# Complex analysis

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# Part I

# Introduction

# Chapter 1

# Preface

One motivation for learning about complex functions is that the theory is often described as being very beautiful. It also shows how certain more difficult integrals can be solved.

Marsden gives three examples that he says are either very difficult or impossible if we are restricted to just the real numbers:

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\, dx = \frac{\pi}{\sin \alpha\pi}$$

$$\int_0^{2\pi} \frac{1}{a+\sin\theta}\, d\theta = \frac{2\pi}{\sqrt{a^2-1}}$$

Here's another from Nahin.

$$\int_{-\infty}^\infty \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{e}$$

Maybe we can learn to solve these before we're done.

# Chapter 2

# Arithmetic

Consider the functions

$$x^2 + 1 = 0$$

and

$$x^2 + x + 1 = 0$$

For the first equation, it is easy to see that there is no solution among the real numbers since $x^2$ is always positive or zero. So adding 1 to $x^2$ cannot bring the sum back to zero.

Visualizing the same function geometrically, this is just the simple parabola $y = x^2$ shifted up by one unit, moving its vertex from $(0, 0)$ to $(0, 1)$. Plotting shows that the graphs of both the above functions never cross the $x$-axis—there are no values that lie on the curve and also on the line $y = 0$.

It is often said that complex numbers arose in the context of finding solutions to such polynomials, however, as Nahin writes in his book *An imaginary tale*, this is not really true. We'll explore some of this history in the next chapter.

The ingenious solution to this problem was to invent a new kind of number

$$i = \sqrt{-1}$$

$$i^2 = -1$$

Once we accept that $i = \sqrt{-1}$ then we can factor

$$(x + i)(x - i) = x^2 - i^2$$

$$= x^2 - (-1) = x^2 + 1$$

so $x = \pm\, i$ are both solutions to the equation

$$(x + i)(x - i) = 0$$

For the second one

$$x^2 + x + 1 = 0$$

we can plot it, or we can recall the quadratic formula for solutions to

$$ax^2 + bx + c = 0$$

for real constants $a$, $b$ and $c$. The formula is

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

When $4ac > b^2$, then the solutions to the quadratic formula involve the square root of a negative number. Here the formula gives

$$x = \frac{1}{2}(-1 \pm \sqrt{-3})$$

Take the positive root and square it

$$x^2 = \frac{1}{4}(-1 + \sqrt{-3})^2$$

$$= \frac{1}{4}(-2 - 2\sqrt{-3})$$
$$= \frac{1}{2}(-1 - \sqrt{-3})$$

Adding this to $x + 1$ we obtain

$$x^2 + x + 1$$
$$= \frac{1}{2}(-1 - \sqrt{-3}) + \frac{1}{2}(-1 + \sqrt{-3}) + 1$$

the terms with $\sqrt{-3}$ cancel, giving

$$= -\frac{1}{2} - \frac{1}{2} + 1 = 0$$

In fact, now that we have $i$ available, any square root like $\sqrt{-(a^2)}$, where $a$ is a real number, can be factored as $\sqrt{-1}\ \sqrt{a^2} = ia$.

**warning**

Note that the converse is not necessarily true. Consider

$$i^2 = \sqrt{-1} \cdot \sqrt{-1} \stackrel{?}{=} \sqrt{(-1) \cdot (-1)} = \sqrt{1}$$

Now, $\sqrt{1}$ has two solutions or roots (since $-1 \times -1$ and $1 \times 1$ are both equal to 1), but we choose the positive root when thinking about $\sqrt{x}$ as a *function*. However, $i^2$ was defined to be equal to $-1$, not 1. What's the deal?

The problem is that the equality with a question mark is not valid

$$\sqrt{-1} \cdot \sqrt{-1} \neq \sqrt{(-1) \cdot (-1)}$$

which explains why this "proof" is erroneous.

Expressions that involve the square root of a negative real number, like $\sqrt{-1} = i$ and $\sqrt{-3} = \sqrt{3}\, i$, are called imaginary (or *purely* imaginary).

Numbers that contain both a real and an imaginary part, like $1 + i$, are termed complex numbers, and imaginary numbers are considered to be complex numbers with the real part equal to 0.

The set of complex numbers $\mathbb{C}$ includes the real numbers:

$$\mathbb{R} \subset \mathbb{C}$$

or, as is written, that $\mathbb{R}$ is a subset of $\mathbb{C}$.

We write complex numbers $z$ as combinations like

$$z = a + ib$$

where $a$ and $b$ are both real numbers. $a$ is the real part, and $b$ the imaginary part of the complex number $z$.

Two useful identities come from factoring $i^2 = -1$:

$$i = -\frac{1}{i}$$

$$-i = \frac{1}{i}$$

It turns out that for much of what is done with complex numbers the fact that $i$ equals $\sqrt{-1}$ is not even relevant.

Instead, we simply think of *ordered pairs* of real numbers $(a, b)$ and the $i$ notation is a bookkeeping device, a marker to remind us that when we multiply two complex numbers

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd$$

the last term gets a minus sign:

$$ib \cdot id = -bd$$

The result of multiplying $ib \cdot id$ is a real number with the sign flipped, while a real number $a$ times an imaginary number $id$ is equal to $iad$ and

$$(a + ib)(c + id) = ac - bd + i(ad + bc)$$

### dual equality

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal

$$z_1 = z_2 \iff a = c \text{ and } b = d$$

*if and only if* both the real and the imaginary parts of $z_1$ and $z_2$ are equal.

### matrix form

Another idea to keep track of the same information is in matrix form, namely:

$$z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Such matrices can be added and multiplied in the normal way and give the desired results for complex numbers. Thus:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \times \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{bmatrix} = \begin{bmatrix} u & -v \\ v & u \end{bmatrix}$$

### Geometric interpretation

Yet another powerful way to think about complex numbers is to use the complex plane (sometimes called the Argand plane), where points are plotted with the real part along the horizontal axis and the imaginary part along the vertical axis.

This figure is from Brown & Churchill.

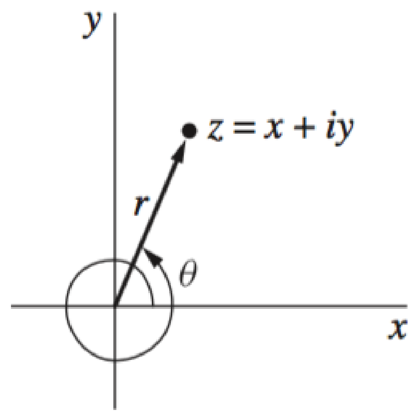



**FIGURE 6**

Looking at the graph, the distance of any point from the origin is denoted by $r$, and $\theta$ is the angle the ray makes with the positive $x$-axis in a CCW direction. This should be familiar from standard polar coordinates.

Switching notation to

$$z = x + iy$$

To plot the complex number $z$ we go out $x$ units along the real (horizontal) axis and then up $y$ units along the imaginary (vertical) axis.

The statement that $\mathbb{R} \subset \mathbb{C}$ is equivalent to the observation that the Argand plane contains the horizontal axis. Real numbers have the form $z = x + i \cdot 0 = x$.

More generally, though

$$x = r\cos\theta$$

$$y = r\sin\theta$$

and

$$\begin{aligned} x + iy &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

$$= re^{i\theta}$$

where the last part makes use of Euler's famous equation. $r$ is called the **modulus** and $\theta$ is called the **argument** or **phase**.

If you look very carefully at the figure above the argument $\theta$ is actually $\theta + 2\pi$.

All multiples $k \cdot 2\pi$ for $k \in 0, \pm 1, \pm 2 \ldots$ are valid.

Depending on the calculation one form is often easier to handle.

Addition is simpler with $a + ib$ (the Cartesian format) since

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

while multiplication is more straightforward with the polar format.

Matrices work well for both addition and multiplication.

Here is multiplication in polar coordinates

$$re^{i\theta}\ \rho e^{i\phi} = r\rho\ e^{i(\theta+\phi)}$$

We multiply the distances and add the angles. Here is the square function:

$$(re^{i\theta})^2 = r^2 e^{i2\theta}$$

Multiplication of $z_1 = r_1 e^{i\theta_1}$ by $z_2 = r_2 e^{i\theta_2}$ stretches $r_1$ (the length of $z_1$) by the factor $r_2$ (the length of $z_2$), and rotates $z_1$ by adding a phase shift of $\theta_2$ to the original angle $\theta_1$.

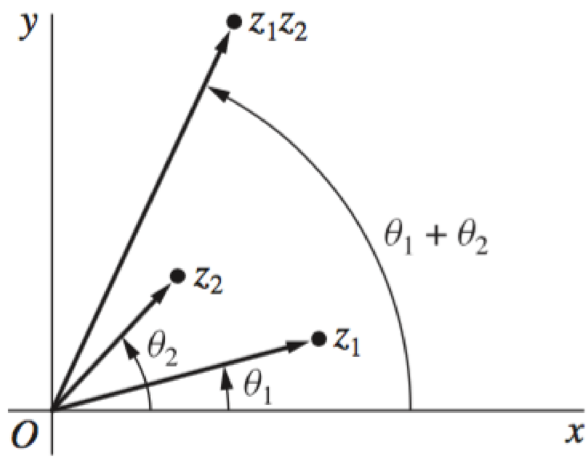



**FIGURE 9**

The person who originally discovered this representation was Caspar Wessel.

Since the calculations can be tedious, I wrote a Python script to do the calculations for roots and powers.

https://gist.github.com/telliott99/916bc75a73e515968debe48ef418d738

# Chapter 3

# Conjugate

Consider the complex number:

$$z = x + iy$$

The complex conjugate of $z$ (called $z*$ or $\bar{z}$) is given by:

$$z* = x - iy$$

The real part of $z*$ is the same as the real part of $z$, while the imaginary part has the sign switched.

### length of $z$

The *length* of $z$ squared is equal to $z$ multiplied by its complex conjugate

$$\begin{gathered} zz* = (x + iy)(x - iy) \\ = x^2 - ixy + ixy - i^2y^2 \\ = x^2 + y^2 \\ = (r\cos\theta)^2 + (r\sin\theta)^2 \\ = r^2 \end{gathered}$$

Again, $r$ is the length of the ray from the origin to $z$ as plotted in the complex plane.

$$r^2 = zz*$$

$$r = \sqrt{zz*}$$

The point corresponding to $z*$ in the complex plane has the same overall distance from the origin and the same $x$-component as $z$, but the sign change on $y$ means that $z*$ is reflected across the $x$-axis from $z$.

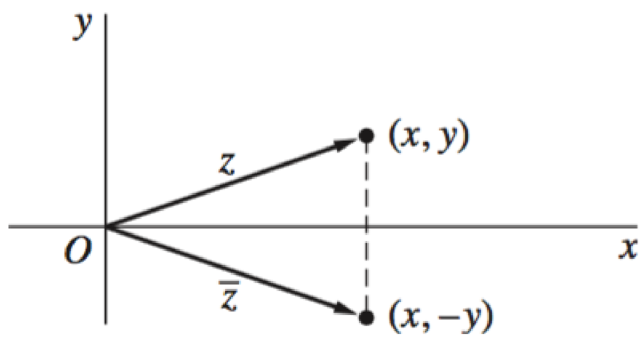



FIGURE 5

In polar coordinates, if $z = re^{i\theta}$ then $z* = re^{i(-\theta)} = re^{-i\theta}$. So

$$zz* = re^{i\theta}\ re^{i-\theta} = r^2e^0 = r^2$$

Multiplication of $z$ by $z*$ makes the product entirely real.

If we consider addition rather than multiplication of the complex conjugate we observe that it also gives an entirely real result:

$$z + z* = x + iy + x - iy = 2x$$

while subtraction gives an entirely imaginary result:

$$z - z* = x + iy - x + iy = i2y$$

**conjugate of several values**

Another result (that we state without proof) is that if we have an expression involving several complex numbers:

$$w = f(z_1, z_2 \dots)$$

we can obtain the complex conjugate of the whole thing by substituting the complex conjugate of each component:

$$w* = f(z_1*, z_2 * \dots)$$

As an example, let us compute the powers of $z$ and $z*$ using the binomial theorem:

$$z = x + iy$$
$$z^2 = x^2 + 2x(iy) + (iy)^2$$
$$z^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$
$$z^4 = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4$$

and the conjugate:

$$z* = x + (-iy)$$
$$(z*)^2 = x^2 + 2x(-iy) + (-iy)^2$$
$$(z*)^3 = x^3 + 3x^2(-iy) + 3x(-iy)^2 + (-iy)^3$$
$$(z*)^4 = x^4 + 4x^3(-iy) + 6x^2(-iy)^2 + 4x(-iy)^3 + (-iy)^4$$

It makes things simpler if we leave the minus signs and the powers of $i$ for the moment.

Now, any even power of $i$ is wholly real. So all we really need to do to form the conjugate is to switch the sign of the odd powers. Since they're odd powers, it makes no difference if we do this inside the parentheses or in front of each term.

So then,

$$\begin{aligned}(z^2)* &= x^2 - 2x(iy) + (iy)^2 \\ &= x^2 + 2x(-iy) + (iy)^2 \\ &= (z*)^2\end{aligned}$$

Furthermore, we can slip an extra minus sign inside any even power without changing the value:

$$\begin{aligned}(z^3)* &= x^3 - 3x^2(iy) + 3x(iy)^2 - (iy)^3 \\ &= x^3 + 3x^2(-iy) + 3x(iy)^2 + (-iy)^3 \\ &= x^3 + 3x^2(-iy) + 3x(-iy)^2 + (-iy)^3 \\ &= (z*)^3\end{aligned}$$

$$\begin{aligned}(z^4)* &= x^4 - 4x^3(iy) + 6x^2(iy)^2 - 4x(iy)^3 + (iy)^4 \\ &= x^4 - 4x^3(iy) + 6x^2(-iy)^2 - 4x(iy)^3 + (-iy)^4 \\ &= (z*)^4\end{aligned}$$

It is clear that this pattern will continue with higher powers.

# Part II

# Differentiation

# Chapter 4

# Difference quotient

This section contains a general discussion of differentiation of complex functions. A complex function takes as input a complex number, and emits as output another complex number. Often $w$ is used for the output:

$$w = f(z)$$

More concretely, a complex number is simply an ordered pair of real numbers, and a complex function is a pair of functions defined on the real numbers:

$$f(z) = f(x + iy) = u(x, y) + i \cdot v(x, y)$$

The functions $u$ and $v$ each take a pair of real numbers and emit one real number. They are connected in $f$ through the fact that $u$ and $v$ have the same input.

Finally, the output of $v$ is multiplied by $i$. Here is an example:

$$z = x + iy$$

$$z^2 = (x + iy)(x + iy)$$

$$= x^2 - y^2 + 2ixy$$

So

$$u(x,y) = x^2 - y^2$$
$$v(x,y) = 2xy$$

Another one is:

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{z*}{z*}$$
$$= \frac{x - iy}{(x+iy)(x-iy)}$$
$$= \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

Often a simplified notation is employed:

$$f(z) = u + i \cdot v$$

Since the inputs cover the entire complex plane, we cannot plot graphs as with real functions. Instead, one version of the complex plane is *mapped* by the function into a different version of the complex plane.

### Cauchy-Riemann

This chapter gives us a first glimpse of the important Cauchy-Riemann conditions and justifies one of the formulas for calculating the derivative

$$f'(z) = u_x + iv_x$$

As an example of its use, consider the complex exponential

$$f(z) = e^z$$

If we write $z = x + iy$ then

$$f(z) = e^{x+iy}$$
$$= e^x e^{iy}$$

and (from Euler):

$$e^{iy} = \cos y + i \sin y$$

so

$$f(z) = e^x \cos y + ie^x \sin y$$

Using the formula, it can be shown easily that the derivative is the same as the function itself, just as for the case of real numbers.

$$u(x, y) = e^x \cos y$$
$$u_x = e^x \cos y = u$$
$$v(x, y) = e^x \sin y$$
$$v_x = e^x \sin y = v$$

Hence

$$f'(z) = u_x + iv_x = z$$

**definition**

We define the derivative $f'(z)$ of a complex function $f(z)$ similarly to the derivative of a real function:

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

if the limit exists.

Alternatively, with $\Delta$ notation, we might write:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

A crucial difference from real functions is that there are only two directions from which to approach a given real number $x$, while there is an infinite number of ways of approaching $z$ in the Argand plane. The limit is over *all* possible ways of approaching $z$.

If the limit exists, the function $f$ is called differentiable and $f'(z)$ is the derivative. Consider

$$f(z) = u(x, y) + iv(x, y)$$

Then

$$f'(z) = \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}$$

**fixed y**

We tame this beast by looking at two specific paths.

Looking at the special path along the $x$-axis where $\Delta y = 0$ we obtain

$$f'(z) = \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x}$$

Rearrange the numerator

$$= \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + \frac{iv(x + \Delta x, y) - iv(x, y)}{\Delta x}$$

The first term is

$$u_x = \frac{\partial u}{\partial x}$$

and the second term is

$$iv_x$$

Hence we conclude that

$$f'(z) = u_x + iv_x$$

**fixed x**

Now look at the special path along the $y$-axis where $\Delta x = 0$:

$$f'(z) = \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y}$$

Rearrange the numerator

$$= \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + \frac{iv(x, y + \Delta y) - iv(x, y)}{i\Delta y}$$

$$= \frac{1}{i} u_y + v_y$$

Recall that $1/i = -i$

$$f'(z) = v_y - iu_y$$

**Putting it together**

We require that the limit be the same regardless of the direction of approach to $z$, so these two expressions for the difference quotient must be equal:

$$f'(z) = u_x + iv_x = -iu_y + v_y$$

Both the real and the imaginary parts must be equal so

$$u_x = v_y$$

$$u_y = -v_x$$

Once differentiability is established, we can use whichever path we want to evaluate the derivative.

As we said at the beginning, in looking at various complex functions we can use this fact:

$$f'(z) = u_x + iv_x$$

One consequence is that
$$\frac{df}{dz} = \frac{\partial f}{\partial x}$$
and since
$$\begin{aligned} &= u_x + iv_x = v_y - iu_y \\ &= -iu_y + v_y \\ &= -i(u_y + iv_y) \\ &= -i\frac{\partial f}{\partial y} \end{aligned}$$
We conclude that
$$\frac{df}{dz} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

**looking ahead**

When we get to integration in a later section we will find that the integral of a complex function is computed as a line integral along a specified curve (often a circle centered either on the origin or on a point $z_0$).

This curve relates the values of $x$ and $y$ and allows us to parametrize either $y$ in terms of $x$ or more generally, both $x$ and $y$ in terms of a single real variable or parameter $t$.

When we have a function of such a variable like

$$f(t) = u(t) + iv(t)$$

then the derivative is defined to be

$$f'(t) = u'(t) + iv'(t)$$

where $u$ and $v$ are real-valued functions of a single real variable and so follow the standard rules from introductory calculus. In particular if

$$w(t) = z_0 f(t)$$

then

$$w'(t) = z_0 f'(t)$$

The derivative of a constant times a function is the constant times the derivative of the function.

**derivative of $z_0$ times a function**

We can show this by using a little algebra:

$$\begin{aligned}
\frac{d}{dt}\ z_0 f(t) &= \ [\ (x_0 + iy_0)(u + iv)\ ]' \\
&= \ [\ (x_0 u - y_0 v) + i(y_0 u + x_0 v)\ ]' \\
&= (x_0 u - y_0 v)' + i(y_0 u + x_0 v)' \\
&= (x_0 u' - y_0 v') + i(y_0 u' + x_0 v') \\
&= (x_0 + iy_0)(u' + iv') \\
&= z_0 \frac{d}{dt}\ f(t)
\end{aligned}$$

Thus

$$\frac{d}{dt}\ z_0 f(t) = z_0 \frac{d}{dt}\ f(t)$$

which is what we just said.

## derivative of exp $z_0$ t

Another expected result is

$$\frac{d}{dt}\ e^{z_0 t} = z_0 e^{z_0 t}$$

where $z_0$ is a complex constant and $t$ is a real variable.

To do this one, refer to the definition

$$f'(t) = u'(t) + iv'(t)$$

And now we need to break up the exponential into its real and imaginary parts.

By Euler's equation, we wrote above

$$e^z = e^{x+iy} = e^x \cos y + ie^x \sin y$$

For the exponential of a real variable, but containing a complex constant we have

$$\begin{aligned} e^{z_0 t} &= e^{(x_0+iy_0)t} \\ &= e^{x_0 t}\ e^{iy_0 t} \\ &= e^{x_0 t}\ (\cos y_0 t + i \sin y_0 t) \\ &= e^{x_0 t} \cos y_0 t + ie^{x_0 t}\ \sin y_0 t \end{aligned}$$

## Substitution

I find this calculation very confusing. Especially the subscripts. Rather than change letters, we will drop the subscripts on $x_0$ and $y_0$ but tell ourselves repeatedly: these are constants. Also, $t$ is a *real* variable.

$$e^{xt} \cos yt + ie^{xt}\ \sin yt$$

Using the definition above we get that the derivative is $u'(t) + iv'(t)$ so the derivative of a sum is the sum of the derivatives.

The first term ($u'$) is (by the product rule):

$$[\ e^{xt} \cos yt\ ]' = xe^{xt} \cos yt - ye^{xt} \sin yt$$

and the second:

$$[\ e^{xt}\ \ \sin yt\ ]' = xe^{xt} \sin yt + ye^{xt} \cos yt$$

Remember that each term in that second one gets an $i$!

$$i[\ e^{xt}\ \ \sin yt\ ]' = ixe^{xt} \sin yt + iye^{xt} \cos yt$$

Combine the first term from each and factor out the $x$:

$$x(e^{xt} \cos yt + ie^{xt} \sin yt)$$

Do the same with the second term:

$$y(ie^{xt} \cos yt - e^{xt} \sin yt)$$

the tricky part

$$= iy(e^{xt} \cos yt + ie^{xt} \sin yt)$$

Putting everything together we have just

$$(x + iy)(e^{xt} \cos yt + ie^{xt} \sin yt)$$

Restoring the original naughts, we have just

$$z_0 e^{z_0 t}$$

As promised.

□

# Chapter 5

# Proofs of CRE

## More proofs of CRE

### difference quotient

We gave a first proof in the section on differentiation which is repeated more briefly here.

The derivative $f'(z)$ is defined to be the limit of the following difference quotient, if the limit exists.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

where $f(z) = u(x, y) + iv(x, y)$.

The difference quotient is rewritten in terms of $u$ and $v$ as:

$$\frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x) - iv(y)}{\Delta x + i\Delta y}$$

Then we consider two special cases, one where $\Delta y = 0$ and a second where $\Delta x = 0$. The first case yields

$$f'(z) = u_x + iv_x$$

and the second yields

$$f'(z) = -iu_y + v_y$$

The derivative is required to be the same for all directions of approach to the point, so we can equate the two expressions

$$u_x + iv_x = -iu_y + v_y$$

Since both the real and the imaginary parts must be equal, we obtain the CRE:

$$u_x = v_y$$

$$u_y = -v_x$$

□

**chain rule**

Here is a second approach:

Write:

$$z = x + iy$$

Clearly,

$$\frac{\partial z}{\partial x} = 1, \qquad \frac{\partial z}{\partial y} = i$$

Now,

$$\begin{aligned} w &= f(z) \\ &= u(x, y) + i\ v(x, y) \end{aligned}$$

where $u$ and $v$ are real functions over $\mathbb{R}^2$.

Recalling the chain rule

$$w = u(x, y) + i\ v(x, y)$$

$$\frac{\partial w}{\partial x} = \frac{dw}{dz}\,\frac{\partial z}{\partial x}$$

by the result immediately above (that $\partial z/\partial x = 1$):

$$\frac{\partial w}{\partial x} = \frac{dw}{dz}$$

Similarly

$$\frac{\partial w}{\partial y} = \frac{dw}{dz}\,\frac{\partial z}{\partial y}$$

$$\frac{\partial w}{\partial y} = i\frac{dw}{dz}$$

Hence we can equate the two expressions for $dw/dz$:

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

Now if we actually compute the partials and plug them in to the last equation, we obtain:

$$u_x + iv_x = -i(u_y + iv_y) = v_y - iu_y$$

Both the real and the imaginary parts must be equal:

$$u_x = v_y$$

$$v_x = -u_y$$

These are (again) the CRE.

