

# Elementary Geometry I

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# Part I

## Lines and angles

# Chapter 1

## Introduction

The image below is a detail from a painting by Raphael entitled “School of Athens”, which was used as the front cover of a wonderful book annotating the Heath translation of Euclid’s *Elements*.



It took a genius to figure it out the first time, but it is within anyone’s grasp to appreciate what they found. I imagine myself looking over Archimedes’ shoulder as he explains the steps of a proof to me.

Most scientists I've met loved geometry in school, as I did. They like how visual it is, and they like clever simple proofs. Geometry should be fun!

A central feature of this book is the relentless use of proof. I emphasize the key insight for each, and have tried to make the proofs simple and as easy to follow as possible. You will notice that we frequently provide multiple proofs (using different methods) for the same theorem.

I express my sincere thanks to the authors of my favorite books, which are listed in the references and mentioned at various places in the text. Everything in here was appropriated from them in one way or another, and styled to my taste.

I offer my profound thanks also to Eugene Colosimo, S.J. He was among the best of a great group of teachers.

This book is the pdf in that repository linked below. Most of the rest is the source files. There are several other books there as well, if you go up one level in Github.

<https://github.com/telliott99/geometry>

Note: the sites referred to by some urls in the text have disappeared from the web and the count of missing pages grows over time. I have left those URLs in, as a kind of protest, and because it is impractical to police them, but also because they may still be useful in connection with the wayback machine.

<https://web.archive.org>

# Chapter 2

## Angles

### On motivation

Modern textbooks make a considerable effort to motivate the student, setting the stage for a problem and attempting to convince her or him that it's worth trying to understand what's being talked about. I usually won't do that.

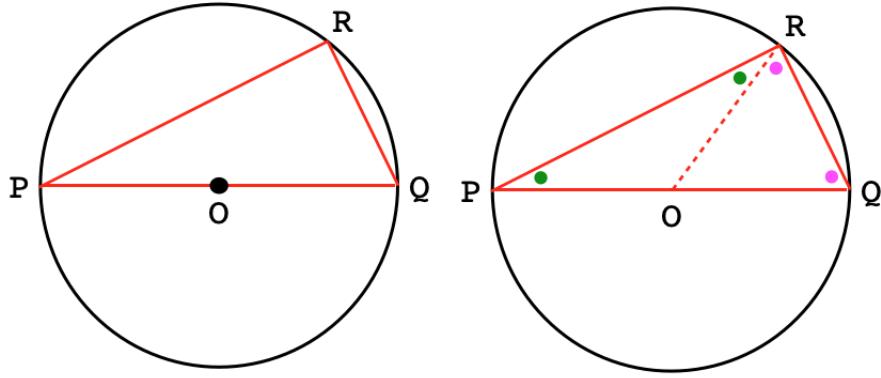
I have tried to achieve simplicity and clarity in the presentation. The subject itself gives us beauty. To see that beauty clearly is my motivation and I hope, yours as well.

In the next chapter, we will prove a beautiful *theorem*: the sum of the angles in a triangle is equal to the angle turned by going halfway around in a circle (like turning from north to south). That's a really remarkable and elegant result, and the proof is simple.

Here is a second beautiful theorem, about circles.

- Any angle inscribed in a semicircle is a right angle.

Think of three points on the circumference of a circle, forming a triangle. If two of the points are on a diameter of the circle (the line joining them passes through the center), then the angle formed at an arbitrary but distinct third point is always a right angle.



In this figure, if  $PQ$  is a diameter, then  $\angle PRQ$  must be a right angle.

*Proof.*

Draw the radius  $OR$  (right panel).

The two smaller triangles produced ( $\triangle OPR$  and  $\triangle OQR$ ) are both isosceles (two sides equal), since two of their sides are radii of the circle. Therefore their base angles are equal (we will see why later).

We can use colored dots to identify angles that are equal. We have for the whole triangle two green dots and two magenta ones, and for the angle at  $R$  one of each. Hence the angle at  $R$  is one-half the total measure of the triangle, namely, one-quarter of a circle. Like turning from north only as far as west.

□

That's a beautiful, simple result. It depends on an idea about isosceles triangles, and it should motivate us to find out more about them.

The □ symbol marks the end of the proof.

## Notation

In the above proof we've mixed together different kinds of notation. The Greeks always named the points at the ends of line segments with letters of the alphabet, using those same points to describe triangles or figures with even more sides. So we can talk about  $PR$  as a line segment or  $PQR$  as an angle ( $\angle PQR$ ) or a triangle ( $\triangle PQR$ ).

However, often it is simpler to just put a label on the angle at the vertex of a triangle,

such as  $P$  or  $R$ , or  $s$  or  $t$  above, or use Greek letters such as  $\theta$  and  $\phi$ ; or even  $\alpha$ ,  $\beta$  and  $\gamma$ .

A third kind of notation is to not even give the angles a lexicographical label, but just mark ones with equal measure by using a colored circle (or open and filled circles).

We use whichever one feels more natural. The simpler the labels, the easier it is to think about what matters in the problem. The right notation frees the mind to concentrate on what's important.

## Euclid and the postulates

Some topics in geometry are as old as civilization: finding the area of a rectangular field or the volume of a cylindrical grain storage building.

However, the Greeks were really the first to treat the subject as an intellectual pursuit, the first to view it systematically and to prove theorems.

Euclid is probably the most famous of Greek geometers because of his book *Elements*. This book is believed to have been a compilation of the accomplishments of more than a dozen other mathematicians. See Chapter 2 of Heath:

<https://www.gutenberg.org/files/35550/35550-h/35550-h.htm>

However, Greek geometry starts several hundred years before the time of Euclid, who was a contemporary of Alexander the Great (356-323 BC).

Some of these early mathematicians were Thales, Eudoxus, and Pythagoras. One of the problems in understanding what happened is that, unfortunately, almost no books survive from this time. We know that extensive histories were written around the time of Euclid, but these are all lost.

As well, about the person Euclid, we also know actually very little. He lived after Plato (died 347 BC), and before Archimedes (born c. 287 BC). He worked in Alexandria, a city founded by Alexander near Cairo in Egypt, and except for that, all other details of his life and death are shrouded in mystery.

After more than 2000 years, *Elements* is still an excellent place to begin surveying the foundations of geometry. It is a quite sophisticated textbook, an organized collection of everything that a well-educated student was expected to know about the subject at the time.

We note in passing and with sadness that so much of what the Greeks wrote and

thought has been lost through the vicissitudes of history but also by the deliberate destruction of libraries. At Alexandria this occurred during war in the time of Julius Caesar, later during riots by Christian mobs, and then later again by Muslim invaders.

*Elements* is known only through later works that reproduced the theorems and proofs with additional commentary. The original source is lost.

## Elements



The *Elements* consists of more than five hundred *propositions*, some of which are constructions (geometric figures) drawn with a pencil on a piece of paper, using a straight-edge or a compass or both. The others are logical proofs of the type we will study in detail.

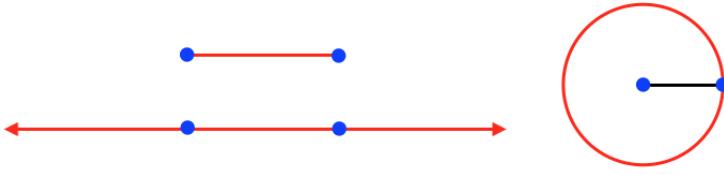
For constructions, there are restrictions on both compass and straight-edge. The straight-edge is unmarked; it cannot be used to measure distances. Sometimes people will say “compass and ruler”, but the ruler involved must not be *ruled*, it must not have divisions (or at least if it does, they are not consulted).

Nearly all proofs of propositions build on previous items in the book.

Euclid does not prove everything. Statements which are assumed to be true are divided into axioms, or common notions, and postulates. Axioms are generally useful, while postulates are specific to the subject of geometry.

Here are some of Euclid's postulates:

- A straight line segment can be drawn joining any two points.
- A line segment can be extended continuously in a straight line.



- Given any straight line segment, a circle can be drawn having the segment as the radius and one endpoint as the center.

Let us assume these as well. We will use them often.

We finesse the difficulty in defining what is meant by *straight* in the real world. If you've ever done any fancy carpentry, you will realize that an unknown edge is checked by comparison with another edge which is known to be straight. We use the known edge to "true" the other.

In geometry, we use an imaginary perfect straight-edge to draw an ideal straight line.

There is another statement commonly claimed, that any straight line segment can be extended indefinitely in a straight line.

According to Morris Kline, this statement does not exist in the *Elements*, and in fact, except when dealing with the question of parallel lines, we never worry about lines to infinity. We use line segments, with defined endpoints, and often refer to these simply as lines, as Euclid did.

People may also talk about a straight line as "the shortest distance between two points". The closest you will find to that is the **triangle inequality**, but that is in our future.

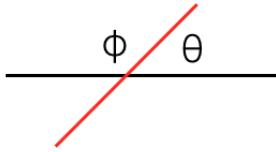
## supplementary angles

We begin geometry with a discussion of angles.

In the diagram below, one line segment is drawn crossing a second one, forming their *intersection* (the points at the ends have been omitted from the drawing).

Four angles are formed at the intersection of two lines or line segments. Two of the angles are labeled in the figure. We can see that one,  $\phi$ , is obviously larger than the other one,  $\theta$ .

We call angles formed on the same side of a line, *supplementary* angles.

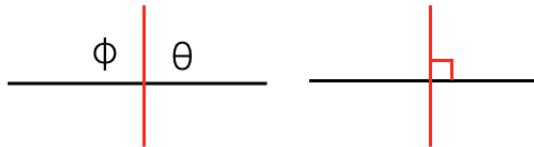


So  $\phi$  and  $\theta$  are supplementary angles because they both lie above the horizontal black line.

## right angles

You've probably seen a definition of a *right angle* as one that contains, or whose measure is, 90 degrees, usually written  $90^\circ$ . But the Greeks would give a different definition:

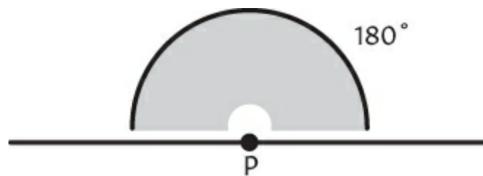
- Two supplementary angles that are equal to each other are both right angles. In the figure below, if  $\phi = \theta$ , then they must both be right angles.



A right angle is frequently designated by drawing a small square, as seen in the right panel.

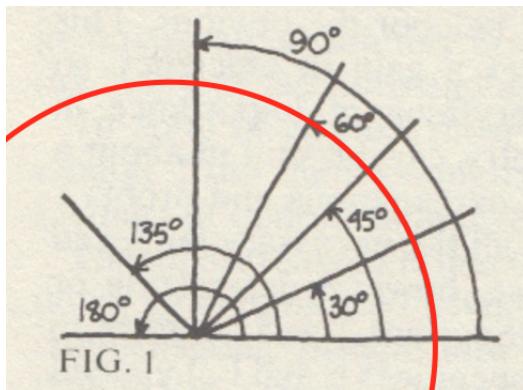
If one of the angles at the intersection of two lines is a right angle, then all four are right angles — we'll see why later in the chapter. The square is only drawn for one, but they are all equal.

In a common system of angle measurement, a right angle is indeed  $90^\circ$ , and there are  $360^\circ$  in a full circle. Supplementary angles sum to  $180^\circ$ .



**Fig. 4** A straight line.

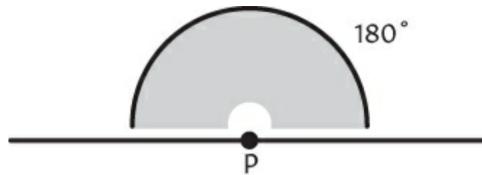
One way to think about the *measure* of an angle is to draw a unit circle (radius equal to 1) and then ask what is the (curved) distance along the circle from one ray of the angle to the second one. In the figure below, we can see angles with various measures drawn.



Imagine drawing a circle (in red) whose center coincides with the origin of the angles and then measuring the distance along that circle starting with the positive horizontal axis on the right and going counter-clockwise. We can take that distance as the measure of the angle.

Another approach is to bisect (cut in half) a straight line. Then, the original straight line is really an "angle" of  $180^\circ$ , and the bisected half-angles are each  $90^\circ$ . Another bisection gives  $45^\circ$ . We will see a method for doing bisection with compass and straight-edge later on. We can also get  $60^\circ$  using a triangle with all sides equal (equilateral). Bisection then gives  $30^\circ$ .

However, the precise measure of an angle is rarely important, especially before trigonometry. We simply use  $= 180$  as a shorthand for *is equal to two right angles*, and in fact, often drop the degree notation.



**Fig. 4** A straight line.

What we most care about is whether one angle is equal to, larger than, or smaller than another one, or whether an angle or some combination of angles is exactly equal

to one right angle or two right angles.

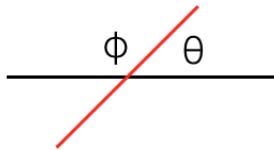
For more about angular measure, there's a short [chapter](#) at the end of the book.

Because of this, we can state the following theorem:

## supplementary angles

- the sum of two angles that are supplementary to each other is equal to two right angles.

This constant sum is correct regardless whether  $\theta$  is equal to  $\phi$ , or one is larger than the other. Here the sum  $\phi + \theta$  is equal to two right angles.

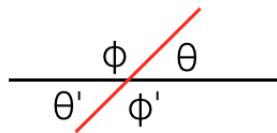


Some textbooks emphasize the arithmetic relationship for supplementary angles: any two angles whose sum is equal to two right angles are supplementary. And then angles that are both adjacent (vertex at the same point) and supplementary are called a *linear pair*.

This distinction seems overly pedantic. We will just use supplementary angles to describe both situations.

## vertical angles

Now, consider the angles lying below the horizontal:



We said that the sum of the two angles  $\phi + \theta$  is equal to two right angles, because those two angles (and no others) lie above the black line. But  $\theta' + \phi$  and  $\theta + \phi'$  are equal to two right angles, for the same reason: they are the angles on one side or the other of the red line.

As a result

$$\phi + \theta = \text{two right angles} = \theta + \phi'$$

By subtracting  $\theta$  from both sides, we conclude that

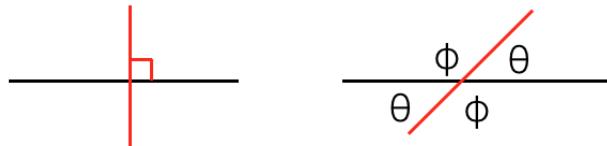
$$\phi = \phi'$$

A similar argument will show that

$$\theta = \theta'$$

This is the *vertical angle theorem*.

- Vertical angles are equal.



The vertical angle theorem is obtained by two successive applications of the supplementary angle theorem. It's powerful because it doesn't matter how large or small the two angles are. The vertical angles, that oppose one another, are always equal.

On the left, one of the angles where two lines cross is a right angle. If one angle at an intersection of two lines is a right angle, all four are right angles, by the supplementary and vertical angle theorems.

## problem

*To prove.*

If two adjacent supplementary angles are bisected, the bisectors are perpendicular to each other, that is, the angle between them is a right angle.

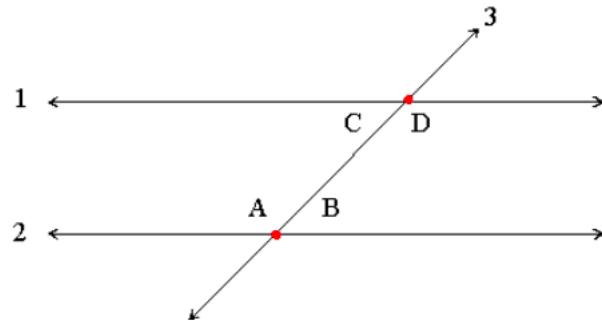
# Chapter 3

## Parallel lines

The theorems from the previous chapter about supplementary and vertical angles seem rather obvious, once you get used to them. Euclid's fifth and final postulate is more subtle.

In the figure, line 1 and line 2 are parallel, *if and only if*

$$B + D = 2 \text{ right angles} = 180^\circ$$



The same statement holds for  $A + C$ , their sum also equals  $180^\circ$ .

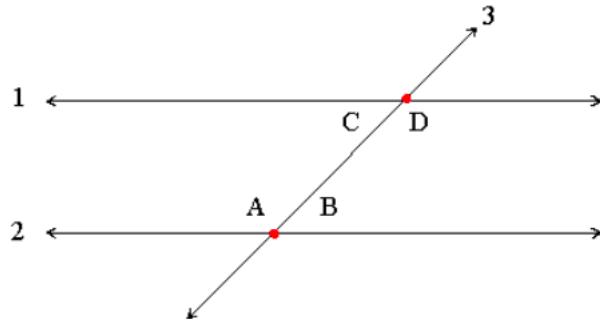
- o If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

This postulate is equivalent to what is known as the parallel postulate.

<http://mathworld.wolfram.com/EuclidsPostulates.html>

## alternate interior angles

In the figure



lines 1 and 2 are parallel. Furthermore, the supplementary angles  $A + B$  add up to  $180^\circ$ . So

$$A + C = 180 = A + B$$

The first equality follows from the parallel postulate and the second is because they are supplementary angles.

But then by subtraction,

$$B = C$$

This is called the theorem on *alternate interior angles* between two parallel lines.

- for a transversal of two parallel lines, alternate interior angles are equal.

The angles are called interior because they lie between the two parallel lines.

## Logic

We drew a third line traversing two other lines. Given that background, we then said that if (P) those two lines never cross (they are *parallel*), then (Q) the sum of the adjacent interior angles is equal to two right angles, which means that alternate interior angles are equal.

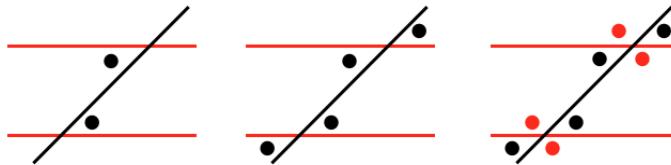
In symbols, we write  $P \Rightarrow Q$ , by which we mean *if P is true, then so is Q*.

However, you may have noticed that what we actually wrote was *P if and only if Q*. In symbols, this is  $P \iff Q$ .

The meaning of *if and only if* is simply that both  $P \Rightarrow Q$  and  $Q \Rightarrow P$ .

If the sum of the adjacent interior angles is equal to two right angles, then the two lines being traversed are parallel.

## extending the result



In the figure above (left panel), we're given that the two horizontal lines are parallel.

The indicated angles are equal because they are alternate interior angles of two parallel lines (parallel postulate).

In the middle panel, two additional equalities are established by the vertical angle theorem. Then on the right, we use the supplementary angle theorem.

Note that the conclusions for the angles marked with a red dot are themselves consistent with the three postulates/theorems that we have so far: supplementary and vertical angles and alternate interior angles.

This postulate is also valid in reverse. If we have a line that traverses two others so as to give interior angles summing to  $180^\circ$ , then the two lines must be parallel.

## symmetry

This is first of all a statement about the world, that we can have two lines that run alongside each other but don't touch. Another way to talk about the situation is to say that if we sight down a pair of parallel lines from left to right, then turn the paper  $180^\circ$  and look again, the picture will look the same.

By rotational symmetry then, we have no reason to distinguish two parallel lines in the forward and reverse directions. It follows that the intersection between two parallel lines and a third line that crosses them should look the same in both directions, the angles involved should have the same measures. That is our theorem.

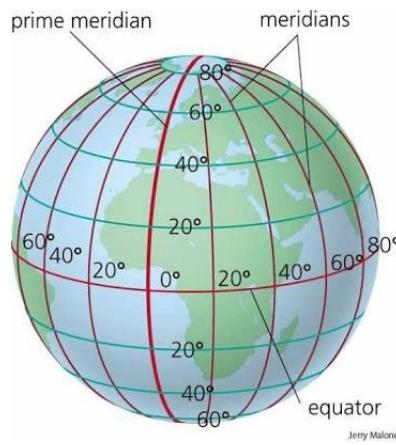
And this reasoning works, just so long as we are talking about lines on a flat piece of paper.

## flat geometry

Without getting too deep in the philosophical weeds, it's clear that mathematics is partly a construction. We must choose rules that work, or at least do not conflict with each other, and adoption of the parallel postulate is a choice.

This choice of definition works for geometry in the flat plane, but not on a curved surface like the earth. That's a familiar situation where our postulate is not appropriate.

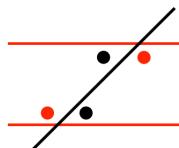
Two adjacent lines of longitude can be drawn so as to cross the equator at right angles, and the lines are parallel there, but they will meet (intersect) at the poles.



The same thing happens if you imagine the earth at the center of the universe, looking out at the stars. Or place yourself inside a globe, looking at the thin skin of the object and thinking about the lines of latitude and longitude as seen from the inside.

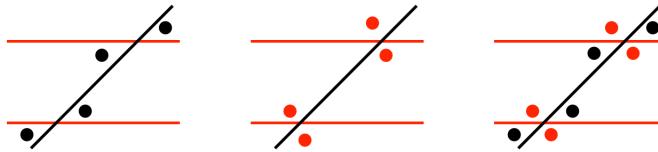
The parallel postulate only holds for geometry on a *flat* surface.

## note on terminology



Technically, it is only the angles in the figure above that are marked with black

dots or red dots and lie *between* the parallel lines, that are equal by the theorem of alternate interior angles. The others in the panel below are equal by the vertical angles theorem.



We will not usually distinguish these cases, and just refer to all of them as "equal by alternate interior angles."

## axioms

Euclid also lists five axioms, things which are assumed. Here are two examples:

- Things that are equal to the same thing are also equal to one another.
- If equals are added to equals, then the wholes are equal.

These seem quite reasonable.

We will see how to proceed from the postulates and axioms to various proofs. Given these *assumptions*, we can prove theorems that must be true.

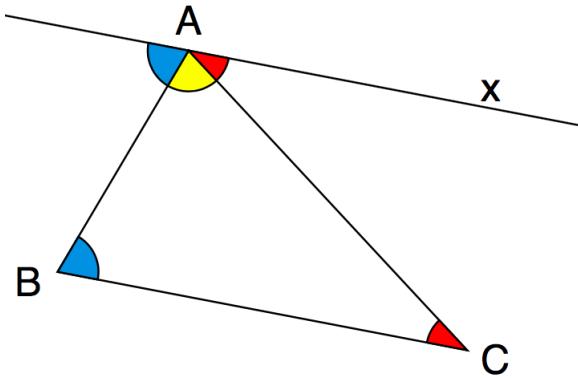
William Dunham has written a lot about the history of mathematics in Greece, starting with Thales (624-546 BC), who was from a Greek town called Miletus on the coast of Asia Minor (modern Turkey). Thales lived long before Euclid (ca. 600 BC, about 300 years before Euclid). Although none of his writing survives, it is believed that Thales proved several early theorems including the ones we saw above.

## Triangle sum theorem

We come to our first truly novel theorem. It relies on everything we've said so far. It is attributed to Thales, one of three elementary but novel theorems for which he is thought to have developed proofs.

### *Triangle sum theorem*

- The sum of the three angles in any triangle is equal to two right angles.



This theorem depends on the ideas we developed above.

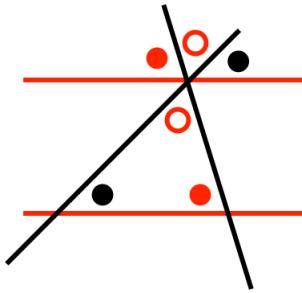
*Proof.*

Draw a line segment through  $A$  parallel to  $BC$ . Now, use alternate interior angles and follow the colors to the result. By the theorem, the two angles marked in blue are equal, as are the two angles marked in red. But the three angles at the point marked  $A$  add up to two right angles.

So the total measure of three angles in a triangle is equal to two right angles.

□

A simple variation on this proof uses the angles above the line.

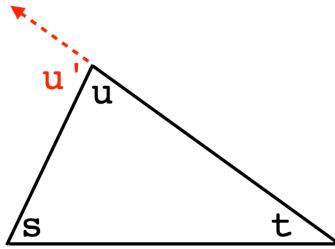


There are two more famous theorems ascribed to Thales, one in the next chapter, and one about circles that we mentioned in the beginning.

Heath says that rather than Thales, Pythagoras is responsible for the triangle sum proof. Since it's not clear that Pythagoras knew any geometry, I'm happy to stick with Thales. No one knows, unfortunately.

## problem

Many geometry books will have you do some arithmetic at this point. For example, in the triangle below, suppose that  $\angle s = 70^\circ$  and  $\angle t = 30^\circ$ . What is the measure of  $\angle u$ ? What is the measure of  $\angle u'$ ?



This is simply addition and subtraction, once you understand about supplementary angles and the sum of angles theorem. Go ahead and do that if it amuses you.

We turn to the **external angle theorem**.  $\angle u'$  is the *external angle* of this triangle. It is related to  $\angle s$  and  $\angle t$ .

Write two equations giving the relationships between the angles:

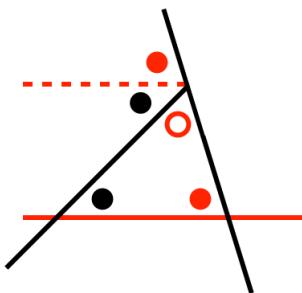
$$u + u' = 180$$

$$s + t + u = 180$$

We used supplementary angles and the sum of angles.

It follows that  $u' = s + t$ .

This figure makes a proof without words:



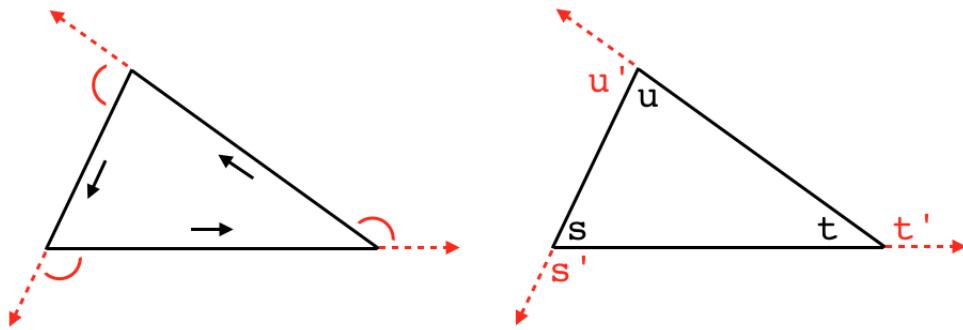
## another proof

Here is a different proof of both of these theorems relating to the angles in a triangle. This one is from Lara Alcock (*Mathematics Rebooted*).

It never hurts to re-prove important results by a different method. This serves as a check on both the result and the methods.

Imagine walking around the perimeter of a triangle in the counter-clockwise direction. At each vertex we turn left by a certain angle, called the exterior angle. After passing through all three vertices, we will end up facing in the same direction as we started.

We have made one complete turn, the sum of the exterior angles is 4 right angles.



$$s' + t' + u' = 4 \text{ right angles}$$

In addition, for each vertex, the interior angle plus the exterior angle add up to 2 right angles. If we add up all three pairs, we obtain 6 right angles.

$$(s + s') + (t + t') + (u + u') = 6 \text{ right angles}$$

Subtract the first equation from the second

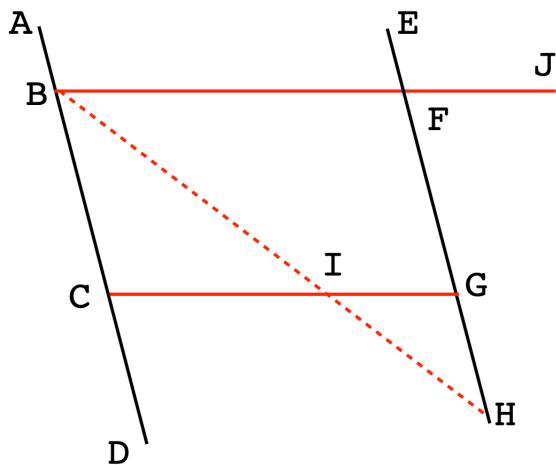
$$s + t + u = 2 \text{ right angles}$$

□

## problem

In the diagram below,  $ABCD$  lie on one line, and it is parallel to  $EFGH$ . We can write this as  $AD \parallel EH$ .

Tell which angles are equal and why.



Which are supplementary, which are vertical? (Refer to the angles in any way you wish, using the points, or with new labels).

## summary

Make sure you learn and understand each of these theorems.

- the sum of two supplementary angles is equal to two right angles ([ref](#)).
- by definition, if two angles are supplementary and also equal, they are both right angles
- vertical angles are equal ([ref](#))
- alternate interior angles of parallel lines are equal ([ref](#))
- the sum of angles in a triangle is equal to two right angles ([ref](#)).

## Eratosthenes

It is often supposed that the ancient world believed the earth to be flat, but this is just wrong. People with any level of sophistication not only knew the earth is roughly spherical but also knew its size within a few percent of the true value.

One likely basis is the false story that Columbus had trouble getting financing for his proposed trip to China because everyone thought he would fall off the edge of

the earth. This was a tall tale invented by Washington Irving, who also made up several remarkable fables about George Washington.

The real reason the Italian and Portuguese bankers from whom Columbus sought financing thought he would fail is that they had a pretty good idea of the size of the spherical earth and thus of the distance to China, while the over-optimistic Columbus believed it was about half the true value. The prospective financiers knew that he was not able to carry the supplies necessary for a trip of this length.

Morris Kline (*Mathematics and the Physical World*) says that the error is due to geographers after Eratosthenes, who reduced the estimated circumference from 24,000 to 17,000 miles.

## Eratosthenes

Views of the Greek philosophers on the earth and its sphericity are detailed here

<https://www.iep.utm.edu/thales/#SH8d>

Here is a partial quotation:

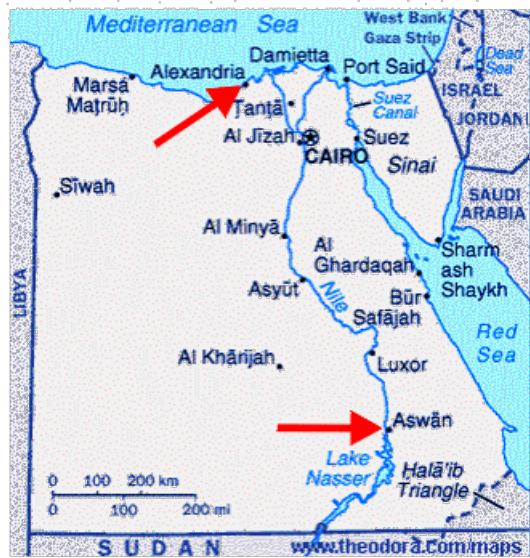
There are several good reasons to accept that Thales envisaged the earth as spherical. Aristotle used these arguments to support his own view [...] . First is the fact that during a solar eclipse, the shadow caused by the interposition of the earth between the sun and the moon is always convex; therefore the earth must be spherical. In other words, if the earth were a flat disk, the shadow cast during an eclipse would be elliptical. Second, Thales, who is acknowledged as an observer of the heavens, would have observed that stars which are visible in a certain locality may not be visible further to the north or south, a phenomenon which could be explained within the understanding of a spherical earth.

<https://en.wikipedia.org/wiki/Eratosthenes>

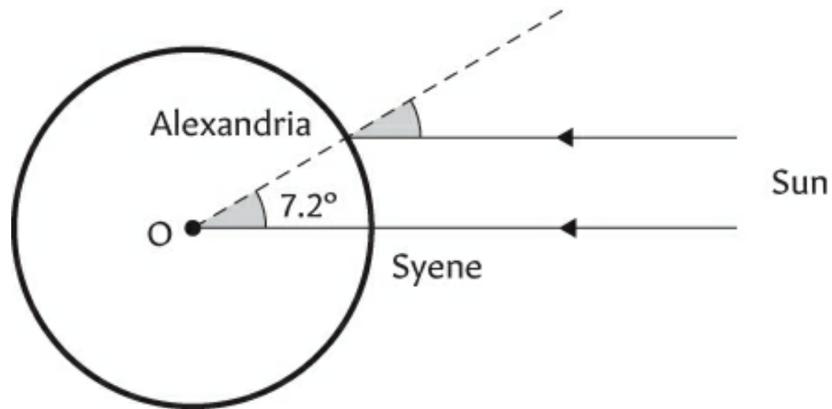
Eratosthenes (ca. 276 - 195 BCE) measured the circumference of the earth from this observation: at high noon on the solstice on June 21st there was no shadow seen at Syene, allegedly from a stick placed vertically in the ground. Some people say a deep well had the bottom illuminated at midday. Acheson says Eratosthenes was born at Syene, so he would probably know!

Syene is presently known as Aswan. It is on the Nile about 150 miles upstream of Luxor, which includes the famous site called the Valley of the Kings, where many

Egyptian Pharaohs were entombed. At 24.1 degrees north latitude, Aswan or Syene has the sun almost directly overhead on June 21. (The "Tropic of Cancer" is at 23 degrees, 26 minutes north).



Alexandria was a famous center of learning of the ancient world, and Eratosthenes was hired by the pharaoh Ptolemy III to be the librarian there in 245 BCE. Alexandria lies on the Mediterranean some 500 miles north of Syene, and anyone there who was looking could observe that at high noon on June 21st there *was a shadow*. This shadow Eratosthenes measured to be some 7 degrees and a bit (7 degrees and 10 minutes).



**Fig. 21** Measuring the Earth.

A full 360 degrees divided by 7 degrees and a bit is approximately 50. So we can calculate on this basis that the circumference of the earth is about  $50 \times 500 = 25000$  miles. That's pretty close to the correct value.

For this calculation, we assume that the sun's rays are effectively parallel (not a bad assumption given a distance of 93 million miles). Then we just use an application of the alternate-interior-angles theorem.

It is curious how the distance from Alexandria to Syene was calculated.

Kline:

Camel trains, which usually traveled 100 stadia a day, took 50 days to reach Syene. Hence the distance was 5000 stadia...It is believed that a stadium was 157 meters.

We obtain

$$157 \times 5000 \times 50 = 39,250 \text{ km}$$

That's a much better estimate than a method that relies on camels really deserves.

# Part II

## Triangles

# Chapter 4

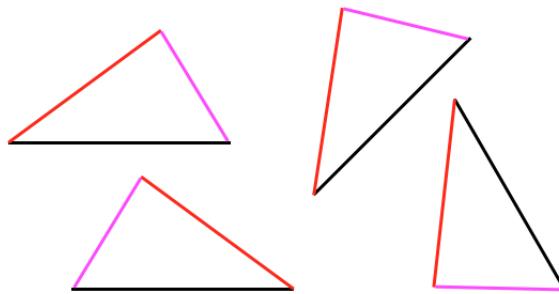
## Congruent triangles

### meaning of congruence

Probably the most fundamental idea concerning triangles is how to decide that two triangles are *congruent*. If so, they have all 3 sides the same length, and all 3 angles the same measure.

However this is logically the converse (backwards) of what we really want to know. We would like to be able to say that that two triangles are congruent *if* they have [ some properties ].

We will find tests that, if passed, imply congruence.



We allow one triangle to be rotated at any angle with respect to the other. An example of this is the two triangles at the top in the figure above.

We will also allow the term congruent to apply to the case of a triangle and its mirror image. All of the triangles shown are congruent, even though two are flipped — they

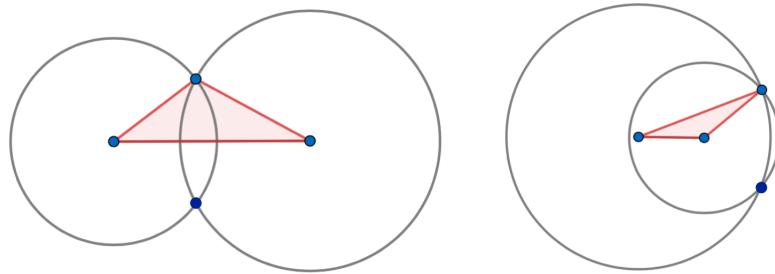
are mirror images.

## SSS test for congruence

Perhaps a practical definition would be that if you used a pair of scissors to cut out one triangle and then lay it on top of the second one so that it superimposes exactly, they would be congruent.

In fact this is how Euclid handles the issue in the first theorem of *Elements*, I.4.

A little thought may convince you that if all three corresponding side lengths are equal, two triangles are congruent. Draw one side of a triangle, and from its endpoints draw circles with radii the length of the other two sides.



The two radii map out all the points that are the same distance from the centers. They include only two possible arrangements for three given side lengths which result in triangles.

We see that the radii cross at two and only two different points, which have mirror image symmetry above and below the original base. Three given side lengths can only be drawn together to give two resulting triangles, and these two shapes are mirror images.

It is certainly possible to come up with side lengths that *cannot* form a triangle. Consider (left panel, above) what would happen if the two shorter sides were exactly equal to the longest one. Then they would meet at a single point on the longest side, and there would be no triangle.

If the sum were shorter, they would not meet at all. This is called the triangle inequality and we'll prove it more formally later.

Thus, we arrive at a fundamental theorem about congruency for triangles:

- Two triangles are *congruent* if they have the same three side lengths.

This test is abbreviated SSS (side-side-side).

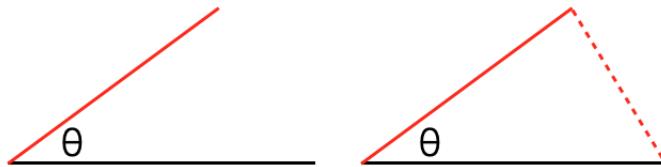
## SAS

In addition to SSS (side-side-side), there are three other conditions that always lead to congruence of two triangles when they are satisfied, namely

- SAS (side-angle-side)
- ASA (angle-side-angle)
- AAS (angle-angle-side)

When we say SAS, we mean that the equal angle we know lies between the pair of equal sides. Similarly, ASA means that we know one side of a triangle and the two angles at either end of that side are the angles in question.

As we did above for SSS, a good way to think about the congruence condition SAS is to imagine trying to construct a triangle from the given information, and ask whether it is uniquely determined.

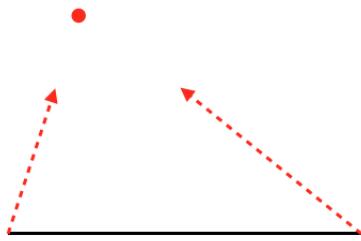


Two sides and the angle between them are given. So draw that part of the triangle. Notice that the second and third vertices are also determined, because they just lie at the ends of the two sides we're given. All that remains is to draw the line segment that joins them.

Of course we might have drawn the shorter side as the horizontal. If we did that, it would generate the mirror image, and still be congruent by our rule.

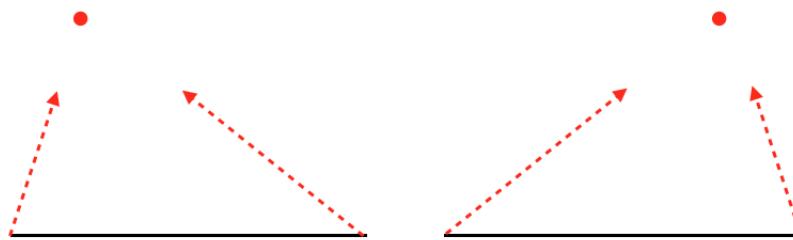
## ASA

The next one is ASA. Since we know two angles, we know the third. Here is a diagram of the situation:



Draw the known side, then using the known angles, start two other sides from the ends of that side. They must cross at a unique point.

But...actually, if we start the two lines from opposite ends of the horizontal



there is another solution, the mirror image. These two triangles are congruent to the one above.

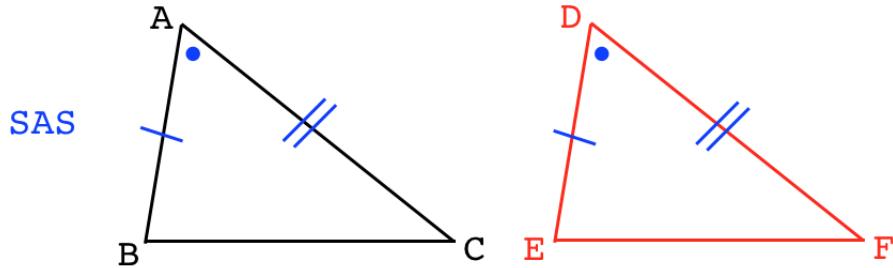
I'm tempted to draw the constructions below the starting line. But this doesn't give anything new. These are merely rotated versions of the ones above. Try it and see. Congruent triangles include a pair of mirror images and that's it.

Now, if we know two angles we also know the third, by the angle sum theorem. For this reason, ASA and AAS mean that we have exactly the same information, because we know all three angles and we know one side.

Crucially, we know *which* two angles flank the known side. Equivalently, it is enough to know which angle is opposite to the known side.

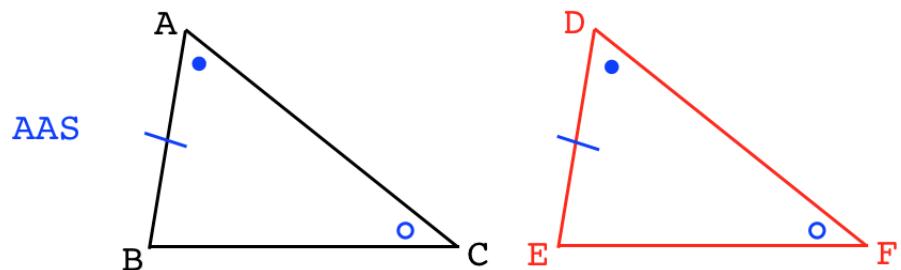
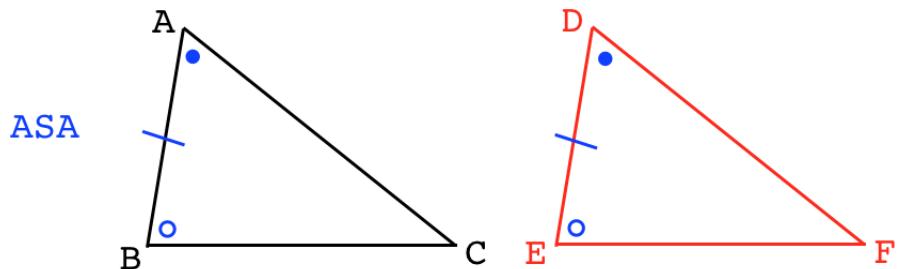
## marks for equality

It is often useful to mark sides and (particularly) angles to show they are equal. Here is



In this diagram, sides of equal length are indicated by one or more short lines called hash marks.

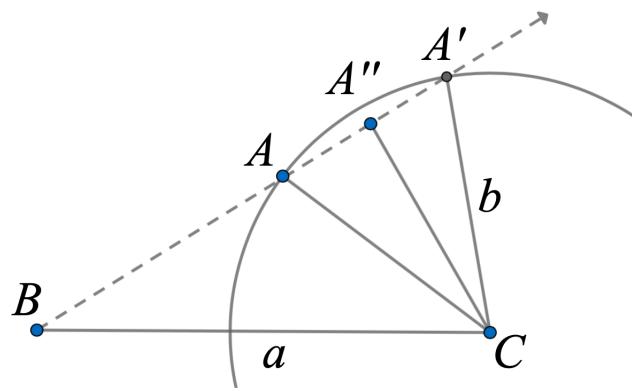
Equal angles are usually indicated by dots in this book. Dots are easier to place on the figures, and lend themselves to color-coding; the common method for pencil and paper is to draw an arc with a hash across it, or just use a filled and open circle.



### but not SSA

There is one set of three that doesn't work in the general case, and that is SSA (side-side-angle).

Suppose we know the lengths of sides  $a$  and  $b$  and the angle at vertex  $B$ , adjacent to  $a$  and opposite side  $b$ .



Since the length of the third side isn't known we make it a dashed line. Let's see if we can construct a triangle from this information.

We do not know the angle at vertex  $C$  between  $a$  and  $b$  so we imagine  $b$  swinging on a hinge there. If  $b$  is too short, there can't be a triangle. If the length of  $b$  is exactly right, we'll have a right angle at  $A''$ .

If  $b$  is longer than this but still  $b < a$ , there are two points  $A$  and  $A'$  where  $b$  can intersect with the side projecting from vertex  $B$ .

This is the *ambiguous case*: we have two possibilities. If two different triangles match by the SSA criterion, we cannot say whether they are congruent or not without more information.

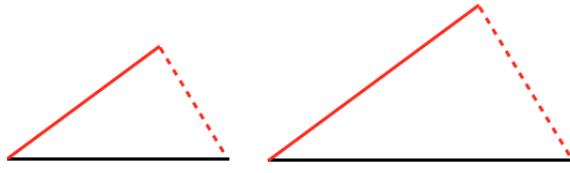
The right angle case is not ambiguous, but we'll save that for the chapter on right triangles.

If you compare this chapter with most others in the book you'll notice that we have not formally proved that any of these methods are correct. Even Euclid encounters some difficulty with this point. He "proves" SAS by a method that is arguably not really a proof.

It won't hurt my feelings if you think of them as axioms. The famous mathematician David Hilbert does this. We will revisit the issue when we look at Euclid's proofs.

## similar triangles have equal angles

Sometimes two triangles are not congruent, but have all three angles the same. We call such triangles *similar*.



Similarity means that all three angles are the same but the triangles are of different overall sizes.

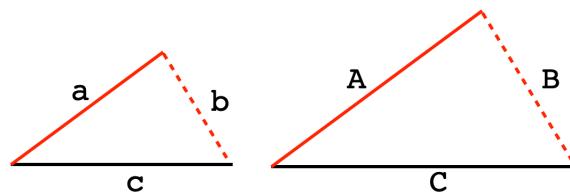
Our basic criterion for similarity is AAA (angle-angle-angle).

However, because of the sum of angles theorem, if any two angles of a pair of triangles are known to be equal, then the third one must be equal as well. We can say that:

- Two triangles are similar if they have at least two angles equal.

## scaling

For similar triangles, the three corresponding pairs of sides are in the same proportions, but re-scaled by a constant factor.



From the above diagram of two similar triangles, similarity implies that (for example)

$$\frac{A}{a} = \frac{B}{b}$$

which can be rearranged to give:

$$\frac{a}{b} = \frac{A}{B}$$

For any pair of similar triangles, there is a constant  $k$  such that

$$k = \frac{A}{a} = \frac{B}{b} = \frac{C}{c}$$

We will come back to proofs about similar triangles in a later chapter. In particular, we will show that AAA implies equal ratios of sides as given here, as well as the converse.

## sides and opposing angles

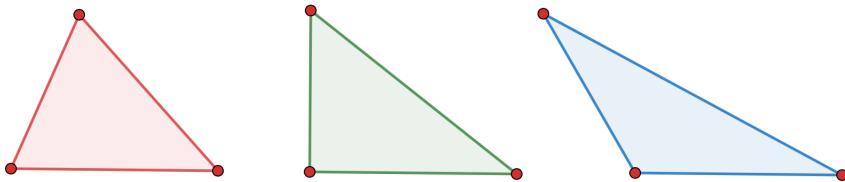
When two triangles are congruent, each of the three angles and all three sides are the same. You might wonder whether the sides could be placed in a different order. We usually draw side of length  $a$  opposite  $\angle A$ , side  $b$  opposite  $\angle B$  and side  $c$  opposite  $\angle C$ . Can we switch the sides so that say, the length  $c$  is opposite  $\angle B$ ?

It turns out not to be possible. Later we will have a theorem which says that in any triangle, the longest side is opposite the largest angle, and the smallest side opposite the smallest angle. If switching two sides were possible, the theorem could not be correct, but it is correct, which means that switching sides is not possible.

# Chapter 5

## Isosceles triangles

Triangles are classified by the largest angle they contain: acute, right, or obtuse.



The acute triangle (left) has all three angles smaller than a right angle.

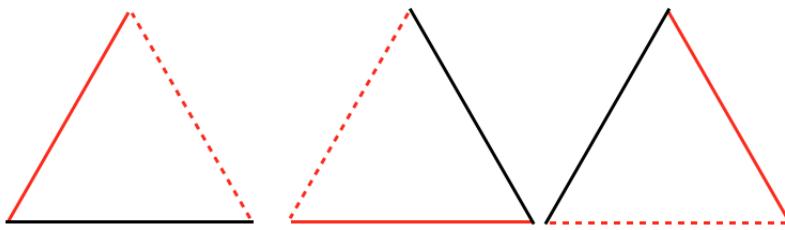
The right triangle, naturally, has one right angle.

An obtuse triangle has one angle larger than a right angle (right panel, above).

### symmetry

One can also talk about the situation where either two sides, or all three sides, have the same length. An *isosceles* triangle has two sides the same length, while an *equilateral* triangle has all three sides the same.

The most important consequence of all three sides equal for an equilateral triangle is three-fold rotational symmetry. Three turns of 120 degrees, and we're back where we started. Each of the two intermediates is identical.



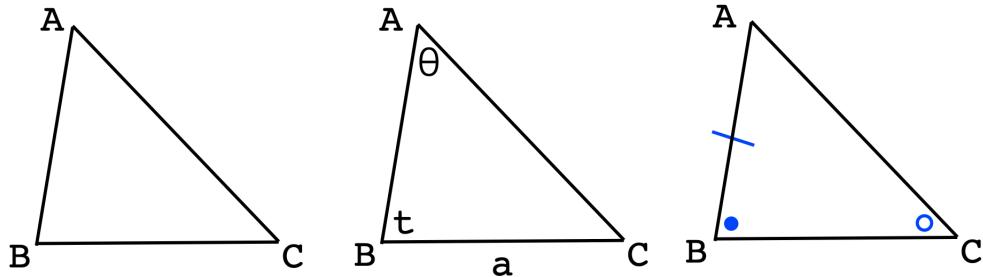
The implication of rotational symmetry is that the three angles are also equal because there is no reason to choose one larger than any other.

Therefore each angle of an equilateral triangle is  $2/3$  of a right angle, or  $60^\circ$ , by the triangle sum theorem ([ref](#)).

It is also true that if all three angles are equal, then the triangle is equilateral (three sides equal). We will show how to prove this later.

## notation

The Greeks, including Euclid, always label points with letters, and line segments are referred to by the endpoints. Angles and triangles are denoted by the line segments from which they are composed, as in  $\angle ABC$ , and triangles by their vertices:  $\triangle ABC$ .



The Greek notation (left panel) is problematic because it is more complicated than need be. I have found myself repeatedly tracing out angles from three points.

Quite often, we will label the side opposite a vertex with a lower case letter: side  $a$  lies opposite  $\angle A$ . We may also use letters like  $\theta$  and  $\phi$  for angles, or even, more boldly,  $s$  and  $t$ .

Sometimes, we will dispense with labels altogether and use colored dots for equal angles.

The image below is from the web, it uses the convention of an arc drawn across equal angles. The curved arcs are common in books.

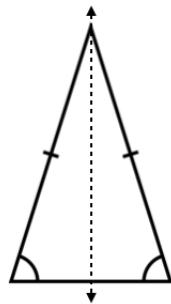
### theorem from Thales

- If a triangle is isosceles (two sides equal), then the base angles are also equal.

The converse is

- If two base angles are equal, then the triangle is isosceles.

My favorite proof of both theorems about isosceles triangles is from reflective or mirror image symmetry.



*Proof.*

Imagine that the triangle sticks straight up from the plane like one of the standing stones at Stonehenge. Think about walking around the triangle.

Looking from behind, it would appear exactly the same. We would say that the left side as viewed from the front is equal to the right side as also viewed from the front, because if we walk around behind the triangle the *right* side becomes the *left* and vice-versa.

Much later than Euclid, Pappus invokes SAS on the mirror image, rather than thinking about the plane being in 3D space.

The top vertex angle is shared. So the triangle as we look from the front is equal by SAS to the one where we look from the back. It follows that the base angles are equal.

□

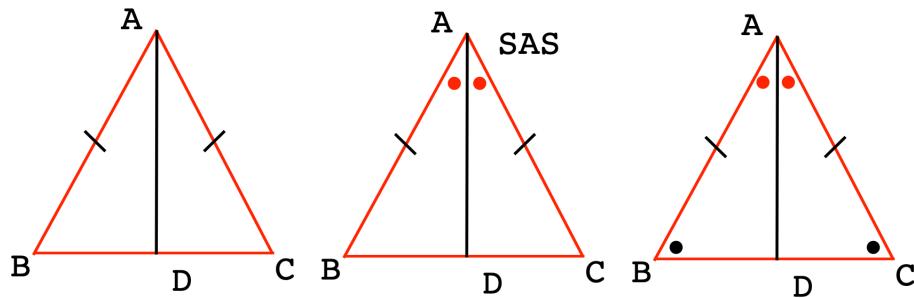
## proofs based on triangle bisection

The argument above is not Euclid's proof, namely, I.5, the second theorem in his book. We will not give that one right now, but instead something simpler.

The forward theorem on isosceles triangles is:

- Two sides equal  $\Rightarrow$  opposite angles equal.

There is a simple demonstration based on angle bisection, and another based on the ability to cut a line segment into two equal parts, also called a bisection.



*Demonstration.*

We are given  $AB = AC$  (left panel).

Draw the bisector of the top angle  $A$  (middle panel). This construction forms equal angles at the top, marked with red dots.

$AB = AC$  and  $\angle BAD = \angle DAC$ .

$AD$  is shared.

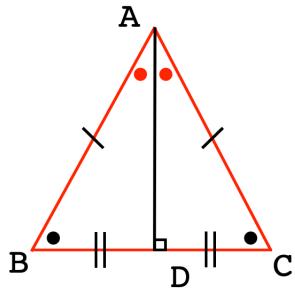
The two smaller triangles  $\triangle ABD$  and  $\triangle ACD$  are congruent by SAS.

We write that as  $\triangle ABD \cong \triangle ACD$ .

□

Therefore (as corresponding parts of congruent triangles) the base angles are equal (right panel). Other corresponding angles and sides are equal as well.

The full set of equal angles and sides is:

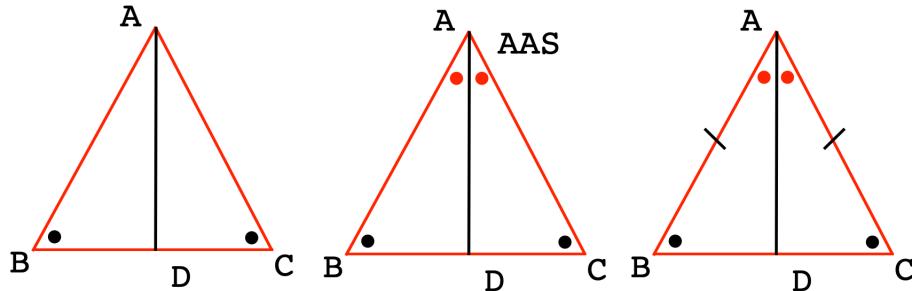


The base is bisected, there is a right angle where the bisector  $AD$  cuts the base, plus of course the central line segment  $AD$  is equal to itself.

### converse

We can also prove the converse theorem:

- Two angles equal  $\Rightarrow$  opposite sides equal.



*Proof.*

We are given that the angles marked with black dots are equal.

We again draw the bisector of the angle  $A$ . Then we have all three angles the same, and the side  $AD$  is shared.

Therefore,  $\triangle ABD \cong \triangle ADC$  by AAS.

It follows that  $AB = AC$ .

As before, there are right angles at the base, and the base is bisected.

□

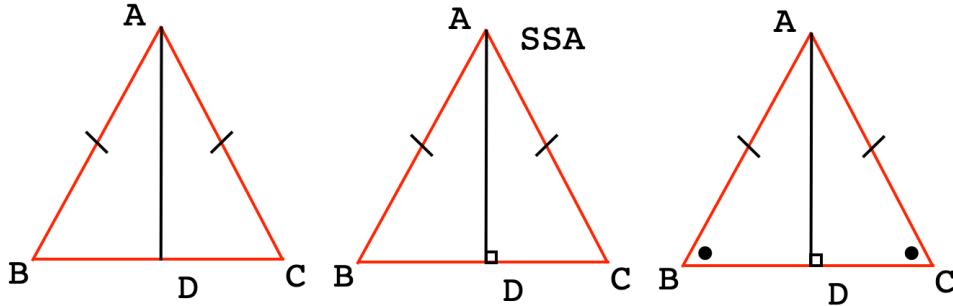
Euclid's proof of the isosceles triangle theorem is more complicated than what we have given above, and there is a good reason for this.

Our demonstration depends on the existence of the angle bisector, but we haven't actually shown how to do that. It will turn out that **construction** of the bisector *depends on* the isosceles triangle theorem.

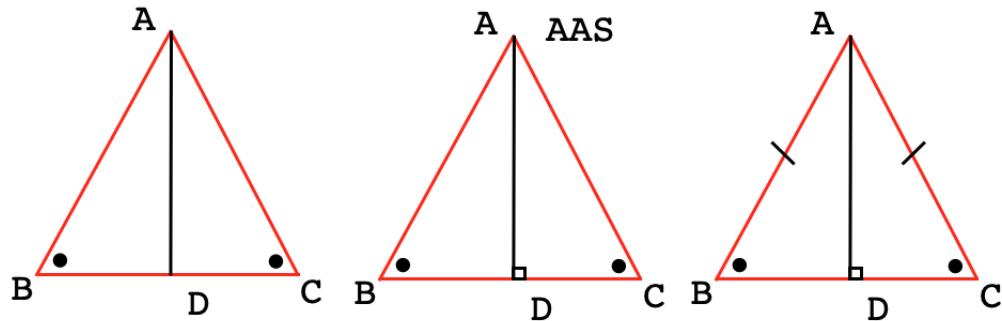
That's a problem because the reasoning is circular, thus invalid. We cannot use  $p$  to prove  $q$  if we have previously used  $q$  to prove  $p$ . That proves nothing. We will fix this problem later.

## other proofs

We used the angle bisector at vertex  $A$  for the proofs above. But we might equally have constructed a right angle at  $D$ , or bisected the base. We will just show diagrams for both proofs adapted to these methods and sketch the idea.

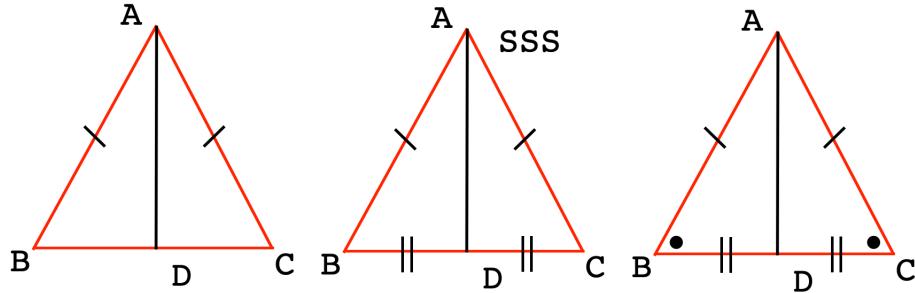


Above, we are given  $AB = AC$  and right angles at  $D$  (and  $BCD$  colinear). We have hypotenuse-leg in a right triangle (HL), so the two small triangles are congruent.

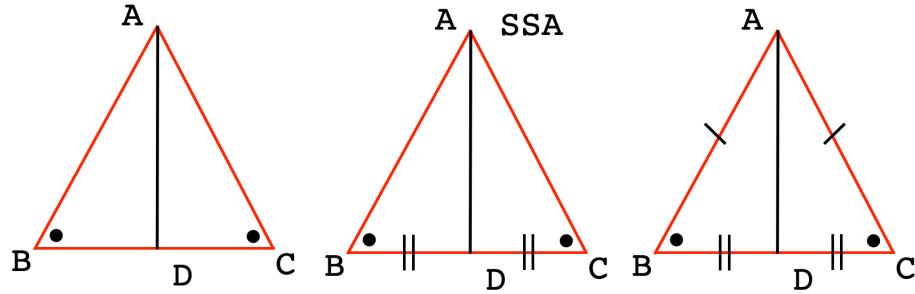


Above, we are given equal angles at  $B$  and  $C$  and right angles at  $D$ . We have three

angles equal plus a shared side, so congruent triangles by AAS (or after application of the triangle sum theorem, by ASA).



Above, we are given  $AB = AC$  and  $BD = DC$ , i.e. the base is bisected. Since  $AD$  is shared we have that  $\triangle ABD \cong \triangle ADC$  by SSS.



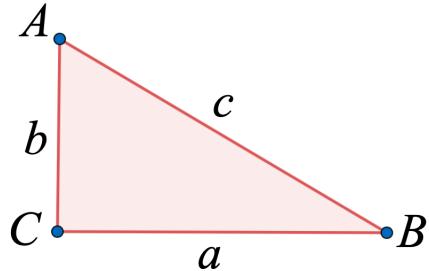
Finally, we are given equal angles at  $B$  and  $C$  and we bisect the base. We have SSA but do not know yet about the right triangle. This presents some difficulties, which we put off until later.

# Chapter 6

## Right triangles

A *right* triangle is a triangle containing one right angle. We saw previously that the Greeks' definition of a right angle is that two of them add up to one straight line or 180 degrees.

No triangle can contain two right angles, because the measure of the third angle would then have to be zero. This goes back to the parallel postulate. Right angles at both ends of a line segment send out parallel lines.



Right angles and right triangles are special in many ways.

In the figure, let us have the angle at vertex  $C$  be a right angle. It is common practice to mark a right angle with a little square, but we often just state the fact. So  $\angle ACB$  is right.

The side opposite vertex  $C$  — it is common practice to label it the same but lowercase, so  $c$  — is called the *hypotenuse*, and the other two sides  $a$  and  $b$  are sometimes called legs.

## complementary angles

Since the sum of angles in a triangle is equal to two right angles, the two acute angles,  $\angle BAC$  and  $\angle ABC$  in the figure above, together equal one right angle, or 90 degrees. Along with with  $\angle ACB$  they sum up to two right angles for the whole triangle.

*Proof.*

This is a direct consequence of the sum of angles theorem applied to a right triangle.

□

The two smaller angles in a right triangle are said to be *complementary*. This fact is often exploited in proofs, since if we know one, we know the other, by sum of angles.

- the sum of the two smaller angles in a right triangle is equal to one right angle.

## Pythagorean theorem

A second very important fact about right triangles is expressed in the Pythagorean theorem. Although we haven't proved it yet, we will do so shortly, and call on it here. Given the hypotenuse  $c$  and two legs of a right triangle,  $a$  and  $b$ , the theorem says that

$$a^2 + b^2 = c^2$$

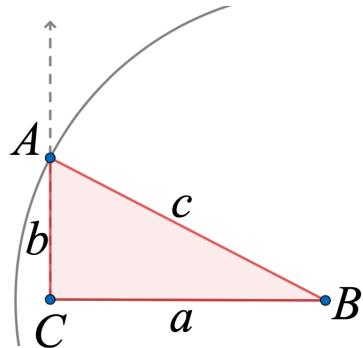
It follows that if we know any two sides in a right triangle, then we know all three sides.

A third fact is that the hypotenuse is the longest side in a right triangle. We will prove later that in any triangle, one side is longer than another in any triangle *if and only if* the angle opposite that side is also larger than the angle opposite the shorter side.

Previously, we gave several ways to prove that two triangles are congruent. These four methods (SAS, SSS, ASA, and AAS) are also useful with right triangles. Some books give them new names in the context of a right triangle. One of these is useful.

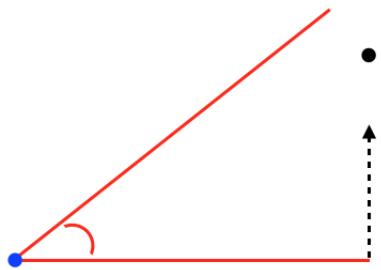
## hypotenuse-leg in a right triangle (HL)

For two right triangles, if one hypotenuse is equal to the other, and also one pair of legs equal, the two triangles are congruent. This condition is called hypotenuse-leg (HL). It is effectively SSA in the case where we know that the angle is a right angle.



Here we know sides  $c$  and  $a$  and also have that the angle at  $C$  is a right angle.

If two right triangles have hypotenuse and leg equal , then they are congruent by HL. We will use this test often.



In the figure, imagine the hypotenuse swinging on the hinge of its vertex with the horizontal base, since we do not know the angle between them.

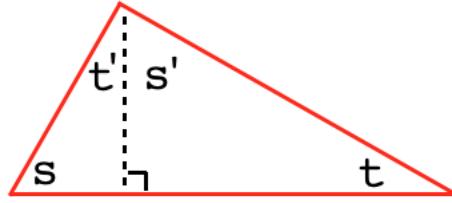
There is only one angle where the hypotenuse will terminate on the vertical extension from the right angle. That point is marked with a black dot.

*Proof.*

An algebraic proof is that the third side is determined by the other two sides by the Pythagorean theorem, since we know which side is the hypotenuse. Therefore we have SSS.

□

## altitudes



In the large right triangle above, drawn in red, we know that

$$s + t = 90$$

When we draw the perpendicular to the hypotenuse that goes through the upper vertex, that is an *altitude* of the triangle. Because of the right angle, we obtain two smaller right triangles. Thus

$$\begin{aligned} s + t' &= 90 \\ s' + t &= 90 \end{aligned}$$

Hence

$$s + t = 90 = s + t'$$

so

$$t = t'$$

and similarly,  $s = s'$ .

This is another really important result: if the altitude to the hypotenuse is drawn in a right triangle, the two smaller right triangles are both similar to the original one. All three triangles have the same angles.

## theorems about right triangles

- In any right triangle, the right angle is larger than either of the other two angles.

*Proof.*

Suppose  $\alpha$  and  $\beta$  are complementary angles in a right triangle. Then  $\alpha + \beta$  is equal to one right angle.

$$\alpha + \beta = 90$$

Both angles  $\alpha$  and  $\beta$  must be non-zero:  $\alpha > 0$  and  $\beta > 0$  (otherwise we do not have a triangle).

$$\alpha > 0$$

$$\alpha + 90 > 90$$

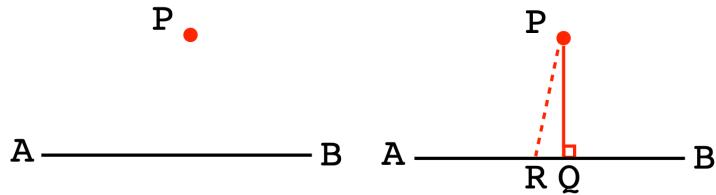
$$90 > 90 - \alpha = \beta$$

The same proof applies starting with  $\beta > 0$ .

□

### only one perpendicular to a line from a point

Suppose we have a line and a point not on the line.



We claim that only one perpendicular can be drawn from the point to the line.

*Proof.*

Assume that two such lines can be drawn. So, in addition to  $PQ \perp AB$ , we draw  $PR$  and claim that it is also perpendicular to  $AB$ .

Then, by the converse of the alternate interior angles theorem,  $PQ \parallel PR$ .

But  $PQ$  and  $PR$  also meet at the point  $P$ . This contradicts the fundamental definition of parallel lines. Our assumption must be false.

Only one such line can be drawn.

□

### hypotenuse longest side

- In any right triangle, the hypotenuse is longer than either side.

*Proof.*

We showed above that in any right triangle, the right angle is larger than either of the other two angles. By **Euclid I.18**: in any triangle, a greater side is opposite a greater angle.

□

Or we look ahead again to the Pythagorean theorem. Since  $a > 0$  and  $b > 0$  and

$$c^2 = a^2 + b^2$$

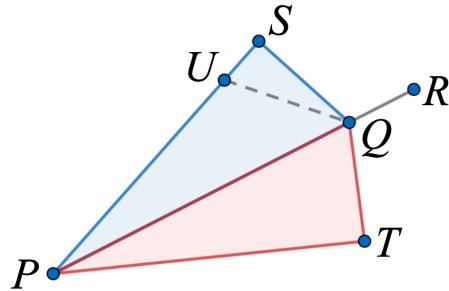
Suppose  $b > a$ . Since  $a > 0$ , it follows that  $c^2 > b^2$  so  $c > b > a$ .

The fact that the hypotenuse is the longest side in a right triangle will come in handy when we investigate the tangent to a circle. It is also useful in the next theorem.

### shortest distance from a point to a line

- The distance from a fixed point to a line is least when the new line segment makes a right angle with the line.

The claim is that if  $QS \perp PS$ , then  $QS$  is the shortest line connecting  $Q$  with  $PS$ .



*Proof.*

Aiming for a contradiction, suppose that  $QU$  is not perpendicular, but it is shorter than  $QS$ .

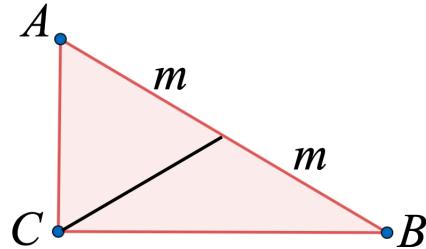
$\triangle QUS$  is a right triangle, with the right angle at  $S$ , so  $QU$  is the hypotenuse of  $\triangle QUS$ .

Since the hypotenuse is the longest side of a right triangle, by the previous theorem, So  $QS$  must be shorter than  $QU$ .

This is a contradiction. Therefore  $QU$  is not shorter than or even equal to  $QS$ .

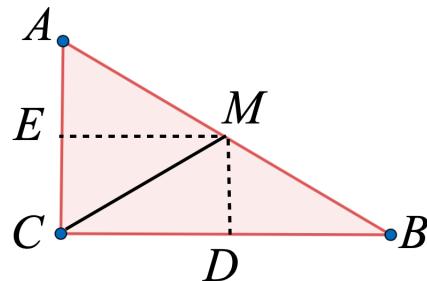
## hypotenuse midpoint theorem

In a right triangle, draw the line segment from the vertex that contains a right angle to the midpoint of the hypotenuse, separating it into two equal lengths  $m$ . We will show that the length of the bisector is also  $m$ .



*Proof.*

Draw the perpendicular from the midpoint  $M$  to the base  $BC$  at  $D$ . Also draw the perpendicular from  $M$  to the base  $AC$  at  $E$ .



$\triangle MDB$  is a right triangle, and so is  $\triangle AEM$ .

By complementary angles the other angles are equal.

We are given that  $AM = MB$ .

It follows that  $\triangle MDB \cong \triangle AEM$  by ASA.

Therefore  $EM = DB$ .

Because it has four right angles at its vertices,  $EMDC$  is a rectangle. (The fourth, at  $M$ , follows by sum of angles).

Thus  $EM = CD = BD$ .

So  $\triangle MDB \cong \triangle MDC$  by SAS.

It follows that  $MC = BM = AM$ .

□

Note that both  $\triangle AMC$  and  $\triangle CMB$  are isosceles.

There are two easier proofs of the above theorem. The first uses Thales' theorem, which we will cover again later, but mentioned at the beginning of the book. Any right triangle can be placed in (inscribed into) a circle, with the hypotenuse as the diameter.  $AM$  as well as  $BM$  and  $CM$  are all radii of this circle.

Another proof uses similar triangles. However, we have not established that theory yet, so we will skip it for now.

### converse

As a converse, if we are given that the line drawn to the midpoint of the longest side of any triangle also has length  $m$ , then the triangle is a right triangle.

*Proof.*

The two smaller triangles are isosceles. Therefore, the total angle at vertex  $C$  is half the total for the triangle, and thus equal to one right angle.

□

# Chapter 7

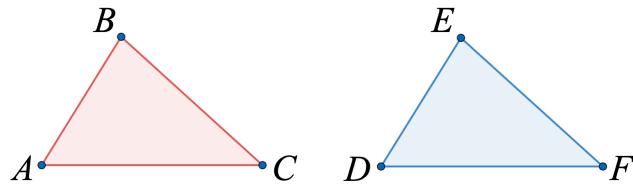
## Introduction to Euclid

In this chapter we will look at some Propositions from Book I of Euclid's *Elements*.

His book was put together as a compendium of geometry for students. One thing we will see is how the propositions build on one another to make a dependent chain. This includes a more sophisticated proof of the isosceles triangle theorem that depends only on SAS.

### Euclid. I.4

If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.



This is a method for proving congruence (equality) of two triangles

$$\triangle ABC \cong \triangle DEF$$

In modern usage, we call the method SAS or *side angle side*. Given that  $AB = DE$  and  $AC = DF$  and that the angles between them at the vertices  $A$  and  $D$  are also equal, the two triangles are congruent: all three angles and all three sides are equal.

Euclid I.4 is a proof that SAS is correct.

*Proof.*

The proof is by superposition. The facts establish the positions of the points  $B$  and  $C$ , which determines  $BC$  and so the angles at vertices  $B$  and  $C$ .

Euclid says that if we lift up  $\triangle ABC$  and lay it on top of  $\triangle DEF$  then  $B$  coincides with  $E$  and  $C$  coincides with  $F$  so  $BC = EF$ .

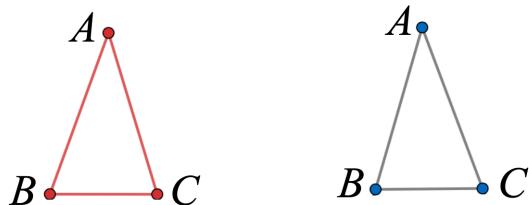
□

This seems perhaps a little shaky logically, and it's not a method of proof that Euclid uses much.

But one might instead have taken this proposition as a postulate. One source says that David Hilbert claims that under the hypotheses of the proposition it is true that the two base angles are equal, and then proves that the sides are equal.

## Euclid I.5

The forward isosceles triangle theorem is that if two sides in a triangle are equal, then so are the opposing angles. It's difficult precisely because Euclid proves it *before* we know how to bisect an angle.

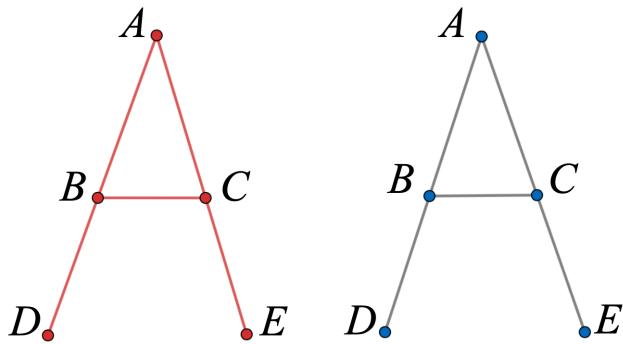


Given  $AB = AC$ . We will prove that the base angles are equal:  $\angle ABC = \angle ACB$ .

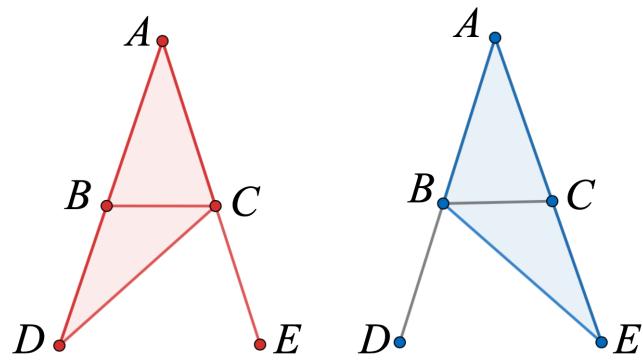
The image shows two copies of the triangle. This is so that we may compare congruent triangles formed within the *same* figure.

*Proof.*

Extend  $AB$  and  $AC$  so that  $AD = AE$ .

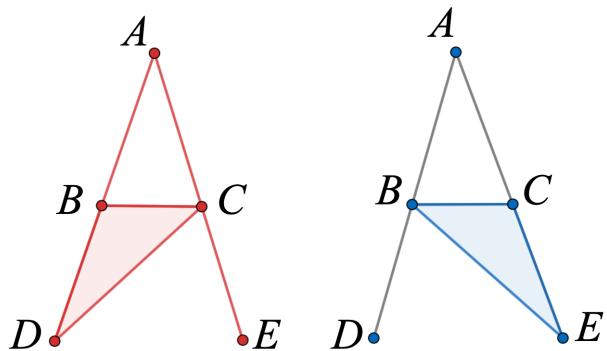


Then  $BD = CE$  by subtraction. Connect  $CD$  and  $BE$ .



$\triangle ACD \cong \triangle ABE$  by SAS ( $AB = AC$ ,  $AD = AE$ , and they share the angle at vertex  $A$ ).

As a result, we have two more pairs of angles equal:  $\angle ADC = \angle AEB$  and  $\angle ACD = \angle ABE$ . Also,  $CD = BE$ .



Since  $\angle ADC = \angle AEB$  and the flanking sides are equal, namely,  $BD = CE$  and  $CD = BE$ , we have  $\triangle BCD \cong \triangle CBE$  by SAS.

It follows that  $\angle DBC = \angle ECB$ . By supplementary angles,  $\angle ABC = \angle ACB$ .

□

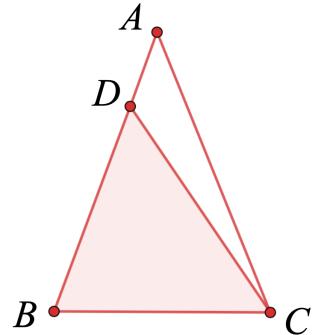
Notice how the original figure is extended to provide auxiliary shapes helpful in the proof. This is a common theme.

We will use both results ( $\angle ABC = \angle ACB$  and  $\angle DBC = \angle ECB$ ), going forward.

To summarize: in a triangle with two equal sides, the angles opposite those sides are also equal, as well as their supplementary angles.

## Euclid I.6

We proved the converse of I.5 previously based on angle bisection. Now that we have proved Euclid I.5, which provides the basis for bisection, this is logically solid. Nevertheless, for completeness, here is Euclid's proof of I.6.



*Proof.*

Given  $\angle ABC = \angle ACB$ . Suppose  $AB \neq AC$ . Then let one be less, say  $AC$ . So cut off from  $AB$  the length  $BD = AC$ .

Compare  $\triangle DBC$  with  $\triangle ABC$ . We have the sides equal, namely  $DB = AC$  and  $BC = BC$ . As well,  $\angle DBC = \angle ACB$ . Therefore  $\triangle ABC \cong \triangle DBC$  by Euclid I.4 (SAS).

But this is absurd. The part cannot be equal to the whole.

Therefore,  $AB = AC$ .

□

↔

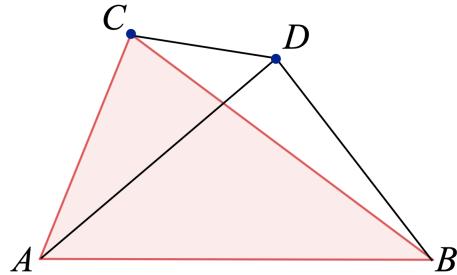
The theorem together with its converse says that, in an isosceles triangle, the base angles are equal  $\iff$  the two sides are equal (not the base).

The symbol  $\iff$  means *if and only if*, so both base angles equal  $\rightarrow$  two sides equal and two sides equal  $\rightarrow$  base angles equal.

We now extend our methods by showing that SSS leads to SAS, providing a second method of proof for triangle congruence.

But first, look at the next theorem after the two on isosceles triangles, namely, Euclid I.7.

## Euclid I.7



- Let  $\triangle ABC$  be drawn and a point  $D$  be chosen on the same side of  $AB$  as  $C$  lies. It cannot be true that both  $AC = AD$  and  $BC = BD$ .

*Proof.*

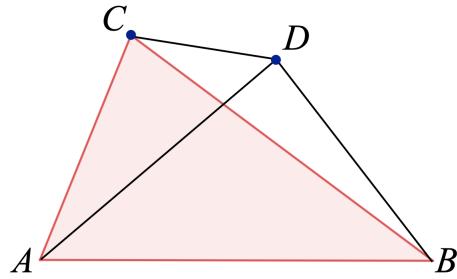
Suppose that both statements are true:  $AC = AD$  and  $BC = BD$ .

Then  $\triangle ACD$  is isosceles so  $\angle ACD = \angle ADC$ .

We notice that

$$\angle BCD < \angle ACD = \angle ADC < \angle BDC$$

But since  $\triangle BCD$  is also isosceles,  $\angle BCD = \angle BDC$ .

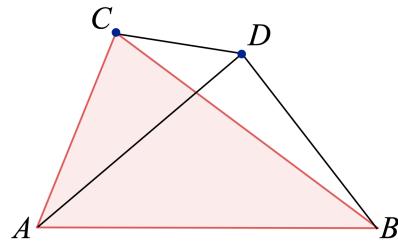


This is a contradiction. Therefore it cannot be that both  $AC = AD$  and  $BC = BD$ .

□

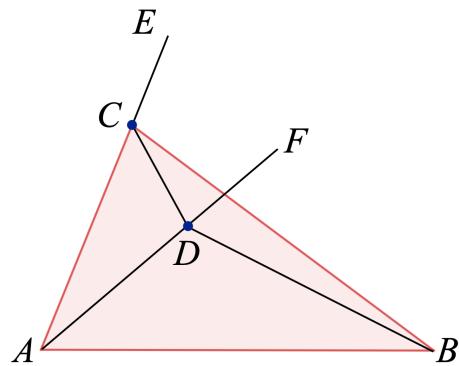
### revisit Euclid I.7

There is a small problem (really, an assumption) with our proof. Recall that  $AC = AD$  and it is supposed that  $BC = BD$ .



$D$  is drawn so that it is not contained within  $\triangle ABC$ . But suppose it were?

The problem is that the proof given before makes no sense if  $D$  lies inside  $\triangle ABC$  (for example, we do not know  $\angle BCD < \angle ACD$ ).



*Proof* (Alternate).

We can find a different proof for this case in several ways. Suppose  $AC = AD$  and  $BC = BD$ . Then  $\triangle ABC \cong \triangle ABD$  by SSS, since  $AB$  is shared.

But this is absurd.  $\triangle ABC$  is contained within and is less than  $\triangle ABC$ .

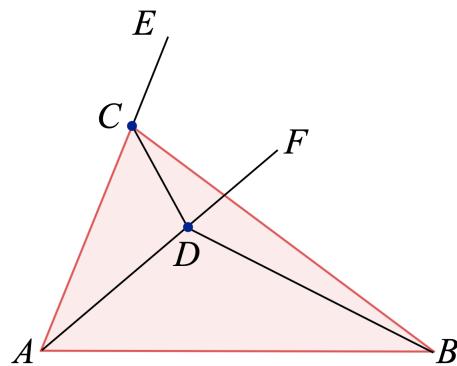
This is a contradiction.

□

All of this leads to a second difficulty: SSS is Euclid I.8, and Euclid actually proves I.8 by relying on I.7 !

However, we have a proof of SSS (below) that does not depend on I.7, so this could still work.

Perhaps a better solution is something like what we did before:



*Proof.*

Suppose that both  $AC = AD$  and  $BC = BD$ .

Then  $\triangle ACD$  is isosceles so  $\angle ACD = \angle ADC$ . Euclid I.5 also says that the supplementary angles are equal.  $\angle ECD = \angle FDC$ .

We notice that

$$\angle BCD < \angle ECD = \angle FDC < \angle BDC$$

But since  $\triangle BCD$  is also isosceles,  $\angle BCD = \angle BDC$ . This is a contradiction. Therefore it cannot be that both  $AC = AD$  and  $BC = BD$ .

□

[ Another idea might be to use Euclid I.21, which says that if  $D$  lies inside  $\triangle ABC$ , then  $AD + BD < AC + BC$ , but that proposition turns out to be dependent through several steps, on I.8. ]

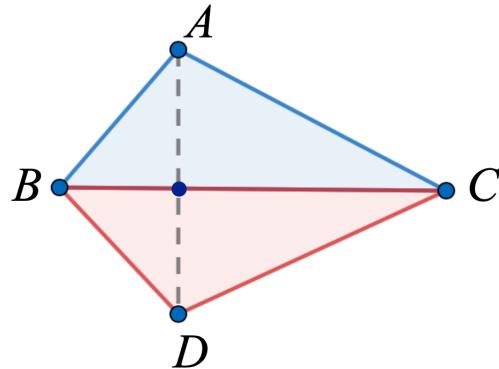
## Euclid I.8: SSS $\rightarrow$ SAS

We will prove that the SSS criterion implies SAS.

*Proof.*

Let all the sides of  $\triangle ABC$  be equal to  $\triangle ADC$  and align the triangles as shown.  $\triangle DEF$  is the mirror image of  $\triangle ABC$  (if not, reflect  $\triangle ADC$  through one of its sides and reposition it.

A mirror image is allowed to be congruent.



$D$  is placed so that two sets of sides are equal, and equal sides are adjacent in the quadrilateral.  $AB = BD$  and  $AC = CD$ . The third side  $BC$  is shared. So we have SSS.

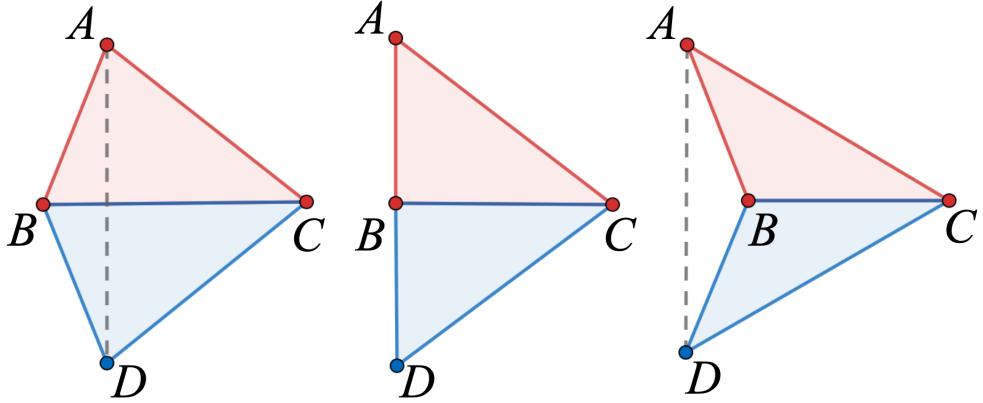
By the forward version of the isosceles triangle theorem,  $\angle BAD = \angle BDA$  and  $\angle CAD = \angle CDA$ . Therefore the total angles at the vertices are equal:  $\angle A = \angle D$ .

We have SAS, so  $\triangle ABC \cong \triangle ABD$ .

□

There is also a problem with this proof that may not be obvious.

We have acute angles on the base ( $\angle ABC$  and  $\angle ACB$ ). If we allow other possibilities, then this version of the proof isn't valid. Luckily, we can extend the proof in the following way.



*Proof.*

We have  $\triangle ABC$  and  $\triangle DEF$  with all three sides equal and superimposed along any one of the sides, say,  $BC = EF$ . We suppress the labels  $E$  and  $F$ .

Now, one of the angles along the base  $BC$  may be right or obtuse, or neither may be (the case already treated).

Then, if  $\angle ABC$  and  $\angle DBC$  are both right, we have SAS immediately.

Alternatively, if  $\angle ABC$  and  $\angle DBC$  are both obtuse (right panel), then draw  $AD$ . We have that both  $\triangle ABD$  and  $\triangle ACD$  are isosceles. By subtraction, we find that  $\angle BAC = \angle BDC$ .

Both pairs of flanking sides are equal, so we have SAS and thus  $\triangle ABC \cong \triangle DEF$  (labeled as  $\triangle DBC$  in the figure).

□

SSS  $\rightarrow$  SAS, and since SAS is sufficient to show congruence, so is SSS.

This illustrates a problem with drawing diagrams, we may introduce unrecognized assumptions into the proof. The most common is to draw an acute triangle and presume it stands for all triangles

# Chapter 8

## Bisection

Here we look at the problem of angle bisection, cutting an angle into two equal parts. We also look at the perpendicular bisector of any line, a perpendicular line that cuts halfway between two end points, or a vertical through any particular point on a line, or third, through a point not on the line.

We will show that every point on the perpendicular bisector is equidistant from the two points used to construct it.

We also prove the converse theorem, that *every* point which is equidistant lies on the bisector.

### constructions

A number of the propositions in book 1 of Euclid concern constructions.



The tools we have are a straight-edge and a compass. The compass is collapsible,

meaning that it cannot be used to transfer distances since it loses its setting when lifted from the page. This is a limitation Euclid solves in the second and third propositions of book I. It's also important that the straight-edge is not a ruler, there is no way to measure distance by reference to marks on it.

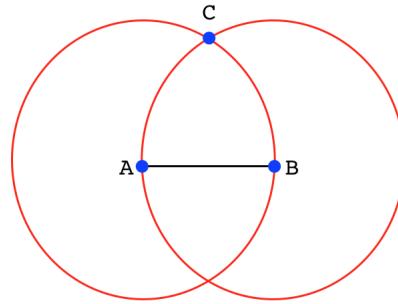
Euclid was smart enough to know about compasses and how to set them. The idea he had was this: to make the fewest possible assumptions. A non-collapsible compass was a luxury he didn't need, since he could accomplish the same end without it.

### **Euclid I.1: equilateral triangle**

To construct an equilateral triangle on a given line segment.



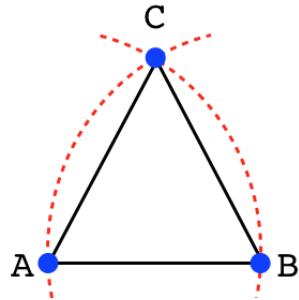
The first step is to draw two circles on centers  $A$  and  $B$ .



The circles are drawn with each radius equal to the line segment  $AB$ . It is a property of circles that all points on the circle are at the same distance from the center. Thus all points on the left-hand circle are equidistant from  $A$ , and all points on the second one are equidistant from  $B$ .

Therefore, the point  $C$  where the circles cross is equidistant from *both*  $A$  and  $B$ .

For this, we don't really need the entire circles, just the part where the arcs cross at  $C$ .



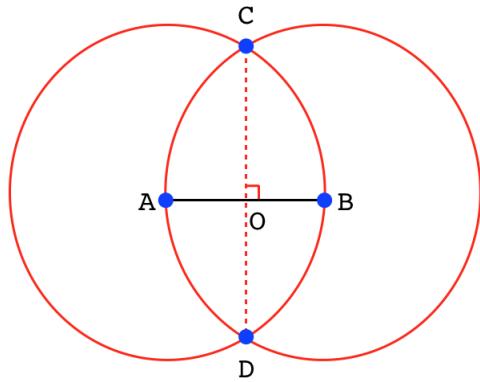
Now use the straight edge to draw  $\triangle ABC$ . Since  $AC = AB$  and  $BC = AB$ , we know that  $AC = BC$ . The triangle is equilateral.

□

The proof doesn't stand on its own. We used one definition (D) and a common notion (CN).

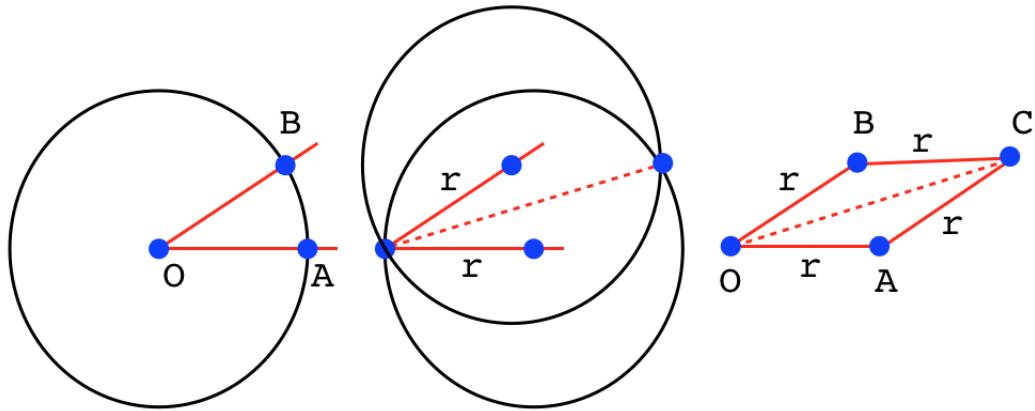
- D I.15 all radii of a circle are equal.
- CN I.1 things which equal the same thing also equal one another.

If we look again at the figure, and label the other point where the circles cross as  $D$ :



### **Euclid I.9: bisection of an angle**

To bisect a given angle.



As radii of a circle on center  $O$ , we first find points  $A$  and  $B$  equidistant from  $O$  (left panel). Let that distance be  $r$ .

We want to draw circles on centers  $A$  and  $B$  that have the same radius, and the easiest way to do that (with a collapsible compass), is to make the radius equal to  $r$ , which we can measure.

As radii of circles on the centers  $A$  and  $B$ , find  $C$  equidistant from  $A$  and  $B$  (middle panel), with radius also equal to  $r$ .

Thus,  $OA = OB = AC = BC$  (right panel).

So  $\triangle OAC \cong \triangle OBC$  by SSS, Euclid I.8.

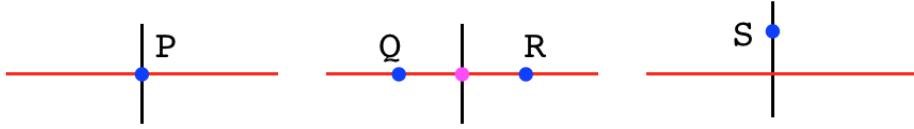
Therefore  $\angle BOC$  is congruent to  $\angle AOC$  and the given angle is bisected.

□

## perpendicular lines

When constructing a line segment perpendicular to another line segment, there are three common situations. We want the perpendicular line to pass:

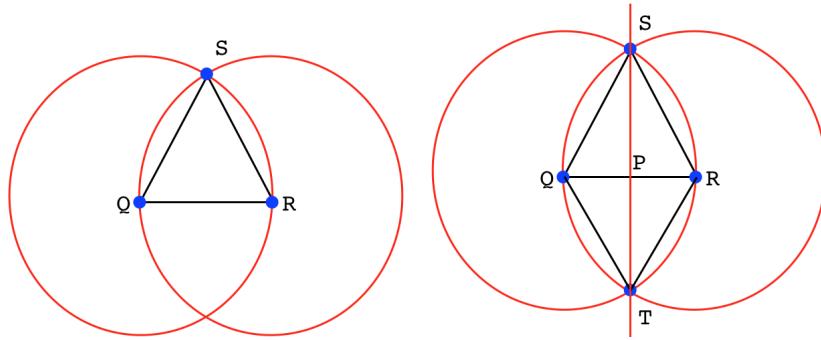
- through a given point  $P$  on the line.
- halfway between two points  $Q$  and  $R$  on the line, bisecting  $QR$
- through the line and also through a point  $S$  not on the line



We solve the second case and then show how the other two can be adapted to it.

### Euclid I.10: perpendicular bisector

Simply construct two circles of equal radius, one centered at  $Q$  and the other at  $R$ . It's easiest to choose the radius to be equal to the length  $QR$  (left panel, below).



The point  $S$  is on both circles, hence it is a radius of both, and therefore equidistant from  $Q$  and  $R$ .  $QS = SR = QR$ .

The three points form an equilateral triangle,  $\triangle QRS$  by I.1.

The point  $T$  below the line segment has the same property, and  $\triangle QRS$  is congruent to  $\triangle QRT$  by SSS.

Furthermore, we claim that the line segment  $ST$  is perpendicular to  $QR$  and the point  $P$  where  $ST$  intersects  $QR$  is such that  $QP = PR$ , and  $SP = PT$ .

Euclid's proof is simple.

*Proof.*

$\triangle QRS$  and  $\triangle QRT$  are both equilateral and congruent by SSS.

Thus  $\triangle QST \cong \triangle RST$  by SSS, and both are isosceles.

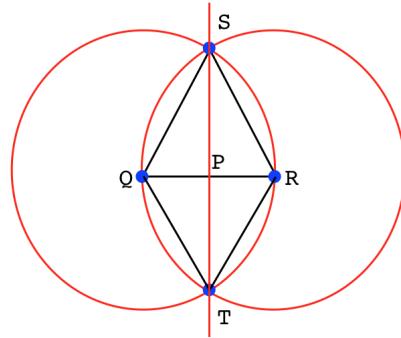
It follows that  $\angle QSP = \angle RSP$ , i.e.  $\angle QSR$  is bisected.

So  $\triangle QPS \cong \triangle RPS$  by SAS.

This gives  $QP = RP$ .

Also, the angles at  $P$ ,  $\angle QPS = \angle RPS$ , so all four are right angles.

□



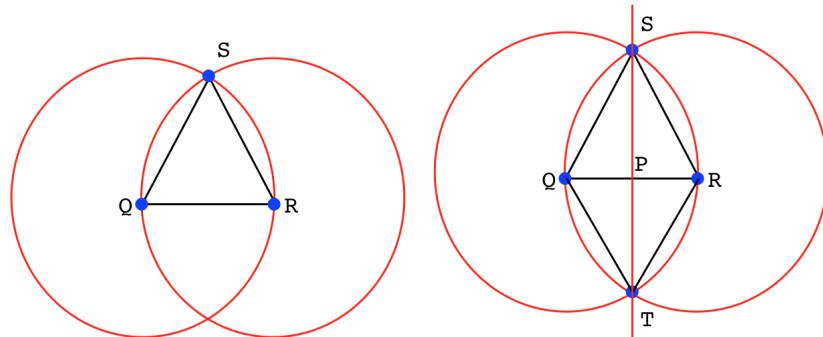
Here is another way to proceed.

*Proof.*

We are given that  $\triangle QRS \cong \triangle QRT$  by SSS (see above). So both angles at  $Q$  (right panel) are equal, and then  $\triangle QPS \cong \triangle QPT$  by SAS.

Therefore the base angles at  $P$  are equal and thus right angles, and therefore all the angles at  $P$  are right angles.

$\triangle PRS \cong \triangle QPS$  by hypotenuse-leg in a right triangle (HL). It follows that  $QP = PR$  so  $SPT$  bisects  $QR$  and also forms a right angle where it intersects at  $P$ .



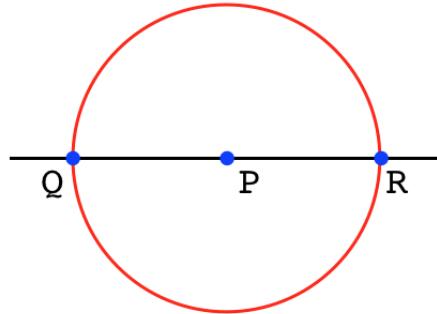
All four smaller triangles are congruent by SAS or SSS or HL.

□

### **Euclid I.11: bisector through a given point on the line**

Suppose we know a point  $P$  on the line and wish to construct the vertical line through  $P$ .

Use the compass to mark off points  $Q$  and  $R$  on both sides of  $P$ , equidistant from it. This can be done by drawing a circle with center at  $P$ .

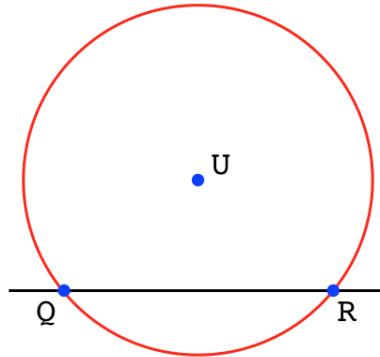


Now, simply proceed as above.

### **Euclid I.12: bisector through a given point not on the line**

Alternatively, suppose we know the line and the point  $U$  but not  $P$ , and we wish to construct a vertical through the line that also passes through  $U$ .

Find  $Q$  and  $R$  on the line an equal distance from  $U$  ( $QU = RU$ ), as radii of a circle centered at  $U$  (left panel, below). Their exact position is unimportant.



Now repeat the previous construction, using  $Q$  and  $R$ . The line segment  $ST$  passes through  $U$ , as required.

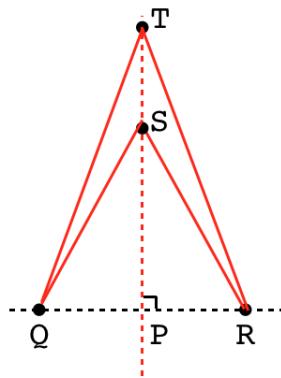
*Proof.* (Sketch).

Since  $QU = RU$ ,  $\triangle QUR$  is isosceles. Therefore the base angles are equal. In an isosceles triangle, the top vertex lies on the perpendicular bisector of the base (see the chapter on isosceles triangles).

