Elementary Geometry

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March 30, 2025

Contents

Part I

Lines and angles

Chapter 1

Introduction

The image below is a detail from a painting by Raphael entitled “School of Athens”, which was used as the front cover of a wonderful book annotating the Heath translation of Euclid’s *Elements*.

It took a genius to figure it out the first time, but it is within anyone’s grasp to appreciate what they found. I imagine myself looking over Archimedes’ shoulder as he explains the steps of a proof to me.

Most scientists I've met loved geometry in school, as I did. They like how visual it is, and they like clever simple proofs. Geometry should be fun!

A central feature of this book is the relentless use of proof. I emphasize the key insight for each, and have tried to make the proofs simple and as easy to follow as possible. You will notice that we frequently provide multiple proofs (using different methods) for the same theorem.

I express my sincere thanks to the authors of my favorite books, which are listed in the references and mentioned at various places in the text. Everything in here was appropriated from them in one way or another, and styled to my taste.

I offer my profound thanks also to Eugene Colosimo, S.J. He was among the best of a great group of teachers.

This book is the pdf in that repository linked below. Most of the rest is the source files. There are several other books there as well, if you go up one level in Github.

https://github.com/telliott99/geometry

Note: the sites referred to by some urls in the text have disappeared from the web and the count of missing pages grows over time. I have left those URLs in, as a kind of protest, and because it is impractical to police them, but also because they may still be useful in connection with the wayback machine.

https://web.archive.org

Chapter 2

Angles

On motivation

Modern textbooks make a considerable effort to motivate the student, setting the stage for a problem and attempting to convince her or him that it's worth trying to understand what's being talked about. I usually won't do that.

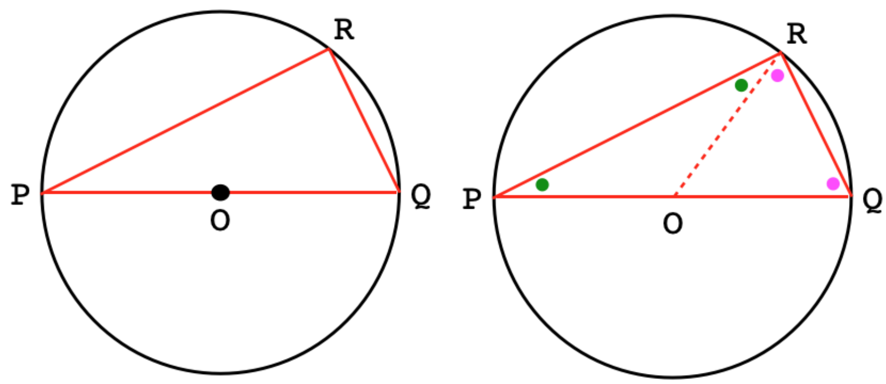
I have tried to achieve simplicity and clarity in the presentation. The subject itself gives us beauty. To see that beauty clearly is my motivation and I hope, yours as well.

In the next chapter, we will prove a beautiful *theorem*: the sum of the angles in a triangle is equal to the angle turned by going halfway around in a circle (like turning from north to south). That's a really remarkable and elegant result, and the proof is simple.

Here is a second beautiful theorem, about circles.

- Any angle inscribed in a semicircle is a right angle.

Think of three points on the circumference of a circle, forming a triangle. If two of the points are on a diameter of the circle (the line joining them passes through the center), then the angle formed at an arbitrary but distinct third point is always a right angle.



In this figure, if PQ is a diameter, then $\angle PRQ$ must be a right angle.

Proof.

Draw the radius OR (right panel).

The two smaller triangles produced ($\triangle OPR$ and $\triangle OQR$) are both isosceles (two sides equal), since two of their sides are radii of the circle. Therefore their base angles are equal (we will see why later).

We can use colored dots to identify angles that are equal. We have for the whole triangle two green dots and two magenta ones, and for the angle at R one of each. Hence the angle at R is one-half the total measure of the triangle, namely, one-quarter of a circle. Like turning from north only as far as west.

□

That's a beautiful, simple result. It depends on an idea about isosceles triangles, and it should motivate us to find out more about them.

The □ symbol marks the end of the proof.

Notation

In the above proof we've mixed together different kinds of notation. The Greeks always named the points at the ends of line segments with letters of the alphabet, using those same points to describe triangles or figures with even more sides. So we can talk about PR as a line segment or PQR as an angle ($\angle PQR$) or a triangle ($\triangle PQR$).

However, often it is simpler to just put a label on the angle at the vertex of a triangle, such as P or R, or s or t above, or use Greek letters such as θ and ϕ; or even α, β and γ.

A third kind of notation is to not even give the angles a lexicographical label, but just mark ones with equal measure by using a colored circle (or open and filled circles).

We use whichever one feels more natural. The simpler the labels, the easier it is to think about what matters in the problem. The right notation frees the mind to concentrate on what's important.

Euclid and the Elements

Some topics in geometry are as old as civilization: finding the area of a rectangular field or the volume of a cylindrical grain storage building.

However, the Greeks were really the first to treat the subject as an intellectual pursuit, the first to view it systematically and to prove theorems.

Euclid is probably the most famous of Greek geometers because of his book *Elements*. This book is believed to have been a compilation of the accomplishments of more than a dozen other mathematicians. See Chapter 2 of Heath:

https://www.gutenberg.org/files/35550/35550-h/35550-h.htm

However, Greek geometry starts several hundred years before the time of Euclid, who was (roughly speaking) a contemporary of Alexander the Great (356-323 BC).

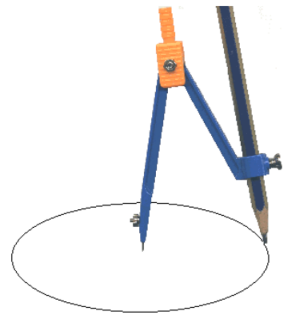
Some of the earliest mathematicians were Thales, Eudoxus, and Pythagoras. One of the problems in understanding what happened is that, unfortunately, almost no books survive from this time. We know that extensive histories were written around the time of Euclid, but these are all lost.

As well, about the person Euclid, we also know actually very little. He lived after Plato (died 347 BC), and before Archimedes (born c. 287 BC). He worked in Alexandria, the city founded by Alexander near Cairo in Egypt, and except for that, all other details of his life and death are shrouded in mystery.

After more than 2000 years, *Elements*, especially the first half dozen books, is still an excellent place to begin surveying the foundations of geometry. It is a quite sophisticated textbook, an organized collection of everything that a well-educated student was expected to know about the subject at the time.

We note in passing and with sadness that so much of what the Greeks wrote and thought has been lost through the vicissitudes of history but also by the deliberate destruction of libraries. At Alexandria this occurred during war in the time of Julius Caesar, later during riots by Christian mobs, and then later again by Muslim invaders.

Elements is known only through later works that reproduced the theorems and proofs with additional commentary. The original source is lost.



The *Elements* consists of more than five hundred *propositions*, some of which are constructions (geometric figures) drawn with a pencil on a piece of paper, using a straight-edge or a compass or both. The others are logical proofs of the type we will study in detail.

For constructions, there are restrictions on both compass and straight-edge. The straight-edge is unmarked; it cannot be used to measure distances. Sometimes people will say "compass and ruler", but the ruler involved must not be *ruled*, it must not have divisions (or at least if it does, they are not consulted).

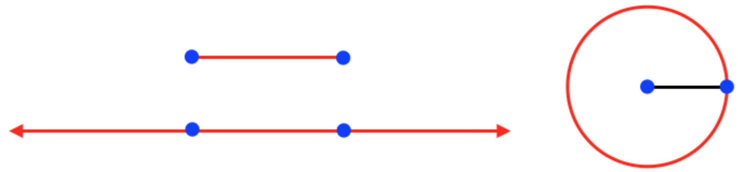
Nearly all proofs of propositions build on previous items in the book.

Euclid does not prove everything. Statements which are assumed to be true are divided into axioms, or common notions, and postulates. Axioms are generally useful, while postulates are specific to the subject of geometry.

Here are some of Euclid's postulates:

◦ A straight line segment can be drawn joining any two points.

◦ A line segment can be extended continuously in a straight line.



∘ Given any straight line segment, a circle can be drawn having the segment as the radius and one endpoint as the center.

Let us assume these as well. We will use them often.

We finesse the difficulty in defining what is meant by *straight* in the real world. If you've ever done any fancy carpentry, you will realize that an unknown edge is checked by comparison with another edge which is known to be straight. We use the known edge to "true" the other.

In geometry, we use an imaginary perfect straight-edge to draw an ideal straight line.

There is another statement commonly claimed, that any straight line segment can be extended indefinitely in a straight line.

According to Morris Kline, this statement does not exist in the *Elements*, and in fact, except when dealing with the question of parallel lines, we never worry about lines to infinity. We use line segments, with defined endpoints, and often refer to these simply as lines, as Euclid did.

People may also talk about a straight line as "the shortest distance between two points". The closest you will find to that is the **triangle inequality**, but that is in our future.

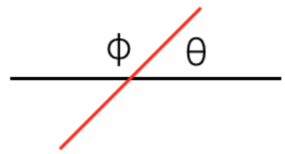
Supplementary angles

We begin geometry with a discussion of angles.

In the diagram below, one line segment is drawn crossing a second one, forming their *intersection* (the points at the ends have been omitted from the drawing).

Four angles are formed at the intersection of two lines or line segments. Two of the angles are labeled in the figure. We can see that one, ϕ, is obviously larger than the other one, θ.

We call angles formed on the same side of a line, *supplementary* angles.

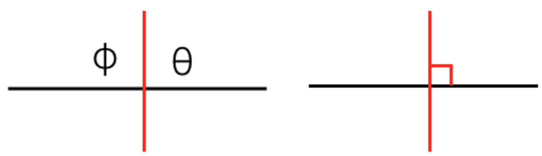


So ϕ and θ are supplementary angles because they both lie above the horizontal black line.

Right angles

You've probably seen a definition of a *right angle* as one that contains, or whose measure is, 90 degrees, usually written 90°. But the Greeks would give a different definition:

- Two supplementary angles that are equal to each other are both right angles. In the figure below, if $\phi = \theta$, then they must both be right angles.



A right angle is frequently designated by drawing a small square, as seen in the right panel.

If one of the angles at the intersection of two lines is a right angle, then all four are right angles — we'll see why later in the chapter. The square is only drawn for one, but they are all equal.

In a common system of angle measurement, a right angle is indeed 90°, and there are 360° in a full circle. Supplementary angles sum to 180°.

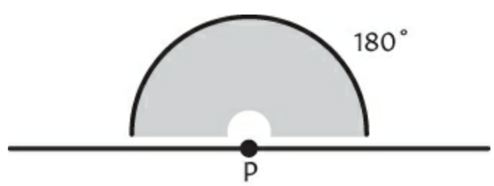
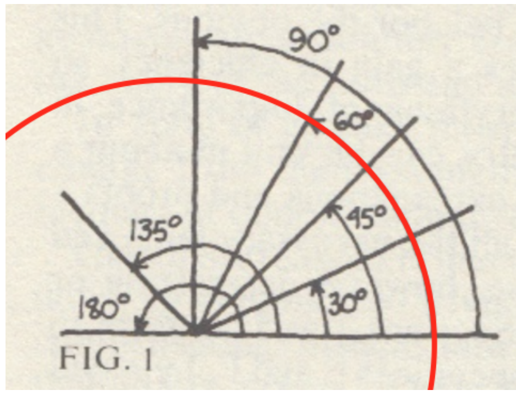


Fig. 4 A straight line.

One way to think about the *measure* of an angle is to draw a unit circle (radius equal to 1) and then ask what is the (curved) distance along the circle from one ray of the angle to the second one. In the figure below, we can see angles with various measures drawn.



Imagine drawing a circle (in red) whose center coincides with the origin of the angles and then measuring the distance along that circle starting with the positive horizontal axis on the right and going counter-clockwise. We can take that distance as the measure of the angle.

Another approach is to bisect (cut in half) a straight line. Then, the original straight line is really an "angle" of 180°, and the bisected half-angles are each 90°. Another bisection gives 45°. We will see a method for doing bisection with compass and straight-edge later on. We can also get 60° using a triangle with all sides equal (equilateral). Bisection then gives 30°.

However, the precise measure of an angle is rarely important, especially before trigonometry. We simply use = 180 as a shorthand for *is equal to two right angles*, and in fact, often drop the degree notation.

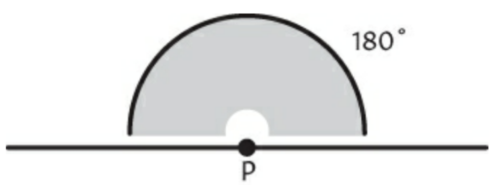


Fig. 4 A straight line.

What we most care about is whether one angle is equal to, larger than, or smaller than another one, or whether an angle or some combination of angles is exactly equal to one right angle or two right angles.

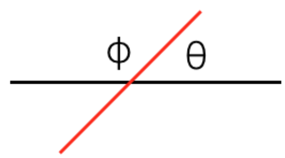
For more about angular measure, there's a short **chapter** at the end of the book.

Because of this, we can state the following theorem:

addition of supplementary angles

- the sum of two angles that are supplementary to each other is equal to two right angles.

This constant sum is correct regardless whether θ is equal to ϕ, or one is larger than the other. Here the sum $\phi + \theta$ is equal to two right angles.

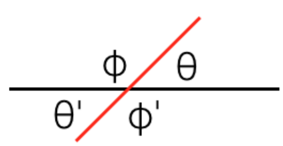


Some textbooks emphasize the arithmetic relationship for supplementary angles: any two angles whose sum is equal to two right angles are supplementary. And then angles that are both adjacent (vertex at the same point) and supplementary are called a *linear pair*.

This distinction seems overly pedantic. We will just use supplementary angles to describe both situations.

Vertical angles

Now, consider the angles lying below the horizontal:



We said that the sum of the two angles $\phi + \theta$ is equal to two right angles, because those two angles (and no others) lie above the black line. But $\theta' + \phi$ and $\theta + \phi'$ are equal to two right angles, for the same reason: they are the angles on one side or the other of the red line.

As a result

$$\phi + \theta = \text{ two right angles } = \theta + \phi'$$

By subtracting θ from both sides, we conclude that

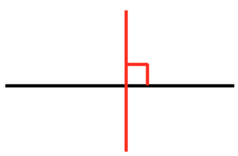
$$\phi = \phi'$$

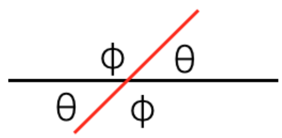
A similar argument will show that

$$\theta = \theta'$$

This is the *vertical angle theorem.*

- Vertical angles are equal.





The vertical angle theorem is obtained by two successive applications of the supplementary angle theorem. It's powerful because it doesn't matter how large or small the two angles are. The vertical angles, that oppose one another, are always equal.

On the left, one of the angles where two lines cross is a right angle. If one angle at an intersection of two lines is a right angle, all four are right angles, by the supplementary and vertical angle theorems.

problem

To prove.

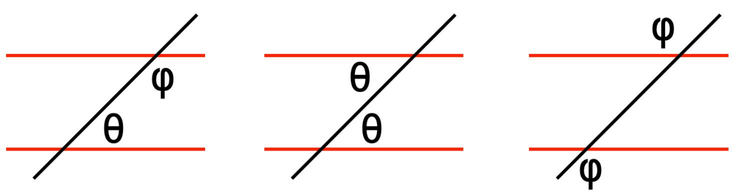
If two adjacent supplementary angles are bisected, the bisectors are perpendicular to each other, that is, the angle between them is a right angle.

Chapter 3

Parallel lines

The theorems from the previous chapter about supplementary and vertical angles seem rather obvious, once you get used to them. Euclid's fifth and final postulate is more subtle.

> If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



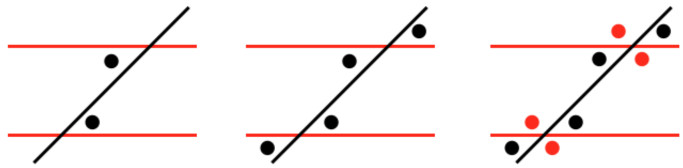
Euclid says that if (in the left panel) $\theta + \phi$ is less than two right angles, the two lines must eventually meet somewhere if extended far enough (to the right).

Of course the fact that they meet means they cannot be parallel lines, since two parallel lines never meet.

He does not worry about the case where the sum of $\theta + \phi$ is greater than two right angles. If the sum is less on the right-hand side of the transversal, it will be greater on the left-hand side. *Problem.* Prove this.

The other two panels also show parallel lines. The known equal angles have different names in these cases, but we will not worry about that. We refer to all of them as examples of lines parallel by equal *alternate interior angles*, even though technically, that only refers to the middle panel.

Here is a graphic that makes it easy to remember:



There is a large philosophical literature on this issue, and there are other ways to say what amounts to the same thing. For example Playfair's axiom, which says that, through any point not on a given line, there is only one line that can be drawn parallel to the first one.

https://proofwiki.org/wiki/Axiom:Euclid%27s_Fifth_Postulate

if and only if

• Two lines are parallel, *if and only if* a third crossing line makes the adjacent interior angles sum to two right angles.

We drew a third line crossing (or making a transversal of) two other lines. Given that background, we then said that if (P) those two lines never cross (they are *parallel*), then (Q) the sum of the adjacent interior angles is equal to two right angles, which means that alternate interior angles are equal.

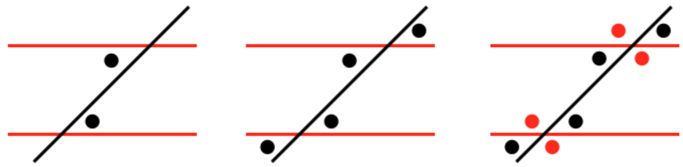
In symbols, we write $P \Rightarrow Q$, by which we mean *if* P is true, *then* so is Q.

However, you may have noticed that what we actually wrote was P *if and only if* Q. In symbols, this is $P \iff Q$.

The meaning of *if and only if* is simply that both $P \Rightarrow Q$ *and* $Q \Rightarrow P$.

If the sum of the adjacent interior angles is equal to two right angles, then the two lines being traversed are parallel.

extending the result



In the figure above (left panel), we're given that the two horizontal lines are parallel.

The indicated angles are equal because they are alternate interior angles of two parallel lines (parallel postulate).

In the middle panel, two additional equalities are established by the vertical angle theorem. Then on the right, we use the supplementary angle theorem.

Note that the conclusions for the angles marked with a red dot are themselves consistent with the three postulates/theorems that we have so far: supplementary and vertical angles and alternate interior angles.

This postulate is also valid in reverse. If we have a line that traverses two others so as to give interior angles summing to 180°, then the two lines must be parallel.

symmetry

This is first of all a statement about the world, that we can have two lines that run alongside each other but don't touch. Another way to talk about the situation is to say that if we sight down a pair of parallel lines from left to right, like a railroad track, then 180° and look again, the picture will be identical.

By rotational symmetry then, we have no reason to distinguish two parallel lines in the forward and reverse directions. It follows that the intersection between two parallel lines and a third line that crosses them should look the same in both directions, the angles involved should have the same measures. That is our theorem.

And this reasoning works, just so long as we are talking about lines on a flat piece of paper.

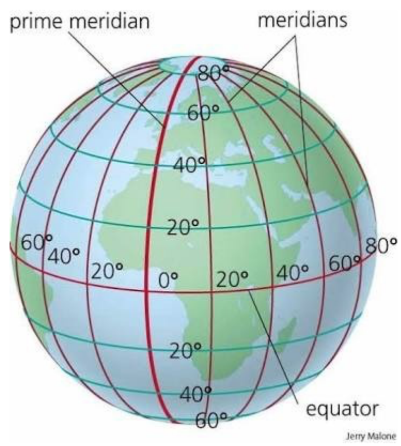
flat geometry

Without getting too deep in the philosophical weeds, it's clear that mathematics is partly a construction. We must choose rules that work, or at least do not conflict with each other, and adoption of the parallel postulate is a choice.

This choice of definition works for geometry in the flat plane, but not on a curved surface like the earth. That's a familiar situation where our postulate is not appropriate.

Lines of longitude go vertically on the globe. They are great circles, they all have the same length and go through the poles. Lines of latitude, on the other hand, form circles of different lengths and get shorter as you near the poles.

Two adjacent lines of longitude can be drawn so as to cross the equator at right angles, and the lines are parallel there, but they will meet (intersect) at the poles.



The same thing happens if you imagine the earth at the center of the universe, looking out at the stars. Or place yourself inside a globe, looking at the thin skin of the object and thinking about the lines of latitude and longitude as seen from the inside.

The parallel postulate only holds for geometry on a *flat* surface.

Axioms

Euclid also lists five axioms, things which are assumed. Here are two examples:

◦ Things that are equal to the same thing are also equal to one another.

◦ If equals are added to equals, then the wholes are equal.

These seem quite reasonable.

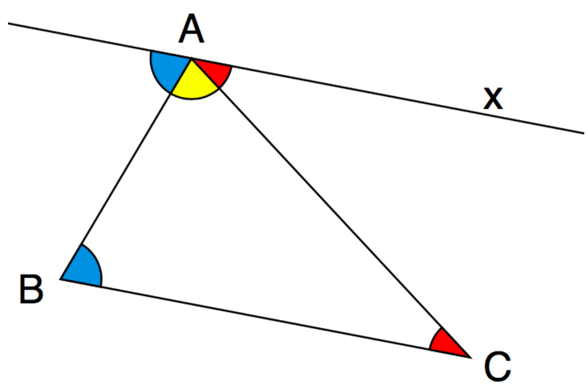
We will see how to proceed from the postulates and axioms to various proofs. Given these *assumptions*, we can prove theorems that must be true.

William Dunham has written a lot about the history of mathematics in Greece, starting with Thales (624-546 BC), who was from a Greek town called Miletus on the coast of Asia Minor (modern Turkey). Thales lived long before Euclid (ca. 600 BC, about 300 years before Euclid). Although none of his writing survives, it is believed that Thales proved several early theorems including the ones we saw above.

Triangle sum theorem

We come to our first truly novel theorem. It relies on everything we've said so far. It is attributed to Thales, one of three elementary but novel theorems for which he is thought to have developed proofs.

- The sum of the three angles in any triangle is equal to two right angles.



This theorem depends on the ideas we developed above.

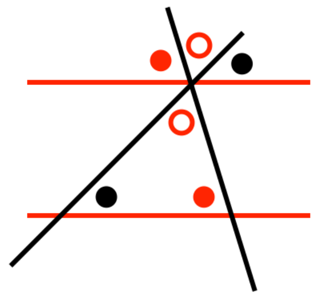
Proof.

Draw a line segment through A parallel to BC. Now, use alternate interior angles and follow the colors to the result. By the theorem, the two angles marked in blue are equal, as are the two angles marked in red. But the three angles at the point marked A add up to two right angles.

So the total measure of three angles in a triangle is equal to two right angles.

□

A simple variation on this proof uses the angles above the line.

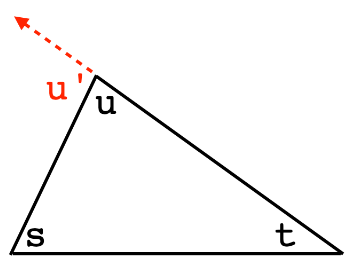


There are two more famous theorems ascribed to Thales, one in the next chapter, and one about circles that we mentioned in the beginning.

Heath says that rather than Thales, Pythagoras is responsible for the triangle sum proof. Since it's not clear that Pythagoras knew any geometry, I'm happy to stick with Thales. No one knows, unfortunately.

problem

Many geometry books will have you do some arithmetic at this point. For example, in the triangle below, suppose that $\angle s = 70°$ and $\angle t = 30°$. What is the measure of $\angle u$? What is the measure of $\angle u'$?



This is simply addition and subtraction, once you understand about supplementary angles and the sum of angles theorem. Go ahead and do that if it amuses you.

We turn to the **external angle theorem**. $\angle u'$ is the *external angle* of this triangle. It is related to $\angle s$ and $\angle t$.

Write two equations giving the relationships between the angles:

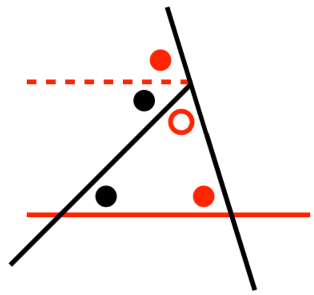
$$u + u' = 180$$

$$s + t + u = 180$$

We used supplementary angles and the sum of angles.

It follows that $u' = s + t$.

This figure makes a proof without words:



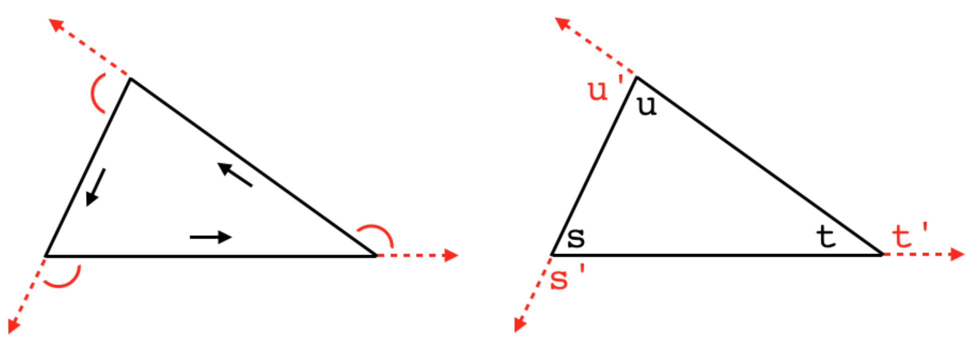
Another proof

Here is a different proof of both of these theorems relating to the angles in a triangle. This one is from Lara Alcock (*Mathematics Rebooted*).

It never hurts to re-prove important results by a different method. This serves as a check on both the result and the methods.

Imagine walking around the perimeter of a triangle in the counter-clockwise direction. At each vertex we turn left by a certain angle, called the exterior angle. After passing through all three vertices, we will end up facing in the same direction as we started.

We have made one complete turn, the sum of the exterior angles is 4 right angles.



$$s' + t' + u' = 4 \text{ right angles}$$

In addition, for each vertex, the interior angle plus the exterior angle add up to 2 right angles. If we add up all three pairs, we obtain 6 right angles.

$$(s + s') + (t + t') + (u + u') = 6 \text{ right angles}$$

Subtract the first equation from the second

$$s + t + u = 2 \text{ right angles}$$

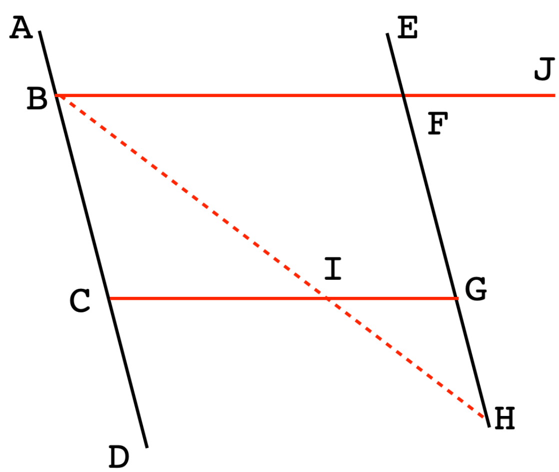
□

Note: both terms, exterior angle and external angle, are used to describe the same angle by different writers. I'm not aware of a reason to prefer one or the other.

problem

In the diagram below, $ABCD$ lie on one line, and it is parallel to $EFGH$. We can write this as $AD \parallel EH$.

Tell which angles are equal and why.



Which are supplementary, which are vertical? (Refer to the angles in any way you wish, using the points, or with new labels).

summary

Make sure you learn and understand each of these theorems.

- the sum of two supplementary angles is equal to two right angles (**ref**).

◦ by definition, if two angles are supplementary and also equal, they are both right angles

- vertical angles are equal (**ref**)
- alternate interior angles of parallel lines are equal (**ref**)
- the sum of angles in a triangle is equal to two right angles (**ref**).

Eratosthenes

It is often supposed that the ancient world believed the earth to be flat, but this is just wrong. People with any level of sophistication not only knew the earth is roughly spherical but also knew its size within a few percent of the true value.

One likely basis is the false story that Columbus had trouble getting financing for his proposed trip to China because everyone thought he would fall off the edge of the earth. This was a tall tale invented by Washington Irving, who also made up several remarkable fables about George Washington.

The real reason the Italian and Portuguese bankers from whom Columbus sought financing thought he would fail is that they had a pretty good idea of the size of the spherical earth and thus of the distance to China, while the over-optimistic Columbus believed it was about half the true value. The prospective financiers knew that he was not able to carry the supplies necessary for a trip of this length.

Morris Kline (*Mathematics and the Physical World*) says that the error is due to geographers after Eratosthenes, who reduced the estimated circumference from 24,000 to 17,000 miles.

Views of the Greek philosophers on the earth and its sphericity are detailed here

`https://www.iep.utm.edu/thales/#SH8d`

Here is a partial quotation:

> There are several good reasons to accept that Thales envisaged the earth as spherical. Aristotle used these arguments to support his own view [...] . First is the fact that during a solar eclipse, the shadow caused by the interposition of the earth between the sun and the moon is always convex; therefore the earth must be spherical. In other words, if the earth were a flat disk, the shadow cast during an eclipse would be elliptical. Second,

> Thales, who is acknowledged as an observer of the heavens, would have observed that stars which are visible in a certain locality may not be visible further to the north or south, a phenomen[on] which could be explained within the understanding of a spherical earth.

https://en.wikipedia.org/wiki/Eratosthenes

Eratosthenes (ca. 276 - 195 BCE) measured the circumference of the earth from this observation: at high noon on the solstice on June 21st there was no shadow seen at Syene, allegedly from a stick placed vertically in the ground. Some people say a deep well had the bottom illuminated at midday. Acheson says Eratosthenes was born at Syene, so he would probably know!

Syene is presently known as Aswan. It is on the Nile about 150 miles upstream of Luxor, which includes the famous site called the Valley of the Kings, where many Egyptian Pharaohs were entombed. At 24.1 degrees north latitude, Aswan or Syene has the sun almost directly overhead on June 21. (The "Tropic of Cancer" is at 23 degrees, 26 minutes north).

Alexandria was a famous center of learning of the ancient world, and Eratosthenes was hired by the pharaoh Ptolemy III to be the librarian there in 245 BCE. Alexandria lies on the Mediterranean some 500 miles north of Syene, and anyone there who was looking could observe that at high noon on June 21st there *was a shadow*. This shadow Eratosthenes measured to be some 7 degrees and a bit (7 degrees and 10 minutes).

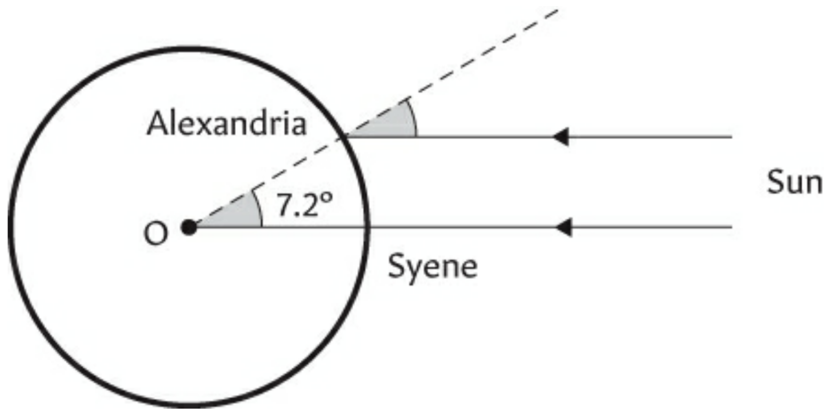


Fig. 21 Measuring the Earth.

A full 360 degrees divided by 7 degrees and a bit is approximately 50. So we can calculate on this basis that the circumference of the earth is about $50 \times 500 = 25000$ miles. That's pretty close to the correct value.

For this calculation, we assume that the sun's rays are effectively parallel (not a bad assumption given a distance of 93 million miles). Then we just use an application of the alternate-interior-angles theorem.

It is curious how the distance from Alexandria to Syene was calculated.

Kline:

> Camel trains, which usually traveled 100 stadia a day, took 50 days to reach Syene. Hence the distance was 5000 stadia...It is believed that a stadium was 157 meters.

We obtain

$$157 \times 5000 \times 50 = 39,250 \text{ km}$$

That's a much better estimate than a method that relies on camels really deserves.

Part II

Triangles

Chapter 4

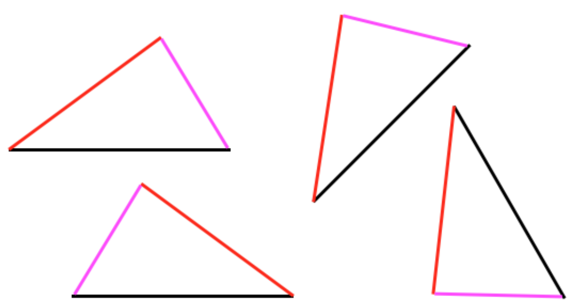
Congruent triangles

meaning of congruence

Probably the most fundamental idea concerning triangles is how to decide that two triangles are *congruent*. If so, they have all 3 sides the same length, and all 3 angles the same measure.

However this is logically the converse (backwards) of what we really want to know. We would like to be able to say that that two triangles are congruent *if* they have some properties.

We will find tests that, if passed, imply congruence.



We allow one triangle to be rotated at any angle with respect to the other. An example of this is the two triangles at the top in the figure above.

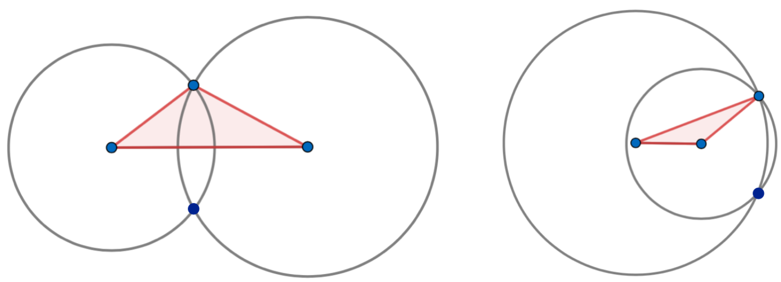
We will also allow the term congruent to apply to the case of a triangle and its mirror image. All of the triangles shown are congruent, even though two are flipped — they are mirror images.

SSS test for congruence

Perhaps a practical definition would be that if you used a pair of scissors to cut out one triangle and then lay it on top of the second one so that it superimposes exactly, they would be congruent.

In fact this is how Euclid handles the issue in the first theorem of *Elements*, I.4. (The first three *propositions* are actually constructions).

A little thought may convince you that if all three corresponding side lengths are equal, two triangles are congruent. Draw one side of a triangle, and from its endpoints draw circles with radii the length of the other two sides.



The two radii map out all the points that are the same distance from the centers. They include only two possible arrangements for three given side lengths which result in triangles.

We see that the radii cross at two and only two different points, which have mirror image symmetry above and below the original base. Three given side lengths can only be drawn together to give two resulting triangles, and these two shapes are mirror images.

It is certainly possible to come up with side lengths that *cannot* form a triangle. Consider (left panel, above) what would happen if the two shorter sides were exactly equal to the longest one. Then they would meet at a single point on the longest side, and there would be no triangle.

If the sum were shorter, they would not meet at all. This is called the triangle inequality: the two shorter sides of a triangle must sum to more than the length of the longest side. We'll prove it more formally later.

Thus, we arrive at a fundamental theorem about congruency for triangles:

• Two triangles are *congruent* if they have the same three equal side lengths.

When we say "equal sides", we are making a comparison *between* two triangles, that corresponding sides are equal.

This test is abbreviated SSS (side-side-side).

SAS

In addition to SSS (side-side-side), there are three other conditions that always lead to congruence of two triangles when they are satisfied, namely

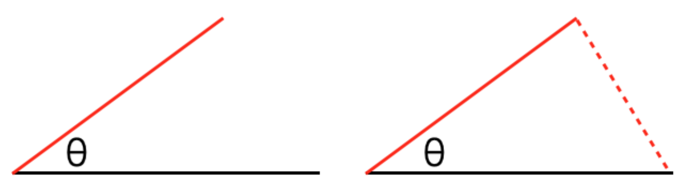
∘ SAS (side-angle-side)

∘ ASA (angle-side-angle)

∘ AAS (angle-angle-side)

When we say SAS, we mean that the equal angle we know lies between the pair of equal sides. Similarly, ASA means that we know one side of a triangle and the two angles at either end of that side are the angles in question.

As we did above for SSS, a good way to think about the congruence condition SAS is to imagine trying to construct a triangle from the given information, and ask whether it is uniquely determined.

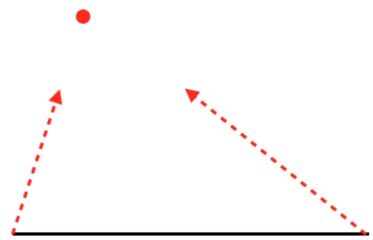


Two sides and the angle between them are given. So draw that part of the triangle. Notice that the second and third vertices are also determined, because they just lie at the ends of the two sides we're given. All that remains is to draw the line segment that joins them.

Of course we might have drawn the shorter side as the horizontal. If we did that, it would generate the mirror image, and still be congruent by our rule.

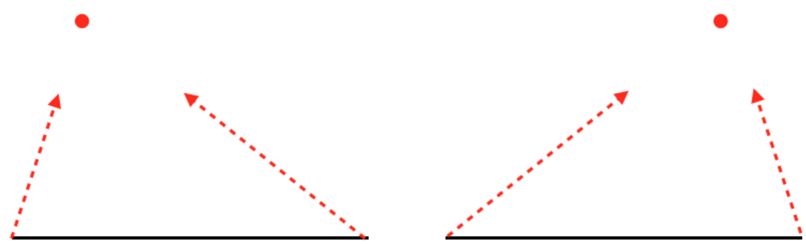
ASA

The next one is ASA. Since we know two angles, we know the third. Here is a diagram of the situation:



Draw the known side, then using the known angles, start two other sides from the ends of that side. They must cross at a unique point.

But...actually, if we start the two lines from opposite ends of the horizontal



there is another solution, the mirror image. These two triangles are congruent to the one above.

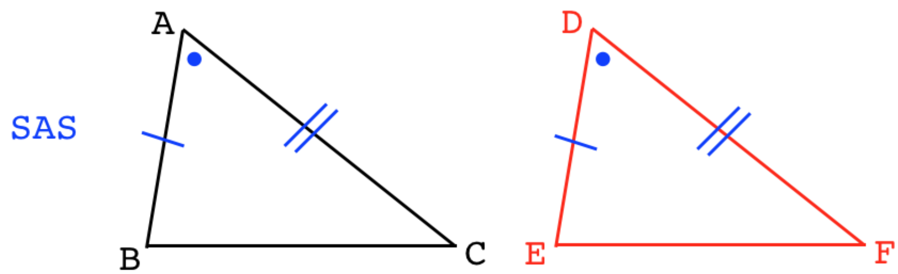
I'm tempted to draw the constructions below the starting line. But this doesn't give anything new. These are merely rotated versions of the ones above. Try it and see. Congruent triangles include a pair of mirror images and that's it.

Now, if we know two angles we also know the third, by the angle sum theorem. For this reason, ASA and AAS mean that we have exactly the same information, because we know all three angles and we know one side.

Crucially, we know *which* two angles flank the known side. Equivalently, it is enough to know which angle is opposite to the known side.

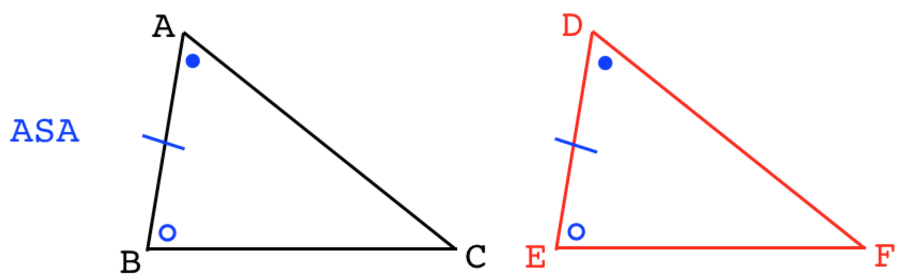
marks for equality

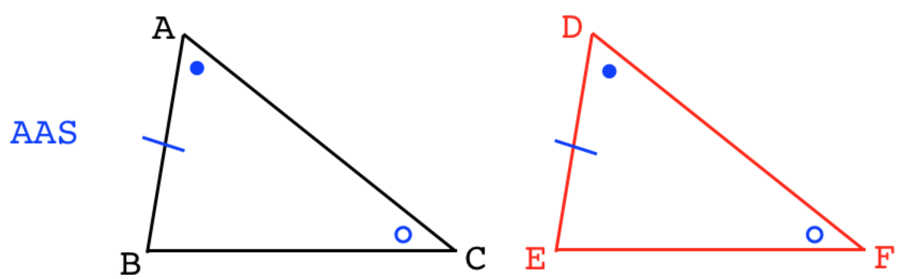
It is often useful to marking sides and (particularly) angles to show they are equal. Here is



In this diagram, sides of equal length are indicated by one or more short lines called hash marks.

Equal angles are usually indicated by dots in this book. Dots are easier to place on the figures, and lend themselves to color-coding; the common method for pencil and paper is to draw arcs with one or more hash across them.

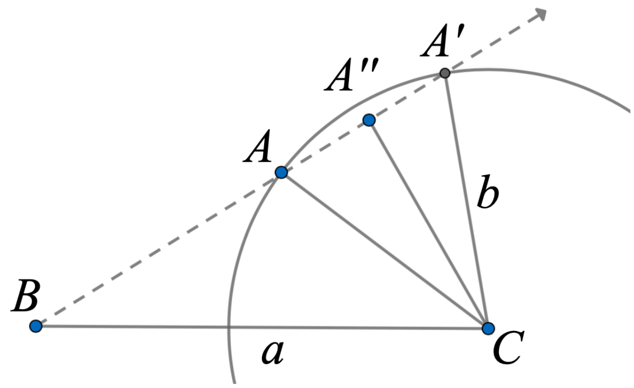




but not SSA

There is one set of three that doesn't work in the general case, and that is SSA (side-side-angle).

Suppose we know the lengths of sides a and b and the angle at vertex B, adjacent to a and opposite side b.



Since the length of the third side isn't known we make it a dashed line. Let's see if we can construct a triangle from this information.

We do not know the angle at vertex C between a and b so we imagine b swinging on a hinge there. If b is too short, there can't be a triangle. If the length of b is exactly right, we'll have a right angle at A''.

If b is longer than this but still $b < a$, there are two points A and A' where b can intersect with the side projecting from vertex B.

This is the *ambiguous case*: we have two possibilities. If two different triangles match by the SSA criterion, we cannot say whether they are congruent or not without more information. It is not that we don't know anything, we just don't know enough to choose one of two possibilities.

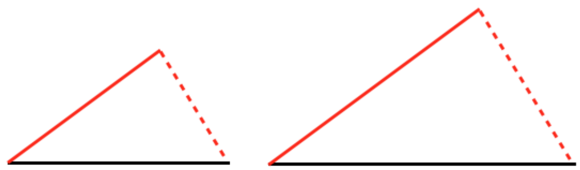
The right angle case is not ambiguous, but we'll save that for the chapter on right triangles.

If you compare this chapter with most others in the book you'll notice that we have not formally proved that any of these methods are correct. Even Euclid encounters some difficulty with this point. He "proves" SAS by a method that is arguably not really a proof.

It won't hurt my feelings if you think of them as axioms. The famous mathematician David Hilbert does this. We will revisit the issue when we look at Euclid's proofs.

similar triangles have equal angles

Sometimes two triangles are not congruent, but have all three angles the same. We call such triangles *similar*. They have the same shape.



Similarity means that all three angles are the same but the triangles are of different overall sizes.

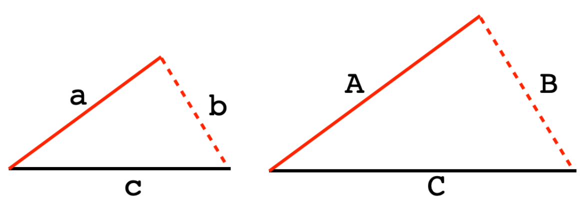
Our basic criterion for similarity is AAA (angle-angle-angle).

However, because of the sum of angles theorem, if any two angles of a pair of triangles are known to be equal, then the third one must be equal as well. We can say that:

- Two triangles are similar if they have at least two angles equal.

scaling

For similar triangles, the three corresponding pairs of sides are in the same proportions, but re-scaled by a constant factor.



From the above diagram of two similar triangles, similarity implies that (for example)

$$\frac{A}{a} = \frac{B}{b}$$

which can be rearranged to give:

$$\frac{a}{b} = \frac{A}{B}$$

For any pair of similar triangles, there is a constant k such that

$$k = \frac{A}{a} = \frac{B}{b} = \frac{C}{c}$$

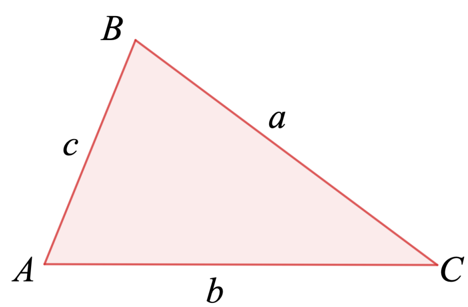
We will come back to proofs about similar triangles in a later chapter. In particular, we will show that AAA implies equal ratios of sides as given here, as well as the converse.

sides and opposing angles

When two triangles are congruent, each of the three angles and all three sides are the same. You might wonder whether the sides could be placed in a different order. We usually draw side of length a opposite $\angle A$, side b opposite $\angle B$ and side c opposite $\angle C$. Can we switch the sides so that say, the length c is opposite $\angle B$?

It turns out not to be possible. Later we will have a theorem which say that in any triangle, the longest side is opposite the largest angle, and the smallest side opposite the smallest angle. If switching two sides were possible, the theorem would be false.

note on notation



Consider $\triangle ABC$. The side opposite the vertex A may be referred to as BC, but CB is also fine. In many cases I prefer a single letter for the label, usually a. If we refer to the *length* of the side, we will prefer a, although elsewhere one may see $|BC|$ or even just, "the length BC".

There is no compelling reason to prefer $\triangle ABC$ to any of the five other permutations of three letters, such as $\triangle BAC$, beyond a liking for alphabetical order. Old texts may use the letters A, B, and G, because the third letter in Greek is Gamma, Γ.

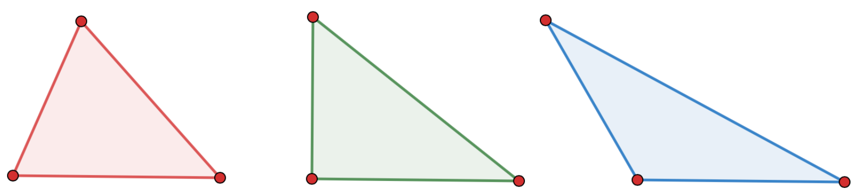
If we have two triangles sharing two vertices, we should usually write those shared letters in the same positions, thus, $\triangle BCA$ and $\triangle DCA$.

When we work with similar triangles and their corresponding angles, we will strive to label the angles in the order of their equality. Thus if $\triangle ABC \sim \triangle QPR$, you should expect that $\angle A = \angle Q$, and so on.

Chapter 5

Isosceles triangles

Triangles are classified by the largest angle they contain: acute, right, or obtuse.



The acute triangle (left) has all three angles smaller than a right angle.

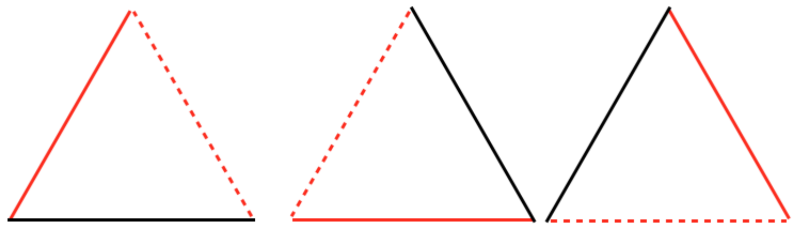
The right triangle, naturally, has one right angle (it cannot have two).

An obtuse triangle has one angle larger than a right angle (right panel, above).

symmetry

One can also talk about the situation where either two sides, or all three sides, have the same length. An *isosceles* triangle has two sides the same length, while an *equilateral* triangle has all three sides the same.

The most important consequence of all three sides equal for an equilateral triangle is three-fold rotational symmetry. Three turns of 120 degrees, and we're back where we started. Each of the two intermediates is identical.



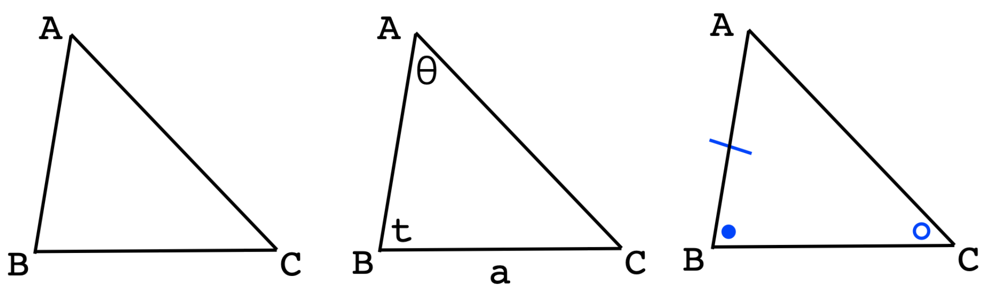
The implication of rotational symmetry is that the three angles are also equal because there is no reason to choose one larger than any other.

Therefore each angle of an equilateral triangle is $2/3$ of a right angle, or $60°$, by the triangle sum theorem (**ref**).

It is also true that if all three angles are equal, then the triangle is equilateral (three sides equal). We will show how to prove this later.

notation

The Greeks, including Euclid, always label points with letters, and line segments are referred to by the endpoints. Angles and triangles are denoted by the line segments from which they are composed, as in $\angle ABC$, and triangles by their vertices: $\triangle ABC$.



The Greek notation (left panel) is problematic because it is more complicated than need be. I have found myself repeatedly tracing out angles from three points.

As mentioned previously, we will usually label the side opposite a vertex with a lower case letter: side a lies opposite $\angle A$. We may also use letters like θ and ϕ for angles, or even, more boldly, s and t.

Sometimes, we will dispense with labels altogether and use colored dots for equal angles.

The image below is from the web, it uses the convention of an arc drawn across equal angles. The curved arcs are common in books.

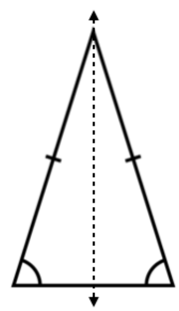
theorem from Thales

- If a triangle is isosceles (two sides equal), then the base angles are also equal.

The converse is

- If two base angles are equal, then the triangle is isosceles.

My favorite proof of both theorems about isosceles triangles is from reflective or mirror image symmetry.



Proof.

Imagine that the triangle sticks straight up from the plane like one of the standing stones at Stonehenge. Imagine walking around the back of it.

Looking from behind, it would appear exactly the same. We would say that the left side as viewed from the front is equal to the right side as also viewed from the front, because if we walk around behind the triangle the *right* side becomes the *left* and vice-versa.

Much later than Euclid, Pappus invokes SAS on the mirror image, rather than thinking about the plane being in 3D space.

The top vertex angle is shared. So the triangle as we look from the front is equal by SAS to the one where we look from the back. It follows that the base angles are equal.

□

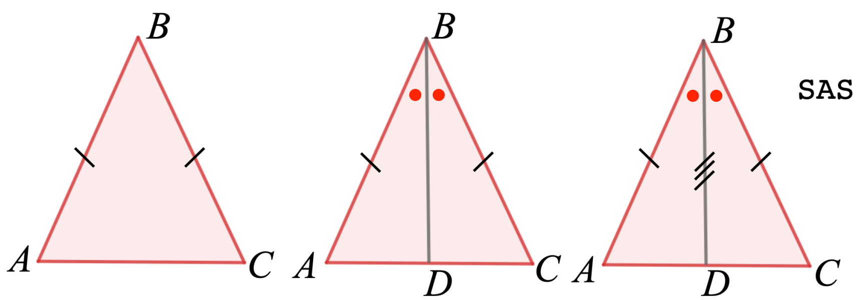
proofs based on triangle bisection

The argument above is not Euclid's proof, namely, I.5, the second theorem in his Book I. We will not give that one right now, but instead something simpler.