□

It is worth taking a breath for a moment and repeating what we just said: the derivative of a differentiable complex function $z$ (what we will call an analytic function) is

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

$$
\begin{aligned}
&= u_x + iv_x \\
&= -i(u_y + iv_y) \\
&= v_y - iu_y
\end{aligned}
$$

**Alder**

A third, very simple proof is given in Alder:

Suppose $f : C \to C$ is a function, taking $x + iy$ to $u(x, y) + iv(x, y)$, then the derivative is a matrix of partial derivatives:

$$
\begin{matrix} u_x & u_y \\ v_x & v_y \end{matrix}
$$

> the above matrix is the two dimensional version of the slope of the tangent line in dimension one. It gives the linear part (corresponding to the slope) of the affine map which best approximates $f$ at each point.
>
> But at any point $x + iy$, if $f$ is differentiable in the *complex* sense, this must be just a linear complex map, i.e. it multiplies by some complex number. So the matrix must be in our set of complex numbers. In other words, for every value of $x$ it looks like

$$
\begin{matrix} a & -b \\ b & a \end{matrix}
$$

for some real numbers $a, b$, which change with $x$.

Of course, this constraint leads directly to the CRE.

□

A very important point is that the CRE and analyticity and differentiability are all related (either a function has all these properties or none of them). For an analytic function, the rules for integration and differentiation are analogous to the real case. For example:

$$\int \frac{1}{3}z^2 \; dz = z^3$$

$$\frac{d}{dz}\;\frac{1}{z-z_0} = -\frac{1}{(z-z_0)^2}$$

We will see a lot more of this.

**McMahon**

Here is yet another proof which I found in McMahon.

https://www.amazon.com/Complex-Variables-Demystified-David-McMahon/dp/007154920X

I include it here because it explains Shankar's statement that by definition an analytic function has no dependence on $z*$. (The $\bar{z}$ notation is used below).

Write

$$z = x + iy, \quad \bar{z} = x - iy$$

so

$$2x = (z + \bar{z})$$

$$x = \frac{1}{2}(z + \bar{z})$$

$$2iy = (z - \bar{z})$$

$$y = \frac{1}{2i}(z - \bar{z}) = -\frac{1}{2}i(z - \bar{z})$$

Take partial derivatives:

$$\frac{\partial x}{\partial z} = \frac{1}{2} = \frac{\partial x}{\partial \bar{z}}$$

and

$$\frac{\partial y}{\partial z} = -\frac{1}{2i} = -\frac{\partial y}{\partial \bar{z}}$$

Then, using the chain rule we write:

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left[\ \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\ \right]$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial x}{\partial \bar{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar{z}}\frac{\partial}{\partial y} = \frac{1}{2}\left[\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\ \right]$$

Now apply the two operators (just a matter of a few minus signs):

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left[\ \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\ \right]\left[\ u + iv\ \right] = \frac{1}{2}\left[\ (u_x + v_y) + i(v_x - u_y)\ \right]$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\ \right]\left[\ u + iv\ \right] = \frac{1}{2}\left[\ (u_x - v_y) + i(v_x + u_y)\ \right]$$

And now to the point: we *require* the last expression to be zero. $f(z)$ must have no dependence on $\bar{z}$.

As usual, both the real and the imaginary parts must vanish.

$$0 = \frac{1}{2}\left[\ (u_x - v_y) + i(v_x + u_y)\ \right]$$

$$u_x = v_y, \quad v_x = -u_y$$

In other words, the CRE apply. And using these conditions, we can rewrite

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left[\ (u_x + v_y) + i(v_x - u_y)\ \right]$$
$$= u_x + iv_x$$

□

To put this another way, if we have already established the CRE, we can run this proof backwards to show that for $f(z)$, $\partial f/\partial \bar{z} = 0$.

Sources:

[1] Alder. *An Introduction to Complex Analysis for Engineers.*

http://www.eee.metu.edu.tr/~ccandan/EE202_summer2004/solutions/An%20Introduction%20to%20Complex%20Analysis%20for%20Engineers%20-%20Michael%20Alder.pdf

# Part III

# Functions

# Chapter 6

# Powers

**square**

Consider

$$f(z) = z^2$$
$$= (x + iy)(x + iy)$$
$$= x^2 - y^2 + i2xy$$
$$u(x, y) = x^2 - y^2$$
$$v(x, y) = 2xy$$

Note that

$$u_x = 2x = v_y$$
$$u_y = -2y = -v_x$$

The Cauchy-Riemann conditions (CRE) hold.

Compute the derivative as follows:

$$\frac{df}{dz} = u_x + iv_x$$
$$= 2x + i2y = 2z$$

or alternatively

$$\frac{df}{dz} = v_y - iu_y$$
$$= 2x - i(-2y) = 2x + i2y = 2z$$

This is the result we would expect to get by simply differentiating f(z) as if it was a real function. For analytic functions this will always be the case.

**cube**

Let

$$f(z) = z^3 = (x + iy)^3$$
$$= (x + iy)(x^2 - y^2 + i2xy)$$
$$= x^3 - xy^2 + i2x^2y + ix^2y - iy^3 - 2xy^2$$
$$= x^3 - 3xy^2 + i(3x^2y - y^3)$$

So

$$u(x, y) = x^3 - 3xy^2$$
$$v(x, y) = 3x^2y - y^3$$

and

$$u_x = 3x^2 - 3y^2$$
$$v_x = 6xy$$

That means

$$\frac{df}{dz} = u_x + iv_x$$
$$= 3x^2 - 3y^2 + i6xy$$
$$= 3\ [x^2 - y^2 + i2xy\ ]$$
$$= 3z^2$$

We could continue and show that $z^n$ is analytic for any positive integer power of $n$. Notice the pattern for $i$:

$$(x+iy)^n = x^n + nx^{n-1}(iy) + (n)(n-1)x^{n-2}(iy)^2 + \dots$$

The progression goes:

$$i^0, i^1, i^2, i^3$$
$$= 1, i, -1, -i$$

and then repeats.

**inverse**

Using the complex conjugate is a good way to work with the inverse function (or with division by any complex number):

$$\frac{1}{z} = \frac{z*}{zz*} = \frac{x-iy}{x^2+y^2}$$

or in polar notation:

$$\frac{1}{z} = \frac{re^{-i\theta}}{r^2\ e^{i\theta}\ e^{-i\theta}} = \frac{1}{r}\ e^{-i\theta}$$

Let's look at what it means to take the inverse for different $z$. In every case, the point is reflected across the $x$-axis (the ray makes an angle $-\theta$ with the $x$-axis).

There is no change in length for $r = 1$. But if say

$$z = 1 + i = (1,1) = \sqrt{2}\ e^{i \cdot \pi/4}$$

then the new point has $r = \frac{1}{\sqrt{2}}$ and it is located at

$$\frac{1}{z} = \frac{1}{\sqrt{2}} e^{-i \cdot \pi/4} = (\frac{1}{2}, -\frac{1}{2}) = \frac{1}{2} - i\frac{1}{2}$$

Differentiate

$$
\begin{aligned}
f(z) &= 1/z \\
&= \frac{z*}{zz*} \\
&= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}
\end{aligned}
$$

Now let's do the partial derivatives. $u(x,y)$ has $x$ in both the numerator and the denominator.

Recall the quotient rule (reusing the symbols $g$ and $h$):

$$(g/h)' = (g'h - gh')/h^2$$

So:

$$
\begin{aligned}
u_x &= \frac{1}{(x^2+y^2)^2}\ (x^2+y^2-2x^2) \\
&= \frac{y^2-x^2}{(x^2+y^2)^2}
\end{aligned}
$$

To do $v_y$ just switch $x$ and $y$ in the result, but remember to then multiply by the leading factor of $-1$:

$$v_y = (-1)\ \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Thus $u_x = v_y$

Now for the other ones we have that

$$u(x,y) = \frac{x}{x^2+y^2}$$

which only involves $y$ in the denominator so the derivative is just

$$u_y = x\ \frac{1}{(x^2+y^2)^2}\ (-1)(2y)$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

For $v_x$ we have

$$v(x,y) = \frac{-y}{x^2+y^2}$$

Again, we have $x$ only in the denominator so the derivative is

$$(-y)\ \frac{1}{(x^2+y^2)^2}\ (-1)(2x)$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

We see that $u_y = -v_x$.

So the CRE are satisfied (except at $z = 0$) and the derivative is

$$\frac{df}{dz} = u_x + iv_x$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2}$$

$$= \frac{1}{(x^2+y^2)^2}\ (y^2 - x^2 + i2xy)$$

We expect that this should be (in disguise) $-1/z^2$. Let's see:

$$\frac{1}{z} = \frac{z*}{zz*}$$

$$\frac{1}{z^2} = \frac{(z*)^2}{(zz*)^2}$$

The denominator is certainly correct since

$$zz* = x^2 + y^2$$

What about

$$(z*)^2 = (x - iy)(x - iy) = x^2 - y^2 - i2xy$$

so

$$-(z*)^2 = (-1)(x - iy)(x - iy) = y^2 - x^2 + i2xy$$

Everything checks.

**powers: de Moivre's formula**

Let $n$ be an integer:

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

Suppose $r = 1$:

$$z^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$$

This is de Moivre's formula.

Suppose $n = 2$, then

$$(\cos\theta + i\sin\theta)^2$$

$$= \cos^2\theta - \sin^2\theta + i\ 2\sin\theta\cos\theta$$

Equating with the right-hand side of de Moivre's formula:

$$\cos^2\theta - \sin^2\theta + i\ 2\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$$

we find that

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

We already know these, they are the double angle formulas.

Suppose $n = 3$, then

$$(\cos\theta + i\sin\theta)^3$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

we find that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

and so on.

We can just check that last one for $\theta = \pi/6$:

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

$$1 = 3(\frac{\sqrt{3}}{2})^2\ \frac{1}{2} - (\frac{1}{2})^3$$

Multiply both sides by $2^3$:

$$8 = 3(\sqrt{3})^2 - 1$$

That looks correct.

# Chapter 7

# Exponential

First of all, we consider the generic complex number

$$z = x + iy$$

and then we write

$$\begin{aligned} f(z) &= e^z \\ &= e^{x+iy} \\ &= e^x e^{iy} \end{aligned}$$

We can thus visualize the complex exponential as having a modulus or length $e^x$ and argument or angle $\theta$ of $y$.

Then, using Euler's formula:

$$\begin{aligned} e^x e^{iy} &= e^x(\cos y + i \sin y) \\ &= e^x \cos y + ie^x \sin y \end{aligned}$$

So the real part of $e^z$ is

$$u(x, y) = e^x \cos y$$

with partial derivatives

$$u_x = e^x \cos y$$

$$u_y = -e^x \sin y$$

and the imaginary part of $e^z$ is

$$v(x, y) = e^x \sin y$$

with partial derivatives

$$v_x = e^x \sin y$$

$$v_y = e^x \cos y$$

Hence

$$u_x = e^x \cos y = v_y$$

$$u_y = -e^x \sin y = -v_x$$

In other words, these two important conditions hold for the complex exponential:

$$u_x = v_y$$

$$u_y = -v_x$$

These are the famous Cauchy-Riemann equations or CR conditions.

When the CRE are satisfied is then the function in question is a "good" function — it is one we can do calculus with. It has a derivative.

For this reason, the complex exponential $e^z$ is said to be analytic.

(Which, according to Shankar, we could have predicted, since it depends only on $z$ and not on $z*$).

We showed in the previous section that we can evaluate the derivative along $\Delta y = 0$ as:

$$f'(z) = u_x + iv_x$$

We obtain

$$= e^x \cos y + ie^x \sin y = z$$

That is: the exponential is its own derivative.

This is very good because we want our definitions for complex functions to give the standard results when $z$ has only a real part, i.e. when $y = 0$.

Now, once more we recall Euler's formula (for a real variable $\theta$ or $x$):

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{ix} = \cos x + i\sin x$$

Substitute $-x$ for $x$:

$$e^{-ix} = \cos -x + i\sin -x$$

$$= \cos x - i\sin x$$

By addition and subtraction we obtain:

$$2\cos x = e^{ix} + e^{-ix}$$

$$\cos x = \frac{1}{2}\,(e^{ix} + e^{-ix})$$

and

$$2i\sin x = e^{ix} - e^{-ix}$$

$$\sin x = \frac{1}{2i}\,(e^{ix} - e^{-ix})$$

**alternative derivations**

Another proof that the derivative of the complex exponential is as we would hope and expect:

$$\frac{d}{dz}e^z = e^z$$

uses a Taylor series. Shankar says to define $e^z$ in the same way as $e^x$. For the real series:

$$e^x = \sum_0^\infty \frac{x^n}{n!}$$

which we know converges, since the ratio of successive terms is

$$R = \frac{x^{n+1}}{(n+1)!} \ \frac{n!}{x^n} = \frac{x}{n+1}$$

We ask, for what values of $x$ is the limit $\lim_{n\to\infty} R = 0$? This is true for all $x$.

For the complex exponential:

$$e^z = \sum_0^\infty \frac{z^n}{n!}$$

and again we see that

$$\frac{d}{dz} \ e^z = e^z$$

differentiating the series term by term.

Another approach (from McMahon) uses the limit definition:

$$\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\frac{d}{dz} e^z = \lim_{\Delta z \to 0} \frac{e^{z+\Delta z} - e^z}{\Delta z}$$

and just as in the real case, we factor out

$$= e^z \lim_{\Delta z \to 0} \frac{e^{\Delta z} - 1}{\Delta z}$$

This limit will turn out to be equal to 1.

Use Euler's formula to get this expression in $x$ and $y$:

$$\frac{e^{\Delta z} - 1}{\Delta z} = \frac{e^{\Delta x + i\Delta y} - 1}{\Delta x + i\Delta y}$$

$$= \frac{e^{\Delta x}(\cos \Delta y + i \sin \Delta y) - 1}{\Delta x + i\Delta y}$$

$$= \frac{(e^{\Delta x} \cos \Delta y - 1) + i e^{\Delta x} \sin \Delta y}{\Delta x + i\Delta y}$$

The real part of the numerator is

$$\lim_{\Delta x, \Delta y \to 0} e^{\Delta x} \cos \Delta y - 1$$

Both the $\Delta x$ and the $\Delta y$ term tend to 1 in the limit, so the entire expression for the real part of the numerator is equal to zero. We are left with

$$\lim_{\Delta x, \Delta y \to 0} \frac{i e^{\Delta x} \sin \Delta y}{\Delta x + i\Delta y}$$

The trick is that we set $x = 0$ *first*

$$\lim_{\Delta y \to 0} \frac{i e^{0} \sin \Delta y}{0 + i\Delta y} = \lim_{\Delta y \to 0} \frac{\sin \Delta y}{\Delta y} = 1$$

where the last part is the famous limit from calculus.

**other properties**

The complex exponential

$$e^z = e^x e^{iy}$$

has some properties that are not shared with the real exponential. As we saw before, the angle $\theta + 2\pi = \theta$ (and $2\pi = 0$), so any number is really a family of numbers with different $\theta + 2\pi k$ for integer $k$.

In particular, $e^z$ is periodic with a period of $2\pi i$. Additionally, it is possible for $e^z$ to be negative. Consider that it is possible that

$$e^z = -1$$

as follows. Let $z = 0 + i\pi$. Then

$$e^x = e^0 = 1$$

and

$$e^{iy} = e^{i\pi} = -1$$

So

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y) = 1(-1) = -1$$

On the other hand, $e^z$ **cannot be zero**.

$$e^z = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y)$$

For $x \in \mathbb{R}$, $e^x > 0$ so the only way this could be zero would be if we can find a $y$ such that $\sin y$ and $\cos y$ were both zero. Since there is no such $y$, we conclude that $e^z$ cannot be equal to zero.

# Chapter 8

# Logarithm

Nearly everything works for the logarithm of $z$ similarly to the real numbers, except for the issue of multiple phase angles or complex arguments. For example

$$\log(z) = \log(re^{i\theta})$$

$$= \log(r) + \log(e^{i\theta})$$

$$= \ln r + i\theta$$

but we may have any multiple of $k \cdot 2\pi$ added to $\theta$

$$\log(z) = \ln r + i(\theta + k2\pi)$$

We call one particular range of $2\pi$ the range for the *principal value* of the function.

For example, here it is natural to make the range go from $-\pi < \theta < \pi$. The reason is that the negative $x$-axis consists of negative real numbers, for which the natural logarithm isn't defined, and neither is the complex logarithm. So we exclude that from the domain of the complex logarithm.

This is called a "branch cut," where we take one particular branch of this multi-valued function.

Here is a derivation.

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$$
$$r = |z| = \sqrt{x^2 + y^2}$$

The logarithm of $z$ is $w$

$$w = \log z \iff e^w = z$$

So what about $w$? Well, in general, it's a complex number

$$w = s + it$$

so

$$e^w = e^{s+it} = e^s(\cos t + i\sin t)$$

Equating the two we get

$$r(\cos\theta + i\sin\theta) = e^s(\cos t + i\sin t)$$

Hence

$$s = \ln r$$
$$t = \theta$$
$$w = \ln r + i\theta$$

**different base**

What is

$$i^i = ?$$

The complex logarithm of $i$ is

$$\log i = \ln r + i\theta = \ln 1 + i\frac{\pi}{2} = i\frac{\pi}{2}$$

Write

$$a^z = (e^{\log a})^z$$

$$i^i = (e^{\log i})^i = (e^{i\pi/2})^i = e^{-\pi/2}$$

Not only is $i$ to the $ith$ power computable, it is entirely real. It is $\approx 0.2079$.

**derivative**

When we study the Cauchy-Riemann equations we will show that if $f(z)$ is differentiable, then the CRE hold. The converse theorem is also true, that if the CRE hold, then $f(z)$ is differentiable, and its derivative is

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

So we have that the logarithm function is

$$\log(z) = \ln|z| + i\theta$$

Rewriting in terms of $x$ and $y$ we have that

$$\log(x+iy) = \ln(\sqrt{x^2+y^2}) + i\tan^{-1}(\frac{y}{x})$$

$$\log(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}(\frac{y}{x})$$

So

$$u(x,y) = \frac{1}{2}\ln(x^2+y^2)$$

$$u_x = \frac{1}{2}\,\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$$

$$u_y = \frac{y}{x^2+y^2}$$

and

$$v(x,y) = \tan^{-1}(\frac{y}{x})$$

$$v_x = \frac{1}{1+(y/x)^2}\ y\ (-\frac{1}{x^2}) = \frac{-y}{x^2+y^2}$$

$$v_y = \frac{1}{1+(y/x)^2}\ \frac{1}{x} = \frac{x}{x^2+y^2}$$

We see that CRE are satisfied and that means that the derivative is

$$[\ \log z\ ]'\ = u_x + iv_x$$

$$= \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

$$= \frac{1}{x^2+y^2}(x-iy)$$

$$= \frac{1}{|z|^2}\ z*$$

$$= \frac{1}{zz*}\ z* = \frac{1}{z}$$

The derivative of the complex logarithm is the inverse of $z$, completely analogous to the real case.

# Chapter 9

# Roots

Consider the square root function $\sqrt{z}$. For the modulus part, we see that $\sqrt{r} \cdot \sqrt{r}$ is obviously equal to $r$, and what we need to determine the argument is to find an angle that is one-half of the original one, which leads us to

$$\sqrt{z} = \sqrt{re^{i\theta}} = \sqrt{r}\ e^{i(\theta/2)}$$

However, recall from trigonometry (and what we said above) that $\theta + 2k\pi$ is the same angle as $\theta$, for integer $k$ ($k \in \mathbb{Z}$).

The same thing is true here. It is allowed that $\theta' = (2k\pi + \theta)$ for $k \in \{0, \pm 1, \pm 2 \dots\}$, since the result maps to the same point in the plane.

This means that a second solution to the square root problem is

$$\sqrt{re^{i\theta}} = \sqrt{r}\ e^{i(\pi + \theta/2)}$$

because, again, $\sqrt{r} \cdot \sqrt{r} = r$ and

$$e^{i(\pi+\theta/2)} \cdot\ e^{i(\pi+\theta/2)} = e^{i(2\pi+\theta)} = e^{i\theta}$$

For the square root, there is only one additional distinct solution, since one-half of $4\pi + \theta = 2\pi + \theta/2$ which is no different than $\theta/2$, but the cube root has 3 solutions and in general the $n^{th}$ root has $n$ solutions.

Consider points on the unit circle with $r = 1$ (so $\sqrt{r} = r$) and suppose

$$\theta = \pi/2$$

so

$$z = e^{i\pi/2}$$

Points with $\theta = \pi/2$ lie directly above the origin on the imaginary axis (there is no real component). This point is one unit from the origin so it is the point $(0 + i \cdot 1) = i$. Thus

$$e^{i\pi/2} = i$$

Note that

$$(e^{i\pi/2})^2 = e^{i(\pi/2+\pi/2)}$$
$$= e^{i\pi} = -1 = i^2$$

We can justify this last step by geometry ($\theta = \pi$), or by using Euler's equation

$$e^{i\theta} = cos\theta + i\sin\theta$$
$$e^{i\pi} = cos\pi + i\sin\pi = -1 + i \cdot 0 = -1$$

$\sqrt{e^{i\pi/2}} = \sqrt{i}$ has two possible values. One is

$$\sqrt{e^{i\pi/2}} = (e^{i\pi/2})^{1/2} = e^{i\pi/4}$$

Let's just check. The point is at a distance 1 from the origin and angle $\theta = \pi/4$. We go equal distances along the real and imaginary axes:

$$x = \cos\theta = \frac{1}{\sqrt{2}}$$

$$y = \sin\theta = \frac{1}{\sqrt{2}}$$

So we have that the square is:

$$\begin{aligned}
(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})^2 &= \frac{1}{2} - \frac{1}{2} + 2i\frac{1}{2} \\
&= 0 + i = i
\end{aligned}$$

the second solution is

$$\sqrt{e^{i\pi/2}} = e^{i\cdot 5/4\pi}$$

which can be plotted as

$$x = \cos\theta = -\frac{1}{\sqrt{2}}$$

$$y = \sin\theta = -\frac{1}{\sqrt{2}}$$

The square is the same except the first term is $(-1/\sqrt{2})^2$, so the result is unchanged. It's a bit counter-intuitive that squaring a number may possibly reduce the phase angle, but you can think of it as modular arithmetic (mod $2\pi$).

In general, if we're working with the complex number

$$re^{i\theta}$$

and we want the nth root, the modulus is just

$$\rho = r^{1/n}$$

And the question always is, what's the angle?

$$\phi = \frac{\theta + 2k\pi}{n}, \quad k = 0, 1, 2 \dots n-1$$

Let's say we want the cube roots of 1. Obviously, all the roots will have length 1. What about the angles? The starting angle $\theta = 0$, so $\phi = 2k\pi/3$ and

$$\phi_1 = \frac{2\pi}{3}$$

$$\phi_2 = \frac{4\pi}{3}$$

$$\phi_3 = \frac{6\pi}{3} = 0$$

Notice that the first and second roots are complex conjugates because

$$\phi_1 + \phi_2 = \frac{6\pi}{3} = 2\pi = 0$$

Suppose our number is $z = -8i$ and we want the cube roots. Writing the number in polar coordinates:

$$z = 8e^{3\pi/2}$$

All of the roots have the same modulus, 2, since $2^3 = 8$. The are three roots which differ in their arguments. Since $\theta = 3\pi/2$, these are:

$$\phi_1 = \frac{\theta}{3} = \frac{\pi}{2}$$

$$\phi_2 = \frac{\theta + 2\pi}{3} = \frac{\pi}{2} + \frac{2\pi}{3} = \frac{5\pi}{6}$$

$$\phi_3 = \frac{\theta + 4\pi}{3} = \frac{\pi}{2} + \frac{4\pi}{3} = \frac{7\pi}{6}$$

Notice that the second and third roots are complex conjugates.

When the argument for $z$ is $\theta_0$, a general formula for the angle of the $nth$ root of $z$ is:

$$\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n} \quad k = 0, \pm 1, \pm 2 \ldots$$

We derive this as follows:

$$z = re^{i\theta}$$

$$z^{1/n} = \left(re^{i(\theta+2k\pi)}\right)^{1/n}$$

Writing only the argument part

$$\left(e^{i(\theta+2k\pi)}\right)^{1/n} = e^{i(\theta/n+2k\pi/n)}$$

**Nahin's puzzle**

In one of his books Nahin starts by posing this question: suppose we are given that

$$x + \frac{1}{x} = 1$$

*Without computing* $x$, find the value of

$$x^7 + \frac{1}{x^7}$$

Nahin says that if you are the type to just start right in trying to figure this out, then you will like his book.

From its placement in this section, you might just guess the answer. First of all, no real $x$ solves the equation

$$x + \frac{1}{x} = 1$$

as you will see if you use the quadratic formula. So let's change nomenclature and call it $z$. (Of course, we were not supposed to *compute* $z$).