Alternatively, find a point below the line equidistant from  $Q$  and  $R$ . Proceed as in the proof above.

### bisector is the altitude of an isosceles triangle

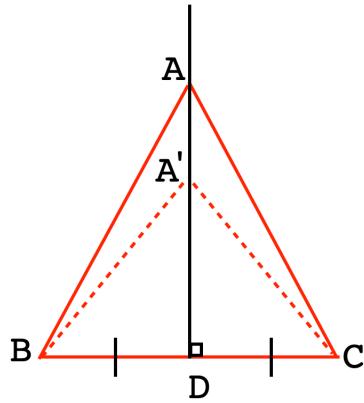
Suppose we know two points  $Q$  and  $R$ . We find the point  $P$  equidistant between them and construct the perpendicular bisector  $PS$ . Then the two sides  $SQ$  and  $SR$  have equal length. Triangle  $\triangle SQR$  is isosceles.



It is true for *any* point on the line drawn through  $S$  and  $P$ . For example,  $TQ = TR$  in the figure above.

### perpendicular bisector converse

Suppose are given that  $BD = DC$  and that  $\angle ADC$  is a right angle. For any point on  $AD$ , we can draw triangles with vertices at  $B$  and  $C$  that are congruent by SAS.



We conclude that every point on  $AD$  (and extensions of  $AD$ ) is equidistant from  $B$  and  $C$ .

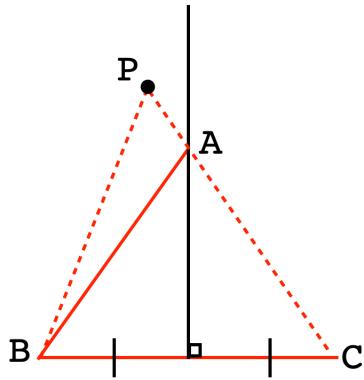
The converse theorem says that *every* point which is equidistant from  $B$  and  $C$  lies on  $AD$ .

Here are two proofs, the first one relying on the **triangle inequality**, which says that in any triangle, the sum of any two sides must be greater than the length of the third side. We will prove this theorem later.

To the current proof:

*Proof.*

Suppose that  $P$  is equidistant from  $B$  and  $C$  but does not lie on the perpendicular bisector. Then, find the point where  $PC$  crosses the bisector at  $A$ .



By the forward theorem,  $AB = AC$ .

We are supposing that  $PB = PC$ . By the triangle inequality

$$PB < AB + AP$$

Since  $AB = AC$ :

$$PB < AC + AP = PC$$

But this is absurd.  $PB$  cannot both be equal to and less than  $PC$ . Therefore, our supposition is incorrect, and there does not exist any such point  $P$ .

□

See [here](#) for a somewhat fuller explanation.

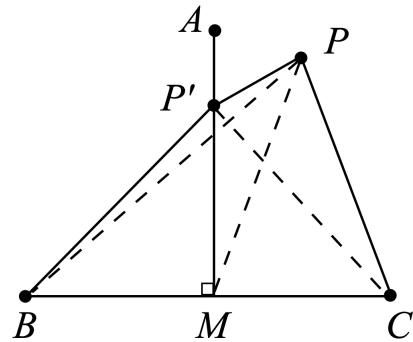
### perpendicular bisector using I.7

We want to prove that every point which is equidistant from two points on a line lies on the perp bisector.

Let  $AM \perp BC$ , and bisect it such that  $BM = MC$ .

$AM$  is the perpendicular bisector of  $BC$ .

Without loss of generality, let  $P$  lie on the same side of  $AM$  as  $C$ .



*Proof.*

Now, suppose  $PM$  is also perpendicular to  $BC$  at  $M$ .

Then  $PM$  bisects  $BC$  so it is a perpendicular bisector of  $BC$ .

By the forward theorem  $PB = PC$ .

Now find  $P'$  on  $AM$  such that  $PC = P'C$ .

We have three equal segments:  $PB = PC = P'C$ .

By the forward theorem, since  $P'$  is a point on the perpendicular bisector,  $P'B = P'C$ .

Now we have four equal segments:  $PB = PC = P'B = P'C$ .

But by Euclid I.7, this is impossible. It cannot be that  $P'B = PB$  and also  $P'C = PC$  on the same side of  $BC$ .

This is a contradiction. There is only one perpendicular through  $BC$  at  $M$ .

□

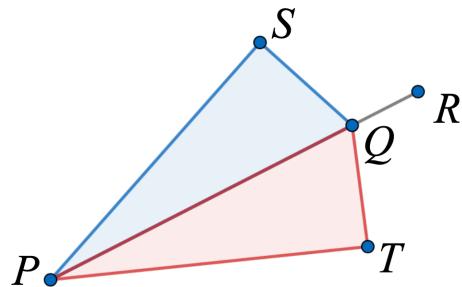
Alternatively, suppose that  $PB = PC$ . Then  $\triangle PBM \cong \triangle PCM$  by SSS.

$\angle PMB = \angle PMC$  and both are right angles, so  $PM$  is also perpendicular to  $BC$  at  $M$ .

Then the proof runs just as before. It cannot be that  $PB = PC$  but  $P$  is not on  $AM$ .

### bisector equidistant from sides

- Any point on the bisector of an angle is equidistant from the sides at the point of closest approach.



*Proof.*

Let  $PR$  bisect  $\angle SPT$ .

Draw perpendiculars  $QS$  and  $QT$ .

The angle at  $P$  is bisected, so  $\triangle PQS$  and  $\triangle PQT$  have two angles equal, and by sum of angles they have three angles equal.

They share the hypotenuse,  $PQ$ .

So  $\triangle PQS \cong \triangle PQT$  by ASA.

It follows that  $QS = QT$ .

□

### equidistant from sides → bisector

- If a point is equidistant from the sides of an angle, then it lies on the angle bisector.

*Proof.*

Given  $QS = QT$ . We have  $PQ$  shared.

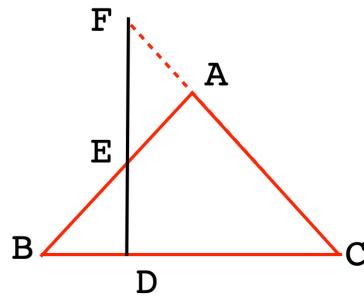
So  $\triangle PQS \cong \triangle PQT$  by HL.

It follows that  $\angle PQS = \angle PQT$ .

□

### problem

Here is a problem to exercise some concepts we've seen to this point: isosceles triangles, complementary angles, and vertical angles.



Given that  $AB = AC$ . Pick any point  $D$  on the base  $BC$  (except directly under  $A$ ), and draw the vertical  $DF$ . Extend  $AC$  to meet that vertical line.

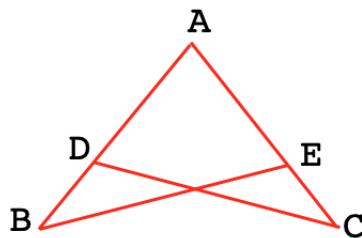
Prove that  $\triangle AEF$  is isosceles.

## problems

To prove:

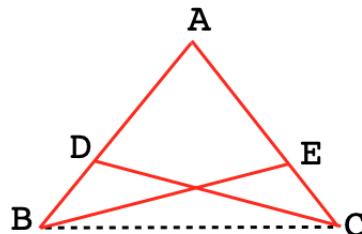
- Prove that for two supplementary angles, the angle bisectors are perpendicular to each other.
- Prove that an equilateral triangle (all 3 sides equal) is equiangular (all 3 angles equal). (Don't just rely on symmetry. Adapt the proofs given in this chapter).
- A line perpendicular to the bisector of an angle cuts off congruent segments on its sides.

In the figure below, given that  $AC = AB$  and  $\angle B = \angle C$ .



Prove that  $BE = DC$ .

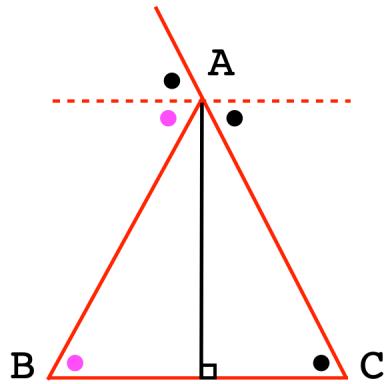
Hint: draw  $BC$  and then mark all the angles that are equal.



- An equilateral triangle has all three angles equal.

## problem

**155.** If one of the equal sides of an isosceles triangle be extended at the vertex and a line be drawn through the vertex parallel to the base, this line will bisect the exterior angle.



The angles labeled with black dots are equal by alternate interior angles and then vertical angles, while the angles labeled magenta are equal by alternate interior angles.

But we are given that this triangle is isosceles, so black and magenta are equal. Therefore the exterior angle at  $A$  is bisected by the horizontal.

Alternatively, use the fact that the exterior angle is the sum of the two base angles, and just use alternate interior angles once.

# **Part III**

## **Area**

# Chapter 9

## Quadrilaterals

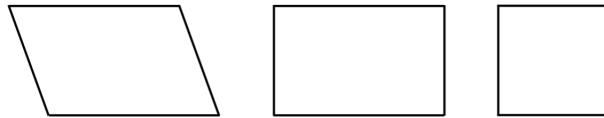
### classification

Polygons are constructed from straight sides. If the sides are all the same length, the figure is a *regular* polygon.

A polygon may have 3 or more sides: triangles (3), hexagons (6), and so on. There is a famous theorem from Gauss that involves the construction of a 17-sided regular polygon.

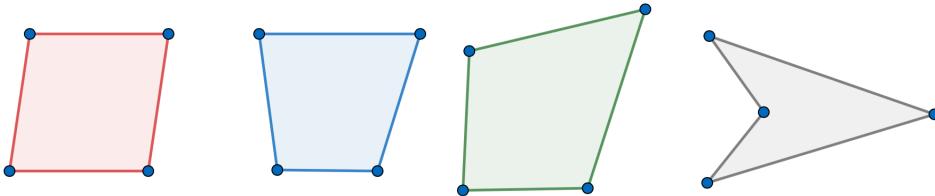
Let us start with four-sided polygons, called quadrilaterals. There are several types, of which the most important are:

- parallelogram: opposite sides equal and parallel
- rectangle: a parallelogram with four right angles
- square: a rectangle with four sides equal



We usually specify these shapes according to the most restrictive conditions they meet. When we think about a parallelogram we mean one without right angles, and when we think about a rectangle, we mean one that is not a square.

There are a few more quadrilaterals to mention (left to right in the figure below):



- rhombus: a square-like parallelogram with all sides equal
- trapezoid: just two sides parallel
- kite: both pairs of *adjacent* sides equal, one pair of opposite angles equal
- dart: like the kite, but one vertex concave ( $\angle > 180$ )

## parallelograms

Parallelograms are four-sided polygons with

- both pairs of opposing sides parallel
- both pairs of opposing sides equal
- both pairs of opposing angles equal

It is convenient to think of parallelograms primarily in terms of the first property. For example, parallel sides is enough to establish the other two properties.

In fact, each can be derived easily from the others. It is also sufficient if one pair of sides is both equal and parallel.

## rectangles

Rectangles are parallelograms with

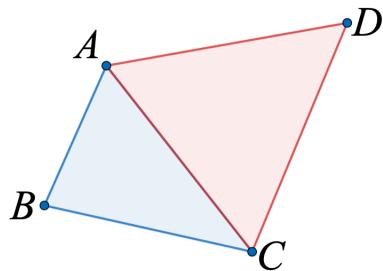
- all four vertices containing right angles

We now look at some fundamental theorems about polygons.

## sum of angles theorem

- The sum of all the internal angles in any quadrilateral is the same as that in two triangles, or four right angles altogether.

*Proof.*



Connect two opposing vertices ( $A$  and  $C$  above) to form two triangles.

Using the triangle sum theorem, add the component angles at all four vertices.

$$\angle B + \angle BAC + \angle BCA = 180$$

$$\angle D + \angle DAC + \angle DCA = 180$$

But

$$\angle A = \angle BAC + \angle DAC$$

$$\angle C = \angle BCA + \angle DCA$$

So by addition:

$$\angle A + \angle B + \angle C + \angle D = 360$$

□

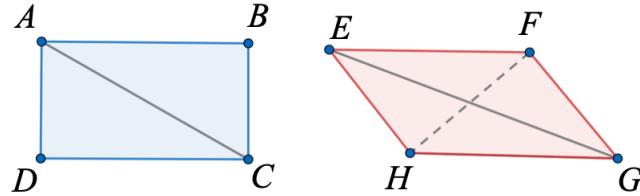
This theorem can be extended to polygons having more sides.

Adding a vertex (with a net addition of one side) is the same as adding another triangle. Correcting for the “base case” we have the sum of angles  $S = (n - 2) \cdot 180$ .

The proof of the extended theorem is famously done by induction. We cover this elsewhere.

## diagonal theorem

- Any diagonal in a parallelogram produces two congruent triangles.



*Proof.*

**rectangle:**  $\triangle ABC \cong \triangle CDA$ .

We can use different properties to obtain different proofs.

Given  $AB = DC$  and  $BC = AD$ , the result follows by SSS.

Given parallel sides, we can use ASA.

**parallelogram:**  $\triangle EFG \cong \triangle GHE$  and  $\triangle EFH \cong \triangle GHF$ .

- Given  $EF = GH$  and  $EH = FG$ , the result follows by SSS.
- Given  $EF \parallel GH$  and  $EH \parallel FG$ , the result follows by ASA.
- Given  $\angle E = \angle G$  and  $\angle F = \angle H$ ,  $\angle E + \angle F$  is equal to two right angles, by the sum of angles. Thus,  $EF \parallel GH$  and  $EH \parallel FG$ , and the result follows.
- Given  $EH \parallel FG$  and  $EH = FG$ .  $\angle GEH = \angle EGF$ . the result follows by SAS.

□

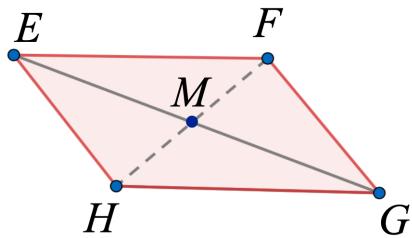
## diagonal theorem: equal angles corollary

- In any parallelogram the opposing angles are equal.

This follows immediately from the main theorem.

## diagonal theorem: bisection corollary

- The two diagonals in a parallelogram bisect one another.



*Proof.*

Given  $EH \parallel FG$  and  $EH = FG$ .

$\angle GEH = \angle EGF$  by alternate interior angles.

$\angle EMH = \angle FMG$  by vertical angles.

$\triangle EMH$  and  $\triangle FMG$  are equiangular by sum of angles in a triangle.

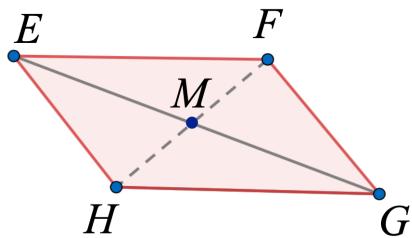
$\triangle EMH \cong \triangle FMG$  by ASA.

It follows that  $EM = GM$  and  $FM = HM$ .

□

### diagonal theorem: converse 1

- If the two diagonals in a quadrilateral bisect each other, the figure is a parallelogram.



*Proof.*

Given  $EM = GM$  and  $FM = HM$ .

$\angle EMH = \angle GMF$  by vertical angles.

$\triangle EMH \cong \triangle GMF$  by SAS.

It follows that  $EH = FG$  and  $\angle GEH = \angle EGF$ .

So  $EH \parallel FG$  by alternate interior angles.

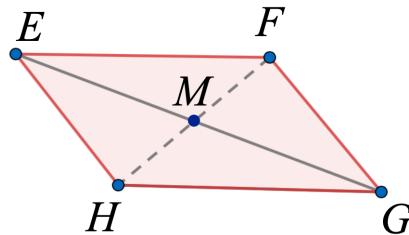
The same logic gives  $EF$  parallel and equal to  $HG$ .

□

Note that the diagonal theorem and its converse together allow us to interconvert all four specifications for a parallelogram.

### diagonal theorem: converse 2

- If the diagonal in a quadrilateral forms congruent triangles, the figure is a parallelogram.



*Proof.*

Given  $\triangle EHG \cong \triangle GFE$

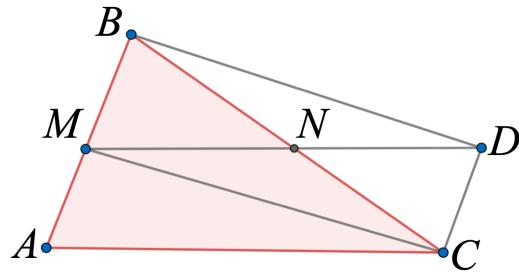
$\angle GEH = \angle EGF$  and  $\angle EGH = \angle GEF$ .

We have both pairs of opposing sides parallel by the converse of alternate interior angles.

□

### midline theorem

- The midline of a triangle is parallel to the third side of that triangle and half its length.



*Proof.*

Given  $AM = BM$  and  $CN = DN$ .

Extend  $MN$  to  $D$  with  $MN = DN$ .

Draw  $BD$ ,  $CM$  and  $CD$ .

$BDCM$  is a parallelogram, by the converse of the diagonal theorem.

$CD = BM = AM$  and  $CD \parallel AMB$ .

It follows that  $MDCA$  is a parallelogram.

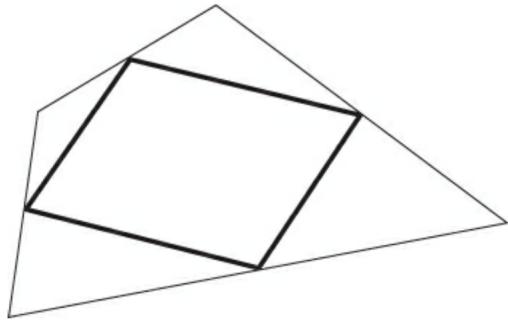
Hence  $AC = MD$  which is twice  $MN$  and  $AC \parallel MND$ .

□

## Varignon's theorem

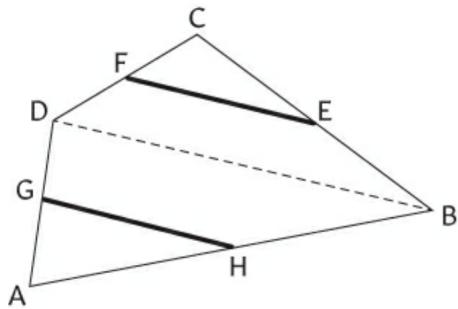
This famous theorem concerns any quadrilateral. Let's start with the four points lying flat in the same plane.

If we draw the lines connecting the midpoints of each side, the result must be a parallelogram. Here is Acheson's figure:



**Fig. 50** Varignon's theorem.

To visualize this, imagine the quadrilateral drawn as two triangles connected at the diagonal.



**Fig. 51** Proof of Varignon's theorem.

This idea about the diagonal contains the germ of the answer. In the second figure, above, by the midline theorem,  $EF \parallel BD$ , but also  $BD \parallel GH$ .

Thus  $EF \parallel GH$ . We have opposite sides parallel. Repeat with  $FG$  and  $EH$  to obtain the result.

Now, if we imagine the quadrilateral folding on a hinge at  $DB$ , we see that the midlines  $EF$  and  $GH$  will remain parallel even if  $C$  is no longer co-planar with  $A$ ,  $D$  and  $B$ .

# Chapter 10

## Rectangles

### minimal specification

In considering a shape that might be a rectangle, and given only some of the properties, the rest may follow. There are too many permutations to go through them all.

First, if we know the figure is a parallelogram, then if it also has at least one right angle, it is a rectangle.

*Proof.*

By the diagonal theorem, there are two opposing angles that are right angles.

Because opposite sides are parallel, it follows that if one of two adjacent angles is a right angle, then so is the other one, by alternate interior angles.

Thus, all four angles are right angles.

□

If we do not know about the sides but only about the right angles, and that there are four of them, then we have a rectangle.

*Proof.*

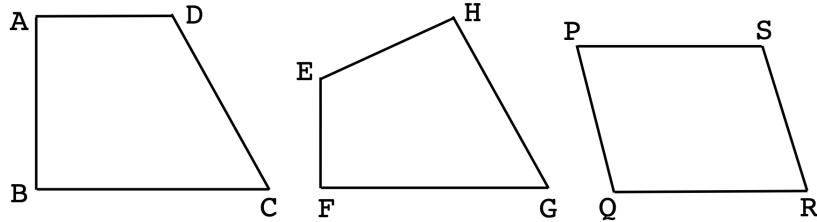
Since the adjacent angles are right, the sides in question are parallel.

A parallelogram with at least one right angle is a rectangle.

□

Even if we know only three angles, by the sum of angles in a quadrilateral, we have four.

However, two right angles are not enough, as counter-examples are easily constructed for both cases (right angles as either neighbors or opposite one another).

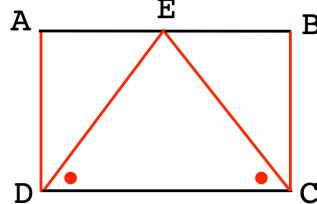


In the left panel  $\angle A$  and  $\angle B$  are right angles, and in the middle panel,  $\angle F$  and  $\angle H$  are right angles. But neither  $ABCD$  nor  $EFGH$  is a rectangle.

If both pairs of sides are equal and parallel, and we don't know there is a right angle, that's not enough by itself.

Here's another permutation.

Let  $AD = BC$  and let  $\angle A$  and  $\angle B$  both be right angles.  $ABCD$  is a rectangle.



*Proof.*

Bisect  $AB$  at  $E$  and draw the half-diagonals.  $\triangle ADE \cong \triangle BCE$  by SAS.

Therefore  $ED = EC$  so the angles marked with a red dot are equal by the forward isosceles theorem.

The other components of  $\angle C$  and  $\angle D$  are equal because of the congruent triangles, so  $\angle C = \angle D$ .

By the angle sum theorem for quadrilaterals and given the angles at  $A$  and  $B$  are right angles:

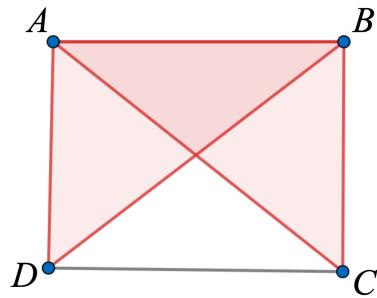
$$\angle C + \angle D = 180$$

Then, both are right angles and thus, all four angles are right angles.

□

## diagonals of a rectangle

The diagonals of a rectangle are equal.



*Proof.*

We consider two overlapping triangles.

$\triangle ABC \cong \triangle BAD$  by SAS.

So  $AC = BD$ .

Since the diagonals cross at their midpoints, by the diagonal bisector corollary.

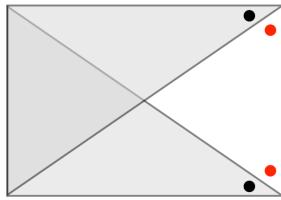
All four half-diagonals are equal.

It follows that the four smaller triangles are all isosceles.

□

We can also invoke symmetry.

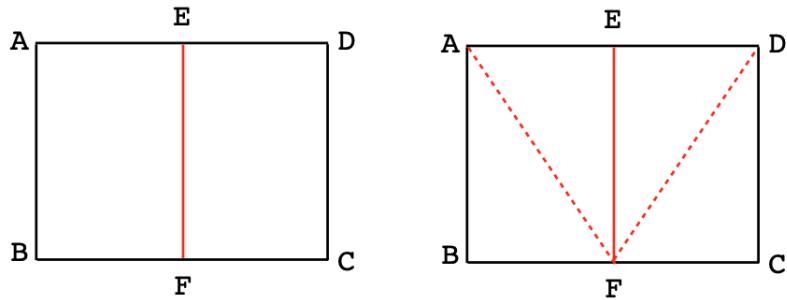
The rectangle has mirror image symmetry by reflection in both the left-right and top-bottom dimensions. As a result, the black-dotted angles in the figure below are equal, and so are the red ones.



And that implies that all four segments from a vertex to the central point are equal in length, by the converse of the isosceles triangle theorem.

### bisection of a rectangle

Suppose we are given that  $ABCD$  is a rectangle and that  $EF$  is the perpendicular bisector of one of the sides, say  $AD$ . Then  $AE$  is also the perpendicular bisector of the other side,  $BC$ .



*Proof.*

Draw the diagonals of the two small quadrilaterals, namely  $AF$  and  $DF$ .

Then  $\triangle AEF \cong \triangle DEF$  by SAS, using the right angles at  $E$ .

But  $\triangle AEF$  is also congruent to  $\triangle ABF$  (by SAS).

Reasoning in the same way, and using transitivity, we have four congruent right triangles.

It follows easily that the two small rectangles  $ABFE$  and  $DCFE$  are congruent, so  $\angle BFE$  and  $\angle CFE$  are right angles, with  $BF = CF$ .

Therefore  $EF$  is the perpendicular bisector of  $BC$ .

□

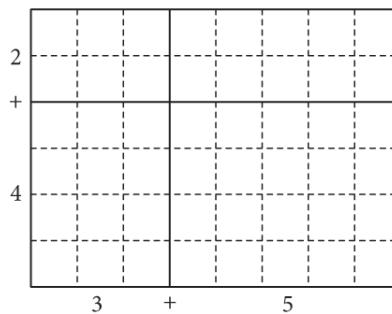
An even simpler proof: given  $EF \parallel DC$  so  $EDCF$  is a parallelogram. Thus  $ED = FC = BF$ .

## area of a rectangle

Let's say a few brief words about rectilinear area, the area of shapes like squares or rectangles with perpendicular sides. We will then extend this to the areas of triangles and parallelograms, which are like squashed rectangles.

To find areas, we must first fix a unit length. For now, in geometry, we will need an even number of units for each dimension. (There is an exception, but it occurs in a case where we only care about the squared area — see the chapter on the Pythagorean theorem).

Suppose that, in the figure below, the small squares have side lengths of 1 cm, and 6 squares stack vertically and 8 horizontally to fill the shape.



Just multiply the width by the height (in cm) to obtain  $48 \text{ cm}^2$ .

But then suppose instead that the squares have side lengths of 2.54 cm. Define 1 in = 2.54 cm. The total area would be  $48 \text{ in}^2$ .

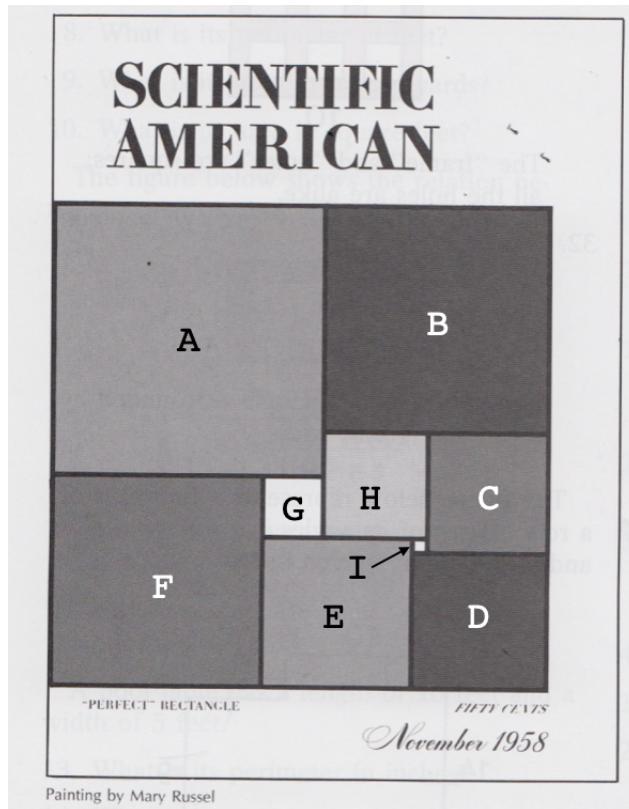
This particular figure above (from Lockhart) shows the distributive law in action:

$$\begin{aligned}
 & (3 + 5) \cdot (4 + 2) \\
 &= 3 \cdot 4 + 3 \cdot 2 + 5 \cdot 4 + 5 \cdot 2 \\
 &= 48
 \end{aligned}$$

Any combination of numbers that add up to 8, times any combination of numbers that add up to 6, gives the same result.

## problem

From Jacobs, chapter 9.



Suppose each of  $A$  through  $I$  is a square, and the areas of squares  $C$  and  $D$  are 64 and 81, respectively.

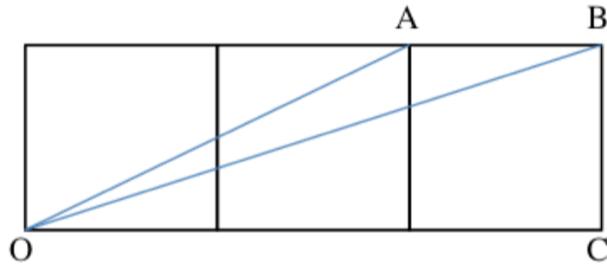
Can you find the areas of all the other squares?

Is the entire figure a square? What is the total area?

## problem

Next is a problem from the web. Given three identical squares arranged as follows:

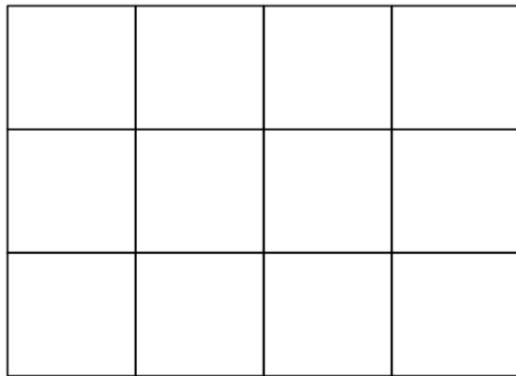
There are three squares in the diagram.



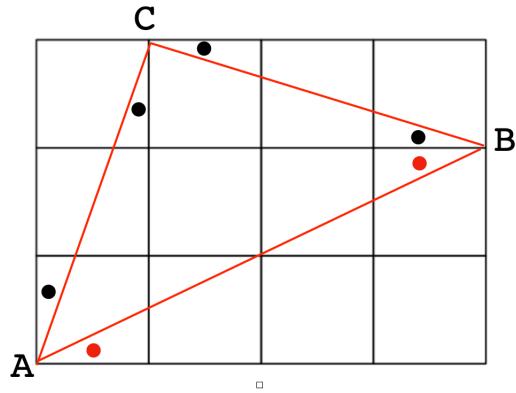
Find  $\angle AOC + \angle BOC$ .

There is a simple solution, to be obtained without measuring or using trigonometry.

As in so many problems, the key is to draw an inspired diagram, one that extends the figure somehow. Here, a major hint was provided, namely, a grid of squares.



So let's draw the same angles using that grid and form a triangle.



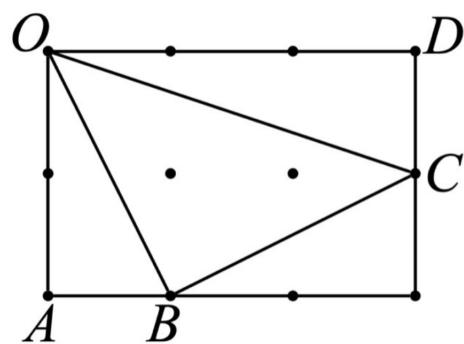
We can identify some equal angles. The two with red dots are equal by alternate interior angles. The two rectangles, one containing the diagonal  $AC$  and the other the diagonal  $BC$  each consist of three unit squares so they are congruent. That plus alternate interior angles accounts for all the black dotted equal angles.

The last thing we can learn from the diagram is that because of congruent rectangles,  $AC = BC$ . So that means  $\angle BAC$  is equal to  $\angle ABC$ .

It follows that  $\angle BAC$  is one-half the total angle at  $A$ , namely one-half of a right angle.

So finally, by sum of angles,  $\triangle ABC$  is a right triangle.

Here is a similar proof, more compactly executed

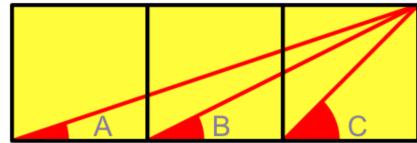


<https://mathenchant.wordpress.com/2022/07/17/twisty-numbers-for-a-screwy-universe/>

(Endnote 5).

□

Martin Gardner has a version of this problem for which he gives this diagram:



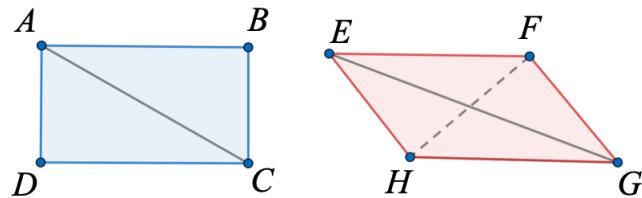
# Chapter 11

## Parallelograms

### diagonal forms congruent triangles

There is a close relationship between parallelograms and triangles. The area of a triangle is one-half the area of the corresponding parallelogram. In the case of a right triangle the corresponding figure is a rectangle.

As we've seen, a line joining opposite vertices of a rectangle or parallelogram is called a *diagonal*. Any diagonal divides a parallelogram into two congruent triangles. The difference is that starting from a rectangle, we obtain right triangles.



$$\triangle ABC \cong \triangle CDA.$$

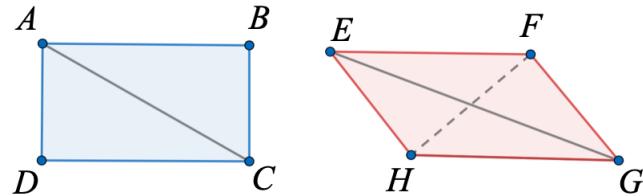
$$\triangle EFG \cong \triangle GHE, \text{ and } \triangle EFH \cong \triangle GFH.$$

*Proof.*

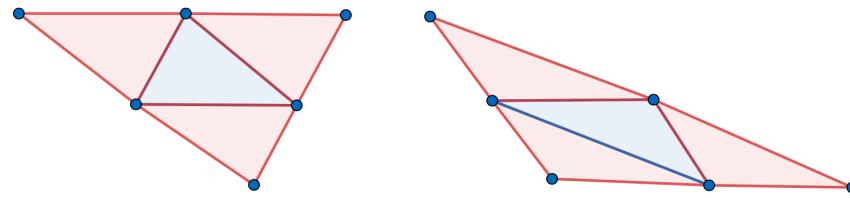
One can use SSS (opposite sides equal), or SAS, or even ASA (opposite sides parallel),

□

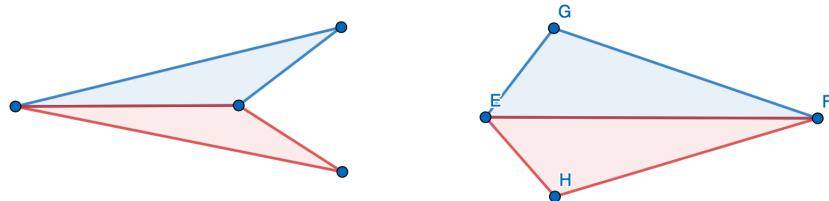
Conversely, two copies of the same triangle (i.e. congruent) can always be joined into a parallelogram, and that figure is a rectangle, if we start with a right triangle.



We can see each of the three possible ways of joining two identical triangles in the figure for the midpoint theorem.



One must be careful, however. The triangles to be joined must not be mirror images, otherwise one may obtain a dart or a kite.



By our fundamental definition of what area is, for a rectangle it is the product of the lengths of two adjacent sides. The area of a parallelogram must be adjusted somehow for the fact that it isn't standing up straight. The easiest way to deal with this is Euclid's approach.

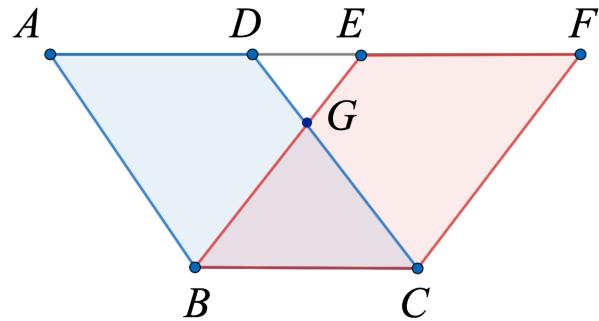
### Euclid I.35

As detailed in Book I of *Elements*, here is Euclid I.35. This proposition (theorem) says that given any two parallelograms  $ABCD$  and  $EBCF$  on the same base  $BC$

and with  $ADEF \parallel BC$ , they have the same area.

The diagram (below) is drawn for the awkward case, where the two parallelograms overlap.

We have  $AB = DC$ ,  $AD = BC$ ,  $BC = EF$  and the sides are parallel as well.



*Proof.*

Because of the shared base,  $AD = EF$ .

By addition:  $AE = DF$ .

We also have  $AB = DC$  and because  $AB \parallel DC$ ,  $\angle EAB = \angle FDC$ .

Thus  $\triangle EAB \cong \triangle FDC$ , by SAS.

Subtract the shared area of  $\triangle DGE$  and add the shared area of  $\triangle GBC$  to obtain equality for the area of the two parallelograms.

□

This result is immediately extended to the case where  $\angle ABC = \angle DCB$  and both are right angles.

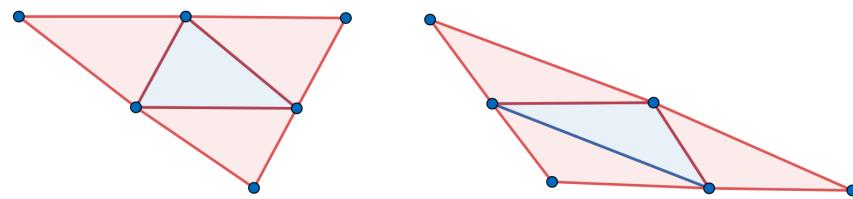
By I.35, this rectangle has the same area as the parallelogram.

Furthermore, the theorem can be invoked sequentially to prove that two parallelograms which do not overlap at all are equal in area, if they lie anywhere on equal bases between two parallel lines.

Roughly speaking, in the figure below, cut off a right triangle from the left side and attach it on the right. The angles add up to form a straight line along the base and a right angle at the upper right. The area is  $h \cdot b$ .

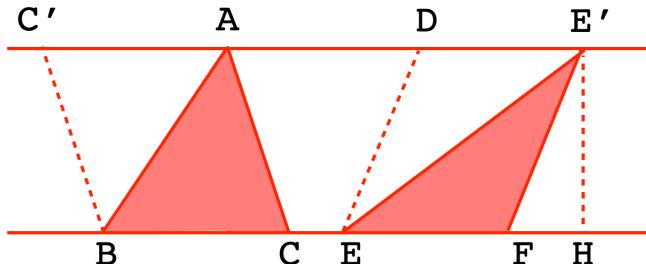


To reiterate, any triangle can be turned into a parallelogram, by attaching a rotated image of itself:



It does not matter which side we choose. Any pair of triangles containing the central one is a parallelogram, since it has both pairs of opposite sides equal. We have shown that this is sufficient to know we have a parallelogram.

In the figure below,  $ACBC'$  is a parallelogram composed of two copies of  $\triangle ABC$ . The area of this parallelogram is twice that of  $\triangle ABC$ . To obtain the value, multiply the base  $BC$  times the "height"  $E'H$ .



$E'H$  is equal in length to the *altitude* of the triangle. That would be a line dropping vertically from  $A$  and making a right angle with the base,  $BC$ .

The area of the triangle is one-half that of the parallelogram that contains two copies of the triangle.

$$\mathcal{A} = \frac{1}{2} \cdot BC \cdot E'H$$

Euclid I.35 and the opposite of dissection, that every triangle can be assembled into a parallelogram that clearly has twice the area, together give everything we need.

If we go back to the idea of cutting off a triangle from one side of a parallelogram and placing it on the other side, to form a rectangle:



it is possible that a parallelogram is particularly skinny for a given height, then it will not be possible to cut off just one triangle.

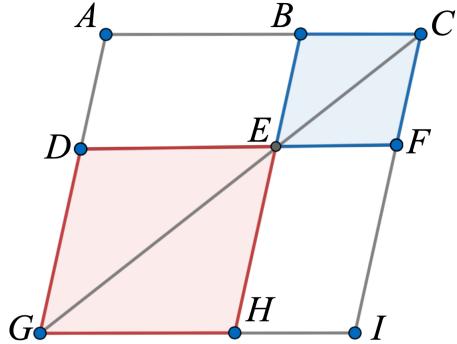
There are at least three solutions to this. One is to rotate the figure so that the sides become the base and the top. A second idea is to slice the parallelogram into horizontal pieces and add them up.



A sophisticated approach is to call on I.43 and change the shape to a different one with equal area.

## Euclid I.43

**Parallelogram complements equal** says that, in the figure below, the two smaller parallelograms not on the diagonal,  $ABED$  and  $EFIH$ , are equal.



*Proof.*

$GEC$  is the diagonal for three parallelograms and cuts each into two congruent triangles with equal area, by the diagonal theorem.

Subtracting

$$(ACG) - (BCE) - (DEG) = (ABED)$$

$$(CIG) - (CFE) - (EHG) = (EFIH)$$

Since the left-hand sides are equal, it follows that  $(ABED) = (EFIH)$ .

□

## squaring figures

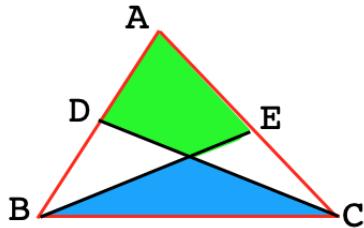
Any parallelogram (such as  $ABED$ ) can be turned into another with the same area but a different shape, simply by inclining the diameter  $CG$  at an appropriate angle in theorem I.43.

Any parallelogram can be turned into a rectangle with the same area, as described in this chapter.

Any rectangle can be turned into a square with the same area. This is Euclid II.14, which we will see later.

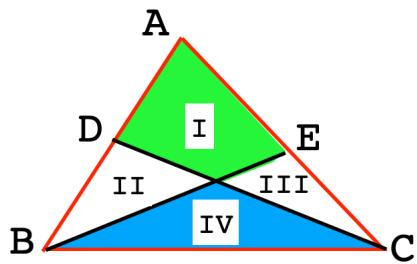
## problem

Given that  $D$  and  $E$  are midpoints of their sides:  $AD = DB$  and  $AE = EC$ . Prove that the colored areas are equal.



*Solution.*

Let the four triangular areas be labeled  $I - IV$ .



Then, by the area-ratio theorem and using  $AC$  as the base, and since the base is bisected:

$$I + II = III + IV$$

But using  $AB$  as the base

$$I + III = II + IV$$

Add the two equations:

$$2 \cdot I + II + III = 2 \cdot IV + II + III$$

$$I = IV$$

□

We might note as well that since  $DE$  bisects the sides it is parallel to the base  $BC$ . (We will get to that, it is a consequence of similar triangles). As a result,  $II + IV$

and  $III + IV$  have the same altitude as well as the same base  $BC$ , so they are equal. In other words, subtracting the two equations:

$$II - III = III - II$$

$$II = III$$

### twice the area

It can be convenient to write the formula as *twice* the area, and we will often do that.

$$2\mathcal{A}_\Delta = ab$$

We rewrite two important formulas from this chapter:

$$d \cdot h + e \cdot h = (d + e) \cdot h = c \cdot h$$

$$\frac{\mathcal{A}_A}{\mathcal{A}_B} = \frac{ah}{bh} = \frac{a}{b}$$

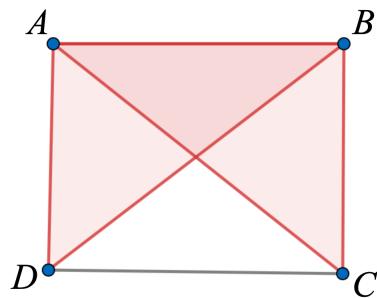
The first one is an equality of different areas, and the second involves a ratio of areas on the left-hand side. The result is unchanged by using twice the area, as long as we are consistent.

# Chapter 12

## Altitudes

### easy theorems

- If a quadrilateral has two adjacent right angles and equal diagonals, it is a rectangle.



*Proof.*

Let  $ABCD$  be a quadrilateral with  $\angle A = \angle B$  and both are right angles.

Let  $AC = BD$  be the two diagonals.

Compare mirror image right triangles  $\triangle ABC$  and  $\triangle BAD$ .

They have an equal hypotenuse and they share the side  $AB$ .

Hence they are congruent by HL, so  $AD = BC$ .

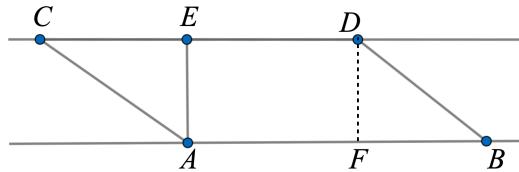
Then  $\triangle ADC \cong \triangle BCD$  by SSS.

Thus  $\angle C = \angle D$  and by sum of angles both are right angles.

So  $ABCD$  has four right angles and thus is a rectangle.

□

- perpendiculars between parallel lines are equal



*Proof.*

Let  $AB$  and  $CD$  be parallel lines.

Let  $AE \perp AB$ , then  $AE \perp CD$  by alternate interior angles.

Draw  $DF \perp AB$ .

$DF$  is also  $\perp CD$  for the same reason.

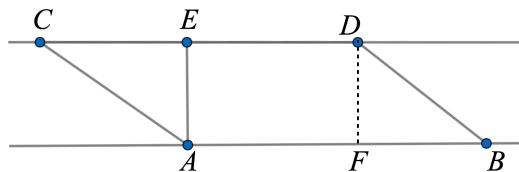
Thus  $AEDF$  has four right angles so it is a rectangle.

It follows that  $AE = DF$ .

□

Any perpendicular is the shortest line segment with one end point on  $AB$  and the other on  $CD$ .

*Proof.*



Aiming for a contradiction, suppose  $BD$  is shorter, but  $BD$  is not  $\perp AB$ .

Draw the line  $AC \parallel BD$  through  $A$ .

Then  $ACDB$  is a parallelogram, so  $AC = BD$ .

Let  $\triangle ACE$  be a right triangle.

The hypotenuse is the longest side in a right triangle, so  $BD = AC > AE$ .

This is a contradiction.

A perpendicular is the shortest line segment connecting two parallel lines, and every perpendicular is equal to every other one.

□

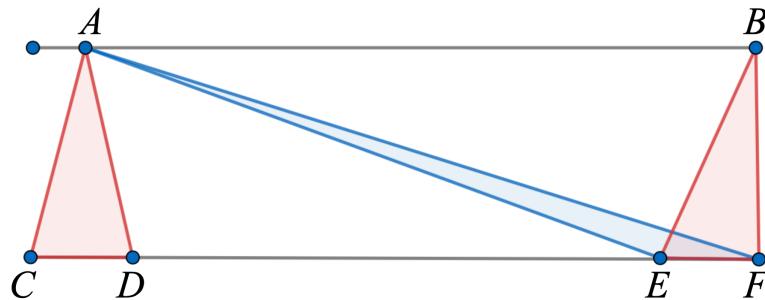
## area only depends on base and altitude

An important consequence is that:

- all triangles with the same base and height have the same area.

In the figure below, we have two parallel lines. Mark off  $CD = EF$ .

Now pick any point on the top and draw the triangle with two equidistant points on the bottom. Any other triangle drawn with an equal base has the same area.



The areas of  $\triangle ACD$ ,  $\triangle AEF$ , and  $\triangle BEF$  are equal, because they have equal bases and equal heights.

There are several different conventions for referring to the area of a triangle. One is just to use the capital letter  $A$  (for area). To make it stand out, we might use a special font:

$$\mathcal{A}_{\triangle ACD} = \mathcal{A}_{\triangle AEF}$$

That helps but can still become awkward when  $\mathcal{A}$  has another meaning in the problem. Some people switch to using  $K$ , but a second approach is to use the  $\triangle$  symbol, as in

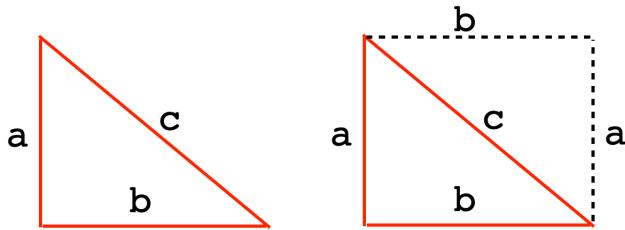
$$\triangle ACD = \triangle AEF = \triangle BEF$$

Euclid always refers to angles as  $\angle ABC$  etc., so when he says  $ABC = DEF$ , he means the *area* of those triangles.

And yet another is to use parentheses. This is a good solution when there are other shapes like rectangles in the problem.

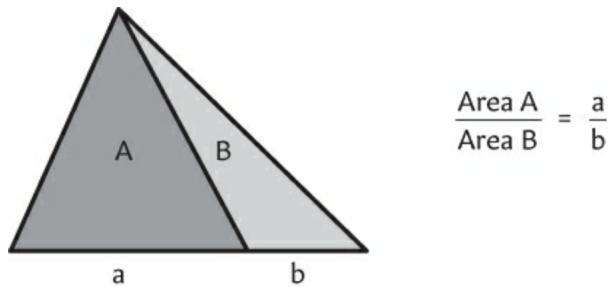
$$(\triangle ACD) = (\triangle AEF)$$

A right triangle has the largest area for a given pair of side lengths. If we imagine a side of length  $a$  tilting right or left, then the resulting triangle will have a smaller area, because the altitude  $h$  will be less than  $a$ .



### area-ratio theorem

If in a triangle we draw the line connecting the upper vertex to any point on the bottom side, then the areas of the two smaller triangles are in the same ratio as the lengths of their bases.



**Fig. 120** An area-ratio theorem.

*Proof.*

The area of  $\triangle A$  is  $ah/2$ , while that of  $\triangle B$  is  $bh/2$ , so the ratio of areas is

$$\frac{\mathcal{A}_A}{\mathcal{A}_B} = \frac{ah/2}{bh/2} = \frac{a}{b}$$

□

It is also the case that the ratio of the area of any sub-triangle to the whole is the same as the proportion of its base length to that of the whole base.

$$\frac{\mathcal{A}_A}{\mathcal{A}_A + \mathcal{A}_B} = \frac{a}{a+b}$$

*Proof.*

Simple algebra: invert, add one to both sides, and invert again. We show only the right-hand side. Start with the fraction  $b/a$  and add 1 to it:

$$\frac{b}{a} + 1 = \frac{b}{a} + \frac{a}{a} = \frac{a+b}{a}$$

Inverted:

$$\frac{a}{a+b}$$

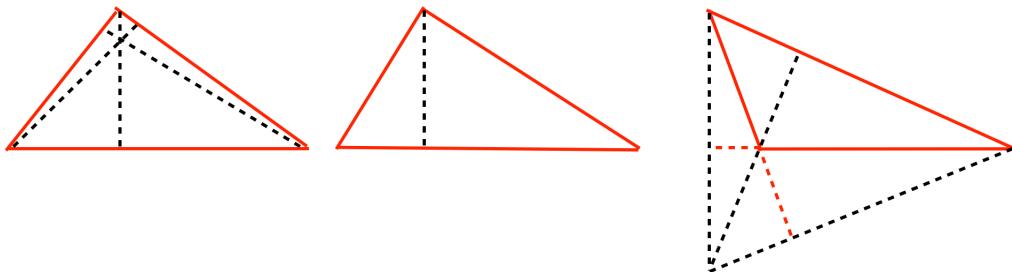
Do the same to both sides and the result follows.

## altitudes and the orthocenter

Each of the three sides of a triangle has a corresponding *altitude*, which is the perpendicular line drawn from a vertex to the side opposite, or its extension.

The three altitudes meet at a point (they are said to be concurrent). We will prove this later. The point is called the *orthocenter*.

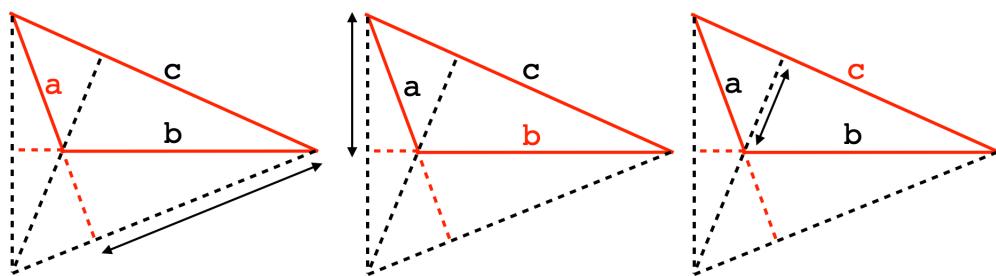
For an acute triangle, the orthocenter lies inside the triangle.



For a right triangle, the orthocenter is just the vertex containing the right angle.

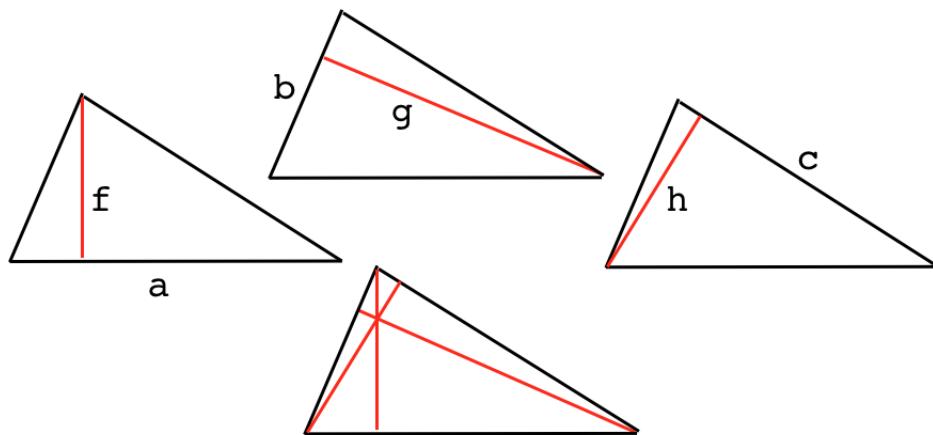
For an obtuse triangle, the orthocenter is external to the triangle, as are two of the altitudes and part of the third.

In the latter case, it may take some thought to determine which altitude goes with which side. The rule is that the altitude forming a right angle with any side originates at the vertex opposite that side. If necessary, the side is extended to meet the altitude at a right angle.



In the figure above, we have one obtuse angle in the triangle. The altitudes to sides  $a$ ,  $b$  and  $c$  are indicated, in turn, by arrows.

### computing triangular area



Again the simple formula: one-half base times height. In the figure above, twice the area is

$$2A = af = bg = ch$$

We can choose any side of the triangle to be the base and then multiply by the height to get twice the area.

We must always get the same answer! The area of the triangle is surely the same no matter how you calculate it.

Here's a proof by counting up the area of smaller triangles. A simpler proof follows, but this gives practice in defining altitudes.

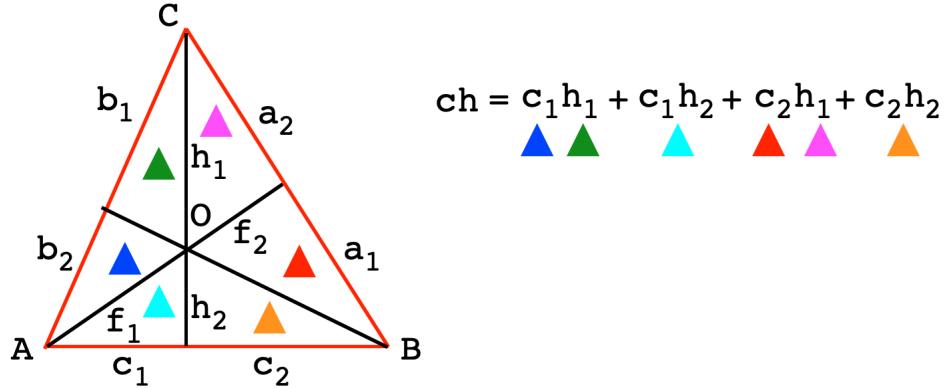
*Proof.*

In  $\triangle ABC$  with sides  $a, b, c$ , drop the three altitudes from the vertices to form right angles on the opposing sides. We label two of them:  $f$  for side  $a$  and  $h$  for side  $c$ .

These altitudes cross at a single point. We look at Newton's proof of this in just a bit ([here](#)).

Each altitude and side is then divided into two parts as shown.

This gives six small triangles. To make it easier to keep track of them, they are labeled with colors.



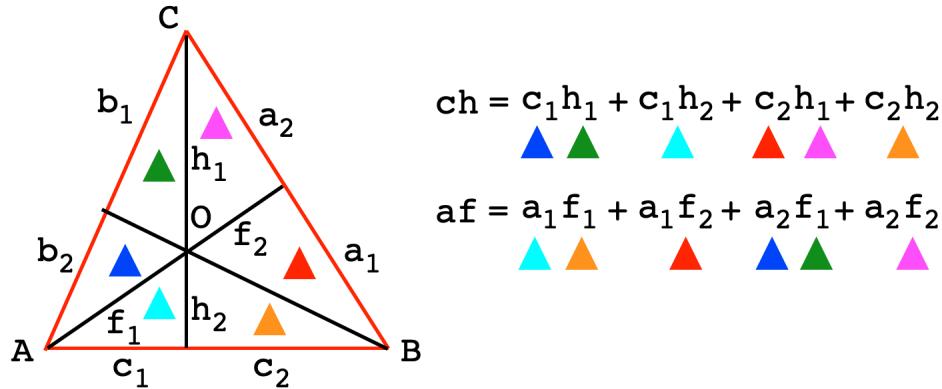
So then twice the area of the whole triangle is  $2A = af = ch$ . Start with  $ch$

$$\begin{aligned} ch &= (c_1 + c_2)(h_1 + h_2) \\ &= c_1h_1 + c_1h_2 + c_2h_1 + c_2h_2 \end{aligned}$$

The single right triangles are easy to see:  $c_1 h_2$  and  $c_2 h_2$ . The other two are composed of two right triangles. For both, the base is  $h_1$ , and then, for green and blue, the height is  $c_1$ , or for red and magenta, the height is  $c_2$ . For these obtuse triangles, the height must be extended to the base to form the right angle.

But the same six triangles can be arranged in a different way so that twice the area of the whole triangle is

$$af = a_1 f_1 + a_1 f_2 + a_2 f_1 + a_2 f_2$$



$f_1$  is the base and  $a_1$  or  $a_2$  the height, for the compound cases.

A similar calculation can be carried out for side  $b$  and altitude  $g$ . The area is the same regardless of which side is chosen as the base.

□

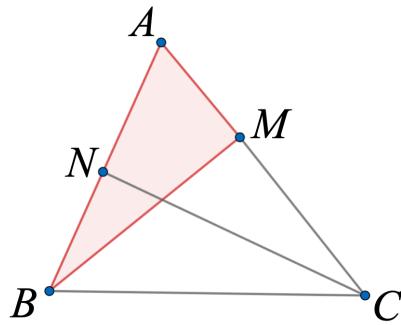
### altitude proof

In  $\triangle ABC$  draw  $BM \perp AC$  and  $CN \perp AB$ .

Then  $AB \cdot CN = AC \cdot BM$ .

*Proof.*

$\triangle AMB$  and  $\triangle ANC$  are both right triangles.



They also share  $\angle A$ , so they are similar.

As similar triangles, corresponding sides are in the same ratio:

$$\frac{AB}{AC} = \frac{BM}{CN}$$

Then  $AB \cdot CN = AC \cdot BM$

We can show that side  $BC$  times its altitude is equal to  $AB \cdot CN$ , in exactly the same way.

So any altitude times the base has the same value, which is twice the area of the triangle.

□

A straightforward corollary of this result is that any triangle with two equal altitudes is isosceles.

*Proof.*

If  $BM = CN$  then

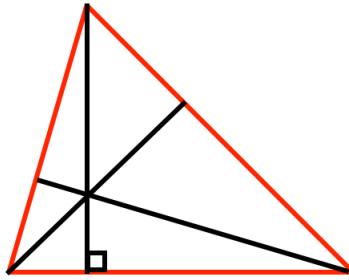
$$\frac{AB}{AC} = 1$$

so  $AB = AC$ .

□

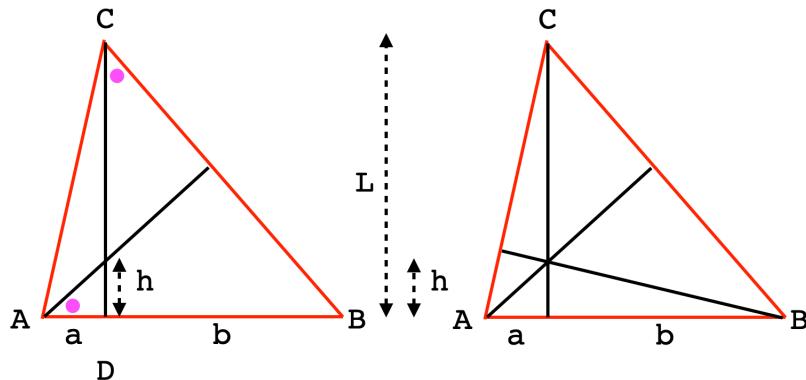
## orthocenter: Newton's proof

As we've said, the orthocenter is the point where all three altitudes cross.



**orthocenter/altitude**

An altitude is a line drawn from any vertex to the opposing side, forming a right angle with the base, thereby dividing the triangle into two right triangles.



In the left panel, we draw the altitude from the vertex  $C$  down in  $\triangle ABC$  to meet the base at a right angle.

The altitude divides the base into lengths  $a$  and  $b$ . Now draw a second altitude from vertex  $A$  to the side opposite (left panel).

What is the height  $h$  above the base where the two lines cross?

The small triangle with sides  $a$  and  $h$  and the large triangle  $\triangle BCD$  are similar, because they are both right triangles that are joined by vertical angles.

This means that the angles marked with magenta dots are equal.

Similar triangles have equal ratios for the corresponding sides. This is a very important theorem which we haven't proved yet, but will soon.

In the small triangle the side opposite the marked angle ( $\angle BAC$ ) has length  $h$ , while

the entire length of the altitude is  $L$ . By similar triangles

$$\frac{h}{a} = \frac{b}{L}$$

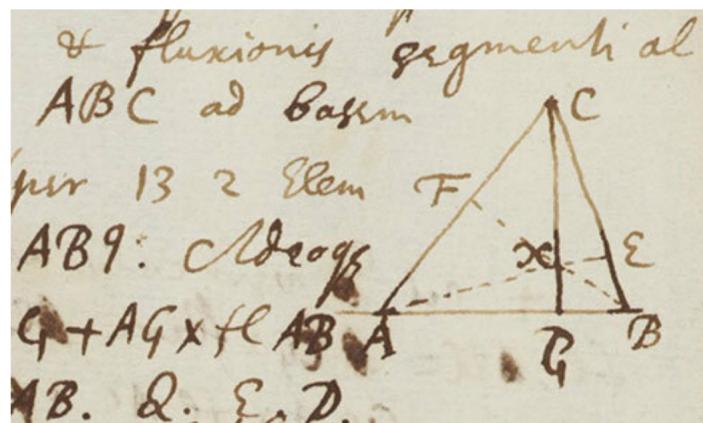
(the side opposing the marked angle is in the numerator on both sides). So the height  $h$  is

$$h = \frac{ab}{L}$$

The formula is noteworthy because it is symmetrical in  $a$  and  $b$  and does not contain any term related to side  $AC$ , opposite vertex  $B$ .

Therefore, if we draw the third altitude to side  $BC$ , opposite vertex  $A$ , we can calculate that it crosses the vertical altitude at the same height  $h = ab/L$  (right panel).

This means that the three altitudes cross at a single point, at height  $h$ .



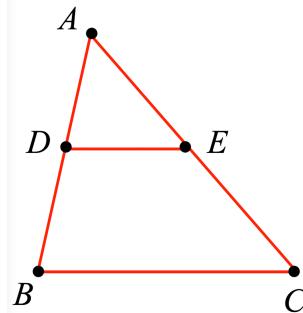
**Fig. 129** The diagram for Newton's proof, from his *Geometria curvilinea and Fluxions*, Ms Add. 3963, p54r.

Newton published this proof about 1680.

# Chapter 13

## Right triangle similarity

In the chapter on congruence, we introduced similar triangles as triangles with the same three angles, but scaled differently.



Let  $\triangle ABC \sim \triangle ADE$ . ( $\sim$  is the symbol for similar). Then they can nestle one inside the other, using any one of the shared angles.

If we do this so that  $\angle ADE = \angle ABC$ , then by alternate interior angles  $DE \parallel BC$ .

The other property is the equal proportions of sides.

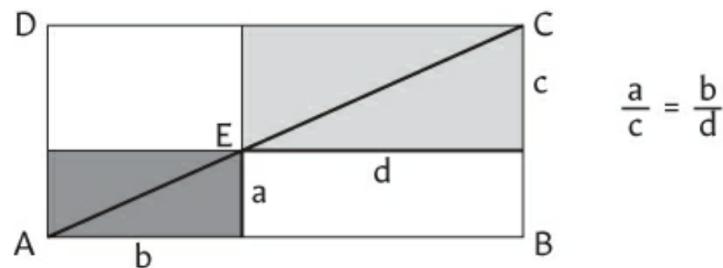
$$\frac{AD}{BD} = \frac{AE}{CE}$$

equal angles and parallel third side  $\iff$  equal ratios of sides.

## similar right triangles

We will show an easy proof that for similar right triangles, all angles equal implies equal ratios of sides. Our approach is from Acheson and is equivalent to a previous result (Euclid I.43).

Draw a rectangle ABCD, and a diagonal AC. Then pick a point E on the diagonal and draw lines through it parallel to the sides.



**Fig. 42** Area and similarity.

All of the right triangles in the figure are similar. Start with the alternate interior angles theorem, then use complementary angles in a right triangle, and finish with vertical angles.

By changing the height of the figure, we can obtain any two complementary angles we wish. And by changing the placement of  $E$  we can get any ratio we like.

So then, the two shaded rectangles are bisected by the diagonal  $AEC$ , by the diagonal theorem. So the two light-gray triangles have equal area, and the two dark gray ones do as well.

But  $\triangle ABC$  and  $\triangle ADC$  also have equal area.

Therefore, we just subtract equal areas to find that the two unshaded rectangles above and below the diagonal are equal in area. The one on top has area  $bc$  and the one below has area  $ad$ . We have

$$bc = ad$$

$$\frac{a}{c} = \frac{b}{d}$$

and also

$$\frac{a}{b} = \frac{c}{d}$$

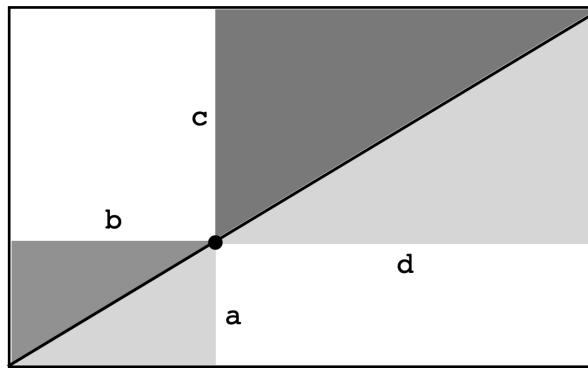
□

Corresponding sides are in the same proportion, but also the ratio of the two sides is the same for similar triangles.

### similar parts and the whole

We showed that the two smaller triangles have corresponding sides in proportion.

But the large triangle (one-half of the entire rectangle) has the same angles, and should have the same ratios. Here is a simple manipulation to obtain that result:



$$bc = ad$$

$$bc + ab = ad + ab$$

$$b(a + c) = a(b + d)$$

$$\frac{a}{b} = \frac{a+c}{b+d}$$

Given any two of these relationships we can derive the third. This is the same math in reverse.

$$\frac{a+c}{b+d} = \frac{c}{d}$$

$$\frac{a+c}{c} = \frac{b+d}{d}$$

$$\frac{a}{c} + 1 = \frac{b}{d} + 1$$

$$ad = bc$$

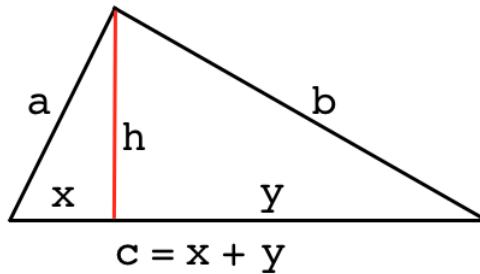
□

## hypotenuse in proportion

It is natural to ask, what about the hypotenuse?

Consider any right triangle. Drop the altitude to the hypotenuse.

Using complementary angles, we can show that the two new smaller triangles formed by the altitude are similar to each other and to the original large one.



Now, write the ratio of the long to the short side *for each one* of the three triangles:

$$\frac{h}{x} = \frac{y}{h} = \frac{b}{a} = k$$

From our previous theorem we know this equality is valid.

However the same statement can be viewed in a different way.

$b/a$  is the ratio for the hypotenuse of the medium triangle compared to the small one.

And  $y/h$  is the ratio of the long side in the medium triangle to the long side in the small one.

They are equal, and this completes the proof!

□

We can also give an algebraic proof, by looking ahead to the Pythagorean theorem. As you likely know, for a right triangle with sides  $a$  and  $b$  and hypotenuse  $g$ :

$$a^2 + b^2 = g^2$$

We can use the Pythagorean theorem to prove that:

$$\frac{a}{c} = \frac{b}{d} = \frac{g}{h}$$

*All* of the sides of two similar right triangles have the same ratio.

We must be careful, however. A deep connection exists between similarity, area and the Pythagorean theorem. It is important that we will have Euclid's proof of the Pythagorean theorem, and that proof depends on SAS rather than on similarity.

Equal ratios extends to the hypotenuse.

*Proof.*

Start with

$$\begin{aligned}\frac{a}{c} &= \frac{b}{d} = k \\ a &= kc, \quad b = kd \\ a^2 + b^2 &= k^2c^2 + k^2d^2 \\ g^2 &= k^2h^2\end{aligned}$$

Since these are lengths, we can take the positive square root and obtain

$$\begin{aligned}\frac{g}{h} &= k \\ &= \frac{a}{c} = \frac{b}{d}\end{aligned}$$

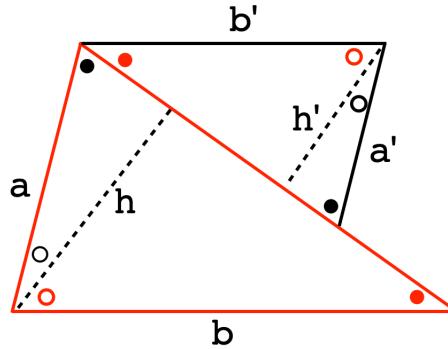
□

Thus, AAA similarity is established for right triangles. If either of the smaller angles matches between two right triangles, then they are not only similar but all the side lengths are in the same ratio as well.

Getting the converse, going from equal ratios to equal angles, uses a result about parallelograms. Also, the proofs apply not just to right triangles, but to triangles

of any type. For that reason, we defer further development of similarity to a later chapter.

We proved this above for right triangles, and could extend the result to all triangles by dissection. We just show the figure and skip the proof.

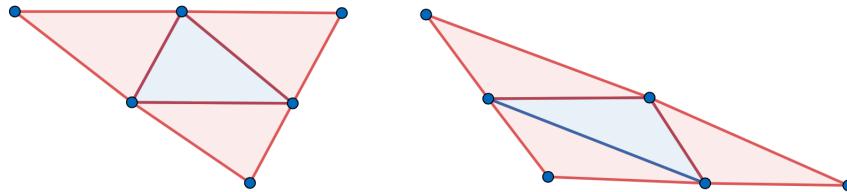


We promise to prove it later when we deal with the theory of similarity more explicitly. Euclid VI.2 does this elegantly.

We can prove it for a special case now.

## triangle dissection

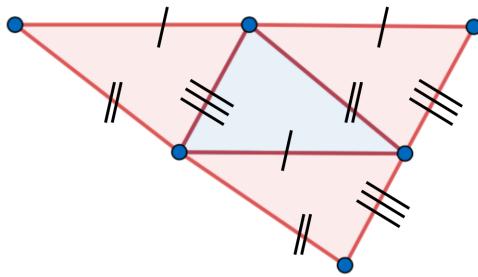
- Any triangle can be dissected into four congruent triangles.



*Proof.*

Find the midpoints of the sides and connect them.

By the midline theorem each side of the central triangle is one-half the length of the side to which it is parallel.



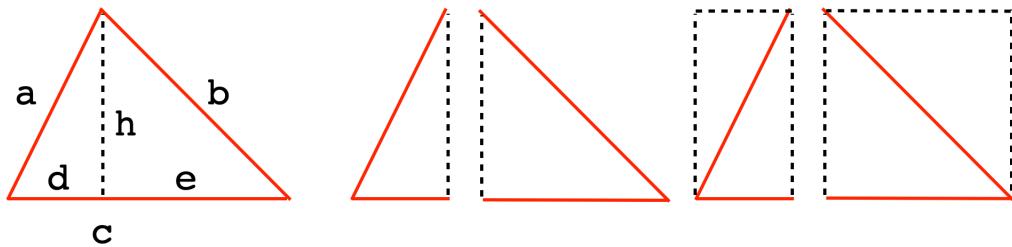
We have all congruent triangles by SSS.

Also, as a result we have 3 parallelograms, each containing two congruent triangles.

□

### dissection into right triangles

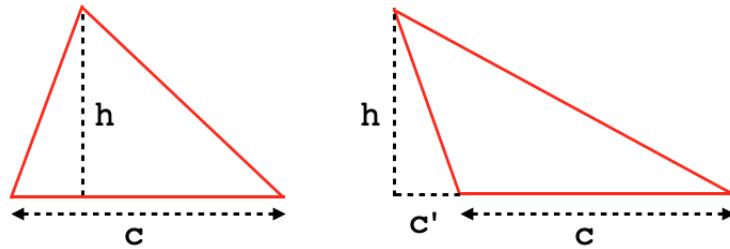
Any triangle can be cut into two right triangles by drawing its *altitude*,  $h$ , where  $h$  is perpendicular to  $c$ .



The area of the triangle with sides  $a, d, h$  is  $dh/2$ , and that with sides  $a, e, h$  is  $eh/2$  so the area of the original triangle is

$$\frac{dh}{2} + \frac{eh}{2} = \frac{(d+e)h}{2} = \frac{ch}{2}$$

This formula is correct even for an obtuse triangle like the one in the right panel, below. The area of the two red triangles is the same:  $ch/2$ .



We get that by computing the area of the large triangle with base  $c + c'$  and then subtracting the area of the skinny triangle with the base  $c'$ :

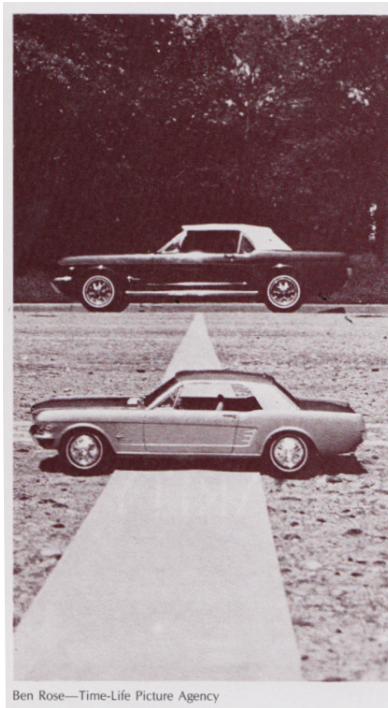
$$\mathcal{A} = \frac{h(c + c') - h(c')}{2} = \frac{hc}{2}$$

Of course, for an obtuse triangle we could choose one of the other sides as the base and proceed in the usual way, but this works as well.

# Chapter 14

## Similar triangles

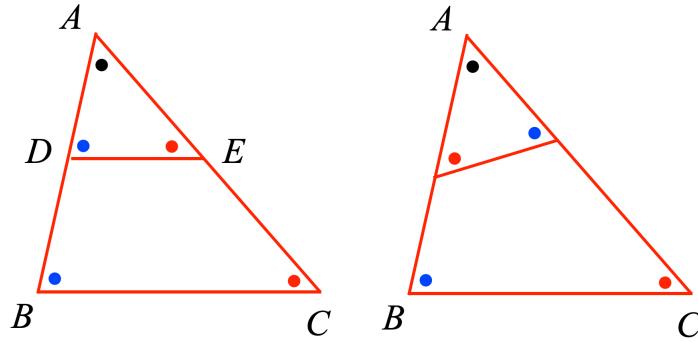
Triangles that are alike, but not congruent, because they are scaled differently, are called similar. We write  $\triangle ABC \sim \triangle PQR$ .



— Figure from Jacobs, chapter 10

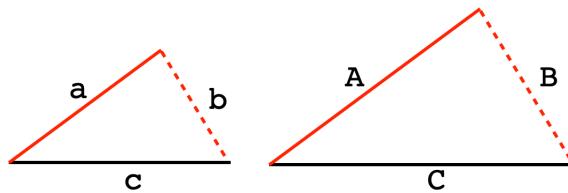
Statements about two similar triangles.

- They have at least two (thus, three) angles known to be equal.
- When superimposed using a shared angle, the third pair of sides, that do not coincide with each other, are nevertheless parallel (you may need the mirror image for one triangle).



- They have corresponding pairs of sides in the same proportions, but scaled by a constant factor.

If any one of these properties hold, they all do.



From the above diagram of two similar triangles, similarity implies that (for example)

$$\frac{A}{a} = \frac{B}{b}$$

For any pair of similar triangles, there is a constant  $k$  such that

$$k = \frac{A}{a} = \frac{B}{b} = \frac{C}{c}$$

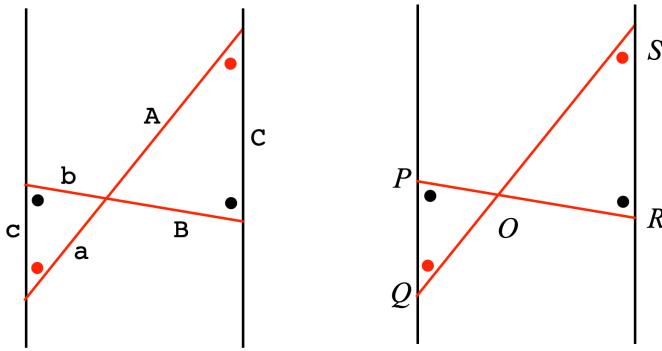
A slight rearrangement gives:

$$\frac{a}{b} = \frac{A}{B}$$

These ratios are obviously different.

As with congruent triangles, our definitions allow one triangle to be flipped before the comparison is made (comparing two triangles, originally the angles were in opposite order, one clockwise and the other counter-clockwise).

In the figures below, the vertical black lines are parallel. Any two lines connecting them that cross, form two similar triangles. The angles marked with dots of the same color are equal by alternate interior angles. We also have vertical angles.

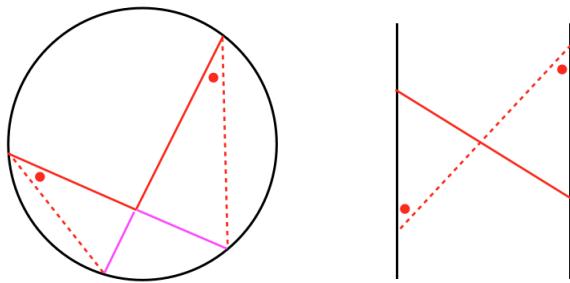


Set up the ratios carefully, by finding sides that lie opposite equal angles.

$$\begin{array}{ccc} a & b & c \\ A & B & C \end{array} \quad \begin{array}{l} \triangle OPQ = \frac{OP}{OR} \quad \frac{PQ}{RS} \quad \frac{QO}{SO} \\ \triangle ORS = \frac{OR}{OP} \quad \frac{RS}{PQ} \quad \frac{SO}{QO} \end{array}$$

For triangles with labeled vertices, I like to do this by naming the triangles in the corresponding order. We start  $\triangle OPQ$  with  $OP$  opposite the red dot, moving counter-clockwise. Do the same with  $\triangle ORS$ .

Later we will have a theorem about similar triangles formed by crossed chords in a circle. In the figure below, the angles marked with a red dot are equal, as are the vertical angles, so the two triangles are similar. In this case we follow equal angles (and sides) around in opposite directions, comparing the two triangles.

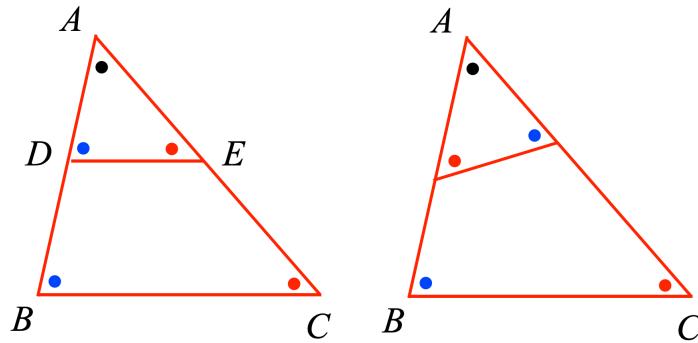


AAA is probably the most common way to establish that two triangles are similar.

## similarity and ratios

Similar triangles are defined to have three angles equal (equiangular), but they are scaled differently and so, not congruent.

If two equiangular triangles are superimposed at one vertex ( $\angle A$ ), so the adjacent sides coincide, and angles lie in the same order (left panel), then the sides opposite  $\angle A$  are parallel ( $DE \parallel BC$ ). In the right panel, the angles are in opposite order and this doesn't work.



This can be demonstrated by employing the parallel postulate, or as Euclid does in VI.2, by an argument based on area and the properties of two parallel lines. Actually, Euclid says to take a triangle and draw the line segment  $DE$  either parallel to the base or with the given angles, but the result is the same.

All angles equal and the third side parallel are closely related and easily proven in both directions.

Equal angles  $\iff$  parallel sides.

We will call this property similarity: all angles equal and the third side parallel.

Another property is that similar triangles have their sides in the same proportion. This comes in two flavors: either two sides in equal proportion flanking an equal angle, often called SAS similarity, or all three sides in the same proportion with no prior knowledge about angles.

We now explore all of these situations:

- $\parallel$  sides and equal  $\angle \iff$  equal ratios

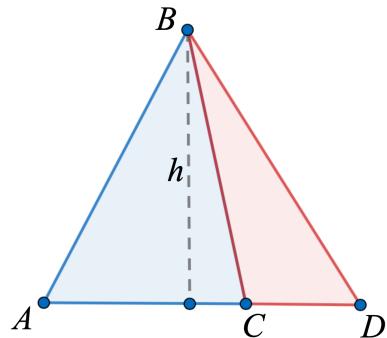
We will call these the *forward* ratio theorem, and the *converse* ratio theorem, even though there are two cases, either SAS or ratios only.

### parallel third side implies equal ratios

As background, we recall two fundamental ideas about triangles.

The first is that if two triangles have their bases on the same line, and they also share the same vertex opposite, then they have the same altitude.

It follows that the areas are in the same proportion as the lengths of the bases. This is the **area-ratio theorem**.



$$2\mathcal{A}_{\triangle BAC} = h \cdot AC$$

$$2\mathcal{A}_{\triangle BCD} = h \cdot CD$$

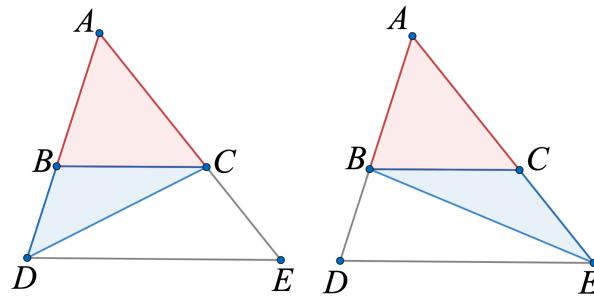
$$\frac{\mathcal{A}_{\triangle BAC}}{\mathcal{A}_{\triangle BCD}} = \frac{AC}{CD}$$

There is only one line that can be drawn from a point vertically to a straight line. So there is only one altitude that can be drawn from a point to a base. Since the triangles share the vertex point, they have the same altitude.

The second idea is that if two triangles share the same base, and the two opposing vertices both lie along a line that is parallel to the base, then the triangles have the same area.

## Euclid VI.2

- $\parallel$  sides and equal  $\angle \rightarrow$  equal ratios



*Proof.*

Start with  $\triangle ADE$  and then draw  $BC \parallel DE$ .

We will show that  $AB : BD = AC : CE$ .

The key to Euclid's proof of this theorem is to observe that the two triangles shaded blue,  $\triangle DBC$  and  $\triangle ECB$ , have the same area.

The reason is that they lie on the same base, and their vertex (either  $D$  or  $E$ ) lies along the same line parallel to that base. Hence the result follows.

But by the area-ratio theorem

$$\frac{\mathcal{A}_{\triangle ABC}}{\mathcal{A}_{\triangle DBC}} = \frac{AB}{BD}$$

and

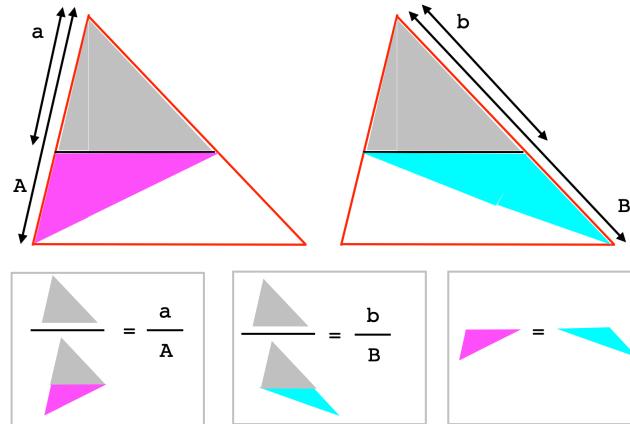
$$\frac{\mathcal{A}_{\triangle ABC}}{\mathcal{A}_{\triangle ECB}} = \frac{AC}{CE}$$

Since the left-hand sides of the two expressions are equal, so are the right-hand sides. Namely

$$\frac{AB}{BD} = \frac{AC}{CE}$$

□

Here is a different pictorial view of the argument.



Therefore

$$\frac{a}{A} = \frac{b}{B}$$

Equal angles (parallel third side) implies equal ratios.

□

This result applies to any vertex of the two similar triangles and its adjacent sides, hence it applies to all three sides of the two triangles.

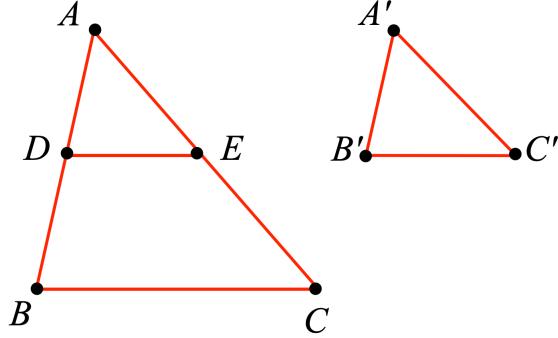
### converse theorem

Euclid proves the converse theorem by using the same steps in reverse. Start with equal areas, form equal ratios of them, and then use the equal altitude to show that  $DE \parallel BC$ .

A somewhat different proof of the converse theorem, that equal ratios implies equal angles, is given by Kiselev.

*Proof.*

Let there be two triangles  $\triangle ABC \sim \triangle A'B'C'$  with three pairs of sides in the same ratio. Let  $\triangle ABC$  be the larger one.



Mark off inside  $\triangle ABC$ , for example,  $AD = A'B'$  and then draw  $DE \parallel BC$ . By the forward theorem,  $\triangle ABC \sim \triangle ADE$ .

We can form the dual equality:

$$\frac{A'B'}{B'C'} = \frac{AB}{BC} = \frac{AD}{DE}$$

since we are given the first (equal ratios), and have the second from similar triangles,  $\triangle ABC \sim \triangle ADE$ .

Equating the first and third terms:

$$\frac{A'B'}{B'C'} = \frac{AD}{DE}$$

But we also have  $A'B' = AD$ , by construction. Cancel to obtain  $B'C' = DE$ .

The same can also be done for the third side. Thus,  $\triangle ADE \cong \triangle A'B'C'$  by SSS.

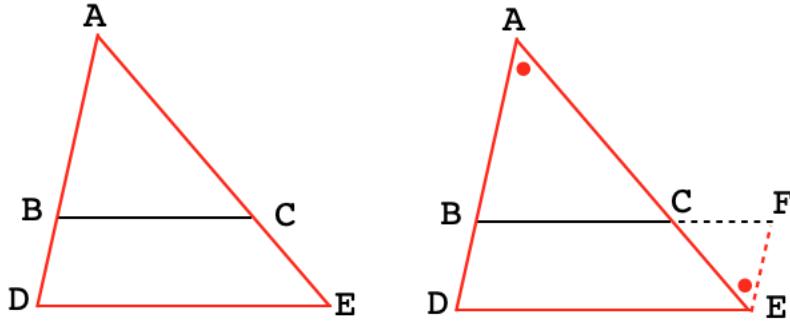
It follows that  $\triangle A'B'C'$  is equiangular with  $\triangle ADE$  and thus with  $\triangle ABC$ .

□

## SAS for similar triangles

Now, we mix and match the conditions, taking two sides in proportion and one angle shared. This is often called SAS similarity.

We will prove that this mix of conditions is enough for two triangles to be similar.



*Proof.*

Given two triangles with the same vertex angle at  $A$  and two of three sides known to be in the same proportion ( $BD/AB = CE/AC = k$ ).

Draw  $EF$  parallel to  $ABD$  and extend  $BC$  to meet it at  $F$ .

$\triangle ABC \sim \triangle CEF$  by vertical angles plus alternate interior angles (red dots). The corresponding sides opposite the vertical angles are  $EF$  and  $AB$ .

By the forward theorem

$$\frac{EF}{AB} = k = \frac{CE}{AC}$$

but

$$\frac{CE}{AC} = \frac{BD}{AB}$$

Therefore,  $BD = EF$ . We are given  $BD \parallel EF$ .  $BDEF$  has one pair of opposing sides equal and parallel, so it is a parallelogram.

It follows that  $BC \parallel DE$  and  $BF = DE$ .

It also follows that  $\angle D = \angle ABC$  by alternate interior angles, so we have two angles equal which means that  $\triangle ABC \sim \triangle ADE$ .

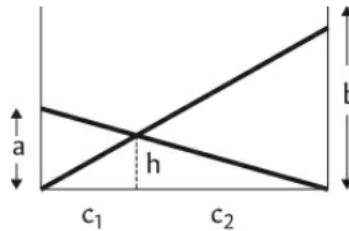
The forward theorem then gives  $(DE - BC)/BC = BD/AB = k$

□

Each of the standard congruence theorems (yes, even SSA) has a similarity version. We won't prove the others.

## problem

The figure below is from Acheson's wonderful book aptly titled *The Wonder Book of Geometry*. He shows this problem, which he says "[goes] back to at least AD 850, when it appeared in a textbook by the Indian mathematician Mahavira."



**Fig. 40** A problem with ladders.

Looking down an alleyway, you see two ladders arranged as shown and wonder about the point where they cross, at height  $h$  and distances  $c_1$  and  $c_2$  from the edges of the alley, where the width of the alley is  $c = c_1 + c_2$ .

By similar triangles

$$\frac{c_1}{h} = \frac{c}{b}$$

Can you see why?

Going the opposite direction

$$\frac{c_2}{h} = \frac{c}{a}$$

Adding the two equations and substituting for  $c_1 + c_2$ :

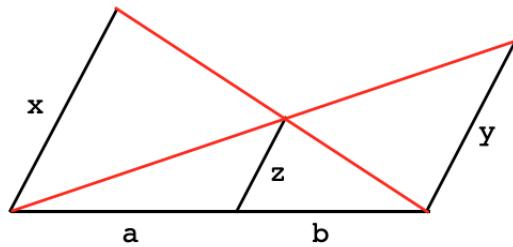
$$\frac{c}{h} = \frac{c}{a} + \frac{c}{b}$$

Thus

$$\frac{1}{h} = \frac{1}{a} + \frac{1}{b}$$

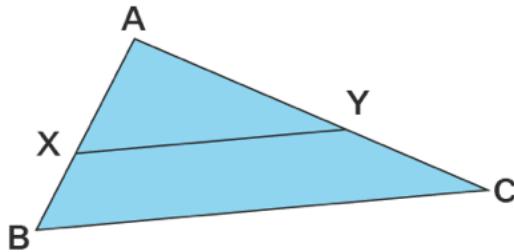
That's a simple and interesting result.  $h$  depends only on  $a$  and  $b$  and not on  $c, c_1$ , or  $c_2$ .

If you think about it, you should see that in the previous problem we never used the information that the sides and the height were vertical, only parallel.



Work through the same proof for the figure above to show that  $1/x + 1/y = 1/z$ .

### problem



It is given that  $XY \parallel BC$  and divides the triangle into two parts of equal areas. Find the ratio  $AX : XB$  using the area of similar triangles theorem.

The problem states that the bases are parallel ( $XY \parallel BC$ ) and also that the subdivision produces *equal areas*. We are asked to find the ratio  $AX : XB$ .

*Solution.*

Recall from a previous chapter that for two similar triangles, the altitudes to corresponding sides are in the same ratio as any of the three pairs of sides themselves.

The altitudes  $h$  and  $H$  to  $BC$  (not drawn) are also in the same ratio as the sides, so let  $H = kh$  —  $H$  lies on  $BC$ .

Then twice the area of the top triangle is  $AX \cdot h$  and twice the area of the whole is  $AB \cdot H = kAB \cdot h$

We have that the whole is twice the smaller area so the ratio is equal to 2:

$$\frac{kAB \cdot h}{AX \cdot h} = 2$$

But  $AB/AX$  is also equal to  $k$  so we have that  $k^2 = 2$  and  $k = \sqrt{2}$ .

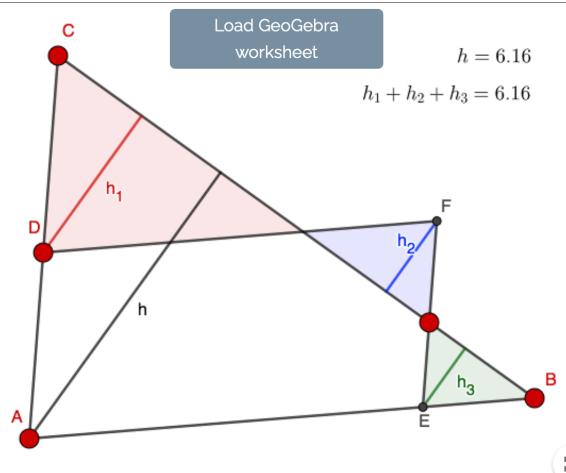
However, we are not asked for  $k$ . Instead, we want the ratio of  $AX$  to the smaller piece along the bottom. Let  $AX = a$ . Then the whole is  $A = ka$ . The difference is the small piece,  $XB = A - a = a(k - 1)$ . We need

$$\frac{a}{a(k-1)} = \frac{1}{k-1} = \frac{1}{\sqrt{2}-1}$$

## problem

Here's a problem from the web. Given that  $AC \parallel EF$  and  $AB \parallel DF$ .

We are to prove that the sum of the altitudes of the small triangles is equal to the altitude of the large one.



Informal solution: The smaller triangles are all similar to  $\triangle ABC$  by the alternate interior angles and vertical angle theorems.

For similar triangles, not only are the sides in the same ratio to each other, but so are other measures like the altitudes to a particular side. So if we label the bases  $b_1$  etc., collectively  $b_i$ , then we have that

$$\frac{b_i}{h_i} = \frac{b}{h}$$

$$b_i = b \cdot h_i/h$$

for each of the  $b_i$ .

But the sum of the  $b_i$  is simply equal to  $b$  so

$$b_1 + b_2 + b_3 = b \cdot (h_1/h + h_2/h + h_3/h)$$

$$b = b \cdot (h_1/h + h_2/h + h_3/h)$$

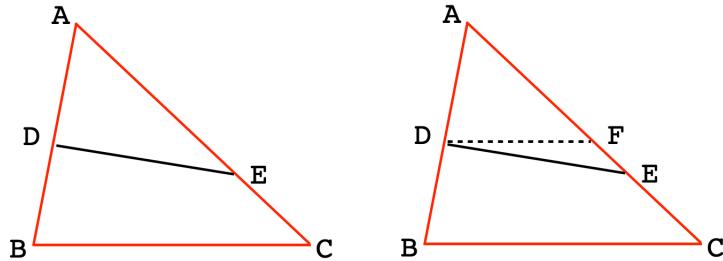
$$1 = h_1/h + h_2/h + h_3/h$$

$$h = h_1 + h_2 + h_3$$

## problem

Given two triangles that are not similar, where the ratio  $AD/AB = r$  and  $AE/AC = s$ . We are asked to show that

$$\frac{\Delta_{ADE}}{\Delta_{ABC}} = rs$$



*Solution.*

Draw  $DF \parallel BC$ . Now  $\triangle ADF \sim \triangle ABC$  so  $AF/AC = r$ .

By our previous result

$$\frac{\Delta_{ADF}}{\Delta_{ABC}} = r^2$$

Since  $AE/AC = s$  and  $AF/AC = r$ ,  $AE/AF = s/r$ . As triangles with a common vertex and bases in that proportion, these areas are in the same proportion:

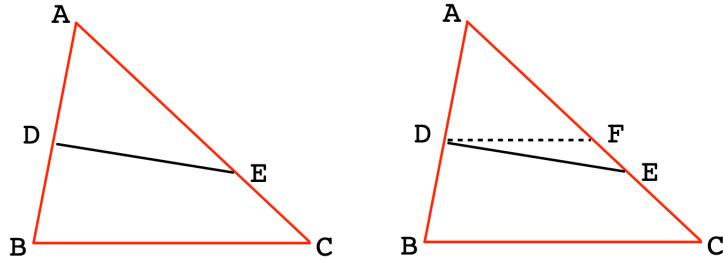
$$\frac{\Delta_{ADE}}{\Delta_{ADF}} = \frac{s}{r}$$

Multiply the two ratios together to obtain

$$\frac{\triangle_{ADE}}{\triangle_{ABC}} = sr$$

□

*Solution.* (alternate).



Looking ahead to trigonometry, in any triangle, twice the area can be computed as the product of two sides flanking an angle times the *sine* of the angle. (The reason is that the altitude to one side, divided by the length of the second side, is defined to be the sine of the angle.)

$$2(ADE) = AD \cdot AE \cdot \sin \angle DAE$$

$$2(ABC) = AB \cdot AC \cdot \sin \angle BAC$$

But  $\angle DAE = \angle BAC$ , so they have the same sine, and the ratio of areas is just

$$\frac{AD \cdot AE}{AB \cdot AC} = rs$$

□

To rework this proof in terms of familiar concepts, draw the altitude from  $D$  to  $AE$ , and also the one from  $B$  to  $AC$

They form similar right triangles including the angle at  $A$ . The altitudes scale like  $AD/AB = r$ . But the bases scale like  $AE/AC = s$ .

And area scales like the product of the two, namely,  $rs$ .

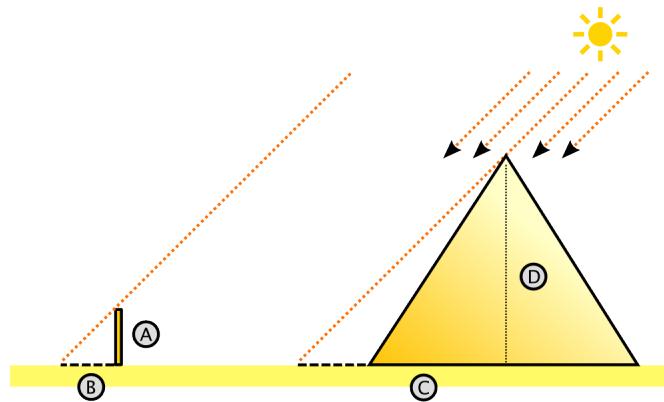
## pyramid height

As we said earlier, Thales was from Miletus and he lived around 600 BC. Thales is believed to have traveled extensively and was likely of Phoenician heritage. As you probably know, the Phoenicians were famous sailors who founded many settlements around the Mediterranean.

