain

$$f'(z) = \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x}$$

Rearrange the numerator

$$= \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + \frac{iv(x + \Delta x, y) - iv(x, y)}{\Delta x}$$

The first term is

$$u_x = \frac{\partial u}{\partial x}$$

and the second term is

$$iv_x$$

Hence we conclude that

$$f'(z) = u_x + iv_x$$

**fixed x**

Now look at the special path along the $y$-axis where $\Delta x = 0$:

$$f'(z) = \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y}$$

Rearrange the numerator

$$= \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + \frac{iv(x, y + \Delta y) - iv(x, y)}{i\Delta y}$$

$$= \frac{1}{i} u_y + v_y$$

Recall that $1/i = -i$

$$f'(z) = v_y - iu_y$$

**Putting it together**

We require that the limit be the same regardless of the direction of approach to $z$, so these two expressions for the difference quotient must be equal:

$$f'(z) = u_x + iv_x = -iu_y + v_y$$

Both the real and the imaginary parts must be equal so

$$u_x = v_y$$

$$u_y = -v_x$$

Once differentiability is established, we can use whichever path we want to evaluate the derivative.

As we said at the beginning, in looking at various complex functions we can use this fact:

$$f'(z) = u_x + iv_x$$

One consequence is that

$$\frac{df}{dz} = \frac{\partial f}{\partial x}$$

and since

$$\begin{aligned} &= u_x + iv_x = v_y - iu_y \\ &= -iu_y + v_y \\ &= -i(u_y + iv_y) \\ &= -i\frac{\partial f}{\partial y} \end{aligned}$$

We conclude that

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

**looking ahead**

When we get to integration in a later section we will find that the integral of a complex function is computed as a line integral along a specified curve (often a circle centered either on the origin or on a point $z_0$).

This curve relates the values of $x$ and $y$ and allows us to parametrize either $y$ in terms of $x$ or more generally, both $x$ and $y$ in terms of a single real variable or parameter $t$.

When we have a function of such a variable like

$$f(t) = u(t) + iv(t)$$

then the derivative is defined to be

$$f'(t) = u'(t) + iv'(t)$$

where $u$ and $v$ are real-valued functions of a single real variable and so follow the standard rules from introductory calculus. In particular if

$$w(t) = z_0 f(t)$$

then

$$w'(t) = z_0 f'(t)$$

The derivative of a constant times a function is the constant times the derivative of the function.

**derivative of $z_0$ times a function**

We can show this by using a little algebra:

$$\begin{aligned}
\frac{d}{dt}\, z_0 f(t) &= \ [\ (x_0 + iy_0)(u + iv)\ ]' \\
&= \ [\ (x_0 u - y_0 v) + i(y_0 u + x_0 v)\ ]' \\
&= (x_0 u - y_0 v)' + i(y_0 u + x_0 v)' \\
&= (x_0 u' - y_0 v') + i(y_0 u' + x_0 v') \\
&= (x_0 + iy_0)(u' + iv') \\
&= z_0 \frac{d}{dt}\ f(t)
\end{aligned}$$

Thus

$$\frac{d}{dt}\ z_0 f(t) = z_0 \frac{d}{dt}\ f(t)$$

which is what we just said.

## derivative of exp $z_0$ t

Another expected result is

$$\frac{d}{dt}\, e^{z_0 t} = z_0 e^{z_0 t}$$

where $z_0$ is a complex constant and $t$ is a real variable.

To do this one, refer to the definition

$$f'(t) = u'(t) + iv'(t)$$

And now we need to break up the exponential into its real and imaginary parts.

By Euler's equation, we wrote above

$$e^z = e^{x+iy} = e^x \cos y + ie^x \sin y$$

For the exponential of a real variable, but containing a complex constant we have

$$\begin{aligned} e^{z_0 t} &= e^{(x_0+iy_0)t} \\ &= e^{x_0 t}\ e^{iy_0 t} \\ &= e^{x_0 t}\ (\cos y_0 t + i \sin y_0 t) \\ &= e^{x_0 t} \cos y_0 t + ie^{x_0 t}\ \sin y_0 t \end{aligned}$$

## Substitution

I find this calculation very confusing. Especially the subscripts. Rather than change letters, we will drop the subscripts on $x_0$ and $y_0$ but tell ourselves repeatedly: these are constants. Also, $t$ is a *real* variable.

$$e^{xt} \cos yt + ie^{xt}\ \sin yt$$

Using the definition above we get that the derivative is $u'(t) + iv'(t)$ so the derivative of a sum is the sum of the derivatives.

The first term ($u'$) is (by the product and chain rules):

$$[\ e^{xt} \cos yt\ ]' = xe^{xt} \cos yt - ye^{xt} \sin yt$$

and the second:

$$[\ e^{xt}\ \ \sin yt\ ]' = xe^{xt} \sin yt + ye^{xt} \cos yt$$

Remember that each term in that second one gets an $i$!

$$i[\ e^{xt}\ \ \sin yt\ ]' = ixe^{xt} \sin yt + iye^{xt} \cos yt$$

Combine the first term from each and factor out the $x$:

$$x(e^{xt} \cos yt + ie^{xt} \sin yt)$$

Do the same with the second term:

$$y(ie^{xt} \cos yt - e^{xt} \sin yt)$$

the tricky part

$$= iy(e^{xt} \cos yt + ie^{xt} \sin yt)$$

Putting everything together we have just

$$(x + iy)(e^{xt} \cos yt + ie^{xt} \sin yt)$$

Restoring the original naughts, we have just

$$z_0 e^{z_0 t}$$

As promised.

$\square$

# Chapter 6

# Proofs of CRE

### difference quotient

We gave a first proof in the section on differentiation which is repeated more briefly here.

The derivative $f'(z)$ is defined to be the limit of the following difference quotient, if the limit exists.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

where $f(z) = u(x, y) + iv(x, y)$.

The difference quotient is rewritten in terms of $u$ and $v$ as:

$$\frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x) - iv(y)}{\Delta x + i\Delta y}$$

Then we consider two special cases, one where $\Delta y = 0$ and a second where $\Delta x = 0$. The first case yields

$$f'(z) = u_x + iv_x$$

and the second yields

$$f'(z) = -iu_y + v_y$$

The derivative is required to be the same for all directions of approach to the point, so we can equate the two expressions

$$u_x + iv_x = -iu_y + v_y$$

Since both the real and the imaginary parts must be equal, we obtain the CRE:

$$u_x = v_y$$

$$u_y = -v_x$$

□

**chain rule**

Here is a second approach:

Write:

$$z = x + iy$$

Clearly,

$$\frac{\partial z}{\partial x} = 1, \qquad \frac{\partial z}{\partial y} = i$$

Now,

$$\begin{aligned} w &= f(z) \\ &= u(x, y) + i\ v(x, y) \end{aligned}$$

where $u$ and $v$ are real functions over $\mathbb{R}^2$.

Recalling the chain rule

$$w = u(x, y) + i\ v(x, y)$$

$$\frac{\partial w}{\partial x} = \frac{dw}{dz}\,\frac{\partial z}{\partial x}$$

by the result immediately above (that $\partial z/\partial x = 1$):

$$\frac{\partial w}{\partial x} = \frac{dw}{dz}$$

Similarly

$$\frac{\partial w}{\partial y} = \frac{dw}{dz}\,\frac{\partial z}{\partial y}$$

$$\frac{\partial w}{\partial y} = i\frac{dw}{dz}$$

Hence we can equate the two expressions for $dw/dz$:

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

Now if we actually compute the partials and plug them in to the last equation, we obtain:

$$u_x + iv_x = -i(u_y + iv_y) = v_y - iu_y$$

Both the real and the imaginary parts must be equal:

$$u_x = v_y$$

$$v_x = -u_y$$

These are (again) the CRE.

□

It is worth taking a breath for a moment and repeating what we just said: the derivative of a differentiable complex function $z$ (what we will call an analytic function) is

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

$$
\begin{aligned}
&= u_x + iv_x \\
&= -i(u_y + iv_y) \\
&= v_y - iu_y
\end{aligned}
$$

**Alder**

A third, very simple proof is given in Alder:

Suppose $f : C \to C$ is a function, taking $x + iy$ to $u(x, y) + iv(x, y)$, then the derivative is a matrix of partial derivatives:

$$
\begin{matrix} u_x & u_y \\ v_x & v_y \end{matrix}
$$

> the above matrix is the two dimensional version of the slope of the tangent line in dimension one. It gives the linear part (corresponding to the slope) of the affine map which best approximates $f$ at each point.
>
> But at any point $x + iy$, if $f$ is differentiable in the *complex* sense, this must be just a linear complex map, i.e. it multiplies by some complex number. So the matrix must be in our set of complex numbers. In other words, for every value of $x$ it looks like

$$
\begin{matrix} a & -b \\ b & a \end{matrix}
$$

for some real numbers $a, b$, which change with $x$.

Of course, this constraint leads directly to the CRE.

□

A very important point is that the CRE and analyticity and differentiability are all related (either a function has all these properties or none of them). For an analytic function, the rules for integration and differentiation are analogous to the real case. For example:

$$\int \frac{1}{3}z^2 \, dz = z^3$$

$$\frac{d}{dz}\,\frac{1}{z-z_0} = -\frac{1}{(z-z_0)^2}$$

We will see a lot more of this.

**McMahon**

Here is yet another proof which I found in McMahon.

https://www.amazon.com/Complex-Variables-Demystified-David-McMahon/dp/007154920X

I include it here because it explains Shankar's statement that by definition an analytic function has no dependence on $z*$. (The $\bar{z}$ notation is used below).

Write

$$z = x + iy, \quad \bar{z} = x - iy$$

so

$$2x = (z + \bar{z})$$

$$x = \frac{1}{2}(z + \bar{z})$$

$$2iy = (z - \bar{z})$$

$$y = \frac{1}{2i}(z - \bar{z}) = -\frac{1}{2}i(z - \bar{z})$$

Take partial derivatives:

$$\frac{\partial x}{\partial z} = \frac{1}{2} = \frac{\partial x}{\partial \bar{z}}$$

and

$$\frac{\partial y}{\partial z} = -\frac{1}{2i} = -\frac{\partial y}{\partial \bar{z}}$$

Then, using the chain rule we write:

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left[\ \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\ \right]$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial x}{\partial \bar{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar{z}}\frac{\partial}{\partial y} = \frac{1}{2}\left[\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\ \right]$$

Now apply the two operators (just a matter of a few minus signs):

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left[\ \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\ \right]\left[\ u + iv\ \right] = \frac{1}{2}\left[\ (u_x + v_y) + i(v_x - u_y)\ \right]$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\ \right]\left[\ u + iv\ \right] = \frac{1}{2}\left[\ (u_x - v_y) + i(v_x + u_y)\ \right]$$

And now to the point: we *require* the last expression to be zero. $f(z)$ must have no dependence on $\bar{z}$.

As usual, both the real and the imaginary parts must vanish.

$$0 = \frac{1}{2}\left[\ (u_x - v_y) + i(v_x + u_y)\ \right]$$

$$u_x = v_y, \quad v_x = -u_y$$

In other words, the CRE apply. And using these conditions, we can rewrite

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{1}{2}\left[\ (u_x + v_y) + i(v_x - u_y)\ \right]\\ &= u_x + iv_x\end{aligned}$$

□

To put this another way, if we have already established the CRE, we can run this proof backwards to show that for $f(z)$, $\partial f/\partial \bar{z} = 0$.

Sources:

[1] Alder. *An Introduction to Complex Analysis for Engineers.*

http://www.eee.metu.edu.tr/~ccandan/EE202_summer2004/solutions/An%20Introduction%20to%20Complex%20Analysis%20for%20Engineers%20-%20Michael%20Alder.pdf

# Chapter 7

# Powers

**square**

Consider

$$
\begin{aligned}
f(z) &= z^2 \\
&= (x+iy)(x+iy) \\
&= x^2 - y^2 + i2xy \\
u(x,y) &= x^2 - y^2 \\
v(x,y) &= 2xy
\end{aligned}
$$

Note that

$$
\begin{aligned}
u_x &= 2x = v_y \\
u_y &= -2y = -v_x
\end{aligned}
$$

The Cauchy-Riemann conditions (CRE) hold.

Compute the derivative as follows:

$$
\begin{aligned}
\frac{df}{dz} &= u_x + iv_x \\
&= 2x + i2y = 2z
\end{aligned}
$$

or alternatively

$$\frac{df}{dz} = v_y - iu_y$$

$$= 2x - i(-2y) = 2x + i2y = 2z$$

This is the result we would expect to get by simply differentiating f(z) as if it was a real function. For analytic functions this will always be the case.

**cube**

Let

$$f(z) = z^3 = (x + iy)^3$$

$$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$

$$= x^3 - 3xy^2 + i\ [\ 3x^2y - y^3\ ]$$

So

$$u(x, y) = x^3 - 3xy^2$$

$$v(x, y) = 3x^2y - y^3$$

and

$$u_x = 3x^2 - 3y^2$$

$$v_x = 6xy$$

That means

$$f'(z) = u_x + iv_x$$

$$= 3x^2 - 3y^2 + i6xy$$

$$= 3\ [x^2 + 2x(iy) + (iy)^2\ ]$$

$$= 3z^2$$

We could continue and show that $z^n$ is analytic for any positive integer power of $n$. Notice the pattern for $i$:

$$(x+iy)^n = x^n + nx^{n-1}(iy) + (n)(n-1)x^{n-2}(iy)^2 + \dots$$

The progression goes:

$$i^0, i^1, i^2, i^3$$
$$= 1, i, -1, -i$$

and then repeats.

**inverse**

Using the complex conjugate is a good way to work with the inverse function (or with division by any complex number):

$$\frac{1}{z} = \frac{z*}{zz*} = \frac{x-iy}{x^2+y^2}$$

or in polar notation:

$$\frac{1}{z} = \frac{re^{-i\theta}}{r^2\ e^{i\theta}\ e^{-i\theta}} = \frac{1}{r}\ e^{-i\theta}$$

Let's look at what it means to take the inverse for different $z$. In every case, the point is reflected across the $x$-axis (the ray makes an angle $-\theta$ with the $x$-axis).

There is no change in length for $r = 1$. But if say

$$z = 1 + i = (1,1) = \sqrt{2}\ e^{i \cdot \pi/4}$$

then the new point has $r = \frac{1}{\sqrt{2}}$ and it is located at

$$\frac{1}{z} = \frac{1}{\sqrt{2}} e^{-i \cdot \pi/4} = (\frac{1}{2}, -\frac{1}{2}) = \frac{1}{2} - i\frac{1}{2}$$

**differentiation**

We wish to differentiate

$$\begin{aligned} f(z) &= 1/z \\ &= \frac{z*}{zz*} \\ &= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \end{aligned}$$

Let's do the partial derivatives. $u(x, y)$ has $x$ in both the numerator and the denominator.

Recall the quotient rule (using unfamiliar symbols $g$ and $h$):

$$(g/h)' = (g'h - gh')/h^2$$

which we check by differentiating $x/1$.

- $u_x$

$$u(x, y) = \frac{x}{x^2+y^2}$$

$$u_x = (x^2 + y^2 - x \cdot 2x) \cdot \frac{1}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

- $v_y$

$$v(x, y) = -\frac{y}{x^2+y^2}$$

To do $v_y$ just switch $x$ and $y$ in the result above, but remember to then multiply by the leading factor of $-1$:

$$v_y = (-1)\ \frac{x^2 - y^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

Thus $u_x = v_y$

- $u_y$

$$u(x,y) = x(x^2+y^2)^{-1}$$
$$u_y = x(-1)\cdot 2y(x^2+y^2)^{-2} = \frac{-2xy}{(x^2+y^2)^2}$$

- $v_x$

$$v(x,y) = -y(x^2+y^2)^{-1}$$
$$v_x = -y(-1)\cdot 2x(x^2+y^2)^{-2} = \frac{2xy}{(x^2+y^2)^2}$$

So we have that the CRE are satisfied (except at $z = 0$) and the derivative is

$$\begin{aligned}\frac{df}{dz} &= u_x + iv_x \\ &= \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2} \\ &= \frac{1}{(x^2+y^2)^2}\,(y^2-x^2+i2xy)\end{aligned}$$

We expect that this should be (in disguise) $-1/z^2$. Let's see:

$$\frac{1}{z} = \frac{z*}{zz*}$$
$$\frac{1}{z^2} = \frac{(z*)^2}{(zz*)^2}$$

The denominator is certainly correct since

$$zz* = x^2+y^2$$

What about the denominator?

$$(z*)^2 = (x - iy)(x - iy) = x^2 - y^2 - i2xy$$

so

$$-(z*)^2 = (-1)(x - iy)(x - iy) = y^2 - x^2 + i2xy$$

Everything checks.

**powers: de Moivre's formula**

Let $n$ be an integer:

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

Suppose $r = 1$:

$$z^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$$

This is de Moivre's formula.

Suppose $n = 2$, then

$$(\cos\theta + i\sin\theta)^2$$

$$= \cos^2\theta - \sin^2\theta + i\ 2\sin\theta\cos\theta$$

Equating with the right-hand side of de Moivre's formula:

$$\cos^2\theta - \sin^2\theta + i\ 2\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$$

we find that

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

We already know these, they are the double angle formulas.

Suppose $n = 3$, then

$$(\cos\theta + i\sin\theta)^3$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

we find that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

and so on.

We can just check that last one for $\theta = \pi/6$:

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$
$$1 = 3(\frac{\sqrt{3}}{2})^2\ \frac{1}{2} - (\frac{1}{2})^3$$

Multiply both sides by $2^3$:

$$8 = 3(\sqrt{3})^2 - 1$$

That looks correct.

# Chapter 8

# Polar CRE

In this chapter we will derive the CRE conditions for polar coordinates and show how to compute the derivative in the same system. The take home lesson is that there is an extra factor for each:

Recall that the CRE in Cartesian coordinates are:

$$u_x = v_y$$

$$v_x = -u_y$$

It turns out that $r, \theta$ is similar except for a factor of $r$, which goes on the partial with respect to $r$

$$ru_r = v_\theta$$

$$rv_r = -u_\theta$$

Also,

$$f'(z) = e^{-i\theta}\ (u_r + iv_r)$$

We'll also see two derivations for each, a simple one and a more careful but also more complicated one.

A clever way to derive the CRE in polar coordinates is to take advantage of the result that we obtained in Cartesian coordinates.

**short and sweet**

The function $f(z) = z = x + iy$ is "analytic" and obeys the CRE.

Write the same function in polar coordinates:

$$z = re^{i\theta}$$

and then separate it into a completely real part $u(r, \theta)$ and a completely imaginary part $v(r, \theta)$:

$$z = r(\cos\theta + i\sin\theta)$$

$$= r\cos\theta + ir\sin\theta$$

Observe that

$$u_r = \cos\theta$$

$$v_\theta = r\cos\theta$$

so we deduce that

$$ru_r = v_\theta$$

Similarly

$$v_r = \sin\theta$$

$$u_\theta = -r\sin\theta$$

so

$$-rv_r = u_\theta$$

There are the CRE in polar coordinates. Carrying out the same computation for *any* analytic function would give the same result (with some other expression in the middle, of course).

$$ru_r = v_\theta$$

$$rv_r = -u_\theta$$

Notice the similar format to the Cartesian version, with the addition of a factor of $r$.

It reminds me of the Jacobian from multi-variable calculus.

**polar derivative**

This will also turn out to be very similar to the Cartesian version, with an extra factor out front:

$$f'(z) = e^{-i\theta} \ [\ u_r + iv_r \ ]$$

Let's just assume that the derivative is equal to what we would expect, within some unknown factor of $k$:

$$z' = k(u_r + iv_r)$$

and now we know that for this function the derivative is equal to 1:

$$f(z) = z$$

$$z' = 1 = k(u_r + iv_r)$$

if we write in polar coordinates:

$$z = r\cos\theta + ir\sin\theta$$

Then

$$u_r + iv_r = \cos\theta + i\sin\theta$$

What is the factor that multiplies this expression to give 1? Clearly

$$e^{-i\theta}(\cos\theta + i\sin\theta) = 1$$

So $k = e^{-i\theta}$.

## CRE derivation by the chain rule

We know equations to go back and forth between $x, y$ and $r, \theta$ so it is not hard to imagine that we can always re-write $u$ and $v$ as

$$z = u\ [\ x(r,\theta), y(r,\theta)\ ] + iv\ [\ x(r,\theta), y(r,\theta)\ ]$$

or more succinctly:

$$z = u(r,\theta) + iv(r,\theta)$$

Now we ask about relations between the partial derivatives. Let us first make a table of them:

$$x = r\cos\theta$$

$$x_r = \cos\theta, \qquad x_\theta = -r\sin\theta$$

$$y = r\sin\theta$$

$$y_r = \sin\theta, \qquad y_\theta = r\cos\theta$$

Clearly, we want expressions involving $u_r$, $v_\theta$ etc. Write:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

or in a convenient shorthand

$$u_r = u_x x_r + u_y y_r = u_x \cos\theta + u_y \sin\theta$$

The other three are

$$u_\theta = u_x x_\theta + u_y y_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta)$$

$$v_r = v_x x_r + v_y y_r = v_x(\cos\theta) + v_y(\sin\theta)$$

$$v_\theta = v_x x_\theta + v_y y_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta)$$

Our key insight is to use the relations given by the CRE in Cartesian coordinates

$$u_x = v_y$$

$$u_y = -v_x$$

Thus, starting with the first expression for partial derivatives above

$$u_r = u_x \cos\theta + u_y \sin\theta$$

Use the CRE in $x, y$ to substitute in terms of $v$:

$$u_r = v_y \cos\theta + (-v_x)\sin\theta$$

we can see that this is different from the fourth expression above

$$v_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta)$$

only by a factor of $r$:

$$ru_r = v_\theta$$

which is what needed to prove.

The other one is:

$$u_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta)$$

$$= v_y(-r\sin\theta) - v_x(r\cos\theta)$$

compare with

$$v_r = v_x(\cos\theta) + v_y(\sin\theta)$$

We need a factor of $-r$:

$$u_\theta = -rv_r$$

Like the original Cartesian version but with an extra factor of $r$ on the partials with respect to $r$.