We may guess that $z$ is a complex number with length 1 so that the lengths don't change with powers or roots.

Then, all that happens is that $\theta$ changes in such a way that

$$7\theta = \theta = \frac{\theta}{7}$$

To actually compute $z$, multiply by $z$, rearrange, and solve:

$$z^2 + 1 = z$$

$$z^2 - z + 1 = 0$$

From the quadratic equation:

$$z = \frac{1 \pm \sqrt{1 - 4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

The square of the length is

$$r^2 = zz*$$
$$= (\frac{1}{2} + i\frac{\sqrt{3}}{2})(\frac{1}{2} - i\frac{\sqrt{3}}{2})$$
$$= \frac{1}{4} + \frac{3}{4} = 1$$

The angle we seek has tangent equal to $1/\sqrt{3}$. You may recognize the sine and cosine of $\pi/3$ as the real and imaginary components of $z$.

So if

$$z = e^{i\pi/3} = (\frac{1}{2} + i\frac{\sqrt{3}}{2})$$
$$\frac{1}{z} = e^{-i\pi/3} = (\frac{1}{2} - i\frac{\sqrt{3}}{2})$$

then when doing the addition the imaginary parts of $z$ cancel and we have that

$$z + \frac{1}{z} = \frac{1}{2} + \frac{1}{2} = 1$$

The other special attribute of this value for $z$ is that the length is 1 so all powers of $r$ are 1. As for the angle, $\pi/3$ is special in that $7 \times \pi/3 = 2\pi + \pi/3 = \pi/3$. Now it's not strictly true that *the* 7th root of $\theta$ is equal to $\theta$ (since there are 7 distinct roots). But I hope you can see that there is at least one such root.

# Chapter 10

# Trig functions

We can define the complex counterparts of the real trigonometric functions by saying that Euler's formula is also good for a complex number $z$ (a math book would define them by their power series). So

$$e^{iz} = \cos z + i \sin z$$

and

$$\begin{aligned} e^{-iz} &= \cos -z + i \sin -z \\ &= \cos z - i \sin z \end{aligned}$$

This leads to:

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

and

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

A nice property for this definition of cosine is that

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

so

$$\cos(z + 2\pi) = \frac{1}{2}\left(e^{iz}e^{i2\pi} + e^{-iz}e^{-2\pi}\right)$$

but

$$e^{i2\pi} = \cos 2\pi + i \sin 2\pi = 1 = e^{-i2\pi}$$

so

$$\cos(z + 2\pi) = \cos z$$

The *period* of the complex cosine and sine is $2\pi$.

Take derivatives

$$\sin z = \frac{1}{2i}\ (e^{iz} - e^{-iz})$$

$$\frac{d}{dz}\ \sin z = \frac{1}{2i} i(e^{iz} + e^{-iz}) = \cos z$$

Similarly

$$\cos z = \frac{1}{2}\ (e^{iz} + e^{-iz})$$

$$\frac{d}{dz}\ \cos z = \frac{i}{2}\ (e^{iz} - e^{-iz})$$

$$= -\frac{1}{2i}\ (e^{iz} - e^{-iz}) = -\sin z$$

Also

$$\sin -z = \frac{1}{2i}\ (e^{-iz} - e^{+iz}) = -\sin z$$

$$\cos -z = \frac{1}{2}\ (e^{-iz} + e^{iz}) = \cos z$$

**hyperbolic connection**

The definitions of the hyperbolic sine and cosine for a real variable are:

$$\cosh x = \frac{1}{2}(e^{x} + e^{-x})$$

$$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$$

Now

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

if we consider $z = iy$ then

$$\cos iy = \frac{e^{i^2 y} + e^{-i^2 y}}{2}$$

$$= \frac{e^{-y} + e^{y}}{2}$$

But this is just $\cosh y$. That is:

$$\cos iy = \cosh y$$

Similarly

$$2i \sin iy = e^{i^2 y} - e^{-i^2 y}$$

$$= e^{-y} - e^{y}$$

$$= -(e^{y} - e^{-y})$$

$$= -2 \sinh y$$

Hence

$$i \sin iy = - \sinh y$$

$$\sin iy = i \sinh y$$

So if we view $z = x + iy$ and use the standard addition formula

$$\cos z = \cos(x + iy)$$

gives

$$\cos z = \cos x \cos iy - \sin x \sin iy$$

$$= \cos x \cosh y - i \sin x \sinh y$$

and what's nice about this is that we have the real and imaginary parts of the complex cosine easily visible. Similarly

$$\sin z = \sin(x + iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

We could also obtain this result by working through the formulas with the complex exponential.

In fact, that is probably worth doing, since we have not proved yet that the sum of angles formulas are valid for complex numbers. If we work through starting from Euler it will amount to a proof of that fact.

To prove:

$$\sin x + iy = \sin x \cos iy + \sin iy \cos x$$

or equivalently

$$\sin x + iy = \sin x \cosh y + i \sinh y \cos x$$

We had above that:

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

and

$$e^{iz} = e^{i(x+iy)} = e^{ix}e^{-y}$$

$$e^{-iz} = e^{-i(x+iy)} = e^{-ix}e^{y}$$

Furthermore

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos -x + i \sin -x = \cos x - i \sin x$$

Putting that all together:

$$\sin z = \frac{1}{2i} \; [ \; (\cos x + i \sin x)e^{-y} - (\cos x - i \sin x)e^{y} \; ]$$

Group the terms with $\sin x$ together

$$= \frac{1}{2i}\ [\ i \sin x(e^{-y} + e^{y}) + \cos x(e^{-y} - e^{y})\ ]$$

$$= \sin x\ \frac{(e^{y} + e^{-y})}{2} - \frac{1}{i}\cos x \frac{(e^{y} - e^{-y})}{i}$$

$$= \sin x \cosh y + i \cos x \sinh y$$

Cosine is perhaps easier

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$= \frac{1}{2}\ [\ (\cos x + i \sin x)e^{-y} + (\cos x - i \sin x)e^{y}\ ]$$

$$= \frac{1}{2}\ [\ (\cos x)(e^{y} + e^{-y}) + i(\sin x)(e^{y} + e^{-y})$$

$$= \cos x \cosh y - \frac{1}{2i}(\sin x)(e^{y} + e^{-y})$$

now we just massage the second term

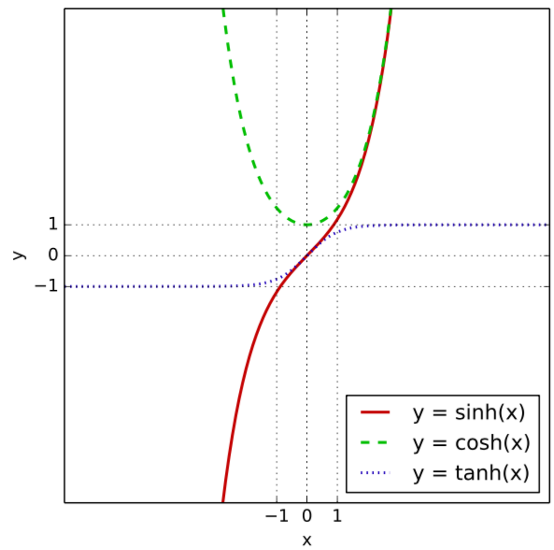
$$= \cos x \cosh y - \frac{1}{i} \sin x \sinh y$$

$$= \cos x \cosh y - \frac{1}{i} \sin x(i \sin iy)$$

$$= \cos x \cosh y - \sin x \sin iy$$

which is what we wanted to prove.

**zeroes**




Notice that cosh is never zero, while only $\sinh 0 = 0$.

So if we look again at

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

and ask, where is this function equal to zero?

Both parts must vanish. Since cosh is never zero, $\sin x$ must be zero. This happens for $x = 2k\pi$.

The cosine of this $x$ is equal to 1, that means $\sinh y$ must be 0 which only happens for $y = 0$.

So the zeroes of the complex sine function are at $z = 2k\pi + 0i$.

Alternatively, go back to the original definition:

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

which vanishes only for

$$e^{iz} = e^{-iz} = \frac{1}{e^{iz}}$$

$$e^{2iz} = \left[\ e^{iz}\ \right]^2\ = 1$$
$$e^{iz} = \pm 1$$
$$e^{i(x+iy)} = \pm 1$$
$$e^{-y}e^{ix} = \pm 1$$
$$e^{-y}(\cos x + i \sin x) = \pm 1$$

The imaginary part must be zero, so $x = 2k\pi$. The other part must be equal to $\pm 1$, so $y = 0$ and $\cos 2k\pi = 1$, which works.

For the cosine

$$\cos z = \frac{1}{2i}(e^{iz} + e^{-iz})$$

This is equal to zero when

$$e^{iz} = -e^{-iz} = -\frac{1}{e^{iz}}$$
$$e^{2iz} = -1$$
$$e^{i(x+iy)} = \pm i$$
$$e^{-y}(\cos x + i \sin x) = \pm i$$

In this case we need $\cos x = 0$ and then $y = 0$ and $\sin x = 1$ will work. $x = (2k + 1)\pi/2$.

Recall that

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

Since cosh is never zero, $\cos x$ must be zero. Then either $\sin x = 0$ or $\sinh y = 0$. Only the latter works for the non-imaginary part, so we have that $y = 0$.

**analyticity**

We proved before that the complex exponential is analytic. There is a theorem that says that if we add two analytic functions together, the result is also analytic. Hence, the trigonometric functions are analytic.

But, just to check this result, let's write them out in terms of $u$ and $v$ and see whether the partial derivatives follow the CRE conditions:

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Taking the derivatives:

$$u(x, y) = \sin x \cosh y$$

$$u_x = \cos x \cosh y$$

$$u_y = \sin x \sinh y$$

and

$$v(x, y) = \cos x \sinh y$$

$$v_x = -\sin x \sinh y$$

$$v_y = \cos x \cosh y$$

So we see that

$$u_x = v_y$$

$$u_y = -v_x$$

The CRE are satisfied and therefore, the complex sine is analytic.

Similarly we have that

$$\cos z = \cos(x + iy)$$

$$= \cos x \cos iy - \sin x \sin iy$$

$$= \cos x \cosh y - i \sin x \sinh y$$

So

$$u(x, y) = \cos x \cosh y$$

$$u_x = -\sin x \cosh y$$

$$u_y = \cos x \sinh y$$

and

$$v(x, y) = -\sin x \sinh y$$

$$v_x = -\cos x \sinh y$$

$$v_y = -\sin x \cosh y$$

So we see that

$$u_x = v_y$$

$$u_y = v_x$$

Thus the complex cosine is also analytic.

We can also prove that:

$$\sin^2 z + \cos^2 z = 1$$

The easy way is

$$\cos^2 z + \sin^2 z = \ [\frac{e^{iz} + e^{-iz}}{2}\ ]^2 + [\frac{e^{iz} - e^{-iz}}{2i}\ ]^2\ ]$$

$$= \frac{e^{2iz} + 2 + e^{-2iz} - e^{2iz} + 2 - e^{-2iz}}{4}$$

$$= 1$$

**series**

On the other hand, Shankar defines the trig functions and the exponential using series in the same way as the real versions:

$$\sin z = \sum_0^\infty (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\cos z = \sum_0^\infty (-1)^n \frac{z^{2n}}{(2n)!}$$

$$\sinh z = \sum_0^\infty \frac{z^{2n+1}}{(2n+1)!}$$

$$\cosh z = \sum_0^\infty \frac{z^{2n}}{(2n)!}$$

and showing that they converge for any $z$.

**alternate derivation for hyperbolic formula**

$$\cos z = \frac{1}{2} \; [\; e^z + e^{-z} \;]$$

$$= \frac{1}{2} \; [\; e^{i(x+iy)} + e^{-i(x+iy)} \;]$$

$$= \frac{1}{2} \; [\; e^{ix-y} + e^{-ix+y} \;]$$

Double the top and the bottom

$$= \frac{1}{4} \; [\; e^{ix-y} + e^{-ix+y} + e^{ix-y} + e^{-ix+y} \;]$$

The pattern in the exponents is $+-\quad -+\quad +-\quad -+$.

We reach a new pattern by first switching the order to $-+ \quad +- \quad -+ \quad +-$

$$= \frac{1}{4} \; [ \; e^{-ix+y} + e^{ix-y} + e^{-ix+y} + e^{ix-y} \; ]$$

then add and subtract terms with $++$ and $--$, like this:

$$++ \;\; -+ \;\; +- \;\; -- \;\; ++ \;\; -+ \;\; +- \;\; --$$

Written out, we have:

$$= \frac{1}{4} \; [ \; e^{ix+y} + e^{-ix+y} + e^{ix-y} + e^{-ix-y} - e^{ix+y} + e^{-ix+y} + e^{ix-y} - e^{-ix-y} \; ]$$

Group the second set of four terms and put a minus sign out front

$$= \frac{1}{4} \; [ \; e^{ix+y} + e^{-ix+y} + e^{ix-y} + e^{-ix-y} \; ] - \frac{1}{4} \; [ \; e^{ix+y} - e^{-ix+y} - e^{ix-y} + e^{-ix-y} \; ]$$

Now we realize that we can factor the first term as:

$$= \frac{1}{2}(e^{y} + e^{-y}) \; \frac{1}{2}(e^{ix} + e^{-ix})$$

$$= \cosh y \cos x$$

The second term is:

$$= -\frac{1}{2}(e^{y} - e^{-y}) \; \frac{1}{2}(e^{ix} - e^{-ix})$$

$$= -i\frac{1}{2}(e^{y} - e^{-y}) \; \frac{1}{2i}(e^{ix} - e^{-ix})$$

$$= -i \sinh y \sin x$$

Putting it all together:

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

That required a lot of bookkeeping, and now we have to go back and repeat it all for the sine. So I prefer the derivation that was given previously.

# Chapter 11

# Polar CRE

A clever way to derive the CRE in polar coordinates is to take advantage of the result that we obtained in Cartesian coordinates.

**short and sweet**

As we've demonstrated, the function

$$f(z) = z = x + iy$$

is analytic, so it is differentiable. If we write the same function in polar coordinates:

$$z = re^{i\theta}$$

and then separate it into a completely real part $u(r, \theta)$ and a completely imaginary part $v(r, \theta)$:

$$z = r\cos\theta + ir\sin\theta$$

Observe that $u_r = \cos\theta$ and $v_\theta = r\cos\theta$ so we deduce that

$$ru_r = r\cos\theta = v_\theta$$

Similarly, $v_r = \sin\theta$ and $u_\theta = -r\sin\theta$ so we deduce that

$$rv_r = r\sin\theta = -u_\theta$$

There are the CRE in polar coordinates. Carrying out the same analysis for *any* analytic function would give the same result (with some other expression in the middle, of course).

$$ru_r = v_\theta$$
$$rv_r = -u_\theta$$

Notice the similar format to the Cartesian version, with the addition of a factor of $r$. Reminds me of the Jacobian from multi-variable calculus.

**by the chain rule**

We know equations to go back and forth between $x, y$ and $r, \theta$ so it is not hard to imagine that we can always re-write $u$ and $v$ as

$$z = u\ [\ x(r,\theta), y(r,\theta)\ ] + iv\ [\ x(r,\theta), y(r,\theta)\ ]$$

or more succinctly:

$$z = u(r,\theta) + iv(r,\theta)$$

Now we ask about relations between the partial derivatives.

Obviously, we want expressions involving $u_r$, $v_\theta$ etc. Write:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

or using the convenient shorthand

$$u_r = u_x x_r + u_y y_r$$

Now, we know

$$x = r\cos\theta$$

$$y = r\sin\theta$$

and the derivatives are

$$x_r = \cos\theta$$

$$x_\theta = -r\sin\theta$$

$$y_r = \sin\theta$$

$$y_\theta = r\cos\theta$$

Now write the four expansions (like the one given above) as:

$$u_r = u_x x_r + u_y y_r = u_x\cos\theta + u_y\sin\theta$$

$$v_r = v_x x_r + v_y y_r = v_x\cos\theta + v_y\sin\theta$$

$$u_\theta = u_x x_\theta + u_y y_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta)$$

$$v_\theta = v_x x_\theta + v_y y_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta)$$

Our key insight is to substitute the relations given by the CRE in Cartesian coordinates ($u_x = v_y$, $u_y = -v_x$). In the first above we get

$$u_r = u_x\cos\theta + u_y\sin\theta = v_y\cos\theta - v_x\sin\theta = \frac{1}{r}v_\theta$$

(where the final step equating to $v_\theta/r$ uses the last equation).

In the second

$$v_r = v_x\cos\theta + v_y\sin\theta = -u_y\cos\theta + u_x\sin\theta = -\frac{1}{r}u_\theta$$

(and the last step uses the third equation).

In other words:

$$ru_r = v_\theta$$

$$rv_r = -u_\theta$$

which is what needed to prove.

Like the original Cartesian version but with an extra factor of $r$.

## polar derivative

This will also turn out to be very similar to the Cartesian version, with a twist, which is that we have an extra factor out front:

$$f'(z) = e^{-i\theta} \ [ \ u_r + iv_r \ ]$$

### short and sweet

Let's just assume that the derivative is equal to what we would expect, within some unknown factor of $k$:

$$z' = k(u_r + iv_r)$$

and now we know that for this function the derivative is equal to 1:

$$f(z) = z$$

$$z' = 1 = k(u_r + iv_r)$$

if we write in polar coordinates:

$$z = r\cos\theta + ir\sin\theta$$

Then

$$u_r + iv_r = \cos\theta + i\sin\theta$$

What is the factor that multiplies this expression to give 1? Clearly

$$e^{-i\theta}(\cos\theta + i\sin\theta) = 1$$

So $k = e^{-i\theta}$.

**more carefully**

To get the derivative, start with the version that we know for $x, y$ coordinates:

$$f'(z) = u_x + iv_x$$

Our problem is to define $f'(z)$ in terms of $u_r$ and $v_r$.

Substitute for $u_x$ first. Go back to the two equations involving $u_x$ above

$$u_r = u_x \cos\theta + u_y \sin\theta$$

$$u_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta)$$

Multiply the first by $\cos\theta$

$$u_r \cos\theta = u_x \cos^2\theta + u_y \sin\theta\cos\theta$$

and the second by $-\sin\theta/r$

$$-\frac{u_\theta}{r}\sin\theta = u_x \sin^2\theta - u_y \sin\theta\cos\theta$$

add

$$u_x = u_r \cos\theta - \frac{u_\theta}{r}\sin\theta$$

Substitute $u_\theta/r = -v_r$

$$u_x = u_r \cos\theta + v_r \sin\theta$$

Now get the second and fourth equations, with $v_x$

$$v_r = v_x \cos\theta + v_y \sin\theta$$

$$v_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta)$$

Multiply the first by $\cos\theta$

$$v_r \cos\theta = v_x \cos^2\theta + v_y \sin\theta\cos\theta$$

and the second by $-\sin\theta/r$:

$$-\frac{v_\theta}{r}\sin\theta = v_x \sin^2\theta - v_y \sin\theta\cos\theta$$

add

$$v_x = v_r\cos\theta - \frac{v_\theta}{r}\sin\theta$$

Subtitute $v_\theta/r = u_r$

$$v_x = v_r\cos\theta - u_r\sin\theta$$

Combine the two results:

$$f'(z) = u_x + iv_x$$

$$= u_r\cos\theta + v_r\sin\theta + i\ [\ v_r\cos\theta - u_r\sin\theta\ ]$$

Group terms with $u_r$ and $v_r$ separately:

$$= u_r(\cos\theta - i\sin\theta) + v_r(\sin\theta + i\cos\theta)$$

Multiply the second term by $1 = -i \cdot i$

$$= u_r(\cos\theta - i\sin\theta) + iv_r(-i\sin\theta + \cos\theta)$$

$$f'(z) = e^{-i\theta}\ [\ u_r + iv_r\ ]$$

And since

$$ru_r = v_\theta$$

$$rv_r = -u_\theta$$

then

$$f'(z) = \frac{1}{r}\ e^{-i\theta}\ [\ v_\theta - iu_\theta\ ]$$

compare this with

$$f'(z) = v_y - iu_y$$

and notice that the factor in front is just $1/z$

**example 1**

Let's see if we can do an example. Suppose

$$f(z) = \sqrt{z}$$

Written in terms of $r, \theta$ we have

$$f(z) = \sqrt{r}e^{i\theta/2}$$

$$= \sqrt{r}\cos\theta/2 + i\sqrt{r}\sin\theta/2$$

Then

$$u_r = \frac{\cos\theta/2}{2\sqrt{r}}$$

$$v_r = \frac{\sin\theta/2}{2\sqrt{r}}$$

and

$$[\sqrt{z}\ ]' = \ [\ \sqrt{r}e^{i\theta/2}\ ]' = e^{-i\theta}\ [\ u_r + iv_r\ ]$$

$$= \frac{1}{2\sqrt{r}}\ [\ e^{-i\theta}(\cos\theta/2 + i\sin\theta/2)\ ]$$

$$= \frac{1}{2\sqrt{r}}e^{-i\theta/2} = \frac{1}{2\sqrt{z}}$$

**example 2**

Let's try

$$f(z) = \frac{1}{z} = \frac{1}{r}e^{-i\theta}$$

$$= \frac{1}{r}\cos -\theta + i\frac{1}{r}\sin -\theta$$

$$= \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$$

So

$$u_r = -\frac{1}{r^2}\cos\theta$$

$$v_r = \frac{1}{r^2}\sin\theta$$

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$\frac{1}{r^2}(e^{-i\theta})(-\cos\theta + i\sin\theta)$$

$$\frac{1}{r^2}(e^{-i\theta})(-\cos -\theta - i\sin -\theta)$$

$$\frac{1}{r^2}(e^{-i\theta})(-1)(e^{-i\theta})$$

$$= -\frac{1}{r^2 e^{i2\theta}} = -\frac{1}{z^2}$$

# Chapter 12

# CRE examples

The function

$$f(x,y) = \frac{x^2}{x^2+y^2}$$

has some problems: first, it is not defined at the origin $(0,0)$ but also, as we approach the origin along the $x$-axis and the $y$-axis we get different limiting values, namely

$$f(x,0) = \frac{x^2}{x^2} = 1$$

$$f(0,y) = \frac{0}{y^2} = 0$$

Rewriting it in polar coordinates ($x = r\cos\theta, r^2 = x^2+y^2$):

$$f(r,\theta) = \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$$

Shankar says: the function $f$ is generally a function of *two* complex variables, $z$ and its complex conjugate:

$$z = x + iy$$

$$z* = x - iy$$

which can be written in terms of $x$ and $y$ as

$$x = \frac{z + z*}{2}$$

$$y = \frac{z - z*}{2i}$$

Generally, the value of $f$ depends on both $z$ and $z*$, but we will be very interested in functions which depend only on $z$ and not $z*$. The reason for this is that only such functions have the property that the derivative at a point does not depend on the direction from which we approach that point.

Consider the function:

$$f(x, y) = x^2 - y^2$$

$$= \frac{(z + z*)^2}{4} + \frac{(z - z*)^2}{4}$$

$$= \frac{1}{4} \; [\; z^2 + 2zz * + z *^2 + z^2 - 2zz * + z*^2 \;]$$

$$= \frac{z^2 + z*^2}{2}$$

This function is not a function only of $z$ but of both $z$ and $z*$.

We say that $f$ is an *analytic* function of $z$ if it does not depend on $z*$. Shankar says this means that "$x$ and $y$ enter $f$ *only* in the combination $x + iy$".

The famous Cauchy-Riemann Equations (CRE) are true for $f \iff f$ is an analytic function of $z$.

For:

$$f(x, y) = u(x, y) + iv(x, y)$$

The CRE conditions are:

$$u_x = v_y$$
$$u_y = -v_x$$

Consider:

$$f(x, y) = x^2 - y^2 + i2xy$$

CRE requires

$$u_x = 2x \stackrel{?}{=} v_y = 2x$$
$$v_x = 2y \stackrel{?}{=} -u_y = 2y$$

The function is analytic. As Shankar says, this is expected because:

$$x^2 - y^2 + 2ixy = (x + iy)(x + iy) = z^2$$

Consider:

$$f(x, y) = \cos y - i \sin y$$

CRE requires:

$$u_x = 0 \stackrel{?}{=} v_y = -\cos y$$
$$v_x = 0 \stackrel{?}{=} -u_y = -\sin y$$

This is "impossible" since there is no $y$ that satisfies both of the conditions. And it's not surprising since

$$y = \frac{z - z*}{2i}$$

Consider:

$$f(x, y) = x^2 + y^2$$

CRE requires:

$$u_x = 2x \stackrel{?}{=} v_y = 2y$$
$$u_y = 0 \stackrel{?}{=} -v_x$$

CRE are only satisfied if $x = y$. Also not surprising since

$$x^2 + y^2 = zz*$$

Consider:

$$f(x, y) = x^2 - y^2$$

CRE requires:

$$u_x = 2x \stackrel{?}{=} v_y = -2y$$

which is true if $x = y$.

$$u_y = 0 \stackrel{?}{=} -v_x = 0$$

But ”no importance is given to functions which obey the CRE only at isolated points or on lines.”

