Plane Geometry: Fundamentals

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Part I

Lines and angles

Chapter 1

Introduction

The image below is a detail from a painting by Raphael entitled "School of Athens", which was used as the front cover of a wonderful book annotating the Heath translation of Euclid's *Elements*.

It took a genius to figure it out the first time, but it is within anyone's grasp to appreciate what they found. I imagine myself looking over Archimedes' shoulder as he explains the steps of a proof to me.

Most scientists I've met loved geometry in school, as I did. They like how visual it is, and they like clever simple proofs. Geometry should be fun!

A central feature of this book is the relentless use of proof. I emphasize the key insight for each, and have tried to make the proofs simple and as easy to follow as possible. You will notice that we frequently provide multiple proofs (using different methods) for the same theorem.

I express my sincere thanks to the authors of my favorite books, which are listed in the references and mentioned at various places in the text. Everything in here was appropriated from them in one way or another, and styled to my taste.

I offer my profound thanks also to Eugene Colosimo, S.J. He was among the best of a great group of teachers.

This book is the pdf in that repository linked below. Most of the rest is the source files. There are several other books there as well, if you go up one level in Github.

https://github.com/telliott99/geometry

Note: the sites referred to by some urls in the text have disappeared from the web and the count of missing pages grows over time. I have left those URLs in, as a kind of protest, and because it is impractical to police them, but also because they may still be useful in connection with the wayback machine.

https://web.archive.org

Chapter 2

Angles

On motivation

Modern textbooks make a considerable effort to motivate the student, setting the stage for a problem and attempting to convince her or him that it's worth trying to understand what's being talked about. I usually won't do that.

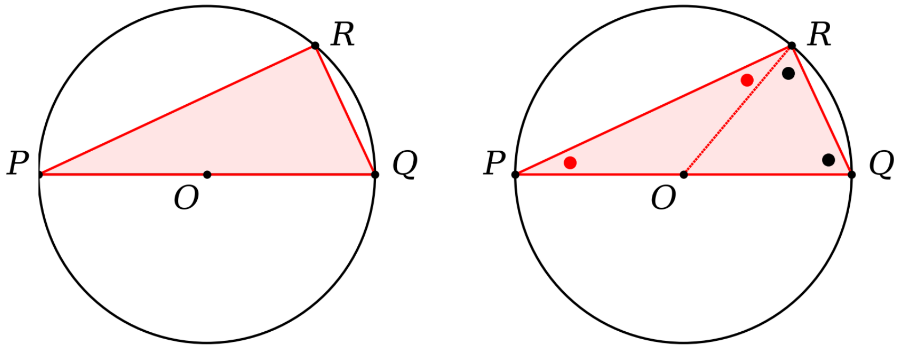
I have tried to achieve simplicity and clarity in the presentation. The subject itself gives us beauty. To see that beauty clearly is my motivation and I hope, yours as well.

In the next chapter, we will prove a beautiful *theorem*: the sum of the angles in a triangle is equal to the angle turned by going halfway around in a circle (like turning from north to south). That's a really remarkable and elegant result, and the proof is simple.

Here is a second beautiful theorem, about circles.

- Any angle inscribed in a semicircle is a right angle.

Think of three points on the circumference of a circle, forming a triangle. If two of the points are on a diameter of the circle (the line joining them passes through the center), then the angle formed at an arbitrary but distinct third point is always a right angle.



In this figure, if PQ is a diameter, then it must be that $\angle PRQ$ is a right angle.

Proof.

Draw the radius OR (right panel).

The two smaller triangles produced ($\triangle OPR$ and $\triangle OQR$) are both isosceles (two sides equal), since two of their sides are radii of the circle. Therefore their base angles are equal (we will see why later).

We can use colored dots to identify angles that are equal. We have for the whole triangle two red dots and two black ones, and for the angle at R one of each.

Hence the total angle at R is one-half the total measure of the triangle, namely, one-quarter of a circle. Like turning from north only as far as west.

$\square$

That's a beautiful, simple result. It depends on an idea about isosceles triangles, which should motivate us to find out more about them.

The $\square$ symbol marks the end of the proof.

Notation

In the above proof we've mixed together different kinds of notation. The Greeks always named the points at the ends of line segments with letters of the alphabet, using those same points to describe triangles or figures with even more sides. So we can talk about PR as a line segment or PQR as an angle ($\angle PQR$) or a triangle ($\triangle PQR$).

However, often it is simpler to just put a label on the angle at the vertex of a triangle, such as P or R, simply s or t, or more stylishly, use Greek letters such as θ and ϕ; or α, β, γ, δ and so on.

With a third kind of notation we may not even give the angles a lexicographical label, but just mark ones with equal measure by using a colored circle (or open and filled circles).

Use whichever one feels more natural. The simpler the labels, the easier it is to think about what matters in the problem. The right notation frees the mind to concentrate on what's important.

Euclid and the Elements

Some topics in geometry are as old as civilization and very practical: finding the area of a rectangular field or the volume of a cylindrical grain storage building, so the ruler can calculate and collect his taxes.

The Greeks were really the first to treat the subject as an intellectual pursuit, the first to view it systematically and to prove theorems.

Euclid is probably the most famous of Greek geometers because of his book *Elements*. His book is believed to have been a compilation of the accomplishments of more than a dozen other mathematicians. See Chapter 2 of Heath:

https://www.gutenberg.org/files/35550/35550-h/35550-h.htm

However, Greek geometry starts several hundred years before the time of Euclid, who was (roughly speaking) a contemporary of Alexander the Great (356-323 BC).

Some of the earliest mathematicians were Thales, Eudoxus, and Pythagoras. One of the problems in understanding what happened is that, unfortunately, almost no books survive from this time. We know that extensive histories were written around the time of Euclid, but these are all lost.

As well, about the person Euclid, we also know actually very little. He lived after Plato (died 347 BC), and before Archimedes (born c. 287 BC). He worked in Alexandria, the city founded by Alexander near Cairo in Egypt, and except for that, all other details of his life and death are shrouded in mystery.

After more than 2000 years, *Elements*, especially the first half dozen books, is still an excellent place to begin surveying the foundations of geometry. It is a sophisticated textbook, an organized collection of everything that a well-educated student was expected to know about the subject at the time. Nothing was left out, until others, such as Archimedes, went further

We note in passing and with sadness that so much of what the Greeks wrote and thought has been lost through the vicissitudes of history but also by the deliberate destruction of libraries. At Alexandria this occurred during war in the time of Julius Caesar, later during riots by Christian mobs outraged by "pagan" tracts, and then later again by Muslim invaders with a similar view.

Elements is known only through later works that reproduced the theorems and proofs with additional commentary. The original source is lost.

The *Elements* consists of more than five hundred *propositions*, some of which are constructions (geometric figures) drawn with a pencil on a piece of paper, using a straight-edge or a compass or both. The others are logical proofs of the type we will study in detail.

For constructions, there are restrictions on both compass and straight-edge. The straight-edge is unmarked; it cannot be used to measure distances. Sometimes people will say "compass and ruler", but the ruler involved must not be *ruled*, it must not have divisions (or at least if it does, they are not consulted).

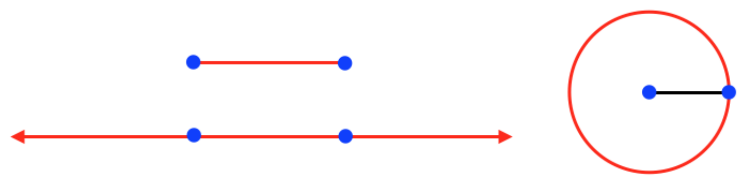
Nearly all proofs of propositions build on previous items in the book.

Euclid does not prove everything. Statements which are assumed to be true are divided into axioms, or common notions, and postulates. Axioms are generally useful, while postulates are specific to the subject of geometry.

Here are some of Euclid's postulates:

◦ A straight line segment can be drawn joining any two points.

◦ A line segment can be extended continuously in a straight line.



○ Given any straight line segment, a circle can be drawn having the segment as the radius and one endpoint as the center.

Let us assume these as well. We will use them often.

We finesse the difficulty in defining what is meant by *straight* in the real world. If you've ever done any fancy carpentry, you will realize that an unknown edge is checked by comparison with another edge which is known to be straight. We use the known edge to "true" the other.

In geometry, we use an imaginary perfect straight-edge to draw an ideal straight line.

There is another statement commonly claimed, that any straight line segment can be extended indefinitely in a straight line.

According to Morris Kline, this statement does not exist in the *Elements*, and in fact, except when dealing with the question of parallel lines, we never worry about lines to infinity. Instead, we have line segments with defined endpoints, but usually refer to these simply as lines, as Euclid did.

People may also talk about a straight line as "the shortest distance between two points". The closest you will find to that is the **triangle inequality**, but we postpone that discussion for now.

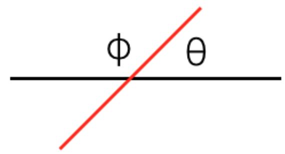
Supplementary angles

We begin with a discussion of angles. Actually, we might start with the plane, the two-dimensional world in which most of our work will live. But let's assume that such a thing makes sense, and move on to angles.

In the diagram below, one line (i.e. a line segment) is drawn crossing a second one, forming their *intersection* (the points at the ends have been omitted from the drawing).

Four angles are formed at the intersection of two lines or line segments. Two of the angles are labeled in the figure. We can see that one, ϕ, is obviously larger than the other one, θ.

We call angles formed on the same side of a line, *supplementary* angles.

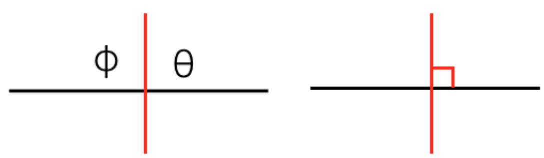


So ϕ and θ are supplementary angles because they both lie above the horizontal black line.

Right angles

A common definition of a *right angle* is as one that contains, or whose measure is, 90 degrees, usually written $90°$. But the Greeks would give a different definition:

- Two supplementary angles that are equal to each other are both right angles. In the figure below, if $\phi = \theta$, then *by definition* they are both right angles.



A right angle is frequently designated by drawing a small square, as seen in the right panel.

If one of the angles at the intersection of two lines is a right angle, then all four are right angles — we'll see why later in the chapter. The square is only drawn for one, but they are all equal.

In a common system of angle measurement, a right angle is indeed $90°$, and there are $360°$ in a full circle. Supplementary angles sum to $180°$.

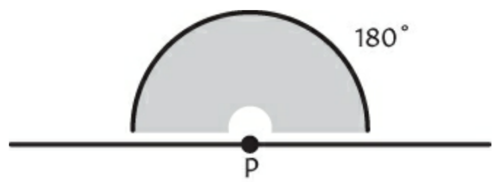
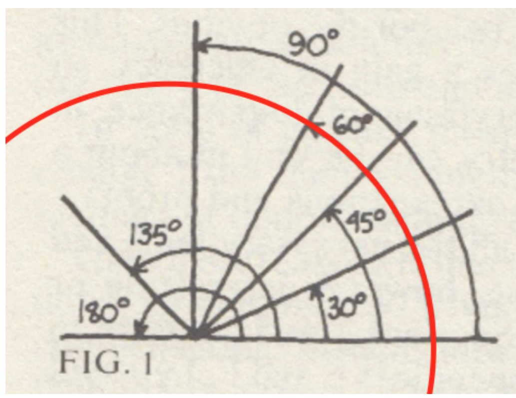


Fig. 4 A straight line.

One way to think about the *measure* of an angle is to draw a unit circle (radius equal to 1) and then ask what is the (curved) distance along the circle from one ray of the angle to the second one. In the figure below, we can see angles with various measures drawn.



Imagine drawing a circle (in red) whose center coincides with the origin of the angles and then measuring the distance along that circle starting with the positive horizontal axis on the right and going counter-clockwise. We can take that distance as the measure of the angle.

Another approach is to bisect (cut in half) a straight line. Then, the original straight line is really an "angle" of 180°, and the bisected half-angles are each 90°. Another bisection gives 45°. We will see a method for doing bisection with compass and straight-edge later on. We can also get 60° using a triangle with all sides equal (equilateral). Bisection then gives 30°.

However, the precise measure of an angle is rarely important, especially before trigonometry. We simply use = 180 as a shorthand for *is equal to two right angles*, and in fact, usually drop the degree notation.

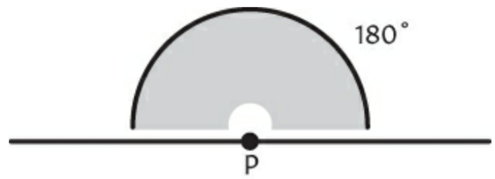


Fig. 4 A straight line.

What we most care about is whether one angle is equal to, larger than, or smaller than another one, or whether an angle or some combination of angles is exactly equal to one right angle or two right angles.

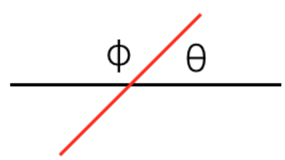
For more about angular measure, there's a short **chapter** at the end of the book.

Because of this, we can state the following theorem:

addition of supplementary angles

- the sum of two angles that are supplementary to each other is equal to the sum of two right angles.

This constant sum is correct regardless whether θ is equal to ϕ, or one is larger than the other. Here the sum $\phi + \theta$ is equal to two right angles.

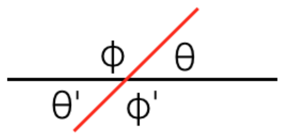


Some textbooks emphasize the arithmetic relationship for supplementary angles: any two angles whose sum is equal to two right angles are supplementary. And then angles that are both adjacent (vertex at the same point) and supplementary are called a *linear pair*.

This distinction seems overly pedantic. We will just use supplementary angles to describe both situations.

Vertical angles

Now, consider the angles lying below the horizontal:



We said that the sum of the two angles $\phi + \theta$ is equal to two right angles, because those two angles (and no others) lie above the black line. But $\theta' + \phi$ and $\theta + \phi'$ are equal to two right angles, for the same reason: they are the angles on one side or the other of the red line.

As a result

$$\phi + \theta = \text{ two right angles } = \theta + \phi'$$

By subtracting θ from both sides, we conclude that

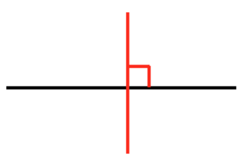
$$\phi = \phi'$$

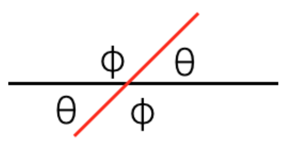
A similar argument will show that

$$\theta = \theta'$$

This is the *vertical angle theorem.*

- Vertical angles are equal.





The vertical angle theorem is obtained by two successive applications of the supplementary angle theorem. It's powerful because it doesn't matter how large or small the two angles are. The vertical angles, that oppose one another, are always equal.

On the left, one of the angles where two lines cross is a right angle. If one angle at an intersection of two lines is a right angle, all four are right angles, by the supplementary and vertical angle theorems.

problem

To prove.

If two adjacent supplementary angles are bisected, the bisectors are perpendicular to each other, that is, the angle between them is a right angle.

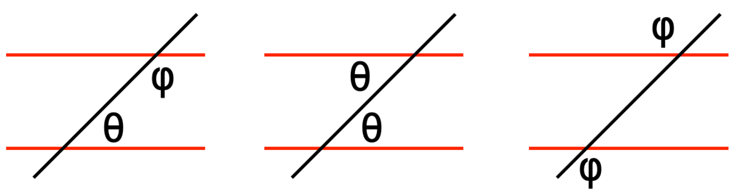
Chapter 3

Parallel lines

Alternate interior angles

The theorems from the previous chapter about supplementary and vertical angles seem rather obvious, once you get used to them. Euclid's fifth and final postulate is more subtle.

> If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



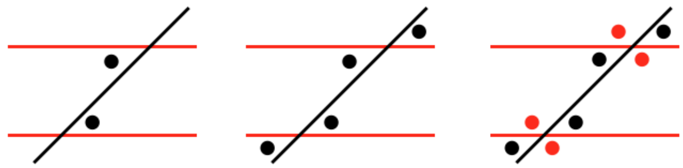
Euclid says that if (in the left panel) $\theta + \phi$ is less than two right angles, the two lines must eventually meet somewhere if extended far enough (to the right).

Of course the fact that they meet means they cannot be parallel lines, since two parallel lines never meet.

He does not worry about the case where the sum of $\theta + \phi$ is greater than two right angles. If the sum is less on the right-hand side of the transversal, it will be greater on the left-hand side. *Problem.* Prove this.

The other two panels also show parallel lines. The known equal angles have different names in these cases, but we will not worry about that. We refer to all of them as examples of lines parallel by equal *alternate interior angles*, even though technically, that only refers to the middle panel.

Here is a graphic that makes it easy to remember:



There is a large philosophical literature on this issue, and there are other ways to say what amounts to the same thing. For example Playfair's axiom, which says that, through any point not on a given line, there is only one line that can be drawn parallel to the first one.

https://proofwiki.org/wiki/Axiom:Euclid%27s_Fifth_Postulate

if and only if

• Two lines are parallel, *if and only if* a third crossing line makes the adjacent interior angles sum to two right angles.

We drew a third line crossing (or making a transversal of) two other lines. Given that background, we then said that if (P) those two lines never cross (they are *parallel*), then (Q) the sum of the adjacent interior angles is equal to two right angles, which means that alternate interior angles are equal.

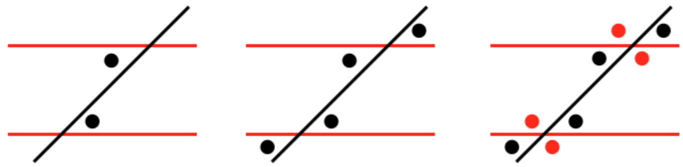
In symbols, we write $P \Rightarrow Q$, by which we mean *if* P is true, *then* so is Q.

However, you may have noticed that what we actually wrote was P *if and only if* Q. In symbols, this is $P \iff Q$.

The meaning of *if and only if* is simply that both $P \Rightarrow Q$ *and* $Q \Rightarrow P$.

If the sum of the adjacent interior angles is equal to two right angles, then the two lines being traversed are parallel.

extending the result



In the figure above (left panel), we're given that the two horizontal lines are parallel.

The indicated angles are equal because they are alternate interior angles of two parallel lines (parallel postulate).

In the middle panel, two additional equalities are established by the vertical angle theorem. Then on the right, we use the supplementary angle theorem.

Note that the conclusions for the angles marked with a red dot are themselves consistent with the three postulates/theorems that we have so far: supplementary and vertical angles and alternate interior angles.

This postulate is also valid in reverse. If we have a line that traverses two others so as to give interior angles summing to 180°, then the two lines must be parallel.

symmetry

This is first of all a statement about the world, that we can have two lines that run alongside each other but don't touch. Another way to talk about the situation is to say that if we sight down a pair of parallel lines from left to right, like a railroad track, then turn 180° and look again, the picture will be identical.

By rotational symmetry then, we have no reason to distinguish two parallel lines in the forward and reverse directions. It follows that the intersection between two parallel lines and a third line that crosses them should look the same in both directions, the angles involved should have the same measures. That is our theorem.

And this reasoning works, just so long as we are talking about lines on a flat piece of paper.

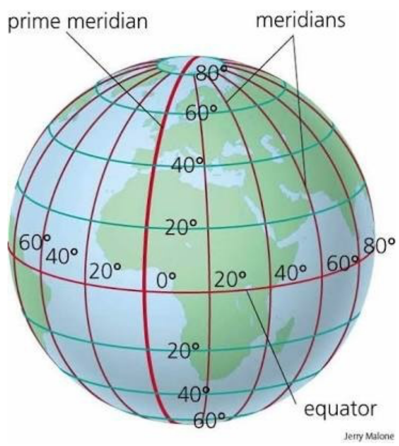
flat geometry

Without getting too deep in the philosophical weeds, it's clear that mathematics is partly a construction. We must choose rules that work, or at least do not conflict with each other, and adoption of the parallel postulate is a choice.

This choice of definition works for geometry in the flat plane, but not on a curved surface like the earth. That's a familiar situation where our postulate is not appropriate.

Lines of longitude go vertically on the globe. They are great circles, they all have the same length and go through the poles. Lines of latitude, on the other hand, form circles of different lengths and get shorter as you near the poles.

Two adjacent lines of longitude can be drawn so as to cross the equator at right angles, and the lines are parallel there, but they will meet (intersect) at the poles.



The same thing happens if you imagine the earth at the center of the universe, looking out at the stars. Or place yourself inside a globe, looking at the thin skin of the object and thinking about the lines of latitude and longitude as seen from the inside.

The parallel postulate only holds for geometry on a *flat* surface.

Axioms

Euclid also lists five axioms, things which are assumed. Here are two examples:

◦ Things that are equal to the same thing are also equal to one another.

◦ If equals are added to equals, then the wholes are equal.

These seem quite reasonable.

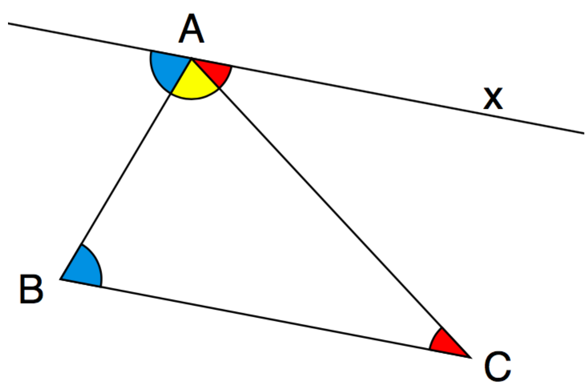
We will see how to proceed from the postulates and axioms to various proofs. Given these *assumptions*, we can prove theorems that must be true.

William Dunham has written a lot about the history of mathematics in Greece, starting with Thales (624-546 BC), who was from a Greek town called Miletus on the coast of Asia Minor (modern Turkey). Thales lived long before Euclid (ca. 600 BC, about 300 years before Euclid). Although none of his writing survives, it is believed that Thales proved several early theorems including the ones we saw above.

Triangle sum theorem

We come to our first truly novel theorem. It relies on everything we've said so far. It is attributed to Thales, one of three elementary but novel theorems for which he is thought to have developed proofs.

- The sum of the three angles in any triangle is equal to two right angles.



This theorem depends on the ideas we developed above.

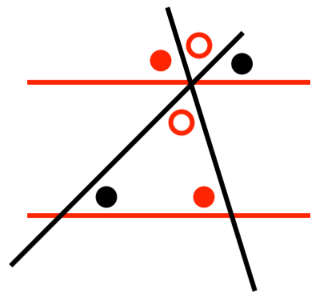
Proof.

Draw a line segment through A parallel to BC. Now, use alternate interior angles and follow the colors to the result. By the theorem, the two angles marked in blue are equal, as are the two angles marked in red. But the three angles at the point marked A add up to two right angles.

So the total measure of three angles in a triangle is equal to two right angles.

□

A simple variation on this proof uses the angles above the line.

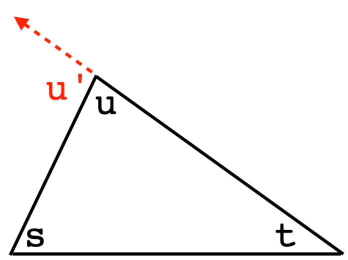


There are two more famous theorems ascribed to Thales, one in the next chapter, and one about circles that we mentioned in the beginning.

Heath says that rather than Thales, Pythagoras is responsible for the triangle sum proof. Since it's not clear that Pythagoras knew any geometry, I'm happy to stick with Thales. No one knows, unfortunately.

problem

Many geometry books will have you do some arithmetic at this point. For example, in the triangle below, suppose that $\angle s = 70°$ and $\angle t = 30°$. What is the measure of $\angle u$? What is the measure of $\angle u'$?



This is simply addition and subtraction, once you understand about supplementary angles and the sum of angles theorem. Go ahead and do that if it amuses you.

We turn to the **external angle theorem**. $\angle u'$ is the *external angle* of this triangle. It is related to $\angle s$ and $\angle t$.

Write two equations giving the relationships between the angles:

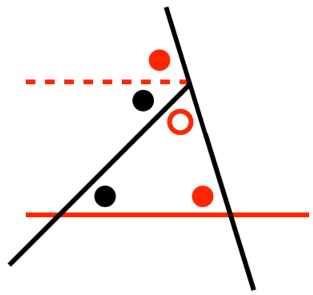
$$u + u' = 180$$

$$s + t + u = 180$$

We used supplementary angles and the sum of angles.

It follows that $u' = s + t$.

This figure makes a proof without words:



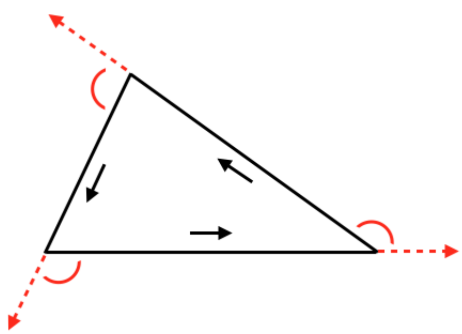
Another proof

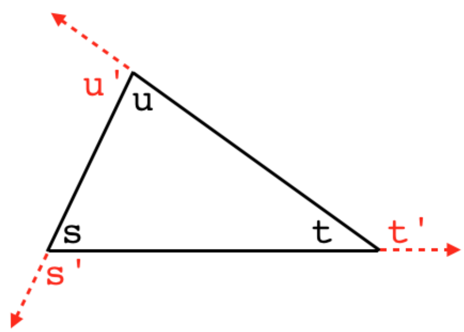
Here is a different proof of both of these theorems relating to the angles in a triangle. This one is from Lara Alcock (*Mathematics Rebooted*).

It never hurts to re-prove important results by a different method. This serves as a check on both the result and the methods.

Imagine walking around the perimeter of a triangle in the counter-clockwise direction. At each vertex we turn left by a certain angle, called the exterior (external) angle. After passing through all three vertices, we will end up facing in the same direction as we started.

We have made one complete turn, the sum of the exterior angles is 4 right angles.





$$s' + t' + u' = 4 \text{ right angles}$$

In addition, for each vertex, the interior angle plus the exterior angle add up to 2 right angles. If we add up all three pairs, we obtain 6 right angles.

$$(s + s') + (t + t') + (u + u') = 6 \text{ right angles}$$

Subtract the first equation from the second

$$s + t + u = 2 \text{ right angles}$$

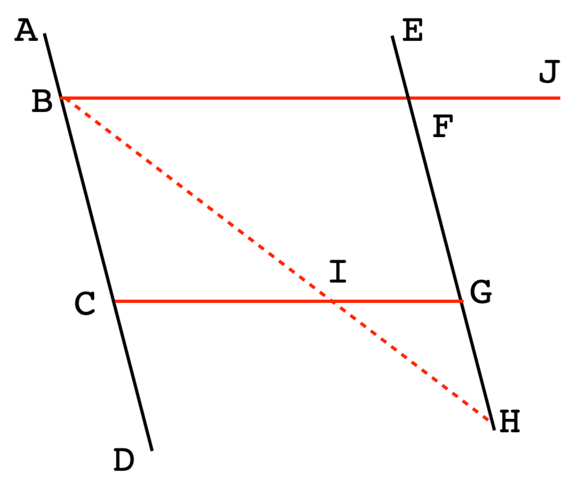
□

Note: both terms, exterior angle and external angle, are used to describe the same angle by different writers. I'm not aware of a reason to prefer one or the other.

problem

In the diagram below, $ABCD$ lie on one line, and it is parallel to $EFGH$. We can write this as $AD \parallel EH$.

Tell which angles are equal and why.



Which are supplementary, which are vertical? (Refer to the angles in any way you wish, using the points, or with new labels).

summary

Make sure you learn and understand each of these theorems.

- the sum of two supplementary angles is equal to two right angles (**ref**).

◦ by definition, if two angles are supplementary and also equal, they are both right angles

- vertical angles are equal (**ref**)
- alternate interior angles of parallel lines are equal (**ref**)
- the sum of angles in a triangle is equal to two right angles (**ref**).

Eratosthenes

It is often supposed that the ancient world believed the earth to be flat, but this is just wrong. People with any level of sophistication not only knew the earth is roughly spherical but also knew its size within a few percent of the true value.

One likely basis is the false story that Columbus had trouble getting financing for his proposed trip to China because everyone thought he would fall off the edge of the earth. This was a tall tale invented by Washington Irving, who also made up several remarkable fables about George Washington.

The real reason the Italian and Portuguese bankers from whom Columbus sought financing thought he would fail is that they had a pretty good idea of the size of the spherical earth and thus of the distance to China, while the over-optimistic Columbus believed it was about half the true value. The prospective financiers knew that he was not able to carry the supplies necessary for a trip of this length.

Morris Kline (*Mathematics and the Physical World*) says that the error is due to geographers after Eratosthenes, who reduced the estimated circumference from 24,000 to 17,000 miles.

Views of the Greek philosophers on the earth and its sphericity are detailed here

`https://www.iep.utm.edu/thales/#SH8d`

Here is a partial quotation:

> There are several good reasons to accept that Thales envisaged the earth as spherical. Aristotle used these arguments to support his own view [...] . First is the fact that during a solar eclipse, the shadow caused by the

interposition of the earth between the sun and the moon is always convex; therefore the earth must be spherical. In other words, if the earth were a flat disk, the shadow cast during an eclipse would be elliptical. Second, Thales, who is acknowledged as an observer of the heavens, would have observed that stars which are visible in a certain locality may not be visible further to the north or south, a phenomen[on] which could be explained within the understanding of a spherical earth.

https://en.wikipedia.org/wiki/Eratosthenes

Eratosthenes (ca. 276 - 195 BCE) measured the circumference of the earth from this observation: at high noon on the solstice on June 21st there was no shadow seen at Syene, allegedly from a stick placed vertically in the ground. Some people say a deep well had the bottom illuminated at midday. Acheson says Eratosthenes was born at Syene, so he could well have personal knowledge of the fact!

Syene is presently known as Aswan. It is on the Nile about 150 miles upstream of Luxor, which includes the famous site called the Valley of the Kings, where many Egyptian Pharaohs were entombed. At 24.1 degrees north latitude, Aswan or Syene has the sun almost directly overhead on June 21. (The ”Tropic of Cancer” is at 23 degrees, 26 minutes north).

Alexandria was a famous center of learning of the ancient world, and Eratosthenes was hired by the pharaoh Ptolemy III to be the librarian there in 245 BCE. Alexandria lies on the Mediterranean some 500 miles north of Syene, and anyone there who was looking could observe that at high noon on June 21st there *was a shadow*. This shadow Eratosthenes measured to be some 7 degrees and a bit (7 degrees and 10 minutes).

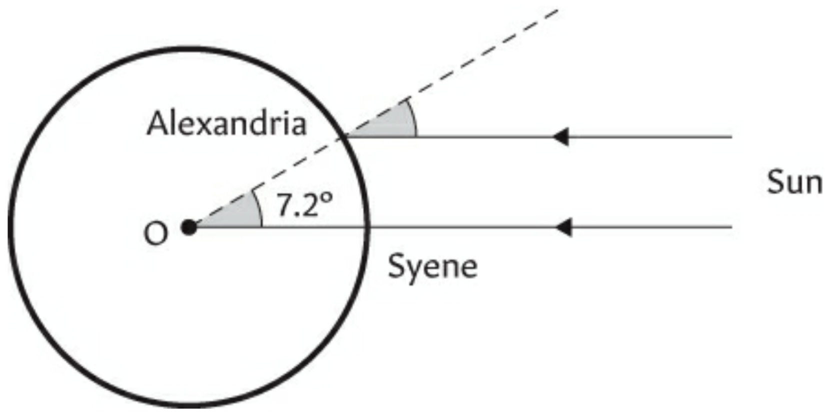


Fig. 21 Measuring the Earth.

A full 360 degrees divided by 7 degrees and a bit is approximately 50. So we can calculate on this basis that the circumference of the earth is about $50 \times 500 = 25000$ miles. That's pretty close to the correct value.

For this calculation, we assume that the sun's rays are effectively parallel (not a bad assumption given a distance of 93 million miles). Then we just use an application of the alternate-interior-angles theorem.

It is curious how the distance from Alexandria to Syene was calculated.

Kline:

> Camel trains, which usually traveled 100 stadia a day, took 50 days to reach Syene. Hence the distance was 5000 stadia...It is believed that a stadium was 157 meters.

We obtain

$$157 \times 5000 \times 50 = 39,250 \text{ km}$$

That's a much better estimate than a method that relies on camels really deserves.

Part II

Congruence

Chapter 4

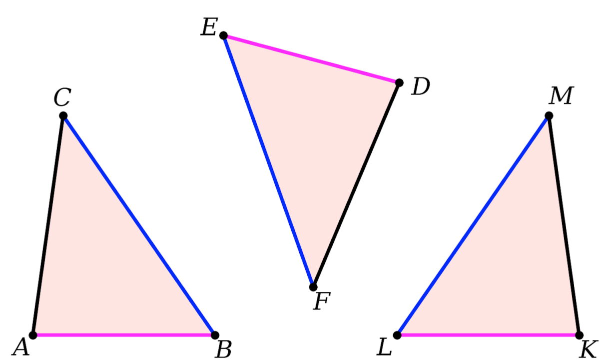
Congruent triangles

meaning of congruence

Probably the most fundamental idea concerning triangles is how to decide that two triangles are *congruent*. If so, they have all 3 sides the same length, and all 3 angles the same measure.

However this is logically the converse (backwards) of what we really want to know. We would like to be able to say that *if* two triangles share certain properties, *then* the two triangles are congruent

We will find tests that, if passed, imply congruence.



We allow one triangle to be rotated at any angle with respect to the other. $\triangle ABC$ and $\triangle DEF$ are an example of this. Equal sides have been colored to show which ones correspond.

We will also use the term congruent to apply to the case of a triangle and its mirror image. All three of the triangles shown above are congruent, even though the one on the right is flipped with respect to the others — they are mirror images.

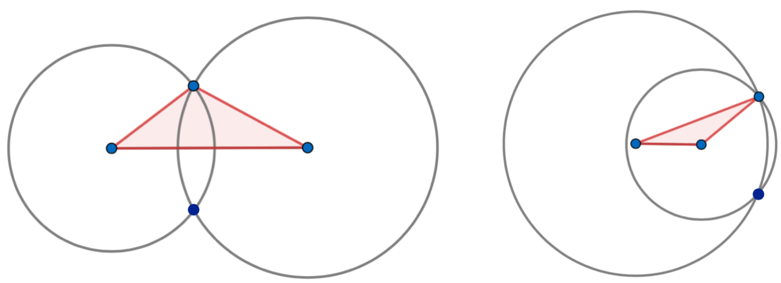
Some authors insist on distinguishing these cases but we will not do so.

SSS test for congruence

Perhaps a practical definition would be that if you used a pair of scissors to cut out one triangle and then lay it on top of the second one so that it superimposes exactly, they would be congruent.

In fact this is how Euclid handles the issue in the first theorem of *Elements*, I.4. (It's the first theorem because the first three *propositions* are actually constructions).

A little thought may convince you that if all three corresponding side lengths are equal, two triangles are congruent. Draw one side of a triangle, and from its endpoints draw circles with radii the length of the other two sides.



The two radii map out all the points that are the same distance from the centers. They include only two possible arrangements for three given side lengths which result in triangles.

We see that the radii cross at two and only two different points, which have mirror image symmetry above and below the original base. Three given side lengths can only be drawn together to give two resulting triangles, and these two shapes are mirror images.

If the base side is one of the shorter ones, then the appropriate diagram is in the right panel. But the result is the same.

It is certainly possible to come up with side lengths that *cannot* form a triangle. Consider (left panel, above) what would happen if the two shorter sides were exactly equal to the longest one. Then they would meet at a single point on the longest side, and there would be no triangle.

If the sum were shorter, they would not meet at all. This is called the triangle inequality: the two shorter sides of a triangle must sum to more than the length of the longest side. We'll prove it more formally later.

Thus, we arrive at a fundamental theorem about congruency for triangles:

• Two triangles are *congruent* if they have the same three equal side lengths.

When we say "equal sides", we are making a comparison *between* two triangles, that corresponding sides are equal.

This test is abbreviated SSS (side-side-side).

SAS

In addition to SSS (side-side-side), there are three other conditions that always lead to congruence of two triangles when they are satisfied, namely

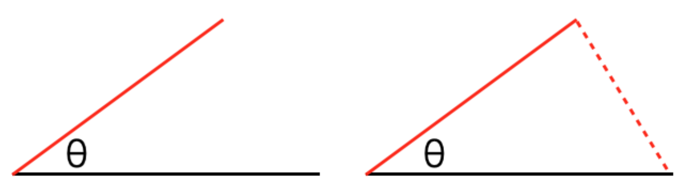
∘ SAS (side-angle-side)

∘ ASA (angle-side-angle)

∘ AAS (angle-angle-side)

When we say SAS, we mean that the angle we know is equal is the one that lies between the pair of equal sides. Similarly, ASA means that we know one side of a triangle and the two angles at either end of that side are the angles in question.

As we did above for SSS, a good way to think about the congruence condition SAS is to imagine trying to construct a triangle from the given information, and ask whether it is uniquely determined.

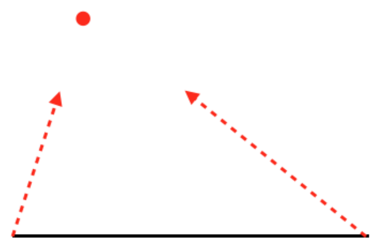


Two sides and the angle between them are given. So draw that part of the triangle. Notice that the second and third vertices are also determined, because they just lie at the ends of the two sides we're given. All that remains is to draw the line segment that joins them.

On a cautionary note, it may happen that a particular diagram incorporates an unspoken *assumption*. We will have to be careful about that and will return to the issue later.

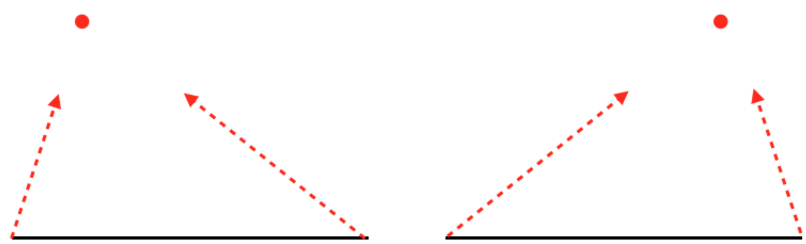
ASA

The next one is ASA. Of course, since we know two angles, we know the third, by sum of angles. However, we do not even need that. Here is a diagram of the situation:



Draw the known side, then using the known angles, start two other sides from the ends of that side. They must cross at a unique point.

But... actually, if we start the two lines from opposite ends of the horizontal



there is another solution, the mirror image. These two triangles are congruent to the one above.

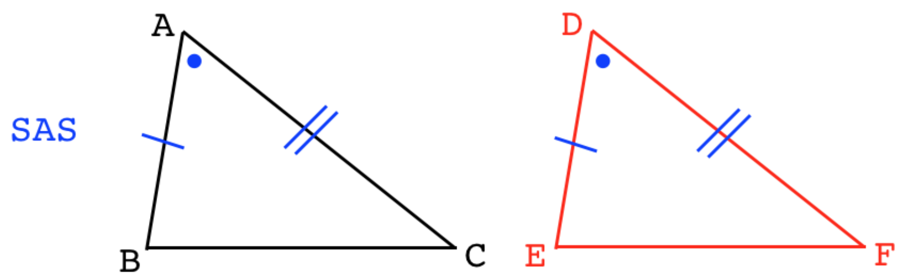
I'm tempted to draw the constructions below the starting line. But this doesn't give anything new. These are merely rotated versions of the ones above. Try it and see. Congruent triangles include a pair of mirror images and that's it.

Now, if we know two angles we also know the third, by the angle sum theorem. For this reason, ASA and AAS mean that we have exactly the same information, because we know all three angles and we know one side.

Crucially, we know *which* two angles flank the known side. Equivalently, it is enough to know which angle is opposite to the known side.

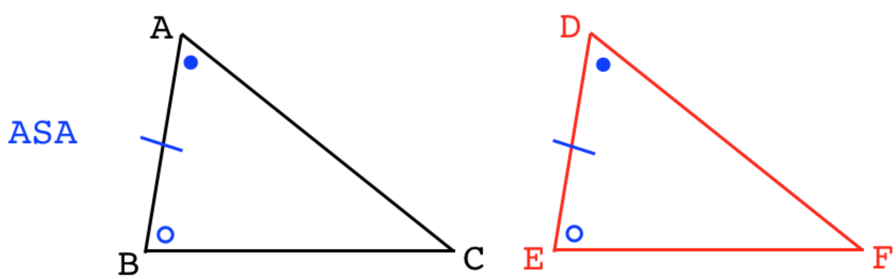
marks for equality

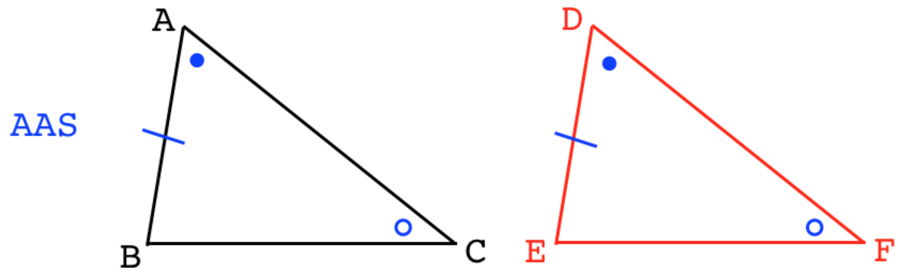
It is often useful to mark sides and (particularly) angles to show they are equal. Here is



In this diagram, sides of equal length are indicated by one or more short lines called hash marks.

Equal angles are usually indicated by dots in this book. Dots are easier to place on the figures, and lend themselves to color-coding; the common method for pencil and paper is to draw arcs with one or more hashes across them.

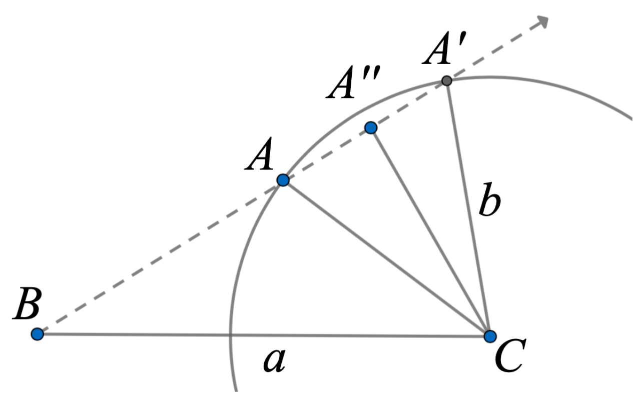




but not SSA

There is one set of three that doesn't work in the general case, and that is SSA (side-side-angle).

Suppose we know the lengths of sides a and b and the angle at vertex B, adjacent to a and opposite side b.



Since the length of the third side isn't known we make it a dashed line. Let's see if we can construct a triangle from this information.

We do not know the angle at vertex C between a and b so we imagine b swinging on a hinge there. If b is too short, there can't be a triangle. If the length of b is exactly right, we'll have a right angle at A''.

If b is longer than this but still $b < a$, there are two points A and A' where b can intersect with the side projecting from vertex B.

This is the *ambiguous case*: we have two possibilities. If two different triangles match by the SSA criterion, we cannot say whether they are congruent or not without more information. It is not that we don't know anything, we just don't know enough to choose one of two possibilities.

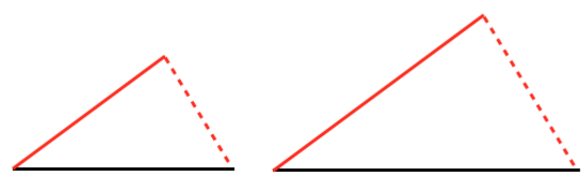
The right angle case is not ambiguous, but we'll save that for the chapter on right triangles.

If you compare this chapter with most others in the book you'll notice that we have not formally proved that any of these methods are correct. Even Euclid encounters some difficulty with this point. He "proves" SAS by a method that is arguably not really a proof.

It won't hurt my feelings if you think of them as axioms. The famous mathematician David Hilbert did this. We will revisit the issue when we look at Euclid's proofs.

similar triangles have equal angles

Sometimes two triangles are not congruent, but have all three angles the same. We call such triangles *similar*. They have the same shape.



Similarity means that all three angles are the same but the triangles are scaled differently, they are of different sizes.

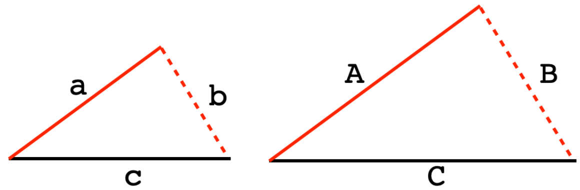
Our basic criterion for similarity is AAA (angle-angle-angle).

However, because of the sum of angles theorem, if any two angles of a pair of triangles are known to be equal, then the third one must be equal as well. We can say that:

- Two triangles are similar if they have at least two angles equal.

scaling

For similar triangles, the three corresponding pairs of sides are in the same proportions, but re-scaled by a constant factor.



From the above diagram of two similar triangles, similarity implies that (for example)

$$\frac{A}{a} = \frac{B}{b}$$

which can be rearranged to give:

$$\frac{a}{b} = \frac{A}{B}$$

For any pair of similar triangles, there is a constant k such that

$$k = \frac{A}{a} = \frac{B}{b} = \frac{C}{c}$$

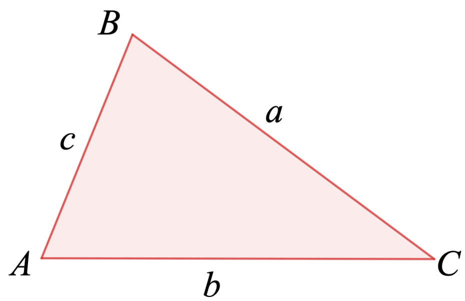
We will come back to proofs about similar triangles in a later chapter. In particular, we will show that AAA implies equal ratios of sides as given here, as well as the converse.

sides and opposing angles

When two triangles are congruent, each of the three angles and all three sides are the same. You might wonder whether the sides could be placed in a different order. We usually draw side of length a opposite $\angle A$, side b opposite $\angle B$ and side c opposite $\angle C$. Can we switch the sides so that say, the length c is opposite $\angle B$?

It turns out not to be possible. Later we will have a theorem which says that in any triangle, the longest side is opposite the largest angle, and the smallest side opposite the smallest angle. If switching two sides were possible, the theorem would be false.

note on notation



Consider $\triangle ABC$. The side opposite the vertex A may be referred to as BC, but CB is also fine. In many cases, especially when doing algebra, I prefer a single letter for the label, usually a. If we refer to the *length* of the side, we will prefer a, although elsewhere one may see $|BC|$ or even just, "the length BC".

There is no compelling reason to prefer $\triangle ABC$ to any of the five other permutations of three letters, such as $\triangle BAC$, beyond a liking for alphabetical order. Old texts may use the letters A, B, and G, because the third letter in Greek is Gamma, Γ.

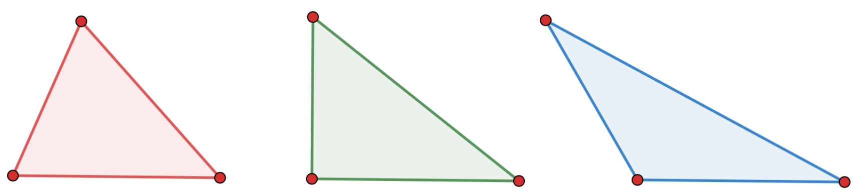
If we have two triangles sharing two vertices, we should usually write those shared letters in the same positions, thus, $\triangle BCA$ and $\triangle DCA$.

When we work with similar triangles and their corresponding angles, we will strive to label the angles in the order of their equality. Thus if $\triangle ABC \sim \triangle QPR$, you should expect that $\angle A = \angle Q$, and so on.

Chapter 5

Isosceles triangles

Triangles are classified by the largest angle they contain: acute, right, or obtuse.



The acute triangle (left) has all three angles smaller than a right angle.

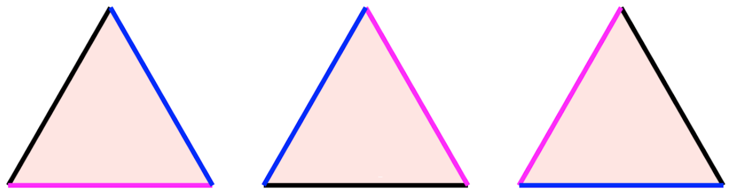
The right triangle, naturally, has one right angle (it cannot have two).

An obtuse triangle has one angle larger than a right angle (right panel, above).

symmetry

One can also talk about the situation where either two sides, or all three sides, have the same length. An *isosceles* triangle has two sides the same length, while an *equilateral* triangle has all three sides the same.

The most important consequence of all three sides equal for an equilateral triangle is three-fold rotational symmetry. Three turns of 120 degrees, and we're back where we started. Each of the two intermediates is identical.

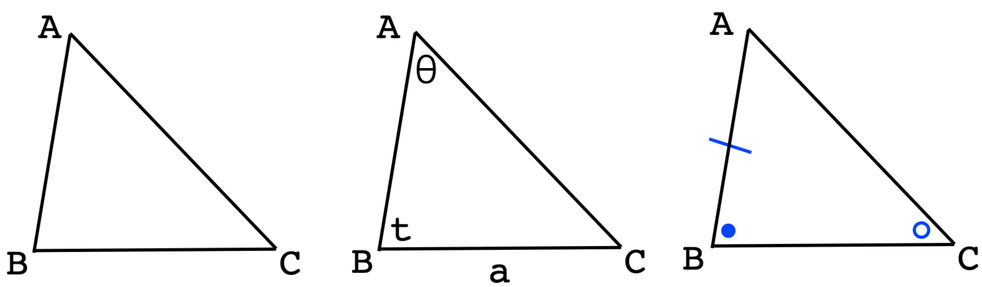


The implication of rotational symmetry is that the three angles are also equal because there is no reason to choose one larger than any other.

Therefore each angle of an equilateral triangle is 2/3 of a right angle, or 60°, by the triangle sum theorem (**ref**).

It is also true that if all three angles are equal, then the triangle is equilateral (three sides equal). We will show how to prove this later.

The Greeks, including Euclid, always label points with letters, and line segments are referred to by the endpoints. Angles and triangles are denoted by the line segments from which they are composed, as in $\angle ABC$, and triangles by their vertices: $\triangle ABC$.



As mentioned previously, we will usually label the side opposite a vertex with a lower case letter: side a lies opposite $\angle A$. We may also use letters like θ and ϕ for angles, or even, more boldly, s and t.

Sometimes, we will dispense with labels altogether and use colored dots for equal angles. The image below is from the web, it uses the convention of an arc drawn across equal angles. The curved arcs are common in books.

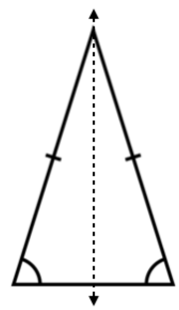
theorem from Thales

- If a triangle is isosceles (two sides equal), then the base angles are also equal.

The converse is

- If two base angles are equal, then the triangle is isosceles.

My favorite proof of both theorems about isosceles triangles is from reflective or mirror image symmetry.



Proof.

Imagine that the triangle sticks straight up from the plane like one of the standing stones at Stonehenge. Imagine walking around the back of it.