## polar derivative, more carefully

To get the derivative, start with the version that we know for $x, y$ coordinates:

$$f'(z) = u_x + iv_x$$

Our problem is to define $f'(z)$ in terms of $u_r$ and $v_r$.

Substitute for $u_x$ first. Go back to the two equations involving $u_x$ above

$$u_r = u_x \cos\theta + u_y \sin\theta$$

$$u_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta)$$

Multiply the first by $\cos\theta$

$$u_r \cos\theta = u_x \cos^2\theta + u_y \sin\theta\cos\theta$$

and the second by $-\sin\theta/r$

$$-\frac{u_\theta}{r}\sin\theta = u_x \sin^2\theta - u_y \sin\theta\cos\theta$$

add

$$u_x = u_r \cos\theta - \frac{u_\theta}{r}\sin\theta$$

Substitute $u_\theta/r = -v_r$

$$u_x = u_r \cos\theta + v_r \sin\theta$$

Now get the second and fourth equations, with $v_x$

$$v_r = v_x \cos\theta + v_y \sin\theta$$

$$v_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta)$$

Multiply the first by $\cos\theta$

$$v_r \cos\theta = v_x \cos^2\theta + v_y \sin\theta\cos\theta$$

and the second by $-\sin\theta/r$:

$$-\frac{v_\theta}{r}\sin\theta = v_x \sin^2\theta - v_y \sin\theta\cos\theta$$

add

$$v_x = v_r\cos\theta - \frac{v_\theta}{r}\sin\theta$$

Subtitute $v_\theta/r = u_r$

$$v_x = v_r\cos\theta - u_r\sin\theta$$

Combine the two results:

$$f'(z) = u_x + iv_x$$

$$= u_r\cos\theta + v_r\sin\theta + i\ [\ v_r\cos\theta - u_r\sin\theta\ ]$$

Group terms with $u_r$ and $v_r$ separately:

$$= u_r(\cos\theta - i\sin\theta) + v_r(\sin\theta + i\cos\theta)$$

Multiply the second term by $1 = -i\cdot i$

$$= u_r(\cos\theta - i\sin\theta) + iv_r(-i\sin\theta + \cos\theta)$$

$$f'(z) = e^{-i\theta}\ [\ u_r + iv_r\ ]$$

And since

$$ru_r = v_\theta$$

$$rv_r = -u_\theta$$

then

$$f'(z) = \frac{1}{r}\ e^{-i\theta}\ [\ v_\theta - iu_\theta\ ]$$

compare this with

$$f'(z) = v_y - iu_y$$

and notice that the factor in front is just $1/z$

**example 1**

Let's see if we can do an example. Suppose

$$f(z) = \sqrt{z}$$

Written in terms of $r, \theta$ we have

$$f(z) = \sqrt{r}e^{i\theta/2}$$

$$= \sqrt{r}\cos\theta/2 + i\sqrt{r}\sin\theta/2$$

Then

$$u_r = \frac{\cos\theta/2}{2\sqrt{r}}$$

$$v_r = \frac{\sin\theta/2}{2\sqrt{r}}$$

and

$$[\sqrt{z}\ ]' = \ [\ \sqrt{r}e^{i\theta/2}\ ]' = e^{-i\theta}\ [\ u_r + iv_r\ ]$$

$$= \frac{1}{2\sqrt{r}}\ [\ e^{-i\theta}(\cos\theta/2 + i\sin\theta/2)\ ]$$

$$= \frac{1}{2\sqrt{r}}e^{-i\theta/2} = \frac{1}{2\sqrt{z}}$$

**example 2**

Let's try

$$f(z) = \frac{1}{z} = \frac{1}{r}e^{-i\theta}$$

$$= \frac{1}{r}\cos -\theta + i\frac{1}{r}\sin -\theta$$

$$= \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$$

So

$$u_r = -\frac{1}{r^2}\cos\theta$$

$$v_r = \frac{1}{r^2}\sin\theta$$

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$= \frac{1}{r^2}(e^{-i\theta})(-\cos\theta + i\sin\theta)$$

$$= \frac{1}{r^2}(e^{-i\theta})(-\cos -\theta - i\sin -\theta)$$

$$= \frac{1}{r^2}(e^{-i\theta})(-1)(e^{-i\theta})$$

$$= -\frac{1}{r^2 e^{i2\theta}} = -\frac{1}{z^2}$$

# Chapter 9

# CRE examples

The function

$$f(x,y) = \frac{x^2}{x^2+y^2}$$

has some problems: first, it is not defined at the origin $(0,0)$ but also, as we approach the origin along the $x$-axis and the $y$-axis we get different limiting values, namely

$$f(x,0) = \frac{x^2}{x^2} = 1$$

$$f(0,y) = \frac{0}{y^2} = 0$$

Rewriting it in polar coordinates ($x = r\cos\theta, r^2 = x^2+y^2$):

$$f(r,\theta) = \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$$

Shankar says: the function $f$ is generally a function of *two* complex variables, $z$ and its complex conjugate:

$$z = x + iy$$

$$z* = x - iy$$

which can be written in terms of $x$ and $y$ as

$$x = \frac{z + z*}{2}$$

$$y = \frac{z - z*}{2i}$$

Generally, the value of $f$ depends on both $z$ and $z*$, but we will be very interested in functions which depend only on $z$ and not $z*$. The reason for this is that only such functions have the property that the derivative at a point does not depend on the direction from which we approach that point.

Consider the function:

$$f(x, y) = x^2 - y^2$$

$$= \frac{(z + z*)^2}{4} + \frac{(z - z*)^2}{4}$$

$$= \frac{1}{4} \left[ \ z^2 + 2zz * + z *^2 + z^2 - 2zz * + z*^2 \ \right]$$

$$= \frac{z^2 + z*^2}{2}$$

This function is not a function only of $z$ but of both $z$ and $z*$.

We say that $f$ is an *analytic* function of $z$ if it does not depend on $z*$. Shankar says this means that "$x$ and $y$ enter $f$ *only* in the combination $x + iy$".

The famous Cauchy-Riemann Equations (CRE) are true for $f \iff f$ is an analytic function of $z$.

For:

$$f(x, y) = u(x, y) + iv(x, y)$$

The CRE conditions are:

$$u_x = v_y$$
$$u_y = -v_x$$

Consider:

$$f(x, y) = x^2 - y^2 + i2xy$$

CRE requires

$$u_x = 2x \stackrel{?}{=} v_y = 2x$$
$$v_x = 2y \stackrel{?}{=} -u_y = 2y$$

The function is analytic. As Shankar says, this is expected because:

$$x^2 - y^2 + 2ixy = (x + iy)(x + iy) = z^2$$

Consider:

$$f(x, y) = \cos y - i \sin y$$

CRE requires:

$$u_x = 0 \stackrel{?}{=} v_y = -\cos y$$
$$v_x = 0 \stackrel{?}{=} -u_y = -\sin y$$

This is "impossible" since there is no $y$ that satisfies both of the conditions. And it's not surprising since

$$y = \frac{z - z*}{2i}$$

Consider:

$$f(x, y) = x^2 + y^2$$

CRE requires:

$$u_x = 2x \stackrel{?}{=} v_y = 2y$$
$$u_y = 0 \stackrel{?}{=} -v_x$$

CRE are only satisfied if $x = y$. Also not surprising since

$$x^2 + y^2 = zz*$$

Consider:

$$f(x, y) = x^2 - y^2$$

CRE requires:

$$u_x = 2x \stackrel{?}{=} v_y = -2y$$

which is true if $x = y$.

$$u_y = 0 \stackrel{?}{=} -v_x = 0$$

But "no importance is given to functions which obey the CRE only at isolated points or on lines."

Consider:

$$f(x, y) = e^x \cos y + ie^x \sin y$$

CRE requires:

$$u_x = e^x \cos y \stackrel{?}{=} v_y = e^x \cos y$$

$$u_y = -e^x \sin y \stackrel{?}{=} -v_x = -\sin y e^x$$

Both are true, so this one does satisfy CRE.

Shankar doesn't mention it here but the last function is special, it is $f(z) = e^z$:

$$e^x \cos y + ie^x \sin y$$

$$= e^x(\cos y + i \sin y)$$

$$= e^x e^{iy}$$

$$= e^{x+iy}$$

$$= e^z$$

and we did this one in the previous section.

For functions of interest, it may often be true that CRE fails at particular points called *singularities*.

Consider:

$$f(x,y) = \frac{1}{z} = \frac{z*}{zz*} = \frac{x - iy}{x^2 + y^2}$$

We need:

$$u_x = \frac{d}{dx}\,\frac{x}{x^2 + y^2} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$v_y = \frac{d}{dy}\,(-\frac{y}{x^2 + y^2}) = -\frac{x^2 - y^2}{(x^2 + y^2)^2} = u_x$$

$$u_y = 0 = v_x$$

But the function blows up at the origin. This described by saying it has a pole at the origin. The function

$$f(z) = \frac{c}{z}$$

where $c$ is a constant, also blows up at the origin. We say that the *residue* of the pole at the origin is $c$.

# Chapter 10

# Roots

Consider the square root function $\sqrt{z}$.

For the modulus part, we see that $\sqrt{r} \cdot \sqrt{r}$ is obviously equal to $r$, and what we need to determine the argument is to find an angle that is one-half of the original one, which leads us to

$$\sqrt{z} = \sqrt{re^{i\theta}}$$

$$= \sqrt{r}\ e^{i(\theta/2)}$$

However, recall from trigonometry that if

$$\theta' = \theta + 2k\pi$$

for integer $k$, then

$$\sin\theta' = \sin\theta$$

We can even say that $\theta'$ is equal to $\theta$ since the result for a given $r$ maps to the same point in the plane.

This means that a second solution to the square root problem is

$$\sqrt{z} = \sqrt{r}\ e^{i(\theta/2+\pi)}$$

because, again, $\sqrt{r} \cdot \sqrt{r} = r$ and

$$[\ e^{i(\theta/2+\pi)} \cdot \ e^{i(\theta/2+\pi)}\ ] = e^{i(\theta/2+\pi)} = e^{i\theta}$$

**example**

Consider

$$z = e^{i\pi/3}$$

We don't have to worry about $r$, since it is equal to 1. One solution to the square root is

$$\sqrt{z} = e^{i\pi/6}$$

The second one is

$$\sqrt{z} = e^{i(\pi/6+\pi)} = e^{i(7\pi/6)}$$

which lies in the third quadrant.

To check this:

$$[\ e^{i(7\pi/6)} \cdot e^{i(7\pi/6)}\ ] = e^{i(14\pi/6)} = e^{i\pi/3}$$

For the square root, there is only one additional distinct solution, since one-half of $4\pi + \theta = 2\pi + \theta/2$ which is no different than $\theta/2$.

However, the cube root has 3 solutions and in general the $n^{th}$ root has $n$ solutions.

Consider points on the unit circle with $r = 1$ (so $\sqrt{r} = r$) and suppose

$$\theta = \pi/2$$

so

$$z = e^{i\pi/2}$$

Points with $\theta = \pi/2$ lie directly above the origin on the imaginary axis (there is no real component). This point is one unit from the origin so it is the point $(0 + i \cdot 1) = i$. Thus

$$e^{i\pi/2} = i$$

Note that

$$\begin{aligned}(e^{i\pi/2})^2 &= e^{i(\pi/2+\pi/2)} \\ &= e^{i\pi} = -1 = i^2\end{aligned}$$

We can justify this last step by geometry ($\theta = \pi$), or by using Euler's equation

$$e^{i\theta} = cos\theta + i\sin\theta$$

$$e^{i\pi} = cos\pi + i\sin\pi = -1 + i \cdot 0 = -1$$

**square root of i**

$\sqrt{e^{i\pi/2}} = \sqrt{i}$ has two possible values. One is

$$\sqrt{e^{i\pi/2}} = (e^{i\pi/2})^{1/2} = e^{i\pi/4}$$

Let's just check. The point is at a distance 1 from the origin and angle $\theta = \pi/4$. We go equal distances along the real and imaginary axes:

$$x = \cos\theta = \frac{1}{\sqrt{2}}$$

$$y = \sin\theta = \frac{1}{\sqrt{2}}$$

So we have that the square is:

$$(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})^2 = \frac{1}{2} - \frac{1}{2} + 2i\frac{1}{2}$$

$$= 0 + i = i$$

the second solution is

$$\sqrt{e^{i\pi/2}} = e^{i \cdot 5/4\pi}$$

which can be plotted as

$$x = \cos\theta = -\frac{1}{\sqrt{2}}$$

$$y = \sin\theta = -\frac{1}{\sqrt{2}}$$

The square is the same except the first term is $(-1/\sqrt{2})^2$, so the result is unchanged. It's a bit counter-intuitive that squaring a number may possibly reduce the phase angle, but you can think of it as modular arithmetic (mod $2\pi$).

In general, if we're working with the complex number

$$re^{i\theta}$$

and we want the nth root, the modulus is just

$$\rho = r^{1/n}$$

And the question always is, what's the angle?

$$\phi = \frac{\theta + 2k\pi}{n}, \quad k = 0, 1, 2 \ldots n - 1$$

**roots of unity**

Let's say we want the cube roots of 1. Obviously, all the roots will have length 1. What about the angles? The starting angle $\theta = 0$, so $\phi = 2k\pi/3$ and

$$\phi_1 = \frac{2\pi}{3}$$

$$\phi_2 = \frac{4\pi}{3}$$
$$\phi_3 = \frac{6\pi}{3} = 0$$

Notice that the first and second roots are complex conjugates because

$$\phi_1 + \phi_2 = \frac{6\pi}{3} = 2\pi = 0$$

Suppose our number is $z = -8i$ and we want the cube roots. Writing the number in polar coordinates:

$$z = 8e^{3\pi/2}$$

All of the roots have the same modulus, 2, since $2^3 = 8$. The are three roots which differ in their arguments. Since $\theta = 3\pi/2$, these are:

$$\phi_1 = \frac{\theta}{3} = \frac{\pi}{2}$$
$$\phi_2 = \frac{\theta + 2\pi}{3} = \frac{\pi}{2} + \frac{2\pi}{3} = \frac{5\pi}{6}$$
$$\phi_3 = \frac{\theta + 4\pi}{3} = \frac{\pi}{2} + \frac{4\pi}{3} = \frac{7\pi}{6}$$

Notice that the second and third roots are complex conjugates.

We take the original angle and multiply by the power that the root corresponds to. Then, divide $2\pi$ up into that many pieces, and add $k$ pieces where $k$ runs from 0 to $r - 1$.

When the argument for $z$ is $\theta_0$, a general formula for the angle of the $nth$ root of $z$ is:

$$\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n} \quad k = 0, \pm 1, \pm 2 \ldots$$

We derive this as follows:

$$z = re^{i\theta}$$

$$z^{1/n} = (re^{i(\theta+2k\pi)})^{1/n}$$

Writing only the argument part

$$(e^{i(\theta+2k\pi)})^{1/n} = e^{i(\theta/n+2k\pi/n)}$$

**Nahin's puzzle**

In one of his books Nahin starts by posing this question: suppose we are given that

$$x + \frac{1}{x} = 1$$

*Without computing* $x$, find the value of

$$x^7 + \frac{1}{x^7}$$

Nahin says that if you are the type to just start right in trying to figure this out, then you will like his book.

From its placement in this section, you might just guess the answer. First of all, no real $x$ solves the equation

$$x + \frac{1}{x} = 1$$

as you will see if you use the quadratic formula. So let's change nomenclature and call it $z$.

(Of course, we were not supposed to *compute* $z$).

We may guess that $z$ is a complex number with length 1 so that the lengths don't change with powers or roots.

Then, all that happens is that $\theta$ changes in such a way that

$$7\theta = \theta = \frac{\theta}{7}$$

To actually compute $z$, multiply by $z$, rearrange, and solve:

$$z^2 + 1 = z$$

$$z^2 - z + 1 = 0$$

From the quadratic equation:

$$z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

The square of the length is

$$r^2 = zz*$$

$$= (\frac{1}{2} + i\frac{\sqrt{3}}{2})(\frac{1}{2} - i\frac{\sqrt{3}}{2})$$

$$= \frac{1}{4} + \frac{3}{4} = 1$$

The angle we seek has tangent equal to $1/\sqrt{3}$. You may recognize the sine and cosine of $\pi/3$ as the real and imaginary components of $z$.

So if

$$z = e^{i\pi/3} = (\frac{1}{2} + i\frac{\sqrt{3}}{2})$$

$$\frac{1}{z} = e^{-i\pi/3} = (\frac{1}{2} - i\frac{\sqrt{3}}{2})$$

then when doing the addition the imaginary parts of $z$ cancel and we have that

$$z + \frac{1}{z} = \frac{1}{2} + \frac{1}{2} = 1$$

The other special attribute of this value for $z$ is that the length is 1 so all powers of $r$ are 1. As for the angle, $\pi/3$ is special in that $7 \times \pi/3 = 2\pi + \pi/3 = \pi/3$. Now it's not strictly true that *the* 7th root of $\theta$ is equal to $\theta$ (since there are 7 distinct roots). But I hope you can see that there is at least one such root.

# Part III

# Transcendentals

# Chapter 11

# Sine and cosine

### cosine and sine

Start by recalling Euler's formula for *real* x:

$$e^{ix} = \cos x + i \sin x$$

Substitute $-x$ for $x$

$$e^{-ix} = \cos -x + i \sin -x$$

$$= \cos x - i \sin x$$

Addition gives:

$$2 \cos x = e^{ix} + e^{-ix}$$

$$\cos x = \frac{1}{2} \left(e^{ix} + e^{-ix}\right)$$

Subtraction:

$$2i \sin x = e^{ix} - e^{-ix}$$

$$\sin x = \frac{1}{2i} \left(e^{ix} - e^{-ix}\right)$$

Our old friends:

$$\cosh x = \frac{1}{2}\ (e^x + e^{-x})$$
$$\sinh x = \frac{1}{2}\ (e^x - e^{-x})$$

**complex versions**

The complex counterparts of the real trigonometric functions can be explained by saying that Euler's formula is also good for a complex number $z$ (a math book would define them by their power series).

By the same algebra, this gives

$$\cos z = \frac{1}{2}\ (e^{iz} + e^{-iz})$$
$$\sin z = \frac{1}{2i}\ (e^{iz} - e^{-iz})$$

Now we see that the complex sine and cosine have properties just like their real cousins.

We will do the complex hyperbolic functions in the next chapter.

**period**

The above definition of cosine is

$$\cos z = \frac{1}{2}\ (e^{iz} + e^{-iz})$$

then

$$cos(z + 2\pi) = \frac{1}{2}\ (e^{iz}e^{i2\pi} + e^{-iz}e^{-i2\pi})$$

but

$$e^{i2\pi} = \cos 2\pi + i \sin 2\pi = 1$$

and the same for $e^{-i2\pi}$, so

$$\cos(z + 2\pi) = \cos z$$

The *period* of the complex cosine and sine is $2\pi$, just as for the real function.

**derivatives**

Take derivatives is straightforward:

$$\sin z = \frac{1}{2i}\ (e^{iz} - e^{-iz})$$

$$\frac{d}{dz}\ \sin z = i \cdot \frac{1}{2i}\ (e^{iz} + e^{-iz}) = \cos z$$

Similarly

$$\cos z = \frac{1}{2}\ (e^{iz} + e^{-iz})$$

$$\frac{d}{dz}\ \cos z = \frac{i}{2}\ (e^{iz} - e^{-iz})$$

$$= -\frac{1}{2i}\ (e^{iz} - e^{-iz}) = -\sin z$$

Also

$$\sin -z = \frac{1}{2i}\ (e^{-iz} - e^{iz}) = -\sin z$$

$$\cos -z = \frac{1}{2}\ (e^{-iz} + e^{iz}) = \cos z$$

**separating real and imaginary parts of trig functions**

Since

$$\cos z = \frac{1}{2}\ (e^{iz} + e^{-iz})$$

if we let $z = iy$ then

$$\cos iy = \frac{1}{2}\ (e^{i^2 y} + e^{-i^2 y})$$

$$= \frac{1}{2}\ (e^{-y} + e^{y}) = \cosh y$$

Similarly

$$\sin iy = \frac{1}{2i}\ (e^{i^2 y} - e^{-i^2 y})$$

$$= \frac{1}{2i}\ (e^{-y} - e^{y})$$

$$= -\frac{1}{2i}\ (e^{y} - e^{-y})$$

$$= -\frac{1}{i} \sinh y = i \sinh y$$

Hence

$$\cos iy = \cosh y$$

$$\sin iy = i \sinh y$$

So now if we let $z = x + iy$ and use the standard addition formula

$$\cos z = \cos(x + iy)$$

gives

$$\cos z = \cos x \cos iy - \sin x \sin iy$$

Since $\cos iy = \cosh y$ and $\sin iy = i \sinh y$:

$$= \cos x \cosh y - i \sin x \sinh y$$

and what's nice about this is that we have the real and imaginary parts of the complex cosine easily visible.

Similarly

$$\sin z = \sin(x+iy)$$

$$\sin z = \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

These are very similar to the sum of angles results for real numbers. It's just that $z = x + iy$ means the $y$ gets the hyperbolic functions and sinh has a leading factor of $i$.

**using the exponential to get cos $z$ and sin $z$**

We can obtain the same results by working through the formulas using the complex exponential.