Consider:

$$f(x, y) = e^x \cos y + ie^x \sin y$$

CRE requires:

$$u_x = e^x \cos y \stackrel{?}{=} v_y = e^x \cos y$$

$$u_y = -e^x \sin y \stackrel{?}{=} -v_x = -\sin y e^x$$

Both are true, so this one does satisfy CRE.

Shankar doesn’t mention it here but the last function is special, it is $f(z) = e^z$:

$$e^x \cos y + ie^x \sin y$$
$$= e^x(\cos y + i \sin y)$$
$$= e^x e^{iy}$$
$$= e^{x+iy}$$
$$= e^z$$

and we did this one in the previous section.

For functions of interest, it may often be true that CRE fails at particular points called *singularities*.

Consider:

$$f(x,y) = \frac{1}{z} = \frac{z*}{zz*} = \frac{x - iy}{x^2 + y^2}$$

We need:

$$u_x = \frac{d}{dx}\ \frac{x}{x^2 + y^2} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$v_y = \frac{d}{dy}\ (-\frac{y}{x^2 + y^2}) = -\frac{x^2 - y^2}{(x^2 + y^2)^2} = u_x$$

$$u_y = 0 = v_x$$

But the function blows up at the origin. This described by saying it has a pole at the origin. The function

$$f(z) = \frac{c}{z}$$

where $c$ is a constant, also blows up at the origin. We say that the *residue* of the pole at the origin is $c$.

# Part IV

# Integration

# Chapter 13

# Integrate

Complex functions are differentiated and integrated in a way that is similar to real functions, with these differences: normally, we restrict our attention to functions that are analytic, and we pay attention to points in the complex plane where they have poles (or singularities).

Also, the integrals that we compute are line integrals.

In general terms, a complex function is a function that produces a complex number as the result. And the most general case is that the input is a complex number as well. Se we could write:

$$w = f(z)$$

where both $w$ and $z$ are complex numbers.

In this case we can look at the parts of $f$ that generate the real and imaginary parts of $w$ as separate functions:

$$\begin{aligned} w &= f(z) \\ &= u(x, y) + iv(x, y) \end{aligned}$$

However, things may be simpler and often are. The reason is that $x$ and $y$ may be related because we are following a curve in the complex plane. We'll see why later. For now, following a curve means that $u$ and $v$ are both parametric equations of a parameter $t$ and that parameter is *real*.

So a lot of the proof relating to this stuff follow directly from calculus.

**getting started**

$$z = x + iy$$
$$dz = dx + idy$$

and the function

$$w = f(z)$$
$$= u(x, y) + iv(x, y)$$

we can write as

$$\int f(z)\ dz = \int (u + iv)(dx + idy)$$

$$= \int (u + iv)\ dx + (-v + iu)\ dy$$

$$= \int u\ dx - v\ dy + iv\ dx + iu\ dy$$

The integral of a complex function is defined as a sum of integrals of two real variables. Just as with line integrals for real functions of $x$ and $y$, the variables are related by the curve over which we will integrate.

Recall that for the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\ dx + N\ dy$$

we parametrize the curve to get the integral over a single variable.

We can view $y$ as a function of $x$ or perhaps, we can parametrize both $x$ and $y$ as functions of $t$.

Suppose our function is simply $z = x + iy$. The integral is

$$\int z\ dz = \int (x + iy)(dx + idy)$$

$$= \int x\ dx - y\ dy + ix\ dy + iy\ dx$$

Now we must get $y$ in terms of $x$ from the curve. Suppose the curve goes from $(1, i)$ to $(3, i)$, then to $(3, 3i)$ and finally back to where we started.

We have three segments. Along the first part, we are moving in the positive $x$ direction, with no change in $y$, so $dy = 0$ and $y = 1$, a constant, and the integral is

$$\int x\ dx - y\ dy + ix\ dy + iy\ dx$$

$$= \int x\ dx + iy\ dx$$

$$= \int_{x=1}^{x=3} x + i\ dx$$

$$= \frac{x^2}{2} + ix\ \Big|_1^3$$

$$= 4 + 2i$$

Along the second part, we are moving in the positive $y$ direction with $dx = 0$ and $x = 3$ so

$$\int x\ dx - y\ dy + ix\ dy + iy\ dx$$

$$= \int_{y=1}^{y=3} -y \ dy + 3i \ dy$$
$$= -\frac{y^2}{2} + 3iy \ \Big|_1^3$$
$$= -4 + 6i$$

And for the third, both $dx$ and $dy$ are non-zero, so we must actually do the parametrization. The curve is $y = x$. Hence $dy = dx$.

$$\int x \ dx - y \ dy + ix \ dy + iy \ dx$$
$$= \int x \ dx - x \ dx + ix \ dx + ix \ dx$$
$$= 2i \int x \ dx$$

Consider first the direction of the path as going from $(1, 1)$ to $(3, 3)$.

$$= 2i \int_{x=1}^{x=3} x \ dx$$
$$= 2i \ \frac{x^2}{2} \ \Big|_1^3$$
$$= 2i \ \frac{8}{2} = 8i$$

However, for the closed path, where we end up back at the starting point, $C3$ should be moving from $(3, 3)$ to $(1, 1)$ so we have

$$2i \ \frac{x^2}{2} \ \Big|_3^1 = 2i \ -\frac{8}{2} = -8i$$

Notice that

$$\int_{C1} + \int_{C2} = 8i = -\int_{C3}$$

If we follow the curve $C3$ from $(3, 3)$ to $(1, 1)$, the whole thing is just zero. Later we'll see that this is not a coincidence.

**example**

Suppose the function is

$$f(z) = y - x - i3x^2$$

and we proceed from the origin to the point $z = 1 + i$ either directly ($C_2$) or by first going up vertically and then across ($C_1$).

For the vertical part of $C_1$ we have that $x = 0$ and $dx = 0$.

$$I = \int (y - x - i3x^2)\ (dx + idy)$$

$$= \int yi\ dy$$

It's important to recognize that although we are proceeding from the point $z = 0$ to the point $z = i$, the upper bound on this integral is not $i$ but $y = 1$! Hence

$$I = i\frac{y^2}{2}\Bigg|_0^1 = \frac{i}{2}$$

For the horizontal part of $C_1$ we have that $y = 1$ and $dy = 0$ so

$$I = \int (y - x - i3x^2)\ (dx + idy)$$

$$= \int (1 - x - i3x^2)\ dx$$

$$= x - \frac{x^2}{2} - ix^3\ \Bigg|_0^1 = \frac{1}{2} - i$$

Therefore the total

$$I = \frac{i}{2} + \frac{1}{2} - i = \frac{1}{2}\ (1 - i)$$

When going directly from the origin to $1+i$ we relate $x$ to $y$ by the equation of the line $y = x$ so $dy = dx$ and

$$I = \int (y - x - i3x^2)\ (dx + idy)$$

$$= \int -i3x^2\ (dx + idx)$$

$$= -i\ x^3 \Big|_0^1 + x^3 \Big|_0^1 = -i \cdot 1 + 1 = 1 - i$$

And around the closed curve going backward along $C_2$:

$$\oint f(z)\ dz = \frac{1}{2}\ (1-i) - (1-i) = -\frac{1}{2}(1+i)$$

# Chapter 14

# Integrate 2

If the contour (curve) of integration $C$ is parametrized in terms of $t$, then

$$\int_C f(z)\ dz = \int_a^b f[z(t)]\ z'(t)\ dt$$

A particularly important parametrization is for circular paths. On such a path, $z$ takes on values with constant $r$ and the only change is in $\theta$. So we have

$$z = re^{i\theta}$$

$$z'(\theta) = iz$$

$$dz = iz\ d\theta$$

As an example, consider $f(z) = z*$.

Note that this function is *not* analytic, because it involves $z*$ rather than $z$, and secondly because

$$z* = x - iy$$

so

$$u_x = 1\ , \quad v_y = -1$$

$$u_x \neq v_y$$

The CRE do not hold.

Suppose our curve is the circle of radius $r$ centered at the origin, and we proceed between the endpoints $z = -ri \to ri$. On this half-circle

$$z = re^{i\theta}$$

we have then

$$dz = i\ re^{i\theta}\ d\theta$$

In radial coordinates

$$z* = re^{-i\theta}$$

so we have

$$\int z*\ dz = \int re^{-i\theta} rie^{i\theta}\ d\theta$$

$$= r^2 i \int_{-\pi/2}^{\pi/2} d\theta = r^2 \pi i$$

Alternatively,

$$zz* = |z|^2 = r^2$$

$$z* = \frac{r^2}{z}$$

$$\int z*\ dz = r^2 \int \frac{1}{z}\ dz$$

Again

$$z = re^{i\theta}$$

$$dz = iz\ d\theta$$

So the integral is just

$$= r^2 \int \frac{1}{z}\ iz\ d\theta$$

$$= r^2 i \int d\theta = r^2 \pi i$$

**example**

Consider $f(z) = z^2$. For the path, take the unit circle over the first quadrant from $(1, 0)$ to $(0, 1)$. There is an easy way to do this, and a hard way. Let's start by checking that this function is analytic, and then doing the hard way first.

Write $z$ in terms of $x$ and $y$:

$$z = x + iy$$

$$z^2 = (x + iy)^2 = x^2 - y^2 + i2xy$$

$$u_x = 2x = v_y$$

$$u_y = -2y = -v_x$$

The CRE hold.

Also

$$dz = dx + i\ dy$$

So

$$\int z^2\ dz = \int (x^2 - y^2 + 2ixy)(dx + i\ dy)$$

$$= \int (x^2 - y^2)\ dx - \int 2xy\ dy + i \int 2xy\ dx + i \int (x^2 - y^2)\ dy$$

As before, we must parametrize this using the relationship between $x$ and $y$ along the curve.

$$x = \cos t$$

$$y = \sin t$$

$$dx = -\sin t\ dt$$

$$dy = \cos t\ dt$$

and then

$$x^2 - y^2 = \cos^2 t - \sin^2 t = \cos 2t$$

$$2xy = 2\cos t \sin t = \sin 2t$$

so the integral is

$$= \int -\cos 2t \;\; \sin t \; dt - \int \sin 2t \;\; \cos t \; dt + \ldots$$

$$+ \; i \; [ \int -\sin 2t \;\; \sin t \; dt + \int \cos 2t \;\; \cos t \; dt$$

Looks pretty wild! In the book they use some trig identities I hadn't seen before, namely starting with the standard

$$\sin s + t = \sin s \cos t + \sin t \cos s$$

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

then, if $s = 2t$ then

$$\sin 3t = \sin 2t \cos t + \sin t \cos 2t$$

$$\cos 3t = \cos 2t \cos t - \sin 2t \sin t$$

Looking at the real part of the integral we had (combining terms)

$$\int -\cos 2t \;\; \sin t - \sin 2t \;\; \cos t \; dt = \int -\sin 3t \; dt = \frac{\cos 3t}{3}$$

and for the imaginary part of the integral

$$i \; [ \int -\sin 2t \;\; \sin t + \cos 2t \;\; \cos t \; dt = i \int \cos 3t \; dt = i \; \frac{\sin 3t}{3}$$

That looks a lot better.

$$\frac{\cos 3t}{3} + i \; \frac{\sin 3t}{3} \; \Big|_0^{\pi/2} = -\frac{1}{3} - i \; \frac{1}{3} = -\frac{1}{3}(1+i)$$

For the easy way, just treat $z$ as if it were a real variable

$$\int z^2 \, dz = \frac{z^3}{3} \bigg|_1^i = -\frac{1}{3}i - \frac{1}{3}$$

Note that if we go all the way around the unit circle the integral is just zero.

Going back to the first example we had

$$\int z \, dz = \frac{z^2}{2} \bigg|_{1+i}^{3+3i}$$

$$= \frac{9 - 9 + 18i - \ [\ 1 - 1 + 2i\ ]}{2}$$
$$= 8i$$

**example**

$$\int_0^{2\pi} \frac{1}{z} \, dz$$

Examining the inverse function, let's first confirm that it is analytic by calculating the partial derivatives. We have

$$\frac{1}{z} = \frac{1}{x + iy}$$

Simplify by multiplying on top and bottom by $z*$:

$$= \frac{1}{x + iy} \, \frac{x - iy}{x - iy}$$

$$= \frac{x - iy}{x^2 + y^2}$$

Thus

$$u = \frac{x}{x^2 + y^2}$$

$$u_x = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$u_y = \frac{-2xy}{(x^2+y^2)^2}$$

And

$$v = \frac{-y}{x^2+y^2}$$
$$v_y = -\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$v_x = \frac{2xy}{(x^2+y^2)^2}$$

CRE are satisfied and the inverse of $z$ is indeed analytic.

If we are on the unit circle, then

$$z = e^{i\theta}$$

$$dz = ie^{i\theta}d\theta = iz\ d\theta$$
$$\int \frac{dz}{z} = \int \frac{iz}{z}\ d\theta = 2\pi i$$

If we're centered on the origin but we don't have a unit circle, there will be an $R$ in both the numerator and the denominator, which cancel.

The result is thus independent of the radius of the circle.

In general

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 0, & n \neq 1 \\ 2\pi i, & n = 1 \end{cases}$$

(We will see examples for $n \neq 1$ below).

We can also integrate the inverse function in terms of $x$ and $y$:

$$\oint \frac{1}{z}\ dz = \oint \frac{dx + idy}{x+iy}$$

$$= \oint \frac{1}{x^2+y^2} \, [\; x \; dx - y \; dy + ix \; dy + iy \; dx \;]$$

Suppose we go on a circle of radius $R$ centered on the origin and parametrize in terms of $\theta$. We obtain:

$$x = R\cos\theta$$

$$y = R\sin\theta$$

$$x^2 + y^2 = R^2$$

$$dx = -R\sin\theta \; d\theta$$

$$dy = R\cos\theta \; d\theta$$

We have for the integral

$$\oint \frac{1}{x^2+y^2} \, [\; x \; dx - y \; dy + ix \; dy + iy \; dx \;]$$

$$= \int \frac{1}{R^2} \, [\, -R^2\cos\theta\sin\theta \; d\theta + R^2\sin\theta\cos\theta \; d\theta + iR^2\cos^2\theta \; d\theta + iR^2\sin^2\theta \; d\theta \,]$$

$$= \int \frac{1}{R^2} \, [\; iR^2\cos^2\theta \; d\theta + iR^2\sin^2\theta \; d\theta \;]$$

$$= \int \frac{1}{R^2} \, [\; iR^2 \; d\theta \;]$$

$$= \int i \; d\theta = 2\pi i$$

Note that if we integrate the same function around a unit square, we run into problems. First let's do $[0, 0 \times 1, 1]$. We have

$$\int u \; dx - \int v \; dy + i \; [\; \int v \; dx + \int u \; dy \;]$$

Along $C1$, $y = 0$ and $dy = 0$ so:

$$\int \frac{x}{x^2+y^2}\ dx + i\ [\ \int \frac{-y}{x^2+y^2}\ dx$$

$$= \int_0^1 \frac{1}{x}\ dx = \ln x\ \Big|_0^1$$

Since $\ln 0$ is not defined, we can't do this.

Logarithms are tricky, no doubt. If the complex logarithm $Logz$ is defined and differentiable along the curve (say the semicircle from $-i$ to $i$), we can do this:

$$I = \int_{-i}^{i} \frac{1}{z}\ dz = \ \text{Log}\ z\ \Big|_{-i}^{i}$$

Recall that $z = re^{i\theta}$ with $r = 1$ so this is

$$= (\ln 1 + i\ \frac{\pi}{2}) - (\ln 1 + i\frac{-\pi}{2}) = 2i\ \frac{\pi}{2} = \pi i$$

For any value of $r$ (except $r = 0$), we get the same answer, since $\ln r - \ln r = 0$.

**example**

We can extend this to

$$\oint \frac{1}{z^2}\ dz$$

As before, on the unit circle

$$z = e^{i\theta}$$

and

$$\frac{dz}{d\theta} = ie^{i\theta}$$

so

$$dz = iz\ d\theta$$

The integral is

$$\int_0^{2\pi} \frac{i}{z}\ d\theta = i \int_0^{2\pi} e^{-i\theta}\ d\theta$$

Now

$$\int e^{-i\theta}\ d\theta = \frac{1}{-i} e^{-i\theta} = ie^{-i\theta}$$

so we have

$$= -e^{-i\theta}\ \Big|_0^{2\pi}$$

Evaluate at the upper bound using Euler's formula:

$$e^{-2\pi i} = \cos -2\pi + i \sin -2\pi$$

$$= \cos 2\pi - i \sin 2\pi = 1$$

At the lower bound we also get 1 so the whole thing is zero.

In fact, for any negative integer power of $z$

$$\int z^{-n}\ dz$$

around the unit circle $z = e^{i\theta}$ we have

$$i \int e^{-i(n-1)\theta}\ d\theta$$

$$= -\frac{1}{n-1}\ e^{-i(n-1)\theta}\ \Big|_0^{2\pi}$$

$$= -\frac{1}{n-1}\ [\ (\cos 2(n-1)\pi - i \sin 2(n-1)\pi)\ \ - 1]$$

$$= -\frac{1}{n-1}\ [\ 1 - 1]\ \ = 0$$

**example**

Consider

$$\int \sqrt{z}\ dz$$

along the half-circle of radius 3 starting from the point $z = R$ on the $x$-axis and proceeding counter-clockwise. We can do this integral even if the "branch" of the square root function that we're using is only defined for $\theta > 0$. We have that

$$z = Re^{i\theta}, \quad \theta = 0 \to \pi$$

$$dz = iz = iRe^{i\theta}\ d\theta$$

$$\sqrt{z} = \sqrt{R}e^{i\theta/2}$$

so

$$I = \int_0^\pi iR\sqrt{R}e^{i3\theta/2}\ d\theta$$

We need

$$\int e^{i3\theta/2}\ d\theta = \frac{2}{3i}e^{i3\theta/2}\ \Big|_0^\pi$$

easiest to write it out as

$$e^{i3\theta/2}\ \Big|_0^\pi = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} - \cos 0 - i\sin 0$$

$$= 0 + i(-1) - 1 - 0 = -(1+i)$$

Going back to pick up all the factors we left behind:

$$I = -iR\sqrt{R}\ \frac{2}{3i}\ (1+i) = -R\sqrt{R}\ \frac{2}{3}\ (1+i)$$

In the problem, $R$ was actually specified as 3, leading to the cancellation:

$$I = -2\sqrt{3}\ (1+i)$$

We can also do this problem by antiderivatives:

$$\int_R^{-R} \sqrt{z}\ dz = \frac{2}{3}\ z^{3/2}\ \Big|_R^{-R}$$

$$= \frac{2}{3}(R^{3/2}e^{i3\pi/2} - R^{3/2}e^{0})$$

$$= \frac{2}{3}R^{3/2}(e^{i3\pi/2} - 1)$$

and, as we showed above:

$$e^{i3\pi/2} = -i$$

If $R = 3$ we get the same answer as before.

# Chapter 15

# Cauchy 1

Cauchy's first theorem says that the integral of an analytic function over a closed path is equal to zero, when the enclosed region does not contain a singularity.

$$\oint_C f(z)\ dz = 0$$

This will turn out to be a consequence of Green's Theorem, which you should remember from multivariable calculus. Let

$$z = x + iy$$

$$dz = dx + idy$$

$$z = f(x, y) = u(x, y) + iv(x, y)$$

Our integral is

$$\oint z\ dz = \oint (u(x, y) + iv(x, y))\ (dx + idy)$$

$$= \oint u(x, y)\ dx - \oint v(x, y)\ dy + i \oint v(x, y)\ dx + i \oint u(x, y)\ dy$$

As before, because we are moving along a curve there is a relationship between $x$ and $y$, so we can either express that relationship (for say, $y$

in terms of $x$), or parametrize the curve in terms of $t$ or $\theta$. In either case, these become integrals in a single variable. We suppress the $(x, y)$ notation and the extra integral signs:

$$= \oint u\ dx - v\ dy + iv\ dx + iu\ dy$$

**proof of Cauchy 1**

Back in vector calculus we proved Green's theorem, which says that for two real functions of $x$ and $y$: $M(x, y)$ and $N(x, y)$:

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\ dx\ dy$$

Back then, $M$ and $N$ were components of a vector field $\mathbf{F}$ and we wrote the shorthand for curl:

$$= \iint_R \nabla \times \mathbf{F}\ dA$$

but the important thing is that they are real-valued functions of two real variables $f : \mathbb{R}^2 \to \mathbb{R}^1$.

In terms of $u$ and $v$ we have for the real part of Cauchy's Theorem that $M = u$ and $N = -v$ (notice the minus sign!).

So:

$$\oint u\ dx - v\ dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\ dx\ dy$$

$$= - \iint_R (v_x + u_y)\ dx\ dy$$

But, according to the CRE

$$u_y = -v_x$$

Hence, this integral is zero.

For the imaginary part we use Green's Theorem again with $N = u$ and $M = v$ (no minus sign here):

$$\oint v\ dx + u\ dy = \iint_R (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})\ dx\ dy$$

$$= \iint_R (u_x - v_y)\ dx\ dy$$

But, again, according to the CRE

$$u_x = v_y$$

So the integral for the imaginary part is also zero, and thus the whole thing is zero as well:

$$\oint u\ dx - \oint v\ dy + i \oint v\ dx + i \oint u\ dy = 0$$

Remember how important it was (for Green's theorem) that the function being integrated be defined everywhere in the region. Well, it's true here as well.

$$\oint_C \frac{1}{z}\ dz \stackrel{?}{=} 0$$

We've already seen by direct calculation that this integral is *not* zero when the curve $C$ includes the origin, but it *is* zero otherwise.

To repeat the demonstration for the former case we use the unit circle centered at the origin. Write

$$z = re^{i\theta}$$

$$\frac{dz}{d\theta} = rie^{i\theta} = iz$$

Hence

$$\oint_C \frac{1}{z}\, dz = \oint_C \frac{1}{z}\, iz\, d\theta$$
$$= i \oint_C d\theta = 2\pi i$$

### Path independence

The theorem that says the integral of an analytic function over a closed path (over a region without a singularity), is equal to zero.

$$\oint_C f(z)\, dz = 0$$

This result means, in turn, that the integral of an analytic function between two points $z_1$ and $z_2$ is independent of the path taken. Call the two paths $C1$ and $C2$.

Form the closed path by going from $z_1$ to $z_2$ over $C1$ and then return to $z_1$ by going backward over $C2$. The total integral is equal to zero by Cauchy's Theorem.

$$\int_{C1} f(z)\, dz + \int_{-C2} f(z)\, dz = 0$$

But the integral over the path $C_2$ in the forward direction is just minus the integral over the reverse path $-C2$.

$$\int_{-C2} f(z)\, dz = -\int_{C2} f(z)\, dz$$

Thus

$$\int_{C1} f(z)\, dz - \int_{C2} f(z)\, dz = 0$$

and

$$\int_{C1} f(z)\, dz = \int_{C2} f(z)\, dz$$

**example**

We will integrate the function $f(z) = z$ over a rectangle $(R = [0, a] \times [b, 0]$. Write

$$z = x + iy$$

$$dz = dx + idy$$

$$f(x, y) = u(x, y) + iv(x, y)$$

Our integral is

$$\int z\ dz = \int (u + iv)\ (dx + idy)$$

$$= \int u\ dx - \int v\ dy + i \int v\ dx + i \int u\ dy$$

Since the whole thing is equal to zero over our closed path, both parts are equal to zero:

$$\int u\ dx - \int v\ dy = 0$$

$$\int v\ dx + \int u\ dy = 0$$

Does this look familiar?

# Chapter 16

# Cauchy 2

If we can write an integral in this form:

$$\oint_C \frac{f(z)}{z - z_0}\ dz$$

where $f(z)$ is analytic and defined everywhere in the domain we care about, with this composite function of course not defined at $z = z_0$, then we will show that the value of the integral is

$$\oint_C \frac{f(z)}{z - z_0}\ dz = 2\pi i f(z_0)$$

This is called Cauchy's Residue Theorem, or simply Cauchy 2.