They competed with the mainland Greeks and later with the Romans for colonies, and their major city, Carthage, was destroyed much later by the Romans, in the third Punic War. Hannibal rode his famous elephants over the Alps in the second Punic war.

During his travels, Thales went to Egypt, home to the great pyramids at Giza, which were already ancient then. They had been built about 2560 BC (dated by reference to Egyptian kings) and were already 2000 years old at that time!

The story is that Thales asked the Egyptian priests about the height of the Great Pyramid of Cheops, and they would not tell him. So he set about measuring it himself. He used similar triangles. I'm sure he wrote down his answer, but I'm not aware that it survives. The current height is 480 feet.



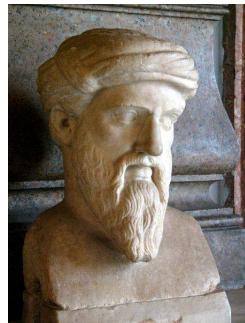
# Part IV

## Pythagorean theorem

# Chapter 15

## Simple proofs

The most famous theorem of Greek geometry is also without a doubt the most useful in calculus.



Like many Greek mathematicians, Pythagoras (c.570 - c.490 BCE) was not from Greece “proper” (i.e. the territory of mainland Greece). Instead, he was from one of the islands. Pythagoras was born on the island of Samos, which lies off the coast of what was called Asia Minor (modern Turkey).

Pythagoras was much younger than Thales but may have encountered him as a youth, since Thales lived on the mainland not too far from Samos, at Miletus. Later, Pythagoras moved to a Greek colony in southern Italy, at Croton.

During his lifetime, Pythagoras was known as a philosopher much more than as a mathematician. For example, he was famous as an expert on the fate of the soul after death.

<https://plato.stanford.edu/entries/pythagoras/>

Wilczek cites Bertrand Russell on Pythagoras:

“A combination of Einstein and Mary Baker Eddy.”

which will be funny if you look up Mary Baker Eddy.

Pythagoras founded a “school” and it is not sure now which of the theorems developed by this school are due to Pythagoras, and which to his disciples. It is not even clear whether the Pythagorean theorem, as we understand it today, was known to Pythagoras. There is no contemporaneous account (say, Plato or Aristotle) connecting Pythagoras with the theorem. Of course, nearly all of the histories that were written are lost.

Regardless of whose idea it was, and who could prove it first, it’s clear that they knew something. The Pythagorean theorem says that if  $c$  is the hypotenuse of any right triangle and  $a$  and  $b$  are the side lengths then

$$a^2 + b^2 = c^2$$

The simplest example in integers is

$$3^2 + 4^2 = 9 + 16 = 25 = 5^2$$

but there are many triplets of integers with this property, for example

$$5^2 + 12^2 = 25 + 144 = 169 = 13^2$$

Not just the classic 3-4-5 right triangle, but a number of other *Pythagorean triples* had been known for a thousand years.

The tablet “Plimpton 322” contains (by extrapolation) the triplet 4601-4800-6649 and it dates to about 1800 BCE. Maor analyzes this in his book on the theorem of Pythagoras.



Line 3 of the tablet contains the numbers 1, 16, 41 and 1, 50, 49 which are in base 60 notation. Thus  $1 \cdot 3600 + 16 \cdot 60 + 41 = 4601$  and  $1 \cdot 3600 + 50 \cdot 60 + 49 = 6649$ . The third number of the triple is missing but it’s obvious since  $6649^2 - 4601^2 = 23040000$ , which is  $4800^2$ .

It seems highly unlikely that these were found by searching randomly among squares. There is even a triple with 5 decimal digits on the next line of the Plimpton 322 tablet.

One should not think that the theorem *only* applies to triangles with integer side lengths. For example, it applies to the isosceles right triangle with side length equal to 1, whose hypotenuse is the real number  $\sqrt{2}$ . Also, any integer solution can be modified. For example

$$0.5^2 + 1.2^2 = 0.25 + 1.44 = 1.69 = 1.3^2$$

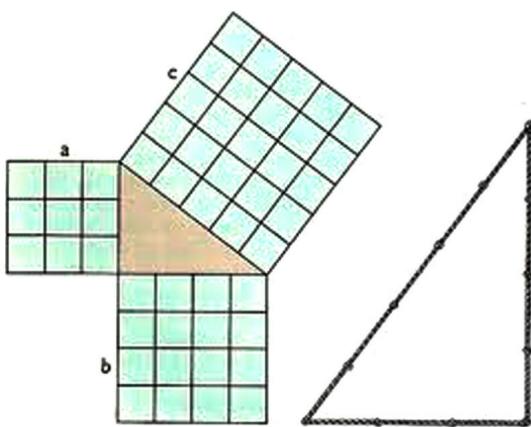
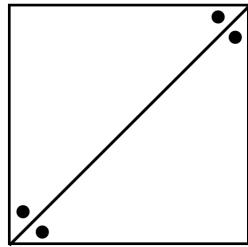


FIGURE 1. THE 3-4-5 RIGHT TRIANGLE, A SIMPLE CASE OF PYTHAGORAS'S THEOREM.

We're going to spend time with a few particular proofs of this theorem (there are literally hundreds of them), so as to examine the ideas from different perspectives. This chapter is an introduction that shows some basic algebraic proofs.

The proof due to the Greeks presented in the next chapter is from Euclid. It employs SAS for triangle congruence, and is probably his own contribution. Later we'll explore how the concept of area is connected to the Pythagorean theorem.

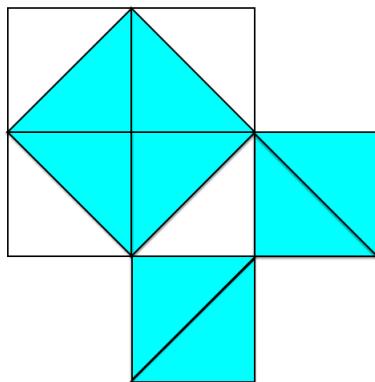
To begin, let's examine a special case, easily proved, for an isosceles right triangle.



We can obtain such a triangle by drawing the diagonal in a square. The black dotted angles in any one triangle are equal, by the isosceles triangle theorem, and the others are equal by alternate interior angles.

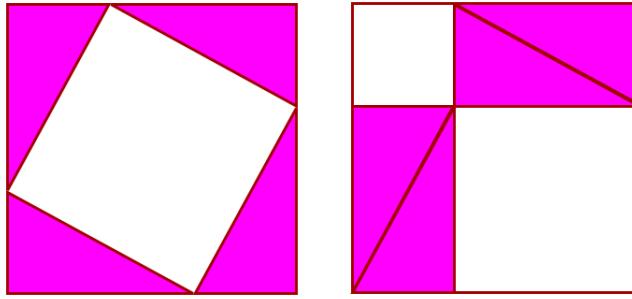
Our hypothesis is that when the lengths of the sides of the triangles are squared and then summed, they will be equal to the square of the diagonal. That is to say, the square of the diagonal should be equal to two of the squares shown above.

Here is a proof without words.



Adding some words: the area of the square on the hypotenuse is equal to one-half of the area of four squares of the sides. This is a proof for the special case where the sides are equal.

The following general proof is sometimes called the “Chinese proof.” I can easily imagine proceeding from the figure above to the left panel below by simply rotating the inner square and collapsing the surrounding one.



It really needs no explanation, but ..

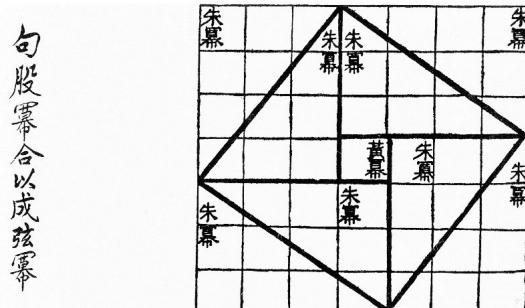
We have a large square box that contains within it a white square, whose side is also the hypotenuse of the four identical right triangles contained inside. Altogether the four triangles plus the white area add up to the total.

We simply rearrange the triangles. Now we evidently have the same area left over from the four triangles, because they still have the same area and the surrounding box has not changed.

But clearly, now the white area is the sum of the squares on the second and third sides of the triangles. Hence the two white squares on the right are equal in area to the large white square on the left.

□

This diagram is contained in the Chinese text Zhoubi Suanjing.



Eight right triangles are formed with sides of 3 and 4 units. There are 7 units on each side of the large square so the total area is 49. Each pair of triangles has area  $12 (3 \cdot 4)$ , so the square in the center really is a unit square with area  $49 - 4 \cdot 12 = 1$ .

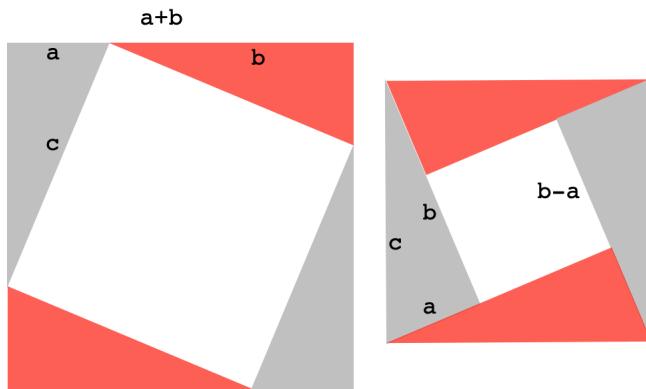
So then, the central square consists of four triangles with total area 24 plus the unit square for 25. This is the square of the long side, namely 5.

[https://en.wikipedia.org/wiki/Zhoubi\\_Suanjing](https://en.wikipedia.org/wiki/Zhoubi_Suanjing)

There is debate about whether this is really a proof, or if it simply presents the arithmetic for the example of a 3-4-5 right triangle.

## simple proofs

Many proofs of the theorem are algebraic. Here are two:



*Proof.*

In the left panel, we have the same arrangement as before, with four identical right triangles. The white square at the center has sides of length  $c$  and angles that are right angles because when summed to two complementary angles, the result is two right angles.

The algebra is

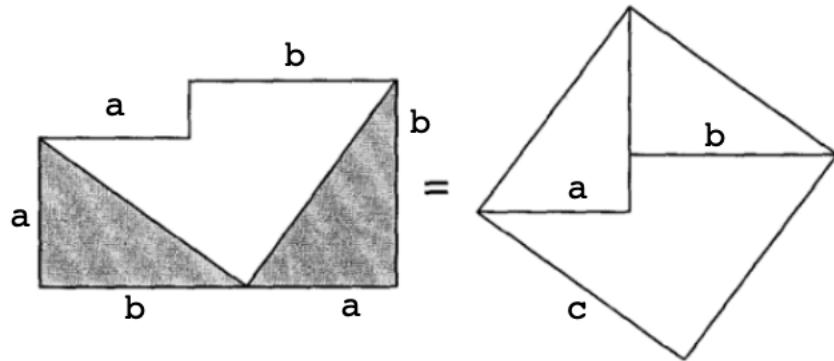
$$\begin{aligned}(a+b)^2 &= 4 \cdot \frac{1}{2} \cdot ab + c^2 \\ a^2 + 2ab + b^2 &= 2ab + c^2\end{aligned}$$

Subtract  $2ab$  from both sides, and we're done.

□

I will leave the right panel to you.

This is a proof from Gelfand and Saul's trigonometry book.

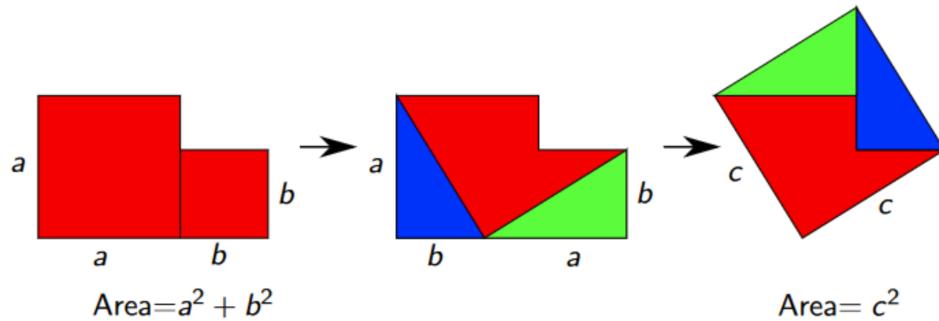


I have added the labels for side length.

If it's not obvious, note that we start with two adjacent squares of area  $a^2 + b^2$ . Two triangles (shaded) are cut off and re-arranged to make a shape whose area is  $c^2$ .

□

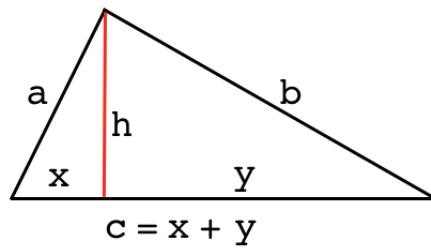
Here is a colored version of the same proof I found on the web.



<https://t.co/xTnARHQNWw>

### similar triangles

Here is a simple classic that depends on the ratios formed for similar right triangles:



*Proof*

We know that when an altitude is drawn in a right triangle, the two resulting right triangles are similar, by complementary angles. Similarity means that we have equal ratios of sides. Here are two sets:

ratio of hypotenuse to short side

$$\frac{a}{x} = \frac{b}{h} = \frac{c}{a}$$

ratio of hypotenuse to long side

$$\frac{a}{h} = \frac{b}{y} = \frac{c}{b}$$

From the first

$$a^2 = cx$$

And from the second

$$b^2 = cy$$

Add them:

$$\begin{aligned} a^2 + b^2 &= cx + cy \\ &= c(x + y) = c^2 \end{aligned}$$

□

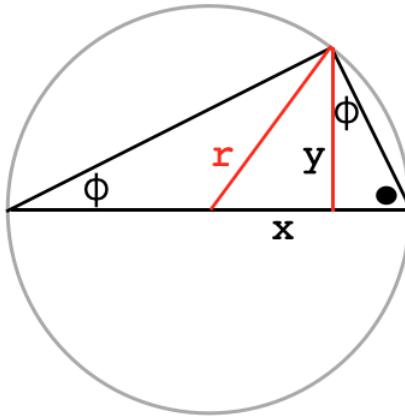
Another relationship from similar triangles is long side to short side:

$$\begin{aligned} \frac{h}{x} &= \frac{y}{h} \\ h^2 &= xy \\ h &= \sqrt{xy} \end{aligned}$$

$h$  is the *geometric mean* of  $x$  and  $y$ .

## proof without words

This one is from Nelsen's *Proof without words*. It depends on Thales' theorem, and also on similarity in right triangles.



The horizontal line is a diagonal of the circle, and the line labeled  $y$  is vertical.

*Proof.*

The two angles labeled  $\phi$  are equal because they are complementary to the angle marked with a black dot.  $r$  is the radius. The larger triangle has sides  $y$  and  $r + x$ , while the other has sides  $r - x$  and  $y$ . By similar triangles:

$$\frac{y}{r+x} = \frac{r-x}{y}$$

$$y^2 = r^2 - x^2$$

$$x^2 + y^2 = r^2$$

□

## converse of Pythagorean theorem

In reasoning deductively, we move from the premise or premises (collection of facts, data given, previous theorems that were proved), and use logic to reach a conclusion. The question arises whether, if we know only that the conclusion is true, does it follow logically that the premises are true? This is the problem of the *converse* of a theorem.

It may be so, or it may not.

We can state the converse of the Pythagorean theorem as follows: suppose we have triangle such that  $a^2 + b^2 = c^2$ . Does it follow that the angle between  $a$  and  $b$  is a right angle?

We profess to not know whether it is or is not a right angle.

*Proof.*

So then, draw another triangle with sides  $a$  and  $b$  that *is* a right triangle. Then, by the forward theorem,  $a^2 + b^2 = c^2$ .

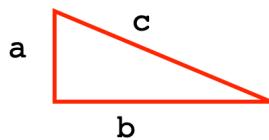
But the second triangle is congruent to the first one by SSS, each side is the same. Since they are equivalent parts of congruent triangles, the angle between  $a$  and  $b$  is a right angle.

□

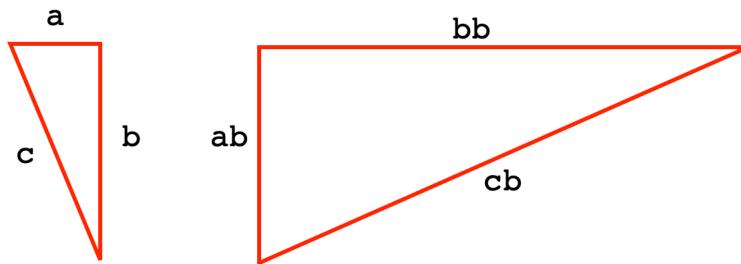
### scaled triangle proof of Pythagoras

*Proof.*

Draw a right triangle and label sides  $a, b$  and  $c$ .



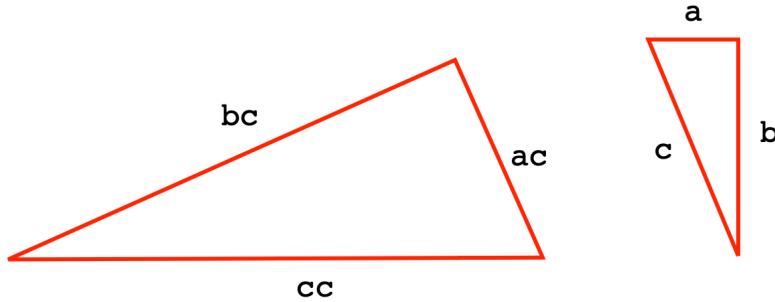
Now, flip and rotate the triangle. Make a copy of that one and rotate the copy so that the shortest side is to the left. Enlarge the copy until the two adjacent sides are equal in length.



These are similar triangles so the angles are all equal, but the sides are proportional, multiplied by a common factor, which here is  $b$ .

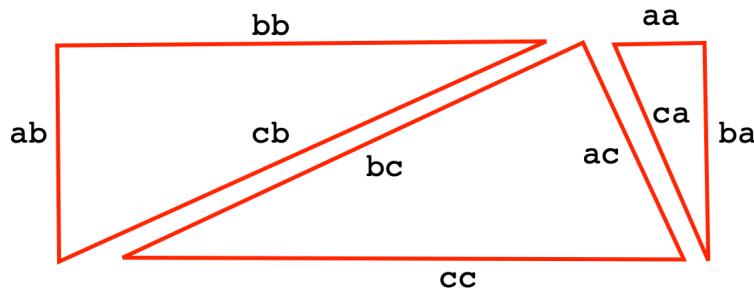
The adjacent sides show that  $ab = b$ , so this choice of scaling defines  $a = 1$ . And since  $a = 1$ , we can also view each side of the original triangle as being scaled by a factor of  $a$ .

We just tried scaling by a factor of  $b$  and then by  $a$ , so that naturally suggests we try scaling by a factor of  $c$ .



All sides of the enlarged triangle have been multiplied by the same factor, namely,  $c$ . Notice that  $c = ac$ , as required.

We're nearly done. Put the three triangles together.



The lower left and right corners are places where two vertices come together. By complementary angles, these sum to be right angles.

Since there are two other right angles at the vertices of the quadrilateral, it is a rectangle. Therefore the place where three vertices come together is a straight line, which we can also verify because there are two complementary angles plus a right angle.

Since the composite figure is a rectangle, opposing sides are equal, so  $a^2 + b^2 = c^2$ .

□

## **problem**

The Russian mathematician V.I. Arnold wrote a famous small book of “problems for children from 5 to 15.” Here is no. 6:

The hypotenuse of a right-angled triangle (in a standard American examination) is 10 inches, the altitude dropped onto it is 6 inches. Find the area of the triangle.

American school students had been coping successfully with this problem over a decade. But then Russian school students arrived from Moscow, and none of them was able to solve it as had their American peers (giving 30 square inches as the answer). Why?

We leave this one as a challenge.

*Hint 1:* it's a joke.

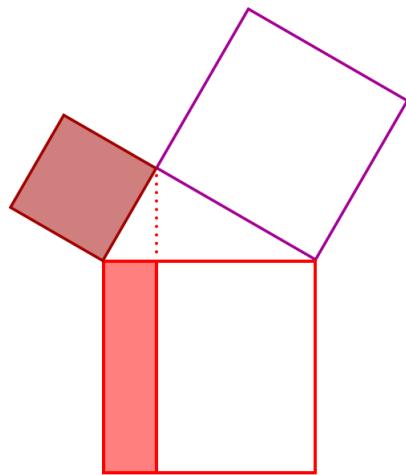
*Hint 2:* For two sides of a fixed total length, the isosceles right triangle has the largest area (why?). Given the hypotenuse, what are the side lengths?

Now use this fact to find the maximum area in terms of the sides, and compare that with the data we are given.

# Chapter 16

## Euclid's proof

My favorite proof of the Pythagorean theorem relies on the construction below, sometimes called the “bridal chair” or the “windmill”, where the central triangle is a right triangle, and the other figures are squares (Euclid I.47).

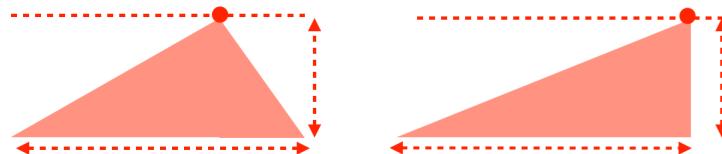


We have the squares on three sides of a right triangle.

What we will show is that the rectangular area which is part of the large square, in red, is equal in area to the entire small square, in maroon.

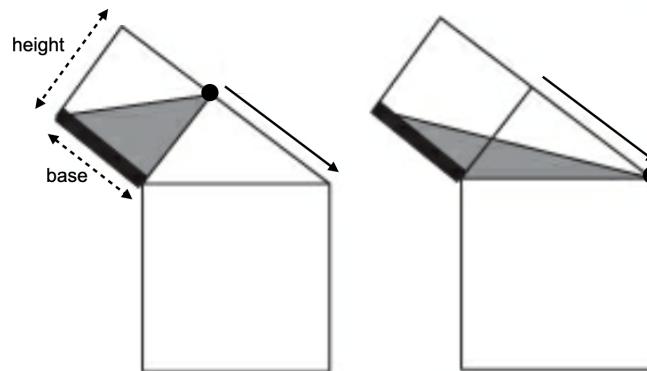
## preliminary

We begin by restating a fundamental idea about the area of triangles, which is that, if two triangles have the same base and the same height, they have the same area. So if we imagine sliding the top vertex of a triangle along a line parallel to the base, the area will not change.



*Proof.*

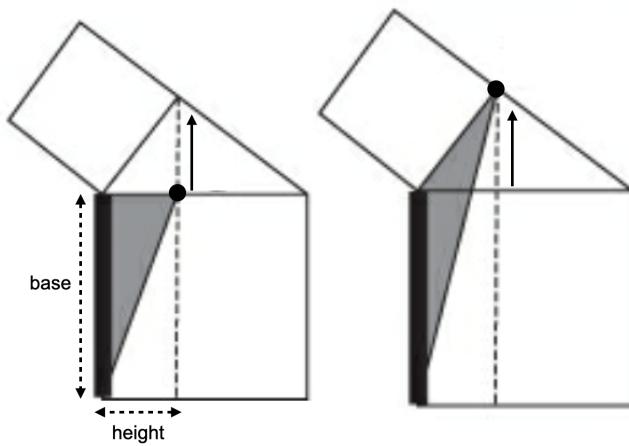
The next figure shows this principle as it comes up in the proof. The gray triangle with the black base is one-half the area of the small square.



Slide the vertex down to the right and the resulting triangle will still have the same area.

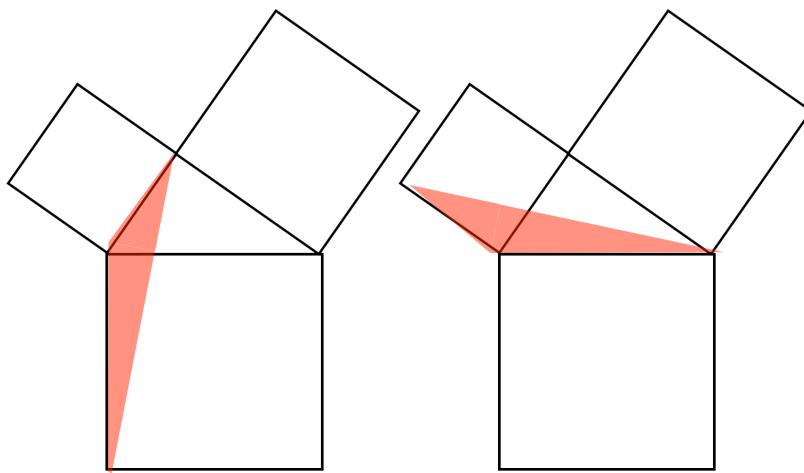
We can do the same thing with a triangle in the large square, below.

The gray triangle has half the area of the part of the large square that is to the left of the dotted line, because its base is equal to the side of the square and the height extends to the right to the dotted line.



Now slide the vertex up. The area is unchanged.

And now, finally, we observe that the two triangles have exactly the same shape. Just rotate one to obtain the other.



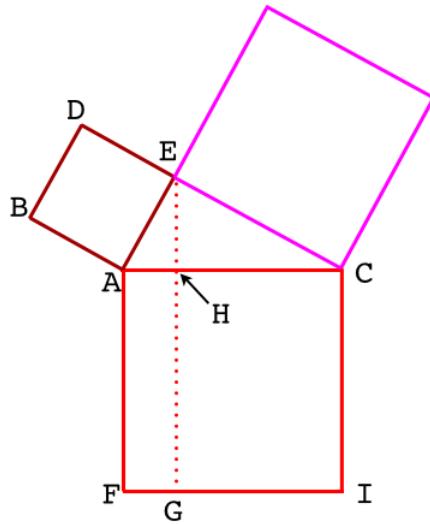
This completes our informal proof.

□

The formal approach follows, based on triangle congruence.

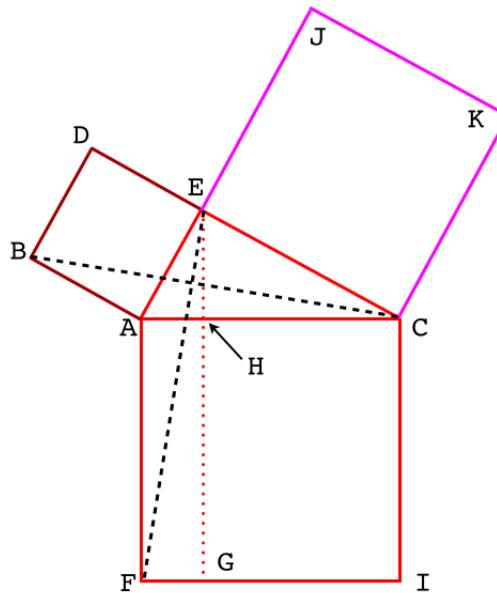
## main

We label some points as shown:



First, drop a vertical line  $EHG$ , constructing the rectangle  $AFGH$ .

Finally, sketch dotted lines for the long sides of two triangles:



*Proof.*

The crucial point is this: we will show that triangle  $\Delta ABC$  is congruent to triangle  $\Delta AEF$ .

Use side-angle-side (SAS). The two sets of sides are evidently equal

$$AB = AE$$

$$AF = AC$$

because, for example  $AB$  and  $AE$  are given as the sides of one square, and  $AC$  and  $AF$  as sides of another.

What about the included angle? The angles  $\angle BAC$  and  $\angle EAF$  each contain a right angle plus the shared angle  $\angle EAC$ .

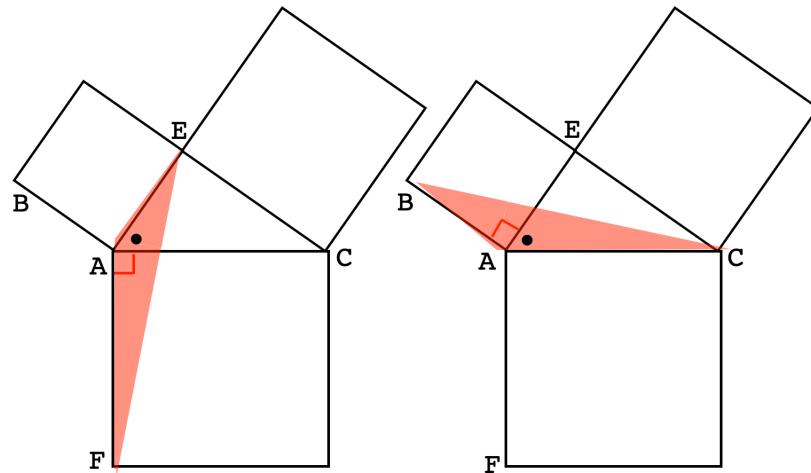
$$\angle BAC = \text{right angle} + \angle EAC$$

$$\angle EAF = \angle EAC + \text{right angle}$$

$$\angle BAC = \angle EAF$$

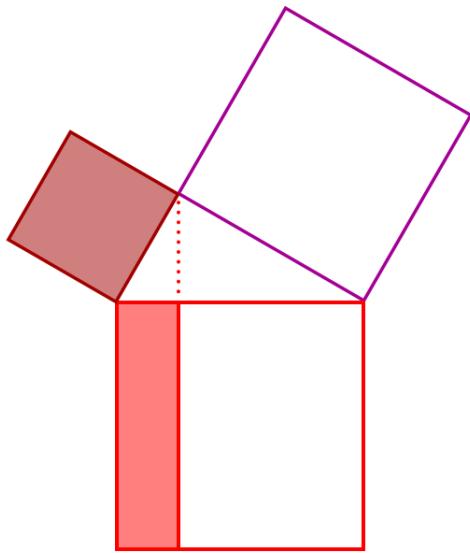
So they are themselves equal, and thus we have proved SAS and thus the congruence relationship:

$$\triangle ABC = \triangle AEF$$



The next part of the proof is to move the vertex of triangle  $\triangle ABC$  to the left and see that it has base  $AB$  and altitude  $AE$  so its area is one-half that of the small square  $ABDE$ . On the other hand triangle  $\triangle AEF$  has base  $AF$  and altitude  $AH$  so its area is one-half that of the rectangle  $AFGH$ .

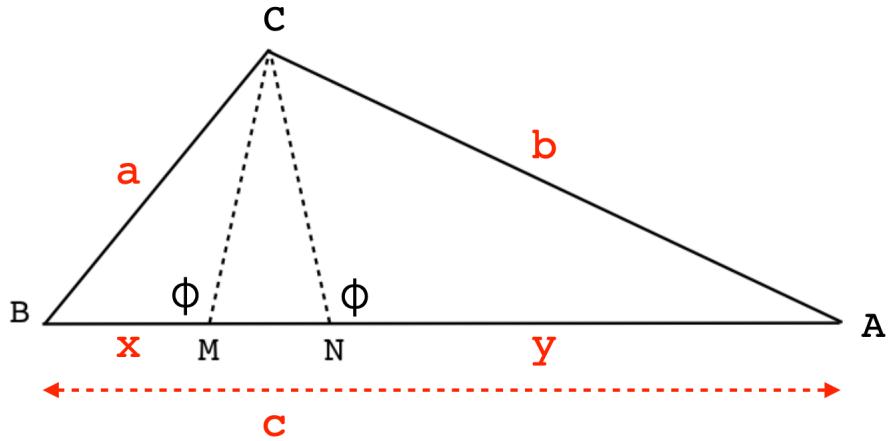
Hence we have proved that the two colored areas in this figure are equal:



Finally, we could proceed to do the same thing on the right side of the figure, but we just appeal to symmetry. All the equivalent relationships will hold.

□

### Quorra's corollary



Let  $\triangle ABC$  be *any* triangle (here it is obtuse). Draw  $CM$  and  $CN$  so that the new angles  $\angle CMB$  and  $\angle CNA$  (labeled  $\phi$ ), are equal to  $\angle C$ . The corresponding triangles are similar to the original, because they share  $\phi$  plus one other from the

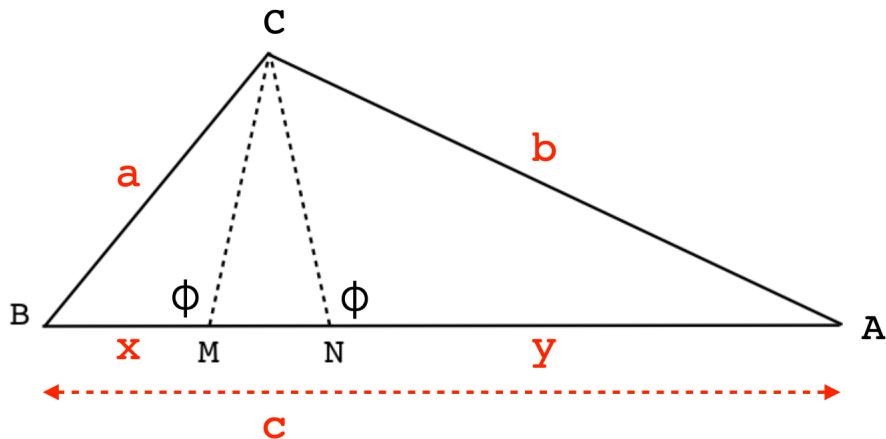
original triangle.

We use single letters for the sides to make the algebra simpler.  $a$  is opposite both  $\angle A$  and  $\angle CMB$ , while  $b$  is opposite both  $\angle B$  and  $\angle CNA$ , and  $c$  is opposite  $\angle C$ .

The shortest side in  $\triangle CMB$  is  $x$  and the longest is  $a$ , while in  $\triangle ABC$  the corresponding sides are  $a$  and  $c$ . So by equal ratios of sides in similar triangles we have that  $x/a = a/c$ . The middle side in  $\triangle CNA$  is  $y$  and by similar logic we have that  $y/b = b/c$ .

$$\begin{aligned} a^2 &= cx, & b^2 &= cy \\ a^2 + b^2 &= c(x + y) \end{aligned}$$

The sum of the squares of the two short sides of a triangle is equal to the product of the third side with the the sum of the two components  $x + y$ , when they are drawn with the angle  $\phi$  as specified.



This is actually a generalization of our original algebraic proof of the Pythagorean theorem.

In the case where the angle at vertex  $C$  is a right angle, then  $M$  coincides with  $N$ , because there is only one vertical to a line from a given point. So then and  $x+y = c$ , and this reduces to the Pythagorean theorem.

There are a large number of proofs of the Pythagorean theorem. Many of them are collected or linked here:

<https://www.cut-the-knot.org/pythagoras/>

# Chapter 17

## Pappus's proof

### Pappus's parallelogram theorem

Pappus came up with a beautiful theorem which includes the Pythagorean theorem as an extension. First, we need a simple lemma about parallelograms.

*Lemma.*

Given two parallel lines:  $AD \parallel EBFC$ .

If two parallelograms  $AEFD$  and  $ABCD$  have opposite sides on the two lines, and those two segments are of equal length, then they have equal areas.



Here is a special case where they have the same base  $AD$ .

We proved earlier that the perpendicular between two parallel lines is of equal length no matter where it is drawn.

So the two parallelograms are each composed of pairs of triangles, which, having the same base and the same altitude, also have equal area.

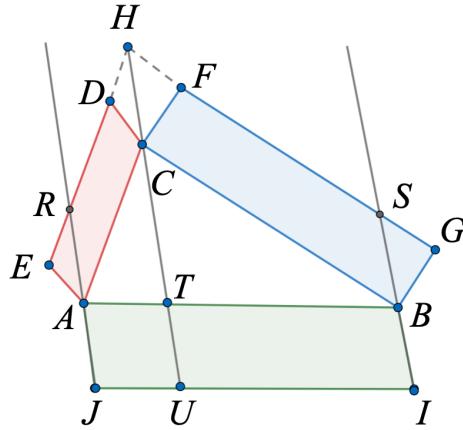
One might also just invoke Euclid I.35.

□

In the figure below, on the two sides of any  $\triangle ABC$  draw any parallelograms  $ACDE$  and  $BCFG$ . Extend the two new sides to meet at  $H$ .

Draw  $HC$  and its extension such that it cuts  $AB$  at  $T$  and then let  $HC = TU$ .

Draw  $AJ \parallel HCTU \parallel BI$ .



*Proof.*

The new parallelogram with  $AC$  as one side and  $RH$  the other, is equal, by our lemma.  $(ACDE) = (ACHR)$ .

Since  $RAJ \parallel HCTU$  and  $RA = HC = TU = AJ$ ,  $(ATUJ) = (ACHR)$  for the same reason.

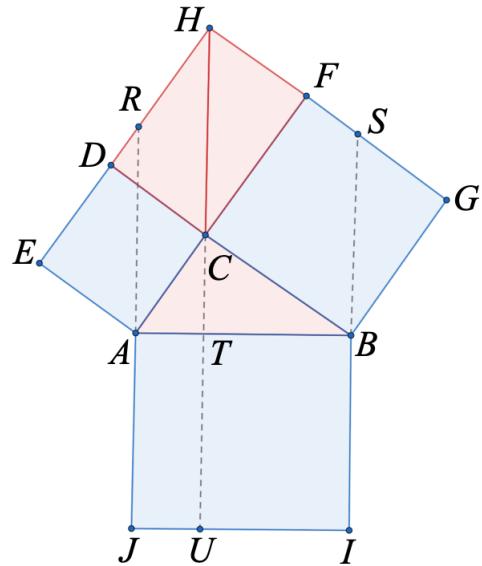
So  $(ATUJ) = (ACDE)$ .

We use the same argument for  $(BCFG) = (TBIU)$  on the right.

Then add the two results:  $(ACDE) + (BCFG) = (ABIJ)$ .

□.

Let  $\angle ACB$  be a right angle, and let the parallelograms be squares.



$\triangle DHC$  and  $\triangle FHC$  are right triangles and they are congruent to  $\triangle ABC$  by SAS.

So  $\angle CHF = \angle CAT$  and we also have vertical angles, so  $\triangle HFC \sim \triangle CTA$ .

It follows that  $\angle HFC = \angle CTA$  and both are right angles.

Further,  $CH = TU = AB$ , so  $ABIJ$  is the square on  $AB$ .

It is easy to see that  $(AEDC) = (ARHC) = (ATUJ)$ , and the rest follows.

Thus Pappus' theorem becomes the Pythagorean theorem as a special case.

[ Reference: George F. Simmons, *Calculus Gems*. ]

# Chapter 18

## Brief review

Let us summarize what we know so far about the basic theorems of geometry.

They are almost all that you should need to attack any of the other problems in the book, and they'll be used repeatedly.

The first two aren't technically theorems but fundamental assumptions that we make about the geometrical world.

- **supplementary angles**
- **alternate interior angles (parallel postulate)**

The next three easily follow from those initial ideas.

- **vertical angles**
- **triangle sum of angles**
- **complementary angles**

We have two basic methods for proving that two triangles are congruent

- **SAS for congruence**
- **ASA for congruence**

and one specifically for right triangles

- **hypotenuse leg in a right triangle (HL)**

(We proved that SSS is equivalent to SAS, and AAS is equivalent to ASA).

- o **SSS implies SAS**

Next we have the powerful fundamental theorems of geometry:

- o **isosceles triangle theorem** (sides  $\rightarrow$  angles) and **converse**
- o **external angle theorem**
- o **Thales' circle theorem** (right angle in a semi-circle)
- o **area ratio theorem**
- o **similar triangles** (similar right  $\triangle$ s, same ratio of sides)
- o **general similarity theorem**
- o **Pythagorean theorem** (similar triangles)

Some people might add a few more. But this is a reasonable number to start with. You should have these instantly available (and it's nice to know how to prove them, as well).

Looking forward, probably the most important we have yet to do is

- o **inscribed angle theorem** (on a circle is one-half central angle)

which has the consequence that inscribed angles on the same arc are all equal.

## **Part V**

### **Circles**

# Chapter 19

## Circles and angles

### diameters

Pick some point to be the *center* of a circle. Then a circle contains all the points a specified distance away from the center. That distance is called the *radius*.

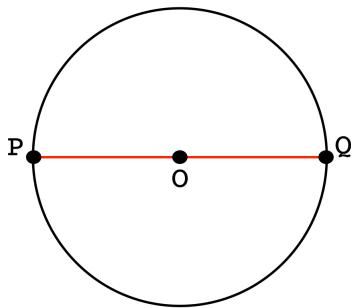
A circle is drawn with a compass, as we mentioned previously.



Euclid says:

- Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

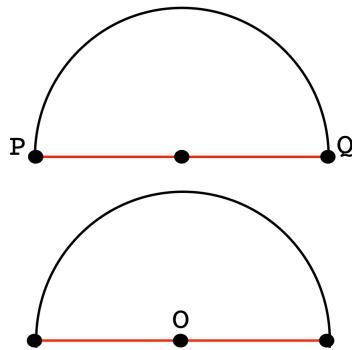
Suppose we start with the line segment  $OP$  in the figure below, and choose  $O$  as the center. Place the needle point of the compass on  $O$  and the pencil on  $P$  and then draw the circle containing all the points the same distance from  $O$  as  $P$  is.



Next, extend the radius  $PO$  to meet the circle on the other side at  $Q$ . This whole line segment  $PQ$  is called a *diameter* of the circle.

- Any diameter divides the circle into two equal parts.

(Recall that we allow a mirror image of something to be called equal to itself). So then, if we take the bottom half and flip it vertically, we claim the two halves are exactly equal.



*Proof.*

The proof is by contradiction. Lay the two pieces on top of one another. The diameters of the half-circle are duplicates of the original. Thus, they are equal to each other and each half is equal to one radius.

We suppose that, somewhere, the two half-circles are not identical.

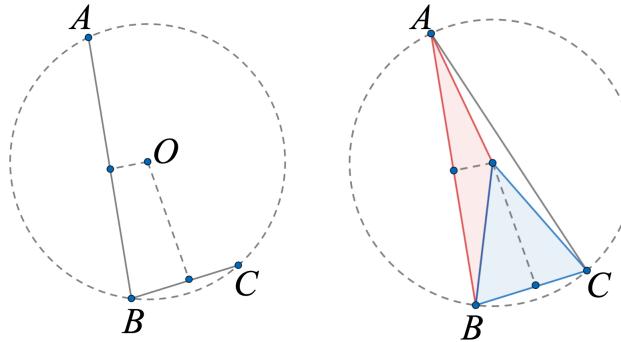
Then there will be some radial line we can draw from  $O$  through the two half circles, where the line meets the two curves at different distances from the center, because they are supposed to be different. This will identify points on the same radius of each half-circle that lie at a different distance from the center.

But then, the original figure could not be a circle, because all radii of a circle are equal.

This is a contradiction. Hence any diameter divides the circle into two equal parts, called semi-circles.

□

### circle containing three points



Suppose we have three arbitrary points in the plane:  $A$ ,  $B$  and  $C$ .

The claim is we can draw a circle that contains all three points on its circumference.

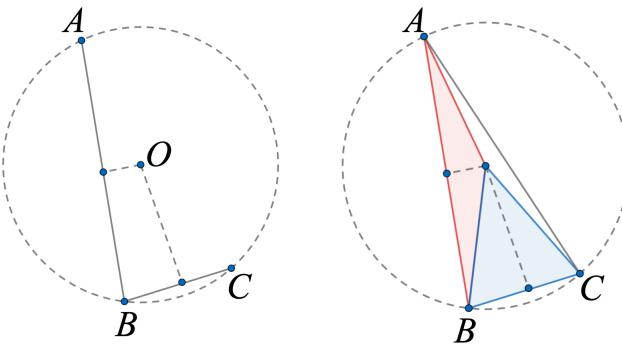
*Proof.*

Pick two pairs of points and draw, say,  $AB$  and  $BC$ .

Find the perpendicular bisector of each. We know that every point on the perpendicular bisector of  $AB$  is equidistant from both  $A$  and  $B$ , while for the latter case  $AC$  every point is equidistant from  $B$  and  $C$ .

If  $O$  is the point where the bisectors meet, that point is such that  $OA = OB = OC$ .

So if we draw the circle on center  $O$  with radius  $OA$  it contains all three points.



□

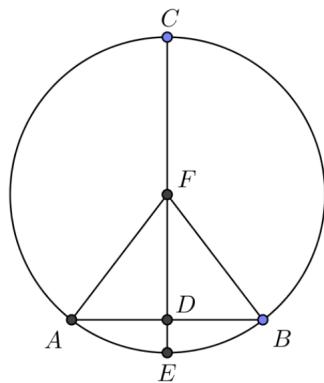
The question arises whether there is a general rule about  $\triangle ABC$  and there is: if the origin of the circle lies on one of the sides of  $\triangle ABC$  then it is a right triangle, that side is a diameter, the center of the circle bisects the side. This depends on the converse of Thales' circle theorem.

If  $\triangle ABC$  contains an angle greater than a right angle, then the center of the circumcircle is not inside the triangle, while if it is acute, the center lies inside the triangle.

We will not prove this now. Euclid's proof for all three cases is III.31.

### **Euclid III.1**

This is a simple method from Euclid to find the center of a circle.



*Proof.*

To find the center of any circle, take two points on the circle and draw the chord connecting them, then erect the perpendicular bisector of the chord. In the diagram above,  $CE \perp AB$  and bisects it.

We showed previously that there does not exist any point which is equidistant from  $A$  and  $B$  and is *not* on this perpendicular bisector. Since the center of the circle  $F$  has the property  $AF = BF$ , it must lie somewhere on  $CE$ .

Thus,  $CE$  is a diameter of the circle. Bisect it to find  $F$ , the center.

□

## internal and external points

As a preliminary to the next section, we claim that

- A line through any internal point of a circle can be extended to intersect the circle at two and only two different points.

*Proof.*

Assume there is a straight line segment that intersects a circle at more than two points, say at three points. By the definition of a circle, those three points are equidistant from the center.

Of the three points, one lies between the other two (going the shortest way around the circle). Use the middle point and (one at a time) the other two points to construct two perpendicular bisectors of the two segments. This is the classic way to find the center of a circle from points on the periphery.

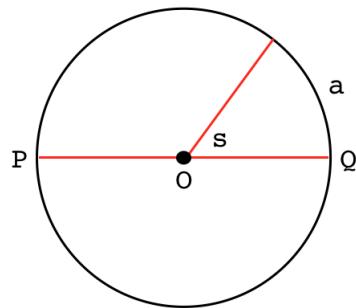
But then, we would have two bisectors that both run through the center. Yet they are parallel to each other, because they have been constructed perpendicular to the same line.

This is a contradiction. There cannot be more than two points on intersection of a line with a circle.

□

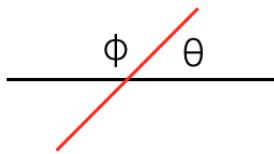
If there is no specification of an internal point, then it is possible for a line to intersect with the circle at only one point, namely a tangent line. We deal with tangents separately in a later chapter.

## radian measure



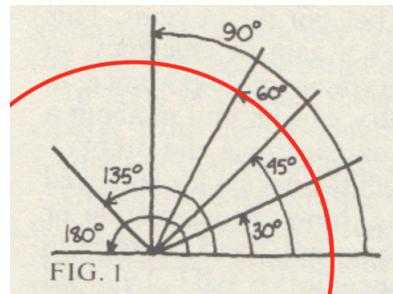
In the figure above, the angle  $s$  is *defined* as the length of arc  $a$  that it sweeps out in a unit circle.

We talked briefly early in the book about the *measure* of an angle. It seems intuitively obvious that, in this figure,  $\theta < \phi$ .



The question is how to quantify this notion.

Effectively what we'll do is imagine that we draw a circle with radius 1, a *unit circle*, which contains the angle as a central angle.



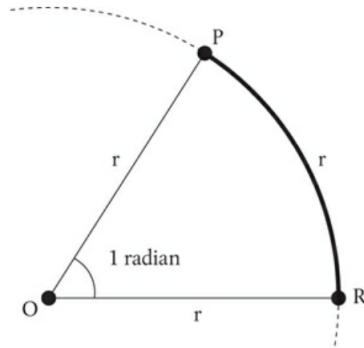
Then, the measure of the angle is the distance traveled around the circle counter-clockwise from the horizontal on the right.

Although this drawing shows angles in degrees, in calculus and analytical geometry angles are defined in terms of radians of arc. For a unit circle with radius = 1, the total circumference is  $2\pi$ .

If  $s$  is the measure in radians and  $D^\circ$  the measure in degrees, then

$$\frac{s}{2\pi} = \frac{D}{360}$$

It seems natural then to adopt the arc length as a measure of the angle, where  $360^\circ$  is equal to  $2\pi$  radians, and an angle of  $90^\circ$ , a right angle, is equal to  $\pi/2$  radians.



72. Definition of a radian.

Divide 360 by  $2\pi$  to find that one radian is approximately  $57^\circ$ .

To convert some more measures of angles in degrees to radians:

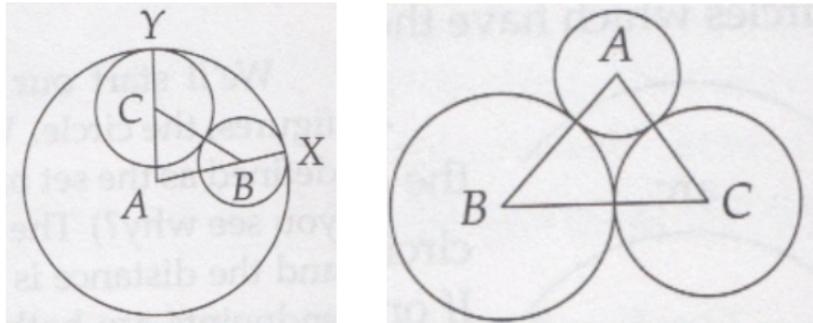
$$180^\circ = \pi, \quad 90^\circ = \frac{\pi}{2}$$

$$60^\circ = \frac{\pi}{3}, \quad 45^\circ = \frac{\pi}{4}, \quad 30^\circ = \frac{\pi}{6}$$

Central angle and the arc that subtends that angle are numerically equal, but remember that they are dimensionally different. An arc is a length, an angle is just an angle and not a length.

## two problems

Here are two problems from *the Art of Problem Solving*.



For the first (left panel), we are given that  $A$ ,  $B$  and  $C$  are the centers of the respective circles, that the points which appear tangent are actually so, and also that  $AB = 6$ ,  $AC = 5$  and  $BC = 9$ .

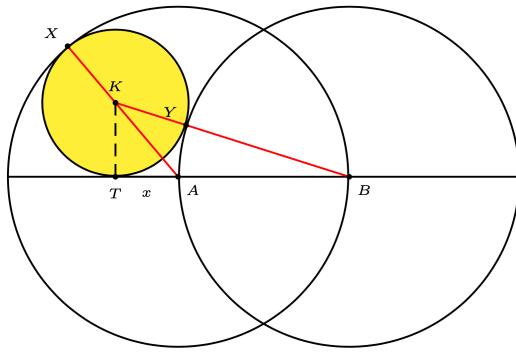
What is  $AX$ , the radius of the large circle?

For the right panel, we are asked to express the radius of circle  $A$  in terms of the sides of  $\triangle ABC$ .

*Hint.* Both problems ask you to exercise your skills in solving simultaneous equations.

## problem

Here is a problem from Paul Yiu. It is really more about the Pythagorean theorem with an equilateral triangle thrown in but I put it here.



We have two larger circles of equal size drawn with  $AB = a$ , the radius, plus another smaller circle drawn tangent to both and to their diameter as shown.

The line through both centers of tangent circles goes through the point of tangency,

so  $KB = r + a$ . Since  $A$  is the center of one circle,  $AX$  is equal to  $a$ , so  $KA = a - r$ . For the two right triangles we have

$$(a+x)^2 + r^2 = (a+r)^2$$

$$x^2 + r^2 = (a-r)^2$$

After cancelations, these (separately) give

$$2ax + x^2 = 2ar$$

$$x^2 = a^2 - 2ar$$

Adding

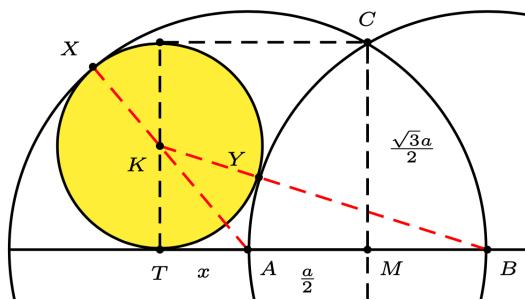
$$x^2 + ax - \frac{a^2}{2} = 0$$

$$x = \frac{-a \pm \sqrt{a^2 + 2a^2}}{2}$$

We take the positive branch for a length

$$x + \frac{a}{2} = \frac{\sqrt{3}}{2}a$$

Now, if  $M$  is the midpoint of  $AB$ , then  $MT = x + a/2 = \sqrt{3}a/2$ .



The classic **construction** of an equilateral triangle on  $AB$  would put its top vertex at the intersection of the two circles, with side length  $a$  and **altitude**  $\sqrt{3}a/2$ , forming a square.

It is also claimed that  $2r = x + a/2$ , which means that the other vertex of the square is also the top of the small circle. This may not be obvious but recall that we added our two equations above. Subtract!

$$2ax = 4ar - a^2$$

$$x = 2r - \frac{a}{2}$$

$$x + \frac{a}{2} = 2r$$

□

# Chapter 20

## Inscribed angles

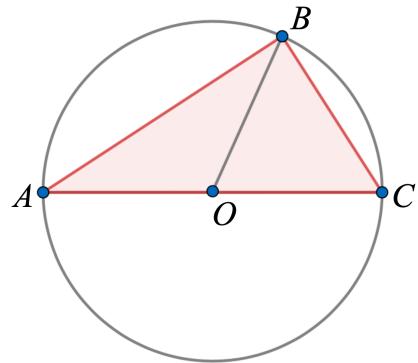
### Thales' circle theorem

In this chapter, we will introduce the inscribed angle theorem. Let's start by revisiting Thales' circle theorem:

- Any angle inscribed in a semicircle is a right angle.

Take a diameter of the circle and any third, distinct point. The three points on the circumference of the circle form a triangle, and the angle at the third vertex is always a right angle.

In the figure below,  $\angle ABC$  is a right angle.



*Proof.*

By I.32, the sum of angles in a triangle is equal to two right angles. So

$$\angle OAB + \angle OBA + \angle OBC + \angle OCB = 180$$

But  $\triangle OAB$  and  $\triangle OBC$  are both isosceles, so the base angles are equal, with

$$\angle OAB = \angle OBA \quad \angle OBC = \angle OCB$$

It follows that

$$2\angle OBA + 2\angle OBC = 180$$

$$\angle OBA + \angle OBC = 90$$

□

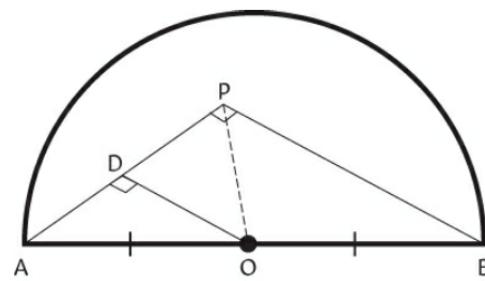
According to Boyer, this result was known to the Egyptians 1000 years before Thales. But it is yet another example of knowing a result before proving that it is so. Archimedes had something to say about the importance of having discovered a fact, before finding a way to prove it.

The converse of Thales' theorem is also true. If the third point of a triangle contains a right angle, then it must lie on the circle where the other two points form the diameter.

### Thales' circle theorem converse

A nice direct proof of this is given in Acheson.

*Proof.*



**Fig. 59** An alternative method.

We are given that  $\angle APB$  is a right angle.

Draw  $OD$  parallel to  $PB$ .  $\triangle AOD$  is similar to  $\triangle ABP$  because they are both right triangles with a shared vertex at  $A$ .

Since  $AO$  is one-half  $AB$ , the scale factor is  $1/2$ . In particular,  $AD = DP$ .

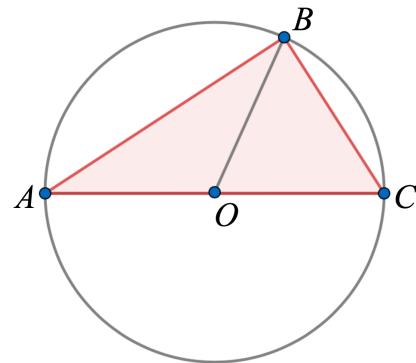
Now draw  $OP$ . The two smaller triangles  $\triangle AOD$  and  $\triangle DOP$  are congruent by SAS. Therefore,  $OP = OA$ .

But  $OA$  is a radius of the circle. Therefore,  $OP$  is also a radius.

It follows that  $P$  must lie on the circle.

□

We can use Thales' theorem to introduce the inscribed angle theorem.



## inscribed angle theorem

An inscribed angle is an angle whose vertex lies on the circle, such as  $\angle BAC$

- An inscribed angle is one-half the corresponding central angle lying on the same arc,  $\angle BOC$ . The central angle is twice the corresponding inscribed angle.

*Proof.*

Euclid's proof uses the external angle theorem.  $\angle BOC$  is the external angle to  $\triangle OAB$ .

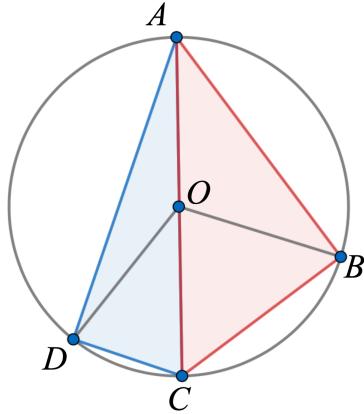
Thus, it is equal to the sum of  $\angle OAB + \angle OBA$ .

The triangle is isosceles, so these two angles are equal.

It follows that  $\angle BOC = 2\angle BAC$ .

□

This proof is short and sweet, but limited by the fact that we used the diameter for one of the arms of the inscribed angle. Here is a more general proof.



The claim is that  $\angle BOD = 2\angle BAD$ .

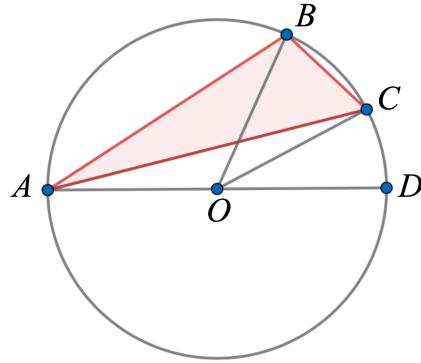
*Proof.*

Just add the results obtained using the previous proof.  $\angle DOC = 2\angle DAC$  and  $\angle BOC = 2\angle BAC$ .

$$\begin{aligned}\angle BOD &= \angle BOC + \angle DOC \\ &= 2\angle BAC + 2\angle DAC = 2\angle BAD\end{aligned}$$

□

The proof is *still* limited, since the angle we looked at includes the center of the circle. There are two ways to fix this. The first is subtraction.



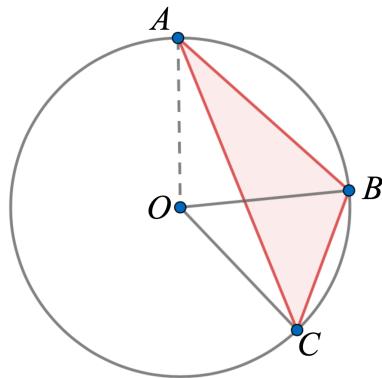
*Proof.*

By the first proof,  $\angle BOD = 2\angle BAD$  and  $\angle COD = 2\angle CAD$ . The angle of interest,  $\angle BOC$ , is the difference:

$$\begin{aligned}\angle BOC &= \angle BOD - \angle COD \\ &= 2\angle BAD - 2\angle CAD = 2\angle BAC\end{aligned}$$

□

A somewhat more elegant proof of the second case is as follows.



*Proof.*

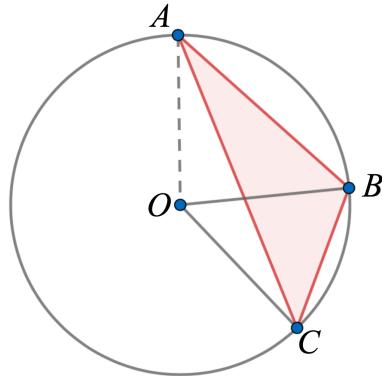
The total angle at A is  $\angle OAB = \angle OAC + \angle BAC$ . Since  $\triangle OAB$  is isosceles, the base angles are equal.

With the addition of the central angle  $\angle AOB$ , by sum of angles we get two right angles for  $\triangle OAB$ .

$$2\angle OAC + 2\angle BAC + \angle AOB = 180$$

In  $\triangle OAC$ , again isosceles, we have by sum of angles

$$\angle AOB + \angle BOC + 2\angle OAC = 180$$



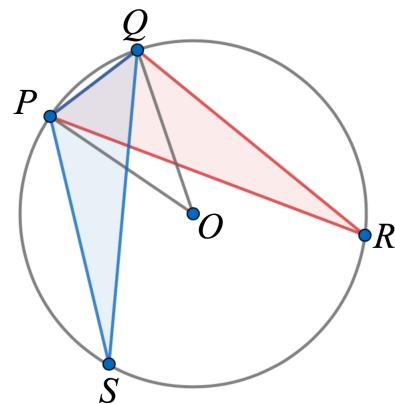
Subtracting

$$2\angle BAC - \angle BOC = 0$$

$$\angle BOC = 2\angle BAC$$

□

**angles on the same arc**



- Angles that lie on the same arc or are subtended by the same chord in the same circle, are equal.

*Proof.*

This is an immediate consequence of the previous theorem, since two such angles are equal to one-half of the *same* central angle.  $\angle POQ$  is the central angle for arc  $PQ$  and hence is twice both  $\angle PRQ$  and  $\angle PSQ$ . Thus,  $\angle PRQ = \angle PSQ$ .

□

This theorem is Euclid III.21. It is such an immediate consequence of the **inscribed angles theorem** (Euclid III.20) that they are sometimes thought the same.

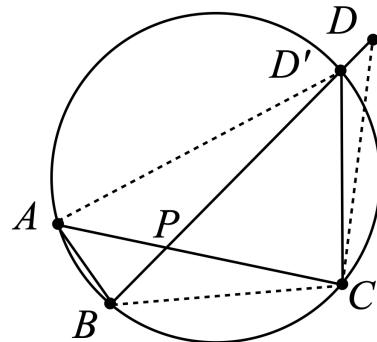
Possibly I have mis-stated this elsewhere in the book. Theorem III.21 will appear often in the pages to come.

### converse

There are several variant arguments from similar setups which are converses of the inscribed angles theorem. Here is one:

Let  $\triangle ABC$  lie on a circle.

Let point  $D$  be such that  $\angle BDC = \angle BAC$ .



Then  $D$  is also on the same circle.

*Proof.*

Aiming for a contradiction, suppose otherwise.

Let  $D$  be external and also let  $\angle BDC = \angle BAC$ .

Find where  $BD$  cuts the circle at  $D'$ .

So  $D'$  is on the circle and  $\angle BD'C$  is subtended by  $BC$ .

By the forward version of the inscribed angle theorem:  $\angle BD'C = \angle BAC$ .

It follows that  $\angle BDC = \angle BD'C$ .

But  $\angle BD'C$  is external to  $\triangle CDD'$ .

So  $\angle BD'C > \angle BDC$ .

This is a contradiction.

A similar argument will show that  $D$  cannot be internal to the circle.

Therefore, it must be that  $D$  is on the circle.

□

## geometric mean

We showed in the chapter on the **Pythagorean theorem** that the altitude of a right triangle is the geometric mean of the two components of the base.

$$h^2 = pq$$

$$h = \sqrt{pq}$$

According to wikipedia:

[https://en.wikipedia.org/wiki/Geometric\\_mean](https://en.wikipedia.org/wiki/Geometric_mean)

The fundamental property of the geometric mean is that (letting  $m$  be the *geometric mean* here):

$$m \left[ \frac{x_i}{y_i} \right] = \frac{m(x_i)}{m(y_i)}$$

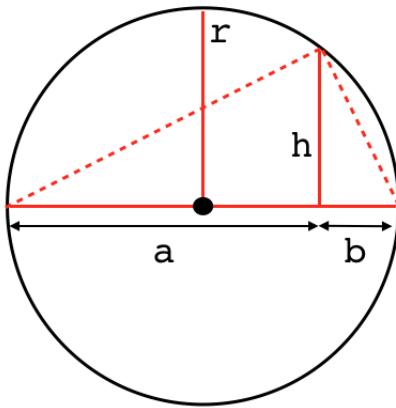
and one consequence is that

This makes the geometric mean the only correct mean when averaging normalized results; that is, results that are presented as ratios to reference values.

A number of examples are given in the article.

We discuss this here because originally, there was a proof-without-words that the geometric mean is always less than or equal to the arithmetic mean.

I decided to add some words.



A right triangle is inscribed in a semicircle.

Let the length of the long dotted line be  $c$ . Then

$$a^2 + h^2 = c^2$$

Let the length of the short dotted line be  $d$ . Then

$$b^2 + h^2 = d^2$$

Adding together

$$a^2 + b^2 + 2h^2 = c^2 + d^2$$

But a third application of the Pythagorean theorem shows that the right-hand side is just  $(a+b)^2$  so

$$a^2 + b^2 + 2h^2 = (a+b)^2$$

$$2h^2 = 2ab$$

$$h^2 = ab$$

□

An even simpler proof is to recognize that the two smaller triangles are similar with equal ratios of sides including:

$$\frac{h}{a} = \frac{b}{h}$$

and the result follows immediately.

This says that the square of the altitude  $h$  is equal to the product of chord segments (we will prove this geometrically in the next chapter as well).

$$h = \sqrt{ab}$$

But we also have that  $a + b = 2r$  and hence

$$r = \frac{a + b}{2}$$

Do you recognize these? The second expression is the arithmetic mean of  $a$  and  $b$ , while the first is the geometric mean.

The geometry shows that  $h \leq r$  so:

$$\sqrt{ab} \leq \frac{a + b}{2}$$

The geometric mean is always less than the arithmetic mean, except when  $a = b$ , where they are equal (or all of  $n$  values are equal).

## problem

Here is a problem I found on the web as a Youtube video:

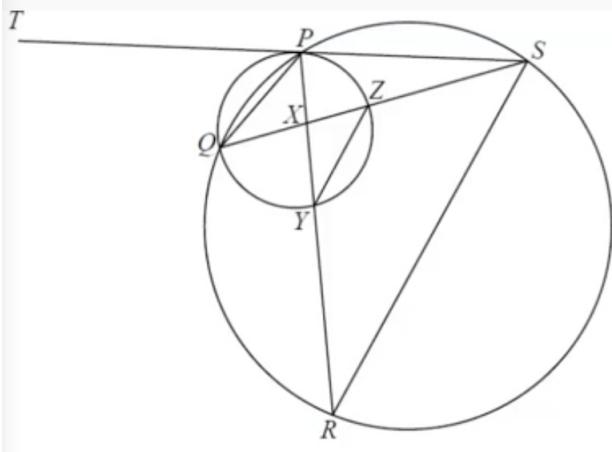
<https://youtu.be/2Jt8lynddQ8>

It is described as a GCE O-Level A-Maths Plane Geometry Question.

The relationships that seem obvious from the diagram are given. Namely,  $PXYR$  and  $QXZS$  are each lie on a straight line (colinear).

And the two circles each have the four points lying on them as shown.

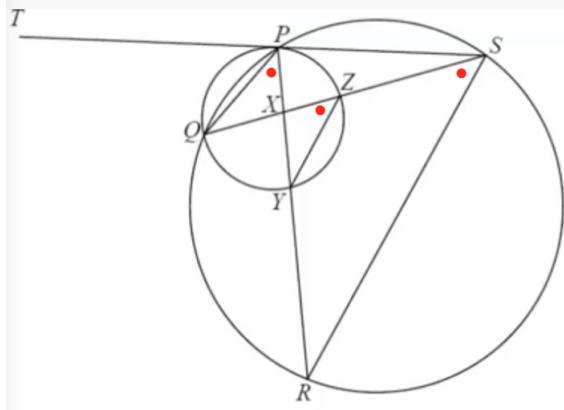
$TPS$  is tangent to the smaller circle at  $P$ .



The problem asks us to show that  $SR$  is parallel to  $ZY$  and hence, *deduce* that  $YX/ZX = YR/SZ$ .

The approach that first occurred to me was to use the similar triangles that arise from crossed chords. However, the problem asks us to start by showing that the given line segments are parallel. This is a hint to a different proof.

The result comes from the theorem which is the basis of this chapter: the inscribed angle theorem.



*Proof.*

The marked angles are all equal. The first two are equal because they correspond to the same arc in the small circle, and the third (at  $S$ ) is equal to the first because they both correspond to the same arc in the large circle.

Therefore, the two line segments are parallel by the converse of the alternate interior angles theorem.

That gives us similar triangles  $\triangle XYZ \sim \triangle XRS$  from which the equal ratios follow almost immediately (see below).

□

The last part of the problem says that given  $SQ = XR$ , prove that  $PS^2 = XS \cdot YR$ . We're not ready to do that yet. It uses the information about the tangent and the **secant-tangent theorem**.

*Proof.* (Alternate).

Here's the first part of the proof by the crossed chord theorem:

$$PX \cdot XY = QX \cdot XZ$$

$$PX \cdot XR = QX \cdot XS$$

It follows that

$$\frac{XZ}{XY} = \frac{XS}{XR}$$

We need some algebra to get to  $YX/ZX = YR/SZ$ . This is a standard parts and the whole manipulation for similar triangles, made complex by the cumbersome notation.

Let  $a = XY$ ,  $A = XR$ ,  $b = XZ$ , and  $B = XS$ . We have

$$\frac{b}{a} = \frac{B}{A}$$

and we want

$$\frac{a}{b} = \frac{A-a}{B-b}$$

The way to get there is:

$$\begin{aligned}\frac{A}{a} &= \frac{B}{b} \\ \frac{A}{a} - 1 &= \frac{B}{b} - 1\end{aligned}$$

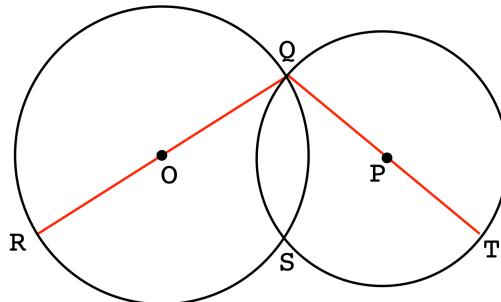
then

$$\frac{A-a}{a} = \frac{B-b}{b}$$

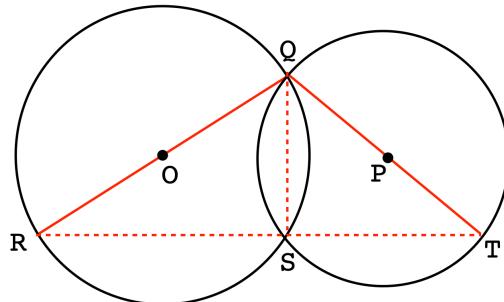
and the result follows easily in one more step.

□

### problem



Two circles meet at  $Q$  and  $S$ .  $QR$  and  $QT$  are diameters of the two circles. Prove that  $RST$  are collinear.



Since  $QR$  is a diameter of the circle centered at  $O$ ,  $\angle QSR$  is a right angle.

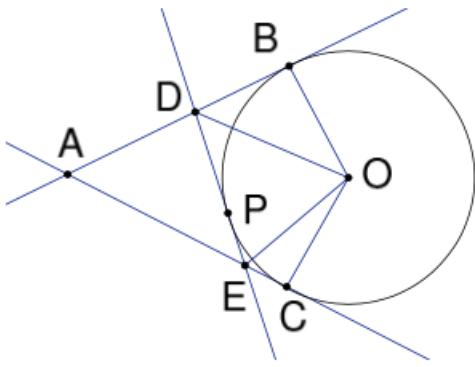
But so is  $\angle QST$ , since  $QT$  is a diameter of the second circle.

Hence the total angle at  $S$  is two right angles or a straight line. Therefore  $RS$  and  $ST$  together form a straight line segment.

### double arc problem

This problem is taken from an online collection by David Surowski.

<https://www.math.ksu.edu/~dbski/writings/further.pdf>



Given that  $AB$  and  $AC$  are tangents to the circle meeting at  $A$ . Given a second tangent  $DE$ , meeting the circle at  $P$ . Prove that the arc that subtends  $\angle BOC$  is twice that which subtends  $\angle DOE$ .

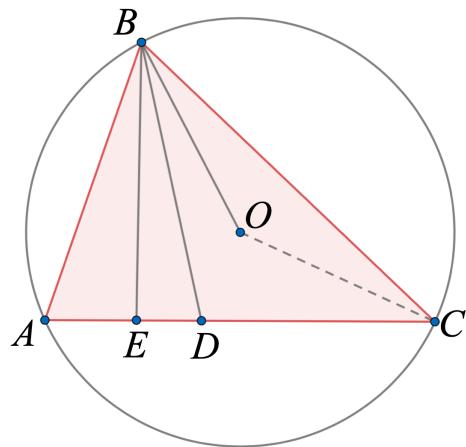
*Proof.*

Notice that  $DB$  and  $DP$  are tangents to the circle meeting at  $D$ . Therefore  $DB = DP$  and then  $\triangle BOD \cong \triangle DOP$ , so  $\angle BOD = \angle DOP$ .

The same argument applies to  $EC$  and  $EP$ . Therefore the inner arc is composed of two angles, while the outer arc has two copies of each of those angles.

□

### problem



Posamentier gives this problem.

Let  $O$  be the center of the circumcircle of  $\triangle ABC$ .

Let  $BE \perp AC$  and  $BD$  bisect  $\angle B$ .

Show that  $BD$  also bisects  $\angle OBE$ .

*Proof.*

Given  $\angle ABD = \angle CBD$ .

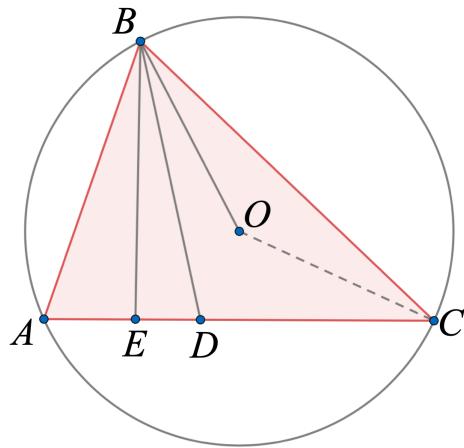
Draw  $OC$ .

The central  $\angle BOC$  is  $2 \angle BAC$ .

But  $\triangle BOC$  is isosceles with equal base angles.

Thus  $\angle BAC + \angle OBC = 90$ .

$\angle ABE$  is also complementary to  $\angle BAE$ , hence it is equal to  $\angle OBC$ .



Subtracting equals:  $\angle EBD = \angle OBD$ .

□

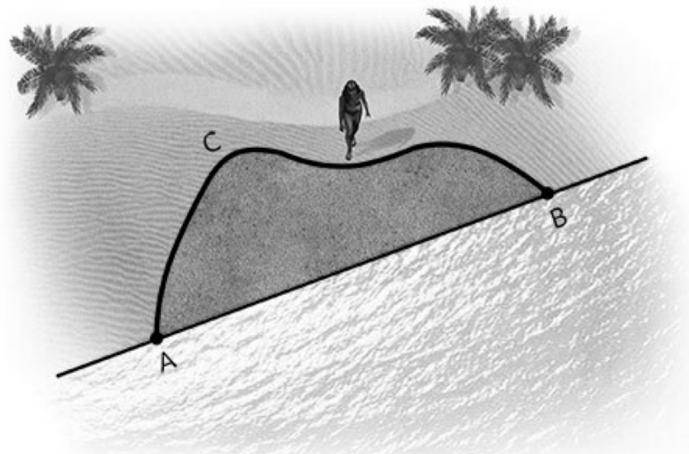
## Queen Dido

The mighty city of Carthage was the capital city of a major Phoenician colony. As Rome grew strong, there was a titanic struggle between the two peoples, which Carthage eventually lost. The ruins of Carthage lie near present-day Tunis.

Queen Dido was the legendary founder of the city of Carthage. She was supposedly

granted as much land as she could encompass with an oxhide. She promptly cut the ox-hide into very thin strips.

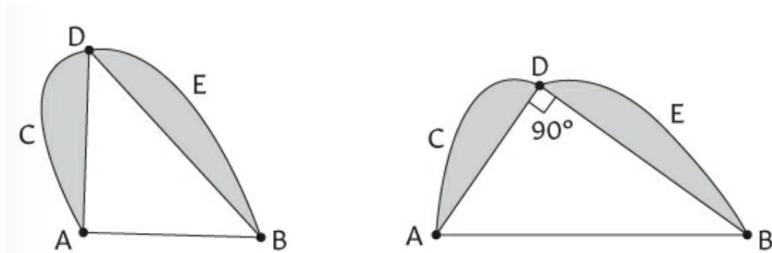
The problem then is to maximize the area enclosed by a curve of fixed length.



**Fig. 138** Dido's problem (with a straight coastline).

In calculus there are a number of problems like this. What's nice is that this problem has a wonderful solution that uses only the tools we have so far. In particular, we need the converse of Thales' circle theorem.

The argument goes as follows. Suppose we have a particular outline for the city limits and we're pretty happy with it. We suppose it is a maximum (left panel).



**Fig. 140** The heart of Simpson's argument.

Then we notice that by rearranging  $AD$  and  $BD$  so they meet at a right angle, the crescent-shaped areas are unchanged, but the area of  $\triangle ABD$  is a maximum. That's because a right triangle, having the two sides at right angles, has area equal to the product of the two sides (divided by 2). No other triangle with the same two sides has as much area.

So the arrangement on the right has a bigger area.

But then, with  $AB$  as the diameter of a circle, if  $\angle ADB$  is a right angle, it must lie on the circumference of that circle, by the converse of Thales' theorem.

And this is true regardless of the relative lengths of  $AD$  and  $BD$ . Therefore the maximum area is obtained when  $D$  traces out a semi-circle.

This example is in Acheson's Geometry.

# Chapter 21

## Tangents

**tangent: perpendicular → touches one point**

We show [here](#) that one can construct a line perpendicular to any given line, and passing through any point whether on the line or not.

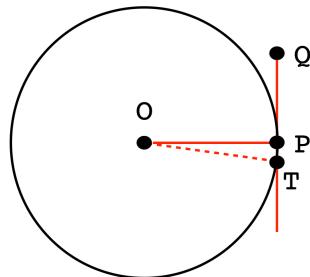
If we make the construction perpendicular to the radius (or diameter) at the point where it meets the circle, the new line is called a tangent line (from the Latin *tangere*, to touch).

- The tangent line, defined as perpendicular to the radius, touches the circle at a single point.

*Proof.*

By definition, the tangent line at  $P$  is perpendicular to the diameter or radius including  $O$  and  $P$ .

Let  $Q$  be some other point on that tangent line. Then  $\angle OPQ$  is a right angle.



Let  $T$  be on the tangent line so  $QPT$  collinear. Draw  $OT$ .

Aiming for a contradiction, suppose that  $OT \perp QPT$ .

By the parallel postulate,  $OT \parallel OP$ .

But  $OT$  meets  $OP$  at  $O$ . This is a contradiction.

$OT$  cannot be perpendicular to  $QPT$ .

It follows that  $OT$  is the hypotenuse of a right triangle  $\triangle OPT$ , so  $OT > OP$  and thus  $T$  cannot be on the circle.

□

### **tangent: touches one point → perpendicular**

An alternative definition of the tangent is that it is a line that touches the circle at just one point. One can use this definition to prove that the angle between the tangent and the radius is a right angle. This is the converse of the previous theorem.

- The tangent line, defined as touching the circle at a single point, is perpendicular to the radius.

*Proof.*

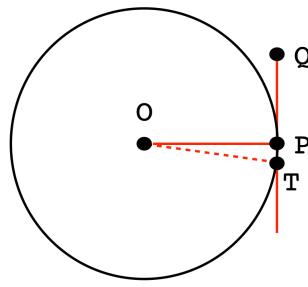
Draw the line that touches the circle at only one point  $P$ , and draw the radius to that point,  $OP$ .

If we assume that  $OP$  is not perpendicular to  $QPT$ , we can derive a contradiction.

Consider a succession of points moving along the line away from  $P$  in either direction. The angle formed with a line drawn through  $O$  gets smaller as the points get farther from  $P$ .

The angle between the tangent and the radius is greater than a right angle on one side of  $P$  (since the angle at  $P$  is not a right angle). On that side, move along the line until we find the point that does form a right angle with a radius of the circle.

Let us suppose the point is  $T$ , in the figure above.  $OTP$  is a right angle and we are in doubt about whether  $T$  lies inside or outside the circle.



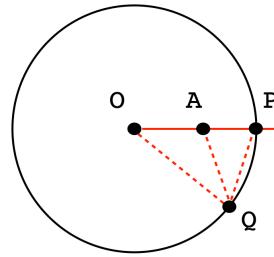
But then  $OTP$  is a right angle, with the side opposite, namely,  $OP$ , the hypotenuse of a right triangle. But the hypotenuse is the longest side in a right triangle, so therefore  $OT < OP$ , so the point  $T$  is *inside* the circle.

We have a line with one point on the circle, and another point inside the circle. Any line through an interior point of a circle must cross the circle at two points, which contradicts the assumption above. Hence the angle at  $P$  is a right angle.

□

### shortest distance to the circle

Let  $A$  be any point inside a circle. Draw the radius that passes through  $A$  to point  $P$  on the circle. I claim that the length  $AP$  is smaller than the distance to *any* other point on the circle, such as  $Q$ .



*Proof.*

Draw the radius  $OQ$ , which is equal to radius  $OP$ .

$$OQ = OP = OA + AP$$

By the **triangle inequality**

$$OA + AQ > OQ$$

so

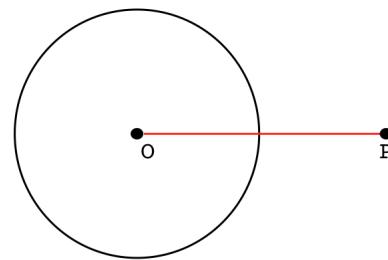
$$OA + AQ > OA + AP$$

$$AQ > AP$$

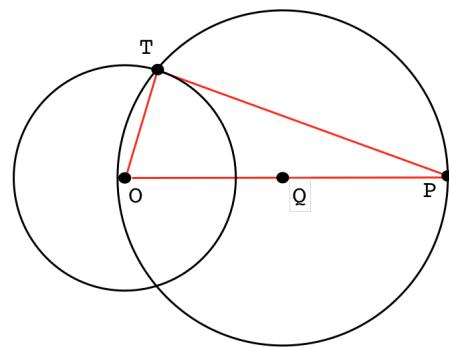
□

## construction of a tangent

**Thales theorem** provides a way to construct the tangent to a circle that passes through any exterior point  $P$  — actually, there are two such lines.

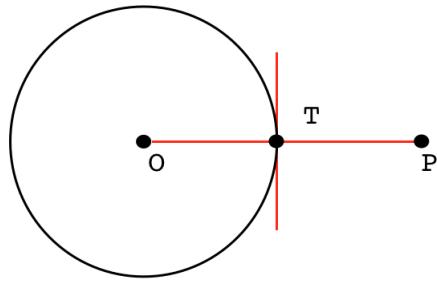


Use  $OP$  as the diameter of a circle. Draw the line segment  $OP$  and divide it in half by erecting the perpendicular bisector at  $Q$ . Use that point  $Q$  as the center of a new circle with radius  $OQ$ . The point  $T$  is the intersection of the two circles.



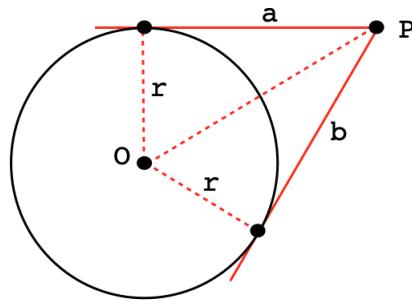
By the theorem,  $\angle OTP$  is a right angle, and since  $OT$  is a radius of the original circle,  $TP$  is the tangent to the smaller circle at the point  $T$ .

To construct a tangent on a circle at a given point  $T$ :



Extend  $OT$  to  $P$  so that  $OP$  is twice the radius  $OT$ . Construct the perpendicular bisector at  $T$ . The bisector is also the tangent of the circle.

- From any external point  $P$ , one can draw two tangents to a circle. These two tangents have the same length.
- From any external point  $P$ , the line to the center of the circle bisects the angle between the two tangents, as well as the angle between the radii drawn to the two points of tangency.



*Proof.*

The angle between a tangent and the radius to the point where it touches the circle, is a right angle. For a pair of tangent lines from a given point, there are two such points on the circle.

The base lengths are both radii, so they are equal, and there is a shared side (the dotted line segment  $OP$ ).

Therefore the two triangles are congruent, by hypotenuse-leg in a right triangle (HL).

The two congruent sides  $a$  and  $b$  are the same length.

$$a = b$$

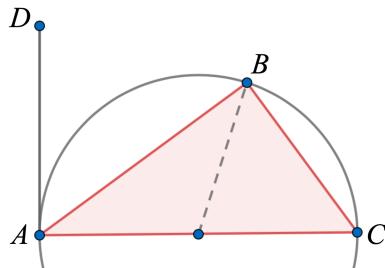
For any point external to a circle, two tangents to the circle can be drawn, of equal length. The line from the point to the center of the circle bisects the angle between the two tangents.

□

### tangent-chord theorem

A tangent is a line that just touches the circle (or another curve, like a parabola). By definition it touches the circle at a single point, and is perpendicular to the radius which extends to that point.

- The arc swept out between a tangent and a chord is equal to the arc lying between the point of tangency and the point where the chord meets the circle.



*Proof.*

By Thales' theorem,  $\angle B$  is right, so  $\angle BAC$  and  $\angle BCA$  are complementary.

The tangent  $DA \perp AC$ , so  $\angle BAC$  is complementary to  $\angle DAB$ .

It follows that  $\angle DAB = \angle BCA$ .

Therefore, they cut the same arc of the circle.

□

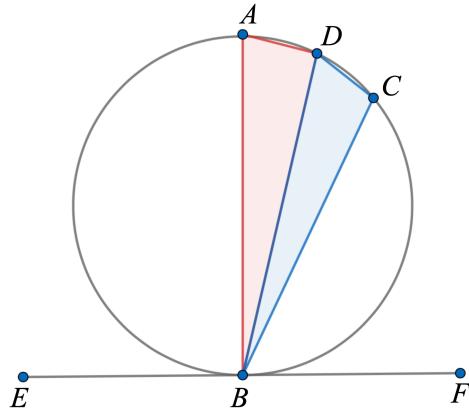
This is Euclid III.32, the tangent-chord theorem. Acheson calls this the alternate segment theorem.

Here is Euclid's proof:

In the figure below, let  $EF$  be tangent to the circle at  $B$  and  $AB$  be a diameter of the circle.  $\angle ABF$  is a right angle.

I claim that  $\angle DBF = \angle BAD$ , and they cut the same arc  $DCB$ .

Also,  $\angle DBE = \angle DCB$ , and they cut the same arc  $DAB$ .



*Proof.*

By Thales' theorem  $\angle ADB$  is right, so by sum of angles  $\angle BAD + \angle ABD$  is right.

Therefore

$$\angle ABD + \angle BAD = \angle ABD + \angle DBF$$

$$\angle BAD = \angle DBF$$

For the second part,  $\angle DBE + \angle DBF$  equals two right angles.

By the **quadrilateral supplementary theorem** (Euclid III.22)

$$\angle BAD + \angle BCD = 180$$

and since  $\angle DBF = \angle BAD$  we have

$$\angle DBE + \angle DBF = \angle DBF + \angle BCD$$

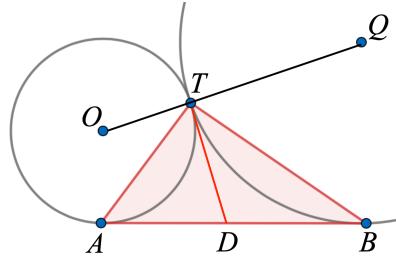
$$\angle DBE = \angle BCD$$

□

## Euclid III.12

Two circles are tangent to each other at  $T$ .

In one circle draw the radius to  $T$  and extend it. This line goes through the center of the second circle.



*Proof.*

Draw the tangent to the first circle at  $T$ . This line goes through a single point ( $T$ ) on both circles.

The line perpendicular to it (the extension through  $T$ ) is a radius of the second circle.

□

Euclid's proof of III.12 is by contradiction.

*Proof.*

Suppose the straight line  $OQ$  does not go through  $T$ .

By the triangle inequality,  $OQ < OT + QT$ , where  $OT$  and  $QT$  are radii of the two circles.

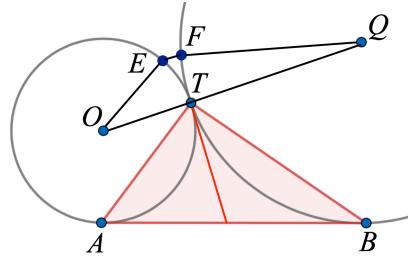
Then  $OQ$  must cut the two circles at different points, say  $E$  and  $F$ .

The straight line is

$$OQ = OE + EF + QF$$

$$OQ > OE + QF$$

where  $OE$  and  $QF$  are radii of the two circles.



But  $OT + QT = OE + QF$

$OQ$  cannot be both less than the sum of radii and greater than the same sum. This is a contradiction.

Therefore  $OTQ$  are collinear.  $OQ$  goes through  $T$ .

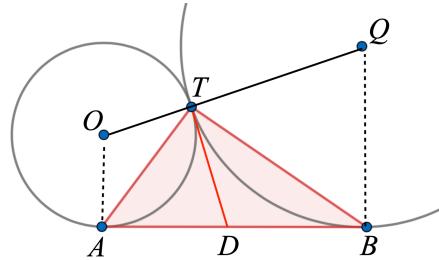
□

## problem

Draw  $AT$  and  $BT$ . Prove that  $\angle ATB$  is a right angle.

*Solution.*

Draw the tangent line to both circles at  $T$ , the red line in the figure below.



Draw radii of both circles to their respective tangent points at  $A$  and  $B$ .

$DA$  and  $DT$  are both tangent to the circle on center  $O$ , while  $DT$  and  $DB$  are both tangent to the circle on center  $Q$

The tangents are all equal in length, so  $\triangle DAT$  and  $\triangle DBT$  are both isosceles.

Since the base angles are equal, we have (by the external angle theorem) that

$$2\angle ATD = \angle BDT$$

$$2\angle BTD = \angle ADT$$

But the sum of the two right-hand sides is equal to two right angles.

Therefore, the sum of the two left-hand sides is also equal to two right angles. Hence

$$\angle ATD + \angle BTD = 90 = \angle ATB$$

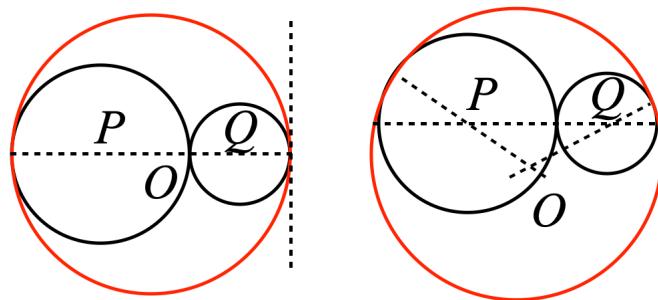
□

Alternatively, since  $\triangle DAT$  and  $\triangle OAT$  are both isosceles, the sum of the angles making up  $\angle OTD$  is equal to the sum of angles making up  $\angle OAD$ .

But  $\angle OAD$  is a right angle, therefore so is  $\angle OTD$ .

## tangents

As indicated in the previous problem, if two circles just touch one another at a single point, and the tangent is drawn at that point, then it is tangent to *both* circles. The perpendicular to the tangent at that point is a radius of both circles. Thus a single line is collinear with both diameters.



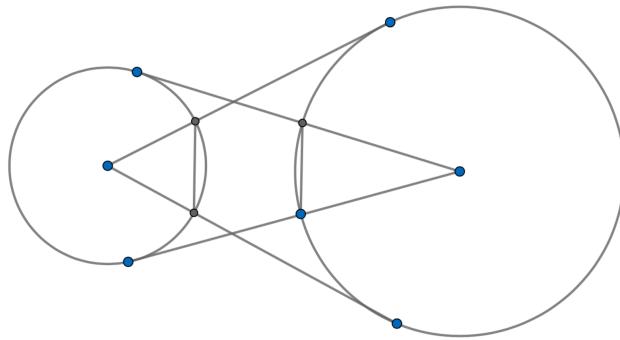
So then, what about three circles? Clearly, the third circle can be drawn with the joined diameters of the smaller two as its diameter. Or it can be drawn somewhat larger (right panel).

The diameter through  $P$  and  $Q$  in the second case must be extended to meet the large circle, so apparently, the arrangement in the left panel is the circle with the *smallest* diameter that is also tangent to the two circles on centers  $P$  and  $Q$ . (See Euclid III.7 for a proof).

## The eyeball theorem

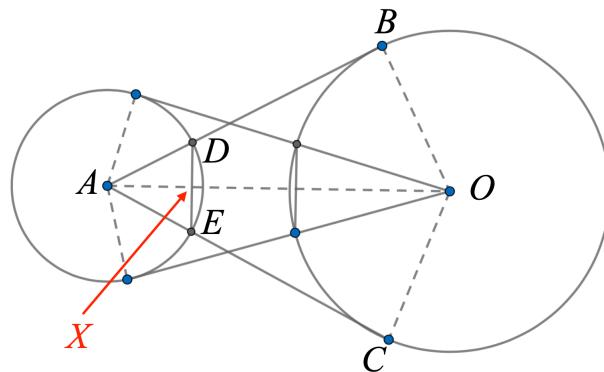
This problem is from Acheson's Geometry book (Fig 131). We have two circles with tangents drawn from the center of each circle to the other one.

We are to show that the lengths of the chords formed by the tangents as they exit the originating circle are equal.



*Solution.*

Let us label some points.



The two triangles on the centerline with tangents as the base are equal:  $\triangle AOB \cong \triangle AOC$  by HL.

Therefore, the centerline bisects  $\angle BAC$ .

So then the two small triangles  $\triangle AXD \cong \triangle AXE$  by SAS.

Therefore the angles formed at X are right.

We have similar right triangles:  $\triangle AXD \sim \triangle ABO$ .

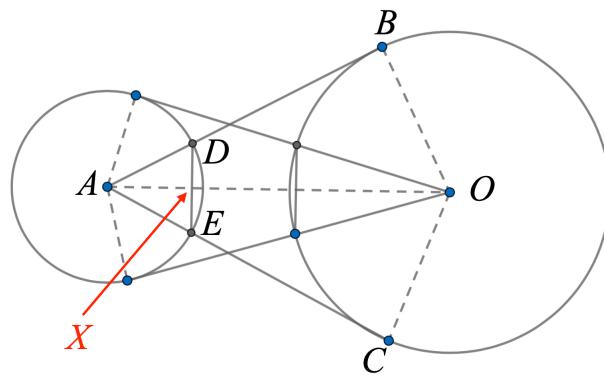
Form the equal ratio of the hypotenuses on the left-hand side and of the shorter legs on the right-hand side:

$$\frac{AD}{AO} = \frac{DX}{OB}$$

$$DX = \frac{AD \cdot OB}{AO}$$

Define  $OB = R$ ,  $AD = r$  and  $AO = d$  so

$$DX = \frac{rR}{d}$$

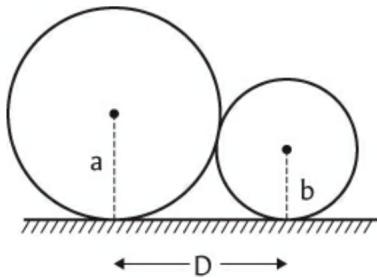


$DX$  is one-half the chord. Its length in the formula is symmetrical in  $r$  and  $R$ . Therefore the half-chord in the other circle has an identical length.

The chords are equal in length and parallel to each other (since they are perpendicular to the centerline, and bisected by it), so they form the sides of a rectangle.

### penny-farthing problem problem

Here is a problem with a fascinating history (see Acheson). Find an expression for  $D$  in terms of  $a$  and  $b$ .



**Fig. 161** A penny-farthing problem.

Its solution is very easy, so I will not give it here but I encourage you to try.

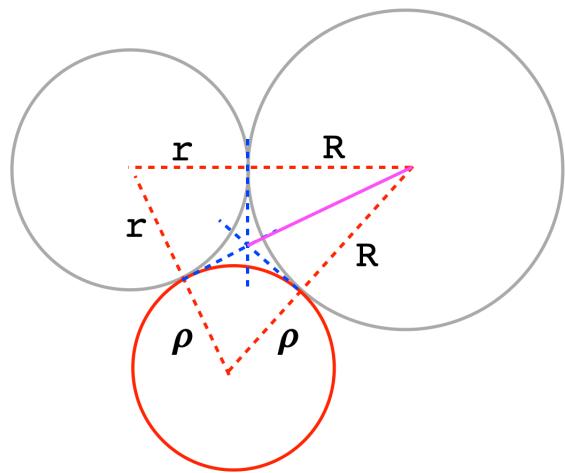
### problem

Harvard 1899 exam:

4. Three unequal circles are so situated that each of them is externally tangent to the other two. At the points of contact tangents are drawn. Prove that these three tangents meet in a point.

*Solution.*

The tangent lines are perpendicular to a radius drawn to the point of tangency. As a result, two corresponding radii meet at the point of tangency and the two corresponding centers are co-linear with the point of tangency.



Therefore, we have a triangle with sides as shown. The tangent lines are perpendicular to the sides of the triangle, but will not, in general, pass through a vertex or center of the third circle, for any pair of circles and their tangent line.

But for each vertex of the triangle, the bisector of the angle gives congruent triangles, in which one of the sides is a blue dotted line, forming a right angle with the radius. The hypotenuse of one such pair is shown in magenta.

The two triangles that share the magenta line as hypotenuse are congruent by hypotenuse-leg in a right triangle (HL), and therefore the two blue dotted lines meeting in the center have equal length. Now do the same for the circle with radius  $r$  and then for the circle with radius  $\rho$ .

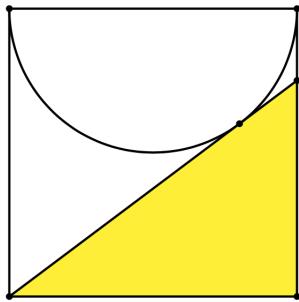
The blue-dotted lines are thus all equal. So they can be used to draw a circle that just touches the sides of the triangle. That circle is called the incircle of the triangle.

The point where these tangents meet is the incenter of the triangle. Since the incenter exists, the three angle bisectors are concurrent, the point where the tangents meet is the same and it also exists.

□

## problem

This is a problem from Paul Yiu. We have a semicircle inside a square with one side of the square as the diameter. The tangent is drawn as shown.



Prove that the triangle is a  $3 - 4 - 5$  right triangle.