Work backward from the answer:

$$\cos x \cosh y = \frac{(e^{ix} + e^{-ix})(e^{y} + e^{-y})}{4}$$

$$= \frac{e^{ix}e^{y} + e^{ix}e^{-y} + e^{-ix}e^{y} + e^{-ix}e^{-y}}{4}$$

and then also

$$i \sin x \sinh y = \frac{(e^{ix} - e^{-ix})(e^{y} - e^{-y})}{4}$$

$$= \frac{(e^{ix}e^{y} - e^{ix}e^{-y} - e^{-ix}e^{y} + e^{-ix}e^{-y})}{4}$$

*Subtraction* gives cancelations:

$$= \frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2}$$

And now there's a trick. The exponents of the first product in the numerator add to give

$$ix - y = i(x + iy) = iz$$

the second is

$$-ix + y = -(ix - y) = -iz$$

So we have just

$$\frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \cos z$$

The sine was

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Working with one term at a time, we have

$$\sin x \cosh y = \frac{(e^{ix} - e^{-ix})}{2i} \cdot \frac{(e^{y} + e^{-y})}{2}$$

$$= \frac{e^{ix}e^{y} + e^{ix}e^{-y} - e^{-ix}e^{y} - e^{-ix}e^{-y}}{4i}$$

and

$$i \cos x \sinh y = -\frac{\cos x \sinh y}{i}$$

$$= -\frac{1}{i} \cdot \frac{(e^{ix} + e^{-ix})}{2} \cdot \frac{(e^{y} - e^{-y})}{2}$$

$$= -\frac{e^{ix}e^{y} - e^{ix}e^{-y} + e^{-ix}e^{y} - e^{-ix}e^{-y}}{4i}$$

$$= \frac{-e^{ix}e^{y} + e^{ix}e^{-y} - e^{-ix}e^{y} + e^{-ix}e^{-y}}{4i}$$

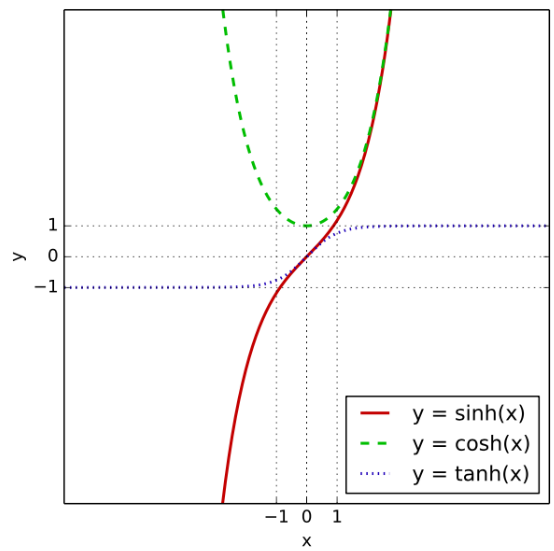
*Addition* gives cancelations:

$$= \frac{e^{ix}e^{-y} - e^{-ix}e^{-y}}{2i}$$

Recall from above that $ix - y = iz$ and $ix + y = -iz$ so

$$= \frac{e^{iz} - e^{-iz}}{2i}$$
$$= \sin z$$

**zeroes**




cosh is never zero, while only $\sinh 0 = 0$.

So if we look again at

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

and ask, where is this function equal to zero?

Both parts must vanish. Since cosh is never zero, $\sin x$ must be zero. This happens for $x = 2k\pi$.

The cosine of this $x$ is equal to 1, that means $\sinh y$ must be 0 which only happens for $y = 0$.

So the zeroes of the complex sine function are at $z = 2k\pi + 0i$.

Alternatively, go back to the original definition:

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

which vanishes only for

$$e^{iz} = e^{-iz} = \frac{1}{e^{iz}}$$

$$e^{2iz} = \left[\ e^{iz}\ \right]^2 = 1$$

$$e^{iz} = \pm 1$$

$$e^{i(x+iy)} = \pm 1$$

$$e^{-y}e^{ix} = \pm 1$$

$$e^{-y}(\cos x + i \sin x) = \pm 1$$

The imaginary part must be zero, so $x = 2k\pi$.

The other part must be equal to $\pm 1$, so $y = 0$ and $\cos 2k\pi = 1$, which works.

For the cosine

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

This is equal to zero when

$$e^{iz} = -e^{-iz} = -\frac{1}{e^{iz}}$$

$$e^{2iz} = -1$$

$$e^{i(x+iy)} = \pm i$$

$$e^{-y}(\cos x + i \sin x) = \pm i$$

In this case we need $\cos x = 0$ and then $y = 0$ and $\sin x = 1$ will work. $x = (2k + 1)\pi/2$.

Recall that

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

Since cosh is never zero, $\cos x$ must be zero. Then either $\sin x = 0$ or $\sinh y = 0$. Only the latter works for the non-imaginary part, so we have that $y = 0$.

**summary**

The definition:

$$\cos z = \frac{1}{2}\ (e^{iz} + e^{-iz})$$

$$\sin z = \frac{1}{2i}\ (e^{iz} - e^{-iz})$$

$$\cosh x = \frac{1}{2}\ (e^{x} + e^{-x})$$

$$\sinh x = \frac{1}{2}\ (e^{x} - e^{-x})$$

A pair of identities

$$\cos iy = \cosh y$$

$$\sin iy = i \sinh y$$

By the sum of angles formula, or by manipulating the exponential

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

# Chapter 12

# More trigonometry

### analyticity

We proved before that the complex exponential obeys the CRE, which means that it is analytic. There is a theorem that says that if we add two analytic functions together, the result is also analytic. Hence, the trigonometric functions are analytic.

But, just to check this result, let's write them out in terms of $u$ and $v$ and see whether the partial derivatives follow the CRE conditions:

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Taking the derivatives:

$$u(x,y) = \sin x \cosh y$$

$$u_x = \cos x \cosh y$$

$$u_y = \sin x \sinh y$$

and

$$v(x,y) = \cos x \sinh y$$

$$v_x = -\sin x \sinh y$$

$$v_y = \cos x \cosh y$$

So we see that indeed

$$u_x = v_y$$

$$u_y = -v_x$$

The CRE are satisfied and therefore, the complex sine is analytic.

Similarly we have that

$$\cos z = \cos(x + iy)$$

$$= \cos x \cos iy - \sin x \sin iy$$

$$= \cos x \cosh y - i \sin x \sinh y$$

So

$$u(x, y) = \cos x \cosh y$$

$$u_x = -\sin x \cosh y$$

$$u_y = \cos x \sinh y$$

and

$$v(x, y) = -\sin x \sinh y$$

$$v_x = -\cos x \sinh y$$

$$v_y = -\sin x \cosh y$$

So we see that

$$u_x = v_y$$

$$u_y = v_x$$

Thus the complex cosine is also analytic.

We can also prove that:

$$\sin^2 z + \cos^2 z = 1$$

The easy way is

$$\cos^2 z + \sin^2 z = \ [\frac{e^{iz} + e^{-iz}}{2}\ ]^2 + [\frac{e^{iz} - e^{-iz}}{2i}\ ]^2\ ]$$
$$= \frac{e^{2iz} + 2 + e^{-2iz} - e^{2iz} + 2 - e^{-2iz}}{4}$$
$$= 1$$

**series**

On the other hand, Shankar defines the trig functions and the exponential using series in the same way as the real versions:

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \cdots = \sum_0^\infty (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \cdots = \sum_0^\infty (-1)^n \frac{z^{2n}}{(2n)!}$$

$$\sinh z = \sum_0^\infty \frac{z^{2n+1}}{(2n+1)!}$$
$$\cosh z = \sum_0^\infty \frac{z^{2n}}{(2n)!}$$

and showing that they converge for any $z$.

**complex hyperbolics**

The definition is analogous to the real case:

$$\cos z = \frac{1}{2}\ [\ e^z + e^{-z}\ ]$$

$$= \frac{1}{2} \left[\ e^{i(x+iy)} + e^{-i(x+iy)}\ \right]$$

$$= \frac{1}{2} \left[\ e^{ix-y} + e^{-ix+y}\ \right]$$

Double the top and the bottom

$$= \frac{e^{ix-y} + e^{-ix+y} + e^{ix-y} + e^{-ix+y}}{4}$$

The pattern in the exponents is

$$+- \quad -+ \quad +- \quad -+$$

We reach a new pattern by first switching the order to

$$-+ \quad +- \quad -+ \quad +-$$

$$= \frac{e^{-ix+y} + e^{ix-y} + e^{-ix+y} + e^{ix-y}}{4}$$

then add and subtract terms with $++$ and $--$, like this:

$$= \frac{e^{ix+y} + e^{-ix+y} + e^{ix-y} + e^{-ix-y}}{4} - \frac{e^{ix+y} - e^{-ix+y} - e^{ix-y} + e^{-ix-y}}{4}$$

Now we realize that we can factor the first term as:

$$= \frac{(e^{y} + e^{-y})}{2} \ \frac{(e^{ix} + e^{-ix})}{2}$$

$$= \cosh y \cos x$$

The second term is:

$$= -\frac{(e^{y} - e^{-y})}{2} \ \frac{(e^{ix} - e^{-ix})}{2}$$

$$
\begin{aligned}
&= -i\frac{(e^{y} - e^{-y})}{2}\ \frac{(e^{ix} - e^{-ix})}{2i} \\
&= -i \sinh y \sin x
\end{aligned}
$$

Putting it all together:

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

That required a lot of bookkeeping, and now we have to go back and repeat it all for the sine. And this is basically a repeat of the derivation in the last chapter. It's just nice to see the factoring trick.

# Chapter 13

# Exponential

Consider first the generic complex number

$$z = x + iy$$

and write

$$\begin{aligned} f(z) &= e^z \\ &= e^{x+iy} \\ &= e^x e^{iy} \end{aligned}$$

We can visualize the complex exponential as having a modulus or length $e^x$ and argument or angle $\theta$ of $y$.

Then, using Euler's formula we can decompose this:

$$\begin{aligned} e^x e^{iy} &= e^x(\cos y + i \sin y) \\ &= e^x \cos y + ie^x \sin y \end{aligned}$$

So if $f(z) = u(x, y) + iv(x, y)$, the real part of $e^z$ is

$$u = e^x \cos y$$

with partial derivatives

$$u_x = e^x \cos y$$

$$u_y = -e^x \sin y$$

and the imaginary part of $e^z$ is

$$v = e^x \sin y$$

with partial derivatives

$$v_x = e^x \sin y$$

$$v_y = e^x \cos y$$

Hence

$$u_x = e^x \cos y = v_y$$

$$u_y = -e^x \sin y = -v_x$$

In other words, these two important conditions hold for the complex exponential:

$$u_x = v_y$$

$$u_y = -v_x$$

These are the famous Cauchy-Riemann equations (CRE) or CR conditions.

When the CRE are satisfied then the function in question is a "good" function — it is one we can do calculus with. It has a derivative.

For this reason, the complex exponential $e^z$ is said to be analytic.

(Which, according to Shankar, we could have predicted, since it depends only on $z$ and not on $z*$).

**derivative**

We showed before that we can evaluate the derivative along $\Delta y = 0$ as:

$$f'(z) = u_x + iv_x$$

We obtain

$$= e^x \cos y + ie^x \sin y = z$$

The exponential is its own derivative.

This is tremendously important because we want our definitions for complex functions to give the standard results when $z$ has only a real part, i.e. when $y = 0$.

Now, once more we recall Euler's formula (for a real variable $\theta$ or $x$):

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{ix} = \cos x + i\sin x$$

Substitute $-x$ for $x$:

$$e^{-ix} = \cos -x + i\sin -x$$

$$= \cos x - i\sin x$$

By addition and subtraction we obtain:

$$2\cos x = e^{ix} + e^{-ix}$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

and

$$2i\sin x = e^{ix} - e^{-ix}$$

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

we'll see a lot more of this coming up.

**alternative derivations**

Another proof that the derivative of the complex exponential is as we would hope and expect:

$$\frac{d}{dz}e^z = e^z$$

uses a Taylor series. Shankar says to define $e^z$ in the same way as $e^x$. For the real series:

$$e^x = \sum_0^\infty \frac{x^n}{n!}$$

which we know converges, since the ratio of successive terms is

$$R = \frac{x^{n+1}}{(n+1)!}\ \frac{n!}{x^n} = \frac{x}{n+1}$$

We ask, for what values of $x$ is the limit

$$\lim_{n\to\infty} R = 0\ ??$$

This is true for all $x$.

For the complex exponential:

$$e^z = \sum_0^\infty \frac{z^n}{n!}$$

and again we see that

$$\frac{d}{dz}\ e^z = e^z$$

differentiating the series term by term.

Another approach (from McMahon) uses the limit definition:

$$\frac{d}{dz}f(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\frac{d}{dz}e^z = \lim_{\Delta z \to 0} \frac{e^{z+\Delta z} - e^z}{\Delta z}$$

and just as in the real case, we factor out

$$= e^z \lim_{\Delta z \to 0} \frac{e^{\Delta z} - 1}{\Delta z}$$

This limit will turn out to be equal to 1.

Use Euler's formula to get this expression in $x$ and $y$:

$$\frac{e^{\Delta z} - 1}{\Delta z} = \frac{e^{\Delta x + i\Delta y} - 1}{\Delta x + i\Delta y}$$

$$= \frac{e^{\Delta x}(\cos \Delta y + i \sin \Delta y) - 1}{\Delta x + i\Delta y}$$

$$= \frac{(e^{\Delta x} \cos \Delta y - 1) + ie^{\Delta x} \sin \Delta y}{\Delta x + i\Delta y}$$

The real part of the numerator is

$$\lim_{\Delta x, \Delta y \to 0} e^{\Delta x} \cos \Delta y - 1$$

Both the $\Delta x$ and the $\Delta y$ term tend to 1 in the limit, so the entire expression for the real part of the numerator is equal to zero. We are left with

$$\lim_{\Delta x, \Delta y \to 0} \frac{ie^{\Delta x} \sin \Delta y}{\Delta x + i\Delta y}$$

The trick is that we actually set $x = 0$ *first*

$$\lim_{\Delta y \to 0} \frac{ie^0 \sin \Delta y}{0 + i\Delta y} = \lim_{\Delta y \to 0} \frac{\sin \Delta y}{\Delta y} = 1$$

and the last part is the famous limit from calculus.

**other properties**

The complex exponential

$$e^z = e^x e^{iy}$$

has some properties that are not shared with the real exponential. As we saw before, the angle $\theta + 2\pi = \theta$ (and $2\pi = 0$), so any angle is really a family of angles with different $\theta + 2\pi k$ for integer $k$.

In particular, $e^z = e^x e^{iy}$ is periodic with a period of $2\pi i$.

Additionally, it is possible for $e^z$ to be negative. Consider that it is possible that

$$e^z = -1$$

as follows. Let

$$z = 0 + i\pi$$

This is a point on the $y$-axis, a distance $\pi$ from the origin, and purely imaginary.

Then

$$e^x = e^0 = 1$$

and

$$e^{iy} = e^{i\pi} = -1$$

So

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y) = 1(-1) = -1$$

On the other hand, $e^z$ **cannot be zero**.

$$e^z = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y)$$

For $x \in \mathbb{R}$, $e^x > 0$.

So the only way this could be zero would be if we can find a $y$ such that $\sin y$ and $\cos y$ were both zero. Since there is no such $y$, we conclude that $e^z$ cannot be equal to zero.

# Chapter 14

# Logarithm

Nearly everything works for the logarithm of $z$ similarly to the real numbers, except for the issue of multiple phase angles or complex arguments. For example

$$\log(z) = \log(re^{i\theta})$$

$$= \log(r) + \log(e^{i\theta})$$

$$= \ln r + i\theta$$

but we may have any multiple of $k \cdot 2\pi$ added to $\theta$

$$\log(z) = \ln r + i(\theta + k2\pi)$$

We call one particular range of $2\pi$ the range for the *principal value* of the function.

Here it is natural to make the range go from $-\pi < \theta < \pi$. The reason is that the negative $x$-axis consists of negative real numbers, for which the natural logarithm isn't defined, and neither is the complex logarithm.

So we exclude that from the domain of the complex logarithm.

This is called a "branch cut," where we take one particular branch of this multi-valued function.

Here is a derivation.

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$r = |z| = \sqrt{x^2 + y^2}$$

The logarithm of $z$ is $w$

$$w = \log z \iff e^w = z$$

So what about $w$? Well, in general, it's a complex number

$$w = s + it$$

so

$$e^w = e^{s+it} = e^s(\cos t + i\sin t)$$

Equating the two we get

$$r(\cos\theta + i\sin\theta) = e^s(\cos t + i\sin t)$$

Hence

$$s = \ln r$$

$$t = \theta$$

$$w = \ln r + i\theta$$

**different base**

What is

$$i^i = ?$$

The complex logarithm of $i$ is

$$\log i = \ln r + i\theta = \ln 1 + i\frac{\pi}{2} = i\frac{\pi}{2}$$

Write

$$a^z = (e^{\log a})^z$$

$$i^i = (e^{\log i})^i = (e^{i\pi/2})^i = e^{-\pi/2}$$

Not only is $i$ to the $ith$ power computable, it is entirely real. It is $\approx 0.2079$.

**derivative**

When we study the Cauchy-Riemann equations we will show that if $f(z)$ is differentiable, then the CRE hold. The converse theorem is also true, that if the CRE hold, then $f(z)$ is differentiable, and its derivative is

$$f'(z) = u_x + iv_x$$

We have that the logarithm function is

$$\log(z) = \ln|z| + i\theta$$

Rewriting in terms of $x$ and $y$ we have that

$$\log(x+iy) = \ln(\sqrt{x^2+y^2}) + i\tan^{-1}(\frac{y}{x})$$

$$\log(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}(\frac{y}{x})$$

So

$$u(x,y) = \frac{1}{2}\ln(x^2+y^2)$$

$$u_x = \frac{1}{2}\ \frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$$

$$u_y = \frac{y}{x^2+y^2}$$

and

$$v(x,y) = \tan^{-1}(\frac{y}{x})$$

$$v_x = \frac{1}{1+(y/x)^2}\ y\ (-\frac{1}{x^2}) = \frac{-y}{x^2+y^2}$$

$$v_y = \frac{1}{1+(y/x)^2}\ \frac{1}{x} = \frac{x}{x^2+y^2}$$

We see that CRE are satisfied and that means that the derivative is

$$[\ \log z\ ]' \ = u_x + i v_x$$

$$= \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

$$= \frac{1}{x^2+y^2}(x - iy)$$

$$= \frac{1}{|z|^2}\ z*$$

$$= \frac{1}{zz*}\ z* = \frac{1}{z}$$

The derivative of the complex logarithm is the inverse of $z$, completely analogous to the real case.

# Chapter 15

# Summary 1

We have expressions $f(z) = u(x, y) + iv(x, y)$ for all standard complex functions.

Powers can be computed easily

$$z = x + iy$$

$$(x + iy)^2 = x + 2x(iy) + (iy)^2$$
$$(x + iy)^3 = x + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$
$$\dots$$

We worked with the inverse function using the complex conjugate $z* = x - iy$ so

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{z*}{z*}$$

Exponential:

$$e^z = e^{x+iy} = e^x(\cos x + i \sin x)$$

Sine and cosine:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$= \cos x \cosh y - i \sin x \sinh y$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$= \sin x \cosh y + i \cos x \sinh y$$

Logarithm:

$$w = \log z = \ln r + i\theta$$

**derivatives**

We showed that all of these functions are analytic, with $u_x = v_y$ and $u_y = -v_x$, so therefore, their derivatives can be computed as $f'(z) = u_x + iv_x$.

When this is done, they turn out to be just what you'd want:

$$(z^n)' = nz^{n-1}$$
$$(e^z)' = e^z$$
$$(\sin z)' = \cos z$$
$$(\cos z)' = -\sin z$$
$$(\log z)' = \frac{1}{z}$$

The only ones we haven't done are the roots.

We will not do a general proof, but let's go through the square root. It will remind us of the special features of polar CRE and derivatives.

**square root**

$$f(z) = \sqrt{z}$$

Use polar notation so $z = re^{i\theta}$ and then

$$f(z) = \sqrt{z} = \sqrt{r}e^{i\theta/2}$$

Using Euler:

$$= \sqrt{r}(\cos\theta/2 + i\sin\theta/2)$$

$$u = \sqrt{r}\cos\theta/2$$

$$v = \sqrt{r}\sin\theta/2$$

The partials are:

$$u_r = \frac{1}{2\sqrt{r}}\cos\theta/2$$

$$u_\theta = \frac{\sqrt{r}}{2}(-\sin\theta/2)$$

and

$$v_r = \frac{1}{2\sqrt{r}}\sin\theta/2$$

$$v_\theta = \frac{\sqrt{r}}{2}\cos\theta/2$$

At first we're worried ($u_r \neq v_\theta$), but then we recall the polar CRE have an extra factor of $r$:

$$ru_r = v_\theta$$

$$rv_r = -u_\theta$$

So the CRE do obtain, and we can get the derivative.

Next, we recall the second unusual thing about the polar derivative:

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

Leave aside the factor of $e^{-i\theta}$ out front and just combine:

$$u_r + iv_r = \frac{1}{2\sqrt{r}} \cos\theta/2 + \frac{1}{2\sqrt{r}} \sin\theta/2$$

$$= \frac{1}{2\sqrt{r}} \ (\cos\theta/2 + i\sin\theta/2)$$

$$= \frac{e^{i\theta/2}}{2\sqrt{r}}$$

which appears problematic, but the extra factor gives us just what we need

$$f'(z) = e^{-i\theta} \cdot \frac{1}{2\sqrt{r}} \ e^{i\theta/2}$$

$$= \frac{1}{2\sqrt{r}} \ e^{-i\theta/2}$$

$$= \frac{1}{2\sqrt{r}} \cdot \frac{1}{e^{i\theta/2}}$$

$$= \frac{1}{2\sqrt{z}}$$

Just exactly analogous to the real function.

# Part IV

# Integration

# Chapter 16

# Integration

**summary**

$$\int f(z)\ dz$$

$$= \int (u + iv)(dx + idy)$$

$$= \int u\ dx - v\ dy + i\ [\ u\ dy + v\ dx\ ]$$

where $x$ and $y$ are *related*, either as parametric equations in $t$ or $\theta$, or simply because $y = f(x)$.

**Introduction**

Complex functions are differentiated and integrated in a way that is similar to real functions, but with some differences. We've already seen that the derivatives are like their real cousins, and the integrals will be too.

However, there are some differences. A main one is that the integrals are line integrals. We will set up integrals which look like multi-variable integrals, but the two variables are connected because they lie on a path. Because they are line integrals and most integrals will evaluate to something like $e^{it}$, going around on a circle in a closed path usually gives a value of zero.