Looking from behind, it would appear exactly the same. We would say that the left side as viewed from the front is equal to the right side as also viewed from the front, because if we walk around behind the triangle the *right* side becomes the *left* and vice-versa.

Much later than Euclid, Pappus invokes SAS on the mirror image, rather than thinking about the plane being in 3D space.

The top vertex angle is shared. So the triangle as we look from the front is equal by SAS to the one where we look from the back. It follows that the base angles are equal.

□

Ideas based on triangle bisection

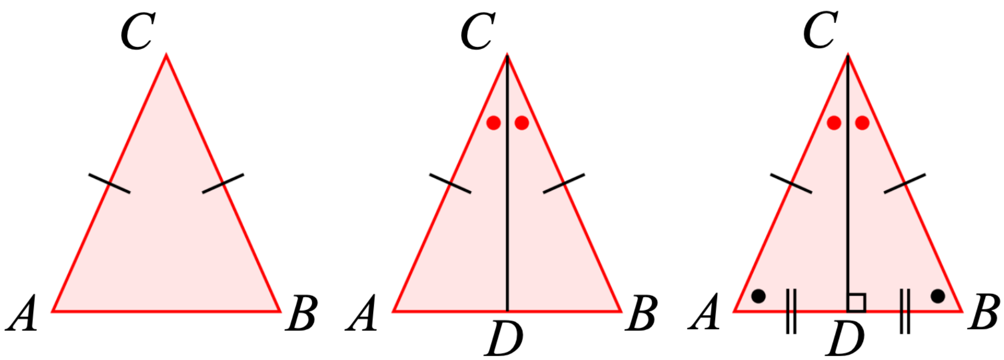
The argument above is not Euclid's proof, namely, I.5, the second theorem in his Book I. We will not show that one right now, but instead try for something simpler.

Again, the *forward* theorem on isosceles triangles is:

- Two sides equal ⇒ opposite angles equal.

We will use several different approaches trying to generate something like the figure given above. We want a triangle divided down the middle, and then to apply one of the congruence tests to the two parts.

Methods to produce two halves include bisecting the angle at the top vertex, as well as bisection of a line segment, cutting the base into two equal parts. Yet another approach is to construct a right angle at the point where the divider reaches the base. Here is the first one:



Forward theorem by angle bisection.

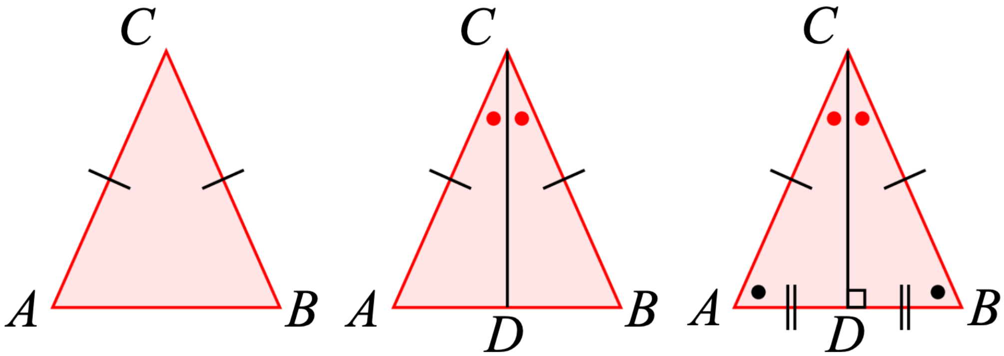
∘ *Given*: $AC = BC$.

∘ Draw the bisector of $\angle C$ (middle panel). This construction forms equal angles at the top, marked with red dots. $\angle ACD = \angle BCD$.

∘ The central line CD is shared, it is equal to itself.

It follows that the two smaller triangles $\triangle ABD$ and $\triangle CBD$ are congruent by SAS.

We write that relationship as $\triangle ACD \cong \triangle BCD$.



Therefore (as corresponding parts of congruent triangles):

○ the base angles $\angle A$ and $\angle B$ are equal.

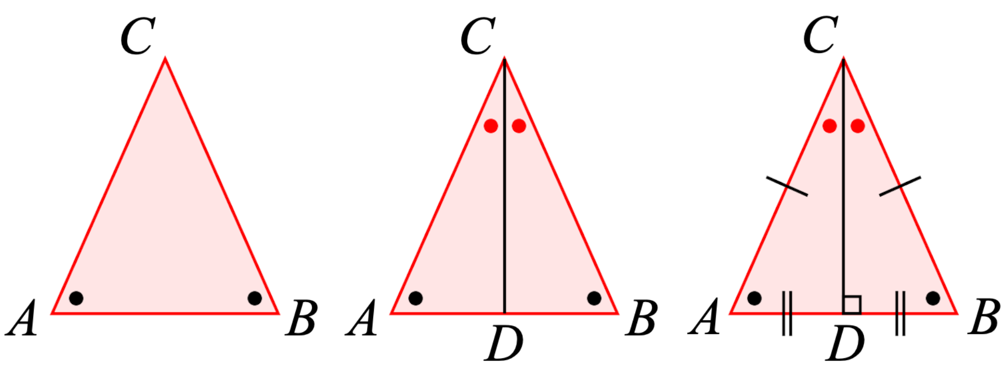
○ the points ADB lie on a straight line, and the two angles $\angle ADC$ and $\angle BDC$ are equal, so they must both right angles.

○ the parts of the base are also equal: $AD = BD$.

converse

We might try to prove the converse theorem by a similar approach:

- Two angles equal $\Rightarrow$ opposite sides equal.



Converse by angle bisection.

○ *Given*: the angles marked with black dots are equal, $\angle A = \angle B$.

○ draw the bisector of angle C.

◦ by sum of angles, we have all three angles the same.

◦ the side CD is shared.

Therefore, $\triangle ABD \cong \triangle ADC$ by AAS and also ASA.

It follows that $AC = BC$. Equal opposing angles means equal sides.

The other properties also follow, such as bisection of the base. Since ADC lie on one straight line (they are *collinear*), there are right angles at the base.

discussion

Euclid's proof of the isosceles triangle theorem is more complicated than what we have given above, and there is a good reason for it.

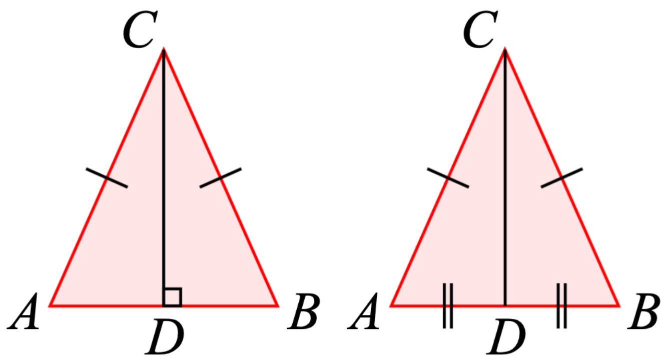
Our demonstration depends on the existence of the angle bisector, but we haven't shown how to sctually bisect an angle. It will turn out that **construction** of the bisector *depends on* the isosceles triangle theorem, indirectly. The angle bisector actually depends on SSS, which in turn depends on the theorem we're trying to prove here.

That's a problem because the reasoning is circular, thus invalid. We cannot use p to prove q if we have previously used q to prove p. That proves nothing. We will fix this problem in the next chapter.

other ideas

Let us try instead to construct a right angle at D, or bisect the base. We will just show diagrams for these methods and sketch the idea.

Forward theorem by right angles or bisecting the base



- *Given*: equal sides.
- In the left panel, draw CD to form a right angle at D.
- Side CD is shared.

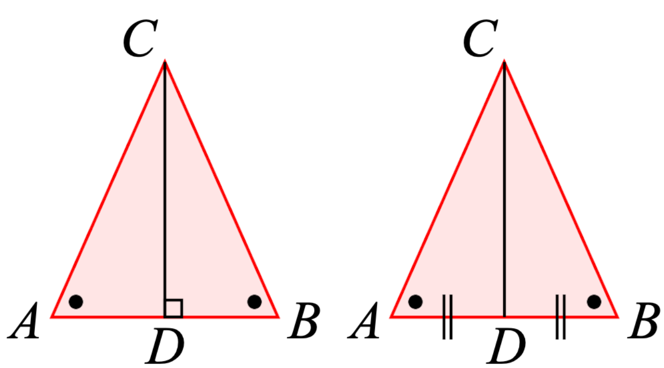
We have SSA, two sides and then the shared right angle. As discussed in the last chapter, SSA is *not enough* for congruence in general. However, it is in the case of a right angle (called HL). This gets a bit complicated so we postpone discussion until later, but these methods also turn out to depend on what we're trying to prove now. We need something else.

Alternatively, in the right panel,

- $AD = BD$, i.e. the base is cut in half.

Since CD is shared we have that $\triangle ACD \cong \triangle BCD$ by SSS. So again, we need something more.

Converse theorem by right angles or bisecting the base



◦ *Given*: equal angles at A and B

◦ construct right angles at D.

◦ we have three angles equal plus a shared side, so congruent triangles by AAS (or after application of the triangle sum theorem, by ASA).

In the right panel,

◦ we have $BA = BC$ and equal angles at A (and B. This is (again) SSA. In general, it is not enough to show congruence.

discussion

There is a problem with all of these constructions, even the ones that seem to work, and this leads me to title them as *Demonstrations* rather than *Proofs*.

We have not show how to perform any of these: angle bisection, bisection of a given line, or construction of a right angle to extend through a given point. This last is called *erecting a perpendicular*.

Angle bisection relies on SSS, but SSS relies on the isosceles triangle theorem! Similarly, bisection of a line and erection of a perpendicular also rely on this theorem. We have to come up with a better argument.

Chapter 6

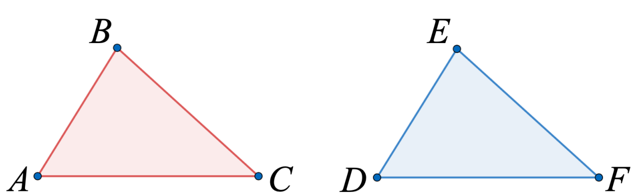
Euclid's proof I.5

In this chapter we will look at some early Propositions from Book I of Euclid's *Elements*.

Elements was put together as a compendium of geometry for students. One thing we will see is how the propositions build on one another to make a dependent chain. This includes a more sophisticated proof of the isosceles triangle theorem that depends only on SAS.

Euclid. I.4

> If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.



This is a method for proving congruence (equality) of two triangles

$$\triangle ABC \cong \triangle DEF$$

In modern usage, we call the method SAS or *side angle side*. Given that $AB = DE$ and $AC = DF$ and that the angles between them at the vertices A and D are also equal, the two triangles are congruent: all three angles and all three sides are equal.

Euclid I.4 is a proof that SAS is correct.

Proof.

The proof is by superposition. The facts establish the positions of the points B and C, which determines BC and so the angles at vertices B and C.

Euclid says that if we lift up $\triangle ABC$ and lay it on top of $\triangle DEF$ then B coincides with E and C coincides with F so $BC = EF$.

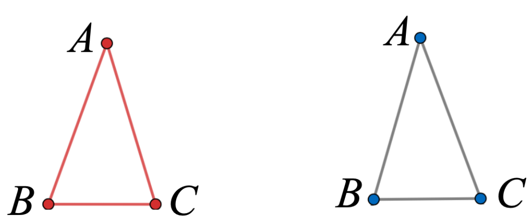
□

This seems perhaps a little shaky logically, and it's not a method of proof that Euclid uses much.

But one might instead have taken this proposition as a postulate. One source says that David Hilbert claims that under the hypotheses of the proposition it is true that the two base angles are equal, and then proves that the sides are equal.

Euclid I.5

The forward isosceles triangle theorem is that if two sides in a triangle are equal, then so are the opposing angles. It's difficult precisely because Euclid proves it *before* we know how to bisect an angle.

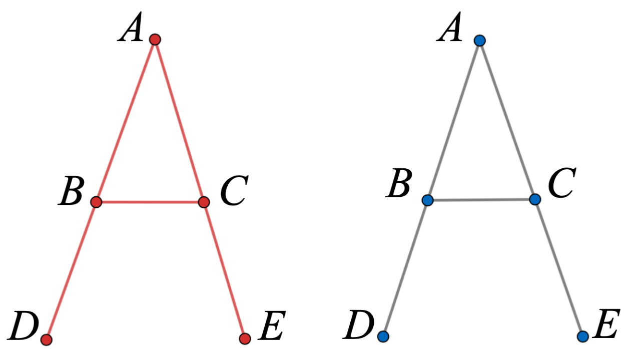


Given $AB = AC$. We will prove that the base angles are equal: $\angle ABC = \angle ACB$.

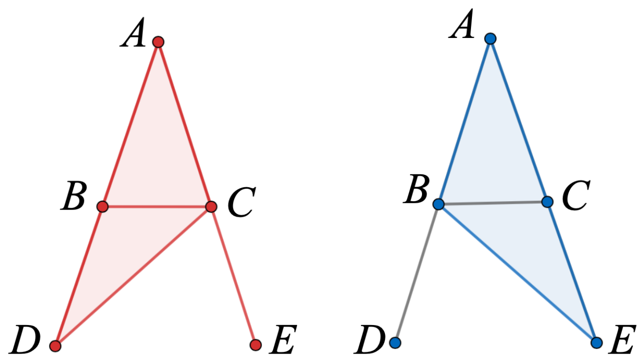
The image shows two copies of the triangle. This is so that we may compare congruent triangles formed within the *same* figure.

Proof.

Extend AB and AC so that $AD = AE$.

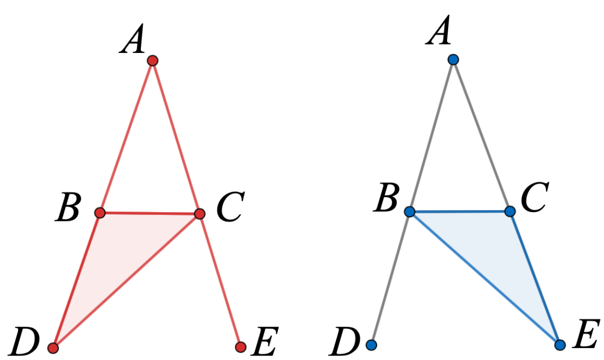


Then $BD = CE$ by subtraction. Connect CD and BE.



$\triangle ACD \cong \triangle ABE$ by SAS ($AB = AC$, $AD = AE$, and they share the angle at vertex A).

As a result, we have two more pairs of angles equal: $\angle ADC = \angle AEB$ and $\angle ACD = \angle ABE$. Also, $CD = BE$.



Since $\angle ADC = \angle AEB$ and the flanking sides are equal, namely, $BD = CE$ and $CD = BE$, we have $\triangle BCD \cong \triangle CBE$ by SAS.

It follows that $\angle DBC = \angle ECB$. By supplementary angles, $\angle ABC = \angle ACB$.

□

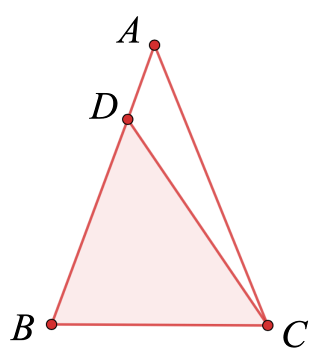
Notice how the original figure is extended to provide auxiliary shapes helpful in the proof. This is a common theme.

We will use both results ($\angle ABC = \angle ACB$ and $\angle DBC = \angle ECB$), going forward.

To summarize: in a triangle with two equal sides, the angles opposite those sides are also equal, as well as their supplementary angles.

Euclid I.6

We proved the converse of I.5 previously based on angle bisection. Now that we have proved Euclid I.5, which provides the basis for bisection, this is logically solid. Nevertheless, for completeness, here is Euclid's proof of I.6. It is a *Proof by Contradiction*.



Proof.

Given $\angle ABC = \angle ACB$. Suppose $AB \neq AC$. Then let one be less, say AC. So cut off from AB the length $BD = AC$.

Compare $\triangle DBC$ with $\triangle ABC$. We have the sides equal, namely $DB = AC$ and $BC = BC$. As well, $\angle DBC = \angle ACB$. Therefore $\triangle ABC \cong \triangle DBC$ by Euclid I.4 (SAS).

But this is absurd. The part cannot be equal to the whole.

Therefore, $AB = AC$.

$\square$

$\iff$

The theorem together with its converse says that, in an isosceles triangle, the base angles are equal $\iff$ the two sides sides are equal (not the base).

The symbol $\iff$ means *if and only if*, so both base angles equal $\rightarrow$ two sides equal *and* two sides equal $\rightarrow$ base angles equal.

Chapter 7

Bisection

Here we look at the problem of angle bisection, cutting an angle into two equal parts. We also look at the perpendicular bisector of any line, a perpendicular line that cuts halfway between two end points, or a vertical through any particular point on a line, or third, through a point not on the line.

We will show that every point on the perpendicular bisector is equidistant from the two points used to construct it.

We also prove the converse theorem, that *every* point which is equidistant lies on the bisector.

constructions

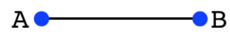
A number of the propositions in book 1 of Euclid concern constructions.

The tools we have are a straight-edge and a compass. The compass is collapsible, meaning that it cannot be used to transfer distances since it loses its setting when lifted from the page. This is a limitation Euclid solves in the second and third propositions of book I. It's also important that the straight-edge is not a ruler, there is no way to measure distance by reference to marks on it.

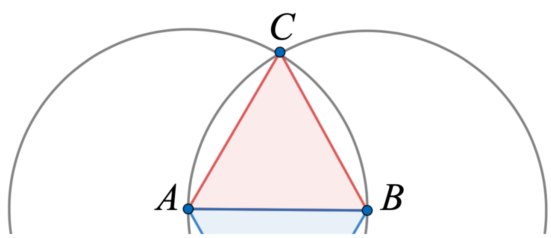
Euclid was smart enough to know about compasses and how to set them. The idea he had was this: to make the fewest possible assumptions. A non-collapsible compass was a luxury he didn't need, since he could accomplish the same end without it.

Euclid I.1: equilateral triangle

To construct an equilateral triangle on a given line segment.



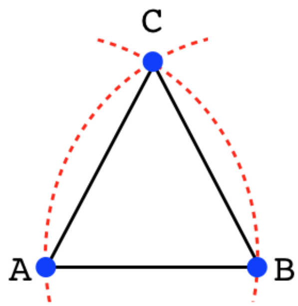
The first step is to draw two circles on centers A and B.



The circles are drawn with each radius equal to the line segment AB. It is a property of circles that all points on the circle are at the same distance from the center. Thus all points on the left-hand circle are equidistant from A, and all points on the second one are equidistant from B.

Therefore, the point C where the circles cross is equidistant from *both* A and B.

Now use the straight edge to draw $\triangle ABC$. Since $AC = AB$ and $BC = AB$, we know that $AC = BC$. The triangle is equilateral.

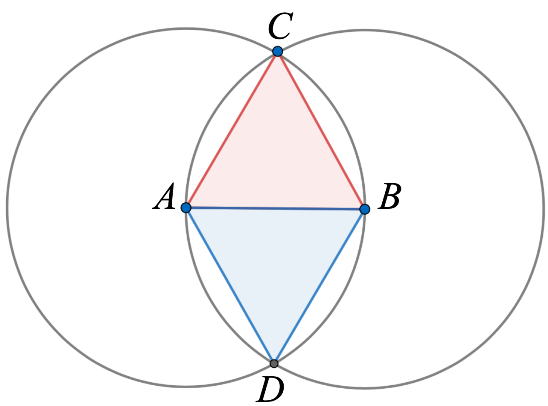


□

The proof doesn't stand on its own. We used one definition (D) and a common notion (CN).

- D I.15 all radii of a circle are equal.
- CN I.1 things which equal the same thing also equal one another.

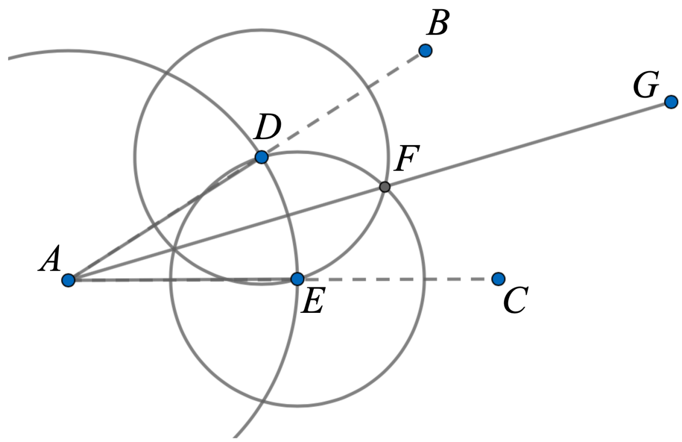
If we look again at the figure, and label the other point where the circles cross as D:



We have a second equilateral triangle, congruent to the first.

Euclid I.9: bisection of an angle

To bisect a given angle. Let $\angle BAC$ be the angle to bisect.

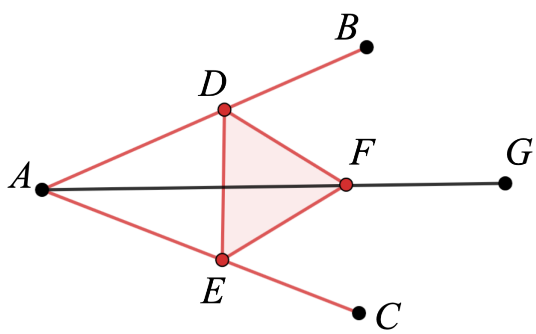


As radii of a circle on center A, we first find points D and E equidistant from A. So $AD = AE$.

Next, we draw circles on centers D and E that have the same radius, and the easiest way to do that (with a collapsible compass), is to make the radius equal to DE.

As radii of circles on the centers D and E, $DE = DF = EF$ and $\triangle DEF$ is equilateral.

Here is a somewhat cleaner view.



AF is shared, so $\triangle ADF \cong \triangle AEF$ by SSS, Euclid I.8.

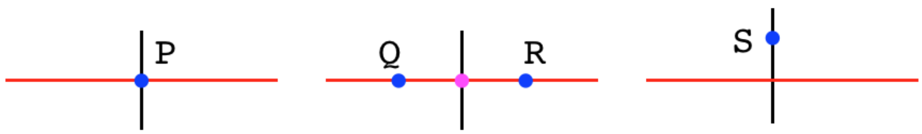
Therefore $\angle BAF$ is congruent to $\angle CAF$ and the given angle $\angle BAC$ is bisected.

$\square$

perpendicular lines

When constructing a line segment perpendicular to another line segment, there are three common situations. We want the perpendicular line to pass:

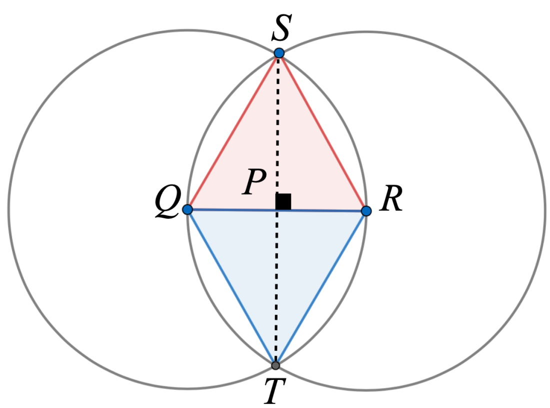
- through a given point P on the line.
- halfway between two points Q and R on the line, bisecting QR
- through the line and also through a point S not on the line



We solve the second case and then show how the other two can be adapted to it.

Euclid I.10: perpendicular bisector

Simply construct two circles of equal radius, one centered at Q and the other at R. It's easiest to choose the radius to be equal to the length QR.

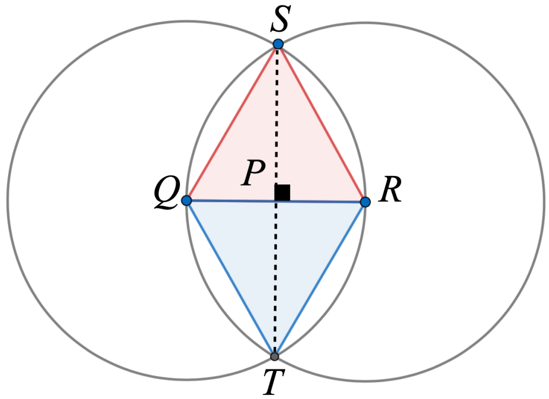


As before, the point S is on both circles, hence it is a radius of both, and therefore equidistant from Q and R. $QS = SR = QR$. The three points form an equilateral triangle, $\triangle QRS$.

The point T below the line segment has the same property, and $\triangle QRS$ is congruent to $\triangle QRT$ by SSS.

Furthermore, we claim that the angles as P are right angles. Thus, $SPT \perp QPR$.

Euclid's proof is simple.



Proof.

$\triangle QST \cong \triangle RST$ by SSS, and both are isosceles.

It follows that $\angle QSP = \angle RSP$, i.e. $\angle QSR$ is bisected.

So $\triangle QPS \cong \triangle RPS$ by SAS.

This gives $QP = RP$.

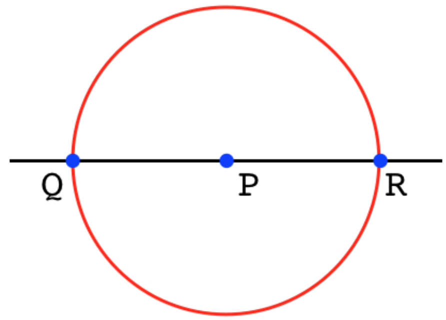
Also, the angles at P, $\angle QPS = \angle RPS$, so all four are right angles.

□

Euclid I.11: bisector through a given point on the line

Suppose we know a point P on the line and wish to construct the vertical line through P.

Use the compass to mark off points Q and R on both sides of P, equidistant from it. This can be done by drawing a circle with center at P.

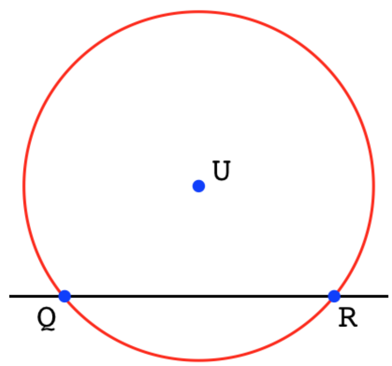


Now, simply proceed as above.

Euclid I.12: bisector through a given point not on the line

Alternatively, suppose we know the line and the point U but not P, and we wish to construct a vertical through the line that also passes through U.

Find Q and R on the line an equal distance from U ($QU = RU$), as radii of a circle centered at U (left panel, below). Their exact position is unimportant.



Now repeat the previous construction, using Q and R. The line segment ST passes through U, as required.

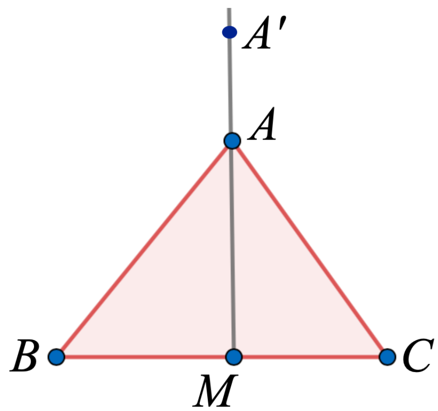
Proof. (Sketch).

Since $QU = RU$, $\triangle QUR$ is isosceles. Therefore the base angles are equal. In an isosceles triangle, the top vertex lies on the perpendicular bisector of the base (see the chapter on isosceles triangles.

Alternatively, find a point below the line equidistant from Q and R. Proceed as in the proof above.

bisector is the altitude of an isosceles triangle

Suppose we know two points B and C. We find the point M equidistant between them and construct the perpendicular bisector AM. Then the two sides AB and AC have equal length. $\triangle ABC$ is isosceles.



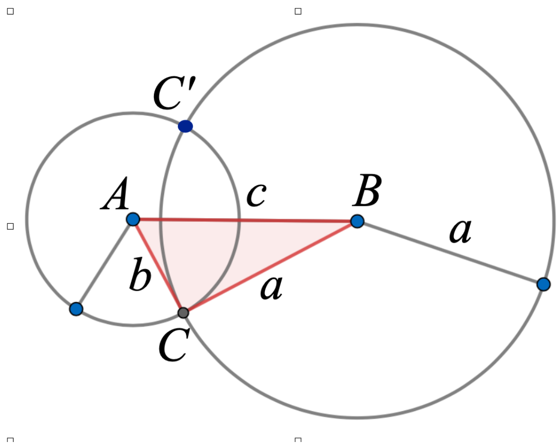
That is true for *any* point on the line extended through A and M. For example, $A'B = A'C$ in the figure above.

perpendicular bisector converse

The converse theorem says that *every* point which is equidistant from B and C lies on AM or an extension of it.

Here are two proofs. The first one relies on the **triangle inequality**, which says that in any triangle, the sum of any two sides must be greater than the length of the third side.

We give a plausible argument for the triangle inequality based on a construction. (We'll look at Euclid's proof later). The theorem is trivially true if the third side is not the longest. Hence we compare the two shorter sides to the longest one.

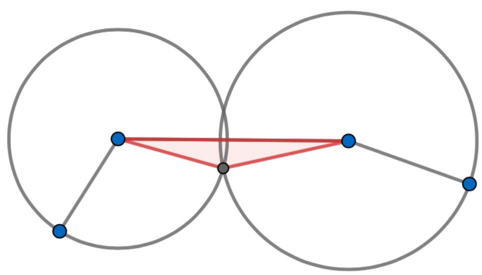


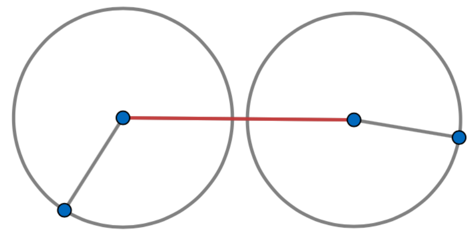
Let AB be the points connected by the longest side c. We claim that the sum of the lengths of the other two sides a and b must be such that:

$$a + b > c$$

Draw two circles, one on center A, and the other on center B, with radii a and b the same as the two shorter sides. There are two points where those circles cross, C and C'. Triangles constructed using either one of those as the third vertex are congruent by SSS. (Mirror images are fine).

Now consider what happens as one of those two sides gets shorter. Let a be the side that is decreased in length.





As the combined length drops to be less than the longest side, it is no longer possible to construct a triangle.

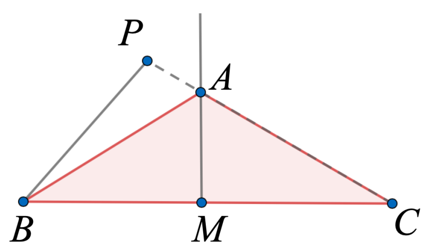
□

To the current proof:

Proof.

Suppose that P is equidistant from B and C but does not lie on the perpendicular bisector. Suppose that P lies on the same side of the bisector as B.

Find the point where PC crosses the bisector at A.



By the forward theorem, $AB = AC$.

We are supposing that $PB = PC$. By the triangle inequality

$$PB < AB + AP$$

Since $AB = AC$:

$$PB < AC + AP = PC$$

But this is absurd. PB cannot both be equal to and less than PC. Therefore, our supposition is incorrect, and there does not exist any such point P.

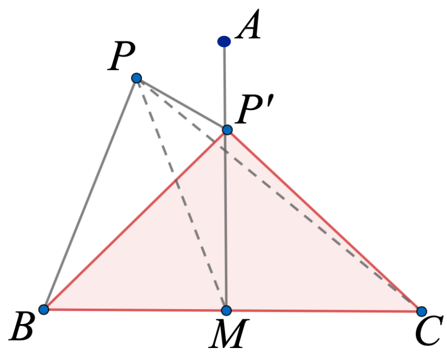
□

See **here** for a somewhat fuller explanation.

perpendicular bisector using I.7

We will prove that every point which is equidistant from two points on a line lies on the perpendicular bisector.

Let $AM \perp BC$, and bisect it such that $BM = CM$. AM is the perpendicular bisector of BC.



Proof.

Suppose P lies on the same side of AM as B.

Now, seeking a contradiction, suppose PM is also perpendicular to BC at M.

PM bisects BC so it is a perpendicular bisector of BC.

By the forward theorem $PB = PC$.

Now find P' on AM such that $P'B = PB$.

We have three equal segments: $PB = PC = P'B$.

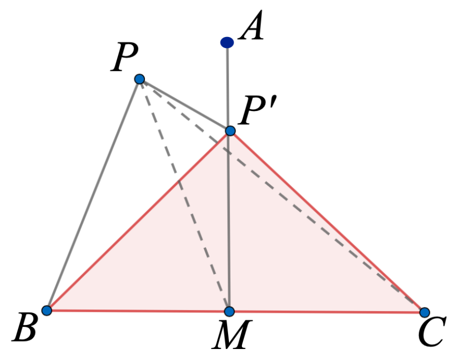
By the forward theorem, since P' is a point on the perpendicular bisector, $P'B = P'C$.

Now we have four equal segments: $PB = PC = P'B = P'C$.

But by Euclid I.7, this is impossible. It cannot be that $P'B = PB$ and also $P'C = PC$ on the same side of BC.

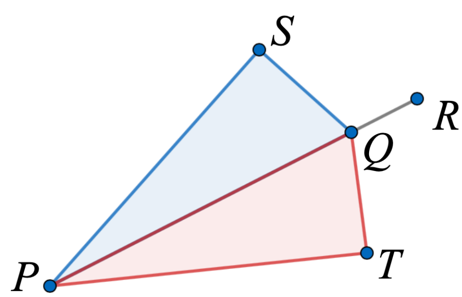
This is a contradiction. There is only one perpendicular through BC at M.

$\square$



bisector equidistant from sides

• Any point on the bisector of an angle is equidistant from the sides at the point of closest approach.



Proof.

Let PR bisect $\angle SPT$.

Draw perpendiculars QS and QT.

The angle at P is bisected, so $\triangle PQS$ and $\triangle PQT$ have two angles equal, and by sum of angles they have three angles equal.

They share the hypotenuse, PQ.

So $\triangle PQS \cong \triangle PQT$ by ASA.

It follows that $QS = QT$.

□

equidistant from sides → bisector

- If a point is equidistant from the sides of an angle, then it lies on the angle bisector.

Proof.

Given $QS = QT$. We have PQ shared.

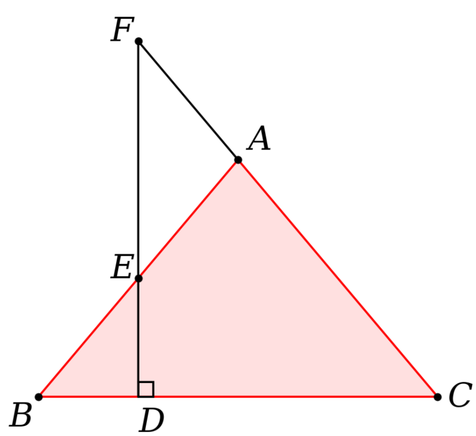
So $\triangle PQS \cong \triangle PQT$ by HL.

It follows that $\angle PQS = \angle PQT$.

□

problem

Here is a problem to exercise some concepts we've seen to this point: isosceles triangles, complementary angles, and vertical angles.



Given that $AB = AC$. Pick any point D on the base BC (except directly under A), and draw the vertical DF. Extend AC to meet that vertical line.

Prove that $\triangle AEF$ is isosceles.

problems

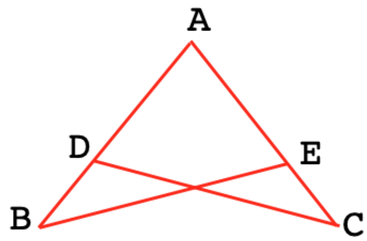
To prove:

- ∘ Prove that for two supplementary angles, the angle bisectors are perpendicular to each other.

∘ Prove that an equilateral triangle (all 3 sides equal) is equiangular (all 3 angles equal). (Don't just rely on symmetry. Adapt the proofs given in this chapter).

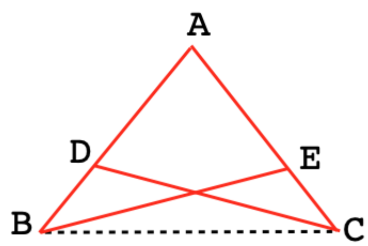
∘ A line perpendicular to the bisector of an angle cuts off congruent segments on its sides.

In the figure below, given that $AC = AB$ and $\angle B = \angle C$.



Prove that $BE = DC$.

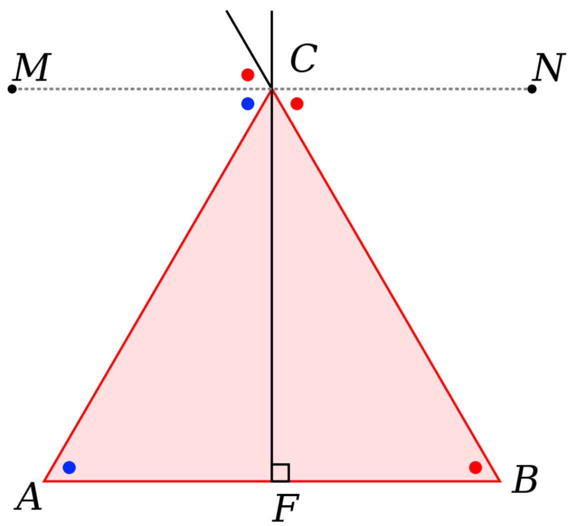
Hint: draw BC and then mark all the angles that are equal.



∘ An equilateral triangle has all three angles equal.

problem

155. If one of the equal sides of an isosceles triangle be extended at the vertex and a line be drawn through the vertex parallel to the base, this line will bisect the exterior angle.

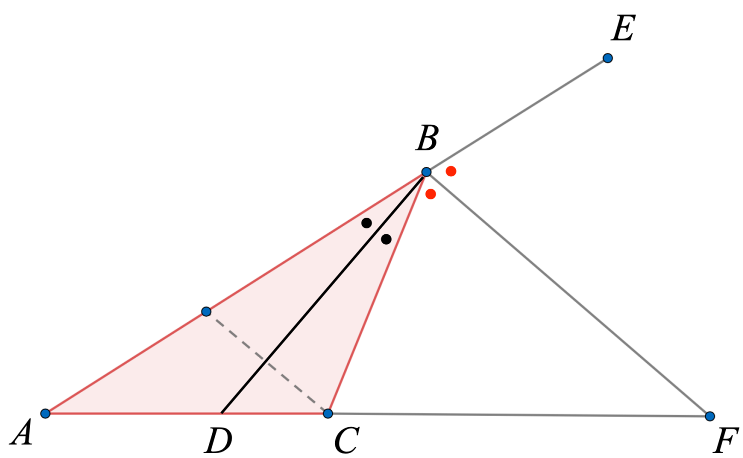


The angles labeled with red dots are equal by alternate interior angles and then vertical angles, while the angles labeled black are equal by alternate interior angles.

But we are given that this triangle is isosceles, so black and red are equal. Therefore the exterior angle at C is bisected by the horizontal.

Alternatively, use the fact that the external angle is the sum of the two base angles, and just use alternate interior angles once.

It is generally true that the adjacent bisectors of an internal and external angle form a right angle. Above, $\angle MCF$ is right.



$\angle DBF$ is right.

Proof.

Twice $\angle DBC$ plus twice $\angle FBC$ is equal to two right angles.

The result follows easily.

□

Chapter 8

Additional constructions

This chapter explores some constructions that come very early in Euclid's *Elements*. These are not strictly necessary to understand the material that follows. However, in Euclid's view they are *logically* necessary, so we include them here.

collapsible compass

There is a famous restriction to a *collapsible* compass, one which loses its setting when lifted from the page. That means that generally, you wouldn't be able to draw two circles of the same radius on different centers.

In erecting a perpendicular bisector, we get around that restriction by drawing the circles on Q and R with the same radius QR, by placing Q and R the same distance apart as the radius to be used.

We will call a compass that is able to hold its setting, a *standard* compass. Within the first few pages of Euclid, it is shown how to use a collapsible compass to carry out the very construction we said we couldn't do, namely, construct two circles on Q and R with equal radius and that radius not equal to QR or QP.

Also, see the video at the url:

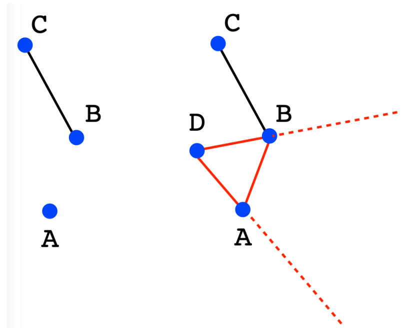
https://www.mathopenref.com/constperpextpoint.html

Euclid I.1: construction of an equilateral triangle

We saw this in the previous chapter.

Euclid I.2: transfer a length

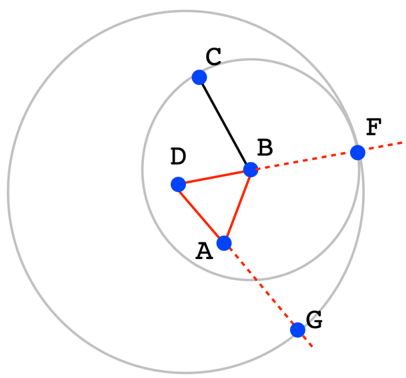
To construct a line segment on a point, equal to a given line segment.



Since this immediately follows the first construction, it seems likely we will need an equilateral triangle. The other thing we know how to do is to draw circles.

Construct a circle of radius BC on center B. Where that intersects the extension of DB at F we have a length equal to one side of the $\triangle$ plus BC.

So then draw a circle on center D of radius DF.



The intersection of that circle with the extension of DA marks off a length equal to one side of the $\triangle$ plus BC.

So $AG = BF = BC$, as required.

If it is desired to draw a line segment of length BC in some other direction from A, we just need another circle, centered at A. That is Proposition 3.

In sum then, we can mark off any length from one line segment onto another, even with just a collapsing compass.

construct a given angle at a given point

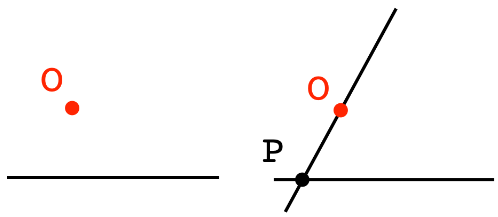
Briefly, construct the triangle containing the desired angle by using SSS. This can be done by transferring lengths as just described above.

Euclid I.31: construct a line parallel to another line

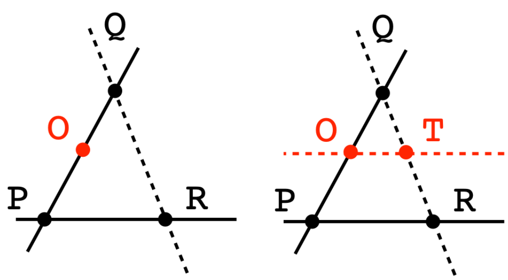
Suppose we are asked to construct a line parallel to a line or line segment, through a given point. We remain true to the Greek ideal, that dividers should not come off the paper.

Proof.

First, pick some point on the line segment P, and draw a line segment through OP.



Find Q on the second line, on the same side as O, such that $QP > OQ$.



Now draw the circle with center Q and radius QP and, at the intersection with the first line, R. Draw the line QR. $\triangle PQR$ is isosceles.

Finally, draw the circle with center Q and radius OQ, and at the intersection of the circle with the last line, find T. $OQ = QT$ and $QP = QR$. Therefore the base angles of $\triangle QOT$ and $\triangle QPR$ are equal.

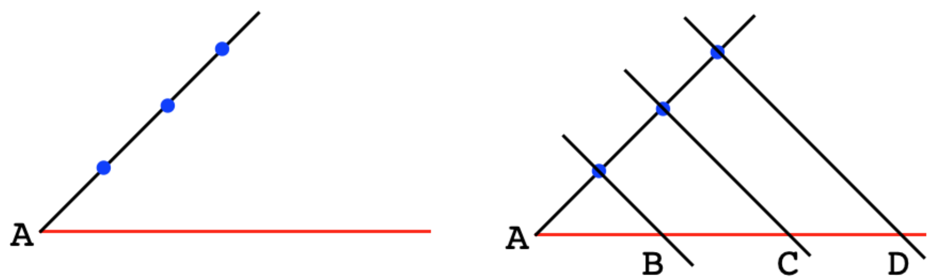
We have that $\triangle OQT$ is also isosceles, and because it shares the angle at the other vertex the two triangles are similar: $\triangle PQR \sim \triangle OQT$.

Therefore the bases PR and OT are parallel, by the converse of the alternate interior angles theorem.

□

Euclid VI.9: division of a line segment into parts

We wish to divide a general segment (in red, below) into an even number of pieces. Suppose that number is three.



Using one end of the target segment, draw any other line, and mark off on that line segments of equal length, using a compass. Even with a collapsible compass, this can be done sequentially by drawing a series of circles..

Then, erect the perpendicular bisector of the black line at each point and extend the bisector to the target red line.

We will have that $AB = BC = CD$. Furthermore, since $AB = BC$, $AB = \frac{1}{2}(AB + BC)$ so $AC = 2AB$.

□

This construction uses properties of *similar* triangles that we have not proved yet. Since the angle at A is shared and the angle with the black line is always the same, all the triangles have the same shape and their sides are in proportion. We have fixed that proportion as an integer: 1, 2 or 3.

Part III

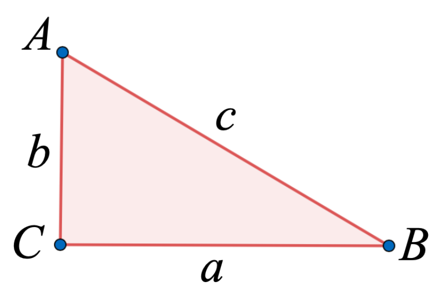
Triangles and Area

Chapter 9

Right triangles

A *right* triangle is a triangle containing one right angle. We saw previously that the Greeks' definition of a right angle is that two of them add up to one straight line or 180 degrees.

No triangle can contain two right angles, because the measure of the third angle would then have to be zero. This goes back to the parallel postulate. Right angles at both ends of a line segment send out parallel lines.



Right angles and right triangles are special in many ways.

In the figure, let us have the angle at vertex C be a right angle. It is common practice to mark a right angle with a little square, but we often just state the fact. So $\angle ACB$ is right.

The side opposite vertex C — it is common practice to label it the same but lowercase, so c — is called the *hypotenuse*, and the other two sides a and b are sometimes called legs.

complementary angles

Since the sum of angles in a triangle is equal to two right angles, the two acute angles, $\angle BAC$ and $\angle ABC$ in the figure above, together equal one right angle, or 90 degrees. Along with with $\angle ACB$ they sum up to two right angles for the whole triangle.

Proof.

This is a direct consequence of the sum of angles theorem applied to a right triangle.

□

The two smaller angles in a right triangle are said to be *complementary*. This fact is often exploited in proofs, since if we know one, we know the other, by sum of angles.

- the sum of the two smaller angles in a right triangle is equal to one right angle.

Pythagorean theorem

A second very important fact about right triangles is expressed in the Pythagorean theorem. Although we haven't proved it yet, we will do so shortly, and call on it here. Given the hypotenuse c and two legs of a right triangle, a and b, the theorem says that

$$a^2 + b^2 = c^2$$

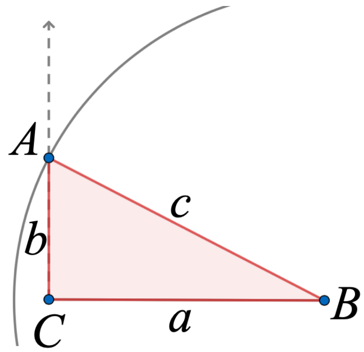
It follows that if we know any two sides in a right triangle, then we know all three sides.

A third fact is that the hypotenuse is the longest side in a right triangle. We will prove later that in any triangle, one side is longer than another in any triangle *if and only if* the angle opposite that side is also larger than the angle opposite the shorter side.

Previously, we gave several ways to prove that two triangles are congruent. These four methods (SAS, SSS, ASA, and AAS) are also useful with right triangles. Some books give them new names in the context of a right triangle. One of these is useful.

hypotenuse-leg in a right triangle (HL)

For two right triangles, if one hypotenuse is equal to the other, and also one pair of legs equal, the two triangles are congruent. This condition is called hypotenuse-leg (HL). It is effectively SSA in the case where we know that the angle is a right angle.



Here we know sides c and a and also have that the angle at C is a right angle.

If two right triangles have hypotenuse and leg equal , then they are congruent by HL. We will use this test often.

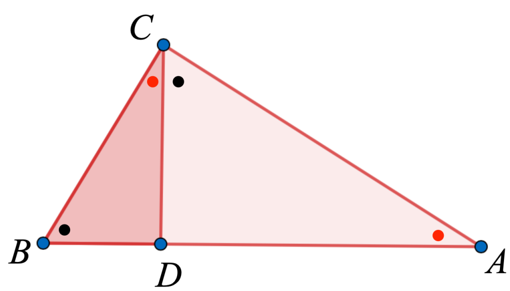
There is only one angle where the hypotenuse will terminate on the vertical extension from the right angle.

Proof.

An algebraic proof is that the third side is determined by the other two sides by the Pythagorean theorem, since we know which side is the hypotenuse. Therefore we have SSS.

□

altitudes



Let $\triangle ABC$ be a right triangle, with a right angle at B. We know that the base angles add up to a right angle.

$$\angle BAD + \angle BCD = 90$$

When we draw the perpendicular to the hypotenuse that goes through the upper vertex, that is an *altitude* of the triangle. Because of the right angle, we obtain two smaller right triangles. Thus

$$\angle BAD + \angle ABD = 90$$

It follows that

$$\angle ABD = \angle BCD$$

(red dots). For the same reason (black dots):

$$\angle BAD = \angle CBD$$

This is a very useful result: if the altitude to the hypotenuse is drawn in a right triangle, the two smaller right triangles are both similar to the original one. All three triangles have the same angles.

theorems about right triangles

- In any right triangle, the right angle is larger than either of the other two angles.

Proof.

Suppose α and β are complementary angles in a right triangle, Then $\alpha + \beta$ is equal to one right angle.

$$\alpha + \beta = 90$$

Both angles α and β must be non-zero: $\alpha > 0$ and $\beta > 0$ (otherwise we do not have a triangle).

$$\alpha > 0$$

$$\alpha + 90 > 90$$

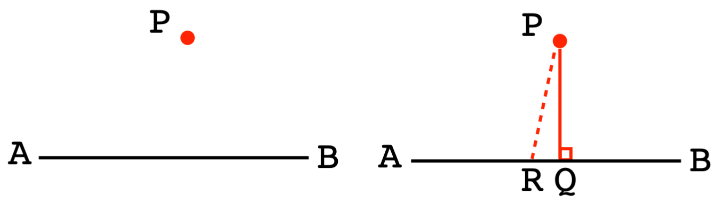
$$90 > 90 - \alpha = \beta$$

The same proof applies starting with $\beta > 0$.

□

only one perpendicular to a line from a point

Suppose we have a line and a point not on the line.



- We claim that only one perpendicular can be drawn from the point to the line.

Proof.

Assume that two such lines can be drawn. So, in addition to $PQ \perp AB$, we draw PR and claim that it is also perpendicular to AB.

Then, by the converse of the alternate interior angles theorem, $PQ \parallel PR$.