The forward theorem on isosceles triangles is:

- Two sides equal $\Rightarrow$ opposite angles equal.

There is a simple demonstration based on angle bisection, and another based on the ability to cut a line segment into two equal parts, also called a bisection.



Demonstration.

In the left panel, we are given $BA = BC$.

Draw the bisector of $\angle B$ (middle panel). This construction forms equal angles at the top, marked with red dots. So $\angle ABD = \angle CBD$.

BD is shared (3 hash marks, right panel).

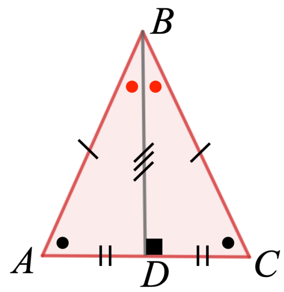
The two smaller triangles $\triangle ABD$ and $\triangle CBD$ are congruent by SAS.

We write that as $\triangle ABD \cong \triangle CBD$.

□

Therefore (as corresponding parts of congruent triangles) the base angles are equal. Other corresponding angles and sides are equal as well.

The full set of equal angles and sides is:

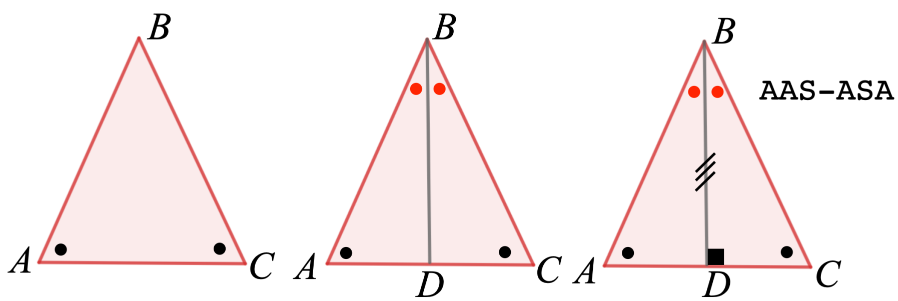


The base is bisected, there is a right angle where the bisector BD cuts the base, plus of course the central line segment BD is equal to itself.

converse

We can also prove the converse theorem by a similar approach:

- Two angles equal $\Rightarrow$ opposite sides equal.



Proof.

We are given that the angles marked with black dots are equal.

We again draw the bisector of the angle B. Then, by sum of angles, we have all three angles the same, and the side AD is shared.

Therefore, $\triangle ABD \cong \triangle ADC$ by AAS.

It follows that $BA = BC$.

Since ADC lie on one straight line (they are *collinear*), there are right angles at the base.

$\square$

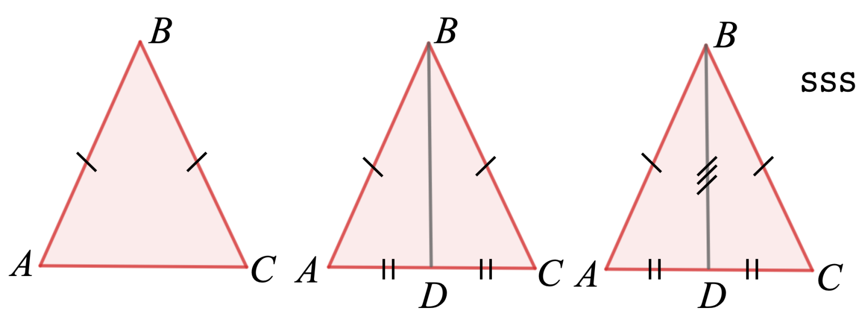
Euclid's proof of the isosceles triangle theorem is more complicated than what we have given above, and there is a good reason for this.

Our demonstration depends on the existence of the angle bisector, but we haven't actually shown how to do that. It will turn out that **construction** of the bisector *depends on* the isosceles triangle theorem.

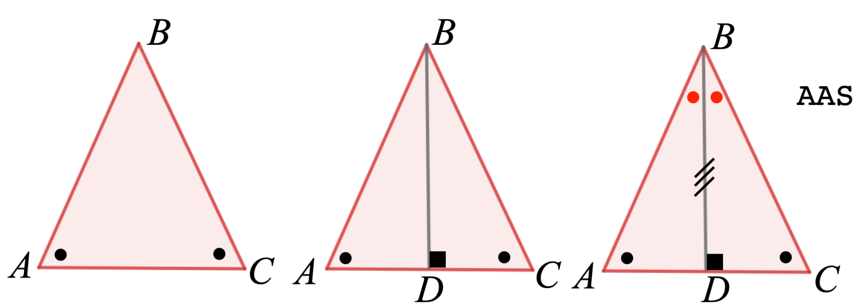
That's a problem because the reasoning is circular, thus invalid. We cannot use p to prove q if we have previously used q to prove p. That proves nothing. We will fix this problem later.

other proofs

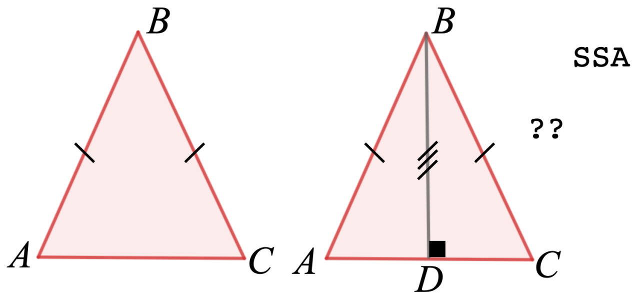
We used the angle bisector at vertex B for the proofs above. But we might equally have constructed a right angle at D, or bisected the base. We will just show diagrams for proofs adapted to these methods and sketch the idea.



Above, we are given $BA = BC$ and $AD = CD$, i.e. the base is bisected. Since BD is shared we have that $\triangle ABD \cong \triangle CBD$ by SSS.



Above, we are given equal angles at A and C and right angles at D. We have three angles equal plus a shared side, so congruent triangles by AAS (or after application of the triangle sum theorem, by ASA).



Above, we are given $BA = BC$ and right angles at D (and BCD *colinear*). This might be called SSA.

However, since the triangles are right triangles, this is a special case called hypotenuse-leg in a right triangle (HL). The two small triangles are congruent. For a proof, look ahead to Pythagoras's theorem: if in a right triangle we know two sides, we know the third.

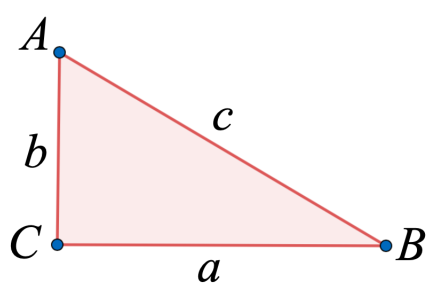
In general, if the known angle were not a right angle, this is not enough to show congruence. However, there is much more to be said about this, as we will see.

Chapter 6

Right triangles

A *right* triangle is a triangle containing one right angle. We saw previously that the Greeks' definition of a right angle is that two of them add up to one straight line or 180 degrees.

No triangle can contain two right angles, because the measure of the third angle would then have to be zero. This goes back to the parallel postulate. Right angles at both ends of a line segment send out parallel lines.



Right angles and right triangles are special in many ways.

In the figure, let us have the angle at vertex C be a right angle. It is common practice to mark a right angle with a little square, but we often just state the fact. So $\angle ACB$ is right.

The side opposite vertex C — it is common practice to label it the same but lowercase, so c — is called the *hypotenuse*, and the other two sides a and b are sometimes called legs.

complementary angles

Since the sum of angles in a triangle is equal to two right angles, the two acute angles, $\angle BAC$ and $\angle ABC$ in the figure above, together equal one right angle, or 90 degrees. Along with with $\angle ACB$ they sum up to two right angles for the whole triangle.

Proof.

This is a direct consequence of the sum of angles theorem applied to a right triangle.

□

The two smaller angles in a right triangle are said to be *complementary*. This fact is often exploited in proofs, since if we know one, we know the other, by sum of angles.

- the sum of the two smaller angles in a right triangle is equal to one right angle.

Pythagorean theorem

A second very important fact about right triangles is expressed in the Pythagorean theorem. Although we haven't proved it yet, we will do so shortly, and call on it here. Given the hypotenuse c and two legs of a right triangle, a and b, the theorem says that

$$a^2 + b^2 = c^2$$

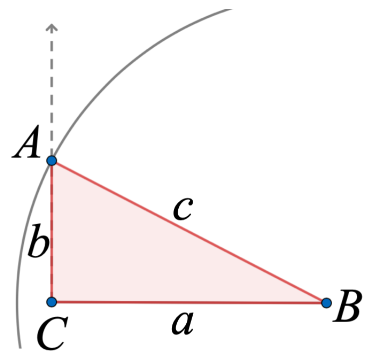
It follows that if we know any two sides in a right triangle, then we know all three sides.

A third fact is that the hypotenuse is the longest side in a right triangle. We will prove later that in any triangle, one side is longer than another in any triangle *if and only if* the angle opposite that side is also larger than the angle opposite the shorter side.

Previously, we gave several ways to prove that two triangles are congruent. These four methods (SAS, SSS, ASA, and AAS) are also useful with right triangles. Some books give them new names in the context of a right triangle. One of these is useful.

hypotenuse-leg in a right triangle (HL)

For two right triangles, if one hypotenuse is equal to the other, and also one pair of legs equal, the two triangles are congruent. This condition is called hypotenuse-leg (HL). It is effectively SSA in the case where we know that the angle is a right angle.



Here we know sides c and a and also have that the angle at C is a right angle.

If two right triangles have hypotenuse and leg equal , then they are congruent by HL. We will use this test often.

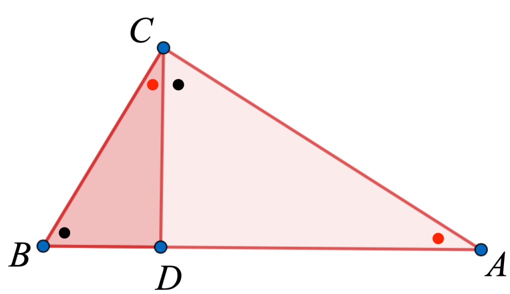
There is only one angle where the hypotenuse will terminate on the vertical extension from the right angle.

Proof.

An algebraic proof is that the third side is determined by the other two sides by the Pythagorean theorem, since we know which side is the hypotenuse. Therefore we have SSS.

□

altitudes



Let $\triangle ABC$ be a right triangle, with a right angle at B. We know that the base angles add up to a right angle.

$$\angle BAD + \angle BCD = 90$$

When we draw the perpendicular to the hypotenuse that goes through the upper vertex, that is an *altitude* of the triangle. Because of the right angle, we obtain two smaller right triangles. Thus

$$\angle BAD + \angle ABD = 90$$

It follows that

$$\angle ABD = \angle BCD$$

(red dots). For the same reason (black dots):

$$\angle BAD = \angle CBD$$

This is a very useful result: if the altitude to the hypotenuse is drawn in a right triangle, the two smaller right triangles are both similar to the original one. All three triangles have the same angles.

theorems about right triangles

- In any right triangle, the right angle is larger than either of the other two angles.

Proof.

Suppose α and β are complementary angles in a right triangle, Then $\alpha + \beta$ is equal to one right angle.

$$\alpha + \beta = 90$$

Both angles α and β must be non-zero: $\alpha > 0$ and $\beta > 0$ (otherwise we do not have a triangle).

$$\alpha > 0$$

$$\alpha + 90 > 90$$

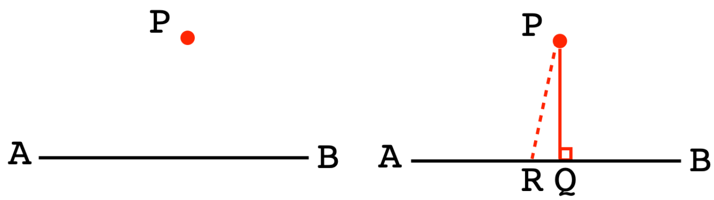
$$90 > 90 - \alpha = \beta$$

The same proof applies starting with $\beta > 0$.

□

only one perpendicular to a line from a point

Suppose we have a line and a point not on the line.



- We claim that only one perpendicular can be drawn from the point to the line.

Proof.

Assume that two such lines can be drawn. So, in addition to $PQ \perp AB$, we draw PR and claim that it is also perpendicular to AB.

Then, by the converse of the alternate interior angles theorem, $PQ \parallel PR$.

But PQ and PR also meet at the point P. This contradicts the fundamental definition of parallel lines. Our assumption must be false.

Only one such line can be drawn.

□

hypotenuse longest side

- In any right triangle, the hypotenuse is longer than either side.

Proof.

We showed above that in any right triangle, the right angle is larger than either of the other two angles. By **Euclid I.18**: in any triangle, a greater side is opposite a greater angle.

□

Or we look ahead again to the Pythagorean theorem. Since $a > 0$ and $b > 0$ and

$$c^2 = a^2 + b^2$$

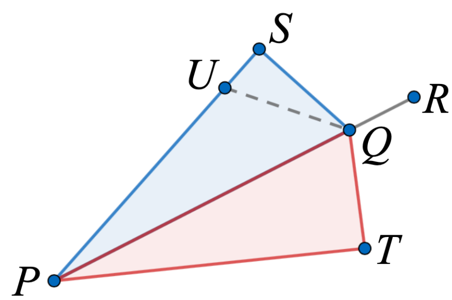
Suppose $b > a$. Since $a > 0$, it follows that $c^2 > b^2$ so $c > b > a$.

The fact that the hypotenuse is the longest side in a right triangle will come in handy when we investigate the tangent to a circle. It is also useful in the next theorem.

shortest distance from a point to a line

• The distance from a fixed point to a line is least when the new line segment makes a right angle with the line.

The claim is that if $QS \perp PS$, then QS is the shortest line connecting Q with PS.



Proof.

Aiming for a contradiction, suppose that QU is not perpendicular, but it is shorter than QS.

$\triangle QUS$ is a right triangle, with the right angle at S, so QU is the hypotenuse of $\triangle QUS$.

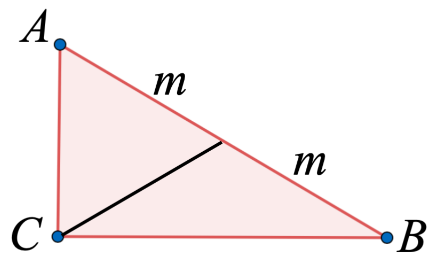
Since the hypotenuse is the longest side of a right triangle, by the previous theorem,

So QS must be shorter than QU.

This is a contradiction. Therefore QU is not shorter than or even equal to QS.

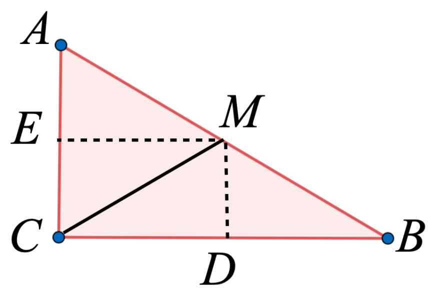
hypotenuse midpoint theorem

In a right triangle, draw the line segment from the vertex that contains a right angle to the midpoint of the hypotenuse, separating it into two equal lengths m. We will show that the length of the bisector is also m.



Proof.

Draw the perpendicular from the midpoint M to the base BC at D. Also draw the perpendicular from M to the base AC at E.



$\triangle MDB$ is a right triangle, and so is $\triangle AEM$.

By complementary angles the other angles are equal.

We are given that $AM = MB$.

It follows that $\triangle MDB \cong \triangle AEM$ by ASA.

Therefore $EM = DB$.

Because it has four right angles at its vertices, $EMDC$ is a rectangle. (The fourth, at M, follows by sum of angles).

Thus $EM = CD = BD$.

So $\triangle MDB \cong \triangle MDC$ by SAS.

It follows that $MC = BM = AM$.

$\square$

Note that both $\triangle AMC$ and $\triangle CMB$ are isosceles.

There are two easier proofs of the above theorem. The first uses Thales' theorem, which we will cover again later, but mentioned at the beginning of the book. Any right triangle can be placed in (inscribed into) a circle, with the hypotenuse as the diameter. AM as well as BM and CM are all radii of this circle.

Another proof uses similar triangles. However, we have not established that theory yet, so we will skip it for now.

converse

As a converse, if we are given that the line drawn to the midpoint of the longest side of any triangle also has length m, then the triangle is a right triangle.

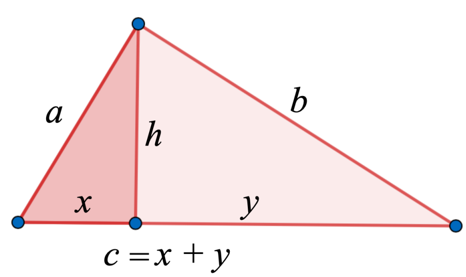
Proof.

The two smaller triangles are isosceles. Therefore, the total angle at vertex C is half the total for the triangle, and thus equal to one right angle.

□

Challenge.

Above we had the result that the vertical line to the hypotenuse (called an *altitude*), forms two smaller right triangles similar to the first. One can construct equal ratios and use them to make a proof of the Pythagorean theorem. We leave this up to you, but will return to it when we have laid out the basic theory of similarity.



Chapter 7

Bisection

Here we look at the problem of angle bisection, cutting an angle into two equal parts. We also look at the perpendicular bisector of any line, a perpendicular line that cuts halfway between two end points, or a vertical through any particular point on a line, or third, through a point not on the line.

We will show that every point on the perpendicular bisector is equidistant from the two points used to construct it.

We also prove the converse theorem, that *every* point which is equidistant lies on the bisector.

constructions

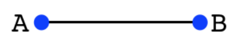
A number of the propositions in book 1 of Euclid concern constructions.

The tools we have are a straight-edge and a compass. The compass is collapsible, meaning that it cannot be used to transfer distances since it loses its setting when lifted from the page. This is a limitation Euclid solves in the second and third propositions of book I. It's also important that the straight-edge is not a ruler, there is no way to measure distance by reference to marks on it.

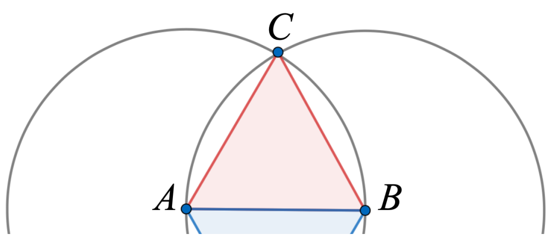
Euclid was smart enough to know about compasses and how to set them. The idea he had was this: to make the fewest possible assumptions. A non-collapsible compass was a luxury he didn't need, since he could accomplish the same end without it.

Euclid I.1: equilateral triangle

To construct an equilateral triangle on a given line segment.



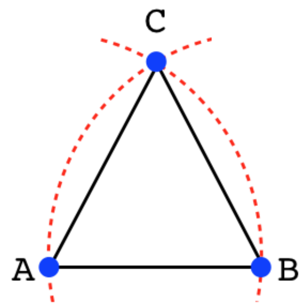
The first step is to draw two circles on centers A and B.



The circles are drawn with each radius equal to the line segment AB. It is a property of circles that all points on the circle are at the same distance from the center. Thus all points on the left-hand circle are equidistant from A, and all points on the second one are equidistant from B.

Therefore, the point C where the circles cross is equidistant from *both* A and B.

Now use the straight edge to draw $\triangle ABC$. Since $AC = AB$ and $BC = AB$, we know that $AC = BC$. The triangle is equilateral.

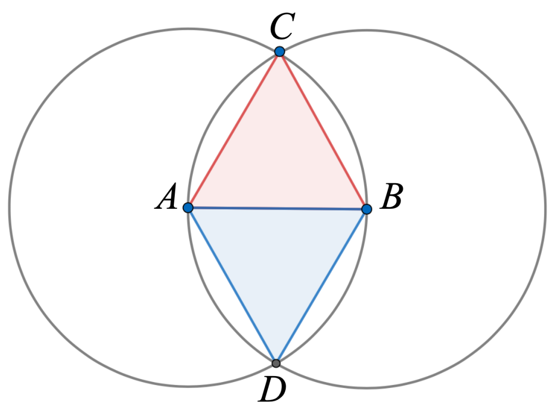


□

The proof doesn't stand on its own. We used one definition (D) and a common notion (CN).

- D I.15 all radii of a circle are equal.
- CN I.1 things which equal the same thing also equal one another.

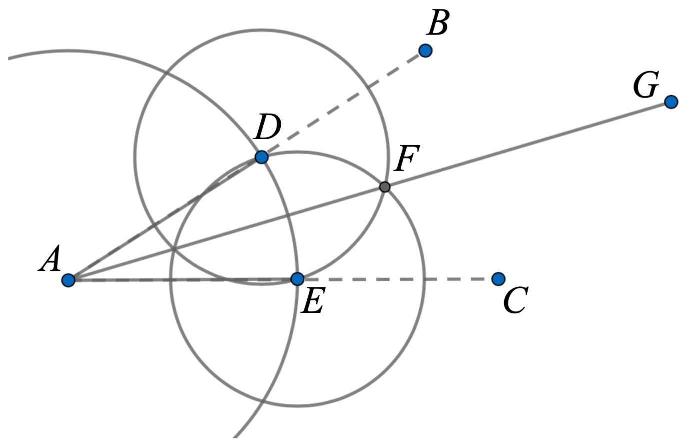
If we look again at the figure, and label the other point where the circles cross as D:



We have a second equilateral triangle, congruent to the first.

Euclid I.9: bisection of an angle

To bisect a given angle. Let $\angle BAC$ be the angle to bisect.

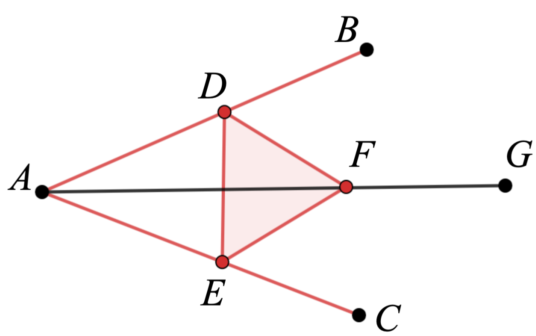


As radii of a circle on center A, we first find points D and E equidistant from A. So $AD = AE$.

Next, we draw circles on centers D and E that have the same radius, and the easiest way to do that (with a collapsible compass), is to make the radius equal to DE.

As radii of circles on the centers D and E, $DE = DF = EF$ and $\triangle DEF$ is equilateral.

Here is a somewhat cleaner view.



AF is shared, so $\triangle ADF \cong \triangle AEF$ by SSS, Euclid I.8.

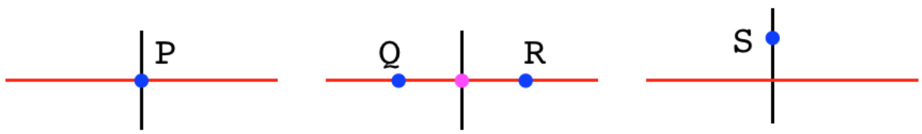
Therefore $\angle BAF$ is congruent to $\angle CAF$ and the given angle $\angle BAC$ is bisected.

$\square$

perpendicular lines

When constructing a line segment perpendicular to another line segment, there are three common situations. We want the perpendicular line to pass:

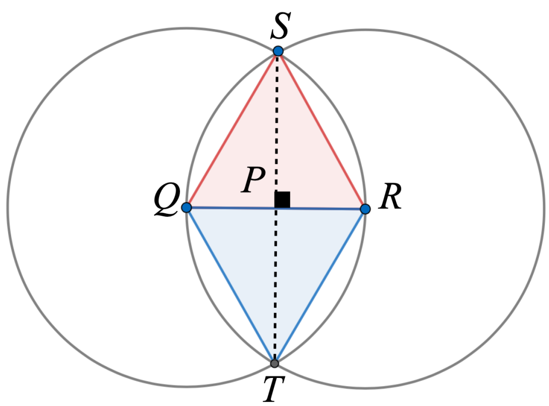
- through a given point P on the line.
- halfway between two points Q and R on the line, bisecting QR
- through the line and also through a point S not on the line



We solve the second case and then show how the other two can be adapted to it.

Euclid I.10: perpendicular bisector

Simply construct two circles of equal radius, one centered at Q and the other at R. It's easiest to choose the radius to be equal to the length QR.

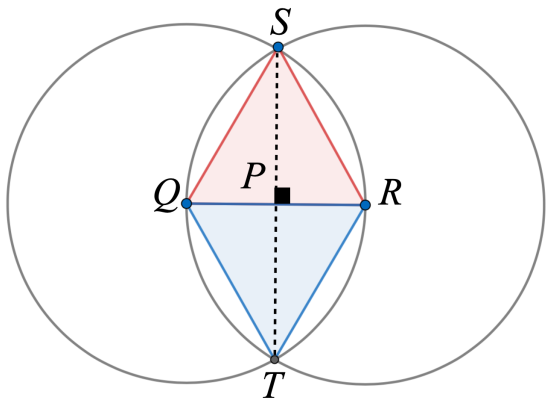


As before, the point S is on both circles, hence it is a radius of both, and therefore equidistant from Q and R. $QS = SR = QR$. The three points form an equilateral triangle, $\triangle QRS$.

The point T below the line segment has the same property, and $\triangle QRS$ is congruent to $\triangle QRT$ by SSS.

Furthermore, we claim that the angles as P are right angles. Thus, $SPT \perp QPR$.

Euclid's proof is simple.



Proof.

$\triangle QST \cong \triangle RST$ by SSS, and both are isosceles.

It follows that $\angle QSP = \angle RSP$, i.e. $\angle QSR$ is bisected.

So $\triangle QPS \cong \triangle RPS$ by SAS.

This gives $QP = RP$.

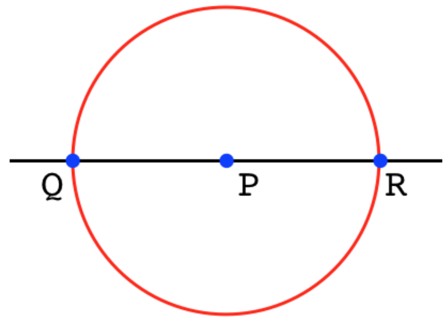
Also, the angles at P, $\angle QPS = \angle RPS$, so all four are right angles.

$\square$

Euclid I.11: bisector through a given point on the line

Suppose we know a point P on the line and wish to construct the vertical line through P.

Use the compass to mark off points Q and R on both sides of P, equidistant from it. This can be done by drawing a circle with center at P.

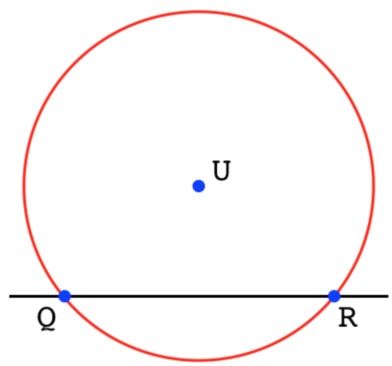


Now, simply proceed as above.

Euclid I.12: bisector through a given point not on the line

Alternatively, suppose we know the line and the point U but not P, and we wish to construct a vertical through the line that also passes through U.

Find Q and R on the line an equal distance from U ($QU = RU$), as radii of a circle centered at U (left panel, below). Their exact position is unimportant.



Now repeat the previous construction, using Q and R. The line segment ST passes through U, as required.

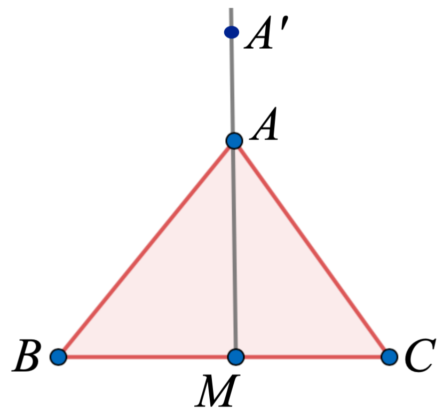
Proof. (Sketch).

Since $QU = RU$, $\triangle QUR$ is isosceles. Therefore the base angles are equal. In an isosceles triangle, the top vertex lies on the perpendicular bisector of the base (see the chapter on isosceles triangles.

Alternatively, find a point below the line equidistant from Q and R. Proceed as in the proof above.

bisector is the altitude of an isosceles triangle

Suppose we know two points B and C. We find the point M equidistant between them and construct the perpendicular bisector AM. Then the two sides AB and AC have equal length. $\triangle ABC$ is isosceles.



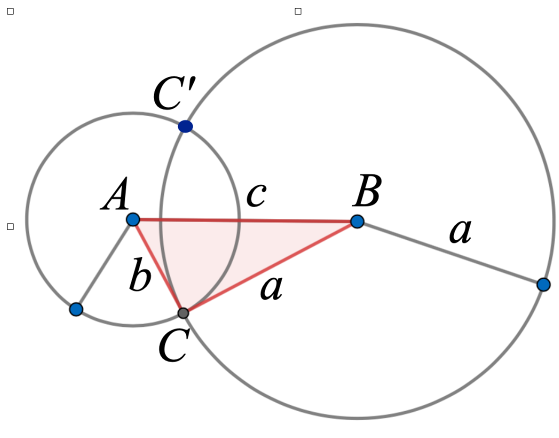
That is true for *any* point on the line extended through A and M. For example, $A'B = A'C$ in the figure above.

perpendicular bisector converse

The converse theorem says that *every* point which is equidistant from B and C lies on AM or an extension of it.

Here are two proofs. The first one relies on the **triangle inequality**, which says that in any triangle, the sum of any two sides must be greater than the length of the third side.

We give a plausible argument for the triangle inequality based on a construction. (We'll look at Euclid's proof later). The theorem is trivially true if the third side is not the longest. Hence we compare the two shorter sides to the longest one.

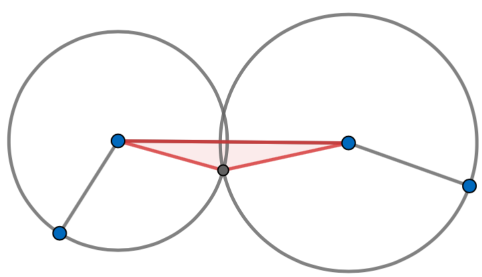


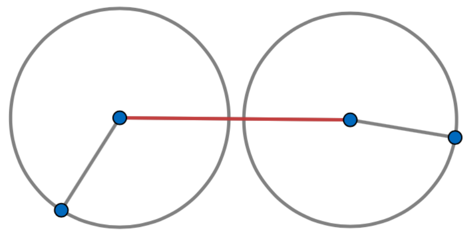
Let AB be the points connected by the longest side c. We claim that the sum of the lengths of the other two sides a and b must be such that:

$$a + b > c$$

Draw two circles, one on center A, and the other on center B, with radii a and b the same as the two shorter sides. There are two points where those circles cross, C and C'. Triangles constructed using either one of those as the third vertex are congruent by SSS. (Mirror images are fine).

Now consider what happens as one of those two sides gets shorter. Let a be the side that is decreased in length.





As the combined length drops to be less than the longest side, it is no longer possible to construct a triangle.

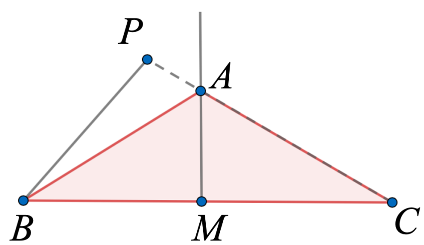
□

To the current proof:

Proof.

Suppose that P is equidistant from B and C but does not lie on the perpendicular bisector. Suppose that P lies on the same side of the bisector as B.

Find the point where PC crosses the bisector at A.



By the forward theorem, $AB = AC$.

We are supposing that $PB = PC$. By the triangle inequality

$$PB < AB + AP$$

Since $AB = AC$:

$$PB < AC + AP = PC$$

But this is absurd. PB cannot both be equal to and less than PC. Therefore, our supposition is incorrect, and there does not exist any such point P.

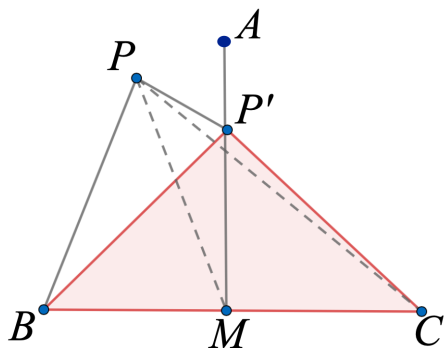
□

See **here** for a somewhat fuller explanation.

perpendicular bisector using I.7

We will prove that every point which is equidistant from two points on a line lies on the perpendicular bisector.

Let $AM \perp BC$, and bisect it such that $BM = CM$. AM is the perpendicular bisector of BC.



Proof.

Suppose P lies on the same side of AM as B.

Now, seeking a contradiction, suppose PM is also perpendicular to BC at M.

PM bisects BC so it is a perpendicular bisector of BC.

By the forward theorem $PB = PC$.

Now find P' on AM such that $P'B = PB$.

We have three equal segments: $PB = PC = P'B$.

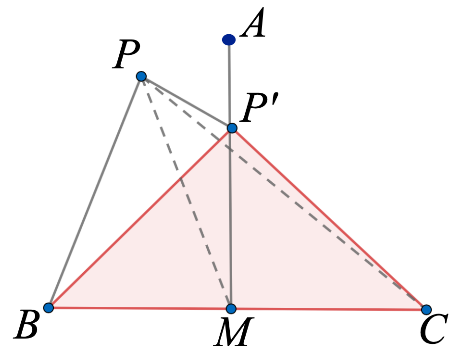
By the forward theorem, since P' is a point on the perpendicular bisector, $P'B = P'C$.

Now we have four equal segments: $PB = PC = P'B = P'C$.

But by Euclid I.7, this is impossible. It cannot be that $P'B = PB$ and also $P'C = PC$ on the same side of BC.

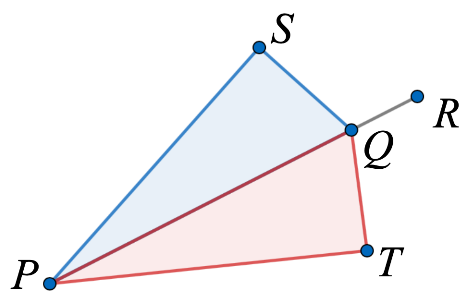
This is a contradiction. There is only one perpendicular through BC at M.

□



bisector equidistant from sides

• Any point on the bisector of an angle is equidistant from the sides at the point of closest approach.



Proof.

Let PR bisect $\angle SPT$.

Draw perpendiculars QS and QT.

The angle at P is bisected, so $\triangle PQS$ and $\triangle PQT$ have two angles equal, and by sum of angles they have three angles equal.

They share the hypotenuse, PQ.

So $\triangle PQS \cong \triangle PQT$ by ASA.

It follows that $QS = QT$.

$\square$

equidistant from sides → bisector

- If a point is equidistant from the sides of an angle, then it lies on the angle bisector.

Proof.

Given $QS = QT$. We have PQ shared.

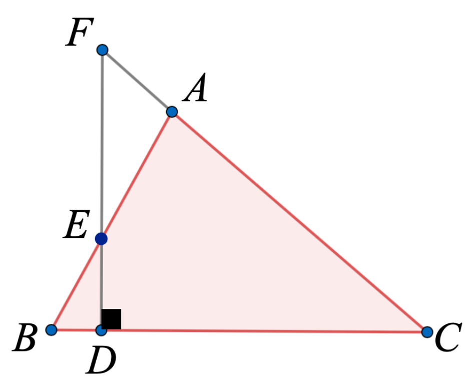
So $\triangle PQS \cong \triangle PQT$ by HL.

It follows that $\angle PQS = \angle PQT$.

□

problem

Here is a problem to exercise some concepts we've seen to this point: isosceles triangles, complementary angles, and vertical angles.



Given that $AB = AC$. Pick any point D on the base BC (except directly under A), and draw the vertical DF. Extend AC to meet that vertical line.

Prove that $\triangle AEF$ is isosceles.

problems

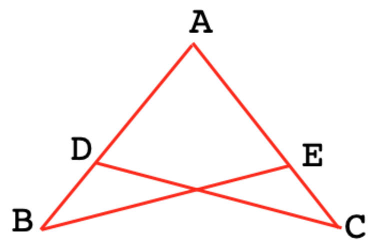
To prove:

○ Prove that for two supplementary angles, the angle bisectors are perpendicular to each other.

○ Prove that an equilateral triangle (all 3 sides equal) is equiangular (all 3 angles equal). (Don't just rely on symmetry. Adapt the proofs given in this chapter).

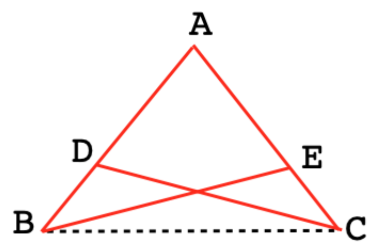
∘ A line perpendicular to the bisector of an angle cuts off congruent segments on its sides.

In the figure below, given that $AC = AB$ and $\angle B = \angle C$.



Prove that $BE = DC$.

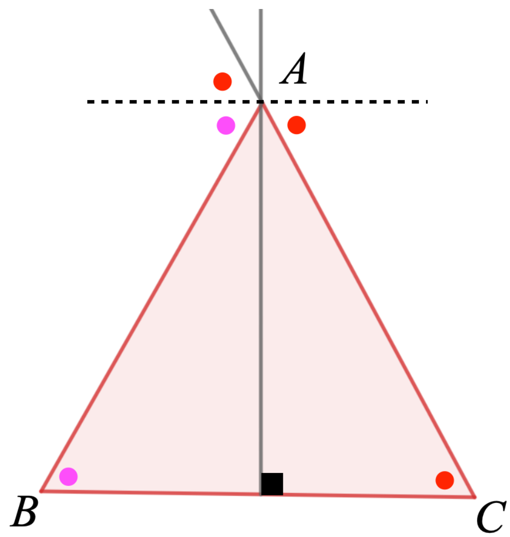
Hint: draw BC and then mark all the angles that are equal.



∘ An equilateral triangle has all three angles equal.

problem

155. If one of the equal sides of an isosceles triangle be extended at the vertex and a line be drawn through the vertex parallel to the base, this line will bisect the exterior angle.

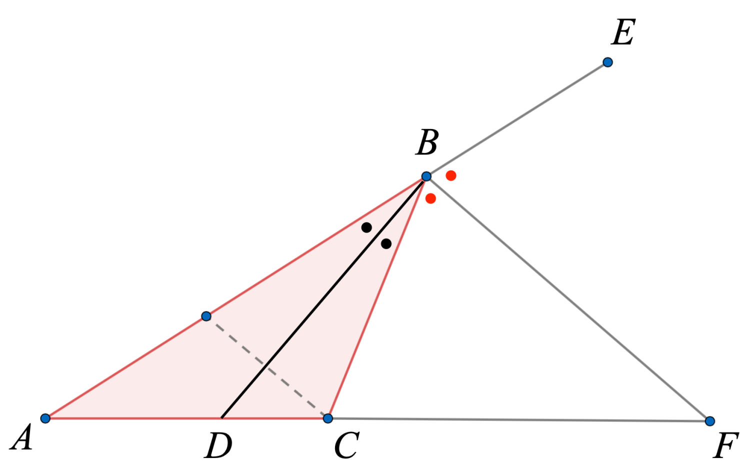


The angles labeled with black dots are equal by alternate interior angles and then vertical angles, while the angles labeled magenta are equal by alternate interior angles.

But we are given that this triangle is isosceles, so black and magenta are equal. Therefore the exterior angle at A is bisected by the horizontal.

Alternatively, use the fact that the exterior angle is the sum of the two base angles, and just use alternate interior angles once.

It is generally true that the adjacent bisectors of an internal and external angle form a right angle.



$\angle DBF$ is right.

Proof.

Twice $\angle DBC$ plus twice $\angle FBC$ is equal to two right angles.

The result follows easily.

□

Part III

Area

Chapter 8

Quadrilaterals

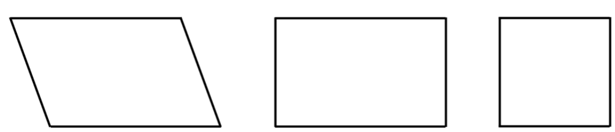
classification

Polygons are constructed from straight sides. If the sides are all the same length, the figure is a *regular* polygon.

A polygon may have 3 or more sides: triangles (3), hexagons (6), and so on. There is a famous theorem from Gauss that involves the construction of a 17-sided regular polygon.

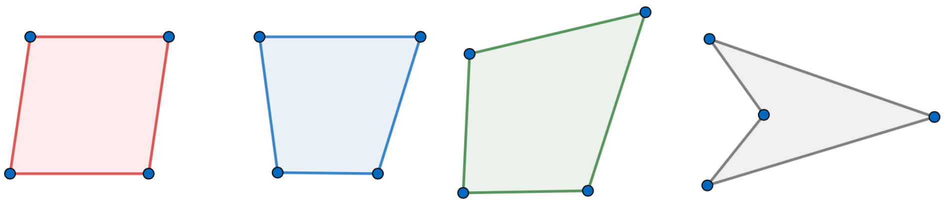
Let us start with four-sided polygons, called quadrilaterals. There are several types, of which the most important are:

- parallelogram: opposite sides equal and parallel
- rectangle: a parallelogram with four right angles
- square: a rectangle with four sides equal



We usually specify these shapes according to the most restrictive conditions they meet. When we think about a parallelogram we mean one without right angles, and when we think about a rectangle, we mean one that is not a square.

There are a few more quadrilaterals to mention in passing (left to right in the figure below):



- rhombus: a square-like parallelogram with all sides equal
- trapezoid: just two sides parallel
- kite: both pairs of *adjacent* sides equal, one pair of opposite angles equal
- dart: like the kite, but one vertex concave (angle > 180)

parallelograms

Parallelograms are four-sided polygons with

- both pairs of opposing sides parallel
- both pairs of opposing sides equal
- both pairs of opposing angles equal

It is convenient to think of parallelograms primarily in terms of the first property. For example, parallel sides is enough to establish the other two properties.

In fact, each can be derived easily from the others. It is also sufficient if one pair of sides is both equal and parallel.

rectangles

Rectangles are parallelograms with

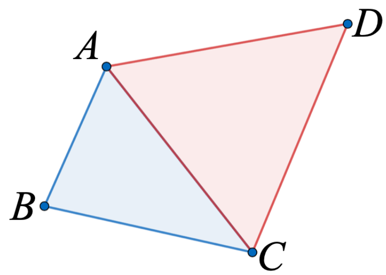
- all four vertices containing right angles

We now look at some fundamental theorems about polygons.

sum of angles theorem

• The sum of all the internal angles in any quadrilateral is the same as that in two triangles, or four right angles altogether.

Proof.



Connect two opposing vertices (A and C above) to form two triangles.

Using the triangle sum theorem, add the component angles at all four vertices.

$$\angle B + \angle BAC + \angle BCA = 180$$
$$\angle D + \angle DAC + \angle DCA = 180$$

But

$$\angle A = \angle BAC + \angle DAC$$
$$\angle C = \angle BCA + \angle DCA$$

So by addition:

$$\angle A + \angle B + \angle C + \angle D = 360$$

□

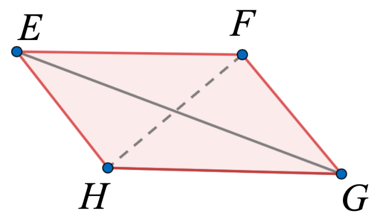
This theorem can be extended to polygons having more sides.

Adding a vertex (with a net addition of one side) is the same as adding another triangle. Correcting for the "base case" we have the sum of angles $S = (n - 2) \cdot 180$.

The proof of the extended theorem is famously done by induction. We cover this elsewhere.

diagonal theorem

- Any diagonal in a parallelogram produces two congruent triangles.



Proof.

parallelogram:

$\triangle EFG \cong \triangle GHE$ and $\triangle EFH \cong \triangle GHF$.

∘ Given $EF = GH$ and $EH = FG$, the result follows by by SSS.

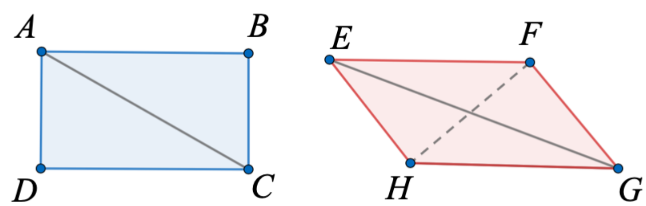
∘ Given $EF \parallel GH$ and $EH \parallel FG$, the result follows by by ASA.

∘ Given $\angle E = \angle G$ and $\angle F = \angle H$, $\angle E + \angle F$ is equal to two right angles, by the sum of angles. Thus, $EF \parallel GH$ and $EH \parallel FG$, and the result follows.

□

Since every rectangle is also a parallelogram, the theorem applies to rectangles.

rectangle: $\triangle ABC \cong \triangle CDA$.



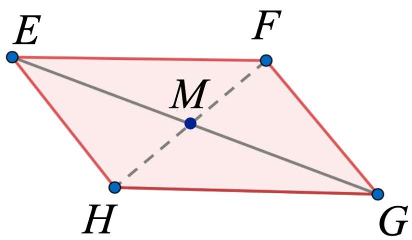
diagonal theorem: equal angles corollary

- In any parallelogram the opposing angles are equal.

This follows immediately from the main theorem.

diagonal theorem: bisection corollary

- The two diagonals in a parallelogram bisect one another.



Proof.

Given $EH \parallel FG$ and $EH = FG$.

$\angle GEH = \angle EGF$ by alternate interior angles.

$\angle EMH = \angle FMG$ by vertical angles.

$\triangle EMH$ and $\triangle FMG$ are equiangular by sum of angles in a triangle.

$\triangle EMH \cong \triangle FMG$ by ASA.

It follows that $EM = GM$ and $FM = HM$.

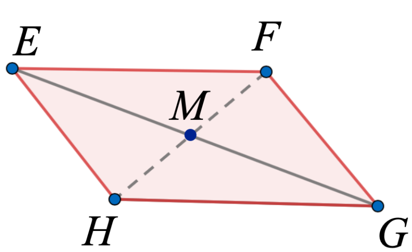
$\square$

special parallelogram theorem

We state this result separately to emphasize it. We will need it for the theory of similar triangles.

In $EFGH$ let $EF = GH$ and $EF \parallel GH$. Then $EFGH$ is a parallelogram.

Proof.



Draw diagonal EG.

By alternate interior angles, $\angle FEG = \angle EGH$.

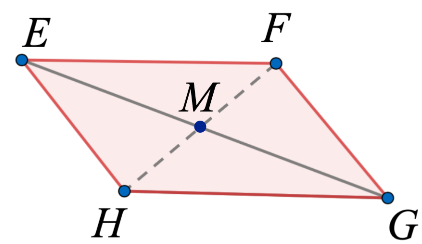
Given $EF = GH$ and EG is shared. $\triangle EGH \cong \triangle GEF$ by SAS.

By the diagonal theorem, it follows that $EFGH$ is a parallelogram.

□

diagonal theorem: converse 1

- If the two diagonals in a quadrilateral bisect each another, the figure is a parallelogram.



Proof.

Given $EM = GM$ and $FM = HM$.

$\angle EMH = \angle GMF$ by vertical angles.

$\triangle EMH \cong \triangle GMF$ by SAS.

It follows that $EH = FG$ and $\angle GEH = \angle EGF$.

So $EH \parallel FG$ by alternate interior angles.

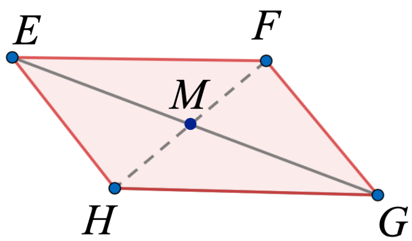
The same logic gives EF parallel and equal to HG.

□

Note that the diagonal theorem and its converse together allow us to interconvert all four specifications for a parallelogram.

diagonal theorem: converse 2

- If the diagonal in a quadrilateral forms congruent triangles, the figure is a parallelogram.



Proof.

Given $\triangle EHG \cong \triangle GFE$

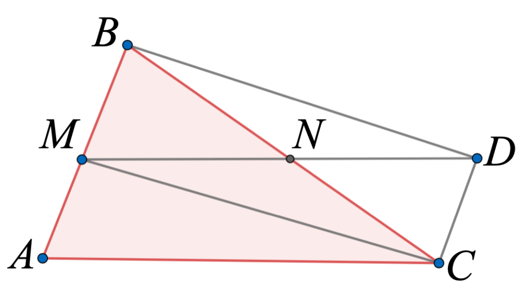
$\angle GEH = \angle EGF$ and $\angle EGH = \angle GEF$.

We have both pairs of opposing sides parallel by the converse of alternate interior angles.

□

midline theorem

- The midline of a triangle is parallel to the third side of that triangle and half its length.



Proof.

Given $AM = BM$ and $BN = CN$. Extend MN to D with $MN = DN$. Draw BD, CM and CD.

$BDCM$ is a parallelogram, by the converse of the diagonal theorem.

$CD = BM = AM$ and $CD \parallel AMB$. It follows that $MDCA$ is a parallelogram.

Hence $AC = MD$ which is twice MN and $AC \parallel MND$.

□

Proof. (Alternate)

With the same construction, $MN = DN$, and given $BN = CN$, and vertical angles, we have $\triangle BMN \cong \triangle CDN$ by SAS.

It follows that $BM = CD$ hence $CD = AM$.

Also from the congruent triangles, $\angle BMN = \angle CDN$. By alternate interior angles, $BMA \parallel CD$.

Thus, $MDCA$ is a parallelogram, with $AC = MD = 2MN$ and $AC \parallel MN$.

□

Varignon's theorem

This famous theorem concerns any quadrilateral. Let's start with the four points lying flat in the same plane.

If we draw the lines connecting the midpoints of each side, the result must be a parallelogram. Here is Acheson's figure:

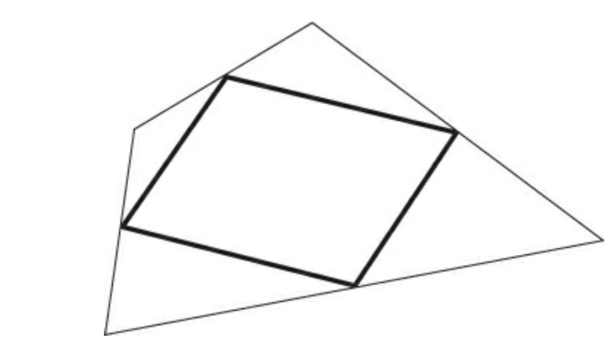


Fig. 50 Varignon's theorem.

To visualize this, imagine the quadrilateral drawn as two triangles connected at the diagonal.

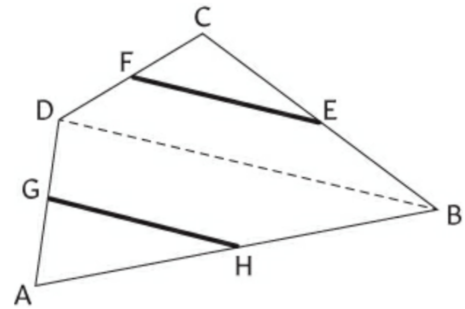


Fig. 51 Proof of Varignon's theorem.

This idea about the diagonal contains the germ of the answer. In the second figure, above, by the midline theorem, $EF \parallel BD$, but also $BD \parallel GH$.

Thus $EF \parallel GH$. We have opposite sides parallel. Repeat with FG and EH to obtain the result.

Now, if we imagine the quadrilateral folding on a hinge at DB, we see that the midlines EF and GH will remain parallel even if C is no longer co-planar with A, D and B.

Chapter 9

Rectangles

minimal specification

In considering a shape that might be a rectangle, and given only some of the properties, the rest may follow. There are too many permutations to go through them all.

First, if we know the figure is a parallelogram, then if it also has at least one right angle, it is a rectangle.

Proof.

By the diagonal theorem, there are two opposing angles that are right angles.

Because opposite sides are parallel, it follows that if one of two adjacent angles is a right angle, then so is the other one, by alternate interior angles.

Thus, all four angles are right angles.

□

If we do not know about the sides but only about the right angles, and that there are four of them, then we have a rectangle.

Proof.