We often but not always restrict our attention to functions that are analytic, paying attention to points in the complex plane where they have poles (or singularities).

A complex function is a function that produces a complex number as the result. The most general case is that the input is a complex number as well.

Se we could write:

$$w = f(z)$$

where both $w$ and $z$ are complex numbers.

The parts of $f$ that generate the real and imaginary parts of $w$ as separate functions of two *real* variables $x$ and $y$:

$$w = f(z) = u(x,y) + iv(x,y)$$

**getting started**

The function is

$$w = f(z) = u(x,y) + iv(x,y)$$

The input, the complex variable $z$ is

$$z = x + iy$$

$$dz = dx + idy$$

The key is to write the integral as

$$\int f(z)\ dz = \int (u + iv)(dx + idy)$$

Now group the pieces as:

$$= \int u\ dx - v\ dy + iv\ dx + iu\ dy$$

The integral of a complex function is defined as a sum of integrals of two real variables. Just as with line integrals for real functions of $x$ and $y$, the variables are related by the curve over which we will integrate.

Recall that for the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\ dx + N\ dy$$

we parametrize the curve to get the integral over a single variable.

We can view $y$ as a function of $x$ or perhaps, we will be able to parametrize both $x$ and $y$ as functions of $t$.

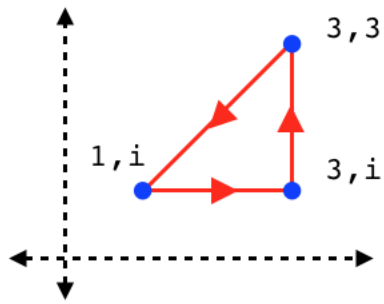
**example:** $z$

Suppose our function is simply $z = x + iy$. The integral is

$$\int z\ dz = \int (x + iy)(dx + idy)$$

$$= \int x\ dx - y\ dy + ix\ dy + iy\ dx$$

Now we must get $y$ in terms of $x$ from the curve. Suppose the curve goes from $(1, i)$ to $(3, i)$, then to $(3, 3i)$ and finally back to where we started.




We have three segments. Along the first part, we are moving in the positive $x$ direction, with no change in $y$, $y = 1$, a constant, so $dy = 0$, and the integral is (writing the non-zero parts only):

$$= \int x \ dx + i \cdot 1 \ dx$$

$$= \int_{x=1}^{x=3} x + i \ dx$$

$$= \frac{x^2}{2} + ix \ \Big|_1^3$$

$$= 4 + 2i$$

Along the second part, we are moving in the positive $y$ direction with $dx = 0$ and $x = 3$ so

$$= \int_{y=1}^{y=3} -y \ dy + 3i \ dy$$

$$= -\frac{y^2}{2} + 3iy \ \Big|_1^3$$

$$= -4 + 6i$$

Along the third path, both $dx$ and $dy$ are non-zero, so we must actually do the parametrization. The curve is $y = x$. Hence $dy = dx$.

$$= \int x\ dx - x\ dx + ix\ dx + ix\ dx$$

$$= 2i \int x\ dx$$

For the closed path, where we end up back at the starting point, $C3$ should be moving from $(3,3)$ to $(1,1)$ so we have

$$2i\ \frac{x^2}{2}\ \Big|_3^1 = 2i(-\frac{8}{2}) = -8i$$

Notice that

$$\int_{C1} + \int_{C2} = 8i = -\int_{C3}$$

If we follow the curve $C3$ from $(3,3)$ to $(1,1)$, the whole thing is just zero. We'll see that this is not a coincidence.

**example 2**

Suppose the function is

$$f(z) = y - x - i3x^2$$

So $u = (y - x)$ and $v = -3x^2$ and the integral is

$$= \int u\ dx - v\ dy + i\ [\ u\ dy + v\ dx\ ]$$

$$\int (y-x)dx + 3x^2 dy + i\ [\ (-3x^2)dx + (y-x)dy\ ]$$

We proceed from the origin to the point $z = 1 + i$ either directly $(C)$ or by first going up vertically $(C1)$ and then across $(C_2)$.

For the vertical part ($C_1$) we have that $x = 0$ and $dx = 0$.

$$\int (y-x)dx + 3x^2dy + i(-3x^2)dx + i(y-x)dy$$

$$I_1 = \int i(y-x)dy = \int iydy$$

It's important to recognize that although we are proceeding from the point $z = 0$ to the point $z = i$, the upper bound on this integral is not $i$ but $y = 1$! Hence

$$I_1 = i\frac{y^2}{2}\Big|_0^1 = \frac{i}{2}$$

For the horizontal part ($C_2$) we have that $y = 1$ and $dy = 0$ so

$$I = \int (y-x)dx + i(-3x^2)dx$$

$$= \int (1-x)dx + i(-3x^2)dx$$

$x$ goes from 0 to 1

$$= x - \frac{x^2}{2} - ix^3\Big|_0^1 = \frac{1}{2} - i$$

Therefore the total

$$I = \frac{i}{2} + \frac{1}{2} - i = \frac{1}{2}\,(1-i)$$

When going directly from the origin to $1+i$ we relate $x$ to $y$ by the equation of the line $y = x$ so $dy = dx$ and

$$I = \int (y-x)dx + 3x^2dy + i(-3x^2)dx + i(y-x)dy$$

$$= \int 3x^2 dx + i(-3x^2) dx$$

$$= x^3 \ - i \ x^3 \ \Big|_0^1 = 1 - i$$

And around the closed curve going backward along $C$:

$$I1 = i \ \frac{1}{2}$$

$$I2 = \frac{1}{2} - i$$

$$I3 = 1 - i$$

That last one is in the direction $0 \to 1$ so we must subtract it:

$$\oint f(z) \ dz = i\frac{1}{2} + \frac{1}{2} - i - (1 - i) = -\frac{1}{2} + i \ \frac{1}{2}$$

This time, even though we returned to our starting point (traversing a *closed* path), the result is not zero.

Notice that

$$f(z) = y - x - i3x^2$$

$$u_x = -1 \neq v_y = 0$$

This function is not analytic.

**example: $z$, revisited**

Above we wrote:

$$\int z \ dz = \int x \ dx - y \ dy + ix \ dy + iy \ dx$$

And then said: now we must get $y$ in terms of $x$ from the curve.

But what if we don't worry about the curve?

Just write $y = f(x)$ and $dy = f'(x)\ dx$ and see what happens:

$$\int x\ dx - f(x)\ f'(x)\ dx + i\ [\ x\ f'(x)\ dx + f(x)\ dx$$

It helps that we know the answer:

$$\frac{1}{2}z^2 = \frac{1}{2}(x^2 - y^2 + i2xy)$$

$$\int x\ dx = \frac{x^2}{2}$$

then

$$\int f(x)\ f'(x)\ dx$$

but this is

$$\frac{1}{2}\ [f(x)]^2 = \frac{1}{2}y^2$$

We're on to something!

For the imaginary part:

$$\int x\ f'(x)\ dx + f(x)\ dx$$

the integrand is the derivative

$$[\ x\ f(x)\ ]'$$

by the product rule. But that's just $xy$, so this is a match.

# Chapter 17

# Circular paths

If the contour (curve) of integration $C$ is parametrized in terms of $t$, then

$$\int_C f(z)\ dz = \int_a^b f[z(t)]\ z'(t)\ dt$$

A particularly important parametrization is for circular paths. On such a path, $z$ takes on values with constant $r$ and the only change is in $\theta$.

Often we are interested in closed circular paths, where we go around the whole circle and end up where we started.

**example:** $z$

Suppose we have

$$z = re^{i\theta}$$

$$dz = rie^{i\theta} d\theta$$

so

$$\int z\ dz = \int re^{i\theta}\ rie^{i\theta} d\theta$$

$$= r^2 i \int e^{i2\theta}\ d\theta$$
$$= r^2 i\ \frac{1}{2i} e^{i2\theta} = \frac{1}{2}\ r^2 e^{i2\theta}$$
$$= \frac{1}{2}\ z^2$$

For any closed path, the starting and ending $z$ are the same, so the value is zero. Alternatively, the result can be written $\rho e^{i\phi}$, but that exponential is really two trig functions, that have a period of $2\pi$ so evaluated over any closed path, the result is zero.

For the unit circle, evaluated between $\theta = 0 \to \pi/2$ we get

$$\frac{1}{2}\ e^{i2\theta}\Big|_0^{\pi/2} = \frac{1}{2}(\ e^{i\pi} - 1)$$
$$= \frac{1}{2}(\cos\pi - i\sin\pi - 1)$$
$$= \frac{1}{2}(-1 - 1) = -1$$

**example: $z$ in terms of $(x, y)$**

Let's repeat the calculation for $\int z\ dz$ in terms of $u, v$ and $dx, dy$:

$$\int u\ dx - v\ dy + i\ [\ u\ dy + v\ dx\ ]$$

Substitute from $u = x, v = y$:

$$\int x\ dx - y\ dy + i\ [\ x\ dy + y\ dx\ ]$$

Consider a path around the unit circle. Then

$$x = \cos t$$

$$y = \sin t$$

$$dx = -\sin t \; dt$$

$$dy = \cos t \; dt$$

substitute in the integral

$$\int \cos t(-\sin t \; dt) - \sin t \cos t \; dt + i \; [\; \cos^2 t \; dt - \sin^2 t \; dt \;]$$

$$-2 \int \sin t \cos t \; dt + i \int \cos^2 t - \sin^2 t \; dt$$

If this were not a unit circle, each term would have a factor of $r$, one for the $u, v$ part and one for the differential, so we would just have a leading factor of $r^2$.

The real part is $\int u \; du = u^2/2$ so

$$I_{re} = -\sin^2 t$$

If you can't remember $\int \cos^2$ why don't we just guess:

$$[\; \sin t \cos t \;]' = \cos^2 t - \sin^2 t$$

Looks pretty good! So

$$I_{im} = \sin t \cos t$$

Putting it together we get

$$\int z \; dz = -\sin^2 t + i \sin t \cos t$$

Since the trig functions have a period of $2\pi$, any path that is closed has that period, and the value of the integral will be zero.

Otherwise, in say, the first quadrant where $t = 0 \to \pi/2$, at the upper bound we have $-1$ and at the lower bound we have 0, so the value of the integral is $-1$, which matches what we got before.

**example:** $\int z^*$

Consider $f(z) = z*$.

This function is of course *not* analytic, because it involves $z*$ rather than $z$, and also because

$$z^* = x - iy$$

so

$$u_x = 1 \neq v_y = -1$$

The CRE do not hold.

Suppose our curve is the circle of radius $r$ centered at the origin, and we proceed between the endpoints $z = -ri \rightarrow ri$ (from due south to east to due north). On this half-circle

$$z = re^{i\theta}$$

we have then

$$dz = i\ re^{i\theta}\ d\theta$$

In radial coordinates

$$z^* = re^{-i\theta}$$

so then

$$\int z^*\ dz = \int re^{-i\theta} rie^{i\theta}\ d\theta$$

$$= r^2 i \int_{-\pi/2}^{\pi/2} d\theta = r^2 \pi i$$

Alternatively,

$$zz^* = |z|^2 = r^2$$

$$z^* = \frac{r^2}{z}$$

$$\int z^*\ dz = r^2 \int \frac{1}{z}\ dz$$

Again

$$z = re^{i\theta}$$

$$dz = iz\ d\theta$$

So the integral is just

$$= r^2 \int \frac{1}{z}\ iz\ d\theta$$

$$= r^2 i \int d\theta = r^2 \pi i$$

over the half-circle

**example:** $\int z^2$

Let $f(z) = z^2$.

For the path, take the unit circle over the first quadrant from $(1, 0)$ to $(0, 1)$. This is only a part of a circle, so the integral will be non-zero.

There is are easy ways to do this, and there is a hard way.

Let's start by re-checking that this function is analytic, and then do the hard way first.

Write $z$ in terms of $x$ and $y$:

$$z = x + iy$$

$$z^2 = (x + iy)^2 = x^2 - y^2 + i2xy$$

$$u = (x^2 - y^2)$$

$$v = 2xy$$

$$u_x = 2x = v_y$$

$$u_y = -2y = -v_x$$

The CRE hold.

Also

$$dz = dx + i\ dy$$

So then

$$\int z^2\ dz = \int u\ dx - v\ dy + i\ [\ v\ dx + u\ dy\ ]$$

$$= \int (x^2 - y^2)\ dx - \int 2xy\ dy + i \int 2xy\ dx + i \int (x^2 - y^2)\ dy$$

As before, we must parametrize this using the relationship between $x$ and $y$ along the curve.

$$x = \cos t$$

$$y = \sin t$$

so

$$x^2 - y^2 = \cos^2 t - \sin^2 t = 2\cos^2 t - 1$$

$$2xy = 2\cos t \sin t$$

and

$$dx = -\sin t\ dt$$

$$dy = \cos t\ dt$$

To keep things straight, write the integral in its four separate parts:

$$I_1 = \int (2\cos^2 t - 1)\ (-\sin t\ dt)$$

$$I_2 = -\int 2\cos t \sin t\ (\cos t\ dt)$$

leaving off the $i$

$$I_3 = \int 2\cos t \sin t(-\sin t\ dt)$$

$$I_4 = \int (2\cos^2 t - 1)\ (\cos t\ dt)$$

$I_4$ can also be written as $(\cos^2 t - \sin^2 t)(\cos t\ dt)$.

It looks pretty wild, but really, these are fairly easy integrals, since we can use $u$ substitution. Let's make a table for reference:

$$\int \cos^2 t\ (-\sin t\ dt) = \frac{1}{3}\cos^3 t$$

$$\int \sin^2 t\ (\cos t\ dt) = \frac{1}{3}\sin^3 t$$

$$\int \cos^3 t\ dt = \int (1 - \sin^2 t)\ \cos t\ dt = \sin t - \frac{1}{3}\sin^3 t$$

So now we can then just copy the results into the integrals we set up:

$$I_1 = \frac{2}{3}\cos^3 t - \cos t$$

$$I_2 = \frac{2}{3}\cos^3 t$$

The real part is then

$$\frac{4}{3}\cos^3 t - \cos t$$

and

$$I_3 = -\frac{2}{3}\sin^3 t$$

$$I_4 = 2\sin t - \frac{2}{3}\sin^3 t - \sin t$$

The imaginary part is then

$$i\ [\ -\frac{4}{3}\sin^3 t + \sin t\ ]$$

There is some symmetry.

At the upper limit, the cosine is zero and the sine is 1 so the imaginary part is only non-zero at the upper limit and we get just $-1/3\ i$.

The real part is only non-zero at the lower limit, that gives $1/3$ but it's the lower limit so we subtract, and the answer is

$$-\frac{1}{3} - \frac{1}{3}i = -\frac{1}{3}(-i-1)$$

**easy way**

One is to just treat $z$ as if it were a real variable

$$\int z^2\ dz = \frac{1}{3}\ z^3\ \Big|_1^i = \frac{1}{3}(-i-1)$$

If we go all the way around the unit circle the integral is zero.

**update**

Going back to the first example in the previous chapter we had

$$\int z\ dz = \frac{z^2}{2}\ \Big|_{1+i}^{3+3i}$$

$$= \frac{9 - 9 + 18i -\ \ [\ 1 - 1 + 2i\ ]}{2}$$

$$= 8i$$

**example:** $1/z$

$$\int_0^{2\pi} \frac{1}{z}\ dz$$

Examining the inverse function, you might want to first confirm that it is analytic by calculating the partial derivatives. We did this already so I'll skip it.

If we are on the unit circle, then

$$z = e^{i\theta}$$

$$dz = ie^{i\theta} d\theta = iz \; d\theta$$
$$\int \frac{dz}{z} = \int \frac{iz}{z} \; d\theta = i \int d\theta = 2\pi i$$

Pretty and pretty easy!

The inverse is an example of a function that *is* analytic, yet the integral around a closed curve that includes the origin is not equal to zero, it is instead equal to $2\pi i$. The reason is connected to the fact that the function is not defined at the $z = 0$.

If we're centered on the origin but we don't have a unit circle, there will be an $R$ in both the numerator and the denominator, which cancel.

*The result is thus independent of the radius of the circle.*

Other powers:

In general
$$\oint_C \frac{dz}{(z - z_0)^n} = \begin{cases} 0, & n \neq 1 \\ 2\pi i, & n = 1 \end{cases}$$
We will see examples for $n \neq 1$ below.

**inverse in terms of $x$ and $y$**

We can also integrate the inverse function in terms of $x$ and $y$:

$$\frac{1}{z} = \frac{z*}{zz*} = \frac{x - iy}{x^2 + y^2}$$

So

$$u = \frac{x}{x^2 + y^2}$$

$$v = \frac{-y}{x^2 + y^2}$$

The integral is

$$\int u\ dx - v\ dy + i\ [v\ dx + u\ dy$$

$$= \int \frac{1}{x^2 + y^2}\ [\ x\ dx + y\ dy + i\ (\ -y\ dx + x\ dy)\ ]$$

Suppose we go on a circle of radius $R$ centered on the origin and parametrize in terms of $\theta$. We obtain:

$$x = R \cos \theta$$

$$y = R \sin \theta$$

$$x^2 + y^2 = R^2$$

and

$$dx = -R \sin \theta\ d\theta$$

$$dy = R \cos \theta\ d\theta$$

We have for the integral

$$\frac{1}{R^2} \int x\ dx + y\ dy + i\ [\ -y\ dx + x\ dy\ ]$$

Each term $x$, $y$, $dx$ and $dy$ has a factor of $R$. So factor that out and cancel what's in front. Then substitute the trig functions:

$$\int \cos \theta\ (-\sin \theta\ d\theta) + \sin \theta\ (\cos \theta\ d\theta) + i\ [\ -\sin \theta\ (-\sin \theta\ d\theta) + \cos \theta\ (\cos \theta\ d\theta)\ ]$$

The real part has a big cancelation:

$$\int i\ [\ -\sin \theta\ (-\sin \theta\ d\theta) + \cos \theta\ (\cos \theta\ d\theta)\ ]$$

and so does the imaginary part:

$$= \int i \ d\theta = 2\pi i$$

Note that if we integrate the same function around a unit square, we run into problems. First let's do $[0, 0 \times 1, 1]$. We have

$$\int u \ dx - \int v \ dy + i \ [ \ \int v \ dx + \int u \ dy \ ]$$

Along $C1$, $y = 0$ and $dy = 0$ so:

$$\int \frac{x}{x^2 + y^2} \ dx + i \ [ \ \int \frac{-y}{x^2 + y^2} \ dx$$

$$= \int_0^1 \frac{1}{x} \ dx = \ln x \ \Big|_0^1$$

Since $\ln 0$ is not defined, we can't do this.

Logarithms are tricky, no doubt.

If the complex logarithm *Log* $z$ is defined and differentiable along the curve (and it is, along the semicircle from $-i$ to $i$), we can do this:

$$I = \int_{-i}^{i} \frac{1}{z} \ dz = \ \text{Log} \ z$$

To evaluate this, recall that $z = e^{i\theta}$ recognize that at $z = -i$, $\theta = -\pi/2$, while for $z = i$, $\theta = \pi/2$:

$$= i\theta \ \Big|_{-\pi/2}^{\pi/2} = i \ \frac{\pi}{2} - i \frac{-\pi}{2} = 2i \ \frac{\pi}{2} = \pi i$$

If we're on the unit circle, then we need not worry about the real part of Log $z = \ln r + i\theta$, since $\ln 1$ is equal to zero. Also, if we're centered on the origin, then $r$ is a constant and we get the same number at both bounds, so the difference is zero even if the path is not a full circle.

**powers**

We can extend this to

$$\int \frac{1}{z^2} \ dz$$

As before

$$dz = iz \ d\theta$$

so this is

$$\int \frac{1}{z^2} \ iz \ d\theta = i \int \frac{1}{z} \ d\theta$$

On the unit circle

$$z = e^{i\theta}$$

The integral is

$$i \int_0^{2\pi} e^{-i\theta} \ d\theta$$

Now from Euler

$$e^{i\theta} = \cos\theta + i\sin\theta$$

For any closed path, the difference between the bounds is $2\pi$ which is also the period of these functions, so the difference is always zero.

In fact, for any negative integer power of $z$

$$\int z^{-n} \ dz$$

around the unit circle $z = e^{i\theta}$ we have

$$i \int e^{-i(n-1)\theta} \ d\theta$$

$$= -\frac{1}{n-1} \ e^{-i(n-1)\theta} \ \Big|_0^{2\pi}$$

It doesn't matter what $n - 1$ is, the value of the integral is just zero.

**example: square root**

Consider

$$\int \sqrt{z}\ dz$$

along the half-circle of radius 3 starting from the point $z = R$ on the $x$-axis and proceeding counter-clockwise. We can do this integral even if the "branch" of the square root function that we're using is only defined for $\theta > 0$.

We have that

$$\begin{aligned} z = Re^{i\theta}, &\quad \theta = 0 \to \pi \\ dz &= iz\ d\theta \\ &= iRe^{i\theta}\ d\theta \end{aligned}$$

$$\sqrt{z} = \sqrt{R}e^{i\theta/2}$$

so

$$I = \int_0^\pi iR\sqrt{R}e^{i3\theta/2}\ d\theta$$

We need

$$\int e^{i3\theta/2}\ d\theta = \frac{2}{3i}e^{i3\theta/2}\ \Big|_0^\pi$$

easiest to write it out as

$$\begin{aligned} e^{i3\theta/2}\ \Big|_0^\pi &= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} - \cos 0 - i\sin 0 \\ &= 0 + i(-1) - 1 - 0 = -(1+i) \end{aligned}$$

Going back to pick up all the factors we left behind:

$$I = -iR\sqrt{R}\ \frac{2}{3i}\ (1+i) = -R\sqrt{R}\ \frac{2}{3}\ (1+i)$$

In the problem, $R$ was actually specified as 3, leading to the cancellation:

$$I = -2\sqrt{3}\ (1+i)$$

We can also do this problem by antiderivatives:

$$\int_R^{-R} \sqrt{z}\ dz = \frac{2}{3}\ z^{3/2}\ \Big|_R^{-R}$$

$$= \frac{2}{3}(R^{3/2}e^{i3\pi/2} - R^{3/2}e^0)$$

$$= \frac{2}{3}R^{3/2}(e^{i3\pi/2} - 1)$$

and, as we showed above:

$$e^{i3\pi/2} = -i$$

If $R = 3$ we get the same answer as before.