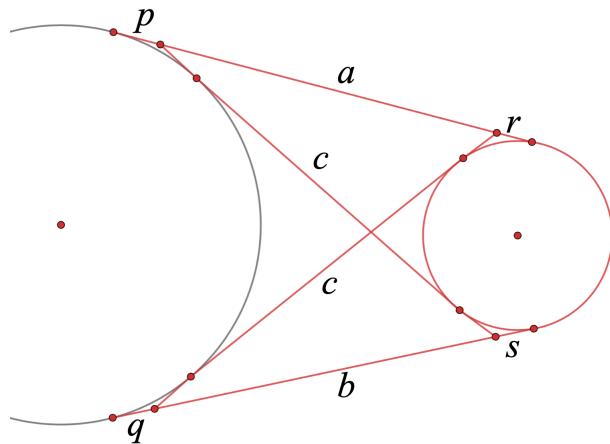
Scale the square so that the side has length 4. We will use the property that the two tangents to any circle from an exterior point are equal. Thus, the distance from the lower left to the point of tangency is 4. Let the other short tangents have length  $x$ .

Then Pythagoras says that:

$$\begin{aligned} 4^2 + (4-x)^2 &= (4+x)^2 \\ 4^2 + 4^2 - 8x + x^2 &= 4^2 + 8x + x^2 \\ 16x = 4^2, \quad x &= 1 \end{aligned}$$

So the sides are indeed  $3 - 4 - 5$ .

### one more



This is another problem from Paul Yiu. Let four tangents to two circles be drawn as shown.

Let the line with length  $a+p+r$  be tangent to both circles, and the same for  $b+q+s$ .

Furthermore draw the crossed tangents of length  $c$  and extend them to meet  $a$  and  $b$ , as shown. The short tangents are all pairs with equal length:  $p, q, r$  and  $s$ .

The tangents labeled  $c$  are equal, because they are composed of two tangents to each circle extended from the intersection in the middle.

We can write other equalities based on tangents from different points:

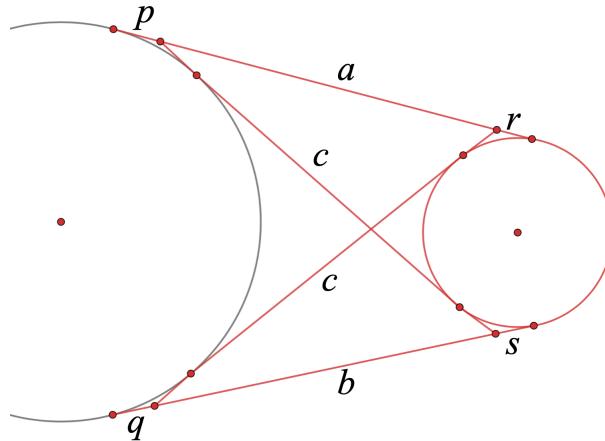
$$p + c = a + r$$

$$p + a = r + c$$

Subtracting

$$c - a = a - c$$

$$c = a$$



Substituting

$$p = r$$

By the same logic  $c = b$ .

Thus  $a = b = c$  and  $q = s$ .

Finally, consider the tangent not drawn. Somewhere offscreen to the right is a point from which two lines can be drawn tangent to *both* circles. The parts not shown are equal (as they are tangents to the right-hand circle), so the parts remaining are also equal, by subtraction.

Namely

$$p + a + r = q + b + s$$

$$2p + a = 2q + b$$

$$p = q$$

So all four of the short segments are also equal:  $p = q = r = s$ .

# Chapter 22

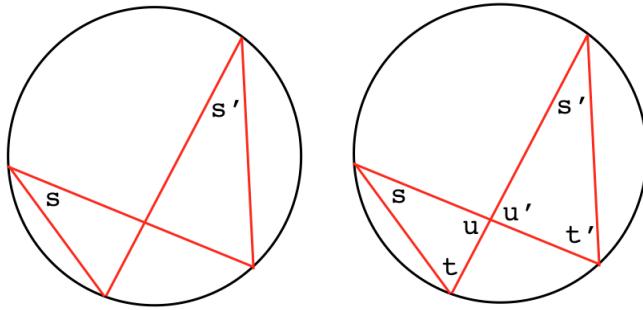
## Arcs of a circle

It is natural to think about angles formed from a vertex lying on the periphery of a circle and ask about their relation to the arcs they cut off.

The inscribed angle theorem says that a vertex placed at any point on the periphery of a circle, forming an angle that corresponds to the same arc as a central angle, is equal to *one-half* the central angle.

A corollary is that whenever two peripheral angles correspond to the same arc, they are equal.

In the figure below,  $s = s'$  (left panel)

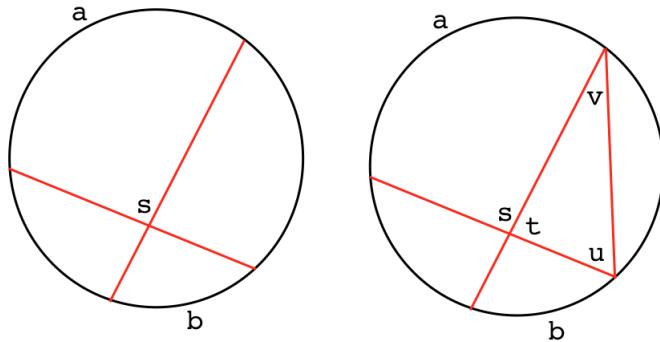


and  $t = t'$  (right panel). So  $u = u'$  by both the vertical angle theorem and by the triangle sum theorem. All three angles in each triangle are equal, so the two triangles are similar.

Later, we will develop this result into the **crossed chord theorem** (product of

lengths).

## Intersecting chords



Given two crossed chords, consider the opposing arcs  $a$  and  $b$

$$s = \frac{1}{2}(a + b)$$

angle  $s$  is the average of the two arc lengths.

*Proof.*

Draw a triangle (right panel, above).

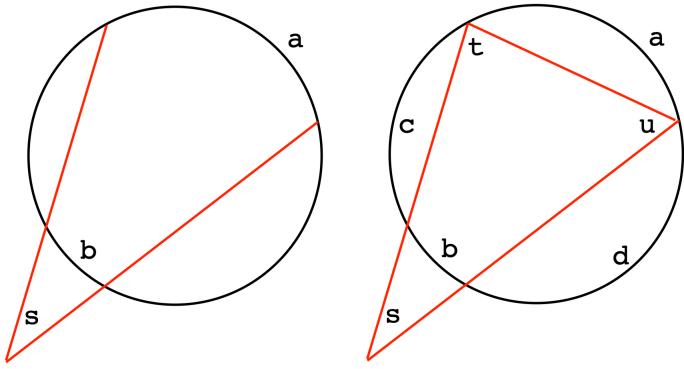
$$u = \frac{a}{2} \quad v = \frac{b}{2}$$

The external angle is the sum of the two opposing interior angles.

$$s = u + v = \frac{1}{2} (a + b)$$

## external vertex

Rather than having the vertex on the circle, it now lies outside (left panel, below). There is a new small piece of arc length  $b$ .



Draw the triangle and do some arithmetic. By the usual theorem we have

$$t = \frac{b+d}{2} \quad u = \frac{b+c}{2}$$

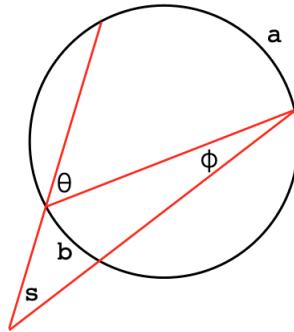
The sum of the arcs in a circle is twice the sum of the angles in a triangle:

$$s + t + u = \frac{a + b + c + d}{2}$$

By subtraction:

$$s = \frac{a - b}{2}$$

□



*Proof.* (Alternative).

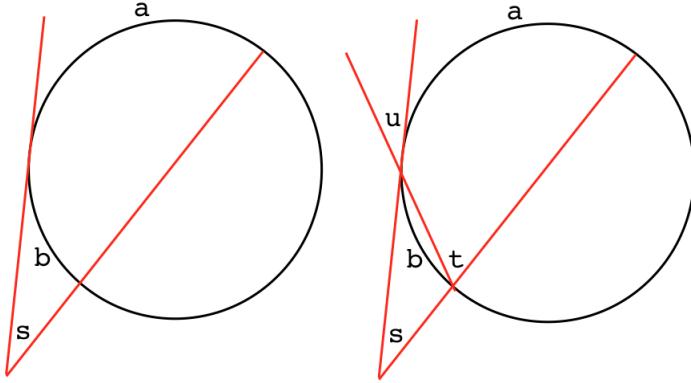
Alternatively, in the figure above  $2\theta = a$  and  $2\phi = b$ , but also  $\theta = s + \phi$ .

$$s = \theta - \phi = \frac{a - b}{2}$$

□

## tangent and secant

Rather than two secants, we now have a secant and a tangent. The result is the same as previously.



$$s = \frac{a - b}{2}$$

*Proof.*

Draw the triangle on the right. We have that:

$$t = \frac{a}{2}$$

Also, we showed that for an angle where one part is the tangent, the expected result holds. So the vertical angle to  $u$  cuts off the arc  $b$

$$u = \frac{b}{2}$$

Alternatively, we could argue that the supplementary angle to  $u$  cuts off everything except  $b$ .

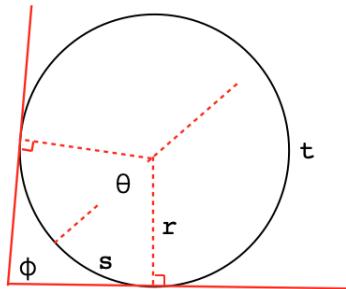
By the exterior angle theorem

$$t = s + u$$

$$s = t - u = \frac{a - b}{2}$$

## two tangents

We showed previously that when two tangents are drawn from an exterior point, one can draw two right triangles that share the hypotenuse and have another side equal to the radius, so they are congruent by hypotenuse-leg in a right triangle (HL).



Let the whole arc between the two right angles be  $s$  the short way and  $t$  the long way around the circle, and let  $\phi$  be the external angle. By analogy with the results above, we expect that

$$\phi = \frac{t - s}{2}$$

*Proof.*

We could use congruent triangles, but instead just note that the sum of angles in any quadrilateral is  $2\pi$ . Hence

$$\phi + \theta = \pi$$

In terms of arc  $\theta = s$  and  $s + t = 2\pi$ . Substituting into the last equation

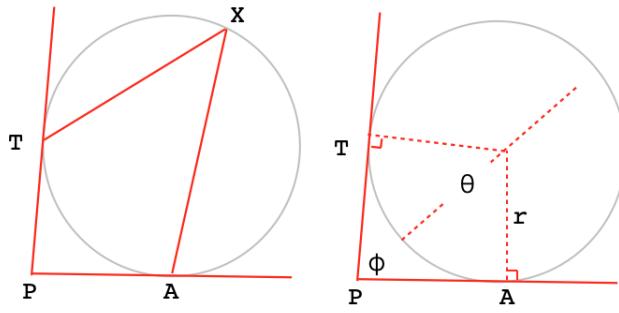
$$\phi + s = \frac{s + t}{2}$$

$$\phi = \frac{t - s}{2}$$

□

## problem

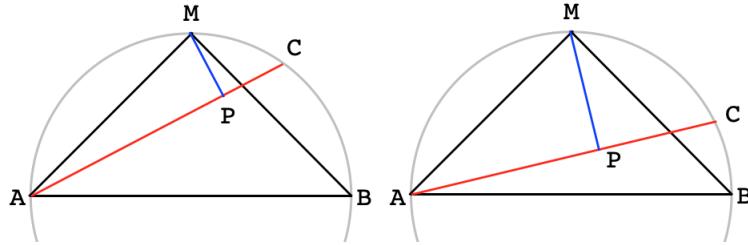
Relate the angle at  $P$  to the one at  $X$ .



By the previous example,  $\theta + \phi = \pi$ . But  $\angle X = \theta/2$ . Hence

$$P = \pi - \theta = \pi - 2\angle X$$

### problem



Given that  $AB$  is a diameter of the circle, and that  $\triangle AMB$  is isosceles. Draw  $MP \perp AC$ .

Show that  $\triangle MPC$  is isosceles.

*Proof.*

Draw  $MC$  (not shown).

Then  $\angle ACM = \angle PCM$  is subtended by chord  $AM$ , which is one-quarter of the circle.

Since  $\angle MPC$  is right,  $\angle PMC$  is also one-quarter of the circle, by sum of angles.

It follows that  $\triangle MPC$  is isosceles.

□

# Chapter 23

## Chords in a circle

Chords are straight lines inside a circle, with endpoints which lie on the circle. If a chord is extended (produced, in Euclid's terminology), then the extended chord becomes a secant, with part of the line lying outside the circle.

- If two arcs are equal, the corresponding chords are also equal, and vice-versa.

*Proof.* (forward)

Draw the radii flanking both arcs. By definition, we have equal angles for the equal arcs, and therefore congruent triangles by SAS. The equal chords are opposing sides to the central angle in the congruent triangles.

□

*Proof.* (converse)

Draw the radii flanking both chords. We have congruent triangles by SSS, so the central angles are equal. By Euclid III.26, we have equal arcs.

□

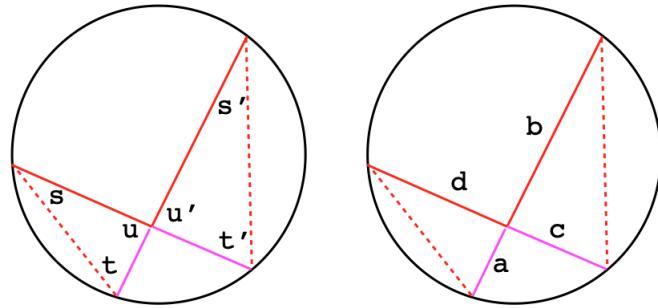
We may occasionally not even point out the equivalence, but just assume that if two arcs are equal, the corresponding chords are also equal, and vice-versa.

### inequality

Suppose we have one angle greater than another, or one chord greater than another?

*Proof.* (Sketch). Start with two chords in a circle. Draw the radii for both. Then, by the **hinge theorem**, Euclid I.24, the greater angle lies opposite the greater third side. Thus, the greater chord lies in the greater circumference because they are connected through the central angle.  $\square$

### crossed chords



Previously, we showed that when two arcs cross in a circle, each of the two equal vertical angles is equal to the *average* of the two arcs they cut out. It is also apparent that the two triangles formed are similar.

In two similar triangles, the sides opposite equal angles are in the same ratio, so:

$$\frac{a}{c} = \frac{d}{b}$$

It follows that

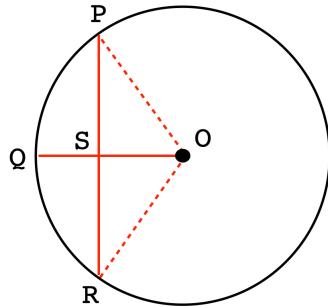
$$ab = cd$$

The products of the two parts of each chord are equal.

### bisected chord theorem

Given any chord of a circle and a line perpendicular to it, if the perpendicular is a radius of the circle then the chord is bisected.

Conversely, if the perpendicular is a bisector, then it is also a radius.



*Proof.*

For the forward theorem: we have that  $OQ$  is a radius (it lies on the diameter), and it is perpendicular to chord  $PR$ .

We have  $OP = OR$ , as radii of the circle. Also, side  $OS$  is shared, and there are right angles at  $S$ . So  $\triangle OPS \cong \triangle ORS$  by hypotenuse-leg in a right triangle (HL).

We conclude that  $PS = SR$ .

□

In addition, since  $\triangle OPS \cong \triangle ORS$ ,  $\angle POS = \angle ROS$ , the arcs  $PQ$  and  $QR$  are cut out by equal central angles, therefore they are equal.

The converse theorem was established when we derived the method to construct a circle given any three points on it.

Briefly, all points that are equidistant from the two ends of a chord must lie on its perpendicular bisector. Since the center of the circle containing the chord forms radii when joined to the two ends, it must also lie on the perpendicular bisector.

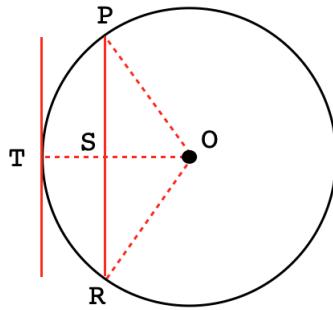
Therefore the perpendicular bisector is a diameter of the circle.

□

One consequence is that if a line is the perpendicular bisector of a chord of the circle, it is also the perpendicular bisector of any other chord parallel to the first.

Furthermore, if we have a tangent and any chord that is parallel to it, then the radius drawn to the point of tangency bisects the chord.

This follows simply from the definition of the tangent as perpendicular to the radius at the point of tangency.



By our previous theorem,  $OT$  bisects  $PR$  so that  $PS = SR$ .

- Given a point of tangency on a radius perpendicular to any chord, not only the chord but the arc lengths are evenly divided by the radius.  $PT = RT$ .

We can also note that if the distance from  $S$  to the periphery of the circle is  $d$  and the height of the half chord is  $h$ , by the crossed chord theorem we have

$$(2r - d)(d) = h^2$$

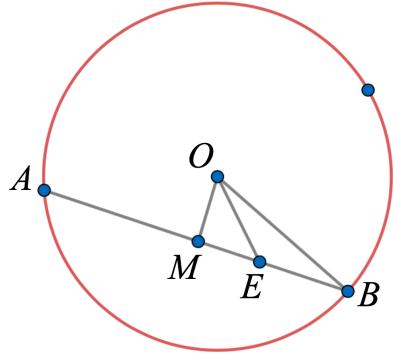
which can be solved as a quadratic in  $d$ .

$$\begin{aligned} d^2 - 2rd + h^2 &= 0 \\ d &= \frac{2r \pm \sqrt{4r^2 - 4h^2}}{2} \\ &= r \pm \sqrt{r^2 - h^2} \end{aligned}$$

which is already fairly obvious since  $OP = r$  and, it's the hypotenuse in a right triangle with  $PS = h$ , so  $OS$  is exactly the square root, while  $r$  minus that is just  $d$ .

## Euclid's proof of crossed-chords

This proof, III.35, uses only the Pythagorean theorem, the previous theorem about perpendicular bisectors, and some algebra from Book II.



First we prove the following:

*Lemma.*

Let  $AB$  be a chord (not diagonal) in a circle on center  $O$ , with  $OM \perp AB$  and bisecting  $AB$  so  $AM = BM$ .

Let  $E$  be an arbitrary point on  $AB$ .

I claim that

$$AE \cdot EB + EM^2 = BM^2$$

This can be proved by geometry (Euclid II.5), but algebra is quicker:

Let  $x = AM = BM$ , and  $y = EM$ .

Then  $EB = x - y$  and  $AE = x + y$ . We have

$$(x + y)((x - y) + y^2 = x^2$$

which is easily verified.

Starting with

$$AE \cdot EB + EM^2 = BM^2$$

Add the square on  $OM$  to both sides.

$$AE \cdot EB + EM^2 + OM^2 = BM^2 + OM^2$$

$$AE \cdot EB + OE^2 = OB^2$$

$$AE \cdot EB = OB^2 - OE^2$$

□

The product of the two parts of a chord cut at any point  $E$  is equal to the difference of squares of the radius of the circle and the distance from the center to the point  $E$ .

This leads to our proof of the crossed-chords theorem.

*Proof.*

So then let some other chord  $CD$  (not diagonal) cut  $AB$  at  $E$ . We will have the same right-hand side.

$$CE \cdot ED = OB^2 - OE^2$$

Thus

$$AE \cdot EB = CE \cdot ED$$

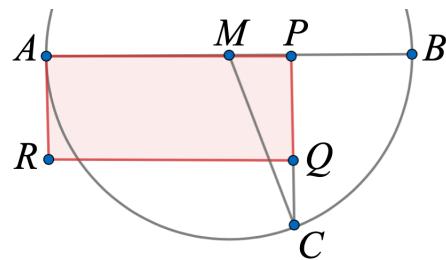
□

## Euclid II.14

As an aside, suppose it is desired to construct the square equal in area to a given rectangle  $APQR$ .

Extend  $AP$  to  $B$  such that  $PB = PQ$ .

Bisect  $AB$  to find center  $M$  for a circle with radius  $AM = BM$ .



By II.5

$$AP \cdot PB = MB^2 - MP^2$$

$$AP \cdot PQ = MC^2 - MP^2 = PC^2$$

Thus  $PC$  is the side of the desired square.

This can also be done by similar triangles, since  $\triangle CAP \sim \triangle BCP$ . Thus  $PB/PC = PC/AP$  so  $AP \cdot PB = PC^2$ .

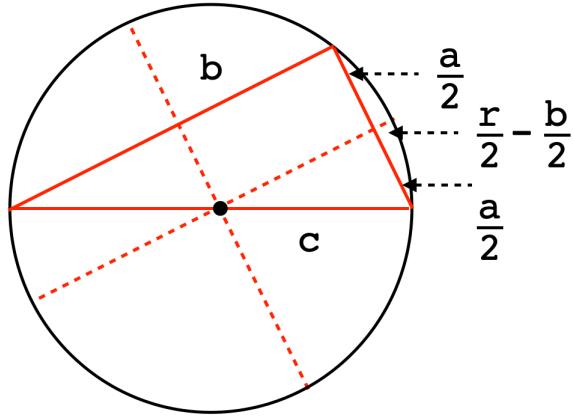
□

Since we can get from Pythagoras to the crossed chord theorem, it may not be so surprising that we can go backward.

## Pythagorean theorem by crossed chords

*Proof.*

A right triangle with sides  $a$ ,  $b$  and  $c$  is inscribed in a circle. By Thales' theorem, the hypotenuse  $c$  is a diagonal. Draw two more diagonals, parallel to the sides  $a$  and  $b$ . We can use similar right triangles to show that these bisect sides  $a$  and  $b$ .



So then, we will use the crossed chords theorem, multiplying the two halves of  $a$  together. The question is, what are the two parts of the dotted diameter that crosses  $a$  at right angles?

The long part is  $c/2 + b/2$ . Can you see why?

The very short part is  $c - (c/2 + b/2) = c/2 - b/2$ . The reason is that when added to the long part, the result is just

$$c/2 + b/2 + c/2 - b/2 = c$$

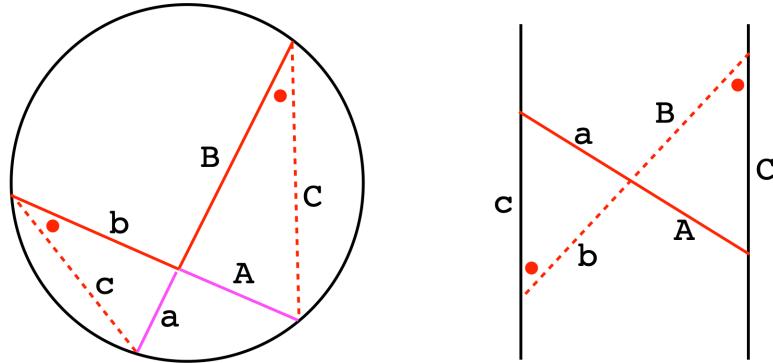
We have

$$\begin{aligned}\frac{a}{2} \cdot \frac{a}{2} &= \left(\frac{c}{2} + \frac{b}{2}\right)\left(\frac{c}{2} - \frac{b}{2}\right) \\ a \cdot a &= (c + b)(c - b) = c^2 - b^2 \\ a^2 + b^2 &= c^2\end{aligned}$$

□

- The crossed chords formed for the vertices of any cyclic quadrilateral form two pairs of similar triangles. The sides that are in the same ratio are on different chords. The fundamental ratio  $a/A = b/B$  leads directly to the crossed chord theorem.

The basic rule is that corresponding sides lie opposite corresponding (equal) angles. In the figure  $a$  and  $A$  lie opposite to the angle marked with a red dot.

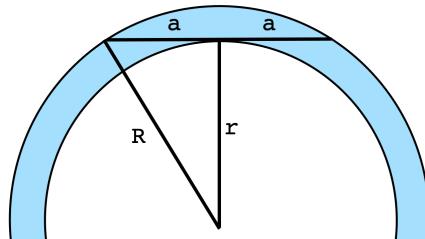


In the left panel (circle) corresponding sides lie on different chords, while in the right panel they lie on the same chord. With crossed chords, the corresponding angles and sides are in clockwise order, while for the other case the order is counter-clockwise.

If in doubt, make sure that the sides you think are corresponding sides lie opposite to equal angles.

### another view

Another way to look at this is shown in the following diagram:



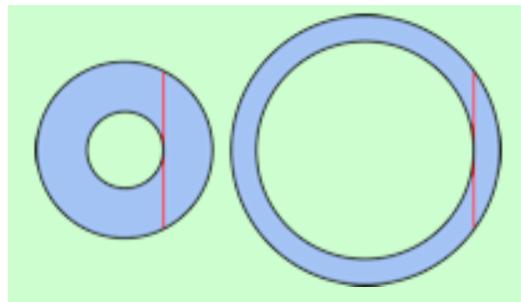
Given a larger circle of radius  $R$  and a chord of it with length  $2a$ , draw the smaller circle of radius  $r$  that just touches the chord. Then, by the Pythagorean theorem we

have that

$$R^2 - r^2 = a^2$$

But  $R^2 - r^2$  (times  $\pi$ ) is the area of the annulus or ring, the difference between the area of the outer circle and that of the inner one. This is equal to  $\pi a^2$ .

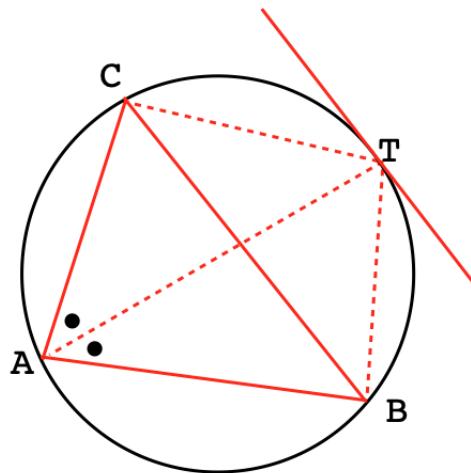
So then, there is family of circles of different sizes that can be drawn, each with a chord of length  $2a$  (as long as the radius is greater than  $a$ ). For each of those circles, one can draw the annulus as we did above, and find that the area is exactly the same.



<https://puzzles.nigelcoldwell.co.uk>

## problem

Given that the line touching the circle at  $T$  is a tangent and that  $BC$  is parallel to it. Show that the angles marked with black dots are equal.



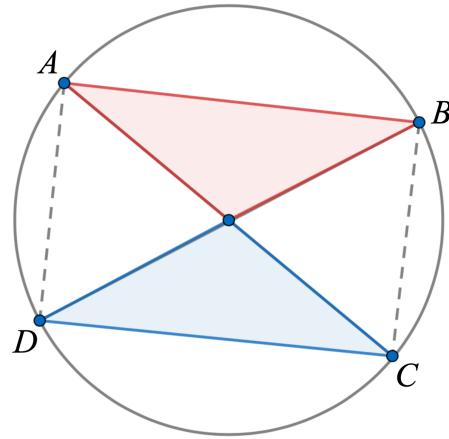
I think the idea of the problem was to use similar triangles somehow. But given our

previous work, this is a trivial consequence of the fact that the arcs  $CT$  and  $TB$  are equal.

So, as peripheral angles corresponding to equal arcs,  $\angle CAT = \angle TAB$ .

### diameters form a rectangle

- When any two diameters of a circle are drawn and consecutive chords on the circle joined together, the result is a rectangle.



*Proof.*

A simple proof relies on Thales' circle theorem. All four vertices are right angles.

It follows that both pairs of opposing sides are parallel. For example

$$AD \perp AB \perp BC \Rightarrow AD \parallel BC$$

So  $ABCD$  is a parallelogram with right-angled vertices, a rectangle.

□

One can, of course, also use the fact that certain angles are subtended by equal arcs, so they are equal (for example  $\angle DAC = \angle DBC$ ). Then use congruent triangles.

### problem

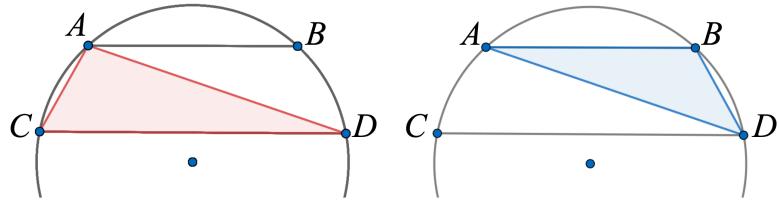
Construct a square circumscribed by a circle, i.e. all four vertices lie on the circle.

*Solution.* (Sketch).

The diagonals originate at vertices, by definition. We also know that the diagonals of a square are perpendicular. Hence, erecting the perpendicular bisector of any diagonal of the circle locates four points on the circle that are vertices of a circumscribed square.

□

### parallel chords joined by equal arcs and chords



Given any two parallel chords in a circle,  $AB$  and  $CD$ . Draw  $AD$ .

The inscribed angles formed,  $\angle ADC$  and  $\angle BAD$ , are equal by alternate interior angles,

Thus arc  $AC = \text{arc } BD$ .

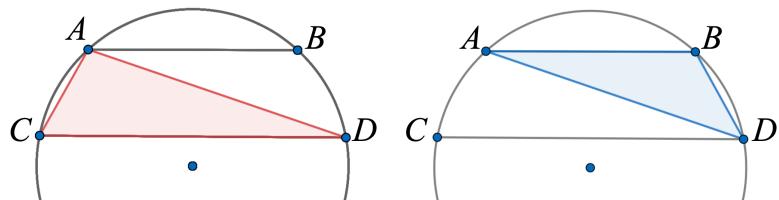
Equal arcs have equal chords so  $AC = BD$ .

□

It is a simple matter to extend this result to triangles such as  $\triangle ABC \cong \triangle BAD$ .

*Proof.*

Since the arcs are equal  $\angle ABC = \angle BAD$ .



But  $\angle BDA$  and  $\angle ACB$  both cut the same arc,  $AB$ .

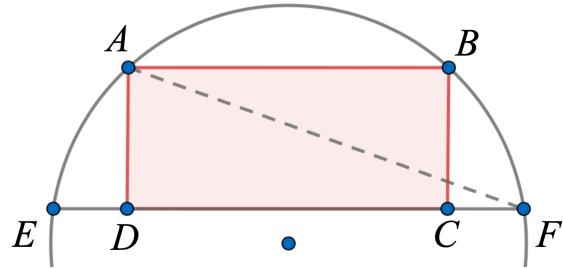
So  $\triangle ABD$  and  $\triangle BAC$  have three angles equal, and they share the same side  $AB$  opposite one pair of equal angles.

$\triangle ABD \cong \triangle BAC$  by ASA.

□

### rectangle side on a circle

We derive a useful theorem about any rectangle in a circle, and its converse.



Let  $ABCD$  be a rectangle such that  $A$  and  $B$  lie on the circle, but  $C$  and  $D$  do not.

Then the extensions of  $DC$  are equal, namely  $DE = CF$ .

*Proof.*

Draw  $AF$ . As before,  $\angle AFE = \angle BAF$ .

Since the inscribed angles are equal, so are the arcs:  $\text{arc } AE = \text{arc } BF$ .

Equal arcs have equal chords, hence  $AE = BF$ .

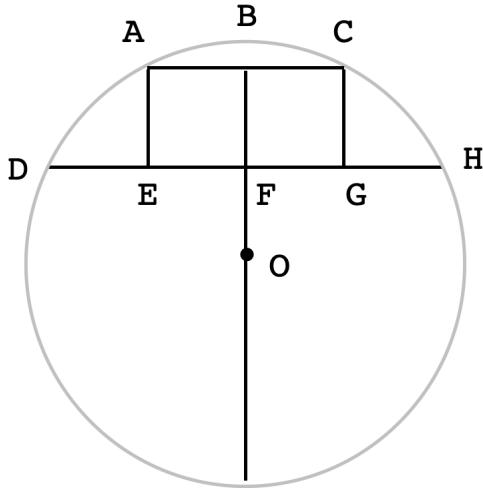
So  $\triangle ADE \cong \triangle BCF$  by HL.

It follows that  $DE = CF$ .

□

*Proof. (Alternate).*

Suppose that  $ACGE$  is a rectangle. Let  $BF$  be the perpendicular bisector of  $AC$ .



Then, the extension of  $BF$  passes through the center of the circle (it is part of a radius), by the converse of the theorem discussed earlier. Every point that is equidistant from  $A$  and  $C$ , including the center of the circle, lies on the extension of  $BF$ .

$OB$  is also the perpendicular bisector of  $DH$ , since  $ACGH$  is a rectangle. So

$$DF = FH$$

but  $AB = BC = EF = FG$ . Subtracting equals from equals:

$$DF - EF = FH - FG$$

$$DE = GH$$

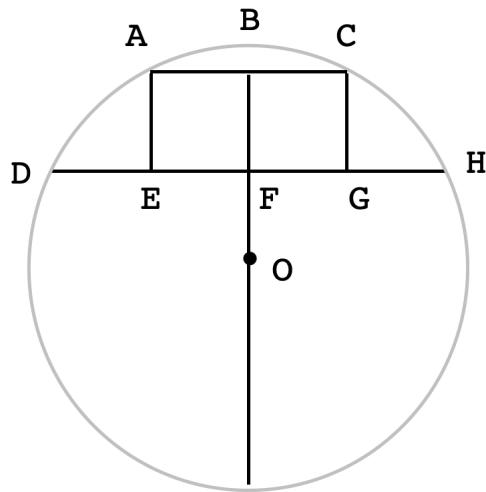
□

For any rectangle with two vertices lying on a circle and two inside the circle, when the side which connects the inside points is extended both ways to reach the circle, the extensions will be equal.

### converse

Above, we started with a rectangle in a circle. Now we start simply with any two parallel chords. Then one will be longer than the other (and may even be a diameter).

Suppose we have  $DH \parallel AC$ .



Drop  $AE \perp DH$  and  $CG \perp DH$ . By alternate interior angles,  $\angle A$  and  $\angle C$  are right angles, as well as  $\angle AEF$  and  $\angle CGF$ . Hence  $ACGE$  is a rectangle.

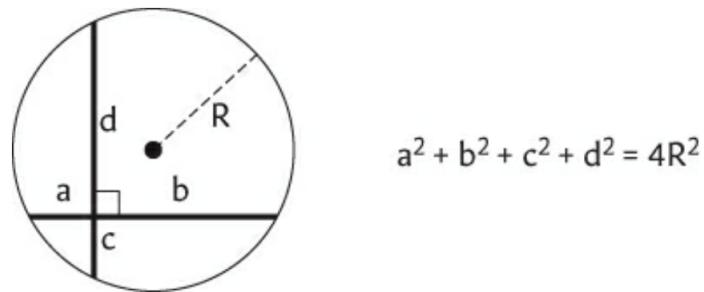
□

Then all the consequences of the forward theorem follow. In particular  $DE = GH$ .

And of course, the perpendicular bisector of  $AC$  is also the perpendicular bisector of  $DH$ . (*Proof.* Draw  $\triangle OAB$  and  $\triangle OCB$  as well as  $\triangle ODF$  and  $\triangle OHF$  to show that all the angles at  $B$  and  $F$  are right angles. □)

### Extraordinary property

According to Acheson, this theorem comes from a book by Malton where it is described as an “extraordinary property of the circle”.



**Fig. 110** An ‘extraordinary property of the Circle’.

Let two chords of a circle meet at right angles, and let the arms of the chords be  $a + b$  and  $c + d$ .

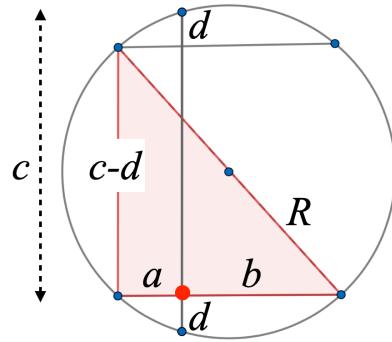
By crossed chords we have  $ab = cd$ .

We will show that the squares of the four components add up to a constant, and that constant is equal to  $(2R)^2$ , twice the radius of the circle, squared.

$$a^2 + b^2 + c^2 + d^2 = (2R)^2 = 4R^2$$

The key to the proof is to form a rectangle.

Pick one of the chords, say  $a + b$ . Draw the diameter of the circle that terminates on one end of each of the chords. The other diameter joins the other ends of chords  $a + b$  but hasn't been drawn.



If the four points at the ends of the two diameters are joined to form a rectangle then the short extension at the top is also equal to  $d$ , so the height of the rectangle is  $c - d$ .

We proved this fact about rectangles in a circle [here](#).

So then:

*Proof.*

This is trivial now that we have  $c - d$ . By the Pythagorean theorem

$$\begin{aligned} (a + b)^2 + (c - d)^2 &= (2R)^2 \\ &= a^2 + b^2 + c^2 + d^2 + 2ab - 2cd \end{aligned}$$

But  $ab = cd$  so

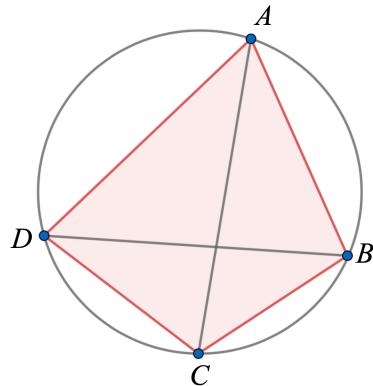
$$4R^2 = a^2 + b^2 + c^2 + d^2$$

□

# Chapter 24

## Cyclic quadrilateral

There is a wonderfully simple theorem about quadrilaterals (Euclid III.22). A cyclic quadrilateral is a four-sided polygon whose four vertices all lie on one circle.



- For *any* cyclic quadrilateral, the opposing angles are supplementary (they sum to two right angles).

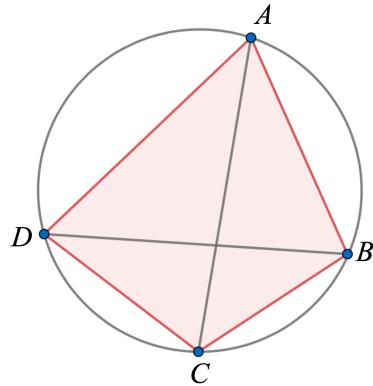
*Proof.*

Together, opposing angles in a cyclic quadrilateral exactly correspond to the whole arc of the circle.

Since the central angle for that arc is four right angles, the sum of opposing inscribed angles is just one-half that or two right angles.

□

Euclid's proof uses sum of angles:



*Proof.*

$$\angle ADC = \angle ADB + \angle BDC, \text{ subtended by arcs } AB \text{ and } BC.$$

As angles on equal arcs (III.21) the latter two angles are equal to  $\angle ACB$  and  $\angle BAC$ .

But by I.32, the same two angles plus the total angle at vertex  $B$  are equal to two right angles.

□

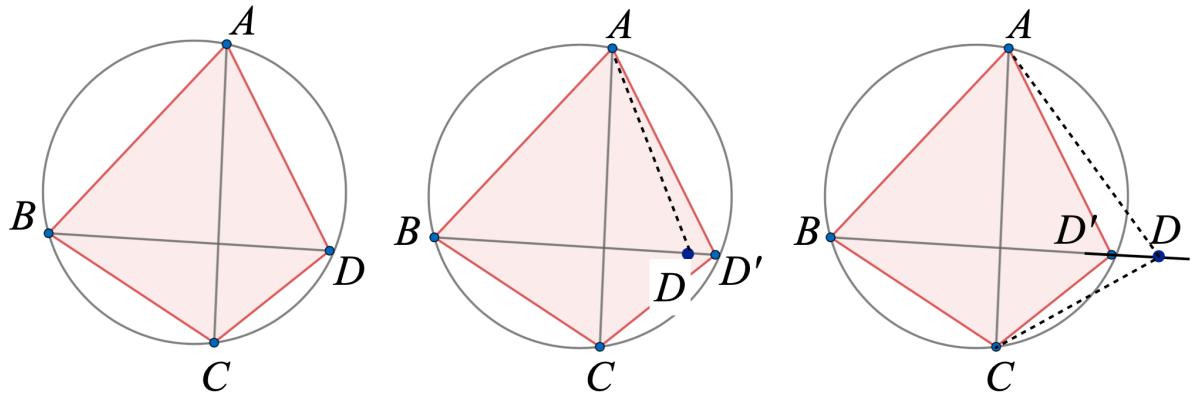
### converse of cyclic quadrilateral theorem

This one uses Euclid III.22 rather than III.21.

Let  $\triangle ABC$  lie on a circle.

Let point  $D$  be such that  $\angle ADC$  is supplementary to  $\angle ABC$ .

Then  $D$  lies on the same circle.



*Proof.*

Aiming for a contradiction, suppose  $D$  does not lie on the circle.

Let  $D$  be external and  $D'$  lie on the point where  $BD$  cuts the circle (right panel).

By the forward theorem,  $\angle AD'C$  is supplementary to  $\angle ABC$  and so equal to  $\angle ADC$ .

But by Euclid I.21,  $\angle ADC < \angle AD'C$ .

This is a contradiction. Therefore  $D$  is not external.

A similar argument will show that  $D$  is not internal.

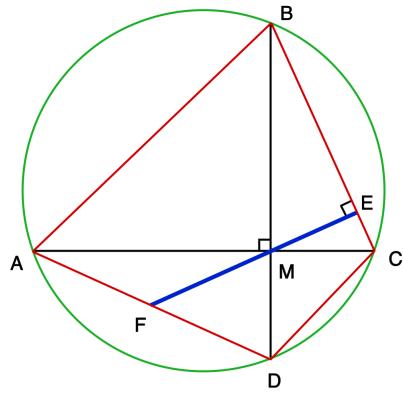
Therefore  $D$  lies on the circle and  $D$  and  $D'$  are the same point.

□

## Brahmagupta's theorem

This is a theorem credited to Brahmagupta.

[https://en.wikipedia.org/wiki/Brahmagupta\\_theorem](https://en.wikipedia.org/wiki/Brahmagupta_theorem)



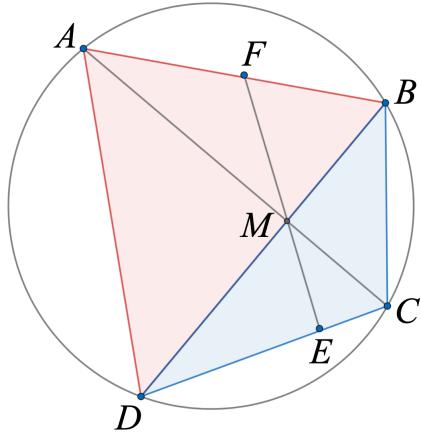
Given  $ABCD$  is a cyclic quadrilateral with diagonals that cross at right angles.

Given  $EF \perp DC$ .

Show that  $AF = BF$ .

*Proof.*

We redraw the figure from Wikipedia slightly



We are given that  $\triangle AMB$  is right.

If  $AF = BF$ , then  $MF = AF = BF$  by the right triangle midpoint theorem.

This suggests we try to show that  $\triangle AFM$  and  $\triangle MFB$  are isosceles.

Start with  $\triangle AFM$ .

We see that  $\angle MAF$  is subtended by chord  $BC$ , but so is  $\angle MDE$ , so they are equal.

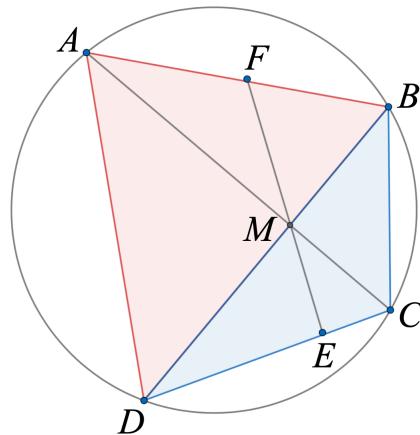
We're given the  $\triangle MDC$  is right and that  $ME$  is the altitude.

It follows that  $\angle CME = \angle MDC$  by similar triangles.

Then  $\angle AMF = \angle CME$  by vertical angles.

We have then,  $\angle AMF = \angle MAF$ .

$\triangle AFM$  is isosceles and so  $AF = MF$ .



For the other one, we need  $\angle FMB = \angle FBM$ .

$\angle MBF = \angle MCD$ , as inscribed angles on the same arc.

And  $\angle MCD = \angle DME$  by similar triangles.

$\angle DME = \angle FMB$  by vertical angles.

It follows that  $\angle FMB = \angle MBF$ .

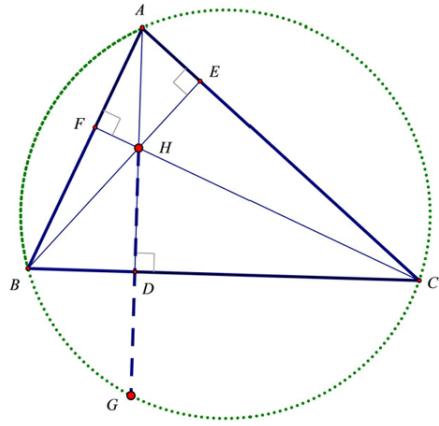
$\triangle MFB$  is isosceles and so  $MF = BF$  by I.6.

Equating equals:  $AF = BF$ .

□

## problem

Arcs of a circle often simplify problems. Draw the altitudes and orthocenter of a triangle and draw its circumcircle.

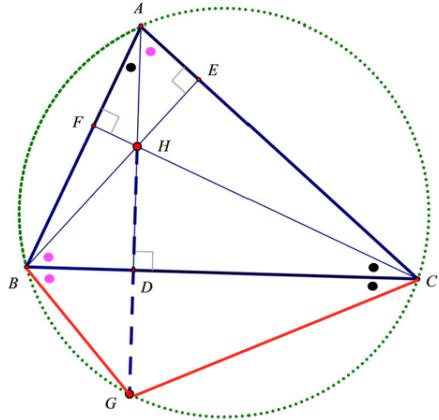


Posamentier gives this relationship. We claim that  $HD = DG$ .

*Proof.*

It seems clear that this must be a consequence of a congruence: it looks like  $\triangle BCH \cong \triangle BCG$ . How to prove that?

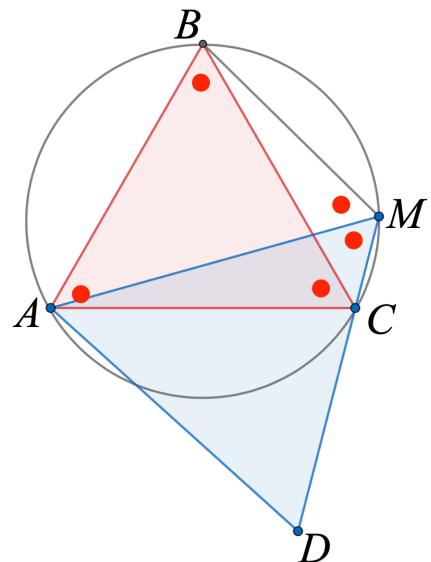
By looking at arcs subtended, we can get relationships for the new angles in  $\triangle BCG$ , and by similar triangles we can get the others shown:



We have congruent triangles by ASA, the component right triangles are congruent for the same reason. Or just say that the altitudes of congruent triangles are also equal.

□

## Van Schooten's theorem



This is given as a problem by Surowski (1.3.6).

Given an equilateral triangle  $ABC$  draw its circumcircle.

Draw an arbitrary line segment from vertex  $A$  through side  $BC$  to meet the circle at  $M$ . Prove that  $AM = BM + MC$ .

This is easy to prove as a special case of Ptolemy's theorem.

Nevertheless, we do it as suggested.

*Proof.*

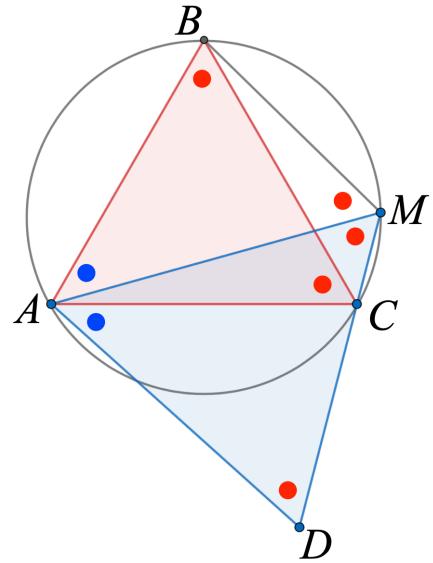
Draw lines from  $M$  to each vertex and extend one to give  $AM = MD$ .

By inscribed angles both angles at  $M$  are equal to two-thirds of a right angle.

$\triangle AMD$  is isosceles with the vertex equal to two-thirds of a right angle.

Thus  $\triangle AMD$  is equilateral.

Subtract the central  $\angle MAC$  from two equal angles to yield  $\angle BAM = \angle DAC$ .



$$\triangle BAM \sim \triangle CAD.$$

But  $BA = CA$  so  $\triangle BAM \cong \triangle CAD$  by ASA.

It follows that  $BM = CD$ .

Thus  $MC + CD = MC + BM = MD = AM$ .

□

I saw this question on the internet: “is a parallelogram a cyclic quadrilateral?”

In a parallelogram, opposing angles are equal. In a cyclic quadrilateral, opposing angles are supplementary. The only supplementary, equal angles are two right angles. So a rectangle is the only parallelogram that is a cyclic quadrilateral.

# **Part VI**

## **Practice with proofs**

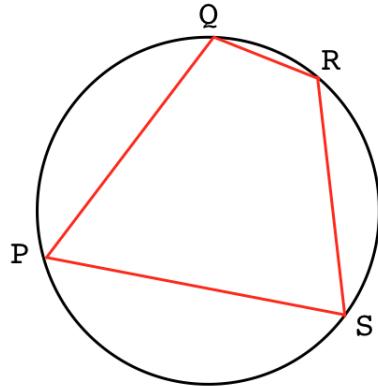
# Chapter 25

## Ptolemy

Ptolemy was a Greek astronomer and geographer who probably lived at Alexandria in the 2nd century AD (died c.168 AD). That is nearly 500 years after Euclid. (Ptolemy was a popular name for Egyptian pharaohs in earlier centuries).

Our Ptolemy is known for many works including his book the *Almagest*, and important to us here, for a theorem in plane geometry concerning cyclic quadrilaterals. These are 4-sided polygons with all four vertices on the same circle.

Recall that any triangle lies on a circle, so this is a restriction on the fourth vertex of the polygon.

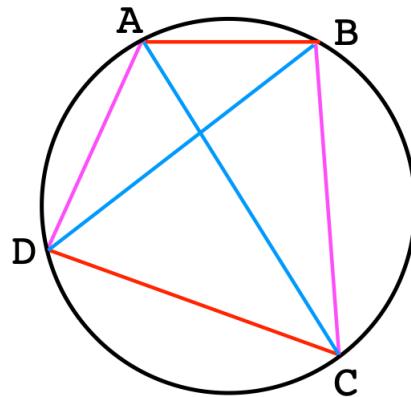


Recall **quadrilateral supplementary theorem** (Euclid III.22):

- For *any* quadrilateral whose four vertices lie on a circle, the opposing angles are

supplementary (they sum to  $180^\circ$ ).

Now, draw the diagonals  $AC$  and  $BD$ . Ptolemy's theorem says that if we take the products of the two pairs of opposing sides and add them, the result is equal to the product of the diagonals.

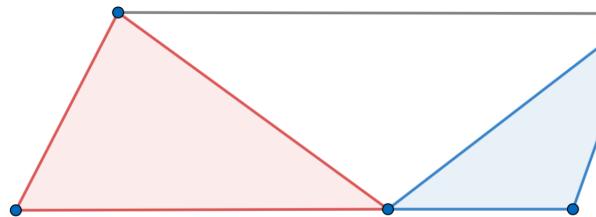


$$\textcolor{red}{AB} \cdot \textcolor{blue}{CD} + \textcolor{blue}{BC} \cdot \textcolor{red}{AD} = \textcolor{blue}{AC} \cdot \textcolor{blue}{BD}$$

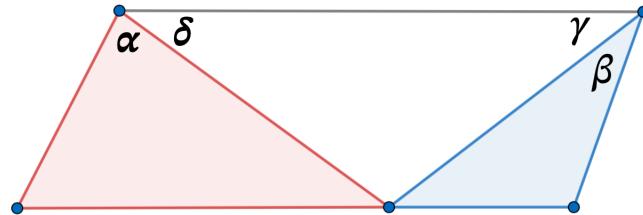
Notice that in the special case of a rectangle, the two diagonals are equal. Given the proof here, Ptolemy's result furnishes a trivial proof of the Pythagorean theorem (see later).

*Proof.*

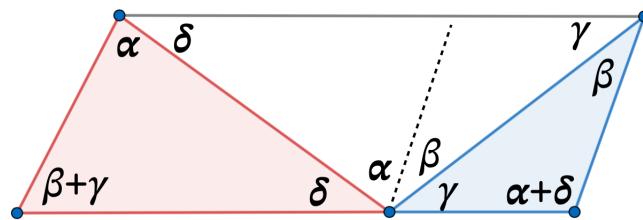
We're going to do a famous dissection of a cyclic quadrilateral as a proof of Ptolemy's theorem, but as a twist on the usual approach we'll do it in reverse, starting with a parallelogram. I'm hoping this will make the whole thing clearer.



We have a parallelogram, and we've picked a point along the bottom edge and drawn lines to the opposing vertices. Let's label some angles.

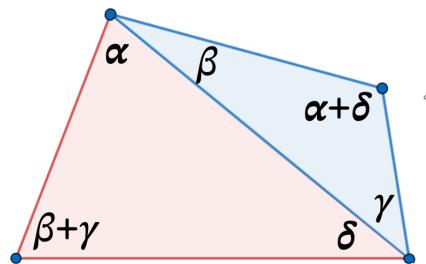


And then use the properties of parallels to get the rest of them labeled as well.

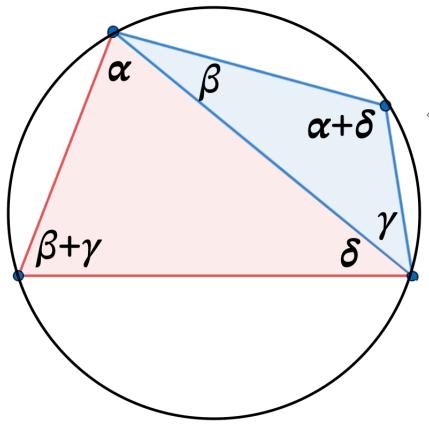


As a parallelogram, we have opposing angles equal, and adjacent angles summing to 180 as they must for parallel lines.

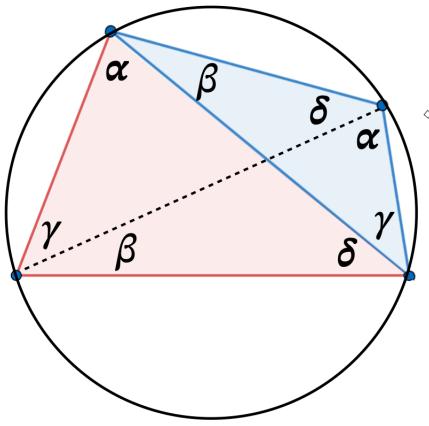
Now do the dissection. Cut out the red triangle and the blue triangle and join them as shown. In general, the scale of one (or even both) will have to be adjusted so that they form a quadrilateral with that edge as the diagonal. We'll return to this point in a minute.



This is a cyclic quadrilateral, by the converse of the theorem on cyclic quadrilaterals, since opposing angles are supplementary.

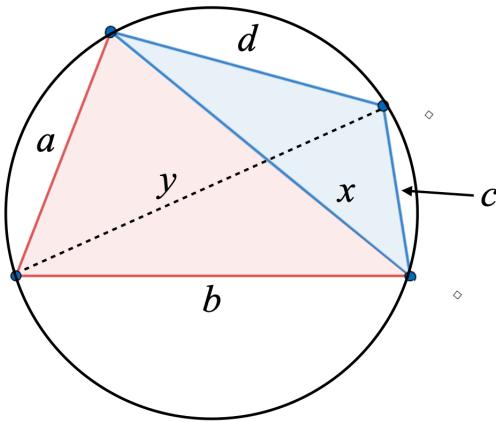


We can use the corollary of the inscribed angle theorem to draw the other diagonal and assign angles.

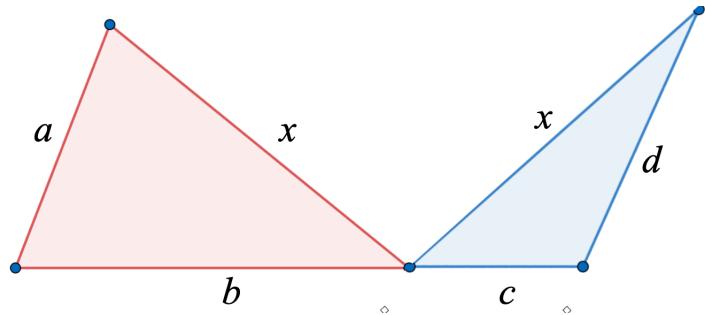


We're going to switch our attention to the lengths of the sides and diagonals, but before we do, notice a very special triangle (partly red and partly blue) with angles  $\gamma, \delta$ , and  $\alpha + \beta$ .

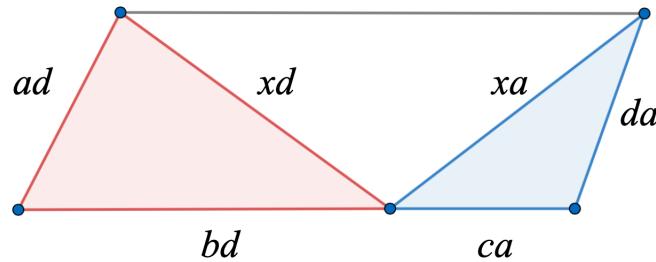
That triangle has sides  $a, d$  and  $y$ .



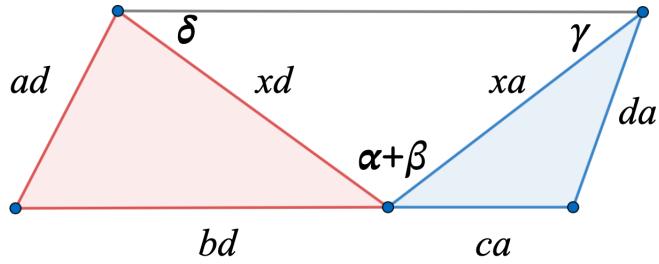
Let's reverse the process, dissecting the cyclic quadrilateral by cutting along the  $x$  diagonal, and arranging the pieces so that the bases are collinear as in the original parallelogram.



In the general case,  $a \neq d$ . But we can scale the two sides to be equal. An easy way to do that is to cross multiply.

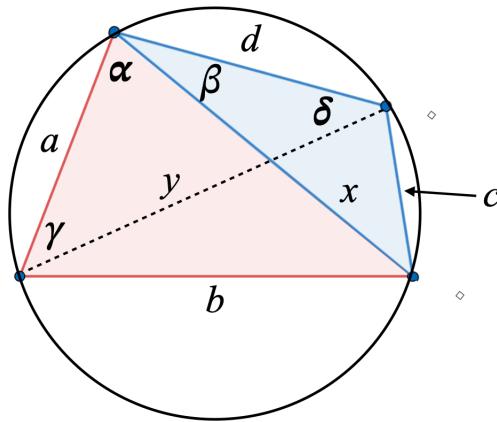


We have a parallelogram again. The angles are correct, and now the sides are equal. So are the top and bottom equal. We focus now on the white triangle.

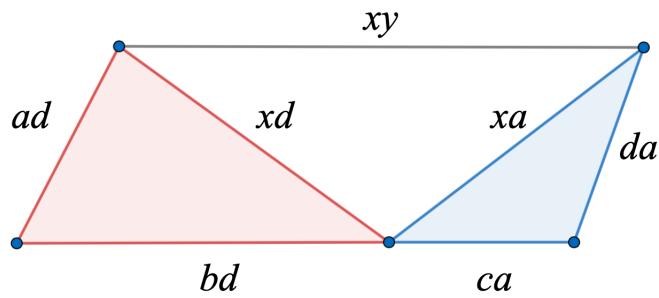


It has two sides with lengths  $a$  and  $d$ , each scaled by a factor of  $x$ , and the two angles opposite those sides are  $\delta$  and  $\gamma$ .

Remember the special triangle?



We can see it in the cyclic quadrilateral. The sides are  $a, d, y$ . In the parallelogram, that white triangle must be scaled by a factor of  $x$ , giving sides  $xa, xd$  and  $xy$ .

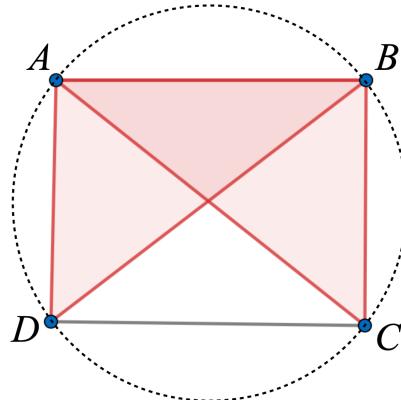


Since this is a parallelogram, the top and bottom are equal. Namely,  $ac + bd = xy$ .

This is Ptolemy's theorem.

□

### Pythagorean theorem from Ptolemy



*Proof.*

Consider an arbitrary rectangle, with all four vertices as right angles, and opposing sides equal.

The diagonals are equal, by our theorem on diagonals of a rectangle, and they bisect one another, by a general theorem about diagonals of a quadrilateral. Therefore, a circle can be drawn whose center is the point where they cross and that circle contains the four vertices.

We note that one can also start with a circle, then draw a diameter and any other point, forming a right triangle, by the converse of Thales' circle theorem. Draw the congruent triangle in the other hemisphere by SSS. We have opposite sides equal and at least one right angle, so the polygon is a rectangle.

Application of Ptolemy's theorem gives:

$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

but by equal sides in a rectangle and diameters in a circle:

$$AB^2 + BC^2 = AC^2$$

□

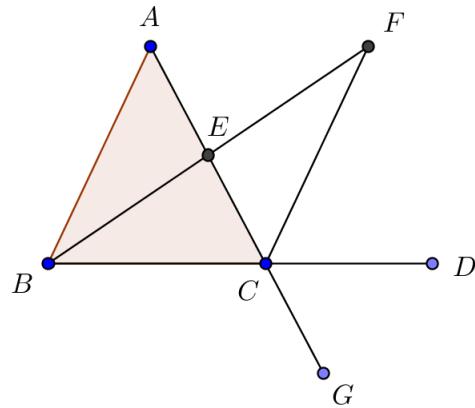
# Chapter 26

## Elements

In this chapter we will study some more *Propositions* from the first volume of Euclid's *Elements*. They are short, sweet and powerful.

### Euclid I.16

- In any triangle, if one of the sides is produced (extended), then the exterior angle is greater than either of the interior and opposite angles.



In  $\triangle ABC$  extend side  $BC$  to  $D$ . We are concerned with the exterior  $\angle ACD$ .

The claim is that  $\angle ACD$  is greater (larger) than either of the interior angles  $\angle ABC$  or  $\angle BAC$ .

*Proof.*

Find the midpoint of side  $AC$  at  $E$  so  $AE = EC$  and then draw  $BECF$  so that  $BE = EF$ . Note that  $\angle ECD$  is the same angle as  $\angle ACD$ .

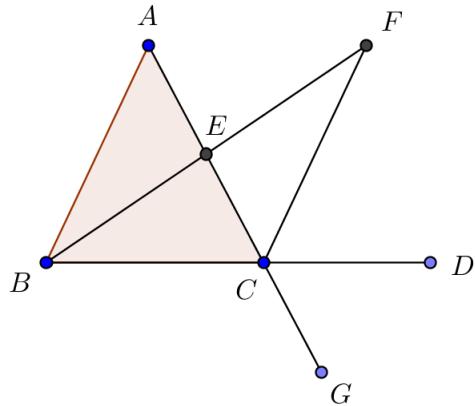
We have equal angles at  $E$  by the vertical angle theorem. So by SAS the two smaller triangles  $\triangle BEA$  and  $\triangle CEF$  to the left and right are congruent.

Thus  $\angle BAE = \angle ECF$ . Since the whole is greater than the part,  $\angle ECD > \angle ECF = \angle BAE$ .

We can make a similar construction and proof for  $\angle ABC$ .

The exterior angle is greater than either of the interior and opposite angles.

□



One might ask why Euclid doesn't use supplementary angles to obtain a stronger proof. The main reason is that he has not yet proved the triangle sum theorem. That is Euclid I.32.

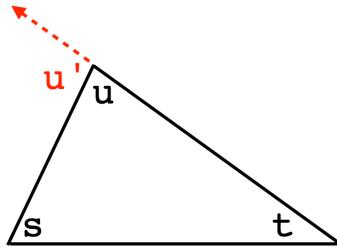
However, we did previously show a proof of the triangle sum theorem, that the sum of angles in any triangle is equal to two right angles.

And you were asked then to find the relationship between an external angle and the angles of the corresponding triangle. Here is our version of that proof.

## external angle theorem

- the external angle is equal to the sum of two internal angles

*Proof.*



As supplementary angles,  $u + u' =$  two right angles. As the three angles of a triangle,  $s + t + u =$  two right angles as well.

But things equal to the same thing are equal to each other:

$$u + u' = s + t + u$$

Subtracting equals from equals:

$$u' = s + t$$

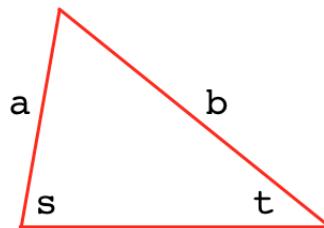
□

This relationship is fundamental.

The next theorem is also extremely useful, and it follows from Euclid I.16.

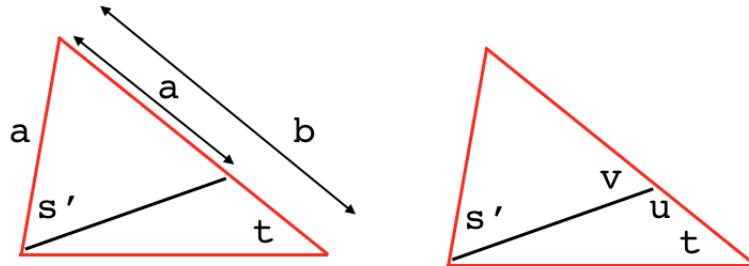
## Euclid I.18

- Comparing two sides in any triangle, if one side is longer than the other, then the angle opposite is larger.



*Proof.*

Given that side  $b > a$ , mark off  $a$  on  $b$ .



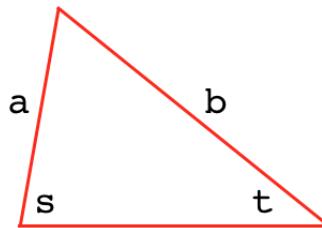
By the external angle theorem (I.16)

$$v > t$$

But  $v = s'$  (by Euclid I.5) so

$$s' > t$$

And since  $s > s'$



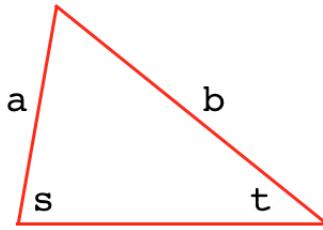
$$s > s' > t$$

□

We get the converse almost for free.

### Euclid I.19

- Comparing two angles in any triangle, if one angle is larger than the other, then the side opposite is longer.



*Proof.*

We are given  $s > t$  and claim that  $b > a$ .

We proceed by eliminating the other two possibilities. Rewrite the order of terms as  $t < s$ , with the claim  $a < b$ .

It cannot be that  $a = b$  because then  $s = t$  by isosceles  $\triangle$  (Euclid I.5), but we are given  $s > t$ .

So then suppose  $a > b$ . By the previous proposition (Euclid I.18), we would have that  $t > s$ . But this is again contrary to what we were given.

Since the other two possibilities are eliminated, we must have  $a < b$ .

□

We have made use of what's called a trichotomy. There are only three possibilities:

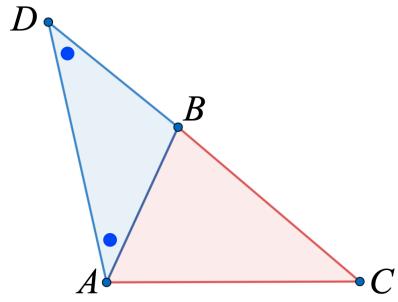
$$a < b, \quad a = b, \quad a > b$$

This applies to line segments and angles as well as many other things.

## Euclid I.20. Triangle inequality

- In any triangle, the longest side is smaller than the sum of the two shorter sides.

Clearly, if a given side is not the longest, so it is shorter than one of the others, then it must also be shorter than the sum of that other one plus the third. For this reason, we consider only the longest side.



Given that side  $AC$  is the longest in  $\triangle ABC$ .

Extend side  $BC$  so that  $BD = AB$ . By Euclid I.5,  $\triangle ABD$  is isosceles, so the angles with blue dots are equal.

Then  $\angle D$  is smaller than  $\angle DAC$  and therefore, by Euclid I.19 just above,  $AC$  is less than  $DC$ .

But  $DC$  is equal to the sum of the two smaller sides of  $\triangle ABC$ . Hence

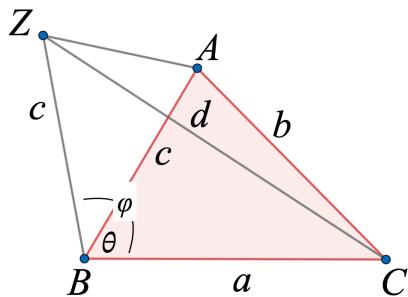
$$AC < AB + BC$$

□

An equivalent statement is the famous “a straight line is the shortest distance between two points”.

The triangle inequality has corresponding statements and proofs in other parts of mathematics including real and complex analysis.

### Euclid I.24. SAS inequality theorem



The next theorem is called the SAS inequality theorem, or informally, the “hinge” theorem.

- If two triangles have two pairs of sides which are the same length, the triangle with the larger included angle also has the larger third side.

We have  $\triangle ABC$  with sides  $a$  and  $c$  flanking  $\angle\theta$  and  $\triangle ZBC$  with sides  $a$  and  $c$  flanking  $\angle\phi$ , and  $\theta < \phi$ . We claim that side  $b$  in  $\triangle ABC$  is smaller than side  $d$  in  $\triangle ZBC$ .

Draw the two triangles nestled inside one another. Notice that  $\triangle AZB$  is isosceles, so  $\angle AZB = \angle ZAB$ .

*Proof.*

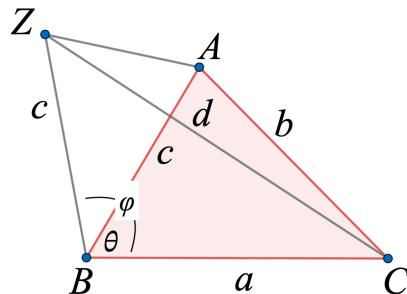
$$\angle AZC < \angle AZB \quad \text{and} \quad \angle ZAB < \angle ZAC$$

From above

$$\angle AZB = \angle ZAB$$

So

$$\angle AZC < \angle ZAC$$



By Euclid I.19, it follows that side  $d$  opposite the larger angle  $\angle ZAC$  is larger than side  $b$  opposite the smaller angle  $\angle AZC$  in  $\triangle AZC$ .

At the same time, in the two triangles  $\triangle ABC$  and  $\triangle ZBC$  the smaller angle  $\theta$  is opposite the smaller side  $b$ , while the larger angle is  $\phi$ , opposite the longer side,  $d$ .

□

[https://proofwiki.org/wiki/Hinge\\_Theorem](https://proofwiki.org/wiki/Hinge_Theorem)

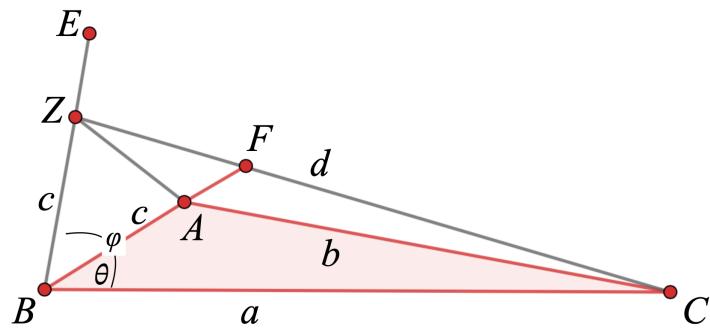
Note that Euclid I.24 says that where the included angle is larger, then the side opposite is larger. The converse is Euclid I.25.

*Proof.* Aiming for a contradiction, given side  $d$  is smaller than side  $b$ , suppose that  $\angle ABC < \angle ZBC$ . The forward theorem gives  $d > b$ . This is a contradiction.  $\square$

Another subtlety is that it cannot be that  $b = d$ , by Euclid I.7, and also then the two triangles would be congruent by SSS, but the angles differ, so this is a contradiction.

One last point: this proof suffers from the same issue that Euclid I.7 does, namely that a different drawing can be made that makes the proof non-sensical. The solution is the same as well.

*Proof.*



Since  $\triangle ABZ$  is isosceles, the base angles are also equal, namely  $\angle AZE = \angle ZAF$ . So then

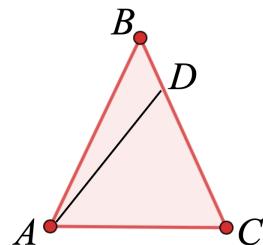
$$\angle AZF < \angle AZE = \angle ZAF < \angle ZAC$$

It follows that since the greater side is opposite the greater angle in  $\triangle ACZ$ ,  $d > b$ .

$\square$

This is enough of the *Elements* to give us a good taste of the basics of Greek geometry of lines and triangles, and methods of proof.

## problem



Let  $\triangle ABC$  be isosceles, and  $D$  lie on  $BC$ . Using Euclid I.19, prove that  $DC < DA$ . Separately, use I.20 for the same proof.

Here is another idea. *Proof.* (Sketch). Let the perpendicular bisector of  $AC$  be  $BE$ . Drop another perpendicular from  $D$  to  $AC$  at  $F$ . So  $E$  is the midpoint and  $F$  lies directly below  $D$ .

Clearly,  $AF > AE > FC$ . So

$$FC^2 < AF^2$$

$$FC^2 + DF^2 < AF^2 + DF^2$$

But by the Pythagorean theorem, the left-hand side is  $DC^2$ , while the right-hand side is  $DA^2$ . Taking side lengths as the positive square root, we have  $DC < DA$ .  $\square$

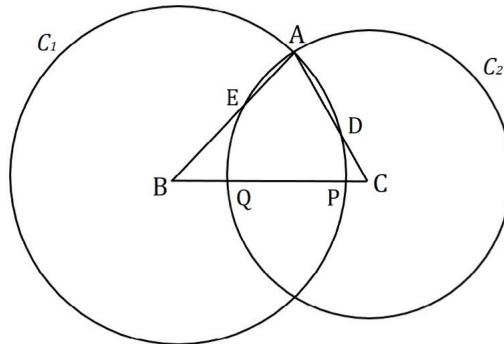
In fact, *every* point which lies on the same side of the perpendicular bisector of  $AC$  as  $C$ , is closer to  $C$  than to  $A$ .

## Triangle inequality by circles

Here is a modern proof of Euclid I.20.

*Proof.*

Consider  $\triangle ABC$ . Draw circle  $\mathcal{C}_1$  on center  $B$  with radius  $AB$  and also circle  $\mathcal{C}_2$  on center  $C$  with radius  $AC$ . The third vertex  $A$  lies on the intersection of the two circles.



If  $BC$  is longer (greater) than the sum of the two radii, then the circles will not cross one another.

If  $BC$  is equal to this sum, then they only touch, on the line  $BC$ . Thus, in order for there to be a triangle,  $BC$  must be less than the sum of the diameters:  $BC < AB + AC$ .

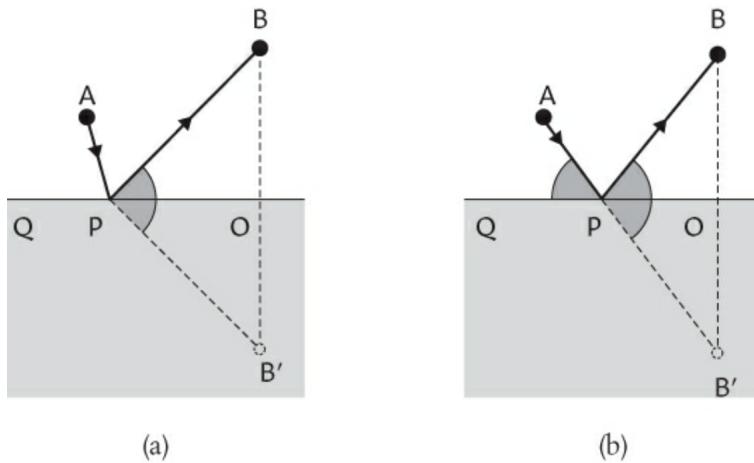
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<https://arxiv.org/pdf/1803.01317>

## angle of reflection

Suppose we shine a light at a mirror, or just look at the reflection of someone else, or even our own outstretched hand. The question arises, what determines the path of the light or the image as it travels to the eye? Heron of Alexandria discovered a proof of the answer in about 100 A.D.

The diagram shows two *possible* paths light might take, but there is only one path that it *does* take, shown in the right panel. Light takes the path where it reaches our eyes the fastest.



**Fig. 46** Finding the shortest path.

What is this path? What is the angle the ray of light makes with the mirror? This angle is called the angle of reflection.

Draw  $\triangle POB'$ , imagined to be on the other side of the mirror, with  $B$  the same distance away from the mirror as  $B'$ , but on the other side. The two triangles  $\triangle BPO$  and  $\triangle B'PO$  are congruent by SAS.

Clearly, the shortest distance from  $A$  to  $B'$  is a straight line, by the triangle inequality.

So the result is that  $\angle APQ$  equals  $\angle B'PO$ , which (by congruent triangles) equals  $\angle BPO$ . This is usually stated as "the angle of incidence is equal to the angle of reflection."

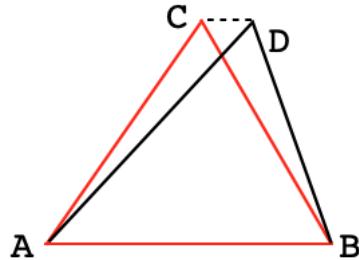
## smallest perimeter

Show that for two triangles with the same area, an isosceles triangle has the *smallest* perimeter.

*Proof.*

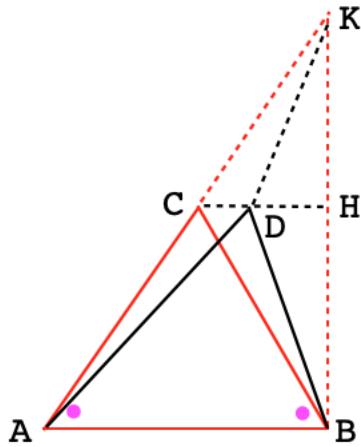
We suppose that the base of the isosceles triangle  $\triangle ABC$  is equal to one of the sides of the other triangle  $\triangle ABD$ . If we would need to re-scale one side to have equality, we could then make a corresponding change in the altitude to that side to maintain equal area.

The equal area constraint means that points  $C$  and  $D$  lie along a horizontal line parallel to the common base  $AB$ .

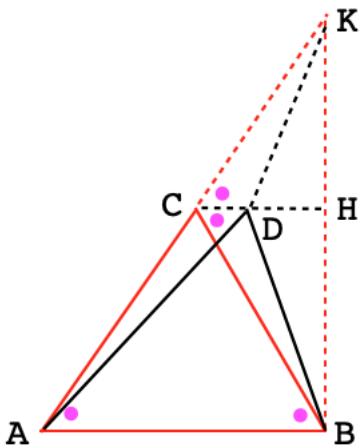


Draw a vertical from  $B$  to meet the extension of  $AC$  at  $K$ . Extend  $CD$  to meet  $KB$  at  $H$  and also draw  $DK$ .

We're given that  $AC = BC$  and so  $\angle CAB = \angle CBA$  (magenta dots).



It follows that other angles are also equal to those two (by alternate interior angles).



$$\angle BCH = \angle ABC \text{ and } \angle KCH = \angle CAB.$$

Since  $\angle CAB = \angle ABC$ , it follows that  $\angle BCH = \angle KCH$ .

The angles at  $H$  are right angles, since  $BK$  is perpendicular to  $AB$  and  $CH$  is parallel to  $AB$ .

Therefore  $\triangle CHK \cong \triangle BCH$  by ASA, so  $BC = CK$ .

Similar reasoning will give that  $BD = DK$ .

But now by the triangle inequality:

$$AC + CK < AD + DK$$

Substituting from above

$$AC + BC < AD + BD$$

Add  $AB$  to both sides

$$AC + BC + AB < AD + BD + AB$$

The perimeter of  $\triangle ABC$  is less than that of  $\triangle ABD$ .

□

# Chapter 27

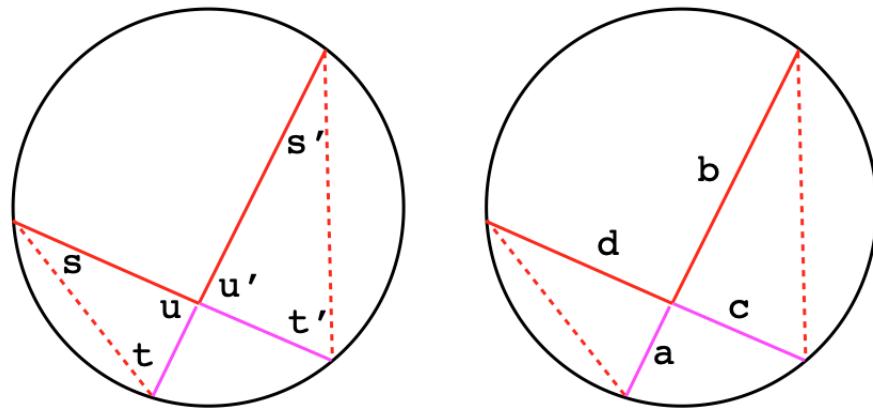
## Secant tangent theorem

### Chord segments

There is a simple algebraic relationship between the line segments formed when any two chords cross in a circle.

Draw two chords of a circle and then draw additional line segments to form two triangles (left panel).

- These two triangles are similar.



*Proof.*

The angles  $s$  is equal to its primed counterpart  $s'$ , because they both are subtended

by the same arc.  $u = u'$  by the vertical angle theorem. We have AAA similarity.

□

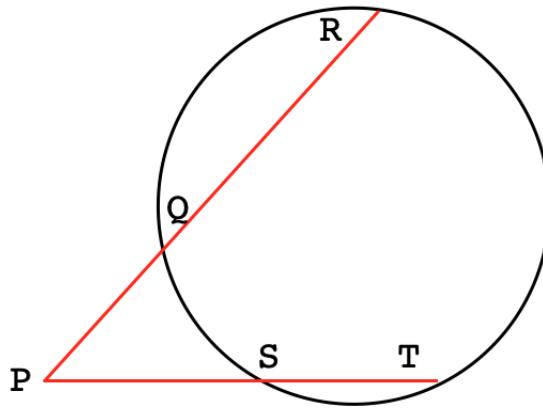
- For any two chords that cross in a circle, the products of the two components of each chord are equal.

In the right panel, the letters refer to sides of the similar triangles. Corresponding sides are indicated by color or style.

$$\begin{aligned}\frac{a}{d} &= \frac{c}{b} \\ ab &= cd\end{aligned}$$

□

Another theorem applies to chords extended from a circle.



$$PQ \cdot PR = PS \cdot PT$$

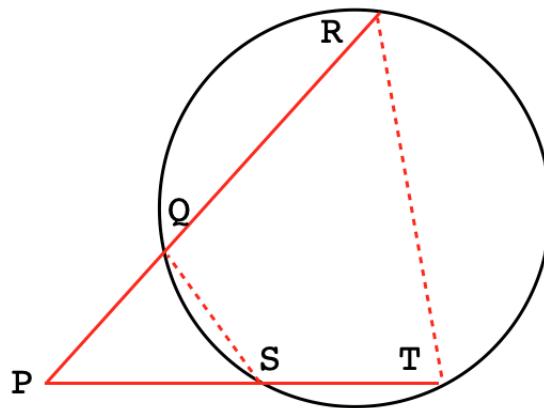
*Hint:* When thinking about such problems, or asked to establish a proof yourself, it is always helpful to consider the form with the ratios:

$$\frac{PQ}{PS} = \frac{PT}{PR}$$

This strongly suggests we look at similar triangles.

*Proof.*

Draw the two triangles.



We showed previously that for any quadrilateral whose four vertices all lie on one circle (a cyclic quadrilateral), the opposing vertices have supplementary angles. Opposing vertices add to  $180^\circ$  because their arc segments add up to one whole circle.

In the figure above, the angle at vertex  $R$  is supplementary to  $\angle QST$ . But  $\angle QST$  is also supplementary to  $\angle QSP$ .

Therefore  $\angle QSP$  is equal to the angle at  $R$ , and  $\triangle PQS$  is similar to  $\triangle PRT$ . The similarity is such that

$$\frac{PS}{PR} = \frac{PQ}{PT}$$

$$PS \cdot PT = PQ \cdot PR$$

Given a point  $P$  outside the circle, the external part of any secant times the entire secant is a constant.

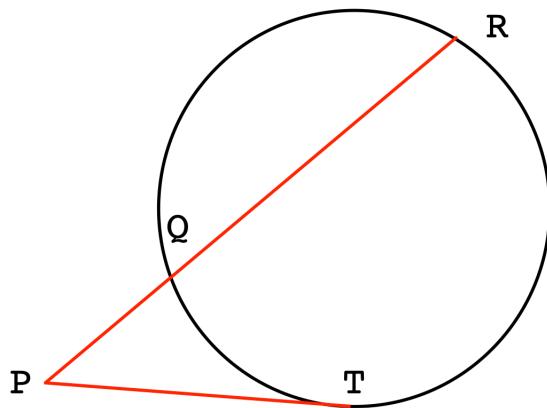
One curious thing about this theorem is that these triangles are similar, and nested, but flipped.

$$\frac{PS}{PQ} = \frac{PR}{PT}$$

□

We can do a bit more.

## Secant-tangent theorem



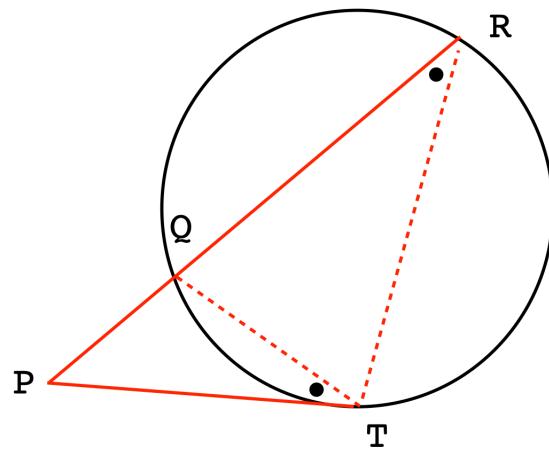
Let the points  $S$  and  $T$  approach each other to become one point. Then  $PT$  will become a tangent of the circle. Previously we had

$$PS \cdot PT = PR \cdot PQ$$

Now we modify it slightly:

$$\begin{aligned} PT \cdot PT &= PR \cdot PQ \\ PT^2 &= PR \cdot PQ \end{aligned}$$

This is the tangent-secant theorem. We can reason our way backward to a sketch of a more formal proof.



$$PT \cdot PT = PR \cdot PQ$$

$$\frac{PT}{PQ} = \frac{PR}{PT}$$

We must have two similar triangles,  $\triangle PQT$  and  $\triangle PRT$ . The angle at vertex  $R$  must correspond to the same arc as  $\angle PTQ$ .

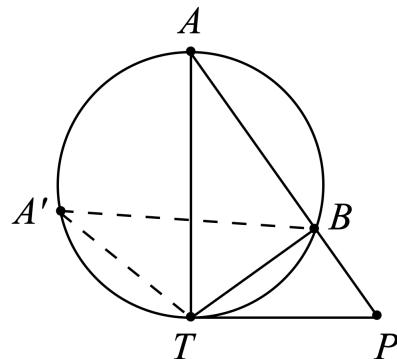
Now run the logic backward and write the proof.

Note: we've seen this sort of reverse logic a few times already. The method has a name! It was called the method of "analysis" by Pappus (320 A.D.), see Posamentier (Introduction).

*Proof.*

The angle  $\angle PTQ$  formed by the tangent  $PT$  and the chord  $QT$  cut off the same arc of the circle as the angle formed at vertex  $R$  (see the last example of the previous chapter), so the two angles marked with black dots are equal.

Here is a proof using similar right triangles.

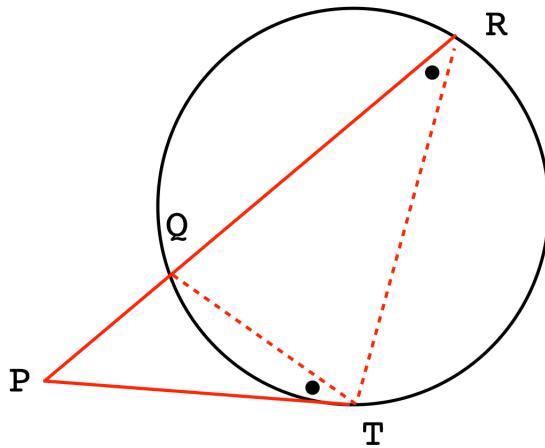


$$\triangle TPA \sim \triangle TBA \sim \triangle PBT$$

so

$$\angle BAT = \angle BTP$$

Therefore, in this figure



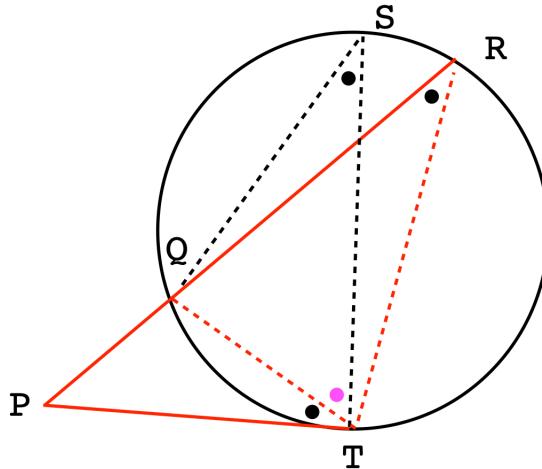
$\triangle PQT$  is similar to  $\triangle PRT$  by AAA, since they also share the angle at  $P$ . This gives

$$\frac{PT}{PQ} = \frac{PR}{PT}$$

which can be rearranged easily to the statement of the theorem.

□

We can simplify this proof even more by drawing a diagonal to the tangent point.



Now it is clear that  $\angle S = \angle R$  because they both sweep out the arc  $QT$ .  $\angle QTS$

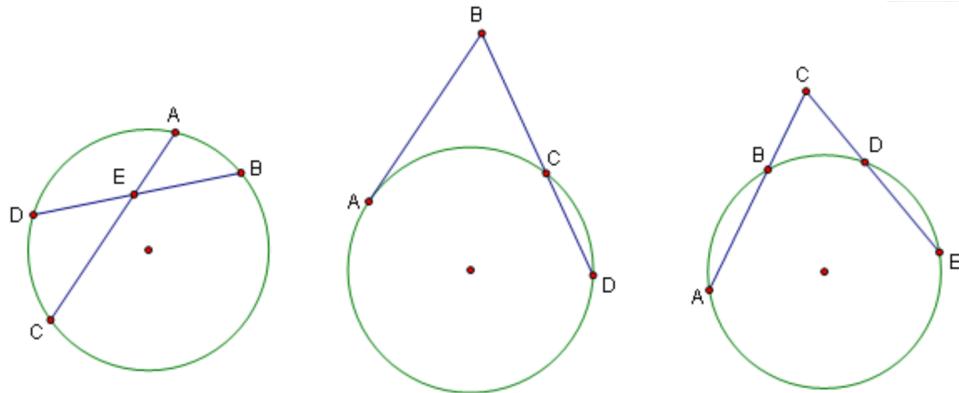
marked in magenta is complementary to the angle at  $S$ , marked with a black dot because  $\angle SQT$  is a right angle, by Thales' theorem.

The same angle marked in magenta is also complementary to  $\angle PTQ$ . Therefore  $\angle PTQ = \angle S = \angle R$ . We continue the proof with analysis of similar triangles, as before.

One application of this theorem is to a determination of the size of the earth. We'll see that later in the book.

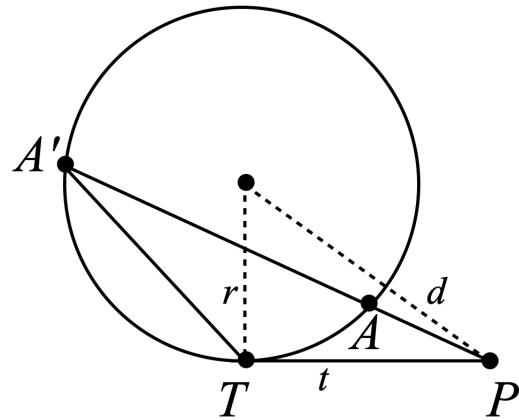
## power of a point

The previous results regarding secants and tangents are sometimes united in a definition called the "power" of a point, which unifies the treatment of points inside and outside the circle.



[https://artofproblemsolving.com/wiki/index.php/Power\\_of\\_a\\_Point\\_Theorem](https://artofproblemsolving.com/wiki/index.php/Power_of_a_Point_Theorem)

Let's start with a point outside.



The length of the tangent and the radius are simply related to the distance from any point  $P$  to the center of the circle,  $d$ :

$$d^2 = t^2 + r^2$$

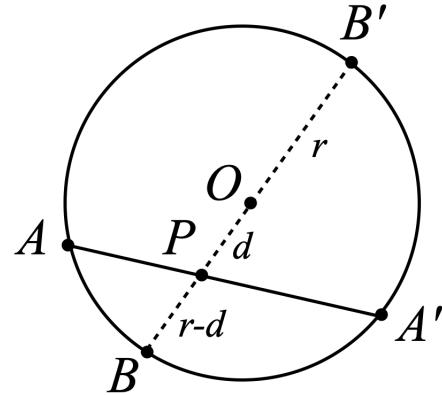
$$t^2 = d^2 - r^2$$

$d^2 - r^2$  is defined as the *power*  $p$  of the point  $P$ .

From our previous work, we know that

$$p = t^2 = PT^2 = PA \cdot PA'$$

The definition also works for a point inside the circle.



By the crossed chords theorem:

$$\begin{aligned} PA \cdot PA' &= (r + d)(r - d) \\ &= r^2 - d^2 = -p \end{aligned}$$

That is, it works if we use *directed line segments*, so that the product  $PA \cdot PA'$  for  $P$  inside the circle has its two components pointing in opposite directions, and thus acquires a minus sign.

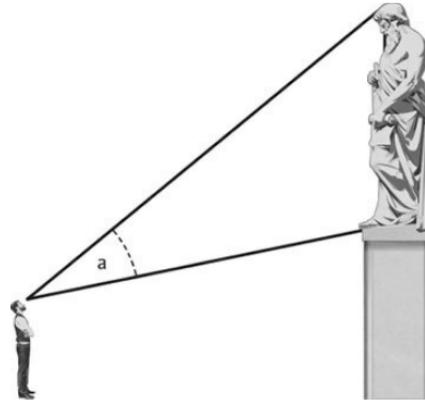
If the point is *on* the circle, it's a bit strange, but we are at the boundary between  $p < 0$  and  $p > 0$ , so it seems reasonable that  $p = 0$  for points on the circle. And there, of course

$$p = d^2 - r^2 = 0$$

## Looking at Euclid

I learned the secant-tangent theorem from Acheson, and continue in the same vein by appropriating an entire example from his book.

Here's the problem. We're looking up at a statue of Euclid on a column.



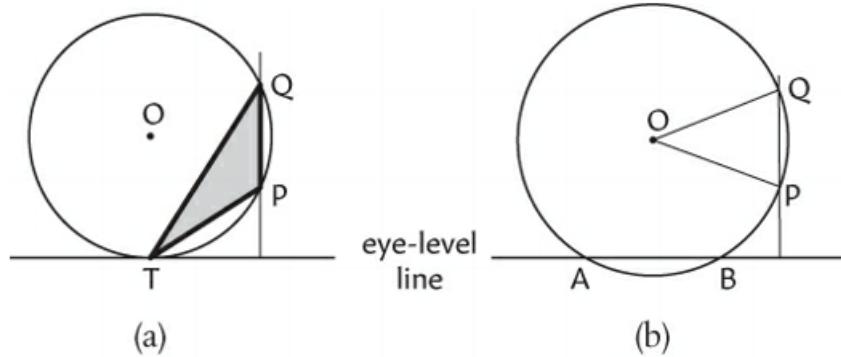
**Fig. 69** Looking at Euclid.

We resist the temptation to make a dumb joke.



In any event, we'd like to get the widest angle view, giving the largest apparent size of the statue. If you get too close, the statue is greatly foreshortened and the angle small, and naturally, it is small at a distance. There must be a best view, in the middle.

Here is Acheson's solution:



**Fig. 70** What's the best view?

Let the foot and head of Euclid be at  $P$  and  $Q$  and draw the circle containing those two points which is also tangent to your eye-level. Then the tangent point provides the best view.

The reason is that a circle through any other horizontal position crosses the eye-level at two points. Such a circle will necessarily be bigger.

Consequently the arc  $PQ$ , which is fixed in size, will be a smaller fraction of the circle.

As a smaller fraction, both the central angle  $\angle POQ$  will be smaller as well as the

angle subtended at  $A$  or  $B$  (one-half of that).

The secant-tangent theorem even gives a quantitative answer.

If  $R$  is the point where the extension of  $QP$  meets eye-level (the ground), then suppose the point  $Q$  is  $h$  units above the ground and  $P$  is  $g$  units. The tangent-secant formula is

$$RT \cdot RT = PR \cdot QR$$

which says that the square of the optimum viewing distance is  $h \cdot g$ . The optimal distance  $d$  is

$$d = \sqrt{hg}$$

I knew I'd seen this problem before, and I find that I wrote it up for my Calculus book. It turns out to be from Acheson's book on Calculus, and is originally about Lord Nelson's statue in Trafalgar Square.

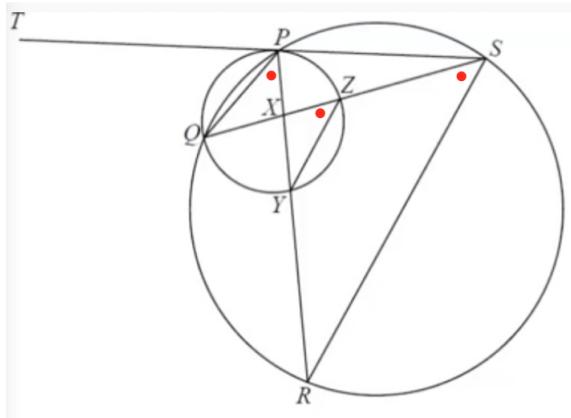
The calculus treatment is not nearly as pretty as reasoning about the tangent; fortunately we came to the same answer.

## Problem

### problem

We continue with a problem that we solved in part [previously](#).

We showed that there are three similar triangles in the figure below



We showed that  $\triangle XYZ \sim \triangle XQP$  and also  $\triangle XYZ \sim \triangle XRS$ .

The last part of the problem says that given  $QS = XR$ , prove that  $PS^2 = XS \cdot YR$ . It is also given that  $TPS$  is tangent to the small circle at  $P$ .