But PQ and PR also meet at the point P. This contradicts the fundamental definition of parallel lines. Our assumption must be false.

Only one such line can be drawn.

□

hypotenuse longest side

- In any right triangle, the hypotenuse is longer than either side.

Proof.

We showed above that in any right triangle, the right angle is larger than either of the other two angles. By **Euclid I.18**: in any triangle, a greater side is opposite a greater angle.

□

Or we look ahead again to the Pythagorean theorem. Since $a > 0$ and $b > 0$ and

$$c^2 = a^2 + b^2$$

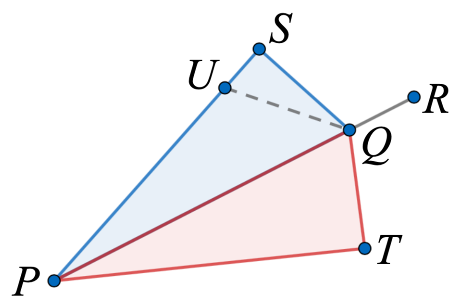
Suppose $b > a$. Since $a > 0$, it follows that $c^2 > b^2$ so $c > b > a$.

The fact that the hypotenuse is the longest side in a right triangle will come in handy when we investigate the tangent to a circle. It is also useful in the next theorem.

shortest distance from a point to a line

• The distance from a fixed point to a line is least when the new line segment makes a right angle with the line.

The claim is that if $QS \perp PS$, then QS is the shortest line connecting Q with PS.



Proof.

Aiming for a contradiction, suppose that QU is not perpendicular, but it is shorter than QS.

$\triangle QUS$ is a right triangle, with the right angle at S, so QU is the hypotenuse of $\triangle QUS$.

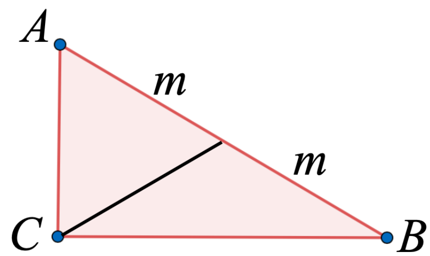
Since the hypotenuse is the longest side of a right triangle, by the previous theorem,

So QS must be shorter than QU.

This is a contradiction. Therefore QU is not shorter than or even equal to QS.

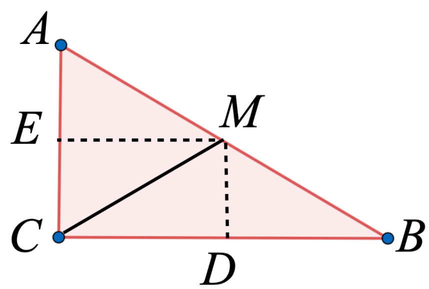
hypotenuse midpoint theorem

In a right triangle, draw the line segment from the vertex that contains a right angle to the midpoint of the hypotenuse, separating it into two equal lengths m. We will show that the length of the bisector is also m.



Proof.

Draw the perpendicular from the midpoint M to the base BC at D. Also draw the perpendicular from M to the base AC at E.



$\triangle MDB$ is a right triangle, and so is $\triangle AEM$.

By complementary angles the other angles are equal.

We are given that $AM = MB$.

It follows that $\triangle MDB \cong \triangle AEM$ by ASA.

Therefore $EM = DB$.

Because it has four right angles at its vertices, $EMDC$ is a rectangle. (The fourth, at M, follows by sum of angles).

Thus $EM = CD = BD$.

So $\triangle MDB \cong \triangle MDC$ by SAS.

It follows that $MC = BM = AM$.

$\square$

Note that both $\triangle AMC$ and $\triangle CMB$ are isosceles.

There are two easier proofs of the above theorem. The first uses Thales' theorem, which we will cover again later, but mentioned at the beginning of the book. Any right triangle can be placed in (inscribed into) a circle, with the hypotenuse as the diameter. AM as well as BM and CM are all radii of this circle.

Another proof uses similar triangles. However, we have not established that theory yet, so we will skip it for now.

converse

As a converse, if we are given that the line drawn to the midpoint of the longest side of any triangle also has length m, then the triangle is a right triangle.

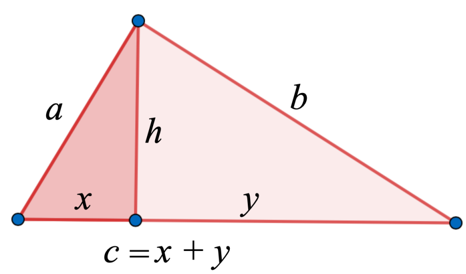
Proof.

The two smaller triangles are isosceles. Therefore, the total angle at vertex C is half the total for the triangle, and thus equal to one right angle.

□

Challenge.

Above we had the result that the vertical line to the hypotenuse (called an *altitude*), forms two smaller right triangles similar to the first. One can construct equal ratios and use them to make a proof of the Pythagorean theorem. We leave this up to you, but will return to it when we have laid out the basic theory of similarity.



Chapter 10

Quadrilaterals

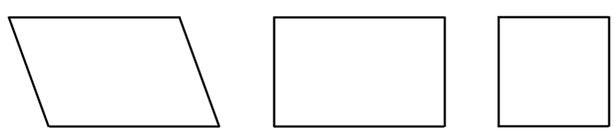
classification

Polygons are constructed from straight sides. If the sides are all the same length, the figure is a *regular* polygon.

A polygon may have 3 or more sides: triangles (3), hexagons (6), and so on. There is a famous theorem from Gauss that involves the construction of a 17-sided regular polygon.

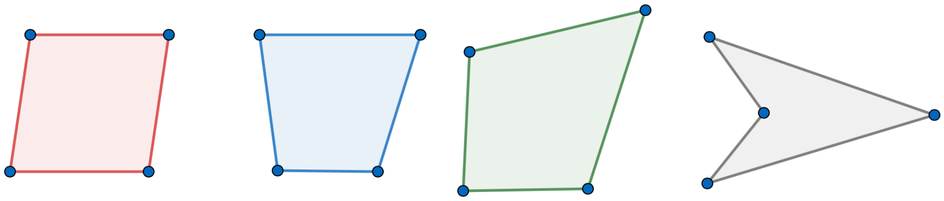
Let us start with four-sided polygons, called quadrilaterals. There are several types, of which the most important are:

- parallelogram: opposite sides equal and parallel
- rectangle: a parallelogram with four right angles
- square: a rectangle with four sides equal



We usually specify these shapes according to the most restrictive conditions they meet. When we think about a parallelogram we mean one without right angles, and when we think about a rectangle, we mean one that is not a square.

There are a few more quadrilaterals to mention in passing (left to right in the figure below):



- rhombus: a square-like parallelogram with all sides equal
- trapezoid: just two sides parallel
- kite: both pairs of *adjacent* sides equal, one pair of opposite angles equal
- dart: like the kite, but one vertex concave (angle > 180)

parallelograms

Parallelograms are four-sided polygons with

- both pairs of opposing sides parallel
- both pairs of opposing sides equal
- both pairs of opposing angles equal

It is convenient to think of parallelograms primarily in terms of the first property. For example, parallel sides is enough to establish the other two properties.

In fact, each can be derived easily from the others. It is also sufficient if one pair of sides is both equal and parallel.

rectangles

Rectangles are parallelograms with

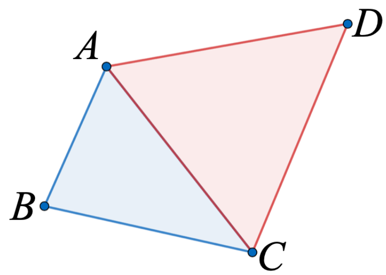
- all four vertices containing right angles

We now look at some fundamental theorems about polygons.

sum of angles theorem

• The sum of all the internal angles in any quadrilateral is the same as that in two triangles, or four right angles altogether.

Proof.



Connect two opposing vertices (A and C above) to form two triangles.

Using the triangle sum theorem, add the component angles at all four vertices.

$$\angle B + \angle BAC + \angle BCA = 180$$
$$\angle D + \angle DAC + \angle DCA = 180$$

But

$$\angle A = \angle BAC + \angle DAC$$
$$\angle C = \angle BCA + \angle DCA$$

So by addition:

$$\angle A + \angle B + \angle C + \angle D = 360$$

□

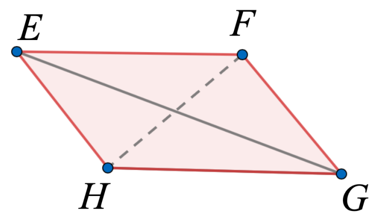
This theorem can be extended to polygons having more sides.

Adding a vertex (with a net addition of one side) is the same as adding another triangle. Correcting for the "base case" we have the sum of angles $S = (n - 2) \cdot 180$.

The proof of the extended theorem is famously done by induction. We cover this elsewhere.

diagonal theorem

- Any diagonal in a parallelogram produces two congruent triangles.



Proof.

parallelogram:

$\triangle EFG \cong \triangle GHE$ and $\triangle EFH \cong \triangle GHF$.

◦ Given $EF = GH$ and $EH = FG$, the result follows by by SSS.

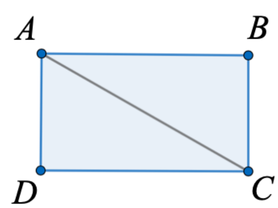
◦ Given $EF \parallel GH$ and $EH \parallel FG$, the result follows by by ASA.

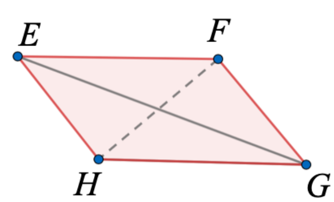
◦ Given $\angle E = \angle G$ and $\angle F = \angle H$, $\angle E + \angle F$ is equal to two right angles, by the sum of angles. Thus, $EF \parallel GH$ and $EH \parallel FG$, and the result follows.

□

Since every rectangle is also a parallelogram, the theorem applies to rectangles.

rectangle: $\triangle ABC \cong \triangle CDA$.





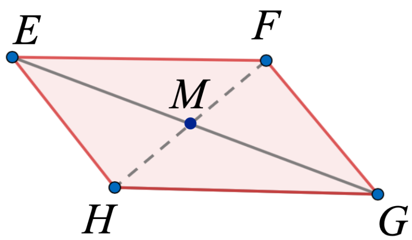
diagonal theorem: equal angles corollary

- In any parallelogram the opposing angles are equal.

This follows immediately from the main theorem.

diagonal theorem: bisection corollary

- The two diagonals in a parallelogram bisect one another.



Proof.

Given $EH \parallel FG$ and $EH = FG$.

$\angle GEH = \angle EGF$ by alternate interior angles.

$\angle EMH = \angle FMG$ by vertical angles.

$\triangle EMH$ and $\triangle FMG$ are equiangular by sum of angles in a triangle.

$\triangle EMH \cong \triangle FMG$ by ASA.

It follows that $EM = GM$ and $FM = HM$.

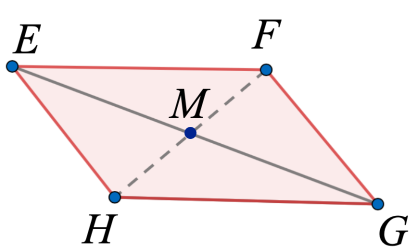
□

special parallelogram theorem

We state this result separately to emphasize it. We will need it for the theory of similar triangles.

In $EFGH$ let $EF = GH$ and $EF \parallel GH$. Then $EFGH$ is a parallelogram.

Proof.



Draw diagonal EG.

By alternate interior angles, $\angle FEG = \angle EGH$.

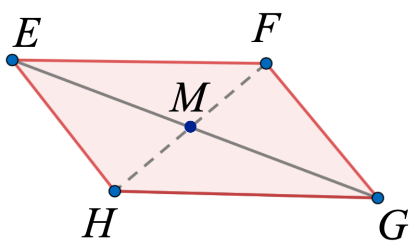
Given $EF = GH$ and EG is shared. $\triangle EGH \cong \triangle GEF$ by SAS.

By the diagonal theorem, it follows that $EFGH$ is a parallelogram.

□

diagonal theorem: converse 1

- If the two diagonals in a quadrilateral bisect each another, the figure is a parallelogram.



Proof.

Given $EM = GM$ and $FM = HM$.

$\angle EMH = \angle GMF$ by vertical angles.

$\triangle EMH \cong \triangle GMF$ by SAS.

It follows that $EH = FG$ and $\angle GEH = \angle EGF$.

So $EH \parallel FG$ by alternate interior angles.

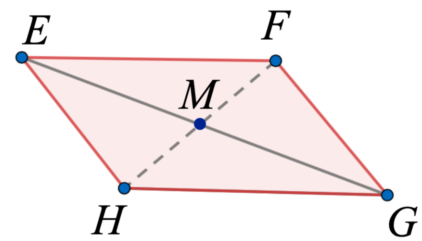
The same logic gives EF parallel and equal to HG.

□

Note that the diagonal theorem and its converse together allow us to interconvert all four specifications for a parallelogram.

diagonal theorem: converse 2

- If the diagonal in a quadrilateral forms congruent triangles, the figure is a parallelogram.



Proof.

Given $\triangle EHG \cong \triangle GFE$

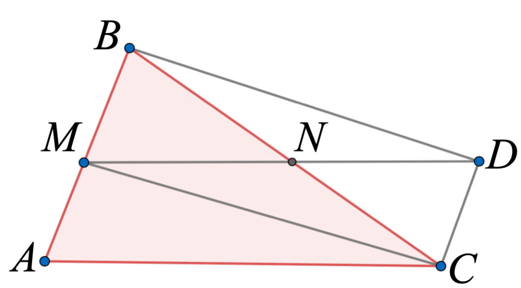
$\angle GEH = \angle EGF$ and $\angle EGH = \angle GEF$.

We have both pairs of opposing sides parallel by the converse of alternate interior angles.

□

midline theorem

- The midline of a triangle is parallel to the third side of that triangle and half its length.



Proof.

Given $AM = BM$ and $BN = CN$. Extend MN to D with $MN = DN$. Draw BD, CM and CD.

$BDCM$ is a parallelogram, by the converse of the diagonal theorem.

$CD = BM = AM$ and $CD \parallel AMB$. It follows that $MDCA$ is a parallelogram.

Hence $AC = MD$ which is twice MN and $AC \parallel MND$.

□

Proof. (Alternate)

With the same construction, $MN = DN$, and given $BN = CN$, and vertical angles, we have $\triangle BMN \cong \triangle CDN$ by SAS.

It follows that $BM = CD$ hence $CD = AM$.

Also from the congruent triangles, $\angle BMN = \angle CDN$. By alternate interior angles, $BMA \parallel CD$.

Thus, $MDCA$ is a parallelogram, with $AC = MD = 2MN$ and $AC \parallel MN$.

□

Varignon's theorem

This famous theorem concerns any quadrilateral. Let's start with the four points lying flat in the same plane.

If we draw the lines connecting the midpoints of each side, the result must be a parallelogram. Here is Acheson's figure:

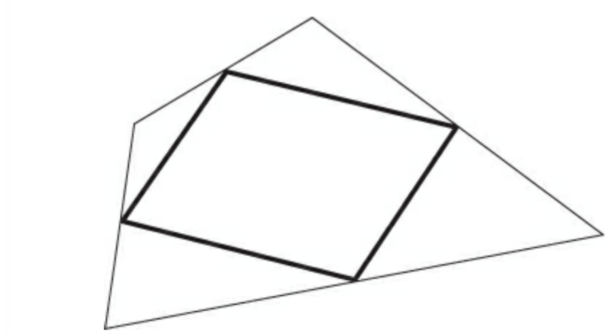
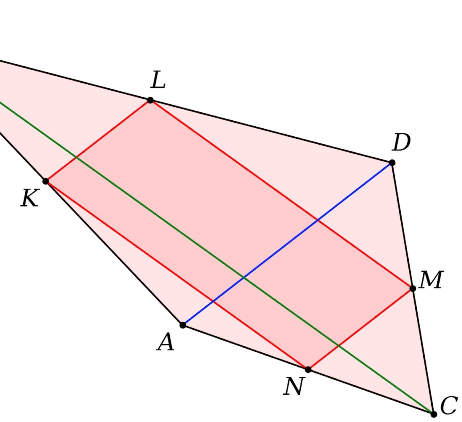


Fig. 50 Varignon's theorem.

which we re-draw

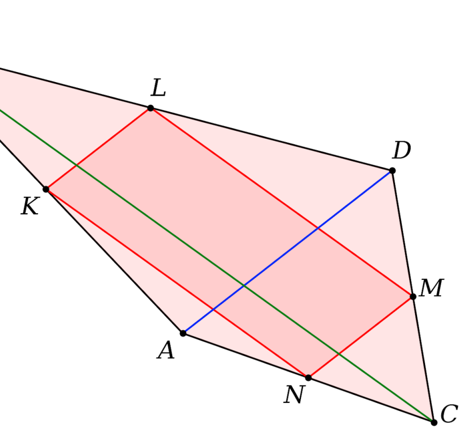


To visualize this, imagine the quadrilateral drawn as two triangles connected at one of the diagonals, say, AD.

The diagonal contains the germ of the answer.

By the midline theorem, the new lines drawn are *parallel* to the base, the diagonal of the original figure. We have $KL \parallel AD \parallel MN$. Two pairs of sides parallel makes a parallelogram.

Furthermore, if we imagine the quadrilateral folding on a hinge at BC or AD, we see that the midlines will remain parallel even if one of the four points is no longer co-planar with its opposite, say A not coplanar with D by bending along BC.



Chapter 11

Rectangles

minimal specification

In considering a shape that might be a rectangle, and given only some of the properties, the rest may follow. There are too many permutations to go through them all.

First, if we know the figure is a parallelogram, then if it also has at least one right angle, it is a rectangle.

Proof.

By the diagonal theorem, there are two opposing angles that are right angles.

Because opposite sides are parallel, it follows that if one of two adjacent angles is a right angle, then so is the other one, by alternate interior angles.

Thus, all four angles are right angles.

□

If we do not know about the sides but only about the right angles, and that there are four of them, then we have a rectangle.

Proof.

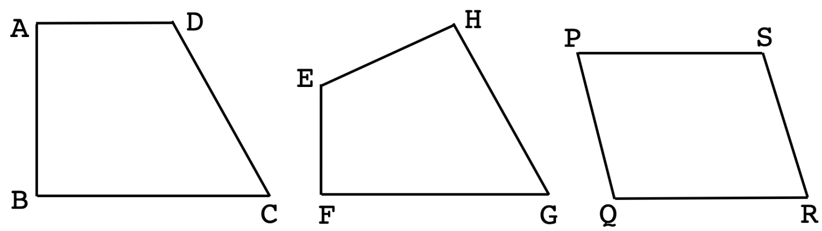
Since the adjacent angles are right, the sides in question are parallel.

A parallelogram with at least one right angle is a rectangle.

□

Even if we know only three angles, by the sum of angles in a quadrilateral, we have four.

However, two right angles are not enough, as counter-examples are easily constructed for both cases (right angles as either neighbors or opposite one another).

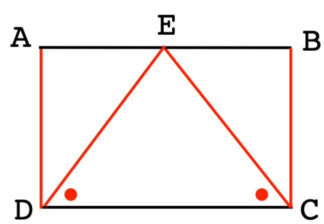


In the left panel $\angle A$ and $\angle B$ are right angles, and in the middle panel, $\angle F$ and $\angle H$ are right angles. But neither $ABCD$ nor $EFGH$ is a rectangle.

If both pairs of sides are equal and parallel, and we don't know there is a right angle, that's not enough by itself.

Here's another permutation.

Let $AD = BC$ and let $\angle A$ and $\angle B$ both be right angles. $ABCD$ is a rectangle.



Proof.

Bisect AB at E and draw the half-diagonals. $\triangle ADE \cong \triangle BCE$ by SAS.

Therefore $ED = EC$ so the angles marked with a red dot are equal by the forward isosceles theorem.

The other components of $\angle C$ and $\angle D$ are equal because of the congruent triangles, so $\angle C = \angle D$.

By the angle sum theorem for quadrilaterals and given the angles at A and B are right angles:

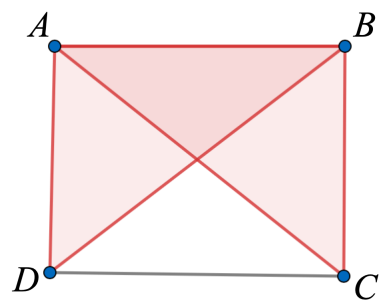
$$\angle C + \angle D = 180$$

Then, both are right angles and thus, all four angles are right angles.

□

diagonals of a rectangle

The diagonals of a rectangle are equal.



Proof.

We consider two overlapping triangles.

$\triangle ABC \cong \triangle BAD$ by SAS.

So $AC = BD$.

Since the diagonals cross at their midpoints, by the diagonal bisector corollary.

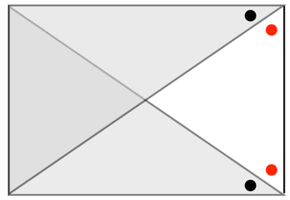
All four half-diagonals are equal.

It follows that the four smaller triangles are all isosceles.

□

We can also invoke symmetry.

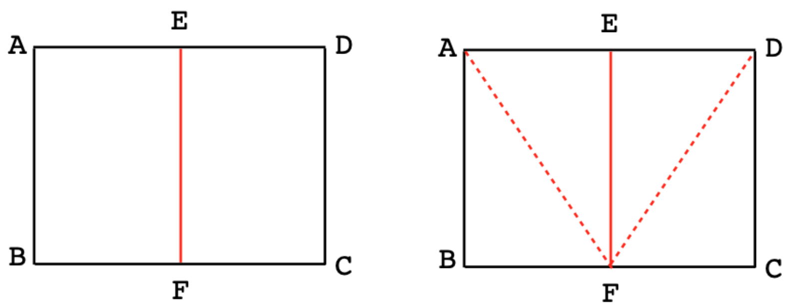
The rectangle has mirror image symmetry by reflection in both the left-right and top-bottom dimensions. As a result, the black-dotted angles in the figure below are equal, and so are the red ones.



And that implies that all four segments from a vertex to the central point are equal in length, by the converse of the isosceles triangle theorem.

bisection of a rectangle

Suppose we are given that $ABCD$ is a rectangle and that EF is the perpendicular bisector of one of the sides, say AD. Then AE is also the perpendicular bisector of the other side, BC.



Proof.

Draw the diagonals of the two small quadrilaterals, namely AF and DF.

Then $\triangle AEF \cong \triangle DEF$ by SAS, using the right angles at E.

But $\triangle AEF$ is also congruent to $\triangle ABF$ (by SAS).

Reasoning in the same way, and using transitivity, we have four congruent right triangles.

It follows easily that the two small rectangles $ABFE$ and $DCFE$ are congruent, so $\angle BFE$ and $\angle CFE$ are right angles, with $BF = CF$.

Therefore EF is the perpendicular bisector of BC.

$\square$

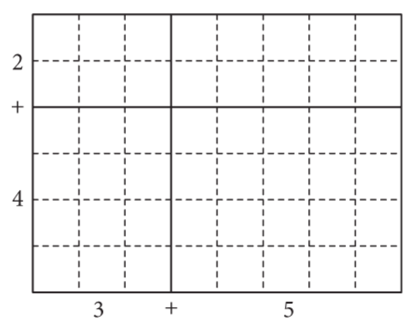
An even simpler proof: given $EF \parallel DC$ so $EDCF$ is a parallelogram. Thus $ED = FC = BF$.

area of a rectangle

Let's say a few brief words about rectilinear area, the area of shapes like squares or rectangles with perpendicular sides. We will then extend this to the areas of triangles and parallelograms, which are like squashed rectangles.

To find areas, we must first fix a unit length. For now, in geometry, we will need an even number of units for each dimension. (There is an exception, but it occurs in a case where we only care about the squared area — see the chapter on the Pythagorean theorem).

Suppose that, in the figure below, the small squares have side lengths of 1 cm, and 6 squares stack vertically and 8 horizontally to fill the shape.



Just multiply the width by the height (in cm) to obtain 48 cm^2.

But then suppose instead that the squares have side lengths of 2.54 cm. Define 1 in = 2.54 cm. The total area would be 48 in^2.

This particular figure above (from Lockhart) shows the distributive law in action:

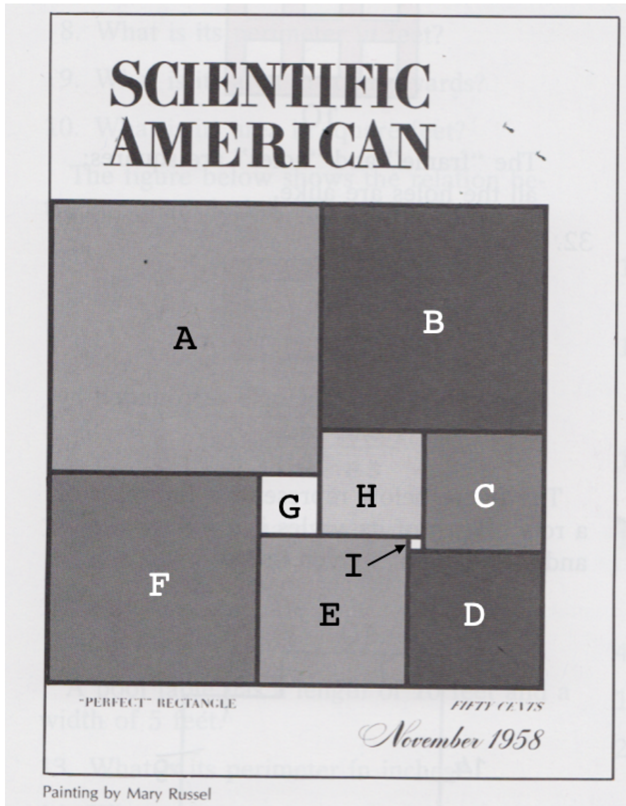
$$(3+5)\cdot(4+2)$$

$$= 3\cdot 4 + 3\cdot 2 + 5\cdot 4 + 5\cdot 2$$
$$= 48$$

Any combination of numbers that add up to 8, times any combination of numbers that add up to 6, gives the same result.

problem

From Jacobs, chapter 9.



Suppose each of A through I is a square, and the areas of squares C and D are 64 and 81, respectively.

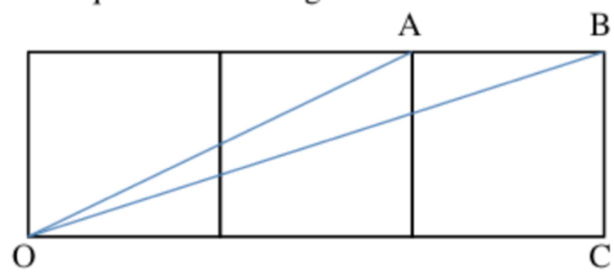
Can you find the areas of all the other squares?

Is the entire figure a square? What is the total area?

problem

Next is a problem from the web. Given three identical squares arranged as follows:

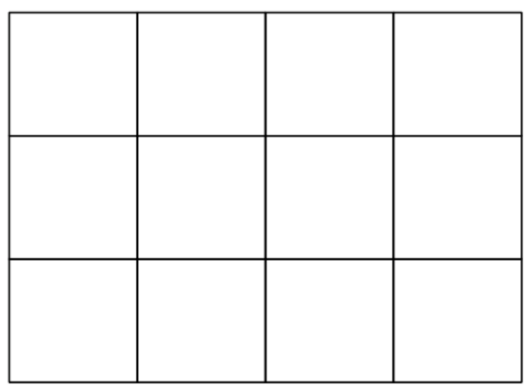
There are three squares in the diagram.



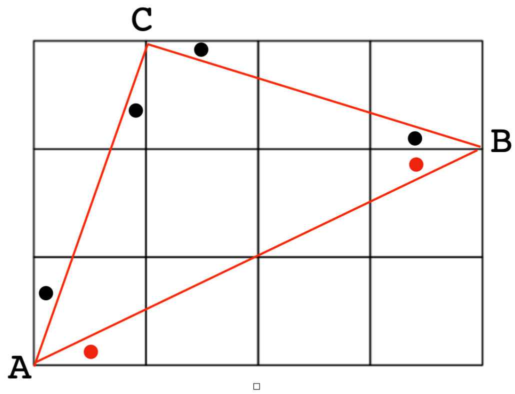
Find $\angle AOC + \angle BOC$.

There is a simple solution, to be obtained without measuring or using trigonometry.

As in so many problems, the key is to draw an inspired diagram, one that extends the figure somehow. Here, a major hint was provided, namely, a grid of squares.



So let's draw the same angles using that grid and form a triangle.



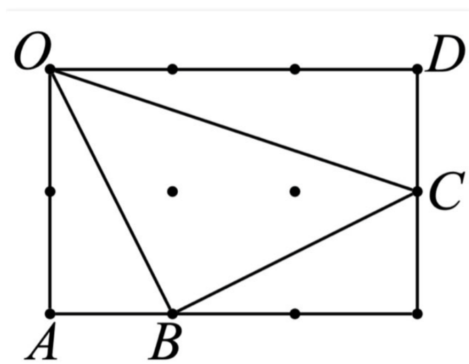
We can identify some equal angles. The two with red dots are equal by alternate interior angles. The two rectangles, one containing the diagonal AC and the other the diagonal BC each consist of three unit squares so they are congruent. That plus alternate interior angles accounts for all the black dotted equal angles.

The last thing we can learn from the diagram is that because of congruent rectangles, $AC = BC$. So that means $\angle BAC$ is equal to $\angle ABC$.

It follows that $\angle BAC$ is one-half the total angle at A, namely one-half of a right angle.

So finally, by sum of angles, $\triangle ABC$ is a right triangle.

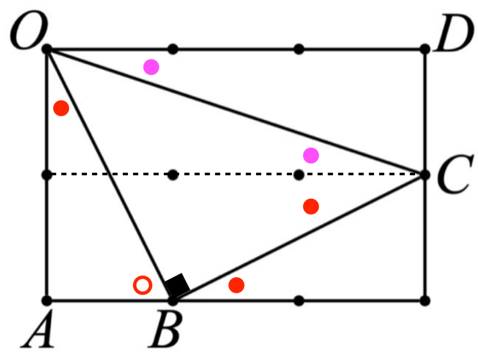
Here is a similar proof, more compactly executed



https://mathenchant.wordpress.com/2022/07/17/twisty-numbers-for-a-screwy-universe/

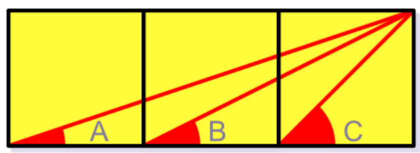
(Endnote 5).

Annotated:



□

Martin Gardner has a version of this problem for which he gives this diagram:



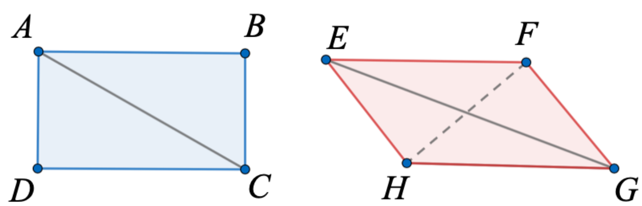
Chapter 12

Parallelograms

diagonal forms congruent triangles

There is a close relationship between parallelograms and triangles. The area of a triangle is one-half the area of the corresponding parallelogram. In the case of a right triangle the corresponding figure is a rectangle.

As we've seen, a line joining opposite vertices of a rectangle or parallelogram is called a *diagonal*. Any diagonal divides a parallelogram into two congruent triangles. The difference is that starting from a rectangle, we obtain right triangles.



$\triangle ABC \cong \triangle CDA$.

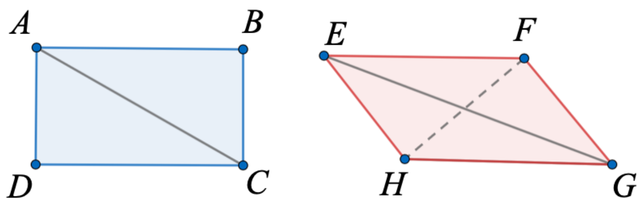
$\triangle EFG \cong \triangle GHE$, and $\triangle EFH \cong \triangle GFH$.

Proof.

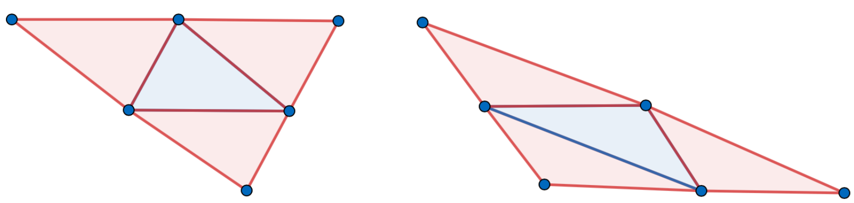
One can use SSS (opposite sides equal), or SAS, or even ASA (opposite sides parallel),

$\square$

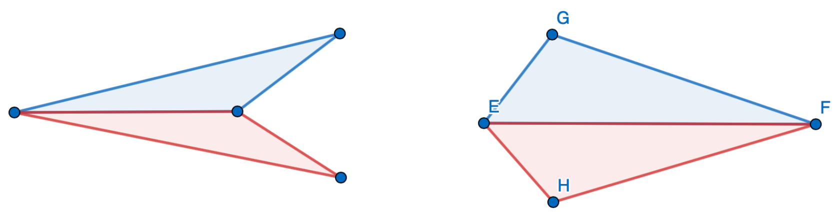
Conversely, two copies of the same triangle (i.e. congruent) can always be joined into a parallelogram, and that figure is a rectangle, if we start with a right triangle.



We can see each of the three possible ways of joining two identical triangles in the figure for the midpoint theorem.



One must be careful, however. The triangles to be joined must not be mirror images, otherwise one may obtain a dart or a kite.



By our fundamental definition of what area is, for a rectangle it is the product of the lengths of two adjacent sides. The area of a parallelogram must be adjusted somehow for the fact that it isn't standing up straight. The easiest way to deal with this is Euclid's approach.

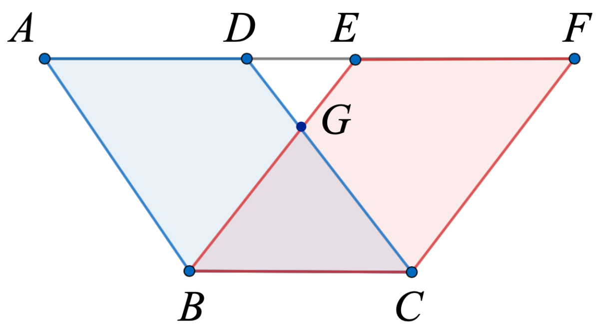
Euclid I.35

As detailed in Book I of *Elements*, here is Euclid I.35. This proposition (theorem) says that given any two parallelograms $ABCD$ and $EBCF$ on the same base BC

and with $ADEF \parallel BC$, they have the same area.

The diagram (below) is drawn for the awkward case, where the two parallelograms overlap.

We have $AB = DC$, $AD = BC$, $BC = EF$ and the sides are parallel as well.



Proof.

Because of the shared base, $AD = EF$.

By addition: $AE = DF$.

We also have $AB = DC$ and because $AB \parallel DC$, $\angle EAB = \angle FDC$.

Thus $\triangle EAB \cong \triangle FDC$, by SAS.

Subtract the shared area of $\triangle DGE$ and add the shared area of $\triangle GBC$ to obtain equality for the area of the two parallelograms.

□

This result is immediately extended to the case where $\angle ABC = \angle DCB$ and both are right angles.

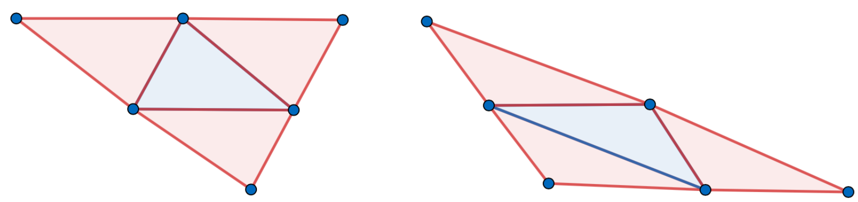
By I.35, this rectangle has the same area as the parallelogram.

Furthermore, the theorem can be invoked sequentially to prove that two parallelograms which do not overlap at all are equal in area, if they lie anywhere on equal bases between two parallel lines.

Roughly speaking, in the figure below, cut off a right triangle from the left side and attach it on the right. The angles add up to form a straight line along the base and a right angle at the upper right. The area is $h \cdot b$.

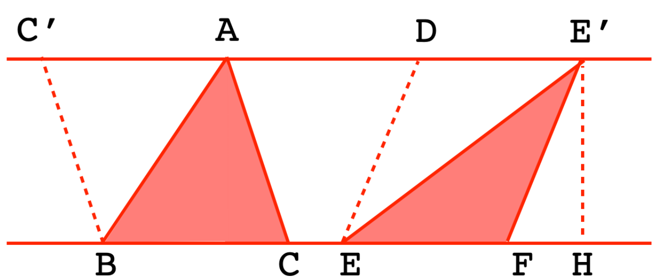
Area of a triangle

To reiterate, any triangle can be turned into a parallelogram, by attaching a rotated image of itself:



It does not matter which side we choose. Any pair of triangles containing the central one is a parallelogram, since it has both pairs of opposite sides equal. We have shown that this is sufficient to know we have a parallelogram.

In the figure below, $ACBC'$ is a parallelogram composed of two copies of $\triangle ABC$. The area of this parallelogram is twice that of $\triangle ABC$. To obtain the value, multiply the base BC times the "height" $E'H$.



$E'H$ is equal in length to the *altitude* of the triangle. That would be a line dropping vertically from A and making a right angle with the base, BC.

The area of the triangle is one-half that of the parallelogram that contains two copies of the triangle.

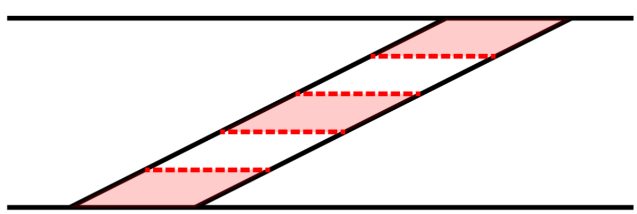
$$\mathcal{A} = \frac{1}{2} \cdot BC \cdot E'H$$

Euclid I.35 and the opposite of dissection, that every triangle can be assembled into a parallelogram that clearly has twice the area, together give everything we need.

If we go back to the idea of cutting off a triangle from one side of a parallelogram and placing it on the other side, to form a rectangle:

it is possible that a parallelogram is particularly skinny for a given height, then it will not be possible to cut off just one triangle.

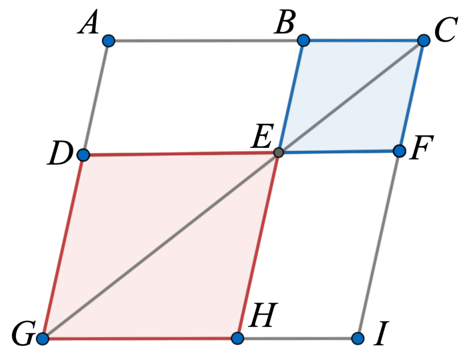
There are at least three solutions to this. One is to rotate the figure so that the sides become the base and the top. A second idea is to slice the parallelogram into identical horizontal pieces, do the operation on each slice, and add them up the result.



A sophisticated approach is to call on I.43 and change the shape to a different one with equal area.

Euclid I.43

This theorem says that, in the figure below, the two smaller parallelograms not on the diagonal, $ABED$ and $EFIH$, are equal.



Proof.

GEC is the diagonal for three parallelograms and cuts each into two congruent triangles with equal area, by the diagonal theorem.

Subtracting

$$(ACG) - (BCE) - (DEG) = (ABED)$$
$$(CIG) - (CFE) - (EHG) = (EFIH)$$

Since the left-hand sides are equal, it follows that $(ABED) = (EFIH)$.

□

squaring figures

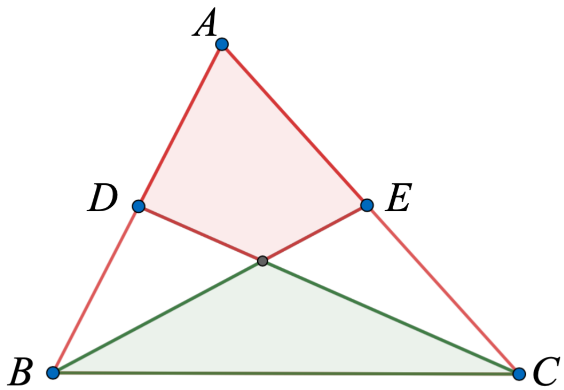
Any parallelogram (such as $ABED$) can be turned into another with the same area but a different shape, simply by inclining the diameter CG at an appropriate angle in theorem I.43.

Any parallelogram can be turned into a rectangle with the same area, as described in this chapter.

Any rectangle can be turned into a square with the same area. This is Euclid II.14, which we will see later.

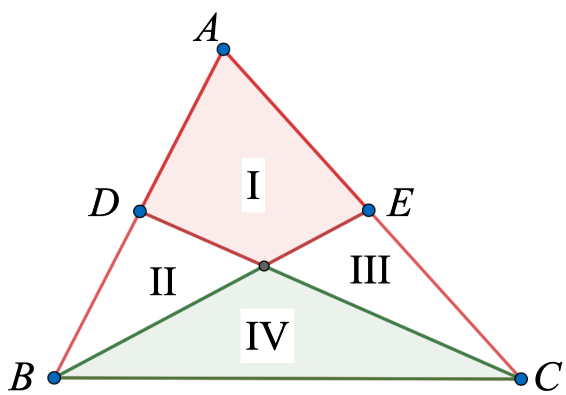
problem

Given that D and E are midpoints of their sides: $AD = DB$ and $AE = EC$. Prove that the colored areas are equal.



Solution.

Let the four triangular areas be labeled $I - IV$.



Then, by the area-ratio theorem and using AC as the base, and since the base is bisected:

$$I + II = III + IV$$

But using AB as the base

$$I + III = II + IV$$

Add the two equations and cancel $II + III$ on both sides:

$$I = IV$$

Subtract the two equations and

$$II - III = III - II$$

$$II = III$$

$\square$

By the **midline theorem**, since DE bisects the sides it is parallel to the base BC.

twice the area

It can be convenient to write the formula as *twice* the area, and we will often do that.

$$2\mathcal{A}_\triangle = ab$$

We rewrite two important formulas from this chapter:

$$d \cdot h + e \cdot h = (d + e) \cdot h = c \cdot h$$

$$\frac{\mathcal{A}_A}{\mathcal{A}_B} = \frac{ah}{bh} = \frac{a}{b}$$

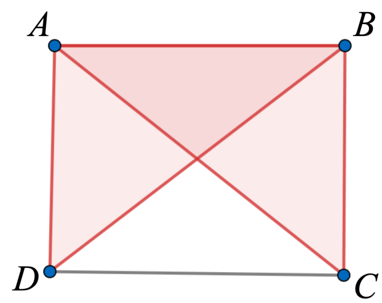
The first one is an equality of different areas, and the second involves a ratio of areas on the left-hand side. The result is unchanged by using twice the area, as long as we are consistent.

Chapter 13

Altitudes

easy theorems

- If a quadrilateral has two adjacent right angles and equal diagonals, it is a rectangle.



Proof.

Let $ABCD$ be a quadrilateral with $\angle A = \angle B$ and both are right angles.

Let $AC = BD$ be the two diagonals.

Compare mirror image right triangles $\triangle ABC$ and $\triangle BAD$.

They have an equal hypotenuse and they share the side AB.

Hence they are congruent by HL, so $AD = BC$.

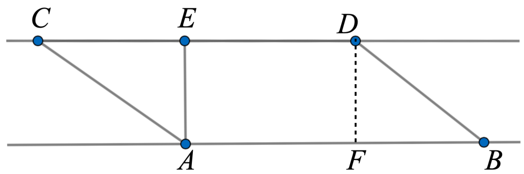
Then $\triangle ADC \cong \triangle BCD$ by SSS.

Thus $\angle C = \angle D$ and by sum of angles both are right angles.

So $ABCD$ has four right angles and thus is a rectangle.

□

- perpendiculars between parallel lines are equal



Proof.

Let AB and CD be parallel lines.

Let $AE \perp AB$, then $AE \perp CD$ by alternate interior angles.

Draw $DF \perp AB$.

DF is also $\perp CD$ for the same reason.

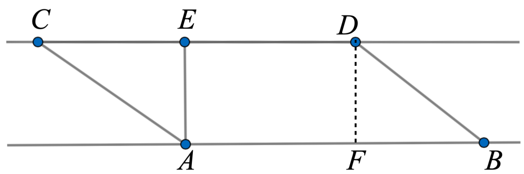
Thus $AEDF$ has four right angles so it is a rectangle.

It follows that $AE = DF$.

□

Any perpendicular is the shortest line segment with one end point on AB and the other on CD.

Proof.



Aiming for a contradiction, suppose BD is shorter, but BD is not $\perp AB$.

Draw the line $AC \parallel BD$ through A.

Then $ACDB$ is a parallelogram, so $AC = BD$.

Let $\triangle ACE$ be a right triangle.

The hypotenuse is the longest side in a right triangle, so $BD = AC > AE$.

This is a contradiction.

A perpendicular is the shortest line segment connecting two parallel lines, and every perpendicular is equal to every other one.

$\square$

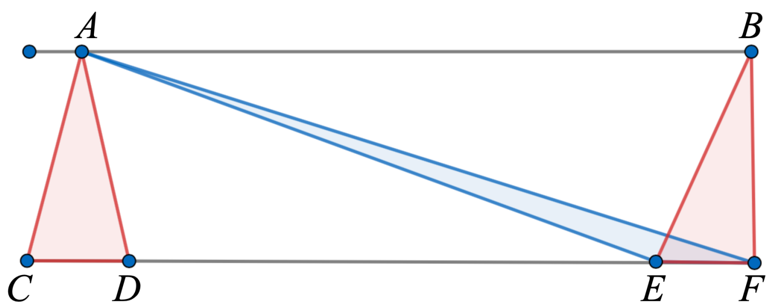
area only depends on base and altitude

An important consequence is that:

- all triangles with the same base and height have the same area.

In the figure below, we have two parallel lines. Mark off $CD = EF$.

Now pick any point on the top and draw the triangle with two equidistant points on the bottom. Any other triangle drawn with an equal base has the same area.



The areas of $\triangle ACD$, $\triangle AEF$, and BEF are equal, because they have equal bases and equal heights.

There are several different conventions for referring to the area of a triangle. One is just to use the capital letter A (for area). To make it stand out, we might use a special font:

$$\mathcal{A}_{\triangle_{ACD}} = \mathcal{A}_{\triangle_{AEF}}$$

That helps but can still become awkward when $\mathcal{A}$ has another meaning in the problem. Some people switch to using K, but a second approach is to use the $\triangle$ symbol, as in

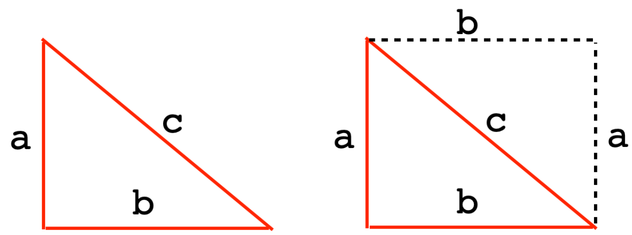
$$\triangle_{ACD} = \triangle_{AEF} = \triangle_{BEF}$$

Euclid always refers to angles as $\angle ABC$ etc., so when he says $ABC = DEF$, he means the *area* of those triangles.

And yet another is to use parentheses. This is a good solution when there are other shapes like rectangles in the problem.

$$(\triangle ACD) = (\triangle AEF)$$

A right triangle has the largest area for a given pair of side lengths. If we imagine a side of length a tilting right or left, then the resulting triangle will have a smaller area, because the altitude h will be less than a.



area-ratio theorem

If in a triangle we draw the line connecting the upper vertex to any point on the bottom side, then the areas of the two smaller triangles are in the same ratio as the lengths of their bases.

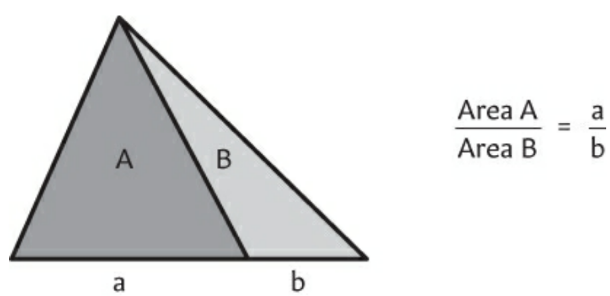


Fig. 120 An area-ratio theorem.

Proof.

The area of $\triangle A$ is $ah/2$, while that of $\triangle B$ is $bh/2$, so the ratio of areas is

$$\frac{\mathcal{A}_A}{\mathcal{A}_B} = \frac{ah/2}{bh/2} = \frac{a}{b}$$

□

It is also the case that the ratio of the area of any sub-triangle to the whole is the same as the proportion of its base length to that of the whole base.

$$\frac{\mathcal{A}_A}{\mathcal{A}_A + \mathcal{A}_B} = \frac{a}{a+b}$$

Proof.

Simple algebra: invert, add one to both sides, and invert again. We show only the right-hand side. Start with the fraction b/a and add 1 to it:

$$\frac{b}{a} + 1 = \frac{b}{a} + \frac{a}{a} = \frac{a+b}{a}$$

Inverted:

$$\frac{a}{a+b}$$

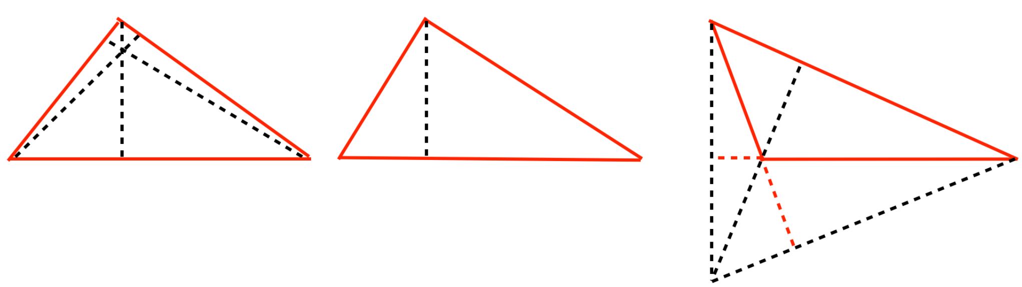
Do the same to both sides and the result follows.

altitudes and the orthocenter

Each of the three sides of a triangle has a corresponding *altitude*, which is the perpendicular line drawn from a vertex to the side opposite, or its extension.

The three altitudes meet at a point (they are said to be concurrent). We will prove this later. The point is called the *orthocenter*.

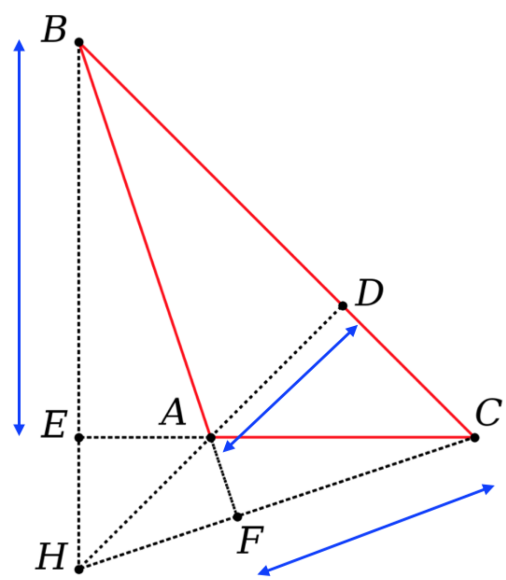
For an acute triangle, the orthocenter lies inside the triangle.