**summary**

For an analytic function, we can compute the integral by analogy with the real numbers: $\int z\ dz = z^2/2$.

For any closed path, the result is just zero, with some special exceptions.

$z^*$ is special because it's not analytic. $1/z$ is said to be special because it's not defined at $z = 0$, though it is analytic, but this begs the question, what about $1/z^2$?

I would rather say that $1/z$ is special because it is $z^*$ in disguise since:

$$\frac{1}{z}\ \frac{z^*}{z^*} = \frac{1}{r^2}\ z^*$$

with $r$ just a constant.

Another reason is that (on the unit circle) we have

$$z = e^{i\theta}$$

so

$$dz = iz\ d\theta$$

and

$$\int \frac{1}{z}\ dz = \int i\ d\theta$$

The $z$ cancels.

For any other power of $z$ we will have some factor of $e^{ki\theta}$ at the end. Since this is a combination of sine and cosine, it will give zero when integrated over a closed path since $\theta = 0 \rightarrow 2\pi$ and the trig functions have a period of $2\pi$ no matter where you pick for the starting point (no matter what the value of $k$ is).

# Chapter 18

# Cauchy theorem

Cauchy's theorem says that the integral of an analytic function over a closed path is equal to zero:

$$\oint_C f(z)\ dz = 0$$

There is an important restriction: the enclosed region must not contain a singularity.

This will turn out to be a consequence of Green's Theorem, which you may remember from multivariable calculus.

Let

$$z = x + iy$$

$$dz = dx + idy$$

$$z = f(x, y) = u(x, y) + iv(x, y)$$

Our integral is

$$= \oint u\ dx - v\ dy + iv\ dx + iu\ dy$$

**proof of Cauchy's theorem**

Back in vector calculus we proved Green's theorem, which says that for two real functions of $x$ and $y$: $M(x, y)$ and $N(x, y)$:

$$\oint_C M dx + N dy = \iint_R (N_x - M_y)\ dx\ dy$$

Back then, $M$ and $N$ were components of a vector field $\mathbf{F}$ and we wrote the shorthand for curl:

$$= \iint_R \nabla \times \mathbf{F}\ dA$$

but the important thing is that the theorem applies to real-valued functions of two real variables $f : \mathbb{R}^2 \to \mathbb{R}^1$, and so it applies to functions like $u(x, y)$ and $v(x, y)$.

Consider the real part of the integral above:

$$I_{Re} = \oint u\ dx - v\ dy$$

Let $M = u$ and let $N = -v$ (notice the minus sign!). Then

$$I_{re} = \oint M\ dx + N\ dy$$

This is equal to a double integral containing:

$$N_x - M_y$$

We have that $N = -v$ so $N_x = -v_x$. And this is equal to $u_y$ by the CRE.

But $M = u$ so $M_y = u_y$. The two terms in the subtraction are equal so the result is zero. Hence, this integral is zero.

For the imaginary part

$$I_{Im} = i \oint v \ dx + u \ dy$$

Let $N = u$ and let $M = v$ (no minus sign!). Then

$$I_{re} = \oint M \ dx + N \ dy$$

This is equal to a double integral containing:

$$N_x - M_y$$

But $N_x = u_x$ and $M_y = v_y$ and these terms are equal by the CRE. Therefore this expression is zero.

So the integral for the imaginary part is also zero, and thus the whole thing is zero as well:

$$\oint u \ dx - v \ dy + i \ [ \ v \ dx + u \ dy \ ] \ = 0$$

Remember how important it was (for Green's theorem) that the function being integrated be defined everywhere in the region. Well, it's true here as well.

$$\oint_C \frac{1}{z} \ dz \stackrel{?}{=}, \qquad \neq 0$$

We've already seen by direct calculation that this integral is *not* zero when the curve $C$ includes the origin, although it zero otherwise.

To repeat the demonstration for the former case we use the unit circle centered at the origin. Write

$$z = re^{i\theta}$$

$$\frac{dz}{d\theta} = rie^{i\theta} = iz$$

Hence

$$\oint_C \frac{1}{z}\ dz = \oint_C \frac{1}{z}\ iz\ d\theta$$
$$= i\oint_C d\theta = 2\pi i$$

**Path independence**

The theorem that says the integral of an analytic function over a closed path (over a region without a singularity), is equal to zero.

$$\oint_C f(z)\ dz = 0$$

This result means, in turn, that the integral of an analytic function between two points $z_1$ and $z_2$ is independent of the path taken. Call the two paths $C1$ and $C2$.

Form the closed path by going from $z_1$ to $z_2$ over $C1$ and then return to $z_1$ by going backward over $C2$. The total integral is equal to zero by Cauchy's Theorem.

$$\int_{C1} f(z)\ dz + \int_{-C2} f(z)\ dz = 0$$

But the integral over the path $C_2$ in the forward direction is just minus the integral over the reverse path $-C2$.

$$\int_{-C2} f(z)\ dz = -\int_{C2} f(z)\ dz$$

Thus

$$\int_{C1} f(z)\ dz - \int_{C2} f(z)\ dz = 0$$

and

$$\int_{C1} f(z)\ dz = \int_{C2} f(z)\ dz$$

# Chapter 19

# Cauchy formula

If we can write an integral in this form:

$$\oint_C \frac{f(z)}{z - z_0} \ dz$$

where $f(z)$ is analytic and defined everywhere in the domain we care about, with this composite function of course not defined at $z = z_0$, we can show that the value of the integral is

$$\oint_C \frac{f(z)}{z - z_0} \ dz = 2\pi i \ f(z_0)$$

This is called the Cauchy Integral formula.

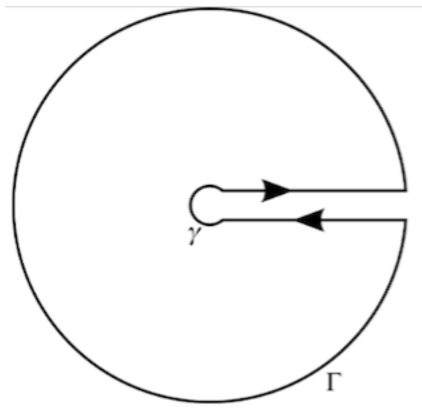
### setup

Suppose $f(z)$ is analytic and defined everywhere within some region *except* at a singularity, $z_0$. For example, suppose we have

$$\frac{f(z)}{z - z_0}$$

We integrate this function around a special closed path in the region of analyticity:

$$\oint \frac{f(z)}{z - z_0}\, dz$$




It's not labeled (I didn't draw the figure) but the singularity $z_0$ is at the center of the two concentric circles. The "keyhole" excludes $z_0$ so $f$ is analytic everywhere in the region enclosed by the path.

Cauchy's integral theorem tells us that the total integral is zero.

The straight line segments are identical but traversed in opposite directions, so the net contribution from them is zero.

Therefore, we have that the integral around the outer ring counter-clockwise + the integral around the inner ring clockwise add up to zero.

But reversing the direction of integration on the inner ring (so both paths go in the counter-clockwise direction) changes the sign of the value, hence we have that

$$\oint_{C \text{ outer}} \frac{f(z)}{z - z_0}\, dz - \oint_{C \text{ inner}} \frac{f(z)}{z - z_0}\, dz = 0$$

and

$$\oint_{C \text{ outer}} \frac{f(z)}{z - z_0}\ dz = \oint_{C \text{ inner}} \frac{f(z)}{z - z_0}\ dz$$

We haven't said anything about the radius of these rings.

What that means is that the value of the integral around a ring enclosing a singularity is not zero, but its value is independent of the radius.

**derivation**

We can parametrize this path by realizing that each point on one of these curves is given by

$$z = z_0 + re^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

$$z - z_0 = re^{i\theta}$$

This is a circle with the center displaced to $z_0$.

Since $z_0$ is a constant

$$dz = ire^{i\theta} d\theta$$

so, substituting for $re^{i\theta}$ above we obtain

$$dz = i(z - z_0)\ d\theta$$

and

$$\oint \frac{f(z)}{z - z_0}\ dz = \oint f(z)\ i\ d\theta$$

$$= i \int_0^{2\pi} f(z)\ d\theta$$

This holds for every circular path enclosing $z_0$. We may choose $r$ as small as we like, and so we choose it very small ($r \to 0$) so

$$f(z) \to f(z_0) = \text{constant}$$

and in that limit, since it's constant we can bring it out from under the integral sign!

$$i \int_0^{2\pi} f(z)\ d\theta$$

$$= i f(z_0) \int_0^{2\pi} d\theta = 2\pi i\ f(z_0)$$

An alternative (equivalent) approach is to say that for a small enough circle

$$\int \frac{f(z)}{z - z_0}\ dz$$

the value of $f(z)$ approaches $f(z_0)$, a constant so this becomes

$$= f(z_0) \int \frac{1}{z - z_0}\ dz$$

We know that integral, it is $2\pi i$.

Summarizing:

$$\oint \frac{f(z)}{z - z_0}\ dz = 2\pi i\ f(z_0)$$

What this means is that we can evaluate the integral in question by simply plugging in the value of the function at $z_0$ and multiplying that by $2\pi i$.

**average**

Since

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\ dz$$

and $z$ can be parametrized as $z - z_0 = re^{i\theta}$ so that $dz = ire^{i\theta}\ d\theta$:

$$
\begin{aligned}
f(z_0) &= \frac{1}{2\pi i} \oint \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} \; ire^{i\theta} \; d\theta \\
&= \frac{1}{2\pi} \oint f(z_0 + re^{i\theta}) \; d\theta \\
&= \frac{1}{2\pi} \oint f(z) \; d\theta
\end{aligned}
$$

*The value of an analytic function at the center of a circle equals the average (arithmetic mean) of the values on the circumference.*

# Chapter 20

# Cauchy corollary

In this section we follow Beck, so I've used their notation. In particular, $w$ is a fixed point inside the region.

The Cauchy Integral formula is:

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-w)}\ dz$$

If $f$ is differentiable for all points in some open disk centered at $w$ then $f$ is holomorphic at $w$. For a holomorphic function $f$, a specific extension of the Cauchy formula is

$$f'(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-w)^2}\ dz$$

This can be obtained by differentiating the original formula under the integral sign on the right-hand side.

**derivative rule**

We show here that

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\, dz$$

and there are more formulas for higher derivatives. Therefore

$$2\pi i\ f'(a) = \oint_C \frac{f(z)}{(z-a)^2}\, dz$$

Again, the Cauchy formula is:

$$\oint_C \frac{f(z)}{z-z_0}\, dz = 2\pi i f(z_0)$$

rewrite with $a$ for $z_0$

$$\oint_C \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$$

We take the partial with respect to $a$ of both sides:

$$\frac{\partial}{\partial a}(\frac{f(z)}{z-a}) = \frac{f(z)}{(z-a)^2}$$

so

$$\oint_C \frac{f(z)}{(z-a)^2}\, dz = 2\pi i f'(a)$$

Thus

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\, dz$$

More generally

$$f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

So

$$\frac{2\pi i}{n!} f^n(a) = \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

**more carefully**

A more formal proof is the following (from Beck).

$$f'(w) = \frac{f(w + \Delta w) - f(w)}{\Delta w}$$

so

$$= \frac{1}{\Delta w} \left[ \; \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - (w + \Delta w)} \; dz - \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - w)} \; dz \; \right]$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - w - \Delta w)(z - w)} \; dz$$

In putting the two fractions over a common denominator we get a factor of $\Delta w$ on top which cancels the other one.

It is now possible to show that the value of this integral approaches what we seek as $\Delta w \to 0$. We will show that the difference of integrals goes to zero as $\Delta w \to 0$.

That difference is

$$= \frac{1}{2\pi i} \int_\gamma ( \frac{f(z)}{(z - w - \Delta w)(z - w)} - \frac{f(z)}{(z - w)^2} \; dz$$

$$= \frac{\Delta w}{2\pi i} \int_\gamma \frac{f(z)}{(z - w - \Delta w)(z - w)^2} \; dz$$

We need to show that the integrand is bounded as $\Delta w \to 0$. Then the $\Delta w$ on top will make the whole thing go to zero.

Let $M$ be the maximum value of the function over the curve. They write:

$$M := \max_{z \in \gamma} |f(z)|$$

Choose $\delta > 0$ such that

$$|z - w| \geq \delta$$

for all $z$ on $\gamma$. Then the reverse triangle equality says that

$$\begin{aligned} |(z-w-\Delta w)(z-w)^2| &\geq (|z-w|-|\Delta w|)|z-w|^2 \\ &\geq (\delta - |\Delta w|)\delta^2 \end{aligned}$$

so

$$\begin{aligned} \frac{f(z)}{|(z-w-\Delta w)(z-w)^2|}| &\leq \frac{|f(z)|}{(|z-w|-|\Delta w|)|z-w|^2} \\ &\leq \frac{M}{(\delta-|\Delta w|)\delta^2} \end{aligned}$$

which certainly stays bounded as $\Delta w \to 0$.

This proves the Cauchy Integral formula for $f'$.

□

The formula for $f''$ is

$$f''(w) = \frac{1}{\pi i} \int_C \frac{f(z)}{(z-w)^3}\ dz$$

Notice the extra factor of 2

The general rule is:

$$f^n(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}}\ dw$$

which can be proved by induction.

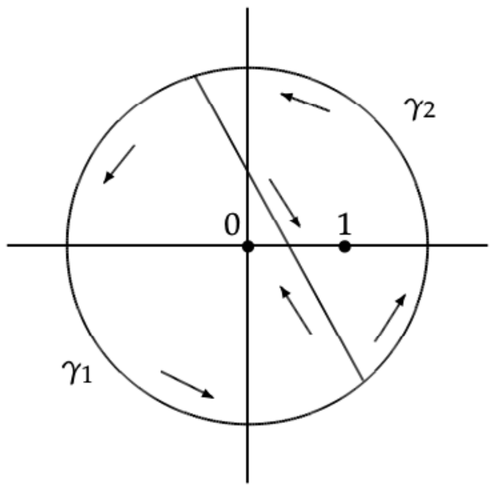
We will also see that any analytic function can be shown to have a power series expansion in a region where it is analytic. So then it's pretty obvious that it is differentiable there. And infinitely so.

Power series is at once a huge complication and yet the source of the most important results in complex function theory. There is just no avoiding them.

**example**

$$\int \frac{1}{z^2(z-1)}\ dz$$

If the region includes both of the singularities $z = 0$ and $z = 1$ we can split that path into two parts as shown in the figure:




Rewrite the integral as

$$\int \frac{1/(z-1)}{z^2}\ dz + \int \frac{1/z^2}{z-1}\ dz$$

For the first path $\gamma_1$, only the first term is non-zero, by the corollary

$$2\pi i f'(w) = \int_{\gamma_1} \frac{f(z)}{(z-w)^2}\ dz$$

$w = 0$ and the integral is:

$$= 2\pi i\ [\ \frac{d}{dz}\ \frac{1}{z-1}\bigg|_{z=0} = 2\pi i\ [\ -\frac{1}{(-1)^2} = -2\pi i\ ]$$

For the second path we use

$$2\pi i f(w) = \int_{\gamma_2} \frac{f(z)}{z-w}\ dz$$

with $w = 1$ so

$$= 2\pi i \left[ \frac{1}{z^2}\bigg|_1 \right] = 2\pi i$$

The total is zero.

**partial fractions**

The previous problem can also be done by partial fractions. We have

$$\frac{1}{z^2(z-1)}$$

$$= \frac{A}{z^2} + \frac{B}{z(z-1)} + \frac{C}{z-1}$$

so

$$A(z-1) + Bz + Cz^2 = 1$$

$$C = 0$$

$$Az + Bz = 0, \qquad A = -B$$

$$A = -1$$

and then

$$-\frac{1}{z^2} + \frac{1}{z(z-1)}$$

which checks.

We're not done, but what's left is easy, it is just

$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

so ultimately:

$$f(z) = -\frac{1}{z^2} - \frac{1}{z} + \frac{1}{z-1}$$

The path includes both $z = 0$ and $z = 1$.

The first integral is zero. The second one is $-2\pi i$ and for the third, just substitute $w = z - 1$, with $dw = dz$. The integral is $2\pi i$ and the total is just zero.

# Chapter 21

# Summary 2

A general complex function $f(z)$ takes a complex number $z$, which is really just an ordered pair $(x, y)$ and feeds that number to a pair of real functions of two real variables, which each output a single real number. So

$$z = x + iy$$

$$f(z) = u(x, y) + v(x, y)$$

$$dz = dx + idy$$

We compute integrals as line integrals along a curve (or contour) by doing

$$\int f(z)\ dz = \int (u + iv)(dx + idy)$$

$$= \int u\ dx - v\ dy + i\ [\ v\ dx + u\ dy\ ]$$

These don't look like it but they are integrals in a single variable, because $x$ and $y$ are related.

There are two kinds of complex functions: *analytic* and otherwise. The analytic functions are *good* functions, they follow the rules we know from basic calculus, and can be differentiated and integrated in analogous forms. We did some examples in $x$ and $y$ like square and triangular paths.

We discovered that integration of analytic functions along a closed path gives a result of zero, except when the function is not defined at some point in the region.

More commonly, integration around a circular contour is of interest, often on a unit circle. In that case, we have a parameter $t$ and the function said to be parametrized.

$$\int_C f(z)\ dz = \int_a^b f\ [\ z\ ]\ z'(t)\ dt$$

For example:

$$z = re^{it}$$

On a unit circle around the origin, $r$ is a constant and

$$dz = r(ie^{it})\ dt = iz\ dt$$

So, for example, the inverse function $1/z$ gives

$$\int \frac{1}{z}\ iz\ dt = i \int dt = 2\pi i$$

Around a closed path, the value of the integral is $2\pi i$.

This occurs despite the fact that $1/z$ obeys the CRE and is analytic. The problem is that it is undefined at the origin.

The factor of $2\pi i$ will come up repeatedly from this point.

Another way to explain this is to say

$$\frac{1}{z} = \frac{z*}{zz*}$$

The denominator is $x^2 + y^2 = r^2$ which is constant for any circular path, so we have

$$\int \frac{1}{z}\ dz = k \int z*\ \ dz$$

and $z*$ is definitely not analytic since $z* = x - iy$ and $u_x = 1 \neq v_y = -1$.

We also did some other examples, such as $1/z^2$. On the unit circle

$$z = e^{it}$$

$$dz = iz\ dt$$

so the integral is

$$\int \frac{1}{z^2}\ iz\ dt = \int \frac{1}{z}\ i\ dt$$

$$= i \int e^{-it}\ dt = i\ \frac{1}{-i}\ e^{-it} = -e^{-it}$$

From Euler

$$e^{ix} = \cos x + i \sin x$$

but evaluated around a closed path, this is zero because the sine and cosine have a period of $2\pi$.

## Cauchy 1

Cauchy's first theorem says that:

$$\oint_C f(z)\ dz = 0$$

for an analytic function around a region without any singularity.

We proved this theorem, it follows very simply from Green's theorem.

A corollary of this theorem is that the result of an integral between any two points over two different paths, is equal.

**Cauchy 2**

Cauchy's second theorem is where things start to get a bit more wild.

If we can write an integral in this form:

$$\oint_C \frac{f(z)}{z - z_0}\ dz$$

where $f(z)$ is analytic and defined everywhere in the domain we care about, with this composite function of course not defined at $z = z_0$.

We parametrize the curve as a circle of radius $r$ around the point $z_0$

$$z = z_0 + re^{it}$$

$z_0$ is a constant so

$$dz = rie^{it}\ dt = i(z - z_0)\ dt$$

and then we can simplify the integral as

$$i \oint_C f(z)\ dt$$

It is easy to show that the value of this integral does not depend on the radius of the path, so we let the radius shrink and approach zero.

The magic thing is that $f(z) \to f(z_0)$, but $f(z_0)$ is a *constant*. It can come out from under the integral:

$$= if(z_0) \int_C dt$$

This integral is just $2\pi$, so the whole thing is $2\pi i f(z_0)$ and we can write:

$$\oint_C \frac{f(z)}{z - z_0}\ dz = 2\pi i \cdot f(z_0)$$

This is Cauchy's second integral theorem.

A simple example is the inverse $1/z$, the numerator is $f(z) = 1$ and $z_0 = 0$ so the result is $2\pi i$ times the value of the function 1 at the origin, which is $2\pi i$ and matches what we got by direct computation.

**extension**

A specific extension of the Cauchy Integral formula is

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} \ dz$$

Generally:

$$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \ dz$$

Rearranged:

$$\frac{2\pi i}{n!} \ f^n(z_0) = \int_C \frac{f(z)}{(z - z_0)^{n+1}} \ dz$$

A function $f$ is said to be differentiable at $z_0$ if the function's domain includes a neighborhood of $z_0$ and the derivative exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

**more**

The existence of the derivative at $z_0$ implies that the function is continuous at that point; however, the converse is not necessarily true.

A function is *analytic* at a point if it has a derivative at that point.

Cauchy's theorem says that the integral around a closed path is zero for a function which is analytic everywhere in a domain, given some conditions.

If such a function is undefined at a limited number of points (e.g. because such values produce zero in the denominator), then those points are called poles or singularities and Cauchy's formula can be used to calculate the value of the integral (called a residue) from the value of the function at those points.

# Part V

# Series

# Chapter 22

# Taylor

### power series

All analytic functions can be expanded as power series around a fixed point $z_0$. Churchill says (sect. 44):

> Suppose that a function $f$ is analytic [i.e. has a derivative] throughout an open disk $|z - z_0| < R_0$, centered at $z_0$ and with radius $R_0$. Then, at each point in that disk, $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

This is a Taylor series. The proof follows, but I will not give it here.