In this chapter, we developed the secant-tangent theorem, which says that the part of the secant outside the circle, multiplied by the whole thing, is equal to the tangent squared:

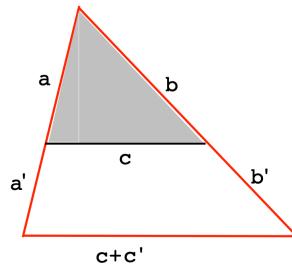
$$QS \cdot ZS = PS^2$$

*Proof.*

By similar triangles, we have that

$$\frac{XZ}{XY} = \frac{XS}{XR} = \frac{ZS}{YR}$$

That last equality requires some algebra. We use this figure again:



We have that

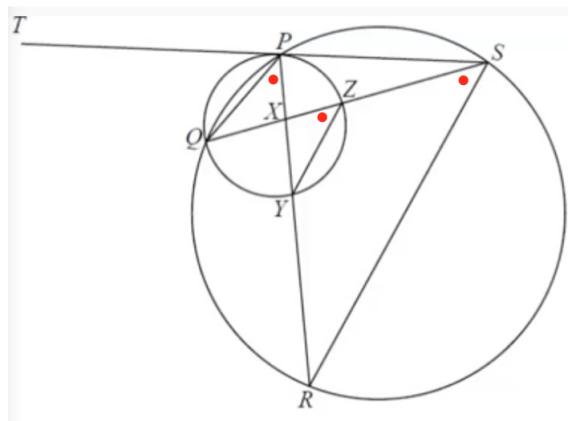
$$\frac{a}{b} = \frac{a + a'}{b + b'}$$

$$\begin{aligned}\frac{a}{a+a'} &= \frac{b}{b+b'} \\ \frac{a+a'}{a} &= \frac{b+b'}{b} \\ \frac{a'}{a} &= \frac{b'}{b} \\ \frac{a}{b} &= \frac{a'}{b'}\end{aligned}$$

The partial sides are in the same ratio as the whole.

Back to our problem. Multiplying by the denominators:

$$XS \cdot YR = XR \cdot ZS$$



We're given that  $XR = QS$

$$XS \cdot YR = QS \cdot ZS$$

but by the secant-tangent theorem

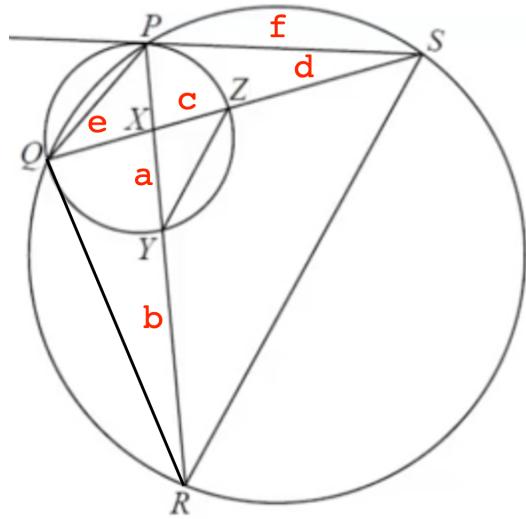
$$QS \cdot ZS = PS^2$$

so we have

$$XS \cdot YR = PS^2$$

□

I really dislike doing algebra with  $XS$  and the rest, so I substituted single letters for the sides and fiddled with the algebra while working backward (analysis) until I could see the answer.



The secant-tangent theorem says that  $f^2 = d \cdot (c+d+e)$ . We're given that  $(c+d+e) = (a+b)$ .

So  $f^2 = d \cdot (a+b)$ . We are asked to prove that this is equal to  $(c+d) \cdot b$ .

Forming ratios we have

$$\frac{b}{d} = \frac{a+b}{c+d}$$

But this is just the result that the partial sides are in the same ratio as the whole.

Reverse (and substitute  $XS$  etc.) to write the proof.

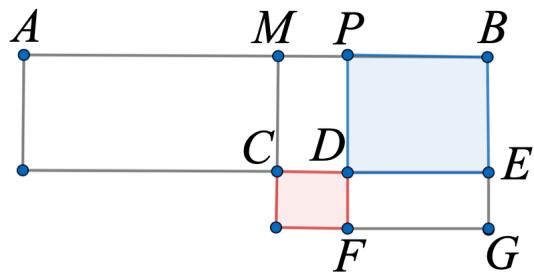
# Chapter 28

## Power of a point

We're going to look at how Euclid was able to establish the **crossed chord theorem** and **secant tangent theorem**.

The main purpose is to appreciate how much he was able to do (up to Book III of *Elements*) without a theory of similar figures, using only the Pythagorean theorem. But we start with two propositions from Book II, which deal with what might be termed geometric algebra.

### Euclid II.5



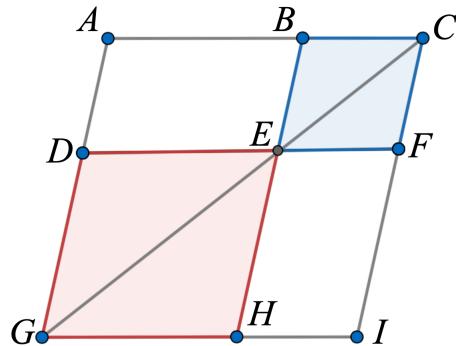
In this figure, the line  $AB$  is bisected at  $M$  and then the point  $P$  placed somewhere within  $MB$ . The first proposition, II.5, says that

$$AP \cdot PB + MP^2 = MB^2$$

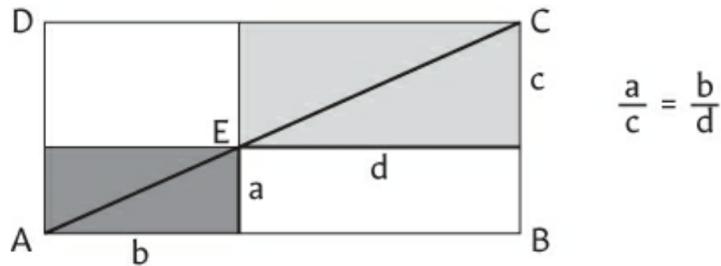
where  $AP \cdot PB$  is the area of rectangle  $AD$ ,  $PE$  is the square on  $PB$ , with  $PB = BE$ , and  $CF$  is the square on  $CD$ , with  $CD = MP = EG$ .

The construction contains two sets of equal rectangles. The first is  $AC = ME = BF$ . And the second is  $MD = DG$ . This is by I.43.

In the figure below, the two white parallelograms are equal.



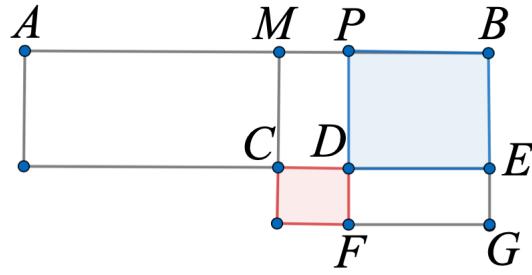
We saw an example very early for rectangles as the [area-atio theorem](#).



**Fig. 42** Area and similarity.

$$ad = bc$$

Returning to the present theorem



As noted, the construction contains two sets of equal rectangles. The first is  $AC = ME = PG$ , while the second is  $MD = DG$ .

Here's the neat idea:  $AC + MD + PE = AE$ .

But this is equal to  $MD + PE + DG$ , since  $AC = DG$ . The latter area is an L-shaped piece called a *gnomon*.

So then if we add  $CF$  (equal to  $MP^2$ ) we have  $MG$  (equal to  $MB^2$ ).

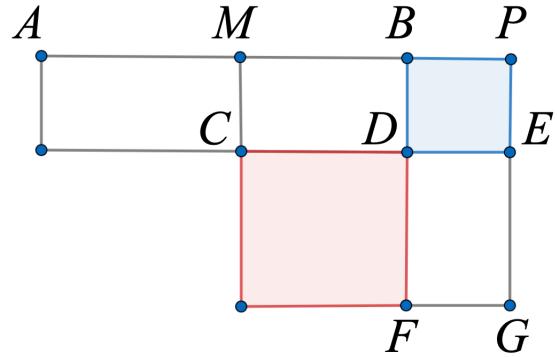
$$AP \cdot PB + MP^2 = MB^2$$

I find this more difficult to remember than I would like. But it is easy to check using algebra. Let  $x = AM$  and  $y = MP$  so  $AP = x + y$  and  $PB = x - y$  and then

$$AP \cdot PB = x^2 - y^2 = MB^2 - MP^2$$

## Euclid II.6

The second theorem is very similar but has the point  $P$  located on an extension of  $AB$ :



Again we have a gnomon and a difference of squares. The gnomon is equal to the whole rectangle  $AP \cdot PB$  and when added to  $MB^2$  we get the whole large square  $MP^2$

$$AP \cdot PB + MB^2 = MP^2$$

In the final formula, we have just switched  $MB$  and  $MP$ , although  $AP$  is not what it used to be.

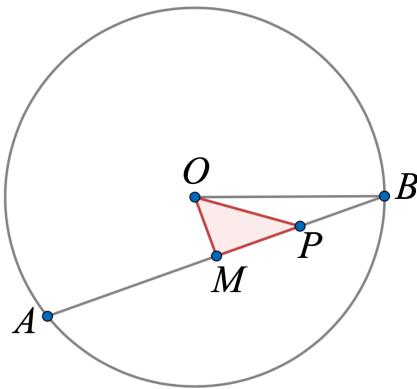
In algebraic terms we let  $AM = x$  and now  $PB = y$  so

$$(2x + y)y + x^2 = 2xy + y^2 + x^2 = (x + y)^2$$

$$AP \cdot PB + MB^2 = MP^2$$

## application to chords

We have a chord of a circle on center  $O$ ,  $AB$ .  $AB$  is bisected at  $M$  and  $P$  is placed somewhere within  $MB$ .



By II.5

$$AP \cdot PB + MP^2 = MB^2$$

$$AP \cdot PB = MB^2 - MP^2$$

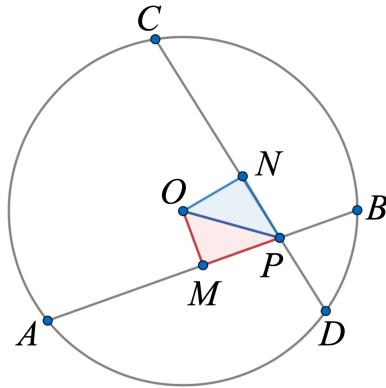
We notice both terms on the right are part of right triangles.

$$\begin{aligned} AP \cdot PB &= (MB^2 + OM^2) - (MP^2 + OM^2) \\ &= r^2 - OP^2 \end{aligned}$$

Let  $d$  be the distance from  $P$  to the center, and this becomes

$$AP \cdot PB = r^2 - d^2$$

This result is *independent* of the particulars of  $AB$  and depends only on the placement of  $P$  (and the requirement that  $AB$  pass through  $P$ ). So any other chord that also passes through  $P$ , say  $CD$ , has the same result.

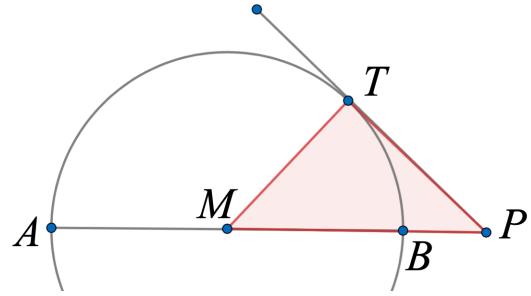


$$AP \cdot PB = CP \cdot PD$$

This, III.35, is just the crossed chord theorem in disguise.

## tangent

The next theorem concerns the tangent.



Here we have  $AB$  bisected at  $M$ , the center of the circle, and then  $P$  placed on the extension of  $AB$ .

By II.6

$$AP \cdot PB + MB^2 = MP^2$$

$MB$  and  $MT$  are radii so

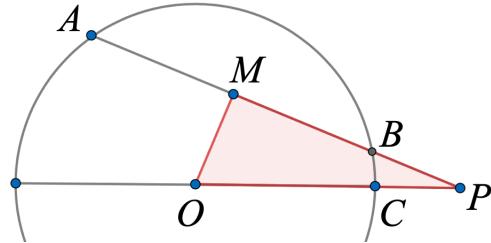
$$AP \cdot PB = MP^2 - MT^2 = PT^2$$

The length of the tangent from point  $P$  is equal  $AP \cdot PB$ .

The two results can be unified by considering  $PA$  and  $PB$  as directed line segments. In the crossed-chords example, they point in opposite directions so their product is negative, while for tangent and secant they point in the same direction.

We can view the result for chords  $r^2 - d^2$  as the same as the result for the tangent,  $d^2 - r^2$ , provided we take account of the sign of  $AP \cdot PB$ .

## secant



As with the tangent, by II.6 we have

$$AP \cdot PB + MB^2 = MP^2$$

and again, we have right triangles so

$$AP \cdot PB + (MB^2 + OM^2) = (MP^2 + OM^2)$$

$$AP \cdot PB + OB^2 = OP^2$$

so again

$$AP \cdot PB = d^2 - r^2$$

By using the previous two results together, we have that for any secant drawn from  $P$ ,  $PA \cdot PB$  is equal to the square of the tangent from the same point  $PT^2$ .

This is the **secant tangent theorem**.

# Chapter 29

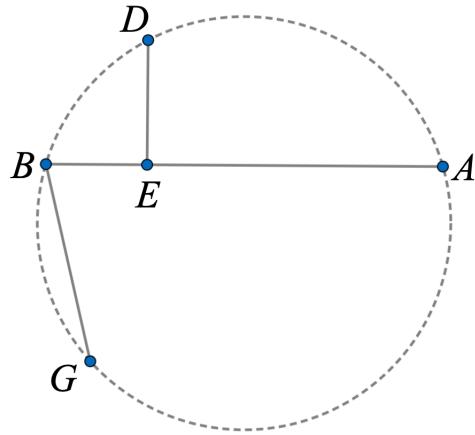
## Broken Chord

The theorem of the "broken chord" is ascribed to Archimedes, although his original work — the *Book of Circles* — has been lost.

It was analyzed in proofs collected by the Arabic mathematician Al Biruni in his *Book on the Derivation of Chords in a Circle*.

The theorem was not simply of academic interest, but related to the construction of tables of chords in the *Almagest* by Pappus (covered elsewhere).

Here is the general setup:



Let  $A$  and  $G$  be any two points on a circle, and let  $D$  be equidistant from both, so that arc  $AD$  is equal to arc  $GD$ . Let  $B$  be another point on the circle, lying between

$G$  and  $D$ .

Drop the perpendicular from  $D$  to  $E$  on  $BC$ .

The claim of the theorem is that  $GB + BE = AE$ .

In this chapter we will look at a number of proofs including several ascribed to Archimedes. This topic is a great one for “elementary” geometry because the proofs are fairly easy and there are literally dozens of them. There is also the sense of adventure from working on a topic which got Archimedes’ attention.

My original source for this problem was

[https://www.uni-miskolc.hu/~matsefi/HMTM\\_2020/papers/HMTM\\_2020\\_Drakaki\\_Broken\\_chord.pdf](https://www.uni-miskolc.hu/~matsefi/HMTM_2020/papers/HMTM_2020_Drakaki_Broken_chord.pdf)

There, it is said that the book contains 22 proofs of the theorem, including 3 different ones due to Archimedes.

Later I found a link to Al Biruni’s book, in Arabic:

[https://tile.loc.gov/storage-services/service/gdc/gdcwdl/wd/l\\_07/46/9/wdl\\_07469/wdl\\_07469.pdf](https://tile.loc.gov/storage-services/service/gdc/gdcwdl/wd/l_07/46/9/wdl_07469/wdl_07469.pdf)

I cannot read Arabic, but the diagrams are reproduced, and using them I worked out a couple more proofs.

Even more recently, I came across a translation of this book, written in German by a historian named Heinrich Suter about 1910:

<http://www.jphogendijk.nl/biruni/Suter-Chords.pdf>

I was able to translate a part of that book into English. I don’t read German either, but the scanned text of Suter is copyable, and Google Translate does a decent job. What makes it hard is that the OCR is so flaky — I’m not complaining — I think it’s amazing that it works at all. But it took quite a bit of editing.

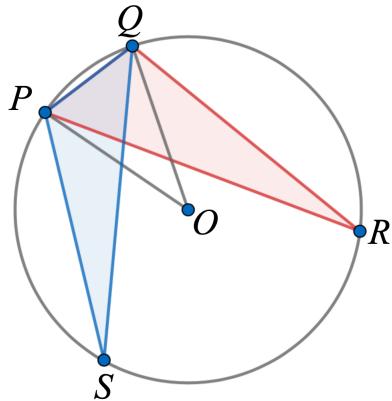
<https://www.dropbox.com/scl/fi/zzuibok8apr6i2trwfb4r/>

I count 23 examples (labeled  $a$  through  $x$ , with no  $j$ ), and four of those are Suter’s own proofs, which leaves a total of 19 for Al Biruni. (Although some of these contain multiple approaches with the same diagram).

## preliminary

Before starting, let's recall the important corollary (Euclid III.21) of the inscribed angle theorem (Euclid III.20).

By definition, the central angle sweeping out a given arc is equal in measure to the length of the arc. Any peripheral angle subtended by the same arc is one-half that central angle. The result is that angles that lie on the same arc or are subtended by the same chord in the same circle, are equal.



$$\angle PRQ = \angle PSQ.$$

We will also need the theorem that, in a given circle, equal arcs correspond to equal chords, and vice-versa, from [here](#).

And we will also use several times the theorem that any point on the perpendicular bisector of a segment forms an isosceles triangle when connected with the endpoints of the segment.

### first proof: Suter (a)

The first and second proofs are attributed to Archimedes.

*Proof.*

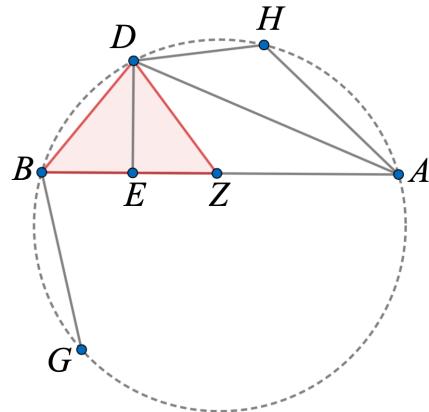
Given arc  $AD = \text{arc } GD$  and  $DE \perp AB$ .

Draw isosceles  $\triangle DBZ$  as follows:

Let  $BE = ZE$ .

So  $\triangle DBE \cong \triangle DZE$  by SAS.

Then  $BD = DZ$ , and  $\angle DBZ = \angle DZB$



Now find  $H$  such that  $BD = HD$ .

Subtracting equals, it follows that  $GB = AH$ .

$\angle A$  is bisected since the halves are subtended by  $BD = HD$ .

We will show that  $\angle ZDA = \angle HDA$ .

- (1)  $\angle DBZ$  corresponds to arc  $DHA = \text{arc } DH + \text{arc } HA$ .
- (2) Since  $HD = BD$ ,  $\angle DBZ$  corresponds to arc  $BD + \text{arc } HA$ .
- (3) Hence  $\angle DBZ = \angle DAB + \angle HDA$ .
- (4) Substituting,  $\angle DZB = \angle DAB + \angle HDA$ .
- (5) But  $\angle DZB$  is external to  $\triangle ZDA$ , so it equals  $\angle DAB + \angle ZDA$ .
- (6) Equating equals,  $\angle ZDA = \angle HDA$ .

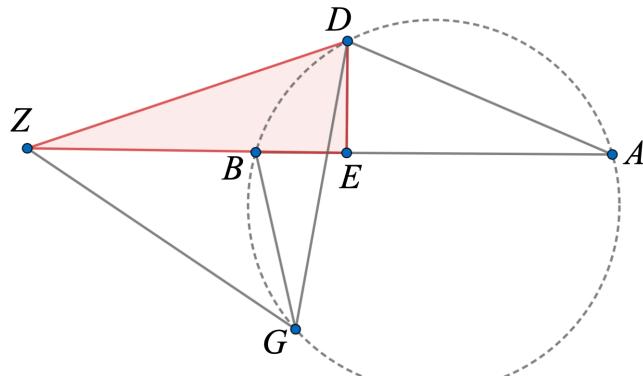
Thus  $\triangle ZDA \cong \triangle HDA$  by ASA.

It follows that  $AZ = AH = GB$ .

Adding equals:  $GB + BE = AZ + ZE = AE$ .

□

### second proof: Suter (c)



*Proof.*

Given arc  $AD = \text{arc } GD$  and  $DE \perp AB$ .

Extend  $AEB$  to  $Z$  such that  $AE = ZE$ .

$\triangle DZE \cong \triangle DAE$  by SAS.

So  $\triangle DZA$  is isosceles and  $DZ = AD = GD$ .

Hence  $\triangle DZG$  is isosceles with base angles equal.

Since  $\triangle DZA$  is isosceles,  $\angle DZE = \angle DAE$ .

Since they correspond to the same arc,  $\angle DAE = \angle DGB$ .

Hence  $\angle DZE = \angle DGB$ .

Subtracting equals we have  $\angle BZG = \angle BGZ$ .

Thus  $\triangle BZG$  is isosceles, and  $ZB = BG$ .

Adding equals:  $BG + BE = ZB + BE = AE$ .

□

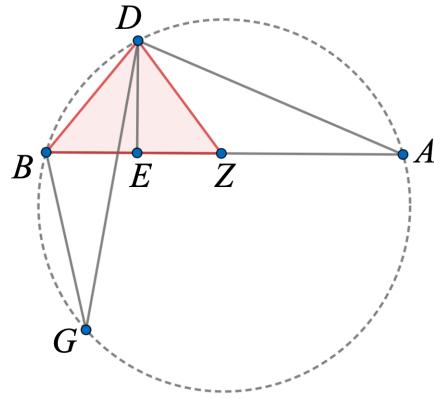
### third proof

This elegant proof is ascribed to Gregg Patruno, a student at Stuyvesant High School in New York (1980). I found it in

[https://www.researchgate.net/publication/341579803\\_FROM\\_THE\\_THEOREM\\_OF\\_THE\\_BROKEN\\_CHORD\\_TO\\_THE\\_BEGINNING\\_OF\\_TRIGONOMETRY](https://www.researchgate.net/publication/341579803_FROM_THE_THEOREM_OF_THE_BROKEN_CHORD_TO_THE_BEGINNING_OF_TRIGONOMETRY)

[Drakaki attributes it to Patruno].

The diagram can be found in Al Biruni's book (p. 15), but, according to Suter's translation, that proof is subtly different. Patruno starts with  $AZ = GB$ , whereas Suter (m) starts with  $DE = DZ$ . We discuss this point at the end of the chapter.



*Proof.*

Find  $Z$  such that  $AZ = GB$ .

$\angle G = \angle A$ , as inscribed angles on the same arc.

We are given  $GD = AD$ .

Hence  $\triangle BDG \cong \triangle ZDA$  by SAS.

So  $BD = ZD$ .

Hence  $\triangle DBZ$  is isosceles.

It follows that  $\triangle BDE \cong \triangle ZDE$  by HL.

So  $BE = ZE$ .

Adding equals:

$$GB + BE = ZE + GB = ZE + AZ = AE.$$

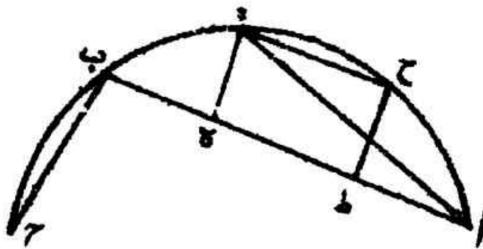
□

## fourth proof: Suter (i)

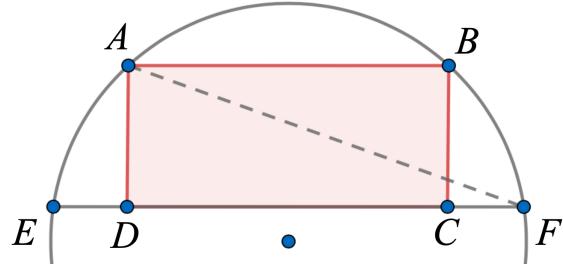
This one is attributed to El-Sidjzi (972), but may actually have been known to Apollonius.

[https://www.researchgate.net/publication/341579803\\_FROM\\_THE\\_THEOREM\\_OF\\_THE\\_BROKEN\\_CHORD\\_TO\\_THE\\_BEGINNING\\_OF\\_TRIGONOMETRY](https://www.researchgate.net/publication/341579803_FROM_THE_THEOREM_OF_THE_BROKEN_CHORD_TO_THE_BEGINNING_OF_TRIGONOMETRY)

This figure from Al Biruni (below) suggests a neat line of attack.

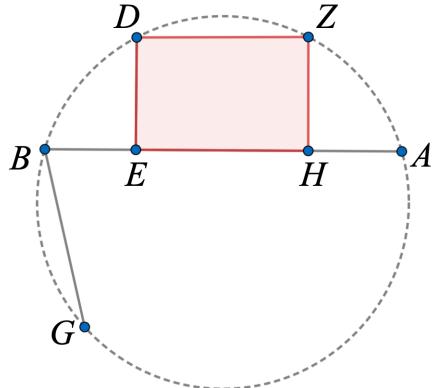


Recall from our work on parallel chords in a circle, a result about the extensions from a rectangle in a circle. In the figure below,  $DE = CF$ .



*Proof.* Let  $ABCD$  be a rectangle. The arc  $AE = \text{arc } BF$ , since  $\angle AFE = \angle BAF$  by alternate interior angles. It follows easily that  $\triangle ADE \cong \triangle BCF$  by HL.  $\square$

So now



*Proof.*

As before,  $\text{arc } GD = \text{arc } AD$ , and  $DE \perp AEB$ .

Erect the perpendicular to  $DE$  at  $D$  and find where it cuts the circle at  $Z$ . Erect the perpendicular  $ZH$ .

Since it has all right angles,  $DZHE$  is a rectangle.

By our preliminary work,  $\triangle DBE \cong \triangle ZAH$ .

Thus  $\text{arc } BD = \text{arc } AZ$  and  $BE = AH$ .

Subtracting equals,  $\text{arc } DZ = \text{arc } GB$ .

It follows that  $GB = DZ = EH$ .

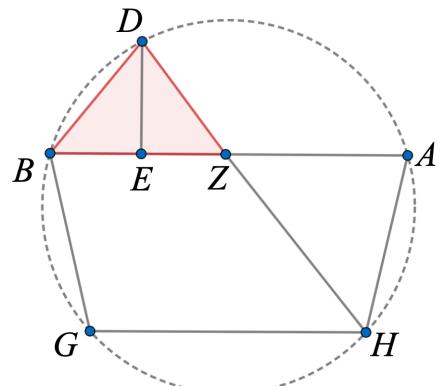
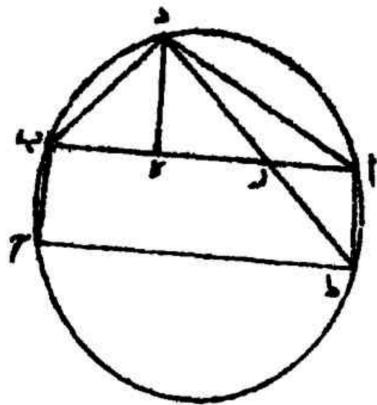
Adding equals:

$$GB + BE = EH + AH = AE.$$

□

## fifth proof

Here is another diagram from the book in Arabic.



As before,  $\triangle DBZ$  is isosceles, with  $BE = ZE$ .

Now  $DZ$  is extended to cut the circle at  $H$ .

$\angle DBA$  and  $\angle AHD$  are both subtended by  $AD$ , so they are equal.

Thus  $\triangle DBZ \sim \triangle AZH$  and both isosceles, with  $AZ = AH$ .

Given arc  $GD = \text{arc } AD$ , it follows that  $\angle H$  is bisected.

But  $\angle ZHA = \angle HZA$ , so altogether  $\angle A + \angle ZHA + \angle ZHG$  are two right angles.

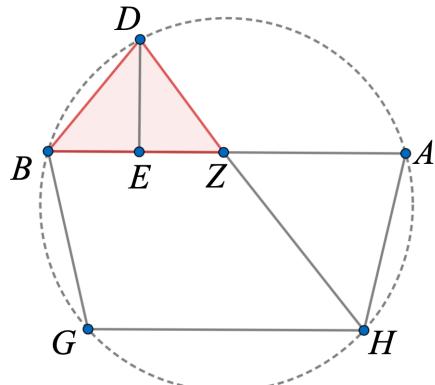
It follows that  $AB \parallel GH$ .

Parallel chords in a circle cut equal arcs and chords.

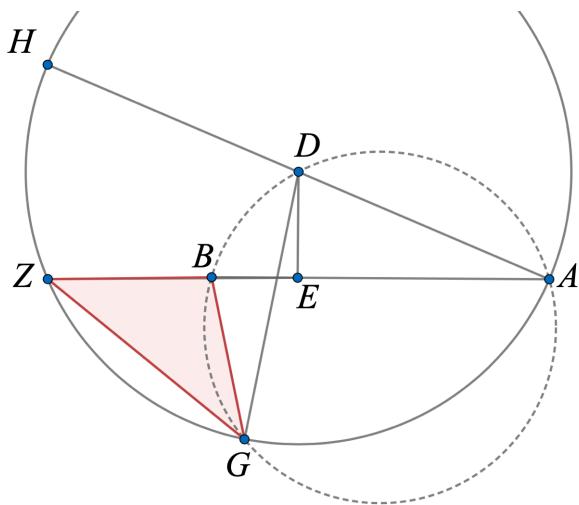
Thus  $GB = AH = AZ$ .

Adding equals:

$$GB + BE = AZ + ZE = AE.$$



### sixth proof: Suter(f)



*Proof.*

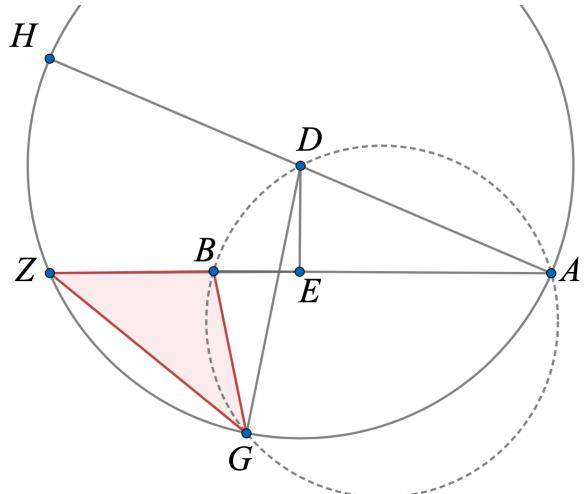
Draw the circle on center  $D$  with radius  $AD$ . Extend  $AD$  to  $H$ . Also extend  $AB$  to  $Z$ . Draw  $GD$ .

$\angle ADG = \angle ABG$  as inscribed angles on the same arc of the original circle.

$\angle ADG$  is the central angle for  $\angle AZG$  in the new circle on  $D$ .

Hence  $\angle ABG = \angle ADG = 2\angle AZG = 2\angle BZG$ .

But  $\angle ABG$  is external to  $\triangle BZG$  so  $\angle ABG = \angle BZG + \angle BGZ$ .



It follows that  $\angle BZG = \angle BGZ$ .

Thus  $\triangle AZG$  is isosceles with  $GB = ZB$ .

$AZ$  is a chord of the circle on  $D$ . The perpendicular  $DE$  goes through the center  $D$ , therefore it bisects  $AZ$ .

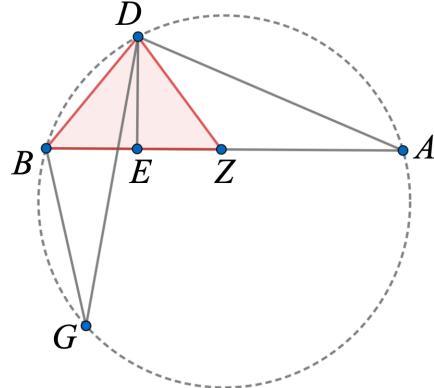
We have  $AE = ZE = ZB + BE = GB + BE$

□

That's a half-dozen proofs of the theorem. There are many more.

## on the value of SSA

Something interesting happens with the third proof if you approach the premises slightly differently.



We draw  $GD$  as before, but we forget to set  $GB = AZ$  and instead put  $BE = ZE$ , as in some other proofs. Then what happens?

We have  $SAS$  in the small right triangles so  $\triangle DBE \cong \triangle DZE$ , which means  $DB = DZ$ . We have  $GD = AD$  as before and  $\angle A = \angle G$ .

We are tempted to compare  $\triangle GBD$  with  $\triangle AZD$ . What we have is  $SSA$ , which — it's been drilled into our heads — is *not enough*, unless there is a right angle, in which case we call it hypotenuse-leg in a right triangle (HL).

But let's take a closer look. For  $SSA$  in the ambiguous case there are only two possibilities. We can see both of them in the figure!

Certainly  $\angle GBD$  is obtuse. It is supplementary to an angle (not drawn) subtended by  $GD$ , which is certainly less than half of the circle (since it equals  $AD$  and there is more than that in addition).

We also know something about  $\angle AZD$ . It is supplementary to  $\angle DZB = \angle DBZ$ , which corresponds to the arc  $AD$  which equals arc  $GD$ , so again,  $\angle AZD$  is everything except that.

Both angles are not only obtuse but clearly equal.

From Suter, here is what Archimedes says:

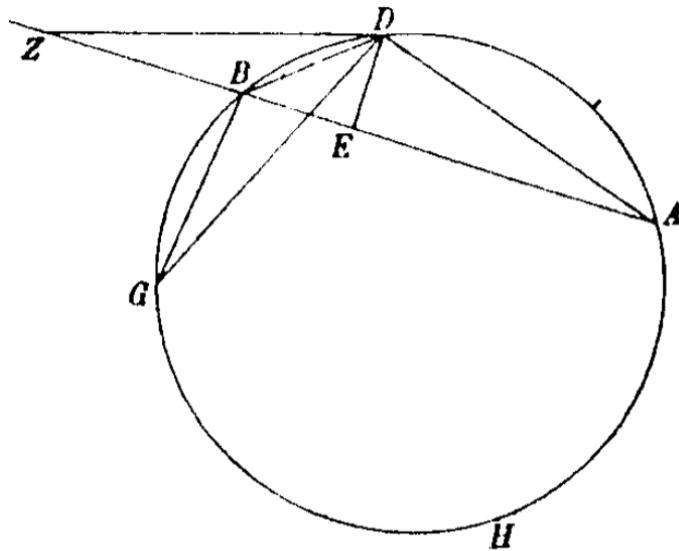


Fig. 1.

(re-phrasing):  $GD = AD$  by the property of the median,  $BD$  is shared, and  $\angle G = \angle A$ , so we have SSA.

Now,  $Z$  has been drawn such that  $ZE = AE$  which means  $ZD = AD$ , so we have actually another triangle sharing SSA with  $\triangle GBD$ .

Archimedes says that  $\angle GBD$  corresponds to everything except arc  $GD$ , i.e.  $DAHG$ .

And the angle supplementary to  $\angle ZBD$  is subtended by arc  $AD$ , so  $\angle ZBD$  itself is subtended by everything except arc  $AD$ , i.e.  $AHGB$ .

The missing parts are equal, so the angles are equal. Since we know two angles, we know three. Therefore,  $\triangle ZBD \cong \triangle GBD$  by ASA.

## more

There is one additional proof of the broken chord theorem that I know about, beyond what is in Suter.

It is on the web attributed to someone named Bùi Quang Tuân. There is another proof from the same source of the **Pythagorean theorem** that I like even better.

<https://www.cut-the-knot.org/pythagoras/BrokenChordPythagoras.shtml>

Google also turns up a blog, but no biographical info.

<https://artofproblemsolving.com/community/c1598>

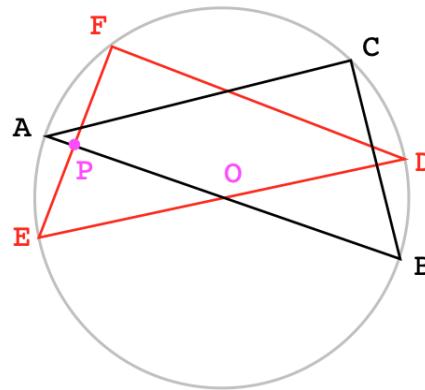
This proof is based on a rectangle, and I leave it to you to see how it relates to the fourth proof, above. I've written about it elsewhere.

Note: some additional material is here:

<https://www.cut-the-knot.org/triangle/BrokenChordmpd1c.shtml>

### *Star of David* proof of the Pythagorean theorem

<https://www.cut-the-knot.org/pythagoras/PythStarOfDavid.shtml>



Draw two congruent mirror-image right triangles in a circle, oriented so that  $EF \perp AB$  (and  $BC \perp DE$ ).

Note that  $AB$  and  $DE$  both pass through  $O$ , the center of the circle, because any right triangle inscribed in a circle has its hypotenuse as a diameter, by the converse of Thales' circle theorem.

$OA$  and  $OD$  are perpendicular by construction and diagonals, so they are perpendicular bisectors. Thus

$$AE = AF = DC = DB$$

Arc  $EC$  plus arc  $CD$  is equal to  $180^\circ$ .

So it is equal to the arc  $EB$ , which added to arc  $DB$  gives  $180$ .

Then  $E$  is the median point between  $B$  and  $C$  and the perpendicular dropped from  $E$  which meets  $AB$  in a right angle, cuts  $AB$  so

$$AP + AC = PB$$

$$AC = PB - AP$$

by the theorem of the broken chord. The two pieces of  $AB$  are

$$AB = AP + PB$$

Putting this together, we have:

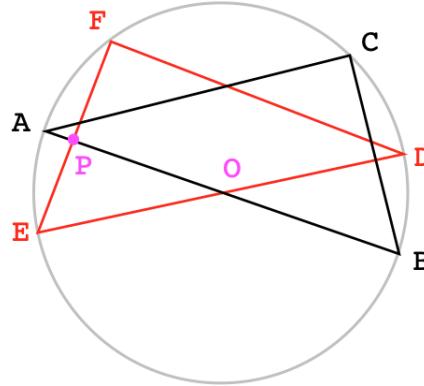
$$AB + AC = 2PB$$

and

$$AB - AC = 2AP$$

Hence

$$PB = \frac{1}{2}(AB + AC), \quad AP = \frac{1}{2}(AB - AC)$$



By the theorem of crossed chords

$$\frac{AB + AC}{2} \cdot \frac{AB - AC}{2} = \left(\frac{EF}{2}\right)^2$$

$$AB^2 - AC^2 = EF^2 = BC^2$$

The result follows.

□

# Chapter 30

## Euler Theorem

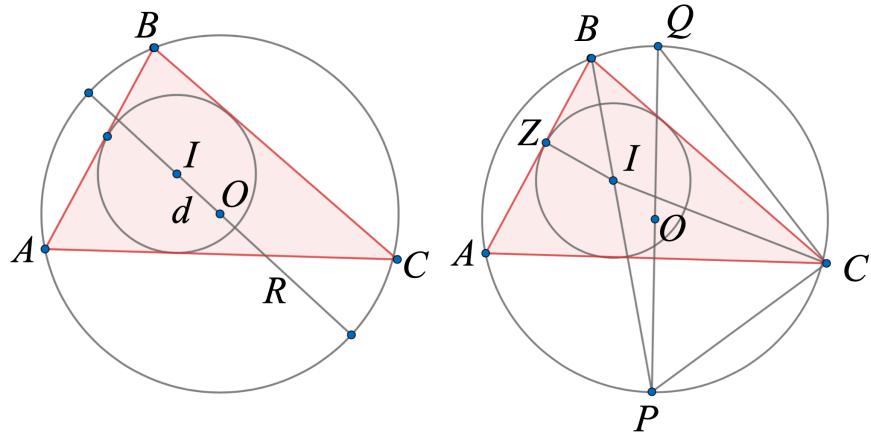
In this chapter we talk about the famous Euler Theorem (the one in geometry, there are others).

Consider  $\triangle ABC$  with its circumcircle on center  $O$  with radius  $R$ , and its incircle on center  $I$  with radius  $r$ .

Let the distance between  $O$  and  $I$  be  $d$ .

Then the distance from  $I$  to the outer circle is  $R + d$  in one direction and  $R - d$  in the other direction. The product of parts of the chord is

$$(R + d)(R - d) = R^2 - d^2$$



We consider another chord drawn through  $I$ , namely the bisector of  $\angle B$ , extended to meet the circumcircle at  $P$ .

Let the bisected angles be  $2\beta = \angle B$  and  $2\gamma = \angle C$ , as usual.

Draw the diameter  $POQ$ . By Thales' theorem,  $\triangle PQC$  is right.

Let  $IZ \perp BZA$ , so  $\triangle IBZ$  is right.

By inscribed angles,  $\angle PQC = \angle PBC = \beta$ .

Thus, the two right triangles are similar, since they both contain  $\beta$ .

Form the ratios:

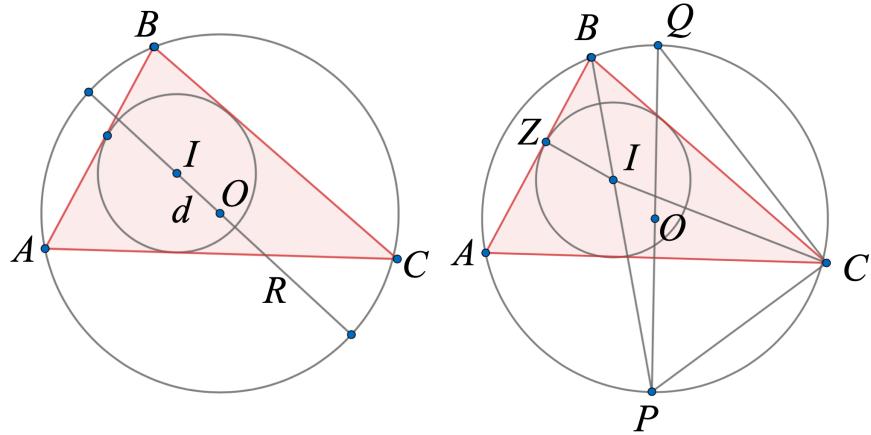
$$\frac{IZ}{IB} = \frac{PC}{PQ}$$

Substituting

$$\begin{aligned}\frac{r}{IB} &= \frac{PC}{2R} \\ 2rR &= PC \cdot IB\end{aligned}$$

If we can show that  $PC = PI$ , we will have that the right-hand side of the previous equation is equal to  $PI \cdot IB = R^2 - d^2$ , by crossed chords.

We consider  $\triangle PCI$ .



$\angle PCI$  is equal to  $\gamma + \beta$  by inscribed angles.

$\angle CPI$  is equal to  $\angle A$ .

By sum of angles,  $\angle PIC$  is also equal to  $\gamma + \beta$ .

With equal base angles, by I.6  $\triangle PCI$  is isosceles.

Thus  $PI = PC$ , and the main result follows:

$$\begin{aligned}2rR &= R^2 - d^2 \\&= (R + d)(R - d)\end{aligned}$$

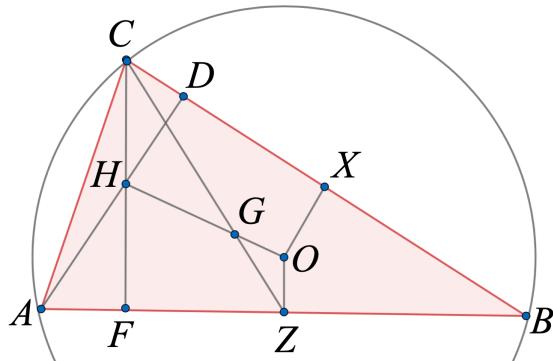
$$\begin{aligned}\frac{1}{r} &= \frac{2R}{(R + d)(R - d)} \\&= \frac{1}{R + d} + \frac{1}{R - d}\end{aligned}$$

# Chapter 31

## Euler Line

In this chapter we talk about the famous Euler Line of a given triangle:

[https://en.wikipedia.org/wiki/Euler\\_line](https://en.wikipedia.org/wiki/Euler_line)



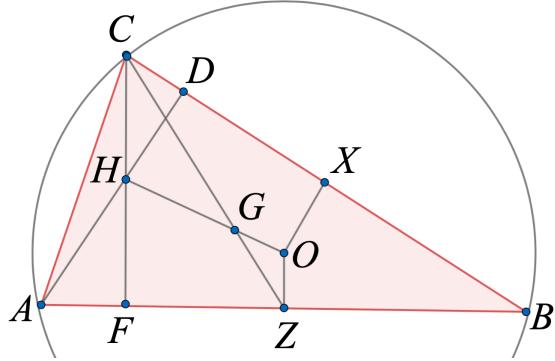
The proof is quite easy (once you know how).

Recall that the circumcenter is the center of the circle that contains all three vertices, and it can be found by erecting the perpendicular bisectors of the sides. Side  $a$  opposite  $\angle A$  is bisected at  $X$ , while side  $c$  opposite  $\angle C$  is bisected at  $Z$ . The perpendicular bisectors  $OX$  and  $OZ$  meet at  $O$ .

The orthocenter  $H$  is the point where altitudes drawn from the three vertices of a triangle cross. Here  $AD$  meets  $CF$  at  $H$ .

The centroid  $G$  is the point where the medians cross, before they bisect the opposite sides.  $CZ$  is one of the medians of  $\triangle ABC$ .

We will assume that these points actually exist (that the three altitudes or three medians are concurrent). That proof is coming later.



Suppose we find  $O$  as described above, draw the median  $CZ$ , find  $G$  arithmetically, and then extend  $OG$  to find  $H$  such that  $HG = 2OG$ .

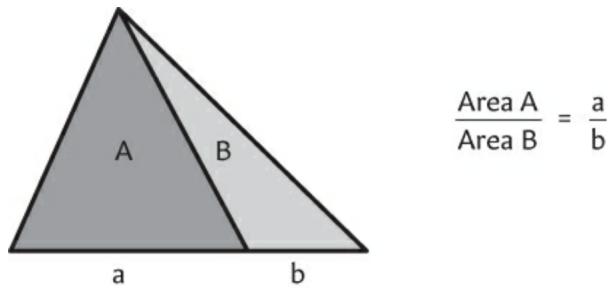
We will show that  $H$  is the orthocenter of  $\triangle ABC$ , that it actually lies on  $CF$ .

We also need a couple of preliminary results. We will prove that the centroid divides the medians by the  $2/3 - 1/3$  rule:  $CG = 2ZG$ . Let's do this below as a lemma.

The second is SAS similarity. In the discussion of similarity for a general or arbitrary triangle, we showed that if two triangles have two pairs of sides in the same proportion, and the angle between is also equal, then they are similar triangles.

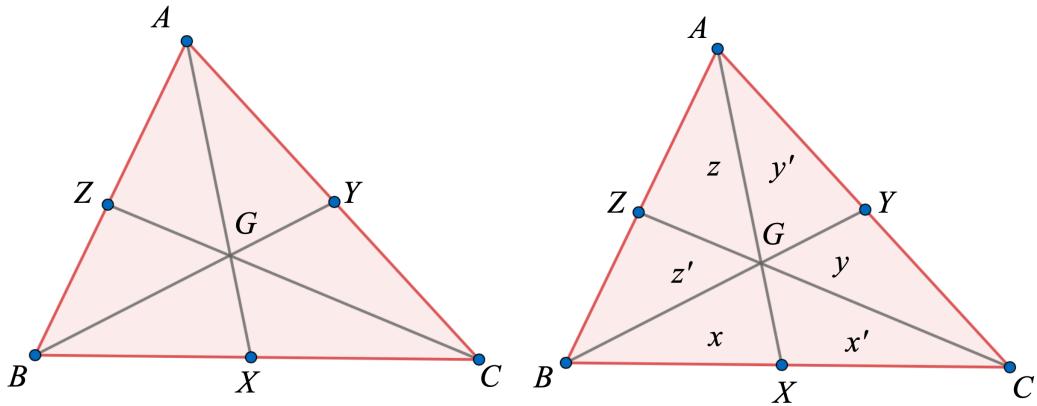
#### *Lemma*

We depend on the area-ratio theorem. If two triangles have one vertex shared and the bases lie on the same line, then their areas are in proportion as the lengths of the bases. The reason is that they have the same altitude, since only one line can be drawn from any point perpendicular to a line.



**Fig. 120** An area-ratio theorem.

So now we consider an arbitrary triangle, divided by its three medians. We will show that each of the small triangles is equal in area.



Compare  $\triangle GBX$  and  $\triangle GCX$  to find that they are equal in area, since  $GX = CX$ . Thus area  $x = \text{area } x'$ . and so on. Similarly, compare  $\triangle ABX$  and  $\triangle ACX$  to find that they are equal in area. It follows that:

$$x + z + z' = x' + y + y'$$

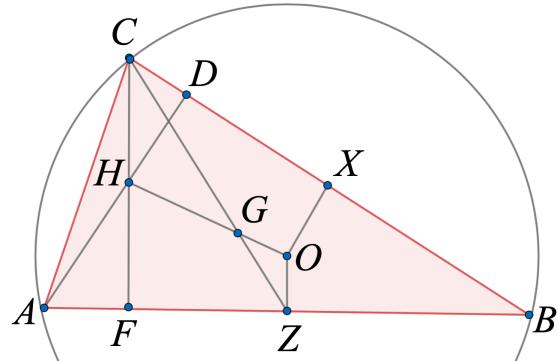
which reduces to  $z = y$ . The same argument will give  $x = y$  and  $x = z$ .

Finally, we use what is technically the converse of the area-ratio theorem. We have that the area of  $\triangle ABG$  is twice that of  $\triangle AYG$ . Since they have the same altitude, it follows that the bases are in proportion:  $BG = 2YG$ . Again, this can be extended to any median by the same argument.

□

*Proof.*

Returning to the main problem, we justify the placement of  $G$  by noting that  $CG = 2ZG$ . We found  $H$  by  $HG = 2OG$ . Finally, we have vertical angles at  $G$ . It follows that  $\triangle CGH \sim \triangle ZGO$  with sides in proportion  $2 : 1$ .



It follows that  $\angle CHG = \angle ZOG$ .

Thus, by alternate interior angles, we find that  $OZ \parallel CHF$ , which means that since  $OZ \perp AFZB$ , so is  $CHF \perp AFZB$ . Thus,  $CF$  is the altitude to the vertex  $C$  in  $\triangle ABC$ .

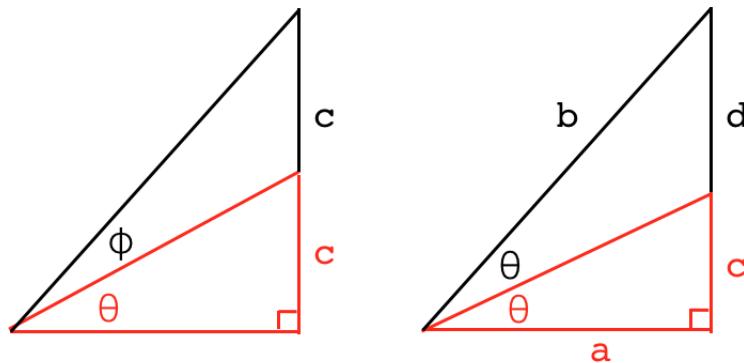
□

Two more quick points. First, in an equilateral triangle, the three points  $O$ ,  $G$  and  $H$  coincide — they are the same point. Second, there is much more to the Euler line, including the center of the *nine point circle*, which we discuss in the other volume.

# Chapter 32

## Angle bisectors

We consider a classic problem: the ratio of sides when there is an angle bisector.



Suppose the large triangle is a right triangle. We draw a line joining the vertex on the left with the side opposite.

This line could in general be drawn anywhere, however two interesting cases are when the side opposite is bisected (left panel), or when the angle at the left is bisected (right panel). These two cases are not the same. In the first  $\phi \neq \theta$  and in the second,  $c \neq d$ .

Let's investigate the second possibility, equal angles. We are in a position to prove an important theorem.

## angle bisector theorem

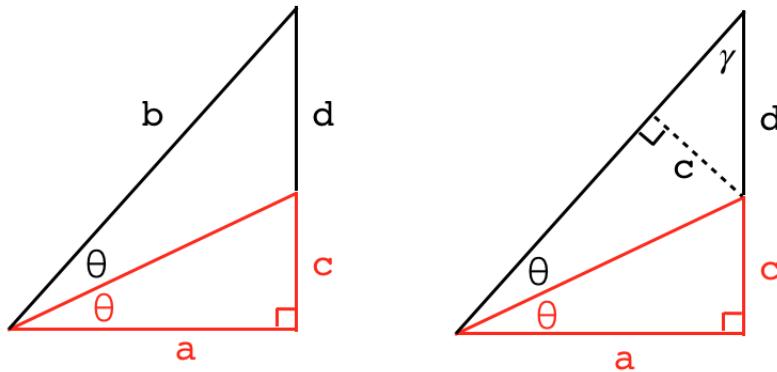
With reference to the right-hand figure above, we are to prove that

$$\frac{d}{b} = \frac{c}{a}$$

- The sides and bases are in proportion for a right triangle with bisected angle.

*Proof.*

Draw an altitude for the upper of the two small triangles, meeting the side of length  $b$ .



The red triangle and the one directly above it are congruent (right panel). They share a side (the hypotenuse of each), and they are right triangles with the same smaller angle  $\theta$ . Therefore, the altitude we just drew has length  $c$ .

The small triangle with sides  $c$  and  $d$  (at the top) is similar to the original large triangle. The reason is that they are both right triangles containing the smaller angle  $\gamma$ .

By similar triangles, we form equal ratios of the side opposite  $\gamma$  to the hypotenuse:

$$\frac{a}{b} = \frac{c}{d}$$

This is rearranged simply to give

$$\frac{d}{b} = \frac{c}{a}$$

which is what we were asked to prove.

The result can be pushed a little further:

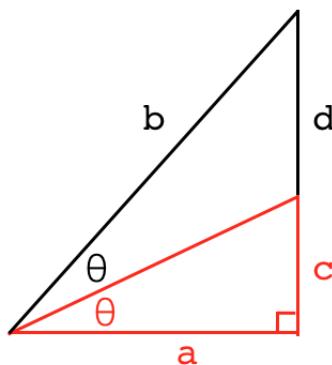
$$\frac{a}{b} = \frac{c}{d}$$

Add 1 to both sides:

$$\frac{a+b}{b} = \frac{c+d}{d}$$

$$\frac{a+b}{c+d} = \frac{b}{d} = \frac{a}{c}$$

□



which is a surprising result and becomes important later in looking at Archimedes method for approximating the value of  $\pi$ .

In the proof just completed, we took advantage of the fact that the large triangle was a right triangle. However, if you think about it for a minute, you should be able to see that the same result holds for an isosceles triangle. There, the two sides are equal, and if the top angle is bisected, so is the base. So the ratio of each side to its part of the base is also equal.

This might lead you to wonder about whether the proof holds for a general triangle. Indeed, we will show later on that the sides and bases are in proportion for any triangle, if the angle is bisected.

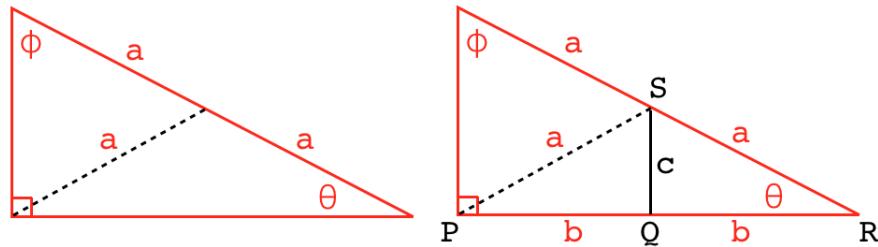
In the formula from above

$$\begin{aligned}\frac{a+b}{c+d} &= \frac{b}{d} = \frac{a}{c} \\ &= \frac{a}{c+d} + \frac{b}{c+d}\end{aligned}$$

we have a relationship between sides of the right triangle with angle  $2\theta$  and the right triangle with angle  $\theta$ . Archimedes uses this in his method to determine the value of  $\pi$ .

## midpoint theorem

The line which bisects the hypotenuse in a right triangle is called the median, but this is commonly called the midpoint theorem. It is a specific case of the theorem for a right triangle.



In a right triangle, draw the line segment from the vertex that contains a right angle to the midpoint of the hypotenuse, separating it into two equal lengths  $a$ . We will show that the length of the bisector is also  $a$ .

This repeats a proof we gave earlier. Following is a simpler proof based on Thales' circle theorem.

*Proof.*

In the right panel, draw the perpendicular from the midpoint  $S$  to the base  $PR$ . The triangle  $SQR$  is similar to the original right triangle by AAA.

Hence the two parts of the base are equal (labeled  $b$ ), because  $a/2a = b/2b = 1/2$ .

Therefore we have two congruent triangles:  $SQR$  and  $PQS$  (by SAS). So the bisector  $PS$  is equal in length to  $SR$ .

Both of the new isosceles triangles formed by the original dashed line have equal base angles.

□

The length  $a$  is the radius of the circle, centered at  $S$ , that contains the three vertices of the original right triangle.

A clever alternative proof of the midpoint theorem is to start with the right triangle and draw a circle centered on the midpoint and size the radius so that the hypotenuse is the diameter.

Then, because the vertex at  $P$  is a right angle, by the converse of Thales' circle theorem,  $P$  is on the circle. Thus,  $PS$  is also a radius.

The converse of this theorem is to start with the right triangle and draw a line from the vertex  $P$  to the hypotenuse, such that the upper segment from the division is equal in length the line just drawn.

We know that this can be done because at the extremes the length of the upper segment is either zero, or it corresponds to the entire hypotenuse. In that case, the new line segment is the existing side, which must be shorter than the hypotenuse.

By this construction, the upper small triangle is isosceles. By the reverse of the isosceles triangle theorem, the upper of the two new angles at  $P$  is  $\phi$ .

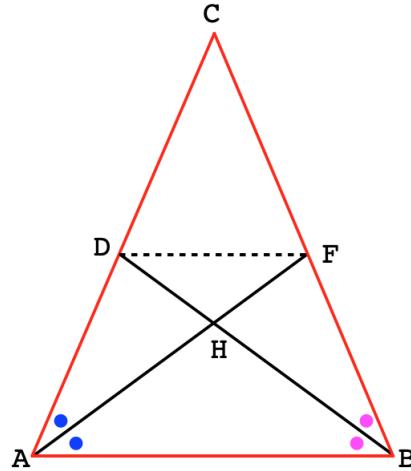
But the total angle at  $P$  is a right angle, so the right angle is now divided into  $\phi + \theta$ , which means in turn that the second small triangle is also isosceles. With equal base angles, the opposing sides are also equal. Thus the point where the new line meets the hypotenuse is the midpoint.

This essentially recapitulates Thales' circle theorem.

□

## **isosceles angle bisector theorem**

The next theorem involves angle bisectors in an isosceles triangle. It is easy in the forward direction, but the converse is very challenging, at least until you draw the right diagram. Then, as usual, it's not so bad.



We are given that  $\triangle ABC$  is isosceles ( $AC = BC$ ), and also that the angles at the base are both bisected.

We claim that the angle bisectors are equal in length:  $AF = BD$ .

*Proof.*

By the forward version of the isosceles triangle theorem, the entire  $\angle A = \angle B$ , so it follows that all four angles dotted blue and magenta are equal.

Then  $\triangle ABD \cong \triangle ABF$  by ASA.

As a result,  $AF = BD$  and  $AD = BF$ .

Furthermore, since the original triangle is isosceles and  $AD = BF$ , the smaller triangle  $\triangle CDF$  is also isosceles, by subtraction. Alternatively,  $\angle C$  is shared, and the long sides are equal so  $\triangle CDB \cong \triangle CFA$  by AAS.

It follows that  $CD = CF$  and  $\triangle CDF$  is isosceles, and therefore,  $\triangle CDF \sim \triangle ABC$  by AAA.

By alternate interior angles,  $DF \parallel AB$ .

□

That's the easy part.

The converse theorem says that if we have angle bisectors and they are equal in length, then the triangle is isosceles. This is called the Steiner-Lehmus Theorem.

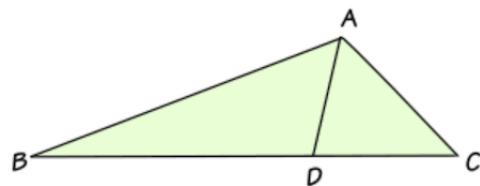
[https://en.wikipedia.org/wiki/Steiner-Lehmus\\_theorem](https://en.wikipedia.org/wiki/Steiner-Lehmus_theorem)

We defer discussion of the **Steiner-Lehmus Theorem** to its own chapter.

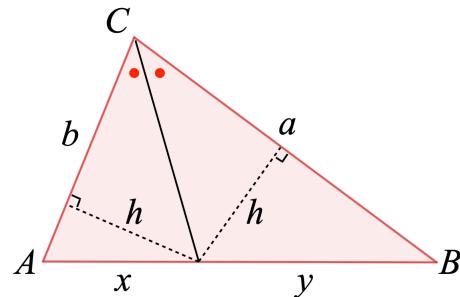
### angle bisector theorem revisited

It turns out that although we used right triangles in our first proof of the angle bisector theorem, that wasn't strictly necessary. We can say something about the general case.

We use the notion of area.



Let's redraw the figure. Below, we have  $\triangle ABC$  with sides  $a, b, c$  and the angle  $C$  is bisected so the two angles marked with red dots are equal.



Start from the point where the bisector meets the side opposite  $\angle C$ , cutting  $c$  into  $x$  and  $y$ . Drop the perpendiculars to the sides  $a$  and  $b$ . By our previous work with bisectors we know these have the same length, marked as  $h$ .

We can compute the ratio of the areas of the left and right-hand sub-triangles in two ways.

The first way uses the area-ratio theorem.

$$\frac{A_L}{A_R} = \frac{x}{y}$$

The two triangles share a common altitude and the area is one-half the base times the altitude. Since the altitude is shared, it cancels, together with the factor of one-half.

The second approach is to use the sides  $a$  and  $b$  as the two bases. We have that the heights  $h$  are equal. Hence

$$\frac{A_L}{A_R} = \frac{b}{a}$$

Equating the two results we have that

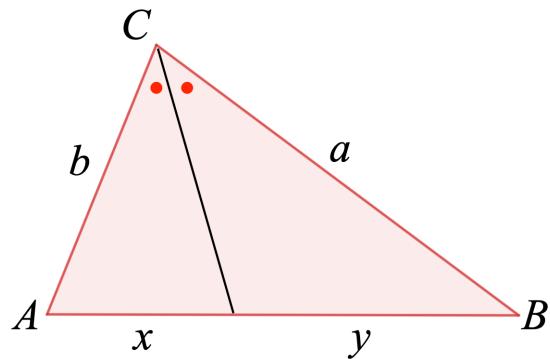
$$\frac{x}{y} = \frac{b}{a}$$

which can be manipulated in various ways but I will just leave it as it is.

The result we obtained previously is a special case where angle  $A$  is a right angle, but the theorem is true generally.

## similar triangles

Generally speaking, any proof by comparing areas can also be done by similar triangles. Here are two proofs of the general angle bisector theorem by similar triangles.



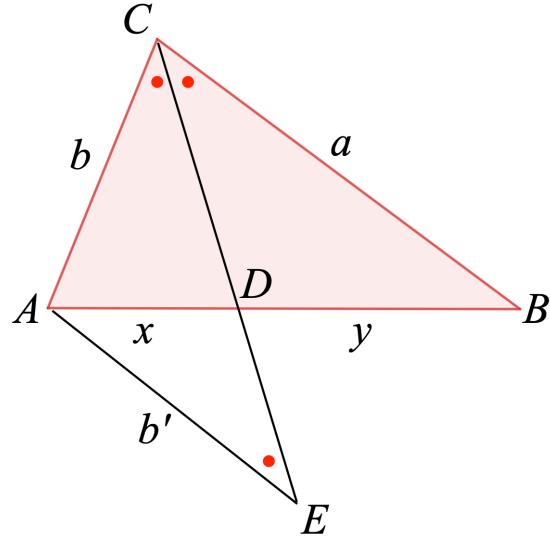
Both depend on extending a parallel line. The first uses a line parallel to one of the sides.

Given that the angle at  $C$  is bisected.

The claim is that  $a/y = b/x$ .

*Proof.*

Draw  $AE$  parallel to  $BC$  and extend the angle bisector to meet it at  $E$ . Then the angle at  $E$  is equal to the half-angle, by alternate interior angles,



We have that  $\triangle ACE$  is isosceles, so  $b = b'$ .

We also have two similar triangles with  $\triangle AED \sim \triangle BCD$ , since the angles at  $D$  are equal by vertical angles.

Form the ratios of the sides opposite vertical angles to the sides opposite the angles marked with red dots:

$$\frac{a}{y} = \frac{b'}{x}$$

But since  $b = b'$ :

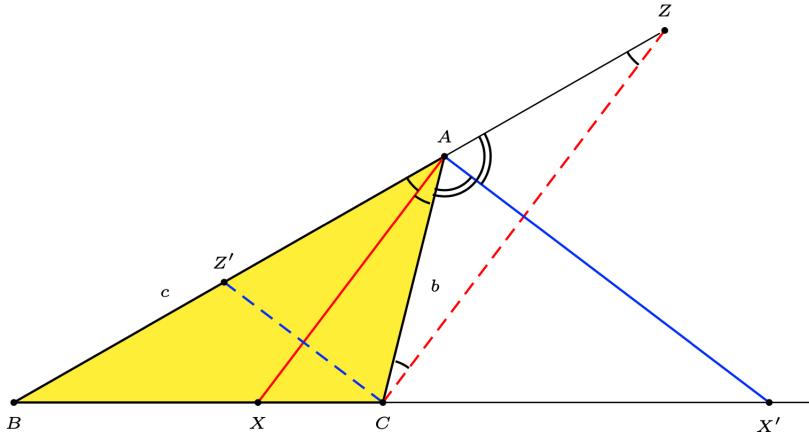
$$\frac{a}{y} = \frac{b}{x}$$

□

### third proof

Here is still another proof, from Paul Yiu, together with a problem. It includes the bisector of the external angle as well.

The proof uses an extended line parallel to the bisector.



$\angle A$  in  $\triangle ABC$  is bisected internally and externally.

Extend  $BA$  and draw  $CZ$  parallel to the bisector  $AX$ . This gives a new  $\triangle AZC$  which is isosceles by two applications of alternate interior angles, so  $AZ = b$ .

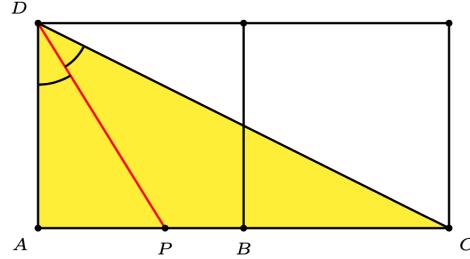
By similar triangles, we have that  $AZ/c = CX/BX$ . Substitute from above to obtain  $CX/BX = b/c$ .

Now for the external angle theorem. We draw  $CZ'$  parallel to the angle bisector  $AX'$ . Again, there is an isosceles triangle formed,  $\triangle ACZ'$ .

We express the theorem as the ratio of the entire base  $BX'$  to the external part  $CX'$ . By similar triangles, this ratio is equal to the ratio of  $c$  to  $AZ'$ , but analogous to what we had before,  $AZ' = b$ . So  $BX'/CX' = c/b$ .

(There is a minus sign on  $b$  if you're using signed lengths, which we do not worry about here).

Then to the problem: find the ratio  $AP/PB$ .



Pythagoras gives  $DC = \sqrt{5}$  and this theorem says that  $PC$  and  $AP$  are in the same ratio as the flanking sides.

Scale the problem so that  $AD = 1$ ,  $AC = 2$  and  $AP = x$  and

$$\frac{PC}{AP} = \frac{2-x}{x} = \sqrt{5}$$

$$\frac{1}{x} = \frac{1+\sqrt{5}}{2}$$

Do not solve for  $x$  yet.  $PB = 1 - x$  so the inverse of the desired ratio is

$$\begin{aligned}\frac{PB}{AP} &= \frac{1-x}{x} = \frac{1}{x} - 1 \\ &= \frac{1+\sqrt{5}}{2} - 1 = \phi - 1 \\ &= \frac{\sqrt{5}-1}{2}\end{aligned}$$

The inverse is

$$\frac{AP}{PB} = \frac{2}{\sqrt{5}-1}$$

# Chapter 33

## Excircles and Heron

I came across some sophisticated “elementary” notes on geometry on the web, including an introduction to *excircles*. We can use excircles to simplify derivation of Heron’s formula. This relates the area of any triangle to one-half of its perimeter, the *semi-perimeter*  $s$ , and the three sides:

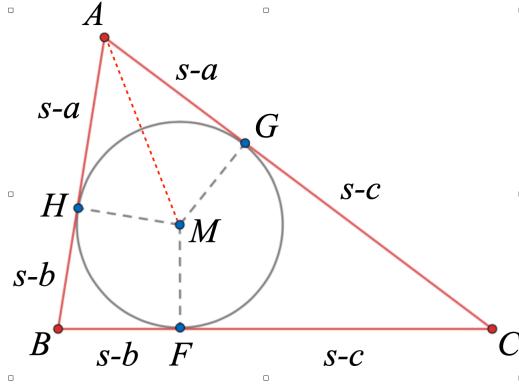
$$A = \sqrt{s(s - a)(s - b)(s - c)}$$

We develop a simple proof of Heron’s theorem in this chapter that utilizes similar triangles and a construction with an excircle. An algebraic excursion is delayed until later, when we also look at Brahmagupta. At the end of the chapter we also look at Heron’s original proof.

### incircle

The *incircle* of a triangle is contained within and just touches (is tangent to) each of the three sides. It can be constructed by bisecting the angle at each vertex, and finding the point  $M$  where the bisectors meet.  $M$  is then the center of the incircle.

*Proof.* Let the triangle be  $\triangle ABC$  with side  $a$  opposite vertex  $A$ , etc., as usual, and let  $AM$  be the bisector of  $\angle A$ ,  $BM$  the bisector of  $\angle B$ , and so on. Drop perpendiculars from  $M$  to the sides at  $F$ ,  $G$  and  $H$ .



Long ago we proved that each pair of perpendiculars from an angle bisector forms two congruent right triangles, by hypotenuse-leg in a right triangle (HL), so, for example,  $\triangle AMG \cong \triangle AMH$ . Thus, the two perpendiculars from any point on the bisector to the rays of the bisected angle are equal ([here](#)).

But we can do the same for any pair of sides. Therefore all three distances are equal, and we can then draw the circle, called the incircle, with that distance as the radius  $r$ .  $MF = MG = MH$ .  $\square$

Let the two congruent right triangles containing the half-angle at vertex  $A$  have base  $AG = AH = x$ , and the two at  $B$  base  $y$  and the two at  $C$  base  $z$ . (These are not labeled as such in the diagram, we will see why in a minute).

The perimeter is  $p = 2x + 2y + 2z$ , and so the *semi-perimeter*  $s$  is

$$s = x + y + z = \frac{a + b + c}{2}$$

Since  $a = y + z$ , using the equation above we can write

$$s = x + a$$

from which

$$x = s - a$$

and this explains the labels in the figure above.

In addition, the area of the triangle is

$$A = xr + yr + zr = r(x + y + z) = rs$$

A commonly used notation for area is  $\mathcal{A} = (ABC)$ , so we rewrite this as

$$(ABC) = rs$$

We will see terms like  $s - a$ ,  $s - b$ , etc. again. We note that

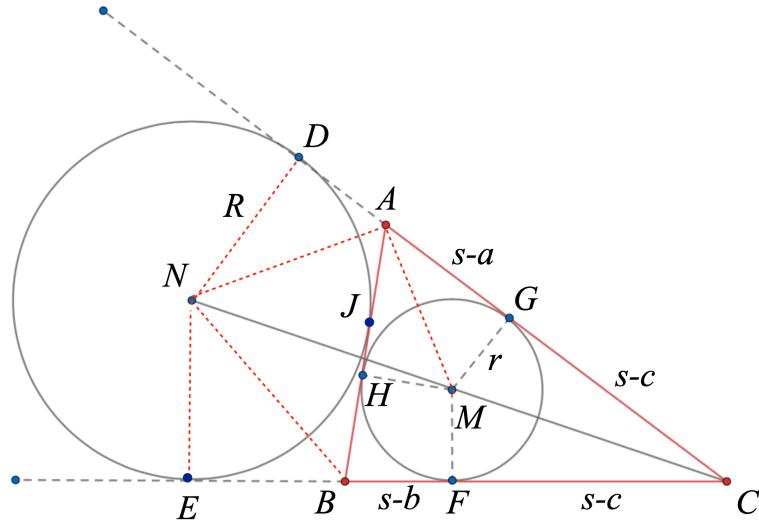
$$s - a = \frac{a + b + c}{2} - a = \frac{-a + b + c}{2}$$

Subtract side  $a$  from  $s$  on the left and change the sign on  $a$  in the formula for the semi-perimeter.

Another useful result that we will come back for later is

$$\begin{aligned} (s - a) + (s - b) + (s - c) &= 3s - (a + b + c) \\ &= s \end{aligned}$$

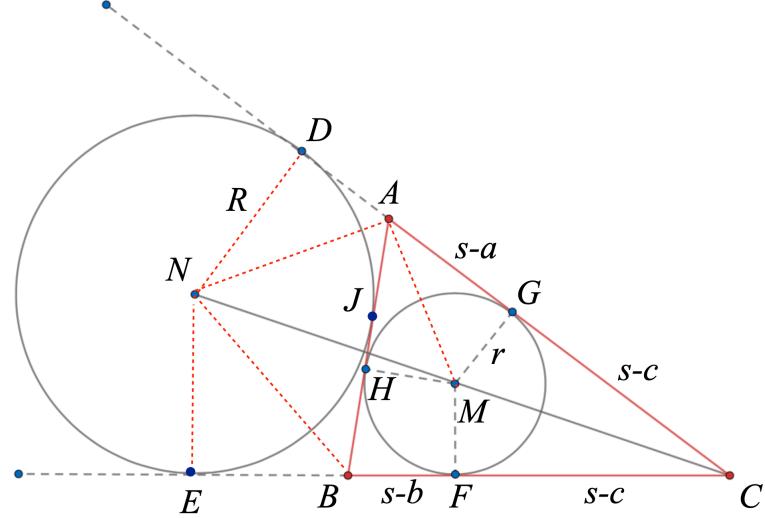
## excircles



Any triangle has a single incircle and three excircles, one for each side of the triangle. The excircle on side  $c$  (opposite  $\angle C$ ) is formed by extending sides  $a$  and  $b$  (i.e.  $BC$  and  $AC$  above) and then finding the circle that is tangent to those two extensions and also tangent to side  $c$ .

Let  $N$  be the center of the excircle on side  $c$ .  $N$  lies on the bisector of the original  $\angle C$ . This follows from the fact that  $CD$  and  $CE$  are tangent to this excircle.

Since  $AD$  and  $AJ$  are also tangents to the excircle from point  $A$ , if the angle supplementary to  $\angle A$  (i.e.  $\angle DAB$ ) is bisected, then the intersection of that bisector with the bisector of  $\angle C$  occurs at  $N$ .



Drop the vertical to side  $c$  at  $J$ , or to either of the extensions at  $D$  and  $E$ , to find the radius of the excircle  $R$ .

$R = ND = NE = NJ$ . (If we were to look at more than one excircle we might use a subscript like  $R_c$ .)

Using the latter notation for one moment, a surprising, simple relationship can be found:

$$R_c \cdot (s - c) = (ABC)$$

This can be done for any side so

$$R_a \cdot (s - a) = R_b \cdot (s - b) = R_c \cdot (s - c)$$

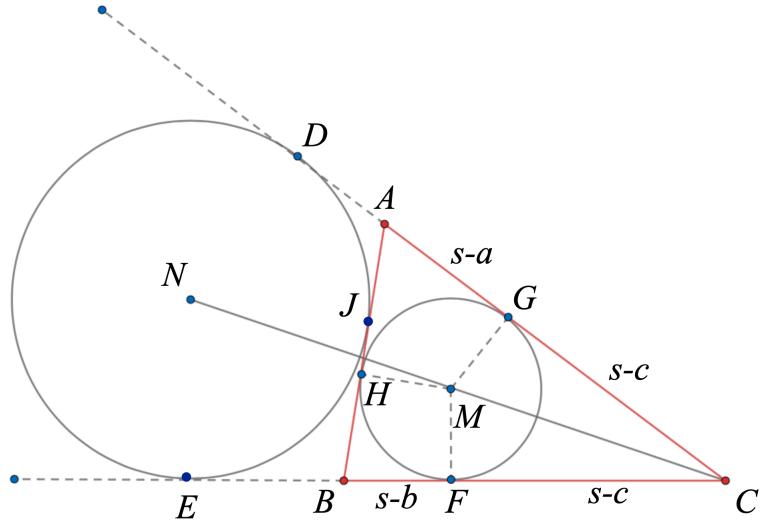
Returning to consideration of  $R_c$  and suppressing the subscript, we now have two expressions for the area

$$rs = R \cdot (s - c)$$

We show how to derive this result below and use it for our key proof.

## tangent length

We can show that the length of the tangents  $CD = CE$  is just  $s$ .



*Proof.*

Let the length of each tangent from  $C$  be  $t$ . Then

$$\begin{aligned} 2t &= CD + CE \\ &= AD + (s - a) + (s - c) + BE + (s - b) + (s - c) \end{aligned}$$

But  $AD = AJ$  and  $BE = BJ$  and also  $AJ + BJ = c$  so that gives

$$\begin{aligned} 2t &= (s - a) + (s - c) + (s - b) + (s - c) + c \\ &= (s - a) + (s - c) + (s - b) + s \\ &= 2s \end{aligned}$$

Hence the length of the entire tangent is just  $s$ .

□

We also have that

$$CD = s = AD + (s - a) + (s - c)$$

Compare with our previous result

$$s = (s - a) + (s - b) + (s - c)$$

Thus

$$AD = s - b$$

It turns out that the lengths of all these tangents, 3 pairs to the excircle and 3 pairs to the incircle, are particularly simple.

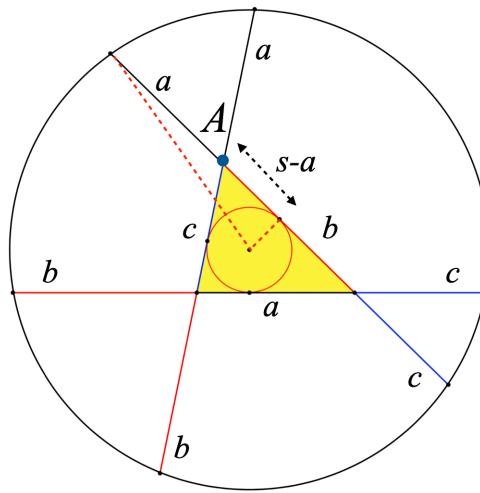
For example, the points of tangency  $J$  and  $H$  divide the side  $c$  into the same two lengths:

$$AD = AJ = BH = s - b$$

and

$$BE = BJ = AH = s - a$$

### another construct



Extend the sides of  $\triangle ABC$  a distance  $a$  past vertex  $A$  and so on.

Then one can draw a circle whose center is also the incenter of the triangle, that passes through the ends of all of the extended line segments.

*Proof.*

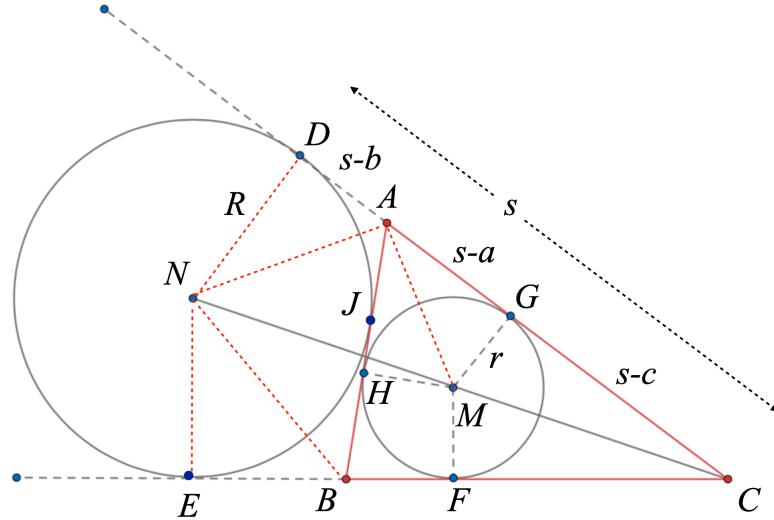
Draw the right triangle with one side equal to  $r$ , the radius of the incircle, and the other side extending side  $b$  through vertex  $A$ , as shown. The second side has length  $s - a + a = s$ .

Therefore the hypotenuse of every such right triangle is the same, and this is the radius of the large circle.

□

## Heron's formula

Since  $ND \perp CD$  and  $MG \perp CG$ , we have a pair of similar right triangles, namely one with sides  $R$  and  $s$ , and the other with sides  $r$  and  $s - c$ .



The proportion is:

$$\frac{r}{R} = \frac{s-c}{s}$$

which justifies what we wrote previously:

$$rs = R(s-c)$$

This leads to a beautiful, simple proof of Heron's formula.

The angle between two tangents from the same point is bisected by the line through the center of the circle. Thus  $\angle DAN$  is bisected by  $AN$  and also  $\angle BAC$  is bisected by  $AM$ .

It follows that  $\angle NAM$  is a right angle.

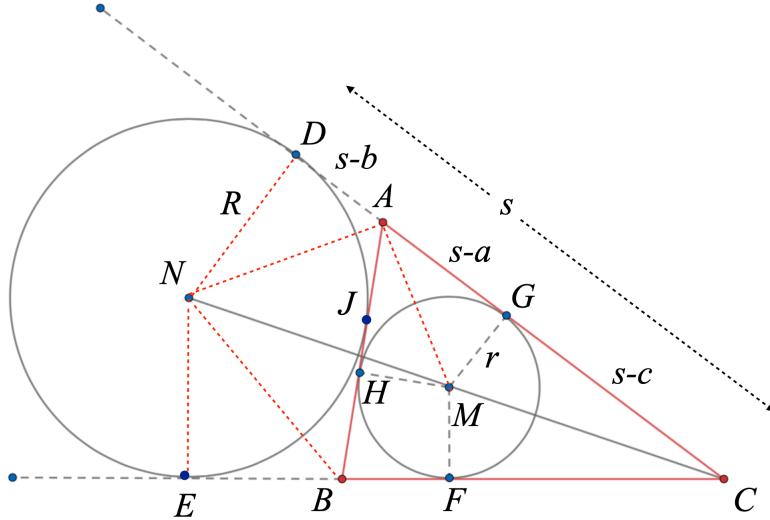
So  $\angle DAN$  is complementary to  $\angle MAG$ , since their sum is one right angle, and therefore,  $\triangle ADN \sim \triangle MGA$ .

The ratios of interest are:

$$\frac{s-b}{R} = \frac{r}{s-a}$$

$$r = \frac{(s-a)(s-b)}{R}$$

$$rs = \frac{s(s-a)(s-b)}{R}$$



Combining with our previous result:

$$(rs)^2 = s(s-a)(s-b)(s-c)$$

But  $rs$  is the area of the triangle  $ABC$ .

$$\mathcal{A}^2 = s(s-a)(s-b)(s-c)$$

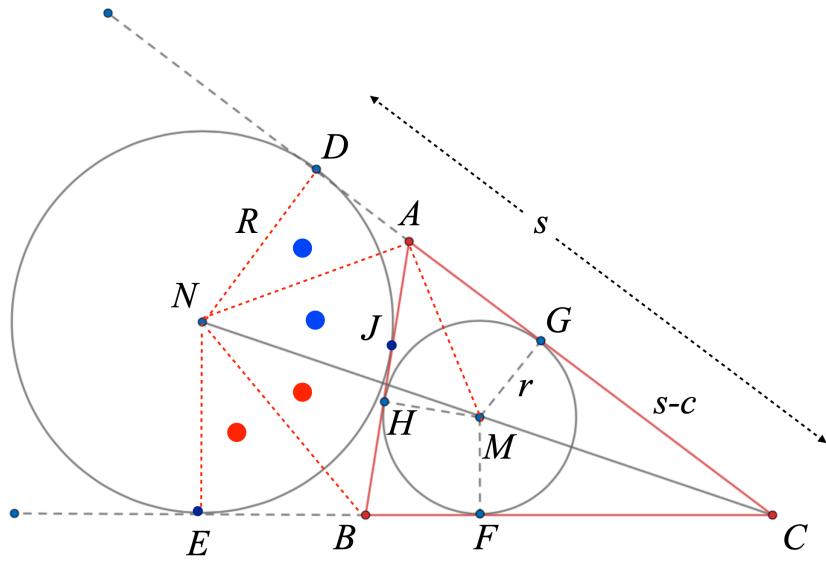
$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

□

This is Heron's formula. Very elegant.

The formula can also be obtained by computing areas in different ways, as described in the link above. Here is an idea of the proof.

*Proof.*



Again using parentheses to signify area, we have that the polygon  $ADNEB$  consists of two pairs of equal triangles, so that its area is twice that of  $\triangle ANB$ :

$$(ADNEB) = 2(\triangle ANB) = Rc$$

The area of quadrilateral  $CDNE$  is twice that of  $\triangle CDN$ :

$$(CDNE) = 2(\triangle CDN) = Rs$$

The area of  $\triangle ABC$  is simply the difference:

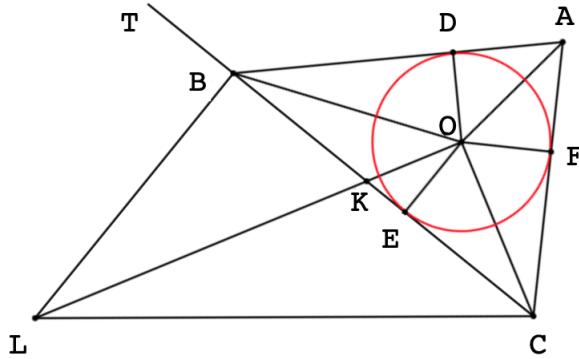
$$\begin{aligned} (\triangle ABC) &= (CDNE) - (ADNEB) \\ &= Rs - Rc = R(s - c) \end{aligned}$$

This is the same relationship we obtained by determining the total length of the tangent  $CD = s$  and then noting similar triangles  $\triangle ADN \sim \triangle MGA$ .

## Heron's proof

Lastly we go through Heron's proof of the eponymous theorem (as given by Paul Yiu). I've stolen the diagram, but switched labels on a few of the points.

The triangle is  $\triangle ABC$  with incircle radii  $OD$ ,  $OE$  and  $OF$ .



$BT$  is drawn equal in length to  $AD$ , so that the whole length  $CT = s$ .

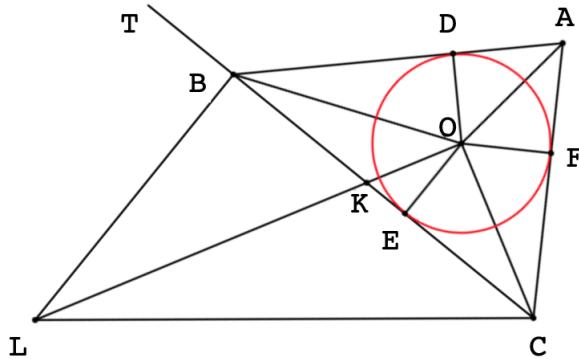
Two right triangles are drawn on the diameter of a circle  $LC$ , so  $B$  and  $O$  lie on the circle and  $\angle LBC = \angle LQC$  are right angles.

As usual, the whole triangle is the sum of three pairs of equal triangles whose total base is the semiperimeter and altitude is equal to the radius  $EO$ .  $CT$  is drawn to be equal to the semiperimeter, hence the area of the triangle is  $CT \cdot OE$ .

The total angle at  $L$  is equal to  $\angle AOD$ .

*Proof.* Because  $\angle L$  and  $\angle BOC$  are opposing angles in a quadrilateral inscribed into a circle, they are supplementary. But the components of  $\angle BOC$  ( $\angle BOE$  and  $\angle COE$ ) are, together with  $\angle AOD$ , one-half the total angle at  $O$ . Thus  $\angle BOC + \angle AOD$  are together equal to two right angles and  $\angle BOC$  is supplementary also to  $\angle AOD$ . Hence  $\angle L = \angle AOD$ .  $\square$

As a result, we have two pairs of similar right triangles:  $\triangle CBL \sim \triangle AOD$  and  $\triangle LBK \sim EKO$  (since  $OE \parallel BL$ ).



Now we follow quite a chain of ratios.

From the first pair of similar triangles ( $\triangle CBL \sim \triangle AOD$ ), we have the ratio of bases as

$$\frac{BC}{BL} = \frac{AD}{DO}$$

Rearranging:

$$\frac{BC}{AD} = \frac{BL}{DO}$$

Substituting  $BT = AD$  and  $EO = DO$

$$\frac{BC}{BT} = \frac{BL}{EO}$$

From the second pair ( $\triangle LBK \sim EKO$ ) we have

$$\frac{BL}{BK} = \frac{OE}{KE}$$

$$\frac{BL}{OE} = \frac{BK}{KE}$$

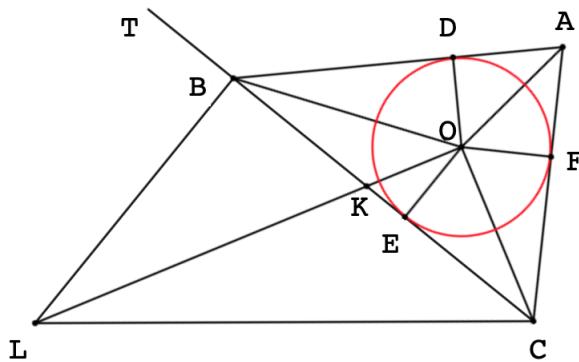
Hence

$$\frac{BC}{BT} = \frac{BK}{KE}$$

The rest is relatively straightforward.

Adding 1 to both sides

$$\begin{aligned}\frac{BC + BT}{BT} &= \frac{BK + KE}{KE} \\ \frac{CT}{BT} &= \frac{BE}{KE}\end{aligned}$$

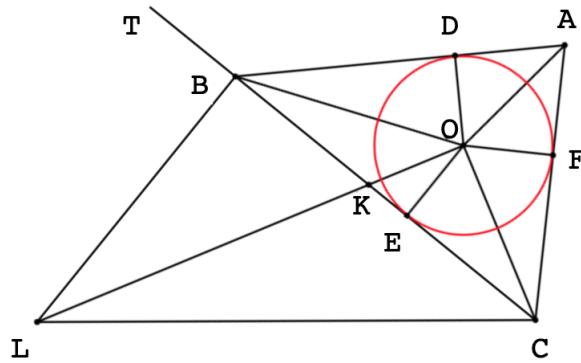


Rearranging

$$CT = BT \cdot \frac{BE}{KE}$$

Multiply top and bottom by  $CE$ :

$$CT = BT \cdot \frac{BE}{KE} \cdot \frac{CE}{CE}$$



$\triangle COK$  is right, thus  $OE^2 = CE \cdot KE$  (in conventional notation  $h^2 = de$ ).

So then

$$OE^2 \cdot CT = BT \cdot BE \cdot CE$$

$$OE^2 \cdot CT = AD \cdot BE \cdot CE$$

$$OE^2 \cdot CT^2 = AD \cdot BE \cdot CE \cdot CT$$

□

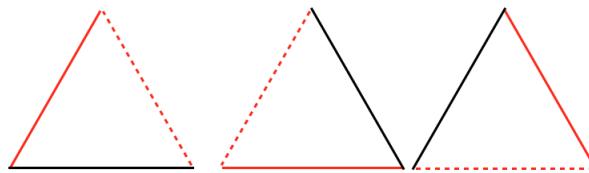
This is Heron's theorem.

# Chapter 34

## Equilateral triangles

### basic triangle

An equilateral triangle has all three sides the same length. By symmetry, as there is no reason to favor one vertex, the three vertices have the same angular measure, namely,  $\pi/3$  (since the total is two right angles).



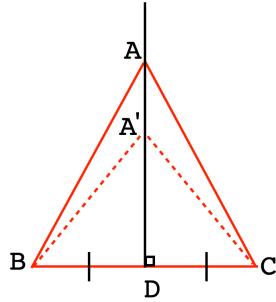
*Proof.*

The **isosceles triangle theorem** (Euclid I.5: equal sides  $\rightarrow$  angles), which can be applied twice to give the result.

We also have **Euclid I.19**, which says that if one angle is larger than another in a triangle, then the corresponding side is also longer. So, assume the angles are not the same, then that implies the sides are not the same length either. Hence the angles are the same.

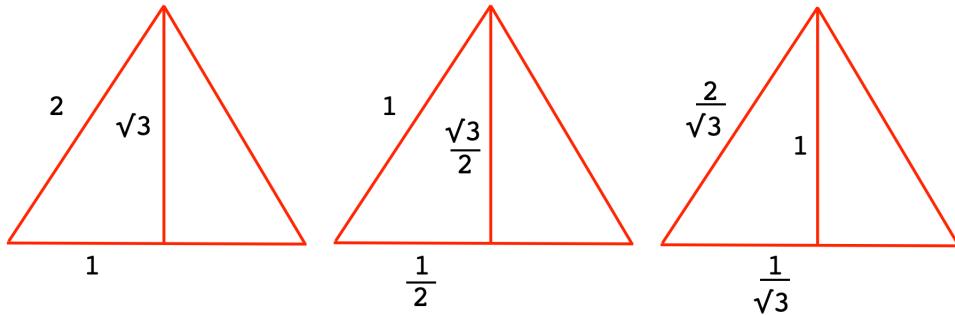
□

An equilateral triangle is also isosceles (three times over), and as a result the altitude dropped from a vertex to the opposing base bisects both that base and the angle at the vertex.



## area

To calculate the area of the equilateral triangle, we could write a general formula, but it is usually easier to go back to basics and derive one.



In the left panel, the result of bisection is that, for a side of length 2, half the base is 1, and then Pythagoras tells us that the altitude is  $\sqrt{3}$ . Using the standard formula, the area of this triangle is also  $\sqrt{3}$ .

To re-scale the triangle, write whichever measurement is given. For example in the middle we're given a side length of 1 so that means that all lengths are multiplied by  $1/2$ .

On the right, we're given that the altitude is 1 so then we must multiply by  $1/\sqrt{3}$ . The logic is that to erase that  $\sqrt{3}$  in the altitude, we must multiply by its inverse.

If you really insist on a general formula...

Suppose the length of the side is  $s$ , then the ratio of the altitude to the side is

$$\frac{h}{s} = \frac{\sqrt{3}}{2}$$

Therefore twice the area of the triangle is

$$2A = \frac{\sqrt{3}}{2} \cdot s^2$$

We can check that against a general trigonometric formula for the area of any triangle. Twice the area is the product of two adjacent sides times the sine of the angle between. Here the angle is  $\pi/3$  and its sine is  $\sqrt{3}/2$  so we have  $2A = \sqrt{3}/2 \cdot s^2$ .

It can be quicker to rely on the fact that the area goes like the square of the side. For side length 2 we had  $A = \sqrt{3}$ . If we shrink the triangle to side length 1, the area goes down by a factor of four, to  $A = \sqrt{3}/4$ .

## circumscribed equilateral triangle

Here is a fun construction based on an equilateral triangle.

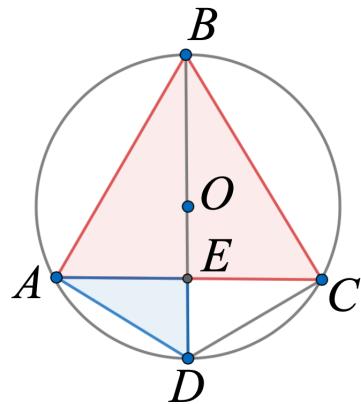
Any triangle fits into a unique circle (see the discussion of circles and bisectors of the chords). Draw the circumcircle for equilateral  $\triangle ABC$ .

Draw a radius to a vertex of the triangle, and extend it as the diameter  $BD$ .

We can prove a number of properties. One is that  $DE = OE$ .

It follows that  $OE$  is one-half of  $OB$  and one-third of  $BE$ .

*Proof.*



(1) Given  $AB = BC = AC$  and all three vertex angles ( $\angle B, \angle BAC, \angle BCA$ ), are equal, so each is two-thirds of a right angle.

(2)  $B$  is equidistant from the ends of  $AC$ , so it lies on the perpendicular bisector of  $AC$ .

(3) It follows that  $AE = CE$

(4) the angles at  $E$  are right angles

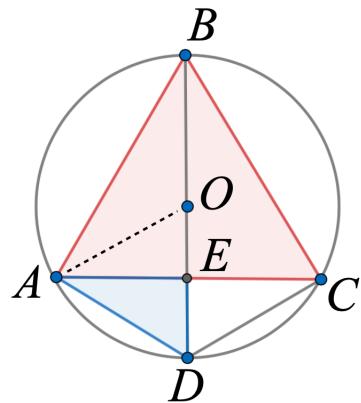
(5) and  $\triangle BAE \cong \triangle BCE$  by SSS or SAS.

(6) So  $\angle B$  is bisected, with  $\angle ABD = \angle CBD$ .

By (3), the ratio of any side of the original triangle to  $AE$  is  $2 : 1$ .

(7) By Thales' theorem,  $\triangle BAD$  and  $\triangle BCD$  are right triangles.

- (8) By (4), all the smaller triangles are right triangles.
- (9) By the corollary of the inscribed angle theorem  $\angle CAD = \angle CBD$  and both are one-third of a right angle.
- (10) For the same reason,  $\angle BDA = \angle BCA$  and by (1) both are two-thirds of a right angle.
- (11) By (3) and (4)  $\triangle DAE \cong \triangle DCE$  by SAS.
- (12) Thus  $AD = DC$ .
- (13) and  $\angle DAE = \angle DCE$ .



- (14) Draw  $OA$ .  $OA$  bisects  $\angle BAC$  (just as  $\angle B$  is bisected by  $OB$ , see above).
- (15) Hence  $\angle OAC$  is one-third of a right angle.
- By (9)  $\angle CAD$  is one-third of a right angle, so by (15)  $\angle OAD$  is two thirds of a right angle.
- (16)  $\triangle AOD$  is (at least) isosceles, with base angles equal. By sum of angles, it follows that  $\triangle AOD$  is equilateral.

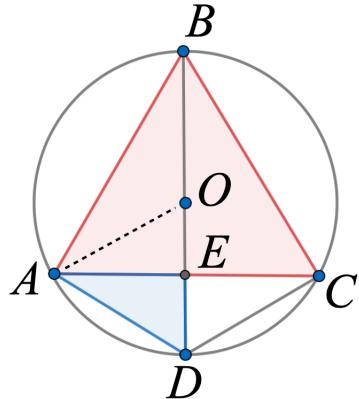
[Possibly easier:  $\triangle AOD$  is isosceles, with two sides being radii of the same circle.  $\angle AOD$  is the central angle for one-half of  $\angle B$ , hence it equals  $\angle B$ . By sum of angles,  $\triangle AOD$  is equi-angular, and then by I.6 it is equilateral.]

- (17) By (16)  $OA = AD$  (i.e.  $AD$  is equal to the radius).
- (18)  $AOCD$  has opposing angles equal, so it is a parallelogram.

(19)  $AOCD$  has adjacent sides equal, so it is a rhombus.

□

*Proof.* (Alternate)



$\triangle ABD$  is a right triangle by Thales' theorem.

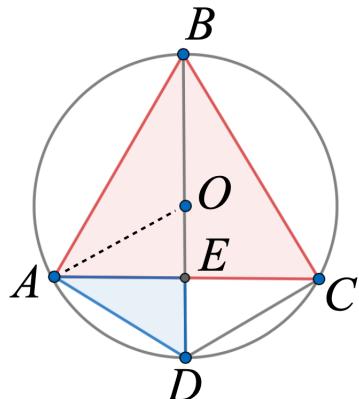
$\angle B$  is bisected so  $\triangle ABD$  is 30-60-90. We know that for such a triangle  $2AD = BD$ .

Thus  $AD = OA = OD$ ,  $\triangle AOD$  is equilateral, and  $AOCD$  has four equal sides, so it is a rhombus.

It follows that the diagonal  $OD$  is bisected by  $AC$ .