For a right triangle, the orthocenter is just the vertex containing the right angle.

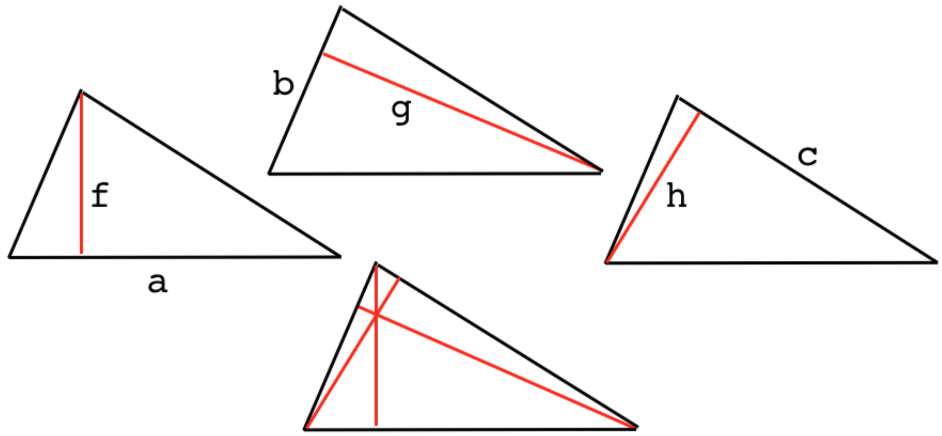
For an obtuse triangle, the orthocenter is external to the triangle, as are two of the altitudes and part of the third.

In the latter case, it may take some thought to determine which altitude goes with which side. The rule is that the altitude forming a right angle with any side originates at the vertex opposite that side. If necessary, the side is extended to meet the altitude at a right angle.



In the figure above, we have one obtuse angle in $\triangle ACB$. The altitudes to sides a, b and c are AD, BE and CF (indicated by blue arrows). The orthocenter is labeled as H. Sides b and c must be extended to show the point of intersection with the corresponding altitude.

computing triangular area



Again the simple formula: one-half base times height. In the figure above, twice the area is

$$2A = af = bg = ch$$

We can choose any side of the triangle to be the base and then multiply by the height to get twice the area.

We must always get the same answer! The area of the triangle is surely the same no matter how you calculate it.

Here's a proof by counting up the area of smaller triangles. A simpler proof follows, but this gives practice in defining altitudes.

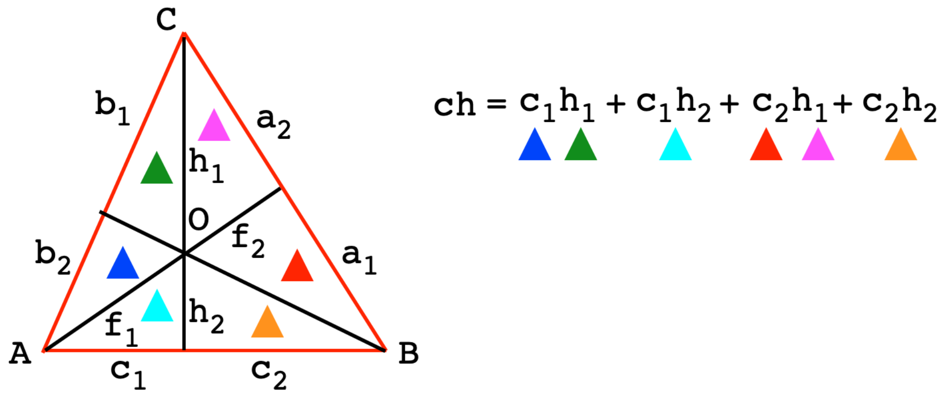
Proof.

In $\triangle ABC$ with sides a, b, c, drop the three altitudes from the vertices to form right angles on the opposing sides. We label two of them: f for side a and h for side c.

These altitudes cross at a single point. We look at Newton's proof of this in just a bit (**here**).

Each altitude and side is then divided into two parts as shown.

This gives six small triangles. To make it easier to keep track of them, they are labeled with colors.



So then twice the area of the whole triangle is $2A = af = ch$. Start with ch

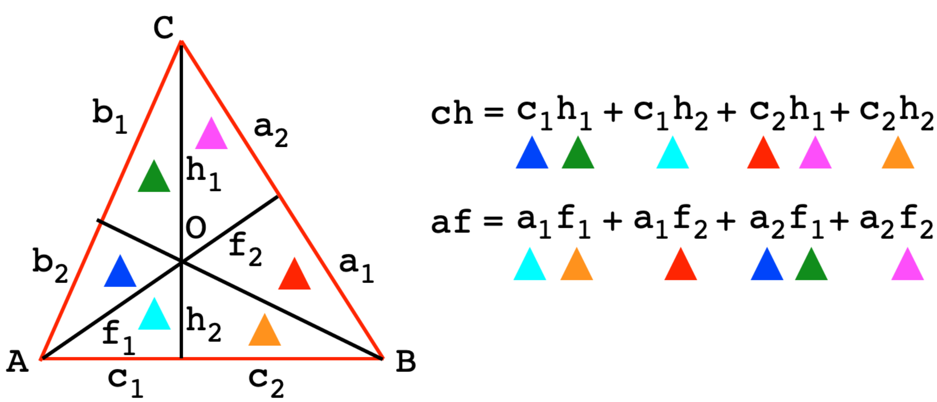
$$ch = (c_1 + c_2)(h_1 + h_2)$$

$$= c_1h_1 + c_1h_2 + c_2h_1 + c_2h_2$$

The single right triangles are easy to see: c_1h_2 and c_2h_2. The other two are composed of two right triangles. For both, the base is h_1, and then, for green and blue, the height is c_1, or for red and magenta, the height is c_2. For these obtuse triangles, the height must be extended to the base to form the right angle.

But the same six triangles can be arranged in a different way so that twice the area of the whole triangle is

$$af = a_1f_1 + a_1f_2 + a_2f_1 + a_2f_2$$



$f1$ is the base and a_1 or a_2 the height, for the compound cases.

A similar calculation can be carried out for side b and altitude g. The area is the same regardless of which side is chosen as the base.

□

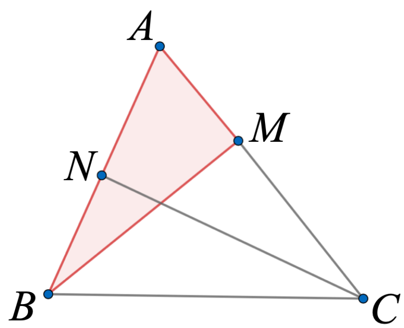
altitude proof

In $\triangle ABC$ draw $BM \perp AC$ and $CN \perp AB$.

Then $AB \cdot CN = AC \cdot BM$.

Proof.

$\triangle AMB$ and $\triangle ANC$ are both right triangles.



They also share $\angle A$, so they are similar.

As similar triangles, corresponding sides are in the same ratio:

$$\frac{AB}{AC} = \frac{BM}{CN}$$

Then $AB \cdot CN = AC \cdot BM$

We can show that side BC times its altitude is equal to $AB \cdot CN$, in exactly the same way.

So any altitude times the base has the same value, which is twice the area of the triangle.

□

A straightforward corollary of this result is that any triangle with two equal altitudes is isosceles.

Proof.

If $BM = CN$ then

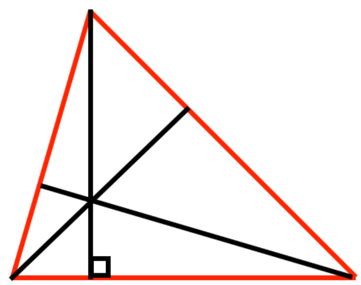
$$\frac{AB}{AC} = 1$$

so $AB = AC$.

□

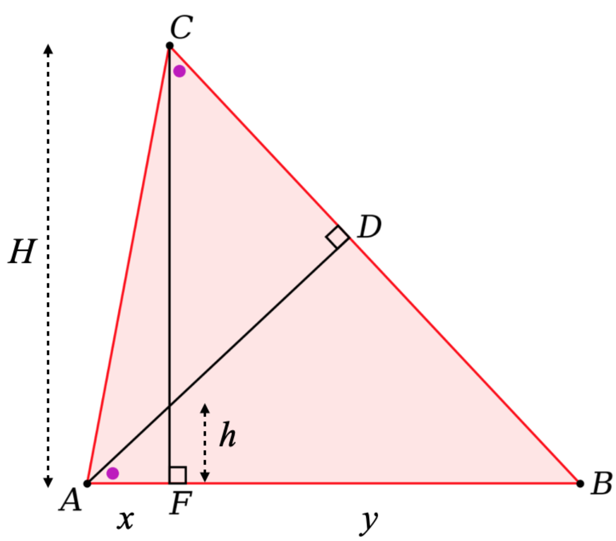
orthocenter: Newton's proof

As we've said, the orthocenter is the point where all three altitudes cross.



orthocenter/altitude

An altitude is a line drawn from any vertex to the opposing side, forming a right angle with the base, thereby dividing the triangle into two right triangles.



Draw the altitude from the vertex C down in $\triangle ABC$ to meet the base in a right angle at F.

The altitude divides the base into lengths x and y. Now draw a second altitude from vertex A to the side opposite at D.

What is the height h above the base where the two lines cross?

The small triangle with sides x and h and the large triangle $\triangle CDA$ are similar, because they are both right triangles with equal vertical angles. This means that the angles marked with magenta dots are equal.

Similar triangles have equal ratios for the corresponding sides. This is a very important theorem which we haven't proved yet, but will soon.

In the small triangle the side opposite the marked angle ($\angle BAC$) has length h, while the entire length of the altitude is H. By similar triangles

$$\frac{h}{x} = \frac{y}{H}$$

(the side opposing the marked angle is in the numerator on both sides). So the height h is

$$h = \frac{xy}{H}$$

The formula is noteworthy because it is symmetrical in x and y and does not contain any term related to side AC, opposite vertex B.

Therefore, if we draw the third altitude to side AC, opposite vertex B, we make exactly the same calculation that it crosses the vertical altitude at the height $h = xy/H$. Therefore, the three altitudes cross at a single point, at height h.

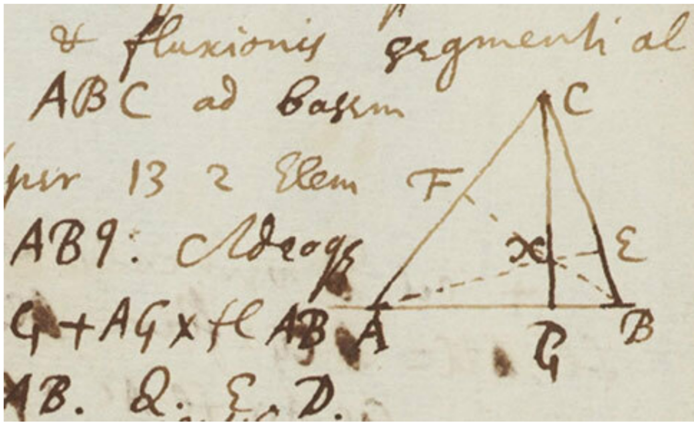


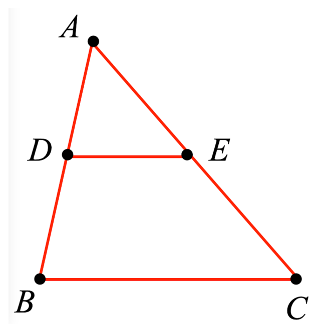
Fig. 129 The diagram for Newton's proof, from his *Geometria curvilinea and Fluxions*, Ms Add. 3963, p54r.

Newton published this proof about 1680.

Chapter 14

Right triangle similarity

In the chapter on congruence, we introduced similar triangles as triangles with the same three angles, but scaled differently.



Let $\triangle ABC \sim \triangle ADE$. ($\sim$ is the symbol for similar). Then they can nestle one inside the other, using any one of the shared angles.

If we do this so that $\angle ADE = \angle ABC$, then by alternate interior angles $DE \parallel BC$.

The other property is the equal proportions of sides.

$$\frac{AD}{BD} = \frac{AE}{CE}$$

equal angles and parallel third side $\iff$ equal ratios of sides.

similar right triangles

We will show an easy proof that for similar right triangles, all angles equal implies equal ratios of sides. Our approach is from Acheson and is equivalent to a previous result (Euclid I.43).

> Draw a rectangle ABCD, and a diagonal AC. Then pick a point E on the diagonal and draw lines through it parallel to the sides.

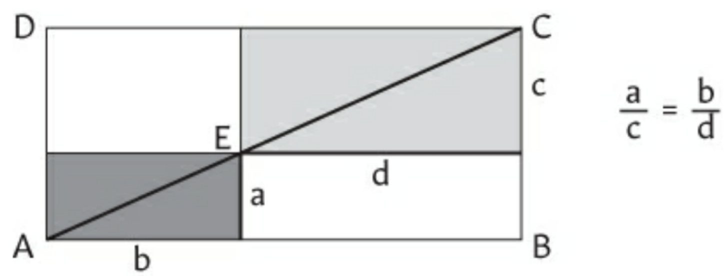


Fig. 42 Area and similarity.

All of the right triangles in the figure are similar. Start with the alternate interior angles theorem, then use complementary angles in a right triangle, and finish with vertical angles.

By changing the height of the figure, we can obtain any ratio a/c that we wish, and by changing the placement of E we can get any ratio a/b that we wish, which amounts to the same thing.

So then, the two shaded rectangles are bisected by the diagonal AEC, by the diagonal theorem. So the two light-gray triangles have equal area, and the two dark gray ones do as well.

But $\triangle ABC$ and $\triangle ADC$ also have equal area.

Therefore, we just subtract equal areas to find that the two unshaded rectangles above and below the diagonal are equal in area. The one on top has area bc and the one below has area ad. We have

$$bc = ad$$
$$\frac{a}{c} = \frac{b}{d}$$

and also

$$\frac{a}{b} = \frac{c}{d}$$

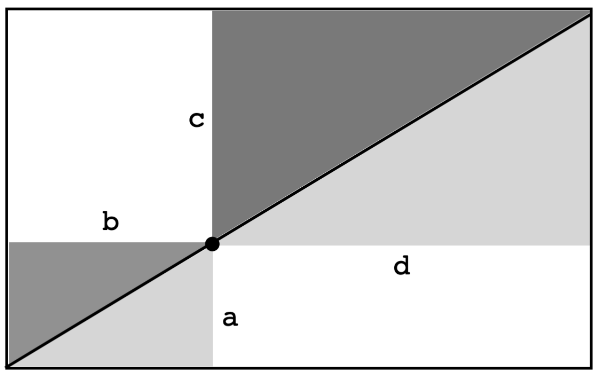
□

Corresponding sides are in the same proportion, but also the ratio of corresponding sides is the same for similar triangles.

similar parts and the whole

We showed that the two smaller triangles have corresponding sides in proportion.

But the large triangle (one-half of the entire rectangle) has the same angles, and should have the same ratios. Here is a simple manipulation to obtain that result:



$$bc = ad$$

$$bc + ab = ad + ab$$

$$b(a + c) = a(b + d)$$

$$\frac{a}{b} = \frac{a + c}{b + d}$$

Given any two of these relationships we can derive the third. This is the same math in reverse.

$$\frac{a + c}{b + d} = \frac{c}{d}$$

$$\frac{a+c}{c} = \frac{b+d}{d}$$
$$\frac{a}{c} + 1 = \frac{b}{d} + 1$$
$$ad = bc$$

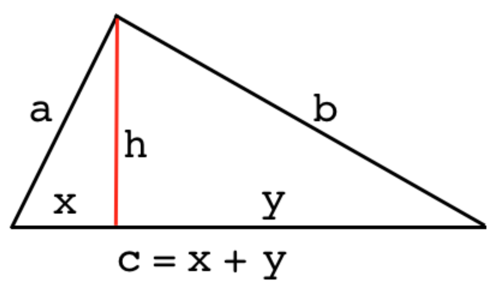
□

hypotenuse in proportion

It is natural to ask, what about the hypotenuse?

Consider any right triangle. Drop the altitude to the hypotenuse.

Using complementary angles, we can show that the two new smaller triangles formed by the altitude are similar to each other and to the original large one.



Now, write the ratio of the long to the short side *for each one* of the three triangles:

$$\frac{h}{x} = \frac{y}{h} = \frac{b}{a} = k$$

From our previous theorem we know this equality is valid.

However the same statement can be viewed in a different way.

b/a is the ratio for the hypotenuse of the medium triangle compared to the small one.

And y/h is the ratio of the long side in the medium triangle to the long side in the small one.

They are equal, and this completes the proof!

□

We can also give an algebraic proof, by looking ahead to the Pythagorean theorem. As you likely know, for a right triangle with sides a and b and hypotenuse g:

$$a^2 + b^2 = g^2$$

We can use the Pythagorean theorem to prove that:

$$\frac{a}{c} = \frac{b}{d} = \frac{g}{h}$$

All of the sides of two similar right triangles have the same ratio.

We must be careful, however. A deep connection exists between similarity, area and the Pythagorean theorem. It is important that we will have Euclid's proof of the Pythagorean theorem, and that proof depends on SAS rather than on similarity.

Equal ratios extends to the hypotenuse.

Proof.

Start with

$$\frac{a}{c} = \frac{b}{d} = k$$
$$a = kc, \qquad b = kd$$
$$a^2 + b^2 = k^2c^2 + k^2d^2$$
$$g^2 = k^2h^2$$

Since these are lengths, we can take the positive square root and obtain

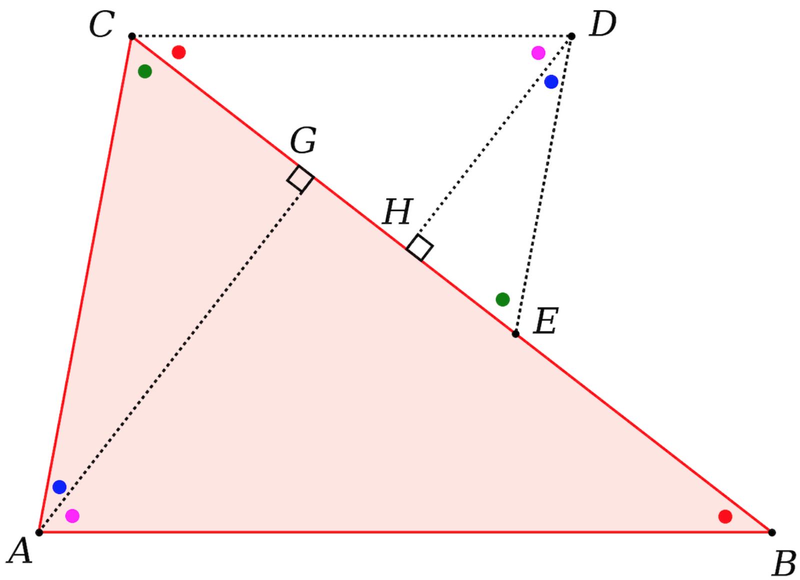
$$\frac{g}{h} = k$$
$$= \frac{a}{c} = \frac{b}{d}$$

□

Thus, AAA similarity is established for right triangles. If either of the smaller angles matches between two right triangles, then they are not only similar but all the side lengths are in the same ratio as well.

Getting the converse, going from equal ratios to equal angles, uses a result about parallelograms. Also, the proofs apply not just to right triangles, but to triangles of any type. For that reason, we defer further development of similarity to a later chapter.

We proved this above for right triangles, and could extend the result to all triangles by dissection. We just show the figure and sketch the proof.



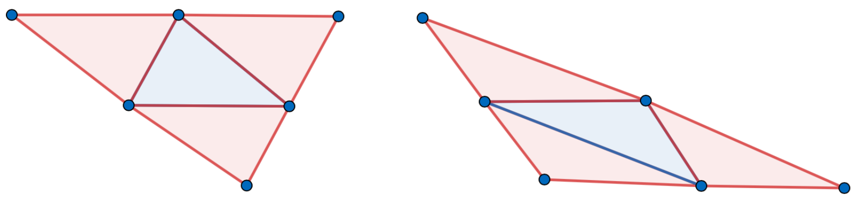
$\triangle ABC$ and $\triangle CDE$ are similar because they have the same angles at all three vertices. It follows that corresponding sides are parallel (e.g. $AB \parallel CD$, $AC \parallel DE$, and also the altitudes $AG \parallel DH$). Equal ratios also comes easily.

We promise to prove this for all triangles later when we deal with the theory of similarity more explicitly. Euclid VI.2 does this elegantly.

We can prove it for a special case now.

triangle dissection

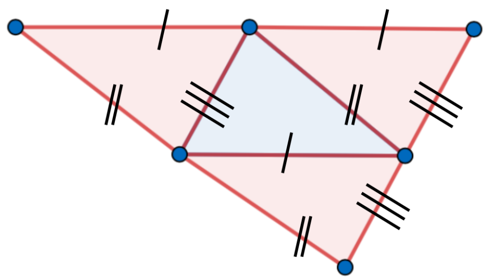
- Any triangle can be dissected into four congruent triangles.



Proof.

Find the midpoints of the sides and connect them.

By the midline theorem each side of the central triangle is one-half the length of the side to which it is parallel.



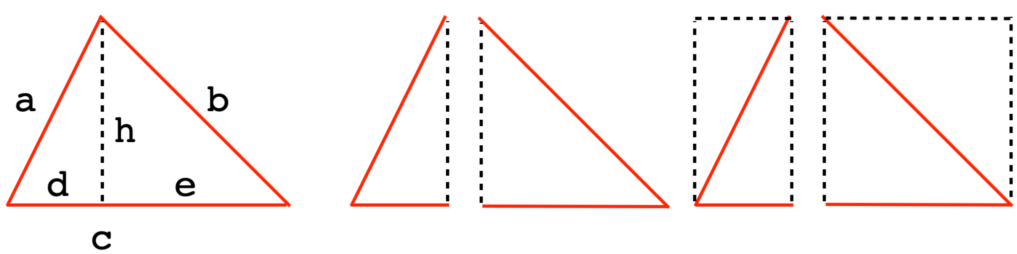
We have all congruent triangles by SSS.

Also, as a result we have 3 parallelograms, each containing two congruent triangles.

□

dissection into right triangles

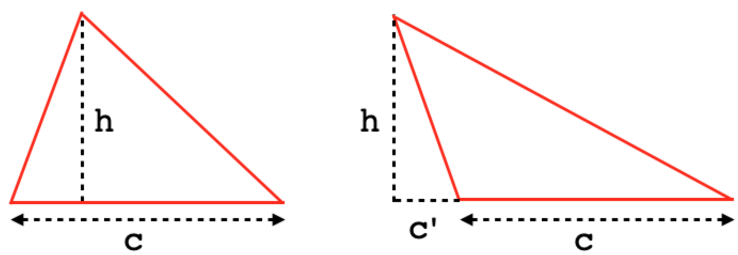
Any triangle can be cut into two right triangles by drawing its *altitude*, h, where h is perpendicular to c.



The area of the triangle with sides a, d, h is $dh/2$, and that with sides a, e, h is $eh/2$ so the area of the original triangle is

$$\frac{dh}{2} + \frac{eh}{2} = \frac{(d+e)h}{2} = \frac{ch}{2}$$

This formula is correct even for an obtuse triangle like the one in the right panel, below. The area of the two red triangles is the same: $ch/2$.



We get that by computing the area of the large triangle with base $c + c'$ and then subtracting the area of the skinny triangle with the base c':

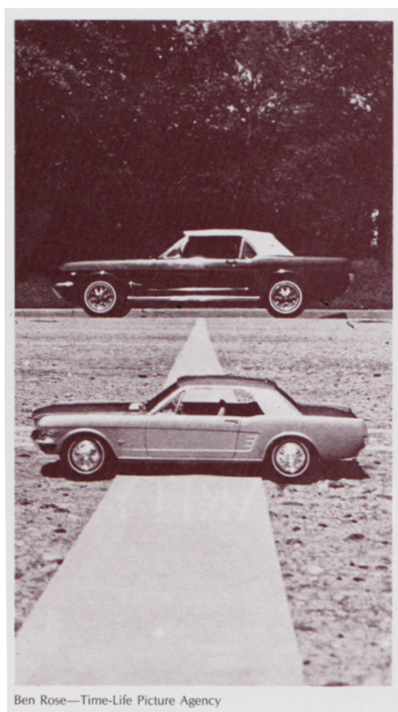
$$\mathcal{A} = \frac{h(c+c') - h(c')}{2} = \frac{hc}{2}$$

Of course, for an obtuse triangle we could choose one of the other sides as the base and proceed in the usual way, but this works as well.

Chapter 15

Similar triangles

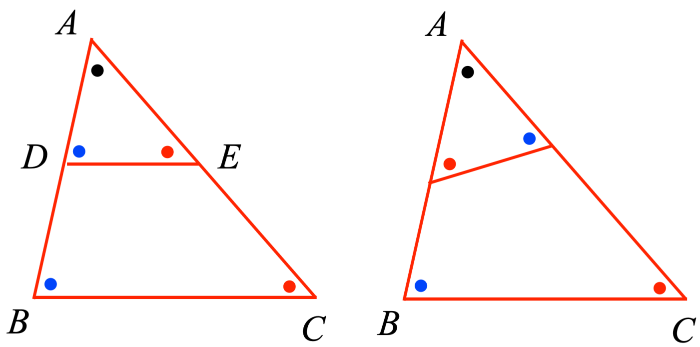
Triangles that are alike, but not congruent, because they are scaled differently, are called similar. We write $\triangle ABC \sim \triangle PQR$.



— Figure from Jacobs, chapter 10

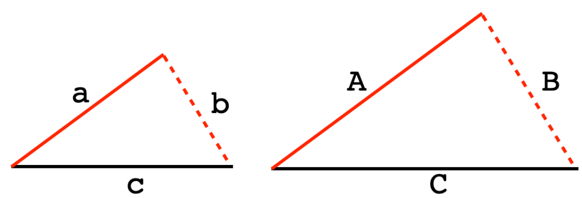
Statements about two similar triangles.

- They have at least two (thus, three) angles known to be equal.
- When superimposed using a shared angle, the third pair of sides, that do not coincide with each other, are nevertheless parallel (you may need the mirror image for one triangle).



- They have corresponding pairs of sides in the same proportions, but scaled by a constant factor.

If any one of these properties hold, they all do.



From the above diagram of two similar triangles, similarity implies that (for example)

$$\frac{A}{a} = \frac{B}{b}$$

For any pair of similar triangles, there is a constant k such that

$$k = \frac{A}{a} = \frac{B}{b} = \frac{C}{c}$$

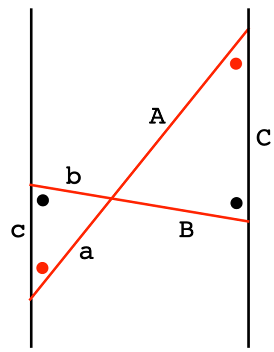
A slight rearrangement gives:

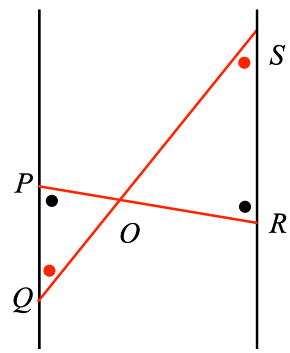
$$\frac{a}{b} = \frac{A}{B}$$

These ratios are obviously different.

As with congruent triangles, our definitions allow one triangle to be flipped before the comparison is made (comparing two triangles, originally the angles were in opposite order, one clockwise and the other counter-clockwise).

In the figures below, the vertical black lines are parallel. Any two lines connecting them that cross, form two similar triangles. The angles marked with dots of the same color are equal by alternate interior angles. We also have vertical angles.



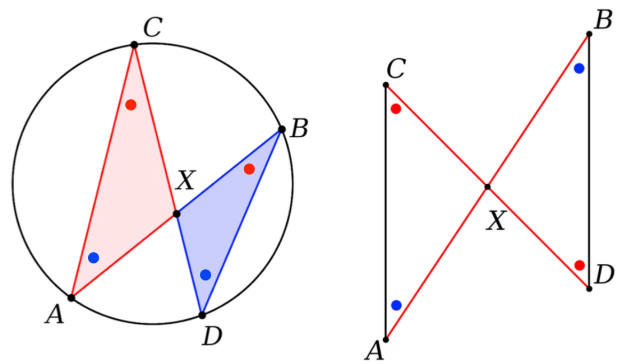


Set up the ratios carefully, by finding sides that lie opposite equal angles.

$$\begin{array}{ccc} a & b & c \\ A & B & C \end{array} \qquad\qquad \begin{array}{rccc} \triangle OPQ = & OP & PQ & QO \\ \triangle ORS = & OR & RS & SO \end{array}$$

For triangles with labeled vertices, I like to do this by naming the triangles in the corresponding order. We start $\triangle OPQ$ with OP opposite the red dot, moving counter-clockwise. Do the same with $\triangle ORS$.

Later we will have a theorem about similar triangles formed by crossed chords in a circle. In the figure below, the angles marked with a red dot are equal, as are the vertical angles, so the two triangles are similar. In this case we follow equal angles (and sides) around in opposite directions, comparing the two triangles.

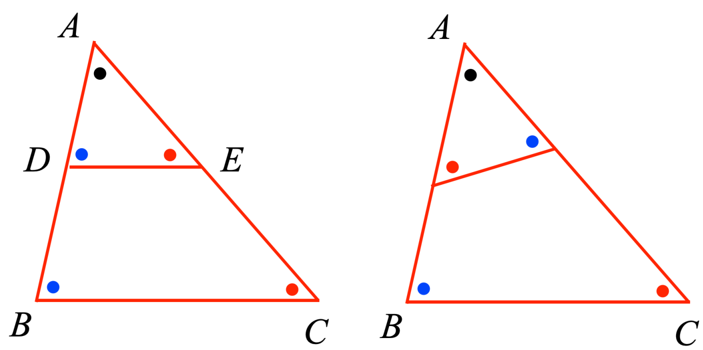


AAA is probably the most common way to establish that two triangles are similar.

similarity and ratios

Similar triangles are defined to have three angles equal (equiangular), but they are scaled differently and so, not congruent.

If two equiangular triangles are superimposed at one vertex ($\angle A$), so the adjacent sides coincide, and angles lie in the same order (left panel), then the sides opposite $\angle A$ are parallel ($DE \parallel BC$). In the right panel, the angles are in opposite order and this doesn't work.



This can be demonstrated by employing the parallel postulate, or as Euclid does in VI.2, by an argument based on area and the properties of two parallel lines. Actually, Euclid says to take a triangle and draw the line segment DE either parallel to the base or with the given angles, but the result is the same.

All angles equal and the third side parallel are closely related and easily proven in both directions.

Equal angles $\iff$ parallel sides.

We will call this property similarity: all angles equal and the third side parallel.

Another property is that similar triangles have their sides in the same proportion. This comes in two flavors: either two sides in equal proportion flanking an equal angle, often called SAS similarity, or all three sides in the same proportion with no prior knowledge about angles.

We now explore all of these situations:

- $\parallel$ sides and equal $\angle \iff$ equal ratios

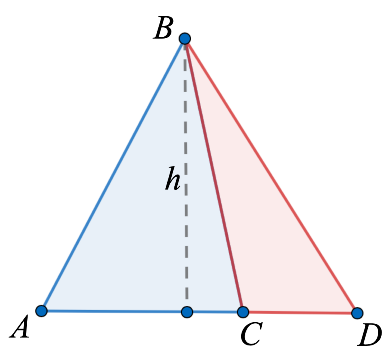
We will call these the *forward* ratio theorem, and the *converse* ratio theorem, even though there are two cases, either SAS or ratios only.

parallel third side implies equal ratios

As background, we recall two fundamental ideas about triangles.

The first is that if two triangles have their bases on the same line, and they also share the same vertex opposite, then they have the same altitude.

It follows that the areas are in the same proportion as the lengths of the bases. This is the **area-ratio theorem**.



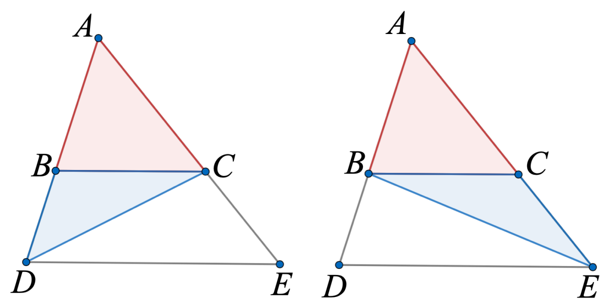
$$2\mathcal{A}_{\triangle BAC} = h \cdot AC$$
$$2\mathcal{A}_{\triangle BCD} = h \cdot CD$$
$$\frac{\mathcal{A}_{\triangle BAC}}{\mathcal{A}_{\triangle BCD}} = \frac{AC}{CD}$$

There is only one line that can be drawn from a point vertically to a straight line. So there is only one altitude that can be drawn from a point to a base. Since the triangles share the vertex point, they have the same altitude.

The second idea is that if two triangles share the same base, and the two opposing vertices both lie along a lie that is parallel to the base, then the triangles have the same area.

Euclid VI.2

- $\parallel$ sides and equal $\angle \to$ equal ratios



Proof.

Start with $\triangle ADE$ and then draw $BC \parallel DE$.

We will show that $AB : BD = AC : CE$.

The key to Euclid's proof of this theorem is to observe that the two triangles shaded blue, $\triangle DBC$ and $\triangle ECB$, have the same area.

The reason is that they lie on the same base, and their vertex (either D or E) lies along the same line parallel to that base. Hence the result follows.

But by the area-ratio theorem

$$\frac{\mathcal{A}_{\triangle ABC}}{\mathcal{A}_{\triangle DBC}} = \frac{AB}{BD}$$

and

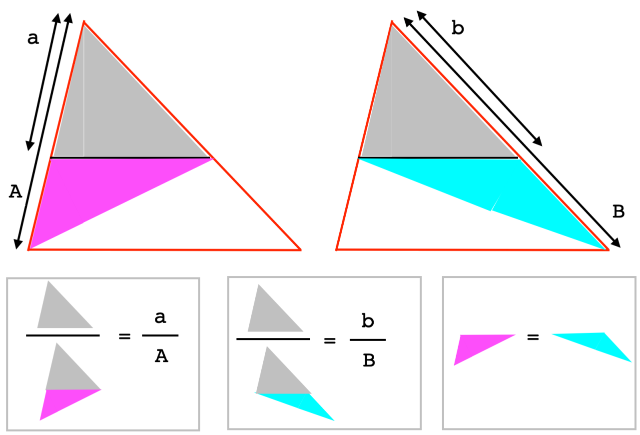
$$\frac{\mathcal{A}_{\triangle ABC}}{\mathcal{A}_{\triangle ECB}} = \frac{AC}{CE}$$

Since the left-hand sides of the two expressions are equal, so are the right-hand sides. Namely

$$\frac{AB}{BD} = \frac{AC}{CE}$$

□

Here is a different pictorial view of the argument.



Therefore

$$\frac{a}{A} = \frac{b}{B}$$

Equal angles (parallel third side) implies equal ratios.

□

This result applies to any vertex of the two similar triangles and its adjacent sides, hence it applies to all three sides of the two triangles.

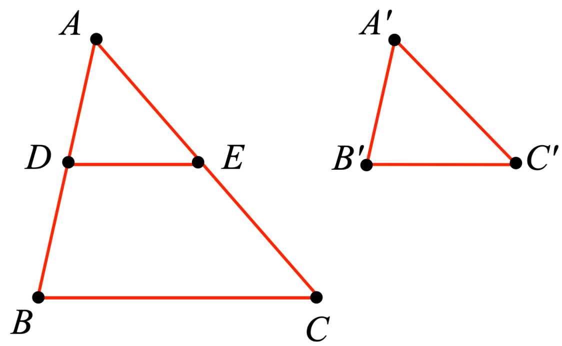
converse theorem

Euclid proves the converse theorem by using the same steps in reverse. Start with equal areas, form equal ratios of them, and then use the equal altitude to show that $DE \parallel BC$.

A somewhat different proof of the converse theorem, that equal ratios implies equal angles, is given by Kiselev.

Proof.

Let there be two triangles $\triangle ABC \sim \triangle A'B'C'$ with three pairs of sides in the same ratio. Let $\triangle ABC$ be the larger one.



Mark off inside $\triangle ABC$, for example, $AD = A'B'$ and then draw $DE \parallel BC$. By the forward theorem, $\triangle ABC \sim \triangle ADE$.

We can form the dual equality:

$$\frac{A'B'}{B'C'} = \frac{AB}{BC} = \frac{AD}{DE}$$

since we are given the first (equal ratios), and have the second from similar triangles, $\triangle ABC \sim \triangle ADE$.

Equating the first and third terms:

$$\frac{A'B'}{B'C'} = \frac{AD}{DE}$$

But we also have $A'B' = AD$, by construction. Cancel to obtain $B'C' = DE$.

The same can also be done for the third side. Thus, $\triangle ADE \cong \triangle A'B'C'$ by SSS.

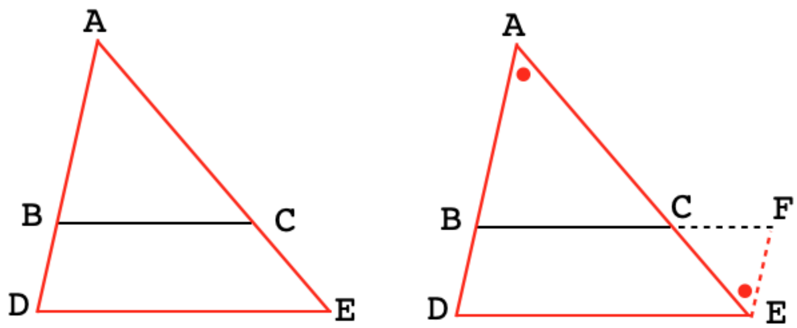
It follows that $\triangle A'B'C'$ is equiangular with $\triangle ADE$ and thus with $\triangle ABC$.

$\square$

SAS for similar triangles

Now, we mix and match the conditions, taking two sides in proportion and one angle shared. This is often called SAS similarity.

We will prove that this mix of conditions is enough for two triangles to be similar.



Proof.

Given two triangles with the same vertex angle at A and two of three sides known to be in the same proportion ($BD/AB = CE/AC = k$).

Draw EF parallel to ABD and extend BC to meet it at F.

$\triangle ABC \sim \triangle CEF$ by vertical angles plus alternate interior angles (red dots). The corresponding sides opposite the vertical angles are EF and AB.

By the forward theorem

$$\frac{EF}{AB} = k = \frac{CE}{AC}$$

but

$$\frac{CE}{AC} = \frac{BD}{AB}$$

Therefore, $BD = EF$. We are given $BD \parallel EF$. $BDEF$ has one pair of opposing sides equal and parallel, so it is a parallelogram.

It follows that $BC \parallel DE$ and $BF = DE$.

It also follows that $\angle D = \angle ABC$ by alternate interior angles, so we have two angles equal which means that $\triangle ABC \sim \triangle ADE$.

The forward theorem then gives $(DE - BC)/BC = BD/AB = k$

□

Each of the standard congruence theorems (yes, even SSA) has a similarity version. We won't prove the others.

problem

The figure below is from Acheson's wonderful book aptly titled *The Wonder Book of Geometry*. He shows this problem, which he says "[goes] back to at least AD 850, when it appeared in a textbook by the Indian mathematician Mahavira."

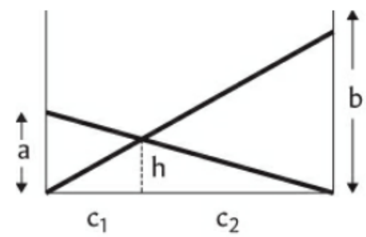


Fig. 40 A problem with ladders.

Looking down an alleyway, you see two ladders arranged as shown and wonder about the point where they cross, at height h and distances c_1 and c_2 from the edges of the alley, where the width of the alley is $c = c_1 + c_2$.

By similar triangles

$$\frac{c_1}{h} = \frac{c}{b}$$

Can you see why?

Going the opposite direction

$$\frac{c_2}{h} = \frac{c}{a}$$

Adding the two equations and substituting for $c_1 + c_2$:

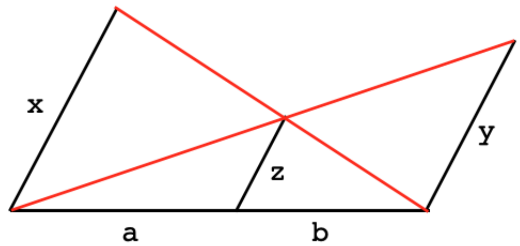
$$\frac{c}{h} = \frac{c}{a} + \frac{c}{b}$$

Thus

$$\frac{1}{h} = \frac{1}{a} + \frac{1}{b}$$

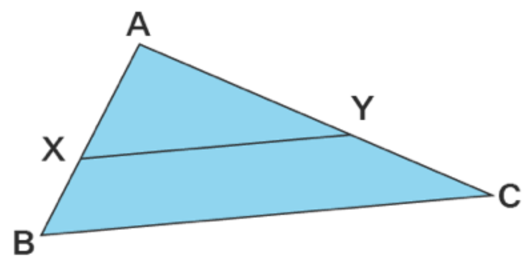
That's a simple and interesting result. h depends only on a and b and not on c, c_1, or c_2.

If you think about it, you should see that in the previous problem we never used the information that the sides and the height were vertical, only parallel.



Work through the same proof for the figure above to show that $1/x + 1/y = 1/z$.

problem



It is given that XY || BC and divides the triangle into two parts of equal areas. Find the ratio AX: XB using the area of similar triangles theorem.

The problem states that the bases are parallel ($XY \parallel BC$) and also that the subdivision produces *equal areas*. We are asked to find the ratio $AX : XB$.

Solution.

Recall from a previous chapter that for two similar triangles, the altitudes to corresponding sides are in the same ratio as any of the three pairs of sides themselves.

The altitudes h and H to BC (not drawn) are also in the same ratio as the sides, so let $H = kh$ — H lies on BC.

Then twice the area of the top triangle is $AX \cdot h$ and twice the area of the whole is $AB \cdot H = kAB \cdot h$

We have that the whole is twice the smaller area so the ratio is equal to 2:

$$\frac{kAB \cdot h}{AX \cdot h} = 2$$

But AB/AX is also equal to k so we have that $k^2 = 2$ and $k = \sqrt{2}$.

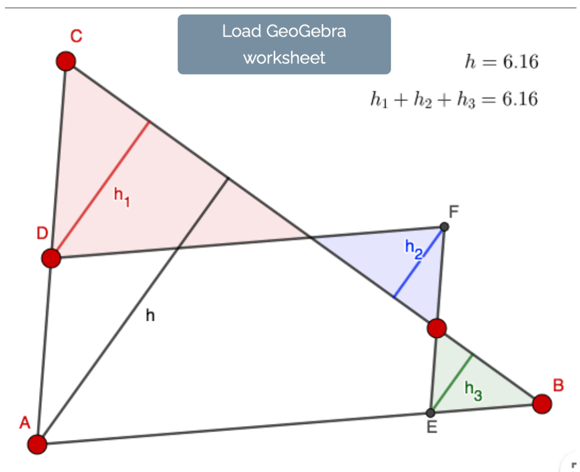
However, we are not asked for k. Instead, we want the ratio of AX to the smaller piece along the bottom. Let $AX = a$. Then the whole is $A = ka$. The difference is the small piece, $XB = A - a = a(k-1)$. We need

$$\frac{a}{a(k-1)} = \frac{1}{k-1} = \frac{1}{\sqrt{2}-1}$$

problem

Here's a problem from the web. Given that $AC \parallel EF$ and $AB \parallel DF$.

We are to prove that the sum of the altitudes of the small triangles is equal to the altitude of the large one.



Informal solution: The smaller triangles are all similar to $\triangle ABC$ by the alternate interior angles and vertical angle theorems.

For similar triangles, not only are the sides in the same ratio to each other, but so are other measures like the altitudes to a particular side. So if we label the bases b_1 etc., collectively b_i , then we have that

$$\frac{b_i}{h_i} = \frac{b}{h}$$

$$b_i = b \cdot h_i/h$$

for each of the b_i.

But the sum of the b_i is simply equal to b so

$$b_1 + b_2 + b_3 = b \cdot (h_1/h + h_2/h + h_3/h)$$

$$b = b \cdot (h_1/h + h_2/h + h_3/h)$$

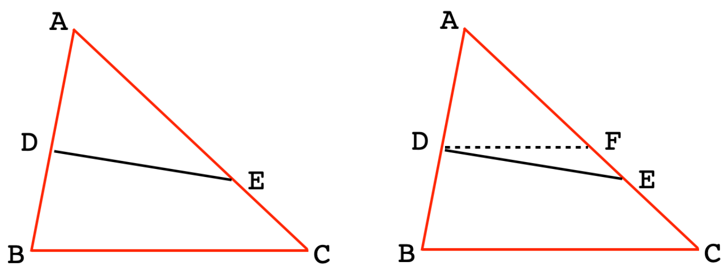
$$1 = h_1/h + h_2/h + h_3/h$$

$$h = h_1 + h_2 + h_3$$

problem

Given two triangles that are not similar, where the ratio $AD/AB = r$ and $AE/AC = s$. We are asked to show that

$$\frac{\triangle_{ADE}}{\triangle_{ABC}} = rs$$



Solution.

Draw $DF \parallel BC$. Now $\triangle ADF \sim \triangle ABC$ so $AF/AC = r$.

By our previous result

$$\frac{\triangle_{ADF}}{\triangle_{ABC}} = r^2$$

Since $AE/AC = s$ and $AF/AC = r$, $AE/AF = s/r$. As triangles with a common vertex and bases in that proportion, these areas are in the same proportion:

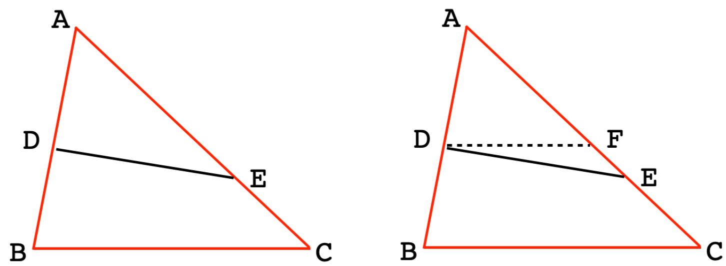
$$\frac{\triangle_{ADE}}{\triangle_{ADF}} = \frac{s}{r}$$

Multiply the two ratios together to obtain

$$\frac{\triangle_{ADE}}{\triangle_{ABC}} = sr$$

□

Solution. (alternate).



Looking ahead to trigonometry, in any triangle, twice the area can be computed as the product of two sides flanking an angle times the *sine* of the angle. (The reason is that the altitude to one side, divided by the length of the second side, is defined to be the sine of the angle.)

$$2(ADE) = AD \cdot AE \cdot \sin \angle DAE$$
$$2(ABC) = AB \cdot AC \cdot \sin \angle BAC$$

But $\angle DAE = \angle BAC$, so they have the same sine, and the ratio of areas is just

$$\frac{AD \cdot AE}{AB \cdot AC} = rs$$

□

To rework this proof in terms of familiar concepts, draw the altitude from D to AE, and also the one from B to AC

They form similar right triangles including the angle at A. The altitudes scale like $AD/AB = r$. But the bases scale like $AE/AC = s$.

And area scales like the product of the two, namely, rs.

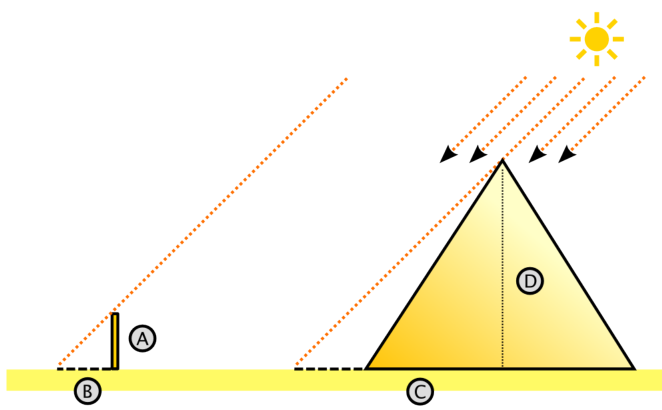
pyramid height

As we said earlier, Thales was from Miletus and he lived around 600 BC. Thales is believed to have traveled extensively and was likely of Phoenician heritage. As you probably know, the Phoenicians were famous sailors who founded many settlements around the Mediterranean.

They competed with the mainland Greeks and later with the Romans for colonies, and their major city, Carthage, was destroyed much later by the Romans, in the third Punic War. Hannibal rode his famous elephants over the Alps in the second Punic war.

During his travels, Thales went to Egypt, home to the great pyramids at Giza, which were already ancient then. They had been built about 2560 BC (dated by reference to Egyptian kings) and were already 2000 years old at that time!

The story is that Thales asked the Egyptian priests about the height of the Great Pyramid of Cheops, and they would not tell him. So he set about measuring it himself. He used similar triangles. I'm sure he wrote down his answer, but I'm not aware that it survives. The height is about 480 feet, which made it the tallest structure in the world until construction of Lincoln Cathedral, finished early in the 14th century. The Washington Monument is taller by about 75 feet.



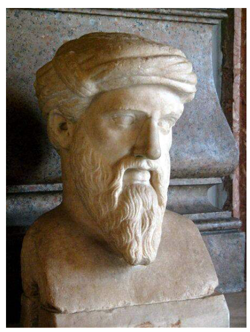
Part IV

Pythagorean theorem

Chapter 16

Simple proofs

The most famous theorem of Greek geometry is also without a doubt the most useful in calculus.



Like many Greek mathematicians, Pythagoras (c.570 - c.490 BCE) was not from Greece "proper" (i.e. the territory of mainland Greece). Instead, he was from one of the islands. Pythagoras was born on the island of Samos, which lies off the coast of what was called Asia Minor (modern Turkey).

Pythagoras was much younger than Thales but may have encountered him as a youth, since Thales lived on the mainland not too far from Samos, at Miletus. Later, Pythagoras moved to a Greek colony in southern Italy, at Croton.

During his lifetime, Pythagoras was known as a philosopher much more than as a mathematician. For example, he was famous as an expert on the fate of the soul after death.

https://plato.stanford.edu/entries/pythagoras/

Wilczek cites Bertrand Russell on Pythagoras:

> "A combination of Einstein and Mary Baker Eddy."

which will be funny if you look up Mary Baker Eddy.

Pythagoras founded a "school" and it is not sure now which of the theorems developed by this school are due to Pythagoras, and which to his disciples. It is not even clear whether the Pythagorean theorem, as we understand it today, was known to Pythagoras. There is no contemporaneous account (say, Plato or Aristotle) connecting Pythagoras with the theorem. Of course, nearly all of the histories that were written are lost.

Regardless of whose idea it was, and who could prove it first, it's clear that they knew something. The Pythagorean theorem says that if c is the hypotenuse of any right triangle and a and b are the side lengths then

$$a^2 + b^2 = c^2$$

The simplest example in integers is

$$3^2 + 4^2 = 9 + 16 = 25 = 5^2$$

but there are many triplets of integers with this property, for example

$$5^2 + 12^2 = 25 + 144 = 169 = 13^2$$

Not just the classic 3-4-5 right triangle, but a number of other *Pythagorean triples* had been known for a thousand years.