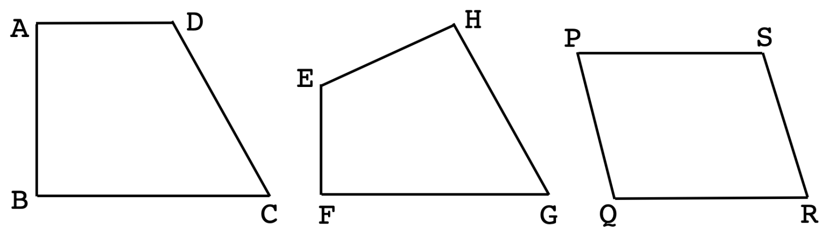
Since the adjacent angles are right, the sides in question are parallel.

A parallelogram with at least one right angle is a rectangle.

□

Even if we know only three angles, by the sum of angles in a quadrilateral, we have four.

However, two right angles are not enough, as counter-examples are easily constructed for both cases (right angles as either neighbors or opposite one another).

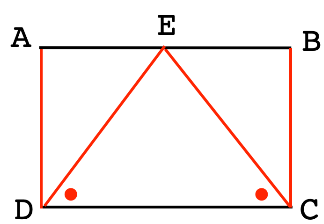


In the left panel $\angle A$ and $\angle B$ are right angles, and in the middle panel, $\angle F$ and $\angle H$ are right angles. But neither $ABCD$ nor $EFGH$ is a rectangle.

If both pairs of sides are equal and parallel, and we don't know there is a right angle, that's not enough by itself.

Here's another permutation.

Let $AD = BC$ and let $\angle A$ and $\angle B$ both be right angles. $ABCD$ is a rectangle.



Proof.

Bisect AB at E and draw the half-diagonals. $\triangle ADE \cong \triangle BCE$ by SAS.

Therefore $ED = EC$ so the angles marked with a red dot are equal by the forward isosceles theorem.

The other components of $\angle C$ and $\angle D$ are equal because of the congruent triangles, so $\angle C = \angle D$.

By the angle sum theorem for quadrilaterals and given the angles at A and B are right angles:

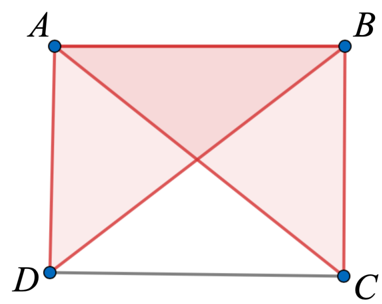
$$\angle C + \angle D = 180$$

Then, both are right angles and thus, all four angles are right angles.

□

diagonals of a rectangle

The diagonals of a rectangle are equal.



Proof.

We consider two overlapping triangles.

$\triangle ABC \cong \triangle BAD$ by SAS.

So $AC = BD$.

Since the diagonals cross at their midpoints, by the diagonal bisector corollary.

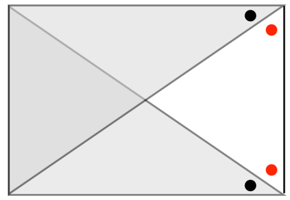
All four half-diagonals are equal.

It follows that the four smaller triangles are all isosceles.

□

We can also invoke symmetry.

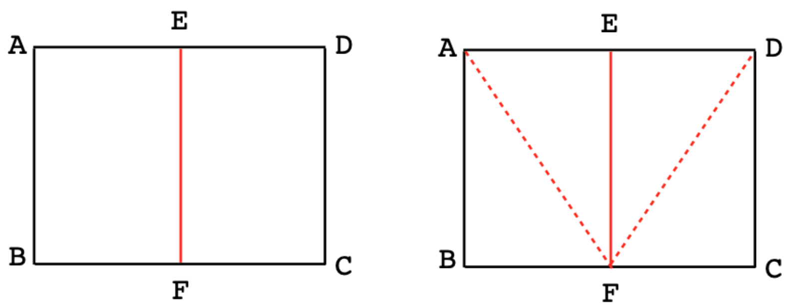
The rectangle has mirror image symmetry by reflection in both the left-right and top-bottom dimensions. As a result, the black-dotted angles in the figure below are equal, and so are the red ones.



And that implies that all four segments from a vertex to the central point are equal in length, by the converse of the isosceles triangle theorem.

bisection of a rectangle

Suppose we are given that $ABCD$ is a rectangle and that EF is the perpendicular bisector of one of the sides, say AD. Then AE is also the perpendicular bisector of the other side, BC.



Proof.

Draw the diagonals of the two small quadrilaterals, namely AF and DF.

Then $\triangle AEF \cong \triangle DEF$ by SAS, using the right angles at E.

But $\triangle AEF$ is also congruent to $\triangle ABF$ (by SAS).

Reasoning in the same way, and using transitivity, we have four congruent right triangles.

It follows easily that the two small rectangles $ABFE$ and $DCFE$ are congruent, so $\angle BFE$ and $\angle CFE$ are right angles, with $BF = CF$.

Therefore EF is the perpendicular bisector of BC.

□

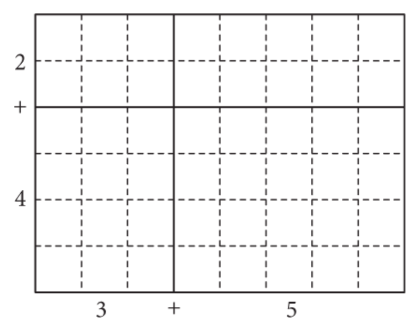
An even simpler proof: given $EF \parallel DC$ so $EDCF$ is a parallelogram. Thus $ED = FC = BF$.

area of a rectangle

Let's say a few brief words about rectilinear area, the area of shapes like squares or rectangles with perpendicular sides. We will then extend this to the areas of triangles and parallelograms, which are like squashed rectangles.

To find areas, we must first fix a unit length. For now, in geometry, we will need an even number of units for each dimension. (There is an exception, but it occurs in a case where we only care about the squared area — see the chapter on the Pythagorean theorem).

Suppose that, in the figure below, the small squares have side lengths of 1 cm, and 6 squares stack vertically and 8 horizontally to fill the shape.



Just multiply the width by the height (in cm) to obtain 48 cm^2.

But then suppose instead that the squares have side lengths of 2.54 cm. Define 1 in = 2.54 cm. The total area would be 48 in^2.

This particular figure above (from Lockhart) shows the distributive law in action:

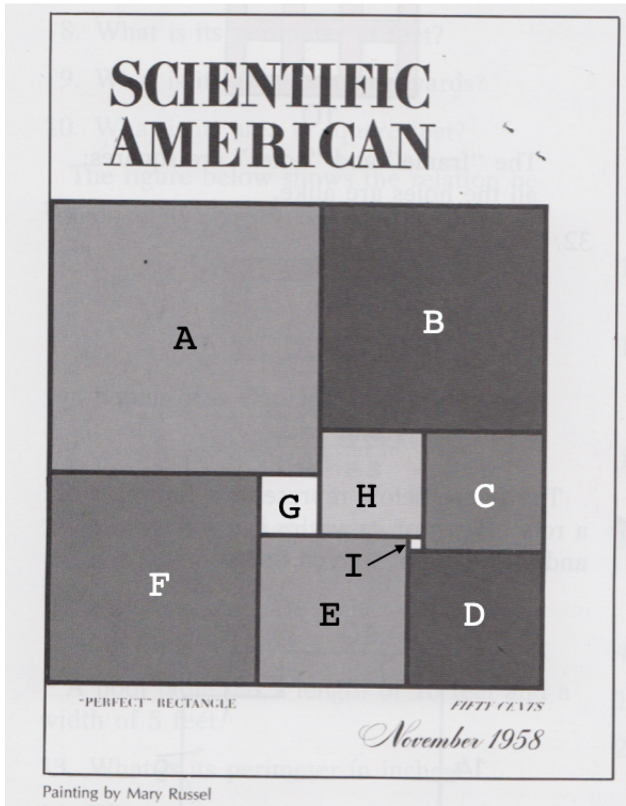
$$(3 + 5) \cdot (4 + 2)$$

$$= 3 \cdot 4 + 3 \cdot 2 + 5 \cdot 4 + 5 \cdot 2$$
$$= 48$$

Any combination of numbers that add up to 8, times any combination of numbers that add up to 6, gives the same result.

problem

From Jacobs, chapter 9.



Suppose each of A through I is a square, and the areas of squares C and D are 64 and 81, respectively.

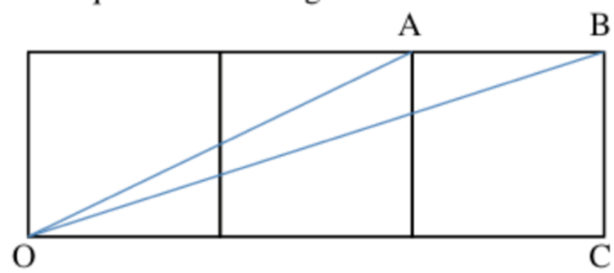
Can you find the areas of all the other squares?

Is the entire figure a square? What is the total area?

problem

Next is a problem from the web. Given three identical squares arranged as follows:

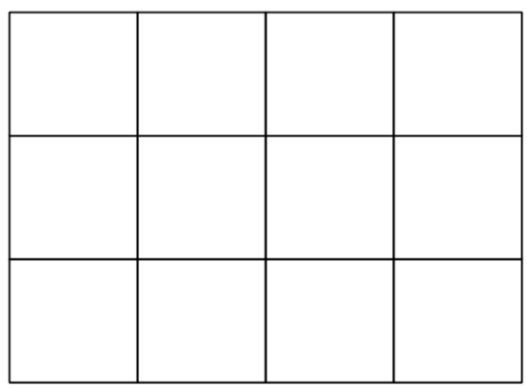
There are three squares in the diagram.



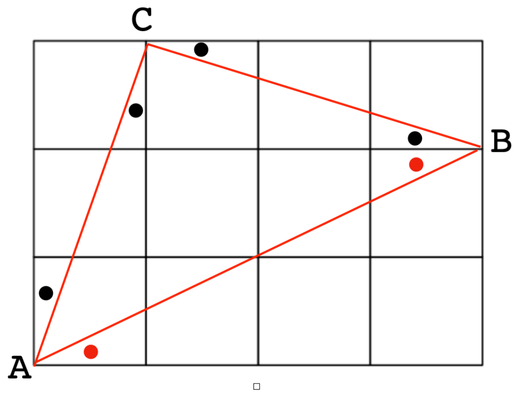
Find $\angle AOC + \angle BOC$.

There is a simple solution, to be obtained without measuring or using trigonometry.

As in so many problems, the key is to draw an inspired diagram, one that extends the figure somehow. Here, a major hint was provided, namely, a grid of squares.



So let's draw the same angles using that grid and form a triangle.



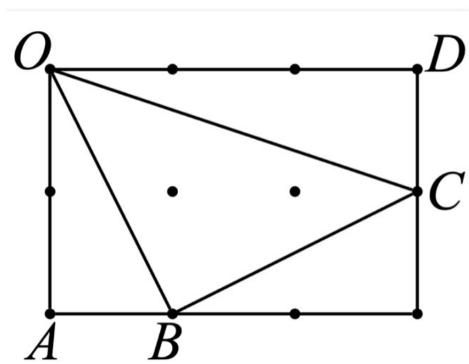
We can identify some equal angles. The two with red dots are equal by alternate interior angles. The two rectangles, one containing the diagonal AC and the other the diagonal BC each consist of three unit squares so they are congruent. That plus alternate interior angles accounts for all the black dotted equal angles.

The last thing we can learn from the diagram is that because of congruent rectangles, $AC = BC$. So that means $\angle BAC$ is equal to $\angle ABC$.

It follows that $\angle BAC$ is one-half the total angle at A, namely one-half of a right angle.

So finally, by sum of angles, $\triangle ABC$ is a right triangle.

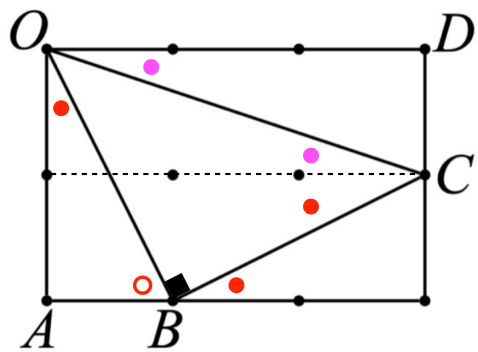
Here is a similar proof, more compactly executed



https://mathenchant.wordpress.com/2022/07/17/twisty-numbers-for-a-screwy-universe/

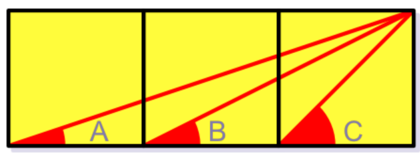
(Endnote 5).

Annotated:



□

Martin Gardner has a version of this problem for which he gives this diagram:



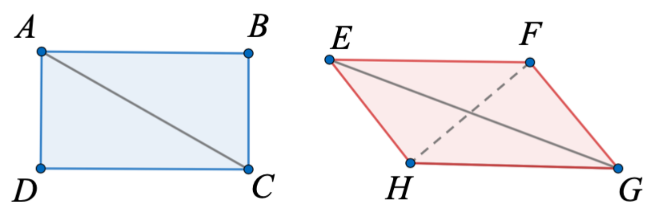
Chapter 10

Parallelograms

diagonal forms congruent triangles

There is a close relationship between parallelograms and triangles. The area of a triangle is one-half the area of the corresponding parallelogram. In the case of a right triangle the corresponding figure is a rectangle.

As we've seen, a line joining opposite vertices of a rectangle or parallelogram is called a *diagonal*. Any diagonal divides a parallelogram into two congruent triangles. The difference is that starting from a rectangle, we obtain right triangles.



$\triangle ABC \cong \triangle CDA$.

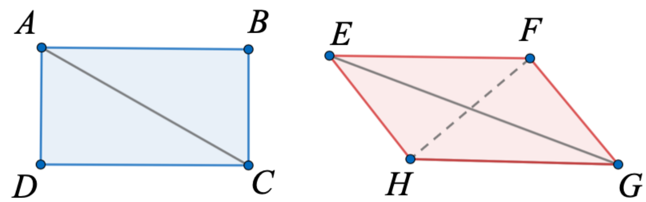
$\triangle EFG \cong \triangle GHE$, and $\triangle EFH \cong \triangle GFH$.

Proof.

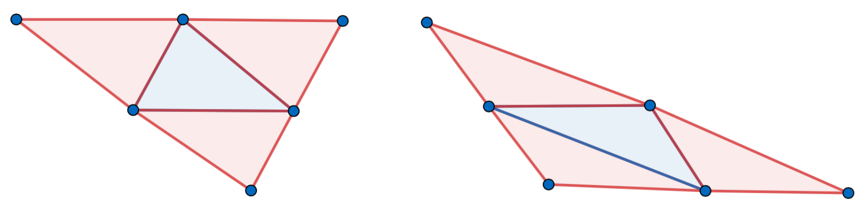
One can use SSS (opposite sides equal), or SAS, or even ASA (opposite sides parallel),

□

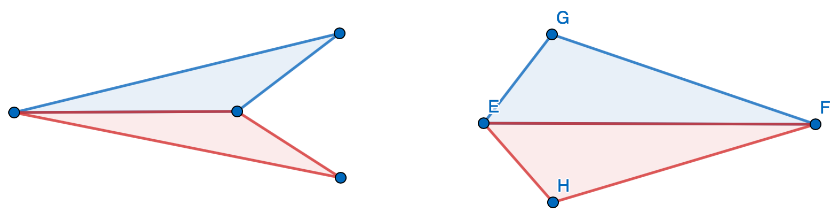
Conversely, two copies of the same triangle (i.e. congruent) can always be joined into a parallelogram, and that figure is a rectangle, if we start with a right triangle.



We can see each of the three possible ways of joining two identical triangles in the figure for the midpoint theorem.



One must be careful, however. The triangles to be joined must not be mirror images, otherwise one may obtain a dart or a kite.



By our fundamental definition of what area is, for a rectangle it is the product of the lengths of two adjacent sides. The area of a parallelogram must be adjusted somehow for the fact that it isn't standing up straight. The easiest way to deal with this is Euclid's approach.

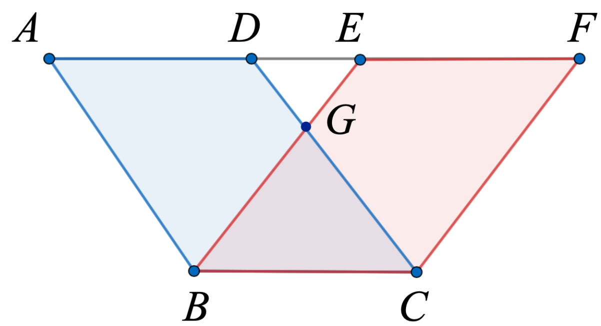
Euclid I.35

As detailed in Book I of *Elements*, here is Euclid I.35. This proposition (theorem) says that given any two parallelograms $ABCD$ and $EBCF$ on the same base BC

and with $ADEF \parallel BC$, they have the same area.

The diagram (below) is drawn for the awkward case, where the two parallelograms overlap.

We have $AB = DC$, $AD = BC$, $BC = EF$ and the sides are parallel as well.



Proof.

Because of the shared base, $AD = EF$.

By addition: $AE = DF$.

We also have $AB = DC$ and because $AB \parallel DC$, $\angle EAB = \angle FDC$.

Thus $\triangle EAB \cong \triangle FDC$, by SAS.

Subtract the shared area of $\triangle DGE$ and add the shared area of $\triangle GBC$ to obtain equality for the area of the two parallelograms.

□

This result is immediately extended to the case where $\angle ABC = \angle DCB$ and both are right angles.

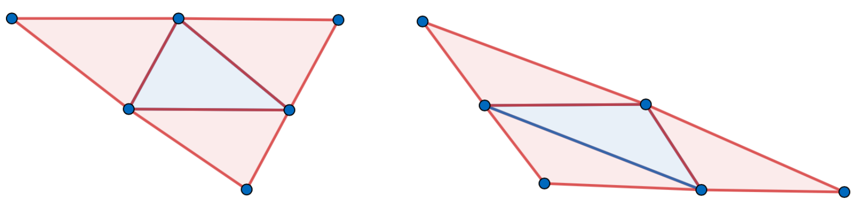
By I.35, this rectangle has the same area as the parallelogram.

Furthermore, the theorem can be invoked sequentially to prove that two parallelograms which do not overlap at all are equal in area, if they lie anywhere on equal bases between two parallel lines.

Roughly speaking, in the figure below, cut off a right triangle from the left side and attach it on the right. The angles add up to form a straight line along the base and a right angle at the upper right. The area is $h \cdot b$.

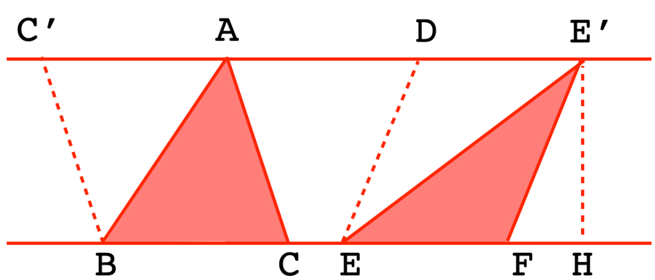
Area of a triangle

To reiterate, any triangle can be turned into a parallelogram, by attaching a rotated image of itself:



It does not matter which side we choose. Any pair of triangles containing the central one is a parallelogram, since it has both pairs of opposite sides equal. We have shown that this is sufficient to know we have a parallelogram.

In the figure below, $ACBC'$ is a parallelogram composed of two copies of $\triangle ABC$. The area of this parallelogram is twice that of $\triangle ABC$. To obtain the value, multiply the base BC times the "height" $E'H$.



$E'H$ is equal in length to the *altitude* of the triangle. That would be a line dropping vertically from A and making a right angle with the base, BC.

The area of the triangle is one-half that of the parallelogram that contains two copies of the triangle.

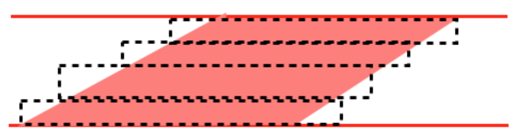
$$\mathcal{A} = \frac{1}{2} \cdot BC \cdot E'H$$

Euclid I.35 and the opposite of dissection, that every triangle can be assembled into a parallelogram that clearly has twice the area, together give everything we need.

If we go back to the idea of cutting off a triangle from one side of a parallelogram and placing it on the other side, to form a rectangle:

it is possible that a parallelogram is particularly skinny for a given height, then it will not be possible to cut off just one triangle.

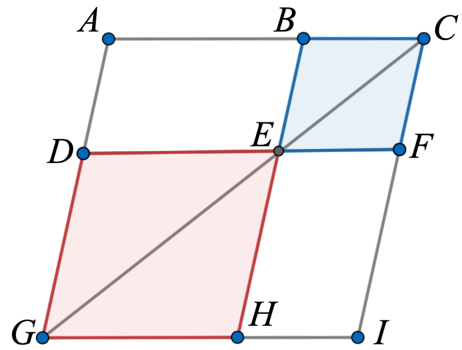
There are at least three solutions to this. One is to rotate the figure so that the sides become the base and the top. A second idea is to slice the parallelogram into horizontal pieces and add them up.



A sophisticated approach is to call on I.43 and change the shape to a different one with equal area.

Euclid I.43

This theorem says that, in the figure below, the two smaller parallelograms not on the diagonal, $ABED$ and $EFIH$, are equal.



Proof.

GEC is the diagonal for three parallelograms and cuts each into two congruent triangles with equal area, by the diagonal theorem.

Subtracting

$$(ACG) - (BCE) - (DEG) = (ABED)$$
$$(CIG) - (CFE) - (EHG) = (EFIH)$$

Since the left-hand sides are equal, it follows that $(ABED) = (EFIH)$.

□

squaring figures

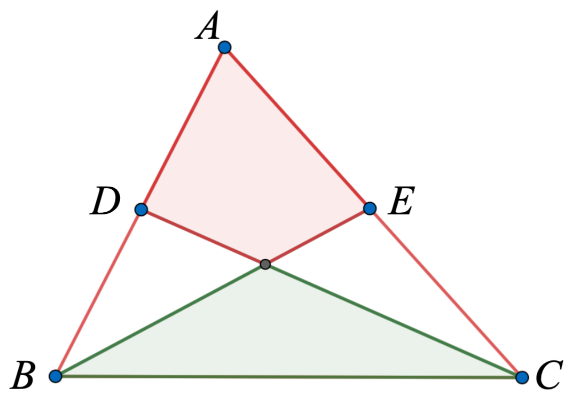
Any parallelogram (such as $ABED$) can be turned into another with the same area but a different shape, simply by inclining the diameter CG at an appropriate angle in theorem I.43.

Any parallelogram can be turned into a rectangle with the same area, as described in this chapter.

Any rectangle can be turned into a square with the same area. This is Euclid II.14, which we will see later.

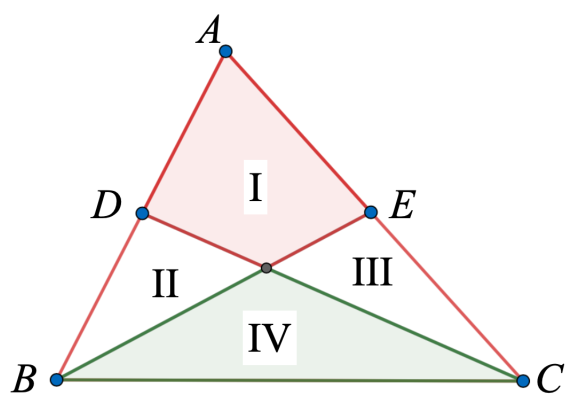
problem

Given that D and E are midpoints of their sides: $AD = DB$ and $AE = EC$. Prove that the colored areas are equal.



Solution.

Let the four triangular areas be labeled $I - IV$.



Then, by the area-ratio theorem and using AC as the base, and since the base is bisected:

$$I + II = III + IV$$

But using AB as the base

$$I + III = II + IV$$

Add the two equations and cancel $II + III$ on both sides:

$$I = IV$$

Subtract the two equations and

$$II - III = III - II$$

$$II = III$$

□

By the **midline theorem**, since DE bisects the sides it is parallel to the base BC.

twice the area

It can be convenient to write the formula as *twice* the area, and we will often do that.

$$2\mathcal{A}_{\triangle} = ab$$

We rewrite two important formulas from this chapter:

$$d \cdot h + e \cdot h = (d + e) \cdot h = c \cdot h$$
$$\frac{\mathcal{A}_A}{\mathcal{A}_B} = \frac{ah}{bh} = \frac{a}{b}$$

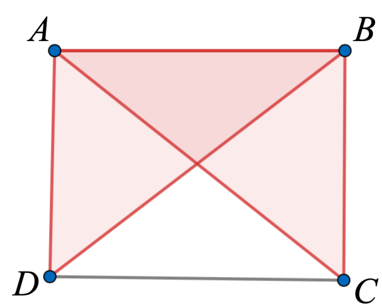
The first one is an equality of different areas, and the second involves a ratio of areas on the left-hand side. The result is unchanged by using twice the area, as long as we are consistent.

Chapter 11

Altitudes

easy theorems

- If a quadrilateral has two adjacent right angles and equal diagonals, it is a rectangle.



Proof.

Let $ABCD$ be a quadrilateral with $\angle A = \angle B$ and both are right angles.

Let $AC = BD$ be the two diagonals.

Compare mirror image right triangles $\triangle ABC$ and $\triangle BAD$.

They have an equal hypotenuse and they share the side AB.

Hence they are congruent by HL, so $AD = BC$.

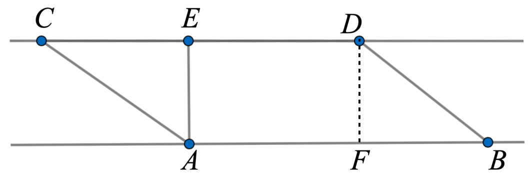
Then $\triangle ADC \cong \triangle BCD$ by SSS.

Thus $\angle C = \angle D$ and by sum of angles both are right angles.

So $ABCD$ has four right angles and thus is a rectangle.

□

- perpendiculars between parallel lines are equal



Proof.

Let AB and CD be parallel lines.

Let $AE \perp AB$, then $AE \perp CD$ by alternate interior angles.

Draw $DF \perp AB$.

DF is also $\perp CD$ for the same reason.

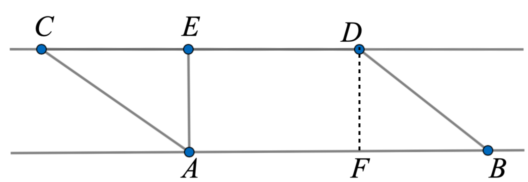
Thus $AEDF$ has four right angles so it is a rectangle.

It follows that $AE = DF$.

□

Any perpendicular is the shortest line segment with one end point on AB and the other on CD.

Proof.



Aiming for a contradiction, suppose BD is shorter, but BD is not $\perp AB$.

Draw the line $AC \parallel BD$ through A.

Then $ACDB$ is a parallelogram, so $AC = BD$.

Let $\triangle ACE$ be a right triangle.

The hypotenuse is the longest side in a right triangle, so $BD = AC > AE$.

This is a contradiction.

A perpendicular is the shortest line segment connecting two parallel lines, and every perpendicular is equal to every other one.

$\square$

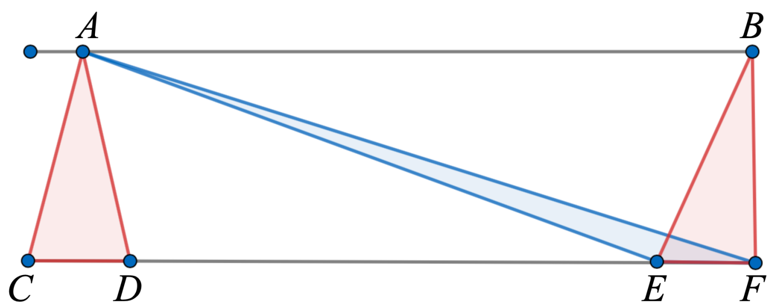
area only depends on base and altitude

An important consequence is that:

- all triangles with the same base and height have the same area.

In the figure below, we have two parallel lines. Mark off $CD = EF$.

Now pick any point on the top and draw the triangle with two equidistant points on the bottom. Any other triangle drawn with an equal base has the same area.



The areas of $\triangle ACD$, $\triangle AEF$, and BEF are equal, because they have equal bases and equal heights.

There are several different conventions for referring to the area of a triangle. One is just to use the capital letter A (for area). To make it stand out, we might use a special font:

$$\mathcal{A}_{\triangle_{ACD}} = \mathcal{A}_{\triangle_{AEF}}$$

That helps but can still become awkward when $\mathcal{A}$ has another meaning in the problem. Some people switch to using K, but a second approach is to use the $\triangle$ symbol, as in

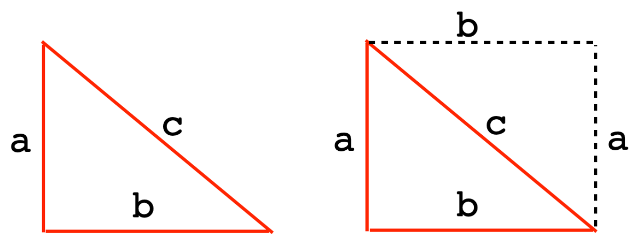
$$\triangle_{ACD} = \triangle_{AEF} = \triangle_{BEF}$$

Euclid always refers to angles as $\angle ABC$ etc., so when he says $ABC = DEF$, he means the *area* of those triangles.

And yet another is to use parentheses. This is a good solution when there are other shapes like rectangles in the problem.

$$(\triangle ACD) = (\triangle AEF)$$

A right triangle has the largest area for a given pair of side lengths. If we imagine a side of length a tilting right or left, then the resulting triangle will have a smaller area, because the altitude h will be less than a.



area-ratio theorem

If in a triangle we draw the line connecting the upper vertex to any point on the bottom side, then the areas of the two smaller triangles are in the same ratio as the lengths of their bases.

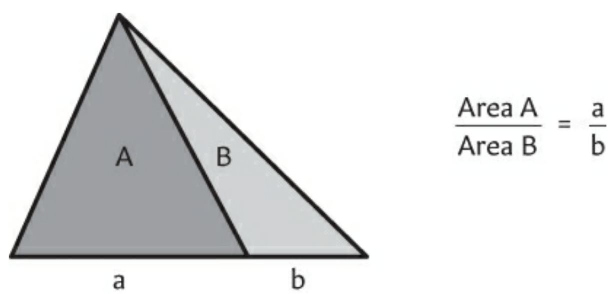


Fig. 120 An area-ratio theorem.

Proof.

The area of $\triangle A$ is $ah/2$, while that of $\triangle B$ is $bh/2$, so the ratio of areas is

$$\frac{\mathcal{A}_A}{\mathcal{A}_B} = \frac{ah/2}{bh/2} = \frac{a}{b}$$

$\square$

It is also the case that the ratio of the area of any sub-triangle to the whole is the same as the proportion of its base length to that of the whole base.

$$\frac{\mathcal{A}_A}{\mathcal{A}_A + \mathcal{A}_B} = \frac{a}{a+b}$$

Proof.

Simple algebra: invert, add one to both sides, and invert again. We show only the right-hand side. Start with the fraction b/a and add 1 to it:

$$\frac{b}{a} + 1 = \frac{b}{a} + \frac{a}{a} = \frac{a+b}{a}$$

Inverted:

$$\frac{a}{a+b}$$

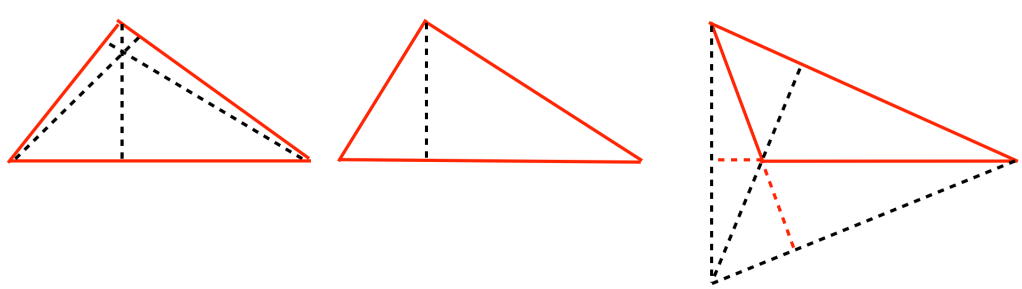
Do the same to both sides and the result follows.

altitudes and the orthocenter

Each of the three sides of a triangle has a corresponding *altitude*, which is the perpendicular line drawn from a vertex to the side opposite, or its extension.

The three altitudes meet at a point (they are said to be concurrent). We will prove this later. The point is called the *orthocenter*.

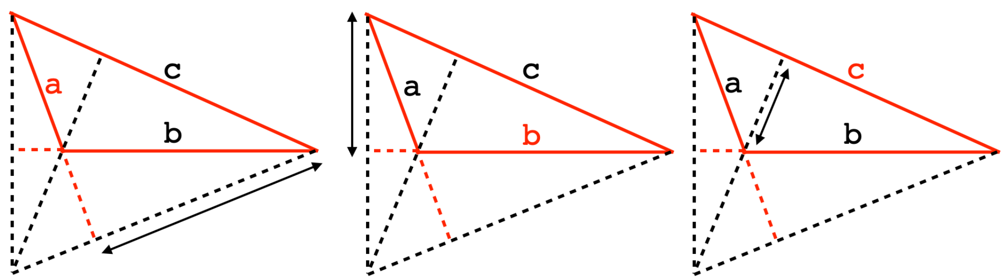
For an acute triangle, the orthocenter lies inside the triangle.



For a right triangle, the orthocenter is just the vertex containing the right angle.

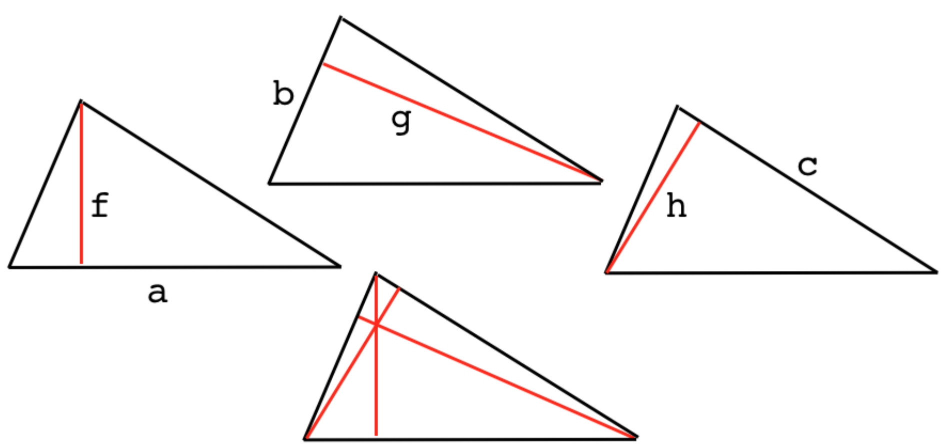
For an obtuse triangle, the orthocenter is external to the triangle, as are two of the altitudes and part of the third.

In the latter case, it may take some thought to determine which altitude goes with which side. The rule is that the altitude forming a right angle with any side originates at the vertex opposite that side. If necessary, the side is extended to meet the altitude at a right angle.



In the figure above, we have one obtuse angle in the triangle. The altitudes to sides a, b and c are indicated, in turn, by arrows.

computing triangular area



Again the simple formula: one-half base times height. In the figure above, twice the area is

$$2A = af = bg = ch$$

We can choose any side of the triangle to be the base and then multiply by the height to get twice the area.

We must always get the same answer! The area of the triangle is surely the same no matter how you calculate it.

Here's a proof by counting up the area of smaller triangles. A simpler proof follows, but this gives practice in defining altitudes.

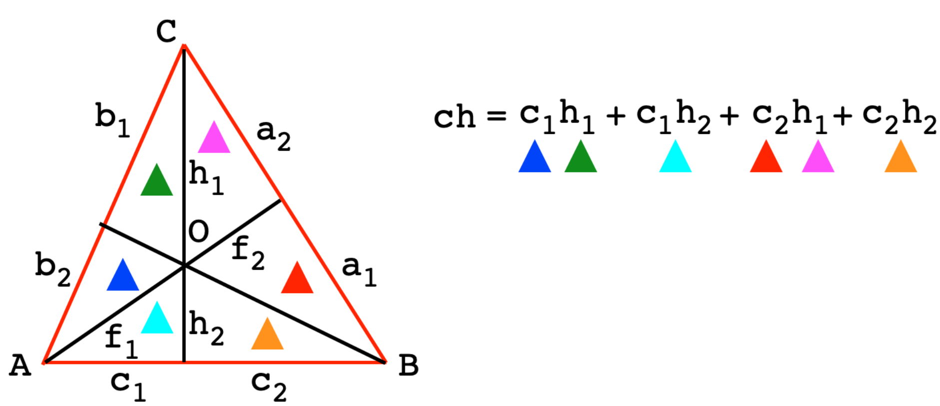
Proof.

In $\triangle ABC$ with sides a, b, c, drop the three altitudes from the vertices to form right angles on the opposing sides. We label two of them: f for side a and h for side c.

These altitudes cross at a single point. We look at Newton's proof of this in just a bit (**here**).

Each altitude and side is then divided into two parts as shown.

This gives six small triangles. To make it easier to keep track of them, they are labeled with colors.



So then twice the area of the whole triangle is $2A = af = ch$. Start with ch

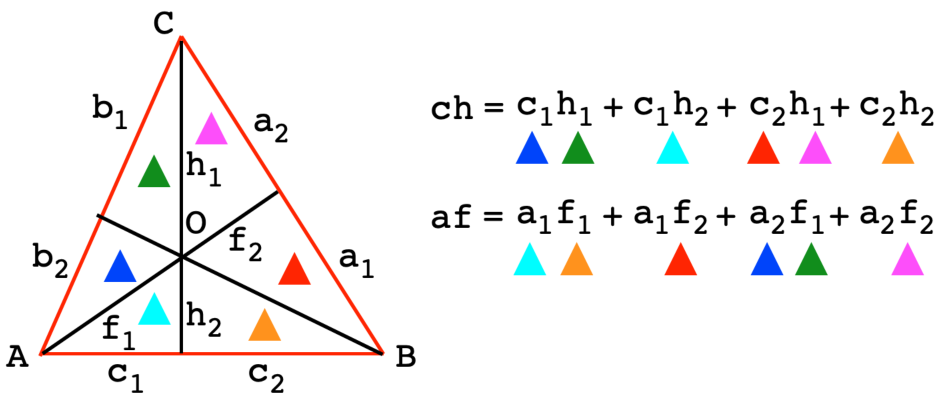
$$ch = (c_1 + c_2)(h_1 + h_2)$$

$$= c_1h_1 + c_1h_2 + c_2h_1 + c_2h_2$$

The single right triangles are easy to see: $c_1 h_2$ and $c_2 h_2$. The other two are composed of two right triangles. For both, the base is h_1, and then, for green and blue, the height is c_1, or for red and magenta, the height is c_2. For these obtuse triangles, the height must be extended to the base to form the right angle.

But the same six triangles can be arranged in a different way so that twice the area of the whole triangle is

$$af = a_1 f_1 + a_1 f_2 + a_2 f_1 + a_2 f_2$$



$f1$ is the base and a_1 or a_2 the height, for the compound cases.

A similar calculation can be carried out for side b and altitude g. The area is the same regardless of which side is chosen as the base.

□

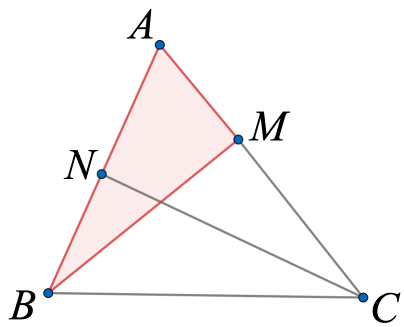
altitude proof

In $\triangle ABC$ draw $BM \perp AC$ and $CN \perp AB$.

Then $AB \cdot CN = AC \cdot BM$.

Proof.

$\triangle AMB$ and $\triangle ANC$ are both right triangles.



They also share $\angle A$, so they are similar.

As similar triangles, corresponding sides are in the same ratio:

$$\frac{AB}{AC} = \frac{BM}{CN}$$

Then $AB \cdot CN = AC \cdot BM$

We can show that side BC times its altitude is equal to $AB \cdot CN$, in exactly the same way.

So any altitude times the base has the same value, which is twice the area of the triangle.

□

A straightforward corollary of this result is that any triangle with two equal altitudes is isosceles.

Proof.

If $BM = CN$ then

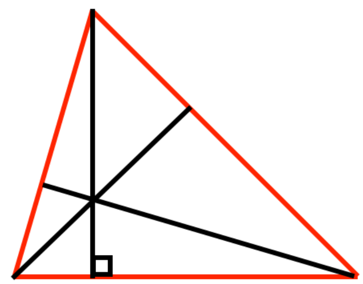
$$\frac{AB}{AC} = 1$$

so $AB = AC$.

□

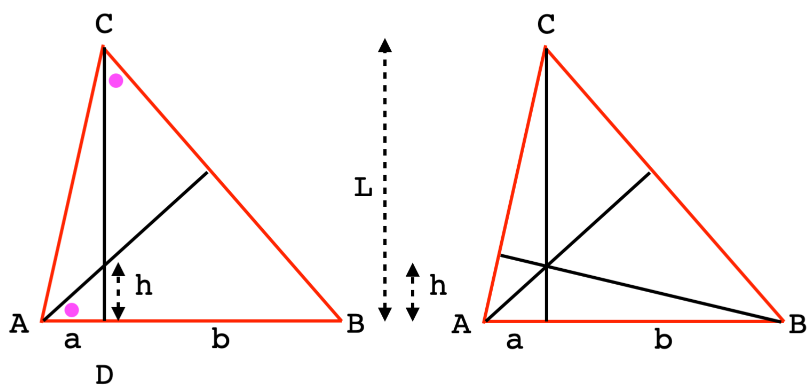
orthocenter: Newton's proof

As we've said, the orthocenter is the point where all three altitudes cross.



orthocenter/altitude

An altitude is a line drawn from any vertex to the opposing side, forming a right angle with the base, thereby dividing the triangle into two right triangles.



In the left panel, we draw the altitude from the vertex C down in $\triangle ABC$ to meet the base at a right angle.

The altitude divides the base into lengths a and b. Now draw a second altitude from vertex A to the side opposite (left panel).

What is the height h above the base where the two lines cross?

The small triangle with sides a and h and the large triangle $\triangle BCD$ are similar, because they are both right triangles that are joined by vertical angles.

This means that the angles marked with magenta dots are equal.

Similar triangles have equal ratios for the corresponding sides. This is a very important theorem which we haven't proved yet, but will soon.

In the small triangle the side opposite the marked angle ($\angle BAC$) has length h, while the entire length of the altitude is L. By similar triangles

$$\frac{h}{a} = \frac{b}{L}$$

(the side opposing the marked angle is in the numerator on both sides). So the height h is

$$h = \frac{ab}{L}$$

The formula is noteworthy because it is symmetrical in a and b and does not contain any term related to side AC, opposite vertex B.

Therefore, if we draw the third altitude to side BC, opposite vertex A, we can calculate that it crosses the vertical altitude at the same height $h = ab/L$ (right panel).

This means that the three altitudes cross at a single point, at height h.

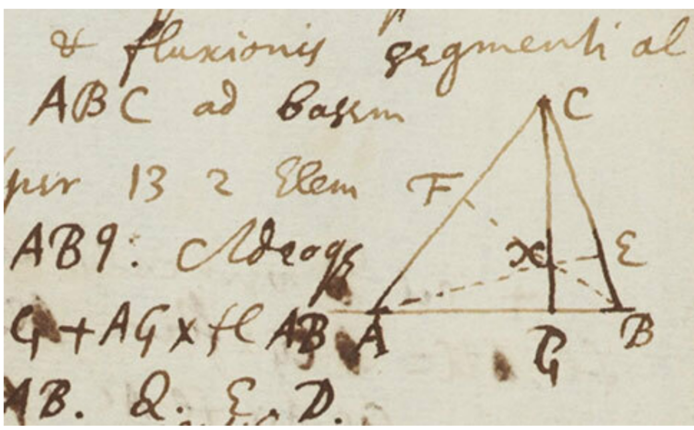


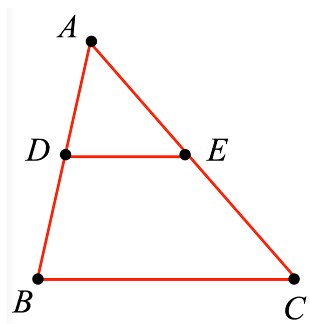
Fig. 129 The diagram for Newton's proof, from his *Geometria curvilinea and Fluxions*, Ms Add. 3963, p54r.

Newton published this proof about 1680.

Chapter 12

Right triangle similarity

In the chapter on congruence, we introduced similar triangles as triangles with the same three angles, but scaled differently.



Let $\triangle ABC \sim \triangle ADE$. ($\sim$ is the symbol for similar). Then they can nestle one inside the other, using any one of the shared angles.

If we do this so that $\angle ADE = \angle ABC$, then by alternate interior angles $DE \parallel BC$.

The other property is the equal proportions of sides.

$$\frac{AD}{BD} = \frac{AE}{CE}$$

equal angles and parallel third side $\iff$ equal ratios of sides.

similar right triangles

We will show an easy proof that for similar right triangles, all angles equal implies equal ratios of sides. Our approach is from Acheson and is equivalent to a previous result (Euclid I.43).

> Draw a rectangle ABCD, and a diagonal AC. Then pick a point E on the diagonal and draw lines through it parallel to the sides.

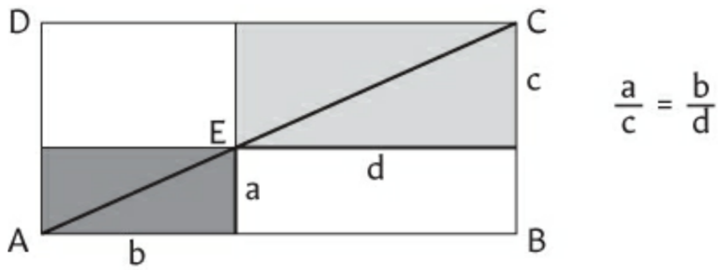


Fig. 42 Area and similarity.

All of the right triangles in the figure are similar. Start with the alternate interior angles theorem, then use complementary angles in a right triangle, and finish with vertical angles.

By changing the height of the figure, we can obtain any ratio a/c that we wish, and by changing the placement of E we can get any ratio a/b that we wish, which amounts to the same thing.

So then, the two shaded rectangles are bisected by the diagonal AEC, by the diagonal theorem. So the two light-gray triangles have equal area, and the two dark gray ones do as well.

But $\triangle ABC$ and $\triangle ADC$ also have equal area.

Therefore, we just subtract equal areas to find that the two unshaded rectangles above and below the diagonal are equal in area. The one on top has area bc and the one below has area ad. We have

$$bc = ad$$
$$\frac{a}{c} = \frac{b}{d}$$

and also

$$\frac{a}{b} = \frac{c}{d}$$

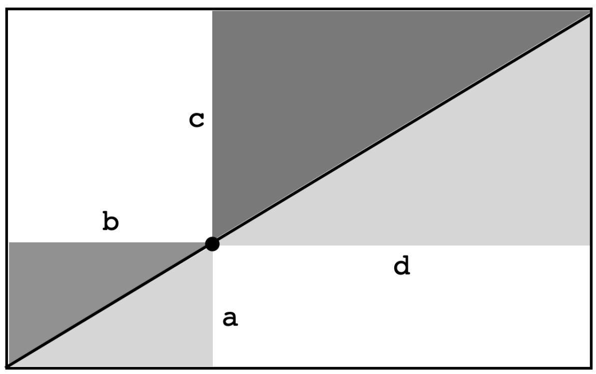
□

Corresponding sides are in the same proportion, but also the ratio of corresponding sides is the same for similar triangles.

similar parts and the whole

We showed that the two smaller triangles have corresponding sides in proportion.

But the large triangle (one-half of the entire rectangle) has the same angles, and should have the same ratios. Here is a simple manipulation to obtain that result:



$$bc = ad$$

$$bc + ab = ad + ab$$

$$b(a + c) = a(b + d)$$

$$\frac{a}{b} = \frac{a + c}{b + d}$$

Given any two of these relationships we can derive the third. This is the same math in reverse.

$$\frac{a + c}{b + d} = \frac{c}{d}$$

$$\frac{a+c}{c} = \frac{b+d}{d}$$
$$\frac{a}{c} + 1 = \frac{b}{d} + 1$$
$$ad = bc$$

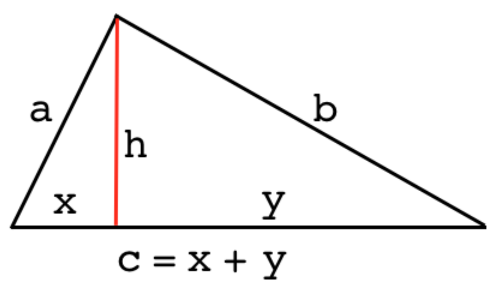
□

hypotenuse in proportion

It is natural to ask, what about the hypotenuse?

Consider any right triangle. Drop the altitude to the hypotenuse.

Using complementary angles, we can show that the two new smaller triangles formed by the altitude are similar to each other and to the original large one.



Now, write the ratio of the long to the short side *for each one* of the three triangles:

$$\frac{h}{x} = \frac{y}{h} = \frac{b}{a} = k$$

From our previous theorem we know this equality is valid.

However the same statement can be viewed in a different way.

b/a is the ratio for the hypotenuse of the medium triangle compared to the small one.

And y/h is the ratio of the long side in the medium triangle to the long side in the small one.

They are equal, and this completes the proof!

□

We can also give an algebraic proof, by looking ahead to the Pythagorean theorem. As you likely know, for a right triangle with sides a and b and hypotenuse g:

$$a^2 + b^2 = g^2$$

We can use the Pythagorean theorem to prove that:

$$\frac{a}{c} = \frac{b}{d} = \frac{g}{h}$$

All of the sides of two similar right triangles have the same ratio.

We must be careful, however. A deep connection exists between similarity, area and the Pythagorean theorem. It is important that we will have Euclid's proof of the Pythagorean theorem, and that proof depends on SAS rather than on similarity.

Equal ratios extends to the hypotenuse.

Proof.

Start with

$$\begin{gathered}\frac{a}{c} = \frac{b}{d} = k \\ a = kc, \qquad b = kd \\ a^2 + b^2 = k^2c^2 + k^2d^2 \\ g^2 = k^2h^2\end{gathered}$$

Since these are lengths, we can take the positive square root and obtain

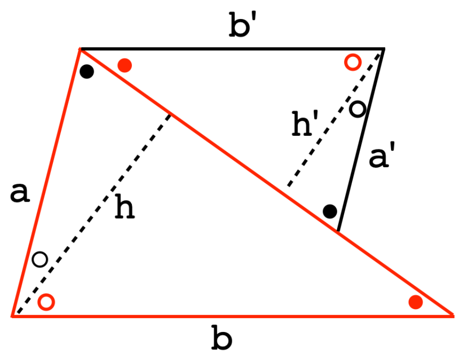
$$\begin{gathered}\frac{g}{h} = k \\ = \frac{a}{c} = \frac{b}{d}\end{gathered}$$

□

Thus, AAA similarity is established for right triangles. If either of the smaller angles matches between two right triangles, then they are not only similar but all the side lengths are in the same ratio as well.

Getting the converse, going from equal ratios to equal angles, uses a result about parallelograms. Also, the proofs apply not just to right triangles, but to triangles of any type. For that reason, we defer further development of similarity to a later chapter.

We proved this above for right triangles, and could extend the result to all triangles by dissection. We just show the figure and skip the proof.

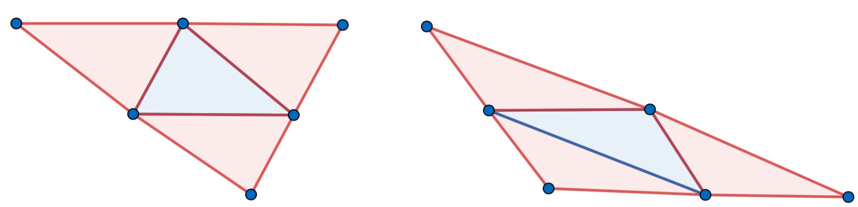


We promise to prove it later when we deal with the theory of similarity more explicitly. Euclid VI.2 does this elegantly.