□

*Proof.* (Alternate)



$\angle AOC$  is the central angle for  $\angle ABC$ , so it is twice  $\angle ABC$ .

$\angle ADB = \angle BCA$  and  $\angle CDB = \angle BAC$  so  $\angle ADC$  is twice  $\angle BCA$  or  $\angle BAC$ .

It follows that  $\angle ADC = \angle AOC$ .

But by Thales' theorem,  $\angle BAD = \angle BCD$  and both are right angles.

Since the opposing angles are equal,  $AOCD$  is a parallelogram.

It follows that  $OA = CD$  and  $OC = AD$ , thus all four sides are equal and  $AOCD$  is a rhombus.

Therefore, the diagonals bisect one another.

□

*Proof. (Alternate).*

Let  $AD = CD = d$  and  $r$  be the radius of the circle.

By Ptolemy's theorem:

$$sd + sd = 2rs$$

$$d = r$$

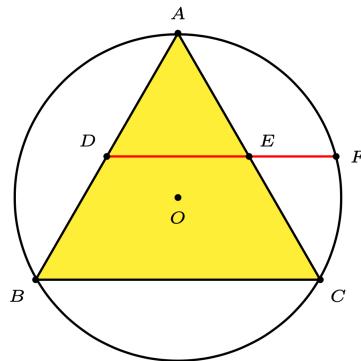
It follows that  $AOCD$  is a parallelogram, etc.

□

Thus, the altitude of the equilateral triangle is  $3/4$  of the diameter of the circle that just encloses it. And the point where the altitudes meet in an equilateral triangle is  $1/3$  of the way up from the base, since  $r = 2a$ .

## problem

Here is a problem from Paul Yiu. Given that  $ABC$  is equilateral and that the points  $D$  and  $E$  are midpoints of the sides, find  $DF/DE$ .



We *could* calculate the length of the perpendicular from  $O$  to  $DE$  and then use the Pythagorean theorem.

A simpler approach is to set the side length of the triangle to 2, so each half is 1, and since  $ADE$  is also equilateral,  $DE = 1$ . Let  $DF$  be  $x$ . Then, by the theorem of crossed chords we have that

$$\begin{aligned} 1 \cdot 1 &= (x - 1) \cdot x \\ x^2 - x - 1 & \\ x &= \frac{1 + \sqrt{5}}{2} = \phi \end{aligned}$$

This is the famous “golden” ratio.

Let’s do it the hard way as well. If we keep the side length of the large triangle as 2, then the small triangle has side length 1, and the altitude is  $\sqrt{3}/2$ .

We need the radius. The altitude of the large triangle is  $\sqrt{3}$  and the diameter is  $4/3$  of that, so the radius is  $2/3$  which makes  $r = 2/\sqrt{3}$ .

The distance from  $O$  to  $DE$  is the difference, and we will need the square of that:

$$\begin{aligned} \delta^2 &= \left(\frac{\sqrt{3}}{2} - \frac{2}{\sqrt{3}}\right)^2 \\ &= \frac{3}{4} + \frac{4}{3} - 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{2}{\sqrt{3}} \\ &= \frac{3}{4} + \frac{4}{3} - 2 \end{aligned}$$

The distance from the midpoint of  $DE$  to  $F$ , squared, is:

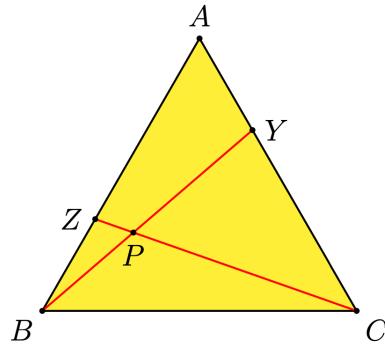
$$\frac{4}{3} - \delta^2 = \frac{4}{3} - \left(\frac{3}{4} + \frac{4}{3} - 2\right) = \frac{5}{4}$$

so the square root is  $\sqrt{5}/2$  and then we must add  $1/2$  to obtain  $DF$ . That matches what we had before.

## problem

Here is a second problem from Dr. Yiu.

Crossing lines are drawn from the vertices in an equilateral triangle, such that the larger pair of angles at  $P$  are each  $120^\circ$ .



We have that, since  $\angle ZPY + \angle A =$  two right angles, the angles inside the quadrilateral at  $Z$  and  $Y$  are supplementary, so the opposing ones are equal.

A consequence of that is  $\triangle AYB$  and  $\triangle BZC$  have all three angles the same, but also share a side, hence they are congruent. Therefore  $AY = BZ$  and  $AZ = YC$ .

By the same reasoning,  $\angle AZC = \angle BYC$ , which means that  $\triangle AZC$  and  $\triangle YCB$  share all three angles plus two sides, so they are congruent. Therefore  $BY = ZC$ .

There are a ridiculous number of similarities and equalities here. The acute angles at  $P$  are  $60^\circ$ , which gives more similar triangles such as  $\triangle PZB \sim \triangle AYB$  and  $\triangle PYC \sim \triangle AZC$ . I write the vertices flanking the short and medium sides in order to make it easy to write the ratios.

We are almost too wealthy. We are asked to show that

$$\frac{AY}{AZ} = \frac{PB}{PC}$$

Now,  $\triangle BPC$  is not similar to anything else. But from the first similarity above ( $\triangle PZB \sim \triangle AYB$ ) we get

$$PZ/ZB/BP = AY/YB/BA$$

so we can write

$$PB = AB \cdot ZB/YB$$

From the second similarity ( $\triangle PYC \sim \triangle AZC$ ) we get

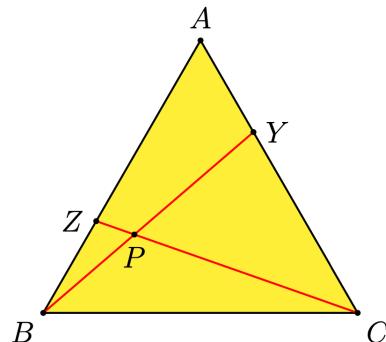
$$PY/YC/CP = AZ/ZC/AC$$

so we can write

$$PC = AC \cdot YC/ZC$$

Then we form the ratio (canceling  $AB = AC$ )

$$\frac{PB}{PC} = \frac{ZB}{YB} \cdot \frac{ZC}{YC}$$



Using the equalities  $ZC = BY$ ,  $AY = ZB$ ,  $AZ = YC$ :

$$\frac{PB}{PC} = \frac{ZB}{YC} = \frac{AY}{AZ}$$

I have no idea if there is an easier solution, just happy to have found one.

## Bertrand's paradox

Grinstead and Snell's wonderful *Introduction to Probability* has this problem (example 2.6). It's called Bertrand's paradox. We are asked to draw a chord of a unit circle randomly.

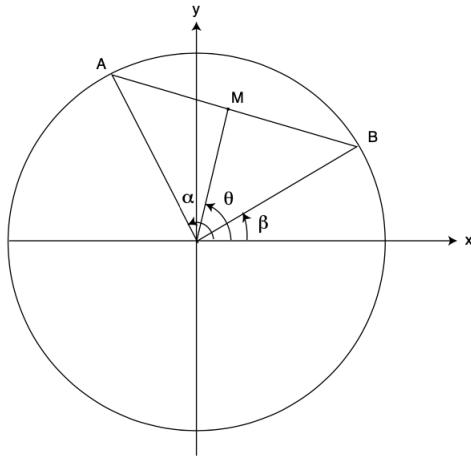


Figure 2.9: Random chord.

Here we might say, let's choose each of three angles  $\alpha$ ,  $\beta$  and  $\theta$  randomly (uniform density) from  $[0, 2\pi]$ .

But there is no reason why the radius to  $B$  cannot lie along the  $x$ -axis, so there are really only two choices.

The question is posed: what is the probability that the length of this random chord is  $> \sqrt{3}$ .

However, there are several different approaches to parametrize the problem, and randomizing the different parameters leads to different results.

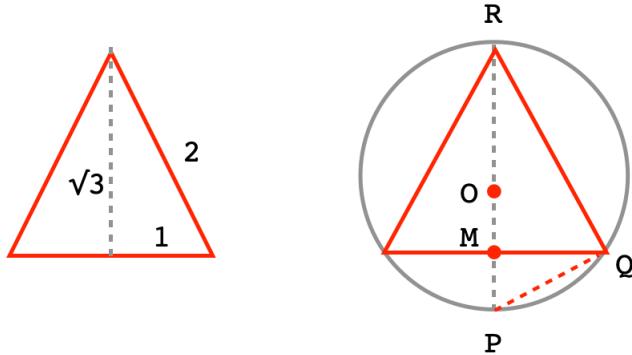
## equilateral triangles

We briefly review some properties of equilateral triangles which we looked at earlier, with a slightly different take.

Drop an altitude and observe the ratio of side lengths. It is convenient to start with a side length of 2 for the original triangle, then in the bisected copies the sides are in the ratio  $1-2-\sqrt{3}$  (the 1 by bisection, and  $\sqrt{3}$  by the Pythagorean theorem).

The angle at each vertex of the original equilateral triangle is  $\pi/3$ , so the new triangles have angles of  $\pi/6$ , both by bisection or because the altitude forms an angle of  $\pi/2$  at the base, so the sum of angles theorem gives us the last angle.

In the right panel, the equilateral triangle is inscribed in a unit circle, so  $OR = OP = OQ = 1$ . We claim that the line segment  $OM$  has a length of  $1/2$ .



*Proof.*

$\angle PQR$  is a right angle, by Thales' circle theorem, and  $\angle MRQ$  is shared, so  $\triangle PQR$  is similar to  $\triangle RMQ$ . Therefore,  $\angle RPQ = \angle MQR = \pi/3$ .

Therefore the sides of  $\triangle PQR$  are also in the ratio  $1-2-\sqrt{3}$ , with  $PQ/PR = 1/2$  and so  $PQ = OP = OQ$ . Thus,  $\triangle OPQ$  is equilateral.

$QM \perp OP$  so  $MQ$  is the bisector of both  $\angle PQO$  and the base  $OP$ .

Therefore,  $OM$  is one-half of  $OP$  and has a length of  $1/2$ .

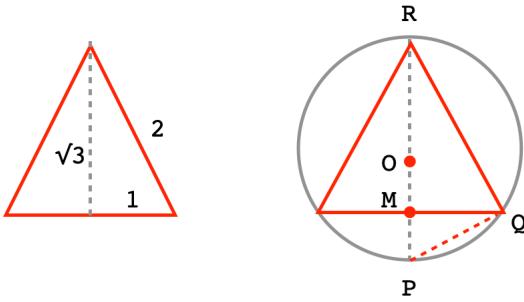
□

## first parametrization

We have just shown that the altitude of the inscribed equilateral triangle in a unit circle has length  $3/2$ . This means that the ratio of the inscribed triangle to the standard one is  $\sqrt{3}/2$ .

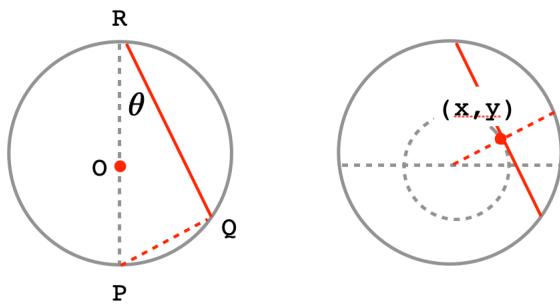
And that means the side length of the inscribed equilateral triangle is  $\sqrt{3}$ .

That explains the length chosen for the chord in this problem. We see that if  $M$  is chosen at random anywhere along  $OP$ , one-half of the time the chord formed will be larger than  $\sqrt{3}$ .



So the probability we are asked to give is just  $1/2$ .

### second parametrization



The second parametrization has the same triangle we just saw,  $\triangle PQR$ . The angle at vertex  $R$  is  $\theta$ .

$\theta$  can lie in the interval  $[0, \pi/2]$ , and in the event that  $\theta < \pi/6$ , the chord length  $RQ > \sqrt{3}$ .

The probability that the chord is greater than  $\sqrt{3}$  in length is  $1/3$ , since  $\pi/6$  is one-third of  $\pi/2$ .

### third parametrization

Finally, we imagine picking two coordinates  $(x, y)$  at random from the interior of the circle. We place the midpoint of the chord  $M$  at  $(x, y)$ .

If  $M$  is such that  $r = \sqrt{x^2 + y^2} < 1/2$ , then  $M$  will be closer to the center of the circle than  $1/2$  and so the chord length will be  $> \sqrt{3}$ .

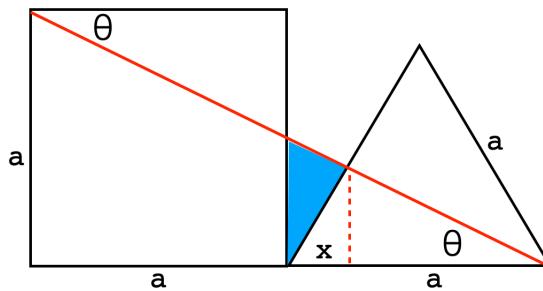
The number of points that have this property is proportional to the relative areas of the inside small circle, and the larger circle around it.

$$\frac{\pi(1/2)^2}{\pi} = \frac{1}{4}$$

We see that, depending on which parameter is randomized, we obtain a probability of  $1/2$ ,  $1/3$  or  $1/4$ .

In Jaynes' words, the problem is not well-formed.

## problem

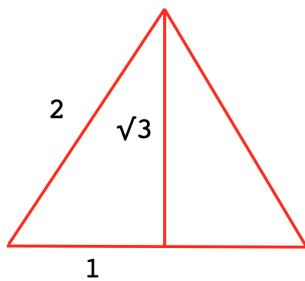


We have a square with side  $a$  and an equilateral triangle, also with side  $a$ . Their bases are colinear and a red line is drawn as shown. What is the area of the blue triangle?

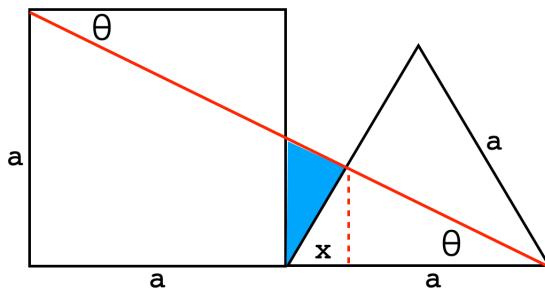
The angles marked  $\theta$  are equal because the base and the top of the square are parallel, but the angle is *not* 30. The red line intersects the side of the square at half the height, so  $a/2$ .

We compute *twice* the area of the combined blue and part of the equilateral triangle as  $a^2/2$  and that of the blue triangle as  $xa/2$ . But what is  $x$ ?

We need a preliminary result. The Pythagorean theorem allows us to calculate that the altitude of an equilateral triangle is  $\sqrt{3}$  times one-half the base.



Back to our problem. The triangle with  $x$  as its base does have an angle of 30 so its height  $h$  is  $\sqrt{3} \cdot x$ . We compute twice the area of the part of the equilateral triangle as  $\sqrt{3}xa$ .



Put the whole thing together as

$$2\mathcal{A} = \frac{a^2}{2} = \frac{xa}{2} + \sqrt{3}xa$$

$$a = x + 2\sqrt{3}x$$

$$x = \frac{a}{1 + 2\sqrt{3}}$$

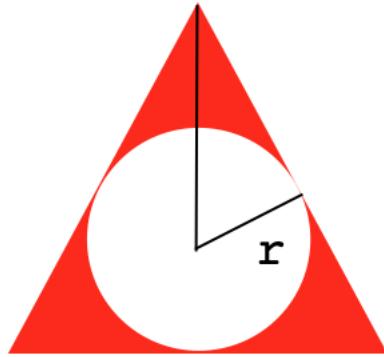
The final result is  $1/2$  times  $a/2$  times what we have above.

## problem

Here is a problem from the entrance exam for Brown, 1901.

4. A regular triangle is circumscribed about a circle whose radius is 7. Find the area of that portion of the triangle which is outside of the circle.

A regular triangle is what we would call an equilateral triangle. Applying the Pythagorean theorem to one-half of such a triangle yields  $1-2-\sqrt{3}$  for the sides, where 2 is the hypotenuse.



From this we deduce that the altitude is  $3r$  and the side length is  $2\sqrt{3}r$ , so the triangle's area is

$$A = \frac{1}{2} \cdot 3 \cdot 2\sqrt{3} \cdot r^2$$

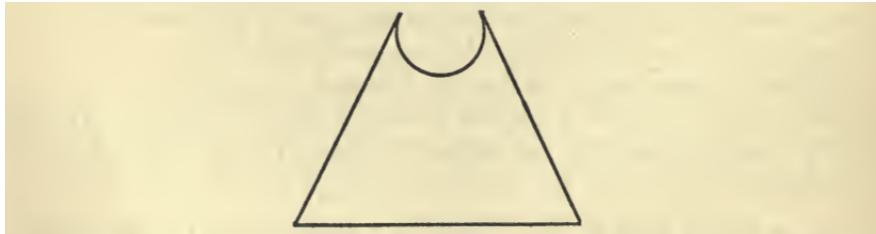
The red part is the difference between this and the circle's area.

$$\text{red} = (3\sqrt{3} - \pi)r^2$$

We're given a radius of 7 but that's arithmetic. We don't need that now.

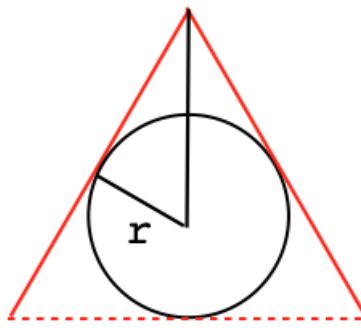
## problem

Here is a problem from the entrance exam for Harvard, 1901.



\*6. A piece is cut out of an equilateral triangle by means of an arc of a circle tangent to two sides. The side of the triangle is 7 inches and the radius of the circle 1 inch. Compute to two decimal places the perimeter and the area of the figure which is left.

The first thing to notice is that the circle is tangent to the sides. The tangent touches the circle at a single point, so the missing part of the diagram looks like this:



The whole angle at the top is  $\pi/3$  so the half-angle is  $\pi/6$ . This right triangle has sides in the ratio  $1-2-\sqrt{3}$ . The radius is given as 1 so the red base side of the triangle is  $\sqrt{3}$ .

The missing perimeter of the triangle is twice this or  $2\sqrt{3}$ . But the added perimeter from the circle is  $2/3 \cdot 2\pi r = 4/3\pi$ .

The area of the triangle is  $1/2 \cdot \sqrt{3} \cdot 1 = \sqrt{3}/2$  and there are two copies so the total missing triangular area is  $\sqrt{3}$ . The added circular area is  $2/3\pi$ .

The original perimeter of the triangle was  $7 \cdot 3 = 21$  and the area was  $\sqrt{3}/4 \cdot s^2$ .

We'll just set up the two calculations:

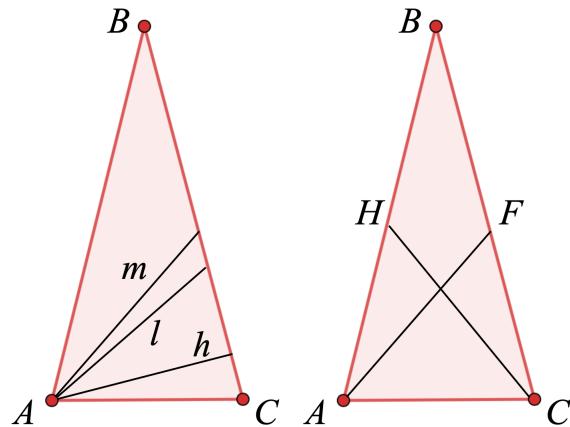
$$A = \frac{\sqrt{3}}{4} \cdot 7^2 - \sqrt{3} + 2/3 \cdot \pi$$

$$P = 7 \cdot 3 - 2\sqrt{3} + 4/3 \cdot \pi$$

It seems that the most efficient way to calculate this is to do  $\sqrt{3} + 2/3 \cdot \pi$ , then double it, then do the rest.

# Chapter 35

## More isosceles



The following statements about isosceles triangles are equivalent:

1. two sides equal
2. base angles equal
3. medians ( $m$ ) equal
4. angle bisectors ( $l$ ) equal
5. altitudes ( $h$ ) equal
6. the bisector, median and altitude at  $b$  coincide

where in (1-5) the medians, bisectors and altitudes in question are those extending to the equal sides.

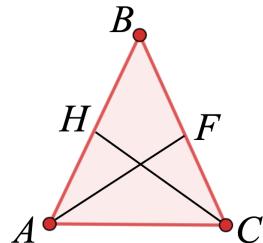
The figure shows (right panel) the case for the medians, but we will use the same labels for the other cases in the discussion below.

The question is then how to prove these one from the other, for example  $1 \Rightarrow 3$  or  $5 \Rightarrow 2$ . Of course, we already have  $1 \Rightarrow 2$  and  $2 \Rightarrow 1$ .

In what we might call the forward direction  $\{ 1,2 \} \Rightarrow \{ 3,4,5,6 \}$ , these proofs are quite easy. I encourage you to try them before reading further.

The converse proofs are another matter.

## forward



$1 \Rightarrow 3$

*Proof.*  $AF$  and  $CH$  are medians in  $\triangle ABC$ . Given  $AB = CB$  and  $\angle A = \angle C$ , it follows that  $AH = CF$  (by the definition of median) and thus by SAS we have  $\triangle ACH \cong \triangle CAF$ . Hence  $AF = CH$ .  $\square$

$1 \Rightarrow 4$

*Proof.*  $AF$  and  $CH$  are angle bisectors in  $\triangle ABC$ . Given  $\angle A = \angle C$ , the half-angles are also equal, so by ASA we have  $\triangle ACH \cong \triangle CAF$ . Hence  $AF = CH$ .  $\square$

$1 \Rightarrow 5$

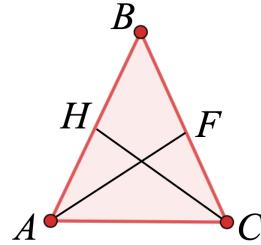
*Proof.*  $AF$  and  $CH$  are altitudes in  $\triangle ABC$ . Twice the area of  $\triangle ABC$  may be computed as  $AF \cdot CB = CH \cdot AB$ . But  $AB = CB$ . Hence  $AF = CH$ .  $\square$

## converse

One of the converse proofs is also easy.

$5 \Rightarrow 1$

*Proof.*  $AF$  and  $CH$  are altitudes in  $\triangle ABC$ . Twice the area of  $\triangle ABC$  may be computed as  $AF \cdot CB = CH \cdot AB$ . But  $AF = CH$ . Hence  $AB = CB$ .  $\square$



$4 \Rightarrow 1$

If the angle bisectors are equal, then the triangle is isosceles. This is a famous theorem, and there is a whole chapter about it here: **Steiner-Lehmus theorem**.

$3 \Rightarrow 1$

For this one, we look ahead to the **Law of cosines**.

*Proof.*

We have that the medians are equal, namely,  $AF = CH$ . Let them be equal to  $m$ . Using the law of cosines, compute the length  $m$  squared of the side opposite  $\angle B$  in two different triangles.

Let side  $a = BC$  (the side opposite  $\angle A$ ), and similarly side  $c = AB$  (the side opposite  $\angle C$ ).

So  $BF = a/2$  and  $BH = c/2$ . Then

$$\begin{aligned} m^2 &= a^2 + (c/2)^2 - 2a \cdot c/2 \cdot \cos B \\ m^2 &= c^2 + (a/2)^2 - 2c \cdot a/2 \cdot \cos B \end{aligned}$$

The last terms on the right-hand side are equal, namely

$$ac \cdot \cos B = ac \cdot \cos B$$

Setting the two expressions equal, it follows that

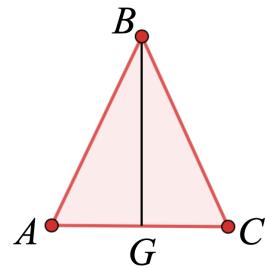
$$a^2 + (c/2)^2 = c^2 + (a/2)^2$$

$$\begin{aligned}\frac{3}{4}a^2 &= \frac{3}{4}c^2 \\ a^2 &= c^2\end{aligned}$$

Since  $a$  and  $c$  are both lengths, we may take the positive square roots for each, and thus  $a = c$ .

□

### median, bisector and altitude at $b$



Let  $b$  be the side opposite  $\angle B$  as usual.

**1 ⇒ 6**

We have that  $AB = CB$  and  $\angle A = \angle C$ .

*Proof.* Let  $BG$  be the median to  $AC$  such that  $AG = GC$ . Then  $\triangle ABG \cong \triangle CBG$  by SSS. Thus  $\angle B$  is bisected and  $\angle AGB = \angle CGB$  and both are right angles. □

*Proof.* Let  $BG$  be the altitude to  $AC$  such that  $\angle AGB = \angle CGB$  and both are right angles. Then  $\triangle ABG \cong \triangle CBG$  by HL. Thus  $\angle B$  is bisected and  $AG = GC$ . □

*Proof.* Let  $BG$  bisect  $\angle B$ . Then  $\triangle ABG \cong \triangle CBG$  by SAS. Thus  $\angle AGB = \angle CGB$  and both are right angles, and  $AG = GC$ . □

**6 ⇒ 1**

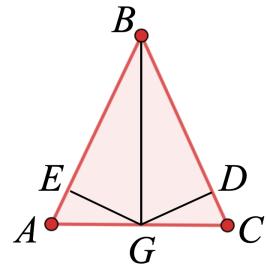
We will prove that if any two of these statements are both true, then  $a = c$ . In what follows, if we have  $\triangle ABG \cong \triangle CBG$ , then (1) and (2) follow immediately.

*Proof.* If the bisector and the altitude at  $b$  coincide, then  $\triangle ABG \cong \triangle CBG$  by ASA.

□

*Proof.* If the median and the altitude at  $b$  coincide: then  $\triangle ABG \cong \triangle CBG$  by SAS.  $\square$

Finally, suppose the bisector and the median at  $b$  coincide. Draw perpendiculars from the midpoint of  $b$  to each of sides  $a$  and  $c$ .



*Proof.*

$BG$  bisects  $\angle B$ ,  $\angle GDB = \angle GEB$  and both are right angles, therefore  $\angle BGE = \angle BGD$  by sum of angles. The hypotenuse  $BG$  is shared. It follows that  $\triangle BEG \cong \triangle BDG$  by ASA. Thus  $GE = GD$ .

Then (since  $AG = GC$ ),  $\triangle AEG \cong \triangle CDG$  by HL. It follows that  $\angle A = \angle C$  and then by  $2 \Rightarrow 1$ , we have  $a = c$ .

$\square$

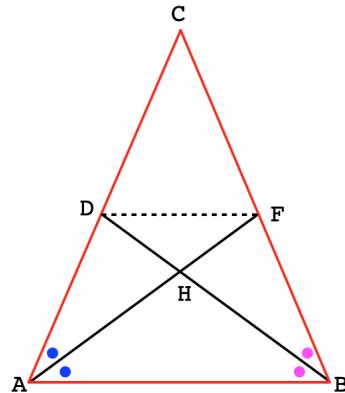
This material is covered in Byer (see **References**).

# Chapter 36

## Steiner Lehmus

This chapter discusses a theorem about angle bisectors in an isosceles triangle. We called the forward version the **isosceles bisector** theorem.

It is easy in the forward direction, but the converse is very challenging, at least until you draw the right diagram. Then, as usual, it's not so bad.



*Proof.* We are given that  $\triangle ABC$  is isosceles ( $AC = BC$ ), and also that the angles at the base are both bisected. It follows that the half-angles are also equal, and thus  $\triangle CDB \cong \triangle CFA$  by ASA. So the angle bisectors are equal in length:  $AF = BD$ .

□

That's the easy part.

## Steiner-Lehmus Theorem

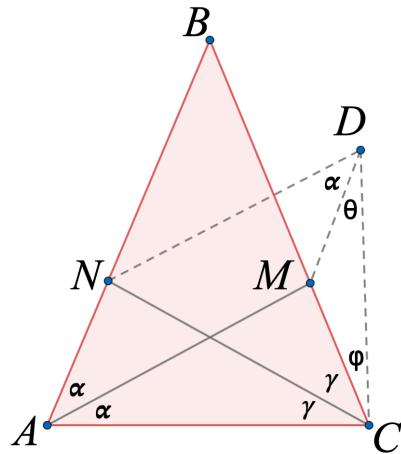
The converse theorem says that if we have angle bisectors and they are equal in length, then the triangle is isosceles.

[https://en.wikipedia.org/wiki/Steiner-Lehmus\\_theorem](https://en.wikipedia.org/wiki/Steiner-Lehmus_theorem)

The problem is that, even though we can draw triangles with two sides equal, we don't know anything about the angles except for some vertical angles, which don't help.

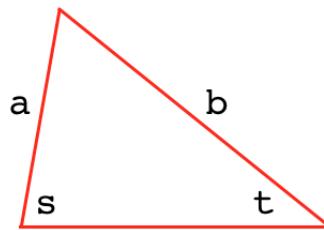
Here is an approach which I found on the web. It's a proof by contradiction.

<https://www.algebra.com/algebra/homework/word/geometry/Angle-bisectors-in-an-isosceles-triangle.lesson>



We claim that if  $AM = CN$  and the angles are bisected, then  $\alpha = \gamma$ .

We rely on Euclid's propositions I.18 and I.19. In any triangle if one side is larger than another, then the angle opposite the longer side is greater (I.18) and conversely, if one angle is larger than another, then the side opposite is greater (I.19).



In the diagram above

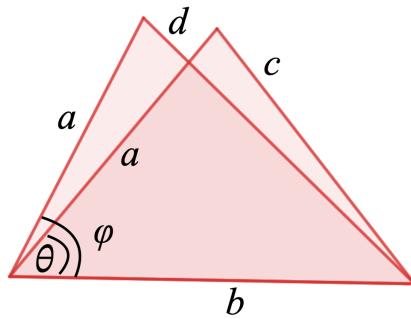
$$s > t \Rightarrow b > a$$

$$b > a \Rightarrow s > t$$

We proved both these theorems [here](#).

The problem is that these are *within-triangle* results. We also need the following result.

If two triangles  $\triangle ABC$  and  $\triangle DEF$  have two pairs of sides equal, and the included angle is greater in one ( $\phi > \theta$ ), then the side opposite  $\phi$  also greater.



*Proof.*

Let  $d$  be opposite  $\phi$  and  $c$  be opposite  $\theta$ . Use the law of cosines:

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

$$d^2 = a^2 + b^2 - 2ab \cos \phi$$

Then if  $d > c$ , so  $d^2 > c^2$ , and

$$a^2 + b^2 - 2ab \cos \phi > a^2 + b^2 - 2ab \cos \theta$$

$$-2ab \cos \phi > -2ab \cos \theta$$

$$\cos \phi < \cos \theta$$

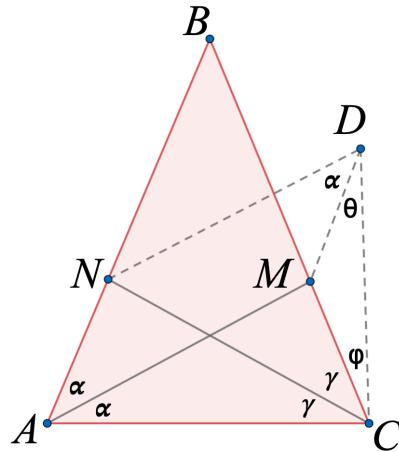
$$\phi > \theta$$

This chain of reasoning works just as well in reverse. So,  $\phi > \theta \Rightarrow d^2 > c^2$ , and then  $d > c$ .  $\square$

This is known as the **hinge theorem**. We showed a different proof earlier. It is Euclid I.24.

□

Back to our problem. We claim that  $\alpha = \gamma$  and the triangle is isosceles. We argue by contradiction.



*Proof.*

In  $\triangle ABC$ , let the base angles be bisected as shown.

Let the bisectors be equal:  $AM = CN$ .

Draw  $ND \parallel AM$  and  $MD \parallel ANB$ .

So  $ANDM$  is a parallelogram.

Thus  $\angle NDM = \alpha$ .

Aiming for a contradiction, suppose  $\gamma > \alpha$ .

By I.24,  $AN > CM$ .

So  $DM > CM$ .

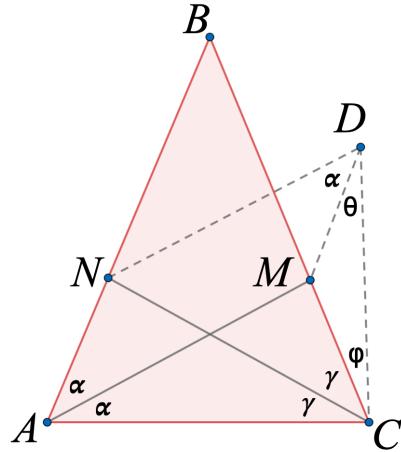
By I.18  $\phi > \theta$ .

By addition of inequalities:

$$\gamma + \phi > \alpha + \theta$$

By I.19,  $ND > CN$ .

But  $AM = ND$  so  $AM > CN$ .



This is a contradiction, we were given that  $AM = CN$ .

□

Therefore, it cannot be that  $\gamma > \alpha$ .

The reverse supposition, that  $\alpha < \gamma$ , also leads to a contradiction by a symmetrical argument, substituting  $<$  for  $>$ .

(Or draw the parallelogram on the other side of  $\triangle ABC$  and use the same argument as previously).

Since  $\alpha$  is neither greater than nor less than  $\gamma$ , we conclude that  $\alpha = \gamma$ .  $\triangle ABC$  is therefore isosceles by I.6.

□.

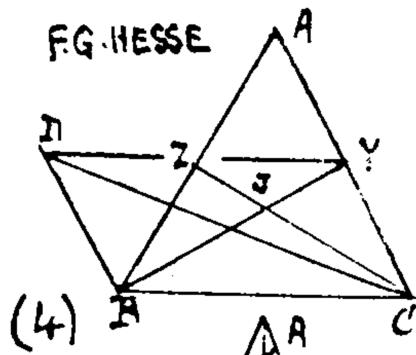
According to the internet, the Steiner-Lehmus theorem is famous for being difficult, for having many different proofs, and for some controversy over whether even one of the proofs is *direct* or not. By direct we mean, not using the technique of proof by contradiction or *reductio ad absurdum*.

I was lucky to find a (non-paywalled) review published on its centenary in 1942.

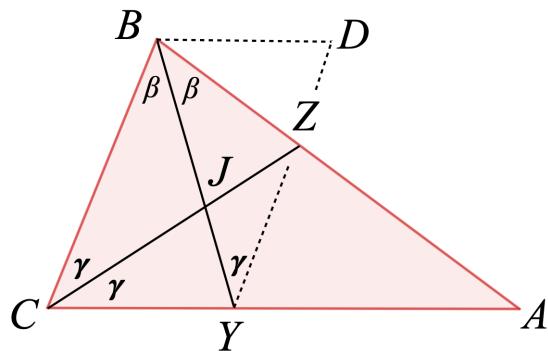
<https://www.cambridge.org/core/services/aop-cambridge-core/content/view/7B625B08567935CAE06A0AC9430477C0/S0950184300000021a.pdf>

It includes this proof.

Hesse (1842)



The construction is to draw  $YD = BC$  and  $BD = BZ$ . Later we will find that  $CD$  divides  $BDYC$  into two congruent triangles. Then it easily follows that  $\beta = \gamma$ .



*Proof.*

The equalities of the construction are  $YD = BC$  and  $BD = BZ$ , and  $BY = CZ$  is given.

(1)  $\triangle D Y B \cong \triangle B C Z$  by SSS.

Thus the corresponding angles of  $\triangle DBY$  and  $\triangle BZC$  are equal, namely:

$$\begin{aligned}\angle D Y B &= \angle B C Z = \gamma \\ \angle B D Y &= \angle Z B C = 2\beta \\ \angle D B Y &= \angle B Z C = \angle A + \gamma\end{aligned}$$

the last by sum of angles.

At this point I had some trouble with the details of the proof, so we may diverge from the source.

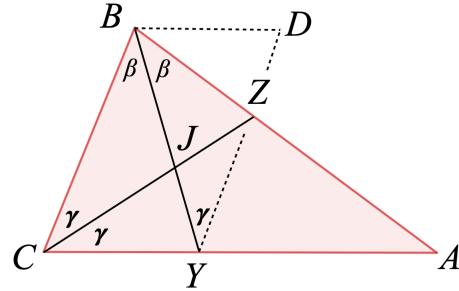
$$(2) \angle BDY = 2\beta$$

so by sum of angles in  $\triangle DBY$  we have:

$$\angle ZBD = A - \beta + \gamma$$

Then

$$\angle CBD = A + \beta + \gamma = 90 + \alpha$$



Now all we need is to get the measure of  $\angle CYD$

(3) By sum of angles  $\angle BJC = \angle A + \beta + \gamma$  so by the external angle theorem:

$$\angle BYC = \angle A + \beta$$

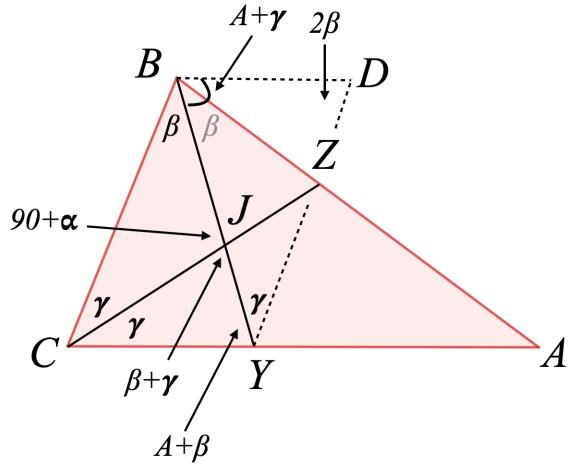
Then

$$\angle CYD = \angle A + \beta + \gamma$$

We have established that

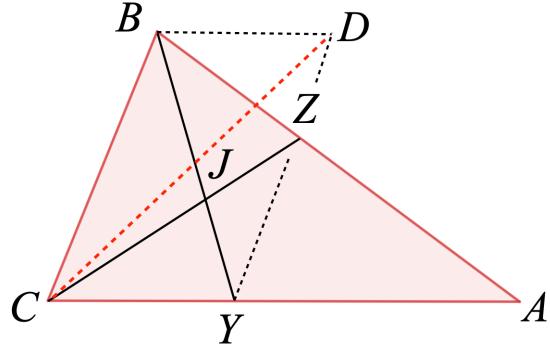
$$\angle CYD = \angle CBD$$

Crucially, they are not only equal but obtuse.



We don't necessarily have a parallelogram yet because we have only one pair of sides equal and one pair of opposing angles equal.

But we can draw  $CD$ .



(4) Comparing  $\triangle CBD$  and  $\triangle DYC$ , we have that  $CD$  is shared, and  $DY = BC$  by construction.  $\angle CYD = \angle CBD$ .

We have SSA and in addition, the angles that we know in each,  $\angle CYD$  and  $\angle CBD$ , are obtuse. It follows that  $\triangle CBD \cong \triangle DYC$ .

At this point we could just invoke the converse of the diagonal theorem for quadrilaterals.

(5) Instead Hesse says:

$$YC = BD = BZ$$

SSS then gives:

$$\triangle ZCB \cong \triangle YBC$$

$$\angle CBZ = \angle BCY$$

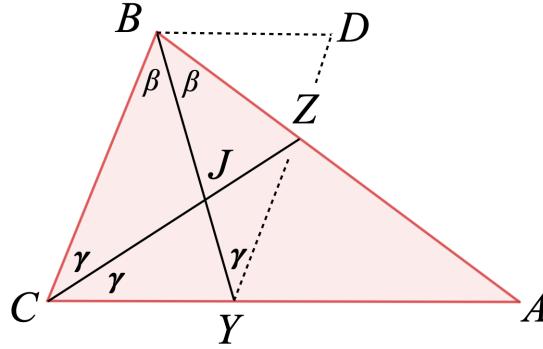
With base angles equal, all we need is I.6.

□

For more about SSA see [here](#).

With the same diagram, I cooked up a proof by careful bookkeeping with the angles.

*Proof.* (Alternate).



Given  $\angle B$  and  $\angle C$  bisected into  $2\beta$  and  $2\gamma$ .

Given  $BY = CZ$ .

Draw  $BD = BZ$  and  $DY = BC$ . (Again, we do not claim  $DZY$  collinear).

$\triangle BYD \cong \triangle ZCB$  by SSS.

So  $\angle BDY = \angle ZBC = 2\beta$  and  $\angle BYD = \angle ZCB = \gamma$ .

At  $J$ , one vertical angle ( $\angle BJC$ ) is  $\angle A + \beta + \gamma$  by sum of angles.

$\angle BJC$  is external to  $\triangle JCY$ , hence  $\angle CYJ = \angle A + \beta$ .

Up to now, nothing has changed. Let  $\angle DBZ = \theta$ .

Summing angles in the quadrilateral  $BDYC$ :

$$\theta + 2\beta + 2\gamma + \angle A + \beta + \gamma + 2\beta = 360$$

$$\theta + \angle A + 5\beta + 3\gamma = 360$$

Since  $\angle A + 2\beta + 2\gamma = 180$ , subtracting

$$\theta + 3\beta + 1\gamma = 180$$

We also have that  $\triangle BDZ$  is isosceles:

$$\theta + 4\beta = 180$$

Subtract again:

$$\begin{aligned} -\beta + \gamma &= 0 \\ \beta &= \gamma \end{aligned}$$

By I.6,  $\triangle ABC$  is isosceles.

□

## afterward

There is some interesting discussion in Coxeter as well. According to what I can find on the web, most of the literature concerns the question of whether it is possible to provide a direct proof of the theorem. The algebraic proof, postponed for now to book II, has been cited as such.

However, that proof depends on Stewart's Theorem, which as we derive it there depends on the Law of Cosines, which depends in turn on the theorem of Pythagoras. And although there are several hundred proofs of Pythagoras most (all?) of them depend on the sum of angles and also on the parallel postulate, which explicitly depends on a proof by contradiction.

The question of a direct proof for Steiner-Lehmus is hard to answer conclusively. I have a write-up from John Conway claiming that it is impossible, but I don't really understand his argument. Unfortunately, nearly all the writing in mathematics journals is paywalled and very expensive.

# **Part VII**

# **Trigonometry**

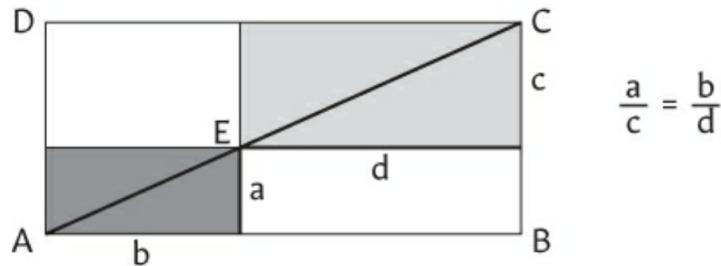
# Chapter 37

## Basic trigonometry

Here we take a look at fundamental ideas from trigonometry and begin to show why they are useful. Let's start by reviewing some previous work.

### similarity

A simple rectangular construction shows that right triangles with the same two complementary angles (*similar* right triangles), have equal ratios of sides.



**Fig. 42** Area and similarity.

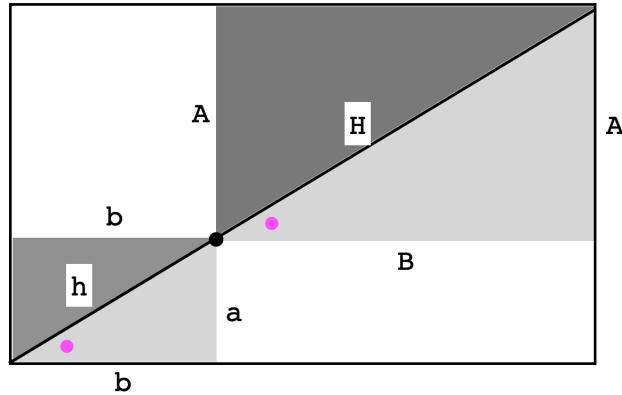
The proof is to use three sets of congruent triangles to show that the white areas are equal to each other which means that

$$bc = ad$$

Two simple rearrangements give different equalities

$$\frac{a}{b} = \frac{c}{d}, \quad \frac{a}{c} = \frac{b}{d}$$

I want to switch notation for a second, to try to make a point. In the figure below, the two similar right triangles have sides  $a, b$  and  $h$ , and then  $A, B$  and  $H$ .



The equality is

$$Ab = aB$$

which can be rearranged to

$$\frac{a}{b} = \frac{A}{B}, \quad \frac{a}{A} = \frac{b}{B}$$

The first one equates ratios of sides within a triangle,  $a$  and  $b$  from the first triangle,  $A$  and  $B$  from the second. The other one equates ratios of *corresponding* sides in different triangles. The sides of a triangle can be ordered by length, where one triangle has sides  $a < b < h$  and a similar triangle has sides  $A < B < H$ .

Let  $k$  be the constant of proportionality between triangles, with  $A = ka$ . Then

$$\begin{aligned} \frac{a}{b} &= \frac{A}{B} = \frac{ka}{B} \\ B &= \frac{ka}{a} \cdot b = kb \end{aligned}$$

So our ratios imply that

$$\frac{A}{a} = k = \frac{B}{b}$$

The factor  $k$  is a scaling factor which says how much bigger the second triangle is than the first.

This result is easily extended to the hypotenuse.

One way is to use the Pythagorean theorem. Let

$$a^2 + b^2 = h^2$$

and

$$A^2 + B^2 = H^2$$

But  $A = ka$  and  $B = kb$  so

$$\begin{aligned} H^2 &= A^2 + B^2 \\ &= (ka)^2 + (kb)^2 \\ &= k^2(a^2 + b^2) = k^2h^2 \end{aligned}$$

$$H = kh$$

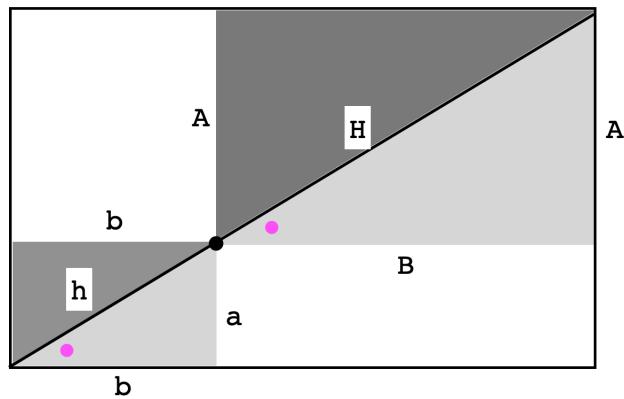
Hence

$$k = \frac{A}{a} = \frac{B}{b} = \frac{H}{h}$$

Now change focus back to comparisons within each triangle, namely

$$\frac{a}{h} = \frac{A}{H}$$

Consider the smaller angle, marked  $\angle CAB$  in the original figure, and labeled by a magenta dot below



This angle is flanked by the hypotenuse  $h$  and the side  $b$ . We will call  $a$  the side *opposite* to  $\angle CAB$  and  $b$  the side *adjacent*.

Then the previous equation

$$\frac{a}{h} = \frac{A}{H}$$

says that the ratio of the opposite side to the hypotenuse is somehow independent of the scale of the triangle. It is the same for both.

A similar thing happens with the *adjacent* sides  $b$  and  $B$ . Their ratios to the hypotenuse are also equal.

$$\frac{b}{h} = \frac{B}{H}$$

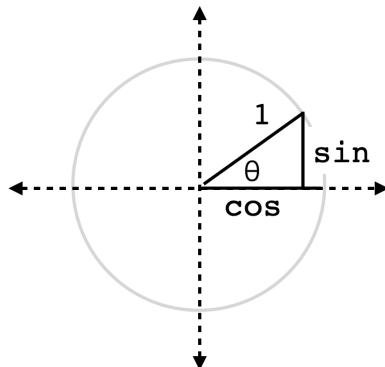
a result which is easily obtained by manipulating what we had above.

These ratios between sides in right triangles with the same angles are somehow characteristic of the angles and independent of the side lengths.

We call the ratio opposite/hypotenuse the *sine* of the angle and the ratio adjacent/hypotenuse the *cosine* and write them like this

$$\sin \angle CAB = \frac{a}{h}, \quad \cos \angle CAB = \frac{b}{h}$$

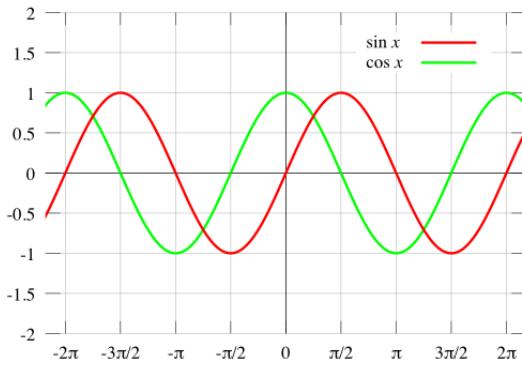
Often, we are working with central angles drawn in a unit circle, with radius 1.



Then the length of the side opposite the central angle is exactly the sine of the angle (since the hypotenuse has length 1), while the side adjacent is the cosine. This is the easiest picture to remember.

From a geometrical perspective, the importance of sine and cosine is that they incorporate the knowledge that these ratios are independent of the size of the triangle. You can think of them as ratios of sides to the hypotenuse for a central angle in a unit circle.

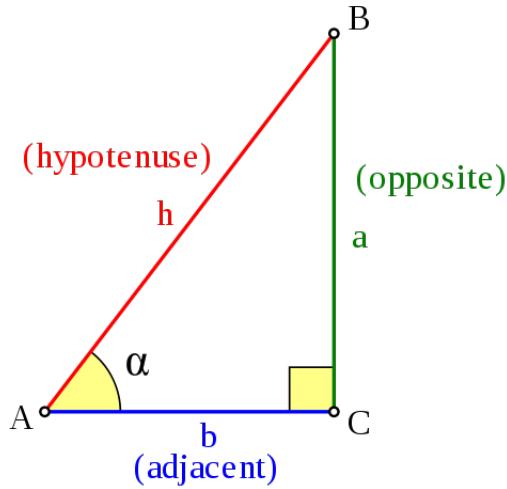
However, they are far more than that. In analytic geometry, we will see sine and cosine as *functions* of the angle  $\theta$ . The curve traced is a periodic one, which repeats as  $\theta$  goes all the way around the circle. This becomes central in calculus, where we will deal often with periodic phenomena modeled using the sine and cosine functions.



That's getting ahead of ourselves.

## sine and cosine

To repeat:



In a right triangle, for angle  $\alpha$ , the sine of the angle is defined as the ratio between the side opposite and the hypotenuse. In the drawing

$$\sin \alpha = \frac{a}{h}$$

while the cosine is

$$\cos \alpha = \frac{b}{h}$$

From our work earlier, we know that any similar right triangle (with the same angles) has its sides in the same ratios. The sine and cosine are functions of the angle, but are independent of the size of the triangle.

Frequently, the hypotenuse is scaled to be equal to 1, so then the opposite side is equal to the sine, and the adjacent side to the cosine.

Switching our focus from  $\alpha$  (the angle at  $A$ ) to the angle at vertex  $B$ , swaps opposite to adjacent and vice-versa. This means that the cosine of an angle is the sine of its complementary angle, and vice-versa.

## radian measure

In this book, we will make an effort not to use degrees for angular measure. They aren't seen much after elementary school. The Greeks thought in terms of one or two right angles, or four right angles for an entire circle.

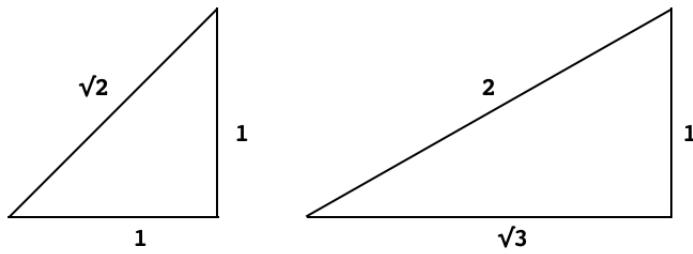
In more advanced mathematics we invariably use radians. Angles are measured in terms of the arc of the unit circle that they are *subtended by* or sweep out. Since the circumference of an entire circle is equal to  $2\pi$ , a right angle is one-quarter of that or  $\pi/2$  radians.

Some other common angles:  $\pi/6 = 30^\circ$ ,  $\pi/4 = 45^\circ$ , and  $\pi/3 = 60^\circ$ .

## particular values

We can easily determine the values for these functions in three special cases.

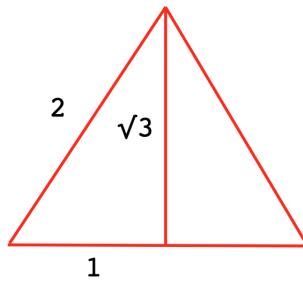
The first is the angle 45 degrees or  $\pi/4$ . Draw an isosceles right triangle with sides of length 1 (left panel).



Then the hypotenuse has length  $\sqrt{2}$  (from Pythagoras) and the values are equal

$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \cos \frac{\pi}{4}$$

If we start with an equilateral triangle (all angles equal to  $60^\circ$  and drop the angle bisector, we get two 30-60-90 triangles.



From the figure, we can read off that the sine of  $30^\circ$  is  $1/2$  and the cosine is  $\sqrt{3}/2$ . The values for  $60^\circ$  are reversed since they are complementary angles.

Hence

$$\sin \pi/6 = \frac{1}{2} = \cos \pi/3$$

$$\cos \pi/6 = \frac{\sqrt{3}}{2} = \sin \pi/3$$

$$\sin \pi/4 = \frac{1}{\sqrt{2}} = \cos \pi/4$$

## extreme values

If you draw a very small angle, it will have a very short length for the vertical,  $y = \sin \theta$ , while the adjacent side becomes nearly equal to the radius. What happens when  $\theta \rightarrow 0$ ? It turns out that

$$\sin 0 = 0, \quad \cos 0 = 1$$

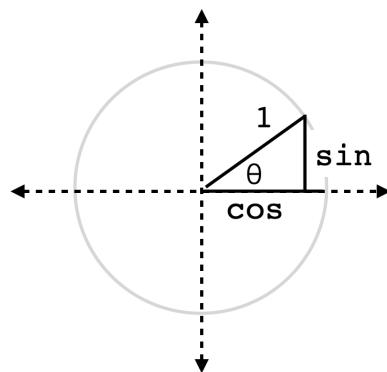
$$\sin \pi/2 = 1, \quad \cos \pi/2 = 0$$

$$\sin \pi = 0, \quad \cos \pi = -1$$

We cannot prove any of this yet (we do not even have a *coordinate system*), but just take it on faith for now, and you will see why when we get farther along.

## a favorite trigonometric identity

Now that we know about the sine and cosine, we can look at what Pythagoras tells us about them:



As we said, in a unit circle, the sine and cosine of an angle are the sides of a right triangle with hypotenuse equal to 1. It follows from the Pythagorean theorem that for any angle  $\theta$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta}$$

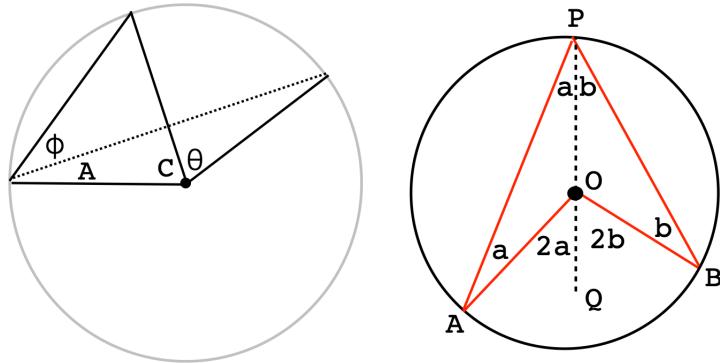
$$\cos \theta = \sqrt{1 - \sin^2 \theta}$$

This identity is fundamental since it provides a way of converting from sine to cosine or vice-versa.

It is traditional to write  $\sin^2 \theta$  rather than  $(\sin \theta)^2$ , but they have the same meaning.

## inscribed angles

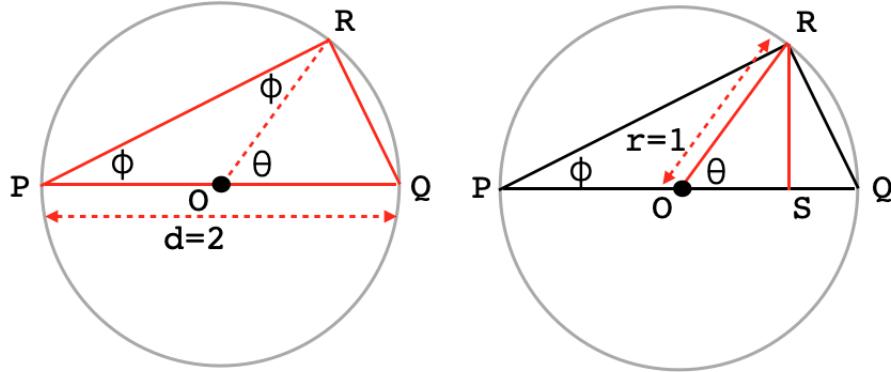
We also established previously the inscribed angle theorem. We showed two proofs that if the same arc on a circle subtends both a central angle  $\theta$  and by a peripheral or inscribed angle  $\phi$ , then  $\theta = 2\phi$ .



The first proof covers the case where the inscribed angle's arc does not include the center of the circle, while in the second case it does. It follows that all inscribed angles with the same arc are equal.

But that means we can study the arc or chord corresponding to any angle (in a given circle) by drawing the angle with one arm as the diagonal of a circle, since the chord is the same no matter where we place the angle.

## inscribed and central angles



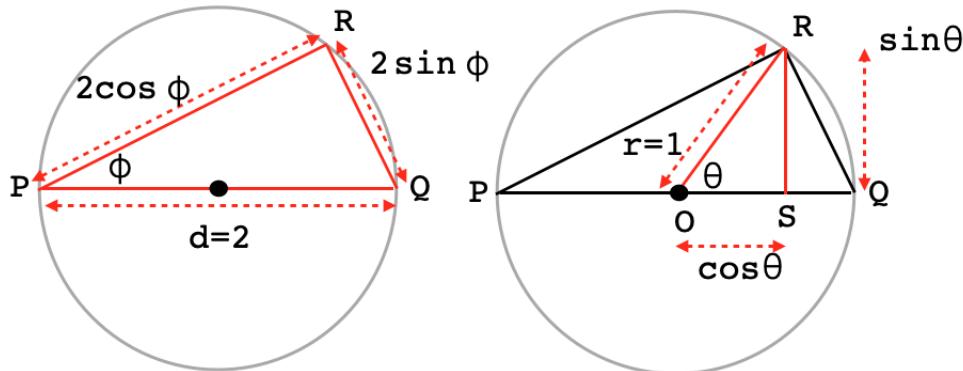
In the figure above, on the left we have drawn  $\triangle PQR$  in a unit circle on center  $O$  with  $\angle\phi$  being the inscribed or peripheral angle and  $\theta$  the corresponding central angle. Since  $\triangle POR$  is isosceles, and  $\theta$  is the external angle to that triangle, this is yet another demonstration that  $\theta = 2\phi$ .

In both panels we have scaled things so that the radius of the circle has length 1 and the diameter is twice that. One right angle is at  $R$ . Use our definitions of sine and cosine to see that

$$\sin \phi = \frac{RQ}{PQ} = \frac{RQ}{2}, \quad \cos \phi = \frac{RP}{PQ} = \frac{RP}{2}$$

while

$$\sin \theta = \frac{RS}{OR} = RS, \quad \cos \theta = \frac{OS}{OR} = OS$$



To repeat, for such a right triangle drawn in a unit circle, the chord  $L$  subtends any peripheral angle  $\phi$  is equal to

$$L = 2 \sin \phi$$

and in general, if the diameter has length  $d = 2r$ , then

$$L = d \sin \phi = 2r \sin \phi$$

## history

The relationship between the chord  $RQ$  as twice the sine of  $\phi$  and  $RS$  as the sine of  $\theta$  and hence that of  $2\phi$ , was of great interest in geometry for centuries, partly for practical reasons. Chords were useful for astronomy, and astronomy was useful in turn for navigation.

Ptolemy constructed a *Table of Chords* containing values for arc lengths in 1 degree increments. The Greeks thought of values in terms of whole numbers or ratios of whole numbers, and since the smallest angle he dealt with was  $1/360$  of a circle, this provides a rationale for why 360 was chosen as the total number of degrees.

It also helps that 360 has so many integer factors:

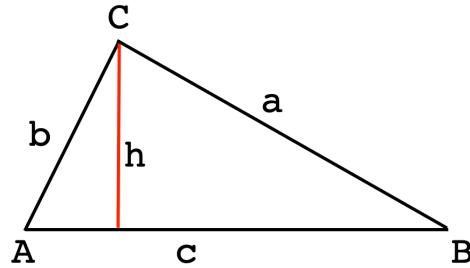
1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18 . . .

so we can talk about the 30 identical triangles that result from cutting a circle into 12 equal parts.

We will now spend some time looking at the "double angle" formula that connects the sine of an angle with that of twice the angle.

## area of any triangle

The altitude to any side of a triangle is equal to the length of the side of a triangle times the sine of the angle that side makes with the base.



$$\frac{h}{b} = \sin A$$

$$h = b \sin A$$

Therefore (if  $\Delta$  is the area of the triangle):

$$2\Delta = hc = bc \sin A$$

One can equally well write

$$h = a \sin B$$

$$2\Delta = hc = ac \sin B$$

Twice the area of any triangle is the product of the two sides times the sine of the angle between them.

## problem

We usually compute the area of a parallelogram in terms of the sides. However, another formula for (twice) the area is:

$$2\Delta = d_1 d_2 \sin \theta$$

where  $d_1$  and  $d_2$  are the two diagonals and  $\theta$  is either one of the central angles.

Derive this formula.

## law of sines

Since the area must be the same no matter how we compute it, this also leads to the equality

$$h = b \sin A = a \sin B$$

$$\frac{a}{b} = \frac{\sin A}{\sin B}$$

which by symmetry we extend to all three angles

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This simple formula is called the law of sines.

The constant ratio has an interesting value, which can be seen by going back to what we said above, namely

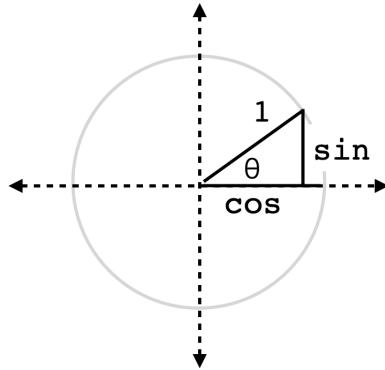
$$L = 2r \sin \phi$$

which, applied to this case, gives

$$\frac{a}{\sin A} = 2r$$

## other functions

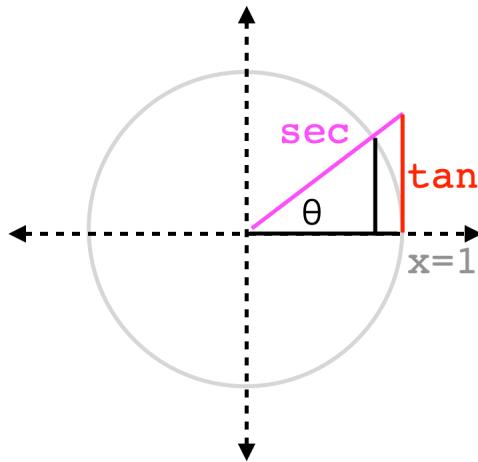
We'll just mention a few other common trig (short for trigonometric) functions that are constructed from sine and cosine, although they aren't used much in this book.



First, the tangent of the angle is defined as the opposite side divided by the adjacent side. In other words, it is equal to the sine divided by the cosine.

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

We can see the tangent as a length. Extend the hypotenuse to make a similar triangle, where the adjacent side has length equal to the radius. Now, opposite over adjacent gives tangent, but adjacent is just 1.



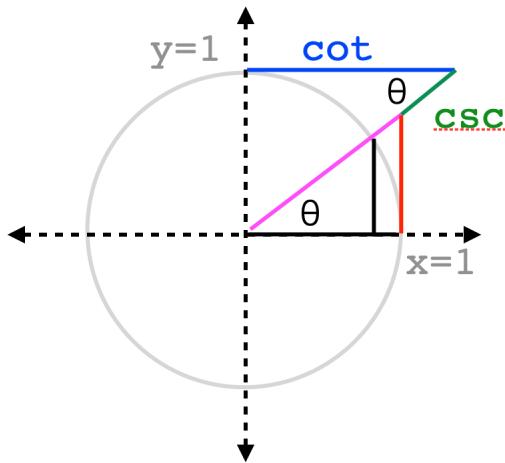
There are also three inverse functions: secant (inverse of the cosine), cosecant (inverse of the sine), and cotangent (inverse of the tangent). From the drawing above, we can get that

$$\frac{1}{\sec \theta} = \cos \theta$$

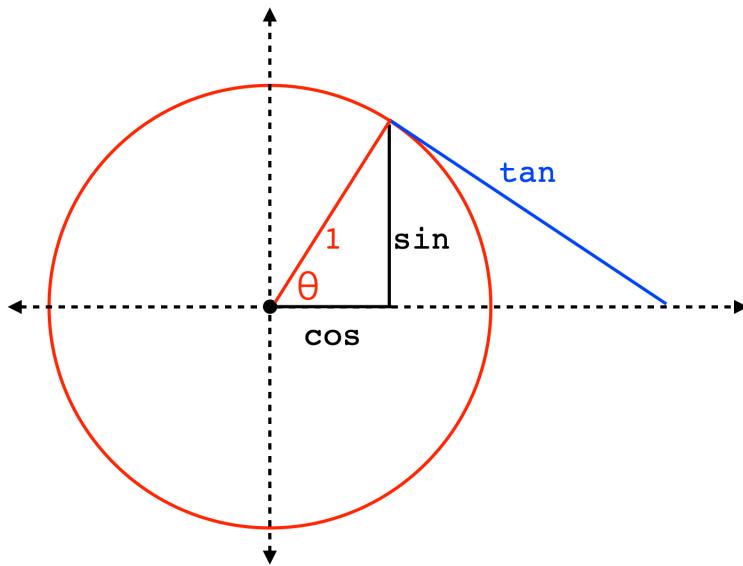
$$\sec \theta = \frac{1}{\cos \theta}$$

(We might also check by looking at the ratio  $\tan \theta / \sec \theta = \sin \theta$ ).

We include figures with the cotangent and cosecant as well, but put off discussion of them for now.



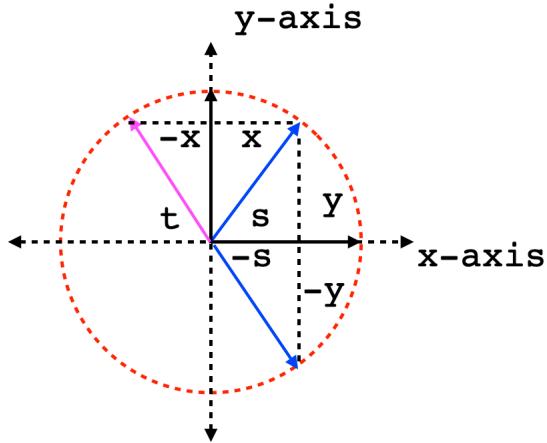
There are several other common representations of the six functions. We'll see more when we get to analytic geometry. Here is a common one where the tangent to the circle, perpendicular to the ray with  $\angle\theta$  is equal to  $\tan\theta$ . See if you can work this out by using the complementary angle to  $\theta$ .



## signed angles

In analytic geometry we will introduce the idea of two axes in the plane, one for  $x$ -values and one for  $y$ -values. The origin will become  $(0, 0)$ , and values will have

signs, i.e.,  $x$ -values to the left of the origin will be minus some number, and  $y$ -values below the origin will be minus some number as well.



Without getting into all the gory details, I hope you can see that when, as here,  $s$  and  $t$  are supplementary angles, then

$$\sin s = \sin t$$

$$\cos s = -\cos t$$

while for  $s$  and  $-s$

$$\sin s = -\sin -s$$

$$\cos s = \cos -s$$

and as we said before, when  $s$  and  $t$  are complementary angles

$$\sin s = \cos t$$

$$\cos s = \sin t$$

This leads to an elementary proof of the **area-ratio theorem**.

*Proof.*

Let a triangle be divided by a line from the upper vertex to the base such that the base is divided into two parts,  $x$  and  $y$ . Call the dividing line  $e$ .

The angles at the base on either side of the intersection point are supplementary, hence they have the same sine. Let those angles be  $\phi$  and  $\phi'$ .

Then twice the area of the left-hand triangle is

$$2A_L = xe \sin \phi$$

while twice the area of the other one is

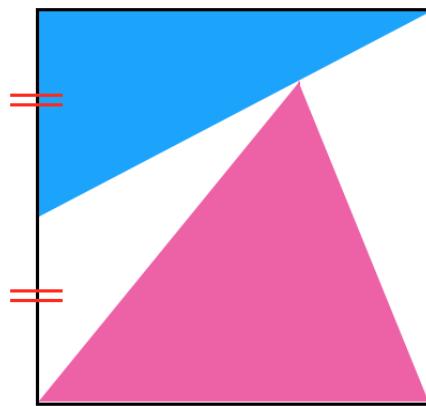
$$2A_R = ye \sin \phi' = ye \sin \phi$$

and the ratio of the areas is simply  $x/y$ .

□

## problem

Here is a nice area problem I saw on Twitter. (The attribution is "Paul Eigenmann, Stuttgart, 1967". Web search finds what looks like a great text, but in German).



Given that the side is bisected and the unshaded areas are equal. What is the area of the triangle shaded magenta?

*Solution.*

Let's say this is a unit square and let the top vertex of the magenta triangle be a distance  $x$  from the left-hand side of the box and a distance  $1 - x$  from the right-hand side. Then the left-hand unshaded area is

$$\frac{1}{2} \cdot x \cdot \frac{1}{2}$$

and the right-hand shaded area is

$$\frac{1}{2} \cdot (1 - x) \cdot 1$$

Set them equal:

$$\frac{x}{4} = \frac{1}{2} - \frac{x}{2}$$

$$x = 2 - 2x$$

$$x = \frac{2}{3}$$

To find the height of the magenta triangle using pure geometry, notice that a line drawn vertically from the vertex forms two similar triangles, therefore  $y$  is a distance  $1/3 \cdot 1/2 = 1/6$  down vertically from the upper boundary.

The height of the magenta triangle is therefore  $5/6$  and its area is one-half that, or  $5/12$  of the unit cube.

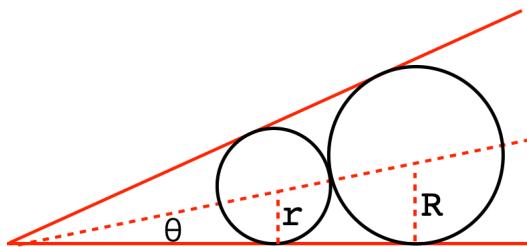
Using analytic geometry, by inspection of the figure, write the equation for the side of the upper triangle as

$$y = \frac{1}{2}x + \frac{1}{2}$$

$$y = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

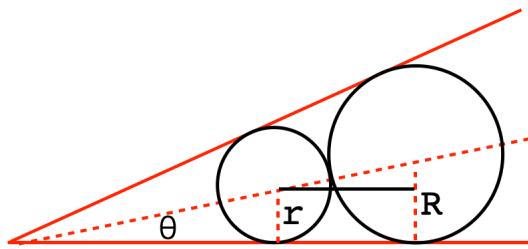
### double scoop problem

We have two lines tangent to two circles that just touch each other, the smaller one of radius  $r$ , and the larger of radius  $R$ .



There is a simple expression for the sine and cosine of  $\theta$ , the angle between the two lines. Recall our introduction to trigonometry [here](#).

The distance between the centers of the two circles is  $r + R$ . Draw a horizontal line through the center of the smaller circle.



We have constructed a right triangle, which is similar to the original one. It includes the angle  $\theta$  and the hypotenuse is the distance between the two centers,  $R + r$ . The opposite side has length  $R - r$  and so

$$\sin \theta = \frac{R - r}{R + r}$$

The adjacent side (the line segment colored black) has its squared length equal to

$$(R + r)^2 - (R - r)^2 = 4Rr$$

thus

$$\cos \theta = \frac{2\sqrt{Rr}}{R + r}$$

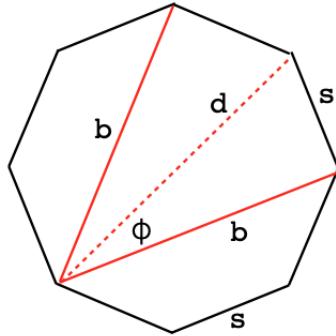
## David Bowie problem

Here's a problem from the web:

Vincent Pantaloni @panlepan · Feb 25  
The octagon that thinks it's David Bowie.  
What fraction of this regular octagon is coloured in red ?  
#GeometrySnacks #ShowYourWork #WIFIS



One way to look at this is to imagine the octagon inscribed in a circle of diameter  $d = 2r$  (below). We reason that the two triangles are right triangles with a shared hypotenuse.



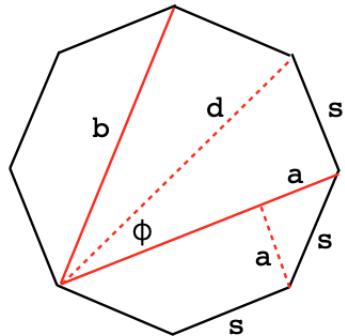
We can apply trigonometry to find that the whole area required is simply

$$A_{\text{region}} = d^2 \sin \phi \cos \phi$$

But we aren't ready to do this kind of calculation yet.

However, it will turn out that the part shaded red is simply one-half of the whole.

Therefore, the three lines divide the octagon into four equal parts. That simple answer is a clue that something important makes the problem easy.



The triangle with small sides  $a$  can be moved so one of the unshaded base parts becomes a rectangle with area

$$a(s + a) = sa + a^2$$

while the base of the triangle is  $s + 2a$  so its area is

$$\frac{1}{2}s(s + 2a) = \frac{1}{2}s^2 + sa$$

It's not obvious at first that these are equal, but working with it, they would be equal if we can show that  $2a^2 = s^2$ .

Of course we can do exactly that, because  $a$  is the side of an isosceles right triangle with hypotenuse  $s$ . By the Pythagorean theorem, we have

$$a^2 + a^2 = s^2$$

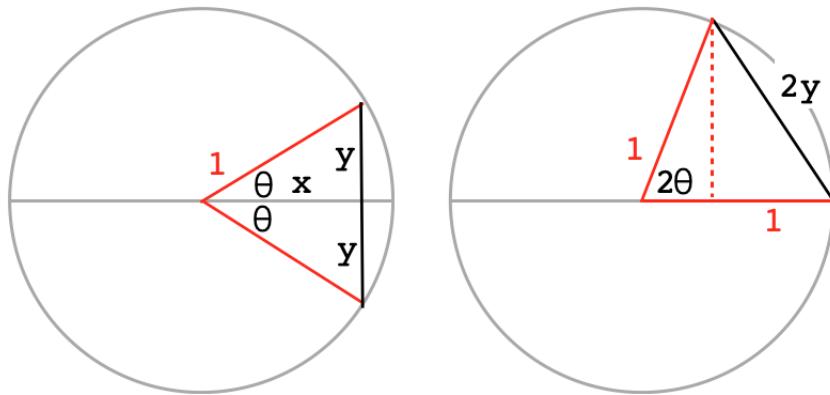
□

# Chapter 38

## Double angle

### double angle formulas

Draw a unit circle with central angle  $\theta$  and then double the central angle below the horizontal diameter. Let  $x$  be the base of the two smaller triangles, the side adjacent to angle  $\theta$ , and  $y$  the side opposite.



Then by the basic definitions of trigonometry, we have that  $x = \cos \theta$  and  $y = \sin \theta$ . The area of each small triangle is  $xy/2$  and the area of the two combined is simply

$$A = xy = \sin \theta \cos \theta$$

Now, rotate the triangle counter-clockwise until the red side on the right lies along

the diagonal. The dotted vertical line is clearly  $\sin 2\theta$  and since the base is length 1, the area of that same triangle is now

$$A = \frac{1}{2} \cdot \sin 2\theta$$

Combining the results

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

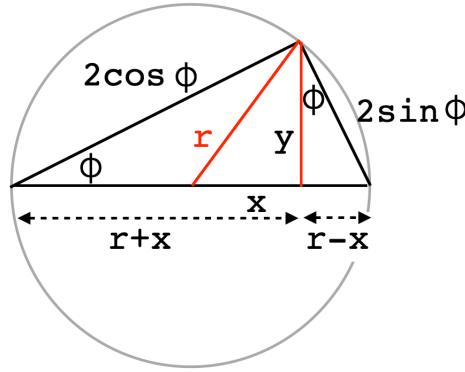
This is the first double angle formula, for sine. We claim that it still works even if the sum is greater than a right angle but keep things simple by not talking about it more.

### double angle for cosine

Consider this right triangle with an inscribed angle of  $\phi$ . Label the sides of a triangle containing the central angle  $2\phi$  as  $x$  and  $y$  to simplify the figure. Let  $r = 1$  so then

$$y = \sin 2\phi, \quad x = \cos 2\phi$$

Whenever we drop an altitude in a right triangle, the smaller triangles are similar to the original. This accounts for the second  $\angle\phi$  in the figure.



We note for later that the base of the altitude divides the base into  $r + x$  and  $r - x$ .

Using the small triangle with angle  $\phi$

$$\frac{y}{2 \sin \phi} = \cos \phi$$

$$y = 2 \sin \phi \cos \phi$$

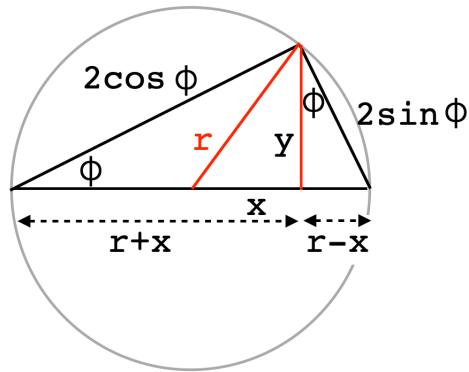
Since  $y = \sin 2\phi$ , this is just what we had before.

Along the base of the triangle extending from the left, we have (again,  $r = 1$ ):

$$\frac{r+x}{2 \cos \phi} = \frac{1+x}{2 \cos \phi} = \cos \phi$$

$$\frac{1+x}{2} = \cos^2 \phi$$

But (from the small triangle again)



$$\frac{r-x}{2 \sin \phi} = \frac{1-x}{2 \sin \phi} = \sin \phi$$

$$\frac{1-x}{2} = \sin^2 \phi$$

And then

$$\frac{1+x}{2} - \frac{1-x}{2} = x$$

but  $x = \cos 2\phi$  so

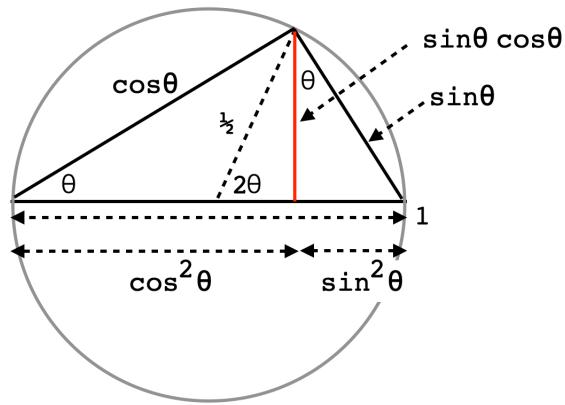
$$\cos 2\phi = \cos^2 \phi - \sin^2 \phi$$

Note also that

$$\cos^2 \phi + \sin^2 \phi = \frac{1+x}{2} + \frac{1-x}{2} = 1$$

which is true for any angle  $\phi$ , a restatement of the Pythagorean theorem.

Here is the same proof but scaled so that the diameter has unit length and we use sine and cosine labels rather than  $x$  and  $y$ . The details are left to you.



## half-angle formulas

The half-angle formulas are easy to derive from the double angle cosine formula, because the sine and cosine terms are separated:

$$\cos 2A = \cos^2 A - \sin^2 A$$

Just use our favorite identity ( $\sin^2 A + \cos^2 A = 1$ ) to obtain

$$\cos 2A = \cos^2 A - (1 - \cos^2 A)$$

or

$$\cos 2A = (1 - \sin^2 A) - \sin^2 A$$

We'll do the sine first

$$2 \sin^2 A = 1 - \cos 2A$$

$$\sin A = \sqrt{\frac{1 - \cos 2A}{2}}$$

which is easily verified using the values for  $30^\circ$  and  $60^\circ$ .

On the other hand:

$$\cos 2A = \cos^2 A + \cos^2 A - 1$$

Solve for  $\cos A$ :

$$\cos A = \sqrt{\frac{1 + \cos 2A}{2}}$$

This is called the half-angle formula since  $A$  is one-half of  $2A$ . That the sum of the squares is equal to 1 is also easily verified. We see that the square roots go away, the  $\cos 2A$  terms will cancel and we end up with  $1/2 + 1/2$ .

## some algebra

You might be tempted to try to derive a formula for  $\sin 2A$  from that for  $\cos 2A$  (or vice-versa), using our favorite identity. Let us re-write the formulas substituting  $s$  for  $\sin A$  and  $c$  for  $\cos A$ :

$$\sin 2A = 2sc, \quad \cos 2A = c^2 - s^2$$

They have different forms: the first mixes  $s$  and  $c$ , while the second has a difference of squares. If we were to take

$$\begin{aligned} (c+s)^2 &= c^2 + 2sc + s^2 \\ (c-s)^2 &= c^2 - 2sc + s^2 \\ (c+s)(c-s) &= c^2 - s^2 \end{aligned}$$

There is some of what we want, but in the formula for  $\cos 2A$  one term is positive and one negative. What's going on?

First, just verify what  $\sin^2 2A + \cos^2 2A$  is equal to:

$$\begin{aligned} &= 4s^2c^2 + c^4 - 2s^2c^2 + s^4 \\ &= c^4 + 2s^2c^2 + s^4 \\ &= (s^2 + c^2)^2 = 1^2 = 1 \end{aligned}$$

That checks out, and it contains a hint to the answer. We used  $4s^2c^2 - 2s^2c^2$  to convert  $-2s^2c^2$  to the positive  $2s^2c^2$ .

Remember, to do the conversion we square the sine or cosine, subtract it from 1 and then take the square root.

$$\begin{aligned} \sin \theta &= \sqrt{1 - \cos^2 \theta} \\ \cos \theta &= \sqrt{1 - \sin^2 \theta} \end{aligned}$$

It is easier to see how this works using the first formula. So we start with the formula for cosine of  $2A$  and plug in:

$$1 - (c^2 - s^2)^2 = 1 - (c^4 - 2s^2c^2 + s^4)$$

Add (and subtract)  $2s^2c^2$  in the parentheses!

$$\begin{aligned} &= 1 - (c^4 + 2s^2c^2 + s^4 - 4s^2c^2) \\ &= 1 - [ (c^2 + s^2)^2 - 4s^2c^2 ] \\ &= 1 - 1 + 4s^2c^2 = 4s^2c^2 \end{aligned}$$

and now the square root is easy and gives exactly what we need.

Then it becomes clear that we can add  $(s^2 + c^2) - 1$  or  $(s^2 + c^2)^2 - 1$  to expressions and if they will simplify, it helps us. Hence to obtain the formula for  $\cos 2A$ , starting with 1 minus sine squared:

$$\begin{aligned} &1 - 4s^2c^2 \\ &= 1 - 4s^2c^2 + (s^2 + c^2)^2 - 1 \\ &= -4s^2c^2 + [ s^4 + 2s^2c^2 - c^4 ] \\ &\quad = s^4 - 2s^2c^2 + c^4 \\ &\quad = (s^2 - c^2)^2 = (c^2 - s^2)^2 \end{aligned}$$

remembering that  $(x - y)^2 = (y - x)^2$ .

Take the square root to finish.

## examples

Restating the double-angle formula for sine

$$\sin 2A = 2 \sin A \cos A$$

Plugging in

$$\sin 60^\circ = 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

and

$$\cos 2A = \cos^2 A - \sin^2 A$$

plugging in again

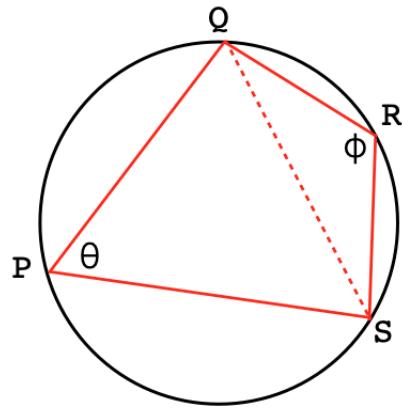
$$\cos 60^\circ = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

They both look good.

We can also use the formulas to check supplementary angles (recall that  $\sin 180 = 0$  and  $\cos 180 = -1$ ):

$$\begin{aligned}\sin 180 - A &= \sin 180 \cos A - \sin A \cos 180 \\&= 0 - (-\sin A) = \sin A \\ \cos 180 - A &= \cos 180 \cos A + \sin 180 \sin A \\&= -\cos A + 0 = -\cos A\end{aligned}$$

Of course, we hardly need algebra for the result about the sine of the supplementary angle. Two inscribed angles with the supplementary arcs clearly have the same sine:



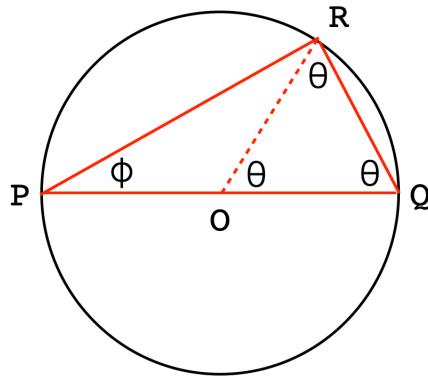
The total arc of the circle is  $2\pi$ , while the two angles  $\theta$  and  $\phi$  together take up that entire arc. As inscribed angles they sum to one-half or  $\pi$ . Hence  $\theta$  and  $\phi$  are supplementary.

But we previously had

$$L = d \sin \phi = 2r \sin \phi$$

where  $L$  is the length of the dotted line, and if the circle is scaled appropriately, it is  $\sin \phi$ . But it is also  $\sin \theta$ .

## equilateral triangle



Here is another way of looking at  $30^\circ$  and  $60^\circ$ . Let the central angle  $\theta$  be such that the triangle formed is equilateral.

Now the central angle is equal to both of the base angles, and the measure of all these angles is  $60^\circ$  or  $\pi/3$ .

Since  $\triangle OQR$  is equilateral, the chord  $QR$  is the same length as the radius. It follows that the sine of the inscribed angle  $\phi$  is  $r/2r = 1/2$ . And of course the length of  $PR$ , which is 2 times the sine of  $60^\circ$ , is  $\sqrt{3}$ .

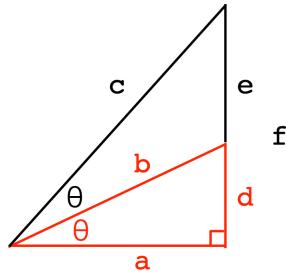
## Archimedes and pi

Later we will explore the math relating to Archimedes' method to approximate  $\pi$ , resulting in the bounds  $3 \frac{10}{71} < \pi < 3 \frac{1}{7}$ .

The lower bound is about 3.1408 and the upper bound about 3.1429 while the true value is about 3.1416. (There is an interesting story relating to a fraction which is a much, much better estimate of  $\pi$ , namely  $355/113$ , good to six digits).

As you probably know, the method involves calculating the length of the perimeter of circumscribed and inscribed regular polygons.

We can build on two basic results from the geometry of right triangles:



The first is that the sum of the cotangent and cosecant for the original angle are equal to the cotangent of the half-angle:

$$\frac{a}{f} + \frac{c}{f} = \frac{a}{d}$$

$$\cot 2\theta + \csc 2\theta = \cot \theta$$

This follows from the angle bisector theorem,  $a/d = c/e$ , with a bit of manipulation.

$$\begin{aligned}\frac{a}{c} &= \frac{d}{e} \\ \frac{a}{c} + \frac{c}{c} &= \frac{d}{e} + \frac{e}{e} \\ \frac{a+c}{d+e} &= \frac{c}{e} = \frac{a}{d}\end{aligned}$$

Adding components in the denominator of the first term:

$$\frac{a+c}{f} = \frac{a}{d}$$

Another way to get there is to use the double angle formulas. We'll abbreviate  $\sin, \cos, \tan$  as  $S, C, T$  and identify the values for the half-angle as lowercase. The formulas are:

$$S = 2sc$$

$$C = c^2 - s^2 = 2c^2 - 1$$

The cotangent of  $2\theta$  is then (cosine/sine):

$$\frac{1}{T} = \frac{C}{S} = \frac{2c^2 - 1}{2sc}$$

$$= \frac{1}{t} - \frac{1}{S}$$

which rearranges to give the result

$$\frac{1}{t} = \frac{1}{T} + \frac{1}{S}$$

# Chapter 39

## Sum of angles by scaling

### sum of angles

In order to gain some practice thinking about sine and cosine, we will derive what are called the sum of angles formulas. These are really for the sum and difference of two angles:

$$\sin(s \pm t), \quad \cos(s \pm t)$$

Previously we talked about formulas for the case where  $s = t$ , but now we want more general expressions. Here is the first, for the difference of angles  $s$  and  $t$

$$\cos s - t = \cos s \cos t + \sin s \sin t$$

By  $\cos s - t$  we mean  $\cos(s - t)$ , but have left off the parentheses.

There are four formulas, and then some special examples. These are used a lot in calculus, not only for solving problems, but most important, in finding an expression for the derivatives of the sine and cosine functions in calculus.

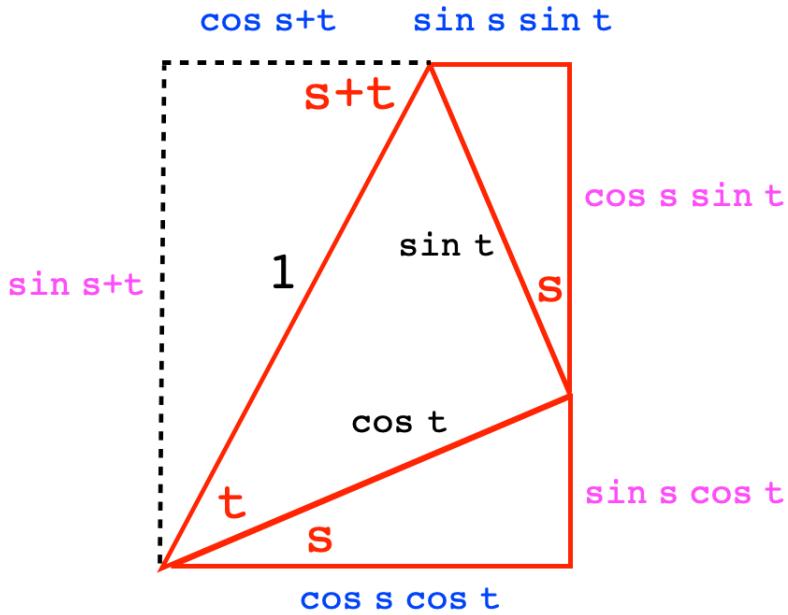
I've memorized only the single equation given above. Say "cos cos" and then recall the difference in sign, minus on the left, plus on the right.

I like this version because it can be checked easily. Just set  $s = t$ . Then  $s - t$  becomes  $s - s$  and we have

$$\cos s - s = \cos 0 = 1 = \cos^2 s + \sin^2 s$$

which is our favorite trigonometric identity and obviously correct.

This diagram shows where we're headed:



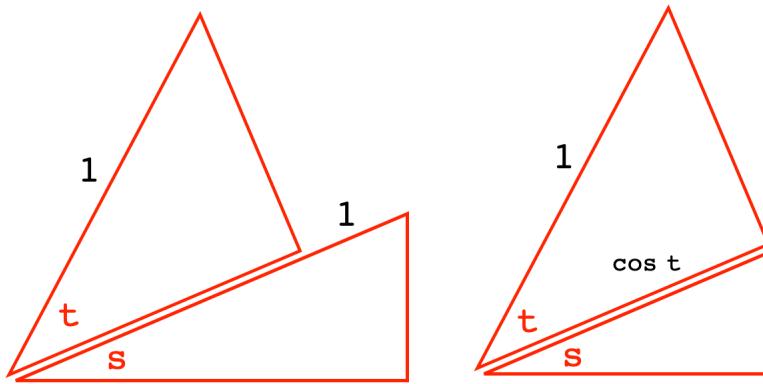
Since the figure is a rectangle, you can read the relevant equalities off the opposite sides.

$$\sin s + t = \sin s \cos t + \cos s \sin t$$

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

As so often, we begin with an inspired diagram. Let's build it up in stages.

On the left, below, we have stacked two right triangles. The one containing angle  $t$  is rotated so that its base is parallel to the hypotenuse of the triangle containing angle  $s$ .



The crucial step is to re-scale the triangle on the bottom so that the parallel line segments are also equal in length (right panel).

By re-scaling, we change the length of the hypotenuse of the triangle containing angle  $s$ . Its hypotenuse is now the length  $\cos t$ .

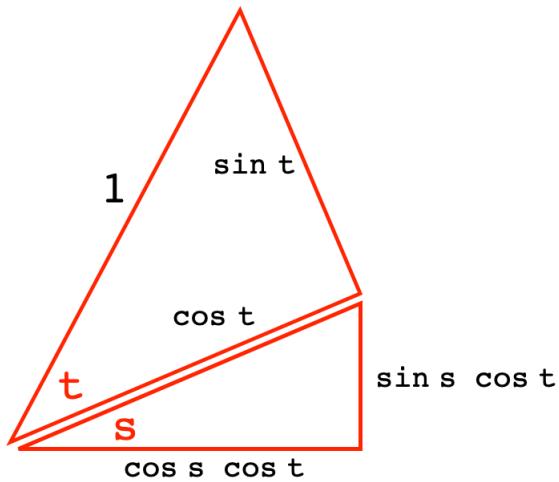
Recall that when the length of the hypotenuse was  $h$ , we divided the length of the adjacent side by the hypotenuse to get the cosine. There we had  $b/h = \cos s$ . Now we have

$$\cos s = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{adjacent}}{\cos t}$$

What should be the length of the base of the triangle with angle  $s$ ? It must be  $\cos s \cos t$ , the product of cosines! Just multiply both sides above by  $\cos t$ .

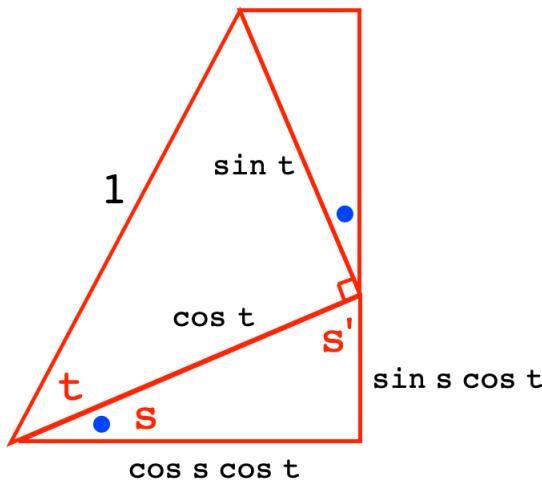
Or recognize that after dividing  $\cos s \cos t$  by the hypotenuse, which is  $\cos t$ , we will then have what we want,  $\cos s$ .

$$\frac{\cos s \cos t}{\cos t} = \cos s$$



By the same reasoning, the opposite side in the triangle with angle  $s$  is  $\sin s \cos t$ , so that after dividing by  $\cos t$  we obtain the correct value,  $\sin s$ .

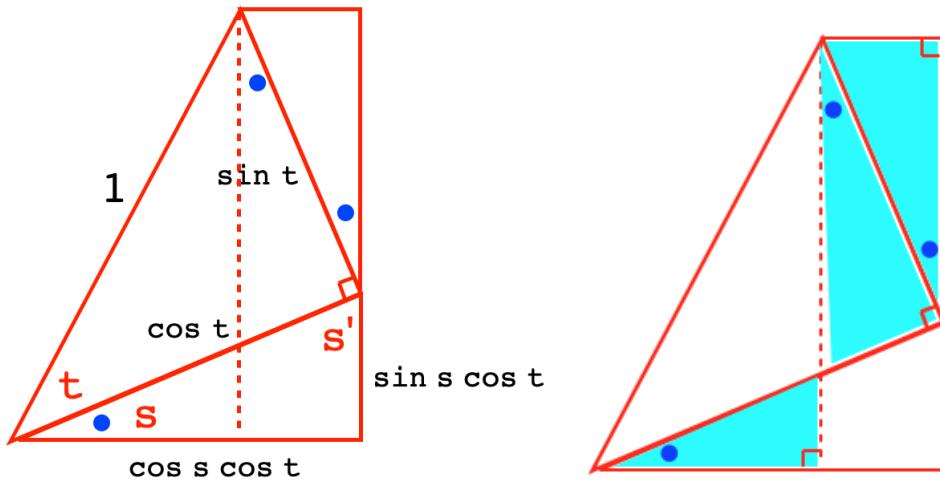
### a similar triangle



We add another right triangle to the figure

We claim that the angles labeled with blue dots are equal.

The easiest way to do that is algebraic. The angle that is complementary to  $s$  in the bottom triangle is  $s'$ . Complementarity means that  $s + s'$  is equal to a right angle.



But the second blue dotted angle plus  $s'$  is also equal to a right angle. The reason is that, when added to another right angle, it makes two right angles or a straight line. Using the language of radian measure

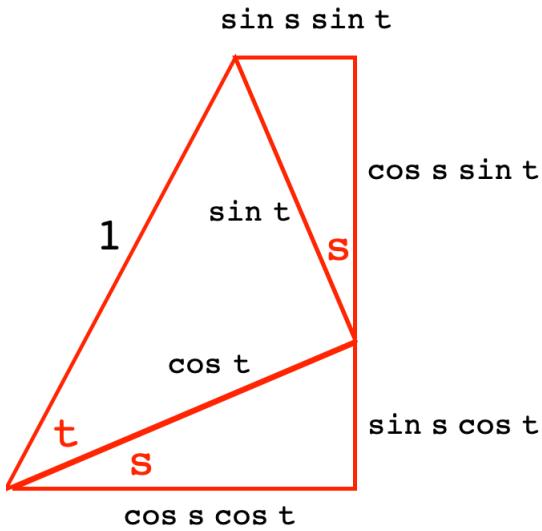
$$s + s' = \frac{\pi}{2}$$

$$s' + \frac{\pi}{2} + \text{blue dot} = \pi$$

$$s' + \text{blue dot} = \frac{\pi}{2}$$

$$\text{blue dot} = s$$

So we add the correct label, and then play the same trick as before.



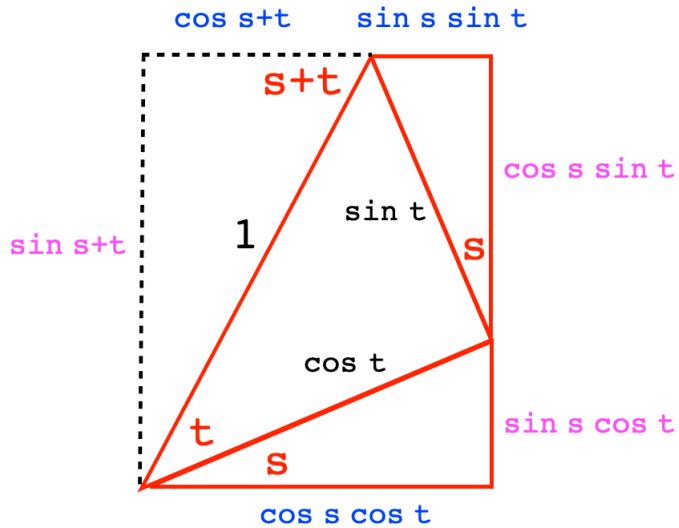
The difference is that now, the length of the hypotenuse of the upper triangle containing angle  $s$  is  $\sin t$ . The two similar triangles are scaled differently from one another.