The tablet "Plimpton 322" contains (by extrapolation) the triplet 4601-4800-6649 and it dates to about 1800 BCE. Maor analyzes this in his book on the theorem of Pythagoras.

Line 3 of the tablet contains the numbers $1, 16, 41$ and $1, 50, 49$ which are in base 60 notation. Thus $1 \cdot 3600 + 16 \cdot 16 + 41 = 4601$ and $1 \cdot 3600 + 50 \cdot 60 + 49 = 6649$. The third number of the triple is missing but it's obvious since $6649^2 - 4601^2 = 23040000$, which is 4800^2.

It seems highly unlikely that these were found by searching randomly among squares.

There is even a triple with 5 decimal digits on the next line of the Plimpton 322 tablet.

One should not think that the theorem *only* applies to triangles with integer side lengths. For example, it applies to the isosceles right triangle with side length equal to 1, whose hypotenuse is the real number $\sqrt{2}$. Also, any integer solution can be modified. For example

$$0.5^2 + 1.2^2 = 0.25 + 1.44 = 1.69 = 1.3^2$$

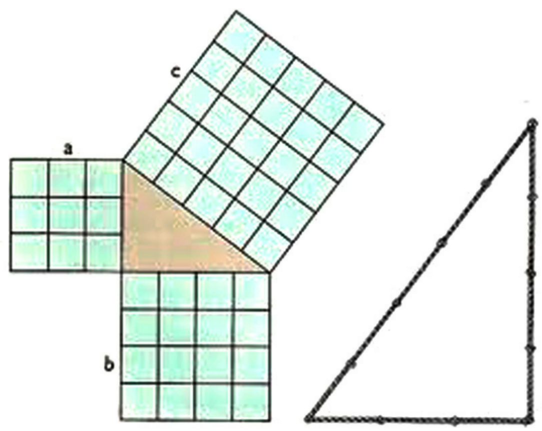
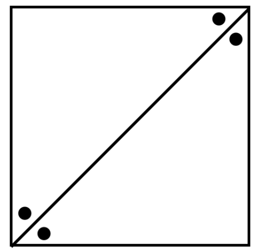


FIGURE 1. THE 3-4-5 RIGHT TRIANGLE, A SIMPLE CASE OF PYTHAGORAS'S THEOREM.

We're going to spend time with a few particular proofs of this theorem (there are literally hundreds of them), so as to examine the ideas from different perspectives. This chapter is an introduction that shows some basic algebraic proofs.

The proof due to the Greeks presented in the next chapter is from Euclid. It employs SAS for triangle congruence, and is probably his own contribution. Later we'll explore how the concept of area is connected to the Pythagorean theorem.

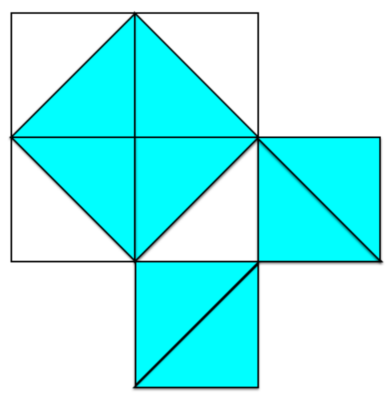
To begin, let's examine a special case, easily proved, for an isosceles right triangle.



We can obtain such a triangle by drawing the diagonal in a square. The black dotted angles in any one triangle are equal, by the isosceles triangle theorem, and the others are equal by alternate interior angles.

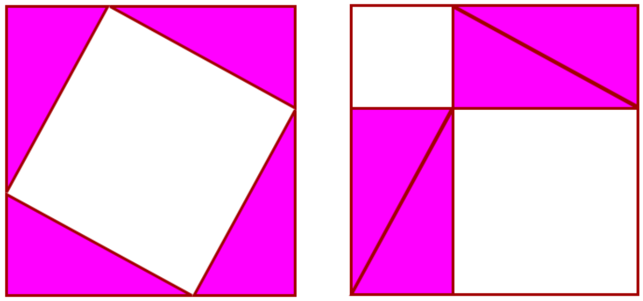
Our hypothesis is that when the lengths of the sides of the triangles are squared and then summed, they will be equal to the square of the diagonal. That is to say, the square of the diagonal should be equal to two of the squares shown above.

Here is a proof without words.



Adding some words: the area of the square on the hypotenuse is equal to one-half of the area of four squares of the sides. This is a proof for the special case where the sides are equal.

The following general proof is sometimes called the "Chinese proof." I can easily imagine proceeding from the figure above to the left panel below by simply rotating the inner square and collapsing the surrounding one.



It really needs no explanation, but ..

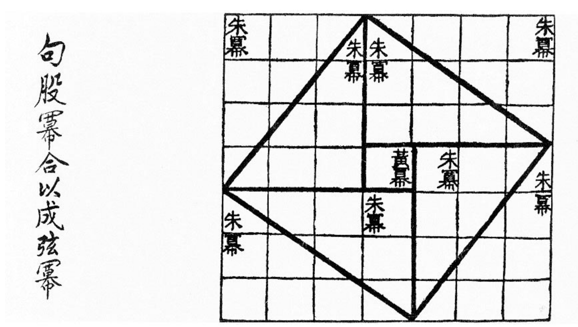
We have a large square box that contains within it a white square, whose side is also the hypotenuse of the four identical right triangles contained inside. Altogether the four triangles plus the white area add up to the total.

We simply rearrange the triangles. Now we evidently have the same area left over from the four triangles, because they still have the same area and the surrounding box has not changed.

But clearly, now the white area is the sum of the squares on the second and third sides of the triangles. Hence the two white squares on the right are equal in area to the large white square on the left.

□

This diagram is contained in the Chinese text Zhoubi Suanjing.



Eight right triangles are formed with sides of 3 and 4 units. There are 7 units on each side of the large square so the total area is 49. Each pair of triangles has area 12 $(3 \cdot 4)$, so the square in the center really is a unit square with area $49 - 4 \cdot 12 = 1$.

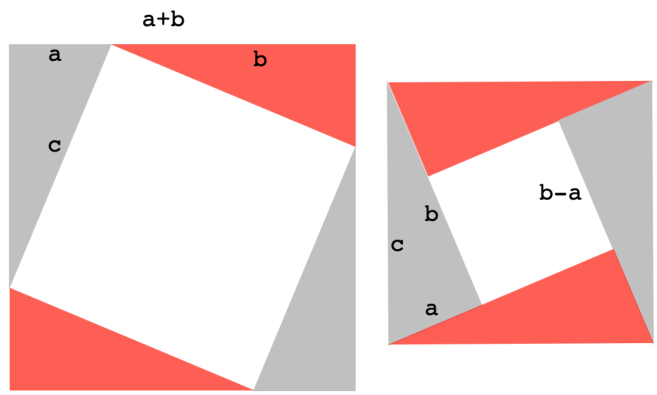
So then, the central square consists of four triangles with total area 24 plus the unit square for 25. This is the square of the long side, namely 5.

https://en.wikipedia.org/wiki/Zhoubi_Suanjing

There is debate about whether this is really a proof, or if it simply presents the arithmetic for the example of a 3-4-5 right triangle.

simple proofs

Many proofs of the theorem are algebraic. Here are two:



Proof.

In the left panel, we have the same arrangement as before, with four identical right triangles. The white square at the center has sides of length c and angles that are right angles because when summed to two complementary angles, the result is two right angles.

The algebra is

$$(a+b)^2 = 4 \cdot \frac{1}{2} \cdot ab + c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

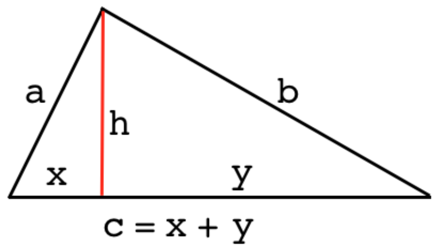
Subtract $2ab$ from both sides, and we're done.

□

I will leave the right panel to you.

similar triangles

Here is a simple classic that depends on the ratios formed for similar right triangles:



Proof

We know that when an altitude is drawn in a right triangle, the two resulting right triangles are similar, by complementary angles. Similarity means that we have equal ratios of sides. Here are two sets:

ratio of hypotenuse to short side

$$\frac{a}{x} = \frac{b}{h} = \frac{c}{a}$$

ratio of hypotenuse to long side

$$\frac{a}{h} = \frac{b}{y} = \frac{c}{b}$$

From the first

$$a^2 = cx$$

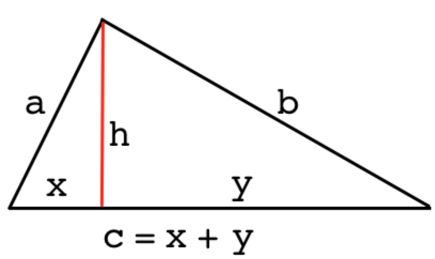
And from the second

$$b^2 = cy$$

Add them:

$$a^2 + b^2 = cx + cy$$
$$= c(x + y) = c^2$$

□



Another relationship from similar triangles is long side to short side:

$$\frac{h}{x} = \frac{y}{h}$$

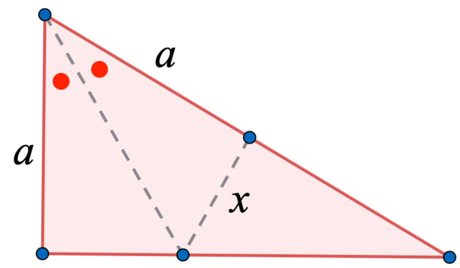
$$h^2 = xy$$

$$h = \sqrt{xy}$$

h is the *geometric mean* of x and y.

Another proof (from Dunham's Problems) also relies on similarity.

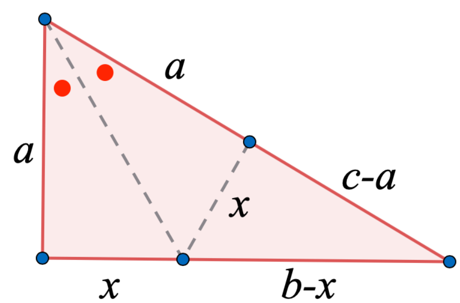
Proof.



In a right triangle, bisect one of the smaller angles, and mark off the same length as the adjacent side a on the hypotenuse. Draw the side x from where the bisector meets side b.

It is easy to see that the two triangles containing the angle bisectors are congruent by SAS.

Therefore, we have that x makes a right angle where it cuts the hypotenuse. Fill in the other sides:



We have that the right triangle with side x is similar to the original. From the corresponding sides we obtain:

$$\frac{x}{a} = \frac{c-a}{b} = \frac{b-x}{c}$$

From the first and second terms:

$$bx = a(c-a) = ac - a^2$$

From the second and third terms:

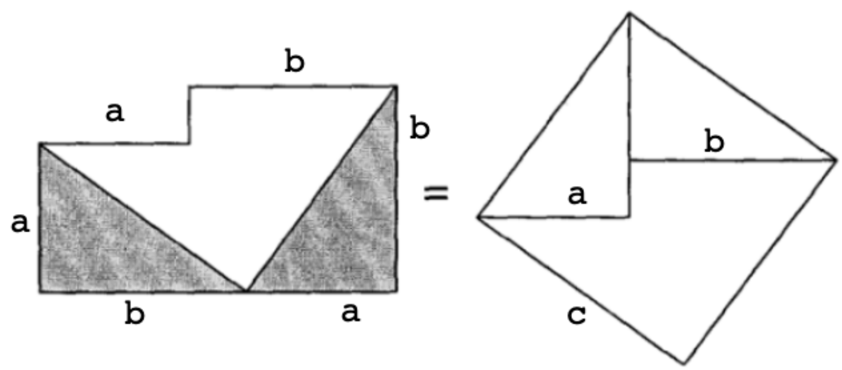
$$c^2 - ac = b^2 - bx$$

Substitute for bx:

$$c^2 - ac = b^2 - ac + a^2$$
$$c^2 = a^2 + b^2$$

□

This is a proof from Gelfand and Saul's trigonometry book.

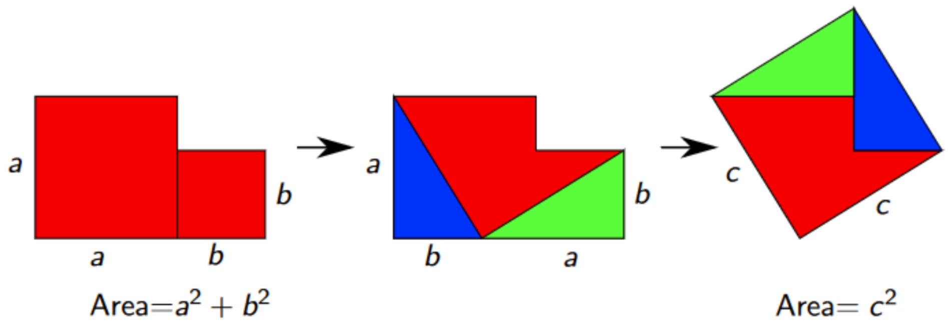


I have added the labels for side length.

If it's not obvious, note that we start with two adjacent squares of area $a^2 + b^2$. Two triangles (shaded) are cut off and re-arranged to make a shape whose area is c^2.

□

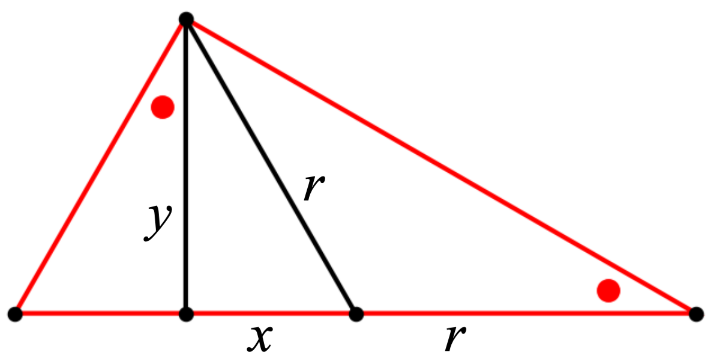
Here is a colored version of the same proof I found on the web.



https://t.co/xTnARHQNWw

proof without words

This one is from Nelsen's *Proof without words*. It depends on Thales' theorem, and also on similarity in right triangles.



The horizontal line is a diagonal of the circle, while the top vertex is on the circle, and the line labeled y is vertical.

Proof.

The two angles marked with red dots are equal because they are both complementary to the angle at the extreme left.

r is the radius. The larger right triangle (with the red dot) has sides y and $r + x$, while the small right triangle on the left (with the red dot) has sides $r - x$ and y.

By similar triangles:

$$\frac{y}{r+x} = \frac{r-x}{y}$$
$$y^2 = r^2 - x^2$$
$$x^2 + y^2 = r^2$$

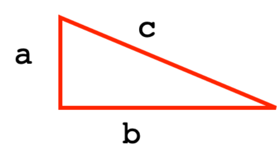
But the small central triangle is also right, with sides x, y and hypotenuse r, and this is the relationship we seek.

□

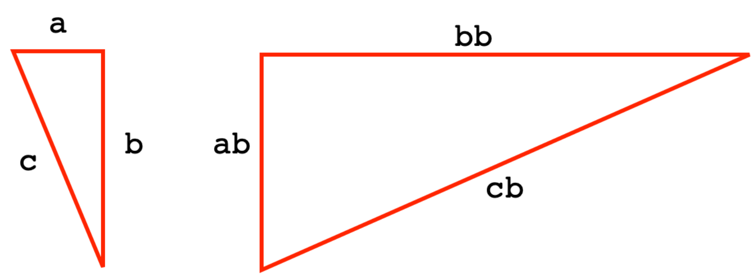
scaled triangle proof of Pythagoras

Proof.

Draw a right triangle and label sides a, b and c.



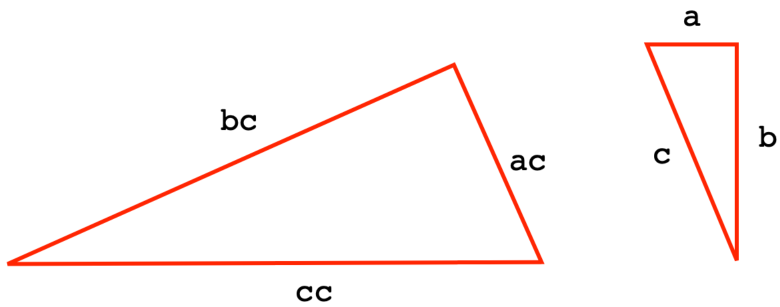
Now, flip and rotate the triangle. Make a copy of that one and rotate the copy so that the shortest side is to the left. Enlarge the copy until the two adjacent sides are equal in length.



These are similar triangles so the angles are all equal, but the sides are proportional, multiplied by a common factor, which here is b.

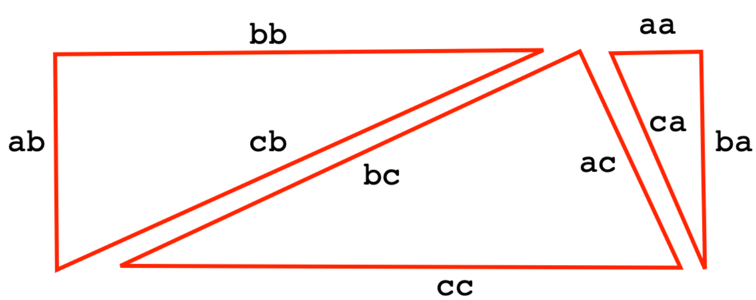
The adjacent sides show that $ab = b$, so this choice of scaling defines $a = 1$. And since $a = 1$, we can also view each side of the original triangle as being scaled by a factor of a.

We just tried scaling by a factor of b and then by a, so that naturally suggests we try scaling by a factor of c.



All sides of the enlarged triangle have been multiplied by the same factor, namely, c. Notice that $c = ac$, as required.

We're nearly done. Put the three triangles together.



The lower left and right corners are places where two vertices come together. By complementary angles, these sum to be right angles.

Since there are two other right angles at the vertices of the quadrilateral, it is a rectangle. Therefore the place where three vertices come together is a straight line, which we can also verify because there are two complementary angles plus a right angle.

Since the composite figure is a rectangle, opposing sides are equal, so $a^2 + b^2 = c^2$.

$\square$

problem

The Russian mathematician V.I. Arnold wrote a famous small book of "problems for children from 5 to 15." Here is no. 6:

> The hypotenuse of a right-angled triangle (in a standard American examination) is 10 inches, the altitude dropped onto it is 6 inches. Find the area of the triangle.
>
> American school students had been coping successfully with this problem over a decade. But then Russian school students arrived from Moscow, and none of them was able to solve it as had their American peers (giving 30 square inches as the answer). Why?

We leave this one as a challenge.

Hint 1: it's a joke.

Hint 2: For two sides of a fixed total length, the isosceles right triangle has the largest area (why?). Given the hypotenuse, what are the side lengths?

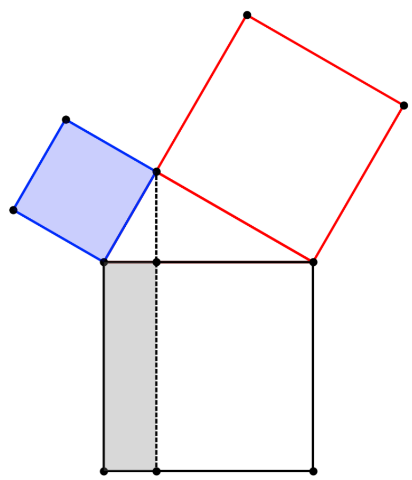
Now use this fact to find the maximum area in terms of the sides, and compare that with the data we are given.

There are a number of other proofs of the Pythagorean theorem later in the book. See the last chapter **here** for links.

Chapter 17

Euclid's proof of Pythagoras

My favorite proof of the Pythagorean theorem relies on the construction below, sometimes called the "bridal chair" or the "windmill", where the central triangle is a right triangle, and the other figures are squares (Euclid I.47).

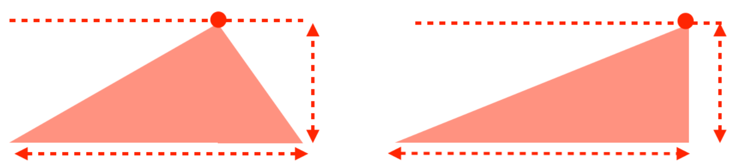


We have the squares on three sides of a right triangle.

What we will show is that the rectangular area which is part of the large square, in gray, is equal in area to the entire small square, in blue.

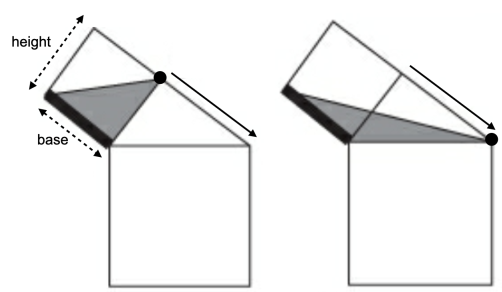
preliminary

We begin by restating a fundamental idea about the area of triangles, which is that, if two triangles have the same base and the same height, they have the same area. So if we imagine sliding the top vertex of a triangle along a line parallel to the base, the area will not change.



Proof.

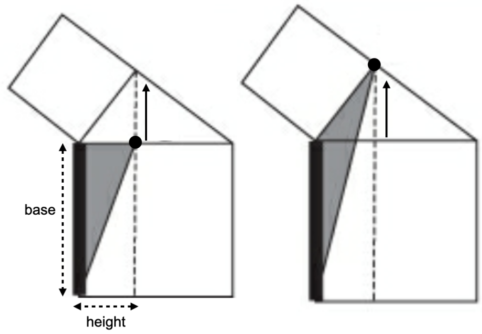
The next figure shows this principle as it comes up in the proof. The gray triangle with the black base is one-half the area of the small square.



Slide the vertex down to the right and the resulting triangle will still have the same area.

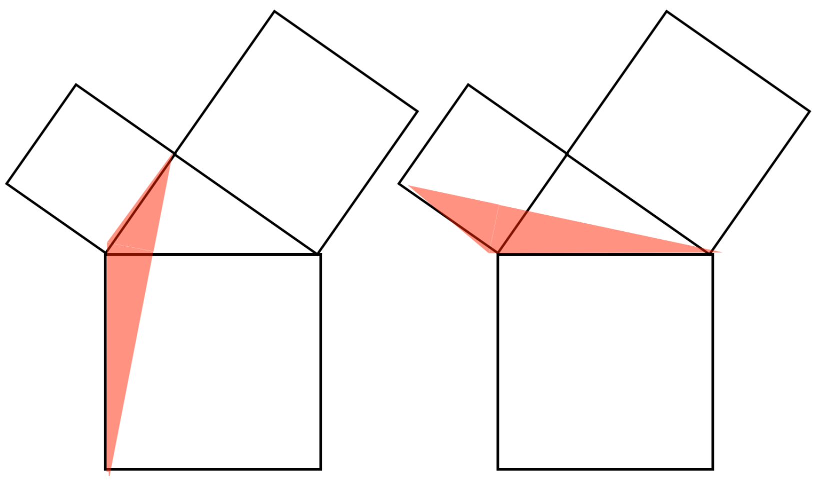
We can do the same thing with a triangle in the large square, below.

The gray triangle has half the area of the part of the large square that is to the left of the dotted line, because its base is equal to the side of the square and the height extends to the right to the dotted line.



Now slide the vertex up. The area is unchanged.

Last, we observe that the two triangles have exactly the same shape. Just rotate one to obtain the other.



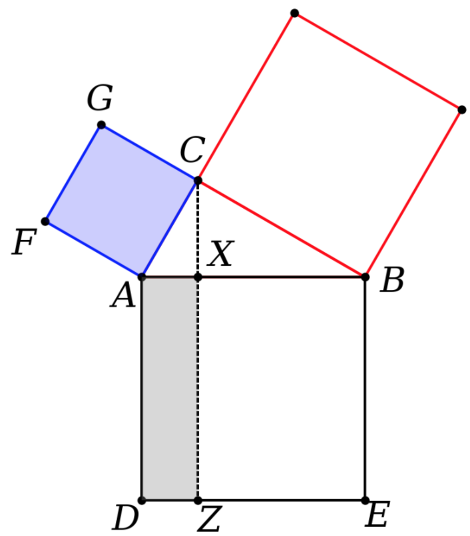
This completes our informal proof.

□

The formal approach follows, based on triangle congruence.

main

We label some points as shown:



As noted above, the figure is divided in half by the vertical down from C to DE. We will show that the area of the square $ACGF$ is equal to the area of the rectangle $ADZX$.

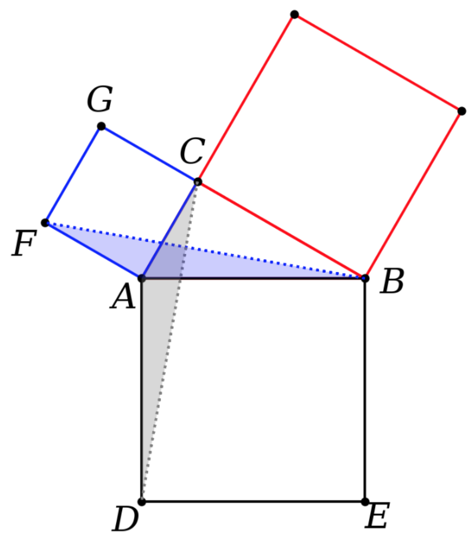
The proof proceeds in stages:

∘ (i) We repeat the observation from above that the area of a triangle does not change if we slide the upper vertex along a line parallel to the base. So $\triangle FAC = \triangle FAB$ in area, and the same for $\triangle DAC = \triangle DAX$.

∘ (ii) We will show that $\triangle BAF \cong \triangle DAC$.

∘ (iii) Finally, the areas of the square and rectangle are twice that of the triangles formed by a diameter and two adjacent sides. So for example, $\triangle FAC$ has one-half the area of the square $FACG$, and $\triangle DAX$ has one-half the area of the rectangle $ADZX$.

Parts 1 and 3 have been covered in such detail that we can accept them at this point. The second claim remains.



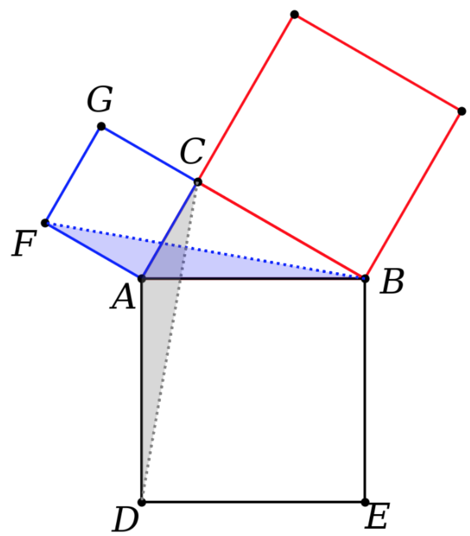
- (ii) $\triangle BAF \cong \triangle DAC$.

These two triangles have same side lengths, namely $AF = AC$ and $AB = AD$. Let us consider the angles in between. One triangle has $\angle BAF$ and the other has $\angle DAC$.

But these two angles share the central part $\angle BAC$, with one right angle, $\angle CAF$ added to make the first, and another right angle, $\angle BAD$ added to make the second.

Thus, the two angles are equal, so the two triangles are congruent by SAS, Euclid I.4.

Hence we have proved that the two colored areas in this figure are equal:



Finally, we could proceed to do the same thing on the right side of the figure, but we just appeal to symmetry. All the equivalent relationships will hold.

Addition yields the final result.

□

One detail worth pointing out: $CD \perp FB$. This proof is left as an exercise.

Converse

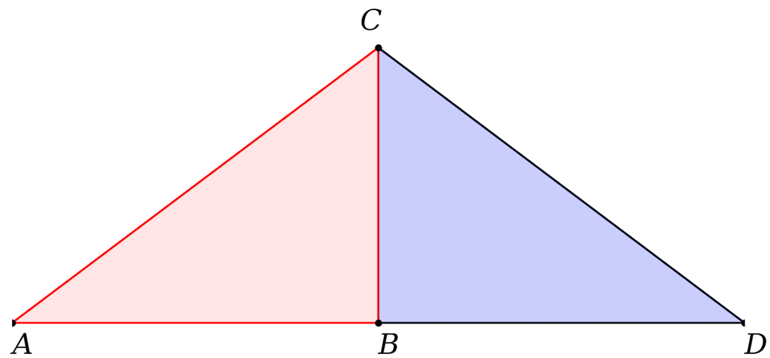
converse of Pythagorean theorem

In reasoning deductively, we move from the premise or premises (collection of facts, data given, previous theorems that were proved), and use logic to reach a conclusion. The question arises whether, if we know only that the conclusion is true, does it follow logically that the premises are true? This is the problem of the *converse* of a theorem.

It may be so, or it may not.

We can state the converse of the Pythagorean theorem as follows: suppose we have triangle such that $a^2 + b^2 = c^2$. Does it follow that the angle between a and b is a right angle?

We profess to not know whether it is or is not a right angle.



Proof.

Suppose we know that $\triangle ABC$ has its sides such that

$$AB^2 + BC^2 = AC^2$$

Draw BD at right angles to BC, and extend it to $D \mid BD = AB$.

We have BC shared.

Since $\angle CBD$ is right, by the forward theorem we have

$$BC^2 + BD^2 = CD^2$$

$$BC^2 + AB^2 = CD^2$$

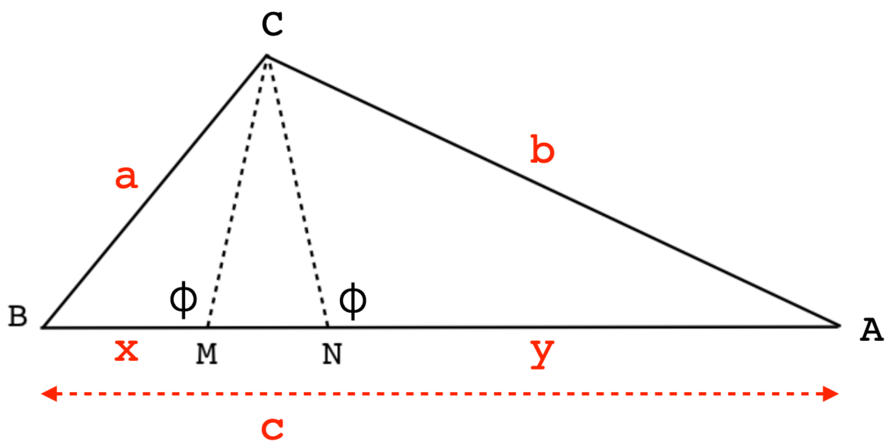
But we're given

$$BC^2 + AB^2 = AC^2$$

It follows that the $AC = DC$, so the two triangles are congruent by SSS, which means that $\angle ABC = \angle DBC$ is a right angle.

$\square$

Quorra's corollary



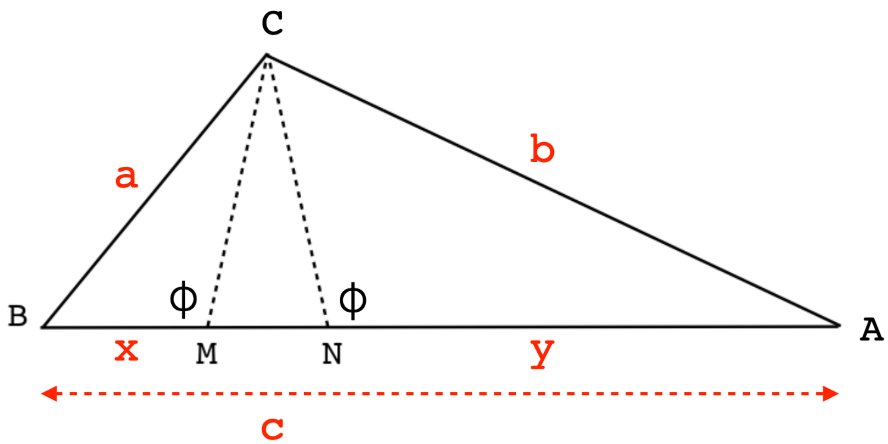
Let $\triangle ABC$ be *any* triangle (here it is obtuse). Draw CM and CN so that the new angles $\angle CMB$ and $\angle CNA$ (labeled ϕ), are equal to $\angle C$. The corresponding triangles are similar to the original, because they share ϕ plus one other from the original triangle.

We use single letters for the sides to make the algebra simpler. a is opposite both $\angle A$ and $\angle CMB$, while b is opposite both $\angle B$ and $\angle CNA$, and c is opposite $\angle C$.

The shortest side in $\triangle CMB$ is x and the longest is a, while in $\triangle ABC$ the corresponding sides are a and c. So by equal ratios of sides in similar triangles we have that $x/a = a/c$. The middle side in $\triangle CNA$ is y and by similar logic we have that $y/b = b/c$.

$$a^2 = cx, \qquad b^2 = cy$$
$$a^2 + b^2 = c(x + y)$$

The sum of the squares of the two short sides of a triangle is equal to the product of the third side with the the sum of the two components $x + y$, when they are drawn with the angle ϕ as specified.



This is actually a generalization of our original algebraic proof of the Pythagorean theorem.

In the case where the angle at vertex C is a right angle, then M coincides with N, because there is only one vertical to a line from a given point. So then and $x + y = c$, and this reduces to the Pythagorean theorem.

There are a large number of proofs of the Pythagorean theorem. Many of them are collected or linked here:

https://www.cut-the-knot.org/pythagoras/

Chapter 18

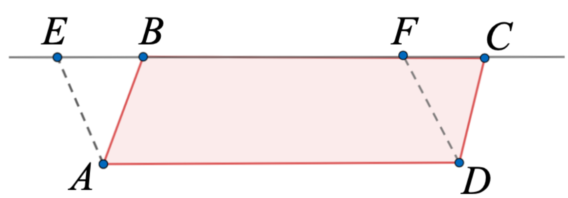
Pappus's proof

Pappus came up with a beautiful theorem which includes the Pythagorean theorem as an extension. First, we need a simple lemma about parallelograms.

Lemma.

Given two parallel lines: $AD \parallel EBFC$.

If two parallelograms $AEFD$ and $ABCD$ have opposite sides on the two lines, and those two segments are of equal length, then they have equal areas.



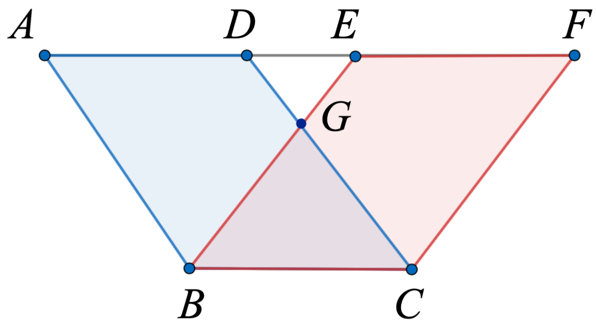
Here is a special case where they have the same base AD.

We proved earlier that the perpendicular between two parallel lines is of equal length no matter where it is drawn.

So the two parallelograms are each composed of pairs of triangles, which, having the same base and the same altitude, also have equal area.

□

One might also just invoke Euclid I.35.

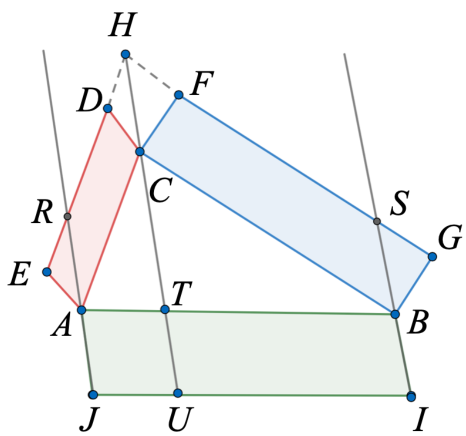


Pappus's parallelogram theorem

In the figure below, on the two sides of any $\triangle ABC$ draw any parallelograms $ACDE$ and $BCFG$. Extend the two new sides to meet at H.

Draw HC and its extension such that it cuts AB at T and then let $HC = TU$.

Draw $AJ \parallel HCTU \parallel BI$.



Proof.

The new parallelogram with AC as one side and RH the other, is equal, by our lemma. $(ACDE) = (ACHR)$.

Since $RAJ \parallel HCTU$ and $RA = HC = TU = AJ$, $(ATUJ) = (ACHR)$ for the same reason.

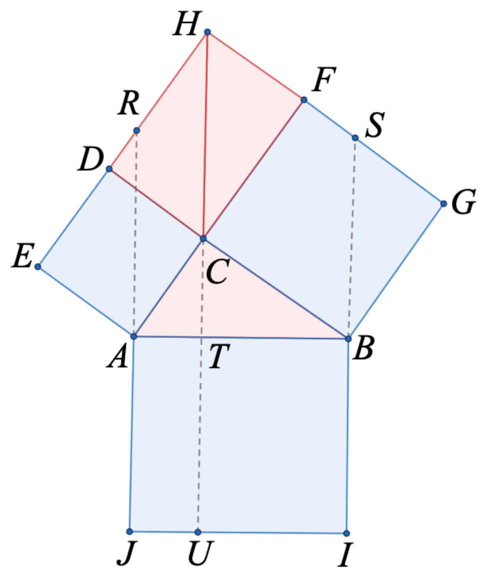
So $(ATUJ) = (ACDE)$.

We use the same argument for $(BCFG) = (TBIU)$ on the right.

Then add the two results: $(ACDE) + (BCFG) = (ABIJ)$.

$\square$.

Let $\angle ACB$ be a right angle, and let the parallelograms be squares.



$\triangle DHC$ and $\triangle FHC$ are right triangles and they are congruent to $\triangle ABC$ by SAS.

So $\angle CHF = \angle CAT$ and we also have vertical angles, so $\triangle HFC \sim \triangle CTA$.

It follows that $\angle HFC = \angle CTA$ and both are right angles.

Further, $CH = TU = AB$, so $ABIJ$ is the square on AB.

It is easy to see that $(AEDC) = (ARHC) = (ATUJ)$, and the rest follows.

Thus Pappus' theorem becomes the Pythagorean theorem as a special case.

[Reference: George F. Simmons, *Calculus Gems*.]

Chapter 19

Brief review

Let us summarize what we know so far about the basic theorems of geometry.

They are almost all that you should need to attack any of the other problems in the book, and they'll be used repeatedly.

The first two aren't technically theorems but fundamental assumptions that we make about the geometrical world.

- **supplementary angles**
- **alternate interior angles (parallel postulate)**

The next three easily follow from those initial ideas.

- **vertical angles**
- **triangle sum of angles**
- **complementary angles**

We have two basic methods for proving that two triangles are congruent

- **SAS for congruence**
- **ASA for congruence**

and one specifically for right triangles

- **hypotenuse leg in a right triangle (HL)**

(We proved that SSS is equivalent to SAS, and AAS is equivalent to ASA).

- **SSS implies SAS**

Next we have the powerful fundamental theorems of geometry:

- **isosceles triangle theorem** (sides $\rightarrow$ angles) and **converse**
- **external angle theorem**
- **Thales' circle theorem** (right angle in a semi-circle)
- **area ratio theorem**
- **similar triangles** (similar right $\triangle$s, same ratio of sides)
- **general similarity theorem**
- **Pythagorean theorem** (similar triangles)

Some people might add a few more. But this is a reasonable number to start with. You should have these instantly available (and it's nice to know how to prove them, as well).

Looking forward, probably the most important we have yet to do is

- **inscribed angle theorem** (on a circle is one-half central angle)

which has the consequence that inscribed angles on the same arc are all equal (**equal angles** $\iff$ **equal arcs**).

Part V

Circles

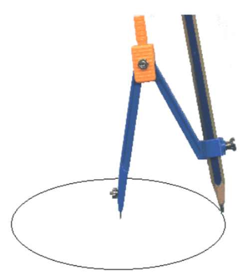
Chapter 20

Circles and angles

diameters

Pick some point to be the *center* of a circle. Then a circle contains all the points a specified distance away from the center. That distance is called the *radius*.

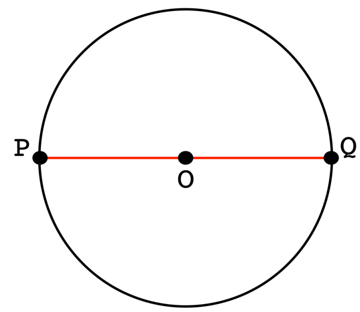
A circle is drawn with a compass, as we mentioned previously.



Euclid says:

• Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

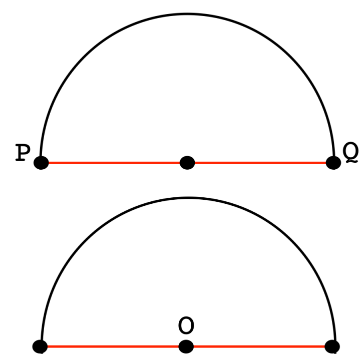
Suppose we start with the line segment OP in the figure below, and choose O as the center. Place the needle point of the compass on O and the pencil on P and then draw the circle containing all the points the same distance from O as P is.



Next, extend the radius PO to meet the circle on the other side at Q. This whole line segment PQ is called a *diameter* of the circle.

• Any diameter divides the circle into two equal parts.

(Recall that we allow a mirror image of something to be called equal to itself). So then, if we take the bottom half and flip it vertically, we claim the two halves are exactly equal.



Proof.

The proof is by contradiction. Lay the two pieces on top of one another. The diameters of the half-circle are duplicates of the original. Thus, they are equal to each other and each half is equal to one radius.

We suppose that, somewhere, the two half-circles are not identical.

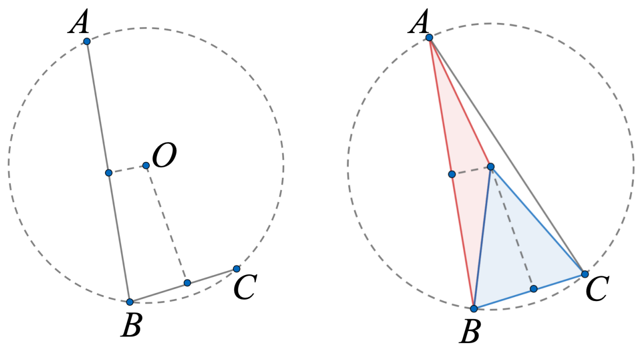
Then there will be some radial line we can draw from O through the two half circles, where the line meets the two curves at different distances from the center, because they are supposed to be different. This will identify points on the same radius of each half-circle that lie at a different distance from the center.

But then, the original figure could not be a circle, because all radii of a circle are equal.

This is a contradiction. Hence any diameter divides the circle into two equal parts, called semi-circles.

□

circle containing three points



Suppose we have three arbitrary points in the plane: A, B and C.

The claim is we can draw a circle that contains all three points on its circumference.

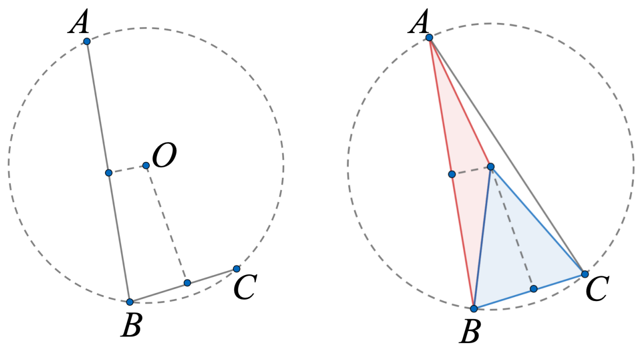
Proof.

Pick two pairs of points and draw, say, AB and BC.

Find the perpendicular bisector of each. We know that every point on the perpendicular bisector of AB is equidistant from both A and B, while for the latter case AC every point is equidistant from B and C.

If O is the point where the bisectors meet, that point is such that $OA = OB = OC$.

So if we draw the circle on center O with radius OA it contains all three points.



□

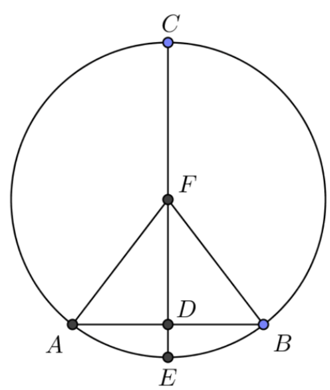
The question arises whether there is a general rule about $\triangle ABC$ and there is: if the origin of the circle lies on one of the sides of $\triangle ABC$ then it is a right triangle, that side is a diameter of the circle and the hypotenuse of the right triangle, and the center of the circle bisects the hypotenuse. This depends on the converse of Thales' circle theorem.

If $\triangle ABC$ contains an angle greater than a right angle, then the center of the circumcircle is not inside the triangle, while if it is acute, the center lies inside the triangle.

We will not prove this now. Euclid's proof for all three cases is III.31.

Euclid III.1

This is a simple method from Euclid to find the center of a circle.



Proof.

To find the center of any circle, take two points on the circle and draw the chord connecting them, then erect the perpendicular bisector of the chord. In the diagram above, $CE \perp AB$ and bisects it.

We showed previously that there does not exist any point which is equidistant from A and B and is *not* on this perpendicular bisector. Since the center of the circle F has the property $AF = BF$, it must lie somewhere on CE.

Thus, CE is a diameter of the circle. Bisect it to find F, the center.

$\square$

internal and external points

As a preliminary to the next section, we claim that

• A line through any internal point of a circle can be extended to intersect the circle at two and only two different points.

Proof.

Assume there is a straight line segment that intersects a circle at more than two points, say at three points. By the definition of a circle, those three points are equidistant from the center.

Of the three points, one lies between the other two (going the shortest way around the circle). Use the middle point and (one at a time) the other two points to construct two perpendicular bisectors of the two segments. This is the classic way to find the center of a circle from points on the periphery.

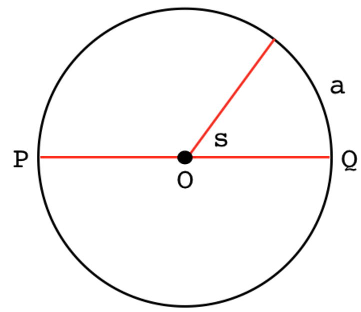
But then, we would have two bisectors that both run through the center. Yet they are parallel to each other, because they have been constructed perpendicular to the same line.

This is a contradiction. There cannot be more than two points on intersection of a line with a circle.

$\square$

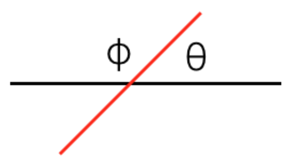
If there is no specification of an internal point, then it is possible for a line to intersect with the circle at only one point, namely a tangent line. We deal with tangents separately in a later chapter.

radian measure



In the figure above, the angle s is *defined* as the length of arc a that it sweeps out in a unit circle.

We talked briefly early in the book about the *measure* of an angle. It seems intuitively obvious that, in this figure, $\theta < \phi$.



The question is how to quantify this notion.

Effectively what we'll do is imagine that we draw a circle with radius 1, a *unit* circle, which contains the angle as a central angle.

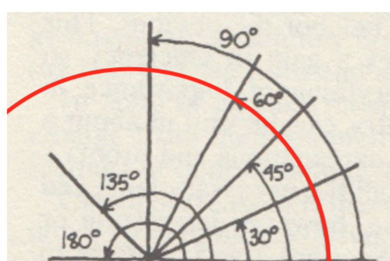


FIG. 1

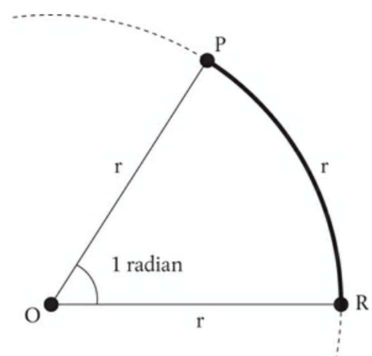
Then, the measure of the angle is the distance traveled around the circle counter-clockwise from the horizontal on the right.

Although this drawing shows angles in degrees, in calculus and analytical geometry angles are defined in terms of radians of arc. For a unit circle with radius $= 1$, the total circumference is 2π.

If s is the measure in radians and $D°$ the measure in degrees, then

$$\frac{s}{2\pi} = \frac{D}{360}$$

It seems natural then to adopt the arc length as a measure of the angle, where 360° is equal to 2π *radians*, and an angle of 90°, a right angle, is equal to $\pi/2$ radians.



72. Definition of a radian.

Divide 360 by 2π to find that one radian is approximately 57°.

To convert some more measures of angles in degrees to radians:

$$180° = \pi,\ 90° = \frac{\pi}{2}$$

$$60° = \frac{\pi}{3},\ 45° = \frac{\pi}{4},\ 30° = \frac{\pi}{6}$$

Central angle and the arc that subtends that angle are numerically equal, but remember that they are dimensionally different. An arc is a length, an angle is just an angle and not a length.

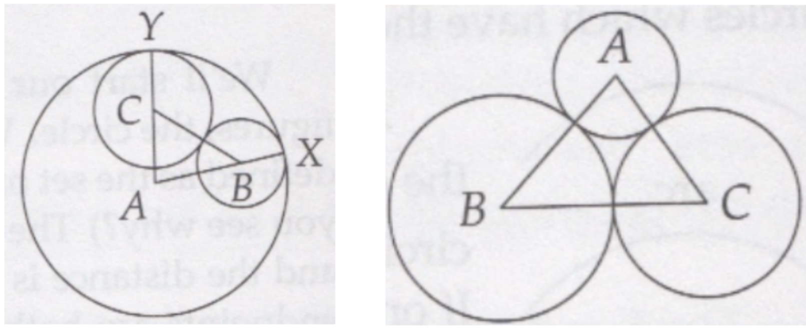
problem

Given two distinct points in the plane and a line l not through either point, find the center of the circle that has its diameter on the line and goes through both points.

There is no solution if a line drawn through the two points is perpendicular to l. Why?

two problems

Here are two problems from *the Art of Problem Solving*.



For the first (left panel), we are given that A, B and C are the centers of the respective circles, that the points which appear tangent are actually so, and also that $AB = 6$, $AC = 5$ and $BC = 9$.

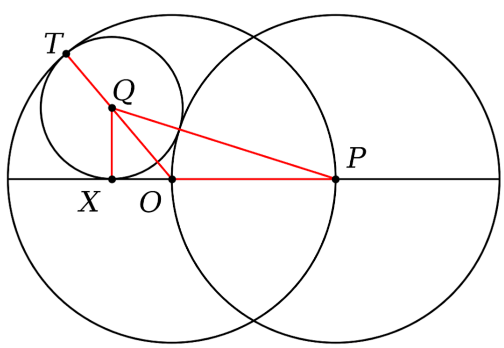
What is AX, the radius of the large circle?

For the right panel, we are asked to express the radius of circle A in terms of the sides of $\triangle ABC$.

Hint. Both problems ask you to exercise your skills in solving simultaneous equations.

problem

Here is a problem from Paul Yiu. It is really more about the Pythagorean theorem with an equilateral triangle thrown in but I put it here.



We have two larger circles of equal size with radius a and the two centers O and P also separated by a. There is another smaller circle on center Q, drawn tangent to both the first two and to their diameter as shown.

In order to even draw the third circle properly, we need to do some algebra, to find the displacement of X from O and then Q from X.

Let the length of OX be x and the radius of Q, QX, be r.