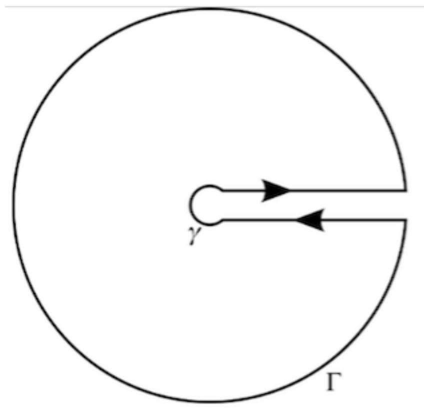
### derivation

Suppose $f(z)$ is analytic and defined everywhere within some region *except* at a singularity, $z_0$. For example, suppose we have

$$\frac{f(z)}{z - z_0}$$

We integrate this function around a special closed path in the region of analyticity:

$$\oint \frac{f(z)}{z - z_0} \, dz$$




It's not labeled (I didn't draw the figure) but the singularity $z_0$ is at the center of the two concentric circles. The "keyhole" excludes $z_0$ so $f$ is analytic everywhere in the region enclosed by the path.

Cauchy's first theorem tells us that the total integral is zero.

The straight line segments are so close to each other as to be equal, but traversed in opposite directions, so the net contribution from them is zero.

Therefore by Cauchy 1 we have that the integral around the outer ring counter-clockwise + the integral around the inner ring clockwise add up to zero.

But reversing the direction of integration on the inner ring (so both paths go in the counter-clockwise direction) changes the sign of the value, hence we have that

$$\oint_{C \text{ outer}} \frac{f(z)}{z - z_0} \, dz - \oint_{C \text{ inner}} \frac{f(z)}{z - z_0} \, dz = 0$$

and

$$\oint_{C\ \text{outer}} \frac{f(z)}{z-z_0}\ dz = \oint_{C\ \text{inner}} \frac{f(z)}{z-z_0}\ dz$$

Notice that we haven't said anything about the radius of these rings.

What this means is that the value of the integral around a ring enclosing a singularity is not zero, but is independent of the radius.

We can parametrize this path by realizing that each point on one of these curves is given by

$$z = z_0 + \rho e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

Since $z_0$ is a constant

$$dz = i\rho e^{i\theta} d\theta$$

But

$$z - z_0 = \rho e^{i\theta}$$

so, substituting for $\rho e^{i\theta}$ above we obtain

$$dz = i(z - z_0)\ d\theta$$

and

$$\oint \frac{f(z)}{z-z_0}\ dz = \oint f(z)\ i\ d\theta$$

$$= i\int_0^{2\pi} f(z)\ d\theta$$

This holds for every circular path enclosing $z_0$. We may choose $\rho$ as small as we like, and so we choose it very small ($\rho \to 0$) so

$$f(z) \to f(z_0) = \text{constant}$$

and since it's constant we can bring it out from under the integral sign!

$$i\int_0^{2\pi} f(z)\ d\theta$$

$$= if(z_0) \int_0^{2\pi} d\theta = 2\pi i f(z_0)$$

Summarizing:

$$\oint \frac{f(z)}{z - z_0} \ dz = 2\pi i f(z_0)$$

What this means is that we can evaluate the integral in question by simply plugging in the value of the function at $z_0$ and multiplying that by $2\pi i$.

**examples**

Our formula is:

$$\oint_C \frac{f(z)}{z - z_0} \ dz = 2\pi i f(z_0)$$

A very simple example is:

$$\oint_C \frac{1}{z} \ dz$$

Here $f(z) = 1$ and $z_0 = 0$ and the result should be

$$I = 2\pi i f(z_0) = 2\pi i$$

We computed this directly using the unit circle centered at $z = 0$. We have that

$$z = e^{i\theta}$$

$$dz = iz \ d\theta$$

so

$$\oint_C \frac{1}{z} \ dz = \int_0^{2\pi} \frac{1}{z} \ iz \ d\theta$$

$$= \int_0^{2\pi} i \ d\theta$$

$$= 2\pi i$$

**translated from the origin**

Suppose we are interested in the value of the integral centered at a point $z_0 \neq 0$. We consider those points $z$ in a circle constructed around $z_0$, that is

$$z = z_0 + re^{it}$$

rearranging

$$z - z_0 = re^{it}$$

We get the derivative of the polar form

$$\begin{aligned} \frac{d}{dt}\, re^{it} &= ire^{it} \\ &= i(z - z_0) \end{aligned}$$

Hence

$$\begin{aligned} \int \frac{1}{(z - z_0)}\ dz &= \int \frac{1}{(z - z_0)}\ i(z - z_0)\ dt \\ &= 2\pi i \end{aligned}$$

However, we can also do this by Cauchy 2. The formula is

$$\int \frac{f(z)}{z - z_0}\ dz = 2\pi i\ f(z_0)$$

here the integrand is

$$\frac{1}{(z - z_0)}$$

so $f(z)$ is just equal to 1, and the answer is $2\pi i$.

**extension**

A specific extension of the Cauchy Integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z}\ dw$$

is

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2}\ dw$$

Brown and Churchill give the proof for this specific case, but not for the general one (which is also true) because it's too hard for their book:

$$f^n(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}}\ dw$$

However they do point out that one simple approach is to differentiate with respect to $z$ under the intnegral sig (without real justification). That gives the result above quite simply.

# Chapter 17

# Partial Fractions

In the previous section, we derived Cauchy 2

$$\oint_C \frac{f(z)}{z - z_0}\ dz = 2\pi i f(z_0)$$

One example is

$$\oint_C \frac{1}{z - z_0}\ dz = 2\pi i$$

which is easy to show even without using the new formula.

### Example 1

This problem is Beck 4.26. Consider

$$\oint f(z)\ dz = \oint \frac{1}{z^2 + 1}\ dz$$

We see that the denominator is zero when

$$z^2 = -1$$

$$z = \pm\ i$$

Therefore we can factor the denominator as

$$z^2 + 1 = (z + i)(z - i)$$

There are a couple of different ways to handle this. One is to use partial fractions:

$$\begin{aligned}\frac{1}{z^2+1} &= \frac{1}{(z+i)(z-i)} \\ &= \frac{1}{2i}\ [\ \frac{1}{z-i} - \frac{1}{z+i}\ ]\end{aligned}$$

So the integral is a sum of two integrals:

$$I = \frac{1}{2i}\ [\oint \frac{1}{z-i}\ dz - \oint \frac{1}{z+i}\ dz\ ]$$

Suppose the curve is the unit circle centered at $i$, designated as $C[i, 1]$. Obviously, this curve contains the singularity $z = i$. The curve goes through the origin, so it does not extend as far as $z = -i$.

Therefore, the second integral is zero (no singularity) and the first is

$$\frac{1}{2i}\ [\ \oint \frac{1}{z-i}\ dz\ ]\ = \frac{1}{2i}\ [\ 2\pi i\ ]$$

by Cauchy 2. Thus the value is just $I = \pi$

According to Beck, as an alternative, rewrite the function as

$$\frac{1}{(z+i)(z-i)} = \frac{(1/z+i)}{z-i}$$

Thus

$$\int \frac{1}{z^2+1}\ dz = \int \frac{(1/z+i)}{z-i}\ dz$$

We have essentially the same thing. The function is

$$\frac{1}{z+i}$$

and when evaluated at $i$, with result $1/2i$, we obtain

$$\oint \frac{f(z)}{z - z_0}\, dz = 2\pi i\ f(z_0)$$

$$= 2\pi i\ \frac{1}{2i} = \pi$$

**partial fractions**

We can also do the curve containing both singularities by using the formal apparatus of partial fractions. Write

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

$$= \frac{A}{z+i} + \frac{B}{z-i}$$

We need to determine $A$ and $B$. When we multiply to put everything over the common denominator $(z^2+1)$ then for the numerators we will have:

$$A(z-i) + B(z+i) = 1$$

All the powers of $z$ must match across the equal sign. So this gives

$$-Ai + Bi = 1$$

$$Az + Bz = 0$$

From the second we get that $A = -B$. And so from the first

$$-Ai - Ai = -2Ai = 1$$

$$A = -\frac{1}{2i}$$

Hence the integrand is

$$\frac{1}{z^2+1} == \frac{A}{z+i} + \frac{B}{z-i}$$

$$= -\frac{1}{2i(z+i)} + \frac{1}{2i(z-i)}$$

which matches what we had above:

$$= \frac{1}{2i} \left[ \frac{1}{z-i} - \frac{1}{z+i} \right]$$

For the curve including $z = i$ but not $z = -i$ we have that the left-hand integral is 0 by Cauchy's Theorem, and for the right hand side the function is

$$f(z = z_0) = \frac{1}{2i}$$

So the value of the integral is

$$2\pi i f(z_0) = 2\pi i \, \frac{1}{2i} = \pi$$

as before. The other pole we would have

$$f(z = z_0) = -\frac{1}{2i}$$

and the result would be $-\pi$. A curve enclosing both poles would have integral equal to zero.

**example**

Consider

$$\int_\gamma \frac{z^2}{4 - z^2} \, dz$$

where $\gamma = |z+1| = 2$. So the denominator of the function is

$$\frac{1}{4-z^2} = \frac{1}{(2+z)(2-z)}$$

It has zeroes as $z = \pm\, 2$. Only the point $z = 2$ is inside our contour. So if we split this by partial fractions

$$\frac{1}{(2+z)(2-z)} = \frac{1}{4}\ [\ \frac{1}{2+z} + \frac{1}{2-z}\ ]$$

so we can rewrite the integral as

$$I = \int_\gamma \frac{z^2}{4}\ [\ \frac{1}{2+z} + \frac{1}{2-z}\ ]\ dz$$

By Cauchy's Theorem, the first term is zero. The second one is:

$$I = \int_\gamma \frac{z^2}{4}\ (\frac{1}{2-z})\ dz$$

and the value of $I$ is

$$I = 2\pi i f(z_0)$$

where

$$f(z_0) = \frac{z^2}{4}\ \Bigg|_{z_0=2} = 1$$

so the integral is just $2\pi i$.

**one from wikipedia**

Consider

$$g(z) = \frac{z^2}{z^2 + 2z + 2}$$

We want to evaluate the integral:

$$I = \oint g(z)\ dz$$

The denominator

$$z^2 + 2z + 2$$

can be factored.

We plug into the quadratic solution:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 4 \cdot 2}}{2}$$
$$= -1 \pm \frac{\sqrt{-4}}{2}$$
$$= -1 \pm i$$

The zeroes of the denominator are

$$-1 + i \ , \quad -1 - i$$

From this we construct the two factors as

$$z - (-1 + i) = z + 1 - i$$

$$z - (-1 - i) = z + 1 + i$$

We confirm that these two factors multiplied together give back what we started with:

$$(z + 1 + i) \ (z + 1 - i)$$
$$= z^2 + z + iz + z + 1 + i + -iz - i + 1$$
$$= z^2 + 2z + 2$$

So we can factor the denominator and write:

$$\frac{1}{z^2 + 2z + 2} = \frac{A}{z + 1 - i} + \frac{B}{z + 1 + i}$$

Putting these two terms over a common denominator means multiplying the two factors and restoring what we started with.

For the numerator we have

$$A(z + 1 + i) + B(z + 1 - i) = 1$$

$$Az + A + iA + Bz + B - iB = 1$$

Equating terms containing the same power of $z$ gives two simultaneous equations:

$$Az + Bz = 0z$$

and

$$A(1+i) + B(1-i) = 1$$

So $A = -B$ and

$$A(1+i) + B(1-i) = 1$$

$$A(1+i) - A(1-i) = 1$$

$$A2i = 1$$

$$A = \frac{1}{2i}, \quad B = -\frac{1}{2i}$$

The integral is

$$\oint \frac{z^2}{2i} \left[ \ \frac{1}{z + (1-i)} - \frac{1}{z + (1+i)} \ \right] dz$$

So we see that we have a sum of integrals of the form

$$\oint \frac{f(z)}{z - z_0}$$

The residues occur at the points

$$z = z_0$$

that is at

$$z = -(1-i) = -1 + i$$

$$z = -(1+i) = -1 - i$$

If the contour is $|z| = 2$ centered at the origin (the circle of radius 2, then both of the points lie within the contour. ($r^2 = 2$ for both).

We evaluate $2\pi i f(z_0)$ for each

$$f(z) = \frac{z^2}{2i}$$

The first term gives

$$f(z_0) = \frac{(-1+i)^2}{2i} = \frac{1-1-2i}{2i} = -1$$

The second term gives

$$\frac{(-1-i)^2}{2i} = \frac{-1-1+2i}{2i} = 1$$

But... this is not quite right. Go back and see that the second term in the integral has a minus sign. Hence the result at this step is $-1$ for both.

Each of these needs to be multiplied by $2\pi i$ and then summed.

$$I = -4\pi i$$

# Part V

# Residue Theory

# Chapter 18

# Residues

Normally, books on complex analysis get into Laurent series and lots of theorems at this point. However, I want to look ahead to Residue theory, which is really the reason for all the series stuff, then we'll spend some time with Taylor and Laurent.

### Residues

By definition the *residue* at a simple pole $z_0$ is defined to be

$$b_1 = \lim_{z \to z_0} (z - z_0)\ f(z)$$

We get there from

$$\oint \frac{f(z)}{z - z_0}\ dz = 2\pi i f(z_0)$$

Just think of it as in the limit that $z \to z_0$, the denominator $z - z_0$ on the left of the first equation is a constant, so we can multiply both sides by $z - z_0$ to obtain the result for the residue.

$$\oint f(z)\ dz = 2\pi i\ \lim_{z \to z_0} (z - z_0)\ f(z)$$

using the definition

$$b_1 = \lim_{z \to z_0} (z - z_0)\ f(z)$$

we obtain

$$\oint f(z)\ dz = 2\pi i\ b_1$$

If there is more than one such point

$$\oint f(z)\ dz = 2\pi i \sum \text{Res}$$

The value of the integral is $2\pi i$ times the sum of all the residues enclosed by the path.

**residues**

We repeat the problem from last time, using residues.

$$\oint f(z)\ dz = \oint \frac{1}{z^2 + 1}\ dz$$

Our formula is:

$$b_1 = \lim_{z \to z_0} (z - z_0)\ f(z)$$

and

$$\oint f(z)\ dz = 2\pi i \sum \text{Res}$$

Evaluate the first equation:

$$b_1 = \lim_{z \to z_0} (z - z_0)\ f(z)$$

$$= \lim_{z \to i} (z - i)\ \frac{1}{(z + i)(z - i)}$$

$$= \frac{1}{2i}$$

And by the second equation:

$$I = \pi$$

as before. Seems a bit easier!

If the unit circle had been centered at $-i$, rewrite the function as

$$f(z) = \frac{1/z - i}{z + i}$$

The value of the function is

$$\frac{1}{z - i}(-i) = -\frac{1}{2i}$$

and that integral is then $-\pi$.

A contour that includes both singularities integrates to zero.

**example**

In this section we get more practice.

$$\oint_C \frac{e^z}{z^2 - 2z - 3} \ dz$$

The denominator can be factored

$$z^2 - 2z - 3 = (z + 1)(z - 3)$$

if our disk contains $|z \leq 2|$ then it includes only $z = -1$ and our formula

$$b_1 = \lim_{z \to z_0} (z - z_0) \ f(z)$$

so

$$b_1 = \lim_{z \to -1} (z + 1) \ \frac{e^z}{(z + 1)(z - 3)}$$

$$= \lim_{z \to -1} \frac{e^z}{(z-3)}$$

$$= \frac{e^{-1}}{-1-3} = -\frac{1}{4e}$$

and

$$I = 2\pi i \; b_1$$

$$= 2\pi i \; (-\frac{1}{4e})$$

$$= -\frac{\pi i}{2e}$$

**example**

$$\int \frac{5z-2}{z(z-1)} \; dz$$

There are two simple poles at $z_0 = 0$ and $z_0 = 1$ and the residues are

$$\text{Res } (0) = \lim_{z \to 0} (z-0) \; \frac{5z-2}{z(z-1)}$$

$$= \lim_{z \to 0} \frac{5z-2}{(z-1)}$$

$$= \frac{5 \cdot 0 - 2}{0-1} = 2$$

$$\text{Res } (1) = \lim_{z \to 1} (z-1) \; \frac{5z-2}{z(z-1)}$$

$$= \lim_{z \to 1} \frac{5z-2}{z}$$

$$= \frac{5 \cdot 1 - 2}{1} = 3$$

Hence the total of all the residues is 5 and $I = 10\pi i$.

**example**

Consider

$$\int \frac{1}{z^4 - 1}\ dz$$

We can factor the denominator as

$$\begin{aligned} z^4 - 1 &= (z^2 - 1)(z^2 + 1) \\ &= (z+1)(z-1)(z+i)(z-i) \end{aligned}$$

We see that there are four poles, and each will have a residue.

$$\begin{aligned} \text{Res}(1) &= \lim_{z \to 1} (z-1)\ \frac{1}{(z+1)(z-1)(z+i)(z-i)} \\ &= \lim_{z \to 1}\ \frac{1}{(z+1)(z+i)(z-i)} \\ &= \frac{1}{2(1+1)} = \frac{1}{4} \\ \text{Res}(i) &= \lim_{z \to i} (z-i)\ \frac{1}{(z+1)(z-1)(z+i)(z-i)} \\ \text{Res}(i) &= \lim_{z \to i}\ \frac{1}{(z+1)(z-1)(z+i)} \\ &= \lim_{z \to i}\ \frac{1}{(z^2-1)(z+i)} \\ &= \frac{1}{(-2)(2i)} = -\frac{1}{4i} = \frac{i}{4} \end{aligned}$$

**Removable singularities**

If the residue turns out to be equal to zero, that is a removable singularity.

$$I = \int_C \frac{\sin \pi z}{z^2 - 1}\ dz$$

The denominator can be factored into

$$z^2 - 1 = (z+1)(z-1)$$

Suppose $C$ includes only $z = 1$, then

$$\text{Res}(1) = \lim_{z \to 1} (z-1) \, \frac{\sin \pi z}{(z+1)(z-1)}$$

$$= \lim_{z \to 1} \, \frac{\sin \pi z}{(z+1)} = \frac{\sin \pi}{2} = 0$$

Here's a trick:

$$f(z) = z^2 \sin \frac{1}{z}$$

Compute Res(0)

$$\sin \frac{1}{z} = \frac{1}{z} - \frac{1}{3!} \, \frac{1}{z^3} + \frac{1}{5!} \, \frac{1}{z^5} \ldots$$

$$z^2 \sin \frac{1}{z} = z - \frac{1}{3!} \, \frac{1}{z} + \frac{1}{5!} \, \frac{1}{z^3} \ldots$$

The only non-zero integral term is

$$-\frac{1}{3!} \, \frac{1}{z}$$

and the residue there is

$$\lim_{z \to 0} \, (z-0)(-\frac{1}{3!} \, \frac{1}{z})$$

$$= -\frac{1}{3!} = -\frac{1}{6}$$

# Chapter 19

# Kaplan

Kaplan gives some rules for computing residues, which we explore in this chapter.

**RULE I** At a simple pole $z_0$ (that is, a pole of first order),

$$\text{Res }[f(z), z_0] = \lim_{z \to z_0} (z - z_0) f(z)$$

**example**

The example is one we already worked.

$$f(z) = \frac{1}{z^2 + 1} = \frac{1}{(z-i)(z+i)}$$

$$\begin{aligned} \text{Res }[f(z), z = i] &= \lim_{z \to i} (z - i) \frac{1}{(z-i)(z+i)} \\ &= \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i} \end{aligned}$$

**review**

To summarize the key points about Cauchy 2 and residues, this is the theorem

$$\oint_C \frac{f(z)}{z - z_0}\ dz = 2\pi i\ f(z_0)$$

By definition the residue at a simple pole is defined to be

$$b_1 = \lim_{z \to z_0} (z - z_0)\ f(z)$$

Just think of it as in the limit that $z \to z_0$, the denominator $z - z_0$ on the left is a constant, so we can multiply both sides by $z - z_0$ to obtain the result for the residue.

$$\oint f(z)\ dz = 2\pi i\ \sum\ \text{Res}$$

The value of the integral is $2\pi i$ times the sum of all the residues enclosed by the path.

**derivative rule**

We show here that

$$f'(a) = \frac{1}{2\pi i}\ \oint_C \frac{f(z)}{(z-a)^2}\ dz$$

and there are more formulas for higher derivatives. Therefore

$$2\pi i\ f'(a) = \oint_C \frac{f(z)}{(z-a)^2}\ dz$$

Again, the Cauchy formula is:

$$\oint_C \frac{f(z)}{z - z_0}\ dz = 2\pi i f(z_0)$$

rewrite with $a$ for $z_0$

$$\oint_C \frac{f(z)}{z-a}\ dz = 2\pi i f(a)$$

We take the partial with respect to $a$ of both sides:

$$\frac{\partial}{\partial a}(\frac{f(z)}{z-a}) = \frac{f(z)}{(z-a)^2}$$

so

$$\oint_C \frac{f(z)}{(z-a)^2}\ dz = 2\pi i f'(a)$$

Thus

$$f'(a) = \frac{1}{2\pi i}\ \oint_C \frac{f(z)}{(z-a)^2}\ dz$$

More generally

$$f^n(a) = \frac{n!}{2\pi i}\ \oint_C \frac{f(z)}{(z-a)^{n+1}}\ dz$$

So

$$\frac{2\pi i}{n!} f^n(a) = \oint_C \frac{f(z)}{(z-a)^{n+1}}\ dz$$

**example**

$$\oint_C \frac{e^z}{z^3 - z^2 - 5z - 3} = \oint_C \frac{e^z}{(z+1)^2(z-3)}$$

Recall the general formula

$$f'(a) = \frac{1}{2\pi i}\ \oint_C \frac{f(z)}{(z-a)^2}\ dz$$

If the contour includes $z = -1$ but not $z = 3$ then

$$f(z) = \frac{e^z}{(z-3)}$$

so

$$f'(z) = \frac{(z-4)e^z}{(z-3)^2}$$

Hence

$$\oint_C \frac{e^z}{z^3 - z^2 - 5z - 3}\ dz = \oint_C \frac{e^z}{(z+1)^2(z-3)}$$

$$= \oint_C \frac{f(z)}{(z+1)^2}\ dz$$

for $f(z) = e^z/(z-3)$ and

$$= 2\pi i f'(-1) = 2\pi i\ \frac{-5}{e}\ \frac{1}{(-4)^2}$$

$$= \frac{-5\pi i}{8e}$$

**Kaplan**

Here is rule II from Kaplan. Rule I is in the previous section.

**RULE II** At a pole of order $N$ $(N = 2, 3, \dots)$,

$$\text{Res}\ [f(z), z_0] = \lim_{z \to z_0} (z - z_0) \frac{g^{(N-1)}(z)}{(N-1)!}$$

where

$$g(z) = (z - z_0)^N f(z)$$

**example**

$$f(z) = \frac{1}{z(z-2)^2}$$

We have a pole of first order at $z_0 = 0$ and one of second order at $z_0 = 2$. At the first

$$\text{Res }(0) = \lim_{z\to 0} \frac{1}{(z-2)^2} = \frac{1}{4}$$

For the other one, remove the factor of $1/(z-2)^2$ and compute the $N-1$ (first) derivative of what's left

$$\frac{d}{dz}\frac{1}{z} = -\frac{1}{z^2}$$

$$\text{Res }(2) = \lim_{z\to 2} -\frac{1}{z^2} = -\frac{1}{4}$$

Don't forget to divide by $(N-1)!$, which is just 1 in this case. The total is just zero.

As a check, let's do this by partial fractions.

$$\frac{1}{z(z-2)^2} = \frac{A}{(z-2)^2} + \frac{B}{z(z-2)} + \frac{C}{z}$$

Hence in putting all terms over a common denominator, for the numerator we have

$$1 = Az + B(z-2) + C(z-2)^2$$

From which we get three equations:

$$-2B + 4C = 1$$

$$Az + Bz - 4Cz = 0$$

$$Cz^2 = 0$$

Hence $C = 0$, so $B = -1/2$ and $A = 1/2$ and we obtain

$$\frac{1}{z(z-2)^2} = \frac{1/2}{(z-2)^2} - \frac{1/2}{z(z-2)}$$

which we check by doing

$$1/2 \cdot z - 1/2 \cdot (z-2) = 1$$

So how to deal with

$$\frac{1/2}{(z-2)^2} - \frac{1/2}{z(z-2)}$$

The first term has a pole of order 2 at $z_0 = 2$. We remove that factor and compute the $N-1$ (first) derivative of what's left, which is just zero.