The cofactors are

$$a_k = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{k+1}} \, dz = \frac{f^k(z_0)}{k!}$$

**example**

The next bit is from HELM:

https://learn.lboro.ac.uk/archive/olmp/olmp_resources/pages/wbooks_fulllist.html

Taylor series have terms of the form

$$a_n(x - x_0)^n$$

where the series is summed over positive integers from $n = 0 \rightarrow \infty$ and the coefficients are

$$a_n = \frac{f^{(n)}}{n!}$$

the nth derivative of $f$ divided by $n!$

As an example consider

$$f(x) = \frac{1}{1-x}$$

This has a singularity at $x = 1$. We can get the Taylor series expanded around 0 for this function (this special form is called the Maclaurin series).

$$f(x) = 1 + x + x^2 + x^3 + \ldots$$

We can show that this series is equal to what we started with

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

by multiplying the right-hand side by $(1-x)$. Imagine two long rows of numbers, one the series itself, and the second containing all the terms of the series multiplied by $-x$. It's clear that everything cancels except the term 1.

Alternatively, we can take derivatives and construct the series formally:

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f'(x) = \frac{1}{(1-x)^2} = (1-x)^{-2}$$

Notice that the minus sign from the exponent cancels the minus sign from the term $(1-x)$ obtained by the chain rule.

$$f''(x) = \frac{2}{(1-x)^3}$$

$$f'''(x) = \frac{3!}{(1-x)^4}$$

and so on.

Now, evaluated at $x_0 = 0$, these derivatives are seen to collapse to just the factorial, so we construct the terms of the series as

$$\begin{aligned} a_n &= \frac{f^{(n)}}{n!} \\ &= n! \, \frac{1}{n!} \end{aligned}$$

and the factorials also cancel. This leaves the particularly simple form:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

**convergence**

For most series the big question is: what is the radius of convergence?

The series expansion for real functions is centered around a fixed point $x_0$ with terms like $(x - x_0)^n$, and the series has a finite sum, only converges for $x$ sufficiently close to $x_0$.

$$|x - x_0| < r$$

Likewise, for complex functions, series expansions will usually only be valid for a circle (or disk, or region) of convergence in the Argand plane with

$$|z - z_0| < R$$

Convergence can be decided by certain tests including the ratio test and the root test (but sometimes the result is not clear).

Consider whether this complex series converges.

$$f(z) = \frac{1}{1-z}$$

$$= \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots$$

Without doing any tests, we see that this is the geometric series with ratio $z$, which is known to converge when $|z| < 1$.

As the source says:

"One of the shortcomings of Taylor series is that the circle of convergence is often only a part of the region in which $f(z)$ is analytic. The Laurent series is an attempt to represent $f(z)$ as a series at as many points as possible. We expand the series around a point of singularity up to, but not including, the singularity itself."

Laurent series involve an annulus, usually called $D$, which is a circle that has an empty small circle in its center, like a slice through a donut.

# Chapter 23

# Laurent theory

Any function that is analytic inside a disk has a power series (a Taylor series) convergent inside that disk.

But what about a punctured disk?

$$0 < |z - z_0| < R$$

or a donut

$$0 < r < |z - z_0| < R$$

The applicable series is then a Laurent series.

The HELM guys say:

> One of the shortcomings of Taylor series is that the circle of convergence is often only a part of the region in which $f(z)$ is analytic.

https://learn.lboro.ac.uk/archive/olmp/olmp_resources/pages/workbooks_1_50_jan2008/Workbook26/26_6_snglrts_n_resdus.pdf

An example is

$$f(z) = \frac{1}{1 - z}$$

This function is analytic everywhere except at the singularity $z = 1$. The Taylor series expanded around $z = 0$ is

$$1 + z + z^2 + z^3 + \dots$$

which converges to $f(z)$ only inside $|z| = 1$.

> The radius of convergence for a series centered on $z = z_0$ is the distance between $z_0$ and the nearest singularity.

Boas:

Let $C1$ and $C2$ be two circles with center at $z_0$. Let $f(z)$ be analytic in the region between the circles. Then $f(z)$ can be expanded in a Laurent series:

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots +$$
$$+\frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots$$

The Taylor part of the series (called the analytic part) usually converges everywhere inside a disk of radius $R$, while the $b$ part, the principal part, usually converges everywhere outside a disk of radius $r$, so the combined series is convergent in the annulus, the area between $r$ and $R$.

Note: if there are several isolated singularities , then there are several annular rings, each with a different Laurent series.

### Laurent's Theorem

If $f(z)$ is analytic through a closed annulus $D$ centered at $z = z_0$, then at any point $z$ inside $D$ we can write:

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

$$+ b_1(z - z_0)^{-1} + b_2(z - z_0)^{-2} + \dots$$

where the coefficients are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{1-n}}\, dz$$

(no, they don't match the powers of $(z - z_0)$).

Any polynomial of $z$ is analytic, and quotients of analytic functions are also analytic.

The end result will be that the integral $\int f(z)\ dz$ may be obtained by integrating the right-hand side, where all the terms except one will have an integral equal to zero.

Out of this entire series given above, only one term matters:

$$b_1(z - z_0)^{-1}$$

This is a consequence of Cauchy's Integral theorem.

**derivation of Laurent series**

We follow

https://www.youtube.com/watch?v=2GC26rJB2L0&list=PLvcbYUQ5t0UFmFXOLwC9E
index=22&t=0s

Fix some particular $z$ in the annulus $0 < r < |z - z_0| < R$.

Choose $r_1$ and $R_1$ just inside (or outside, respectively) the boundaries of the donut:

$$0 < r < r_1 < |z - z_0| < R_1 < R$$

Let $\gamma_1$ go along (let $w$ take on the values) $|w - z_0| = R_1$ counter-clockwise, with the interior on the left, and let $\gamma_2$ go along $|w - z_0| = r_1$, in the opposite direction to $\gamma_1$.

Set up a keyhole contour. Then $f$ is analytic in the domain which the line integral encloses, which is required to use Cauchy's formula:

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}\ dw + \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}\ dw$$

The path that links the two circles cancels because we traverse it in opposite directions.

Now rewrite

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)}$$

$$= \frac{1}{w-z_0}\ [\ \frac{1}{1-(z-z_0)/(w-z_0)}\ ]$$

This is a geometric series with initial term

$$\frac{1}{w-z_0}$$

divided by (or multipied by the inverse of) one minus the common ratio

$$\frac{z-z_0}{w-z_0}$$

Since

$$|\frac{z-z_0}{w-z_0}| < 1$$

For the big circle $\gamma_1$ we have that this ratio is less than 1 ($w$ runs along $R$), so the series converges absolutely.

The corresponding series is

$$\sum_{k=0}^{\infty} \frac{(z-z_0)^k}{(w-z_0)^{k+1}}$$

For the small circle, $\gamma_2$: the ratio $1/w - z$ is equal to

$$-\frac{1}{z-z_0} \left[ \frac{1}{1-(w-z_0)/(z-z_0)} \right]$$

For the small circle $\gamma_2$ the flipped ratio is less than 1 since $w$ runs along $r$, so again the series converges absolutely.

The series is

$$-\sum_{0}^{\infty} \frac{(w-z_0)^j}{(z-z_0)^{j+1}}$$

Rewriting

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_1} f(w) \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{(w-z_0)^{k+1}} \, dw$$

$$-\frac{1}{2\pi i} \int_{\gamma_2} f(w) \sum_{j=0}^{\infty} \frac{(w-z_0)^j}{(z-z_0)^{j+1}} \, dw$$

The sums can come out from under the integral so

$$f(z) = \sum_{k=0}^{\infty} (z-z_0)^k \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(w-z_0)^{k+1}} \, dw$$

$$+\sum_{j=0}^{\infty} z-z_0)^{-j-1} (-\frac{1}{2\pi i}) \int_{\gamma_2} f(w) \; (w-z_0)^j \, dw$$

We can clean up the indices for $f_2$. Instead of running from $j = 0 \to \infty$ with power $-j-1$, let it be $k = -\infty \to -1$ with power $k$.

We now have $f_1(z) + f_2(z)$. Now we can write the coefficients for both in the same way:

$$a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}}\, dw$$

and

$$f_1(z) = \sum_0^\infty a_k (z - z_0)^k$$

$f_1$ is convergent when $|z - z_0| < R$.

$$f_2(z) = \sum_{-\infty}^{-1} a_k (z - z_0)^k$$

$f_2$ is convergent when $|z - z_0| > r$.

A Laurent series is the combination.

$$\sum_{-\infty}^{\infty 1} a_k (z - z_0)^k$$

It is convergent when both criteria are met.

The principal part of the Laurent series consists of the terms with negative exponents. We can also write that part as

$$\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

Although it seems like we've made things *really* complicated, we haven't really, because when such a series is integrated, the only non-zero term is the $k = -1$ term.

Let's sidestep the problem of determining the coefficients using the formulas given above.

Instead, just say that we seek a series expansion using negative powers of $z$, and hope to find that it will be valid in the region $|z| > 1$.

# Chapter 24

# Laurent practice

### geometric series

The initial examples are all based on the geometric series:

$$a + ar + ar^2 + ar^3 \dots$$

The ratio test shows that this infinite series is convergent for $|r| < 1$. In that case it has a sum and we can write:

$$S = a(1 + r + r^2 + r^3 \dots$$

$$\frac{S}{a}r - \frac{S}{a} = 1$$

$$S = \frac{a}{1-r}$$

So any time you see this form think geometric series.

$$\frac{1}{1-r}$$

Switch to $z$ as the variable:

$$\frac{1}{1-z}$$

There is a trick to getting a *different* geometric series. Just write

$$\frac{1}{1-z} = (-\frac{1}{z}) \cdot \frac{1}{1-1/z}$$

That's also a geometric series, but in $1/z$. It converges when $|1/z| < 1$ which means $|z| > 1$!!

And it is equal to

$$(-\frac{1}{z}) \cdot (1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} \dots$$

$$-\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} \dots$$

Note that we can substitute $-w = z$ and get

$$\frac{1}{1+w} = \frac{1}{w} \; [\; 1 - \frac{1}{w} + \frac{1}{w^2} - \frac{1}{w^3} + \dots \;]$$

$$\frac{1}{1+w} = \frac{1}{w} - \frac{1}{w^2} + \frac{1}{w^3} + \dots$$

go back to $z$ as the variable

$$\frac{1}{1+z} = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \dots$$

Check by multiplying the right-hand side by $z$ and see all the cancellations after the first term.

**Boas example**

$$f(z) = \frac{12}{z(2-z)(1+z)}$$

So this function has three isolated singularities (at $z = 0, 2, -1$). And expanded around $z_0 = 0$, there will be three regions in which we have *different* series: namely $0 < |z| < 1$, $1 < |z| < 2$ and $|z| > 2$. There will be three series all together.

Start by using partial fractions to obtain:

$$= \frac{4}{z} \cdot (\frac{1}{2-z} + \frac{1}{1+z})$$

Start with the inner punctured disk. We want convergence for the region $|z| < 1$. Since it's less than, we use standard geometric series.

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 \dots$$
$$\frac{1}{2-z} = \frac{1}{2} \cdot \frac{1}{1-z/2} = \frac{1}{2} \left[ 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} \dots \right]$$

Add them together

$$= \frac{3}{2} - \frac{3z}{4} + \frac{9z^2}{8} - \frac{15z^3}{16} \dots$$

and multiply by $4/z$ to obtain:

$$= \frac{6}{z} - 3 + \frac{9z}{2} - \frac{15z^2}{4} \dots$$

This is the Laurent series valid in the innermost region.

For the outer region, manipulate each fraction:

$$\frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+1/z}$$

$$\frac{1}{2-z} = -\frac{1}{z} \cdot \frac{1}{1-2/z}$$

We do this so that the geometric series will be convergent for $|z| > 2$ (the ratio is $2/z$ in the second one).

Again, geometric series. Write them separately:

$$\frac{1}{z}\ [\ 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} + \dots\ ]$$

$$-\frac{1}{z}\ [\ 1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \frac{16}{z^4} \dots\ ]$$

Move the minus sign inside:

$$\frac{1}{z}\ [\ -1 - \frac{2}{z} - \frac{4}{z^2} - \frac{8}{z^3} - \frac{16}{z^4} \dots\ ]$$

Add

$$\frac{1}{z}\ [-\frac{3}{z} - \frac{3}{z^2} - \frac{9}{z^3} - \frac{15}{z^4} + \dots$$

Recall the leading factor of $4/z$, and get another factor of $-3/z$ giving what it has in the book:

$$-\frac{12}{z^3}\ [\ 1 + \frac{1}{z} + \frac{3}{z^2} + \frac{5}{z^3} + \dots\ ]$$

The last part is the annulus in the middle. For this we want convergence for $|z| > 1$ and for $|z| < 2$. Hence we want

$$\frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+1/z}$$

$$= \frac{1}{z}\ [\ 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} + \dots\ ]$$

and

$$\frac{1}{2-z} = \frac{1}{2} \cdot \frac{1}{1-z/2} = \frac{1}{2}\ [\ 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} \dots\ ]$$

I need a factor of $1/z$ on the latter (and move the factor of 2 inside:

$$\frac{1}{z}\ [\ \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \frac{z^4}{16} \dots\ ]$$

We can add them and remember the leading factor of $4/z$ so it's

$$\frac{4}{z^2}\ [\frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \frac{z^4}{16} \cdots + 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} + \dots]$$

I try to verify one power, pick $z^{-1}$

$$\frac{6}{z} + \frac{2}{z}$$

and that, unfortunately is not a match. But I know we're close, and you can see the general idea.

**example**

Consider:

$$f(z) = \frac{z}{(z-1)(z-3)}$$

and say we need the series around $0 \le |z-1| \le 2$, also written as $C[1,2]$, a circle of radius 2 around the point $z_0 = 1$.

One way to do this is to make substitution $x = z - 1$, so $z = x + 1$ and we have

$$= \frac{x+1}{(x)(x-2)}$$

Factor out the $1/x$

$$= \frac{1}{x}\ (\frac{x+1}{x-2})$$

(this is easy to restore at the end by multiplying by $1/x$).

And then our goal is to get something like $1/1 - r$.

That means we want $x - 2$ on top

$$= \frac{1}{x}\ (\frac{x-2+3}{x-2})$$

$$= \frac{1}{x}\ (1 + \frac{3}{x-2})$$
$$= \frac{1}{x}\ (1 - \frac{3}{2-x})$$
$$= \frac{1}{x}\ (1 - \frac{3/2}{1-x/2})$$
$$= \frac{1}{x}\ (1 - \frac{3}{2} \cdot \frac{1}{1-x/2})$$

and now the

$$\frac{1}{1-x/2}$$

can be expanded because that's a geometric series

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \dots$$

so

$$\frac{1}{1-x/2} = 1 + \frac{x}{2} + (\frac{x}{2})^2 + \dots$$

which gives

$$= \frac{1}{x}\ [1 - \frac{3}{2} \cdot (1 + \frac{x}{2} + (\frac{x}{2})^2 + \dots)\ ]$$

Now, multiplying through by $1/x$ gives

$$-\frac{1}{2x} + \dots$$

and *nothing else matters.* Reverse the change of variable:

$$= -\frac{1}{2(z-1)} + \dots$$

which we will integrate as

$$\oint \frac{-1/2}{z-1}\ dz$$

over $C[1,2]$.

Recall the formula for residues:

$$b_1 = \lim_{z \to z_0} (z - z_0)\ f(z)$$

So

$$\text{Res } (1) = \lim_{z \to 1} (z-1)\ \frac{-1/2}{z-1} = -\frac{1}{2}$$

Multiply by $2\pi i$ to obtain $I = -\pi i$.

As a check on this go back to

$$f(z) = \frac{z}{(z-1)(z-3)}$$

$$\begin{aligned} \text{Res } (1) &= \lim_{z \to 1} (z-1)\ \frac{z}{(z-1)(z-3)} \\ &= \lim_{z \to 1}\ \frac{z}{z-3} \\ &= -\frac{1}{2} \end{aligned}$$

**example**

These examples can get complicated. Here is one from

http://zimmer.csufresno.edu/~doreendl/128.13f/handouts/Lseriesex.pdf

$$f(z) = \frac{1}{(z-2)(z-1)}$$

This function has poles at $z = 1$ and $z = 2$. If we are asked to write expansions around $z_0 = 0$, then we have three regions of interest and three different expansions.

The first region is the circle of radius 1: $|z| < 1$, the second is $1 < |z| < 2$ and then finally $|z| > 2$.

**region 1**

Use partial fractions to write:

$$\frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$$

Considering the second term, we bring the minus sign inside

$$= \frac{1}{z-2} + \frac{1}{1-z}$$

We have the classic

$$\frac{1}{1-z} = 1 + z + z^2 = \sum_{n=0}^{\infty} z^n$$

which we know this is valid for $|z| < 1$, the region of interest.

For the other term

$$\frac{1}{z-2} = -\frac{1}{2-z}$$

Our goal is to convert this into something like the geometric series. Factor out the 2 on the bottom like so

$$= -\frac{1}{2}\ [\ \frac{1}{1-z/2}\ ]$$

We can do a formal substitution or recognize that this is the geometric series

$$= -\frac{1}{2}\ [\ \sum_{n=0}^{\infty}(z/2)^n\ ]$$

We can rewrite this slightly by pulling out the factor of $2^n$ on the bottom and combining it with the factor of 2 out front:

$$= \sum_{n=0}^{\infty} \left[ \frac{-1}{2^{n+1}} \right] z^n$$

which converges for $0 < |z/2| < 1 \Rightarrow 0 < |z| < 2$.

Our series is the sum of these two series, which can be combined as

$$\sum_{n=0}^{\infty} \left[ 1 - \frac{1}{2^{n+1}} \right] z^n$$

**region 2**

This is the annulus $1 < |z| < 2$. Thus

$$|\frac{1}{z}| < 1 \quad \text{and} \quad |\frac{z}{2}| < 1$$

What they do is to work on the right-hand term of

$$\frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$$

and, as we saw in the previous section transform it into something containing $1/z$, which will be valid in the region $|z| > 1$.

So let's do it:

$$\begin{aligned} \frac{1}{1-z} &= -\frac{1}{z-1} \\ &= -\frac{1}{z} \cdot \frac{1}{1-1/z} \end{aligned}$$

leaving aside the leading factor this is

$$= 1 + \frac{1}{z} + \frac{1}{z^2} \ldots$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

add back that factor

$$-\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n}$$

The left-hand term is exactly what we had before:

$$\sum_{n=0}^{\infty} [\ \frac{-1}{2^{n+1}}\ ]\ z^n$$

so we combine them

$$\sum_{n=0}^{\infty} [\ \frac{-1}{2^{n+1}}\ ]\ z^n - \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n}$$

and then just bring that $z$ in the second term inside

$$\sum_{n=0}^{\infty} [\ \frac{-1}{2^{n+1}}\ ]\ z^n - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

or change the index

$$= \sum_{n=0}^{\infty} [\ \frac{-1}{2^{n+1}}\ ]\ z^n - \sum_{n=1}^{\infty} \frac{1}{z^n}$$

**region 3**

We do the $1/z$ trick with both terms

$$\frac{1}{z-2} - \frac{1}{z-1}$$

Start with the first one:

$$\frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-2/z}$$

The series is

$$\frac{1}{z} \ \cdot \ \sum_{n=0}^{\infty} \ [ \ \frac{2}{z} \ ]^n$$

$$= \sum_{n=0}^{\infty} \ \frac{2^n}{z^{n+1}}$$

The second term is (leaving off the factor of $-1$)

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1 - 1/z}$$

The series is

$$\frac{1}{z} \ \cdot \ \sum_{n=0}^{\infty} \ [ \ \frac{1}{z} \ ]^n$$

$$= \sum_{n=0}^{\infty} \ \frac{1}{z^{n+1}}$$

Combining the two results and bringing back the factor we get

$$\sum_{n=0}^{\infty} \ \frac{2^n}{z^{n+1}} - \sum_{n=0}^{\infty} \ \frac{1}{z^{n+1}}$$

$$= \sum_{n=0}^{\infty} \ (2^n - 1) \ \frac{1}{z^{n+1}}$$

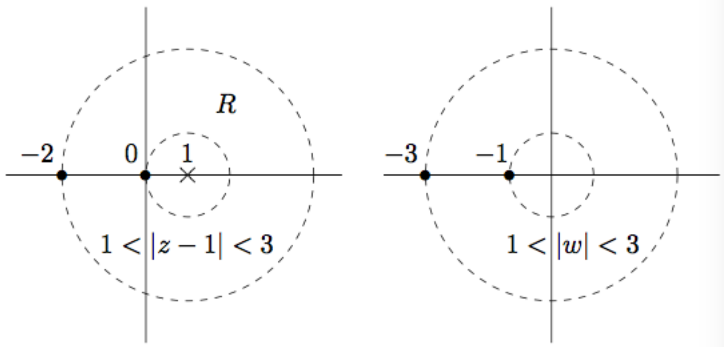
adjust the index

$$= \sum_{n=1}^{\infty} \ (2^{n-1} - 1) \ \frac{1}{z^n}$$

**example**

$$f(z) = \frac{1}{z(z+2)}$$

Suppose the region of interest is an annulus centered on $z = 1$ with $1 < |z-1| < 3$.