The length of PX is $a + x$ and (since the circles are tangent), the length PQ is the sum of the radii, namely, $a + r$

Finally, OQ is a little trickier, but we have that OT is a radius of the first circle, so equal to a, and that means that $OQ = a - r$.

Then, we have two right triangles $\triangle XOQ$ and $\triangle XPQ$.

From the Pythagorean theorem, the first one gives:

$$x^2 + r^2 = (a - r)^2$$

$$= a^2 - 2ar + r^2$$

$$x^2 = a^2 - 2ar$$

while the second one gives:

$$(a + x)^2 + r^2 = (a + r)^2$$

$$a^2 + 2ax + x^2 + r^2 = a^2 + 2ar + r^2$$

$$2ax + x^2 = 2ar$$

Adding the two results:

$$2ax + 2x^2 = a^2$$

We will need this later so let us now subtract the first equation from the second one.

$$2ax = 4ar - a^2$$

$$4r = a + 2x$$

The first is a quadratic in x with standard form:

$$x^2 + ax - \frac{a^2}{2} = 0$$

and roots

$$\begin{aligned}2x &= -a \pm \sqrt{a^2 + 2a^2} \\ &= -a \pm a \cdot \sqrt{3}\end{aligned}$$

For a length we take the positive root:

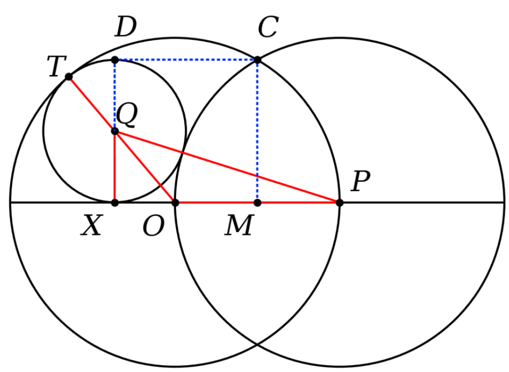
$$2x = a \cdot (\sqrt{3} - 1)$$

$$x = \frac{\sqrt{3} - 1}{2} \cdot a$$

Going back to solve for r

$$\begin{aligned}4r = a + 2x &= a + a \cdot (\sqrt{3} - 1) \\ &= a \cdot \sqrt{3} \\ r &= \frac{\sqrt{3}}{4} \cdot a\end{aligned}$$

There is more to the problem. Find M as the midpoint of OP and then erect the perpendicular to find C.



The length OM is $a/2$. The radius PC is a so the vertical is $a \cdot \sqrt{3}/2$.

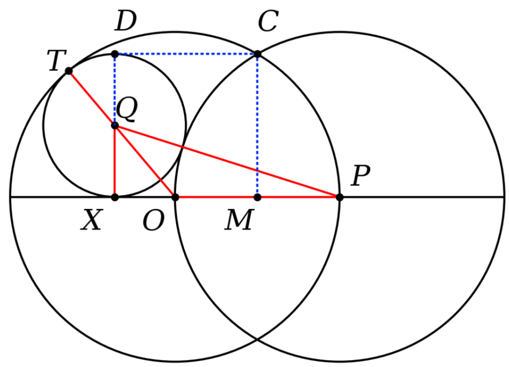
Recall that we had

$$x = \frac{\sqrt{3} - 1}{2} \cdot a$$

The length of MX is

$$x + a/2 = a \cdot \sqrt{3}/2$$

In other words, $MX = MC = 2r$ and we have a square.



We notice that Euclid I.1 constructs an equilateral triangle in exactly this way. $OC = CP = OP = a$.

□

The last mystery concerns T, which we found as the extension of OQ. The two circles on O and Q just touch at T. They are *tangent*. A line through the centers of two circles that are tangent to one another (whether internally, like this, or externally like P and Q), goes through the point of tangency.

Chapter 21

Inscribed angles

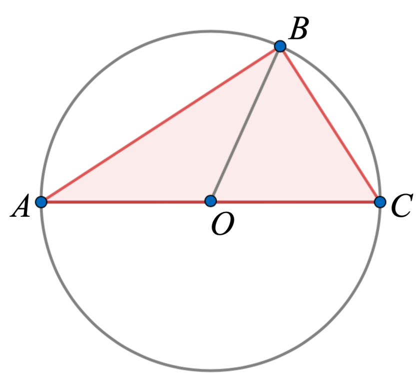
Thales' circle theorem

In this chapter, we will introduce the inscribed angle theorem. Let's start by revisiting Thales' circle theorem:

- Any angle inscribed in a semicircle is a right angle.

Take a diameter of the circle and any third, distinct point. The three points on the circumference of the circle form a triangle, and the angle at the third vertex is always a right angle.

In the figure below, $\angle ABC$ is a right angle.



Proof.

By I.32, the sum of angles in a triangle is equal to two right angles. So

$$\angle OAB + \angle OBA + \angle OBC + \angle OCB = 180$$

But $\triangle OAB$ and $\triangle OBC$ are both isosceles, so the base angles are equal, with

$$\angle OAB = \angle OBA \qquad \angle OBC = \angle OCB$$

It follows that

$$2\angle OBA + 2\angle OBC = 180$$

$$\angle OBA + \angle OBC = 90$$

□

According to Boyer, this result was known to the Egyptians 1000 years before Thales. But it is yet another example of knowing a result before proving that it is so. Archimedes had something to say about the importance of having discovered a fact, before finding a way to prove it.

The converse of Thales' theorem is also true. If the third point of a triangle contains a right angle, then it must lie on the circle where the other two points form the diameter.

Thales' circle theorem converse

A nice direct proof of this is given in Acheson.

Proof.

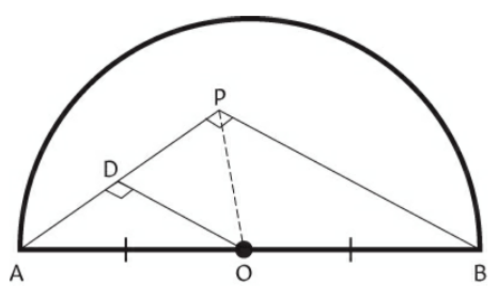


Fig. 59 An alternative method.

We are given that $\angle APB$ is a right angle.

Draw OD parallel to PB. $\triangle AOD$ is similar to $\triangle ABP$ because they are both right triangles with a shared vertex at A.

Since AO is one-half AB, the scale factor is $1/2$. In particular, $AD = DP$.

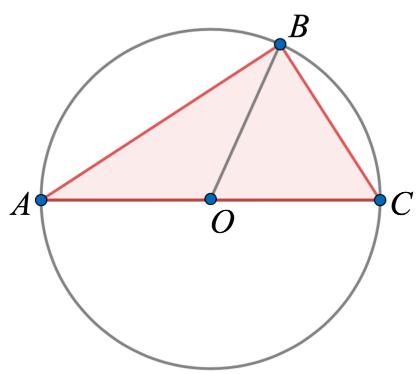
Now draw OP. The two smaller triangles $\triangle AOD$ and $\triangle DOP$ are congruent by SAS. Therefore, $OP = OA$.

But OA is a radius of the circle. Therefore, OP is also a radius.

It follows that P must lie on the circle.

□

We can use Thales' theorem to introduce the inscribed angle theorem.



inscribed angle theorem

An inscribed angle is an angle whose vertex lies on the circle, such as $\angle BAC$

• An inscribed angle is one-half the corresponding central angle lying on the same arc, $\angle BOC$. The central angle is twice the corresponding inscribed angle.

Proof.

Euclid's proof (the theorem is proposition 20 of book III), uses the external angle theorem. $\angle BOC$ is the external angle to $\triangle OAB$.

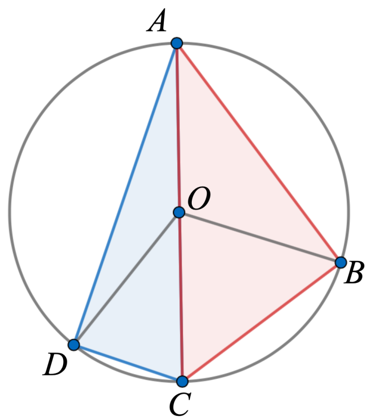
Thus, it is equal to the sum of $\angle OAB + \angle OBA$.

The triangle is isosceles, so these two angles are equal.

It follows that $\angle BOC = 2\angle BAC$.

$\square$

This proof is short and sweet, but limited by the fact that we used the diameter for one of the arms of the inscribed angle. Here is a more general proof.



The claim is that $\angle BOD = 2\angle BAD$.

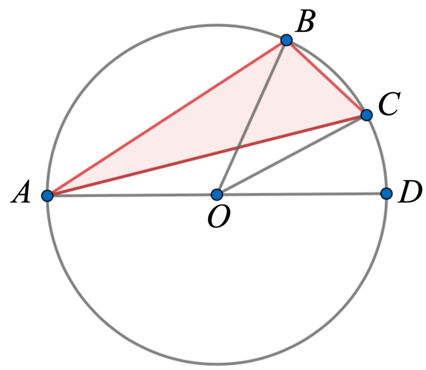
Proof.

Just add the results obtained using the previous proof. $\angle DOC = 2\angle DAC$ and $\angle BOC = 2\angle BAC$.

$$\angle BOD = \angle BOC + \angle DOC$$
$$= 2\angle BAC + 2\angle DAC = 2\angle BAD$$

$\square$

The proof is *still* limited, since the angle we looked at includes the center of the circle. There are two ways to fix this. The first is subtraction.



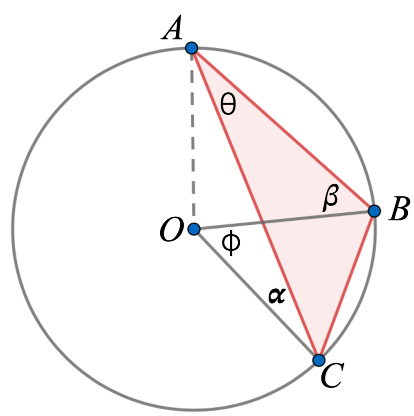
Proof.

By the first proof, $\angle BOD = 2\angle BAD$ and $\angle COD = 2\angle CAD$. The angle of interest, $\angle BOC$, is the difference:

$$\angle BOC = \angle BOD - \angle COD$$
$$= 2\angle BAD - 2\angle CAD = 2\angle BAC$$

□

An elegant proof of the second case is as follows.



Proof.

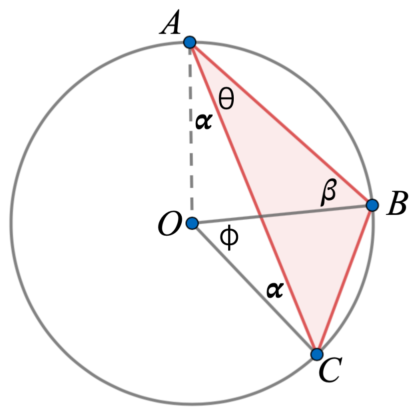
We claim that angle ϕ is twice the inscribed angle θ on the same arc.

Labeling additional angles α and β, we see two triangles with angles ϕ and θ that share vertical angles. So by sum of angles and vertical angles we have that

$$\phi + \alpha = \theta + \beta$$

But $\triangle OAB$ is isosceles with

$$\beta = \alpha + \theta$$



By addition, canceling $\alpha + \beta$, we obtain the result:

$$\phi = 2\theta$$

□

angles on the same arc

• Angles that lie on the same arc or are subtended by the same chord in the same circle, are equal.

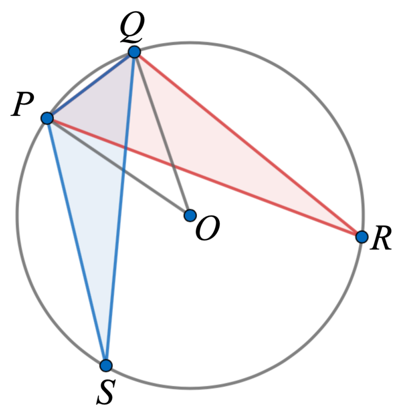
This theorem is Euclid III.21.

The previous **inscribed angle theorem** is Euclid III.20, which says that the central angle on any chord or arc or "segment" of a circle, is twice an arbitrary peripheral (inscribed) angle on the same segment of the same or equal circle.

Euclid III.21 is an immediate consequence of the previous one, because the peripheral angle in III.20 is *arbitrary*.

The two theorems are sometimes thought to be the same. III.20 might be better referred to as the central angle theorem, and III.21 as the inscribed angle theorem. However, that is not the customary usage.

III.21 will appear often in the pages to come. We will refer to it as **equal angles $\iff$ equal arcs** or more simply, equal angles *on* equal arcs, unless we slip, and call it the inscribed angle(s) theorem.



Proof.

This is an immediate consequence of the previous theorem, since two such angles are equal to one-half of the *same* central angle. $\angle POQ$ is the central angle for arc PQ and hence is twice both $\angle PRQ$ and $\angle PSQ$. Thus, $\angle PRQ = \angle PSQ$.

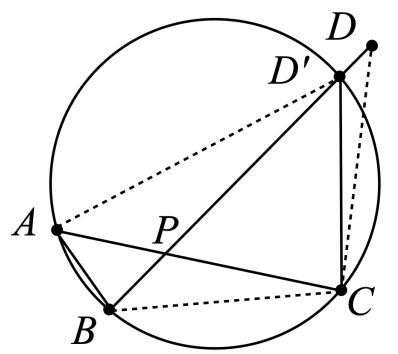
□

converse

There are several variant arguments from similar setups which are converses of the inscribed angles theorem. Here is one:

Let $\triangle ABC$ lie on a circle.

Let point D be such that $\angle BDC = \angle BAC$.



Then D is also on the same circle.

Proof.

Aiming for a contradiction, suppose otherwise.

Let D be external and also let $\angle BDC = \angle BAC$.

Find where BD cuts the circle at D'.

So D' is on the circle and $\angle BD'C$ is subtended by BC.

By the forward version of the inscribed angle theorem: $\angle BD'C = \angle BAC$.

It follows that $\angle BDC = \angle BD'C$.

But $\angle BD'C$ is external to $\triangle CDD'$.

So $\angle BD'C > \angle BDC$.

This is a contradiction.

A similar argument will show that D cannot be internal to the circle.

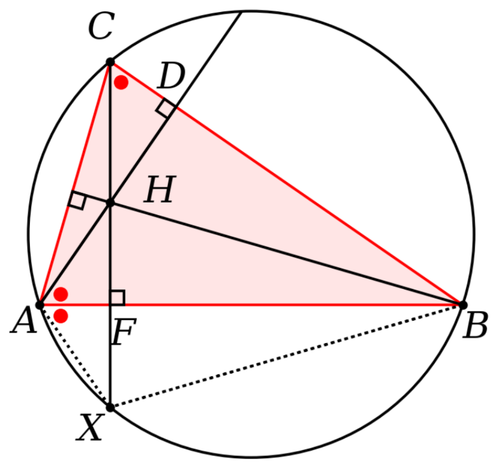
Therefore, it must be that D *is on the circle*.

□

extended altitude

- An altitude extended to the circumcircle of a triangle forms congruent triangles.

Proof.



In $\triangle ABC$ draw the altitudes including AD and CF and find the orthocenter. Draw the circumcircle an extend AD to meet the circle at X..

The angles at D and F are right. Since they are both right triangles and share vertical angles we have

$$\triangle AFH \sim \triangle CDH$$

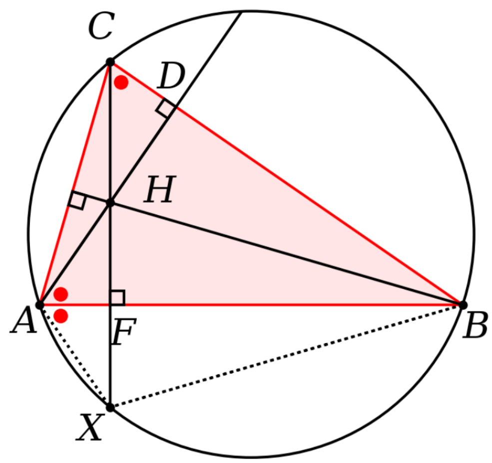
So $\angle DCF = \angle FAD$

But $\angle BAX$ cuts the same arc. Therefore the three angles marked with red dots are all equal.

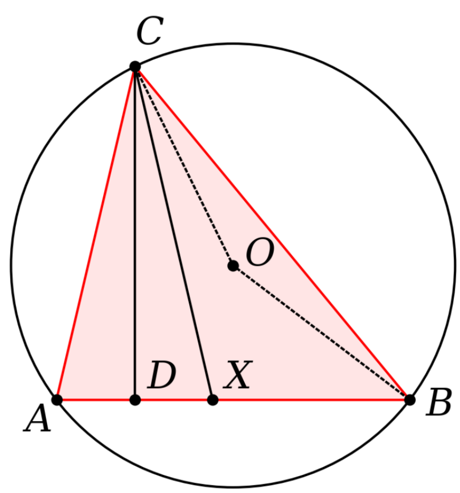
$\triangle AFH \cong \triangle AFX$ (ASA).

It follows that $FX = FH$ and $\triangle GAX$ is isosceles as well. $\triangle ABH \cong \triangle ABX$ by SAS.

$\square$



problem



Posamentier gives this problem. Let O be the center of the circumcircle of $\triangle ABC$.

Let $CD \perp AB$ and CX bisect $\angle C$.

Show that CD also bisects $\angle ACX$.

Proof.

Given $\angle ACX = \angle BCX$.

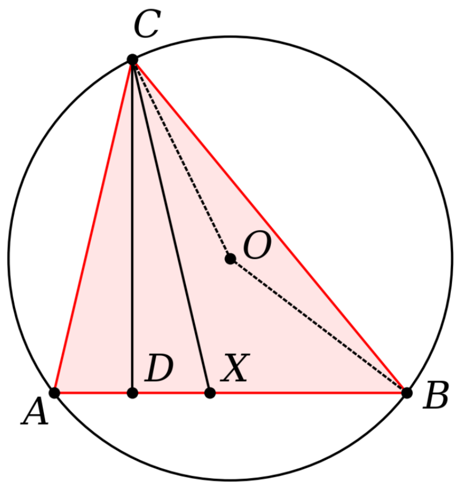
Draw OC.

The central $\angle BOC$ is $2\ \angle CAB$.

But $\triangle BOC$ is isosceles with equal base angles.

Thus $\angle CAB$ and $\angle OCB$ are complementary.

$\angle ACD$ is also complementary to $\angle CAB$, hence $\angle CAB$ is equal to $\angle OCB$.



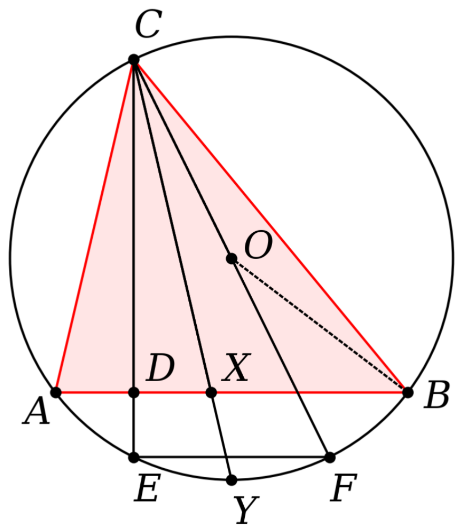
Subtracting equals: $\angle BCX = \angle OCX$.

$\square$

The angle bisector at C also bisects the angle between the altitude and the radius from C.

Proof. (Alternate).

I have redrawn the figure slightly.



Claim: the altitude and diameter of the circumcircle through C form equal angles with the sides CA and CB.

Let $\triangle ABC$ have its circumcircle on center O, so CO is a radius. Extend CO to form the diameter CF.

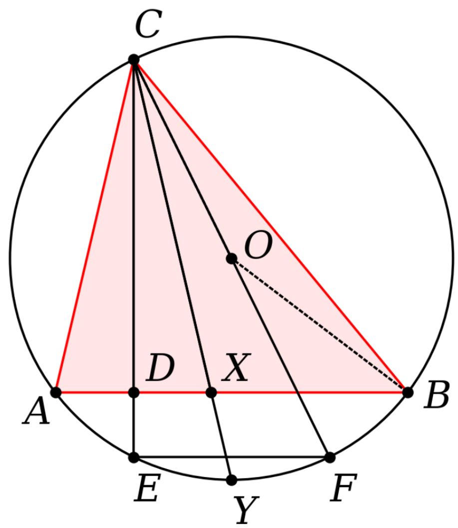
Let CD be the altitude from C, so it cuts AB at a right angle and extends to the circle at E.

By Thales' circle theorem, $\angle CEF$ is right. It follows that $EF \parallel AB$.

Parallel chords in a circle cut equal arcs. (Note: we prove this elsewhere. Hint: connect the opposing ends of the two chords to give equal inscribed angles).

Hence $AE = BF$ and then by inscribed angles, $\angle ACE = \angle BCF$.

□



By subtraction, the bisector also bisects the angle between the altitude and the radius AO.

Alternatively, find Y as the midpoint of arc AB.

Then we have equal arcs $AY = BY$, $AE = BF$, and $EY = FY$. Results about the angles follow easily.

geometric mean

We showed in the chapter on the **Pythagorean theorem** that the altitude of a right triangle is the geometric mean of the two components of the base.

$$h^2 = pq$$
$$h = \sqrt{pq}$$

According to wikipedia:

https://en.wikipedia.org/wiki/Geometric_mean

The fundamental property of the geometric mean is that (letting m be the *geometric mean* here):

$$m\ [\ \frac{x_i}{y_i}\ ]\ = \frac{m(x_i)}{m(y_i)}$$

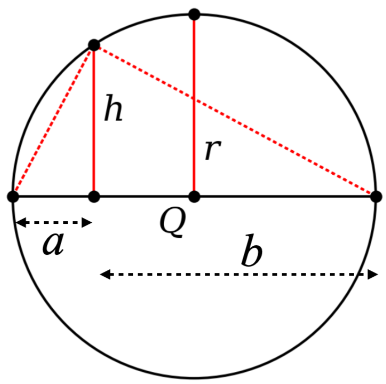
and one consequence is that

> This makes the geometric mean the only correct mean when averaging normalized results; that is, results that are presented as ratios to reference values.

A number of examples are given in the article.

We discuss this here because originally, there was a proof-without-words that the geometric mean is always less than or equal to the arithmetic mean.

I decided to add some words.



A right triangle is inscribed in a semicircle.

We can use the Pythagorean theorem three times or just rely on the fact that the two smaller triangles are similar with equal ratios of sides including:

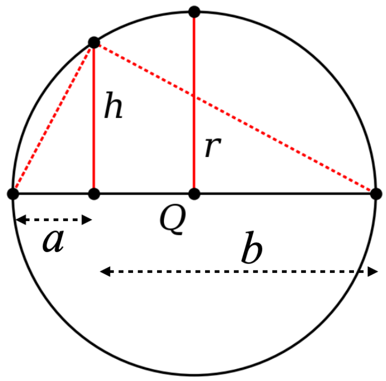
$$\frac{h}{a} = \frac{b}{h}$$

and the result follows immediately:

$$ab = h^2$$

This says that the square of the altitude h is equal to the product of chord segments (we will prove this geometrically in the next chapter as well).

$$h = \sqrt{ab}$$



But we also have that $a + b = 2r$ and hence

$$r = \frac{a+b}{2}$$

Do you recognize these? The second expression is the arithmetic mean of a and b, while the first is the geometric mean.

The geometry shows that $h \le r$ so:

$$\sqrt{ab} \le \frac{a+b}{2}$$

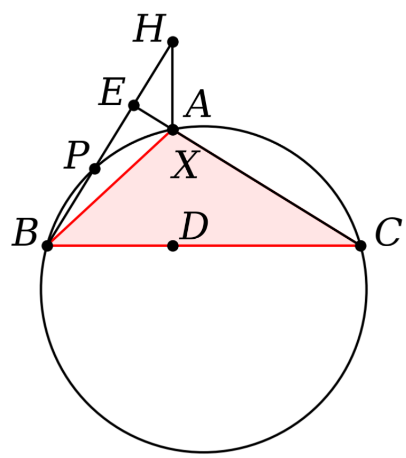
The geometric mean is always less than the arithmetic mean, except when $a = b$, where they are equal (or all of n values are equal, when there are more than two values).

problem

I have lost track of where I found this problem.

Given $\triangle ABC$ with its circumcircle. Given AD is an altitude.

Extend AD to meet the circle at X, and also extend AD vertically to $H \mid HD = DX$. Join HB and extend CA to E. Show that $CAE \perp HB$.



Hint: draw point P where HB cuts the circle. Show that arc PAC = arc CX.

Ignoring the hint for now, we are given that BC is a perpendicular bisector of HX.

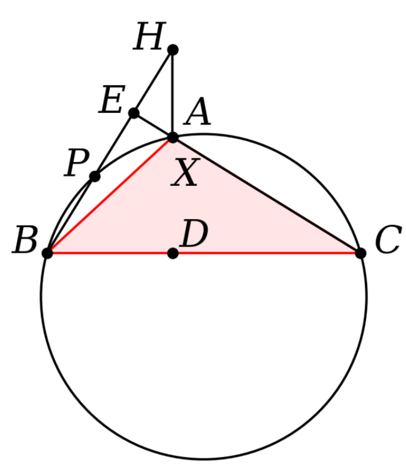
Hence $\triangle HBX$ is isosceles with $HB = BX$ and $\angle BXD = \angle H$.

But $\angle BXD$ and $\angle C$ cut the same arc AB. So $\angle H = \angle C$.

We also have vertical angles so it follows that $\triangle AEH \sim \triangle ADC$. Thus $\angle AEH$ is right.

□

Going back to the hint about P. Again, $\triangle HBX$ is isosceles.



$\angle PBC = \angle XBC$, so arc $PAC = XC$.

Then $\angle HAE = \angle DAC$ and both equal $\angle PBC$ since they stand on equal arcs.

$\angle H$ is shared. It follows that $\triangle HAE \sim \triangle EBC$. Since $BPEH$ are collinear, and the angles at E are equal, they are both right.

problem

Here is a problem I found on the web as a Youtube video:

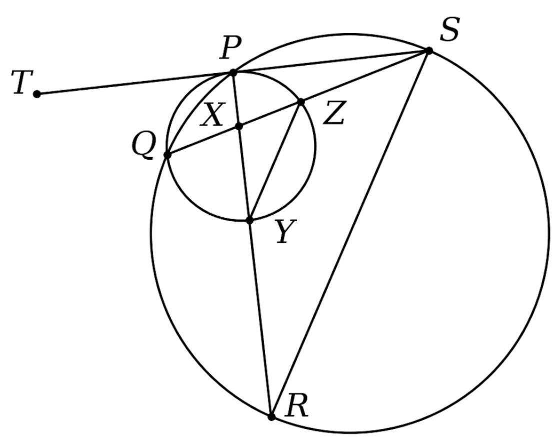
https://youtu.be/2Jt8lynddQ8

It is described as a GCE O-Level A-Maths Plane Geometry Question.

The relationships that seem obvious from the diagram are given. Namely, $PXYR$ and $QXZS$ are each lie on a straight line (colinear).

And the two circles each have the four points lying on them as shown.

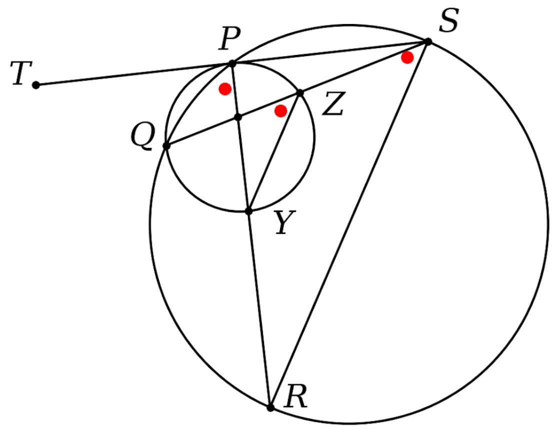
TPS is tangent to the smaller circle at P.



The problem asks us to show that SR is parallel to ZY and hence, *deduce* that $YX/ZX = YR/SZ$.

The approach that first occurred to me was to use the similar triangles that arise from crossed chords. However, the problem asks us to start by showing that the given line segments are parallel. This is a hint to a different proof.

The result comes from the theorem which is the basis of this chapter: the inscribed angle theorem.



Proof.

The marked angles are all equal. The first two are equal because they correspond to the same arc in the small circle, and the third (at S) is equal to the first because they both correspond to the same arc in the large circle.

Therefore, the two line segments are parallel by the converse of the alternate interior angles theorem.

That gives us similar triangles $\triangle XYZ \sim \triangle XRS$ from which the equal ratios follow almost immediately (see below).

□

The last part of the problem says that given $SQ = XR$, prove that $PS^2 = XS \cdot YR$ We're not ready to do that yet. It uses the information about the tangent and the **tangent-secant theorem**.

Proof. (Alternate).

Here's the first part of the proof by the crossed chord theorem:

$$PX \cdot XY = QX \cdot XZ$$

$$PX \cdot XR = QX \cdot XS$$

It follows that

$$\frac{XZ}{XY} = \frac{XS}{XR}$$

We need some algebra to get to $YX/ZX = YR/SZ$. This is a standard parts and the whole manipulation for similar triangles, made complex by the cumbersome notation.

Let $a = XY$, $A = XR$, $b = XZ$, and $B = XS$. We have

$$\frac{b}{a} = \frac{B}{A}$$

and we want

$$\frac{a}{b} = \frac{A-a}{B-b}$$

The way to get there is:

$$\frac{A}{a} = \frac{B}{b}$$

$$\frac{A}{a} - 1 = \frac{B}{b} - 1$$

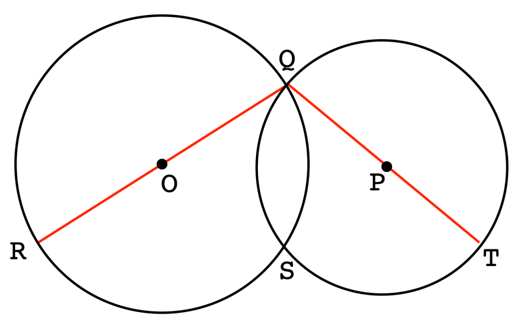
then

$$\frac{A-a}{a} = \frac{B-b}{b}$$

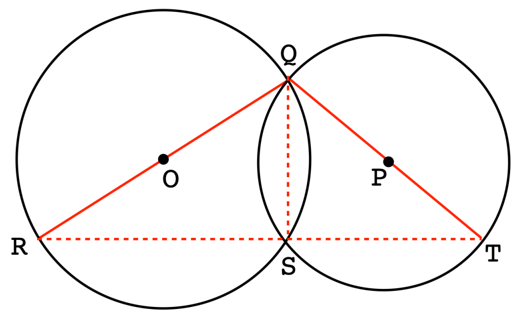
and the result follows easily in one more step.

□

problem



Two circles meet at Q and S. QR and QT are diameters of the two circles. Prove that RST are colinear.



Since QR is a diameter of the circle centered at O, $\angle QSR$ is a right angle.

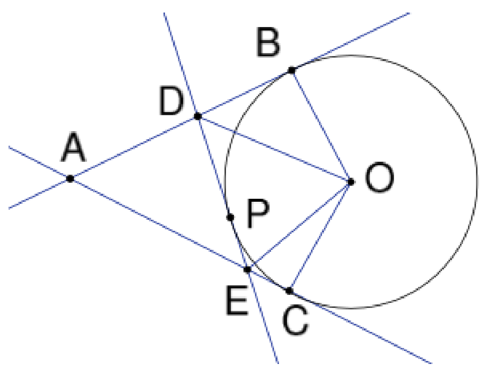
But so is $\angle QST$, since QT is a diameter of the second circle.

Hence the total angle at S is two right angles or a straight line. Therefore RS and ST togethere form a straight line segment.

double arc problem

This problem is taken from an online collection by David Surowski.

https://www.math.ksu.edu/~dbski/writings/further.pdf



Given that AB and AC are tangents to the circle meeting at A. Given a second tangent DE, meeting the circle at P. Prove that the arc that subtends $\angle BOC$ is twice that which subtends $\angle DOE$.

Proof.

Notice that DB and DP are tangents to the circle meeting at D. Therefore $DB = DP$ and then $\triangle BOD \cong \triangle DOP$, so $\angle BOD = \angle DOP$.

The same argument applies to EC and EP. Therefore the inner arc is composed of two angles, while the outer arc has two copies of each of those angles.

□

Queen Dido

The mighty city of Carthage was the capital city of a major Phoenician colony. As Rome grew strong, there was a titanic struggle between the two peoples, which Carthage eventually lost. The ruins of Carthage lie near present-day Tunis.

Queen Dido was the legendary founder of the the city of Carthage. She was supposedly

> granted as much land as she could encompass with an oxhide. She promptly cut the ox-hide into very thin strips.

The problem then is to maximize the area enclosed by a curve of fixed length.

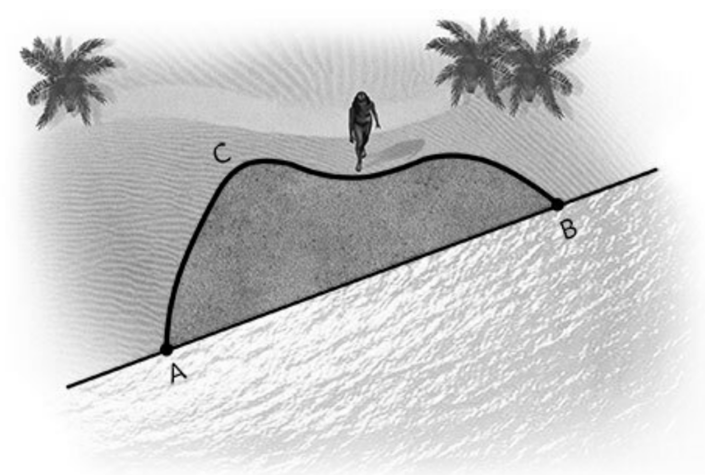


Fig. 138 Dido's problem (with a straight coastline).

In calculus there are a number of problems like this. What's nice is that this problem has a wonderful solution that uses only the tools we have so far. In particular, we need the converse of Thales' circle theorem.

The argument goes as follows. Suppose we have a particular outline for the city limits and we're pretty happy with it. We suppose it is a maximum (left panel).

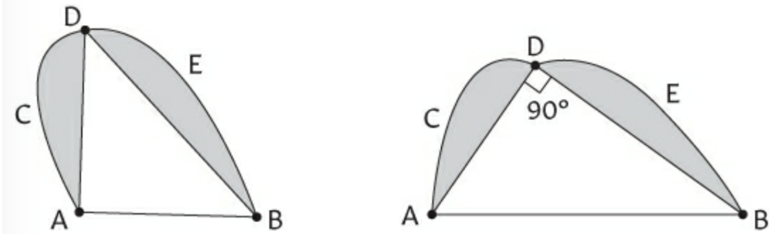


Fig. 140 The heart of Simpson's argument.

Then we notice that by rearranging AD and BD so they meet at a right angle, the crescent-shaped areas are unchanged, but the area of $\triangle ABD$ is a maximum. That's because a right triangle, having the two sides at right angles, has area equal to the product of the two sides (divided by 2). No other triangle with the same two sides has as much area.

So the arrangement on the right has a bigger area.

But then, with AB as the diameter of a circle, if $\angle ADB$ is a right angle, it must lie on the circumference of that circle, by the converse of Thales' theorem.

And this is true regardless of the relative lengths of AD and BD. Therefore the maximum area is obtained when D traces out a semi-circle.

This example is in Acheson's Geometry.

Chapter 22

Tangents

tangent: perpendicular $\rightarrow$ touches one point

We show **here** that one can construct a line perpendicular to any given line, and passing through any point whether on the line or not.

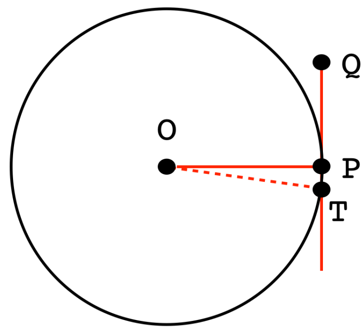
If we make the construction perpendicular to the radius (or diameter) at the point where it meets the circle, the new line is called a tangent line (from the Latin *tangere*, to touch).

• The tangent line, defined as perpendicular to the radius, touches the circle at a single point.

Proof.

By definition, the tangent line at P is perpendicular to the diameter or radius including O and P.

Let Q be some other point on that tangent line. Then $\angle OPQ$ is a right angle.



Let T be on the tangent line so QPT collinear. Draw OT.

Aiming for a contradiction, suppose that $OT \perp QPT$.

By the parallel postulate, $OT \parallel OP$.

But OT meets OP at O. This is a contradiction.

OT cannot be perpendicular to QPT.

It follows that OT is the hypotenuse of a right triangle $\triangle OPT$, so $OT > OP$ and thus T cannot be on the circle.

□

tangent: touches one point → perpendicular

An alternative definition of the tangent is that it is a line that touches the circle at just one point. One can use this definition to prove that the angle between the tangent and the radius is a right angle. This is the converse of the previous theorem.

• The tangent line, defined as touching the circle at a single point, is perpendicular to the radius.

Proof.

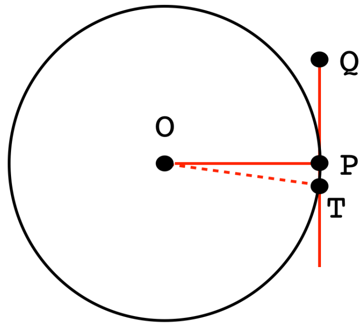
Draw the line that touches the circle at only one point P, and draw the radius to that point, OP.

If we assume that OP is not perpendicular to QPT, we can derive a contradiction.

Consider a succession of points moving along the line away from P in either direction. The angle formed with a line drawn through O gets smaller as the points get farther from P.

The angle between the tangent and the radius is greater than a right angle on one side of P (since the angle at P is not a right angle). On that side, move along the line until we find the point that does form a right angle with a radius of the circle.

Let us suppose the point is T, in the figure above. OTP is a right angle and we are in doubt about whether T lies inside or outside the circle.



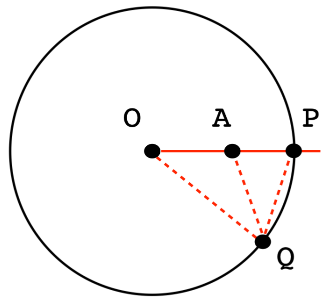
But then OTP is a right angle, with the side opposite, namely, OP, the hypotenuse of a right triangle. But the hypotenuse is the longest side in a right triangle, so therefore $OT < OP$, so the point T is *inside* the circle.

We have a line with one point on the circle, and another point inside the circle. Any line through an interior point of a circle must cross the circle at two points, which contradicts the assumption above. Hence the angle at P is a right angle.

□

shortest distance to the circle

Let A be any point inside a circle. Draw the radius that passes through A to point P on the circle. I claim that the length AP is smaller than the distance to *any* other point on the circle, such as Q.



Proof.

Draw the radius OQ, which is equal to radius OP.

$$OQ = OP = OA + AP$$

By the **triangle inequality**

$$OA + AQ > OQ$$

so

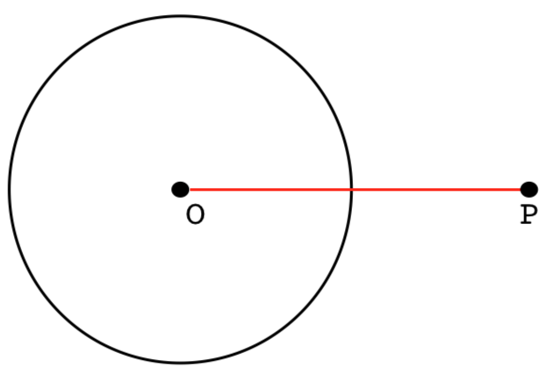
$$OA + AQ > OA + AP$$

$$AQ > AP$$

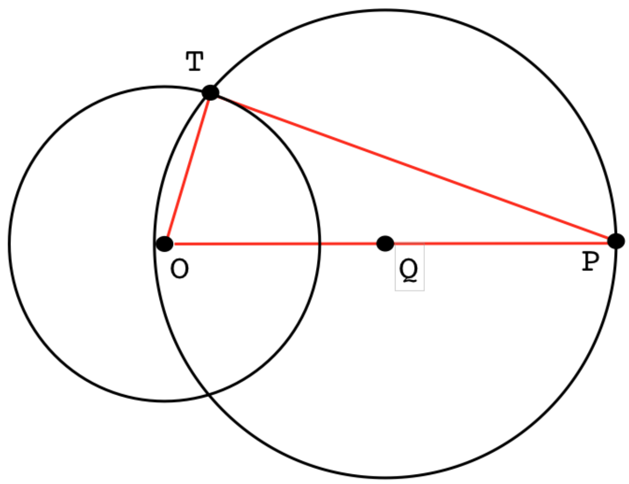
□

construction of a tangent

Thales theorem provides a way to construct the tangent to a circle that passes through any exterior point P — actually, there are two such lines.

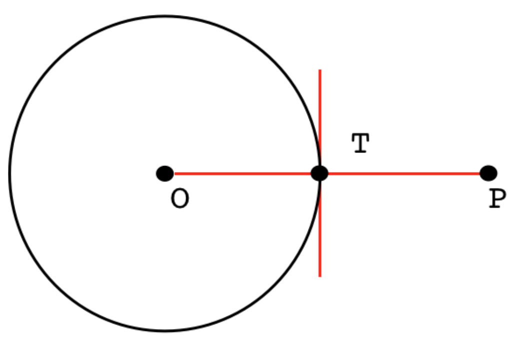


Use OP as the diameter of a circle. Draw the line segment OP and divide it in half by erecting the perpendicular bisector at Q. Use that point Q as the center of a new circle with radius OQ. The point T is the intersection of the two circles.



By the theorem, $\angle OTP$ is a right angle, and since OT is a radius of the original circle, TP is the tangent to the smaller circle at the point T.

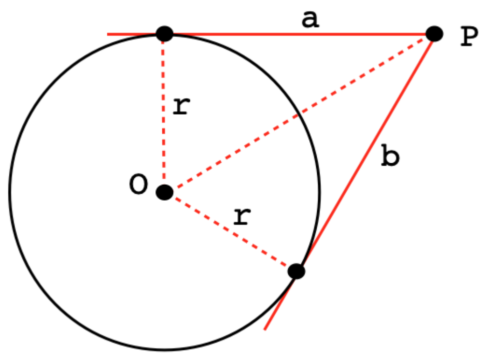
To construct a tangent on a circle at a given point T:



Extend OT to P so that OP is twice the radius OT. Construct the perpendicular bisector at T. The bisector is also the tangent of the circle.

• From any external point P, one can draw two tangents to a circle. These two tangents have the same length.

• From any external point P, the line to the center of the circle bisects the angle between the two tangents, as well as the angle between the radii drawn to the two points of tangency.



Proof.

The angle between a tangent and the radius to the point where it touches the circle, is a right angle. For a pair of tangent lines from a given point, there are two such points on the circle.

The base lengths are both radii, so they are equal, and there is a shared side (the dotted line segment OP).

Therefore the two triangles are congruent, by hypotenuse-leg in a right triangle (HL).

The two congruent sides a and b are the same length.

$$a = b$$

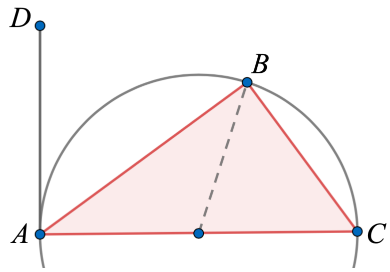
For any point external to a circle, two tangents to the circle can be drawn, of equal length. The line from the point to the center of the circle bisects the angle between the two tangents.

□

tangent-chord theorem

A tangent is a line that just touches the circle (or another curve, like a parabola). By definition it touches the circle at a single point, and is perpendicular to the radius which extends to that point.

• The arc swept out between a tangent and a chord is equal to the arc lying between the point of tangency and the point where the chord meets the circle.



Proof.

By Thales' theorem, $\angle B$ is right, so $\angle BAC$ and $\angle BCA$ are complementary.

The tangent $DA \perp AC$, so $\angle BAC$ is complementary to $\angle DAB$.

It follows that $\angle DAB = \angle BCA$.

Therefore, they cut the same arc of the circle.

□

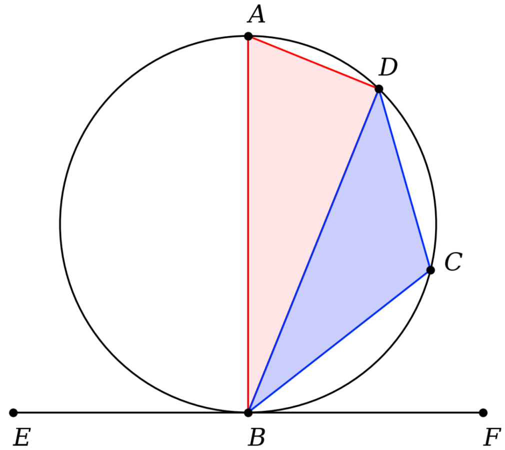
This is Euclid III.32, the tangent-chord theorem. Acheson calls this the alternate segment theorem.

Here is Euclid's proof:

In the figure below, let EF be tangent to the circle at B and AB be a diameter of the circle. $\angle ABF$ is a right angle.

I claim that $\angle DBF = \angle BAD$, and they cut the same arc DCB.

Also, $\angle DBE = \angle DCB$, and they cut the same arc DAB.



Proof.

By Thales' theorem $\angle ADB$ is right, so by sum of angles $\angle BAD + \angle ABD$ is right.

Therefore

$$\angle ABD + \angle BAD = \angle ABD + \angle DBF$$

$$\angle BAD = \angle DBF$$

For the second part, $\angle DBE + \angle DBF$ equals two right angles.

By the **quadrilateral supplementary theorem** (Euclid III.22)

$$\angle BAD + \angle BCD = 180$$

and since $\angle DBF = \angle BAD$ we have

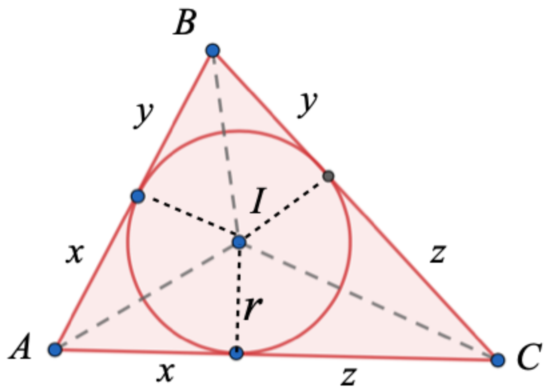
$$\angle DBE + \angle DBF = \angle DBF + \angle BCD$$

$$\angle DBE = \angle BCD$$

□

incircle

For any triangle it is possible to draw a circle, called its incircle, which is tangent to all three sides. When stated in this way, it does not seem obvious how to actually do it.



Proof.

Let AI be the bisector of $\angle A$ and BI be the bisector of $\angle B$, meeting at I.

Recall that if we construct the bisector of an angle, say AI bisecting $\angle A$, then every point on the bisector is equidistant from the sides of the original angle.

So one can draw two right triangles with sides r and AI, which are congruent by HL, explaining the labels x.

The point I is equidistant from side c opposite $\angle C$ and side b opposite $\angle B$.

But the same thing can be done for BI bisecting $\angle B$.

Then draw the circle on center I with radius r, and we're done. This circle is the incircle for $\triangle ABC$.

r is a radius and at the point where it touches the side, perpendicular. Thus the three sides are each tangent to the incircle.

$\square$

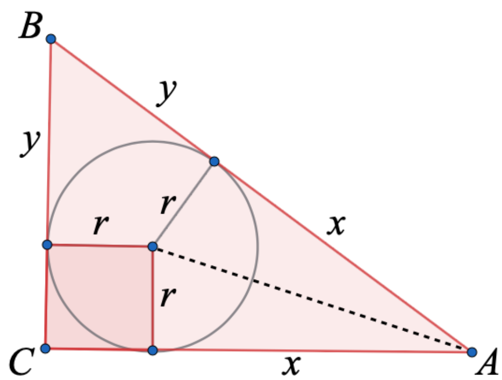
We can obtain a simple result about the area of the triangle. It is the sum of two copies of each of the three types of right triangle: $\mathcal{A} = rx + ry + rz$. But the perimeter of the triangle is twice $x + y + z$, the semi-perimeter $s = x + y + z$ and then $\mathcal{A} = rs$.

Pythagoras incircle proof

Dunham gives this as a problem.

Let right $\triangle ABC$ have sides opposite a, b and c, as usual.

Let the parts of the sides be x and y. Note that since this is a right triangle, what would be z becomes r, the radius of the incircle. Also, $c = x + y$ is the hypotenuse.



Walk around the triangle. The perimeter p is

$$y + x + x + r + r + y$$

$$p = 2x + 2y + 2r$$

But $x + y = c$ and $p = a + b + c$ so

$$a + b + c = 2c + 2r$$

$$2r = a + b - c$$

Now, the area of the whole $\triangle ABC$ is

$$K = \frac{1}{2}ab$$

From its three components, the area is also

$$K = r^2 + rx + ry$$

$$= r^2 + rc$$

Hence

$$\frac{1}{2}ab = r^2 + rc$$

$$2ab = (2r)^2 + 4rc$$

Substitute for $2r$ and crank through some algebra:

$$2ab = (a + b - c)(a + b - c) + 2(a + b - c)(c)$$

$$= a^2 + b^2 + c^2 + 2ab - 2ac - 2bc + 2ac + 2bc - 2c^2$$

Hence

$$0 = a^2 + b^2 + c^2 - 2ac - 2bc + 2ac + 2bc - 2c^2$$
$$0 = a^2 + b^2 + c^2 - 2c^2$$
$$0 = a^2 + b^2 - c^2$$
$$a^2 + b^2 = c^2$$

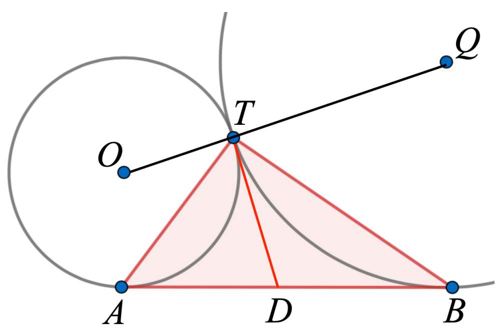
□

Not exactly pretty, but a clever construct, and it works.

Euclid III.12

Two circles are tangent to each other at T.

In one circle draw the radius to T and extend it. This line goes through the center of the second circle.



Proof.

Draw the tangent to the first circle at T. This line goes through a single point (T) on both circles.

The line perpendicular to it (the extension through T) is a radius of the second circle.

□

Euclid's proof of III.12 is by contradiction.

Proof.

Suppose the straight line OQ does not go through T.

By the triangle inequality, $OQ < OT + QT$. where OT and QT are radii of the two circles.

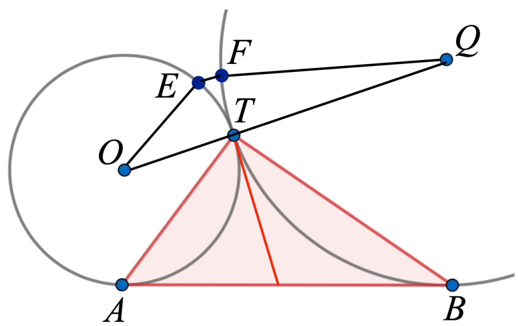
Then OQ must cut the two circles at different points, say E and F.