For the second term, we have two simple poles at $z_0 = 0$ and $z_0 = 2$. The residues are

$$\text{Res }(0) = \lim_{z\to 0} z \cdot \frac{1}{z(z-2)} = -\frac{1}{2}$$

$$\text{Res }(2) = \lim_{z\to 2} (z-2) \cdot \frac{1}{z(z-2)} = \frac{1}{2}$$

which adds up to zero.

**example**

$$f(z) = \frac{1}{z^4 + z^3 - 2z^2}$$

where $C$ is the circle $|z| = 3$ with positive orientation.

The denominator can be factored as

$$z^2(z^2 + z - 2) = z^2(z+2)(z-1)$$

so

$$f(z) = \frac{1}{z^2(z+2)(z-1)}$$

There is a pole of order 2 at the origin and simple poles at 1 and -2. All of these lie within the contour $|z| = 3$.

$$\text{Res } (1) = \lim_{z \to 1} (z-1)\ f(z) = \lim_{z \to 1} \frac{1}{z^2(z+2)} = \frac{1}{3}$$

$$\text{Res } (-2) = \lim_{z \to -2} (z+2)\ f(z) = \lim_{z \to -2} \frac{1}{z^2(z-1)} = -\frac{1}{12}$$

For the double pole, we remove the factor of $1/z^2$, take the first derivative of what's left

$$\frac{d}{dz}\ \frac{1}{z^2+z-2} = \frac{(-1)(2z+1)}{(z^2+z-2)^2}$$

Evaluate at zero and obtain.

$$\text{Res } (0) = -\frac{1}{4}$$

The total of the residues is

$$\frac{1}{3} - \frac{1}{12} - \frac{1}{4} = 0$$

As Mathews and Howell say:

> The value 0 for the integral is not an obvious answer, and all of the preceding calculations are required to find it.

**example**

$$f(z) = \frac{1+e^z}{z^2} + \frac{2}{z}$$

We can break this up into its two component parts. For the first term, the pole is of order $m = 2$ at $z_0 = 0$. We remove the $z^2$ term and take the $m - 1 = 1$ derivative

$$(1+e^z)' = e^z$$

Remember to divide by $(m-1)!$, leaving $e^z$ which is evaluated at the pole giving a residue

$$\text{Res }(0) = e^0 = 1$$

The other term is just 2 times the standard

$$\oint \frac{1}{z}\ dz = 2\pi i$$

Here $I = 4\pi i$ and the residue is 2. Alternatively just use

$$I = 2\pi i f(z_0) = 4\pi i$$

where $f = 2$.

The total of the residues is 3 and the value of the integral is $6\pi i$.

**example**

$$f(z) = \frac{e^z}{z(z-1)^2}$$

We have a pole of first order at $z = 0$ and one of second order at $z = 1$. At the first

$$\text{Res }[f(z), z = 0] = \lim_{z\to 0} \frac{e^z}{(z-1)^2} = 1$$

For the other one, remove the factor of $1/(z-1)^2$ and compute the $N-1$ (first) derivative of what's left

$$\text{Res }[f(z), z = 1] = \lim_{z\to 1}\ [\ \frac{e^z}{z}\ ]'$$

$$= \frac{e^z z - e^z}{z^2}\ \bigg|_1 = 0$$

Hence

$$\oint f(z)\ dz = 2\pi i\ [\ \sum \text{Res}\ \ ]\ = 2\pi i$$

**example**

$$f(z) = \frac{1}{z(z-2)^4}$$

We have a pole of first order at $z = 0$ and one of fourth order at $z = 2$. At the first

$$\text{Res } [f(z), z = 0] = \lim_{z \to 0} z \ \frac{1}{z(z-2)^4}$$

$$= \lim_{z \to 0} \ \frac{1}{(z-2)^4} = \frac{1}{(-2)^4} = \frac{1}{16}$$

For the other pole recall that

$$\frac{2\pi i}{n!} f^n(a) = \oint_C \frac{f(z)}{(z-a)^{n+1}} \ dz$$

We remove the factor of $1/(z{-}2)^4$ leaving $f(z) = 1/z$ and then compute the $N-1$ (third) derivative of what's left

$$\text{Res } [f(z), z = 2] = \frac{1}{n!} \ \lim_{z \to 2} \ [\ \frac{1}{z} \ ]'''$$

$$f(z) = z^{-1}$$

$$f'(z) = -z^{-2}$$

$$f''(z) = 2z^{-3}$$

$$f'''(z) = -6z^{-4}$$

$$\lim_{z \to 2} \ [\ \frac{1}{z} \ ]''' = -\frac{6}{16}$$

Don't forget to divide by $(N-1)!$, which is $3! = 6$ in this case. That leaves

$$\text{Res } [f(z), z = 2] = -\frac{1}{16}$$

The total of the residues is just zero.

This problem is from Brown and Churchill (p. 234), which they work by doing Laurent series. They get a different answer, namely $-\pi i/8$.

The reason is that they integrate over the contour $0 < |z-2| < 2$, which includes the second pole, but not the first. Multiplying by $2\pi i$ gives their result.

# Chapter 20

# quotients

**RULE III** If $A(z)$ and $B(z)$ are analytic in a neighborhood of $z_0$, $A(z_0) \neq 0$, and $B(z)$ has a zero at $z_0$ of order 1, then

$$f(z) = \frac{A(z)}{B(z)}$$

has a pole of first order at $z_0$ and

$$\text{Res}\ [f(z), z_0] = \frac{A(z_0)}{B'(z_0)}$$

**example**

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$$

$$B = z^2+1, \quad B' = 2z$$

$$\text{Res}\ [f(z), z=i] = \frac{1}{2i}$$

$$\text{Res}\ [f(z), z=-i] = -\frac{1}{2i}$$

**example**

$$f(z) = \frac{ze^z}{z^2 - 1}$$

Both top and bottom are analytic. The poles of $B(z)$ are at $\pm$ 1. $A(z) \neq 0$ at those points. We have

$$\frac{A(z)}{B'(z)} = \frac{ze^z}{2z}$$

$$\left.\frac{ze^z}{2z}\right|_{z=1} = \frac{e}{2}$$

$$\left.\frac{ze^z}{2z}\right|_{z=-1} = e^{-1}$$

**RULE IV** If $A(z)$ and $B(z)$ are analytic in a neighborhood of $z_0$, $A(z_0) \neq 0$, and $B(z)$ has a zero at $z_0$ of order 2, then

$$\text{Res } [f(z), z_0] = \frac{6A'B'' - 2AB'''}{3B''^2}$$

**example**

$$f(z) = \frac{e^z}{z(z-1)^2}$$

$$B = z(z^2 - 2z + 1) = z^3 - 2z^2 + z$$

$$B' = 3z^2 - 4z + 1$$

$$B'' = 6z - 4$$

$$B''' = 6$$

So

$$\frac{6A'B'' - 2AB'''}{3B''^2} = \frac{6(e^z)(6z-4) - 2e^z(6)}{3(6z-4)^2}$$

Evaluate at $z = 1$:
$$\frac{6e(2) - 2e(6)}{12} = 0$$
So only the pole at $z = 0$ contributes.

# Chapter 21

# Definitions

### Neighborhood

A neighborhood or $\epsilon$ neighborhood of $z_0$ means simply all

$$z : |z - z_0| < \epsilon$$

### Deleted neighborhood

A deleted neighborhood does not include the point $z_0$:

$$z : 0 < |z - z_0| < \epsilon$$

### Limit

Let a function $f$ be defined at all points $z$ in some deleted neighborhood of $z_0$, then the statement

$$\lim_{z \to z_0} f(z) = w_0$$

means that the point $w = f(z)$ can be made arbitrarily close to $w_0$ if we choose the point $z$ close enough to $z_0$ (though distinct from it).

Formally, for each positive number $\epsilon$, there exists a positive number $\delta$ such that

$$|z - z_0| < \delta \quad \Rightarrow |f(z) - w_0| < \epsilon$$

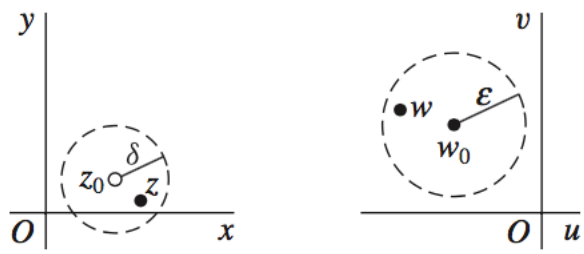



**FIGURE 23**

If the limit of a function exists at a point, it is unique.

## Continuity

A function $f$ is continuous as a point $z_0$ if all three conditions hold:

$$\lim_{z \to z_0} f(z) = f(z_0)$$

which of course requires

$$f(z_0) \text{ exists}$$

$$\lim_{z \to z_0} f(z) \text{ exists}$$

## Differentiable

A function $f$ is said to be differentiable at if the function's domain includes a neighborhood of $z_0$ and the derivative exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

The existence of the derivative at $z_0$ implies that the function is continuous at that point; however, the converse is not true.

### Analytic

A function is analytic at a point if it has a derivative at that point.

### Entire

An entire function is a function that is analytic at each point in the entire finite plane.

### Singular point

A point $z_0$ is called a singular point of a function $f$ if $f$ fails to be analytic at $z_0$ but is analytic at some point in every neighborhood of $z_0$ .

A singular point $z_0$ is said to be isolated if, in addition, there is a deleted neighborhood of $z_0$ throughout which $f$ is analytic.

### Pole

An isolated singular point is called a pole. For example

$$\frac{b_1}{z - z_0}$$

has a pole at $z_0$, since it is undefined there. A pole of order $m$ would be

$$\frac{b_1}{(z - z_0)^m}$$

### Holomorphic and meromorphic

Holomorphic is used as a synonym for analytic. A function $f$ is said to be meromorphic in a domain $D$ if it is analytic throughout $D$ except for poles.

## Limit of a sequence

An infinite sequence of complex numbers $z_1 \dots z_n$ has a limit $L$ if, for each positive number $\epsilon$, there exists a positive integer $n_0$ such that $|z_n - L| < \epsilon$ whenever $n > n_0$.

## Cauchy sequence

# Part VI

# Series

# Chapter 22

# Series

Cauchy's theorem says that the integral around a closed path for an analytic function is zero, given some conditions.

If such a function is undefined at a limited number of points (e.g. because such values produce zero in the denominator), then those points are called poles or singularities and Cauchy 2 can be used to calculate the value of the integral (called a residue) from the value of the function at those points.

All complex functions can be expanded as power series around a fixed point $z_0$. This is useful like if the radius of convergence is large enough to include the contour you want to integrate around. Unlike with Taylor series, Laurent series contain terms with negative powers. So, when integrating a function by integrating its series, we obtain a bunch of terms in different integral powers of $(z - z_0)^n$.

But by the reasoning we've given, only the term with $(z - z_0)^{-1}$ has a non-zero integral.

**HELM**

With this writeup, we'll start working through the last section of material on complex functions produced by the HELM project.

Recall that for (most) real functions we can write them as Taylor series with terms of the form

$$a_n(x - x_0)^n$$

where the series is summed over positive integers from $n = 0 \to \infty$ and the coefficients are

$$a_n = \frac{f^{(n)}}{n!}$$

the nth derivative of $f$ divided by $n!$

As an example consider

$$f(x) = \frac{1}{1-x}$$

This has a singularity at $x = 1$. We can get the Taylor series expanded around 0 for this function (this special form is called the Maclaurin series).

$$f(x) = 1 + x + x^2 + x^3 + \dots$$

We can show that this series is equal to what we started with

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

by multiplying the right-hand side by $(1-x)$. Imagine two long rows of numbers, one the series itself, and the second containing all the terms of the series multiplied by $-x$. It's clear that everything cancels except the term 1.

Alternatively, we can take derivatives and construct the series formally:

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f'(x) = \frac{1}{(1-x)^2} = (1-x)^{-2}$$

Notice that the minus sign from the exponent cancels the minus sign from the term $(1-x)$ obtained by the chain rule.

$$f''(x) = \frac{2}{(1-x)^3}$$

$$f'''(x) = \frac{3!}{(1-x)^4}$$

and so on.

Now, evaluated at $x_0 = 0$, these derivatives are seen to collapse to just the factorial, so we construct the terms of the series as

$$a_n = \frac{f^{(n)}}{n!}$$

$$= n! \, \frac{1}{n!}$$

and the factorials also cancel. This leaves the particularly simple form:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

**Convergence**

For most series the big question is: what is the radius of convergence?

The series expansion for real functions is centered around a fixed point $x_0$ with terms like $(x - x_0)^n$, and the series has a finite sum, only converges for $x$ sufficiently close to $x_0$.

$$|x - x_0| < r$$

Likewise, for complex functions, series expansions will usually only be valid for a circle (or disk, or region) of convergence in the Argand plane with

$$|z - z_0| < R$$

Convergence can be decided by certain tests including the ratio test and the root test (but sometimes the result is not clear).

Consider whether this complex series converges.

$$f(z) = \frac{1}{1-z}$$

$$= \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots$$

Without doing any tests, we see that this is the geometric series with ratio $z$, which is known to converge when $|z| < 1$.

As the source says:

"One of the shortcomings of Taylor series is that the circle of convergence is often only a part of the region in which $f(z)$ is analytic. The Laurent series is an attempt to represent $f(z)$ as a series at as many points as possible. We expand the series around a point of singularity up to, but not including, the singularity itself."

Laurent series involve an annulus, usually called $D$, which is a circle that has an empty small circle in its center, like a slice through a donut.

**Laurent's Theorem**

If $f(z)$ is analytic through a closed annulus $D$ centered at $z = z_0$, then at any point $z$ inside $D$ we can write:

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

$$+ b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \dots$$

where the coefficients are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\ dz$$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{1-n}}\ dz$$

Any polynomial of $z$ is analytic, and quotients of analytic functions are also analytic. The end result will be that the integral $\int f(z)\ dz$ may be obtained by integrating the right-hand side, where all the terms except one will have an integral equal to zero.

That is, out of this entire series given above, the only term that matters is:

$$\oint b_1(z-z_0)^{-1}\ dz$$

**Actually writing a Laurent Series**

Their example is the same function as before.

$$f(z) = \frac{1}{1-z}$$

Let's sidestep the problem of determining the coefficients using the formulas given above.

Instead, just say that we seek a series expansion using negative powers of $z$, and hope to find that it will be valid in the region $|z| > 1$.

Notice that we can factor

$$1 - z = -z(1 - \frac{1}{z})$$

So rewrite

$$f(z) = \frac{1}{1-z} = -\frac{1}{z(1-\frac{1}{z})}$$

Consider just this part

$$\frac{1}{1-\frac{1}{z}}$$

The trick is to see that this is equal to

$$1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$$

One way is to say that we had before

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

Just substitute $1/z$ for $x$. Or, perform the multiplication as we did before.

Thus we have that

$$f(z) = -\frac{1}{z}\ [\ 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots\ ]$$

$$\frac{1}{1-z} = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} + \dots$$

So we have a new series, which converges in a different region.

Namely, this is a geometric series with ratio $1/z$, so it converges when

$$\frac{1}{|z|} < 1$$

that is, when $|z| > 1$!

Note that we can substitute $-w = z$ and get

$$\frac{1}{1+w} = \frac{1}{w}\ [\ 1 - \frac{1}{w} + \frac{1}{w^2} - \frac{1}{w^3} + \dots\ ]$$

$$\frac{1}{1+w} = \frac{1}{w} - \frac{1}{w^2} + \frac{1}{w^3} + \dots$$

go back to $z$ as the variable

$$\frac{1}{1+z} = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \dots$$

Check by multiplying the right-hand side by $z$ and see all the cancellations after the first term.

### Poles and singularities

The *principal part* of the Laurent series is the part containing negative powers of $(z - z_0)$. The series for different functions may have a finite number of terms or they may not.

If the number of terms is finite like

$$\frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots + \frac{b_m}{(z - z_0)^m}$$

then we say that $f(z)$ has a pole of order $m$ at $z = z_0$.

If there is an infinite number of terms then $z = z_0$ is called an isolated essential singularity of $f(z)$. Also, some complex functions have non-isolated singularities called branch points. An example of such a function is $\sqrt{z}$.

A pole of order 1 is called a simple pole, and a pole of order 2 is called a double pole. For example the function

$$f(z) = \frac{i}{z(z-i)} = \frac{1}{z-i} - \frac{1}{z}$$

has a simple pole at $z = 0$ and another simple pole at $z = i$.

**problem**

Expand

$$f(z) = \frac{1}{2-z}$$

Factor out the 1/2

$$= \frac{1}{2}\ \frac{1}{(1-z/2)}$$

and substitute $w = z/2$. The second term is then the same series as before

$$\frac{1}{1-w} = -\frac{1}{w} - \frac{1}{w^2} - \frac{1}{w^3} + \dots$$

substitute back for $w = z/2$

$$\frac{1}{1-z/2} = -\frac{2}{z} - \frac{2^2}{z^2} - \frac{2^3}{z^3} + \dots$$

and multiply by 1/2

$$= -\frac{1}{z} - \frac{2}{z^2} - \frac{2^2}{z^3} + \dots$$

This is a geometric series with ratio $2/z$ so it is valid when

$$|\frac{2}{z}| < 1$$

$$|z| > 2$$

**The Residue Theorem**

Suppose that $f(z)$ is a function which is analytic inside and on a closed contour $C$, except for a pole of order $m$ at $z = z_0$, which lies inside $C$.

To evaluate $\oint_C f(z)\ dz$, we can expand $f(z)$ in a Laurent series in powers of $(z - z_0)$.

If we let $\Gamma$ be a circle of center $z_0$ lying inside $C$ then

$$\oint_C f(z)\ dz = \int_\Gamma f(z)\ dz$$

We know that the integral of each of the powers of $(z - z_0)$ is zero except

$$I = \int \frac{b_1}{(z - z_0)}\ dz$$

and the value of this integral is $2\pi i\ b_1$ by Cauchy 2.

Since it is the only coefficient remaining after integration it is called the *residue* of $f(z)$ at $z = z_0$. We rearrange the above to give

$$b_1 = \frac{1}{2\pi i}\ \oint_C f(z)\ dz$$

**Finding residues**

This gets a little confusing. Let's see if we can clarify things by working through an extremely simple example.

As the theorem describes, suppose we find a Laurent series for $f(z)$ as

$$f(z) = \frac{b_1}{(z - z_0)} + \sum_{n=0}^{\infty} \frac{a_n}{(z - z_0)^n}$$

with no terms in higher negative powers of $(z - z_0)$, then when we integrate

$$\oint_\Gamma f(z)\ dz = \oint \frac{b_1}{(z - z_0)}\ dz$$

because all the other powers drop out. Furthermore, although we may not know what $b_1$ is at this point, it is a *constant*. Therefore, we can easily use Cauchy2 to integrate the right hand side.

$$\oint \frac{b_1}{(z - z_0)}\ dz = 2\pi i\ b_1$$

we just write a constant complex function

$$g(z) = b_1$$

so

$$\oint \frac{g(z)}{z - z_0)} = 2\pi i \; g(z_0) = 2\pi i \; b_1$$

and

$$b_1 = \frac{1}{2\pi i} \; \oint_\Gamma \frac{f(z)}{z - z_0)}$$

**example**

Let's say $f(z)$ has a simple pole at $z = z_0$. For simplicity, say that

$$f(z) = \frac{1}{z^2 + 1}$$

The way I would solve this is to factor

$$= \frac{A}{z - i} + \frac{B}{z + i}$$

If you do the usual manipulation, it turns out that $A = -B = 1/2i$ so

$$\frac{1}{z^2 + 1} = \frac{1}{2i} \; [\frac{1}{z - i} - \frac{1}{z + i} \;]$$

as we can easily check.

Thus, we have two single poles at $z \pm i$.

So then if I want to integrate around a contour which includes (say) the upper pole at $z = i$, the term with $z + i$ in the denominator is not a singularity and its integral is zero. The non-zero one is

$$\int \frac{1}{2i} \; [\frac{1}{z - i} \;] \; dz$$

We use Cauchy 2 to find that the value is

$$= 2\pi i \ f(z_0) = 2\pi i \ f(i)$$

$$= 2\pi i \ \frac{1}{2i} = \pi$$

That is

$$\oint_C \frac{1}{z^2+1} \ dz = \pi$$

for $C$ enclosing $z = i$ but not $z = -i$.

Now, according to the notes I'm following, the value of the residue $b_1$ for this problem is

$$b_1 = \frac{1}{2i}$$

Thus

$$I = \oint_C f(z) \ dz = 2\pi i \ b_1$$

The way they do this is to take the Laurent series

$$f(z) = \frac{b_1}{(z - z_0)} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

multiply both sides

$$(z - z_0) \ f(z) = b_1 + \sum_{n=0}^{\infty} a_n (z - z_0)^{n+1}$$

So the $n = 0$ term all subsequent terms under the sum have a factor of at least $(z - z_0)$ and they disappear when we take the limit as $z \to z_0$, leaving only:

$$\lim_{z \to z_0} (z - z_0) \ f(z) = b_1$$

Thus

$$b_1 = \lim_{z \to z_0} (z - z_0) \ \frac{1}{(z-i)(z+i)}$$

In the case of the pole at $z = i$ we set $z_0 = i$ and then have

$$= (z-i)\ \frac{1}{(z-i)(z+i)}$$

$$= \frac{1}{z+i} = \frac{1}{2i}$$

To recap

$$\oint_C \frac{1}{z^2+1}\ dz = \pi$$

for $C$ enclosing $z = i$ but not $z = -i$. The residue $b_1$ is defined to be

$$b_1 = \lim_{z \to z_0} (z - z_0)\ f(z)$$

which turns out to have the value

$$b_1 = \frac{1}{2i}$$

so we see that

$$\oint_C f(z)\ dz = 2\pi i\ b_1$$

which looks a lot like Cauchy 2.

$$\oint_C \frac{f(z)}{z - z_0}\ dz = 2\pi i\ f(z_0)$$

Just think of it as in the limit that $z \to z_0$, the denominator on the left is a constant, so we can multiply both sides by $z - z_0$ to obtain the result for the residue.

**example**

https://math.berkeley.edu/~nikhil/courses/121a/laurent.pdf

Here is another example where we know the Laurent series:

$$f(z) = ze^{1/z}$$

Use the known series expansion for the exponential to write:

$$= z\ [\ 1 + (1/z) + \frac{(1/z)^2}{2!} + \ldots\ ]$$

$$= z + 1 + \frac{(1/z)}{2!} + \ldots\ ]$$

So the integral around

$$\oint f(z)\ dz = \oint \frac{(1/z)}{2!}$$

Our formulas are:

$$b_1 = \lim_{z \to z_0} (z - z_0)\ f(z)$$

$$\oint f(z)\ dz = 2\pi i\ \sum\ \text{Res}$$

so the residue is

$$b_1 = \lim_{z \to 0} (z)\ \frac{(1/z)}{2!} = \frac{1}{2}$$

and

$$\oint f(z)\ dz = 2\pi i\ \frac{1}{2} = \pi i$$

We don't need to actually know the Laurent series, only that it exists. The form is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m}$$

Now there are three steps. Multiply by $(z - z_0)^m$

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + b_1 (z - z_0)^{m-1} + \cdots + b_m$$

Second, lose the $b_2, \ldots b_m$ terms by differentiating $m - 1$ times. (Note the exponents for terms beyond $b_1$ are *smaller*, so they disappear...

$$\frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) = \sum_{n=0}^{\infty} a_n \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^{n+m} + (m-1)!\; b_1$$

Kill off al the $a_n$ terms by taking the limit $z \to z_0$

$$\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) = (m-1)!\; b_1$$

$$b_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z)$$

**example**

# Chapter 23

# Write series

**example**

Suppose

$$f(z) = \frac{z}{(z-1)(z-3)}$$

and say we need the series around $0 \leq |z-1| \leq 2$, a circle of radius 2 around the point $z_0 = 1$.