We can prove it for a special case now.

triangle dissection

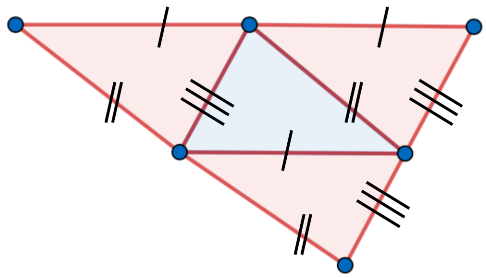
- Any triangle can be dissected into four congruent triangles.



Proof.

Find the midpoints of the sides and connect them.

By the midline theorem each side of the central triangle is one-half the length of the side to which it is parallel.



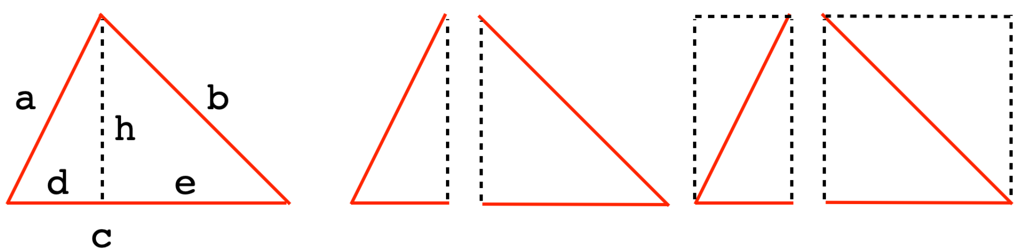
We have all congruent triangles by SSS.

Also, as a result we have 3 parallelograms, each containing two congruent triangles.

□

dissection into right triangles

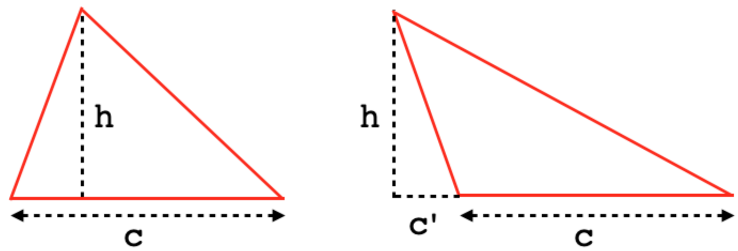
Any triangle can be cut into two right triangles by drawing its *altitude*, h, where h is perpendicular to c.



The area of the triangle with sides a, d, h is $dh/2$, and that with sides a, e, h is $eh/2$ so the area of the original triangle is

$$\frac{dh}{2} + \frac{eh}{2} = \frac{(d+e)h}{2} = \frac{ch}{2}$$

This formula is correct even for an obtuse triangle like the one in the right panel, below. The area of the two red triangles is the same: $ch/2$.



We get that by computing the area of the large triangle with base $c + c'$ and then subtracting the area of the skinny triangle with the base c':

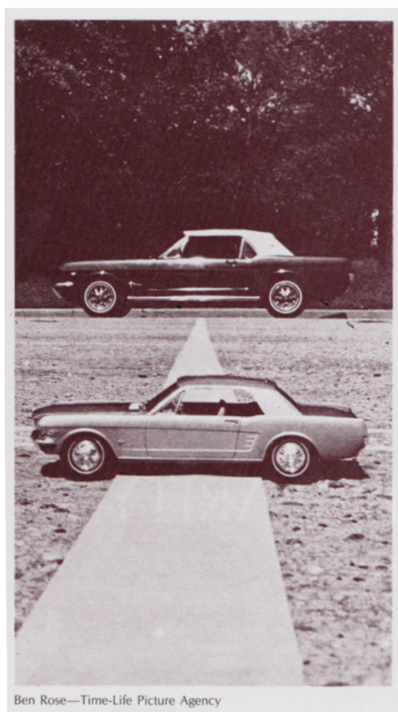
$$\mathcal{A} = \frac{h(c + c') - h(c')}{2} = \frac{hc}{2}$$

Of course, for an obtuse triangle we could choose one of the other sides as the base and proceed in the usual way, but this works as well.

Chapter 13

Similar triangles

Triangles that are alike, but not congruent, because they are scaled differently, are called similar. We write $\triangle ABC \sim \triangle PQR$.

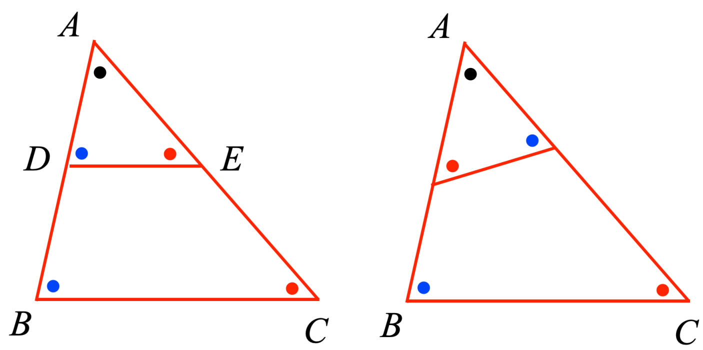


— Figure from Jacobs, chapter 10

Statements about two similar triangles.

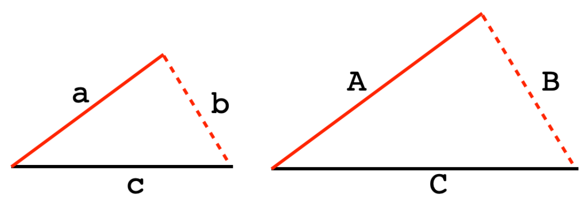
• They have at least two (thus, three) angles known to be equal.

• When superimposed using a shared angle, the third pair of sides, that do not coincide with each other, are nevertheless parallel (you may need the mirror image for one triangle).



• They have corresponding pairs of sides in the same proportions, but scaled by a constant factor.

If any one of these properties hold, they all do.



From the above diagram of two similar triangles, similarity implies that (for example)

$$\frac{A}{a} = \frac{B}{b}$$

For any pair of similar triangles, there is a constant k such that

$$k = \frac{A}{a} = \frac{B}{b} = \frac{C}{c}$$

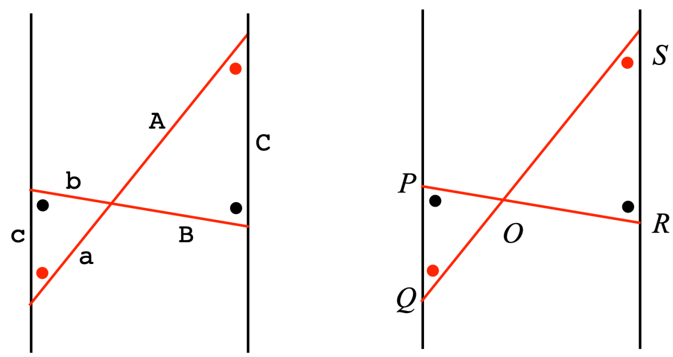
A slight rearrangement gives:

$$\frac{a}{b} = \frac{A}{B}$$

These ratios are obviously different.

As with congruent triangles, our definitions allow one triangle to be flipped before the comparison is made (comparing two triangles, originally the angles were in opposite order, one clockwise and the other counter-clockwise).

In the figures below, the vertical black lines are parallel. Any two lines connecting them that cross, form two similar triangles. The angles marked with dots of the same color are equal by alternate interior angles. We also have vertical angles.

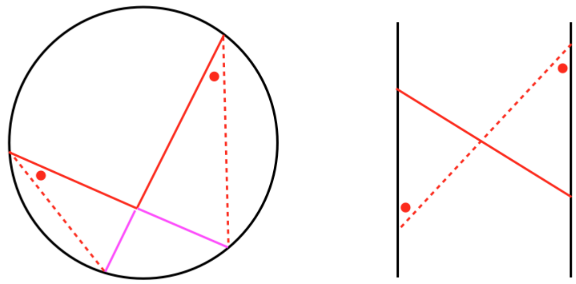


Set up the ratios carefully, by finding sides that lie opposite equal angles.

$$\begin{array}{ccccccccc} a & b & c & \qquad & \triangle OPQ = & OP & PQ & QO \\ A & B & C & \qquad & \triangle ORS = & OR & RS & SO \end{array}$$

For triangles with labeled vertices, I like to do this by naming the triangles in the corresponding order. We start $\triangle OPQ$ with OP opposite the red dot, moving counter-clockwise. Do the same with $\triangle ORS$.

Later we will have a theorem about similar triangles formed by crossed chords in a circle. In the figure below, the angles marked with a red dot are equal, as are the vertical angles, so the two triangles are similar. In this case we follow equal angles (and sides) around in opposite directions, comparing the two triangles.

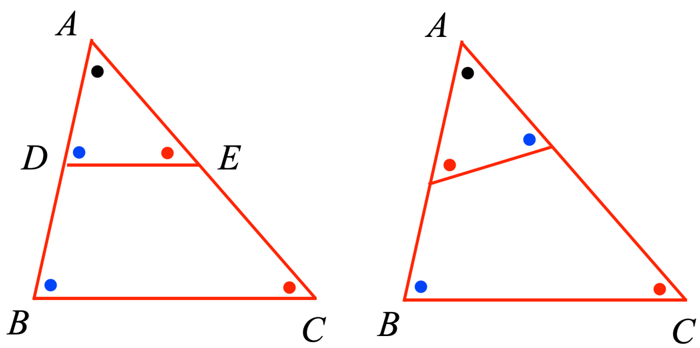


AAA is probably the most common way to establish that two triangles are similar.

similarity and ratios

Similar triangles are defined to have three angles equal (equiangular), but they are scaled differently and so, not congruent.

If two equiangular triangles are superimposed at one vertex ($\angle A$), so the adjacent sides coincide, and angles lie in the same order (left panel), then the sides opposite $\angle A$ are parallel ($DE \parallel BC$). In the right panel, the angles are in opposite order and this doesn't work.



This can be demonstrated by employing the parallel postulate, or as Euclid does in VI.2, by an argument based on area and the properties of two parallel lines. Actually, Euclid says to take a triangle and draw the line segment DE either parallel to the base or with the given angles, but the result is the same.

All angles equal and the third side parallel are closely related and easily proven in both directions.

Equal angles $\iff$ parallel sides.

We will call this property similarity: all angles equal and the third side parallel.

Another property is that similar triangles have their sides in the same proportion. This comes in two flavors: either two sides in equal proportion flanking an equal angle, often called SAS similarity, or all three sides in the same proportion with no prior knowledge about angles.

We now explore all of these situations:

- $\parallel$ sides and equal $\angle \iff$ equal ratios

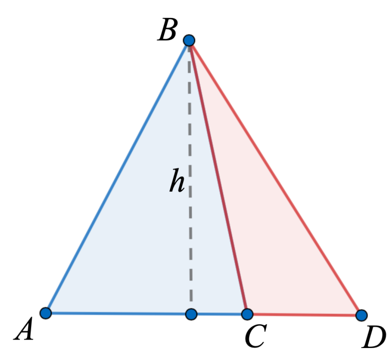
We will call these the *forward* ratio theorem, and the *converse* ratio theorem, even though there are two cases, either SAS or ratios only.

parallel third side implies equal ratios

As background, we recall two fundamental ideas about triangles.

The first is that if two triangles have their bases on the same line, and they also share the same vertex opposite, then they have the same altitude.

It follows that the areas are in the same proportion as the lengths of the bases. This is the **area-ratio theorem**.



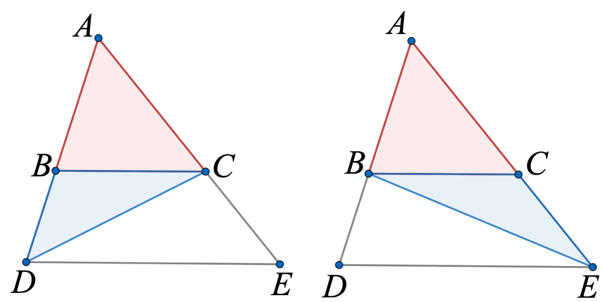
$$2\mathcal{A}_{\triangle BAC} = h \cdot AC$$
$$2\mathcal{A}_{\triangle BCD} = h \cdot CD$$
$$\frac{\mathcal{A}_{\triangle BAC}}{\mathcal{A}_{\triangle BCD}} = \frac{AC}{CD}$$

There is only one line that can be drawn from a point vertically to a straight line. So there is only one altitude that can be drawn from a point to a base. Since the triangles share the vertex point, they have the same altitude.

The second idea is that if two triangles share the same base, and the two opposing vertices both lie along a lie that is parallel to the base, then the triangles have the same area.

Euclid VI.2

- $\parallel$ sides and equal $\angle \rightarrow$ equal ratios



Proof.

Start with $\triangle ADE$ and then draw $BC \parallel DE$.

We will show that $AB : BD = AC : CE$.

The key to Euclid's proof of this theorem is to observe that the two triangles shaded blue, $\triangle DBC$ and $\triangle ECB$, have the same area.

The reason is that they lie on the same base, and their vertex (either D or E) lies along the same line parallel to that base. Hence the result follows.

But by the area-ratio theorem

$$\frac{\mathcal{A}_{\triangle ABC}}{\mathcal{A}_{\triangle DBC}} = \frac{AB}{BD}$$

and

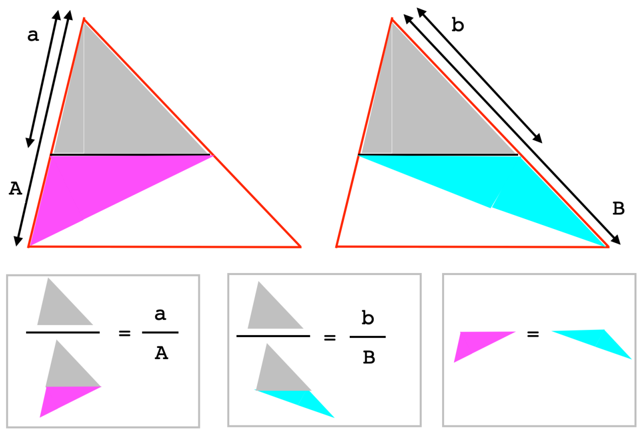
$$\frac{\mathcal{A}_{\triangle ABC}}{\mathcal{A}_{\triangle ECB}} = \frac{AC}{CE}$$

Since the left-hand sides of the two expressions are equal, so are the right-hand sides. Namely

$$\frac{AB}{BD} = \frac{AC}{CE}$$

□

Here is a different pictorial view of the argument.



Therefore

$$\frac{a}{A} = \frac{b}{B}$$

Equal angles (parallel third side) implies equal ratios.

□

This result applies to any vertex of the two similar triangles and its adjacent sides, hence it applies to all three sides of the two triangles.

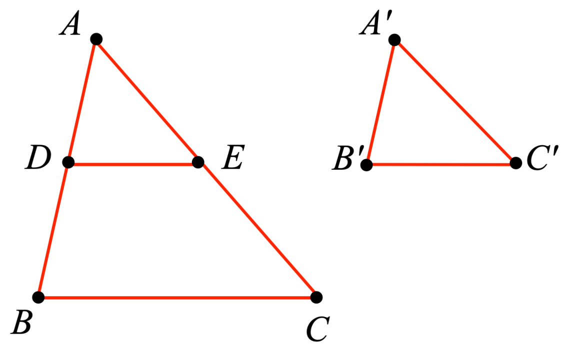
converse theorem

Euclid proves the converse theorem by using the same steps in reverse. Start with equal areas, form equal ratios of them, and then use the equal altitude to show that $DE \parallel BC$.

A somewhat different proof of the converse theorem, that equal ratios implies equal angles, is given by Kiselev.

Proof.

Let there be two triangles $\triangle ABC \sim \triangle A'B'C'$ with three pairs of sides in the same ratio. Let $\triangle ABC$ be the larger one.



Mark off inside $\triangle ABC$, for example, $AD = A'B'$ and then draw $DE \parallel BC$. By the forward theorem, $\triangle ABC \sim \triangle ADE$.

We can form the dual equality:

$$\frac{A'B'}{B'C'} = \frac{AB}{BC} = \frac{AD}{DE}$$

since we are given the first (equal ratios), and have the second from similar triangles, $\triangle ABC \sim \triangle ADE$.

Equating the first and third terms:

$$\frac{A'B'}{B'C'} = \frac{AD}{DE}$$

But we also have $A'B' = AD$, by construction. Cancel to obtain $B'C' = DE$.

The same can also be done for the third side. Thus, $\triangle ADE \cong \triangle A'B'C'$ by SSS.

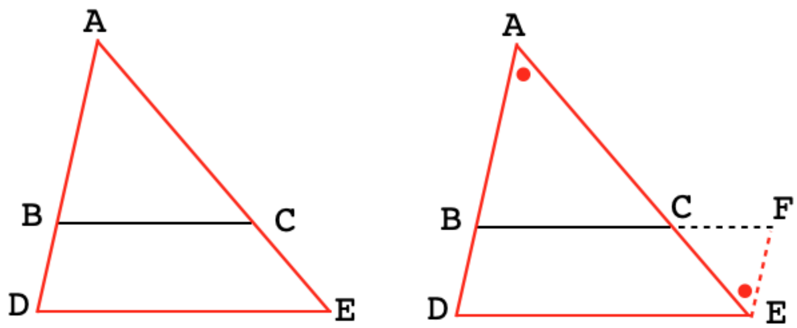
It follows that $\triangle A'B'C'$ is equiangular with $\triangle ADE$ and thus with $\triangle ABC$.

$\square$

SAS for similar triangles

Now, we mix and match the conditions, taking two sides in proportion and one angle shared. This is often called SAS similarity.

We will prove that this mix of conditions is enough for two triangles to be similar.



Proof.

Given two triangles with the same vertex angle at A and two of three sides known to be in the same proportion ($BD/AB = CE/AC = k$).

Draw EF parallel to ABD and extend BC to meet it at F.

$\triangle ABC \sim \triangle CEF$ by vertical angles plus alternate interior angles (red dots). The corresponding sides opposite the vertical angles are EF and AB.

By the forward theorem

$$\frac{EF}{AB} = k = \frac{CE}{AC}$$

but

$$\frac{CE}{AC} = \frac{BD}{AB}$$

Therefore, $BD = EF$. We are given $BD \parallel EF$. $BDEF$ has one pair of opposing sides equal and parallel, so it is a parallelogram.

It follows that $BC \parallel DE$ and $BF = DE$.

It also follows that $\angle D = \angle ABC$ by alternate interior angles, so we have two angles equal which means that $\triangle ABC \sim \triangle ADE$.

The forward theorem then gives $(DE - BC)/BC = BD/AB = k$

$\square$

Each of the standard congruence theorems (yes, even SSA) has a similarity version. We won't prove the others.

problem

The figure below is from Acheson's wonderful book aptly titled *The Wonder Book of Geometry*. He shows this problem, which he says "[goes] back to at least AD 850, when it appeared in a textbook by the Indian mathematician Mahavira."

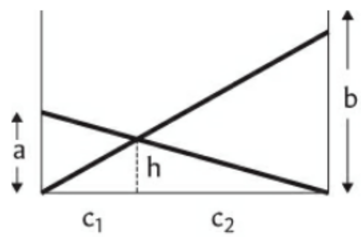


Fig. 40 A problem with ladders.

Looking down an alleyway, you see two ladders arranged as shown and wonder about the point where they cross, at height h and distances c_1 and c_2 from the edges of the alley, where the width of the alley is $c = c_1 + c_2$.

By similar triangles

$$\frac{c_1}{h} = \frac{c}{b}$$

Can you see why?

Going the opposite direction

$$\frac{c_2}{h} = \frac{c}{a}$$

Adding the two equations and substituting for $c_1 + c_2$:

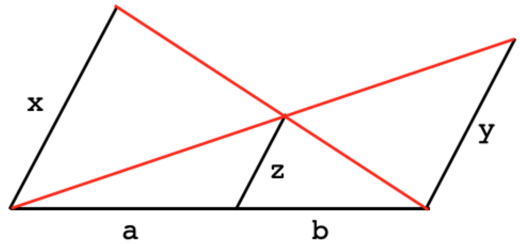
$$\frac{c}{h} = \frac{c}{a} + \frac{c}{b}$$

Thus

$$\frac{1}{h} = \frac{1}{a} + \frac{1}{b}$$

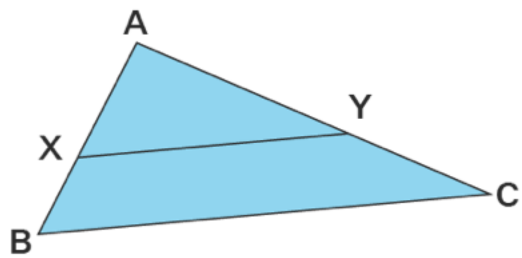
That's a simple and interesting result. h depends only on a and b and not on c, c_1, or c_2.

If you think about it, you should see that in the previous problem we never used the information that the sides and the height were vertical, only parallel.



Work through the same proof for the figure above to show that $1/x + 1/y = 1/z$.

problem



It is given that XY || BC and divides the triangle into two parts of equal areas. Find the ratio AX: XB using the area of similar triangles theorem.

The problem states that the bases are parallel ($XY \parallel BC$) and also that the subdivision produces *equal areas*. We are asked to find the ratio $AX : XB$.

Solution.

Recall from a previous chapter that for two similar triangles, the altitudes to corresponding sides are in the same ratio as any of the three pairs of sides themselves.

The altitudes h and H to BC (not drawn) are also in the same ratio as the sides, so let $H = kh$ — H lies on BC.

Then twice the area of the top triangle is $AX \cdot h$ and twice the area of the whole is $AB \cdot H = kAB \cdot h$

We have that the whole is twice the smaller area so the ratio is equal to 2:

$$\frac{kAB \cdot h}{AX \cdot h} = 2$$

But AB/AX is also equal to k so we have that $k^2 = 2$ and $k = \sqrt{2}$.

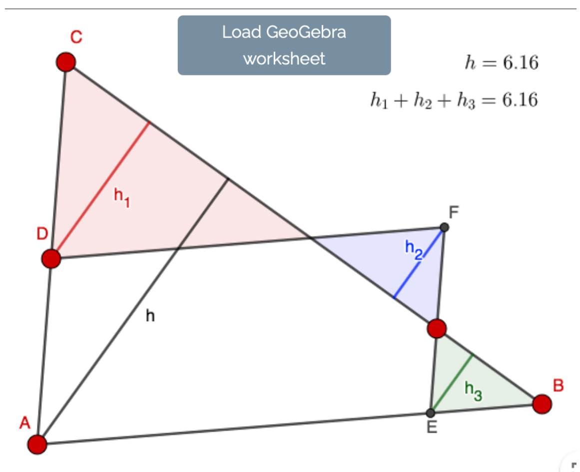
However, we are not asked for k. Instead, we want the ratio of AX to the smaller piece along the bottom. Let $AX = a$. Then the whole is $A = ka$. The difference is the small piece, $XB = A - a = a(k - 1)$. We need

$$\frac{a}{a(k-1)} = \frac{1}{k-1} = \frac{1}{\sqrt{2}-1}$$

problem

Here's a problem from the web. Given that $AC \parallel EF$ and $AB \parallel DF$.

We are to prove that the sum of the altitudes of the small triangles is equal to the altitude of the large one.



Informal solution: The smaller triangles are all similar to $\triangle ABC$ by the alternate interior angles and vertical angle theorems.

For similar triangles, not only are the sides in the same ratio to each other, but so are other measures like the altitudes to a particular side. So if we label the bases b_1 etc., collectively b_i , then we have that

$$\frac{b_i}{h_i} = \frac{b}{h}$$

$$b_i = b \cdot h_i / h$$

for each of the b_i.

But the sum of the b_i is simply equal to b so

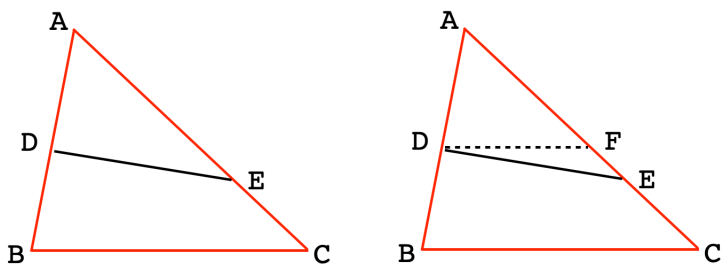
$$b_1 + b_2 + b_3 = b \cdot (h_1/h + h_2/h + h_3/h)$$

$$b = b \cdot (h_1/h + h_2/h + h_3/h)$$
$$1 = h_1/h + h_2/h + h_3/h$$
$$h = h_1 + h_2 + h_3$$

problem

Given two triangles that are not similar, where the ratio $AD/AB = r$ and $AE/AC = s$. We are asked to show that

$$\frac{\triangle_{ADE}}{\triangle_{ABC}} = rs$$



Solution.

Draw $DF \parallel BC$. Now $\triangle ADF \sim \triangle ABC$ so $AF/AC = r$.

By our previous result
$$\frac{\triangle_{ADF}}{\triangle_{ABC}} = r^2$$

Since $AE/AC = s$ and $AF/AC = r$, $AE/AF = s/r$. As triangles with a common vertex and bases in that proportion, these areas are in the same proportion:

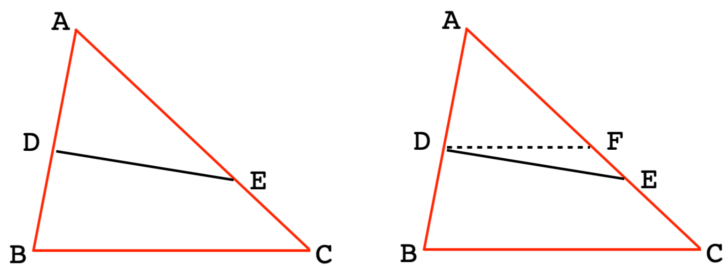
$$\frac{\triangle_{ADE}}{\triangle_{ADF}} = \frac{s}{r}$$

Multiply the two ratios together to obtain

$$\frac{\triangle_{ADE}}{\triangle_{ABC}} = sr$$

□

Solution. (alternate).



Looking ahead to trigonometry, in any triangle, twice the area can be computed as the product of two sides flanking an angle times the *sine* of the angle. (The reason is that the altitude to one side, divided by the length of the second side, is defined to be the sine of the angle.)

$$2(ADE) = AD \cdot AE \cdot \sin \angle DAE$$
$$2(ABC) = AB \cdot AC \cdot \sin \angle BAC$$

But $\angle DAE = \angle BAC$, so they have the same sine, and the ratio of areas is just

$$\frac{AD \cdot AE}{AB \cdot AC} = rs$$

□

To rework this proof in terms of familiar concepts, draw the altitude from D to AE, and also the one from B to AC

They form similar right triangles including the angle at A. The altitudes scale like $AD/AB = r$. But the bases scale like $AE/AC = s$.

And area scales like the product of the two, namely, rs.

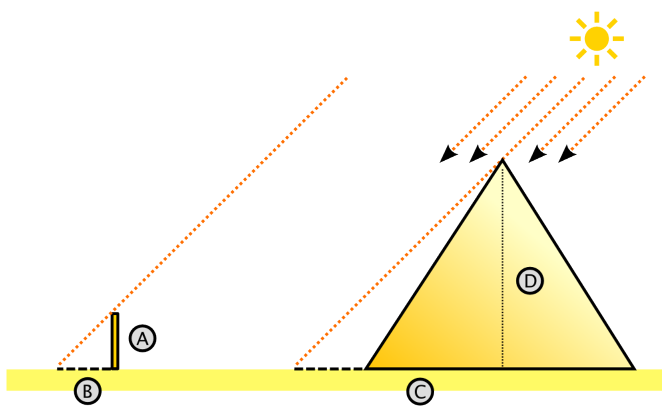
pyramid height

As we said earlier, Thales was from Miletus and he lived around 600 BC. Thales is believed to have traveled extensively and was likely of Phoenician heritage. As you probably know, the Phoenicians were famous sailors who founded many settlements around the Mediterranean.

They competed with the mainland Greeks and later with the Romans for colonies, and their major city, Carthage, was destroyed much later by the Romans, in the third Punic War. Hannibal rode his famous elephants over the Alps in the second Punic war.

During his travels, Thales went to Egypt, home to the great pyramids at Giza, which were already ancient then. They had been built about 2560 BC (dated by reference to Egyptian kings) and were already 2000 years old at that time!

The story is that Thales asked the Egyptian priests about the height of the Great Pyramid of Cheops, and they would not tell him. So he set about measuring it himself. He used similar triangles. I'm sure he wrote down his answer, but I'm not aware that it survives. The current height is 480 feet.



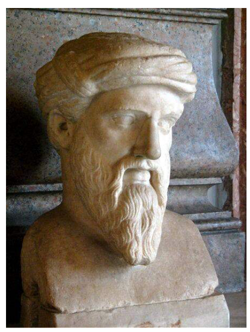
Part IV

Pythagorean theorem

Chapter 14

Simple proofs

The most famous theorem of Greek geometry is also without a doubt the most useful in calculus.



Like many Greek mathematicians, Pythagoras (c.570 - c.490 BCE) was not from Greece "proper" (i.e. the territory of mainland Greece). Instead, he was from one of the islands. Pythagoras was born on the island of Samos, which lies off the coast of what was called Asia Minor (modern Turkey).

Pythagoras was much younger than Thales but may have encountered him as a youth, since Thales lived on the mainland not too far from Samos, at Miletus. Later, Pythagoras moved to a Greek colony in southern Italy, at Croton.

During his lifetime, Pythagoras was known as a philosopher much more than as a mathematician. For example, he was famous as an expert on the fate of the soul after death.

https://plato.stanford.edu/entries/pythagoras/

Wilczek cites Bertrand Russell on Pythagoras:

> “A combination of Einstein and Mary Baker Eddy.”

which will be funny if you look up Mary Baker Eddy.

Pythagoras founded a “school” and it is not sure now which of the theorems developed by this school are due to Pythagoras, and which to his disciples. It is not even clear whether the Pythagorean theorem, as we understand it today, was known to Pythagoras. There is no contemporaneous account (say, Plato or Aristotle) connecting Pythagoras with the theorem. Of course, nearly all of the histories that were written are lost.

Regardless of whose idea it was, and who could prove it first, it’s clear that they knew something. The Pythagorean theorem says that if c is the hypotenuse of any right triangle and a and b are the side lengths then

$$a^2 + b^2 = c^2$$

The simplest example in integers is

$$3^2 + 4^2 = 9 + 16 = 25 = 5^2$$

but there are many triplets of integers with this property, for example

$$5^2 + 12^2 = 25 + 144 = 169 = 13^2$$

Not just the classic 3-4-5 right triangle, but a number of other *Pythagorean triples* had been known for a thousand years.

The tablet “Plimpton 322” contains (by extrapolation) the triplet 4601-4800-6649 and it dates to about 1800 BCE. Maor analyzes this in his book on the theorem of Pythagoras.

Line 3 of the tablet contains the numbers $1, 16, 41$ and $1, 50, 49$ which are in base 60 notation. Thus $1 \cdot 3600 + 16 \cdot 16 + 41 = 4601$ and $1 \cdot 3600 + 50 \cdot 60 + 49 = 6649$. The third number of the triple is missing but it’s obvious since $6649^2 - 4601^2 = 23040000$, which is 4800^2.

It seems highly unlikely that these were found by searching randomly among squares.

There is even a triple with 5 decimal digits on the next line of the Plimpton 322 tablet.

One should not think that the theorem *only* applies to triangles with integer side lengths. For example, it applies to the isosceles right triangle with side length equal to 1, whose hypotenuse is the real number $\sqrt{2}$. Also, any integer solution can be modified. For example

$$0.5^2 + 1.2^2 = 0.25 + 1.44 = 1.69 = 1.3^2$$

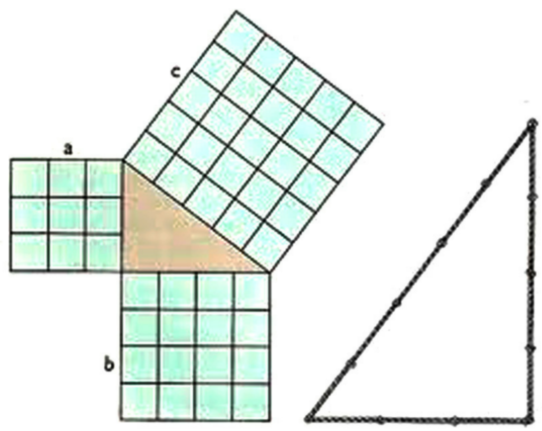
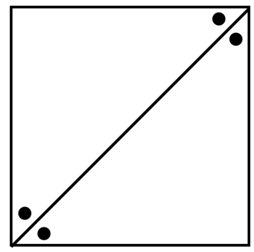


FIGURE 1. THE 3-4-5 RIGHT TRIANGLE, A SIMPLE CASE OF PYTHAGORAS'S THEOREM.

We're going to spend time with a few particular proofs of this theorem (there are literally hundreds of them), so as to examine the ideas from different perspectives. This chapter is an introduction that shows some basic algebraic proofs.

The proof due to the Greeks presented in the next chapter is from Euclid. It employs SAS for triangle congruence, and is probably his own contribution. Later we'll explore how the concept of area is connected to the Pythagorean theorem.

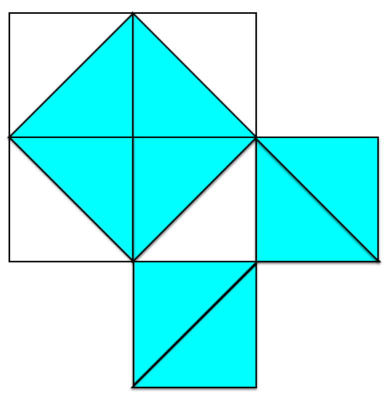
To begin, let's examine a special case, easily proved, for an isosceles right triangle.



We can obtain such a triangle by drawing the diagonal in a square. The black dotted angles in any one triangle are equal, by the isosceles triangle theorem, and the others are equal by alternate interior angles.

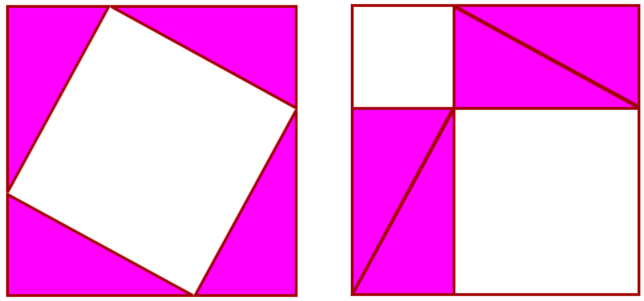
Our hypothesis is that when the lengths of the sides of the triangles are squared and then summed, they will be equal to the square of the diagonal. That is to say, the square of the diagonal should be equal to two of the squares shown above.

Here is a proof without words.



Adding some words: the area of the square on the hypotenuse is equal to one-half of the area of four squares of the sides. This is a proof for the special case where the sides are equal.

The following general proof is sometimes called the "Chinese proof." I can easily imagine proceeding from the figure above to the left panel below by simply rotating the inner square and collapsing the surrounding one.



It really needs no explanation, but ..

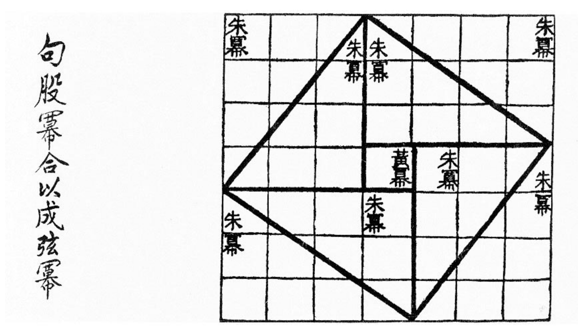
We have a large square box that contains within it a white square, whose side is also the hypotenuse of the four identical right triangles contained inside. Altogether the four triangles plus the white area add up to the total.

We simply rearrange the triangles. Now we evidently have the same area left over from the four triangles, because they still have the same area and the surrounding box has not changed.

But clearly, now the white area is the sum of the squares on the second and third sides of the triangles. Hence the two white squares on the right are equal in area to the large white square on the left.

□

This diagram is contained in the Chinese text Zhoubi Suanjing.



Eight right triangles are formed with sides of 3 and 4 units. There are 7 units on each side of the large square so the total area is 49. Each pair of triangles has area 12 $(3 \cdot 4)$, so the square in the center really is a unit square with area $49 - 4 \cdot 12 = 1$.

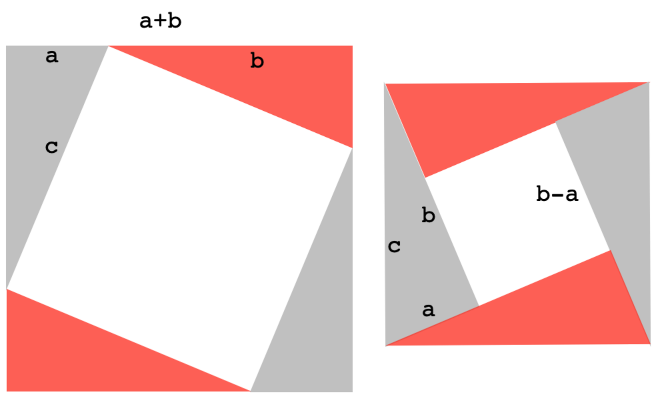
So then, the central square consists of four triangles with total area 24 plus the unit square for 25. This is the square of the long side, namely 5.

https://en.wikipedia.org/wiki/Zhoubi_Suanjing

There is debate about whether this is really a proof, or if it simply presents the arithmetic for the example of a 3-4-5 right triangle.

simple proofs

Many proofs of the theorem are algebraic. Here are two:



Proof.

In the left panel, we have the same arrangement as before, with four identical right triangles. The white square at the center has sides of length c and angles that are right angles because when summed to two complementary angles, the result is two right angles.

The algebra is

$$(a+b)^2 = 4 \cdot \frac{1}{2} \cdot ab + c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

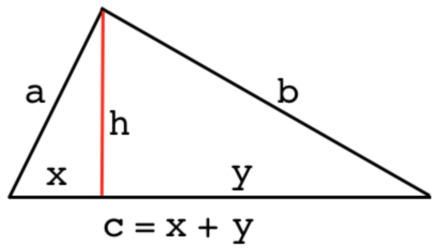
Subtract $2ab$ from both sides, and we're done.

□

I will leave the right panel to you.

similar triangles

Here is a simple classic that depends on the ratios formed for similar right triangles:



Proof

We know that when an altitude is drawn in a right triangle, the two resulting right triangles are similar, by complementary angles. Similarity means that we have equal ratios of sides. Here are two sets:

ratio of hypotenuse to short side

$$\frac{a}{x} = \frac{b}{h} = \frac{c}{a}$$

ratio of hypotenuse to long side

$$\frac{a}{h} = \frac{b}{y} = \frac{c}{b}$$

From the first

$$a^2 = cx$$

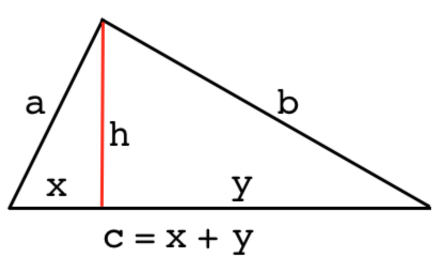
And from the second

$$b^2 = cy$$

Add them:

$$a^2 + b^2 = cx + cy$$
$$= c(x + y) = c^2$$

□



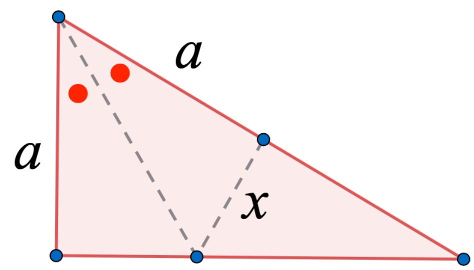
Another relationship from similar triangles is long side to short side:

$$\frac{h}{x} = \frac{y}{h}$$
$$h^2 = xy$$
$$h = \sqrt{xy}$$

h is the *geometric mean* of x and y.

Another proof (from Dunham's Problems) also relies on similarity.

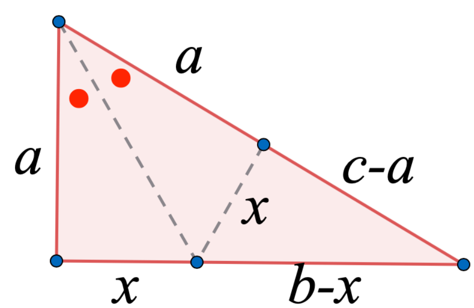
Proof.



In a right triangle, bisect one of the smaller angles, and mark off the same length as the adjacent side a on the hypotenuse. Draw the side x from where the bisector meets side b.

It is easy to see that the two triangles containing the angle bisectors are congruent by SAS.

Therefore, we have that x makes a right angle where it cuts the hypotenuse. Fill in the other sides:



We have that the right triangle with side x is similar to the original. From the corresponding sides we obtain:

$$\frac{x}{a} = \frac{c-a}{b} = \frac{b-x}{c}$$

From the first and second terms:

$$bx = a(c-a) = ac - a^2$$

From the second and third terms:

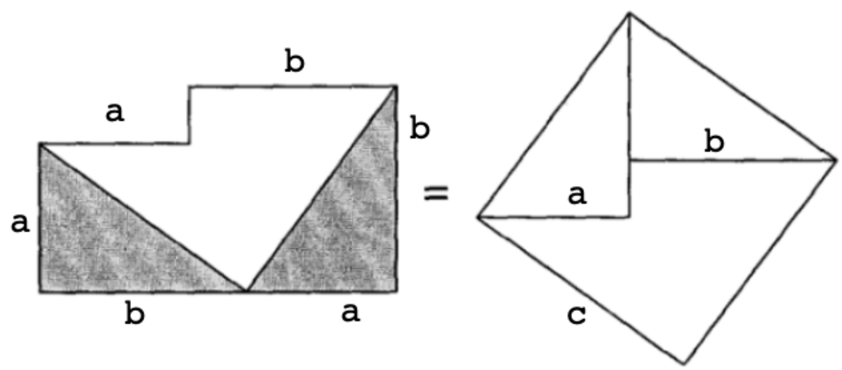
$$c^2 - ac = b^2 - bx$$

Substitute for bx:

$$c^2 - ac = b^2 - ac + a^2$$
$$c^2 = a^2 + b^2$$

□

This is a proof from Gelfand and Saul's trigonometry book.

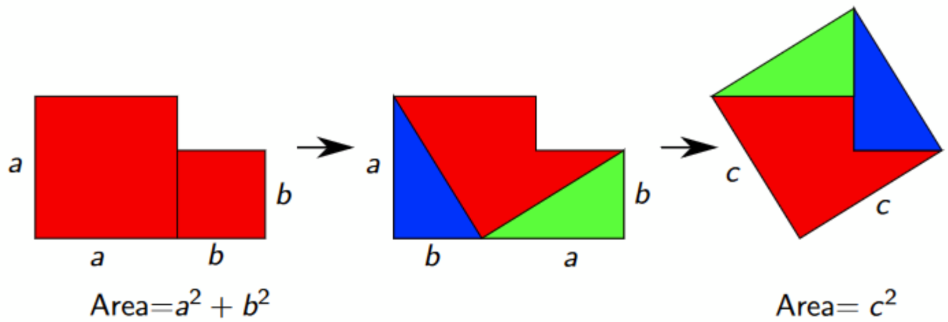


I have added the labels for side length.

If it's not obvious, note that we start with two adjacent squares of area $a^2 + b^2$. Two triangles (shaded) are cut off and re-arranged to make a shape whose area is c^2.

□

Here is a colored version of the same proof I found on the web.



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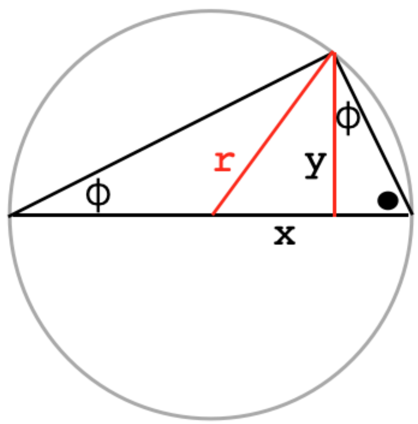
proof without words

This one is from Nelsen's *Proof without words*. It depends on Thales' theorem, and also on similarity in right triangles.



The horizontal line is a diagonal of the circle, and the line labeled y is vertical.

Proof.

The two angles labeled ϕ are equal because they are complementary to the angle marked with a black dot. r is the radius. The larger triangle has sides y and $r + x$, while the other has sides $r - x$ and y. By similar triangles:

$$\frac{y}{r+x} = \frac{r-x}{y}$$

$$y^2 = r^2 - x^2$$

$$x^2 + y^2 = r^2$$

□

converse of Pythagorean theorem

In reasoning deductively, we move from the premise or premises (collection of facts, data given, previous theorems that were proved), and use logic to reach a conclusion. The question arises whether, if we know only that the conclusion is true, does it follow logically that the premises are true? This is the problem of the *converse* of a theorem.

It may be so, or it may not.

We can state the converse of the Pythagorean theorem as follows: suppose we have triangle such that $a^2 + b^2 = c^2$. Does it follow that the angle between a and b is a right angle?

We profess to not know whether it is or is not a right angle.

Proof.

So then, draw another triangle with sides a and b that *is* a right triangle. Then, by the forward theorem, $a^2 + b^2 = c^2$.

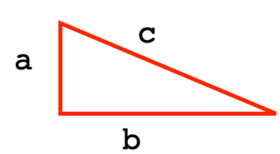
But the second triangle is congruent to the first one by SSS, each side is the same. Since they are equivalent parts of congruent triangles, the angle between a and b is a right angle.

□

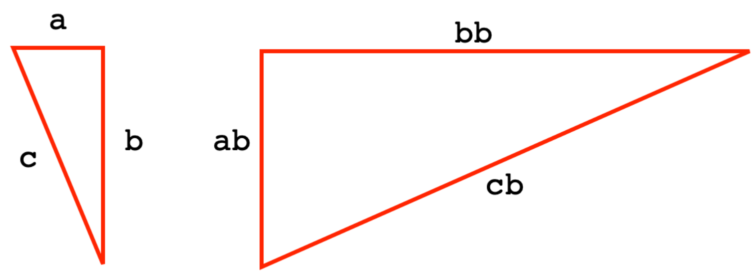
scaled triangle proof of Pythagoras

Proof.

Draw a right triangle and label sides a, b and c.



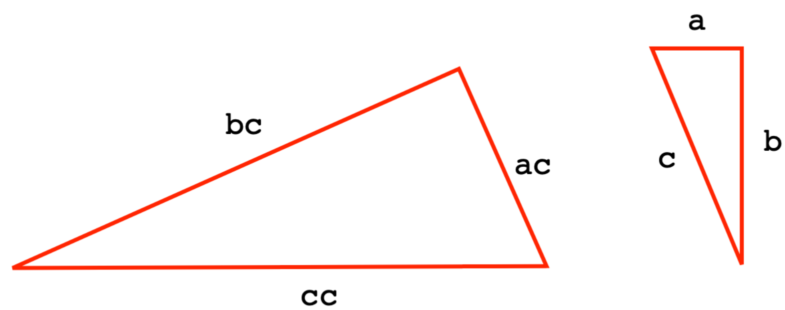
Now, flip and rotate the triangle. Make a copy of that one and rotate the copy so that the shortest side is to the left. Enlarge the copy until the two adjacent sides are equal in length.



These are similar triangles so the angles are all equal, but the sides are proportional, multiplied by a common factor, which here is b.

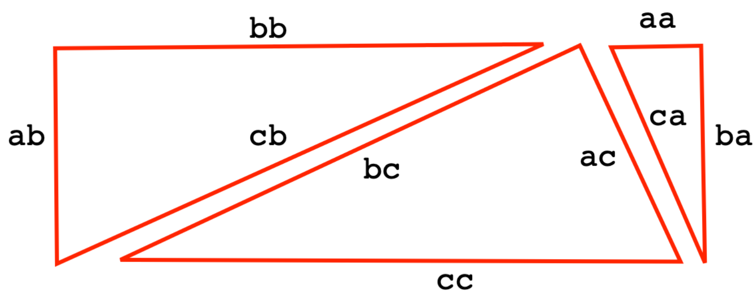
The adjacent sides show that $ab = b$, so this choice of scaling defines $a = 1$. And since $a = 1$, we can also view each side of the original triangle as being scaled by a factor of a.

We just tried scaling by a factor of b and then by a, so that naturally suggests we try scaling by a factor of c.



All sides of the enlarged triangle have been multiplied by the same factor, namely, c. Notice that $c = ac$, as required.

We're nearly done. Put the three triangles together.



The lower left and right corners are places where two vertices come together. By complementary angles, these sum to be right angles.

Since there are two other right angles at the vertices of the quadrilateral, it is a rectangle. Therefore the place where three vertices come together is a straight line, which we can also verify because there are two complementary angles plus a right angle.

Since the composite figure is a rectangle, opposing sides are equal, so $a^2 + b^2 = c^2$.

$\square$

problem

The Russian mathematician V.I. Arnold wrote a famous small book of "problems for children from 5 to 15." Here is no. 6:

> The hypotenuse of a right-angled triangle (in a standard American examination) is 10 inches, the altitude dropped onto it is 6 inches. Find the area of the triangle.
>
> American school students had been coping successfully with this problem over a decade. But then Russian school students arrived from Moscow, and none of them was able to solve it as had their American peers (giving 30 square inches as the answer). Why?

We leave this one as a challenge.

Hint 1: it's a joke.

Hint 2: For two sides of a fixed total length, the isosceles right triangle has the largest area (why?). Given the hypotenuse, what are the side lengths?

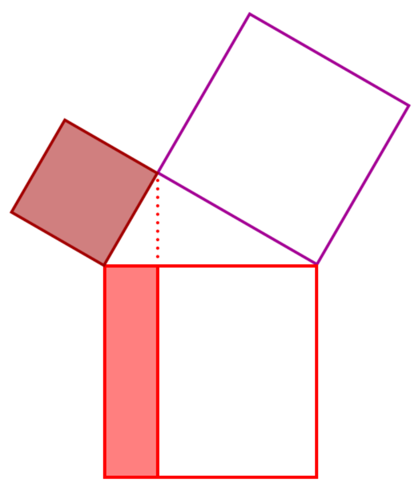
Now use this fact to find the maximum area in terms of the sides, and compare that with the data we are given.

There are a number of other proofs of the Pythagorean theorem later in the book. See the last chapter **here** for links.

Chapter 15

Euclid's proof

My favorite proof of the Pythagorean theorem relies on the construction below, sometimes called the "bridal chair" or the "windmill", where the central triangle is a right triangle, and the other figures are squares (Euclid I.47).

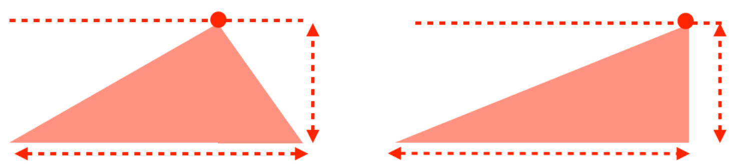


We have the squares on three sides of a right triangle.

What we will show is that the rectangular area which is part of the large square, in red, is equal in area to the entire small square, in maroon.

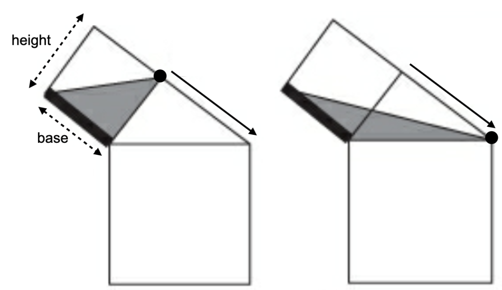
preliminary

We begin by restating a fundamental idea about the area of triangles, which is that, if two triangles have the same base and the same height, they have the same area. So if we imagine sliding the top vertex of a triangle along a line parallel to the base, the area will not change.



Proof.

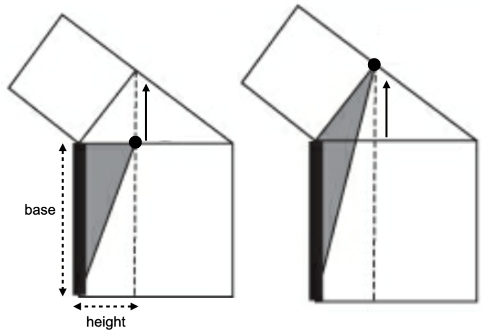
The next figure shows this principle as it comes up in the proof. The gray triangle with the black base is one-half the area of the small square.