The straight line is

$$OQ = OE + EF + QF$$

$$OQ > OE + QF$$

where OE and QF are radii of the two circles.



But $OT + QT = OE + QF$

OQ cannot be both less than the sum of radii and greater than the same sum. This is a contradiction.

Therefore OTQ are collinear. OQ goes through T.

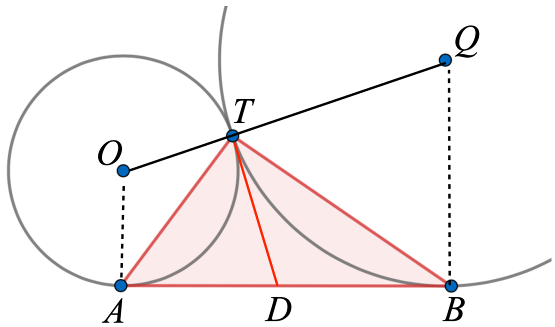
□

problem

Draw AT and BT. Prove that $\angle ATB$ is a right angle.

Solution.

Draw the tangent line to both circles at T, the red line in the figure below.



Draw radii of both circles to their respective tangent points at A and B.

DA and DT are both tangent to the circle on center O, while DT and DB are both tangent to the circle on center Q

The tangents are all equal in length, so $\triangle DAT$ and $\triangle DBT$ are both isosceles.

Since the base angles are equal, we have (by the external angle theorem) that

$$2\angle ATD = \angle BDT$$

$$2\angle BTD = \angle ADT$$

But the sum of the two right-hand sides is equal to two right angles.

Therefore, the sum of the two left-hand sides is also equal to two right angles. Hence

$$\angle ATD + \angle BTD = 90 = \angle ATB$$

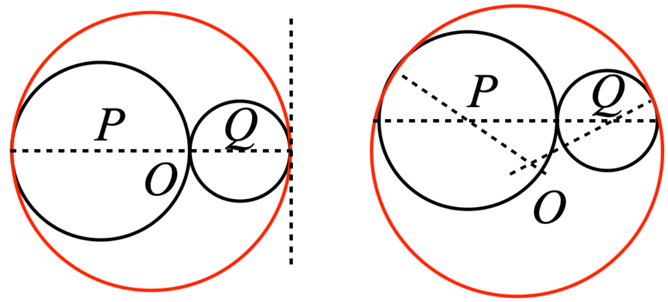
□

Alternatively, since $\triangle DAT$ and $\triangle OAT$ are both isosceles, the sum of the angles making up $\angle OTD$ is equal to the sum of angles making up $\angle OAD$.

But $\angle OAD$ is a right angle, therefore so is $\angle OTD$.

tangents

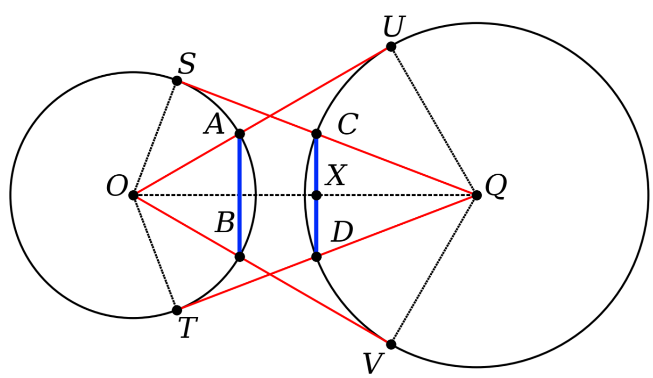
As indicated in the previous problem, if two circles just touch one another at a single point, and the tangent is drawn at that point, then it is tangent to *both* circles. The perpendicular to the tangent at that point is a radius of both circles. Thus a single line is collinear with both diameters.



So then, what about three circles? Clearly, the third circle can be drawn with the joined diameters of the smaller two as its diameter. Or it can be drawn somewhat larger (right panel).

The diameter through P and Q in the second case must be extended to meet the large circle, so apparently, the arrangement in the left panel is the circle with the *smallest* diameter that is also tangent to the two circles on centers P and Q. (See Euclid III.7 for a proof).

The eyeball theorem



This problem is from Acheson's Geometry book (Fig 131). We have two circles with tangents drawn from the center of each circle to the other one.

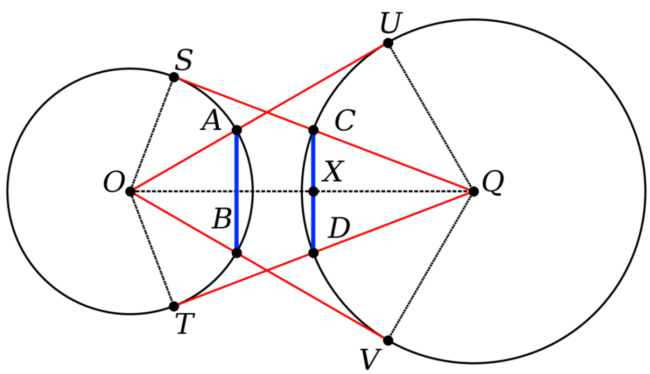
We are to show that the lengths of the chords formed by the tangents as they exit the originating circle are equal.

Solution.

We have labeled a number of points.

$$\triangle OSQ \cong \triangle OTQ$$

by HL. So $\angle OQS = \angle OQT$. As radii, $CQ = DQ$.



Thus $\triangle CXQ \cong \triangle DXQ$ by SAS. They are two right triangles and have $CX = DX$. If the bisector of a chord goes through the circle center it is perpendicular.

Let the distance between centers be d. Then we have similar triangles such as

$$\triangle OSQ \sim \triangle CXQ$$

Let $CX = x$ and the radii be R and r. Similar ratios gives:

$$\frac{x}{r} = \frac{R}{d}$$

Rearranging $x = Rr/d$. But this is symmetric in R and r which implies that $AB = CD$, since we have the same formula for both.

penny-farthing problem problem

Here is a problem with a fascinating history (see Acheson). Find an expression for D in terms of a and b.

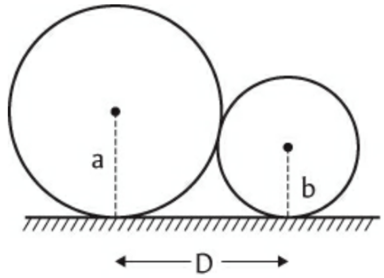


Fig. 161 A penny-farthing problem.

Its solution is very easy, so I will not give it here but I encourage you to try.

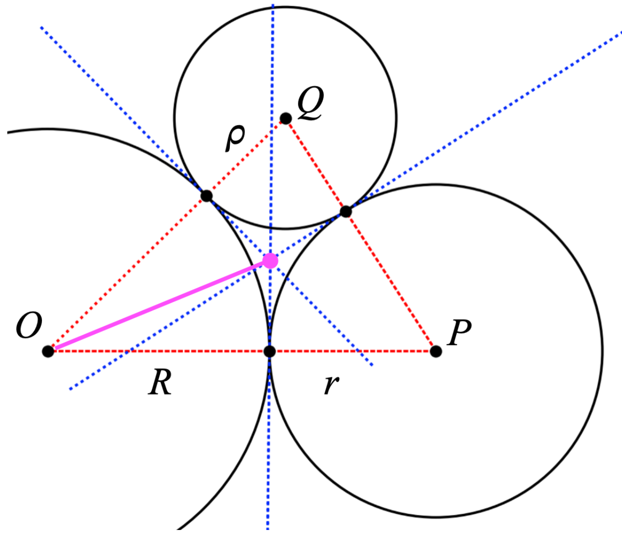
problem

Harvard 1899 exam:

> **4. Three unequal circles are so situated that each of them is externally tangent to the other two. At the points of contact tangents are drawn. Prove that these three tangents meet in a point.**

Solution.

The tangent lines are perpendicular to a radius drawn to the point of tangency. As a result, two corresponding radii meet at the point of tangency and the two corresponding centers are co-linear with the point of tangency.



Therefore, we have a triangle with sides as shown. The tangent lines are perpendicular to the sides of the triangle, but will not, in general, pass through a vertex or center of the third circle, for any pair of circles and their tangent line.

But for each vertex of the triangle, the bisector of the angle gives congruent triangles, in which one of the sides is a blue dotted line, forming a right angle with the radius. The hypotenuse of one such pair is shown in magenta.

The two triangles that share the magenta line as hypotenuse are congruent by hypotenuse-leg in a right triangle (HL), and therefore the two blue dotted lines meeting in the center have equal length. Now do the same for the circle with radius r and then for the circle with radius ρ.

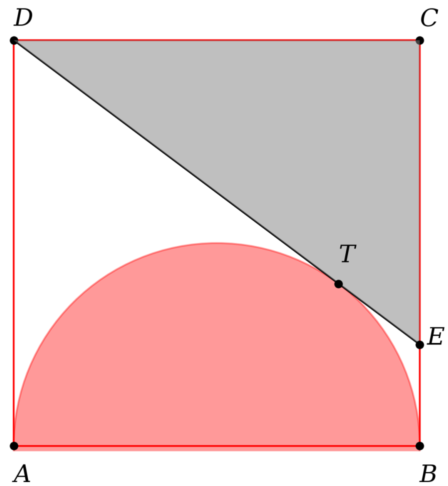
The blue-dotted lines are thus all equal. So they can be used to draw a circle that just touches the sides of the triangle. That circle is called the incircle of the triangle.

The point where these tangents meet is the incenter of the triangle. Since the incenter exists, the three angle bisectors are concurrent, the point where the tangents meet is the same and it also exists.

□

problem

Here is a problem from Paul Yiu. We have a semicircle inside a square with one side of the square as the diameter. The tangent is drawn as shown.



Prove that the triangle is a $3-4-5$ right triangle.

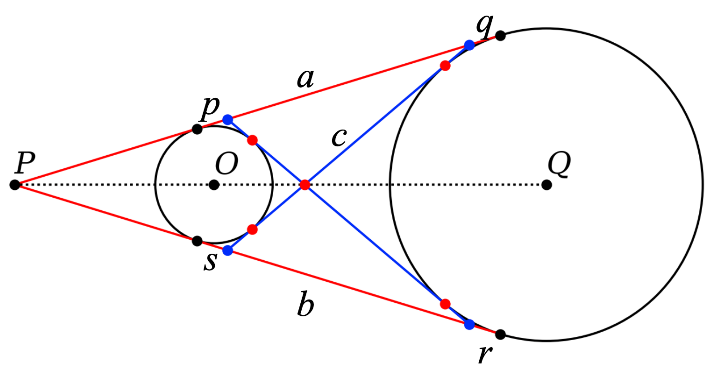
Scale the square so that the side has length s. We will use the property that the two tangents to any circle from an exterior point are equal. Thus, the distance from the Dt to the point of tangency at T is s. Let the other short tangents have length x.

Then Pythagoras says that:

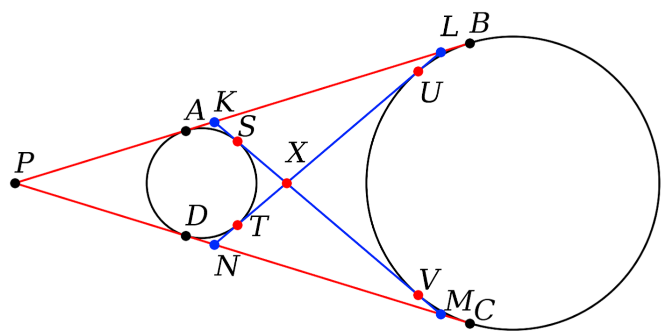
$$s^2 + (s-x)^2 = (s+x)^2$$
$$s^2 = 4sx$$
$$s = 4x$$

So $CD = DT = 4x$, and $CE = s - x = 3x$. This is a right triangle with sides in the ratio 3:4 so it is a 3:4:5 right triangle.

double tangents



This is another problem from Paul Yiu. Let lines be drawn, each one tangent to both of two circles. There are four such lines. The diagram above has simplified labels. In the next one, the points are labeled for reference.



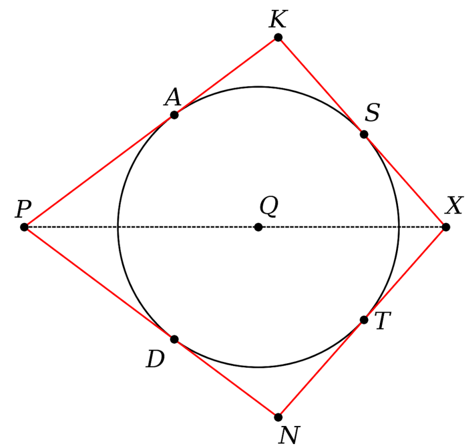
Let PB and PC be tangent to both circles. Furthermore draw the crossed tangents KM and LN.

The four pairs of short tangents have lengths p, q, r and s. For a single pair originating from the same point, the two of them are equal.

We want to determine the relationships between the lengths a, b, c, p, q, r, s.

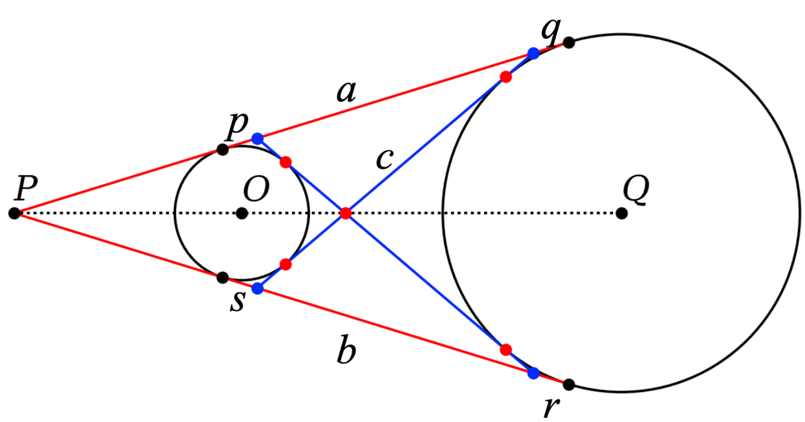
The tangents crossing at X are labeled c in the first diagram and SV and TU in the second. These are equal, because they are composed of two tangents to each circle extended from the intersection in the middle at X.

Before going further, let's look at a simplified diagram.



If PX is drawn through the center of the circle at Q, the angles at P and X are bisected and then $\triangle PKX \cong \triangle PNX$ by ASA. It follows that $XK = XN$, and since $XS = XT$, we have $SK = TN$. Similarly, $KA = ND$.

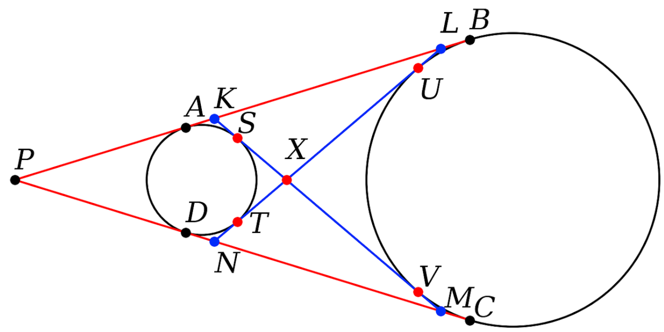
So, looking again at the problem



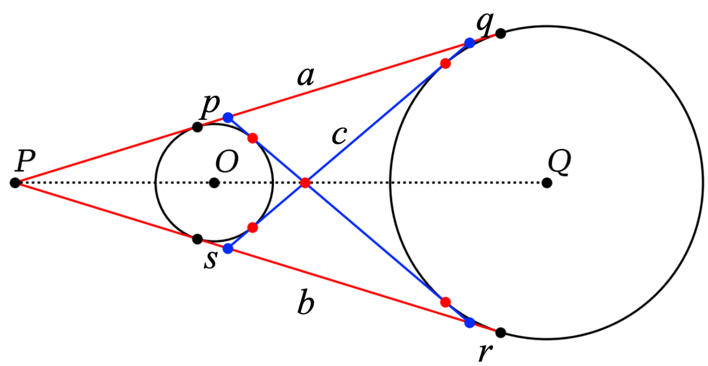
we see that $p = s$ and $q = r$. The next question is the relationship between p and q. We can write other equalities based on tangents from different points:

$$LA = p + a = q + c = LT$$

$$KV = p + c = a + q = KB$$



Adding and cancelling $a + c$, we have that $p = q$.



So all four of the short segments are also equal: $p = q = r = s$. And substituting above, we find that $a = c$.

Finally, we derive the position of P, the origin point of the tangents that do not cross. We also derive the position of X, the intersection of the tangents that do cross,

Since both lie on the line between centers, we worry only about the horizontal dimension. Let $PO = y$. Let the radius of the circle on center O be R and that of the other circle on Q be ρ.

Let the distance between the two circles be d. Then we have two similar right triangles, with

$$\frac{y}{R} = \frac{y + d}{\rho}$$

$$\frac{\rho}{R} = \frac{y + d}{y} = 1 + \frac{d}{y}$$

which is easily solved for y.

$$y = \frac{d}{\rho/R - 1}$$

To find $OX = x$, we have a closely related equation

$$\frac{x}{R} = \frac{d - x}{\rho}$$

which is worked out by similar algebra.

$$y = \frac{d}{\rho/R + 1}$$

Chapter 23

Arcs of a circle

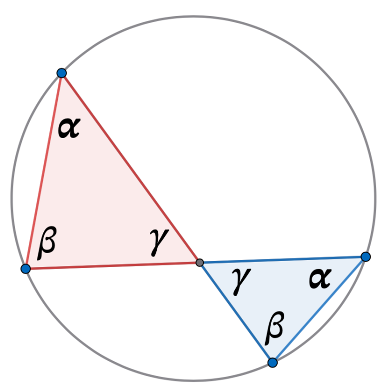
It is natural to think about angles formed from a vertex lying on the periphery of a circle and ask about their relation to the arcs they cut off.

The **inscribed angle theorem** says that a vertex placed at any point on the periphery of a circle, forming an angle that corresponds to the same arc as a central angle, is equal to *one-half* the central angle.

Since the measure of the central angle is equal to the measure of the arc, by definition, we have that twice the peripheral angle is equal to the arc that subtends it.

A corollary is that whenever two peripheral angles correspond to the same arc, they are equal (**equal angles** $\iff$ **equal arcs**).

In the figure below, the two angles marked α are equal, because they correspond to the same arc of the same circle.

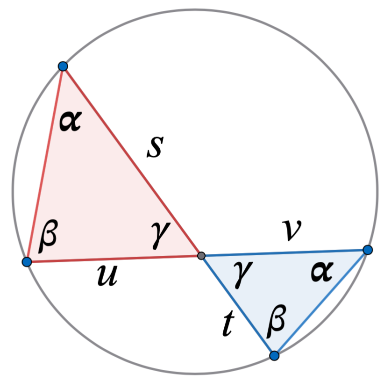


Similarly, the two angles marked β are equal, for the same reason. Then by sum of angles (or vertical angles), the two angles marked γ are also equal. We have two similar triangles.

If two chords of the circle cross, the product of the components is a constant.

Proof.

If we label the sides of the triangles



u and t are the sides opposite $\angle\alpha$ so we have

$$\frac{u}{t} = \frac{s}{v}$$

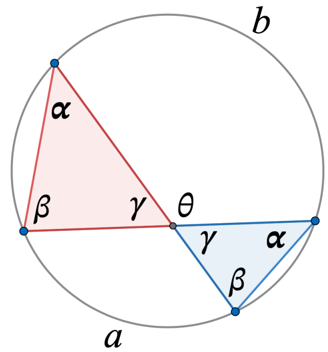
rearranging

$$u \cdot v = s \cdot t$$

□

This is the **crossed chord theorem** (product of lengths).

Arcs of intersecting chords



Given two crossed chords, θ is the average of the opposing arcs a and b.

Proof.

Take the same two similar triangles and consider the angle θ as shown in the figure above. By the corollary of the inscribed angle theorem, we have that

$$2\alpha = a \qquad 2\beta = b$$

But θ is the external angle to both triangles, so $\theta = \alpha + \beta$ and then

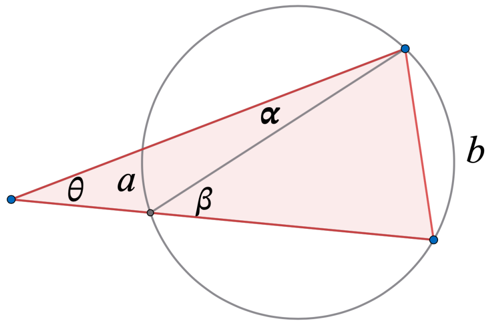
$$2\theta = 2\alpha + 2\beta = a + b$$

$$\theta = \frac{a+b}{2}$$

□

external vertex

Rather than having the vertex on the circle, suppose it lies outside.



If the angle lies outside the circle, then twice its measure is the difference between the two arcs, the one farther away minus the closer one.

Proof.

Again, α corresponds to arc a and β to arc b.

$$2\alpha = a \qquad 2\beta = b$$

But β is the external angle to the small triangle with θ, such that

$$\beta = \theta + \alpha$$

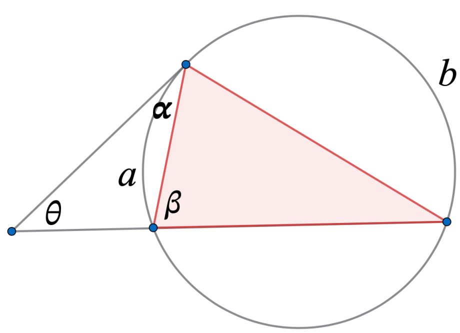
$$2\theta = 2\beta - 2\alpha$$

$$2\theta = b - a$$

$$\theta = \frac{b-a}{2}$$

□

tangent and secant



Rather than two secants, we now have a secant and a tangent. The result is the same as previously.

$$\theta = \frac{b-a}{2}$$

Proof.

We rely on the tangent-chord theorem, which says that if α is the angle between a chord and a tangent, then the corresponding arc has the same relationship as for two chords emanating from the vertex, namely:

$$2\alpha = a$$

$$2\beta = b$$

But

$$\beta = \alpha + \theta$$

$$2\theta = 2\beta - 2\alpha$$

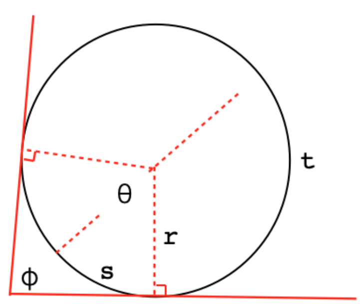
$$2\theta = b - a$$

$$\theta = \frac{b-a}{2}$$

□

two tangents

We showed previously that when two tangents are drawn from an exterior point, one can draw two right triangles that share the hypotenuse and have another side equal to the radius, so they are congruent by hypotenuse-leg in a right triangle (HL).



Let the whole arc between the two right angles be s the short way and t the long way around the circle, and let ϕ be the external angle. By analogy with the results above, we expect that

$$\phi = \frac{t-s}{2}$$

Proof.

We could use congruent triangles, but instead just note that the sum of angles in any quadrilateral is equal to four right angles. Hence

$$\phi + \theta = \text{ two right angles}$$

In terms of arc $\theta = s$ and $s+t =$ two right angles. Substituting into the last equation

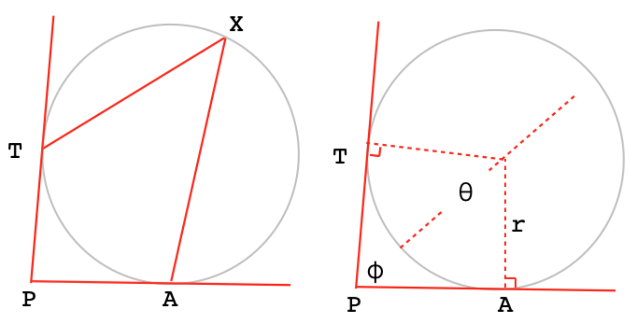
$$\phi + s = \frac{s+t}{2}$$

$$\phi = \frac{t-s}{2}$$

□

problem

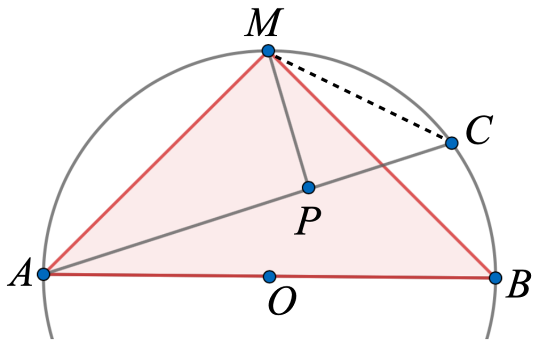
Relate the angle at P to the one at X.



By the previous example, $\theta + \phi = 180$. But $\angle X = \theta/2$. Hence

$$P = 180 - \theta = 180 - 2\angle X$$

problem



Let AB be a diameter of the circle on center O. Let M be found such that $AM = BM$, so $\angle MAB = \angle MBA$ and both are one-half of a right angle.

Draw an arbitrary chord from A such as AC. Draw $MP \perp AC$.

$\triangle MPC$ is isosceles.

Proof.

Draw MC.

$\angle ACM = \angle PCM$ is subtended by chord AM, which is one-quarter of the circle.

Since $\angle MPC$ is right, $\angle PMC$ is also one-half of a right angle, by sum of angles.

It follows that $\triangle MPC$ is isosceles and $MP = CP$.

□

Chapter 24

Chords in a circle

Chords are straight lines inside a circle, with endpoints which lie on the circle. If a chord is extended (produced, in Euclid's terminology), then the extended chord becomes a secant, with part of the line lying outside the circle.

- If two arcs are equal, the corresponding chords are also equal, and vice-versa.

Proof. (forward)

Draw the radii flanking both arcs. By definition, we have equal angles for the equal arcs, and therefore congruent triangles by SAS. The equal chords are opposing sides to the central angle in the congruent triangles.

□

Proof. (converse)

Draw the radii flanking both chords. We have congruent triangles by SSS, so the central angles are equal. By Euclid III.26, we have equal arcs.

□

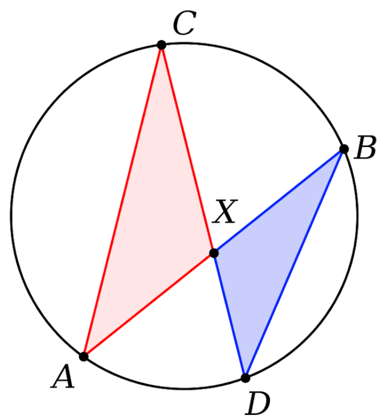
We may occasionally not even point out the equivalence, but just assume that if two arcs are equal, the corresponding chords are also equal, and vice-versa.

inequality

Suppose we have one angle greater than another, or one chord greater than another?

Proof. (Sketch). Start with two chords in a circle. Draw the radii for both. Then, by the **hinge theorem**, Euclid I.24, the greater angle lies opposite the greater third side. Thus, the greater chord lies in the greater circumference because they are connected through the central angle. □

crossed chords



Previously, we showed that when two arcs cross in a circle, each of the two equal vertical angles is equal to the *average* of the two arcs they cut out. It is also apparent that the two triangles formed are similar.

Proof. $\angle A$ and $\angle D$ cut the same arc, and there are vertical angles at X. Hence $\triangle AXD \sim \triangle DXB$.

In two similar triangles, the sides opposite equal angles are in the same ratio, so:

$$\frac{AC}{BD} = \frac{CX}{BX} = \frac{AX}{DX}$$

It follows that

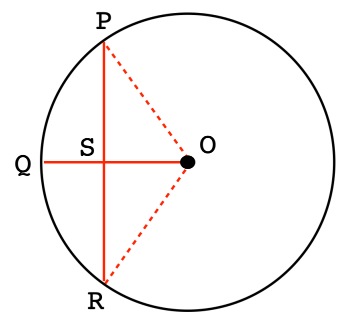
$$AX \cdot BX = CX \cdot DX$$

The products of the two parts of each chord are equal.

bisected chord theorem

Given any chord of a circle and a line perpendicular to it, if the perpendicular is a radius of the circle then the chord is bisected.

Conversely, if the perpendicular is a bisector, then it is also a radius.



Proof.

For the forward theorem: we have that OQ is a radius (it lies on the diameter), and it is perpendicular to chord PR.

We have $OP = OR$, as radii of the circle. Also, side OS is shared, and there are right angles at S. So $\triangle OPS \cong \triangle ORS$ by hypotenuse-leg in a right triangle (HL).

We conclude that $PS = SR$.

□

In addition, since $\triangle OPS \cong \triangle ORS$, $\angle POS = \angle ROS$, the arcs PQ and QR are cut out by equal central angles, therefore they are equal.

The converse theorem was established when we derived the method to construct a circle given any three points on it.

Briefly, all points that are equidistant from the two ends of a chord must lie on its perpendicular bisector. Since the center of the circle containing the chord forms radii when joined to the two ends, it must also lie on the perpendicular bisector.

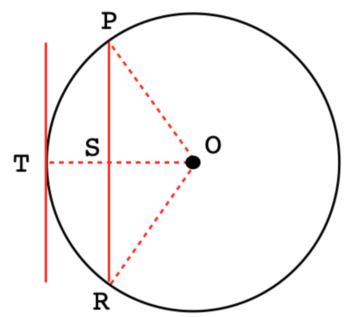
Therefore the perpendicular bisector is a diameter of the circle.

□

One consequence is that if a line is the perpendicular bisector of a chord of the circle, it is also the perpendicular bisector of any other chord parallel to the first.

Furthermore, if we have a tangent and any chord that is parallel to it, then the radius drawn to the point of tangency bisects the chord.

This follows simply from the definition of the tangent as perpendicular to the radius at the point of tangency.



By our previous theorem, OT bisects PR so that $PS = SR$.

∘ Given a point of tangency on a radius perpendicular to any chord, not only the chord but the arc lengths are evenly divided by the radius. $PT = RT$.

We can also note that if the distance from S to the periphery of the circle is d and the height of the half chord is h, by the crossed chord theorem we have

$$(2r - d)(d) = h^2$$

which can be solved as a quadratic in d.

$$d^2 - 2rd + h^2 = 0$$

$$\begin{aligned} d &= \frac{2r \pm \sqrt{4r^2 - 4h^2}}{2} \\ &= r \pm \sqrt{r^2 - h^2} \end{aligned}$$

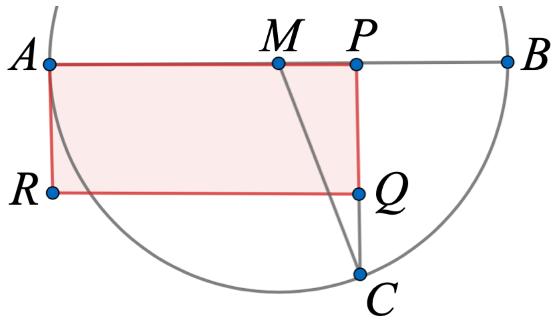
which is already fairly obvious since $OP = r$ and, it's the hypotenuse in a right triangle with $PS = h$, so OS is exactly the square root, while r minus that is just d.

Euclid II.14

As an aside, suppose it is desired to construct the square equal in area to a given rectangle $APQR$.

Extend AP to B such that PB = PQ.

Bisect AB to find center M for a circle with radius $AM = BM$.



By II.5

$$AP.PB = MB^2 - MP^2$$

$$AP \cdot PQ = MC^2 - MP^2 = PC^2$$

Thus PC is the side of the desired square.

This can also be done by similar triangles, since $\triangle CAP \sim \triangle BCP$. Thus $PB/PC = PC/AP$ so $AP \cdot PB = PC^2$.

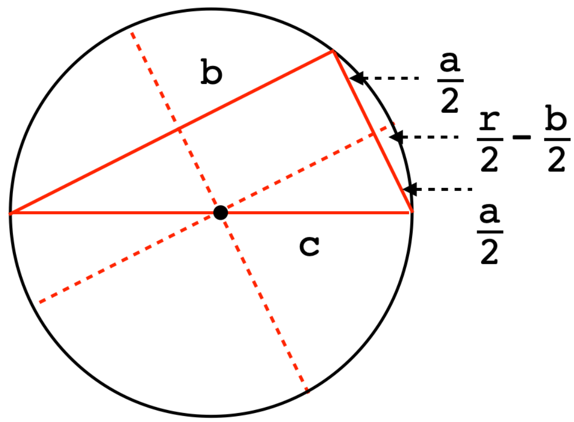
□

Since we can get from Pythagoras to the crossed chord theorem, it may not be so surprising that we can go backward.

Pythagorean theorem by crossed chords

Proof.

A right triangle with sides a, b and c is inscribed in a circle. By Thales' theorem, the hypotenuse c is a diagonal. Draw two more diagonals, parallel to the sides a and b. We can use similar right triangles to show that these bisect sides a and b.



So then, we will use the crossed chords theorem, multiplying the two halves of a together. The question is, what are the two parts of the dotted diameter that crosses a at right angles?

The long part is $c/2 + b/2$. Can you see why?

The very short part is $c - (c/2 + b/2) = c/2 - b/2$. The reason is that when added to the long part, the result is just

$$c/2 + b/2 + c/2 - b/2 = c$$

We have

$$\frac{a}{2} \cdot \frac{a}{2} = (\frac{c}{2} + \frac{b}{2})(\frac{c}{2} - \frac{b}{2})$$

$$a \cdot a = (c + b)(c - b) = c^2 - b^2$$

$$a^2 + b^2 = c^2$$

□

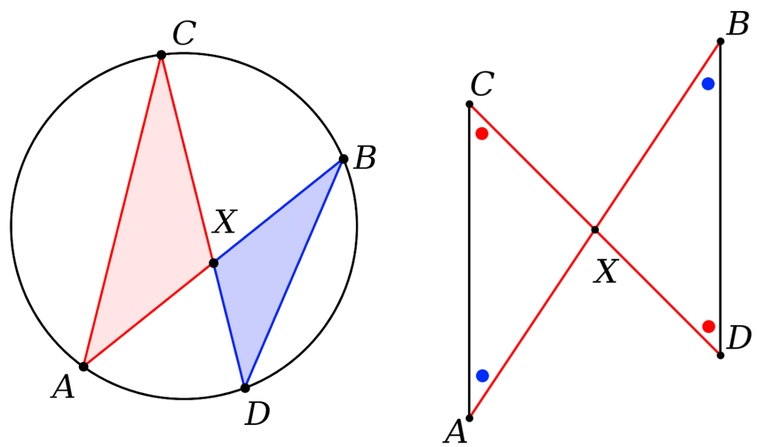
• The crossed chords formed for the vertices of any cyclic quadrilateral form two pairs of similar triangles. The sides that are in the same ratio lie on different chords.

From above, for the cyclic quadrilateral:

$$\frac{AC}{BD} = \frac{CX}{BX} = \frac{AX}{DX}$$

$$AX \cdot BX = CX \cdot DX$$

The products of the two parts of each chord are equal.



For parallel lines, it is different.

$$\frac{AC}{BD} = \frac{CX}{DX} = \frac{AX}{BX}$$

$$AX \cdot DX = CX \cdot BX$$

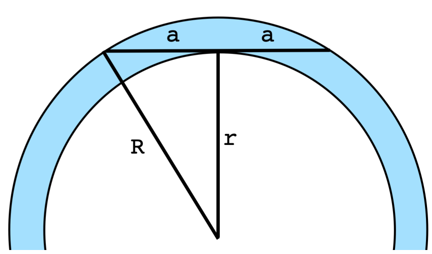
The products of the two parts of the crossing lines are *not* equal.

If $AC \parallel BD$ in the circle, then the triangles are isosceles so it all works out.

If in doubt, make sure that the sides you think are corresponding sides lie opposite equal angles.

another view

Another way to look at this is shown in the following diagram:

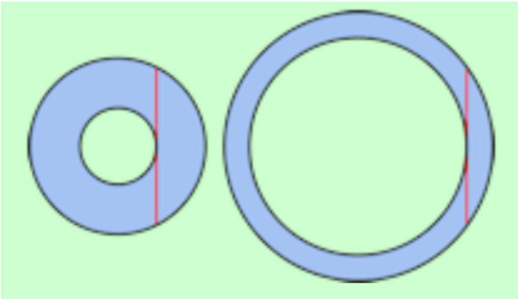


Given a larger circle of radius R and a chord of it with length $2a$, draw the smaller circle of radius r that just touches the chord. Then, by the Pythagorean theorem we have that

$$R^2 - r^2 = a^2$$

But $R^2 - r^2$ (times π) is the area of the annulus or ring, the difference between the area of the outer circle and that of the inner one. This is equal to $\pi\ a^2$.

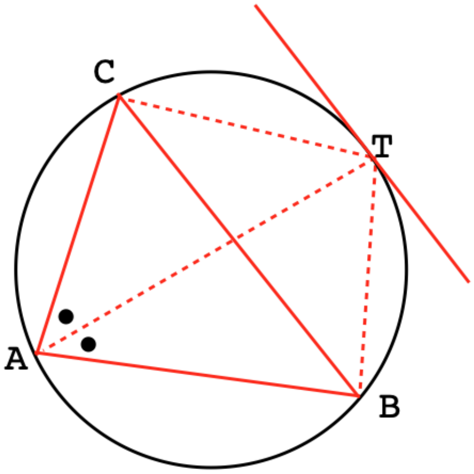
So then, there is family of circles of different sizes that can be drawn, each with a chord of length $2a$ (as long as the radius is greater than a). For each of those circles, one can draw the annulus as we did above, and find that the area is exactly the same.



https://puzzles.nigelcoldwell.co.uk

problem

Given that the line touching the circle at T is a tangent and that BC is parallel to it. Show that the angles marked with black dots are equal.

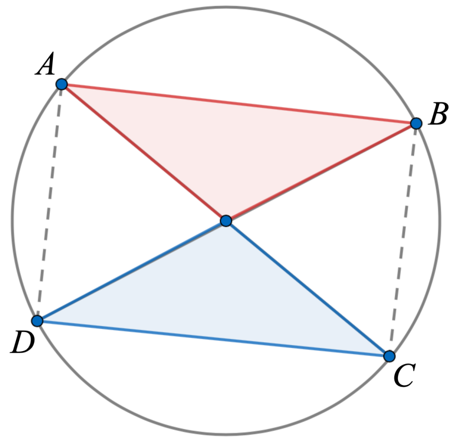


I think the idea of the problem was to use similar triangles somehow. But given our previous work, this is a trivial consequence of the fact that the arcs CT and TB are equal.

So, as peripheral angles corresponding to equal arcs, $\angle CAT = \angle TAB$.

diameters form a rectangle

∘ When any two diameters of a circle are drawn and consecutive chords on the circle joined together, the result is a rectangle.



Proof.

A simple proof relies on Thales' circle theorem. All four vertices are right angles.

It follows that both pairs of opposing sides are parallel. For example

$$AD \perp AB \perp BC \Rightarrow AD \parallel BC$$

So $ABCD$ is a parallelogram with right-angled vertices, a rectangle.

□

One can, of course, also use the fact that certain angles are subtended by equal arcs, so they are equal (for example $\angle DAC = \angle DBC$). Then use congruent triangles.

problem

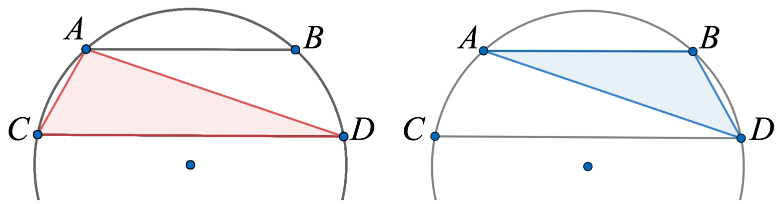
Construct a square circumscribed by a circle, i.e. all four vertices lie on the circle.

Solution. (Sketch).

The diagonals originate at vertices, by definition. We also know that the diagonals of a square are perpendicular. Hence, erecting the perpendicular bisector of any diagonal of the circle locates four points on the circle that are vertices of a circumscribed square.

□

parallel chords joined by equal arcs and chords



Given any two parallel chords in a circle, AB and CD. Draw AD.

The inscribed angles formed, $\angle ADC$ and $\angle BAD$, are equal by alternate interior angles,

Thus arc AC = arc BD.

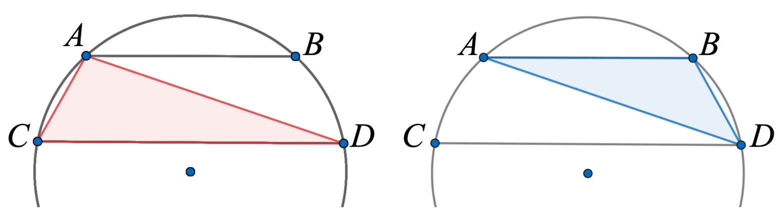
Equal arcs have equal chords so $AC = BD$.

□

It is a simple matter to extend this result to triangles such as $\triangle ABC \cong \triangle BAD$.

Proof.

Since the arcs are equal $\angle ABC = \angle BAD$.



But $\angle BDA$ and $\angle ACB$ both cut the same arc, AB.

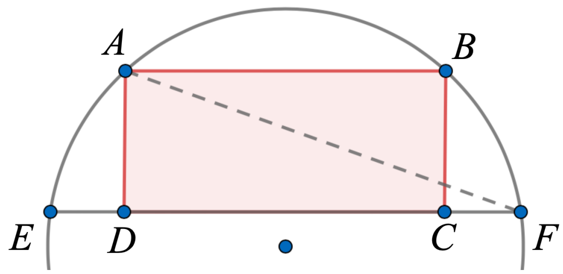
So $\triangle ABD$ and $\triangle BAC$ have three angles equal, and they share the same side AB opposite one pair of equal angles.

$\triangle ABD \cong \triangle BAC$ by ASA.

□

rectangle side on a circle

We derive a useful theorem about any rectangle in a circle, and its converse.



Let $ABCD$ be a rectangle such that A and B lie on the circle, but C and D do not.

Then the extensions of DC are equal, namely $DE = CF$.

Proof.

Draw AF. As before, $\angle AFE = \angle BAF$.

Since the inscribed angles are equal, so are the arcs: arc AE = arc BF.

Equal arcs have equal chords, hence $AE = BF$.

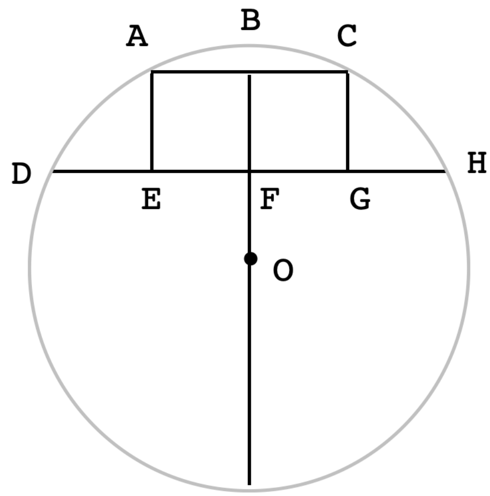
So $\triangle ADE \cong \triangle BCF$ by HL.

It follows that $DE = CF$.

□

Proof. (Alternate).

Suppose that $ACGE$ is a rectangle. Let BF be the perpendicular bisector of AC.



Then, the extension of BF passes through the center of the circle (it is part of a radius), by the converse of the theorem discussed earlier. Every point that is equidistant from A and C, including the center of the circle, lies on the extension of BF.

OB is also the perpendicular bisector of DH, since $ACGH$ is a rectangle. So

$$DF = FH$$

but $AB = BC = EF = FG$. Subtracting equals from equals:

$$DF - EF = FH - FG$$

$$DE = GH$$

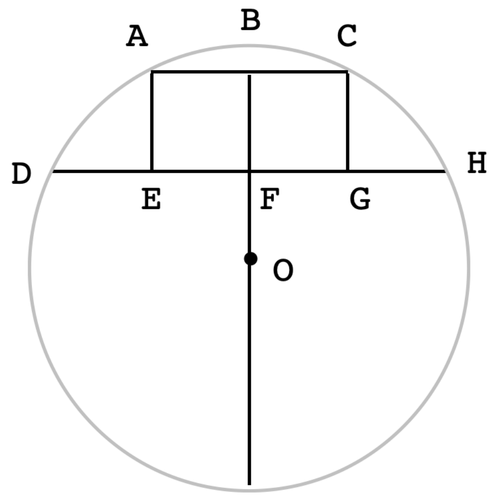
□

For any rectangle with two vertices lying on a circle and two inside the circle, when the side which connects the inside points is extended both ways to reach the circle, the extensions will be equal.

converse

Above, we started with a rectangle in a circle. Now we start simply with any two parallel chords. Then one will be longer than the other (and may even be a diameter).

Suppose we have $DH \parallel AC$.



Drop $AE \perp DH$ and $CG \perp DH$. By alternate interior angles, $\angle A$ and $\angle C$ are right angles, as well as $\angle AEF$ and $\angle CGF$. Hence $ACGE$ is a rectangle.

□

Then all the consequences of the forward theorem follow. In particular $DE = GH$.

And of course, the perpendicular bisector of AC is also the perpendicular bisector of DH. (*Proof.* Draw $\triangle OAB$ and $\triangle OCB$ as well as $\triangle ODF$ and $\triangle OHF$ to show that all the angles at B and F are right angles. □)

Extraordinary property

According to Acheson, this theorem comes from a book by Malton where it is described as an "extraordinary property of the circle".

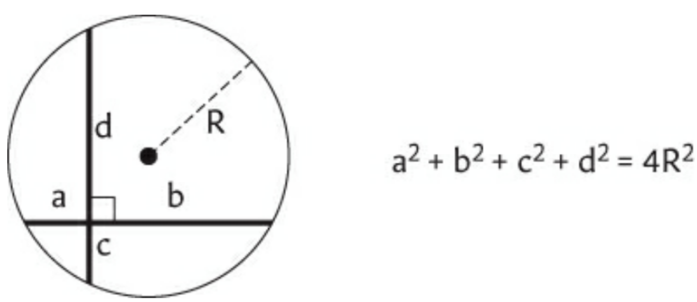


Fig. 110 An 'extraordinary property of the Circle'.

Let two chords of a circle meet at right angles, and let the arms of the chords be $a + b$ and $c + d$.

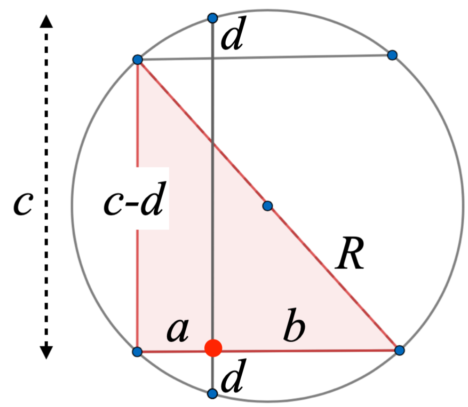
By crossed chords we have $ab = cd$.

We will show that he squares of the four components add up to a constant, and that constant is equal to $(2R)^2$, twice the radius of the circle, squared.

$$a^2 + b^2 + c^2 + d^2 = (2R)^2 = 4R^2$$

The key to the proof is to form a rectangle.

Pick one of the chords, say $a + b$. Draw the diameter of the circle that terminates on one end of each of the chords. The other diameter joins the other ends of chords $a + b$ but hasn't been drawn.



If the four points at the ends of the two diameters are joined to form a rectangle then the short extension at the top is also equal to d, so the height of the rectangle is $c - d$.

We proved this fact about rectangles in a circle **here**.

So then:

Proof.

This is trivial now that we have $c - d$. By the Pythagorean theorem

$$(a + b)^2 + (c - d)^2 = (2R)^2$$

$$= a^2 + b^2 + c^2 + d^2 + 2ab - 2cd$$

But $ab = cd$ so

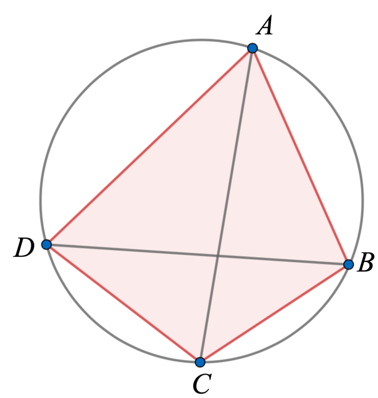
$$4R^2 = a^2 + b^2 + c^2 + d^2$$

$\square$

Chapter 25

Cyclic quadrilateral

There is a wonderfully simple theorem about quadrilaterals (Euclid III.22). A cyclic quadrilateral is a four-sided polygon whose four vertices all lie on one circle.



• For *any* cyclic quadrilateral, the opposing angles are supplementary (they sum to two right angles).

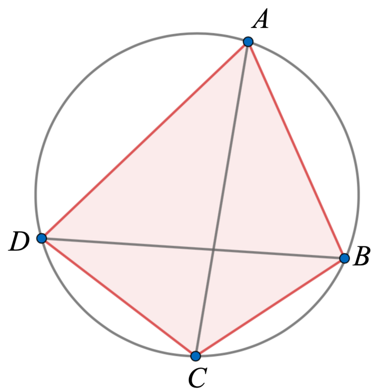
Proof.

Together, opposing angles in a cyclic quadrilateral exactly correspond to the whole arc of the circle.

Since the central angle for that arc is four right angles, the sum of opposing inscribed angles is just one-half that or two right angles.

□

Euclid's proof uses sum of angles:



Proof.

$\angle ADC = \angle ADB + \angle BDC$, subtended by arcs AB and BC.

As angles on equal arcs (III.21) the latter two angles are equal to $\angle ACB$ and $\angle BAC$.

But by I.32, the same two angles plus the total angle at vertex B are equal to two right angles.

□

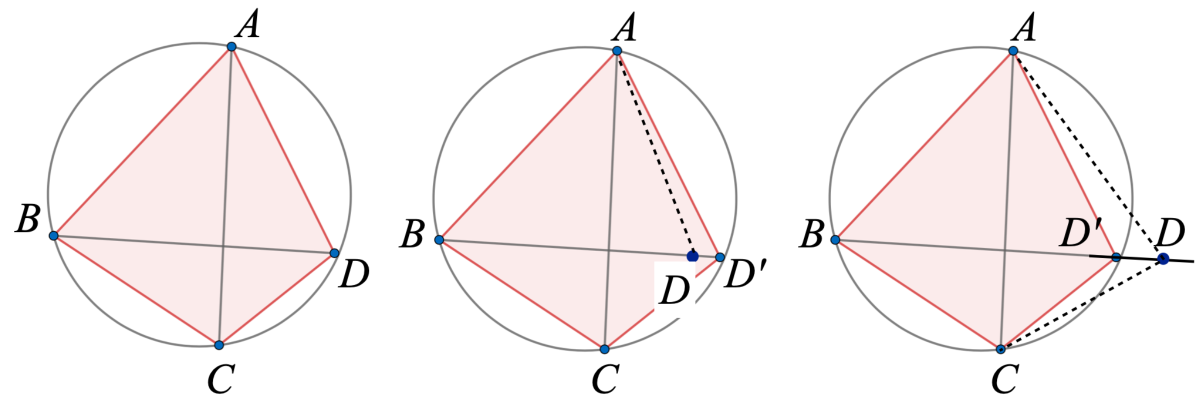
converse of cyclic quadrilateral theorem

This one uses Euclid III.22 rather than III.21.

Let $\triangle ABC$ lie on a circle.

Let point D be such that $\angle ADC$ is supplementary to $\angle ABC$.

Then D lies on the same circle.