One way is to do the substitution $x = z - 1$, so $z = x + 1$ and we have

$$= \frac{x+1}{(x)(x-2)}$$

and then our goal is to get something like $1/1 - x$. Factor out the $1/x$

$$= \frac{1}{x} \, (\frac{x+1}{x-2})$$

Get $x - 2$ on top

$$= \frac{1}{x} \, (\frac{x-2+3}{x-2})$$
$$= \frac{1}{x} \, (1 + \frac{3}{x-2})$$

$$= \frac{1}{x}\ (1 - \frac{3}{2-x})$$

$$= \frac{1}{x}\ (1 - \frac{3/2}{1-x/2})$$

$$= \frac{1}{x}\ (1 - \frac{3}{2} \cdot \frac{1}{1-x/2})$$

and now the

$$\frac{1}{1-x/2}$$

can be expanded because

$$\frac{1}{1-y} = 1 + y + y^2 + y^3$$

so

$$\frac{1}{1-x/2} = 1 + \frac{x}{2} + (\frac{x}{2})^2 + \dots$$

which gives

$$= \frac{1}{x}\ [1 - \frac{3}{2} \cdot (1 + \frac{x}{2} + (\frac{x}{2})^2 + \dots)\ ]$$

Now, multiplying through by $1/x$ gives

$$-\frac{1}{2x} + \dots$$

$$= -\frac{1}{2(z-1)} + \dots$$

which we will integrate as

$$\oint \frac{-1/2}{z-1}\ dz$$

$$\text{Res }(1) = \lim_{z \to 1} (z-1)\ \frac{-1/2}{z-1}$$

$$= -\frac{1}{2}$$

Multiply by $2\pi i$ to obtain $I = \pi i$.

As a check on this go back to

$$f(z) = \frac{z}{(z-1)(z-3)}$$

$$\begin{aligned}
\text{Res }(1) &= \lim_{z\to 1}(z-1)\ \frac{z}{(z-1)(z-3)} \\
&= \lim_{z\to 1}\ \frac{z}{z-3} \\
&= -\frac{1}{2}
\end{aligned}$$

**example**

These examples can get really complicated. Here is one from

http://zimmer.csufresno.edu/~doreendl/128.13f/handouts/Lseriesex.pdf

Consider the very simple example:

$$f(z) = \frac{1}{(z-2)(z-1)}$$

This function has poles at $z = 1$ and $z = 2$. If we are asked to write expansions around $z_0 = 0$, then we have three regions of interest and three different expansions.

The first region is the circle of radius 1: $|z| < 1$, the second is $1 < |z| < 2$ and then finally $|z| > 2$.

**region 1**

Use partial fractions to write:

$$\frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$$

Considering the second term, we bring the minus sign inside

$$= \frac{1}{z-2} + \frac{1}{1-z}$$

We have the classic

$$\frac{1}{1-z} = 1 + z + z^2 = \sum_{n=0}^{\infty} z^n$$

which we know this is valid for $|z| < 1$, the region of interest.

For the other term

$$\frac{1}{z-2} = -\frac{1}{2-z}$$

Our goal is to convert this into something like the geometric series. Factor out the 2 on the bottom like so

$$= -\frac{1}{2} \; [ \; \frac{1}{1 - z/2} \; ]$$

We can do a formal substitution or recognize that this is the geometric series

$$= -\frac{1}{2} \; [ \; \sum_{n=0}^{\infty} (z/2)^n \; ]$$

We can rewrite this slightly by pulling out the factor of $2^n$ on the bottom and combining it with the factor of 2 out front:

$$= \sum_{n=0}^{\infty} \; [ \; \frac{-1}{2^{n+1}} \; ] \; z^n$$

which converges for $0 < |z/2| < 1 \Rightarrow 0 < |z| < 2$.

Our series is the sum of these two series, which can be combined as

$$\sum_{n=0}^{\infty} [\ 1 - \frac{1}{2^{n+1}}\ ]\ z^n$$

**region 2**

This is the annulus $1 < |z| < 2$. Thus

$$|\frac{1}{z}| < 1 \quad \text{and} \quad |\frac{z}{2}| < 1$$

What they do is to work on the right-hand term of

$$\frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$$

and, as we saw in the previous section transform it into something containing $1/z$, which will be valid in the region $|z| > 1$.

So let's do it:

$$\frac{1}{1-z} = -\frac{1}{z-1}$$
$$= -\frac{1}{z} \cdot \frac{1}{1-1/z}$$

leaving aside the leading factor this is

$$= 1 + \frac{1}{z} + \frac{1}{z^2} \dots$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

add back that factor

$$-\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n}$$

The left-hand term is exactly what we had before:

$$\sum_{n=0}^{\infty} [\ \frac{-1}{2^{n+1}}\ ]\ z^n$$

so we combine them

$$\sum_{n=0}^{\infty} [\ \frac{-1}{2^{n+1}}\ ]\ z^n - \frac{1}{z}\ \sum_{n=0}^{\infty} \frac{1}{z^n}$$

and then just bring that $z$ in the second term inside

$$\sum_{n=0}^{\infty} [\ \frac{-1}{2^{n+1}}\ ]\ z^n - \ \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

or change the index

$$= \sum_{n=0}^{\infty} [\ \frac{-1}{2^{n+1}}\ ]\ z^n - \ \sum_{n=1}^{\infty} \frac{1}{z^n}$$

**region 3**

We do the $1/z$ trick with both terms

$$\frac{1}{z-2} - \frac{1}{z-1}$$

Start with the first one:

$$\frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-2/z}$$

The series is

$$\frac{1}{z}\ \cdot\ \sum_{n=0}^{\infty} [\ \frac{2}{z}\ ]^n$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$$

The second term is (leaving off the factor of $-1$)

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1 - 1/z}$$

The series is

$$\frac{1}{z} \cdot \sum_{n=0}^{\infty} [\, \frac{1}{z} \,]^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

Combining the two results and bringing back the factor we get

$$\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

$$= \sum_{n=0}^{\infty} (2^n - 1) \frac{1}{z^{n+1}}$$

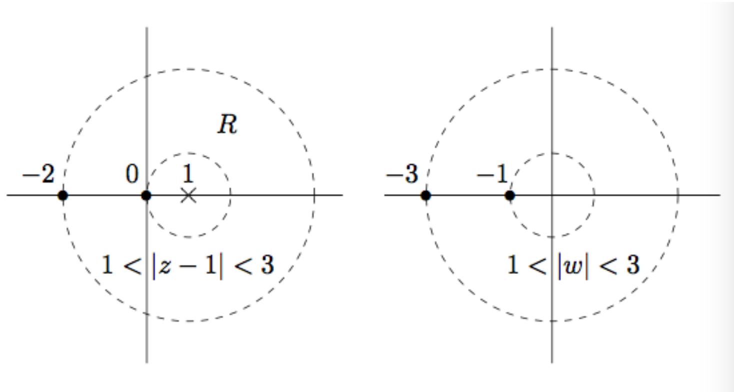
adjust the index

$$= \sum_{n=1}^{\infty} (2^{n-1} - 1) \frac{1}{z^n}$$

**example**

$$f(z) = \frac{1}{z(z+2)}$$

Suppose the region of interest is an annulus centered on $z = 1$ with $1 < |z - 1| < 3$.




The first thing to do is make a substitution that translates the region so that it becomes centered on the origin: $w = z-1$. Then the function becomes

$$\frac{1}{(w+1)(w+3)}$$

The next thing is to write partial fractions. For the numerator we get

$$A(w+3) + B(w+1) = 1$$

$$A = -B = \frac{1}{2}$$

Hence

$$\frac{1}{2} \cdot \left[ \frac{1}{w+1} - \frac{1}{w+3} \right]$$

The third step is to convert each of these fractions into something like $1/1-x$.

$$\frac{1}{w+1} = \frac{1}{1-(-w)}$$

$$\frac{1}{w+3} = \frac{1}{3} \cdot \frac{1}{1-(-w/3)}$$

And then the fourth step is to write the series, recalling that we want different forms depending on whether we are in a circle or an annulus.

$$\frac{1}{1-(-w)}$$

$$= \sum_{n=0}^{\infty} (-w)^n = \sum_{n=0}^{\infty} (-1)^n (w)^n, \quad |w| < 1$$

$$= -\sum_{n=1}^{\infty} \frac{1}{(-w)^n} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{w^n}, \quad |w| > 1$$

We pick the second form because our region is $1 < |z-1| < 3$

A similar thing can be done for the other term. We show only the first series since we are inside the circle.

$$\frac{1}{3} \cdot \frac{1}{1-(-w/3)}$$

$$= \frac{1}{3} \cdot \sum_{n=0}^{\infty} (-1)^n (\frac{w}{3})^n, \quad |w| < 3$$

$$= \sum_{n=0}^{\infty} (-1)^n \ \frac{1}{3^{n+1}} \ w^n, \quad |w| < 3$$

Add the two series together (remembering the minus sign on the second term)

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{w^n} - \sum_{n=0}^{\infty} (-1)^n \ \frac{1}{3^{n+1}} \ w^n$$

and then picking up the leading factor from

$$\frac{1}{2} \cdot \ [\ \frac{1}{w+1} - \frac{1}{w+3}\ ]$$

so

$$\frac{1}{2}\ [\ -\sum_{n=1}^{\infty} \frac{(-1)^n}{w^n} - \sum_{n=0}^{\infty} (-1)^n \ \frac{1}{3^{n+1}} \ w^n\ ]$$

The last step is to reverse the substitution: $w = z - 1$ and bring the minus sign out front

$$f(z) = -\frac{1}{2} \; [ \; \sum_{n=1}^{\infty} \frac{(-1)^n}{(z-1)^n} \sum_{n=0}^{\infty} (-1)^n \; \frac{1}{3^{n+1}} \; (z-1)^n \; ]$$

I don't know if I could ever learn to do this well, but at least the explanations make sense.

Now, if we were to integrate $f(z)$, we would have only one term that gives a non-zero result, namely the first term with $n = 1$

$$-\frac{1}{2}(-1)\frac{1}{z-1}$$

$$\text{Res } (1) = \lim_{z \to 1} \frac{1}{2} = \frac{1}{2}$$

Multiply by $2\pi i$ to obtain $\pi i$.

# Chapter 24

# Trig Integrals

Karkhar gives this problem

$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\ d\theta$$

Before we start I'd just point out that the equation for an ellipse in polar coordinates (with one focus at the origin) is

$$r = \frac{b^2}{a - c\cos\theta}$$

If we neglect the minus sign (which just flips the orientation along the $x$-axis), let $a = 2$ and $c = 1$ and

$$b^2 = a^2 - c^2 = 3$$

rewrite

$$\int_0^{2\pi} \frac{3}{2+\cos\theta}\ d\theta$$

What this looks like to me is the integral of $r\ d\theta$ around an ellipse with $a = 2$ and $b = \sqrt{3}$. This would be $rd\theta$ added up over the perimeter of that ellipse, i.e. the area.

Go back to the given problem. Let

$$z = e^{i\theta}$$

$$dz = iz\ d\theta$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

Use this result, but go back to $z$:

$$\int_{|z|=1} \frac{1}{2 + (1/2)(z + 1/z)}\ \frac{1}{iz}\ dz$$

$$= \frac{1}{i}\int_{|z|=1} \frac{1}{2z + (1/2)(z^2+1)}\ dz$$

$$= \frac{2}{i}\int_{|z|=1} \frac{1}{z^2+4z+1}\ dz$$

The roots of the denominator are $-2 \pm \sqrt{3}$. One of these roots ($-2 + \sqrt{3}$) lies within our contour, which is just the unit circle.

Carry out partial fractions:

$$\frac{1}{z^2+4z+1} = \frac{A}{z - (-2+\sqrt{3})} + \frac{B}{z - (-2-\sqrt{3})}$$

$$Az + A2 + A\sqrt{3} + Bz + B2 - B\sqrt{3} = 1$$

Hence $A = -B$ and

$$A2 + A\sqrt{3} - A2 + A\sqrt{3} = 1$$

$$2A\sqrt{3} = 1$$

$$A = \frac{1}{2\sqrt{3}}$$

The term we want is the one with $z_0 = (-2+\sqrt{3})$ and that has coefficient $A$. Hence the value is

$$2\pi i \ (\frac{1}{2\sqrt{3}}) = \frac{\pi i}{\sqrt{3}}$$

Pick up the leading factor of $2/i$ and obtain $2\pi/\sqrt{3}$.

Going back to the argument about the ellipse at the beginning, multiplied by 3 gives $2\sqrt{3}\pi$. This is exactly the area of an ellipse with $a = 2$ and $b = \sqrt{3}$.

**residues**

The zeros are at

$$-2 \pm \ \sqrt{3}$$

If we call these two values $a_1$ and $a_2$ with

$$a_1 = -2 + \sqrt{3}$$

$$a_2 = -2 - \sqrt{3}$$

then

$$f(z) = \frac{1}{(z-a_1)(z-a_2)}$$

We see that only $a_1 = -2+\sqrt{3}$ is within the contour over which we're integrating, so using the formula for residues

$$\text{Res } (a_1) = \lim_{z \to a_1} (z - a_1) \ \frac{1}{(z-a_1)(z-a_2)}$$

$$= \lim_{z \to a_1} \frac{1}{z-a_2}$$

$$= \frac{1}{a_1 - a_2}$$

Now

$$\begin{aligned} a_1 - a_2 &= (-2 + \sqrt{3}) - (-2 - \sqrt{3}) \\ &= 2\sqrt{3} \end{aligned}$$

so

$$\frac{1}{a_1 - a_2} = \frac{1}{2\sqrt{3}}$$

To get the value of the whole integral we have to pick up the leading factor of $2/i$, giving

$$\frac{2}{i} \cdot \frac{1}{2\sqrt{3}}$$

we also need to multiply the result by $2\pi i$

$$I = 2\pi i \cdot \frac{2}{i} \cdot \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

As we said before, if we get back to the argument about the ellipse at the beginning, multiplying by 3 gives $2\sqrt{3}\pi$, which is exactly the area of an ellipse with $a = 2$ and $b = \sqrt{3}$.

# Chapter 25

# Real Integrals

### inverse tangent

Note the complex function

$$\oint_C^\infty \frac{1}{1+z^2}\ dz$$

We solved this one previously. Basically we factor the integrand as

$$-\frac{1}{2i}\ [\ \frac{1}{z+i} - \frac{1}{z-i}\ ]$$

We will integrate over a path that includes only the pole at $z = i$, so only the second term contributes, and by Cauchy 2 the value is

$$I = 2\pi i f(z_0) = 2\pi i \cdot (-\frac{1}{2i}) \cdot (-1) = \pi$$

Now we do the same integral on the real axis, with different limits:

$$\int_0^\infty \frac{1}{1+x^2}\ dx$$

We know the answer to this one, it is

$$\tan^{-1} x \Big|_0^\infty = \frac{\pi}{2}$$

Since $f(x)$ is an even function, the integral over $-\infty \to \infty = \pi$.

We feel there ought to be a connection between the two results: real and complex.

Suppose we draw a different curve (contour) extending on its base from $-\infty \to \infty$: the real axis. That integral is $\int f(z)\ dz$ but $y$ and $dy$ are both zero so it becomes just $\int f(x)\ dx$ with the result shown.

How to complete the contour? Imagine a semicircle in the upper half-plane with $R \to \infty$. That is, parametrize

$$\gamma(\theta) = Re^{i\theta}, \quad \theta \in [0, \pi]$$
$$\gamma\,'(\theta) = iRe^{i\theta}$$

The integral is

$$\int_0^\pi \frac{1}{1 + R^2 e^{i2\theta}}\ iRe^{i\theta}\ d\theta$$
$$= i \int_0^\pi \frac{1}{1/Re^{i\theta} + Re^{i\theta}}\ \ d\theta$$

Now what?

We'll try to be more formal later, but just for now, it's clear that as $R \to \infty$, this integrand goes to 0. So we have that the total integral for the complex case is equal to the integral for the real part plus this extra half-circle which is zero.

What this means is that if we had not know the result for the real integral, we could deduce it from the fact that the whole complex integral has value equal to $\pi$, and the part over this complex half-circle is zero.

**Gaussian**

**application of Cauchy 1**

The function we'll be working with is one we introduced before:

$$u(x,y) = e^{-x^2} e^{y^2} \cos 2xy$$

$$v(x,y) = e^{-x^2} e^{y^2} (-\sin 2xy)$$

Everything will simplify pretty quickly. Divide the path into its four parts and compute each separately: Over $C1$, $y = 0$ and $dy = 0$ so we have:

$$\int_{C1} = \int u \ dx = \int_0^a e^{-x^2} e^0 \cos 0 \ dx = \int_0^a e^{-x^2} \ dx$$

C2 ($x = a$, $dx = 0$):

$$\int_{C2} = -\int_0^b e^{-a^2} e^{y^2} (-\sin 2ay) \ dy$$

C3 ($y = a$, $dy = 0$):

$$\int_{C3} = \int_a^0 e^{-x^2} e^{b^2} (\cos 2bx) \ dx$$

C4 ($x = 0$, $dx = 0$):

$$\int_{C4} = \int_b^0 e^{y^2} (-\sin 0) \ dy = 0$$

So all together:

$$\int_0^a e^{-x^2} \ dx - \int_0^b e^{-a^2} e^{y^2} (-\sin 2ay) \ dy + \int_a^0 e^{-x^2} e^{b^2} \cos 2bx \ dx = 0$$

$$\int_0^a e^{-x^2} \ dx = e^{-a^2} \int_0^b e^{y^2} (-\sin 2ay) \ dy + e^{b^2} \int_0^a e^{-x^2} \cos 2bx \ dx$$

Let $a \to \infty$. Then

$$e^{-a^2} \to 0$$

so the first term on the right side goes to zero and we have:

$$\int_0^\infty e^{-x^2}\ dx = e^{b^2} \int_0^\infty e^{-x^2} \cos 2bx\ dx$$

But we know the value of the left-hand side, it is

$$\int_0^\infty e^{-x^2}\ dx = \frac{\sqrt{\pi}}{2}$$

so

$$\int_0^\infty e^{-x^2} \cos 2bx\ dx = \frac{\sqrt{\pi}}{2}\ e^{-b^2}$$

The Gaussian that we know, is a special case of this general form.

**example**

We will develop a proof that

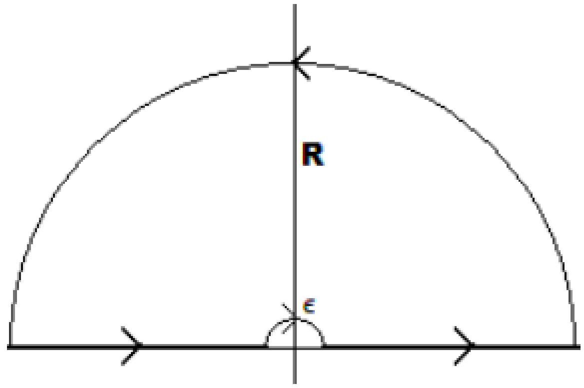
$$\int_{-\infty}^\infty \frac{\sin x}{x}\ dx = \frac{\pi}{2}$$

Start by considering the function

$$f(z) = \frac{e^{iz}}{z}$$

Obviously there is a pole at the origin.

Now we consider a specially designed contour that avoids the origin by going around it clockwise in a semi-circle of radius $\epsilon$.




Since the integral avoids the pole, its value is zero.

**small semi-circle**

Consider the pieces next, starting with the small semi-circle. In the limit as $\epsilon \to 0$

$$f(z) = \frac{e^{iz}}{z}$$

$$\text{Res } (0) = \lim_{z \to 0} z \ \frac{e^{iz}}{z}$$

$$= \lim_{z \to 0} e^{iz}$$

$$= 1$$

We multiply by $\pi i$ for the half-circular path, and put in a minus sign since we are going counter-clockwise.

$$I = -\pi i$$

As a check write

$$e^{iz} = \cos z + i \sin z$$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots + iz - i\frac{z^3}{3!}$$

Multiply by $1/z$ to obtain

$$= \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} \cdots + i - i\frac{z^3}{3!}$$

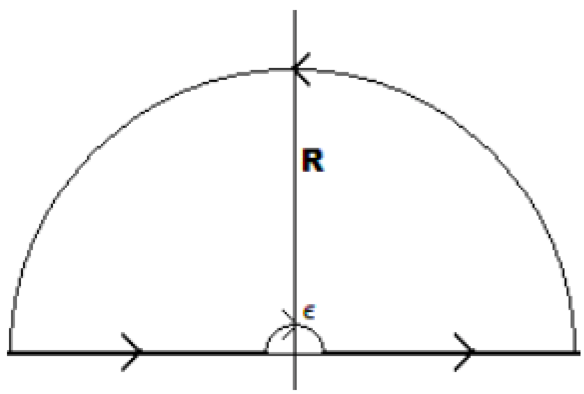
In the limit $z \to 0$,

$$= \frac{1}{z} + i$$

Or just look at the integral term by term of the series. The only non-zero term is

$$\int_{\pi}^{0} \frac{1}{z}\, dz = -\pi i$$

**large semi-circle**




For the large semi-circle at radius $R$ we have

$$z = Re^{i\theta}$$

$$\int \frac{e^{iz}}{z}\, dz$$

The absolute value of the denominator is

$$|z| = |Re^{i\theta}| = R$$

The numerator is

$$e^{iz} = e^{iRe^{i\theta}}$$
$$= e^{R(i\cos\theta - \sin\theta)}$$
$$= e^{-R\sin\theta}\ e^{Ri\cos\theta}$$

Go back to the fundamental definition of length for a complex number: $|w|^2 = ww*$. The square of the length of the numerator is

$$|e^{-R\sin\theta}|^2\ [\ e^{Ri\cos\theta}e^{R(-i)\cos\theta}\ ]$$
$$= |e^{-R\sin\theta}|^2$$

So the absolute value of the numerator is just

$$e^{-R\sin\theta}$$

and the absolute value of

$$|\frac{e^{iz}}{z}| = \frac{e^{-R\sin\theta}}{R}$$

now

$$dz = Re^{i\theta}\ d\theta$$
$$|dz| = |Rd\theta|$$

so

$$\int |\frac{e^{iz}}{z}\ dz| = \int \frac{e^{-R\sin\theta}}{R}\ Rd\theta$$
$$= \int e^{-R\sin\theta}\ d\theta$$

but in the limit as $R \to \infty$, the integrand goes to zero.

The last two segments lie on the real axis. The first

$$\int_{x=-R}^{-\epsilon} \frac{e^{ix}}{x}\ dx$$

(since $y$ and $dy$ are both zero. Now, tsubstitute $-x$ for x

$$\int_{-x=-R}^{-\epsilon} \frac{e^{-ix}}{-x}\ d-x$$
$$= \int_{x=R}^{\epsilon} \frac{e^{-ix}}{x}\ dx$$

the second one is just

$$\int_{x=\epsilon}^{R} \frac{e^{ix}}{x}\ dx$$

so then in the limit as $\epsilon \to 0$ and $R \to \infty$ this is

$$\int_0^\infty \frac{e^{ix}}{x} + \frac{e^{-ix}}{x}\ dx$$
$$= \int_0^\infty 2i\ \frac{\sin x}{x}\ dx$$

Adding together all four pieces and equating them to the first result for the whole contour (zero), we have

$$0 = -\pi i + 0 + 2i \int_0^\infty \frac{\sin x}{x}\ dx$$

Rearranging

$$\pi i = 2i \int_0^\infty \frac{\sin x}{x}\ dx$$
$$\int_0^\infty \frac{\sin x}{x}\ dx = \frac{\pi}{2}$$

**Complicated example using Cauchy 2**

Consider a semicircle of radius $R$ lying in the first two quadrants with its diameter on the real $x$-axis, and a function $f(z)$. We wish to evaluate:

$$= \oint_C \frac{e^{iaz}}{b^2 + z^2}\ dz$$

where $a$ and $b$ are positive constants.

It is apparent that $f(z)$ has singularities at $z = \pm ib$, where $b < R$. In particular, we are interested in what happens as $R \to \infty$.

Along the $x$-axis, we have $y = 0$ and $dy = 0$, so $dz = dx$ and

$$\int_{C1} = \int_{-R}^{R} \frac{e^{iax}}{b^2 + x^2} \ dx$$

Along the semi-circular arc, we have $r = R$ and $\theta = 0 \to \pi$ and

$$z = Re^{i\theta}$$

$$dz = iRe^{i\theta} \ d\theta$$

$$\int_{C2} = \int_{0}^{\pi} \frac{e^{ia(Re^{i\theta})}}{b^2 + R^2 e^{i2\theta}} \ iRe^{i\theta} \ d\theta$$

Thus

$$\oint_C z \ dz = \oint_C \frac{e^{iaz}}{b^2 + z^2} \ dz$$

$$= \int_{-R}^{R} \frac{e^{iax}}{b^2 + x^2} \ dx + \int_{0}^{\pi} \frac{e^{ia(Re^{i\theta})}}{b^2 + R^2 e^{i2\theta}} \ iRe^{i\theta} \ d\theta$$

Rewriting the integrand for the integral on the left-hand side:

$$\frac{e^{iaz}}{b^2 + z^2} = \frac{e^{iaz}}{(z + ib)(z - ib)} = \frac{e^{iaz}}{i2b}\left(\frac{1}{z - ib} - \frac{1}{z + ib}\right)$$

Having factored out $1/i2b$, rewrite the integral as

$$\frac{1}{i2b}\left(\oint \frac{e^{iaz}}{z - ib} \, dz - \oint \frac{e^{iaz}}{z + ib} \, dz\right) = \int_{-R}^{R} \frac{e^{iax}}{b^2 + x^2} \, dx + \int_{0}^{\pi} \frac{e^{ia(Re^{i\theta})}}{b^2 + R^2 e^{i2\theta}} \, iRe^{i\theta} \, d\theta$$

That's quite a mouthful!