Slide the vertex down to the right and the resulting triangle will still have the same area.

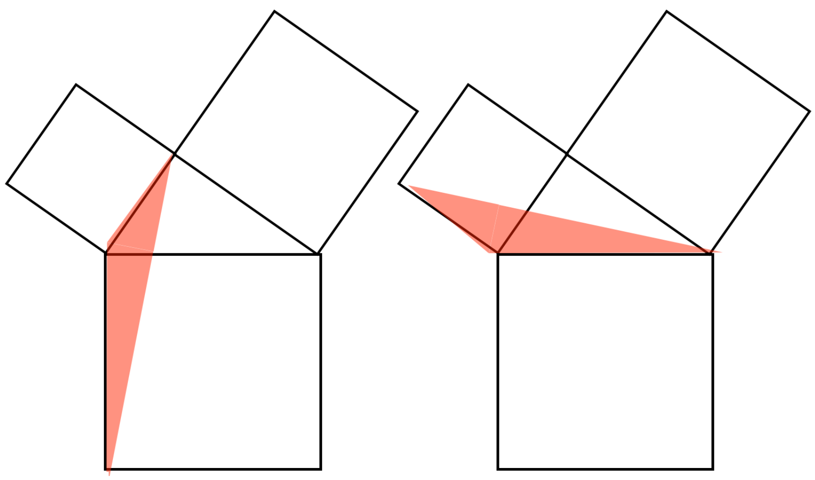
We can do the same thing with a triangle in the large square, below.

The gray triangle has half the area of the part of the large square that is to the left of the dotted line, because its base is equal to the side of the square and the height extends to the right to the dotted line.



Now slide the vertex up. The area is unchanged.

And now, finally, we observe that the two triangles have exactly the same shape. Just rotate one to obtain the other.



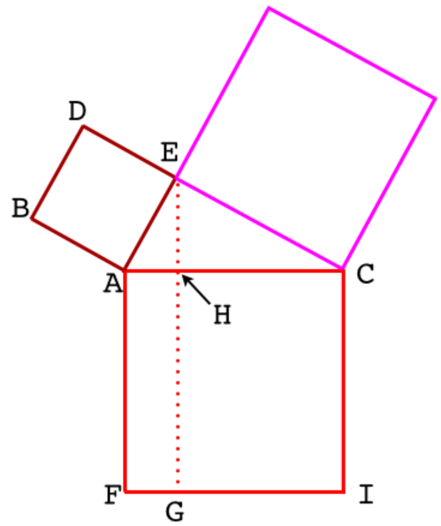
This completes our informal proof.

□

The formal approach follows, based on triangle congruence.

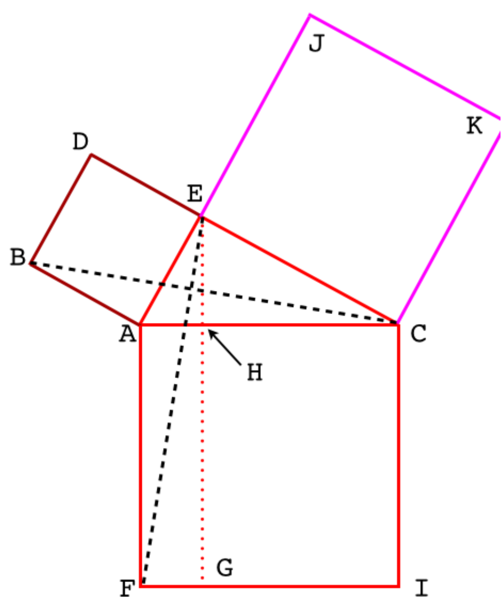
main

We label some points as shown:



First, drop a vertical line EHG, constructing the rectangle $AFGH$.

Finally, sketch dotted lines for the long sides of two triangles:



Proof.

The crucial point is this: we will show that triangle ΔABC is congruent to triangle ΔAEF.

Use side-angle-side (SAS). The two sets of sides are evidently equal

$$AB = AE$$

$$AF = AC$$

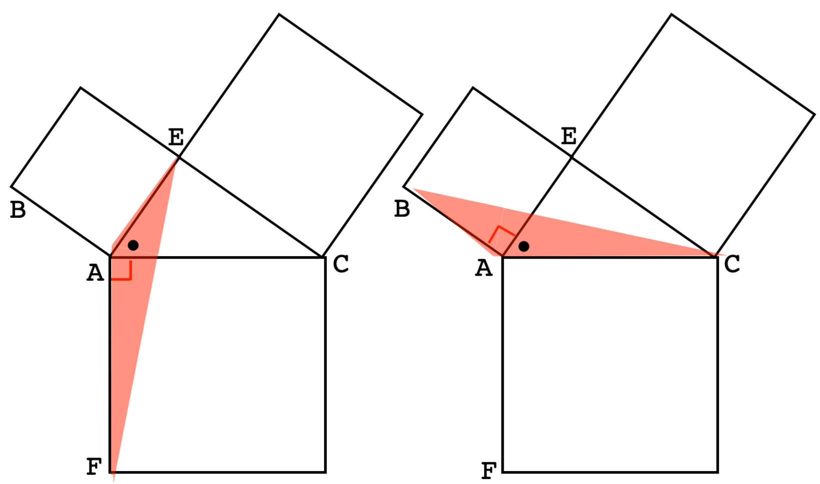
because, for example AB and AE are given as the sides of one square, and AC and AF as sides of another.

What about the included angle? The angles $\angle BAC$ and $\angle EAF$ each contain a right angle plus the shared angle $\angle EAC$.

$$\angle BAC = \text{ right angle } + \angle EAC$$
$$\angle EAF = \angle EAC + \text{ right angle}$$
$$\angle BAC = \angle EAF$$

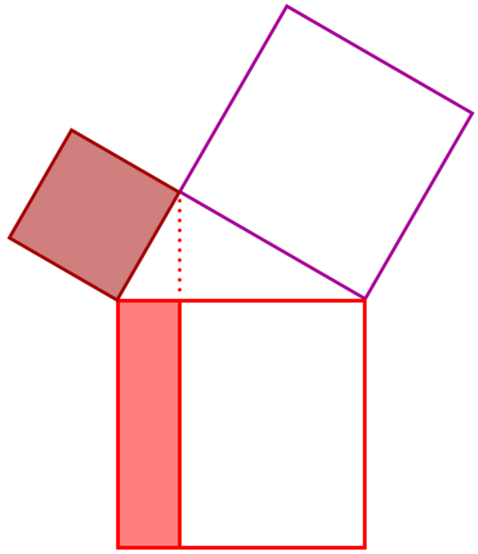
So they are themselves equal, and thus we have proved SAS and thus the congruence relationship:

$$\triangle ABC = \triangle AEF$$



The next part of the proof is to move the vertex of triangle ΔABC to the left and see that it has base AB and altitude AE so its area is one-half that of the small square $ABDE$. On the other hand triangle ΔAEF has base AF and altitude AH so its area is one-half that of the rectangle $AFGH$.

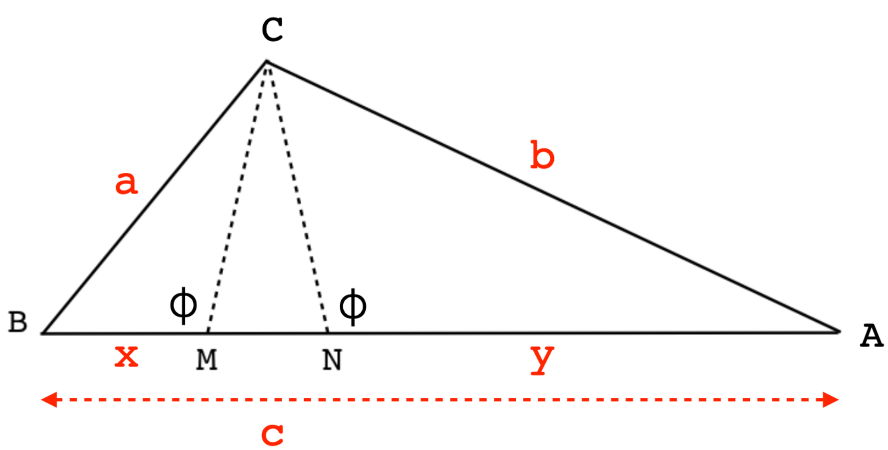
Hence we have proved that the two colored areas in this figure are equal:



Finally, we could proceed to do the same thing on the right side of the figure, but we just appeal to symmetry. All the equivalent relationships will hold.

$\square$

Quorra's corollary



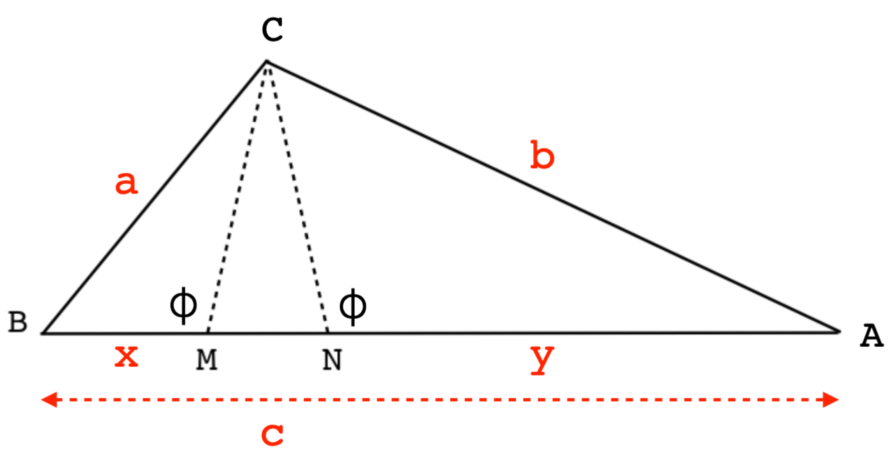
Let $\triangle ABC$ be *any* triangle (here it is obtuse). Draw CM and CN so that the new angles $\angle CMB$ and $\angle CNA$ (labeled ϕ), are equal to $\angle C$. The corresponding triangles are similar to the original, because they share ϕ plus one other from the original triangle.

We use single letters for the sides to make the algebra simpler. a is opposite both $\angle A$ and $\angle CMB$, while b is opposite both $\angle B$ and $\angle CNA$, and c is opposite $\angle C$.

The shortest side in $\triangle CMB$ is x and the longest is a, while in $\triangle ABC$ the corresponding sides are a and c. So by equal ratios of sides in similar triangles we have that $x/a = a/c$. The middle side in $\triangle CNA$ is y and by similar logic we have that $y/b = b/c$.

$$a^2 = cx, \qquad b^2 = cy$$
$$a^2 + b^2 = c(x + y)$$

The sum of the squares of the two short sides of a triangle is equal to the product of the third side with the the sum of the two components $x + y$, when they are drawn with the angle ϕ as specified.



This is actually a generalization of our original algebraic proof of the Pythagorean theorem.

In the case where the angle at vertex C is a right angle, then M coincides with N, because there is only one vertical to a line from a given point. So then and $x+y = c$, and this reduces to the Pythagorean theorem.

There are a large number of proofs of the Pythagorean theorem. Many of them are collected or linked here:

https://www.cut-the-knot.org/pythagoras/

Chapter 16

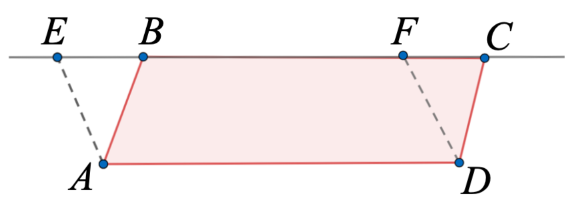
Pappus's proof

Pappus came up with a beautiful theorem which includes the Pythagorean theorem as an extension. First, we need a simple lemma about parallelograms.

Lemma.

Given two parallel lines: $AD \parallel EBFC$.

If two parallelograms $AEFD$ and $ABCD$ have opposite sides on the two lines, and those two segments are of equal length, then they have equal areas.



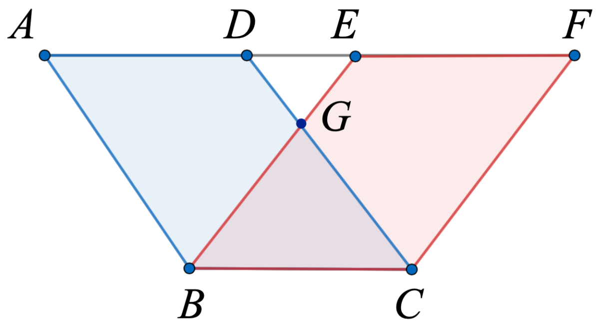
Here is a special case where they have the same base AD.

We proved earlier that the perpendicular between two parallel lines is of equal length no matter where it is drawn.

So the two parallelograms are each composed of pairs of triangles, which, having the same base and the same altitude, also have equal area.

□

One might also just invoke Euclid I.35.

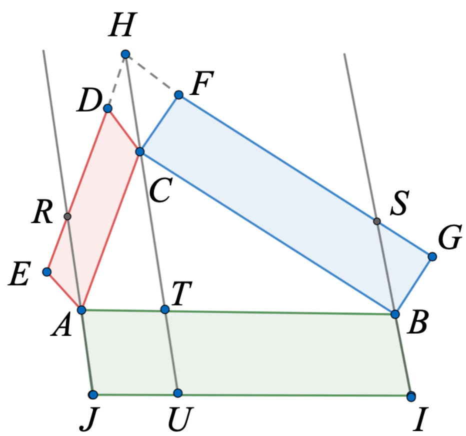


Pappus's parallelogram theorem

In the figure below, on the two sides of any $\triangle ABC$ draw any parallelograms $ACDE$ and $BCFG$. Extend the two new sides to meet at H.

Draw HC and its extension such that it cuts AB at T and then let $HC = TU$.

Draw $AJ \parallel HCTU \parallel BI$.



Proof.

The new parallelogram with AC as one side and RH the other, is equal, by our lemma. $(ACDE) = (ACHR)$.

Since $RAJ \parallel HCTU$ and $RA = HC = TU = AJ$, $(ATUJ) = (ACHR)$ for the same reason.

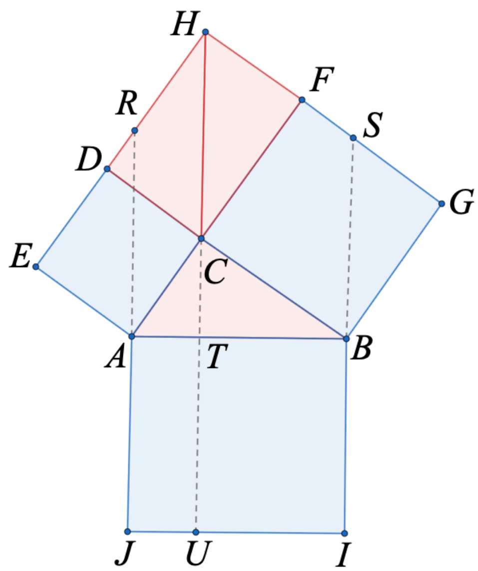
So $(ATUJ) = (ACDE)$.

We use the same argument for $(BCFG) = (TBIU)$ on the right.

Then add the two results: $(ACDE) + (BCFG) = (ABIJ)$.

$\square$.

Let $\angle ACB$ be a right angle, and let the parallelograms be squares.



$\triangle DHC$ and $\triangle FHC$ are right triangles and they are congruent to $\triangle ABC$ by SAS.

So $\angle CHF = \angle CAT$ and we also have vertical angles, so $\triangle HFC \sim \triangle CTA$.

It follows that $\angle HFC = \angle CTA$ and both are right angles.

Further, $CH = TU = AB$, so $ABIJ$ is the square on AB.

It is easy to see that $(AEDC) = (ARHC) = (ATUJ)$, and the rest follows.

Thus Pappus' theorem becomes the Pythagorean theorem as a special case.

[Reference: George F. Simmons, *Calculus Gems.*]

Chapter 17

Brief review

Let us summarize what we know so far about the basic theorems of geometry.

They are almost all that you should need to attack any of the other problems in the book, and they'll be used repeatedly.

The first two aren't technically theorems but fundamental assumptions that we make about the geometrical world.

- **supplementary angles**
- **alternate interior angles (parallel postulate)**

The next three easily follow from those initial ideas.

- **vertical angles**
- **triangle sum of angles**
- **complementary angles**

We have two basic methods for proving that two triangles are congruent

- **SAS for congruence**
- **ASA for congruence**

and one specifically for right triangles

- **hypotenuse leg in a right triangle (HL)**

(We proved that SSS is equivalent to SAS, and AAS is equivalent to ASA).

- **SSS implies SAS**

Next we have the powerful fundamental theorems of geometry:

- **isosceles triangle theorem** (sides $\rightarrow$ angles) and **converse**
- **external angle theorem**
- **Thales' circle theorem** (right angle in a semi-circle)
- **area ratio theorem**
- **similar triangles** (similar right $\triangle$s, same ratio of sides)
- **general similarity theorem**
- **Pythagorean theorem** (similar triangles)

Some people might add a few more. But this is a reasonable number to start with. You should have these instantly available (and it's nice to know how to prove them, as well).

Looking forward, probably the most important we have yet to do is

- **inscribed angle theorem** (on a circle is one-half central angle)

which has the consequence that inscribed angles on the same arc are all equal (**equal angles** $\iff$ **equal arcs**).

Part V

Circles

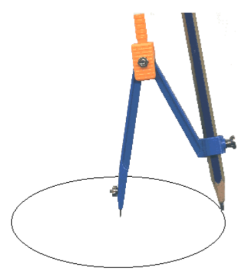
Chapter 18

Circles and angles

diameters

Pick some point to be the *center* of a circle. Then a circle contains all the points a specified distance away from the center. That distance is called the *radius*.

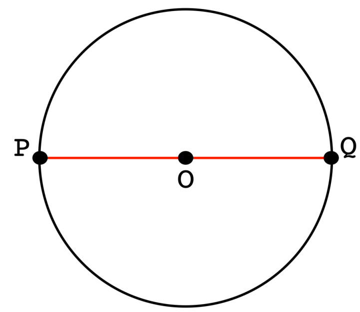
A circle is drawn with a compass, as we mentioned previously.



Euclid says:

• Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

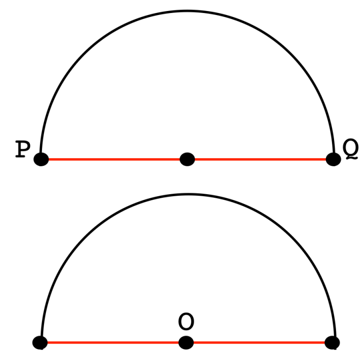
Suppose we start with the line segment OP in the figure below, and choose O as the center. Place the needle point of the compass on O and the pencil on P and then draw the circle containing all the points the same distance from O as P is.



Next, extend the radius PO to meet the circle on the other side at Q. This whole line segment PQ is called a *diameter* of the circle.

• Any diameter divides the circle into two equal parts.

(Recall that we allow a mirror image of something to be called equal to itself). So then, if we take the bottom half and flip it vertically, we claim the two halves are exactly equal.



Proof.

The proof is by contradiction. Lay the two pieces on top of one another. The diameters of the half-circle are duplicates of the original. Thus, they are equal to each other and each half is equal to one radius.

We suppose that, somewhere, the two half-circles are not identical.

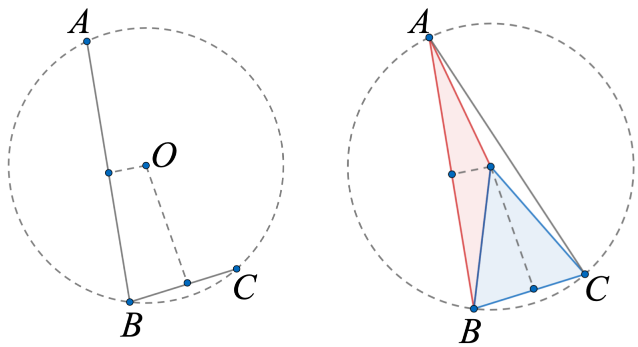
Then there will be some radial line we can draw from O through the two half circles, where the line meets the two curves at different distances from the center, because they are supposed to be different. This will identify points on the same radius of each half-circle that lie at a different distance from the center.

But then, the original figure could not be a circle, because all radii of a circle are equal.

This is a contradiction. Hence any diameter divides the circle into two equal parts, called semi-circles.

□

circle containing three points



Suppose we have three arbitrary points in the plane: A, B and C.

The claim is we can draw a circle that contains all three points on its circumference.

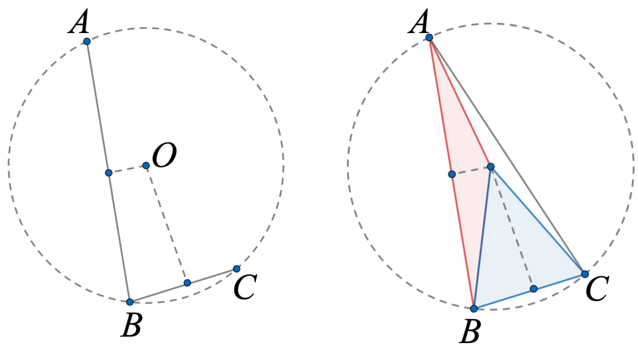
Proof.

Pick two pairs of points and draw, say, AB and BC.

Find the perpendicular bisector of each. We know that every point on the perpendicular bisector of AB is equidistant from both A and B, while for the latter case AC every point is equidistant from B and C.

If O is the point where the bisectors meet, that point is such that $OA = OB = OC$.

So if we draw the circle on center O with radius OA it contains all three points.



□

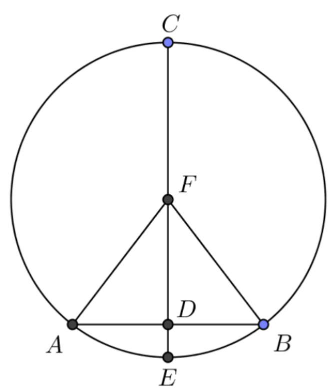
The question arises whether there is a general rule about $\triangle ABC$ and there is: if the origin of the circle lies on one of the sides of $\triangle ABC$ then it is a right triangle, that side is a diameter of the circle and the hypotenuse of the right triangle, and the center of the circle bisects the hypotenuse. This depends on the converse of Thales' circle theorem.

If $\triangle ABC$ contains an angle greater than a right angle, then the center of the circumcircle is not inside the triangle, while if it is acute, the center lies inside the triangle.

We will not prove this now. Euclid's proof for all three cases is III.31.

Euclid III.1

This is a simple method from Euclid to find the center of a circle.



Proof.

To find the center of any circle, take two points on the circle and draw the chord connecting them, then erect the perpendicular bisector of the chord. In the diagram above, $CE \perp AB$ and bisects it.

We showed previously that there does not exist any point which is equidistant from A and B and is *not* on this perpendicular bisector. Since the center of the circle F has the property $AF = BF$, it must lie somewhere on CE.

Thus, CE is a diameter of the circle. Bisect it to find F, the center.

□

internal and external points

As a preliminary to the next section, we claim that

• A line through any internal point of a circle can be extended to intersect the circle at two and only two different points.

Proof.

Assume there is a straight line segment that intersects a circle at more than two points, say at three points. By the definition of a circle, those three points are equidistant from the center.

Of the three points, one lies between the other two (going the shortest way around the circle). Use the middle point and (one at a time) the other two points to construct two perpendicular bisectors of the two segments. This is the classic way to find the center of a circle from points on the periphery.

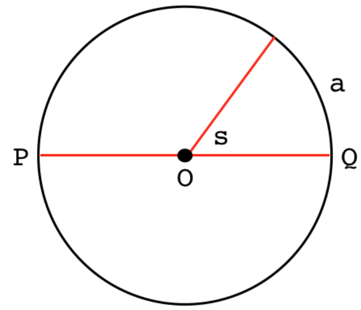
But then, we would have two bisectors that both run through the center. Yet they are parallel to each other, because they have been constructed perpendicular to the same line.

This is a contradiction. There cannot be more than two points on intersection of a line with a circle.

□

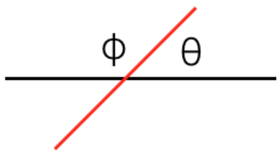
If there is no specification of an internal point, then it is possible for a line to intersect with the circle at only one point, namely a tangent line. We deal with tangents separately in a later chapter.

radian measure



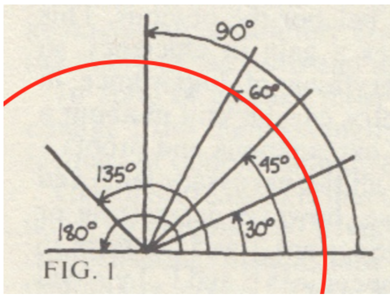
In the figure above, the angle s is *defined* as the length of arc a that it sweeps out in a unit circle.

We talked briefly early in the book about the *measure* of an angle. It seems intuitively obvious that, in this figure, $\theta < \phi$.



The question is how to quantify this notion.

Effectively what we'll do is imagine that we draw a circle with radius 1, a *unit* circle, which contains the angle as a central angle.



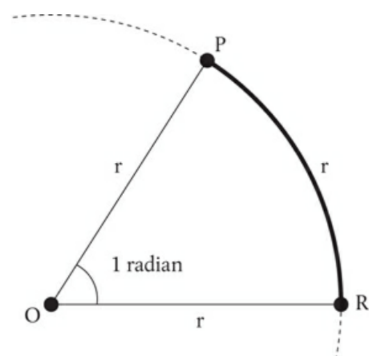
Then, the measure of the angle is the distance traveled around the circle counter-clockwise from the horizontal on the right.

Although this drawing shows angles in degrees, in calculus and analytical geometry angles are defined in terms of radians of arc. For a unit circle with radius = 1, the total circumference is 2π.

If s is the measure in radians and D° the measure in degrees, then

$$\frac{s}{2\pi} = \frac{D}{360}$$

It seems natural then to adopt the arc length as a measure of the angle, where 360° is equal to 2π *radians*, and an angle of 90°, a right angle, is equal to $\pi/2$ radians.



72. Definition of a radian.

Divide 360 by 2π to find that one radian is approximately 57°.

To convert some more measures of angles in degrees to radians:

$$180^\circ = \pi, \; 90^\circ = \frac{\pi}{2}$$

$$60^\circ = \frac{\pi}{3}, \; 45^\circ = \frac{\pi}{4}, \; 30^\circ = \frac{\pi}{6}$$

Central angle and the arc that subtends that angle are numerically equal, but remember that they are dimensionally different. An arc is a length, an angle is just an angle and not a length.

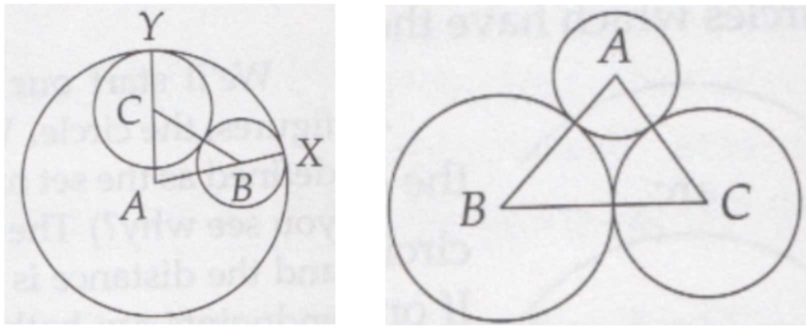
problem

Given two distinct points in the plane and a line l not through either point, find the center of the circle that has its diameter on the line and goes through both points.

There is no solution if a line drawn through the two points is perpendicular to l. Why?

two problems

Here are two problems from *the Art of Problem Solving*.



For the first (left panel), we are given that A, B and C are the centers of the respective circles, that the points which appear tangent are actually so, and also that $AB = 6$, $AC = 5$ and $BC = 9$.

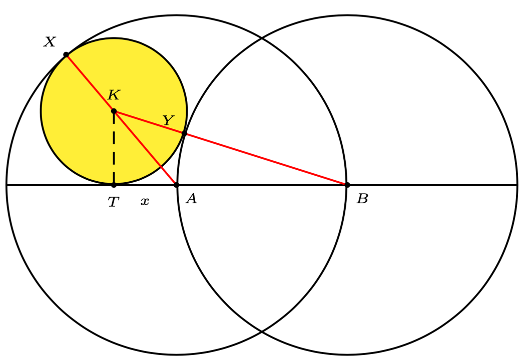
What is AX, the radius of the large circle?

For the right panel, we are asked to express the radius of circle A in terms of the sides of $\triangle ABC$.

Hint. Both problems ask you to exercise your skills in solving simultaneous equations.

problem

Here is a problem from Paul Yiu. It is really more about the Pythagorean theorem with an equilateral triangle thrown in but I put it here.



We have two larger circles of equal size drawn with $AB = a$, the radius, plus another smaller circle drawn tangent to both and to their diameter as shown.

The line through both centers of tangent circles goes through the point of tangency, so $KB = r + a$. Since A is the center of one circle, AX is equal to a, so $KA = a - r$. For the two right triangles we have

$$(a+x)^2 + r^2 = (a+r)^2$$
$$x^2 + r^2 = (a-r)^2$$

After cancelations, these (separately) give

$$2ax + x^2 = 2ar$$

$$x^2 = a^2 - 2ar$$

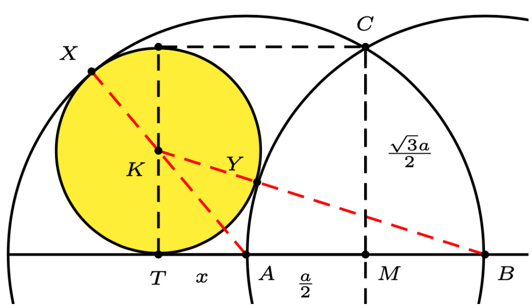
Adding

$$x^2 + ax - \frac{a^2}{2} = 0$$
$$x = \frac{-a \pm \sqrt{a^2 + 2a^2}}{2}$$

We take the positive branch for a length

$$x + \frac{a}{2} = \frac{\sqrt{3}}{2}a$$

Now, if M is the midpoint of AB, then $MT = x + a/2 = \sqrt{3}a/2$.



The classic **construction** of an equilateral triangle on AB would put its top vertex at the intersection of the two circles, with side length a and **altitude** $\sqrt{3}a/2$, forming a square.

It is also claimed that $2r = x + a/2$, which means that the other vertex of the square is also the top of the small circle. This may not be obvious but recall that we added our two equations above. Subtract!

$$\begin{aligned} 2ax &= 4ar - a^2 \\ x &= 2r - \frac{a}{2} \\ x + \frac{a}{2} &= 2r \end{aligned}$$

$\square$

Chapter 19

Inscribed angles

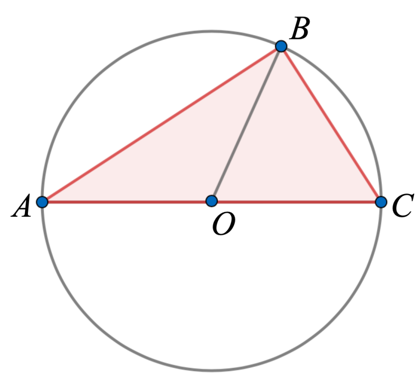
Thales' circle theorem

In this chapter, we will introduce the inscribed angle theorem. Let's start by revisiting Thales' circle theorem:

- Any angle inscribed in a semicircle is a right angle.

Take a diameter of the circle and any third, distinct point. The three points on the circumference of the circle form a triangle, and the angle at the third vertex is always a right angle.

In the figure below, $\angle ABC$ is a right angle.



Proof.

By I.32, the sum of angles in a triangle is equal to two right angles. So

$$\angle OAB + \angle OBA + \angle OBC + \angle OCB = 180$$

But $\triangle OAB$ and $\triangle OBC$ are both isosceles, so the base angles are equal, with

$$\angle OAB = \angle OBA \qquad \angle OBC = \angle OCB$$

It follows that

$$2\angle OBA + 2\angle OBC = 180$$
$$\angle OBA + \angle OBC = 90$$

□

According to Boyer, this result was known to the Egyptians 1000 years before Thales. But it is yet another example of knowing a result before proving that it is so. Archimedes had something to say about the importance of having discovered a fact, before finding a way to prove it.

The converse of Thales' theorem is also true. If the third point of a triangle contains a right angle, then it must lie on the circle where the other two points form the diameter.

Thales' circle theorem converse

A nice direct proof of this is given in Acheson.

Proof.

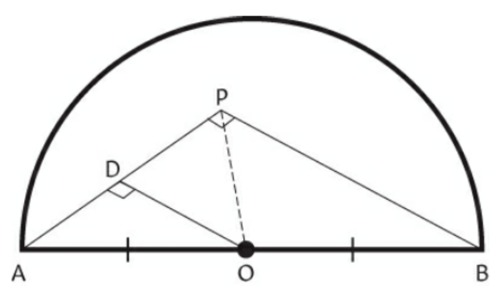


Fig. 59 An alternative method.

We are given that $\angle APB$ is a right angle.

Draw OD parallel to PB. $\triangle AOD$ is similar to $\triangle ABP$ because they are both right triangles with a shared vertex at A.

Since AO is one-half AB, the scale factor is $1/2$. In particular, $AD = DP$.

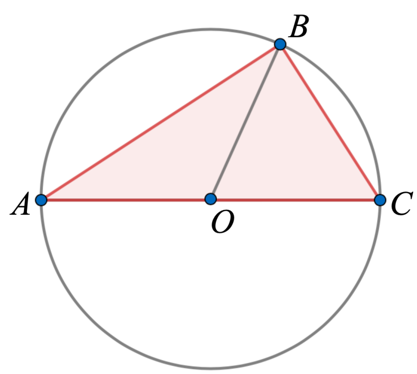
Now draw OP. The two smaller triangles $\triangle AOD$ and $\triangle DOP$ are congruent by SAS. Therefore, $OP = OA$.

But OA is a radius of the circle. Therefore, OP is also a radius.

It follows that P must lie on the circle.

□

We can use Thales' theorem to introduce the inscribed angle theorem.



inscribed angle theorem

An inscribed angle is an angle whose vertex lies on the circle, such as $\angle BAC$

• An inscribed angle is one-half the corresponding central angle lying on the same arc, $\angle BOC$. The central angle is twice the corresponding inscribed angle.

Proof.

Euclid's proof (the theorem is proposition 20 of book III), uses the external angle theorem. $\angle BOC$ is the external angle to $\triangle OAB$.

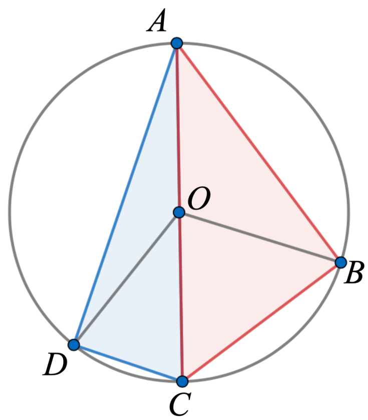
Thus, it is equal to the sum of $\angle OAB + \angle OBA$.

The triangle is isosceles, so these two angles are equal.

It follows that $\angle BOC = 2\angle BAC$.

$\square$

This proof is short and sweet, but limited by the fact that we used the diameter for one of the arms of the inscribed angle. Here is a more general proof.



The claim is that $\angle BOD = 2\angle BAD$.

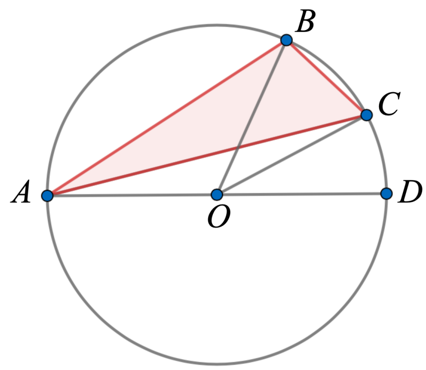
Proof.

Just add the results obtained using the previous proof. $\angle DOC = 2\angle DAC$ and $\angle BOC = 2\angle BAC$.

$$\angle BOD = \angle BOC + \angle DOC$$
$$= 2\angle BAC + 2\angle DAC = 2\angle BAD$$

$\square$

The proof is *still* limited, since the angle we looked at includes the center of the circle. There are two ways to fix this. The first is subtraction.



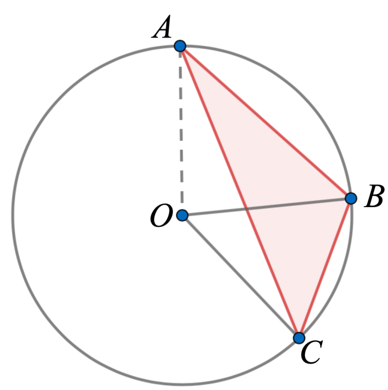
Proof.

By the first proof, $\angle BOD = 2\angle BAD$ and $\angle COD = 2\angle CAD$. The angle of interest, $\angle BOC$, is the difference:

$$\angle BOC = \angle BOD - \angle COD$$
$$= 2\angle BAD - 2\angle CAD = 2\angle BAC$$

□

An elegant proof of the second case is as follows.



Proof.

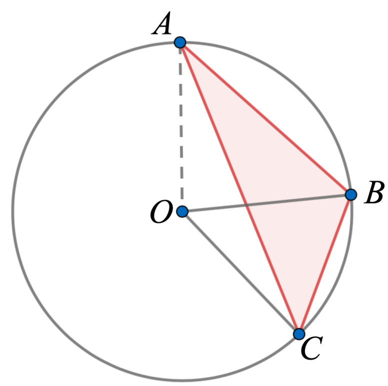
The total angle at A is $\angle OAB = \angle OAC + \angle BAC$. Since $\triangle OAB$ is isosceles, the base angles are equal.

With the addition of the central angle $\angle AOB$, by sum of angles we get two right angles for $\triangle OAB$.

$$2\angle OAC + 2\angle BAC + \angle AOB = 180$$

In $\triangle OAC$, again isosceles, we have by sum of angles

$$\angle AOB + \angle BOC + 2\angle OAC = 180$$



Subtracting

$$2\angle BAC - \angle BOC = 0$$

$$\angle BOC = 2\angle BAC$$

$\square$

angles on the same arc

• Angles that lie on the same arc or are subtended by the same chord in the same circle, are equal.

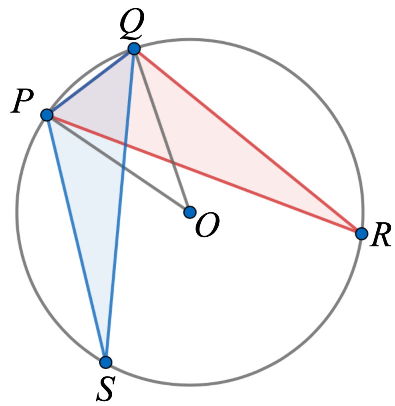
This theorem is Euclid III.21.

The previous **inscribed angle theorem** is Euclid III.20, which says that the central angle on any chord or arc or "segment" of a circle, is twice an arbitrary peripheral (inscribed) angle on the same segment of the same or equal circle.

Euclid III.21 is an immediate consequence of the previous one, because the peripheral angle in III.20 is *arbitrary*.

The two theorems are sometimes thought to be the same. III.20 might be better referred to as the central angle theorem, and III.21 as the inscribed angle theorem. However, that is not the customary usage.

III.21 will appear often in the pages to come. We will refer to it as **equal angles $\iff$ equal arcs** or more simply, equal angles *on* equal arcs, unless we slip, and call it the inscribed angle(s) theorem.



Proof.

This is an immediate consequence of the previous theorem, since two such angles are equal to one-half of the *same* central angle. $\angle POQ$ is the central angle for arc PQ and hence is twice both $\angle PRQ$ and $\angle PSQ$. Thus, $\angle PRQ = \angle PSQ$.

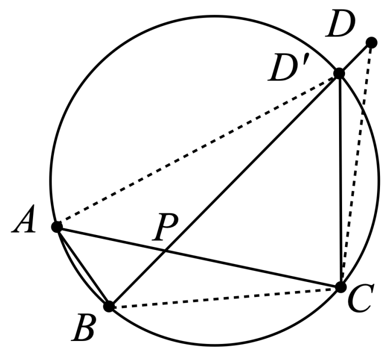
□

converse

There are several variant arguments from similar setups which are converses of the inscribed angles theorem. Here is one:

Let $\triangle ABC$ lie on a circle.

Let point D be such that $\angle BDC = \angle BAC$.



Then D is also on the same circle.

Proof.

Aiming for a contradiction, suppose otherwise.

Let D be external and also let $\angle BDC = \angle BAC$.

Find where BD cuts the circle at D'.

So D' is on the circle and $\angle BD'C$ is subtended by BC.

By the forward version of the inscribed angle theorem: $\angle BD'C = \angle BAC$.

It follows that $\angle BDC = \angle BD'C$.

But $\angle BD'C$ is external to $\triangle CDD'$.

So $\angle BD'C > \angle BDC$.

This is a contradiction.

A similar argument will show that D cannot be internal to the circle.

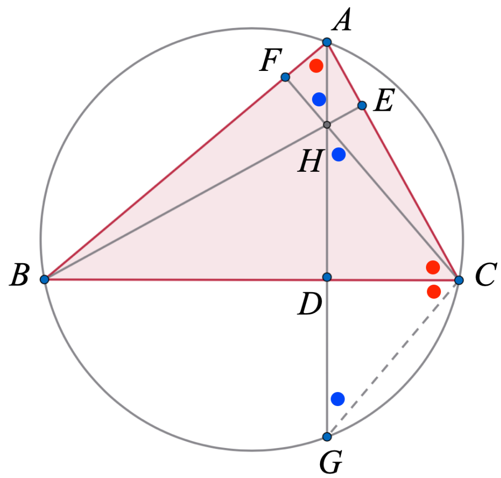
Therefore, it must be that D *is on the circle.*

□

extended altitude

- An altitude extended to the circumcircle of a triangle forms congruent triangles.

Proof.



In $\triangle ABC$ draw altitudes AD and CF. The angles at D and F are right. Since they are both right triangles and share vertical angles we have

$$\triangle AFH \sim \triangle CDH$$

So $\angle DCH = \angle FAH = \angle BAG$.

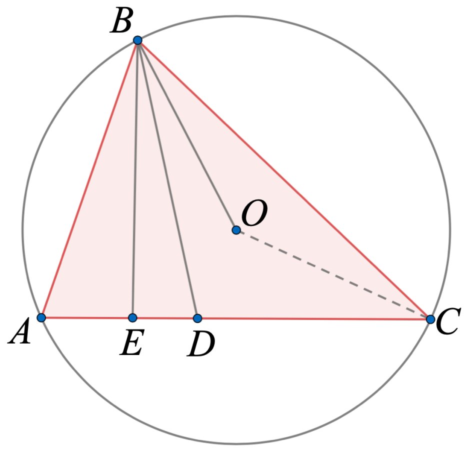
But $\angle BCG$ cuts the same arc as $\angle BAG$, so $\angle BCG = \angle DCG = \angle DCH$.

$\triangle CDG \cong \triangle CDH$ (ASA).

It follows that $DG = DH$ and $\triangle CHG$ is isosceles as well.

$\square$

problem



Posamentier gives this problem.

Let O be the center of the circumcircle of $\triangle ABC$.

Let $BE \perp AC$ and BD bisect $\angle B$.

Show that BD also bisects $\angle OBE$.

Proof.

Given $\angle ABD = \angle CBD$.

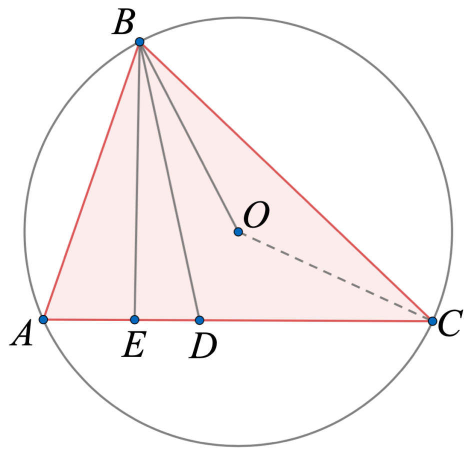
Draw OC.

The central $\angle BOC$ is 2 $\angle BAC$.

But $\triangle BOC$ is isosceles with equal base angles.

Thus $\angle BAC + \angle OBC = 90$.

$\angle ABE$ is also complementary to $\angle BAE$, hence $\angle BAE$ is equal to $\angle OBC$.

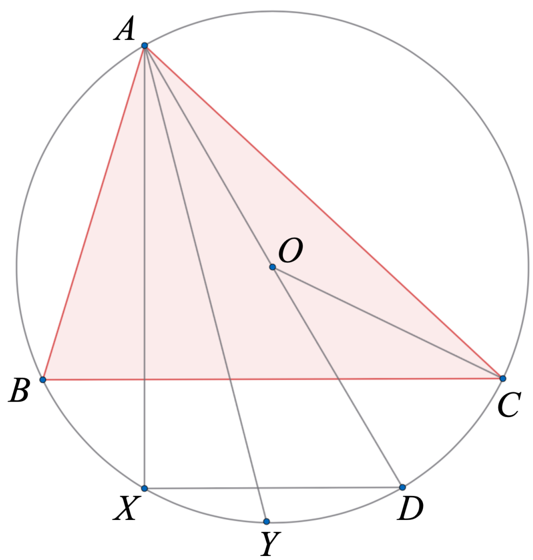


Subtracting equals: $\angle EBD = \angle OBD$.

$\square$

Proof. (Alternate).

I have redrawn the figure slightly.



Claim: the altitude and diameter of the circumcircle through A form equal angles with the sides AB and AC.

Let $\triangle ABC$ have its circumcircle on center O, so AO is a radius. Extend AO to form the diameter AD.

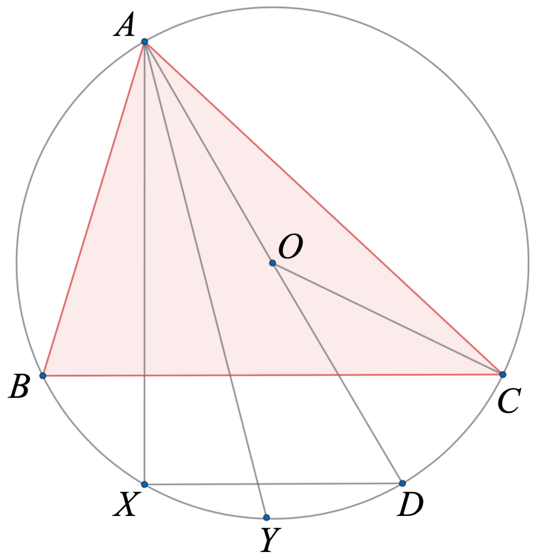
Let AX be the altitude to A, so it cuts BC at a right angle and extends to the circle at X.

By Thales' circle theorem, $\angle AXD$ is right. It follows that $XD \parallel BC$.

Parallel chords in a circle cut equal arcs. (Note: we prove this elsewhere. Hint: connect the opposing ends of the two chords to give equal inscribed angles).

Hence $BX = CD$ and then by inscribed angles, $\angle BAX = \angle CAD$.

$\square$



As a corollary, if AY bisects $\angle A$, by subtraction, the bisector also bisects the angle between the altitude and the radius AO.

Alternatively, find Y as the midpoint of arc XD.

Then we have equal arcs $XY = DY$, $BY = CY$, and $BX = CD$. Results about the angles follow easily.

geometric mean

We showed in the chapter on the **Pythagorean theorem** that the altitude of a right triangle is the geometric mean of the two components of the base.

$$h^2 = pq$$

$$h = \sqrt{pq}$$

According to wikipedia:

https://en.wikipedia.org/wiki/Geometric_mean

The fundamental property of the geometric mean is that (letting m be the *geometric mean* here):

$$m \left[\frac{x_i}{y_i} \right] = \frac{m(x_i)}{m(y_i)}$$

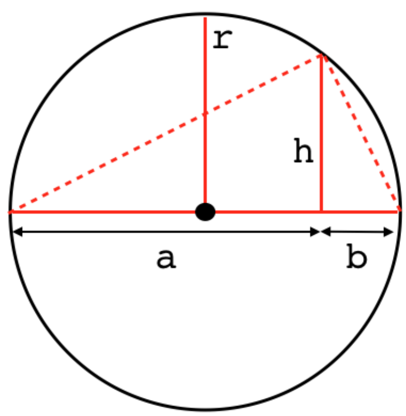
and one consequence is that

> This makes the geometric mean the only correct mean when averaging normalized results; that is, results that are presented as ratios to reference values.

A number of examples are given in the article.

We discuss this here because originally, there was a proof-without-words that the geometric mean is always less than or equal to the arithmetic mean.

I decided to add some words.



A right triangle is inscribed in a semicircle.

Let the length of the long dotted line be c. Then

$$a^2 + h^2 = c^2$$

Let the length of the short dotted line be d. Then

$$b^2 + h^2 = d^2$$

Adding together

$$a^2 + b^2 + 2h^2 = c^2 + d^2$$

But a third application of the Pythagorean theorem shows that the right-hand side is just $(a+b)^2$ so

$$\begin{aligned} a^2 + b^2 + 2h^2 &= (a+b)^2 \\ 2h^2 &= 2ab \\ h^2 &= ab \end{aligned}$$

□

An even simpler proof is to recognize that the two smaller triangles are similar with equal ratios of sides including:

$$\frac{h}{a} = \frac{b}{h}$$

and the result follows immediately.

This says that the square of the altitude h is equal to the product of chord segments (we will prove this geometrically in the next chapter as well).

$$h = \sqrt{ab}$$

But we also have that $a + b = 2r$ and hence

$$r = \frac{a+b}{2}$$

Do you recognize these? The second expression is the arithmetic mean of a and b, while the first is the geometric mean.

The geometry shows that $h \leq r$ so:

$$\sqrt{ab} \leq \frac{a+b}{2}$$

The geometric mean is always less than the arithmetic mean, except when $a = b$, where they are equal (or all of n values are equal).

problem

Here is a problem I found on the web as a Youtube video:

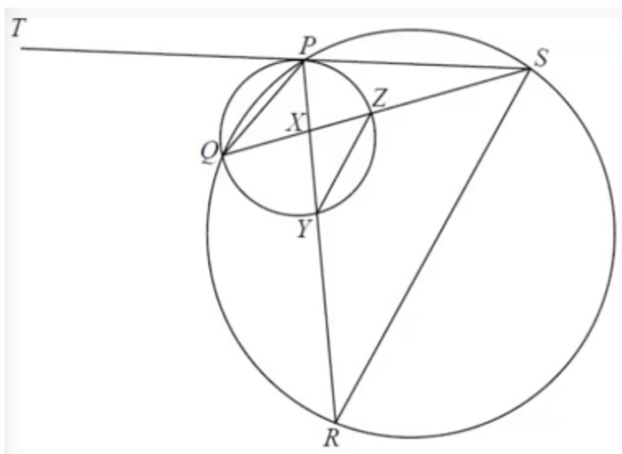
https://youtu.be/2Jt8lynddQ8

It is described as a GCE O-Level A-Maths Plane Geometry Question.

The relationships that seem obvious from the diagram are given. Namely, $PXYR$ and $QXZS$ are each lie on a straight line (colinear).

And the two circles each have the four points lying on them as shown.

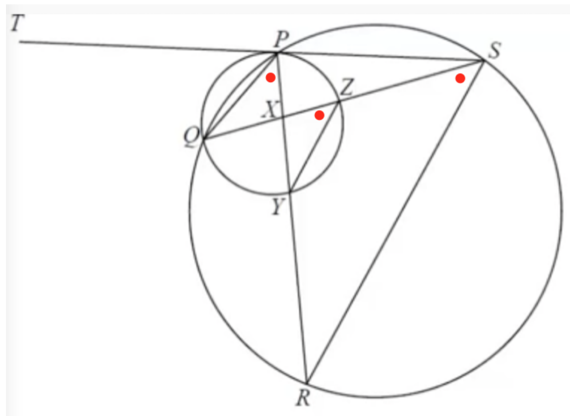
TPS is tangent to the smaller circle at P.



The problem asks us to show that SR is parallel to ZY and hence, *deduce* that $YX/ZX = YR/SZ$.

The approach that first occurred to me was to use the similar triangles that arise from crossed chords. However, the problem asks us to start by showing that the given line segments are parallel. This is a hint to a different proof.