We obtain  $\sin s \sin t$  and  $\cos s \sin t$  for the sides of the upper triangle, so that after dividing by the hypotenuse, which is  $\sin t$ , we obtain the correct values for the sine and cosine of angle  $s$ .

## finale

Why have we gone to the trouble of doing all this?

The two angles,  $s$  and  $t$ , taken together, are equal to the angle at the top of the figure labeled, naturally,  $s + t$ .



We fill in lengths for the dotted lines of the fourth right triangle in the figure below. This forms a rectangle (all four corners are right angles). Therefore, opposite sides are equal.

We just write down the formula by reading off the figure:

$$\sin s + t = \sin s \cos t + \cos s \sin t$$

and

$$\cos s + t + \sin s \sin t = \cos s \cos t$$

which can be rearranged to give

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

These are the sum of angles formulas.

### change signs

For  $\cos s - t$ , flip the sign on the second term.

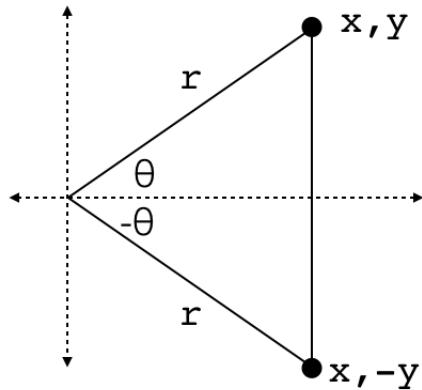
$$\cos s - t = \cos s \cos t + \sin s \sin t$$

That's because

$$\cos -\theta = \cos \theta$$

$$\sin -\theta = -\sin \theta$$

To show this we need to invoke analytic geometry (there's no such thing as a negative angle in classical geometry).



The diagram shows the reason:

$$\cos \theta = x/r = \cos -\theta$$

while

$$\sin \theta = y/r = -(\sin -\theta) = -(-y/r)$$

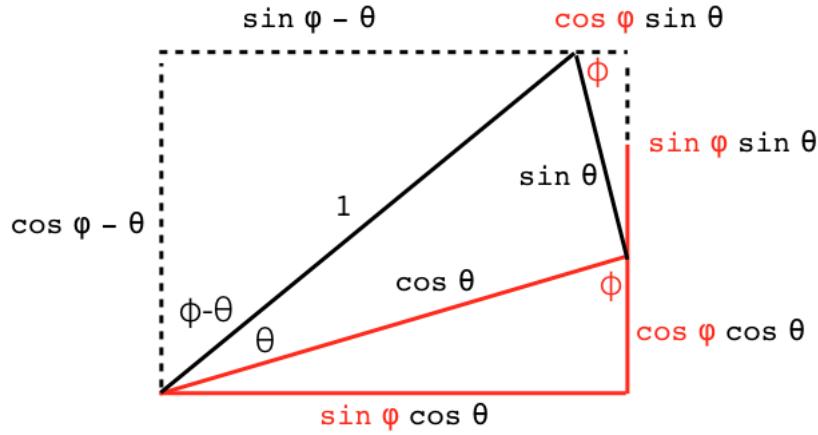
Substitute  $-\sin \theta$  for  $\sin -\theta$  and  $\cos \theta$  for  $\cos -\theta$ :

$$\begin{aligned} \cos s - t &= \cos s \cos -t + \sin s \sin -t \\ &= \cos s \cos t - \sin s \sin t \end{aligned}$$

and

$$\sin s - t = \sin s \cos t - \cos s \sin t$$

It's kind of overkill, but still worth noting that a simple change to the figure we had above will give the difference formulas:



We've changed symbols to  $\theta$  and  $\phi$  for the complementary angles.

We can justify the label  $\phi - \theta$  for the angle at the lower left in various ways, for example, by adding up the three angles at that corner:

$$(\phi - \theta) + \theta + (90 - \phi) = 90$$

Switch the labels appropriately (it's easy since this  $\phi$  is the complement of the old one).

Read the result:

$$\sin \phi - \theta = \sin \phi \cos \theta - \cos \phi \sin \theta$$

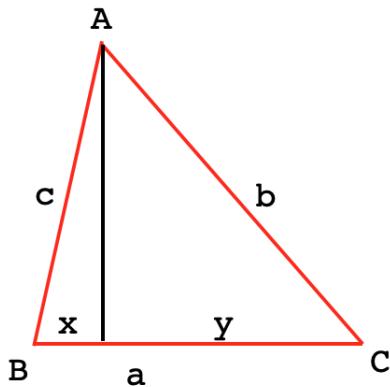
$$\cos \phi - \theta = \cos \phi \cos \theta + \sin \phi \sin \theta$$

### alternative derivation

There are many derivations of the sum of angles formulas. Here is an algebraic one based on the [law of sines](#).

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Start with this triangle



$$a = x + y = c \cos B + b \cos C$$

From the law of sines:  $\sin A = (a/b) \sin B$ . Substituting for  $a$ :

$$\begin{aligned} \sin A &= \frac{c \cos B + b \cos C}{b} \sin B \\ &= \frac{c}{b} \sin B \cos B + \sin B \cos C \end{aligned}$$

Again from the law of sines:  $\sin B = (b/c) \sin C$ , so

$$\sin A = \sin C \cos B + \sin B \cos C$$

But since  $A$  and  $B + C$  are supplementary, their sines are equal, thus

$$\sin B + C = \sin C \cos B + \sin B \cos C$$

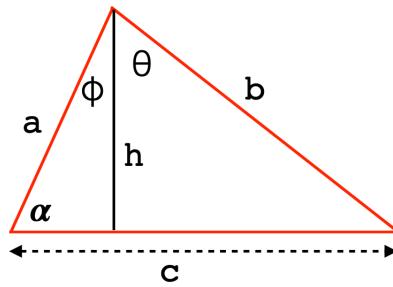
□

The proof for an obtuse angle is left as an exercise.

### general formula for sine

Here is a related, simple proof of the formula for sine.

Consider the following triangle, where the angle at one vertex is divided by the altitude to the opposing side, forming angles  $\theta$  and  $\phi$ . We are interested in finding the sine of the two angles added together.



But first we must introduce a standard formula for the area of a triangle, illustrating it by the angle  $\alpha$ . From our previous work we know that twice the area of the triangle is  $hc$ , but  $h$  is also part of a formula for sine, namely  $h/a = \sin \alpha$ , which can be rearranged to give  $h = a \sin \alpha$ . In other words

$$2A = hc = ac \sin \alpha$$

In general, twice the area of any triangle is the product of two sides times the sine of the angle between. So in this case, we also have that

$$2A = ab \sin(\phi + \theta)$$

Now we just calculate the area of the two smaller triangles and add them together. We have

$$2A = ha \sin \phi + hb \sin \theta$$

I'm going to rearrange this slightly

$$2A = a \sin \phi \cdot h + b \sin \theta \cdot h$$

For these angles,  $h$  is connected to a trig function, but this time it's the cosine.

$$h = a \cos \phi = b \cos \theta$$

Substituting two times into the previous equation, we obtain

$$2A = a \sin \phi \cdot b \cos \theta + b \sin \theta \cdot a \cos \phi$$

and equate it to the first result

$$ab \sin(\phi + \theta) = a \sin \phi \cdot b \cos \theta + b \sin \theta \cdot a \cos \phi$$

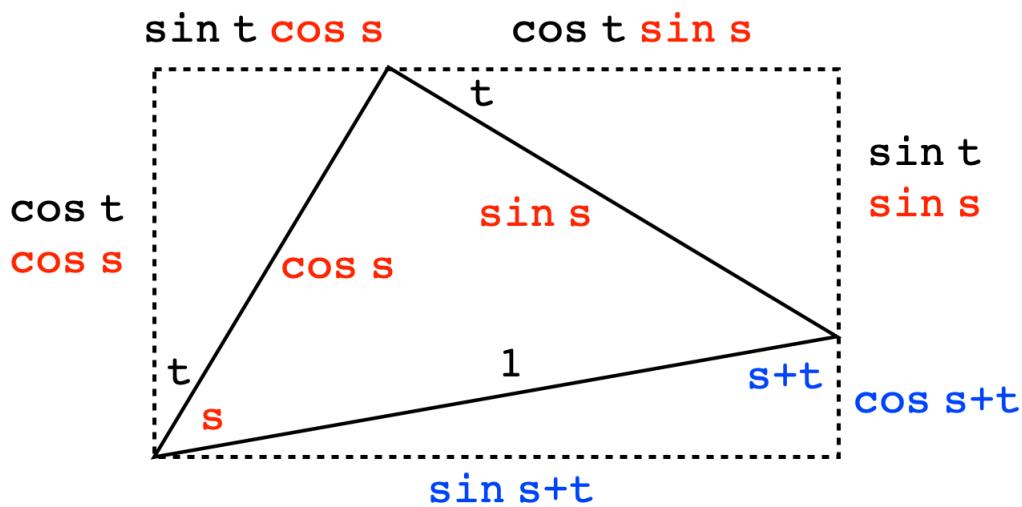
We can cancel  $ab$  from all three terms:

$$\sin(\phi + \theta) = \sin \phi \cos \theta + \sin \theta \cos \phi$$

This is the general form for the sine of the sum of two angles. It is apparent that it reduces to the previous double angle formula in the case where  $\theta = \phi$ .

### alternate version

Just to be complete, here is an alternative version I found that may be even simpler.



### yet another proof of Pythagoras's Theorem

Consider the right triangle in a unit circle with opposite side  $\sin \theta$  and adjacent side  $\cos \theta$ . We will prove that

$$1 = \cos^2 \theta + \sin^2 \theta$$

*Proof.*

We have the formula above:

$$\cos A + B = \cos A \cos B - \sin A \sin B$$

Let  $A = \theta$  and  $B = -\theta$ .

$$\cos \theta + -\theta = \cos \theta \cos -\theta - \sin \theta \sin -\theta$$

For the left-hand side, we have

$$\cos \theta + -\theta = \cos 0 = 1$$

And for the right-hand side we use the fact that cosine is an odd function ( $\cos -s = \cos s$ ). Sine is an even function ( $\sin s = -\sin -s$ ) so that gives

$$\begin{aligned} & \cos \theta \cos \theta - \sin \theta (-\sin \theta) \\ &= \cos^2 \theta + \sin^2 \theta \end{aligned}$$

Bringing back the left-hand side

$$1 = \cos^2 \theta + \sin^2 \theta$$

□

# Chapter 40

## Sum of angles by Ptolemy

### general sum of angles

Previously, we saw the double angle formulas

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

We'd like to find a general formula for the sine of two angles added together.  $\sin A + B = ?$

Of course, if  $A = B$  then the formula must reduce to what we have above, so it seems there are two reasonable possibilities if we keep things simple:

$$\sin A + B \stackrel{?}{=} \sin A \cos A + \sin B \cos B$$

$$\sin A + B = \sin A \cos B + \sin B \cos A$$

Without explaining it at present, we will just assume it is known that  $\sin 0 = 0$  and  $\cos 0 = 1$ . It is easy to see that the sine and cosine approach these values for very small angles but a proper explanation will have to wait for analytic geometry.

If you start with any angle  $A$ , set  $B = 0$ , and then use the first formula it gives  $\sin A + 0 = \sin A \cos A + 0 \cdot 1$  which means  $\sin A = \sin A \cos A$  which makes no sense. Indeed, we will now derive the second formula as the correct result.

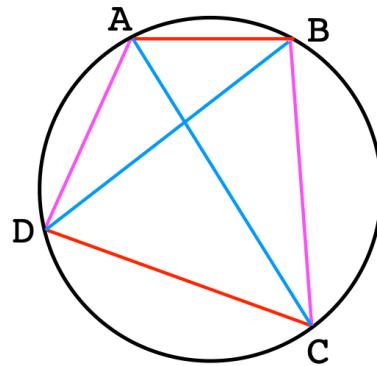
If we start with the cosine formula there seems to be only one simple candidate, namely

$$\cos A + \cos B = \cos A \cos B - \sin A \sin B$$

We use Ptolemy's theorem, which we proved previously using the properties of parallelograms. In fact, a big reason for including this section now is to illustrate the power of this theorem, which is normally thought to be difficult, but since we found an easy proof we are in great shape.

Ptolemy's theorem says that if a four-sided figure, a quadrilateral, has all of its vertices on a circle, then we can form the two products of opposing sides and add them together to obtain the product of the diagonals.

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$



$$\textcolor{red}{AB} \cdot \textcolor{red}{CD} + \textcolor{magenta}{BC} \cdot \textcolor{magenta}{AD} = \textcolor{blue}{AC} \cdot \textcolor{blue}{BD}$$

Our first proof of this is [here](#). And we'll revisit the topic later.

Notice that if the quadrilateral is a rectangle, then the diagonals of the rectangle are also diagonals of the circle ([here](#)), so this theorem gives a simple proof of the Pythagorean theorem.

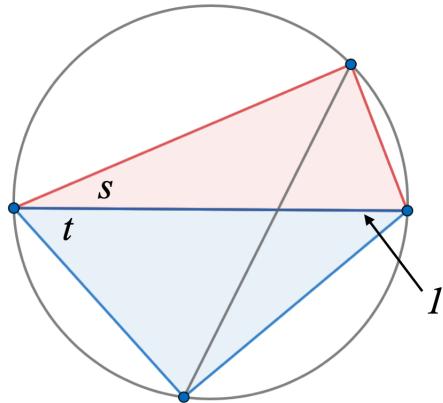
## sum of angles

Ptolemy's theorem can be used to give direct proofs of the sum (and difference) of angles formulas for both sine and cosine. It's a fun exercise because the results come easily from inspired diagrams with slight changes between them.

It helps that we have an idea about what we want.

We need to say a bit more before we start. Each diagram contains a diameter of the circle, scaled so its length is 1.

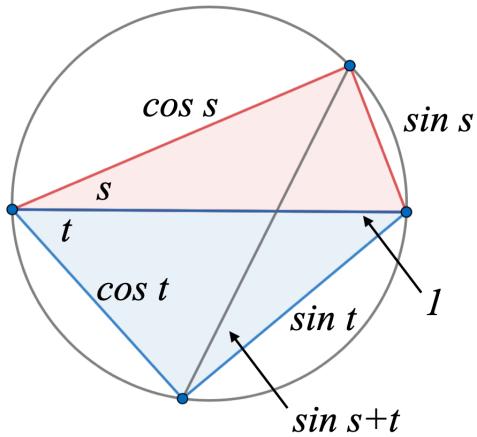
In two cases that length is also a diagonal of the quadrilateral, and in the other two it is a side. In the first case, the product with the other diagonal of the quadrilateral is just equal to whatever value that second diagonal has. Here is the first figure:



There are two right angles on the circle with sides extending to the ends of the diameter. This makes the setup a natural for sine and cosine formulas.

Second, a peripheral or inscribed angle  $\theta$  is related to its chord length by  $L = 2R \sin \theta$ . We saw this important result previously ([here](#)). Since we have  $2R = 1$ , this simplifies to  $L = \sin \theta$ , where in this case  $\theta = s + t$ .

We start with the sum of sines. The idea is that  $\sin s + \sin t$  should be the second diagonal.



From there the formula basically writes itself.

$$\sin s + t = \sin s \cos t + \sin t \cos s$$

Our analysis of what was likely to be the form of the final result turns out to be correct.

### sine of the difference

Algebraically, the difference of sines is easily derived using the fact that cosine is an even function,  $\cos(x) = \cos(-x)$  while sine is odd, so  $\sin(x) = -\sin(-x)$ .

Thus, substituting  $-t$  for  $t$  changes the "sign" of the second term but not the first.

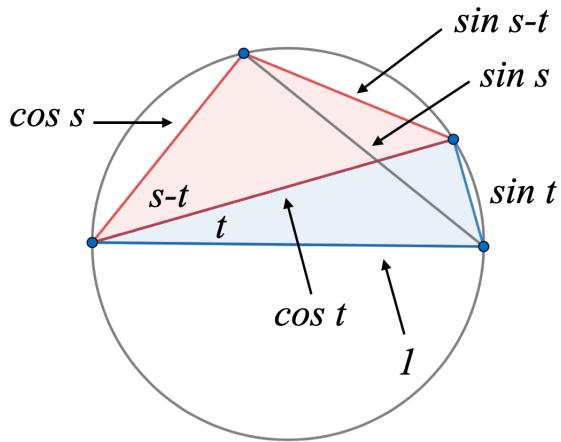
$$\begin{aligned} \sin s + (-t) &= \sin s \cos(-t) + \sin(-t) \cos s \\ &= \sin s \cos t - \sin t \cos s \end{aligned}$$

However, we can use a diagram (kind of) like the one above to do this formula too. See:

[https://www.cut-the-knot.org/proofs/sine\\_cosine.shtml](https://www.cut-the-knot.org/proofs/sine_cosine.shtml)

The trick is that, because of the minus sign on the right-hand side in the final formula, we want  $\sin s - t$  to be one of the sides. So the diagonal of the circle must be opposite, and also be a side of the quadrilateral.

Now,  $s$  is the whole angle, i.e.  $s - t + t = s$ . So one diagonal is  $\sin s$  and the other is  $\cos t$ .



And again, the formula writes itself:

$$\sin(s - t) + \sin t \cos s = \sin s \cos t$$

which rearranges to give

$$\sin s - t = \sin s \cos t - \sin t \cos s$$

### cosine of the sum

For the cosine formulas, we'll need to relate them to the sine of another angle. Recall that if

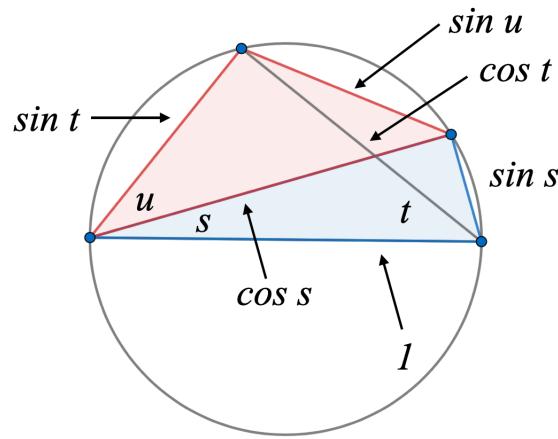
$$s + t + u = 90$$

then

$$\cos(s + t) = \sin u = \sin 90 - (s + t)$$

So for the first cosine formula we add an additional angle  $u$ , and use the fact that  $s + t$  is complementary to  $u$ .

We make  $\sin u = \cos s + t$  one of the sides, because we know the formula has a minus sign in it.



$$\sin s \sin t + \cos(s + t) = \cos s \cos t$$

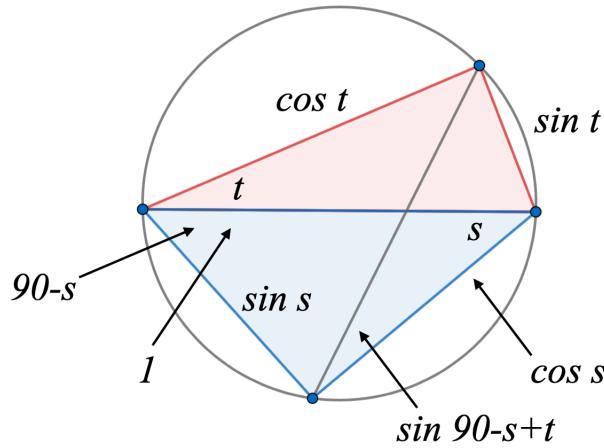
$$\cos s + t = \cos s \cos t - \sin s \sin t$$

### cosine of the difference

Last is the difference formula for cosine. Again, we can derive this formula easily by using the fact that cosine is even and sine is odd.

Knowing where we're headed, we want opposing sides to be both sine or both cosine, and we have them as sides, because they add in the final formula.

So then, somehow, the diagonal must be  $\cos s - t$ .



The complementary angle to  $s$  is  $90 - s$ . Adding it to the adjacent angle  $t$  we have that the diagonal is  $\sin(90 - s + t)$ . But that is

$$\cos 90 - (90 - s + t) = \cos s - t$$

which is just what we need.

$$\cos s - t = \cos s \cos t + \sin s \sin t$$

Perhaps you may object that possibly it could be that  $s < t$  so  $t - s < 0$ . But we can also use the angle on the opposite side and write:

$$\begin{aligned}\sin s + (90 - t) &= \cos 90 - [ (90 - t) + s ] \\ &= \cos t - s\end{aligned}$$

Again, this works because cosine is an even function so  $\cos s - t = \cos t - s$ , or if you prefer you can use the fact that the sine of an angle is equal to the sine of its supplementary angle, which is obvious from the diagram.

$$\sin s + (90 - t) = \sin 180 - [ s + (90 - t) ] = \sin t + (90 - s)$$

# Chapter 41

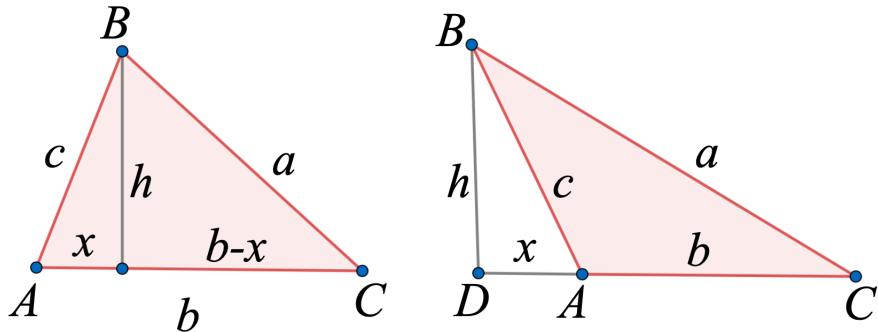
## Law of Cosines

This theorem is used extensively in geometry. It relates the side lengths of any triangle to the cosine of an angle. For example, if the side opposite  $\angle A$  is  $a$ , then  $a^2 = b^2 + c^2 - 2bc \cos A$ .

Rewritten appropriately, it can give any side in terms of the other two.

Consider  $\triangle ABC$  in two versions. In one (left panel),  $\angle A$  is acute, and in the other (right panel)  $\angle A$  is obtuse. Note that the supplementary  $\angle BAD$  on the right is equal to  $\angle A$  from the acute case. In particular, it has the same cosine,  $x/h$ .

Let the sides opposite be  $a, b, c$ , as usual.



For the acute case, we have

$$a^2 = h^2 + (b - x)^2$$

$$c^2 = h^2 + x^2$$

Subtracting:

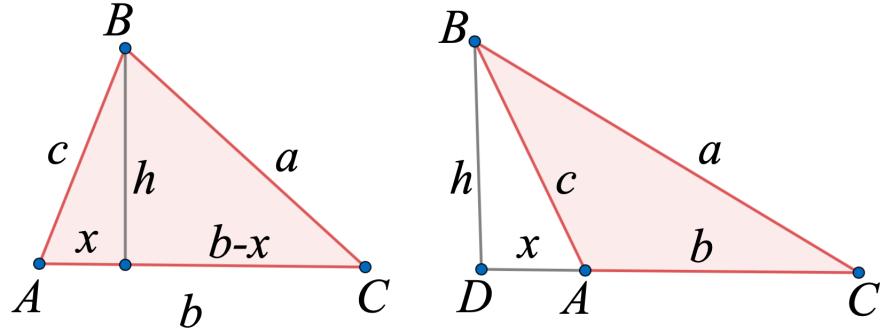
$$a^2 - c^2 = (b - x)^2 - x^2 = b^2 - 2xb$$

Rearranging

$$a^2 = b^2 + c^2 - 2xb$$

$x/c$  is the cosine of  $\angle A$  so  $x = c \cos A$  so then finally

$$a^2 = b^2 + c^2 - 2bc \cos A$$



For the obtuse case the initial arithmetic has a change of sign. ( $a$  is still the side opposite  $\angle A$ , but it is obviously bigger now).

$$a^2 = h^2 + (b + x)^2$$

$$c^2 = h^2 + x^2$$

Subtracting as before:

$$a^2 - c^2 = (b + x)^2 - x^2 = b^2 + 2xb$$

Rearranging

$$a^2 = b^2 + c^2 + 2xb$$

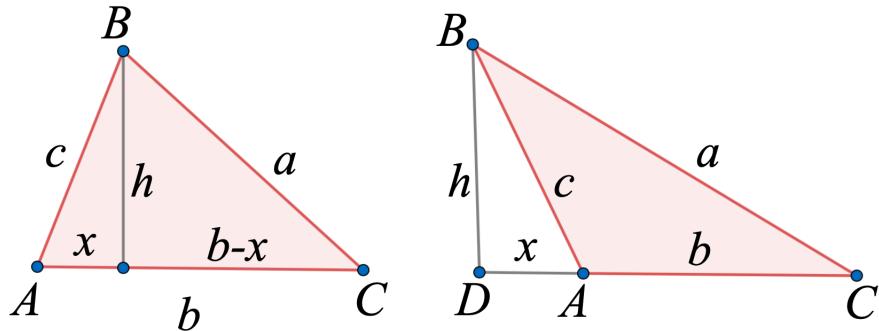
There is a change of sign for the last term.

However, there is another difference for the obtuse case. Now the angle *inside* the triangle is supplementary to the angle whose cosine is  $x/c$ .

Recall the sum of angles formula for cosine. Let  $\theta$  and  $\phi$  be supplementary so  $\theta + \phi = \pi$ . Then

$$\begin{aligned} \cos \theta &= \cos \pi - \phi \\ &= \cos \pi \cos \phi + \sin \pi \sin \phi = -\cos \phi \end{aligned}$$

Thus, when we think about  $\angle A$  as the angle at vertex  $A$  *inside* the triangle



for the obtuse case we have

$$\cos A = -\frac{x}{c} \quad -c \cos A = x$$

giving the same formula for both cases:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

where  $\angle A$  is  $\angle BAC$  for the acute case and  $\angle B'AC$  for the obtuse case.

This is the law of cosines. We compute the length of one side  $a$  in terms of the two other sides  $b$  and  $c$  and the angle between them,  $\angle A$ . This proof is identical to Euclid Book II.12 and II.13, except for the last step.

Using the cosine in the formula is just a form of shorthand for the ratio  $x/c$  and gets rid of that pesky term  $x$ .

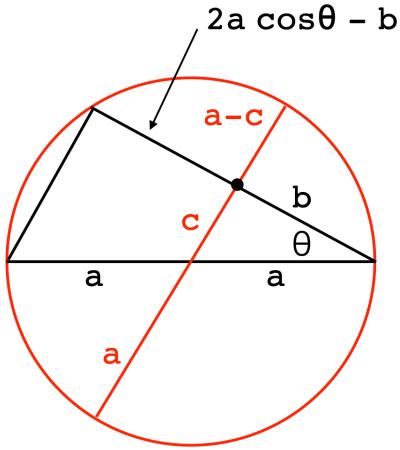
This is Pythagorean theorem with an added correction factor,  $-2bc \cos A$ , that depends on the angle opposite the hypotenuse and which disappears when that angle is a right angle, since the cosine is zero.

The factor is negative for an acute angle, which makes sense, since the smaller angle squeezes the hypotenuse to be smaller as well, while it is positive for an obtuse angle, giving a longer side opposite the greater angle.

## alternative proof

Here is an alternative derivation based on the products of parts of two secants, for the special case of a right triangle.

Draw a right triangle on one diameter in a circle of radius  $a$ . Draw a second diameter such that it crosses the base of the right triangle at a right angle, forming a smaller, similar right triangle.



The smaller triangle has sides  $a, b$  and  $c$ . The lengths of the other parts are easy to compute. Now multiply

$$(a + c)(a - c) = b(2a \cos \theta - b)$$

$$a^2 - c^2 = 2ab \cos \theta - b^2$$

The result follows immediately.

$$a^2 - c^2 = 2ab \cos \theta - b^2$$

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

□

### algebraic proof

In  $\triangle ABC$  drop the altitude from vertex  $A$  to side  $a$  opposite then

$$a = b \cos C + c \cos B$$

In the same way:

$$b = a \cos C + c \cos A$$

$$c = a \cos B + b \cos A$$

Multiply the first by  $a$

$$a^2 = ab \cos C + ac \cos B$$

In the same way

$$b^2 = ab \cos C + bc \cos A$$

$$c^2 = ac \cos B + bc \cos A$$

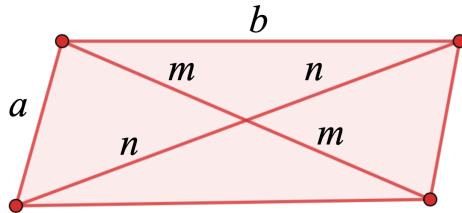
Subtract the first and second from the third:

$$c^2 - a^2 - b^2 = -2ab \cos C$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

## parallelogram sides

The law of cosines leads to an interesting relationship between the sides and diagonals of a parallelogram. Recall that the diagonals divide the figure into two congruent triangles. They also bisect one another.



*Proof.*

For convenience we label the half-diagonals as  $m$  and  $n$ . Applying the theorem twice we have

$$a^2 = m^2 + n^2 - 2mn \cos \theta$$

$$b^2 = m^2 + n^2 + 2mn \cos \theta$$

The sign change on the last term arises because the angles at the center are supplementary, so their cosines are negatives. By addition:

$$a^2 + b^2 = 2(m^2 + n^2)$$

But if  $e$  and  $f$  are the diagonals then  $m^2 = e^2/4$  and  $n^2 = f^2/4$  so

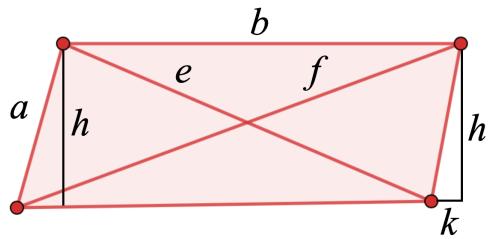
$$2(a^2 + b^2) = e^2 + f^2$$

The sum of the squares of all four sides is equal to the sum of the squares of the diagonals.

Of course this is correct for a rectangle (by Pythagoras's theorem), but it remains true when the shape leans to one side.

□

An alternative proof (Byer) is to apply Pythagoras directly



*Proof.*

$$\begin{aligned} a^2 &= h^2 + k^2 \\ e^2 &= h^2 + (b - k)^2 \\ f^2 &= h^2 + (b + k)^2 \\ e^2 + f^2 &= 2h^2 + 2b^2 + 2k^2 \\ &= 2(a^2 + b^2) \end{aligned}$$

□

## philosophy

Trigonometry is not just about problems like finding the measure of the angle complementary to  $23^\circ$  as  $67^\circ$ .

Instead, trigonometry uses formulas like the sum of angles, and especially, the law of cosines, to solve problems in calculus.

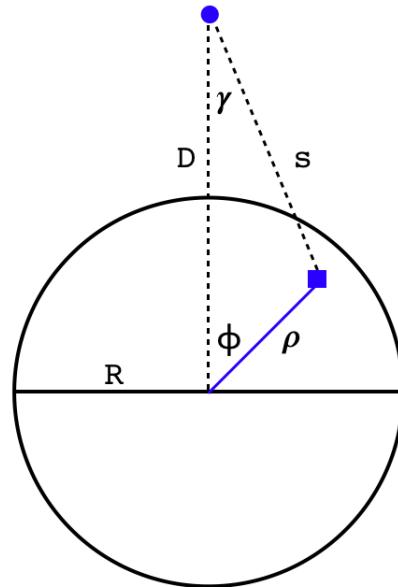
One of the most famous applications came when Newton derived Kepler's laws about the orbits of the planets. Originally, to do that he made the approximation that the mass of the earth acts *as if* it were concentrated at a single point corresponding to the center of the earth, and likewise for the sun.

However, for a rigorous demonstration he needed to prove that this approximation is correct. We do not have the tools yet to see how he did that, but here are two equations from my write-up:

$$\rho^2 = D^2 + s^2 - 2Ds \cos \gamma$$

$$\cos \gamma = \frac{D^2 + s^2 - \rho^2}{2Ds}$$

and the relevant diagram:



You can probably recognize the law of cosines at work.

Trigonometry is hugely important in math and science. Although it has (simple) applications for activities like surveying that is not at all what it is about.

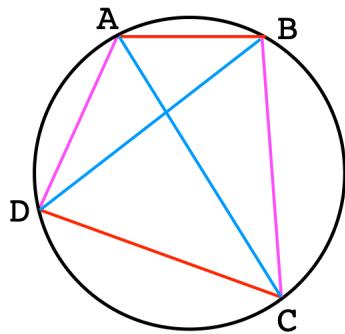
# **Part VIII**

## **Applications of similarity**

# Chapter 42

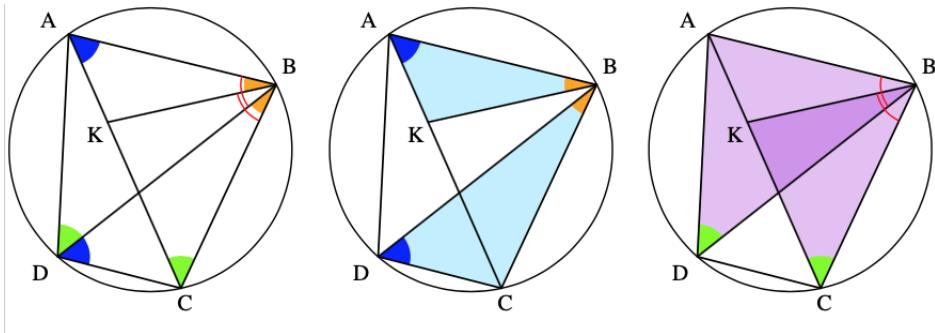
## Ptolemy revisited

In a previous [chapter](#) we introduced Ptolemy's theorem.



$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

Here we provide two more proofs of this theorem, as examples of wonderful proofs, and then explore some consequences.



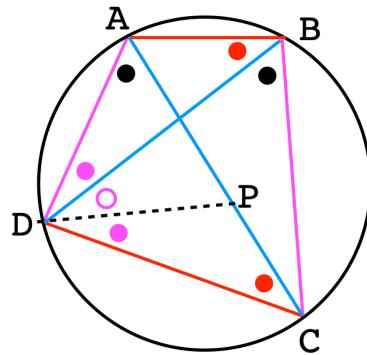
Above is a graphic from wikipedia that shows where we're going in the first proof. We will form two sets of similar triangles and use our knowledge about corresponding ratios.

[https://en.wikipedia.org/wiki/Ptolemy%27s\\_theorem](https://en.wikipedia.org/wiki/Ptolemy%27s_theorem)

### Ptolemy's theorem from similar triangles

Find  $P$  on the diagonal such that  $\angle ADB = \angle PDC$ . We are ignoring the central angle  $\angle BDP$  for now — the open magenta dot.

We can show that  $\triangle ABD \sim \triangle PCD$  (see the angles marked equal in the diagram — two equal is enough). We have the components of  $\angle D$  and also  $\angle ABD = \angle PCD$  by the inscribed angle theorem.



The ratios are  $AB/BD/AD = PC/CD/PD$ , and in particular

$$\frac{AB}{BD} = \frac{PC}{CD}$$

$$AB \cdot CD = PC \cdot BD$$

At the same time,  $\triangle APD \sim \triangle BCD$ . Now the extra angle in the center of  $\angle D$  counts in both:  $\angle BDC = \angle ADP$ . Also,  $\angle DAC = \angle DBC$  by the inscribed angle theorem.

The ratios are  $AP/PD/AD = BC/CD/BD$ , and in particular

$$\frac{AP}{AD} = \frac{BC}{BD}$$

$$BC \cdot AD = AP \cdot BD$$

Combining the two equations

$$AB \cdot CD + BC \cdot AD = (AP + PC) \cdot BD$$

$$= AC \cdot BD$$

□

Yiu also proves the converse theorem.

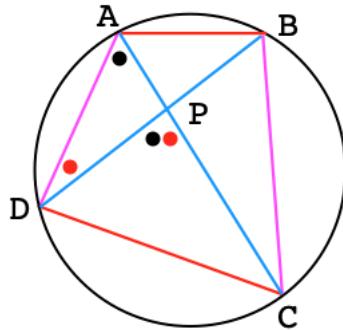
The dots in this proof make it clear which two pairs of triangles are similar, and I've taken care to list the vertices of the triangles in the same order as the sides in each pair of similar triangles, from smallest to largest.

For an alternate notation see [here](#).

## proof by switching sides

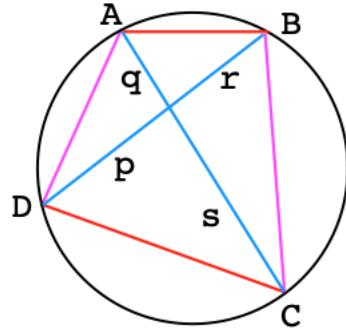
*Proof.* (adapted from wikipedia).

[https://en.wikipedia.org/wiki/Ptolemy%27s\\_theorem](https://en.wikipedia.org/wiki/Ptolemy%27s_theorem)



Let the angle  $s$  (red dot) be subtended by arc  $AB$  and the angle  $t$  (black dot) be subtended by arc  $CD$ . Then the central  $\angle DPC = s + t$  and it has  $\sin s + t$ . The other central  $\angle APD$  has the same sine, as it is supplementary to  $s + t$ .

Let the components of the diagonals be  $AC = q + s$  and  $BD = p + r$ .



Twice the areas of the four small triangles will then be equal to

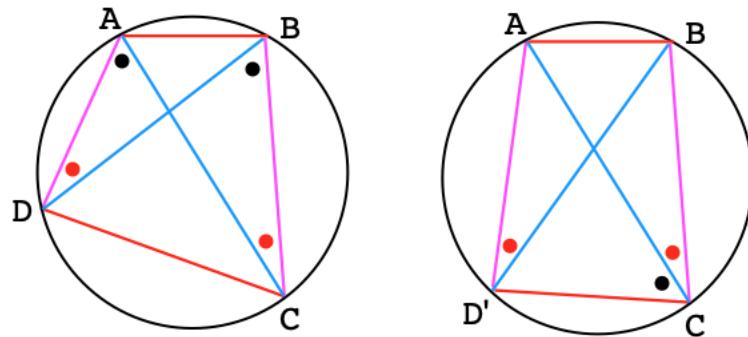
$$2A = (pq + qr + rs + sp) \sin s + t$$

Simple algebra will show that

$$\begin{aligned} (pq + qr + rs + sp) &= (p + r)(q + s) \\ &= AC \cdot BD \end{aligned}$$

The product of the diagonals times the sine of either central angle is equal to twice the area of the quadrilateral.

We're on to something. Now, the great idea.



Move  $D$  to  $D'$ , such that  $AD' = CD$  and  $CD' = AD$ . We know that the arithmetic works out because if two pairs of arcs span the same whole, then their sum is equal, and so is the sum of the corresponding chords.

$\triangle ACD \cong \triangle ACD'$  by SSS, so they have the same area. Therefore the area of  $ABCD$  is equal to the area of  $ABCD'$ .

Some of the angles switch with the arcs. In particular, angle  $t$  (black dot) now corresponds to arc  $AD'$ . As a result  $s + t$  is the measure of the whole angle at vertex  $C$ . The whole angle at vertex  $A$  is supplementary, and the sine of the whole angle at vertex  $A$  is equal to that at  $C$ .

So twice the area of  $\triangle ABD'$  is  $AB \cdot AD' \cdot \sin s + t$ , and twice that of  $\triangle BCD'$  is  $BC \cdot CD' \cdot \sin s + t$ . Add these two areas, equate them with the previous result, and factor out the common term  $\sin s + t$ :

$$AC \cdot BD = AB \cdot AD' + BC \cdot CD'$$

But  $AD' = CD$  and  $CD' = AD$  so

$$AC \cdot BD = AB \cdot CD + BC \cdot AD$$

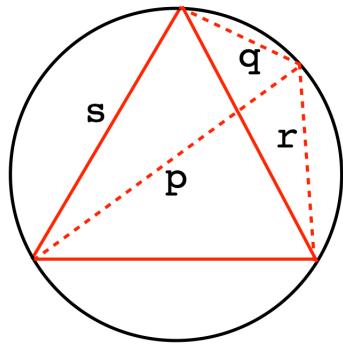
This is Ptolemy's theorem.

□

### corollaries

Here are just a few of the results that follow from this remarkable theorem.

## equilateral triangle



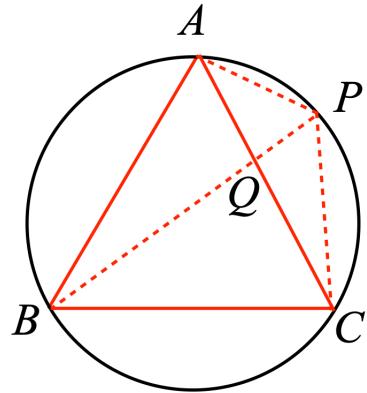
Inscribe an equilateral triangle in a circle and pick any point on the circle.

$$ps = qs + rs$$

$$p = q + r$$

We proved this earlier, without using Ptolemy's theorem, as Van Schooten's theorem.

Here's a different problem from basically the same diagram (Coxeter).



$$\frac{1}{PA} + \frac{1}{PC} = \frac{1}{PQ}$$

*Proof.*

Going back to similar triangles:

$$\triangle AQB \sim \triangle PQC, \quad \triangle AQP \sim \triangle BQC$$

We have then:

$$AQ : BQ : AB = PQ : QC : PC$$

$$PQ : AQ : PA = QC : BQ : BC$$

From the first:

$$\frac{AB}{PC} = \frac{AQ}{PQ}$$

and then

$$\frac{PQ}{PA} = \frac{QC}{BC}$$

Hence

$$\begin{aligned} \frac{1}{PC} + \frac{1}{PA} &= \frac{AQ}{AB \cdot PQ} + \frac{QC}{BC \cdot PQ} \\ &= \frac{1}{PQ} \cdot \left( \frac{AQ}{AB} + \frac{QC}{BC} \right) \end{aligned}$$

But of course  $AB = BC$  and  $AQ + QC = AC = AB = BC$ , so finally:

$$\frac{1}{PC} + \frac{1}{PA} = \frac{1}{PQ}$$

## Pythagorean theorem

Let the quadrilateral be a rectangle. The the sum of squares of opposing sides is

$$a^2 + b^2$$

Triangles made by opposing diagonals are congruent, so the diagonals are equal in length. The diagonal is the hypotenuse, hence

$$a^2 + b^2 = c^2$$

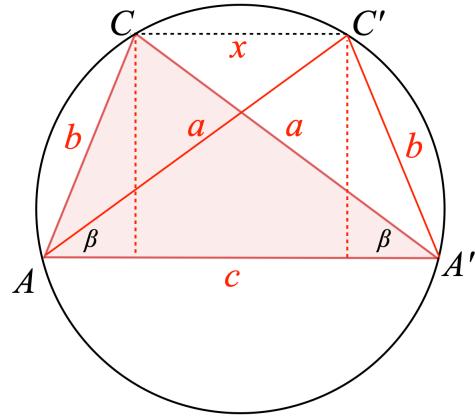
We saw this proof previously ([here](#)).

## Law of Cosines

Draw  $\triangle ABC$  (suppress the  $B$  label) and then draw another triangle congruent with it, with a shared base, and all four points in a circle, forming a cyclic quadrilateral.

Relying on previous work with a rectangle in a circle ([here](#)), we know this construction is possible.

The points are  $A, A', C, C'$ .  $\beta$  marks the original  $\angle B$ , but will not be used.



We need an expression for  $x$ . We have that the base of the altitude from  $C$  to side  $c$  is a distance from  $A$  equal to  $(c - x)/2$ . It follows that

$$\frac{c - x}{2} \div b = \cos A$$

$$c - x = 2b \cos A$$

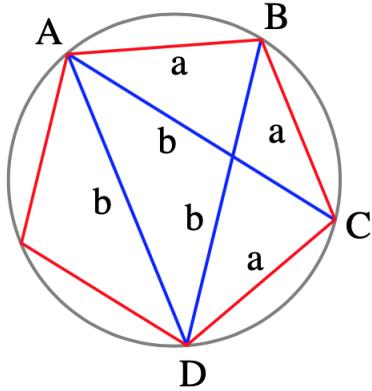
$$x = c - 2b \cos A$$

Now, apply Ptolemy's Theorem. We have:

$$\begin{aligned} a^2 &= b^2 + cx \\ &= b^2 + c(c - 2b \cos A) \\ &= b^2 + c^2 - 2bc \cos A \end{aligned}$$

□

## golden mean in the pentagon



Take four vertices of the regular pentagon and draw two diagonals. From the theorem, we have

$$b \cdot b = a \cdot a + a \cdot b$$

$$\frac{b^2}{a^2} = 1 + \frac{b}{a}$$

Rather than use the quadratic equation, rearrange and add  $1/4$  to both sides to “complete the square”:

$$\frac{b^2}{a^2} - \frac{b}{a} + \frac{1}{2^2} = 1 + \frac{1}{2^2}$$

So

$$\left(\frac{b}{a} - \frac{1}{2}\right)^2 = \frac{5}{4}$$

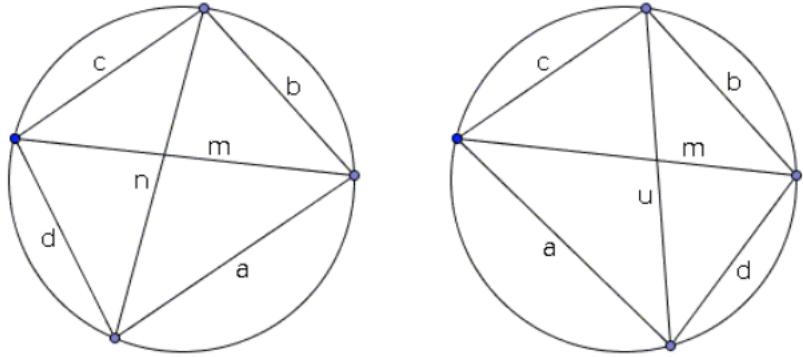
$$\frac{b}{a} - \frac{1}{2} = \pm \frac{\sqrt{5}}{2}$$

$$\frac{b}{a} = \frac{1 \pm \sqrt{5}}{2}$$

This ratio  $b/a$  is known as  $\phi$ , the golden mean.

## diagonals

Let us look at something like what we used for the proof of Ptolemy’s theorem in the beginning.

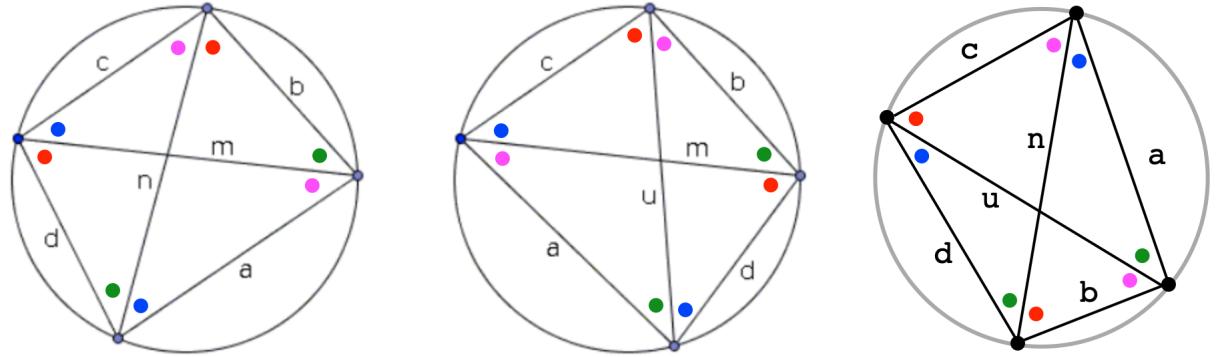


$$nm = ac + bd$$

We move one of the points, exchanging sides  $a$  and  $d$ . Then, one of the diameters,  $n$ , changes length to  $u$ .

$$mu = ab + cd$$

If, instead, we exchange sides  $a$  and  $b$ , the old  $m$  changes to  $u$ . Why?



Mark the peripheral angles with equal arcs ( $abcd$ : red, blue, green, magenta).

The triangle with sides  $b$  and  $d$  in the middle, and magenta plus red for the vertex angle, is congruent to one in the right panel. So their long sides are equal, both have length  $u$ .

Thus,

$$nu = ad + bc$$

We get a formula for the square of the diagonal:

$$m^2 = \frac{(mu)(nm)}{nu} = \frac{(ab + cd)(ac + bd)}{(ad + bc)}$$

There is a similar formula for  $n^2$ . These formulas are sometimes attributed to Brahmagupta. This beautiful proof is due to Paramesvara (14th century).

<https://www.cut-the-knot.org/proofs/PtolemyDiagonals.shtml>

The ratio is

$$\frac{m}{n} = \frac{ab + cd}{ad + bc}$$

which is referred to as Ptolemy's second theorem.

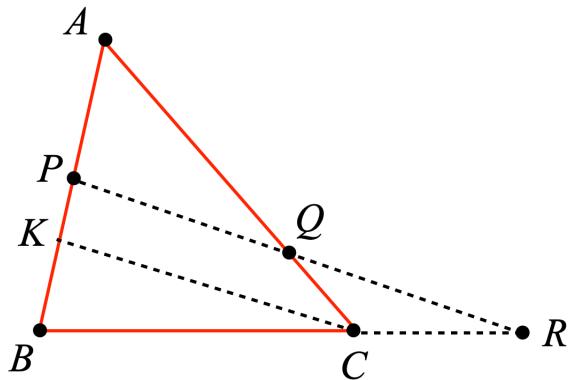
# Chapter 43

## Menelaus's theorem

In this chapter we establish a simple but very valuable result called **Menelaus's theorem**, due to Menelaus of Alexandria (the geometer, not the mythological figure). He lived about 100 CE. It is supposed that Menelaus grew up in Alexandria but is known to have lived in Rome.

### Menelaus's theorem

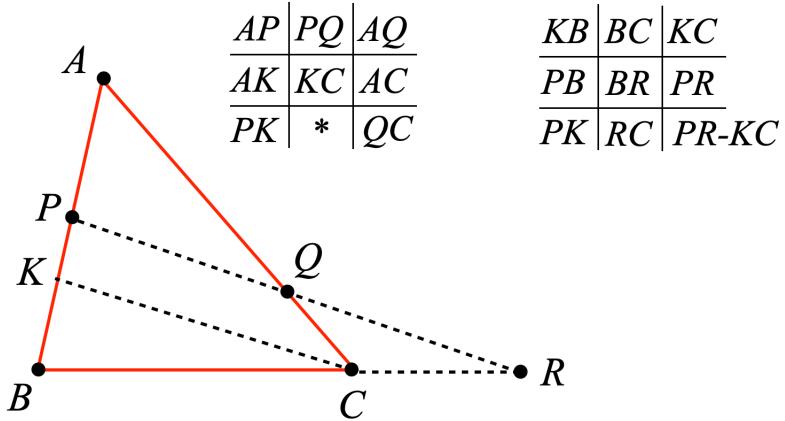
Consider  $\triangle ABC$  and a line that goes through the triangle (not through a vertex and not through a side), called a transversal. There is one side, which when it is extended, meets the transversal. Here the transversal  $PQR$  meets  $BC$  at  $R$ .



*Proof.*

Draw a line parallel to the transversal,  $CK$ . We have two pairs of similar triangles.

The first is  $\triangle APQ \sim \triangle AKC$ , and the second is  $\triangle KBC \sim \triangle PBR$ . I will write the ratios as shown in the figure. I call these ratio boxes.



From the first set:

$$\frac{AP}{AQ} = \frac{PK}{QC}$$

And from the second set:

$$\frac{PB}{BR} = \frac{PK}{RC}$$

Isolate  $PK$  and equate:

$$\frac{AP}{AQ} \cdot QC = \frac{PB}{BR} \cdot RC$$

$$\frac{AP}{PB} \cdot \frac{BR}{RC} \cdot \frac{QC}{AQ} = 1$$

□

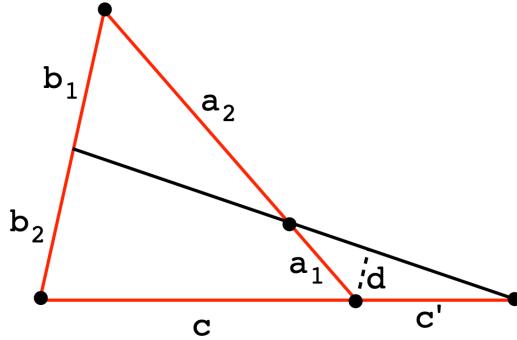
We will show several different proofs, for practice.

In the figure below, subscripts indicate the parts of a side, for example,  $a_1$  and  $a_2$  are the components of side  $a$ .

$c + c'$  is the length of the whole side *plus the extension*.

We will show that the product of three ratios is equal to 1:

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c + c'}{c'} = 1$$



*Proof.*

Draw the dotted line segment parallel to  $b$  and label it  $d$ . We have two pairs of similar triangles. The first pair has side ratios

$$\frac{d}{a_1} = \frac{b_1}{a_2}$$

while the second has

$$\frac{d}{c'} = \frac{b_2}{c + c'}$$

Combining the two results:

$$d = \frac{a_1 b_1}{a_2} = \frac{b_2 c'}{c + c'}$$

So

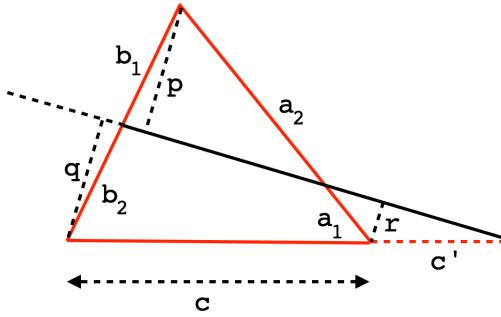
$$\frac{a_1 \cdot b_1 \cdot c + c'}{a_2 \cdot b_2 \cdot c'} = 1$$

□

## Einstein's proof of Menelaus

There is a story that Einstein disliked the proofs of Menelaus's theorem based on similar triangles.

He said they were ugly, and didn't involve the vertices symmetrically. I read that Einstein's proof starts by dropping altitudes to the transversal, and then uses similar triangles. Here's what I came up with.



*Proof.*

The right triangle with  $c+c'$  as hypotenuse is similar to the one with  $c'$  as hypotenuse. This gives

$$\frac{q}{r} = \frac{c+c'}{c'}$$

The right triangles with  $a_1$  and  $a_2$  as hypotenuse are similar. We form the ratio of altitudes:

$$\frac{r}{p} = \frac{a_1}{a_2}$$

Finally, the right triangles with  $b_1$  and  $b_2$  as hypotenuse are similar. We form the ratio of altitudes again:

$$\frac{p}{q} = \frac{b_1}{b_2}$$

Multiply the left-hand sides all together to obtain 1.

$$\frac{q}{r} \cdot \frac{r}{p} \cdot \frac{p}{q} = 1$$

Therefore, the product of the right-hand sides is also 1:

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c+c'}{c'} = 1$$

□

According to

<https://www.cut-the-knot.org/Generalization/MenelausByEinstein.shtml>

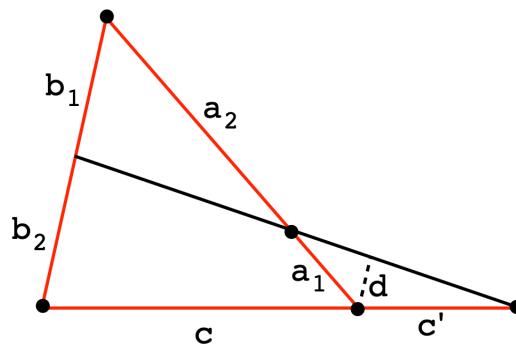
... [in] correspondence between Albert Einstein and a friend of his, Max Wertheimer. In the first letter, Einstein apparently continues a discussion on elegance of mathematical proofs. A proof may require introduction of additional elements, like line AP in the first of the cited proofs. In Einstein's opinion, "... we are completely satisfied only if we feel of each intermediate concept that it has to do with the proposition to be proved."

As an illustration of his viewpoint, Einstein gives two proofs of the same proposition - one ugly, the other elegant. Curiously, the proposition he proves is that of the Menelaus theorem, and the proof ugly in his view is the first of the cited proofs. He writes, "Although the first proof is somewhat simpler, it is not satisfying. For it uses an auxiliary line which has nothing to do with the content of the proposition to be proved, and the proof favors, for no reason, the vertex A, although the proposition is symmetrical in relation to A, B, and C. The second proof, however, is symmetrical, and can be read off directly from the figure."

## converse

We've drawn  $\triangle ABC$  as an acute triangle, with the transversal crossing through points on two sides and on the extension of the third. There are other possibilities. However, if we restrict ourselves to this case, then the converse can be stated as

Given that the product of ratios is equal to 1, the three points are collinear.



*Proof.*

An easy proof draws on the forward theorem. Assume that the product of ratios is 1.

Suppose the points are not colinear, so the case with product 1 has  $c''$  and the case with colinear points has  $c'$ . We are given that

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c + c''}{c''} = 1$$

But, by the forward theorem

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c + c'}{c'} = 1$$

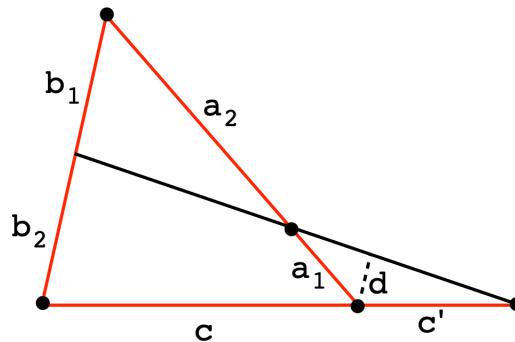
We have

$$\begin{aligned}\frac{c + c''}{c''} &= \frac{c + c'}{c'} \\ \frac{c}{c''} &= \frac{c}{c'} \\ \frac{1}{c''} &= \frac{1}{c'}\end{aligned}$$

We conclude that  $c' = c''$ . Hence, if the product is equal to 1, the points lie on the same line.

□

## more about Menelaus



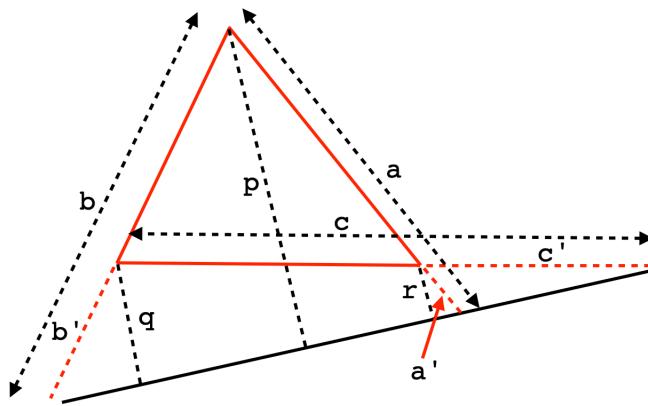
We showed that the product of three ratios is equal to 1:

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c + c'}{c'} = 1$$

One way to remember the terms is that, going around in any direction, we take the first length we run into and put it into the numerator. It doesn't matter which direction you go, as long as you take  $c'$  first when going clockwise and  $c + c'$  first when going counter-clockwise.

Take the extension first if you run into it first.

We also need to prove the theorem for an alternative setup. Even though the black line in the figure below does not go through the triangle, it is technically a transversal.



Notice that we are re-defining  $a$ ,  $b$  and  $c$ .

$$\frac{r}{p} \cdot \frac{p}{q} \cdot \frac{q}{r} = \frac{a'}{a} \cdot \frac{b}{b'} \cdot \frac{c}{c'}$$

Once again, we can remember this as, take the first length we run into, and put it on top.

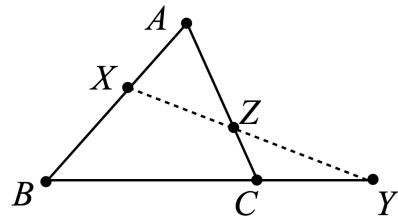
Menelaus is a preliminary lemma to an easy proof of Ceva's theorem, which becomes important in thinking about the concurrency of lines in a triangle, like those which bisect the opposing sides and meet at the centroid. Ceva's theorem gives a condition under which three lines like this are concurrent, or meet at a point.

Menelaus's theorem can be seen as in some sense as similar to Ceva, because it provides conditions under which three points can be proved collinear, or on one line.

It will come in handy in another problem, called Pappus's Theorem.

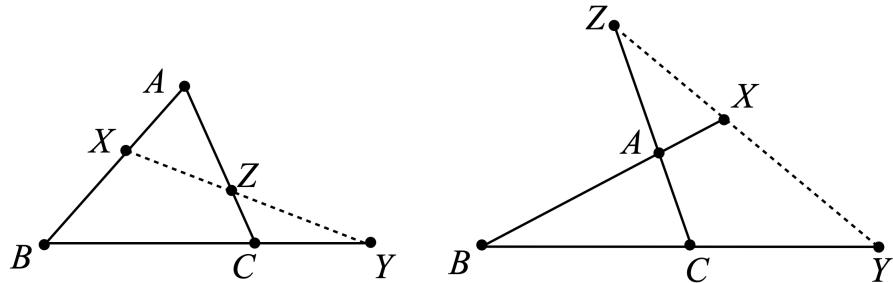
### alternative definition

There is a somewhat more sophisticated version of Menelaus's Theorem which views the path around a triangle as consisting of *directed* (that is, *signed*) line segments.



The transversal is  $XZY$ , which meets the extension of  $BC$  at  $Y$ . Since  $Y$  does not lie between  $B$  and  $C$ , the line segments  $BY$  and  $YC$  point in opposite directions. The ratio  $BY : YC$  thereby acquires a minus sign.

Every transversal has this property. Here (on the right) is a transversal that does not go through the triangle at all.



The path around the triangle has three parts:

$$A \text{ to } X \text{ to } B \Rightarrow AX : XB$$

$$B \text{ to } Y \text{ to } C \Rightarrow BY : YC$$

$$C \text{ to } Z \text{ to } A \Rightarrow CZ : ZA$$

Each ratio has a minus sign so the total product also has a minus sign. Menelaus's Theorem says:

$$\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZA} = -1$$

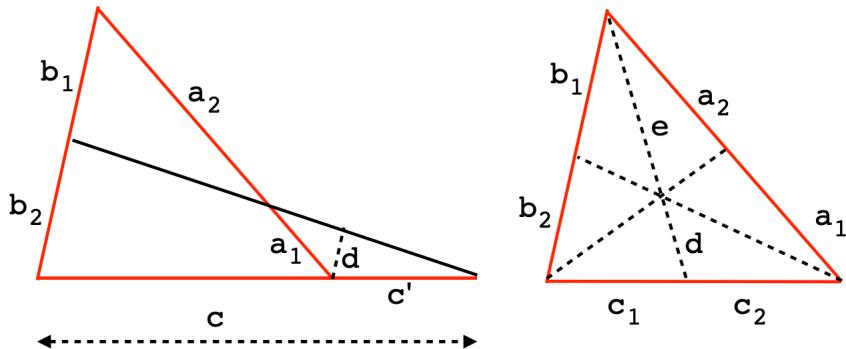
# Chapter 44

## Ceva's Theorem

### Ceva's theorem

We previously showed a proof of Menelaus' theorem. Let us continue with Ceva's theorem (right panel, below). We will show that if the *three lines* are *concurrent* (they all cross at the same point), then

$$\frac{a_1 \cdot b_1 \cdot c_1}{a_2 \cdot b_2 \cdot c_2} = 1$$



We ignore here the signed or directed line segments of the original, which give minus 1 as the product above. These would cancel at the next step.

*Proof.*

The left panel is from the proof of Menelaus' theorem. In the right panel, consider

the left half-triangle with a side composed of  $e + d$ . Apply Menelaus' theorem once.

$$\frac{d}{e} \cdot \frac{b_1}{b_2} \cdot \frac{c}{c_2} = 1$$

For the right half-triangle, apply Menelaus again but go clockwise from the middle:

$$\frac{d}{e} \cdot \frac{a_2}{a_1} \cdot \frac{c}{c_1} = 1$$

Combine the two results:

$$\frac{b_1}{b_2} \cdot \frac{c}{c_2} = \frac{a_2}{a_1} \cdot \frac{c}{c_1}$$

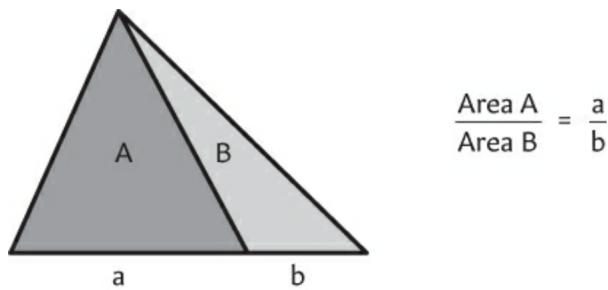
$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1$$

□

## proof based on area

We will look at two more, very classic proofs of Ceva's theorem. These again show the close relationship between similarity in triangles, and area.

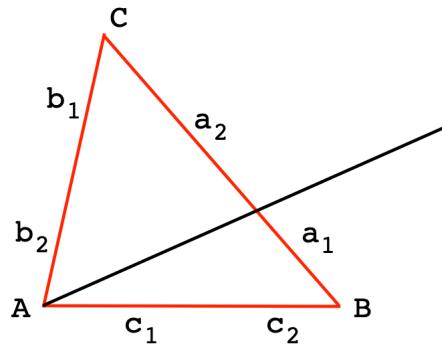
The second one depends on **area-ratio theorem** we developed previously.



**Fig. 120** An area-ratio theorem.

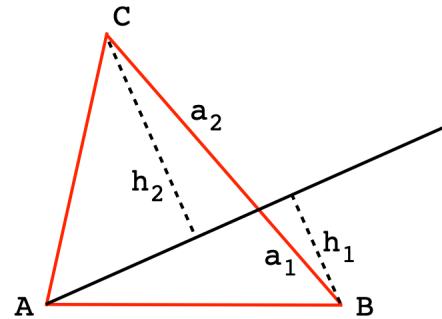
This is simple algebra based on the area formula: one-half the base times the height. Since the two different shaded triangles have the same altitude, the ratio of their areas is the same as the ratio of the length of the bases.

In the triangle shown below, draw a line from the vertex  $A$  to the opposite side  $BC$ , dividing it into lengths  $a_1$  and  $a_2$ .



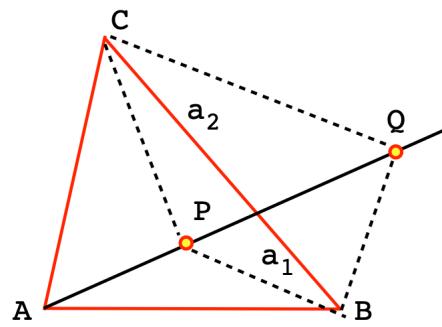
Drop the altitudes from the other vertices to the line. We have vertical angles as well as the corresponding right angles, hence the two triangles are similar with ratios:

$$\frac{a_1}{a_2} = \frac{h_1}{h_2}$$



Hence the altitudes are in the same ratio as the parts of the side  $a_1/a_2$ .

Now, pick a point *anywhere* on that line.

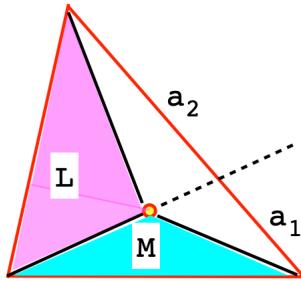


We'll focus on  $P$ , but  $Q$  would give the same result. The triangle  $\triangle APB$  can be visualized as having base  $AP$  and altitude  $h_1$ , while  $\triangle APC$  has the same base  $AP$  and altitude  $h_2$ .

Since they have the identical base, the areas are in same the ratio as the altitudes, but these are, in turn, in the same ratio as the parts of the side.

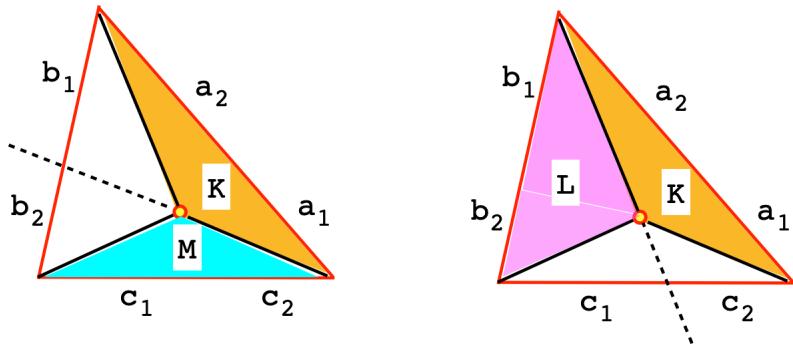
$$\frac{\Delta_{APB}}{\Delta_{APC}} = \frac{h_1}{h_2} = \frac{a_1}{a_2}$$

Let us abbreviate those areas as  $L = \Delta_{APC}$  and  $M = \Delta_{APB}$ . We have



$$\frac{M}{L} = \frac{a_1}{a_2}$$

Now we just appeal to symmetry.



$$\frac{K}{M} = \frac{b_1}{b_2}$$

$$\frac{L}{K} = \frac{c_1}{c_2}$$

Multiply together to obtain a big cancelation:

$$\begin{aligned}\frac{M}{L} \cdot \frac{K}{M} \cdot \frac{L}{K} &= \frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} \\ 1 &= \frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2}\end{aligned}$$

This is the forward version of Ceva's theorem. If we draw the three lines to be concurrent, then the parts of the sides are in this ratio, which we can also write

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} = \frac{c_2}{c_1}$$

□

The proof also works in reverse,

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1 \iff \text{3 lines cross at point } P$$

*Proof.*

Everything is the same as before, except we suppose that the ratio of the parts of  $a$  has to be slightly different in order to obtain 1 when we multiply everything together.

Of course, this would mean that the line from  $A$  would no longer go through  $P$ . Suppose the new line gives  $a'_1$  and  $a'_2$  as the components of the side  $a$ .

$$a_1 \neq a'_1, \quad a_2 \neq a'_2$$

But, by the forward theorem we have that the line which *does* go through  $P$  divides the side into  $a_1$  and  $a_2$  such that

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1$$

We conclude that

$$\frac{a_1}{a_2} = \frac{a'_1}{a'_2}$$

Since they are parts of the same length, the components are individually equal. That is, if the whole side is  $a$  then  $a_2 = a - a_1$  and  $a'_2 = a - a'_1$  so

$$\frac{a_1}{a - a_1} = \frac{a'_1}{a - a'_1}$$

$$a_1a - a_1a'_1 = a'_1a - a_1a'_1 \\ a_1 = a'_1$$

□

## centroid

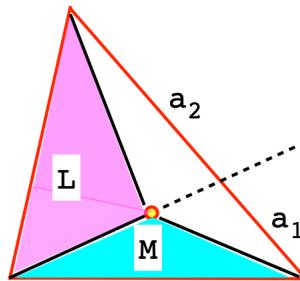
In the general case, the crossing lines are called cevians. In the case of medians, they divide the sides opposite in half, and then the central point is called the centroid. Recall that we had

$$\frac{M}{L} = \frac{a_1}{a_2} \\ \frac{K}{M} = \frac{b_1}{b_2}$$

But if  $a_1 = a_2$  and  $b_1 = b_2$ , then  $K = L = M$ .

Furthermore, the two component triangles of region  $K$  have equal area, because they share the vertex at  $P$  and their bases are equal. (Which also implies the previous result).

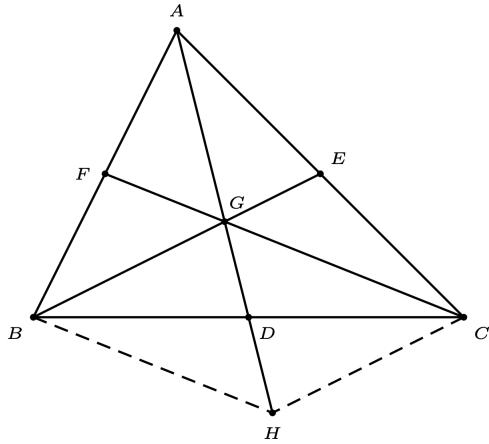
Therefore, in the case of medians, all six small triangles are equal in area.



For a physical object, the point  $P$  would be the *center of mass*.

## Centroid

Consider the  $\triangle ABC$ . Draw the medians  $BE$  and  $CF$  (bisectors of the sides). Extend  $AG$  through the intersection of the two medians at  $G$  to  $D$  and then finally to  $H$ , where  $CH$  is drawn parallel to  $BGE$ . We profess not to know anything about  $BH$  yet.



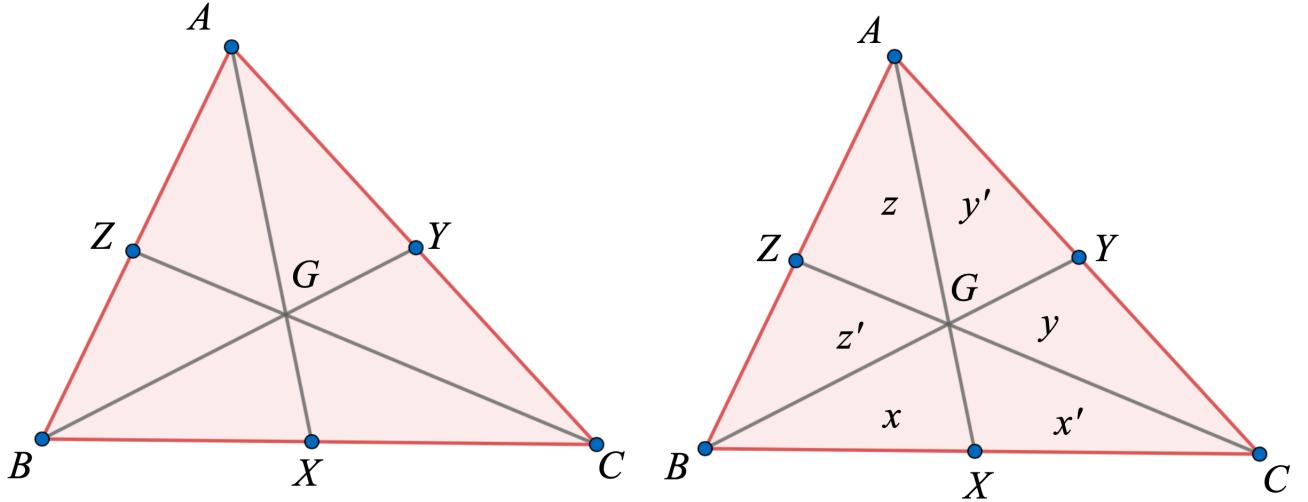
Since  $CH \parallel BGE$  and  $AE = EC$ , we have that  $\triangle AGE \sim \triangle AHC$  with ratio 2 by the midpoint theorem (or what we've called SAS similarity). Thus,  $AG = GH$ .

But we also have  $AF = FB$ , which means that  $\triangle AFG \sim \triangle ABH$  with ratio 2. This gives  $BH \parallel FGC$ .

Therefore,  $BGCH$  has two pairs of opposing sides parallel so it is a parallelogram. The diagonals cross at their midpoints, which means that  $AD$  is a median ( $BC$  is bisected at  $D$ ).

In addition,  $GD = DH$ . Thus,  $GD$  is one-half of  $GH$ , and  $GH$  is equal to  $AG$ , so  $GD$  is one-quarter of  $AH$  and one-third of  $AD$ .  $G$  lies on all three medians, and is called the *centroid* of the triangle. For a physical triangle it would be the center of mass.

## Centroid alternate proof



The medians divide any triangle into six equal small triangles.

*Proof.*

$(\triangle GBX) = (\triangle GCX)$  by the **area-ratio theorem**. Hence the two small triangles on any one side have equal area:  $x = x'$ ,  $y = y'$ ,  $z = z'$ .

Also by the same theorem,  $(\triangle ABX) = (\triangle ACX)$ . So by subtraction  $(\triangle AGB) = (\triangle AGC)$ :  $y + y' = z + z'$ . Thus  $y = z$ . But this is true for any side. Hence all six triangles are equal in area.

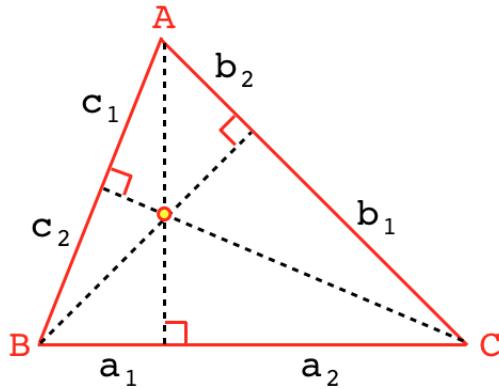
□

Consider  $\triangle GBX$  and  $\triangle ABX$  on base  $XGA$ . They have the same altitude, from the base to the vertex at  $B$ . The ratio of areas is  $1 : 3$ . So that must also be the ratio  $XG : XA$ .

The centroid lies one-third of the way up the median from the side.

## orthocenter

Consider this triangle in which we have drawn the altitudes to each side. We claim that they cross at a single point, called the orthocenter.



Let the angles be  $A, B, C$  as labeled, and the sides opposite be  $a, b, c$ , subdivided as shown.

Then  $\angle A$  is part of two right triangles, and by similar triangles we have that

$$\frac{c_1}{b} = \frac{b_2}{c} \quad \rightarrow \quad \frac{b_2}{c_1} = \frac{c}{b}$$

Similarly for  $\angle B$

$$\frac{a_1}{c} = \frac{c_2}{a} \quad \rightarrow \quad \frac{c_2}{a_1} = \frac{a}{c}$$

And  $\angle C$

$$\frac{b_1}{a} = \frac{a_2}{b} \quad \rightarrow \quad \frac{a_2}{b_1} = \frac{b}{a}$$

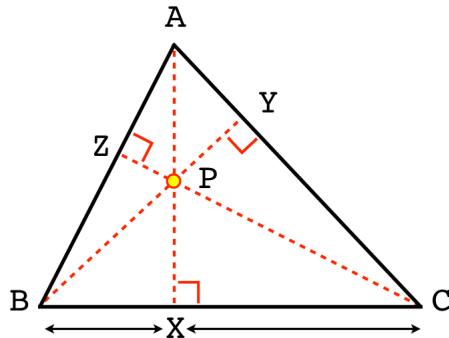
The product of the three right-hand sides above is  $c/b \cdot a/c \cdot b/a = 1$ . Therefore the product of the left-hand sides is also 1:

$$\frac{b_2}{c_1} \cdot \frac{c_2}{a_1} \cdot \frac{a_2}{b_1} = 1$$

Invert and re-order the terms

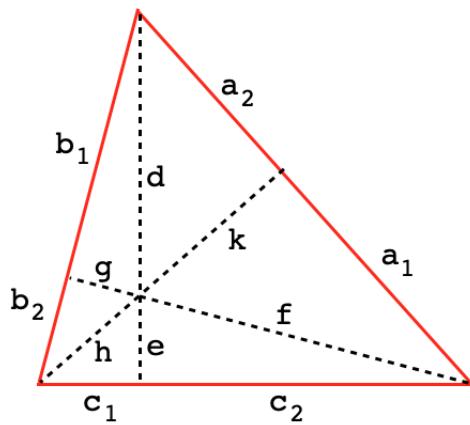
$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1$$

□



Since we have satisfied Ceva's condition, the 3 altitudes all cross at a single point. That point is the orthocenter, and this is a proof that it exists.

### another approach to the orthocenter



There is a different set of similar triangles one can use for the orthocenter. The triangle with sides  $a_2$ ,  $d$  and  $k$  is similar to the triangle with sides  $c_1$ ,  $h$ , and  $e$ . For one pair, we obtain

$$\frac{e}{k} = \frac{c_1}{a_2}$$

You should be able to use the other two pairs to construct Ceva's ratio equal to

$$\frac{e}{k} \cdot \frac{k}{g} \cdot \frac{g}{e}$$

which is, of course, equal to 1.

Another easily derived relationship is that

$$\frac{e}{k} = \frac{h}{d}$$

so

$$de = hk$$

Going around the triangle we will get

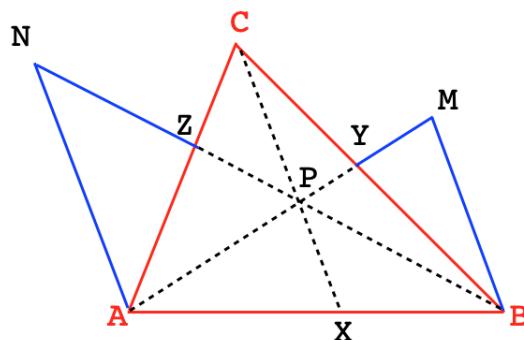
$$de = fg = hk$$

This occurs because the position of  $P$  with respect to each altitude is the same fraction of the whole.

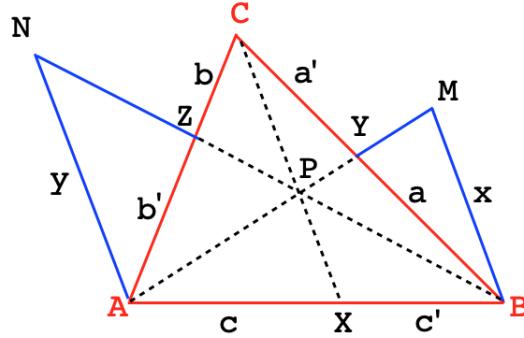
### **another proof of Ceva's theorem**

I found an alternative approach to proving Ceva's theorem as an exercise in a geometry textbook by Jurgensen *et al*. It's a bit weird because this is the only mention of the theorem in the book. I like the proof, which is only hinted at, because we enhance the original diagram, and it provides exercise using similar triangles.

The problem says to extend line segments parallel to  $CX$  from  $A$  and  $B$  and form triangles with the extensions of  $BZ$  and  $AY$ .



The hint given is to consider similar triangles. There are a number of them, due to the three parallel lines. I found the algebra to be a little cleaner with different notation, using lowercase letters for most of the sides.



We have that  $\triangle CYP \sim \triangle MBY$  because  $CX \parallel MB$ . So

$$\frac{x}{CP} = \frac{a}{a'}$$

With similar logic, we find that

$$\frac{y}{CP} = \frac{b'}{b}$$

Hence

$$\frac{x}{y} = \frac{a}{a'} \cdot \frac{b}{b'}$$

We also have similar triangles that involve the base  $AB$ , such as the one with vertex  $N$  compared to vertex  $P$ .  $\triangle ABN \sim \triangle XBP$  so that

$$\frac{y}{XP} = \frac{c + c'}{c'}$$

With similar logic, we find that

$$\frac{x}{XP} = \frac{c + c'}{c}$$

So that

$$\frac{x}{y} = \frac{c'}{c}$$

Combined with the previous result:

$$\frac{x}{y} = \frac{a}{a'} \cdot \frac{b}{b'} = \frac{c'}{c}$$

Thus

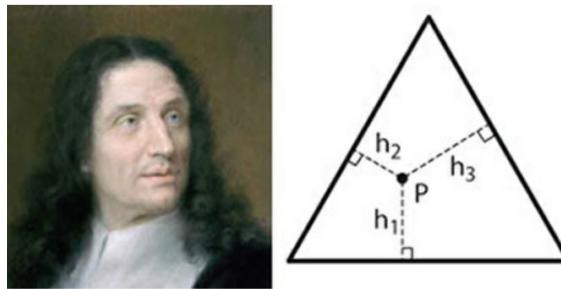
$$\frac{a}{a'} \cdot \frac{b}{b'} \cdot \frac{c}{c'} = 1$$

## Viviani

Here's a problem from Acheson that looks challenging, but yields easily to the right perspective.

### Viviani's theorem

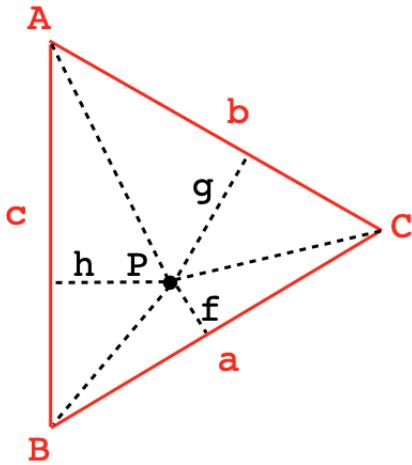
This states that for any internal point  $P$  of an *equilateral* triangle the sum of the perpendicular distances from the sides is a constant, independent of the position of  $P$  ([Fig. 145](#)).



**Fig. 145** Viviani's theorem: in an equilateral triangle,  $h_1 + h_2 + h_3 = \text{constant}$ .

Rather than letting  $P$  be a special point, it can be anywhere inside the triangle.  $P$  is the opposite of a special point, it is completely general.

However, we are also given that the triangle is equilateral.



Note that  $AP$  is not the continuation of the vertical down to side  $a$ . If you look closely, there is a kink, a change of direction, at  $P$ . None of the dotted lines is parallel to any other line.

The area of  $\triangle APB = hc/2$ , and so on for the sub-triangles, so the area of the whole triangle is

$$2\Delta = hc + fa + gb$$

But since the triangle is equilateral,  $a = b = c = s$ .

$$2\Delta = s(f + g + h)$$

$$\frac{2\Delta}{s} = f + g + h$$

From the properties of equilateral triangles, we also know that for a triangle with sides of unit length, the altitude has length  $\sqrt{3}/2$ , so twice the area  $2\Delta = \sqrt{3}/2$ ,

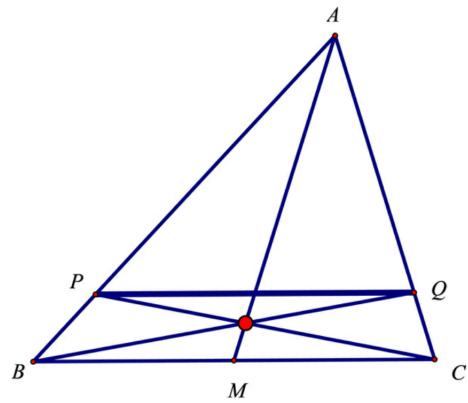
and then finally

$$\frac{\sqrt{3}}{2} = f + g + h$$

and that is true no matter where  $P$  is placed inside this equilateral triangle.

## problem

Draw  $PQ$  parallel to  $BC$ , then draw  $CP$  and  $BQ$ . If  $AM$  is the median of  $\triangle ABC$ , these lines are concurrent.



By (parts of) similar triangles

$$\frac{CQ}{QA} = \frac{PB}{AP}$$

$$\frac{CQ}{QA} \cdot \frac{AP}{PB} = 1$$

since  $BM = MC$

$$\frac{CQ}{QA} \cdot \frac{AP}{PB} \cdot \frac{BM}{MC} = 1$$

By the converse of Ceva's theorem, these lines are concurrent.

# Chapter 45

## Pythagoras by area

### Einstein

Late in life, Albert Einstein wrote that he had two experiences as a youth that influenced him tremendously. The first was when he received a compass as a child and was intrigued by the question of what "force" made the needle turn to the north.

The second was that, while studying geometry at about age 12, he had developed on his own a proof of the Pythagorean theorem. He said the achievement had a profound effect on him, illustrating the power of pure thought.

Strogatz says that the proof relied in a simple way on symmetry.

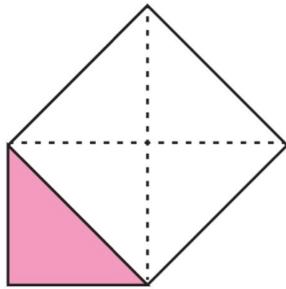
<https://www.newyorker.com/tech/annals-of-technology/einsteins-first-proof-pythagorean-theorem>

### areas in proportion

We will prove the part that Strogatz says was obvious to Einstein from symmetry, which he may have assumed. The idea is that for two similar right triangles, the area of each triangle is in a constant ratio to the product on any two sides, including the square on any one side.

So the area is proportional to the hypotenuse squared.

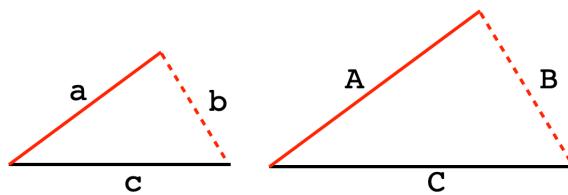
This is easy to see for an isosceles right triangle where we can calculate simply that  $k = 1/4$ .



But it is true for similar right triangles in general.

*Proof.*

We assume that the sides of similar triangles are in proportion. This fundamental proof has been given elsewhere.



$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = k$$

For right triangles, twice the area is the product of the two sides, hence the ratio of the areas is

$$\frac{ab}{AB} = \frac{a}{A} \cdot \frac{b}{B} = k^2$$

But  $k = c/C$  so

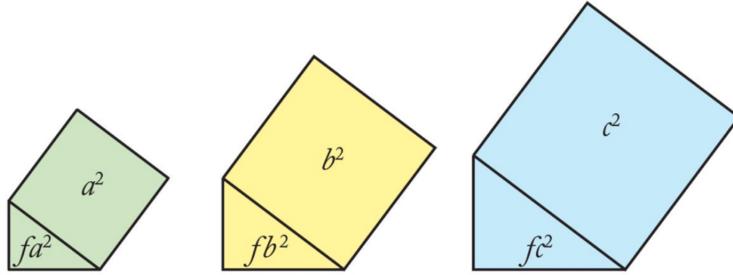
$$\frac{ab}{AB} = \frac{c^2}{C^2}$$

which can be rearranged to give

$$\frac{ab}{c^2} = \frac{AB}{C^2}$$

We conclude that the area of each right triangle is in a constant ratio to the square on any side, including the hypotenuse.

□



## Einstein's proof

Given this symmetry, here is what is thought to be Einstein's proof of the Pythagorean theorem.

*Proof.*

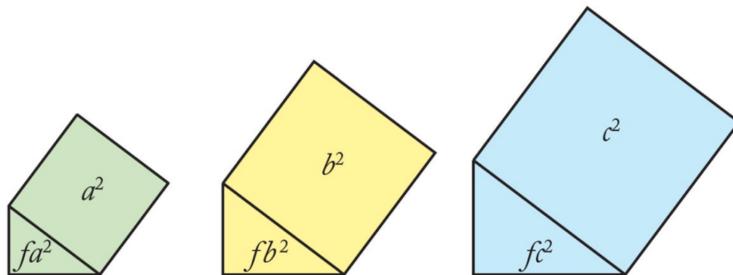
Let the constant of proportionality between area and hypotenuse squared be  $f$ .

Form two smaller triangles with hypotenuse of lengths  $a$  and  $b$ . These are the two triangles formed by dropping an altitude to the hypotenuse in the original right triangle.

The two smaller triangles have areas  $fa^2$  and  $fb^2$  but they add to give the larger one so:

$$fa^2 + fb^2 = fc^2$$

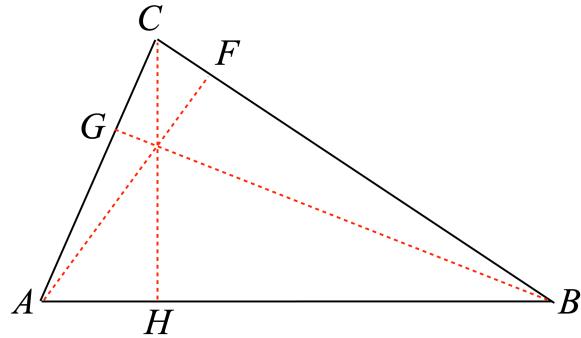
Divide by  $f$  and we're done.



□

We have done the arithmetic to prove this relationship, but an appeal to symmetry abstracts away the underlying arithmetic of ratios. A true believer would simply write the last equation and then divide by  $f$ .

### extension of Pythagoras



Draw the altitudes in a triangle such as  $\triangle ABC$ . We can form pairs of similar triangles by sharing one of the vertex angles. For example  $\triangle AGB \sim \triangle AHC$  and  $\triangle BFA \sim \triangle BHC$ . Form one ratio for each pair:

$$\frac{AG}{AB} = \frac{AH}{AC} \quad \frac{BF}{AB} = \frac{BH}{BC}$$

Be careful to pick the side *not* opposite the shared angle.

Cross multiply:

$$AG \cdot AC = AB \cdot AH$$

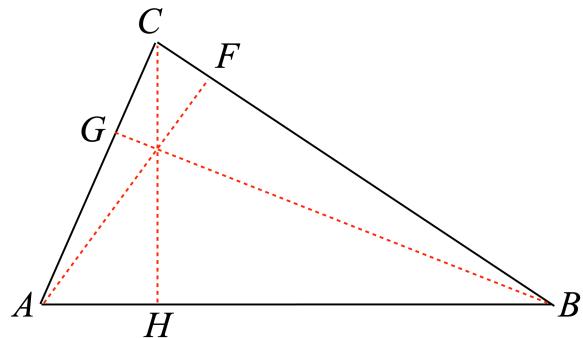
$$BF \cdot BC = AB \cdot BH$$

and add:

$$AG \cdot AC + BF \cdot BC = AB \cdot (AH + BH)$$

A small rearrangement gives a general extension of the Pythagorean Theorem.

$$AG \cdot AC + BF \cdot BC = AB \cdot AB$$



Now let the angle at vertex  $C$  become a right angle.  $AG \rightarrow AC$  and  $BF \rightarrow BC$  so

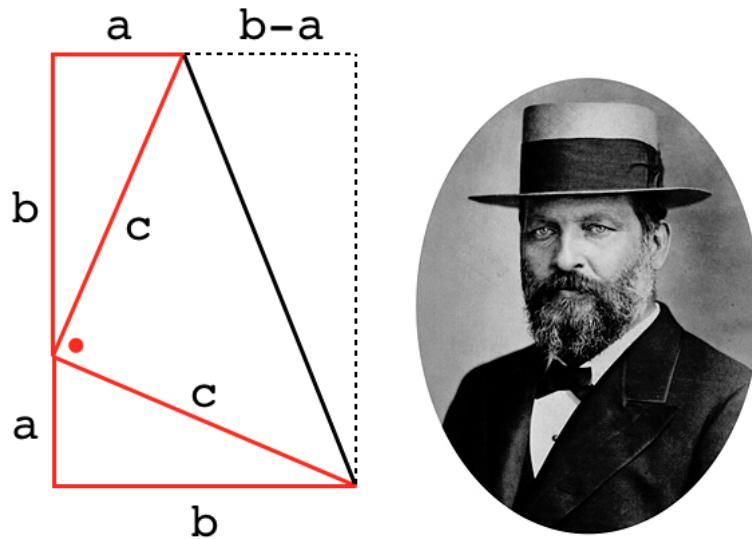
$$AC \cdot AC + BC \cdot BC = AB \cdot AB$$

□

It would be worth thinking about whether this proof extends to the case of an obtuse triangle. We leave that as an exercise.

### Garfield's proof

Here is a proof by a future President of the United States, James A. Garfield. (He was a congressman at the time).



*Proof.*

Draw a right triangle with sides  $a, b$  and  $c$ , and a second, rotated copy as shown. The angles opposite sides  $a$  and  $b$  are complementary angles. So the angle marked with a dot is a right angle, and the triangle with sides labeled  $c$  is a right triangle.

The area of the entire quadrilateral is the product of the left side ( $a + b$ ) and the *average* of  $a$  and  $b$  (top and bottom). This can be seen intuitively.

The halfway point of the solid red line has horizontal dimension  $(a + b)/2$ . Hence

$$A = (a + b) \cdot \frac{1}{2}(a + b)$$

If you're worried about that argument, just subtract the area of the triangle with two dotted sides from the quadrilateral that includes it:

$$\begin{aligned} A &= (a + b)b - \frac{(a + b)(b - a)}{2} \\ &= (a + b)\left(b - \frac{b}{2} + \frac{a}{2}\right) \\ &= (a + b) \cdot \frac{1}{2}(a + b) \end{aligned}$$

which is just what we said.

So now:

$$A = \frac{a^2}{2} + ab + \frac{b^2}{2}$$

But we can also calculate the area of the quadrilateral as the sum of the three triangles:

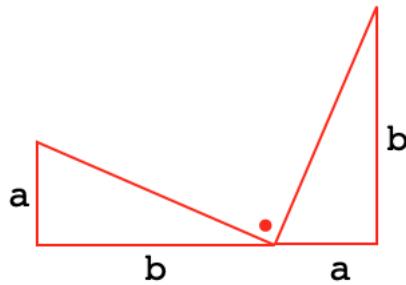
$$A = \frac{ab}{2} + \frac{ab}{2} + \frac{c^2}{2}$$

Equate the two and the result follows almost immediately.

□

## product of slopes

Let's take Garfield's basic figure and turn it sideways.



As we said, since the triangles are right triangles, the angle marked with a red dot is also a right angle.

Later, in analytical geometry, we will define the slope of a line as *rise over run*. So, for example, the slope of the hypotenuse of the right-hand triangle is  $b/a$ .

In a similar way, the slope of the hypotenuse of the left-hand triangle is  $-a/b$ . We think of the "run" of a line as going from left to right. This line heads down as we go to the right, hence the minus sign.

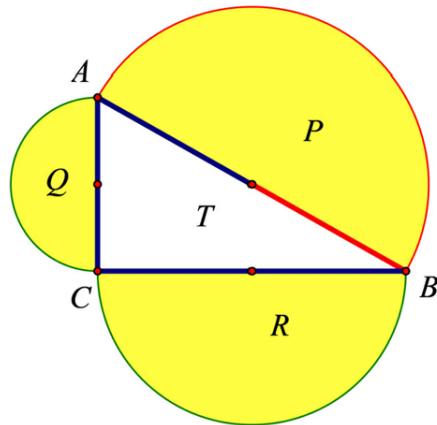
So then the product of slopes is

$$\frac{-a}{b} \cdot \frac{b}{a} = -1$$

This is a proof that the product of the slopes of two line segments that meet at a right angle is equal to  $-1$ .

## **lunes**

Anything else that goes like the square of the side has the same relationship:



**Figure 5.12.**

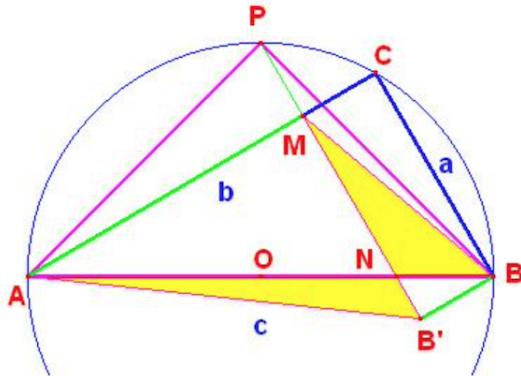
The areas add:  $Q + R = P$ .

These semi-circular areas are called lunes.

### fancy proof

Tuân extended the broken chord theorem of Archimedes to a proof of the Pythagorean theorem in a very clever way.

Here is a diagram.



We start with two right triangles ( $AB$  is a diameter of the circle). One of the triangles,  $\triangle APB$ , is isosceles.

The sides of  $\triangle ABC$  are labeled as  $a, b$  and  $c$ , opposite the corresponding vertices. Side  $BC$  has length  $a$ .