The first thing to do is make a substitution that translates the region so that it becomes centered on the origin: $w = z - 1$. Then the function becomes

$$\frac{1}{(w+1)(w+3)}$$

The next thing is to write partial fractions. For the numerator we get

$$A(w+3) + B(w+1) = 1$$

$$A = -B = \frac{1}{2}$$

Hence

$$\frac{1}{2} \cdot \left[ \frac{1}{w+1} - \frac{1}{w+3} \right]$$

The third step is to convert each of these fractions into something like $1/1 - x$.

$$\frac{1}{w+1} = \frac{1}{1-(-w)}$$

$$\frac{1}{w+3} = \frac{1}{3} \cdot \frac{1}{1-(-w/3)}$$

And then the fourth step is to write the series, recalling that we want different forms depending on whether we are in a circle or an annulus.

$$\frac{1}{1-(-w)}$$

$$= \sum_{n=0}^{\infty} (-w)^n = \sum_{n=0}^{\infty} (-1)^n (w)^n, \quad |w| < 1$$

$$= -\sum_{n=1}^{\infty} \frac{1}{(-w)^n} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{w^n}, \quad |w| > 1$$

We pick the second form because our region is $1 < |z-1| < 3$

A similar thing can be done for the other term. We show only the first series since we are inside the circle.

$$\frac{1}{3} \cdot \frac{1}{1-(-w/3)}$$

$$= \frac{1}{3} \cdot \sum_{n=0}^{\infty} (-1)^n (\frac{w}{3})^n, \quad |w| < 3$$

$$= \sum_{n=0}^{\infty} (-1)^n \ \frac{1}{3^{n+1}} \ w^n, \quad |w| < 3$$

Add the two series together (remembering the minus sign on the second term)

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{w^n} - \sum_{n=0}^{\infty} (-1)^n \ \frac{1}{3^{n+1}} \ w^n$$

and then picking up the leading factor from

$$\frac{1}{2} \cdot \ [\ \frac{1}{w+1} - \frac{1}{w+3}\ ]$$

so

$$\frac{1}{2}\ [\ -\sum_{n=1}^{\infty}\frac{(-1)^n}{w^n}-\sum_{n=0}^{\infty}(-1)^n\ \frac{1}{3^{n+1}}\ w^n\ ]$$

The last step is to reverse the substitution: $w = z - 1$ and bring the minus sign out front

$$f(z) = -\frac{1}{2}\ [\ \sum_{n=1}^{\infty}\frac{(-1)^n}{(z-1)^n}\sum_{n=0}^{\infty}(-1)^n\ \frac{1}{3^{n+1}}\ (z-1)^n\ ]$$

I don't know if I could ever learn to do this well, but at least the explanations make sense.

Now, if we were to integrate $f(z)$, we would have only one term that gives a non-zero result, namely the first term with $n = 1$

$$-\frac{1}{2}(-1)\frac{1}{z-1}$$

$$\text{Res }(1) = \lim_{z\to 1}\frac{1}{2} = \frac{1}{2}$$

Multiply by $2\pi i$ to obtain $\pi i$.

**simpler view**

On the other hand, I would just write the partial fraction:

$$\int \frac{1}{2}\ [\ \frac{1}{z} - \frac{1}{z+2}\ ]dz$$

The curve $C[1,3]$ includes the singularity at $z = 0$, but $z = -2$ is on the boundary, not inside the region. The curve $C[1,1]$ includes neither point.

So for the second curve the integral is zero and for the first one only the $\int 1/z\ dz$ matters. It's an old friend, with value $2\pi i$, the value of the integral is thus $\pi i$.

**example**

Here are two examples from Brown and Churchill.

$$\int \frac{1}{z(z-2)^4}\ dz$$

with singularities at $z = 0$ and $z = 2$.

Use the $1/z$ part to get a geometric series:

$$\frac{1}{z(z-2)^4} = \frac{1}{(z-2)^4} \cdot \frac{1}{(2+z-2)}$$

$$= \frac{1}{(z-2)^4} \cdot \frac{1}{2} \cdot \frac{1}{(1-(-(z-2))/2)}$$

The third term gives the geometric series with common ratio $(z-2)/2$. Those $z-2$ terms will cancel the leading factor. The only term that matters is the cube, which gives:

$$\frac{1}{(z-2)} \cdot \frac{1}{2} \cdot \frac{1}{(-2)^3}$$

We have that $b_1 = -1/16$ so $I = -\pi i/8$.

The second one is:

$$\int e^{1/z^2}\ dz$$

We use the standard series for $e^z$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

substituting $1/z^2$

$$1 + \frac{1}{z^2} + \frac{z^r}{2!} + \frac{z^6}{3!} + \dots$$

Since there's no $z$ with a power $n = -1$, the value of the integral is just zero.

# Part VI

# Residue Theory

# Chapter 25

# Partial Fractions

In the previous section, we derived Cauchy's formula:

$$\oint_C \frac{f(z)}{z - z_0}\ dz = 2\pi i f(z_0)$$

One example is

$$\oint_C \frac{1}{z - z_0}\ dz = 2\pi i$$

which is easy to show even without using the new formula.

**Example 1**

This problem is Beck 4.26. Consider

$$\oint f(z)\ dz = \oint \frac{1}{z^2 + 1}\ dz$$

We see that the denominator is zero when

$$z^2 = -1$$

$$z = \pm\ i$$

Therefore we can factor the denominator as

$$z^2 + 1 = (z+i)(z-i)$$

There are a couple of different ways to handle this. One is to use partial fractions:

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$
$$= \frac{1}{2i}\ [\ \frac{1}{z-i} - \frac{1}{z+i}\ ]$$

So the integral is a sum of two integrals:

$$I = \frac{1}{2i}\ [\oint \frac{1}{z-i}\ dz - \oint \frac{1}{z+i}\ dz\ ]$$

Suppose the curve is the unit circle centered at $i$, designated as $C[i,1]$.

Obviously, this curve contains the singularity $z = i$. The curve goes through the origin, so it does not extend as far as $z = -i$.

Therefore, the second integral is zero (no singularity) and the first is

$$\frac{1}{2i}\ [\ \oint \frac{1}{z-i}\ dz\ ]\ = \frac{1}{2i}\ [\ 2\pi i\ ]$$

by Cauchy's formula because $f(z_0)$ is 1. Thus the value is just $I = \pi$

According to Beck, as an alternative, rewrite the function as

$$\frac{1}{(z+i)(z-i)} = \frac{(1/z+i)}{z-i}$$

Thus

$$\int \frac{1}{z^2+1}\ dz = \int \frac{(1/z+i)}{z-i}\ dz$$

We have essentially the same thing. The function is

$$\frac{1}{z+i}$$

and when evaluated at $i$, with result $1/2i$, we obtain

$$\oint \frac{f(z)}{z-z_0}\,dz = 2\pi i\ f(z_0)$$

$$= 2\pi i\ \frac{1}{2i} = \pi$$

Above we have a constant of $1/2i$ which can either be factored out of the integral, or be part of $f(z_0)$. Either way, it's the same result.

**partial fractions**

We can also do the curve containing both singularities by using the formal apparatus of partial fractions. Write

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

$$= \frac{A}{z+i} + \frac{B}{z-i}$$

We need to determine $A$ and $B$. When we multiply to put everything over the common denominator $(z^2+1)$ then for the numerators we will have:

$$A(z-i) + B(z+i) = 1$$

All the powers of $z$ must match across the equal sign. Since there is no power of $z$ on the right-hand side, this gives

$$Az + Bz = 0$$

and

$$-Ai + Bi = 1$$

From the first we get that $A = -B$. And so from the second

$$-Ai - Ai = -2Ai = 1$$
$$A = -\frac{1}{2i}$$

Hence the integrand is

$$\frac{1}{z^2+1} == \frac{A}{z+i} + \frac{B}{z-i}$$
$$= -\frac{1}{2i(z+i)} + \frac{1}{2i(z-i)}$$

which matches what we had above:

$$= \frac{1}{2i} \left[ \frac{1}{z-i} - \frac{1}{z+i} \right]$$

For the curve including $z = i$ but not $z = -i$ we have that the right-hand integral is 0 by Cauchy's Theorem, and for the left hand side the function is

$$f(z = z_0) = \frac{1}{2i}$$

So the value of the integral is

$$2\pi i f(z_0) = 2\pi i \ \frac{1}{2i} = \pi$$

as before.

The other pole would have

$$f(z = z_0) = -\frac{1}{2i}$$

and the result would be $-\pi$.

A curve enclosing both poles would have for the value of the integral the sum of the two, which is equal to zero.

**example**

Consider

$$\int_\gamma \frac{z^2}{4 - z^2}\ dz$$

where $\gamma$ is $|z + 1| = 2$.

Recall that the definition of a circle around $z_0$ is $|z - z_0| = r$, where $r$ is the radius of the circle. Thus the circle is centered at $z_0 = -1$, which can be checked by looking for values on the real number line that satisfy the equality (yielding $-3$ and $1$).

The denominator of the function can be factored

$$\frac{1}{4 - z^2} = \frac{1}{(2 + z)(2 - z)}$$

It has zeroes at $z = \pm\ 2$. Only the point $z = -2$ is inside our contour.

So if we split this by partial fractions

$$\frac{1}{(2 + z)(2 - z)} = \frac{1}{4}\ [\ \frac{1}{2 + z} + \frac{1}{2 - z}\ ]$$

we can rewrite the integral as

$$I = \int_\gamma \frac{z^2}{4}\ [\ \frac{1}{2 + z} + \frac{1}{2 - z}\ ]\ dz$$

By Cauchy's Theorem, the second term is zero (no singularity).

The first one is:

$$I = \int_\gamma \frac{z^2}{4}\ (\frac{1}{2 + z})\ dz$$

and the value of $I$ is

$$I = 2\pi i f(z_0)$$

where

$$f(z_0) = \frac{z^2}{4}\bigg|_{z_0=-2} = 1$$

so the integral is just $2\pi i$.

**one from wikipedia**

Consider

$$g(z) = \frac{z^2}{z^2+2z+2}$$

We want to evaluate the integral:

$$I = \oint g(z)\ dz$$

The denominator

$$z^2+2z+2$$

can be factored.

We plug into the quadratic solution:

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2\pm\sqrt{4-4\cdot 2}}{2}$$

$$= -1 \pm \ \frac{\sqrt{-4}}{2}$$

$$= -1 \pm i$$

The zeroes of the denominator are $z$ equal to

$$-1+i\ , \quad -1-i$$

From this we construct the two factors as

$$z-(-1+i) = z+1-i$$

$$z - (-1 - i) = z + 1 + i$$

We confirm that these two factors multiplied together give back what we started with:

$$\begin{aligned} &(z + 1 + i)\ (z + 1 - i) \\ &= z^2 + z + iz + z + 1 + i + -iz - i + 1 \\ &= z^2 + 2z + 2 \end{aligned}$$

So we can factor the denominator and write:

$$\frac{1}{z^2 + 2z + 2} = \frac{A}{z + 1 - i} + \frac{B}{z + 1 + i}$$

Putting these two terms over a common denominator means multiplying the two factors and restoring what we started with.

For the numerator we have

$$A(z + 1 + i) + B(z + 1 - i) = 1$$

$$Az + A + iA + Bz + B - iB = 1$$

Equating terms containing the same power of $z$ gives two simultaneous equations:

$$Az + Bz = 0z$$

and

$$A(1 + i) + B(1 - i) = 1$$

So $A = -B$ and

$$A(1 + i) + B(1 - i) = 1$$

$$A(1 + i) - A(1 - i) = 1$$

$$A2i = 1$$

$$A = \frac{1}{2i}, \quad B = -\frac{1}{2i}$$

The integral is

$$\oint \frac{z^2}{2i} \left[ \ \frac{1}{z+(1-i)} - \frac{1}{z+(1+i)} \ \right] dz$$

So we see that we have a sum of integrals of the form

$$\oint \frac{f(z)}{z-z_0}$$

The residues occur at the points

$$z = z_0$$

that is at

$$z = -(1-i) = -1+i$$
$$z = -(1+i) = -1-i$$

If the contour is $|z| = 2$ centered at the origin (the circle of radius 2, then both of the points lie within the contour. ($r^2 = 2$ for both).

We evaluate $2\pi i f(z_0)$ for each

$$f(z) = \frac{z^2}{2i}$$

The first term gives

$$f(z_0) = \frac{(-1+i)^2}{2i} = \frac{1-1-2i}{2i} = -1$$

The second term gives

$$\frac{(-1-i)^2}{2i} = \frac{1+2i-1}{2i} = 1$$

But... this is not quite right. Go back and see that the second term in the integral has a minus sign. Hence the result at this step is $-1$ for both.

Each of these needs to be multiplied by $2\pi i$ and then summed.

$$I = -4\pi i$$

# Chapter 26

# Poles and residues

Boas **definitions**:

Consider the Laurent series for $f(z)$ inside some $C$ centered on $z_0$. Let $z_0$ be either a regular point or an isolated singular point and there are no other singular points inside $C$. Then:

∘ If all the $b$ coefficients are zero, $f(z)$ is analytic at $z = z_0$ and we call $z_0$ a *regular point*.

∘ If $b_n \neq 0$ but all the $b$'s after $b_n$ are zero, then $f(z)$ is said to have a *pole of order* $n$ at $z = z_0$. If $n = 1$ it is called a *simple pole*.

∘ If there are an infinite number of $b$'s different from zero, then $f(z)$ has an *essential singularity* at $z = z_0$.

∘ The coefficient $b_1$ of $1/(z - z_0)$ is called the *residue* of $f(z)$ at $z = z_0$.

**example**

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

There are no $b$'s, $e^z$ is analytic, and the residue at $z = z_0$ is 0.

$$\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!z} + \frac{1}{3!} \dots$$

The pole is of order 3, the residue at $z = z_0$ is $1/2!$.

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

There is an essential singularity at $z = z_0$, and the residue at $z = z_0$ is 1.

**The residue theorem**

Let $z_0$ be an isolated singular point of $f(z)$, and expand the Laurent series about $z = z_0$ for $f$. We want to find $\oint f(z)\ dz$. By Cauchy's integral theorem, the integral of the analytical part of the Laurent series is zero (non-negative exponents for $z - z_0$).

It is easy to show that the negative exponents of power $-2$ and higher also give rise to zero when integrated, because they retain some power of $e^{i\theta}$ which multiplies everything by zero for a closed path.

The exception is the $(z - z_0)$ term, where stuff cancels:

$$\oint \frac{b_1}{z - z_0}\ dz = b_1 \int_0^{2\pi} \frac{rie^{i\theta}}{re^{i\theta}}\ d\theta = 2\pi i\ b_1$$

$b_1$ is called the residue of $f(z)$ at the singular point inside $C$. If there is more than one isolated singularities, the value of the integral is $2\pi i$ times the sum of the residues.

The trick of course, is to know what $b_1$ is equal to.

If we have the Laurent series, then $b_1$ is the coefficient of the $\frac{1}{z-z_0}$ term.

**example**

$$f(z) = \frac{e^z}{(z-1)}$$

$$= \frac{e}{(z-1)} \cdot e^{(z-1)} = \frac{e}{(z-1)} \; [\; 1 + (z-1) + \frac{(z-1)^2}{2!} + \dots$$

$$= e(\frac{1}{(z-1)} + 1 + \frac{(z-1)}{2!} + \dots$$

Then

$$\oint f(z) \; dz = 2\pi i \cdot e$$

The residue is the coefficient of $1/(z-1)$.

**simple pole**

If $f(z)$ has a simple pole at $z = z_0$ we find the residue by a trick with Cauchy's integral formula:

$$\int \frac{f(z)}{z - z_0} \; dz = 2\pi i \cdot f(z_0)$$

in the limit that $z \to z_0$ it can come out from the integral sign:

$$\int f(z) \; dz = \lim_{z \to z_0} (z - z_0) \cdot 2\pi i \cdot f(z_0)$$

Stated in terms of the residue:

$$R = \lim_{z \to z_0} (z - z_0) f(z_0)$$

**examples**

$$f(z) = \frac{z}{(2z+1)(5-z)}$$

The poles are at $z_0 = -1/2$ and $z_0 = 5$.

First multiply the function by $z - z_0$ That gives

$$R(-1/2) = \frac{z}{2(5-z)}\bigg|_{z=-1/2} = \frac{-1/2}{2(11/2)} = -\frac{1}{22}$$

And

$$R(5) = -\frac{z}{2z+1}\bigg|_5 = -\frac{5}{11}$$

For

$$f(z) = \frac{\cos z}{z}$$

The pole is at $z_0 = 0$ so multiply by $z$:

$$R(0) = \cos z\bigg|_0 = 1$$

By definition the *residue* at a simple singularity or pole $z_0$ is defined to be

$$b_1 = \lim_{z \to z_0} (z - z_0)\ f(z)$$

We can get there from Cauchy's integral formula:

$$\oint \frac{f(z)}{z - z_0}\ dz = 2\pi i f(z_0)$$

Just think of it as in the limit that $z \to z_0$, the denominator $z - z_0$ on the left of the first equation is a constant, so we can multiply both sides by $z - z_0$ to obtain the result for the residue.

$$\oint f(z)\ dz = 2\pi i \ \lim_{z\to z_0}(z-z_0)\ f(z)$$

using the definition

$$b_1 = \lim_{z\to z_0}(z-z_0)\ f(z)$$

we obtain

$$\oint f(z)\ dz = 2\pi i\ b_1$$

If there is more than one such point

$$\oint f(z)\ dz = 2\pi i \sum \text{Res}$$

The value of the integral is $2\pi i$ times the sum of all the residues enclosed by the path.

There is no longer a factor of $1/z - z_0$ in the integral, just $f(z)$.

**residues**

We repeat the problem from last time, using residues.

$$\oint f(z)\ dz = \oint \frac{1}{z^2+1}\ dz$$

Our formula is:

$$b_1 = \lim_{z\to z_0}(z-z_0)\ f(z)$$

and

$$\oint f(z)\ dz = 2\pi i \sum \text{Res}$$

Evaluate the formula. Our path includes $i$ but not $-i$.

$$b_1 = \lim_{z\to z_0}(z-z_0)\ f(z)$$

$$= \lim_{z \to i} (z - i) \ \frac{1}{(z+i)(z-i)}$$

$$= \lim_{z \to i} \ \frac{1}{z+i} = \frac{1}{2i}$$

And by the second equation:

$$I = \pi$$

as before. Seems a bit easier!

If the unit circle had been centered at $-i$, rewrite the function as

$$f(z) = \frac{1/z - i}{z + i}$$

The value of the function is

$$\frac{1}{z-i}(-i) = -\frac{1}{2i}$$

and that integral is then $-\pi$.

A contour that includes both singularities integrates to zero.

**example**

In this section we get more practice.

$$\oint_C \frac{e^z}{z^2 - 2z - 3} \ dz$$

The denominator can be factored

$$z^2 - 2z - 3 = (z+1)(z-3)$$

if our disk contains $|z \le 2|$ then it includes only $z = -1$ and our formula

$$b_1 = \lim_{z \to z_0} (z - z_0) \ f(z)$$

so

$$
\begin{aligned}
b_1 &= \lim_{z \to -1} (z+1) \ \frac{e^z}{(z+1)(z-3)} \\
&= \lim_{z \to -1} \ \frac{e^z}{(z-3)} \\
&= \frac{e^{-1}}{-1-3} = -\frac{1}{4e}
\end{aligned}
$$

and

$$
\begin{aligned}
I &= 2\pi i \ b_1 \\
&= 2\pi i \ (-\frac{1}{4e}) \\
&= -\frac{\pi i}{2e}
\end{aligned}
$$

**example**

$$
\int \frac{5z-2}{z(z-1)} \ dz
$$

There are two simple poles at $z_0 = 0$ and $z_0 = 1$ and the residues are

$$
\begin{aligned}
\text{Res } (0) &= \lim_{z \to 0} (z-0) \ \frac{5z-2}{z(z-1)} \\
&= \lim_{z \to 0} \ \frac{5z-2}{(z-1)} \\
&= \frac{5 \cdot 0 - 2}{0-1} = 2 \\
\text{Res } (1) &= \lim_{z \to 1} (z-1) \ \frac{5z-2}{z(z-1)} \\
&= \lim_{z \to 1} \ \frac{5z-2}{z} \\
&= \frac{5 \cdot 1 - 2}{1} = 3
\end{aligned}
$$

Hence the total of all the residues is 5 and $I = 10\pi i$.

**example**

Consider

$$\int \frac{1}{z^4 - 1}\ dz$$

We can factor the denominator as

$$z^4 - 1 = (z^2 - 1)(z^2 + 1)$$

$$= (z+1)(z-1)(z+i)(z-i)$$

We see that there are four poles, and each will have a residue.

$$\text{Res}(1) = \lim_{z \to 1} (z-1)\ \frac{1}{(z+1)(z-1)(z+i)(z-i)}$$

$$= \lim_{z \to 1}\ \frac{1}{(z+1)(z+i)(z-i)}$$

$$= \frac{1}{2(1+1)} = \frac{1}{4}$$

$$\text{Res}(i) = \lim_{z \to i} (z-i)\ \frac{1}{(z+1)(z-1)(z+i)(z-i)}$$

$$\text{Res}(i) = \lim_{z \to i}\ \frac{1}{(z+1)(z-1)(z+i)}$$

$$= \lim_{z \to i}\ \frac{1}{(z^2-1)(z+i)}$$

$$= \frac{1}{(-2)(2i)} = -\frac{1}{4i} = \frac{i}{4}$$

**Removable singularities**

If the residue turns out to be equal to zero, that is a removable singularity.

$$I = \int_C \frac{\sin \pi z}{z^2 - 1}\ dz$$

The denominator can be factored into

$$z^2 - 1 = (z+1)(z-1)$$

Suppose $C$ includes only $z = 1$, then

$$\text{Res}(1) = \lim_{z \to 1}(z-1)\ \frac{\sin \pi z}{(z+1)(z-1)}$$

$$= \lim_{z \to 1}\ \frac{\sin \pi z}{(z+1)} = \frac{\sin \pi}{2} = 0$$

Here's a trick:

$$f(z) = z^2 \sin \frac{1}{z}$$

Compute Res(0)

$$\sin \frac{1}{z} = \frac{1}{z} - \frac{1}{3!}\ \frac{1}{z^3} + \frac{1}{5!}\ \frac{1}{z^5} \ldots$$

$$z^2 \sin \frac{1}{z} = z - \frac{1}{3!}\ \frac{1}{z} + \frac{1}{5!}\ \frac{1}{z^3} \ldots$$

The only non-zero integral term is

$$-\frac{1}{3!}\ \frac{1}{z}$$

and the residue there is

$$\lim_{z \to 0}\ (z-0)(-\frac{1}{3!}\ \frac{1}{z})$$

$$= -\frac{1}{3!} = -\frac{1}{6}$$

# Chapter 27

# Kaplan

We've seen a little bit about Laurent series, which can be written for $f(z)$ analytic in an annulus, in the form

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n + \sum_{n=1}^{\infty} b_n \cdot (z - z_0)^{1-n}$$

Kaplan gives some rules for computing residues, which we explore in this chapter.