The result comes from the theorem which is the basis of this chapter: the inscribed angle theorem.



Proof.

The marked angles are all equal. The first two are equal because they correspond to the same arc in the small circle, and the third (at S) is equal to the first because they both correspond to the same arc in the large circle.

Therefore, the two line segments are parallel by the converse of the alternate interior angles theorem.

That gives us similar triangles $\triangle XYZ \sim \triangle XRS$ from which the equal ratios follow almost immediately (see below).

□

The last part of the problem says that given $SQ = XR$, prove that $PS^2 = XS \cdot YR$ We're not ready to do that yet. It uses the information about the tangent and the **tangent-secant theorem**.

Proof. (Alternate).

Here's the first part of the proof by the crossed chord theorem:

$$PX \cdot XY = QX \cdot XZ$$

$$PX \cdot XR = QX \cdot XS$$

It follows that

$$\frac{XZ}{XY} = \frac{XS}{XR}$$

We need some algebra to get to $YX/ZX = YR/SZ$. This is a standard parts and the whole manipulation for similar triangles, made complex by the cumbersome notation.

Let $a = XY$, $A = XR$, $b = XZ$, and $B = XS$. We have

$$\frac{b}{a} = \frac{B}{A}$$

and we want

$$\frac{a}{b} = \frac{A - a}{B - b}$$

The way to get there is:

$$\frac{A}{a} = \frac{B}{b}$$

$$\frac{A}{a} - 1 = \frac{B}{b} - 1$$

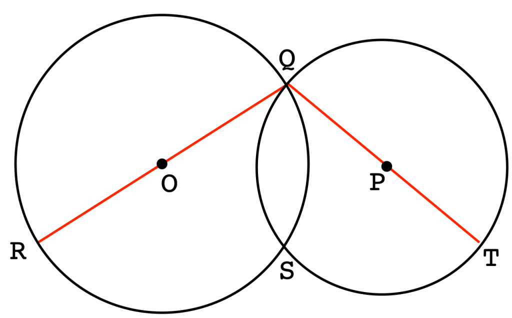
then

$$\frac{A - a}{a} = \frac{B - b}{b}$$

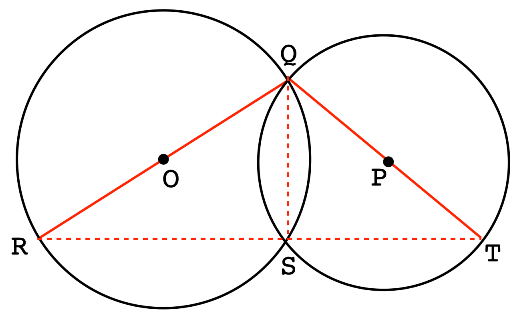
and the result follows easily in one more step.

□

problem



Two circles meet at Q and S. QR and QT are diameters of the two circles. Prove that RST are colinear.



Since QR is a diameter of the circle centered at O, $\angle QSR$ is a right angle.

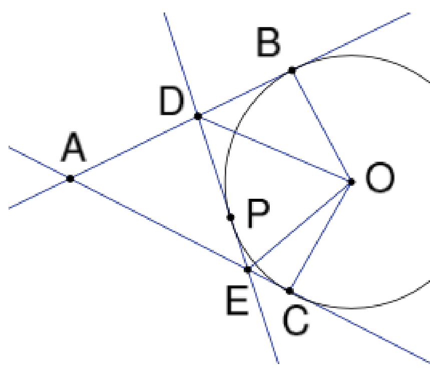
But so is $\angle QST$, since QT is a diameter of the second circle.

Hence the total angle at S is two right angles or a straight line. Therefore RS and ST togethere form a straight line segment.

double arc problem

This problem is taken from an online collection by David Surowski.

https://www.math.ksu.edu/~dbski/writings/further.pdf



Given that AB and AC are tangents to the circle meeting at A. Given a second tangent DE, meeting the circle at P. Prove that the arc that subtends $\angle BOC$ is twice that which subtends $\angle DOE$.

Proof.

Notice that DB and DP are tangents to the circle meeting at D. Therefore $DB = DP$ and then $\triangle BOD \cong \triangle DOP$, so $\angle BOD = \angle DOP$.

The same argument applies to EC and EP. Therefore the inner arc is composed of two angles, while the outer arc has two copies of each of those angles.

□

Queen Dido

The mighty city of Carthage was the capital city of a major Phoenician colony. As Rome grew strong, there was a titanic struggle between the two peoples, which Carthage eventually lost. The ruins of Carthage lie near present-day Tunis.

Queen Dido was the legendary founder of the the city of Carthage. She was supposedly

> granted as much land as she could encompass with an oxhide. She promptly cut the ox-hide into very thin strips.

The problem then is to maximize the area enclosed by a curve of fixed length.

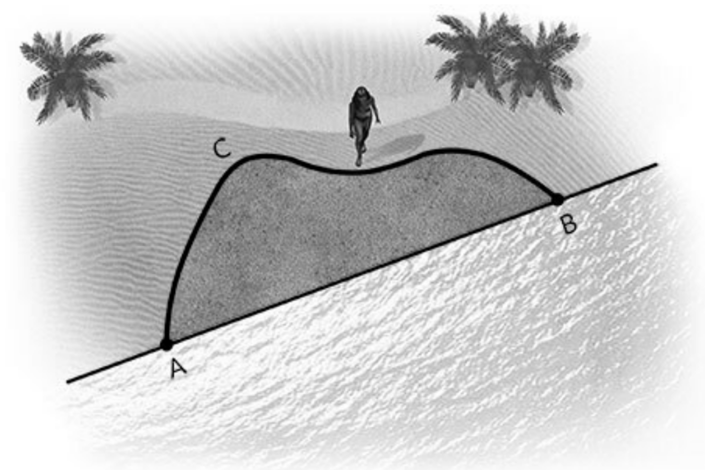


Fig. 138 Dido's problem (with a straight coastline).

In calculus there are a number of problems like this. What's nice is that this problem has a wonderful solution that uses only the tools we have so far. In particular, we need the converse of Thales' circle theorem.

The argument goes as follows. Suppose we have a particular outline for the city limits and we're pretty happy with it. We suppose it is a maximum (left panel).

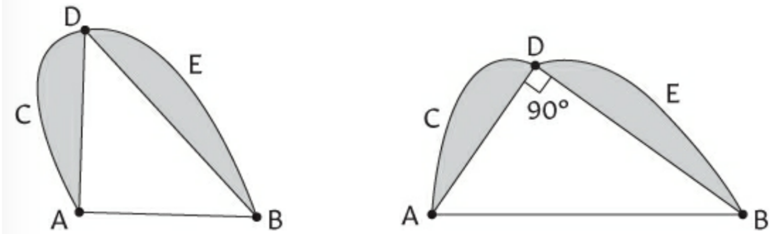


Fig. 140 The heart of Simpson's argument.

Then we notice that by rearranging AD and BD so they meet at a right angle, the crescent-shaped areas are unchanged, but the area of $\triangle ABD$ is a maximum. That's because a right triangle, having the two sides at right angles, has area equal to the product of the two sides (divided by 2). No other triangle with the same two sides has as much area.

So the arrangement on the right has a bigger area.

But then, with AB as the diameter of a circle, if $\angle ADB$ is a right angle, it must lie on the circumference of that circle, by the converse of Thales' theorem.

And this is true regardless of the relative lengths of AD and BD. Therefore the maximum area is obtained when D traces out a semi-circle.

This example is in Acheson's Geometry.

Chapter 20

Tangents

tangent: perpendicular → touches one point

We show **here** that one can construct a line perpendicular to any given line, and passing through any point whether on the line or not.

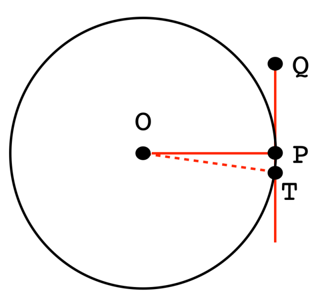
If we make the construction perpendicular to the radius (or diameter) at the point where it meets the circle, the new line is called a tangent line (from the Latin *tangere*, to touch).

• The tangent line, defined as perpendicular to the radius, touches the circle at a single point.

Proof.

By definition, the tangent line at P is perpendicular to the diameter or radius including O and P.

Let Q be some other point on that tangent line. Then $\angle OPQ$ is a right angle.



Let T be on the tangent line so QPT collinear. Draw OT.

Aiming for a contradiction, suppose that $OT \perp QPT$.

By the parallel postulate, $OT \parallel OP$.

But OT meets OP at O. This is a contradiction.

OT cannot be perpendicular to QPT.

It follows that OT is the hypotenuse of a right triangle $\triangle OPT$, so $OT > OP$ and thus T cannot be on the circle.

□

tangent: touches one point → perpendicular

An alternative definition of the tangent is that it is a line that touches the circle at just one point. One can use this definition to prove that the angle between the tangent and the radius is a right angle. This is the converse of the previous theorem.

• The tangent line, defined as touching the circle at a single point, is perpendicular to the radius.

Proof.

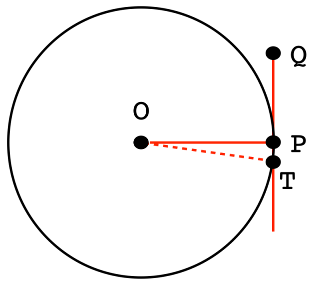
Draw the line that touches the circle at only one point P, and draw the radius to that point, OP.

If we assume that OP is not perpendicular to QPT, we can derive a contradiction.

Consider a succession of points moving along the line away from P in either direction. The angle formed with a line drawn through O gets smaller as the points get farther from P.

The angle between the tangent and the radius is greater than a right angle on one side of P (since the angle at P is not a right angle). On that side, move along the line until we find the point that does form a right angle with a radius of the circle.

Let us suppose the point is T, in the figure above. OTP is a right angle and we are in doubt about whether T lies inside or outside the circle.



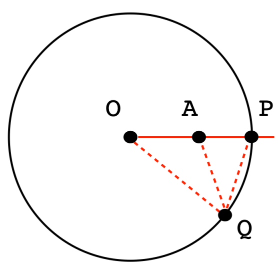
But then OTP is a right angle, with the side opposite, namely, OP, the hypotenuse of a right triangle. But the hypotenuse is the longest side in a right triangle, so therefore $OT < OP$, so the point T is *inside* the circle.

We have a line with one point on the circle, and another point inside the circle. Any line through an interior point of a circle must cross the circle at two points, which contradicts the assumption above. Hence the angle at P is a right angle.

□

shortest distance to the circle

Let A be any point inside a circle. Draw the radius that passes through A to point P on the circle. I claim that the length AP is smaller than the distance to *any* other point on the circle, such as Q.



Proof.

Draw the radius OQ, which is equal to radius OP.

$$OQ = OP = OA + AP$$

By the **triangle inequality**

$$OA + AQ > OQ$$

so

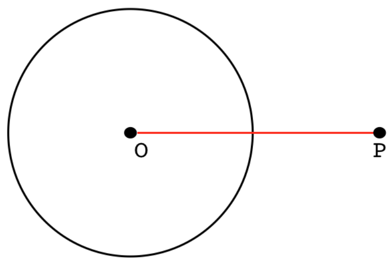
$$OA + AQ > OA + AP$$

$$AQ > AP$$

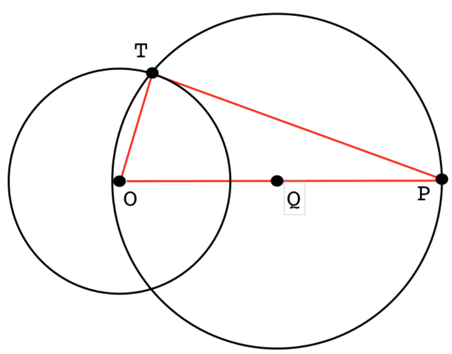
□

construction of a tangent

Thales theorem provides a way to construct the tangent to a circle that passes through any exterior point P — actually, there are two such lines.

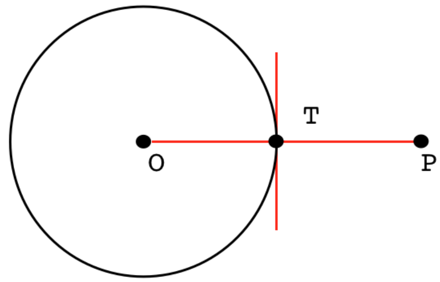


Use OP as the diameter of a circle. Draw the line segment OP and divide it in half by erecting the perpendicular bisector at Q. Use that point Q as the center of a new circle with radius OQ. The point T is the intersection of the two circles.



By the theorem, $\angle OTP$ is a right angle, and since OT is a radius of the original circle, TP is the tangent to the smaller circle at the point T.

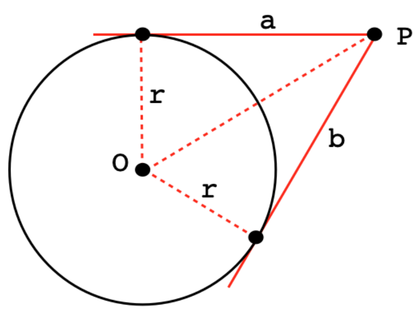
To construct a tangent on a circle at a given point T:



Extend OT to P so that OP is twice the radius OT. Construct the perpendicular bisector at T. The bisector is also the tangent of the circle.

• From any external point P, one can draw two tangents to a circle. These two tangents have the same length.

• From any external point P, the line to the center of the circle bisects the angle between the two tangents, as well as the angle between the radii drawn to the two points of tangency.



Proof.

The angle between a tangent and the radius to the point where it touches the circle, is a right angle. For a pair of tangent lines from a given point, there are two such points on the circle.

The base lengths are both radii, so they are equal, and there is a shared side (the dotted line segment OP).

Therefore the two triangles are congruent, by hypotenuse-leg in a right triangle (HL).

The two congruent sides a and b are the same length.

$$a = b$$

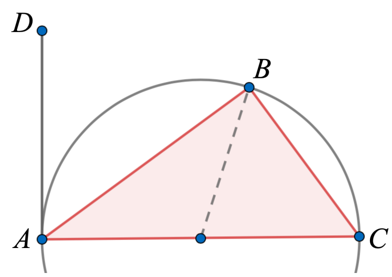
For any point external to a circle, two tangents to the circle can be drawn, of equal length. The line from the point to the center of the circle bisects the angle between the two tangents.

□

tangent-chord theorem

A tangent is a line that just touches the circle (or another curve, like a parabola). By definition it touches the circle at a single point, and is perpendicular to the radius which extends to that point.

• The arc swept out between a tangent and a chord is equal to the arc lying between the point of tangency and the point where the chord meets the circle.



Proof.

By Thales' theorem, $\angle B$ is right, so $\angle BAC$ and $\angle BCA$ are complementary.

The tangent $DA \perp AC$, so $\angle BAC$ is complementary to $\angle DAB$.

It follows that $\angle DAB = \angle BCA$.

Therefore, they cut the same arc of the circle.

□

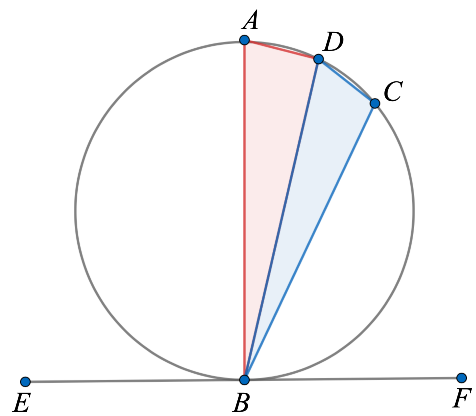
This is Euclid III.32, the tangent-chord theorem. Acheson calls this the alternate segment theorem.

Here is Euclid's proof:

In the figure below, let EF be tangent to the circle at B and AB be a diameter of the circle. $\angle ABF$ is a right angle.

I claim that $\angle DBF = \angle BAD$, and they cut the same arc DCB.

Also, $\angle DBE = \angle DCB$, and they cut the same arc DAB.



Proof.

By Thales' theorem $\angle ADB$ is right, so by sum of angles $\angle BAD + \angle ABD$ is right.

Therefore

$$\angle ABD + \angle BAD = \angle ABD + \angle DBF$$

$$\angle BAD = \angle DBF$$

For the second part, $\angle DBE + \angle DBF$ equals two right angles.

By the **quadrilateral supplementary theorem** (Euclid III.22)

$$\angle BAD + \angle BCD = 180$$

and since $\angle DBF = \angle BAD$ we have

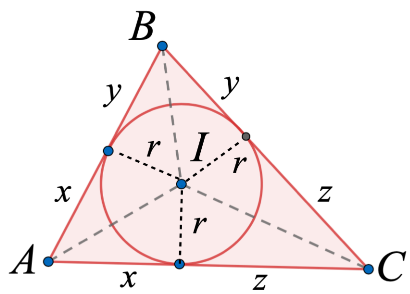
$$\angle DBE + \angle DBF = \angle DBF + \angle BCD$$

$$\angle DBE = \angle BCD$$

□

the incircle

For any triangle it is possible to draw a circle, called its incircle, which is tangent to all three sides. When stated in this way, it does not seem obvious how to actually do it.



Proof.

Let AI be the bisector of $\angle A$ and BI be the bisector of $\angle B$, meeting at I.

Recall that if we construct the bisector of an angle, say AI bisecting $\angle A$, then every point on the bisector is equidistant from the sides of the original angle.

So one can draw two right triangles with sides r and AI, which are congruent by HL, explaining the labels x.

The point I is equidistant from side c opposite $\angle C$ and side b opposite $\angle B$.

But the same thing can be done for BI bisecting $\angle B$.

Then draw the circle on center I with radius r, and we're done. This circle is the incircle for $\triangle ABC$.

r is a radius and at the point where it touches the side, perpendicular. Thus the three sides are each tangent to the incircle.

□

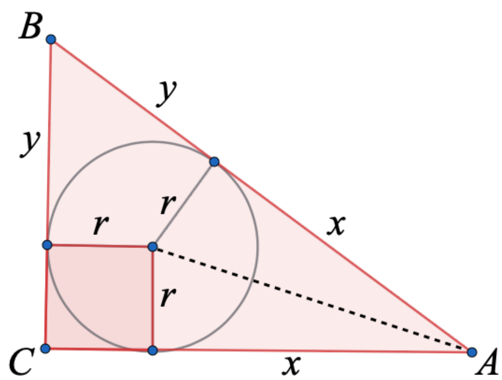
We can obtain a simple result about the area of the triangle. It is the sum of two copies of each of the three types of right triangle: $\mathcal{A} = rx+ry+rz$. But the perimeter of the triangle is twice $x + y + z$, the semi-perimeter $s = x + y + z$ and then $\mathcal{A} = rs$.

Pythagoras incircle proof

Dunham gives this as a problem.

Let right $\triangle ABC$ have sides opposite a, b and c, as usual.

Let the parts of the sides be x and y. Note that since this is a right triangle, what would be z becomes r, the radius of the incircle. Also, $c = x + y$ is the hypotenuse.



Walk around the triangle. The perimeter p is

$$y + x + x + r + r + y$$

$$p = 2x + 2y + 2r$$

But $x + y = c$ and $p = a + b + c$ so

$$a + b + c = 2c + 2r$$

$$2r = a + b - c$$

Now, the area of the whole $\triangle ABC$ is

$$K = \frac{1}{2}ab$$

From its three components, the area is also

$$K = r^2 + rx + ry$$

$$= r^2 + rc$$

Hence

$$\frac{1}{2}ab = r^2 + rc$$

$$2ab = (2r)^2 + 4rc$$

Substitute for $2r$ and crank through some algebra:

$$2ab = (a + b - c)(a + b - c) + 2(a + b - c)(c)$$

$$= a^2 + b^2 + c^2 + 2ab - 2ac - 2bc + 2ac + 2bc - 2c^2$$

Hence

$$0 = a^2 + b^2 + c^2 - 2ac - 2bc + 2ac + 2bc - 2c^2$$

$$0 = a^2 + b^2 + c^2 - 2c^2$$

$$0 = a^2 + b^2 - c^2$$

$$a^2 + b^2 = c^2$$

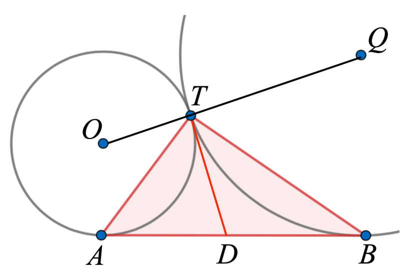
□

Not exactly pretty, but a clever construct, and it works.

Euclid III.12

Two circles are tangent to each other at T.

In one circle draw the radius to T and extend it. This line goes through the center of the second circle.



Proof.

Draw the tangent to the first circle at T. This line goes through a single point (T) on both circles.

The line perpendicular to it (the extension through T) is a radius of the second circle.

□

Euclid's proof of III.12 is by contradiction.

Proof.

Suppose the straight line OQ does not go through T.

By the triangle inequality, $OQ < OT + QT$. where OT and QT are radii of the two circles.

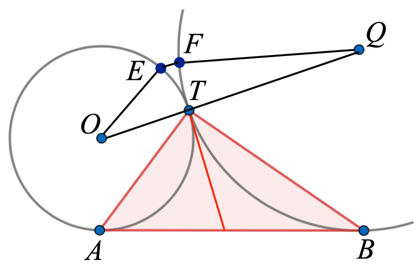
Then OQ must cut the two circles at different points, say E and F.

The straight line is

$$OQ = OE + EF + QF$$

$$OQ > OE + QF$$

where OE and QF are radii of the two circles.



But $OT + QT = OE + QF$

OQ cannot be both less than the sum of radii and greater than the same sum. This is a contradiction.

Therefore OTQ are collinear. OQ goes through T.

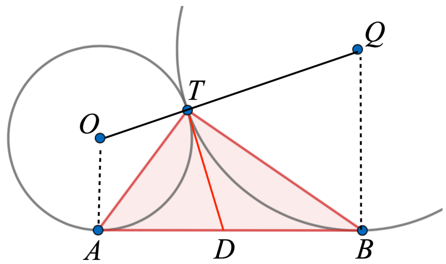
□

problem

Draw AT and BT. Prove that $\angle ATB$ is a right angle.

Solution.

Draw the tangent line to both circles at T, the red line in the figure below.



Draw radii of both circles to their respective tangent points at A and B.

DA and DT are both tangent to the circle on center O, while DT and DB are both tangent to the circle on center Q

The tangents are all equal in length, so $\triangle DAT$ and $\triangle DBT$ are both isosceles.

Since the base angles are equal, we have (by the external angle theorem) that

$$2\angle ATD = \angle BDT$$

$$2\angle BTD = \angle ADT$$

But the sum of the two right-hand sides is equal to two right angles.

Therefore, the sum of the two left-hand sides is also equal to two right angles. Hence

$$\angle ATD + \angle BTD = 90 = \angle ATB$$

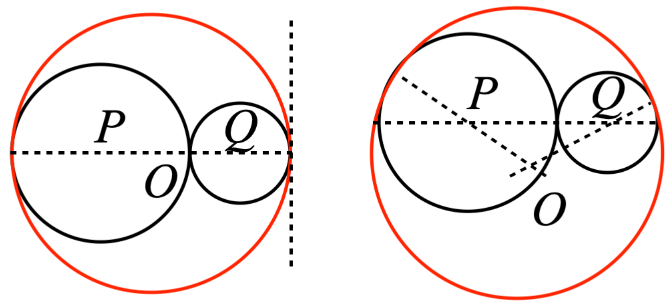
$\square$

Alternatively, since $\triangle DAT$ and $\triangle OAT$ are both isosceles, the sum of the angles making up $\angle OTD$ is equal to the sum of angles making up $\angle OAD$.

But $\angle OAD$ is a right angle, therefore so is $\angle OTD$.

tangents

As indicated in the previous problem, if two circles just touch one another at a single point, and the tangent is drawn at that point, then it is tangent to *both* circles. The perpendicular to the tangent at that point is a radius of both circles. Thus a single line is collinear with both diameters.



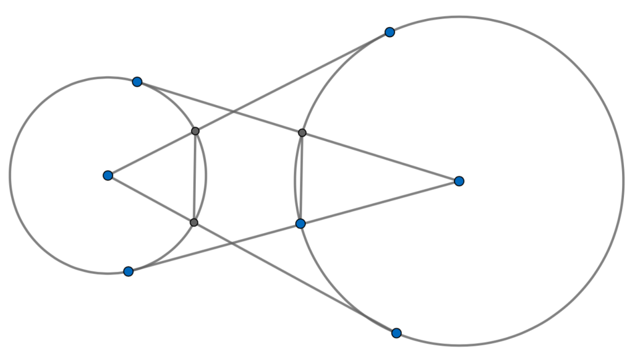
So then, what about three circles? Clearly, the third circle can be drawn with the joined diameters of the smaller two as its diameter. Or it can be drawn somewhat larger (right panel).

The diameter through P and Q in the second case must be extended to meet the large circle, so apparently, the arrangement in the left panel is the circle with the *smallest* diameter that is also tangent to the two circles on centers P and Q. (See Euclid III.7 for a proof).

The eyeball theorem

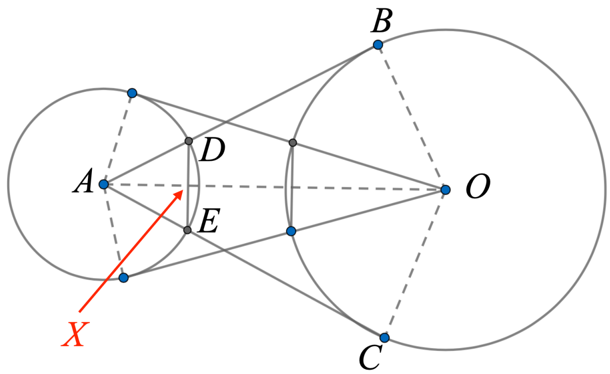
This problem is from Acheson's Geometry book (Fig 131). We have two circles with tangents drawn from the center of each circle to the other one.

We are to show that the lengths of the chords formed by the tangents as they exit the originating circle are equal.



Solution.

Let us label some points.



The two triangles on the centerline with tangents as the base are equal: $\triangle AOB \cong \triangle AOC$ by HL.

Therefore, the centerline bisects $\angle BAC$.

So then the two small triangles $\triangle AXD \cong \triangle AXE$ by SAS.

Therefore the angles formed at X are right.

We have similar right triangles: $\triangle AXD \sim \triangle ABO$.

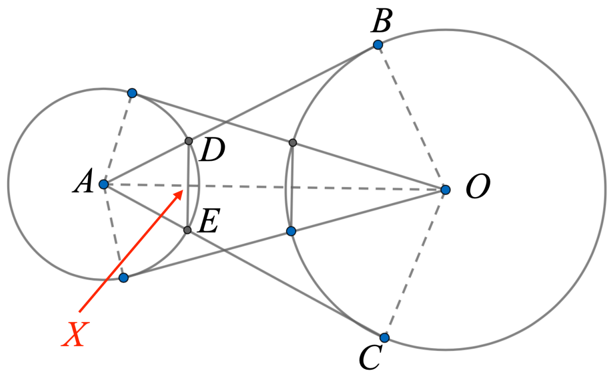
Form the equal ratio of the hypotenuses on the left-hand side and of the shorter legs on the right-hand side:

$$\frac{AD}{AO} = \frac{DX}{OB}$$

$$DX = \frac{AD \cdot OB}{AO}$$

Define $OB = R$, $AD = r$ and $AO = d$ so

$$DX = \frac{rR}{d}$$



DX is one-half the chord. Its length in the formula is symmetrical in r and R. Therefore the half-chord in the other circle has an identical length.

The chords are equal in length and parallel to each other (since they are perpendicular to the centerline, and bisected by it), so they form the sides of a rectangle.

penny-farthing problem problem

Here is a problem with a fascinating history (see Acheson). Find an expression for D in terms of a and b.

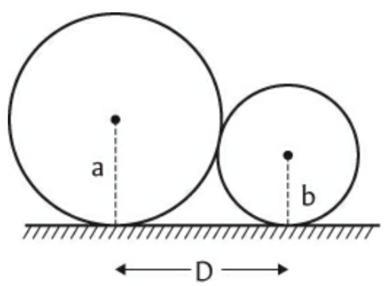


Fig. 161 A penny-farthing problem.

Its solution is very easy, so I will not give it here but I encourage you to try.

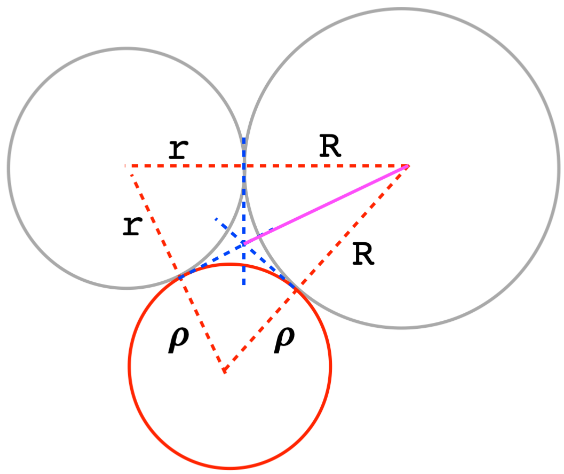
problem

Harvard 1899 exam:

4. Three unequal circles are so situated that each of them is externally tangent to the other two. At the points of contact tangents are drawn. Prove that these three tangents meet in a point.

Solution.

The tangent lines are perpendicular to a radius drawn to the point of tangency. As a result, two corresponding radii meet at the point of tangency and the two corresponding centers are co-linear with the point of tangency.



Therefore, we have a triangle with sides as shown. The tangent lines are perpendicular to the sides of the triangle, but will not, in general, pass through a vertex or center of the third circle, for any pair of circles and their tangent line.

But for each vertex of the triangle, the bisector of the angle gives congruent triangles, in which one of the sides is a blue dotted line, forming a right angle with the radius. The hypotenuse of one such pair is shown in magenta.

The two triangles that share the magenta line as hypotenuse are congruent by hypotenuse-leg in a right triangle (HL), and therefore the two blue dotted lines meeting in the center have equal length. Now do the same for the circle with radius r and then for the circle with radius ρ.

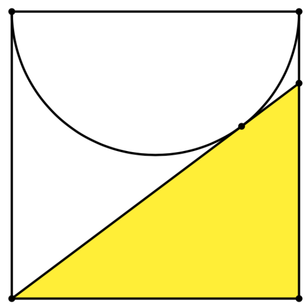
The blue-dotted lines are thus all equal. So they can be used to draw a circle that just touches the sides of the triangle. That circle is called the incircle of the triangle.

The point where these tangents meet is the incenter of the triangle. Since the incenter exists, the three angle bisectors are concurrent, the point where the tangents meet is the same and it also exists.

□

problem

This is a problem from Paul Yiu. We have a semicircle inside a square with one side of the square as the diameter. The tangent is drawn as shown.



Prove that the triangle is a $3-4-5$ right triangle.

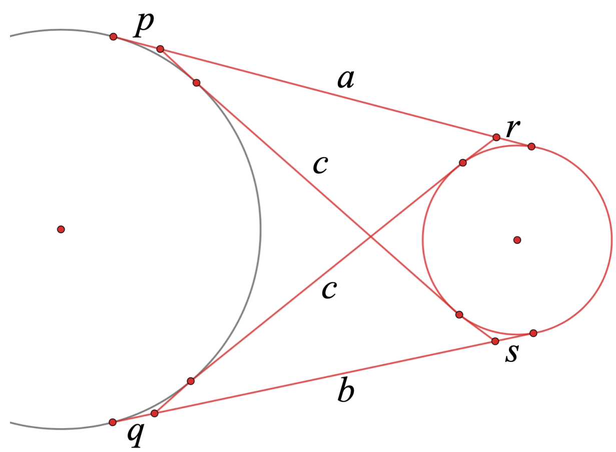
Scale the square so that the side has length 4. We will use the property that the two tangents to any circle from an exterior point are equal. Thus, the distance from the lower left to the point of tangency is 4. Let the other short tangents have length x.

Then Pythagoras says that:

$$4^2 + (4-x)^2 = (4+x)^2$$
$$4^2 + 4^2 - 8x + x^2 = 4^2 + 8x + x^2$$
$$16x = 4^2, \qquad x = 1$$

So the sides are indeed $3-4-5$.

one more



This is another problem from Paul Yiu. Let four tangents to two circles be drawn as shown.

Let the line with length $a+p+r$ be tangent to both circles, and the same for $b+q+s$.

Furthermore draw the crossed tangents of length c and extend them to meet a and b, as shown. The short tangents are all pairs with equal length: p, q, r and s.

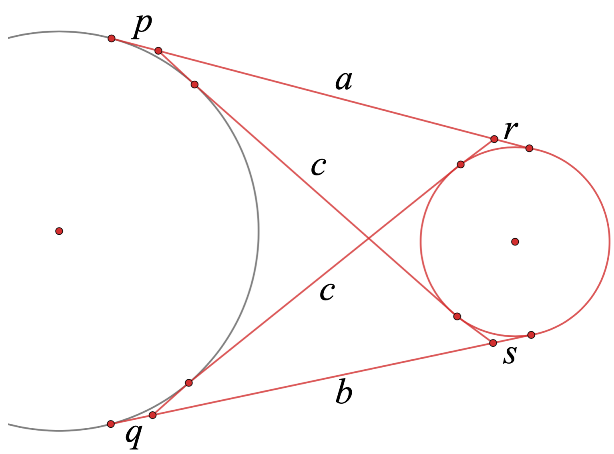
The tangents labeled c are equal, because they are composed of two tangents to each circle extended from the intersection in the middle.

We can write other equalities based on tangents from different points:

$$p + c = a + r$$
$$p + a = r + c$$

Subtracting

$$c - a = a - c$$
$$c = a$$



Substituting

$$p = r$$

By the same logic $c = b$.

Thus $a = b = c$ and and $q = s$.

Finally, consider the tangent not drawn. Somewhere offscreen to the right is a point from which two lines can be drawn tangent to *both* circles. The parts not shown are equal (as they are tangents to the right-hand circle), so the parts remaining are also equal, by subtraction.

Namely

$$p + a + r = q + b + s$$

$$2p + a = 2q + b$$

$$p = q$$

So all four of the short segments are also equal: $p = q = r = s$.

Chapter 21

Arcs of a circle

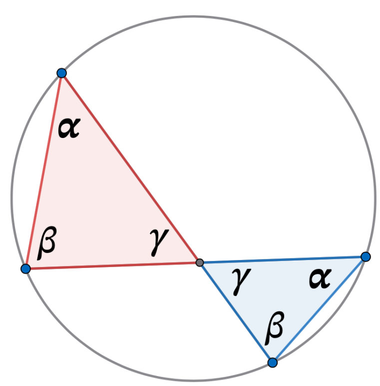
It is natural to think about angles formed from a vertex lying on the periphery of a circle and ask about their relation to the arcs they cut off.

The **inscribed angle theorem** says that a vertex placed at any point on the periphery of a circle, forming an angle that corresponds to the same arc as a central angle, is equal to *one-half* the central angle.

Since the measure of the central angle is equal to the measure of the arc, by definition, we have that twice the peripheral angle is equal to the arc that subtends it.

A corollary is that whenever two peripheral angles correspond to the same arc, they are equal (**equal angles** $\iff$ **equal arcs**).

In the figure below, the two angles marked α are equal, because they correspond to the same arc of the same circle.

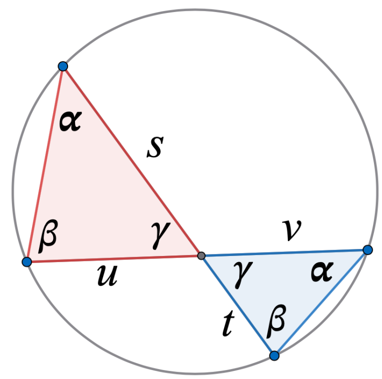


Similarly, the two angles marked β are equal, for the same reason. Then by sum of angles (or vertical angles), the two angles marked γ are also equal. We have two similar triangles.

If two chords of the circle cross, the product of the components is a constant.

Proof.

If we label the sides of the triangles



u and t are the sides opposite $\angle\alpha$ so we have

$$\frac{u}{t} = \frac{s}{v}$$

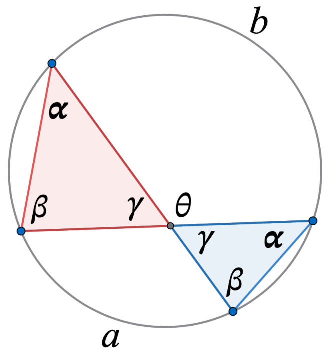
rearranging

$$u \cdot v = s \cdot t$$

□

This is the **crossed chord theorem** (product of lengths).

Arcs of intersecting chords



Given two crossed chords, θ is the average of the opposing arcs a and b.

Proof.

Take the same two similar triangles and consider the angle θ as shown in the figure above. By the corollary of the inscribed angle theorem, we have that

$$2\alpha = a \qquad 2\beta = b$$

But θ is the external angle to both triangles, so $\theta = \alpha + \beta$ and then

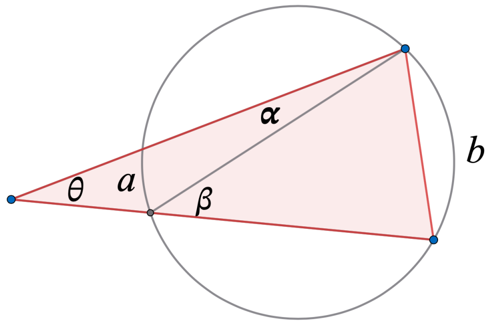
$$2\theta = 2\alpha + 2\beta = a + b$$

$$\theta = \frac{a+b}{2}$$

□

external vertex

Rather than having the vertex on the circle, suppose it lies outside.



If the angle lies outside the circle, then twice its measure is the difference between the two arcs, the one farther away minus the closer one.

Proof.

Again, α corresponds to arc a and β to arc b.

$$2\alpha = a \qquad 2\beta = b$$

But β is the external angle to the small triangle with θ, such that

$$\beta = \theta + \alpha$$

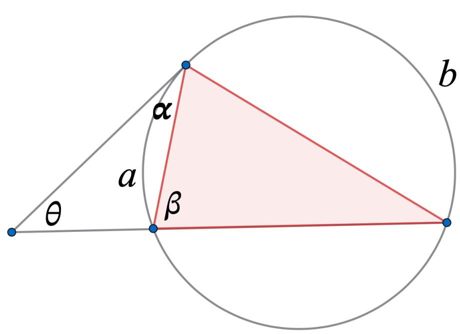
$$2\theta = 2\beta - 2\alpha$$

$$2\theta = b - a$$

$$\theta = \frac{b-a}{2}$$

□

tangent and secant



Rather than two secants, we now have a secant and a tangent. The result is the same as previously.

$$\theta = \frac{b-a}{2}$$

Proof.

We rely on the tangent-chord theorem, which says that if α is the angle between a chord and a tangent, then the corresponding arc has the same relationship as for two chords emanating from the vertex, namely:

$$2\alpha = a$$

$$2\beta = b$$

But

$$\beta = \alpha + \theta$$

$$2\theta = 2\beta - 2\alpha$$

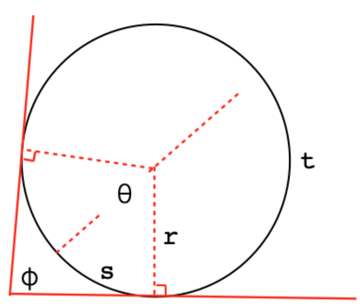
$$2\theta = b - a$$

$$\theta = \frac{b-a}{2}$$

□

two tangents

We showed previously that when two tangents are drawn from an exterior point, one can draw two right triangles that share the hypotenuse and have another side equal to the radius, so they are congruent by hypotenuse-leg in a right triangle (HL).



Let the whole arc between the two right angles be s the short way and t the long way around the circle, and let ϕ be the external angle. By analogy with the results above, we expect that

$$\phi = \frac{t-s}{2}$$

Proof.

We could use congruent triangles, but instead just note that the sum of angles in any quadrilateral is 2π. Hence

$$\phi + \theta = \pi$$

In terms of arc $\theta = s$ and $s + t = 2\pi$. Substituting into the last equation

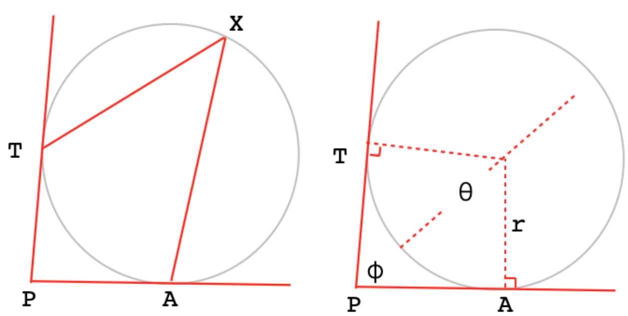
$$\phi + s = \frac{s+t}{2}$$

$$\phi = \frac{t-s}{2}$$

$\square$

problem

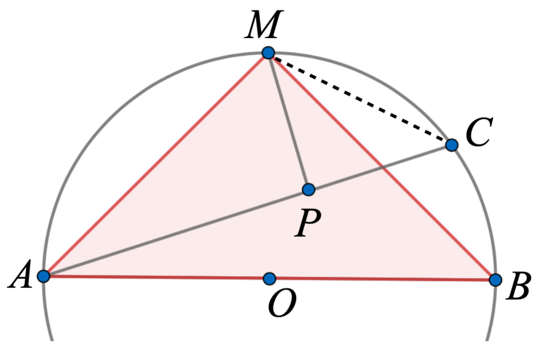
Relate the angle at P to the one at X.



By the previous example, $\theta + \phi = \pi$. But $\angle X = \theta/2$. Hence

$$P = \pi - \theta = \pi - 2\angle X$$

problem



Let AB be a diameter of the circle on center O. Let M be found such that $AM = BM$, so $\angle MAB = \angle MBA$ and both are one-half of a right angle.

Draw an arbitrary chord from A such as AC. Draw $MP \perp AC$.

$\triangle MPC$ is isosceles.

Proof.

Draw MC.

$\angle ACM = \angle PCM$ is subtended by chord AM, which is one-quarter of the circle.

Since $\angle MPC$ is right, $\angle PMC$ is also one-half of a right angle, by sum of angles.

It follows that $\triangle MPC$ is isosceles and $MP = CP$.

□

Chapter 22

Chords in a circle

Chords are straight lines inside a circle, with endpoints which lie on the circle. If a chord is extended (produced, in Euclid's terminology), then the extended chord becomes a secant, with part of the line lying outside the circle.

- If two arcs are equal, the corresponding chords are also equal, and vice-versa.

Proof. (forward)

Draw the radii flanking both arcs. By definition, we have equal angles for the equal arcs, and therefore congruent triangles by SAS. The equal chords are opposing sides to the central angle in the congruent triangles.

□

Proof. (converse)

Draw the radii flanking both chords. We have congruent triangles by SSS, so the central angles are equal. By Euclid III.26, we have equal arcs.

□

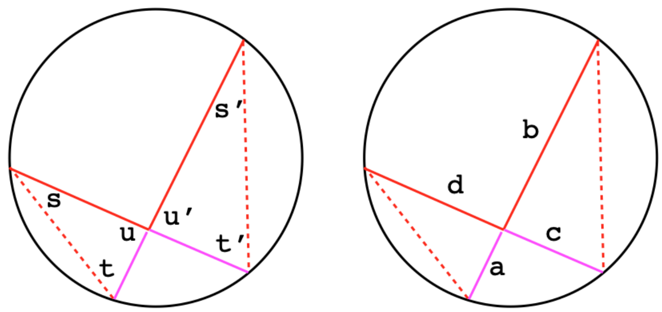
We may occasionally not even point out the equivalence, but just assume that if two arcs are equal, the corresponding chords are also equal, and vice-versa.

inequality

Suppose we have one angle greater than another, or one chord greater than another?

Proof. (Sketch). Start with two chords in a circle. Draw the radii for both. Then, by the **hinge theorem**, Euclid I.24, the greater angle lies opposite the greater third side. Thus, the greater chord lies in the greater circumference because they are connected through the central angle. □

crossed chords



Previously, we showed that when two arcs cross in a circle, each of the two equal vertical angles is equal to the *average* of the two arcs they cut out. It is also apparent that the two triangles formed are similar.

In two similar triangles, the sides opposite equal angles are in the same ratio, so:

$$\frac{a}{c} = \frac{d}{b}$$

It follows that

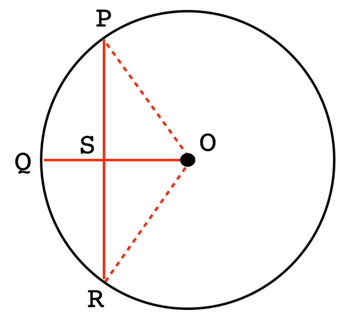
$$ab = cd$$

The products of the two parts of each chord are equal.

bisected chord theorem

Given any chord of a circle and a line perpendicular to it, if the perpendicular is a radius of the circle then the chord is bisected.

Conversely, if the perpendicular is a bisector, then it is also a radius.



Proof.

For the forward theorem: we have that OQ is a radius (it lies on the diameter), and it is perpendicular to chord PR.

We have $OP = OR$, as radii of the circle. Also, side OS is shared, and there are right angles at S. So $\triangle OPS \cong \triangle ORS$ by hypotenuse-leg in a right triangle (HL).

We conclude that $PS = SR$.

□

In addition, since $\triangle OPS \cong \triangle ORS$, $\angle POS = \angle ROS$, the arcs PQ and QR are cut out by equal central angles, therefore they are equal.

The converse theorem was established when we derived the method to construct a circle given any three points on it.

Briefly, all points that are equidistant from the two ends of a chord must lie on its perpendicular bisector. Since the center of the circle containing the chord forms radii when joined to the two ends, it must also lie on the perpendicular bisector.

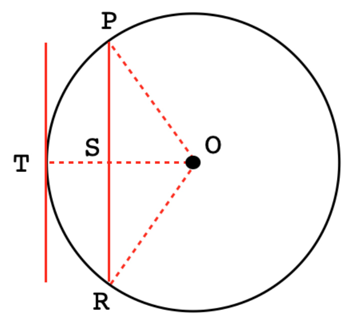
Therefore the perpendicular bisector is a diameter of the circle.

□

One consequence is that if a line is the perpendicular bisector of a chord of the circle, it is also the perpendicular bisector of any other chord parallel to the first.

Furthermore, if we have a tangent and any chord that is parallel to it, then the radius drawn to the point of tangency bisects the chord.

This follows simply from the definition of the tangent as perpendicular to the radius at the point of tangency.



By our previous theorem, OT bisects PR so that $PS = SR$.

∘ Given a point of tangency on a radius perpendicular to any chord, not only the chord but the arc lengths are evenly divided by the radius. $PT = RT$.

We can also note that if the distance from S to the periphery of the circle is d and the height of the half chord is h, by the crossed chord theorem we have

$$(2r - d)(d) = h^2$$

which can be solved as a quadratic in d.

$$d^2 - 2rd + h^2 = 0$$

$$\begin{aligned} d &= \frac{2r \pm \sqrt{4r^2 - 4h^2}}{2} \\ &= r \pm \sqrt{r^2 - h^2} \end{aligned}$$

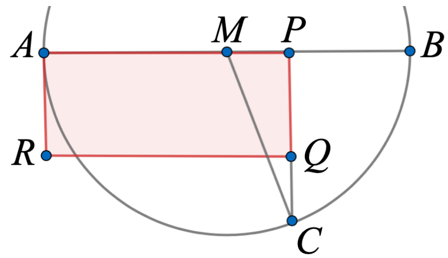
which is already fairly obvious since $OP = r$ and, it's the hypotenuse in a right triangle with $PS = h$, so OS is exactly the square root, while r minus that is just d.

Euclid II.14

As an aside, suppose it is desired to construct the square equal in area to a given rectangle $APQR$.

Extend AP to B such that PB = PQ.

Bisect AB to find center M for a circle with radius $AM = BM$.



By II.5

$$AP.PB = MB^2 - MP^2$$

$$AP \cdot PQ = MC^2 - MP^2 = PC^2$$

Thus PC is the side of the desired square.

This can also be done by similar triangles, since $\triangle CAP \sim \triangle BCP$. Thus $PB/PC = PC/AP$ so $AP \cdot PB = PC^2$.

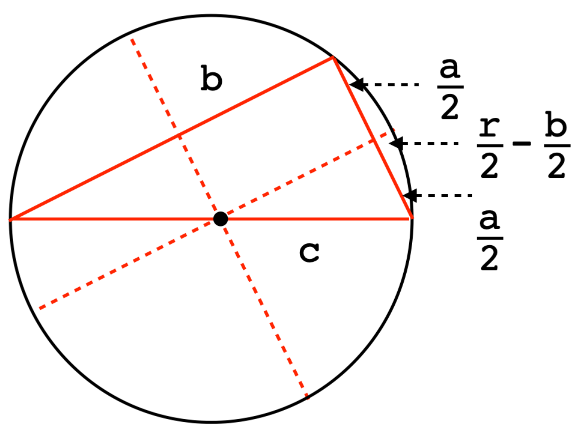
□

Since we can get from Pythagoras to the crossed chord theorem, it may not be so surprising that we can go backward.

Pythagorean theorem by crossed chords

Proof.

A right triangle with sides a, b and c is inscribed in a circle. By Thales' theorem, the hypotenuse c is a diagonal. Draw two more diagonals, parallel to the sides a and b. We can use similar right triangles to show that these bisect sides a and b.



So then, we will use the crossed chords theorem, multiplying the two halves of a together. The question is, what are the two parts of the dotted diameter that crosses a at right angles?

The long part is $c/2 + b/2$. Can you see why?

The very short part is $c - (c/2 + b/2) = c/2 - b/2$. The reason is that when added to the long part, the result is just

$$c/2 + b/2 + c/2 - b/2 = c$$

We have

$$\frac{a}{2} \cdot \frac{a}{2} = (\frac{c}{2} + \frac{b}{2})(\frac{c}{2} - \frac{b}{2})$$

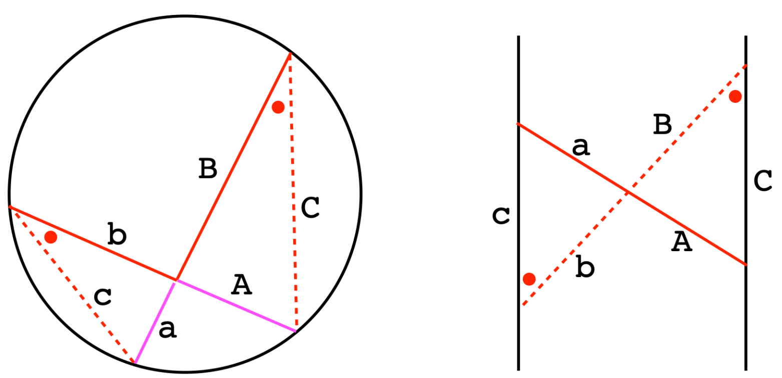
$$a \cdot a = (c + b)(c - b) = c^2 - b^2$$

$$a^2 + b^2 = c^2$$

□

• The crossed chords formed for the vertices of any cyclic quadrilateral form two pairs of similar triangles. The sides that are in the same ratio are on different chords. The fundamental ratio $a/A = b/B$ leads directly to the crossed chord theorem.

The basic rule is that corresponding sides lie opposite corresponding (equal) angles. In the figure a and A lie opposite to the angle marked with a red dot.

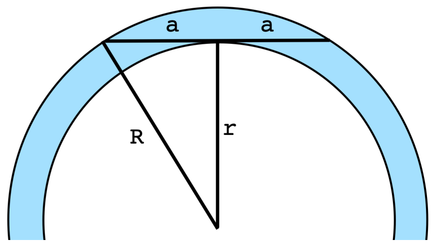


In the left panel (circle) corresponding sides lie on different chords, while in the right panel they lie on the same chord. With crossed chords, the corresponding angles and sides are in clockwise order, while for the other case the order is counter-clockwise.