$PM$  is drawn perpendicular to  $AC$ . By the broken chord theorem,

$$AM = MC + BC$$

Twice that is

$$AM + MC + BC = AC + BC = b + a$$

so

$$AM = \frac{b + a}{2}$$

while

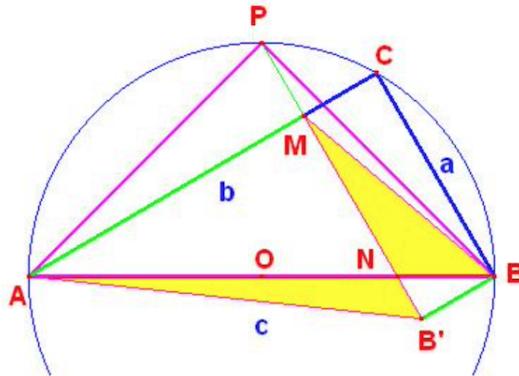
$$\begin{aligned} MC &= AM - BC \\ &= \frac{b + a}{2} - a = \frac{b - a}{2} \end{aligned}$$

$PM$  is extended to meet the diagonal at  $N$  and past it to  $B'$ .  $B'$  is chosen so that  $B'BCM$  is a rectangle. Thus side  $MB'$  is equal to  $BC$  and so to  $a$ .

We make two preliminary claims. The first is that  $\triangle PMC$  is a right *isosceles* triangle. Let us accept that provisionally.

$$PM = MC = \frac{b - a}{2}$$

The second is that the areas of the two triangles shaded yellow are equal.



We reason as follows. Add  $\triangle NB'B$  to both.

$\triangle AB'B$  and  $\triangle MB'B$  have the same base  $BB'$ , and the opposing vertices  $A$  and  $M$  both lie on  $AC$ , which is parallel to the base  $B'B$ . Hence the two triangles have the same altitude, namely,  $a$ , so they have the same area.

By subtraction of  $(\triangle B'BN)$  we obtain:

$$(\triangle AB'N) = (\triangle MNB)$$

Triangle area is indicated by the parentheses.

Now we find the area of  $\triangle APB$  in two different ways.

The first is  $c^2/4$ , since it is one-quarter of a square with sides  $c$ .

The second way is as the sum of smaller triangles:

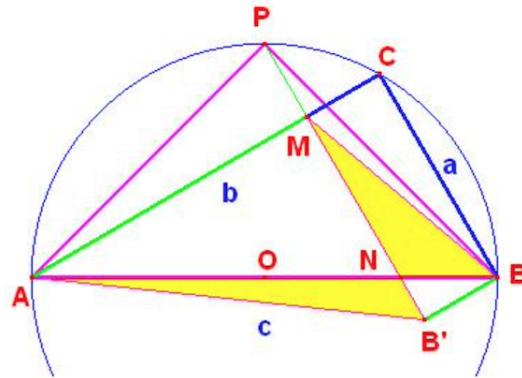
$$(\triangle APM) + (\triangle PMB) + (\triangle AMN) + (\triangle MNB)$$

Since  $(\triangle AB'N) = (\triangle MNB)$ :

$$(\triangle APM) + (\triangle PMB) + (\triangle AMN) + (\triangle AB'N)$$

And since  $(\triangle AMN) + (\triangle AB'N) = (\triangle AMB')$ :

$$(\triangle APM) + (\triangle PMB) + (\triangle AMB')$$



So then the areas are

$$(\triangle APM) = \frac{1}{2} \cdot AM \cdot MC = \frac{1}{2} \cdot \frac{b+a}{2} \cdot \frac{b-a}{2}$$

$$(\triangle PMB) = \frac{1}{2} \cdot PM \cdot MC = \frac{1}{2} \cdot \frac{(b-a)}{2} \cdot \frac{(b-a)}{2}$$

$$(\triangle AMB') = \frac{1}{2} \cdot AM \cdot MB' = \frac{1}{2} \cdot a \cdot \frac{b+a}{2}$$

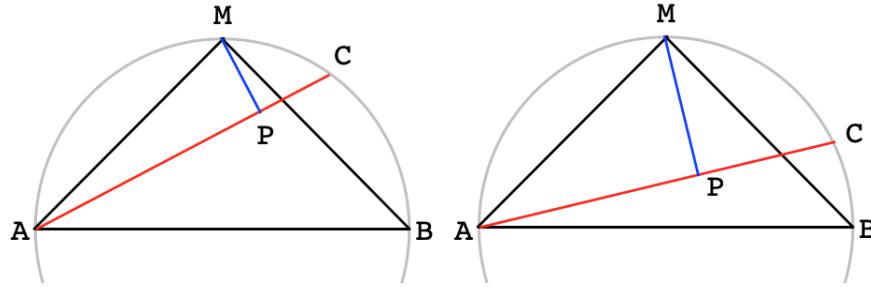
We compute 8 times the sum, so as not to have to deal with fractions:

$$(b^2 - a^2) + (b^2 - 2ab + a^2) + (2ab + 2a^2) = 2b^2 + 2a^2$$

Previously, we calculated the area as  $c^2/4$ , and 8 times that is  $2c^2$ . The result follows immediately.

### last step

However, the proof is not yet complete. We must show that  $\triangle PMC$  is isosceles. Simplifying the figure:

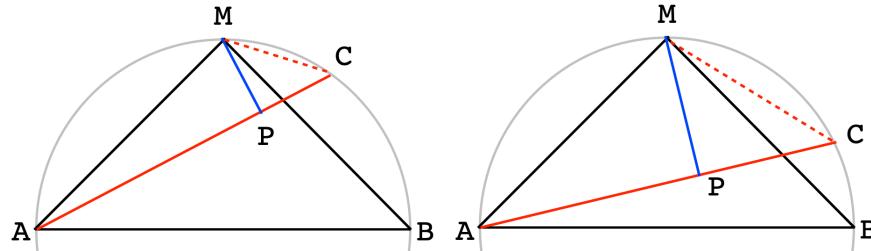


Let  $AB$  be a diameter of the circle and  $\triangle AMB$  isosceles. Let  $C$  be any point on the perimeter, with  $MP \perp AC$ . Then, we claim that  $MP = PC$ .

It seems reasonable. If  $C \rightarrow B$ , the  $P$  becomes the origin and the statement is true, while if  $C \rightarrow M$ , both vanish. I spent some time fooling around with similar triangles before insight came.

*Proof.*

Connect the two vertices by drawing  $MC$ .



Clearly  $\angle MCA$  is one-half of a right angle since it intercepts the same arc as  $\angle ABM$ , by the inscribed angle theorem. Since  $\angle MPC$  is right, it follows that  $\triangle PMC$  is

isosceles (by complementary angles) and so  $MP = PC$  (by the converse of the isosceles triangle theorem).

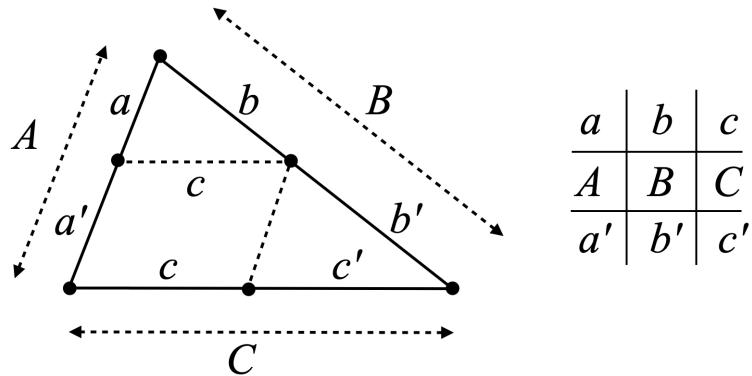
□

Notice how we incorporate the information that  $\triangle AMB$  is isosceles.

# Chapter 46

## Ratio Boxes

In this short chapter, we will look at a device which, at least for now, I'm going to call *ratio boxes*. Here's the idea:



We start with two similar triangles,  $\triangle abc$  nested inside  $\triangle ABC$ .

One part of the definition is that the sides have equal ratios. For example,  $a : A = b : B$ , also written as

$$\frac{a}{A} = \frac{b}{B}$$

Now, you can look at the figure and take away that result, easily enough. But sometimes you need more than one relationship, and if the vertices have the labels, then it's more complicated.

You may notice that there are two similar triangles but nine entries in the box. The

reason is the following:

$$\begin{aligned}\frac{a}{b} &= \frac{A}{B} = \frac{a+a'}{b+b'} \\ \frac{a+a'}{a} &= \frac{b+b'}{b} \\ \frac{a'}{a} &= \frac{b'}{b} \\ \frac{a'}{b'} &= \frac{a}{b} = \frac{A}{B}\end{aligned}$$

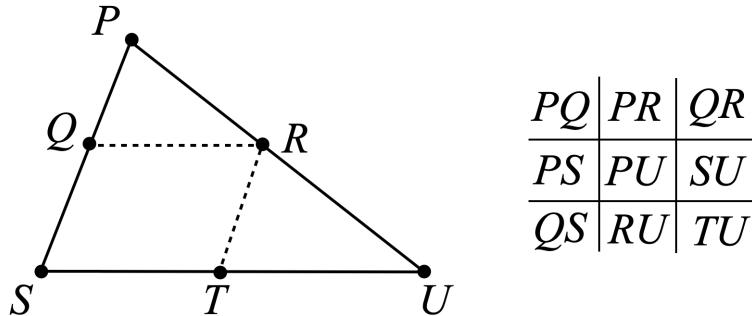
I like to write the sides that are in proportion as shown in the rectangular grid.

$a$	$b$	$c$
$A$	$B$	$C$
$a'$	$b'$	$c'$

Orient it as you like. I usually proceed from the shortest side to the longest side. Sometimes that is hard to see so we start matching sides with the angle opposite.

But however, you do it, there are nine sides or differences of sides. Here the ratios *within* triangles go across, and the ones *between* triangles go down.

The next example uses the traditional Greek notation (well, with Roman letters).



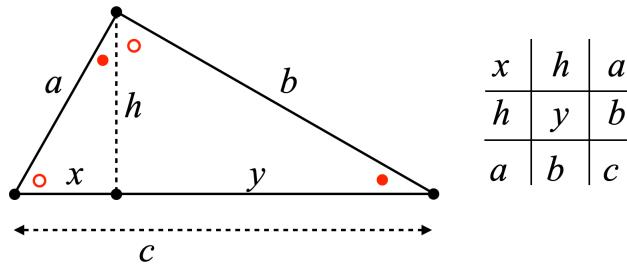
This is exactly the same as before, except the sides are labeled by the flanking vertices. It is fairly easy to go through a figure and make such a box.

The value is this: any four entries that are in the shape of a rectangle form a valid ratio. For example:  $PQ : PS = QR : SU$ . You can also see immediately that  $PQ \cdot SU = PS \cdot QR$ .

$PQ$	$PR$	$QR$
$PS$	$PU$	$SU$
$QS$	$RU$	$TU$

If you have three picked you know the fourth. If you have two picked, and they are not related horizontally or vertically (like  $PQ$  and  $SU$ ), then you obtain the other two partners immediately.

### right triangles



As you know, if you draw the (one) altitude in a right triangle, it forms two smaller similar triangles. The reason is complementary angles, as shown by the red dots marking equal angles.

Say we want a proof of Pythagoras's Theorem. We need  $a^2$  and  $b^2$ . They jump right out of the box:

$$a^2 = xc$$

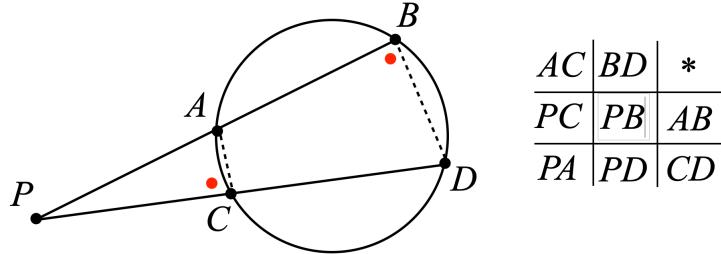
$$b^2 = yc$$

$$a^2 + b^2 = xc + yc = c^2$$

□

We also see, without ever referring back to the figure, that  $h^2 = xy$ .  $h$  is the *geometric mean* of  $x$  and  $y$ .

## secants



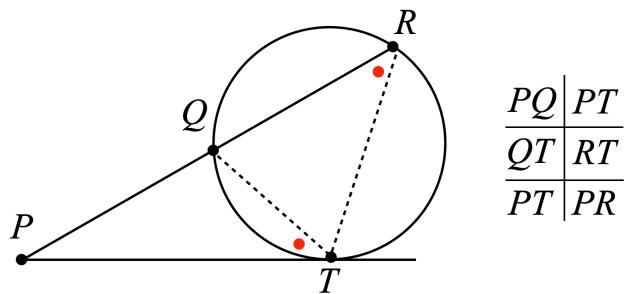
$PAB$  and  $PCD$  are secants of this circle. The red dots show equal angles. The reason is that  $\angle B$  and  $\angle ACD$  are supplementary, because they are opposite angles of a *cyclic quadrilateral*. Together they correspond to a complete arc of the circle.

But  $\angle PCA$  and  $\angle ACD$  are supplementary too. So the marked angles are equal. And since two pairs of angles are equal, we have similar triangles:  $\triangle PCA \sim \triangle PBD$ . I try to remember to write the vertices in the order of similarity, so for example:  $PC : CA = PB : BD$ .

Go through each triangle and write the sides that have equal angles opposite. The famous result is:  $PA \cdot PB = PC \cdot PD$ .

I have left out the entry in the starred box, because it looks funny. It is  $BD - AC$ .

## tangent-secant



Here we have a secant  $PQR$  and a tangent,  $PT$ . The angle that the tangent makes with a chord is easily shown to be equal to any inscribed angle subtended by that chord. That accounts for the red dots.

Again we have similar triangles, this time, one is nested inside another, so there are

only six terms in the box. Here I followed the angles, first red then  $\angle P$ .

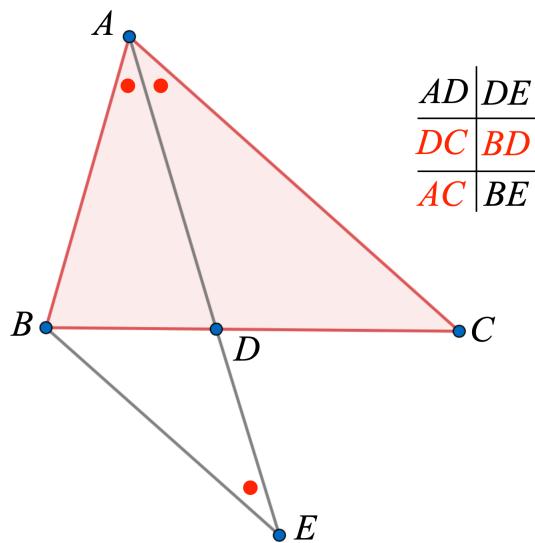
The famous result is the Secant Tangent Theorem (or Tangent Secant Theorem):  $PQ \cdot PR = PT^2$ .

It writes itself.

## Angle Bisector Theorem

We revisit this theorem, from [here](#). We are given that the angle at  $A$  ( $\angle BAC$ ), is bisected.

So we draw a line segment  $BE$  parallel to side  $AC$  and extend the bisector to meet it.



Because of the parallel lines,  $\angle BED$  gets a red dot as well. And that means  $\triangle BED \sim \triangle DAC$ , which gives the ratio box in the figure. Here we want to be careful to match the sides with equal angles opposite.

Highlight the segments we think might be interesting. Finally, notice that  $\triangle ABE$  is isosceles, so  $AB = BE$ . We obtain:

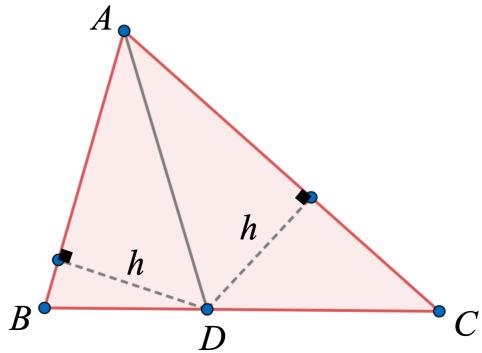
$$\frac{AC}{DC} = \frac{BE}{BD} = \frac{AB}{BD}$$

□

The sides and divisions of the base are in equal proportion. This can also be written as

$$\frac{AB}{AC} = \frac{BD}{DC}$$

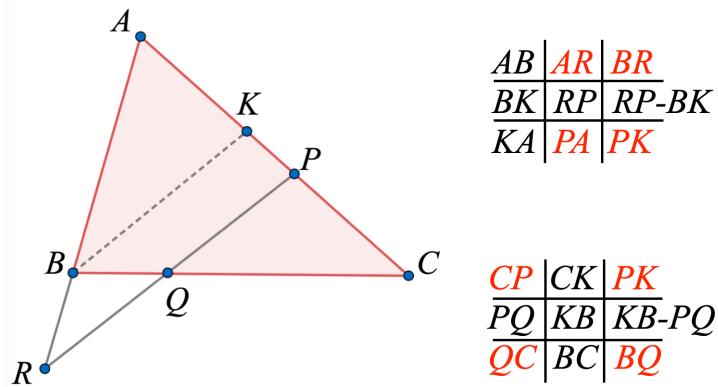
This one can also be done by areas (next figure).  $\triangle ABD$  and  $\triangle ACD$  have the same altitude  $h$ . So the areas are in the ratio  $AB : AC$ .



But looked at another way, they have bases  $BD$  and  $DC$  on the same parallel, so they have the same height drawn to vertex  $A$ . Hence, the areas are in the ratio  $BD : DC$ . Thus

$$\frac{AB}{AC} = \frac{BD}{DC}$$

### Menelaus's Theorem



The next example is one where the method really proves its worth. We prove Menelaus's theorem (we will look at it in more detail later [here](#)).

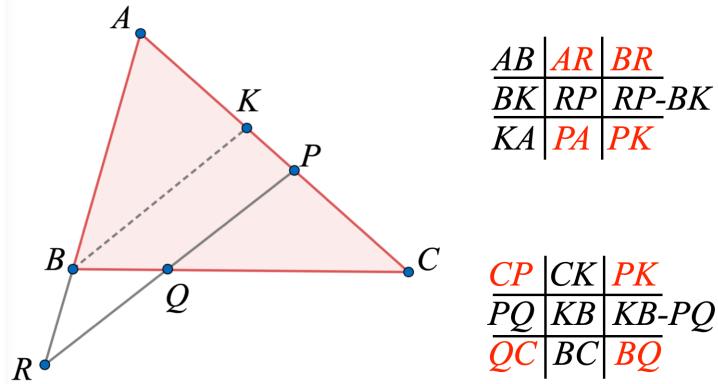
In  $\triangle ABC$  we have the transversal  $RQP$ . Draw an internal line segment  $BK$  parallel to the transversal.

Here, it is usual to label vertices rather than sides.

For the moment, each vertex is listed in order it is encountered going around a triangle, starting at the same vertex each time (since we have parallel bases).

The values in the right-hand columns of the two ratio boxes are all subtractions, but for one there is no labeled segment to equate to the result.

The segments needed for the proof are shown in red.



It writes itself! Since

$$\frac{PK}{BR} = \frac{PA}{AR}$$

$$PK = BR \cdot \frac{PA}{AR}$$

Similarly

$$\frac{PK}{CP} = \frac{BQ}{QC}$$

$$PK = CP \cdot \frac{BQ}{QC}$$

Combining the two results:

$$CP \cdot \frac{BQ}{QC} = PK = BR \cdot \frac{PA}{AR}$$

$$\frac{BQ}{QC} \cdot \frac{CP}{PA} \cdot \frac{AR}{BR} = 1$$

□

Finally, it is usual to view each segment as having a direction or sign. Namely

$$BQ \Rightarrow QC \Rightarrow CP \Rightarrow PA \Rightarrow AR \Rightarrow RB$$

Since  $RB$  and  $BR$  are in opposite directions,  $RB = -BR$ . Substitute  $RB$  for  $BR$  and adjust the sign of the result accordingly.

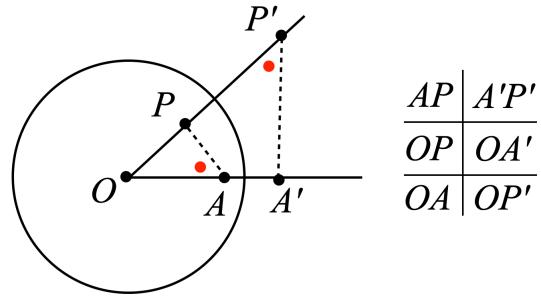
$$\frac{BQ}{QC} \cdot \frac{CP}{PA} \cdot \frac{AR}{RB} = -1$$

## **inversion**

Some good examples involve a topic called *Inversive Geometry*. Given a circle of radius  $r$ , any point (except the origin), like  $A$ , can be transformed into its *image under an inverse transformation*, resulting in  $A'$ . Draw the line from  $O$  through  $A$  and calculate the length of  $OA'$  by this rule:

$$OA \cdot OA' = r^2$$

Since  $P$  and  $P'$  are related by the same transformation, we have  $OA \cdot OA' = OP \cdot OP'$ . Since  $\angle O$  is shared, we again have similar triangles:  $\triangle OAP \sim \triangle OP'A'$ .



You can read the rule right off the box.

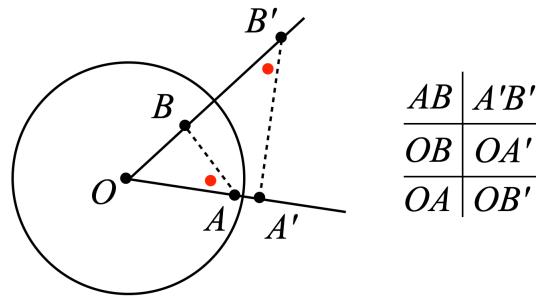
And if  $OAA' \perp A'P'$  then the small  $\triangle OAP$  is a right triangle. By the converse of Thales' Theorem, we can draw a circle with diameter  $OA$  and  $P$  will lie on the circle.

This is true for *any* point on  $A'P'$ . Say  $Q'$  is on  $A'P'$ , then if  $Q$  is the image of  $Q'$ ,  $Q$  lies on the same circle, the one with radius  $OA$ .

As a result, the *image* of any point on the line  $A'P'$  lies on the circle with radius  $OA$ . We say that the image of the line is the circle (and it goes through  $O$ ).

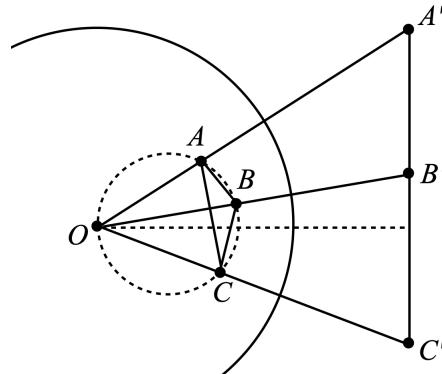
The transformation is an *involution*, so the converse is also true: the image of a circle through  $O$  is a line not through  $O$ .

A second example from inversion is more general.



You can read the rule right off the box, again. What this means is that if we take any two points  $A$  and  $B$  and their images  $A'$  and  $B'$ , we get similar triangles.

## Ptolemy's theorem



We will prove a famous theorem. All we need are some boxes and the previous result.

$AB$	$A'B'$	$AC$	$A'C'$	$BC$	$B'C'$
$OB$	$OA'$	$OC$	$OA'$	$OC$	$OB'$
$OA$	$OB'$	$OA$	$OC'$	$OB$	$OC'$

I didn't even have to think about it. I just copied the box from before, and substituted  $C$  for  $B$  in the middle, then  $B$  for  $A$  on the right.

We see that that the transformed circle is a line with  $A'B' + B'C' = A'C'$ . We can find expressions for those lengths in our boxes. We know that eventually we will want things like  $AB$ ,  $BC$  and  $AC$ , as well as  $OB$ , etc.

I get

$$A'B' = \frac{AB \cdot OA'}{OB} \quad A'C' = \frac{AC \cdot OA'}{OC} \quad B'C' = \frac{BC \cdot OB'}{OC}$$

Again, these may be obtained mechanically, by straight substitution. The one for  $A'C'$  first, by substituting  $C'$  for  $B'$  in the one on the left. Wash, rinse, repeat.

Substitute and then clear the denominator:

$$AB \cdot OA' \cdot OC + BC \cdot OB' \cdot OB = AC \cdot OA' \cdot OB$$

Divide by  $OA'$ :

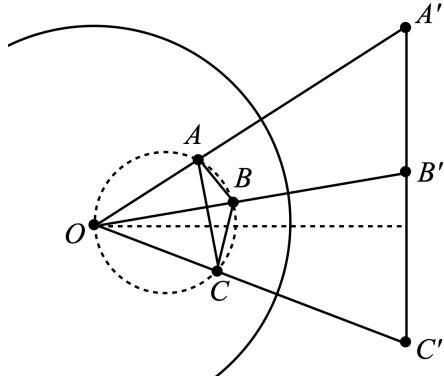
$$AB \cdot OC + BC \cdot \frac{OB'}{OA'} \cdot OB = AC \cdot OB$$

Can you find the four entries we need in the first box on the left and complete the proof?

$AB$	$A'B'$	$AC$	$A'C'$	$BC$	$B'C'$
$OB$	$OA'$	$OC$	$OA'$	$OC$	$OB'$
$OA$	$OB'$	$OA$	$OC'$	$OB$	$OC'$

The final result is

$$AB \cdot OC + BC \cdot OA = AC \cdot OB$$



We have four vertices on a circle, another cyclic quadrilateral. Take the product of opposing sides, add them, and obtain the product of the two diagonals.

This is Ptolemy's Theorem.

$$AB \cdot OC + BC \cdot OA = AC \cdot OB$$

□

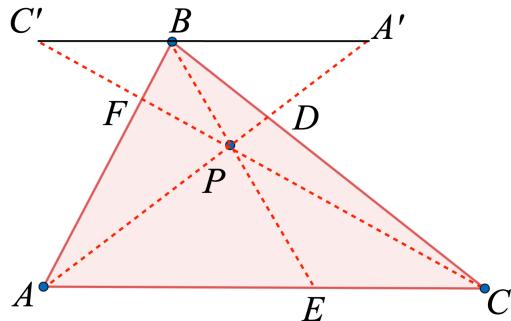
### Ceva's Theorem by parallel lines

Here is another proof of [Ceva's theorem](#).

*Proof.*

Consider  $\triangle ABC$ . Draw  $AD$ ,  $BE$  and  $CF$  concurrent at  $P$ . Draw a line through  $B$  parallel to  $AEC$  and extend  $APDA'$  and  $CPFC'$ .

We have five pairs of similar triangles.



Three with vertical angles at  $P$ :

$$(1) \triangle APC \sim \triangle A'PC' \Rightarrow \frac{AP}{A'P} = \frac{PC}{PC'} = \frac{CA}{C'A'}$$

$$(2) \triangle APE \sim \triangle A'PB \Rightarrow \frac{AP}{A'P} = \frac{PE}{PB} = \frac{EA}{BA'}$$

$$(3) \triangle CPE \sim \triangle C'PB \Rightarrow \frac{CP}{C'P} = \frac{PE}{PB} = \frac{EC}{BC'}$$

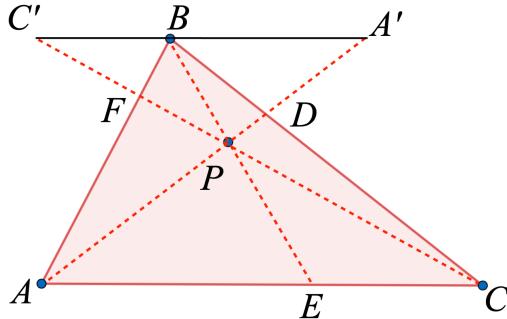
And two more with vertical angles at  $D$  and  $F$ :

$$(4) \triangle ADC \sim \triangle A'DB \Rightarrow \frac{AD}{A'D} = \frac{DC}{DB} = \frac{CA}{BA'}$$

$$(5) \triangle AFC \sim \triangle BFC' \Rightarrow \frac{AF}{BF} = \frac{FC}{FC'} = \frac{CA}{C'B}$$

It is helpful to remember that when similar triangles are formed by an “X” containing vertical angles lying between two parallel lines, corresponding vertices go in mirror image order, and the point at the “X” is a corresponding point in each triangle.

We label the triangles so that corresponding vertices match. Also, do not worry yet about the order in each line segment so for example,  $AB = BA$ .



If the triangles are labeled carefully it is easy to make the ratio boxes:

$$\begin{array}{|c|c|} \hline AP & A'P \\ \hline PC & PC' \\ \hline CA & C'A' \\ \hline \end{array}$$

(1)

$$\begin{array}{|c|c|} \hline AP & A'P \\ \hline PE & PB \\ \hline EA & BA' \\ \hline \end{array}$$

(2)

$$\begin{array}{|c|c|} \hline CP & C'P \\ \hline PE & PB \\ \hline EC & BC' \\ \hline \end{array}$$

(3)

$$\begin{array}{|c|c|} \hline AD & A'D \\ \hline DC & DB \\ \hline CA & BA' \\ \hline \end{array}$$

(4)

$$\begin{array}{|c|c|} \hline AF & BF \\ \hline FC & FC' \\ \hline CA & C'B \\ \hline \end{array}$$

(5)

Now look for the ratios we need namely:

$$\frac{CD}{DB} = \frac{CA}{BA'} \quad \frac{BF}{FA} = \frac{C'B}{CA}$$

We are encouraged by the fact that  $CA$  cancels in the product.

$$\frac{CD}{DB} \cdot \frac{BF}{FA} = \frac{C'B}{BA'}$$

The third ratio is  $AE/EC$ .  $AE$  and  $EC$  do not occur in the same box (they do not lie in similar triangles), but in (2) and (3) we find:

$$AE = A'B \cdot \frac{PE}{PB} \quad EC = BC' \cdot \frac{PE}{PB}$$

so the ratio is just

$$\frac{AE}{EC} = \frac{A'B}{BC'}$$

which is exactly what we need to cancel!

$$\frac{CD}{DB} \cdot \frac{BF}{FA} \cdot \frac{AE}{EC} = \frac{C'B}{BA'} \cdot \frac{A'B}{BC'} = 1$$

□

# Chapter 47

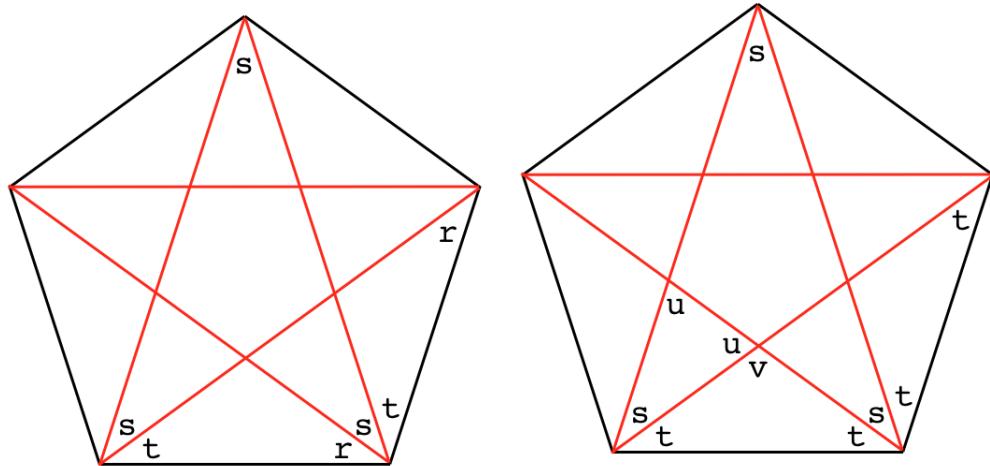
## Pentagon

### pentagons

In this chapter we explore some properties of a regular pentagon. A pentagon has 5 sides, and a regular polygon has all sides equal.

The regular pentagon has five-fold rotational symmetry. Draw all of the internal chords of the figure and label a few angles.

By rotational symmetry each of the five vertices of the pentagon has the same three components, the central one labeled  $s$ , and two flanking ones  $r$  and  $t$ .



Mirror image symmetry shows that  $r = t$ , by reflection across the midline. Alter-

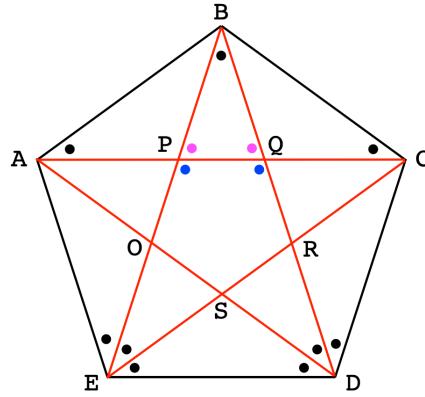
natively, we may invoke the properties of isosceles triangles, using two sides of the pentagon. Hence we relabel the diagram (right panel).

Next, we compute two triangle sums:

$$3s + 2t = 4t + s$$

$$2s = 2t$$

Hence,  $s = t$ . Relabel the diagram with black dots for the angles previously marked  $s$ ,  $r$  or  $t$ .



We observe that five copies of  $s$  add up to one triangle (i.e.  $\pi$ ).

From here, we might proceed by extending the base  $ED$  past point  $D$ . The newly formed angle will be supplementary to three copies of  $s$ , and therefore we have alternate interior angles when comparing it with  $\angle ACD$ .

It follows that  $AC \parallel ED$ .

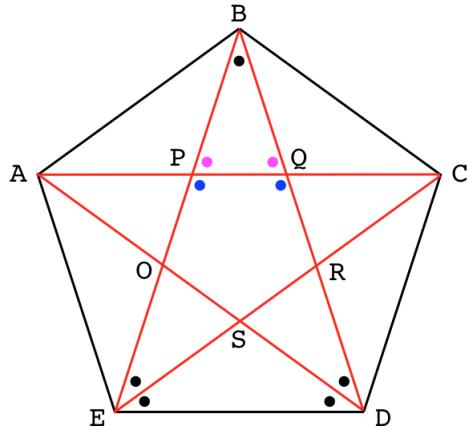
Therefore the angle with the magenta dot is worth  $2s$  and the angle with the blue dot is worth  $3s$ .

Alternatively, compute two triangle sums:

$$5s = v + 2s, \quad v = 3s$$

$$5s = 2u + s, \quad u = 2s$$

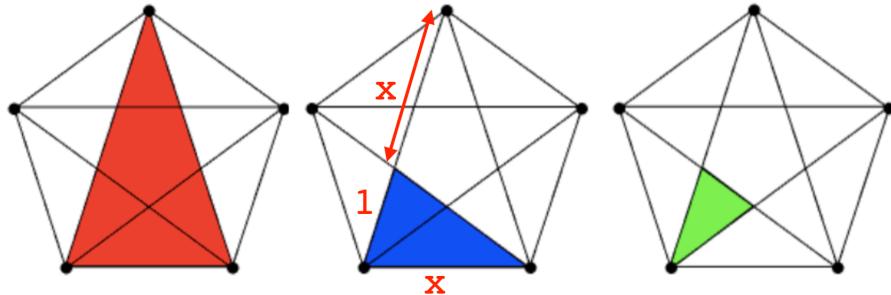
Since  $v = 3s$ , its measure is the same as the vertex angle of the pentagon. Thus the inner figure is also a regular pentagon.



Our drawing is filled with regular parallelograms, with four sides equal (*proof*: use ASA to show that, e.g.  $\triangle CEP \cong \triangle CDE$ . So we have two pairs of opposing sides equal).

One can draw two types of isosceles triangles using the chords and sides of the pentagon. One is tall and skinny, with  $2s$  at each of the base angles, while the other type is short and fat, with  $s$  for the base angles.

Here are three examples of the tall skinny type:

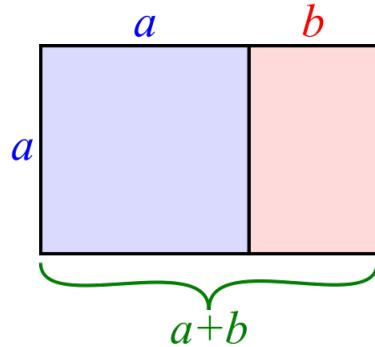


If we look at the medium-sized blue (tall, skinny) triangle, we can scale them so that the base is equal to 1 and then label the long sides as  $x$ . The ratio of the long side to the base is  $x : 1$  or just  $x$

But these triangles are all similar. In particular, the large red ones have long sides equal to  $x + 1$  and base equal to  $x$ . The ratio of side lengths is the same for similar triangles, so form the ratio  $\phi$  of the base to the long side (red on the left, blue on the right):

$$\frac{x+1}{x} = \frac{x}{1}$$

Does that look familiar? Here is a picture from wikipedia.



We draw a square with sides  $a$  and then extend two parallel sides to make a large rectangle and a small one at the same time. We require the rectangles to have the same ratio of sides.

$$\frac{a+b}{a} = \frac{a}{b}$$

Scale so that  $b = 1$  and change notation, using  $x$  instead of  $a$ :

$$\frac{x+1}{x} = \frac{x}{1}$$

This is our equation from above.

To solve it, rearrange:

$$x^2 - x - 1 = 0$$

Substitute into the quadratic equation:

$$x = \frac{1 \pm \sqrt{5}}{2}$$

We are talking about a length. Noticing that one of the solutions is negative, we ignore it for the moment, and take the positive branch of the square root.

$x$  is usually called  $\phi$ , the famous *golden ratio*:

$$\phi = \frac{1 + \sqrt{5}}{2}$$

Check that  $\phi$  really does solve the equation:

$$\begin{aligned}\phi^2 &= \frac{1 + \sqrt{5}}{2} \cdot \frac{1 + \sqrt{5}}{2} \\ &= \frac{1}{4}(1 + 2\sqrt{5} + 5) \\ &= 1 + \frac{2 + 2\sqrt{5}}{4} = 1 + \phi\end{aligned}$$

That checks.

### Aside on $\phi$ and the Fibonacci sequence

The Fibonacci sequence is defined as  $F_{n+2} = F_{n+1} + F_n$ , starting with 1.

The first ten numbers in the sequence are:

1 1 2 3 5 8 13 21 34 55 ...

Recall that  $\phi^2 = 1 + \phi$ . The powers of  $\phi$  generate an interesting pattern:

$$\begin{aligned}\phi^2 &= \phi \cdot \phi = 1 + \phi \\ \phi^3 &= \phi \cdot \phi^2 = \phi + \phi^2 = 1 + 2\phi \\ \phi^4 &= \phi \cdot \phi^3 = \phi + 2\phi^2 = 2 + 3\phi \\ \phi^5 &= \phi \cdot \phi^4 = 2\phi + 3\phi^2 = 3 + 5\phi\end{aligned}$$

Both the first term and the cofactors generate the elements of the Fibonacci sequence from the powers of  $\phi$

The reason is that  $\phi^n + \phi^{n+1} = \phi^{n+2}$ , which is the same as the definition for the Fibonacci numbers.

Going back to the solution that we left behind, take the negative branch of the square root, and let us call that other solution  $\psi$ .

$$\psi = \frac{1 - \sqrt{5}}{2}$$

If you look closely, you can easily see that

$$\psi + \phi = 1$$

Since  $\psi$  is also a solution of the original equation:

$$\psi^2 = 1 + \psi$$

Furthermore

$$(\phi + \psi)^2 = \phi^2 + 2\phi \cdot \psi + \psi^2$$

Now, the left-hand side is just 1, since  $\phi + \psi = 1$ . Furthermore  $\phi^2 = 1 + \phi$  and  $\psi^2 = 1 + \psi$  so

$$1 = 1 + \phi + 2\phi \cdot \psi + 1 + \psi$$

$$1 = 3 + 2\phi \cdot \psi$$

$$\phi \cdot \psi = -1$$

Thus,  $\psi$  is the negative inverse of  $\phi$ .

$\psi$  comes in handy in the following way. Since  $\psi$  solves our original equation, that means the powers of  $\psi$  are just like the powers of  $\phi$

$$\psi^2 = 1 + \psi$$

$$\psi^3 = 1 + 2\psi$$

$$\psi^4 = 2 + 3\psi$$

$$\psi^5 = 3 + 5\psi$$

So

$$\phi^5 - \psi^5 = 5(\phi - \psi)$$

$$\frac{\phi^5 - \psi^5}{\phi - \psi} = 5$$

5 is the fifth Fibonacci number. If  $F_n$  is the nth Fibonacci number

$$\frac{\phi^n - \psi^n}{\phi - \psi} = F_n$$

This is called Binet's formula. If you work out the denominator you find that it is just  $\sqrt{5}$ .

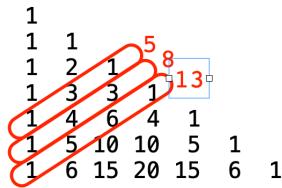
The general equation is

$$F_n = \frac{1}{\sqrt{5}} \cdot (\phi^n - \psi^n)$$

This formula is quite surprising, because the Fibonacci numbers  $F_n$  on the left-hand side are *integers*, and yet the first factor on the right-hand side is the inverse of a square root, which is definitely not an integer or even a rational number.

But it turns out that the differences  $\phi^n - \psi^n$  contain only odd powers of  $\sqrt{5}$ . So after multiplying by  $1/\sqrt{5}$ , we end up only with even powers, which are whole numbers.

In fact, there is a connection between the Fibonacci sequence and Pascal's triangle.



If we let  $f = \sqrt{5}$  then Binet's formula says the  $n$ th Fibonacci number is equal to

$$\left(\frac{1}{2}\right)^n \cdot \frac{1}{f} \cdot [ (1+f)^n - (1-f)^n ]$$

If you do the two binomial expansions, they are the same except that each term of the second one has a factor of  $(-1)^n$ . As a result, the odd powers survive, as twice the value. In the case of  $n = 5$  we would have

$$\left(\frac{1}{2}\right)^5 \cdot \frac{1}{f} \cdot [ 10f + 20f^3 + 2f^5 ]$$

$$\left(\frac{1}{2}\right)^5 \cdot [ 10 + 20f^2 + 2f^4 ]$$

The coefficients of the powers of  $f$  are twice the alternate coefficients in the binomial expansion for  $(1+f)^5$ : 5, 10 and 1. If you work through more examples you'll see there is a cancellation that happens with  $1/2^n$  so that this always results in an integer.

# **Part IX**

## **Addendum**

# Chapter 48

## Author's notes

A central feature of this book is the relentless use of proof. I emphasize the key insight for each, and have tried to make the proofs simple and as easy to follow as possible.

This volume is distinguished from most other texts, since they maintain that a proper proof should be watertight, with each step carefully justified and following closely from the one before. I don't deny that rigor has its proper place in math education, but I also think that this rigidity obscures the core insights. Our purpose here is to view beauty clearly.

We prefer instead to be like the famous mountaineer Ueli Steck. Reach the summit quickly and emphasize the key steps. You should be able to fill in the details if there are loose ends.

Multiple proofs for important theorems are sometimes given, because proof is our stock in trade, and different approaches shed light on how proofs may be found and developed.

Trigonometry comes early. It is often held back for another course, but it is extremely useful to abstract the notion of ratios of sides, so we do it in the first part of the book.

Another distinguishing feature is a set of simple proofs based on scaling of triangles. This happens for the Pythagorean theorem and for Ptolemy's theorem, as well as the sum of angles theorems and then later, a fairly sophisticated theorem of Euler's.

Recently I came across a fantastic book by Acheson, called *The Wonder Book of*

*Geometry*. I helped myself to some of his examples, and now have more than a dozen. Please go find Acheson, and buy it. It's truly magical. In fact, all of his books are wonderful!

A saying attributed to Manaechmus, speaking to Alexander the Great, is that “there is no royal road to geometry”. Others write that this was actually Euclid, speaking to Ptolemy I of Egypt. Since the two sources lived some 700 years after the fact, it is difficult to know.

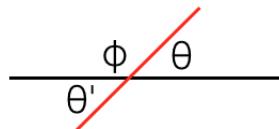
Practically, this means that in learning mathematics you must follow the argument with pencil and paper and work out each step yourself, to your own satisfaction. That is the only way of really learning, and at heart, a principal reason why I wrote this book.

Having read a chapter, see if you can prove the theorems yourself, without looking at the text.

There are a few problems listed in the later chapters, perhaps thirty or more altogether. Most of them have worked out solutions. It is highly recommended that you attempt each problem yourself before reading my answer. Since the crucial point is often to draw an inspired diagram, you must stop reading as soon as the problem is stated!

# Chapter 49

## Angular measure



We can see that obviously

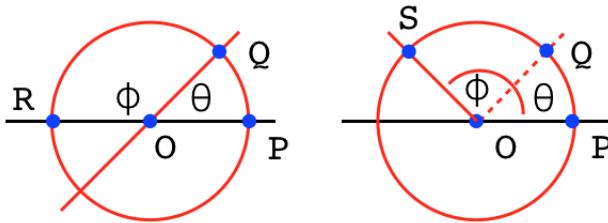
$$\phi > \theta$$

Writing the statement  $\phi > \theta$  is easy, but for this to make sense (what if they are almost equal?), we must have some way of taking the measure of an angle. We can't just rely on a picture.

Our answer is to construct a circle around the central point.

- The measure of the angle is the distance along the circumference between the points where the lines cross the circle.

If that distance along the edge is larger for  $\phi$  than for  $\theta$ , then  $\phi > \theta$ . In the left panel, the arc between  $Q$  and  $R$  (call it arc  $QR$ ) is larger than arc  $PQ$ .



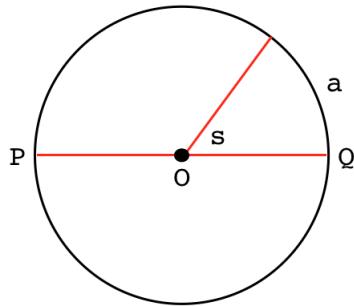
If we lay off the arcs starting from the same point  $P$ , the arc  $PS$  is longer than the arc  $PQ$  (right panel).

We don't need to actually measure the arc itself to do this. Measuring curved lengths is a bit tricky to do.

Instead we can use a standard compass to lay off the linear distance from  $P$  to  $S$  and compare that with the distance from  $P$  to  $Q$ . Since the distance from  $P$  to  $S$  is more than from  $P$  to  $Q$ ,  $\phi > \theta$ .

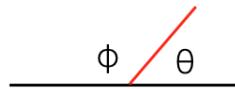
This doesn't work very well as  $S$  approaches the diameter through  $O$  and  $P$ , but we could lay off the distance in two (or more than two) parts and that would be fine. For example, take a unit circle, draw a diameter and the perpendicular bisector to form four right angles. Then bisect each right angle and bisect again. That would divide the circle into 16 arcs of equal length.

No matter how the measurement is to be performed, we *define* the measure of an angle  $s$  to be equal to the arc  $a$  it sweeps out, or is subtended by, in a circle of radius 1, a *unit* circle. Angles are not lengths, but numerically, the measure of the angle is the measure of the arc.



## right angles

- the sum of all the angles on one side of a line (at a given point), is equal to two right angles.



This is simply a matter of symmetry and subtraction. If the sum of all the angles at a point is equal to four right angles, then the sum of the angles on each side of a line through that point is equal to two right angles.

The convention that there are  $360^\circ$  total in a circle dates to the time of the Babylonians (c. 2400 B.C.).

In degrees, a right angle is  $90^\circ$  and two supplementary angles measure  $180^\circ$ .

There is nothing particularly special about using  $90^\circ$  as the measure of a right angle, or  $360^\circ$  for one whole turn. Well, here is one thing: there are *approximately* 360 days in a year, which marks the sun's track across the sky. Another idea is that 360 is special because it has so many factors, which makes it possible to divide up a circle evenly in 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20 . . . 180, 360 parts.

Euclid just talks about angle measurements in terms of right angles. For example, that the sum of two supplementary angles is equal to two right angles.

In his book, *Measurement*, Lockhart adopts the convention that a whole turn is equal to 1.

## radians

We'll just mention here that one whole turn can be defined using a different unit of measure as  $2\pi$  radians, and that convention turns out to be quite important for calculus.

It is based on two ideas: the first, from above, that angular measure is numerically equal to arc length, and second, that in a unit circle, the circumference or total of the arc length is equal to  $2\pi$ .

In calculus, all angles will be in radians. One big reason for that comes in working out the *derivative* of sine and cosine. There, it will be important to consider the

following expression:

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

The notation on the left asks us to consider what happens as  $\theta$  gets close to the value 0, and the rest of it states that the ratio  $\sin \theta / \theta$  is equal to 1. As a result, the derivative of  $\sin \theta$  is simply  $\cos \theta$ .

Well, if you're working in degrees *that's not true*. There's an awkward constant of proportionality. So we work in radians.

It also makes plots prettier, since the sine of  $x$  goes from 0 to 1 as  $x$  goes from 0 to  $\pi/2 \approx 1.57$ . The sine of 1 degree is effectively zero, so that wouldn't look so nice.

# Chapter 50

## Additional constructions

### collapsible compass

We note briefly that there is a restriction in Euclid's *Elements* to a *collapsible* compass, one which loses its setting when lifted from the page. That means that generally, you wouldn't be able to draw two circles of the same radius on different centers.

We get around that restriction by drawing the circles on  $Q$  and  $R$  with the same radius  $QR$ .

We will call a compass that is able to hold its setting, a *standard* compass. Within the first few pages of that book, it is shown how to use a collapsible compass to carry out the very construction we said we couldn't do, namely, construct two circles on  $Q$  and  $R$  with equal radius and that radius not equal to  $QR$  or  $QP$ .

Also, see the video at the url:

<https://www.mathopenref.com/constperpextpoint.html>

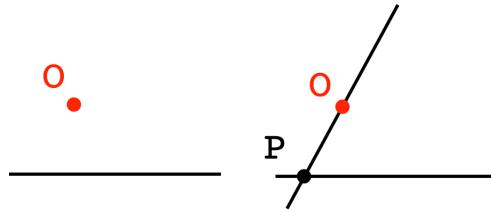
We have skipped that part here.

### Euclid I.31: construct a line parallel to another line

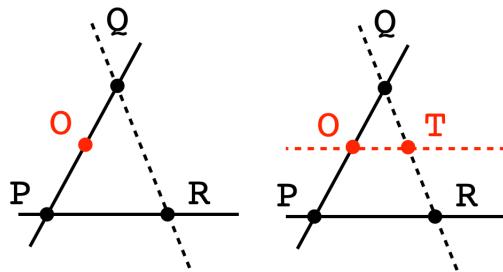
Suppose we are asked to construct a line parallel to a line or line segment, through a given point. We remain true to the Greek ideal, that dividers should not come off the paper.

*Proof.*

First, pick some point on the line segment  $P$ , and draw a line segment through  $OP$ .



Find  $Q$  on the second line such that  $QP > OQ$ .



Now draw the circle with center  $Q$  and radius  $QP$  and, at the intersection with the first line,  $R$ . Draw the line  $QR$ .

Finally, draw the circle with center  $Q$  and radius  $OQ$ , and at the intersection of the circle with the last line, find  $T$ .

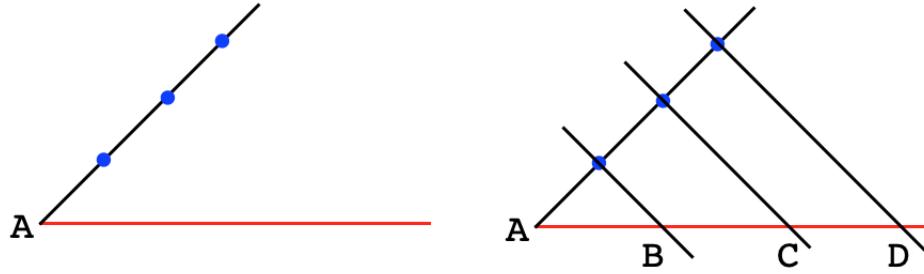
We have that  $OQ = QT$  and  $QP = QR$ . Therefore the base angles of  $\triangle QOT$  and  $\triangle QPR$  are equal and therefore the triangles are similar by AAA.

Therefore the bases  $PR$  and  $OT$  are parallel, by the converse of the alternate interior angles theorem.

□

### Euclid VI.9: division of a line segment into parts

We wish to divide a general segment (in red, below) into an even number of pieces. Suppose that number is three.



Using one end of the target segment, draw any other line, and mark off on that line segments of equal length, using a compass. (Even with a collapsible compass, this can be done sequentially by moving the fixed point).

Then, erect the perpendicular bisector of the black line at each point and extend the bisector to the target red line.

We will have that  $AB = BC = CD$ . Furthermore, since  $AB = BC$ ,  $AB = \frac{1}{2}(AB + BC)$  so  $AC = 2AB$ .

□

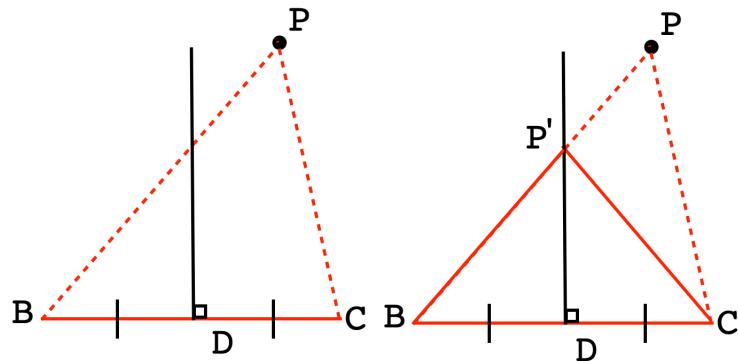
This construction uses properties of *similar* triangles that we have not explained yet. Since the angle at  $A$  is shared and the angle with the black line is always the same, all the triangles have the same shape and their sides are in proportion. We have fixed that proportion as an integer: 1, 2 or 3.

# Chapter 51

## Additional proofs

### perpendicular bisector (converse)

The argument in this section is a little complicated.



Suppose a given point  $P$  does not lie on the vertical bisector. We claim that any such point cannot be equidistant from  $B$  and  $C$ .

*Proof.*

We will use an argument by contradiction. We assume the opposite of the statement we want to prove, and show that it leads to a contradiction and so cannot be correct.

Now, suppose that  $P$  is equidistant, with  $PB = PC$ . By the forward theorem we have that  $\angle PBC = \angle PCB$ .

Draw the perpendicular bisector and find point  $P'$  on both  $PB$  and the perpendicular

bisector.

By the forward theorem we have that  $\angle P'BC = \angle P'CB$ . But  $\angle P'BC$  and  $\angle PBC$  are the same angle.

Therefore

$$\angle PBC = \angle P'CB = \angle PCB$$

But clearly  $\angle PCB > \angle P'CB$ .

This is a contradiction. Therefore,  $PB \neq PC$ .

□

If we raise the perpendicular bisector from a line segment, *every* point on the bisector is equidistant from the ends of the line segment, and when the points are connected, forms an isosceles triangle.

No other point not on the perpendicular bisector can be equidistant from the two endpoints.

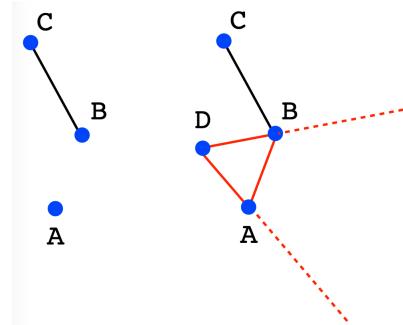
*Proof.* (Alternate).

There is also a much easier proof that relies on **Euclid I.7**, which says there cannot be two points on the same side of  $BC$  above which have the same distance to  $B$  and  $C$ .

Yet that is exactly what this situation calls for.  $P$  is claimed *not* to be on the bisector, with  $PB = PC$ . Yet by the forward theorem, there must be another point  $Q$  on the bisector with  $QB = QC$ . By **Euclid I.7**, this is impossible. □.

## Euclid I.2: transfer a length

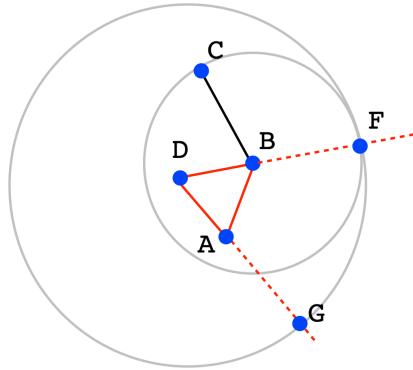
To construct a line segment on a point, equal to a given line segment.



Since this immediately follows the first construction, it seems likely we will need an equilateral triangle. The other thing we know how to do is to draw circles.

We notice that we can construct a circle of radius  $BC$  on center  $B$ . Where that intersects the extension of  $DB$  at  $F$  we have a length equal to one side of the  $\triangle$  plus  $BC$ .

So then draw a circle on center  $D$  of radius  $DF$ .



The intersection of that circle with the extension of  $DA$  marks off a length equal to one side of the  $\triangle$  plus  $BC$ .

So  $AG = BF = BC$ , as required.

If it is desired to draw a line segment of length  $BC$  in some other direction from  $A$ , we just need another circle, centered at  $A$ . That is Proposition 3.

In sum then, we can mark off any length from one line segment onto another, even with just a collapsing compass.

## Euclid I.6

We proved the converse of I.5 early in the book based on angle bisection. Later, we gave another proof by contradiction.

- **isosceles triangle theorem** (Euclid I.6: angles → sides)

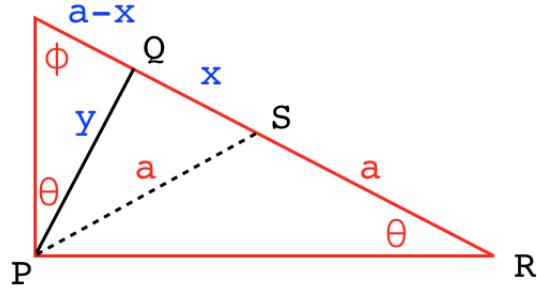
## tangents of similar triangles

Here is a mixed geometric/algebraic proof of the Pythagorean theorem.

*Proof.*

Let  $S$  be the midpoint of the hypotenuse in a right triangle, and draw  $PS$  connecting the midpoint to the vertex at  $P$ . If the hypotenuse has length  $2a$ , then the length of  $PS$  is  $a$ , by the **midpoint theorem** that we talked about previously.

[Quick proof: inscribe the original triangle in a circle, with the hypotenuse of the right triangle as the diameter. The length  $a$  is the radius of the circle centered at  $S$  that contains the three vertices of the original right triangle.]



Now, draw the altitude from the right angle at  $P$  to the hypotenuse  $PQ$ . Suppose  $QS = x$ , then the length  $a$  is divided into  $x$  and  $a - x$  as shown.

The angle  $\phi$  and the angle at vertex  $R$  labeled  $\theta$  are complementary angles in a right triangle, they add up to one right angle. Therefore both angles labeled  $\theta$  are equal.

So we can form the equal ratios of sides:

$$\begin{aligned} \frac{a-x}{y} &= \frac{y}{a+x} \\ (a-x)(a+x) &= y^2 \\ x^2 + y^2 &= a^2 \end{aligned}$$

□

The second line from the last:

$$(a-x)(a+x) = y^2$$

is a statement of our theorem that the altitude of a right triangle is the geometric mean of the two sections of the base.

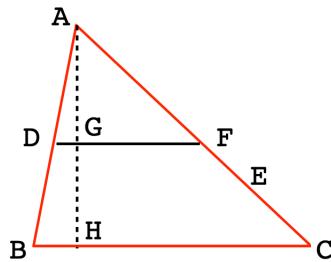
## similar right triangles

### altitudes in proportion

Note that, as well as the sides, the altitudes are also in proportion with the same ratio.

One way to see this is to drop an altitude and then consider the two similar right triangles on one side of it. The altitudes are sides of this triangle.

For similar triangles, where the sides are in proportion  $k$ , the areas are in proportion  $k^2$ . The reason is that the altitudes are in the same proportion, namely  $k$ .



Given that  $DF \parallel BC$ , so  $\triangle ADF \sim \triangle ABC$ .

Drop the altitude  $AGH$ .

Now we see that  $\triangle ADG \sim \triangle ABH$  and  $\triangle AGF \sim \triangle AHC$ , with the same proportionality constant, *since they share the sides AG and AH*.

Suppose that  $AD/AB = DF/BC = k$ . Then  $AG/AH$  also is equal to  $k$ .

The ratio of areas is then

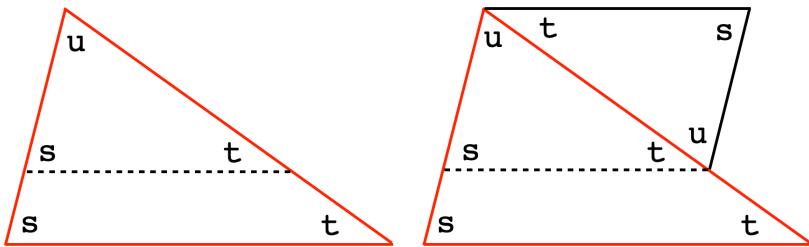
$$\frac{\Delta_{ADF}}{\Delta_{ABC}} = \frac{\frac{1}{2}DF \cdot AG}{\frac{1}{2}BC \cdot AH} = k^2$$

## all triangles

Any triangle can be decomposed into two right triangles.

We previously proved the equal angles for two similar right triangles implies equal ratios of sides. Now we combine the results for the two sub-triangles, both right triangles, and we will have the result for the general case.

We use a flipped and rotated copy of the smaller triangle. Start with two triangles similar because the angles are the same (left panel).



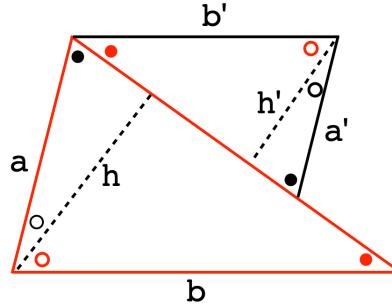
Make a copy of the smaller triangle and rotate it and then attach at the top (forming a parallelogram). The original small triangle and the flipped version are congruent by our construction.

From congruent triangles we get the angle equalities in the right panel.

We also have two pairs of parallel sides, either by alternate interior angles or because  $s + t + u$  is equal to two right angles.

Now, draw the two altitudes, label the sides, and suppress the labels for the angles but just mark them with colored circles.

The angles marked with filled dots are equal (black and black, and red and red) by parallel sides as we just said, and the open dots with the same colors are equal because they are complementary angles in a right triangle.



Thus, we have two different pairs of similar right triangles.

$$\frac{a}{h} = \frac{a'}{h'}$$

$$\frac{b}{h} = \frac{b'}{h'}$$

So then

$$\frac{h}{h'} = \frac{a}{a'} = \frac{b}{b'}$$

Corresponding sides of the two original triangles have equal ratios of sides. (It is important that we extended the equal ratios result to the hypotenuse for this to work).

But there is nothing special about this pair of sides, we could have chosen any other pair, either  $a$  and  $c$  or  $b$  and  $c$ , and have the same result.

Therefore if any two triangles have three angles the same, the side lengths are all in the same proportion.

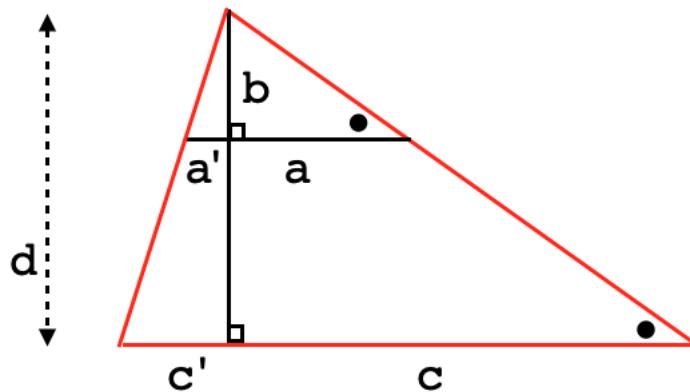
□

## similar triangles

Euclid VI.2 gives us all we need. However, let's take a moment to look at a different proof.

We showed previously that all the statements about similarity that we made in this chapter apply to similar right triangles.

But we can cut any triangle into two right triangles.



If these are right triangles, then the two on the right are similar as well as the two on the left, using complementary angles.

We have

$$\frac{a}{c} = \frac{b}{d}$$

but also

$$\frac{b}{d} = \frac{a'}{c'}$$

So then

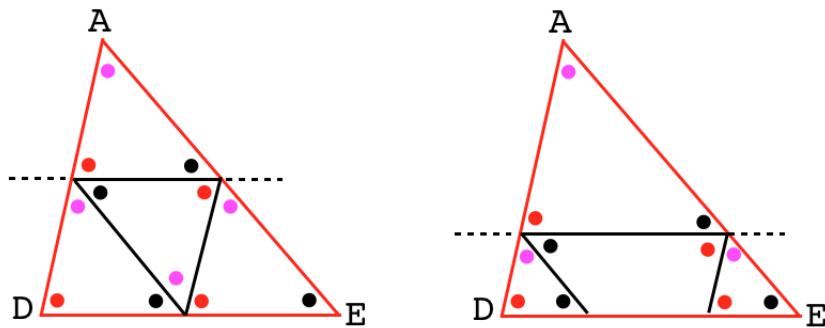
$$\frac{a}{c} = \frac{a'}{c'}$$

and then

$$\begin{aligned}\frac{a}{a'} &= \frac{c}{c'} \\ \frac{a+a'}{a'} &= \frac{c+c'}{c'} \\ \frac{a+a'}{c+c'} &= \frac{a'}{c'} = \frac{a}{c} = \frac{b}{d}\end{aligned}$$

□

### AAA similarity theorem

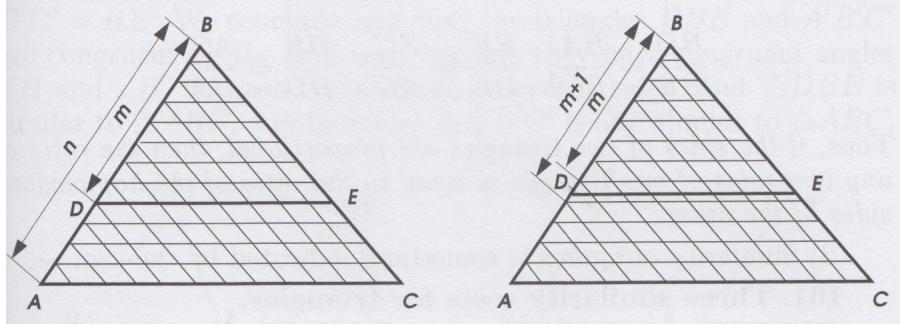


On the left is the easy case where  $AB = BD$ .

We will show that the sides are in proportion even when that proportion is not  $1 : 2$ , as on the right.

Note: we proved this theorem for right triangles already, based on an idea I found in Acheson's book, and combined it with an extension to all triangles.

If the horizontal bisector is parallel to the base, then the triangles are similar. We will have AAA. This is true regardless of which side of the large triangle we choose to be the base.



His notation is different than what we used above, drawing  $\triangle BDE$  smaller than  $\triangle BAC$ . We follow Kiselev for this section.

There are two cases. The first is when the lengths of  $BA$  and  $BD$  are commensurable.

Two lengths are commensurable when there is some small length  $\ell$  that we can define as one unit, such that for integers  $m$  and  $n$ ,  $BD = m\ell = m$  and  $BA = n\ell = n$ .

Divide the side as shown. Draw lines parallel to  $AC$  and also those parallel to  $BC$ .

Then  $BE$  and  $BC$  will be divided into congruent parts, numbering  $m$  and  $n$  for each, respectively. The same thing happens on the bottom. It is clear that

$$\frac{m}{n} = \frac{BD}{BA} = \frac{DE}{AC} = \frac{BE}{BC}$$

The second, harder, case is shown in the right panel above.

$BD$  and  $BA$  are not commensurate and there is some small remainder when dividing the first into the second. Put another way, if  $BA = n\ell$ , then there are two integers  $m$  and  $m + 1$  such that

$$m\ell < BD < (m + 1)\ell$$

But if  $\ell$  is small, with  $n$  and  $m$  large,

$$\frac{m}{n} \approx \frac{BD}{BA}, \quad \frac{m}{n} \approx \frac{DE}{AC}, \quad \frac{m}{n} \approx \frac{BE}{BC}$$

Crucially, by choosing the unit length  $\ell$  smaller and smaller, and thus  $n$  being larger and larger, we can make the remainder  $BD - m\ell$  as small as we like.

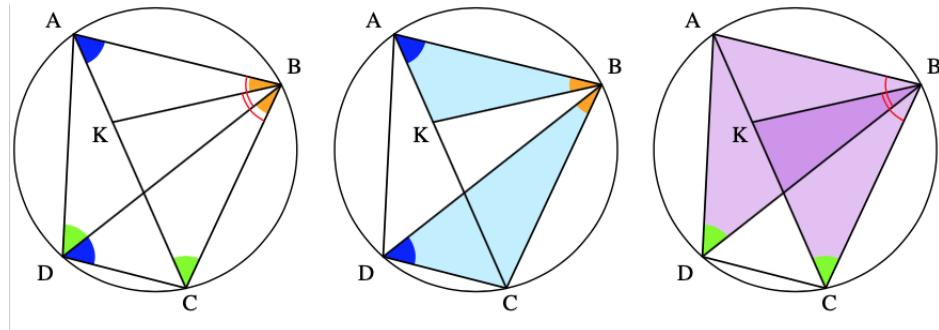
As  $n$  gets very large we approach equality:

$$\frac{m}{n} = \frac{BD}{BA} = \frac{DE}{AC} = \frac{BE}{BC}$$

for the second case as well.

In calculus we say that, in the limit, as  $n \rightarrow \infty$ , they become equal. If this seems strange, wait for the discussion of the limit concept, in calculus.

## Ptolemy's by similar triangles



Above is a graphic from wikipedia that shows where we're going in the first proof. We will form two sets of similar triangles and use our knowledge about corresponding ratios.

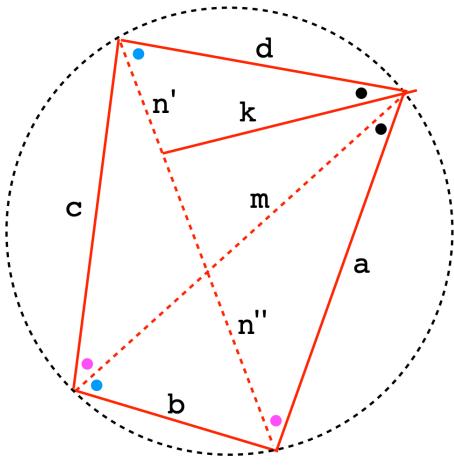
[https://en.wikipedia.org/wiki/Ptolemy%27s\\_theorem](https://en.wikipedia.org/wiki/Ptolemy%27s_theorem)

*Proof.*

It is often easier to adopt a modern notation, designating side lengths by single letters. In this problem, we have sides  $a, b, c, d$  and diagonals  $m$  and  $n$ .

The angles marked with magenta dots are equal as peripheral angles subtended by the same arc, and the same with the blue dots.

The key insight is to draw the line segment marked  $k$ , separating  $n$  into two parts,  $n'$  and  $n''$ . The line is drawn so that the angles marked with black dots are equal.



Other pairs of angles are equal by the inscribed angle theorem (blue and magenta).

The first pair of similar triangles has one vertex with the black dotted angles, and a second vertex with blue dots. Taking the opposite sides in the same order, we have the ratios:

$$\frac{n'}{b} = \frac{k}{a} = \frac{d}{m}$$

The second pair of similar triangles contain a vertex consisting of the black dotted angle *plus* the central angle between the two black dots. Refer to the wikipedia figure if this sounds confusing. There these angles are marked with red arcs.

This pair of triangles also has a second vertex with magenta dots. Taking the opposite sides in the same order we have

$$\frac{n''}{c} = \frac{k}{d} = \frac{a}{m}$$

The second trick is to pick the right relationships to manipulate. We know we don't want  $k$  in the answer, (and we do want all of  $abcd$  plus  $mn'$  and  $n''$ ), so choose from the first:

$$\frac{n'}{b} = \frac{d}{m} \quad \Rightarrow \quad bd = mn'$$

and from the second:

$$\frac{n''}{c} = \frac{a}{m} \quad \Rightarrow \quad ac = mn''$$

Simply add the two equations

$$ac + bd = m(n' + n'') = mn$$

□

## difference of sines

The theorem of the broken chord can be used to derive the formula for the sine of the difference of two angles.

We first recall fundamental result about chords.

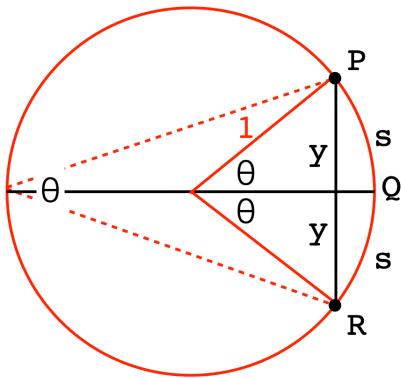
*Lemma.*

Consider any chord of a circle. Place another point on the circle in the larger arc to form a triangle.. All such triangles have the same peripheral angle, by the inscribed angle theorem.

Then choose the vertex for the peripheral angle such that it is bisected by a diameter of the circle and form the central angle corresponding to the same chord.

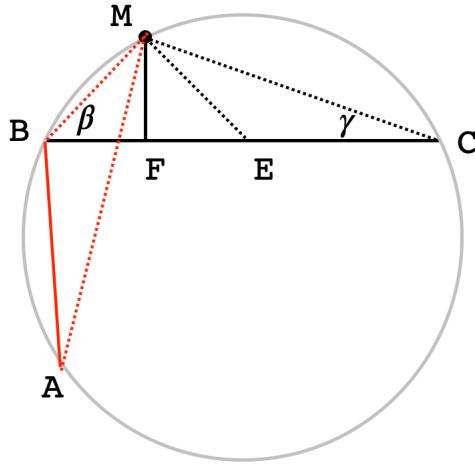
One-half of that central angle, equal to the peripheral angle, has a sine that is equal to one-half the chord.

In other words, the chord corresponding to any peripheral angle is twice the sine of that angle.



We now derive the formula for the sine of a difference of angles.

*Proof.*



Consider  $\angle MBF$  at vertex  $B$ , let us call that angle  $\beta$  for convenience.

Then the lemma says that

$$MC = 2 \sin \beta$$

Similarly, let us label the angle at vertex  $C$  as  $\gamma$ . Then

$$BM = 2 \sin \gamma$$

Now, we can also use the right triangle to find

$$\cos \beta = \frac{BF}{BM}$$

so

$$\begin{aligned} BF &= BM \cos \beta \\ &= 2 \sin \gamma \cos \beta \end{aligned}$$

And

$$\cos \gamma = \frac{FC}{MC}$$

so

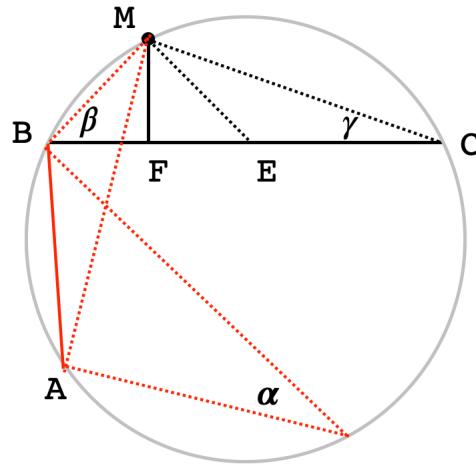
$$\begin{aligned} FC &= MC \cos \gamma \\ &= 2 \sin \beta \cos \gamma \end{aligned}$$

which starts to look familiar.

By the theorem of the broken chord:

$$FC - BF = AB$$

$$AB = 2 \sin \beta \cos \gamma - 2 \sin \gamma \cos \beta$$



What can we do with  $AB$ ? We also know the sum of arcs, namely

$$\text{arc } AB + \text{arc } BM = \text{arc } MC$$

which means that if  $\alpha$  is any peripheral angle subtended by arc  $AB$ :

$$\alpha + \gamma = \beta$$

$$\alpha = \beta - \gamma$$

and by the lemma

$$AB = 2 \sin \alpha = 2 \sin(\beta - \gamma)$$

So finally,

$$2 \sin(\beta - \gamma) = 2 \sin \beta \cos \gamma - 2 \sin \gamma \cos \beta$$

$$\sin(\beta - \gamma) = \sin \beta \cos \gamma - \sin \gamma \cos \beta$$

□

Sine is an odd function ( $f(x) = -f(-x)$ ), so  $\sin -x = -\sin x$ .

Cosine is even so  $\cos x = \cos -x$ . Thus,

$$\sin(\beta + \gamma) = \sin \beta \cos \gamma + \sin \gamma \cos \beta$$

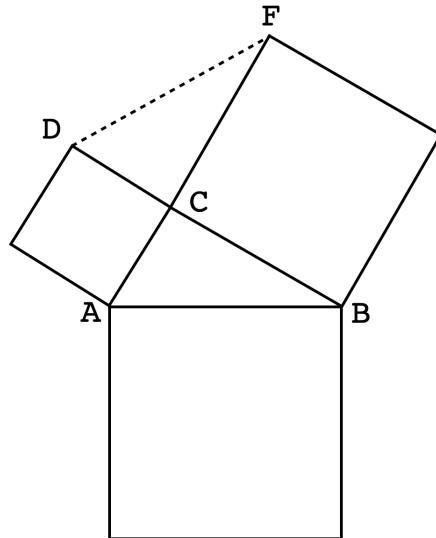
# Chapter 52

## Two final proofs

### Condit proof

Here is a proof by a student named Ann Condit, who was a sophomore in high school at the time. It is one that Euclid would approve of.

We start by drawing the right  $\triangle ABC$  and the squares on each side, as with Euclid *I.47*.

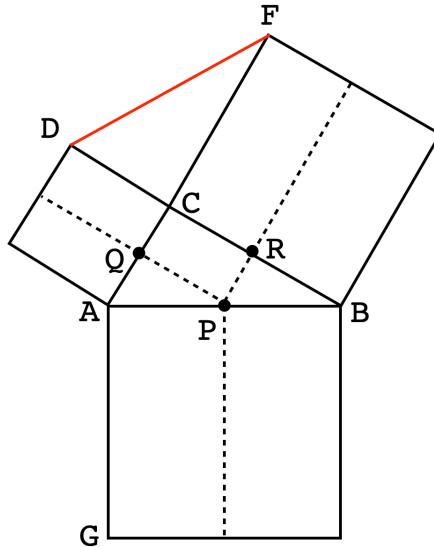


Next, join two nearby points on the two smaller squares to form  $DF$  as shown.

(1)  $\triangle CDF$  is congruent to the original triangle  $ABC$ , by SAS.

We number our conclusions to keep track of everything, because the proof is a bit longer than many.

Next, find the midpoints of each side of  $\triangle ABC$  at  $P$ ,  $Q$  and  $R$ .



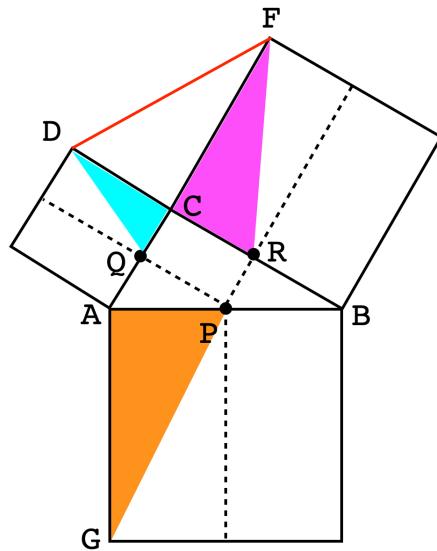
The line segment joining the midpoints of opposite sides forms a smaller triangle similar to the larger one.  $\triangle BPR$  and  $\triangle APQ$  are both similar to  $\triangle ABC$  and congruent to each other. Therefore

(2)  $PQ$  and  $PR$  form right angles with the sides  $AC$  and  $CB$ , respectively.

(3) Thus,  $CRPQ$  is a rectangle (four right angles), so opposing sides are equal.

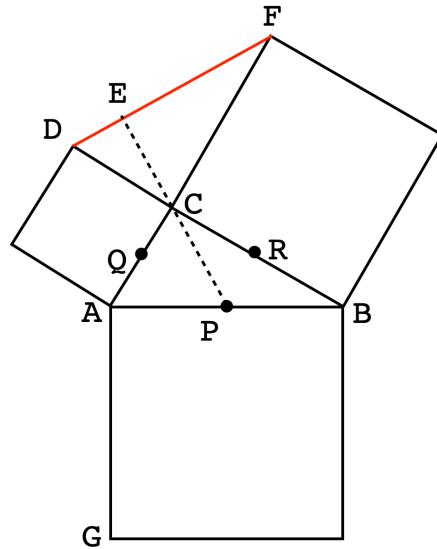
We will erase the dotted lines to reduce clutter, but remember the essential property that  $P$ ,  $Q$  and  $R$  are midpoints; as well as that  $QP$  and  $CR$  are parallel sides of a rectangle and equal in length.

We will show that the area of  $\triangle APG$  is the sum of the area of  $\triangle DQC$  plus that of  $\triangle FRC$ .



Since each triangle is one-fourth the area of its respective square, this proves that the sum the two smaller squares is equal in area to the largest square. This is the theorem of Pythagoras.

We need one more line segment.



Draw  $PC$  and extend it to the opposite side  $DF$  ending at  $E$ .

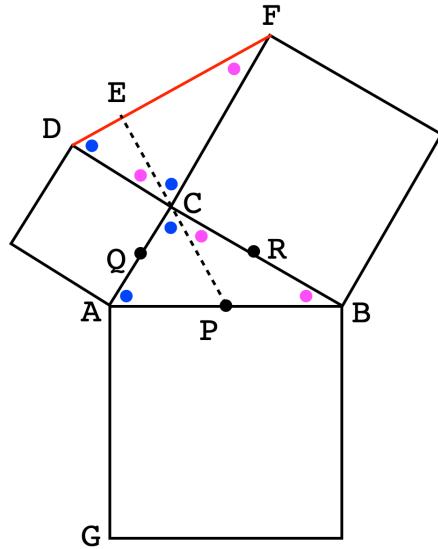
## main idea of the proof

We claim that  $PC$  extended to  $CE$ , is perpendicular to  $DF$ .

Recall that when the side of the hypotenuse is bisected in a right triangle, as we did by drawing  $CP$ , the two resulting smaller triangles are both isosceles.

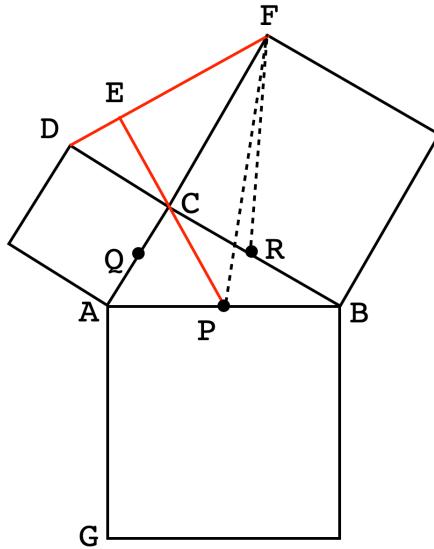
(4) Because of this,  $\angle ACP$  is equal to the angle at the vertex  $A$  ( $\angle CAB$ ) of the original  $\triangle ABC$ .  $\angle ACP$  is also equal to  $\angle EDC$ , by congruent triangles.

(5)  $\angle PCB$  is equal to the angle at vertex  $B$  ( $\angle CBA$ ) as well as to  $\angle EFC$ , for the same reason.



(6) Therefore  $\triangle CDE$  is similar to  $\triangle CEF$ , which means that the angles at  $E$  are identical and so are right angles.

This is the key step of the proof. Now we just look at the areas of different triangles.



First, the area of  $\triangle FRC$  (one-quarter of the square of  $BC$ ) is equal to the area of  $\triangle FCP$ , because they share the base  $CF$ , and  $QP$  is parallel to  $CR$  and equal in length to it (opposite sides of a rectangle).

But the area of  $\triangle FCP$  is equal to  $PC \cdot EF$ , because  $EF$  is the altitude of  $\triangle FCP$  (since  $\angle FEP$  is a right angle).

By symmetry, the one-quarter of the small square's area is equal to that of  $PC \cdot DE$ , so together they equal

$$PC \cdot DE + PC \cdot EF = PC \cdot DF$$

But we have that  $AP = CP$  and  $AG = AB = DF$ , so they have the same base and the same height and therefore the same area.

Thus, the two triangles together have the same area as the triangle with base  $AG$  and height  $AP$ .

□

### yet another proof

Recently (April, 2023) I came across an article by Keith McNulty, referenced on Hacker News. It's an analysis of a very recent proof of the Pythagorean theorem by two teenagers from New Orleans named Calcea Johnson and Ne'Kiya Jackson.

<https://keith-mcnulty.medium.com/heres-how-two-new-orleans-teenagers-found-a-new-proof-of-the-pythagorean-theorem-b4f6e7e9ea2d>

I don't have a reference to an original publication, the article simply refers to a presentation at a meeting.

McNulty claims that the proof involves trigonometry and that this is novel — “[it] might make a few established mathematicians eat their words...because their proof uses trigonometry.”

Trigonometry contains two sorts of theorems, those which explicitly depend on the Pythagorean theorem, and those which do not, including some which use similar triangles. (We hedge the statement because of the complication that similar triangles can be used to prove the theorem).

The law of cosines is an example of a theorem that depends on Pythagoras, while the law of sines does not. Naturally, one could not use anything which depends on the Pythagorean theorem to prove the theorem, that would be circular logic.

What this proof mainly does is to use similar triangles, plus the law of sines. Therefore I think McNulty's claim is wrong.

It's still worth taking a look. The really novel part is the use of infinite series. We're going to use a geometric series:

$$S = 1 + r + r^2 + \cdots + r^n + r^{n+1} + \dots$$

The sum of the first  $n$  terms of such a series is

$$S_n = 1 + r + r^2 + \cdots + r^n$$

Provided  $|r| < 1$ , as  $n$  gets large, the terms beyond  $r^n$  become negligible, so  $S_n$  gets closer and closer to the true sum,  $S$ . Now,

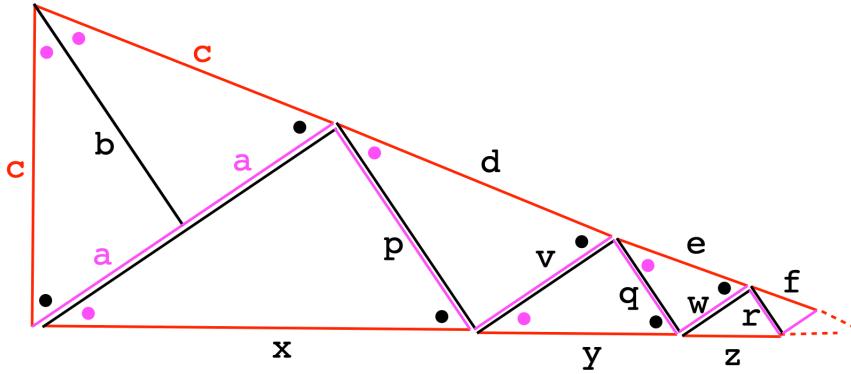
$$(1 - r) \cdot S_n = 1 - r^{n+1}$$

a classic "telescoping" series, and if  $r^{n+1}$  and subsequent terms are small enough then

$$S \approx S_n \approx \frac{1}{1 - r}$$

## construction

The idea of the new proof is that a right triangle with angle  $\theta$  (magenta dot) can be combined with scaled versions of itself to give a different right triangle with angle  $2\theta$ , as shown in the figure.



All the smaller triangles are similar. We will work out the lengths of these pieces in terms of the original triangle's sides.

Let the secant (sec  $\theta$ ) be  $S = c/b$  and the tangent  $T = a/b$ . Then the small triangles have sides:

$$\begin{aligned} x &= 2aS, & p &= 2aT \\ d &= pS = 2aST, & v &= pT = 2aT^2 \\ y &= vS = 2aST^2, & q &= vT = 2aT^3 \\ e &= qS = 2aST^3, & w &= qT = 2aT^4 \\ z &= wS = 2aST^4, & r &= wT = 2aT^5 \end{aligned}$$

The last value to check is

$$f = rS = 2aST^5$$

Let us call the top side of the whole large right triangle  $C$  and the bottom side  $A$ . We have that

$$A = x + y + z + \dots$$

$$C = c + d + e + f \dots$$

Looking at the above results, A and C (minus the first term,  $c$ ) are geometric series with the same ratio,  $T^2$ .

$$\begin{aligned}x &= 2aS, & y &= 2aST^2, & z &= 2aST^4 \\d &= 2aST, & e &= 2aST^3, & f &= 2aST^5\end{aligned}$$

As we said, the sum of a geometric series with ratio  $r$  and initial term 1 is  $1/(1-r)$ , provided that  $|r| < 1$ . Here  $T = a/b$ . If  $b > a$  then  $T > 1$  and then  $1/T^2 < 1$ , so the series converges. If  $a > b$  then just swap  $a$  and  $b$ . We deal with the case  $a = b$  at the end.

The sum for initial term  $k$  is

$$S = k \cdot \frac{1}{1-r}$$

This series has ratio  $T^2$  so its sum is

$$A = 2aS \cdot \frac{1}{1-T^2}$$

Recalling that  $S = c/b$  and  $T = a/b$  so

$$aS = ac/b = cT$$

and then

$$A = 2cT \cdot \frac{1}{1-T^2}$$

The other series is

$$d + e + f + \dots = 2aST \cdot \frac{1}{1-T^2}$$

Using again the relation  $aS = ac/b = cT$  we obtain

$$d + e + f + \dots = \frac{2cT^2}{1-T^2}$$

The whole of the top side  $C$  is  $c + d + e + f + \dots$

$$C = c + \frac{2cT^2}{1-T^2}$$

$$= c \left(1 + \frac{2T^2}{1 - T^2}\right) = c \cdot \frac{1 + T^2}{1 - T^2}$$

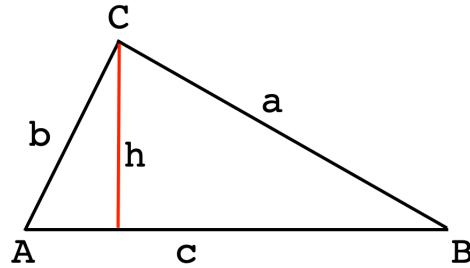
We can finally form the ratio  $A/C$ , canceling the factor of  $1 - T^2$ .

$$\begin{aligned} \frac{A}{C} &= \frac{2cT}{c(1 + T^2)} \\ &= \frac{2a/b}{1 + a^2/b^2} \\ &= \frac{2ab}{a^2 + b^2} \end{aligned}$$

The ratio  $A/C$  is also the sine of the double angle  $2\theta$ .

### law of sines

In any triangle, drop the altitude from vertex  $C$  to the opposite side.



$$\sin A = \frac{h}{b}$$

$$\sin B = \frac{h}{a}$$

So

$$b \sin A = a \sin B$$

The law of sines follows:

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

Apply the law of sines to the isosceles (double) triangle with angle  $2\theta$  and sides  $c$  and  $2a$ :

$$\frac{\sin 2\theta}{2a} = \frac{\sin(90 - \theta)}{c} = \frac{\cos \theta}{c} = \frac{b}{c^2}$$

$$\sin 2\theta = \frac{2ab}{c^2}$$

Substituting the ratio of  $A/C$  for the left-hand side:

$$\frac{2ab}{a^2 + b^2} = \frac{2ab}{c^2}$$

The Pythagorean theorem immediately follows.

Finally, we must handle the case where  $a = b$ . Then the small triangles are all isosceles as well. So  $\theta = 45^\circ$  and  $2\theta = 90^\circ$  and a square is formed with sides  $c$ .

The ratio of opposing sides  $C/A = c/c = 1 = \sin 2\theta$ .

The law of sines gives

$$\frac{\sin 2\theta}{2a} = \frac{\sin \theta}{c}$$

Since  $\sin 2\theta = 1$  and  $\sin \theta = a/c$

$$\begin{aligned}\frac{1}{2a} &= \frac{a/c}{c} \\ c^2 &= 2a^2 \\ &= a^2 + b^2\end{aligned}$$

□

# Chapter 53

## Resources

I have given very few problems, and solved nearly all of those, but you will need to do a lot of problems to really learn the material.

Start with Simmons. He summarizes Algebra, Geometry, Analytic Geometry and Trigonometry in a little more than one hundred pages. The book can easily be found used, and I highly recommend it.

David Acheson's books are wonderful, including *The Wonder Book of Geometry*.

Another resource that I enthusiastically recommend is a website with a highly annotated version of Euclid's *Elements*.

<https://mathcs.clarku.edu/~djoyce/elements/elements.html>

You owe it to yourself to work your way through the first book, at least.

Here is an old textbook by Hopkins that I found online:

<https://archive.org/details/inductiveplanege00hopkrich>

As a unique feature it contains more than a dozen entrance examinations for major colleges from the years around 1900.

Two examples from Dartmouth:

DARTMOUTH—1900.

1. Prove that the three perpendicular bisectors of the sides of any triangle meet in a common point.
2. Circumscribe a circle about a given triangle.
3. Construct a circle having its center in a given line, and passing through two given points.
4. Prove, algebraically, that the square of the side opposite the obtuse angle of an obtuse-angled triangle is equal to the sum of the squares of the other sides plus the product of one of these sides and the projection of the other side upon it.
5. If the side of an equilateral triangle is  $a$ , find its area.
6. Prove that the area of a circle is equal to half the product of its radius and circumference.
7. The radius of a circle is 2; find the area of the regular inscribed dodecagon.
8. The radius of a circle is  $R$ ; what is the radius of a concentric circle which divides it into two equivalent parts?

DARTMOUTH—1901.

1. Prove that the diagonals of a parallelogram bisect each other. When are they equal?
2. How many degrees in one angle of a regular decagon? of a regular dodecagon? What is the largest number of degrees possible in one angle of a regular polygon?
3. One angle between two chords intersecting within a circle is  $50^\circ$ ; its intercepted arc is  $10^\circ$ . How many degrees in the arc intercepted by its vertical angle?
4. Prove that the bisector of an angle of a triangle divides the opposite side into segments proportional to the sides of the angle.
5. The diagonals of a rhombus are 9 and 12. Find its perimeter and area.
6. In a circle whose radius is  $R$ , show that the area of the inscribed square is  $2 R^2$ , and of the inscribed regular dodecagon is  $3 R^2$ .
7. Draw the tangents to a circle from a point outside the circle, and prove that they are equal.

# Chapter 54

## List of theorems

proofs of the Pythagorean theorem

- Pythagorean Thm: similar triangles
- Pythagorean Thm: Euclid I.47
- Pythagorean Thm: scaled triangles
- Pythagorean Thm: sum of angles
- Pythagorean Thm: area
- Pythagorean Thm: Garfield
- Pythagorean Thm: Pappus
- Pythagorean Thm: crossed chords
- Pythagorean Thm: Star of David, Anderson
- Pythagorean Thm: Ptolemy
- Pythagorean Thm: Condit
- Pythagorean Thm: Tuan
- Pythagorean Thm: Quorra corollary
- Pythagorean Thm: converse

## proofs from Euclid

- **construct an equilateral triangle** (Euclid I.1)
- **side angle side (SAS)** (Euclid I.4)
- **isosceles triangle theorem** (Euclid I.5: equal sides → angles)
- **isosceles triangle theorem converse** (Euclid I.6: equal angles → sides)
- **Preliminary to SSS** (Euclid I.7)
- **angle bisection** (Euclid I.9)
- **perpendicular bisector** (Euclid I.10)
- **perpendicular through a point** (Euclid I.11)
- **perpendicular to a point** (Euclid I.12)
- **external angle inequality** (Euclid I.16)
- **longer side → larger angle** (Euclid I.18)
- **larger angle → longer side** (Euclid I.19)
- **triangle inequality** (Euclid I.20)
- **hinge theorem** (Euclid I.24)
- **ASA for congruence** (Euclid I.26)
- **line parallel to another line** (Euclid I.31)
- **parallelogram area** (Euclid I.35)
- **parallelogram complements equal** (Euclid I.43)
- **Pythagorean theorem** (Euclid I.47)
- $(x+y)(x-y) = x^2 - y^2$  (Euclid II.5)
- $(2x+y)y + x^2 = (x+y)^2$  (Euclid II.6)
- **Law of cosines, obtuse case** (Euclid II.12)
- **Law of cosines, acute case** (Euclid II.13)
- **square of a rectangle** (Euclid II.14)
- **find circle center** (Euclid III.1)

- **find circle center** (Euclid III.12)
- **inscribed angle theorem** (Euclid III.20)
- **same arc → equal angles** (Euclid III.21)
- **quadrilateral supplementary theorem** (Euclid III.22)
- **equal angles, on same circle** (Euclid III.26 converse)
- **Thales' theorem** (Euclid III.31)
- **tangent-chord theorem** (Euclid III.32)
- **crossed chord theorem** (Euclid III.35)
- **tangent-secant theorem** (Euclid III.36)
- **similarity: AAA → equal ratios** (Euclid VI.2)
- **equal divisions of a line segment** (Euclid VI.9)

## proofs of other theorems

- **alternate interior angles**
- **angle bisector theorem** (right triangle)
- **angle bisector theorem** (general)
- **ASA for congruence**
- **area ratio theorem**
- **bisector equidistant from sides**
- **equidistant from sides → bisector**
- **Centroid is one-third of cevian**
- **Ceva's theorem**
- **Ceva's theorem** (Menelaus)
- **Ceva's theorem** (by area)
- **Ceva's theorem** (alternate proof)
- **Ceva's theorem by parallel lines**

- **crossed chord theorem** (product of lengths)
- **complementary angles**
- **cyclic quadrilateral** (opposing angles are supplementary)
- **cyclic quadrilateral** (converse)
- **diameter divides circle in half**
- **diameters form a rectangle**
- **equal arcs  $\iff$  equal chords**
- **eyeball theorem**
- **excircle theorems**
- **external angle theorem**
- **extraordinary property of the circle**
- **Heron's formula by excircles**
- **Heron's formula, Heron's proof**
- **hypotenuse-leg in a right triangle (HL)**
- **hypotenuse longest side in a triangle**
- **incenter** (incenter: angle bisectors meet at a point)
- **inscribed angle theorem** (on a circle is one-half central angle)
- **inscribed angles converse**
- **isosceles triangle theorem** (sides  $\rightarrow$  angles)
- **isosceles triangle theorem** (angles  $\rightarrow$  sides)
- **Law of cosines**
- **Law of cosines, algebraic proof**
- **Law of cosines, Ptolemy**
- **Menelaus' theorem**
- **midpoint theorem** (right triangle)
- **orthocenter exists** (Newton)

- **Pappus parallelogram theorem**
- **parallelogram theorems**
- **special parallelogram theorem** (one pair of sides)
- **circumcenter** (perpendicular bisectors of a chord is diameter)
- **circumcenter** (perpendicular bisectors meet at a point)
- **Ptolemy's theorem, by cutting**
- **Ptolemy's theorem, similar triangles**
- **Ptolemy's theorem, switch sides**
- **Ptolemy's theorem, by inversion**
- **rectangle in a circle**
- **right angle is largest in a triangle**
- **secant tangent theorem**
- **shortest distance from a point to a line**
- **supplementary angles**
- **similar right triangles**
- **midpoint theorem** (similar triangles)
- **similar triangles** (ratio of sides)
- **similar triangles** (right triangle composition)
- **AAA similarity theorem** (Kiselev)
- **SAS for congruence**
- **SAS inequality, hinge theorem**
- **SAS to establish similarity**
- **SSS implies SAS**
- **supplementary angle theorem**
- **supplementary angles equal to two right angles**
- **sum of angles**

- **tangent theorem** (right angle → touches one point)
- **tangent-chord theorem**
- **tangent theorem** (touches one point → right angle)
- **tangent construction**
- **Thales' circle theorem** (right angle in a semi-circle)
- **Thales circle theorem: converse**
- **triangular area**
- **triangle inequality** (triangle inequality)
- **triangles are similar if two angles equal**
- **Varignon's theorem**
- **vertical angle theorem**

# Chapter 55

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