**RULE I** At a simple pole $z_0$ (that is, a pole of first order),

$$\text{Res}\,[f(z), z_0] = \lim_{z \to z_0} (z - z_0) f(z_0)$$

**example**

The example is one we already worked.

$$f(z) = \frac{1}{z^2 + 1} = \frac{1}{(z - i)(z + i)}$$

$$\text{Res}\,[f(z), z = i] = \lim_{z \to i} (z - i) \frac{1}{(z - i)(z + i)}$$

$$= \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}$$

**review**

To summarize the key points about Cauchy's Integral formula and residues, this is the formula

$$\oint_C \frac{f(z)}{z - z_0} \ dz = 2\pi i \ f(z_0)$$

By definition the residue at a simple pole is defined to be

$$b_1 = \lim_{z \to z_0} (z - z_0) \ f(z_0)$$

Just think of it as in the limit that $z \to z_0$, the denominator $z - z_0$ on the left is a constant, so we can multiply both sides by $z - z_0$ to obtain the result for the residue.

$$\oint f(z) \ dz = 2\pi i \ \sum \ \text{Res}$$

The value of the integral is $2\pi i$ times the sum of all the residues enclosed by the path.

**example**

$$\oint_C \frac{e^z}{z^3 - z^2 - 5z - 3} = \oint_C \frac{e^z}{(z+1)^2(z-3)}$$

Recall the general formula

$$f'(a) = \frac{1}{2\pi i} \ \oint_C \frac{f(z)}{(z-a)^2} \ dz$$

If the contour includes $z = -1$ but not $z = 3$ then

$$f(z) = \frac{e^z}{(z-3)}$$

so

$$f'(z) = \frac{(z-4)e^z}{(z-3)^2}$$

Hence

$$\oint_C \frac{e^z}{z^3 - z^2 - 5z - 3}\ dz = \oint_C \frac{e^z}{(z+1)^2(z-3)}$$

$$= \oint_C \frac{f(z)}{(z+1)^2}\ dz$$

for $f(z) = e^z/(z-3)$ and

$$= 2\pi i f'(-1) = 2\pi i\ \frac{-5}{e}\ \frac{1}{(-4)^2}$$

$$= \frac{-5\pi i}{8e}$$

**Kaplan**

Here is rule II from Kaplan. Rule I is in the previous section.

**RULE II** At a pole of order $N$ $(N = 2, 3, \dots)$,

$$\text{Res}\ [f(z), z_0] = \lim_{z \to z_0} (z - z_0) \frac{g^{(N-1)}(z)}{(N-1)!}$$

where

$$g(z) = (z - z_0)^N f(z)$$

**example**

$$f(z) = \frac{1}{z(z-2)^2}$$

We have a pole of first order at $z_0 = 0$ and one of second order at $z_0 = 2$. At the first

$$\text{Res }(0) = \lim_{z \to 0} \frac{1}{(z-2)^2} = \frac{1}{4}$$

For the other one, remove the factor of $1/(z-2)^2$ and compute the $N-1$ (first) derivative of what's left

$$\frac{d}{dz}\frac{1}{z} = -\frac{1}{z^2}$$

$$\text{Res }(2) = \lim_{z \to 2} -\frac{1}{z^2} = -\frac{1}{4}$$

Don't forget to divide by $(N-1)!$, which is just 1 in this case. The total is just zero.

As a check, let's do this by partial fractions.

$$\frac{1}{z(z-2)^2} = \frac{A}{(z-2)^2} + \frac{B}{z(z-2)} + \frac{C}{z}$$

Hence in putting all terms over a common denominator, for the numerator we have

$$1 = Az + B(z-2) + C(z-2)^2$$

From which we get three equations:

$$-2B + 4C = 1$$

$$Az + Bz - 4Cz = 0$$

$$Cz^2 = 0$$

Hence $C = 0$, so $B = -1/2$ and $A = 1/2$ and we obtain

$$\frac{1}{z(z-2)^2} = \frac{1/2}{(z-2)^2} - \frac{1/2}{z(z-2)}$$

which we check by doing

$$1/2 \cdot z - 1/2 \cdot (z-2) = 1$$

So how to deal with

$$\frac{1/2}{(z-2)^2} - \frac{1/2}{z(z-2)}$$

The first term has a pole of order 2 at $z_0 = 2$. We remove that factor and compute the $N-1$ (first) derivative of what's left, which is just zero.

For the second term, we have two simple poles at $z_0 = 0$ and $z_0 = 2$. The residues are

$$\text{Res } (0) = \lim_{z \to 0} z \cdot \frac{1}{z(z-2)} = -\frac{1}{2}$$

$$\text{Res } (2) = \lim_{z \to 2} (z-2) \cdot \frac{1}{z(z-2)} = \frac{1}{2}$$

which adds up to zero.

**example**

$$f(z) = \frac{1}{z^4 + z^3 - 2z^2}$$

where $C$ is the circle $|z| = 3$ with positive orientation.

The denominator can be factored as

$$z^2(z^2 + z - 2) = z^2(z+2)(z-1)$$

so

$$f(z) = \frac{1}{z^2(z+2)(z-1)}$$

There is a pole of order 2 at the origin and simple poles at 1 and -2. All of these lie within the contour $|z| = 3$.

$$\text{Res } (1) = \lim_{z \to 1} (z-1)\ f(z) = \lim_{z \to 1} \frac{1}{z^2(z+2)} = \frac{1}{3}$$

$$\text{Res } (-2) = \lim_{z \to -2} (z+2)\ f(z) = \lim_{z \to -2} \frac{1}{z^2(z-1)} = -\frac{1}{12}$$

For the double pole, we remove the factor of $1/z^2$, take the first derivative of what's left

$$\frac{d}{dz}\ \frac{1}{z^2+z-2} = \frac{(-1)(2z+1)}{(z^2+z-2)^2}$$

Evaluate at zero and obtain.

$$\text{Res } (0) = -\frac{1}{4}$$

The total of the residues is

$$\frac{1}{3} - \frac{1}{12} - \frac{1}{4} = 0$$

As Mathews and Howell say:

> The value 0 for the integral is not an obvious answer, and all of the preceding calculations are required to find it.

**example**

$$f(z) = \frac{1+e^z}{z^2} + \frac{2}{z}$$

We can break this up into its two component parts. For the first term, the pole is of order $m = 2$ at $z_0 = 0$. We remove the $z^2$ term and take the $m - 1 = 1$ derivative

$$(1 + e^z)' = e^z$$

Remember to divide by $(m - 1)!$, leaving $e^z$ which is evaluated at the pole giving a residue

$$\text{Res }(0) = e^0 = 1$$

The other term is just 2 times the standard

$$\oint \frac{1}{z}\, dz = 2\pi i$$

Here $I = 4\pi i$ and the residue is 2. Alternatively just use

$$I = 2\pi i f(z_0) = 4\pi i$$

where $f = 2$.

The total of the residues is 3 and the value of the integral is $6\pi i$.

**example**

$$f(z) = \frac{e^z}{z(z-1)^2}$$

We have a pole of first order at $z = 0$ and one of second order at $z = 1$. At the first

$$\text{Res }[f(z), z = 0] = \lim_{z \to 0} \frac{e^z}{(z-1)^2} = 1$$

For the other one, remove the factor of $1/(z - 1)^2$ and compute the $N - 1$ (first) derivative of what's left

$$\text{Res }[f(z), z = 1] = \lim_{z \to 1} [\ \frac{e^z}{z}\ ]'$$

$$= \frac{e^z z - e^z}{z^2} \Big|_1 = 0$$

Hence

$$\oint f(z)\ dz = 2\pi i\ [\ \sum \text{Res}\ \ ]\ = 2\pi i$$

**example**

$$f(z) = \frac{1}{z(z-2)^4}$$

We have a pole of first order at $z = 0$ and one of fourth order at $z = 2$. At the first

$$\text{Res}\ [f(z), z = 0] = \lim_{z \to 0} z\ \frac{1}{z(z-2)^4}$$

$$= \lim_{z \to 0}\ \frac{1}{(z-2)^4} = \frac{1}{(-2)^4} = \frac{1}{16}$$

For the other pole recall that

$$\frac{2\pi i}{n!} f^n(a) = \oint_C \frac{f(z)}{(z-a)^{n+1}}\ dz$$

We remove the factor of $1/(z-2)^4$ leaving $f(z) = 1/z$ and then compute the $N-1$ (third) derivative of what's left

$$\text{Res}\ [f(z), z = 2] = \frac{1}{n!}\ \lim_{z \to 2}\ [\ \frac{1}{z}\ ]'''$$

$$f(z) = z^{-1}$$

$$f'(z) = -z^{-2}$$

$$f''(z) = 2z^{-3}$$

$$f'''(z) = -6z^{-4}$$

$$\lim_{z \to 2} \left[ \frac{1}{z} \right]''' = -\frac{6}{16}$$

Don't forget to divide by $(N-1)!$, which is $3! = 6$ in this case. That leaves

$$\text{Res}\, [f(z), z = 2] = -\frac{1}{16}$$

The total of the residues is just zero.

This problem is from Brown and Churchill (p. 234), which they work by doing Laurent series. They get a different answer, namely $-\pi i/8$.

The reason is that they integrate over the contour $0 < |z-2| < 2$, which includes the second pole, but not the first. Multiplying by $2\pi i$ gives their result.

# Chapter 28

# Summary 3

Cauchy's residue formula was

$$\oint \frac{f(z)}{z - z_0}\ dz = 2\pi i \cdot f(z_0)$$

Two corollaries:

$$\oint \frac{f(z)}{(z - w)^2}\ dz = 2\pi i \cdot f'(z_0)$$

generally

$$\oint \frac{f(z)}{(z - z_0)^{n+1}}\ dz = \frac{1}{n!}\ 2\pi i \cdot f^n(z_0)$$

The residue of $f(z)$ at a singularity $z_0$ is the coefficient of the $(z - z_0)^{-1}$ term in the Laurent series for $f(z)$, if we can write it.

It is also given by

$$\begin{aligned} R &= \lim_{z \to z_0} (z - z_0) \cdot f(z_0) \\ &= \frac{1}{2\pi i} \oint f(z)\ dz \end{aligned}$$

# Part VII

# Addendum

# Chapter 29

# Extra

In what follows $z = re^{is}$ or $z = x + iy$, as convenient.

We also use $w = \rho e^{it}$ or $w = u + iv$.

Properties of the conjugate: P1

$$(z + w)^* = x - iy + u - iv = z^* + w^*$$

P2

$$(zw^*)^* = (re^{is}\rho e^{-it})^* = (r\rho e^{i(s-t)})^* = r\rho e^{i(t-s)} = z^* w$$

P3

$$z + z* = x + iy + x - iy = 2x = 2Re\ z$$

P4

$$|z| = |z^*|$$

Length of a product:

$$|z||w| = |re^{is}|\ |\rho e^{-it}| = r\ \rho = |zw|$$

### Triangle Inequality

$$|z + w|^2 = (z + w)(z + w)^*$$

$z + w$ is a complex number. The square of the length of a complex number is equal to the number multiplied by its modulus.

$$= (z + w)(z^* + w^*)$$

P1 above.

$$= zz^* + zw^* + z^*w + ww^*$$

Distributivity of multiplication.

$$= zz^* + zw^* + (zw^*)^* + ww^*$$

P2 above.

$$= |z|^2 + 2Re(zw*) + |w|^2$$

From the definition of the conjugate and P3.

Now we transition to the inequality

$$\leq |z|^2 + 2|zw^*| + |w|^2$$

since the real part of a complex number is less than or equal to its length (only equal if it is purely real).

$$= |z|^2 + 2|z||w^*| + |w|^2$$

See the length of a product, above.

$$= |z|^2 + 2|z||w| + |w|^2$$

P4 above.

$$= (|z| + |w|)^2$$

from basic multiplication.

The inequality follows from taking the square root of the first and last expressions:

$$|z + w| \leq |z| + |w|$$

# Chapter 30

# Cubics

### quadratic

Some of the earliest examples of problems where the square root of a negative number arises involve a right triangle of a specified area and perimeter.

Nahin says, suppose a right triangle has area 7 and perimeter 12. Find the two sides.

Label the sides as $a$ and $b$.

We can get some idea of where this problem is headed by supposing that the triangle is also isosceles with $a = b$. Then

$$\frac{1}{2}ab = 7$$

$$ab = 14$$

$$a^2 = 14$$

so $a = \sqrt{14}$, and the perimeter is

$$p = a + b + \sqrt{a^2 + b^2}$$

$$= 2\sqrt{14} + \sqrt{14 + 14} = 12.77$$

The perimeter we are given is smaller than that

However, an isosceles right triangle has the smallest possible perimeter for a given area (the largest area for a given perimeter), hence there is no such pair $a, b$. The problem as posed has no solution.

Proof:

Let $k$ be a constant and $x$ and $k - x$ be the given sides. The area is

$$A = \frac{1}{2}x(k - x)$$

$$= -\frac{1}{2}x^2 + \frac{k}{2}x$$

The extreme point is

$$\frac{dA}{dx} = 0 = -x + \frac{k}{2}$$

$$x = \frac{k}{2}$$

The second derivative is $-1 < 0$, which shows that this is a minimum. We can also see the same thing from the negative cofactor of $x^2$ in the equation.

$$A = -\frac{1}{2}x^2 + \frac{k}{2}x$$

Doing the algebra of the original problem anyway, we solve two simultaneous equations

$$a \cdot b = 14$$

$$p = a + b + \sqrt{a^2 + b^2} = 12$$

Isolate and then remove the square root in the second one

$$a^2 + b^2 = (12 - a - b)^2$$

$$= 12^2 - 12a - 12b - 12a + a^2 + ab - 12b + ab + b^2$$

Collect terms and cancel $a^2$ and $b^2$

$$0 = 12^2 - 24a - 24b + 2ab$$

$$0 = 72 - 12a - 12b + ab$$

Substituting from the first equation given above

$$0 = 72 - 12a - 12 \cdot \frac{14}{a} + 14$$

$$0 = 36 - 6a - 6 \cdot \frac{14}{a} + 7$$

$$-6a^2 + 43a - 84 = 0$$

$$6a^2 - 43a + 84 = 0$$

To solve this, use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

However, $b^2 = 43^2 = 1849$ is less than $4ac = 4 \cdot 6 \cdot 84 = 2016$. We end up with

$$x = \frac{43 \pm \sqrt{-167}}{12}$$

The classic answer at this point is just to say, these values do not exist. The graph would be a parabola opening up ($a > 0$) whose vertex lies above the $x$-axis.

But suppose these two *complex* roots do have meaning.

They are, of course, complex conjugates: $p + iq$ and $p - iq$. If they are substituted into the factored form of the quadratic:

$$y = \;[\; x - (p + iq) \;]\; [\; x - (p - iq) \;]$$

$$y = \ [\ x - p - iq)\ ]\ [\ x - p + iq)\ ]$$
$$= x^2 - px + iqx - px + p^2 - ipq - iqx + ipq + q^2$$
$$= x^2 - 2px + p^2 + q^2$$
$$y = (x - p)^2 + q^2$$

$p$ is the value of $x$ at the vertex, corresponding to the minimum value of $y$, which is equal at that point to $q^2$.

Recall that the slope is

$$y' = 2ax + b$$

at the minimum, it equals zero so

$$0 = 2ax + b$$
$$x = -\frac{b}{2a}$$

This value of $x$ makes the factored form equal to zero. It is also the first term in

$$-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

**cubics**

In general, people just ignored problems with negative square roots — sometimes explicitly — until Cardano came to cubic polynomials.

Briefly, he discovered that any cubic like

$$x^3 + ax^2 + bx + c = 0$$

can be converted to a *depressed* cubic of the form

$$x^3 + px + q = 0$$

Any cubic has either one real root and two complex ones, or else three real roots.

We need to look at Cardano's formula to solve the depressed cubic. He was actually solving a problem like

$$x^3 + mx = n$$

(with $m$ and $n$ both positive), we re-write this as

$$x^3 + mx - n = 0$$

Define

$$r = \frac{n}{2}, \qquad s = \frac{m^3}{27}$$

Then Cardano showed that a real root (the only one or one of the three) is

$$x = \; [\; r + \sqrt{r^2 + s}\; ]^{1/3} + \; [\; r - \sqrt{r^2 + s}\; ]^{1/3}$$

Clearly, depending on the values of $m$ and $n$, and thus $r$ and $s$, $r^2 + s$ may be a negative number.

Cardano could not just ignore this issue, because the formula works to give a real result. He struggled with this.

Even today, it is hard to see the resolution because of the cube root.

Write what is in the brackets as a generalized complex number, in polar format

$$x = \; [\; re^{i\theta}\; ]^{1/3} + \; [\; re^{-i\theta}\; ]^{1/3}$$

Recall that complex multiplication goes like so:

$$r_1 e^{i\theta} \; r_2 e^{i\phi} = r_1 r_2 e^{i(\theta + \phi)}$$

So a cubic is

$$re^{i\theta} \cdot re^{i\theta} \cdot re^{i\theta} = r^3 e^{i3\theta}$$

With a change of variable, complex eponentiation is as follows:

$$[\ re^{i\theta}\ ]^{1/3} = r^{1/3}\ e^{i\theta/3}$$

$$[\ re^{-i\theta}\ ]^{1/3} = r^{1/3}\ e^{-i\theta/3}$$

The cube roots of complex conjugates are also complex conjugates!

When added together, the imaginary parts cancel, leaving an entirely real result.

$$x = r^{1/3}\ (e^{i\theta/3} + e^{-i\theta/3})$$

The term in brackets is clearly a sum $z + z*$, which is real, with the value twice the real component of the complex number.

**example**

Let's figure out an example arithmetically. The math is a little messy but we'll try to get through it. One of the problems studied by Cardano is

$$x^3 = 15x + 4$$

All terms are positive, which is typical for the time. We try the solution $x = 4$ and find it works out.

The Tartaglia formula gives

$$r = 4/2 = 2$$

$$s = (-15)^3/27 = -125$$

so we have that

$$x = \ [\ r + \sqrt{r^2 + s}\ ]^{1/3} + \ [\ r - \sqrt{r^2 + s}\ ]^{1/3}$$

$$x = \ [\ 2 + \sqrt{-121}\ ]^{1/3} + \ [\ 2 - \sqrt{-121}\ ]^{1/3}$$

This is easy to solve if one happens to know that

$$(2 \pm \sqrt{-1})^3 = 2 \pm \sqrt{-121}$$

Hence

$$x = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4$$

Let's try to calculate this:

$$(2 + \sqrt{-1})^3 = 2 + \sqrt{-121}$$

Usually, we would think that the polar format would make for easier calculation. However, let's go forward using the Cartesian format

$$\begin{aligned}(2+i)^3 &= (3+4i)(2+i)\\ &= 6 - 4 + 11i\\ &= 2 + 11i\end{aligned}$$

Pretty easy.

To use the polar format, let's compute the cube root:

$$(2 + 11i)^{1/3} = ?$$

We need the polar form of $2 + 11i$. We obtain

$$r = \sqrt{2^2 + 11^2} = \sqrt{125}$$

$$\theta = \tan^{-1} 11/2 = 1.391$$

Then

$$r' = r^{1/3} = \sqrt{5}$$

$$\theta' = \theta/3 = 0.46346$$

To convert back to Cartesian coordinates:

$$x = r \cdot \cos\theta = \sqrt{5} \cdot 0.8944 = 2.0$$

$$y = r \cdot \sin\theta = \sqrt{5} \cdot 0.4472 = 1.0$$

The result is $2 + i$, as expected.

**example**

Here is another problem from Nahin showing that the real component of a complex solution may have application in the real world.

> Imagine that a man is running at his top speed of $v$ feet per second, to catch a bus that is stopped at a traffic light. When he is still a distance of $d$ feet from the bus, the light changes and the bus starts to move away from the running man with a constant acceleration of $a$ feet per second per second. When will the man catch the bus?

Let the origin of coordinates be the traffic light and $x_m$ and $x_b$ be the positions of the man and the bus. At $t = 0$, $x_b = 0$ and $x_m = -d$. For an arbitrary time $t$

$$x_b = \frac{1}{2}at^2$$
$$x_m = -d + vt$$

If the man is to catch the bus at $t = T$, the positions are the same

$$x_m(T) = x_b(T)$$

$$-d + vT = \frac{1}{2}aT^2$$

This is a quadratic

$$\frac{1}{2}aT^2 - vT + d = 0$$

In general, the solution for $T$ may be complex, if

$$v^2 - 2ad < 0$$

Rearranging

$$d > v^2/2a$$

For such values there is no catching the bus.

Nahin rearranges the equation to give

$$T^2 - 2\frac{v}{a}T + 2\frac{d}{a} = 0$$

The quadratic formula gives

$$\begin{aligned} T &= \frac{2v/a \pm \sqrt{4v^2/a^2 - 8d/a}}{2} \\ &= \frac{v}{a} \pm \sqrt{v^2/a^2 - 2d/a} \end{aligned}$$

Even for a complex result, the real part is

$$T = \frac{v}{a}$$

But notice: the separation between the man and the bus is

$$\begin{aligned} s &= x_b - x_m \\ &= \frac{1}{2}at^2 + d - vt \end{aligned}$$

At what time is the man closest to the bus? That occurs when

$$\frac{ds}{dt} = at - v = 0$$

$$t = \frac{v}{a}$$

This is the real part of the result above.

If the man does catch the bus ($\sqrt{v^2/a^2 - 2d/a}$ is real), it worth thinking about the two solutions to the quadratic. Which is the correct one and what is the meaning of the second?

# Chapter 31

# References


- Beck et al. *A first course in complex analysis.*
- Boas. *Mathematical methods in the physical sciences.*
- Brown and Churchill. *Complex variables and applications.*