If in doubt, make sure that the sides you think are corresponding sides lie opposite to equal angles.

another view

Another way to look at this is shown in the following diagram:

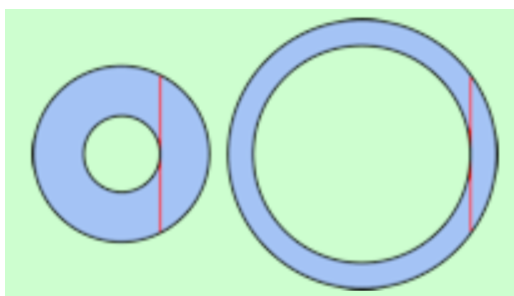


Given a larger circle of radius R and a chord of it with length $2a$, draw the smaller circle of radius r that just touches the chord. Then, by the Pythagorean theorem we have that

$$R^2 - r^2 = a^2$$

But $R^2 - r^2$ (times π) is the area of the annulus or ring, the difference between the area of the outer circle and that of the inner one. This is equal to $\pi\ a^2$.

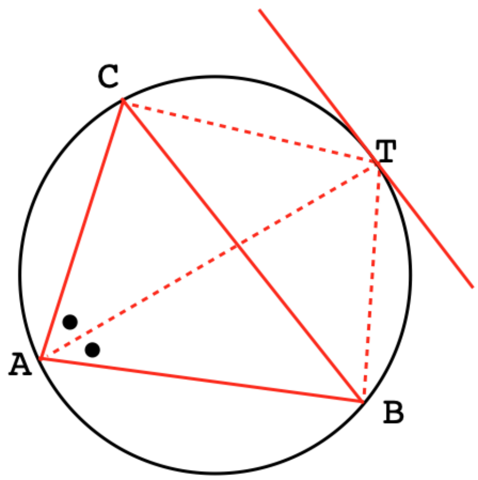
So then, there is family of circles of different sizes that can be drawn, each with a chord of length $2a$ (as long as the radius is greater than a). For each of those circles, one can draw the annulus as we did above, and find that the area is exactly the same.



https://puzzles.nigelcoldwell.co.uk

problem

Given that the line touching the circle at T is a tangent and that BC is parallel to it. Show that the angles marked with black dots are equal.

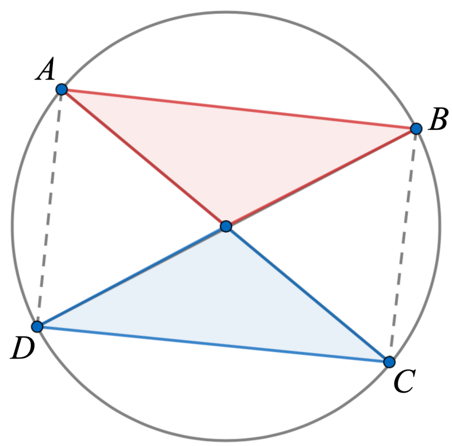


I think the idea of the problem was to use similar triangles somehow. But given our previous work, this is a trivial consequence of the fact that the arcs CT and TB are equal.

So, as peripheral angles corresponding to equal arcs, $\angle CAT = \angle TAB$.

diameters form a rectangle

∘ When any two diameters of a circle are drawn and consecutive chords on the circle joined together, the result is a rectangle.



Proof.

A simple proof relies on Thales' circle theorem. All four vertices are right angles.

It follows that both pairs of opposing sides are parallel. For example

$$AD \perp AB \perp BC \Rightarrow AD \parallel BC$$

So $ABCD$ is a parallelogram with right-angled vertices, a rectangle.

□

One can, of course, also use the fact that certain angles are subtended by equal arcs, so they are equal (for example $\angle DAC = \angle DBC$). Then use congruent triangles.

problem

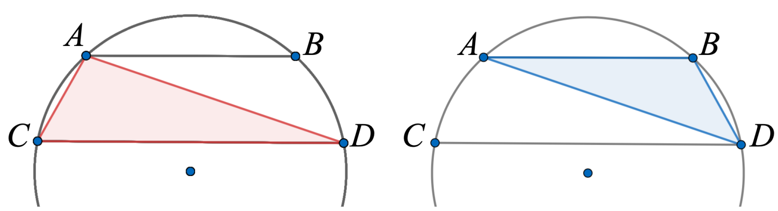
Construct a square circumscribed by a circle, i.e. all four vertices lie on the circle.

Solution. (Sketch).

The diagonals originate at vertices, by definition. We also know that the diagonals of a square are perpendicular. Hence, erecting the perpendicular bisector of any diagonal of the circle locates four points on the circle that are vertices of a circumscribed square.

□

parallel chords joined by equal arcs and chords



Given any two parallel chords in a circle, AB and CD. Draw AD.

The inscribed angles formed, $\angle ADC$ and $\angle BAD$, are equal by alternate interior angles,

Thus arc AC = arc BD.

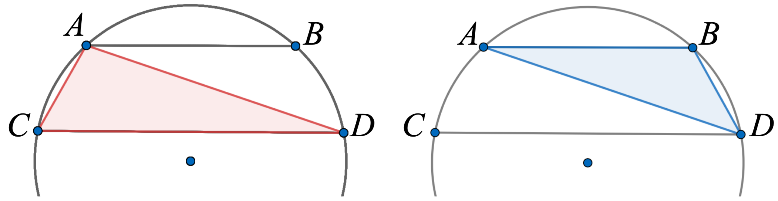
Equal arcs have equal chords so $AC = BD$.

□

It is a simple matter to extend this result to triangles such as $\triangle ABC \cong \triangle BAD$.

Proof.

Since the arcs are equal $\angle ABC = \angle BAD$.



But $\angle BDA$ and $\angle ACB$ both cut the same arc, AB.

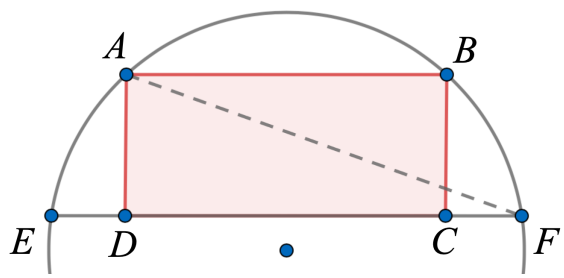
So $\triangle ABD$ and $\triangle BAC$ have three angles equal, and they share the same side AB opposite one pair of equal angles.

$\triangle ABD \cong \triangle BAC$ by ASA.

□

rectangle side on a circle

We derive a useful theorem about any rectangle in a circle, and its converse.



Let $ABCD$ be a rectangle such that A and B lie on the circle, but C and D do not.

Then the extensions of DC are equal, namely $DE = CF$.

Proof.

Draw AF. As before, $\angle AFE = \angle BAF$.

Since the inscribed angles are equal, so are the arcs: arc AE = arc BF.

Equal arcs have equal chords, hence $AE = BF$.

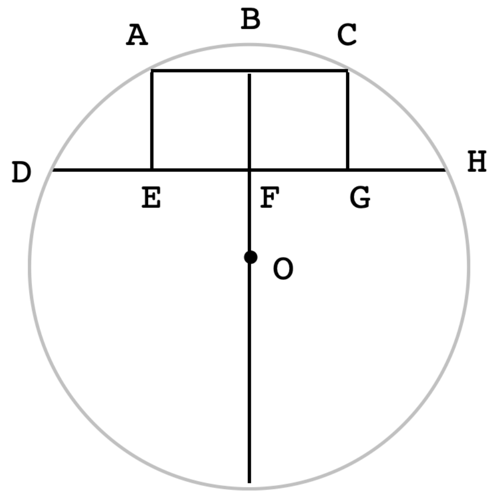
So $\triangle ADE \cong \triangle BCF$ by HL.

It follows that $DE = CF$.

□

Proof. (Alternate).

Suppose that $ACGE$ is a rectangle. Let BF be the perpendicular bisector of AC.



Then, the extension of BF passes through the center of the circle (it is part of a radius), by the converse of the theorem discussed earlier. Every point that is equidistant from A and C, including the center of the circle, lies on the extension of BF.

OB is also the perpendicular bisector of DH, since $ACGH$ is a rectangle. So

$$DF = FH$$

but $AB = BC = EF = FG$. Subtracting equals from equals:

$$DF - EF = FH - FG$$

$$DE = GH$$

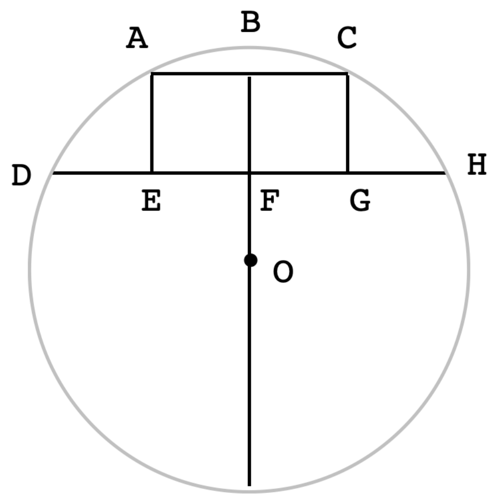
□

For any rectangle with two vertices lying on a circle and two inside the circle, when the side which connects the inside points is extended both ways to reach the circle, the extensions will be equal.

converse

Above, we started with a rectangle in a circle. Now we start simply with any two parallel chords. Then one will be longer than the other (and may even be a diameter).

Suppose we have $DH \parallel AC$.



Drop $AE \perp DH$ and $CG \perp DH$. By alternate interior angles, $\angle A$ and $\angle C$ are right angles, as well as $\angle AEF$ and $\angle CGF$. Hence $ACGE$ is a rectangle.

$\square$

Then all the consequences of the forward theorem follow. In particular $DE = GH$.

And of course, the perpendicular bisector of AC is also the perpendicular bisector of DH. (*Proof.* Draw $\triangle OAB$ and $\triangle OCB$ as well as $\triangle ODF$ and $\triangle OHF$ to show that all the angles at B and F are right angles. $\square$)

Extraordinary property

According to Acheson, this theorem comes from a book by Malton where it is described as an "extraordinary property of the circle".

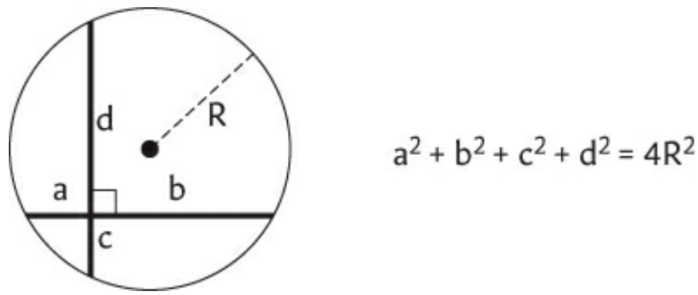


Fig. 110 An 'extraordinary property of the Circle'.

Let two chords of a circle meet at right angles, and let the arms of the chords be $a + b$ and $c + d$.

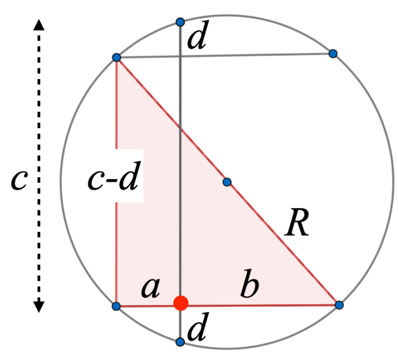
By crossed chords we have $ab = cd$.

We will show that he squares of the four components add up to a constant, and that constant is equal to $(2R)^2$, twice the radius of the circle, squared.

$$a^2 + b^2 + c^2 + d^2 = (2R)^2 = 4R^2$$

The key to the proof is to form a rectangle.

Pick one of the chords, say $a + b$. Draw the diameter of the circle that terminates on one end of each of the chords. The other diameter joins the other ends of chords $a + b$ but hasn't been drawn.



If the four points at the ends of the two diameters are joined to form a rectangle then the short extension at the top is also equal to d, so the height of the rectangle is $c - d$.

We proved this fact about rectangles in a circle **here**.

So then:

Proof.

This is trivial now that we have $c - d$. By the Pythagorean theorem

$$(a + b)^2 + (c - d)^2 = (2R)^2$$

$$= a^2 + b^2 + c^2 + d^2 + 2ab - 2cd$$

But $ab = cd$ so

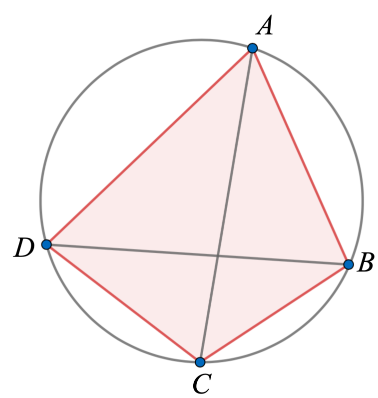
$$4R^2 = a^2 + b^2 + c^2 + d^2$$

□

Chapter 23

Cyclic quadrilateral

There is a wonderfully simple theorem about quadrilaterals (Euclid III.22). A cyclic quadrilateral is a four-sided polygon whose four vertices all lie on one circle.



- For *any* cyclic quadrilateral, the opposing angles are supplementary (they sum to two right angles).

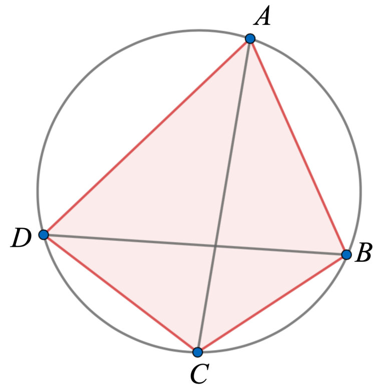
Proof.

Together, opposing angles in a cyclic quadrilateral exactly correspond to the whole arc of the circle.

Since the central angle for that arc is four right angles, the sum of opposing inscribed angles is just one-half that or two right angles.

□

Euclid's proof uses sum of angles:



Proof.

$\angle ADC = \angle ADB + \angle BDC$, subtended by arcs AB and BC.

As angles on equal arcs (III.21) the latter two angles are equal to $\angle ACB$ and $\angle BAC$.

But by I.32, the same two angles plus the total angle at vertex B are equal to two right angles.

□

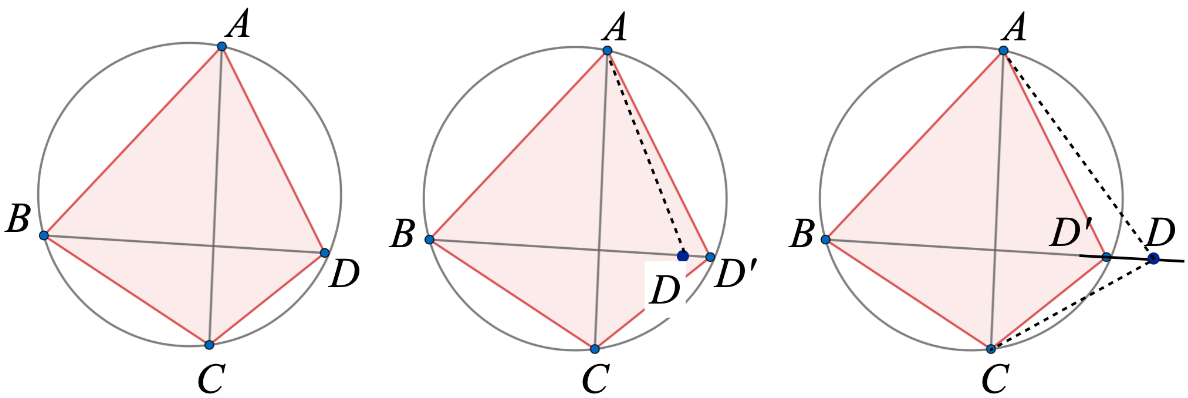
converse of cyclic quadrilateral theorem

This one uses Euclid III.22 rather than III.21.

Let $\triangle ABC$ lie on a circle.

Let point D be such that $\angle ADC$ is supplementary to $\angle ABC$.

Then D lies on the same circle.



Proof.

Aiming for a contradiction, suppose D does not lie on the circle.

Let D be external and D' lie on the point where BD cuts the circle (right panel).

By the forward theorem, $\angle AD'C$ is supplementary to $\angle ABC$ and so equal to $\angle ADC$.

But by Euclid I.21, $\angle ADC < \angle AD'C$.

This is a contradiction. Therefore D is not external.

A similar argument will show that D is not internal.

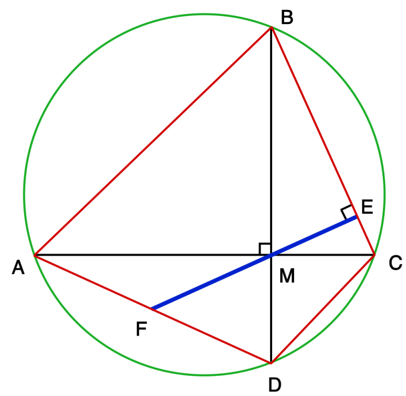
Therefore D lies on the circle and D and D' are the same point.

□

Brahmagupta's theorem

This is a theorem credited to Brahmagupta.

https://en.wikipedia.org/wiki/Brahmagupta_theorem



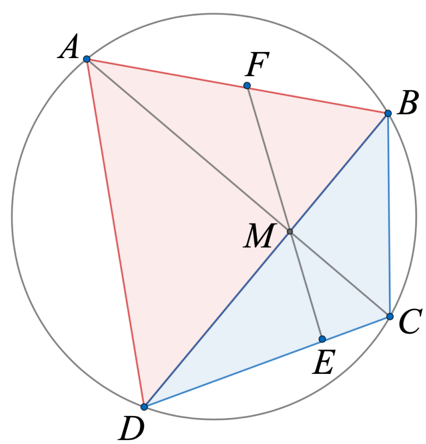
Given $ABCD$ is a cyclic quadrilateral with diagonals that cross at right angles.

Given $EF \perp DC$.

Show that $AF = DF$.

Proof.

We redraw the figure from Wikipedia slightly (so now we have to show $AF = BF$).



We are given that $\triangle AMB$ is right.

If $AF = BF$, then $MF = AF = BF$ by the right triangle midpoint theorem.

This suggests we try to show that $\triangle AFM$ and $\triangle MFB$ are isosceles.

Start with $\triangle AFM$.

We see that $\angle MAF$ is subtended by chord BC, but so is $\angle MDE$, so they are equal.

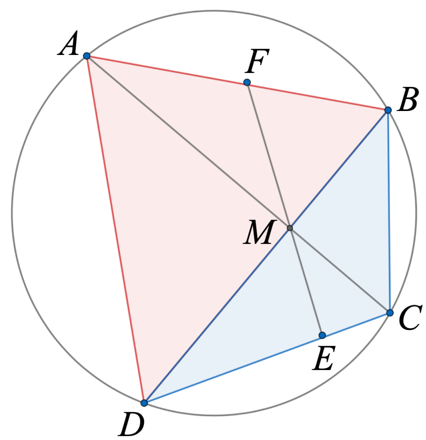
We're given the $\triangle MDC$ is right and that ME is the altitude.

It follows that $\angle CME = \angle MDC$ by similar triangles.

Then $\angle AMF = \angle CME$ by vertical angles.

We have then, $\angle AMF = \angle MAF$.

$\triangle AFM$ is isosceles and so $AF = MF$.



For the other one, we need $\angle FMB = \angle FBM$.

$\angle MBF = \angle MCD$, as inscribed angles on the same arc.

And $\angle MCD = \angle DME$ by similar triangles.

$\angle DME = \angle FMB$ by vertical angles.

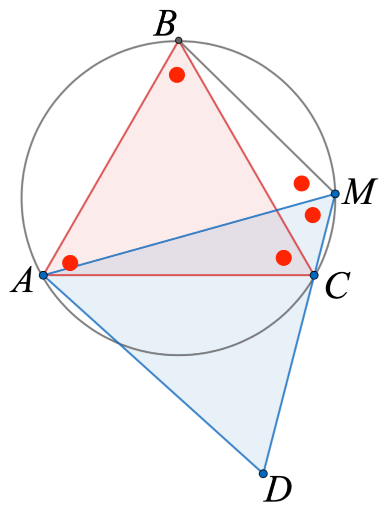
It follows that $\angle FMB = \angle MBF$.

$\triangle MFB$ is isosceles and so $MF = BF$ by I.6.

Equating equals: $AF = BF$.

$\square$

Van Schooten's theorem



This is given as a problem by Surowski (1.3.6).

Given an equilateral triangle ABC draw its circumcircle.

Draw an arbitrary line segment from vertex A through side BC to meet the circle at M. Prove that $AM = BM + MC$.

This is easy to prove as a special case of Ptolemy's theorem.

Nevertheless, we do it as suggested.

Proof.

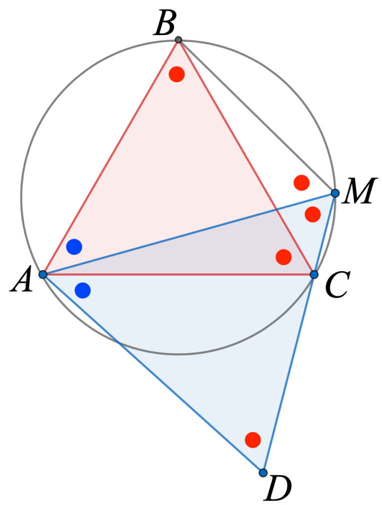
Draw lines from M to each vertex and extend one to give $AM = MD$.

By inscribed angles both angles at M are equal to two-thirds of a right angle.

$\triangle AMD$ is isosceles with the vertex equal to two-thirds of a right angle.

Thus $\triangle AMD$ is equilateral.

Subtract the central $\angle MAC$ from two equal angles to yield $\angle BAM = \angle DAC$.



$\triangle BAM \sim \triangle CAD$.

But $BA = CA$ so $\triangle BAM \cong \triangle CAD$ by ASA.

It follows that $BM = CD$.

Thus $MC + CD = MC + BM = MD = AM$.

□

I saw this question on the internet: "is a parallelogram a cyclic quadrilateral?"

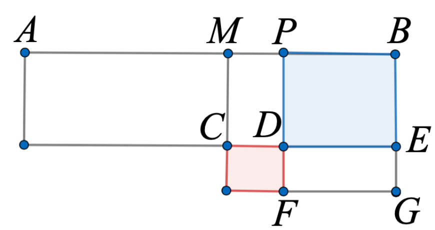
In a parallelogram, opposing angles are equal. In a cyclic quadrilateral, opposing angles are supplementary. The only supplementary, equal angles are two right angles. So a rectangle is the only parallelogram that is a cyclic quadrilateral.

Chapter 24

Tangent secant theorem

Let us start with two propositions from Book II of *Elements*, which deal with what might be termed geometric algebra.

Euclid II.5



In this figure, the line AB is bisected at M with $AM = MB$.

and then the point P placed somewhere within the segment MB. II.5, says that

$$AP \cdot PB + MP^2 = MB^2$$

It may make more sense if we try algebra first. Let $x = AM$ and $y = MP$ so $AP = x + y$ and $PB = x - y$ and then

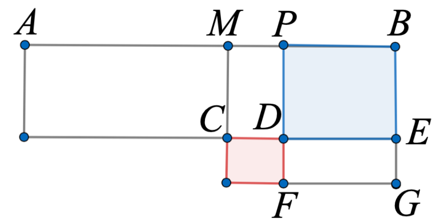
$$AP \cdot PB = x^2 - y^2$$

and we can read these off the diagram:

$$AP \cdot PB = MB^2 - MP^2$$

$$AP \cdot PB + MP^2 = MB^2$$

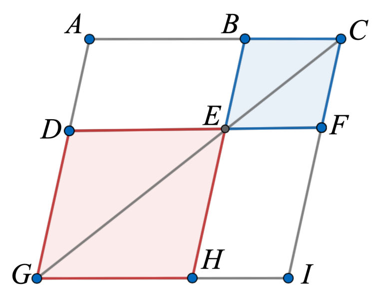
$AP \cdot PB$ is the area of rectangle AD.



PE is the square on PB, with $PB = BE$, and CF is the square on CD, with $CD = MP = EG$.

The construction contains two sets of equal rectangles. The first is $AC = ME = BF$. And the second is $MD = DG$. This is by I.43.

In the figure below, the two white parallelograms are equal.



We saw an example very early for rectangles as the **area-atio theorem**.

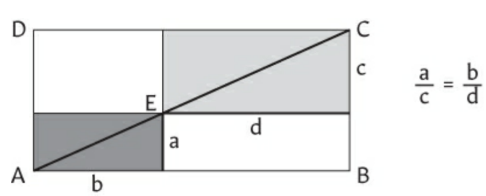
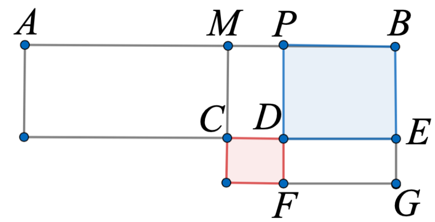


Fig. 42 Area and similarity.

$$ad = bc$$

Returning to the present theorem



As noted, the construction contains two sets of equal rectangles. The first is $AC = ME = PG$, while the second is $MD = DG$.

Here's the neat idea: the rectangle PG contains the blue rectangle plus DG. But since $MD = DG$ this is also ME and $ME = AC$.

So then finally AD is equal in area to the L-shaped piece called a *gnomon*.

And if we add CF (equal to MP^2) to that we have MG (equal to MB^2).

$$AP \cdot PB + MP^2 = MB^2$$

In Joyce's annotated *Elements* he says (paraphrasing) that this solves a quadratic. Suppose we are asked

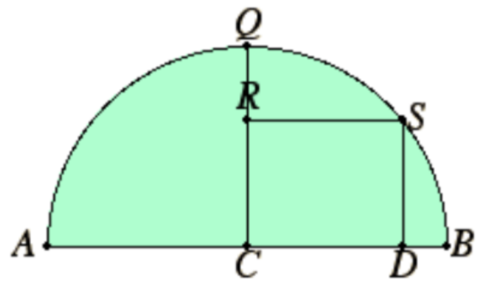
> Find two numbers x and y so that their sum is a known value b and their product is a known value c^2.

In modern terms:

$$x(b - x) = c^2$$

$$x^2 - bx + c^2 = 0$$

$$x = \frac{b}{2} \pm \sqrt{(\frac{b}{2})^2 - c^2}$$



Here

$$AB = b \qquad BC = b/2 = CS \qquad DS = CR = c$$

so

$$CD^2 = (\frac{b}{2})^2 - c^2$$

and

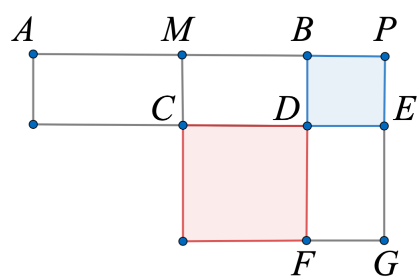
$$AD = AC + CD$$
$$= \frac{b}{2} \pm \sqrt{(\frac{b}{2})^2 - c^2}$$

is equal to x.

This problem is also called the mean proportion, and is solved easily (in modern times) by the same construction, where we can show that $DS^2 = AD \cdot DB$ by using similar right triangles. We looked at this in the chapter on simple proofs of the Pythagorean theorem.

Euclid II.6

The second theorem is very similar but has the point P located on an extension of AB:



The result is nearly the same as before, just with the squares switched:

$$AP \cdot PB + MB^2 = MP^2$$

although AP is not what it used to be. One way to remember: the difference of squares must be positive. In the first case $MB > MP$, and in the second, $MP > MB$.

Again we have a gnomon and a difference of squares. The gnomon is equal to the whole rectangle $AP \cdot PB$ and when added to MB^2 we get the whole large square MP^2

$$AP \cdot PB + MB^2 = MP^2$$

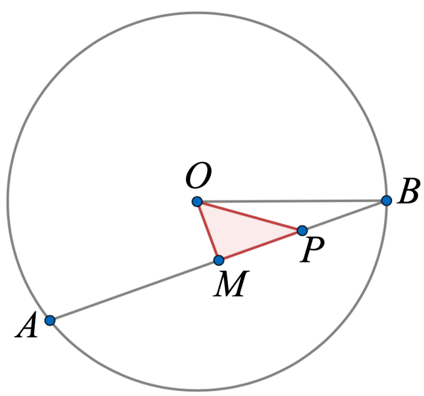
In algebraic terms we let $AM = x$ and now $PB = y$ so

$$(2x + y)y + x^2 = 2xy + y^2 + x^2 = (x + y)^2$$

$$AP \cdot PB + MB^2 = MP^2$$

Euclid III.35: application to chords

We have a chord AB of a circle on center O. AB is bisected at M and P is placed somewhere within MB.



By II.5

$$AP \cdot PB + MP^2 = MB^2$$

We notice both squares are part of right triangles.

$$MP^2 = OP^2 - OM^2$$

$$MB^2 = OB^2 - OM^2$$

Substituting into the first equation, OM cancels, leaving

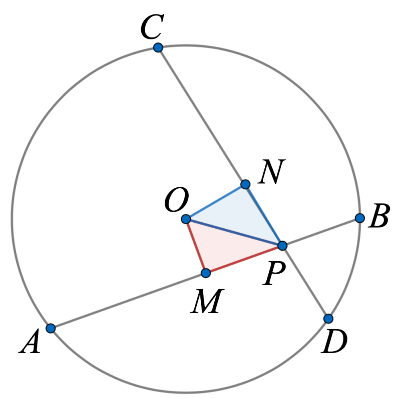
$$AP \cdot PB + OP^2 = OB^2$$

Let $OP = d$, the distance from P to the center, and OB is the radius, so

$$AP \cdot PB = r^2 - d^2$$

This result is *independent* of the particulars of AB and depends only on the placement of P in the circle (and the requirement that AB pass through P).

So any other chord that also passes through P, say CD, has the same result.



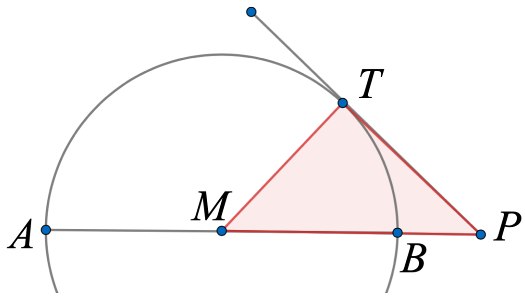
$$AP \cdot PB = CP \cdot PD$$

This is just the crossed chord theorem in disguise.

A modern proof would use similar triangles, but this requires a theory of proportions that Euclid doesn't have yet in book III. We'll see that below.

tangent

The next theorem concerns the tangent.



Here we have AB bisected at M, the center of the circle, and then P placed on the extension of AB.

By II.6

$$AP \cdot PB + MB^2 = MP^2$$

MB and MT are radii so

$$AP \cdot PB = MP^2 - MT^2 = PT^2$$

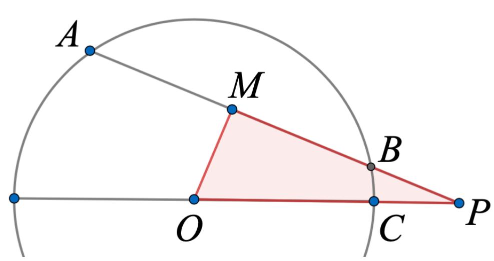
The length of the tangent from point P, squared, is equal $AP \cdot PB$.

We can again generalize the result by letting d be the distance of P from the center of the circle, MP, and t be the length of the tangent, PT. Then

$$d^2 - r^2 = t^2$$

The result has the same magnitude as for crossed chords, but with a minus sign. And again, this makes sense since the difference of squares must be positive. Before we had $r > d$, now we have $d > r$.

secant



As with the tangent, by II.6 we have

$$AP \cdot PB + MB^2 = MP^2$$

and again, we have right triangles so

$$AP \cdot PB + (MB^2 + OM^2) = (MP^2 + OM^2)$$

$$AP \cdot PB + OB^2 = OP^2$$

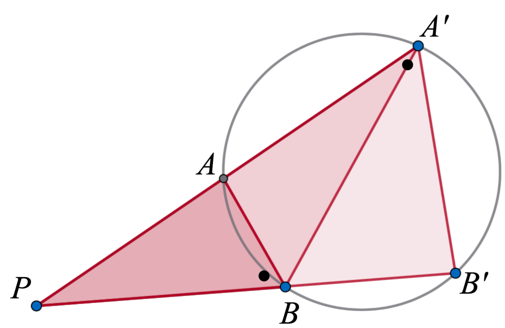
so again

$$AP \cdot PB = d^2 - r^2$$

By using the previous two results together, we have that for any secant drawn from P, $PA \cdot PB$ is equal to the square of the tangent from the same point PT^2.

This is the **tangent-secant theorem**.

proofs based on similar triangles



$$PA \cdot PA' = PB \cdot PB'$$

When thinking about such problems, or asked to establish a proof yourself, it is always helpful to consider the form with the ratios:

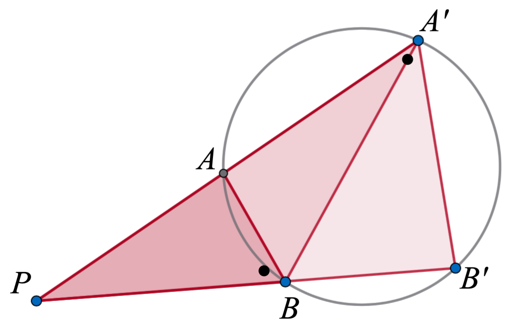
$$\frac{PA}{PB} = \frac{PB'}{PA'}$$

This strongly suggests we look at similar triangles.

Proof.

We showed previously that for any quadrilateral whose four vertices all lie on one circle (a cyclic quadrilateral), the opposing vertices have supplementary angles. Opposing vertices add to 180° because their arc segments add up to one whole circle.

In the figure above, $\angle ABB'$ supplementary to both $\angle PBA$ and $\angle PA'B'$.



Therefore $\triangle PBA$ is similar to $\triangle PA'B$. The similarity is such that

$$\frac{PA}{PB} = \frac{PB'}{PA'}$$

Given a point P outside the circle, the external part of any secant times the entire secant is a constant.

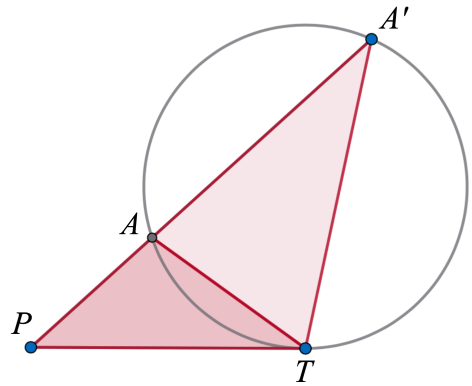
One curious thing about this theorem is that these triangles are similar, and nested, but flipped.

$$PA \cdot PA' = PB \cdot PB'$$

□

We can do a bit more.

Tangent-secant theorem



Let the points B and B' approach each other to become one point, re-labeled as T. Then PT will be a tangent of the circle. Previously we had

$$PA \cdot PA' = PB \cdot PB'$$

Now we modify it slightly:

$$PA \cdot PA' = PT \cdot PT$$

$$PA \cdot PA' = PT^2$$

This is the tangent-secant theorem.

We must have two similar triangles, $\triangle PAT$ and $\triangle PTA'$. The whole angle at vertex T must correspond to $\angle PAT$.

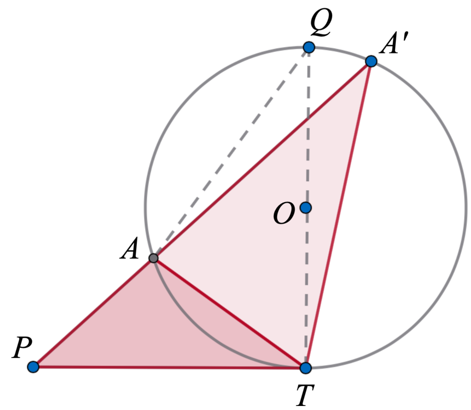
Now run the logic backward and write the proof.

Note: we've seen this sort of reverse logic a few times already. The method has a name! It was called the method of "analysis" by Pappus (320 A.D.), see Posamentier (Introduction).

To do this explicitly, show that since $\angle PTA$ includes the tangent, it cuts arc AT just as $\angle PA'T$ does.

Instead, we take advantage of the tangent point to do something new.

Proof.



Draw the diameter $QOT \perp PT$ and also draw QA. Since they correspond to equal arcs, the angles at Q and A' are equal.

By Thales' circle theorem, $\angle QAT$ is right. So $\angle Q$ is complementary to $\angle ATQ$.

But the diameter is perpendicular to the tangent. So $\angle ATQ$ is complementary to $\angle PTA$.

It follows that $\angle PTA = \angle PA'T$.

$\triangle PAT$ is similar to $\triangle PTA'$ by AAA, since they also share the angle at P. This gives

$$\frac{PT}{PA} = \frac{PA'}{PT}$$

which can be rearranged to the statement of the theorem:

$$PA \cdot PA' = PT^2$$

□

One application of this theorem is to a determination of the size of the earth.

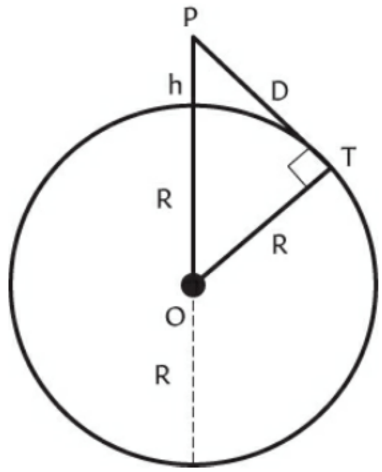


Fig. 68 Measuring the Earth.

In the figure, the circle is the earth, of radius R, h is the height of a convenient mountain, and D is the distance to the horizon, which is tangent to the earth's radius.

Recall from the tangent-secant theorem

$$D^2 = h(2R + h)$$

We neglect h^2 compared to the other term so

$$D^2 \approx 2Rh$$

About 1019 C.E., finding h and D, Al-Biruni computed a value for R equivalent to 3939 miles.

Looking at Euclid

We're looking up at a statue of Euclid on a column.

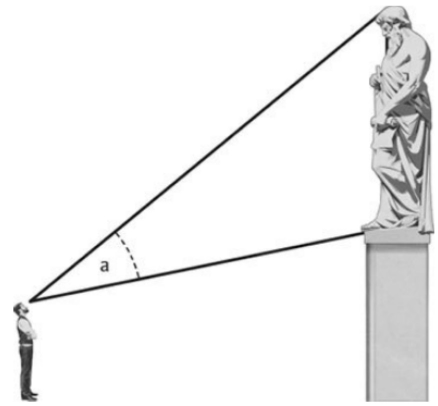


Fig. 69 Looking at Euclid.

We resist the temptation to make a dumb joke.

In any event, we'd like to get the widest angle view, giving the largest apparent size of the statue. If you get too close, the statue is greatly foreshortened and the angle small, and naturally, it is small at a distance. There must be a best view, in the middle.

Here is Acheson's solution:

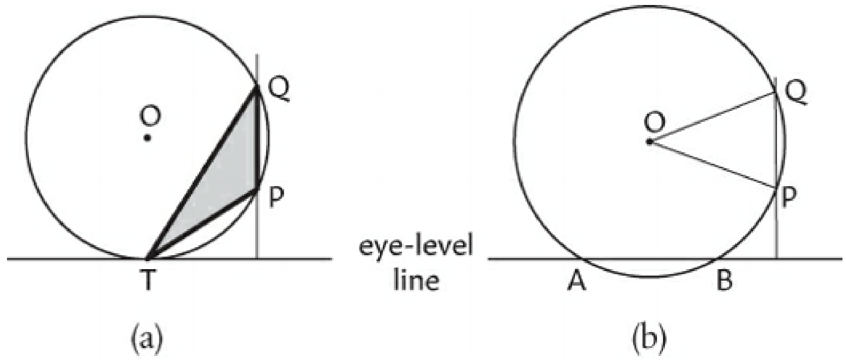


Fig. 70 What's the best view?

Let the foot and head of Euclid be at P and Q and draw the circle containing those two points which is also tangent to your eye-level. Then the tangent point provides the best view.

The reason is that a circle through any other horizontal position crosses the eye-level at two points. Such a circle will necessarily be bigger.

Consequently the arc PQ, which is fixed in size, will be a smaller fraction of the circle.

As a smaller fraction, both the central angle $\angle POQ$ will be smaller as well as the angle subtended at A or B (one-half of that).

The tangent-secant theorem even gives a quantitative answer.

If R is the point where the extension of QP meets eye-level (the ground), then suppose the point Q is h units above the ground and P is g units. The tangent-secant formula is

$$RT \cdot RT = PR \cdot QR$$

which says that the square of the optimum viewing distance is $h \cdot g$. The optimal distance d is

$$d = \sqrt{hg}$$

I knew I'd seen this problem before, and I find that I wrote it up for my Calculus book. It turns out to be from Acheson's book on Calculus, and is originally about Lord Nelson's statue in Trafalgar Square.

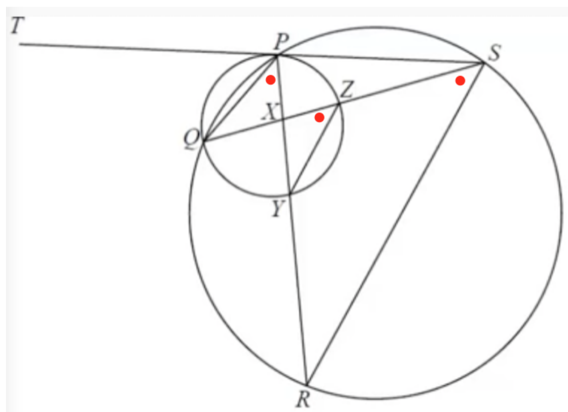
The calculus treatment is not nearly as pretty as reasoning about the tangent; fortunately we came to the same answer.

Problem

problem

We continue with a problem that we solved in part **previously**.

We showed that there are three similar triangles in the figure below



We showed that $\triangle XYZ \sim \triangle XQP$ and also $\triangle XYZ \sim \triangle XRS$.

The last part of the problem says that given $QS = XR$, prove that $PS^2 = XS \cdot YR$. It is also given that TPS is tangent to the small circle at P.

In this chapter, we developed the tangent-secant theorem, which says that the part of the secant outside the circle, multiplied by the whole thing, is equal to the tangent squared:

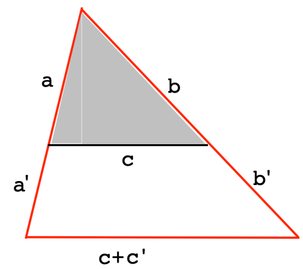
$$QS \cdot ZS = PS^2$$

Proof.

By similar triangles, we have that

$$\frac{XZ}{XY} = \frac{XS}{XR} = \frac{ZS}{YR}$$

That last equality requires some algebra. We use this figure again:



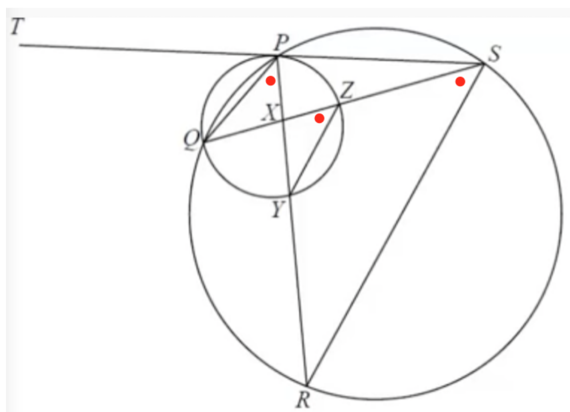
We have that

$$\frac{a}{b} = \frac{a+a'}{b+b'}$$
$$\frac{a}{a+a'} = \frac{b}{b+b'}$$
$$\frac{a+a'}{a} = \frac{b+b'}{b}$$
$$\frac{a'}{a} = \frac{b'}{b}$$
$$\frac{a}{b} = \frac{a'}{b'}$$

The partial sides are in the same ratio as the whole.

Back to our problem. Multiplying by the denominators:

$$XS \cdot YR = XR \cdot ZS$$



We're given that $XR = QS$

$$XS \cdot YR = QS \cdot ZS$$

but by the tangent-secant theorem

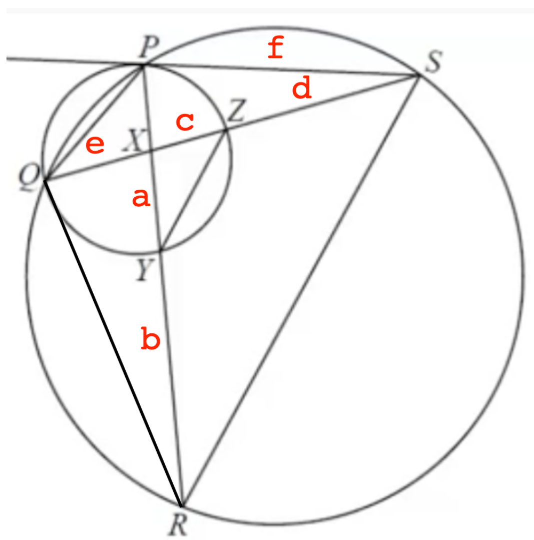
$$QS \cdot ZS = PS^2$$

so we have

$$XS \cdot YR = PS^2$$

□

I really dislike doing algebra with XS and the rest, so I substituted single letters for the sides and fiddled with the algebra while working backward (analysis) until I could see the answer.



The tangent-secant theorem says that $f^2 = d \cdot (c+d+e)$. We're given that $(c+d+e) = (a+b)$.

So $f^2 = d \cdot (a+b)$. We are asked to prove that this is equal to $(c+d) \cdot b$.

Forming ratios we have

$$\frac{b}{d} = \frac{a+b}{c+d}$$

But this is just the result that the partial sides are in the same ratio as the whole.

Reverse (and substitute XS etc.) to write the proof.

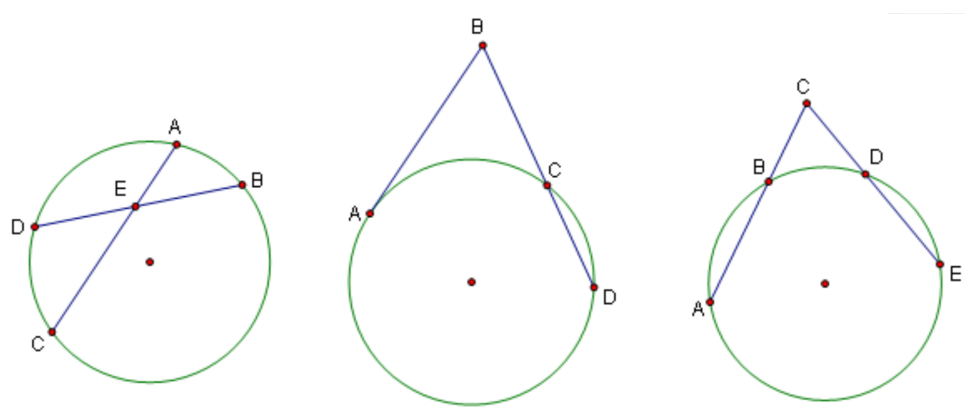
Chapter 25

Power of a point

The crossed chord and tangent-secant theorems can be unified by considering PA and PB as directed line segments. In the crossed-chords example, they point in opposite directions so their product is negative, while for tangent and secant they point in the same direction.

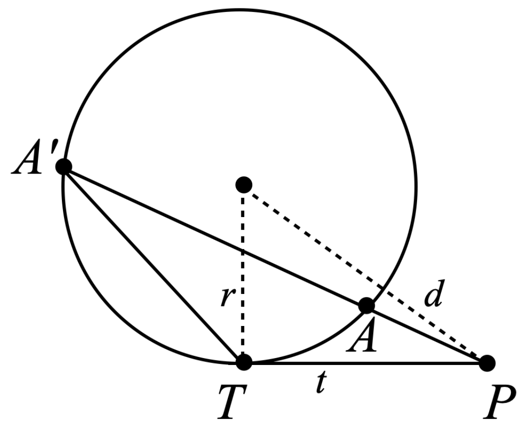
We can view the result for chords $r^2 - d^2$ as the same as the result for the tangent, $d^2 - r^2$, provided we take account of the sign of $AP \cdot PB$.

The previous results regarding secants and tangents are sometimes described in a definition called the "power" of a point, which unifies the treatment of points inside and outside the circle.



https://artofproblemsolving.com/wiki/index.php/Power_of_a_Point_Theorem

Let's start with a point outside.



The length of the tangent and the radius are simply related to the distance from any point P to the center of the circle, d:

$$d^2 = t^2 + r^2$$

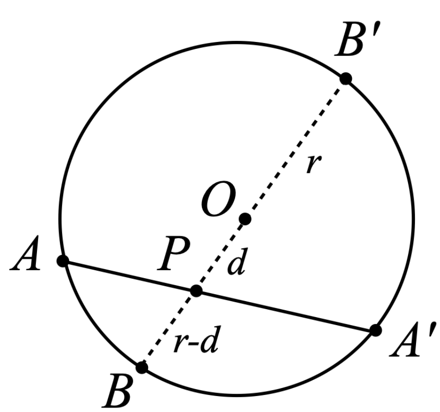
$$t^2 = d^2 - r^2$$

$d^2 - r^2$ is defined as the *power* p of the point P.

From our previous work, we know that

$$p = t^2 = PT^2 = PA \cdot PA'$$

The definition also works for a point inside the circle.



By the crossed chords theorem:

$$PA \cdot PA' = (r+d)(r-d)$$

$$= r^2 - d^2 = -p$$

That is, it works if we use *directed line segments*, so that the product $PA \cdot PA'$ for P inside the circle has its two components pointing in opposite directions, and thus acquires a minus sign.

If the point is *on* the circle, it's a bit strange, but we are at the boundary between $p < 0$ and $p > 0$, so it seems reasonable that $p = 0$ for points on the circle. And there, of course

$$p = d^2 - r^2 = 0$$

Part VI

Practice with proofs

Chapter 26

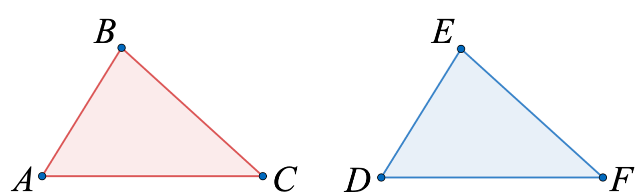
Introduction to Euclid

In this chapter we will look at some Propositions from Book I of Euclid's *Elements*.

Elements was put together as a compendium of geometry for students. One thing we will see is how the propositions build on one another to make a dependent chain. This includes a more sophisticated proof of the isosceles triangle theorem that depends only on SAS.

Euclid. I.4

> If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.



This is a method for proving congruence (equality) of two triangles

$$\triangle ABC \cong \triangle DEF$$

In modern usage, we call the method SAS or *side angle side*. Given that $AB = DE$ and $AC = DF$ and that the angles between them at the vertices A and D are also equal, the two triangles are congruent: all three angles and all three sides are equal.

Euclid I.4 is a proof that SAS is correct.

Proof.

The proof is by superposition. The facts establish the positions of the points B and C, which determines BC and so the angles at vertices B and C.

Euclid says that if we lift up $\triangle ABC$ and lay it on top of $\triangle DEF$ then B coincides with E and C coincides with F so $BC = EF$.

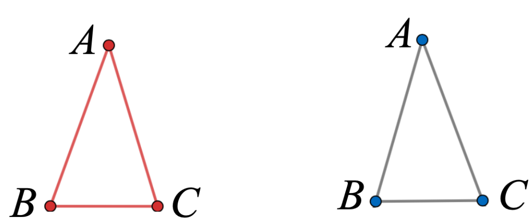
□

This seems perhaps a little shaky logically, and it's not a method of proof that Euclid uses much.

But one might instead have taken this proposition as a postulate. One source says that David Hilbert claims that under the hypotheses of the proposition it is true that the two base angles are equal, and then proves that the sides are equal.

Euclid I.5

The forward isosceles triangle theorem is that if two sides in a triangle are equal, then so are the opposing angles. It's difficult precisely because Euclid proves it *before* we know how to bisect an angle.

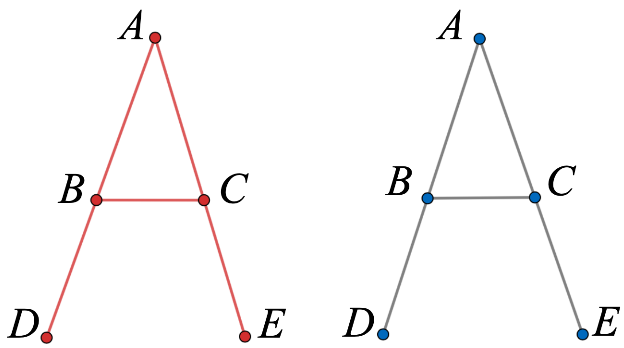


Given $AB = AC$. We will prove that the base angles are equal: $\angle ABC = \angle ACB$.