The second term on the left-hand side has a singularity at $z = -ib$, which is *outside* the region (actually, below it) and hence by Cauchy 1 that integral is zero.

So now

$$\frac{1}{i2b} \oint \frac{e^{iaz}}{z - ib}\ dz = \int_{-R}^{R} \frac{e^{iax}}{b^2 + x^2}\ dx + \int_0^{\pi} \frac{e^{ia(Re^{i\theta})}}{b^2 + R^2 e^{i2\theta}}\ iRe^{i\theta}\ d\theta$$

Looking at the other contour integral

$$\frac{1}{i2b} \oint \frac{e^{iaz}}{z - ib}\ dz$$

we have a singularity at $z = z_0 = ib$, which, as $R$ becomes large, is inside the semicircular region and thus by Cauchy 2 the integral is equal to $2\pi i\ f(z_0)$ where

$$f(z_0) = e^{iaz_0} = e^{iaib} = e^{-ab}$$

and so we have

$$\begin{aligned}\frac{1}{i2b} \oint \frac{e^{iaz}}{z - ib}\ dz &= \frac{1}{i2b} 2\pi i e^{-ab}, \quad R > b \\ &= \frac{\pi}{b} e^{-ab}\end{aligned}$$

Putting it all together

$$\frac{\pi}{b} e^{-ab} = \int_{-R}^{R} \frac{e^{iax}}{b^2 + x^2}\ dx + \int_0^{\pi} \frac{e^{ia(Re^{i\theta})}}{b^2 + R^2 e^{i2\theta}}\ iRe^{i\theta}\ d\theta$$

Nahin shows that the second integral on the right-hand side vanishes as $R \to \infty$. The reason is that we have $R^2$ in the denominator and only $R$ in the numerator.

So

$$\frac{\pi}{b}e^{-ab} = \int_{-\infty}^{\infty} \frac{e^{iax}}{b^2 + x^2}\ dx$$

$$\frac{\pi}{b}e^{-ab} = \int_{-\infty}^{\infty} \frac{\cos(ax)}{b^2 + x^2}\ dx + i\int_{-\infty}^{\infty} \frac{\sin(ax)}{b^2 + x^2}\ dx$$

The imaginary part of the left-hand side is zero, so by the equality we must have that

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{b^2 + x^2}\ dx = 0$$

"which is no surprise since the integrand is an odd function of $x$". But the other result (from the real part) is:

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{b^2 + x^2}\ dx = \frac{\pi}{b}e^{-ab}$$

In the special case $a = b = 1$ we obtain

$$\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2}\ dx = \frac{\pi}{e} = 1.15572735$$

which is not only an integral we didn't know how to do before, but a remarkable fraction as the result.

# Part VII

# Mapping

# Chapter 26

# Conformal mapping

### conformal mapping

At this point let's just remind ourselves how the visual representation of a complex function differs from the more familiar case of a function $f : \mathbb{R}^1 \to \mathbb{R}^1$.

In the real case, the first dimension is the independent variable $x$ and the second is $y = f(x)$ and the derivative is the *slope* of the curve produced by plotting pairs of $x, f(x)$.

In the complex case, our numbers $z$ are points in the complex plane. They are *mapped* to other complex numbers in a different complex plane, which is often called $w$, where $w = f(z) = u(x, y) + iv(x, y)$.

The derivative does not have any notion of slope. Our requirement for differentiability included the constraint that the derivative of the function at a point $z_0$ must be the same no matter from what direction we approach that point. This leads to the CRE and the study of only analytic functions. Many such functions may have isolated points at which they are not defined, and that will still be OK.

In the figure, is shown the mapping corresponding to the complex function $w = f(z) = z^2$.

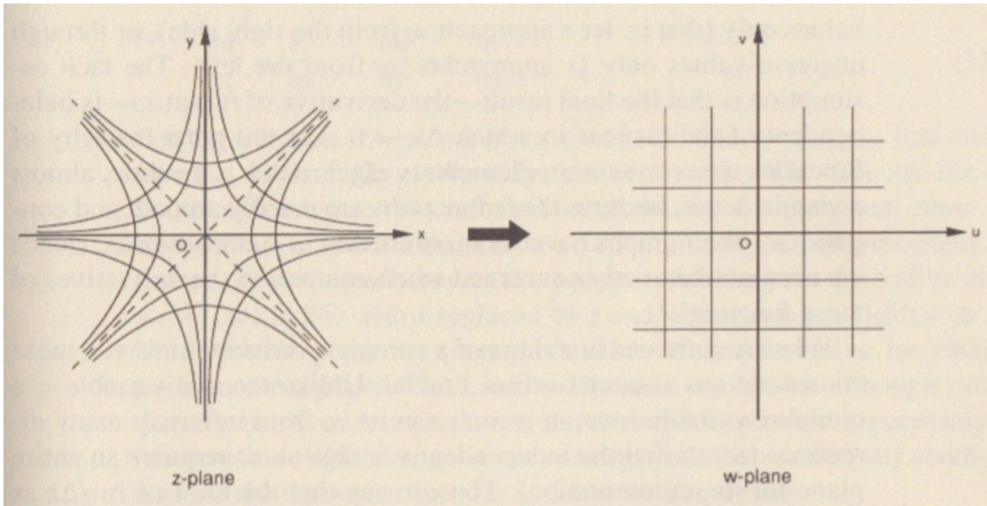



**FIG. 71.** Mapping by the complex function $w = z^2$.

$$z = x + iy$$
$$z^2 = x^2 - y^2 + i2xy$$

The functions $u = x^2 - y^2$ and $v = 2xy$ are both hyperbolas. Consider what happens in the case where $u = c$ where $c$ is a constant. The values of $x$ and $y$ that satisfy this constrain lie on hyperbolas in the $z$-plane. For example, the points on the curve $xy = 1/2$ correspond to the points on the curve $v = 1$, which is a straight vertical line in the $w$ plane.

In this sense, the hyperbolic curves in the $z$-plane shown in the figure are mapped into rectangular grid in the $w$-plane. An important note here is that the angles where these curves meet are the same in both the $z$-plane and the $w$-plane. In both cases the lines meet at right angles.

According to wolfram

http://mathworld.wolfram.com/ConformalMapping.html

A conformal mapping, also called a **conformal map**, conformal transformation, angle-preserving transformation, or biholomorphic map, is a transformation that preserves local angles. An analytic function is conformal at any point where it has a nonzero derivative.

### looking ahead

As motivation to do the work that is coming, consider these statements from the summary article in wikipedia:

One of the central tools in complex analysis is the line integral. The line integral around a closed path of a function that is holomorphic everywhere inside the area bounded by the closed path is always zero, which is what the Cauchy integral theorem states. The values of such a holomorphic function inside a disk can be computed by a path integral on the disk's boundary, as shown in (Cauchy's integral formula).

Path integrals in the complex plane are often used to determine complicated real integrals, and here the theory of residues among others is applicable (see methods of contour integration). A "pole" (or isolated singularity) of a function is a point where the function's value becomes unbounded, or "blows up". If a function has such a pole, then one can compute the function's residue there, which can be used to compute path integrals involving the function; this is the content of the powerful residue theorem.

### harmonics

Boas says this about analytic functions (that satisfy the CRE).

"If $f(z) = u = iv$ is analytic in a region, then $u$ and $v$ satisfy Laplace's equation, that is, $u$ and $v$ are harmonic functions..."

Laplace's equation is:

$$\nabla^2 f = 0$$

Consider the function

$$u(x, y) = x^2 - y^2$$
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
$$= 2 - 2 = 0$$

To find the function $v(x, y)$ such that $z = u + iv$ is analytic, use the CRE:

$$u = x^2 - y^2$$

$$v_y = u_x = 2x$$

$$v_x = -u_y = 2y$$

So it looks like $2xy$ will work. In particular

$$z = x^2 - y^2 + i2xy + \text{constant}$$

But of course

$$x^2 - y^2 + i2xy = (x + iy)^2 = z^2$$

which does not depend on $z*$.

To explore why this is true, take the second derivatives of the CRE:

$$u_x = v_y$$

$$u_y = -v_x$$

$$u_{xx} = v_{yx}$$

$$u_{xy} = v_{yy}$$
$$u_{yx} = -v_{xx}$$
$$u_{yy} = -v_{xy}$$

But the mixed partials must be equal so

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy}$$
$$u_{xx} + u_{yy} = 0$$
$$v_{xx} = -u_{yx} = -u_{xy} = -v_{yy}$$
$$v_{xx} + v_{yy} = 0$$

# Chapter 27

# End

So now, let's go back to the integrals we listed at the very beginning:

$$\int_0^\infty \frac{\sin^2 x}{x^2}\ dx = \frac{\pi}{2}$$
$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\ dx = \frac{\pi}{\sin \alpha\pi}$$
$$\int_0^{2\pi} \frac{1}{a+\sin\theta}\ d\theta = \frac{2\pi}{\sqrt{a^2-1}}$$

Can we solve these now?

### problem 1

We solved a problem related to the first integral previously:

$$\int_0^\infty \frac{\sin x}{x}\ dx = \frac{\pi}{2}$$

How to make use of that?

Start with the sum of angles:

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$

$$\begin{aligned}\cos(s-t) &= \cos s\cos(-t) - \sin s\sin(-t)\\ &= \cos s\cos t + \sin s\sin t\end{aligned}$$

Adding the minus the first to the second gives:

$$\cos(s-t) - \cos(s+t) = 2\sin s\sin t$$

For the general problem, we obtain

$$2\ \frac{\sin(ax)\sin(bx)}{x^2} = \frac{\cos((a-b)x) - \cos((a+b)x)}{x^2}$$

Now if we consider a function of $y$

$$\frac{\sin xy}{x}\ dy$$

the integral of this function is

$$\int \frac{\sin xy}{x}\ dy = -\frac{1}{x^2}\ [\cos xy\ ] + C$$

Suppose we use as bounds $a-b$ and $a+b$

$$\int_{a-b}^{a+b} \frac{\sin xy}{x}\ dy = -\frac{1}{x^2}\ [\cos(a+b)x - \cos(a-b)x\ ]$$

$$\int_{a-b}^{a+b} \frac{\sin xy}{x}\ dy = \frac{1}{x^2}\ [\cos(a-b)x + \cos(a+b)x\ ]$$

Hence, going back to our problem

$$2\ \frac{\sin(ax)\sin(bx)}{x^2} = \frac{\cos((a-b)x) - \cos((a+b)x)}{x^2}$$

and integrating both sides we see that

$$2\int_0^\infty \frac{\sin ax\sin bx}{x^2}\ dx = \int_0^\infty \int_{a-b}^{a+b} \frac{\sin xy}{x}\ dy\ dx$$

Change the order of integration:

$$= \int_{a-b}^{a+b} \int_0^{\infty} \frac{\sin xy}{x} \; dx \; dy$$

for the inner integral $y$ is constant. Substitute $t = xy$, so $1/x = y/t$ and $dt = y \; dx$ and we have

$$\int_0^{\infty} \frac{\sin xy}{x} \; dx = y \frac{\sin t}{t} \; \frac{1}{y} \; dt$$

$$= \int_0^{\infty} \frac{\sin t}{t} \;\; dt = \frac{\pi}{2}$$

so the outer integral is

$$\frac{\pi}{2} \int_{a-b}^{a+b} \; dy = \frac{\pi}{2} \; 2b = \pi b$$

In this problem, $a = b = 1$ so

$$2 \int_0^{\infty} \frac{\sin^2 x}{x^2} \; dx = \pi$$

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} \; dx = \frac{\pi}{2}$$

That was a little harder than I expected! Notice the result that

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} \; dx = \int_0^{\infty} \frac{\sin x}{x} \; dx$$

**problem 2**

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} \; dx = \frac{\pi}{\sin \alpha\pi}$$

Write the complex integral

$$\int \frac{z^{\alpha-1}}{1+z}\, dz$$

This has a simple pole at $z = -1$, so any integral that includes that pole will have the value

$$\text{Res}\,(-1) = (-1)^{\alpha-1}$$

But we want the base of our contour to include $0 \to \infty$ (we don't need to include the pole), hence

$$\oint f(z)\, dz = 0$$

$$= \int_0^\infty \frac{x^{\alpha-1}}{1+x}\, dx + \int_0^{\pi/2} \frac{z^{\alpha-1}}{1+z}\, dz$$

say that

$$z = Re^{i\theta}$$

$$dz = iz\, d\theta$$

??

**problem 3**

$$\int_0^{2\pi} \frac{1}{a+\sin\theta}\, d\theta = \frac{2\pi}{\sqrt{a^2-1}}$$

Previously we solved a similar problem

$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta$$

Recall there that we said that the equation for an ellipse in polar coordinates (with one focus at the origin) is

$$r = \frac{b^2}{a - c\cos\theta}$$

Neglect the minus sign (which just flips the orientation along the $x$-axis) and write

$$r = \frac{b^2}{a + c\cos\theta}$$

Let $a = 2$ and $c = 1$ and

$$b^2 = a^2 - c^2 = 3$$

Substituting

$$r = \frac{3}{2 + \cos\theta}$$

$$3r = \frac{1}{2 + \cos\theta}$$

What the integral seems to be is the integral of $r\ d\theta$ around an ellipse with $a = 2$ and $b = \sqrt{3}$. This would be $rd\theta$ added up over the perimeter of that ellipse, i.e. the area.

The area would be $\pi ab = \pi 2\sqrt{3}$. 3 times the value of the integral is equal to this so

$$I = \frac{1}{3}\ \pi 2\sqrt{3} = \pi\frac{2}{\sqrt{3}}$$

Substitution of $\sin\theta$ for $\cos\theta$ just rotates the ellipse.

Let's solve the given problem:

$$\int_0^{2\pi} \frac{1}{a + \sin\theta}\ d\theta$$

write

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Subtract

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$
$$\sin\theta = \frac{1}{2i}\,(e^{i\theta} - e^{-i\theta})$$

Suppose our contour is the unit disk centered at zero. Then

$$\frac{1}{a+\sin\theta} = \frac{1}{a + 1/2i(z+1/z)}$$

and

$$dz = iz\ d\theta$$

as we've seen before, so we have

$$\int_0^{2\pi} \frac{1}{a+\sin\theta}\ d\theta = \oint \frac{1}{a+1/2i(z+1/z)}\ \frac{1}{iz}\ dz$$
$$= \oint \frac{1}{iaz - 1/2(z^2+1)}\ \ dz$$

# Part VIII

# History

# Chapter 28

# Cubics

### quadratic

Some of the earliest examples of problems where the square root of a negative number arises involve a right triangle of a specified area and perimeter.

Nahin says, suppose a right triangle has area 7 and perimeter 12. Find the two sides.

Label the sides as $a$ and $b$.

We can get some idea of where this problem is headed by supposing that the triangle is also isosceles with $a = b$. Then

$$\frac{1}{2}ab = 7$$

$$ab = 14$$

$$a^2 = 14$$

so $a = \sqrt{14}$, and the perimeter is

$$p = a + b + \sqrt{a^2 + b^2}$$

$$= 2\sqrt{14} + \sqrt{14 + 14} = 12.77$$

The perimeter we are given is smaller than that

However, an isosceles right triangle has the smallest possible perimeter for a given area (the largest area for a given perimeter), hence there is no such pair $a, b$. The problem as posed has no solution.

Proof:

Let $k$ be a constant and $x$ and $k - x$ be the given sides. The area is

$$A = \frac{1}{2}x(k - x)$$

$$= -\frac{1}{2}x^2 + \frac{k}{2}x$$

The extreme point is

$$\frac{dA}{dx} = 0 = -x + \frac{k}{2}$$

$$x = \frac{k}{2}$$

The second derivative is $-1 < 0$, which shows that this is a minimum. We can also see the same thing from the negative cofactor of $x^2$ in the equation.

$$A = -\frac{1}{2}x^2 + \frac{k}{2}x$$

Doing the algebra of the original problem anyway, we solve two simultaneous equations

$$a \cdot b = 14$$

$$p = a + b + \sqrt{a^2 + b^2} = 12$$

Isolate and then remove the square root in the second one

$$a^2 + b^2 = (12 - a - b)^2$$

$$= 12^2 - 12a - 12b - 12a + a^2 + ab - 12b + ab + b^2$$

Collect terms and cancel $a^2$ and $b^2$

$$0 = 12^2 - 24a - 24b + 2ab$$

$$0 = 72 - 12a - 12b + ab$$

Substituting from the first equation given above

$$0 = 72 - 12a - 12 \cdot \frac{14}{a} + 14$$

$$0 = 36 - 6a - 6 \cdot \frac{14}{a} + 7$$

$$-6a^2 + 43a - 84 = 0$$

$$6a^2 - 43a + 84 = 0$$

To solve this, use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

However, $b^2 = 43^2 = 1849$ is less than $4ac = 4 \cdot 6 \cdot 84 = 2016$. We end up with

$$x = \frac{43 \pm \sqrt{-167}}{12}$$

The classic answer at this point is just to say, these values do not exist. The graph would be a parabola opening up ($a > 0$) whose vertex lies above the $x$-axis.

But suppose these two *complex* roots do have meaning.

They are, of course, complex conjugates: $p + iq$ and $p - iq$. If they are substituted into the factored form of the quadratic:

$$y = \; [\; x - (p + iq) \;] \; [\; x - (p - iq) \;]$$

$$y = \ [\ x - p - iq)\ ]\ [\ x - p + iq)\ ]$$
$$= x^2 - px + iqx - px + p^2 - ipq - iqx + ipq + q^2$$
$$= x^2 - 2px + p^2 + q^2$$
$$y = (x-p)^2 + q^2$$

$p$ is the value of $x$ at the vertex, corresponding to the minimum value of $y$, which is equal at that point to $q^2$.

Recall that the slope is

$$y' = 2ax + b$$

at the minimum, it equals zero so

$$0 = 2ax + b$$

$$x = -\frac{b}{2a}$$

This value of $x$ makes the factored form equal to zero. It is also the first term in

$$-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

**cubics**

In general, people just ignored problems with negative square roots — sometimes explicitly — until Cardano came to cubic polynomials.

Briefly, he discovered that any cubic like

$$x^3 + ax^2 + bx + c = 0$$

can be converted to a *depressed* cubic of the form

$$x^3 + px + q = 0$$

Any cubic has either one real root and two complex ones, or else three real roots.

We need to look at Cardano's formula to solve the depressed cubic. He was actually solving a problem like

$$x^3 + mx = n$$

(with $m$ and $n$ both positive), we re-write this as

$$x^3 + mx - n = 0$$

Define

$$r = \frac{n}{2}, \qquad s = \frac{m^3}{27}$$

Then Cardano showed that a real root (the only one or one of the three) is

$$x = \ [\ r + \sqrt{r^2 + s}\ ]^{1/3} + \ [\ r - \sqrt{r^2 + s}\ ]^{1/3}$$

Clearly, depending on the values of $m$ and $n$, and thus $r$ and $s$, $r^2 + s$ may be a negative number.

Cardano could not just ignore this issue, because the formula works to give a real result. He struggled with this.

Even today, it is hard to see the resolution because of the cube root.

Write what is in the brackets as a generalized complex number, in polar format

$$x = \ [\ re^{i\theta}\ ]^{1/3} + \ [\ re^{-i\theta}\ ]^{1/3}$$

Recall that complex multiplication goes like so:

$$r_1 e^{i\theta}\ r_2 e^{i\phi} = r_1 r_2 e^{i(\theta+\phi)}$$

So a cubic is

$$re^{i\theta} \cdot re^{i\theta} \cdot re^{i\theta} = r^3 e^{i3\theta}$$

With a change of variable, complex eponentiation is as follows:

$$[\ re^{i\theta}\ ]^{1/3} = r^{1/3}\ e^{i\theta/3}$$

$$[\ re^{-i\theta}\ ]^{1/3} = r^{1/3}\ e^{-i\theta/3}$$

The cube roots of complex conjugates are also complex conjugates!

When added together, the imaginary parts cancel, leaving an entirely real result.

$$x = r^{1/3}\ (e^{i\theta/3} + e^{-i\theta/3})$$

The term in brackets is clearly a sum $z + z*$, which is real, with the value twice the real component of the complex number.

**example**

Let's figure out an example arithmetically. The math is a little messy but we'll try to get through it. One of the problems studied by Cardano is

$$x^3 = 15x + 4$$

All terms are positive, which is typical for the time. We try the solution $x = 4$ and find it works out.

The Tartaglia formula gives

$$r = 4/2 = 2$$

$$s = (-15)^3/27 = -125$$

so we have that

$$x = \ [\ r + \sqrt{r^2 + s}\ ]^{1/3} + \ [\ r - \sqrt{r^2 + s}\ ]^{1/3}$$

$$x = \ [\ 2 + \sqrt{-121}\ ]^{1/3} + \ [\ 2 - \sqrt{-121}\ ]^{1/3}$$

This is easy to solve if one happens to know that

$$(2 \pm \sqrt{-1})^3 = 2 \pm \sqrt{-121}$$

Hence

$$x = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4$$

Let's try to calculate this:

$$(2 + \sqrt{-1})^3 = 2 + \sqrt{-121}$$

Usually, we would think that the polar format would make for easier calculation. However, let's go forward using the Cartesian format

$$\begin{aligned}(2+i)^3 &= (3+4i)(2+i) \\ &= 6 - 4 + 11i \\ &= 2 + 11i\end{aligned}$$

Pretty easy.

To use the polar format, let's compute the cube root:

$$(2 + 11i)^{1/3} = ?$$

We need the polar form of $2 + 11i$. We obtain

$$r = \sqrt{2^2 + 11^2} = \sqrt{125}$$

$$\theta = \tan^{-1} 11/2 = 1.391$$

Then

$$r' = r^{1/3} = \sqrt{5}$$

$$\theta' = \theta/3 = 0.46346$$

To convert back to Cartesian coordinates:

$$x = r \cdot \cos\theta = \sqrt{5} \cdot 0.8944 = 2.0$$

$$y = r \cdot \sin\theta = \sqrt{5} \cdot 0.4472 = 1.0$$

The result is $2 + i$, as expected.

**example**

Here is another problem from Nahin showing that the real component of a complex solution may have application in the real world.

> Imagine that a man is running at his top speed of $v$ feet per second, to catch a bus that is stopped at a traffic light. When he is still a distance of $d$ feet from the bus, the light changes and the bus starts to move away from the running man with a constant acceleration of $a$ feet per second per second. When will the man catch the bus?

Let the origin of coordinates be the traffic light and $x_m$ and $x_b$ be the positions of the man and the bus. At $t = 0$, $x_b = 0$ and $x_m = -d$. For an arbitrary time $t$

$$x_b = \frac{1}{2}at^2$$
$$x_m = -d + vt$$

If the man is to catch the bus at $t = T$, the positions are the same

$$x_m(T) = x_b(T)$$

$$-d + vT = \frac{1}{2}aT^2$$

This is a quadratic

$$\frac{1}{2}aT^2 - vT + d = 0$$

In general, the solution for $T$ may be complex, if

$$v^2 - 2ad < 0$$

Rearranging

$$d > v^2/2a$$

For such values there is no catching the bus.

Nahin rearranges the equation to give

$$T^2 - 2\frac{v}{a}T + 2\frac{d}{a} = 0$$

The quadratic formula gives

$$T = \frac{2v/a \pm \sqrt{4v^2/a^2 - 8d/a}}{2}$$
$$= \frac{v}{a} \pm \sqrt{v^2/a^2 - 2d/a}$$

Even for a complex result, the real part is

$$T = \frac{v}{a}$$

But notice: the separation between the man and the bus is

$$s = x_b - x_m$$
$$= \frac{1}{2}at^2 + d - vt$$

At what time is the man closest to the bus? That occurs when

$$\frac{ds}{dt} = at - v = 0$$
$$t = \frac{v}{a}$$

This is the real part of the result above.

If the man does catch the bus ($\sqrt{v^2/a^2 - 2d/a}$ is real), it worth thinking about the two solutions to the quadratic. Which is the correct one and what is the meaning of the second?