Proof.

Aiming for a contradiction, suppose D does not lie on the circle.

Let D be external and D' lie on the point where BD cuts the circle (right panel).

By the forward theorem, $\angle AD'C$ is supplementary to $\angle ABC$ and so equal to $\angle ADC$.

But by Euclid I.21, $\angle ADC < \angle AD'C$.

This is a contradiction. Therefore D is not external.

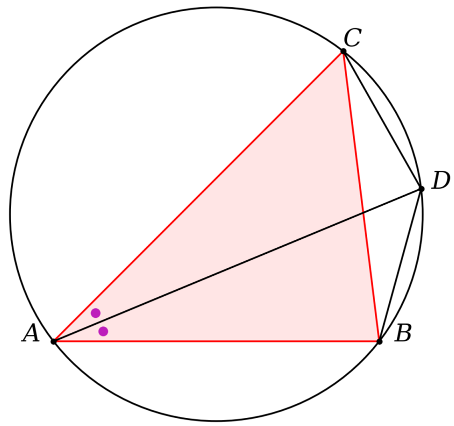
A similar argument will show that D is not internal.

Therefore D lies on the circle and D and D' are the same point.

□

bisected angles and SSA

Take an arbitrary triangle and draw its circumcircle. Bisect one angle say $\angle A$, and find where the bisector cuts the circle at D.



We notice that since the arcs are equal, $CD = DB$, and with a shared side, we have SSA for the two triangles: $\triangle ADC$ and $\triangle ABD$.

These two triangles are obviously *not* congruent.

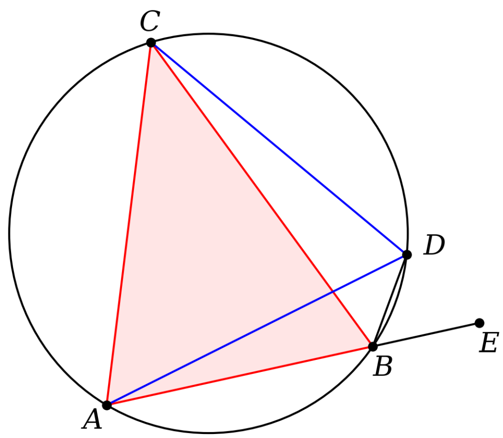
Yet they are related by SSA and if not congruent, we expect that the angles $\angle ACD$ and $\angle ABD$ are supplementary (see the chapter on congruence tests).

But of course they are, since they are opposing vertices in a cyclic quadrilateral.

We also have that $\triangle BCD$ is isosceles.

If the arc for any of the angles is bisected then the line drawn to that point from the vertex is the angle bisector.

The next problem is related to the previous discussion.



Given that $ABDC$ is a cyclic quadrilateral. Let BD bisect $\angle CBE$. Prove that $DC = DA$.

Proof.

Let β be one-half the bisected external angle so $\beta = \angle DBE = \angle DBC$.

$\angle DAC$ is on arc CD so $\angle DAC = \beta$ by inscribed angles on equal arcs.

$\angle DCA$ is supplementary to $\angle ABD$ by the fundamental theorem of the cyclic quadrilateral.

And then $\angle ABD$ is supplementary to β on the straight line ABE.

So $\angle DCA = \beta = \angle DAC$.

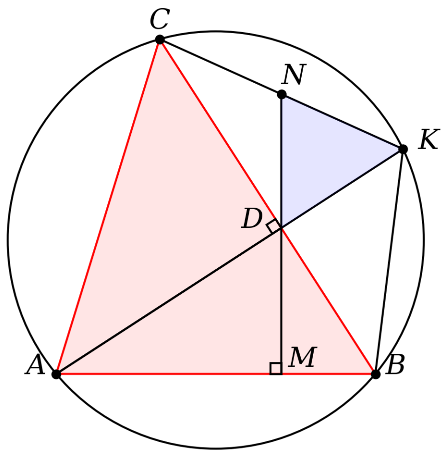
It follows that $\triangle DCA$ is isosceles, with $DC = DA$.

□

Brahmagupta's theorem

I have redrawn the figure from wikipedia.

https://en.wikipedia.org/wiki/Brahmagupta_theorem



$ABCD$ is a cyclic quadrilateral with diagonals that cross at right angles.

Given $NM \perp AB$. Show that CK is bisected, with $CN = NK$.

If the conclusion is true, then since $\triangle CDK$ is right, we should have that the median ND is equal to the two halves of CK. This suggests we try to prove that the two smaller triangles are isosceles.

Proof.

The perpendicular DM in right triangle BDA forms two smaller similar right triangles: $\triangle BDA \sim \triangle DMA \sim \triangle MBD$.

In $\triangle DKN$, $\angle DKN = \angle ABC$ by inscribed angles on equal arcs.

But $\angle ABC = \angle ADM$ by the similar triangles just given, and then $\angle ADM = \angle NDK$ by vertical angles.

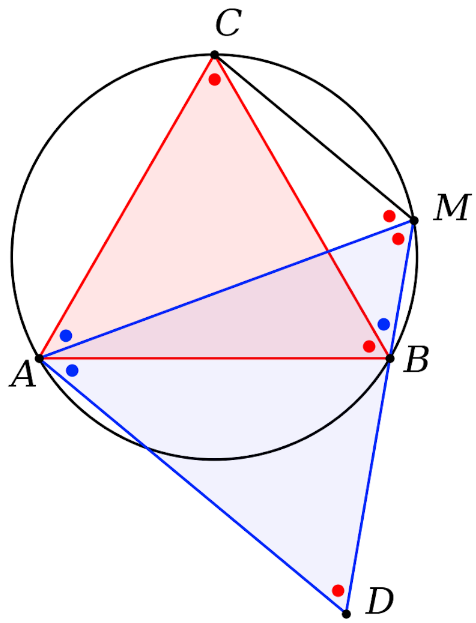
Thus $\triangle NDK$ is isosceles, with $NK = ND$.

For the same reason, $\triangle CND$ is also isosceles. So then

$$CN = ND = NK$$

□

Van Schooten's theorem



This is given as a problem by Surowski (1.3.6).

Given an equilateral $\triangle ABC$ draw its circumcircle.

Draw an arbitrary line segment from vertex A through side BC to meet the circle at M. Prove that $AM = BM + MC$.

Proof.

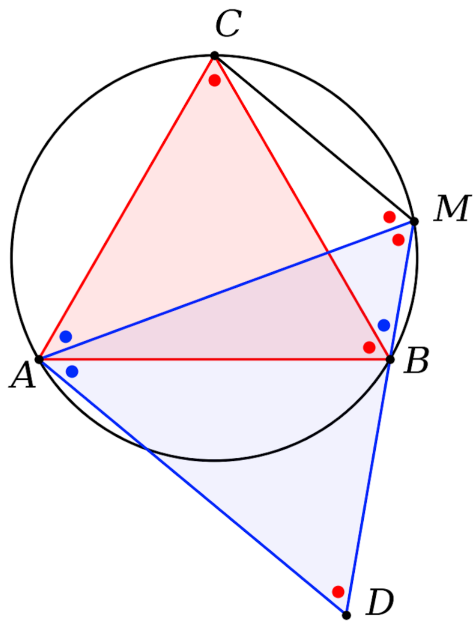
Draw lines from M to each vertex of $\triangle ABC$ and extend MB to give $MD = AM$. Join AD.

By inscribed angles both angles at M are equal to two-thirds of a right angle.

Since $MD = AM$, $\triangle AMD$ is isosceles with the vertex equal to two-thirds of a right angle.

Thus $\triangle AMD$ is equilateral. This accounts for all of the red dots. ($\angle A$ is not marked, to keep the diagram simpler).

Subtract the central $\angle MAB$ from two equal angles to yield $\angle CAM = \angle BAD$.



$\triangle CAM \sim \triangle BAD$.

But $AB = AC$ and also $AM = AD$, so $\triangle BAD \cong \triangle CAM$ by ASA or SAS.

It follows that $BD = MC$.

Thus $BD + MB = MC + MB$ and the former is equal to MD.

$\square$

This theorem is easy to prove using Ptolemy's theorem. Ptolemy says that, in a cyclic quadrilateral, multiply the lengths of opposing sides and add the two products, and that is equal to the product of the diagonals.

Here, let $AB = AC = BC = 1$. Then Ptolemy says:

$$MC \cdot 1 + MB \cdot 1 = AM \cdot 1$$

$$MC + MB = AM$$

However, we don't have that quite yet. So we did it as suggested.

we have questions

I saw this question on the internet: “is a parallelogram a cyclic quadrilateral?”

In a parallelogram, opposing angles are equal, while in a cyclic quadrilateral, opposing angles are supplementary.

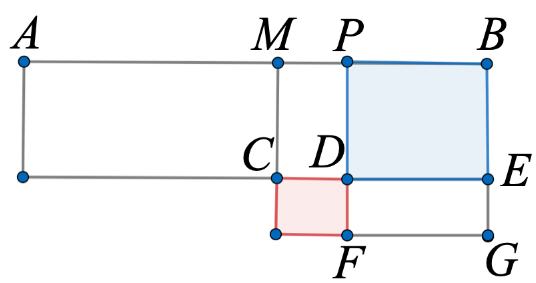
The only supplementary, equal angles are two right angles. So a rectangle is the only parallelogram that is a cyclic quadrilateral.

Chapter 26

Tangent secant theorem

Let us start with two propositions from Book II of *Elements*, which deal with what might be termed geometric algebra.

Euclid II.5



In this figure, the line AB is bisected at M with $AM = MB$.

and then the point P placed somewhere within the segment MB. II.5, says that

$$AP \cdot PB + MP^2 = MB^2$$

It may make more sense if we try algebra first. Let $x = AM$ and $y = MP$ so $AP = x + y$ and $PB = x - y$ and then

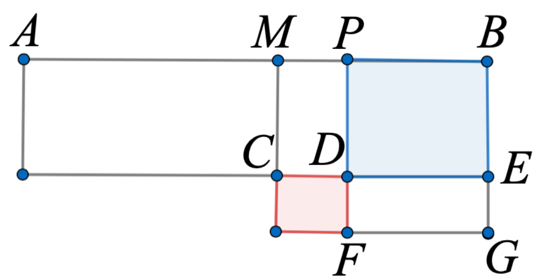
$$AP \cdot PB = x^2 - y^2$$

and we can read these off the diagram:

$$AP \cdot PB = MB^2 - MP^2$$

$$AP \cdot PB + MP^2 = MB^2$$

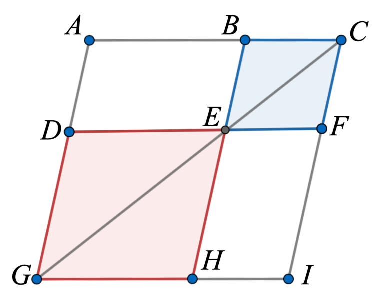
$AP \cdot PB$ is the area of rectangle AD.



PE is the square on PB, with $PB = BE$, and CF is the square on CD, with $CD = MP = EG$.

The construction contains two sets of equal rectangles. The first is $AC = ME = BF$. And the second is $MD = DG$. This is by I.43.

In the figure below, the two white parallelograms are equal.



We saw an example very early for rectangles as the **area-atio theorem**.

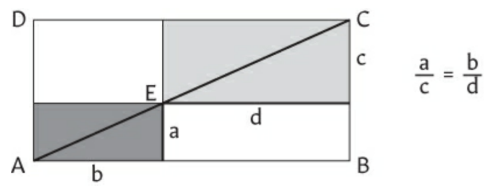
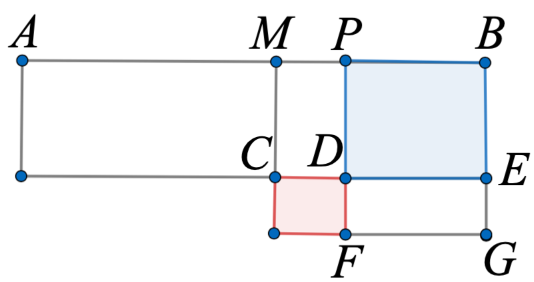


Fig. 42 Area and similarity.

$$ad = bc$$

Returning to the present theorem



As noted, the construction contains two sets of equal rectangles. The first is $AC = ME = PG$, while the second is $MD = DG$.

Here's the neat idea: the rectangle PG contains the blue rectangle plus DG. But since $MD = DG$ this is also ME and $ME = AC$.

So then finally AD is equal in area to the L-shaped piece called a *gnomon*.

And if we add CF (equal to MP^2) to that we have MG (equal to MB^2).

$$AP \cdot PB + MP^2 = MB^2$$

In Joyce's annotated *Elements* he says (paraphrasing) that this solves a quadratic. Suppose we are asked

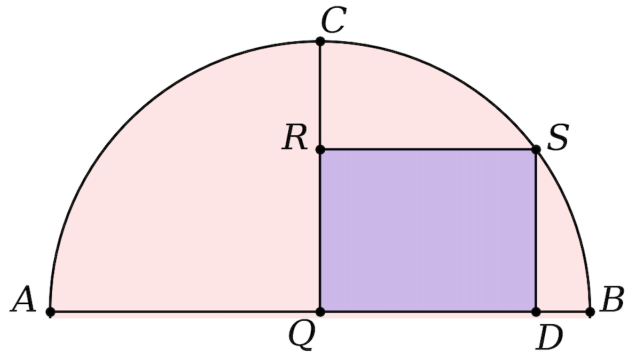
> Find two numbers x and y so that their sum is a known value b and their product is a known value c^2.

In modern terms:

$$x(b - x) = c^2$$

$$x^2 - bx + c^2 = 0$$

$$x = \frac{b}{2} \pm \sqrt{(\frac{b}{2})^2 - c^2}$$



To solve this geometrically, draw the circle on center Q with radius $QB = b/2 = QS$, and $AB = b$.

Let $c = DS$.

$\triangle QDS$ is right, so from Pythagoras, the distance QD^2 is:

$$QD^2 = (\frac{b}{2})^2 - DS^2$$

By the principle of the mean proportion, $DS^2 = AD \cdot BD = x \cdot (b - x)$.

and

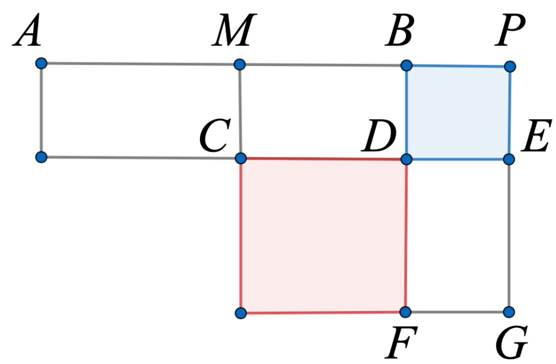
$$AD = AQ + QD$$

$$= \frac{b}{2} + \sqrt{(\frac{b}{2})^2 - c^2}$$

is equal to x.

Euclid II.6

The second theorem is very similar but has the point P located on an extension of AB:



The result is nearly the same as before, just with the squares switched:

$$AP \cdot PB + MB^2 = MP^2$$

although AP is not what it used to be. One way to remember: the difference of squares must be positive. In the first case $MB > MP$, and in the second, $MP > MB$.

Again we have a gnomon and a difference of squares. The gnomon is equal to the whole rectangle $AP \cdot PB$ and when added to MB^2 we get the whole large square MP^2

$$AP \cdot PB + MB^2 = MP^2$$

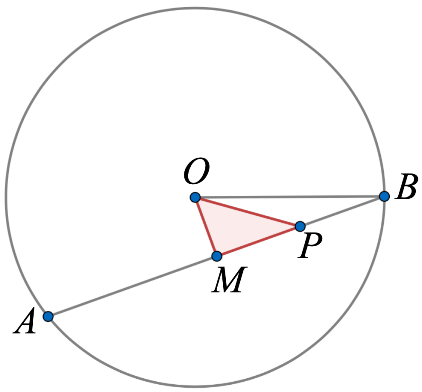
In algebraic terms we let $AM = x$ and now $PB = y$ so

$$(2x + y)y + x^2 = 2xy + y^2 + x^2 = (x + y)^2$$

$$AP \cdot PB + MB^2 = MP^2$$

Euclid III.35: application to chords

We have a chord AB of a circle on center O. AB is bisected at M and P is placed somewhere within MB.



By II.5

$$AP \cdot PB + MP^2 = MB^2$$

We notice both squares are part of right triangles.

$$MP^2 = OP^2 - OM^2$$

$$MB^2 = OB^2 - OM^2$$

Substituting into the first equation, OM cancels, leaving

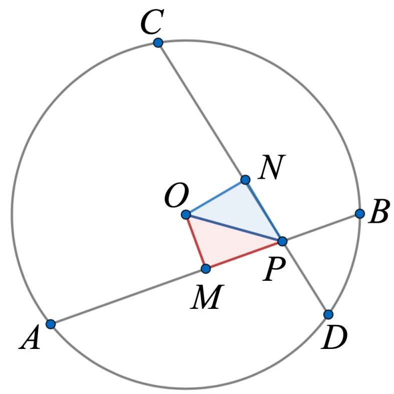
$$AP \cdot PB + OP^2 = OB^2$$

Let $OP = d$, the distance from P to the center, and OB is the radius, so

$$AP \cdot PB = r^2 - d^2$$

This result is *independent* of the particulars of AB and depends only on the placement of P in the circle (and the requirement that AB pass through P).

So any other chord that also passes through P, say CD, has the same result.



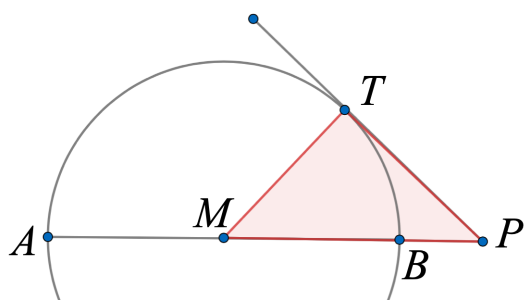
$$AP \cdot PB = CP \cdot PD$$

This is just the crossed chord theorem in disguise.

A modern proof would use similar triangles, but this requires a theory of proportions that Euclid doesn't have yet in book III. We'll see that below.

tangent

The next theorem concerns the tangent.



Here we have AB bisected at M, the center of the circle, and then P placed on the extension of AB.

By II.6
$$AP \cdot PB + MB^2 = MP^2$$

MB and MT are radii so

$$AP \cdot PB = MP^2 - MT^2 = PT^2$$

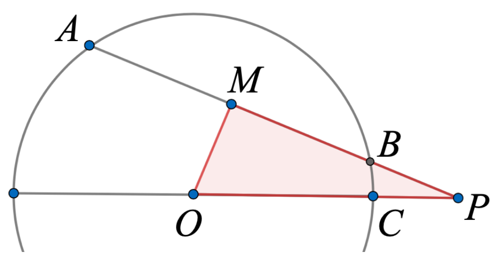
The length of the tangent from point P, squared, is equal $AP \cdot PB$.

We can again generalize the result by letting d be the distance of P from the center of the circle, MP, and t be the length of the tangent, PT. Then

$$d^2 - r^2 = t^2$$

The result has the same magnitude as for crossed chords, but with a minus sign. And again, this makes sense since the difference of squares must be positive. Before we had $r > d$, now we have $d > r$.

secant



As with the tangent, by II.6 we have

$$AP \cdot PB + MB^2 = MP^2$$

and again, we have right triangles so

$$AP \cdot PB + (MB^2 + OM^2) = (MP^2 + OM^2)$$

$$AP \cdot PB + OB^2 = OP^2$$

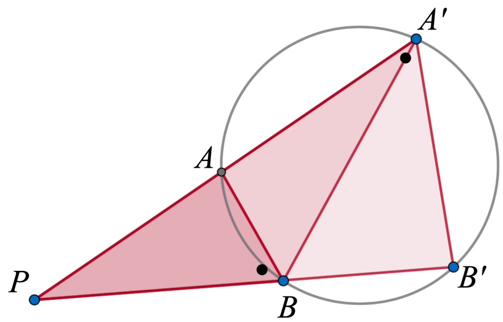
so again

$$AP \cdot PB = d^2 - r^2$$

By using the previous two results together, we have that for any secant drawn from P, $PA \cdot PB$ is equal to the square of the tangent from the same point PT^2.

This is the **tangent-secant theorem**.

proofs based on similar triangles



$$PA \cdot PA' = PB \cdot PB'$$

When thinking about such problems, or asked to establish a proof yourself, it is always helpful to consider the form with the ratios:

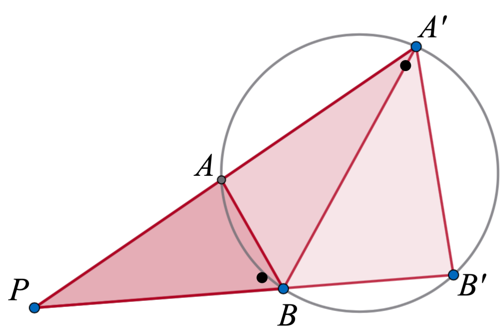
$$\frac{PA}{PB} = \frac{PB'}{PA'}$$

This strongly suggests we look at similar triangles.

Proof.

We showed previously that for any quadrilateral whose four vertices all lie on one circle (a cyclic quadrilateral), the opposing vertices have supplementary angles. Opposing vertices add to 180° because their arc segments add up to one whole circle.

In the figure above, $\angle ABB'$ is supplementary to both $\angle PBA$ and $\angle PA'B'$.



Therefore $\triangle PBA$ is similar to $\triangle PA'B$. The similarity is such that

$$\frac{PA}{PB} = \frac{PB'}{PA'}$$

Given a point P outside the circle, the external part of any secant times the entire secant is a constant.

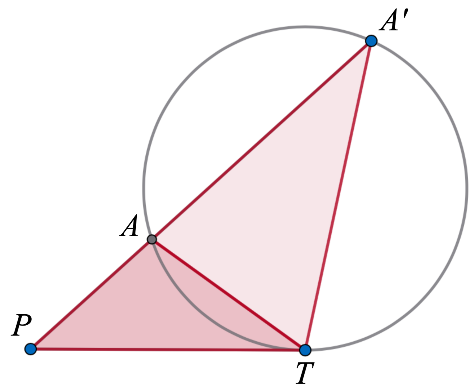
One curious thing about this theorem is that these triangles are similar, and nested, but flipped.

$$PA \cdot PA' = PB \cdot PB'$$

□

We can do a bit more.

Tangent-secant theorem



Let the points B and B' approach each other to become one point, re-labeled as T. Then PT will be a tangent of the circle. Previously we had

$$PA \cdot PA' = PB \cdot PB'$$

Now we modify it slightly:

$$PA \cdot PA' = PT \cdot PT$$

$$PA \cdot PA' = PT^2$$

This is the tangent-secant theorem.

We must have two similar triangles, $\triangle PAT$ and $\triangle PTA'$. The whole angle at vertex T must correspond to $\angle PAT$.

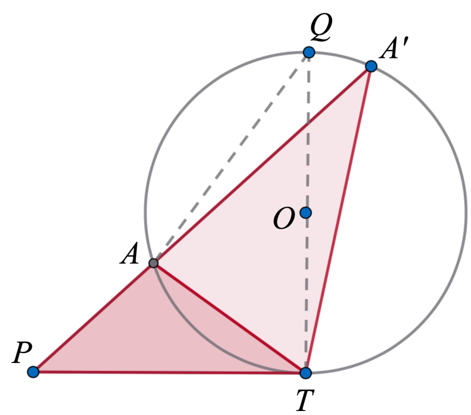
Now run the logic backward and write the proof.

Note: we've seen this sort of reverse logic a few times already. The method has a name! It was called the method of "analysis" by Pappus (320 A.D.), see Posamentier (Introduction).

To do this explicitly, show that since $\angle PTA$ includes the tangent, it cuts arc AT just as $\angle PA'T$ does.

Instead, we take advantage of the tangent point to do something new.

Proof.



Draw the diameter $QOT \perp PT$ and also draw QA. Since they correspond to equal arcs, the angles at Q and A' are equal.

By Thales' circle theorem, $\angle QAT$ is right. So $\angle Q$ is complementary to $\angle ATQ$.

But the diameter is perpendicular to the tangent. So $\angle ATQ$ is complementary to $\angle PTA$.

It follows that $\angle PTA = \angle PA'T$.

$\triangle PAT$ is similar to $\triangle PTA'$ by AAA, since they also share the angle at P. This gives

$$\frac{PT}{PA} = \frac{PA'}{PT}$$

which can be rearranged to the statement of the theorem:

$$PA \cdot PA' = PT^2$$

□

One application of this theorem is to a determination of the size of the earth.

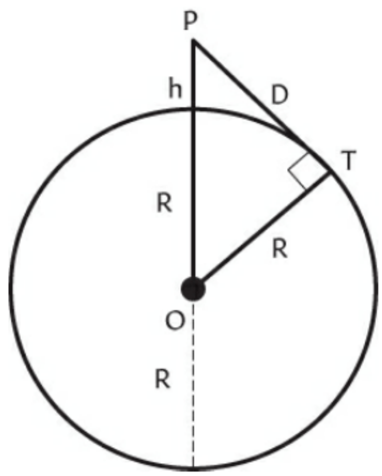


Fig. 68 Measuring the Earth.

In the figure, the circle is the earth, of radius R, h is the height of a convenient mountain, and D is the distance to the horizon, which is tangent to the earth's radius.

Recall from the tangent-secant theorem

$$D^2 = h(2R + h)$$

We neglect h^2 compared to the other term so

$$D^2 \approx 2Rh$$

About 1019 C.E., finding h and D, Al-Biruni computed a value for R equivalent to 3939 miles.

Looking at Euclid

We're looking up at a statue of Euclid on a column.

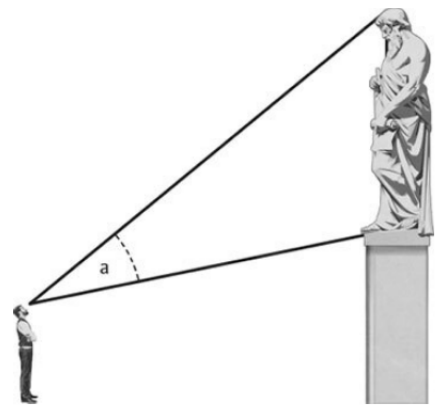


Fig. 69 Looking at Euclid.

We resist the temptation to make a dumb joke.

In any event, we'd like to get the widest angle view, giving the largest apparent size of the statue. If you get too close, the statue is greatly foreshortened and the angle small, and naturally, it is small at a distance. There must be a best view, in the middle.

Here is Acheson's solution:

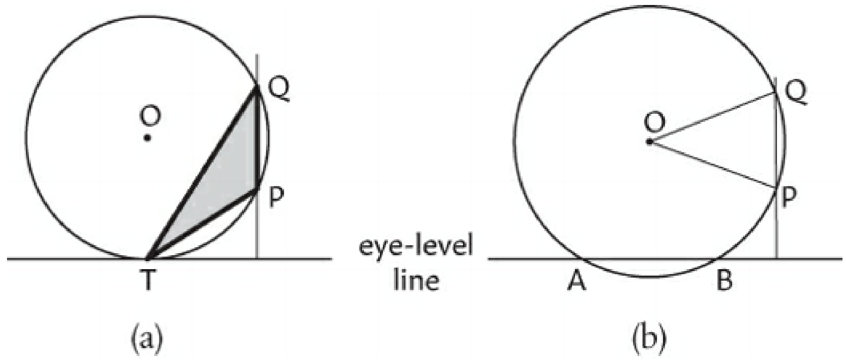


Fig. 70 What's the best view?

Let the foot and head of Euclid be at P and Q and draw the circle containing those two points which is also tangent to your eye-level. Then the tangent point provides the best view.

The reason is that a circle through any other horizontal position crosses the eye-level at two points. Such a circle will necessarily be bigger.

Consequently the arc PQ, which is fixed in size, will be a smaller fraction of the circle.

As a smaller fraction, both the central angle $\angle POQ$ will be smaller as well as the angle subtended at A or B (one-half of that).

The tangent-secant theorem even gives a quantitative answer.

If R is the point where the extension of QP meets eye-level (the ground), then suppose the point Q is h units above the ground and P is g units. The tangent-secant formula is

$$RT \cdot RT = PR \cdot QR$$

which says that the square of the optimum viewing distance is $h \cdot g$. The optimal distance d is

$$d = \sqrt{hg}$$

I knew I'd seen this problem before, and I find that I wrote it up for my Calculus book. It turns out to be from Acheson's book on Calculus, and is originally about Lord Nelson's statue in Trafalgar Square.

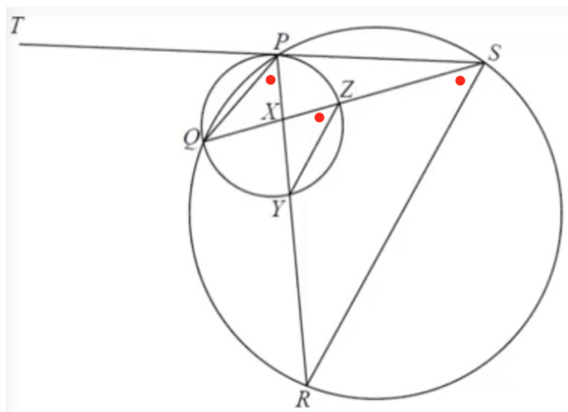
The calculus treatment is not nearly as pretty as reasoning about the tangent; fortunately we came to the same answer.

Problem

problem

We continue with a problem that we solved in part **previously**.

We showed that there are three similar triangles in the figure below



We showed that $\triangle XYZ \sim \triangle XQP$ and also $\triangle XYZ \sim \triangle XRS$.

The last part of the problem says that given $QS = XR$, prove that $PS^2 = XS \cdot YR$. It is also given that TPS is tangent to the small circle at P.

In this chapter, we developed the tangent-secant theorem, which says that the part of the secant outside the circle, multiplied by the whole thing, is equal to the tangent squared:

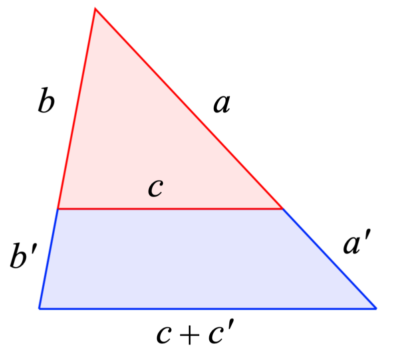
$$QS \cdot ZS = PS^2$$

Proof.

By similar triangles, we have that

$$\frac{XZ}{XY} = \frac{XS}{XR} = \frac{ZS}{YR}$$

That last equality requires some algebra. We use this figure again:



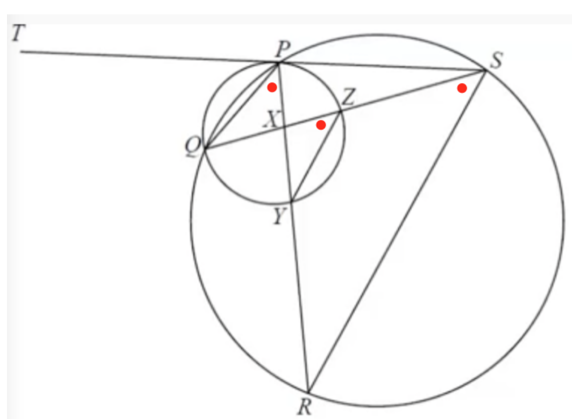
We have that

$$\frac{a}{b} = \frac{a+a'}{b+b'}$$
$$\frac{a}{a+a'} = \frac{b}{b+b'}$$
$$\frac{a+a'}{a} = \frac{b+b'}{b}$$
$$\frac{a'}{a} = \frac{b'}{b}$$
$$\frac{a}{b} = \frac{a'}{b'}$$

The partial sides are in the same ratio as the whole.

Back to our problem. Multiplying by the denominators:

$$XS \cdot YR = XR \cdot ZS$$



We're given that $XR = QS$

$$XS \cdot YR = QS \cdot ZS$$

but by the tangent-secant theorem

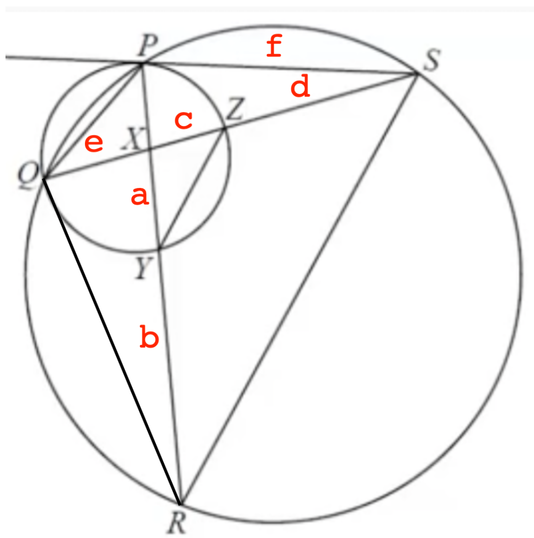
$$QS \cdot ZS = PS^2$$

so we have

$$XS \cdot YR = PS^2$$

$\square$

I really dislike doing algebra with XS and the rest, so I substituted single letters for the sides and fiddled with the algebra while working backward (analysis) until I could see the answer.



The tangent-secant theorem says that $f^2 = d \cdot (c+d+e)$. We're given that $(c+d+e) = (a+b)$.

So $f^2 = d \cdot (a+b)$. We are asked to prove that this is equal to $(c+d) \cdot b$.

Forming ratios we have

$$\frac{b}{d} = \frac{a+b}{c+d}$$

But this is just the result that the partial sides are in the same ratio as the whole.

Reverse (and substitute XS etc.) to write the proof.

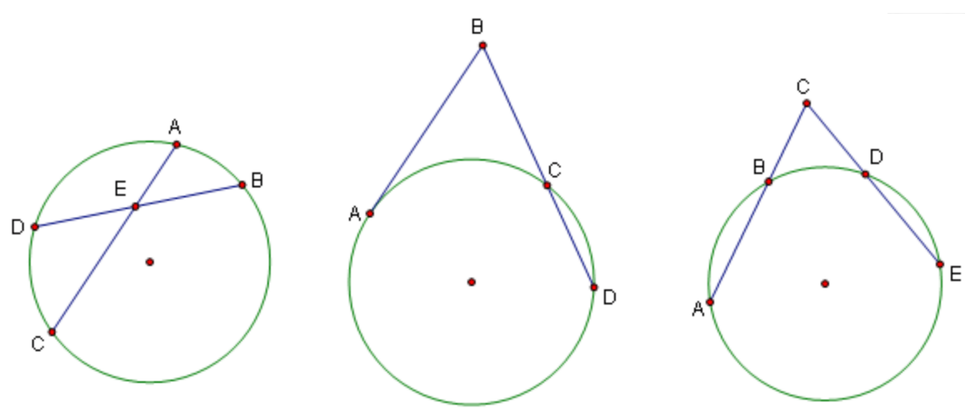
Chapter 27

Power of a point

The crossed chord and tangent-secant theorems can be unified by considering PA and PB as directed line segments. In the crossed-chords example, they point in opposite directions so their product is negative, while for tangent and secant they point in the same direction.

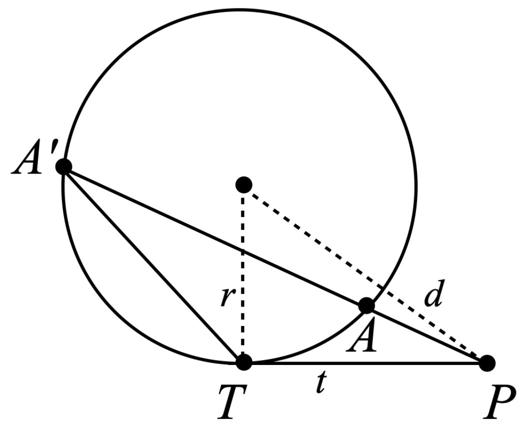
We can view the result for chords $r^2 - d^2$ as the same as the result for the tangent, $d^2 - r^2$, provided we take account of the sign of $AP \cdot PB$.

The previous results regarding secants and tangents are sometimes described in a definition called the "power" of a point, which unifies the treatment of points inside and outside the circle.



https://artofproblemsolving.com/wiki/index.php/Power_of_a_Point_Theorem

Let's start with a point outside.



The length of the tangent and the radius are simply related to the distance from any point P to the center of the circle, d:

$$d^2 = t^2 + r^2$$

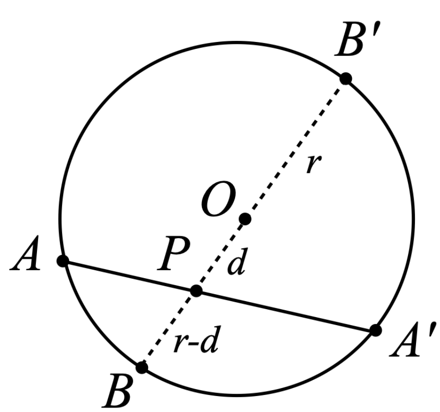
$$t^2 = d^2 - r^2$$

$d^2 - r^2$ is defined as the *power* p of the point P.

From our previous work, we know that

$$p = t^2 = PT^2 = PA \cdot PA'$$

The definition also works for a point inside the circle.



By the crossed chords theorem:

$$PA \cdot PA' = (r + d)(r - d)$$

$$= r^2 - d^2 = -p$$

That is, it works if we use *directed line segments*, so that the product $PA \cdot PA'$ for P inside the circle has its two components pointing in opposite directions, and thus acquires a minus sign.

If the point is *on* the circle, it's a bit strange, but we are at the boundary between $p < 0$ and $p > 0$, so it seems reasonable that $p = 0$ for points on the circle. And there, of course

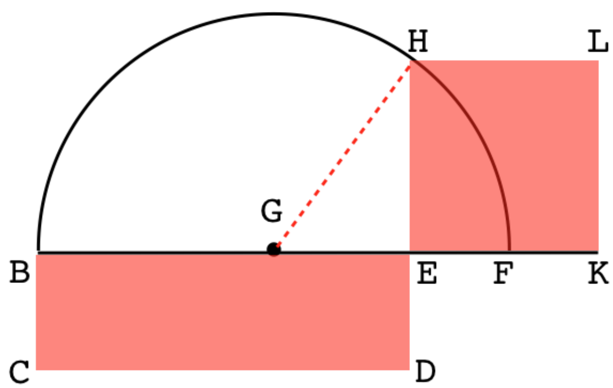
$$p = d^2 - r^2 = 0$$

Chapter 28

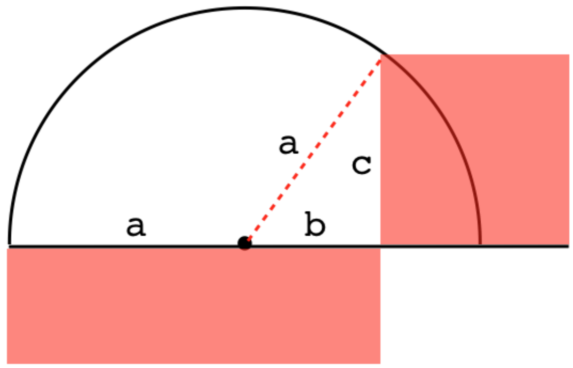
Hippocrates

Hippocrates of Chios (470-410 BC) was a major figure in Greek geometry. (Not to be confused with the physician of the same name, from Kos). In the work that we know about, Hippocrates focused on quadrature, the process of constructing (with straight-edge and compass) a square with area equal to a given geometric figure, particularly curved figures, called lunes.

Here is one of the first of these—construction of the square equivalent to a given rectangle.



The construction says to: (i) extend BE horizontally, (ii) mark off the same distance as DE to construct EF, (iii) find the midpoint G of BF, (iv) draw the half-circle of radius BG, (v) extend DE up to meet the circle at H, construct the square of side the same length as EH.



As suggested by the dotted line in the figure, the proof invokes the Pythagorean theorem. The long side of the rectangle is $a+b$, while its short side is $a-b$, so the area is

$$A = (a+b)(a-b) = a^2 - b^2$$

but Pythagoras says that is equal to c^2.

□

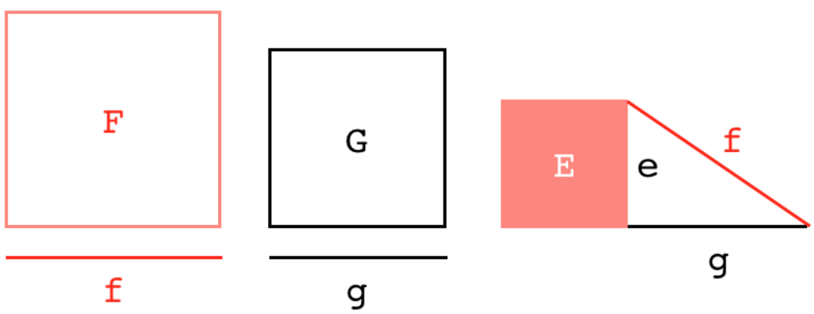
This is a slight restatement of our proof about the geometric mean.

The side of the square, c is the geometric mean of the sides of the rectangle.

$$c = \sqrt{(a+b)(a-b)}$$

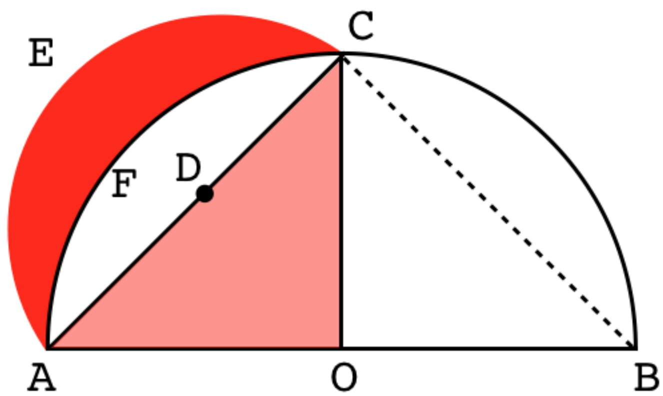
other constructions

Hippocrates "squared" rectangles, triangles and polygons. A lot of his constructions depended on Pythagoras as suggested by this figure:



where two squares resulting from manipulation of part of a polygon need to be subtracted to obtain the final result.

Hippocrates moved on to curves, trying to find squares with area equal to that under or between two curves. That turns out to be a class of problems where few have solutions (in fact, only five, according to Dunham). Famously, it is impossible to square the circle. However, here is one that is possible, it is an example of (the) quadrature of the lune.



We will prove that the two shaded regions are equal in area.

Consider the smaller semicircle with base ADC, which is also the hypotenuse of the right triangle. Let radius AD be equal to r. Let the large semicircle have radius AO equal to R. Pythagoras tells us that

$$R^2 + R^2 = (2r)^2$$

$$R^2 = 2r^2$$

Let the area of the triangle be T. (Its value is $R^2/2$ but that's not needed).

The segment of the larger semicircle (white) is the area of the quadrant minus the area of the triangle

$$\pi \frac{R^2}{4} - T$$

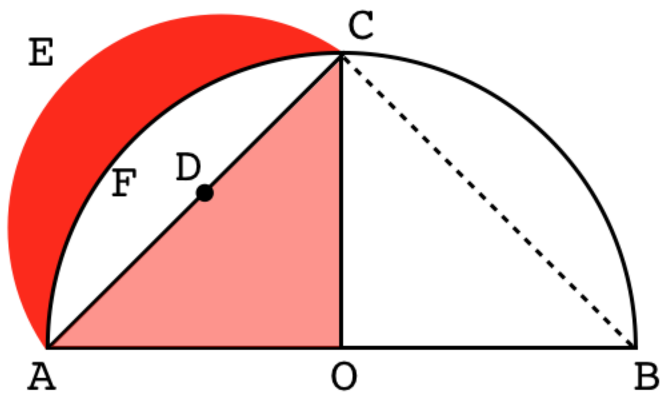
The area of the red lune is the area of the small semicircle minus the white area

$$\pi \frac{r^2}{2} - \ [\ \pi \frac{R^2}{4} - T\ ]$$

$$= \pi \frac{r^2}{2} - \pi \frac{2r^2}{4} + T$$

$$= T$$

Which is just the area of the triangle.



Part VI

Addendum

Chapter 29

List of Theorems

The geometry book has recently been split into parts. If a proof is listed here but the link is broken, it's very likely to be in a different volume. I have left the broken ones in for reference.

proofs of the Pythagorean theorem

- **Pythagorean Thm: Condit**
- **Pythagorean Thm: Tuan**
- **Pythagorean Thm: Quorra corollary**
- **Pythagorean Thm: converse**

proofs from Euclid

- **construct an equilateral triangle** (Euclid I.1)
- **side angle side (SAS)** (Euclid I.4)
- **isosceles triangle theorem** (Euclid I.5: equal sides → angles)
- **isosceles triangle theorem converse** (Euclid I.6: equal angles → sides)
- **Preliminary to SSS** (Euclid I.7)
- **angle bisection** (Euclid I.9)
- **perpendicular bisector** (Euclid I.10)
- **perpendicular through a point** (Euclid I.11)
- **perpendicular to a point** (Euclid I.12)
- **external angle inequality** (Euclid I.16)
- **longer side → larger angle** (Euclid I.18)
- **larger angle → longer side** (Euclid I.19)
- **triangle inequality** (Euclid I.20)
- **hinge theorem** (Euclid I.24)
- **ASA for congruence** (Euclid I.26)
- **line parallel to another line** (Euclid I.31)
- **parallelogram area** (Euclid I.35)
- **parallelogram complements equal** (Euclid I.43)
- **Pythagorean theorem** (Euclid I.47)
- $(x+y)(x-y) = x^2 - y^2$ (Euclid II.5)

- $(2x + y)y + x^2 = (x + y)^2$ (Euclid II.6)
- **Law of cosines, obtuse case** (Euclid II.12)
- **Law of cosines, acute case** (Euclid II.13)
- **square of a rectangle** (Euclid II.14)
- **find circle center** (Euclid III.1)
- **find circle center** (Euclid III.12)
- **inscribed angle theorem** (Euclid III.20)
- **same arc → equal angles** (Euclid III.21)
- **quadrilateral supplementary theorem** (Euclid III.22)
- **equal angles, on same circle** (Euclid III.26 converse)
- **Thales' theorem** (Euclid III.31)
- **tangent-chord theorem** (Euclid III.32)
- **crossed chord theorem** (Euclid III.35)
- **tangent-secant theorem** (Euclid III.36)
- **similarity: AAA → equal ratios** (Euclid VI.2)
- **equal divisions of a line segment** (Euclid VI.9)

proofs of other theorems

- **alternate interior angles**
- **angle bisector theorem** (right triangle)
- **angle bisector theorem** (general)
- **ASA for congruence**
- **area ratio theorem**
- **bisector equidistant from sides**
- **equidistant from sides → bisector**
- **Centroid is one-third of cevian**

- **Ceva's theorem**
- **Ceva's theorem** (Menelaus)
- **Ceva's theorem** (by area)
- **Ceva's theorem** (alternate proof)
- **Ceva's theorem by parallel lines**
- **crossed chord theorem** (product of lengths)
- **complementary angles**
- **cyclic quadrilateral** (opposing angles are supplementary)
- **cyclic quadrilateral** (converse)
- **diameter divides circle in half**
- **diameters form a rectangle**
- **equal arcs** $\iff$ **equal chords**
- **equal angles** $\iff$ **equal arcs**
- **eyeball theorem**
- **excircle theorems**
- **extended altitude theorem**
- **external angle theorem**
- **extraordinary property of the circle**
- **Heron's formula by excircles**
- **Heron's formula, Heron's proof**
- **hypotenuse-leg in a right triangle (HL)**
- **hypotenuse longest side in a triangle**
- **incenter** (incenter: angle bisectors meet at a point)
- **inscribed angle theorem** (on a circle is one-half central angle)
- **inscribed angles converse**
- **isosceles triangle theorem** (sides $\rightarrow$ angles)

- **isosceles triangle theorem** (angles $\to$ sides)
- **Law of cosines**
- **Law of cosines, algebraic proof**
- **Law of cosines, Ptolemy**
- **Menelaus' theorem**
- **midline theorem**
- **midpoint theorem** (right triangle)
- **orthocenter exists** (Newton)
- **Pappus parallelogram theorem**
- **parallelogram theorems**
- **special parallelogram theorem** (one pair of sides)
- **circumcenter** (perpendicular bisectors of a chord is diameter)
- **circumcenter** (perpendicular bisectors meet at a point)
- **Ptolemy's theorem, by cutting**
- **Ptolemy's theorem, similar triangles**
- **Ptolemy's theorem, switch sides**
- **Ptolemy's theorem, by inversion**
- **rectangle in a circle**
- **right angle is largest in a triangle**
- **tangent-secant theorem**
- **shortest distance from a point to a line**
- **supplementary angles**
- **similar right triangles**
- **midline theorem** (similar triangles)
- **similar triangles** (ratio of sides)
- **similar triangles** (right triangle composition)

- **AAA similarity theorem** (Kiselev)
- **SAS for congruence**
- **SAS inequality, hinge theorem**
- **SAS to establish similarity**
- **SSS implies SAS**
- **supplementary angle theorem**
- **supplementary angles equal to two right angles**
- **tangent theorem** (right angle → touches one point)
- **tangent-chord theorem**
- **tangent construction**
- **tangent theorem** (touches one point → right angle)
- **tangent-secant theorem**
- **Thales' circle theorem** (right angle in a semi-circle)
- **Thales circle theorem: converse**
- **triangular area**
- **triangle inequality** (triangle inequality)
- **sum of angles**
- **triangles are similar if two angles equal**
- **Varignon's theorem**
- **vertical angle theorem**

Chapter 30

Author's Notes

A central feature of this book is the relentless use of proof. I emphasize the key insight for each, and have tried to make the proofs simple and as easy to follow as possible.

This volume is distinguished from most other texts, since they maintain that a proper proof should be watertight, with each step carefully justified and following closely from the one before. I don't deny that rigor has its proper place in math education, but I also think that this rigidity obscures the core insights. Our purpose here is to view beauty clearly.

We prefer instead to be like the famous mountaineer Ueli Steck. Reach the summit quickly and emphasize the key steps. You should be able to fill in the details if there are loose ends.

Multiple proofs for important theorems are sometimes given, because proof is our stock in trade, and different approaches shed light on how proofs may be found and developed.

Another distinguishing feature is a set of simple proofs based on scaling of triangles. This happens for the Pythagorean theorem and for Ptolemy's theorem, as well as the sum of angles theorems and then later, a fairly sophisticated theorem of Euler's.

Recently I came across a fantastic book by Acheson, called *The Wonder Book of Geometry*. I helped myself to some of his examples, and now have more than a dozen. Please go find Acheson, and buy it. It's truly magical. In fact, all of his books are wonderful!

A saying attributed to Manaechmus, speaking to Alexander the Great, is that "there is no royal road to geometry". Others write that this was actually Euclid, speaking to Ptolemy I of Egypt. Since the two sources lived some 700 years after the fact, it is difficult to know.

Practically, this means that in learning mathematics you must follow the argument with pencil and paper and work out each step yourself, to your own satisfaction. That is the only way of really learning, and at heart, a principal reason why I wrote this book.

Having read a chapter, see if you can prove the theorems yourself, without looking at the text.

There are a few problems listed in the later chapters, perhaps thirty or more altogether. Most of them have worked out solutions. It is highly recommended that you attempt each problem yourself before reading my answer. Since the crucial point is often to draw an inspired diagram, you must stop reading as soon as the problem is stated!