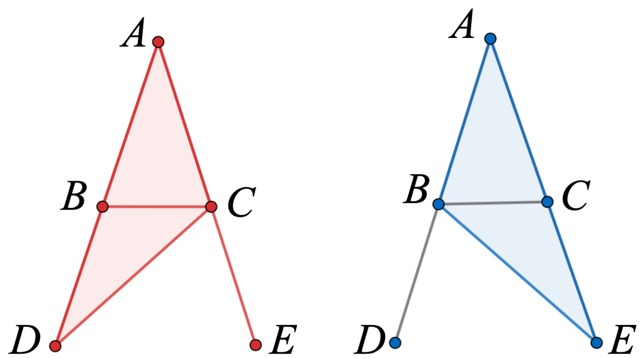
The image shows two copies of the triangle. This is so that we may compare congruent triangles formed within the same figure.

Proof.

Extend AB and AC so that $AD = AE$.

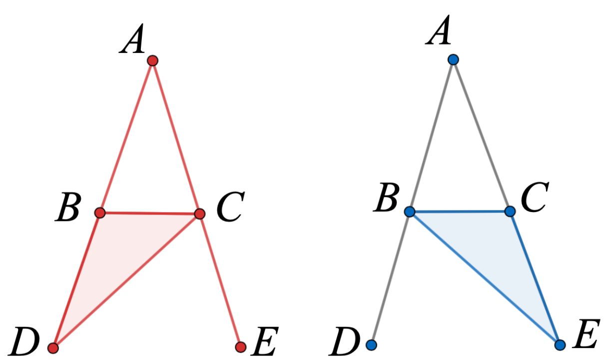


Then $BD = CE$ by subtraction. Connect CD and BE.



$\triangle ACD \cong \triangle ABE$ by SAS ($AB = AC$, $AD = AE$, and they share the angle at vertex A).

As a result, we have two more pairs of angles equal: $\angle ADC = \angle AEB$ and $\angle ACD = \angle ABE$. Also, $CD = BE$.



Since $\angle ADC = \angle AEB$ and the flanking sides are equal, namely, $BD = CE$ and $CD = BE$, we have $\triangle BCD \cong \triangle CBE$ by SAS.

It follows that $\angle DBC = \angle ECB$. By supplementary angles, $\angle ABC = \angle ACB$.

$\square$

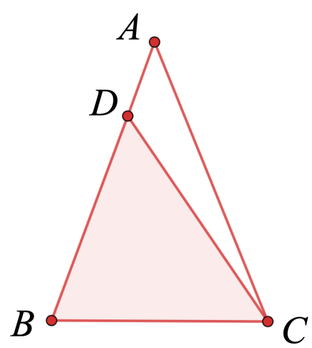
Notice how the original figure is extended to provide auxiliary shapes helpful in the proof. This is a common theme.

We will use both results ($\angle ABC = \angle ACB$ and $\angle DBC = \angle ECB$), going forward.

To summarize: in a triangle with two equal sides, the angles opposite those sides are also equal, as well as their supplementary angles.

Euclid I.6

We proved the converse of I.5 previously based on angle bisection. Now that we have proved Euclid I.5, which provides the basis for bisection, this is logically solid. Nevertheless, for completeness, here is Euclid's proof of I.6.



Proof.

Given $\angle ABC = \angle ACB$. Suppose $AB \neq AC$. Then let one be less, say AC. So cut off from AB the length $BD = AC$.

Compare $\triangle DBC$ with $\triangle ABC$. We have the sides equal, namely $DB = AC$ and $BC = BC$. As well, $\angle DBC = \angle ACB$. Therefore $\triangle ABC \cong \triangle DBC$ by Euclid I.4 (SAS).

But this is absurd. The part cannot be equal to the whole.

Therefore, $AB = AC$.

$\square$

$\iff$

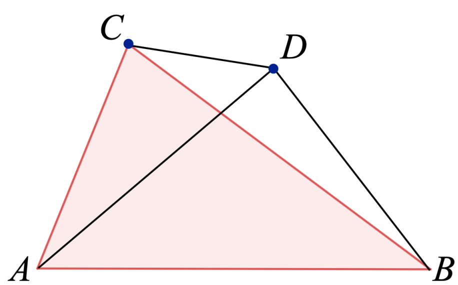
The theorem together with its converse says that, in an isosceles triangle, the base angles are equal $\iff$ the two sides sides are equal (not the base).

The symbol $\iff$ means *if and only if*, so both base angles equal $\rightarrow$ two sides equal *and* two sides equal $\rightarrow$ base angles equal.

We now extend our methods by showing that SSS leads to SAS, providing a second method of proof for triangle congruence.

But first, look at the next theorem after the two on isosceles triangles, namely, Euclid I.7.

Euclid I.7



- Let $\triangle ABC$ be drawn and a point D be chosen on the same side of AB as C lies. It cannot be true that both $AC = AD$ and $BC = BD$.

Proof.

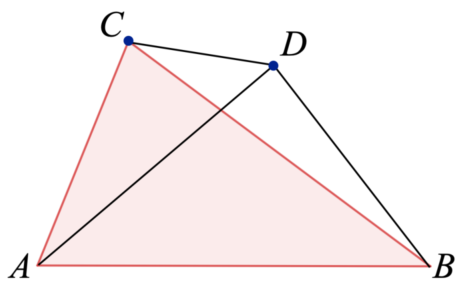
Suppose that both statements are true: $AC = AD$ and $BC = BD$.

Then $\triangle ACD$ is isosceles so $\angle ACD = \angle ADC$.

We notice that

$$\angle BCD < \angle ACD = \angle ADC < \angle BDC$$

But since $\triangle BCD$ is also isosceles, $\angle BCD = \angle BDC$.

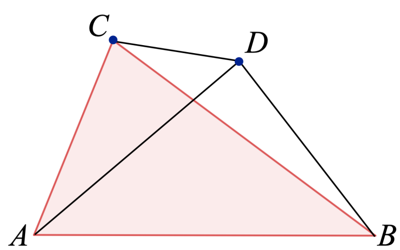


This is a contradiction. Therefore it cannot be that both $AC = AD$ and $BC = BD$.

□

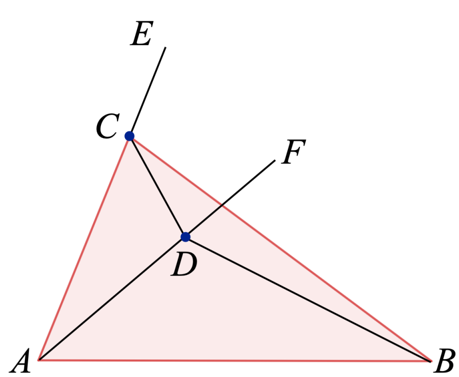
revisit Euclid I.7

There is a small problem (really, an assumption) with our proof. Recall that $AC = AD$ and it is supposed that $BC = BD$.



D is drawn so that it is not contained within $\triangle ABC$. But suppose it were?

The problem is that the proof given before makes no sense if D lies inside $\triangle ABC$ (for example, we do not know $\angle BCD < \angle ACD$).



Proof (Alternate).

We can find a different proof for this case in several ways. Suppose $AC = AD$ and $BC = BD$. Then $\triangle ABC \cong \triangle ABD$ by SSS, since AB is shared.

But this is absurd. $\triangle ABC$ is contained within and is less than $\triangle ABC$.

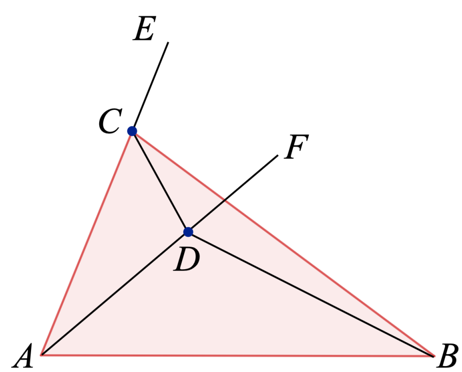
This is a contradiction.

□

All of this leads to a second difficulty: SSS is Euclid I.8, and Euclid actually proves I.8 by relying on I.7 !

However, we have a proof of SSS (below) that does not depend on I.7, so this could still work.

Perhaps a better solution is something like what we did before:



Proof.

Suppose that both $AC = AD$ and $BC = BD$.

Then $\triangle ACD$ is isosceles so $\angle ACD = \angle ADC$. Euclid I.5 also says that the supplementary angles are equal. $\angle ECD = \angle FDC$.

We notice that
$$\angle BCD < \angle ECD = \angle FDC < \angle BDC$$

But since $\triangle BCD$ is also isosceles, $\angle BCD = \angle BDC$. This is a contradiction. Therefore it cannot be that both $AC = AD$ and $BC = BD$.

□

[Another idea might be to use Euclid I.21, which says that if D lies inside $\triangle ABC$, then $AD+BD < AC+BC$, but that proposition turns out to be dependent through several steps, on I.8.]

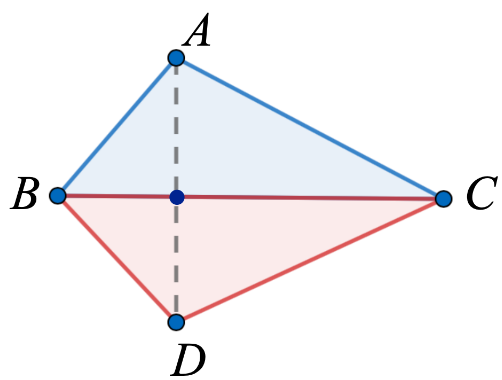
Euclid I.8: SSS implies SAS

We will prove that the SSS criterion implies SAS.

Proof.

Let all the sides of $\triangle ABC$ be equal to $\triangle ADC$ and align the triangles as shown. $\triangle DEF$ is the mirror image of $\triangle ABC$ (if not, reflect $\triangle ADC$ through one of its sides and reposition it.

A mirror image is allowed to be congruent.



D is placed so that two sets of sides are equal, and equal sides are adjacent in the quadrilateral. $AB = BD$ and $AC = CD$. The third side BC is shared. So we have SSS.

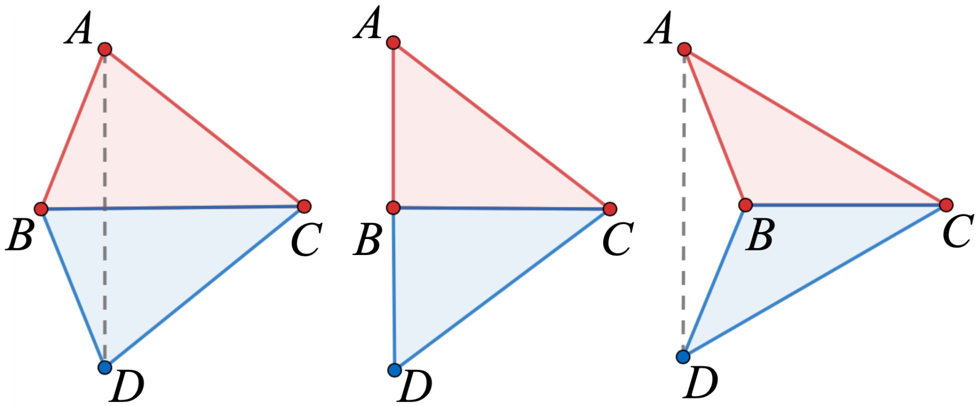
By the forward version of the isosceles triangle theorem, $\angle BAD = \angle BDA$ and $\angle CAD = \angle CDA$. Therefore the total angles at the vertices are equal: $\angle A = \angle D$.

We have SAS, so $\triangle ABC \cong \triangle ABD$.

□

There is also a problem with this proof that may not be obvious.

We have acute angles on the base ($\angle ABC$ and $\angle ACB$). If we allow other possibilities, then this version of the proof isn't valid. Luckily, we can extend the proof in the following way.



Proof.

We have $\triangle ABC$ and $\triangle DEF$ with all three sides equal and superimposed along any one of the sides, say, $BC = EF$. We suppress the labels E and F.

Now, one of the angles along the base BC may be right or obtuse, or neither may be (the case already treated).

Then, if $\angle ABC$ and $\angle DBC$ are both right, we have SAS immediately.

Alternatively, if $\angle ABC$ and $\angle DBC$ are both obtuse (right panel), then draw AD. We have that both $\triangle ABD$ and $\triangle ACD$ are isosceles. By subtraction, we find that $\angle BAC = \angle BDC$.

Both pairs of flanking sides are equal, so we have SAS and thus $\triangle ABC \cong \triangle DEF$ (labeled as $\triangle DBC$ in the figure).

□

SSS $\rightarrow$ SAS, and since SAS is sufficient to show congruence, so is SSS.

This illustrates a problem with drawing diagrams, we may introduce unrecognized assumptions into the proof. The most common is to draw an acute triangle and presume it stands for all triangles

Chapter 27

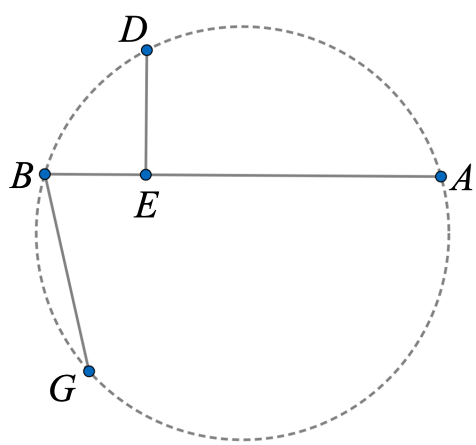
Broken Chord

The theorem of the "broken chord" is ascribed to Archimedes, although his original work — the *Book of Circles* — has been lost.

It was analyzed in proofs collected by the Arabic mathematician Al Biruni in his *Book on the Derivation of Chords in a Circle*.

The theorem was not simply of academic interest, but related to the construction of tables of chords in the *Almagest* by Pappus (covered elsewhere).

Here is the general setup:



Let A and G be any two points on a circle, and let D be equidistant from both, so that arc AD is equal to arc GD. Let B be another point on the circle, lying between

G and D.

Drop the perpendicular from D to E on BC.

The claim of the theorem is that $GB + BE = AE$.

(I wondered about the choice of G as one of the letters but then remembered that the Greek alphabet proceeds: α, β, $\gamma \dots$ Later we will see Z and H, and T — zeta, eta, theta).

In this chapter we will look at a number of proofs including several ascribed to Archimedes. This topic is a great one for "elementary" geometry because the proofs are fairly easy and there are literally dozens of them. There is also the sense of adventure from working on a topic which got Archimedes' attention.

My original source for this problem was

https://www.uni-miskolc.hu/~matsefi/HMTM_2020/papers/HMTM_2020_Drakaki_Broken_chord.pdf

There, it is said that the book contains 22 proofs of the theorem, including 3 different ones due to Archimedes.

Later I found a link to Al Biruni's book, in Arabic:

https://tile.loc.gov/storage-services/service/gdc/gdcwdl/wd/l_/07/46/9/wdl_07469/wdl_07469.pdf

I cannot read Arabic, but the diagrams are reproduced, and using them I worked out a couple more proofs.

Even more recently, I came across a translation of this book, written in German by a historian named Heinrich Suter about 1910:

http://www.jphogendijk.nl/biruni/Suter-Chords.pdf

I was able to translate a part of that book into English. I don't read German either, but the scanned text of Suter is copyable, and Google Translate does a decent job. What makes it hard is that the OCR is so flaky — I'm not complaining — I think it's amazing that it works at all. But it took quite a bit of editing.

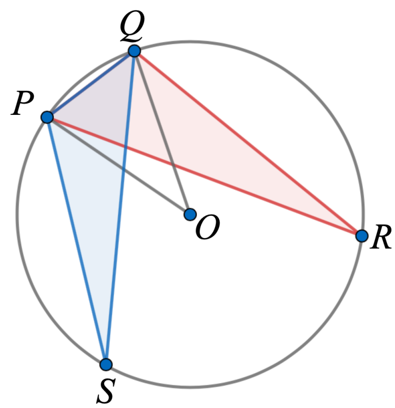
https://www.dropbox.com/scl/fi/zzuibok8apr6i2trwfb4r/

I count 23 examples (labeled a through x, with no j), and four of those are Suter's own proofs, which leaves a total of 19 for Al Biruni. (Although some of these contain multiple approaches with the same diagram).

inscribed angles

Before starting, let's recall the important corollary (Euclid III.21) of the inscribed angle theorem (Euclid III.20).

By definition, the central angle sweeping out a given arc is equal in measure to the length of the arc. Any peripheral angle subtended by the same arc is one-half that central angle. The result is that angles that lie on the same arc or are subtended by the same chord in the same circle, are equal.



$\angle PRQ = \angle PSQ$.

We will also need the theorem that, in a given circle, equal arcs correspond to equal chords, and vice-versa, from **here**.

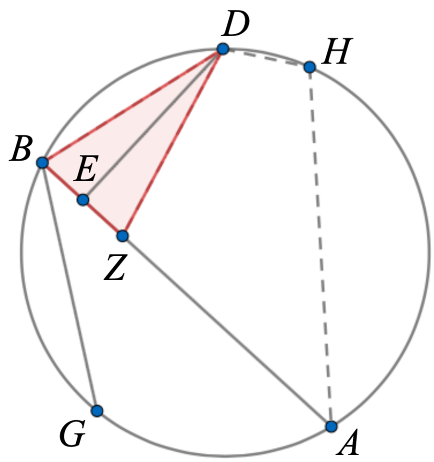
And we will also use several times the theorem that any point on the perpendicular bisector of a segment forms an isosceles triangle when connected with the endpoints of the segment.

preliminary work

We are given arc AD = arc GD.

Given B lies on arc GD (the minor arc, since $BD < GD <$ one half of the circle).

$DE \perp AB$.



Many constructions draw the isosceles triangle DBZ as shown above. Note that $\angle DBZ$ cuts the arc AD so it is equal to any other angle that might be drawn cutting AD. And since $AD = GD$, any angle cutting GD is also equal..

$\angle DBZ = \angle DZB$, corresponding to the combined arcs AD and GD, which leaves arc AG. It follows easily that $\angle BDZ = \angle ABG$

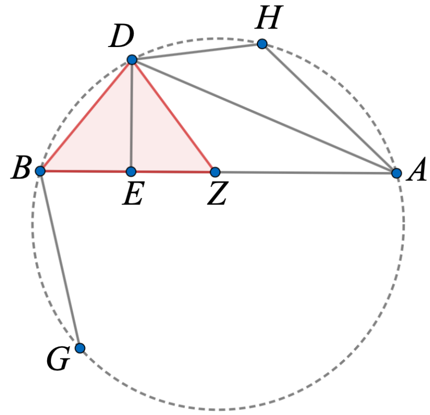
Several angles are equal to arc $AG + GD$. These include $\angle DZA$ (external to $\angle DBZ$ plus $\angle BDZ$), $\angle GBD$ (supplementary to arc GD) and $\angle AHD$ (supplementary to arc AD).

first proof: Suter (a)

The first and second proofs are attributed to Archimedes.

Proof.

Draw the isosceles $\triangle DBZ$, with $\angle DBZ = \angle DZB$.



Now place H such that $BD = HD$.

Subtracting equals, it follows that $GB = AH$. (Equally, we could find the latter first).

$\angle A$ is bisected since the two parts are subtended by $BD = HD$.

From the preliminaries, we have $\angle AZD = \angle AHD$

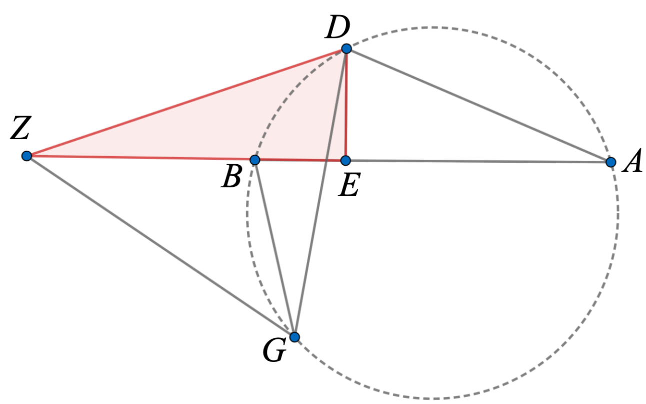
Thus $\triangle ZDA \cong \triangle HDA$ by ASA.

It follows that $AZ = AH = GB$.

Adding equals: $GB + BE = AZ + ZE = AE$.

$\square$

second proof: Suter (c)



Proof.

Given arc $AD =$ arc GD, B on arc GD, and $DE \perp AB$.

So $GD = AD$.

Extend AEB to Z such that $AE = ZE$.

$\triangle DZE \cong \triangle DAE$ by SAS.

So $\triangle DZA$ is isosceles and $DZ = AD = GD$.

Hence $\triangle DZG$ is isosceles with base angles equal.

Since $\triangle DZA$ is isosceles, $\angle DZE = \angle DAE$.

Since they correspond to the same arc, $\angle DAE = \angle DGB$.

Hence $\angle DZE = \angle DGB$.

Subtracting equals we have $\angle BZG = \angle BGZ$.

Thus $\triangle BZG$ is isosceles, and $ZB = BG$.

Adding equals: $BG + BE = ZB + BE = AE$.

□

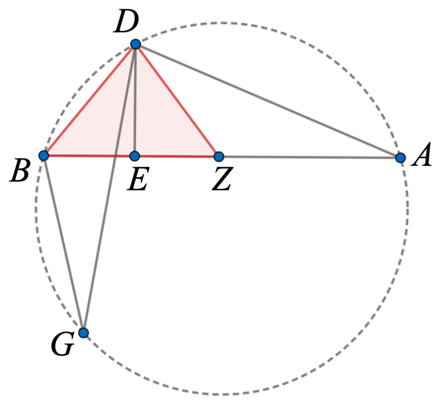
third proof

This elegant proof is ascribed to Gregg Patruno, a student at Stuyvesant High School in New York (1980). I found it in

https://www.researchgate.net/publication/341579803_FROM_THE_THEOREM_OF_THE_BROKEN_CHORD_TO_THE_BEGINNING_OF_TRIGONOMETRY

[Drakaki attributes it to Patruno].

The diagram can be found in Al Biruni's book (p. 15), but, according to Suter's translation, that proof is subtly different. Patruno starts with $AZ = GB$, whereas Suter (m) starts with $DB = DZ$. We discuss this point at the end of the chapter.



Proof.

Find Z such that $AZ = GB$.

$\angle G = \angle A$, as inscribed angles on the same arc.

We are given $GD = AD$.

Hence $\triangle BDG \cong \triangle ZDA$ by SAS.

So $BD = ZD$.

Hence $\triangle DBZ$ is isosceles.

It follows that $\triangle BDE \cong \triangle ZDE$ by HL.

So $BE = ZE$.

Adding equals:

$GB + BE = ZE + GB = ZE + AZ = AE$.

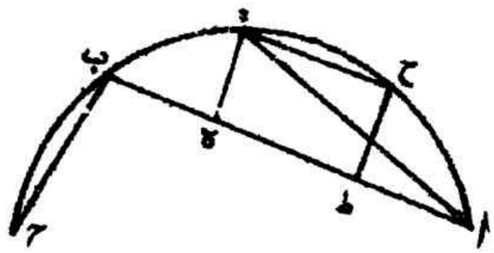
$\square$

fourth proof: Suter (i)

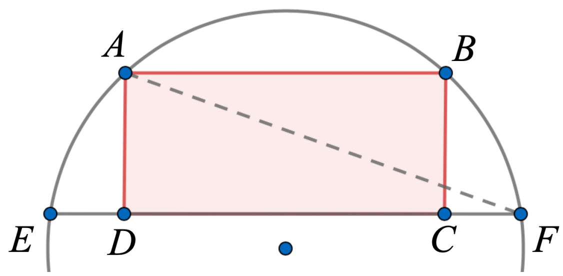
This one is attributed to El-Sidjzi (972), but may actually have been known to Apollonius.

https://www.researchgate.net/publication/341579803_FROM_THE_THEOREM_OF_THE_BROKEN_CHORD_TO_THE_BEGINNING_OF_TRIGONOMETRY

This figure from Al Biruni (below) suggests a neat line of attack.



Recall from our work on parallel chords in a circle, a result about the extensions from a rectangle in a circle. In the figure below, $DE = CF$.



Proof.

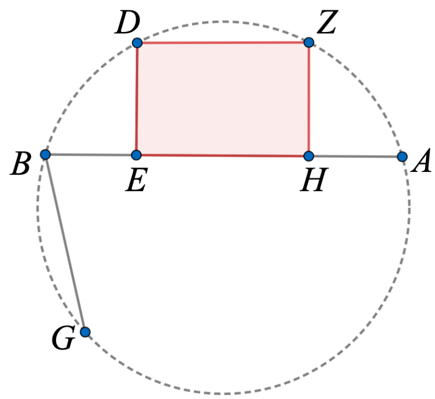
Let AB and EF be two parallel chords in a circle with unequal lengths. Draw the perpendiculars AD and BC. Then $ABCD$ is a rectangle.

$\angle AFE = \angle BAF$ by alternate interior angles. So arc $AE =$ arc BF.

It follows easily that $\triangle ADE \cong \triangle BCF$ by HL.

$\square$

So now



Proof.

As before, arc GD = arc AD, and $DE \perp AEB$.

Erect the perpendicular to DE at D and find where it cuts the circle at Z. This always forms a polygon since $BD < AD$.

Erect the perpendicular ZH.

Since it has all right angles, $DZHE$ is a rectangle.

By our preliminary work, $\triangle DBE \cong \triangle ZAH$.

Thus arc BD = arc AZ and $BE = AH$.

Subtracting equals, arc DZ = arc GB.

It follows that $GB = DZ = EH$.

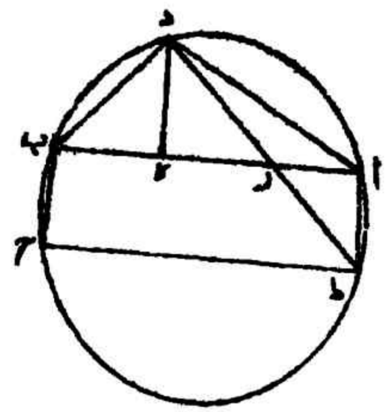
Adding equals:

$GB + BE = EH + AH = AE$.

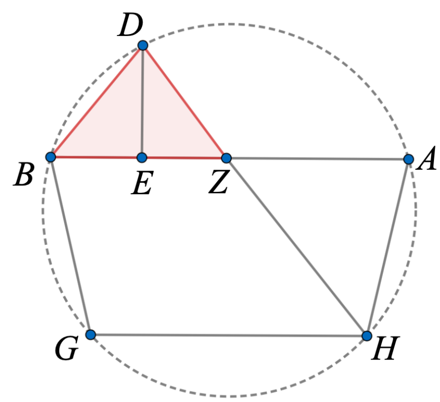
□

fifth proof

Here is another diagram from Al Biruni's book in Arabic.



Re-rendered:



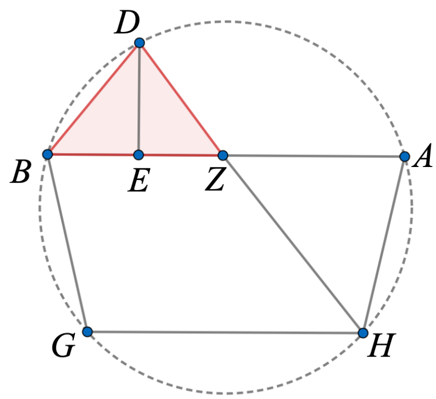
$\triangle DBZ$ is isosceles, as usual, and DZ is extended to meet the circle at H. To make the discussion simpler, let us refer to the measure of $\angle DBZ$ as α and that of $\angle BDZ$ as β.

There are four other angles all equal to α, $\angle DZB$ (by construction), $\angle AZH$ (vertical angles), $\angle AHZ$ (cuts the same arc AD), and $\angle GHD$ (cuts arc $GD =$ arc AD).

We also have the third angle in $\triangle DBZ$, namely $\angle BDZ = \beta$ which cuts arc BGH. This angle is equal to $\angle BAH$ (cuts the same arc).

It follows that the sum of all the angles at A and H is equal to 180, from which we obtain $AB \parallel GH$.

Parallel chords in a circle cut equal arcs and chords.



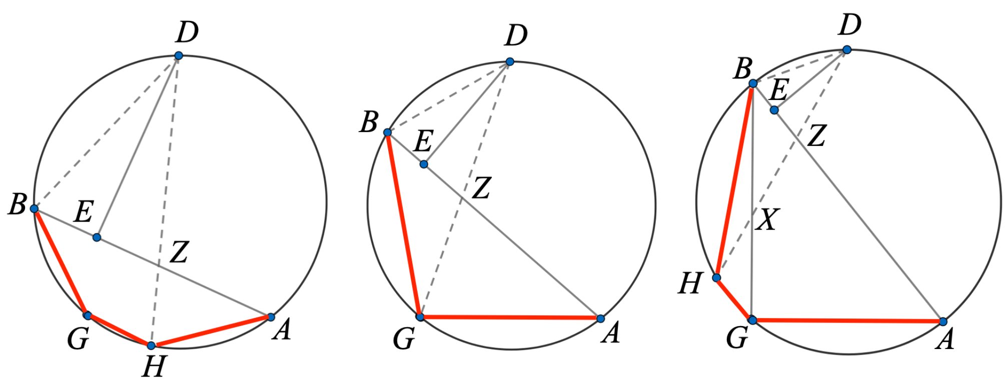
Thus $GB = AH$. And since $\triangle AZH$ is isosceles, $GB = AH = AZ$.

Adding equals:

$GB + BE = AZ + ZE = AE$.

□

However, this proof is incomplete, since for a given arrangement of G, A and D, one can draw H in three different arrangements — coincident with G, or lying on either side of it.



The left panel is our original diagram. In the right panel, it appears that $GH \parallel AB$. Can the proof be rescued?

In a word, yes. There are various approaches. Here is a sketch of a simple proof.

Proof. (Sketch).

Draw the tangent at D and connect AD and GD.

Since the two other angles formed by the tangent cut the arcs GD and AD, they are equal to $\angle DBZ = \alpha$.

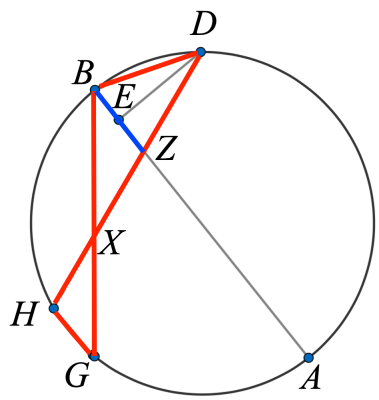
Then, $\angle ADG$ is equal to $\angle BDH$,

We obtain $AG = BH$ as arcs of equal angles, and thus $GH \parallel AB$ and then proceed as in the first case.

□

We might also proceed by showing that the obtuse angles at X, Z and B are all equal.

Since the acute angles at X are equal, it follows that $\triangle BDX \sim \triangle BZX \sim \triangle GHX$.

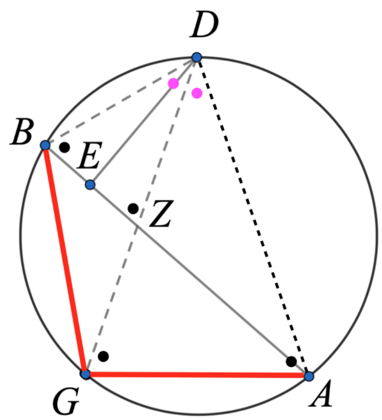


We still have to show that $AH = AZ$, but this follows since $\angle AHD$ cuts arc AD so $\triangle AHZ$ is isosceles.

□

The **third case** is where the extension of DZ terminates exactly on point G. H coincides with G.

A simple proof is to complete $\triangle DGA$.



The base angles cut the arcs AD and GD, so the triangle is similar to $\triangle BDZ$. In particular, the base angles are equal to α, so they cut equal arcs. Thus $AG = BG$.

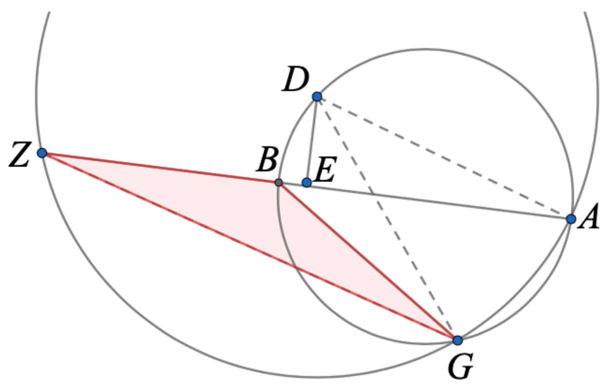
Since $\triangle AZG$ is also isosceles, we have $AG = AZ$ so $AZ = BG$, and the result follows easily.

□

sixth proof: Suter(f)

Proof.

Draw the circle on center D with radius AD. Extend AB to Z. Draw AD and GD.



AZ is a chord of the circle on D. The perpendicular DE goes through the center D, therefore it bisects AZ.

We have $AE = ZE = ZB + BE$.

Three angles intercept an arc between points A and G, $\angle BZG$ on the big circle, $\angle ABG$ on the small circle, and $\angle ADG$ on both.

Since $\angle ADG$ is a central angle in the big circle on D, it is twice $\angle AZG$. But $\angle ADG = \angle ABG$.

Hence $\angle ABG$ is external to $\triangle BZG$ and twice the measure of $\angle BZG$. It follows that $\angle BZG = \angle BGZ$.

Thus $\triangle AZG$ is isosceles with $GB = ZB$.

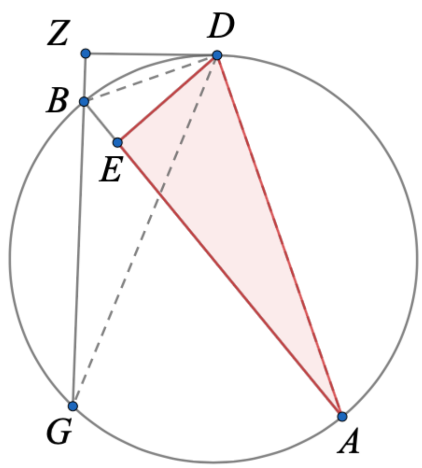
Substituting into what we had above: $AE = GB + BE$.

□

seventh proof: Suter(p)

Proof.

Draw GD and AD. Extend GB to Z and connect to D such that Z is a right angle.



$\angle ZGD = \angle BAD$ because they cut the same arc. We get the third angle by sum of angles and also $GD = AD$ so $\triangle GZD \cong \triangle AED$ by ASA.

$DZ = DE$ and then $\triangle DBZ \cong \triangle DBE$ by HL. So $BZ = BE$.

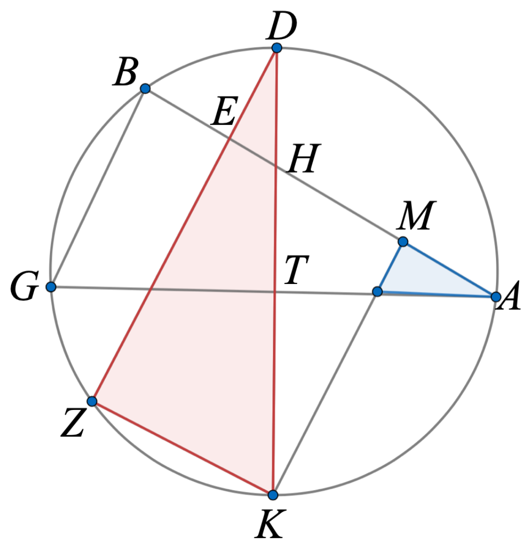
From the first congruence $GZ = AE$.

$GZ = GB + BZ = GB + BE$.

□

eighth proof: Suter(n)

As before, $AD = GD$ and $DE \perp AB$.



Proof.

Extend DE to cut the circle at Z.

Draw AG and its perpendicular bisector HTK. It is a diameter of the circle and goes through D. Why?

Connect ZK and then draw $MK \parallel DEZ$.

Since $DHTK$ is a diameter, $\angle Z$ is right. We're given that $\angle DEA$ is right.

Since $MK \parallel DEZ$, $EMKZ$ is a rectangle.

By similar right triangles, $\angle D = \angle HKM = \angle A$.

So $BG = ZK = EM$.

From rectangles in a circle we know that $BE = AM$.

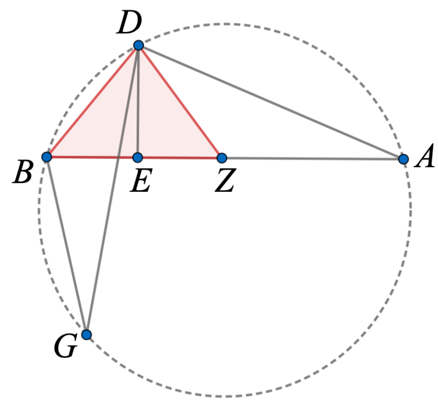
Add equals to equals to obtain the result.

□

That's enough proofs of this theorem. There are quite a few more. Each one follows naturally from an inspired diagram, and they are nearly all different.

on the value of SSA

Something interesting happens with the third proof if you approach the premises slightly differently.



We draw GD as before, but we forget to set $GB = AZ$ and instead put $BE = ZE$, as in some other proofs. Then what happens?

We have SAS in the small right triangles so $\triangle DBE \cong \triangle DZE$, which means $DB = DZ$. We have $GD = AD$ as before and $\angle A = \angle G$.

We are tempted to compare $\triangle GBD$ with $\triangle AZD$. What we have is SSA, which — it's been drilled into our heads — is *not enough*, unless there is a right angle, in which case we call it hypotenuse-leg in a right triangle (HL).

But let's take a closer look. For SSA in the ambiguous case there are only two possibilities. We can see both of them in the figure!

Certainly $\angle GBD$ is obtuse. It is supplementary to an angle (not drawn) subtended by GD, which is certainly less than half of the circle (since it equals AD and there is more than that in addition).

We also know something about $\angle AZD$. It is equal to the sum of two angles which add up to more than a right angle. It follows that $\angle AZD$ is obtuse.

Therefore, the angles are not only obtuse but clearly equal.

From Suter, here is what Archimedes says:

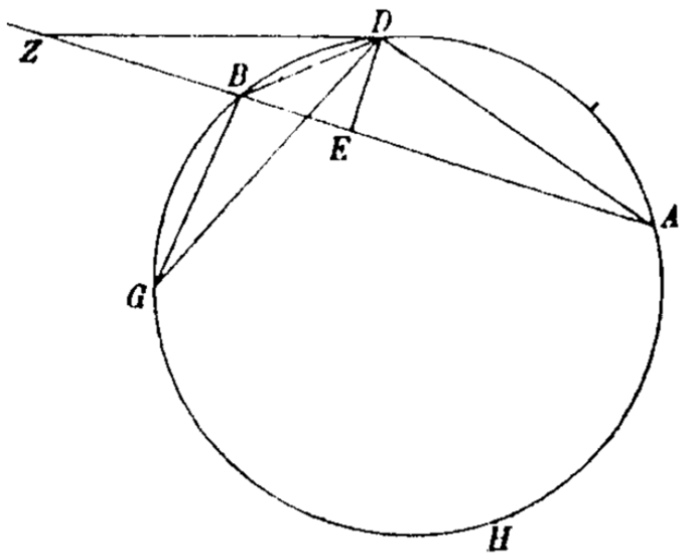


Fig. 1.

(re-phrasing): $GD = AD$ by the property of the median, BD is shared, and $\angle G = \angle A$, so we have SSA.

Now, Z has been drawn such that $ZE = AE$ which means $ZD = AD$, so we have actually another triangle sharing SSA with $\triangle GBD$.

Archimedes says that $\angle GBD$ corresponds to everything except arc GD, i.e. $DAHG$.

And $\angle DBZ$ is external to $\triangle ABD$, supplementary to $\angle DBA$ subtended by arc AD, so $\angle DBZ$ is corresponds to everything except arc AD, i.e. $AHGB$.

The missing parts are equal, so the angles are equal. Since we know two angles, we know three. Therefore, $\triangle ZBD \cong \triangle GBD$ by ASA.

more

There is one additional proof of the broken chord theorem that I know about, beyond what is in Suter.

It is on the web attributed to someone named Bùi Quang Tuần. There is another proof from the same source of the **Pythagorean theorem** that I like even better.

`https://www.cut-the-knot.org/pythagoras/BrokenChordPythagoras.shtml`

Google also turns up a blog, but no biographical info.

https://artofproblemsolving.com/community/c1598

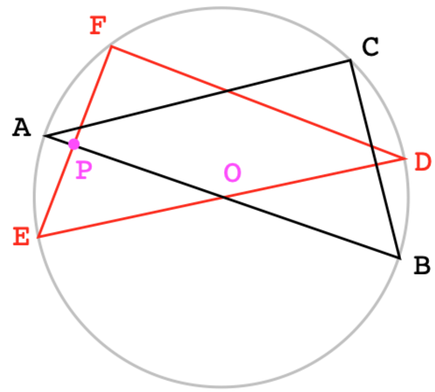
This proof is based on a rectangle, and I leave it to you to see how it relates to the fourth proof, above. I've written about it elsewhere.

Note: some additional material is here:

https://www.cut-the-knot.org/triangle/BrokenChordmpdlc.shtml

Star of David proof of the Pythagorean theorem

https://www.cut-the-knot.org/pythagoras/PythStarOfDavid.shtml



Draw two congruent mirror-image right triangles in a circle, oriented so that $EF \perp AB$ (and $BC \perp DE$).

Note that AB and DE both pass through O, the center of the circle, because any right triangle inscribed in a circle has its hypotenuse as a diameter, by the converse of Thales' circle theorem.

OA and OD are perpendicular by construction and diagonals, so they are perpendicular bisectors. Thus

$$AE = AF = DC = DB$$

Arc EC plus arc CD is equal to 180°.

So it is equal to the arc EB, which added to arc DB gives 180.

Then E is the median point between B and C and the perpendicular dropped from E which meets AB in a right angle, cuts AB so

$$AP + AC = PB$$

$$AC = PB - AP$$

by the theorem of the broken chord. The two pieces of AB are

$$AB = AP + PB$$

Putting this together, we have:

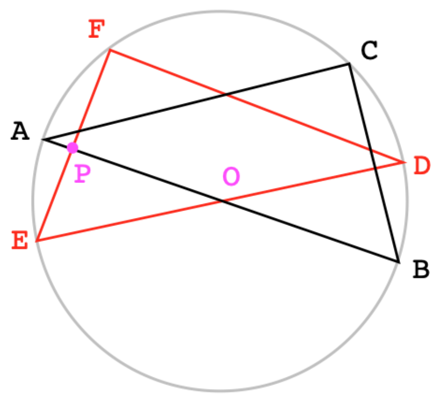
$$AB + AC = 2PB$$

and

$$AB - AC = 2AP$$

Hence

$$PB = \frac{1}{2}(AB + AC), \qquad AP = \frac{1}{2}(AB - AC)$$



By the theorem of crossed chords

$$\frac{AB + AC}{2} \cdot \frac{AB - AC}{2} = (\frac{EF}{2})^2$$

$$AB^2 - AC^2 = EF^2 = BC^2$$

The result follows.

□

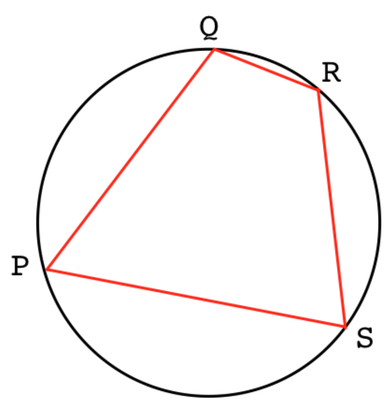
Chapter 28

Ptolemy

Ptolemy was a Greek astronomer and geographer who probably lived at Alexandria in the 2nd century AD (died c.168 AD). That is nearly 500 years after Euclid. (Ptolemy was a popular name for Egyptian pharaohs in earlier centuries).

Our Ptolemy is known for many works including his book the *Almagest*, and important to us here, for a theorem in plane geometry concerning cyclic quadrilaterals. These are 4-sided polygons with all four vertices on the same circle.

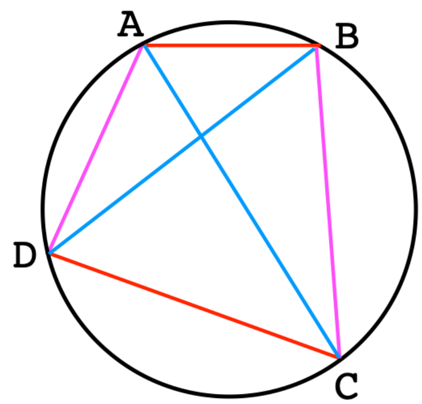
Recall that any triangle lies on a circle, so this is a restriction on the fourth vertex of the polygon.



Recall **quadrilateral supplementary theorem** (Euclid III.22):

- For *any* quadrilateral whose four vertices lie on a circle, the opposing angles are supplementary (they sum to 180°).

Now, draw the diagonals AC and BD. Ptolemy's theorem says that if we take the products of the two pairs of opposing sides and add them, the result is equal to the product of the diagonals.

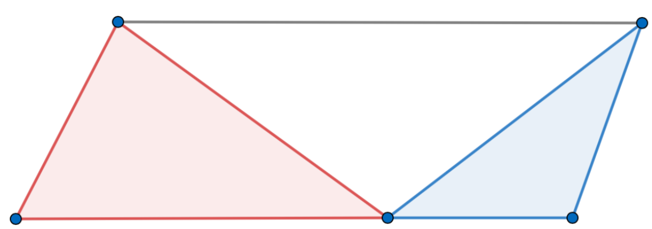


AB CD + BC AD = AC BD

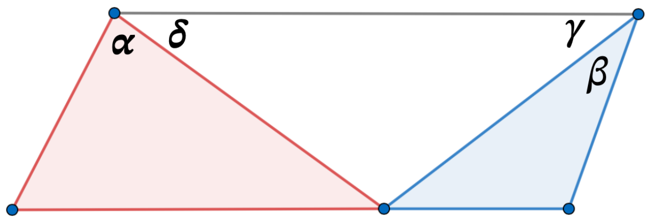
Notice that in the special case of a rectangle, the two diagonals are equal. Given the proof here, Ptolemy's result furnishes a trivial proof of the Pythagorean theorem (see later).

Proof.

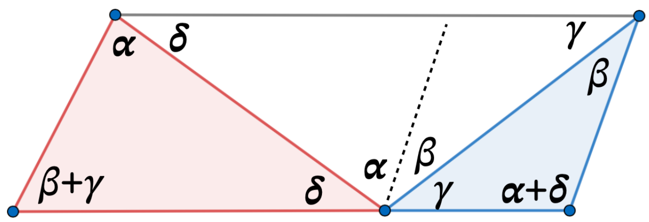
We're going to do a famous dissection of a cyclic quadrilateral as a proof of Ptolemy's theorem, but as a twist on the usual approach we'll do it in reverse, starting with a a parallelogram. I'm hoping this will make the whole thing clearer.



We have a parallelogram, and we've picked a point along the bottom edge and drawn lines to the opposing vertices. Let's label some angles.

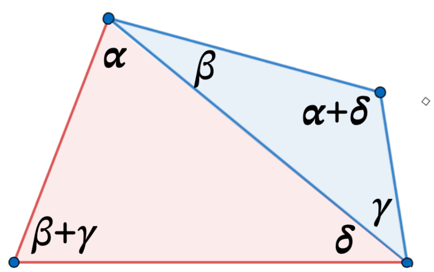


And then use the properties of parallels to get the rest of them labeled as well.

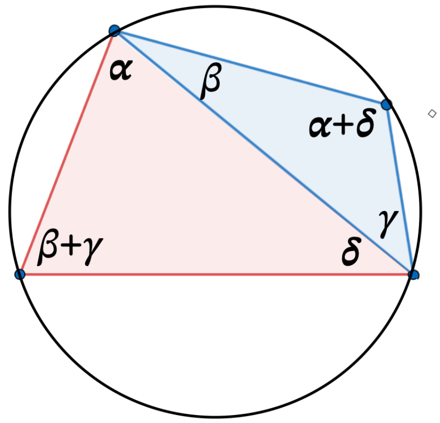


As a parallelogram, we have opposing angles equal, and adjacent angles summing to 180 as they must for parallel lines.

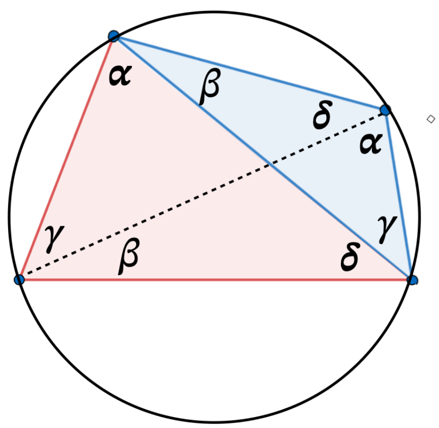
Now do the dissection. Cut out the red triangle and the blue triangle and join them as shown. In general, the scale will have to be adjusted so that they form a quadrilateral with that edge as the diagonal. We'll return to this point in a minute.



This is a cyclic quadrilateral, by the converse of the theorem on cyclic quadrilaterals, since opposing angles are supplementary.

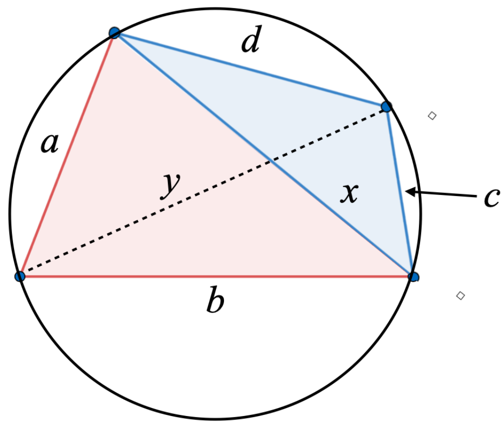


We can use the corollary of the inscribed angle theorem to draw the other diagonal and assign angles.

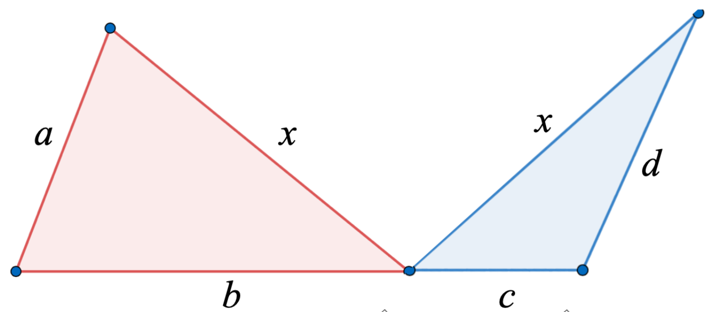


We're going switch our attention to the lengths of the sides and diagonals, but before we do, notice a very special triangle (partly red and partly blue) with angles γ, δ, and $\alpha + \beta$.

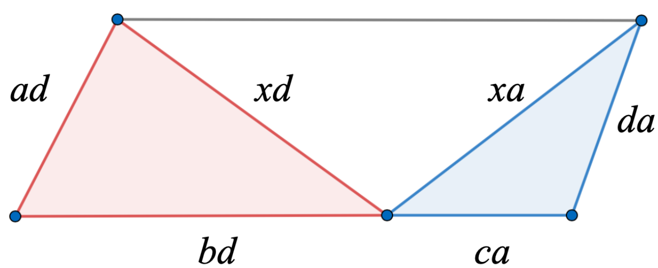
That triangle has sides a, d and y.



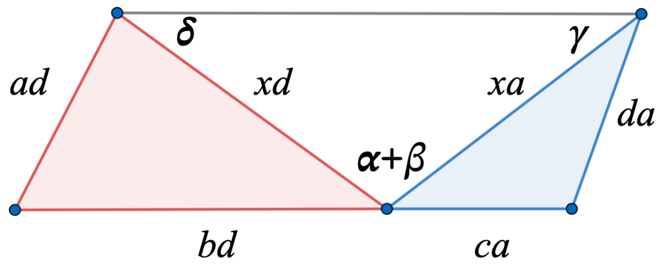
Let's reverse the process, dissecting the cyclic quadrilateral by cutting along the x diagonal, and arranging the pieces so that the bases are collinear as in the original parallelogram.



In the general case, $a \neq d$. But we can scale the two sides to be equal. An easy way to do that is to cross multiply.

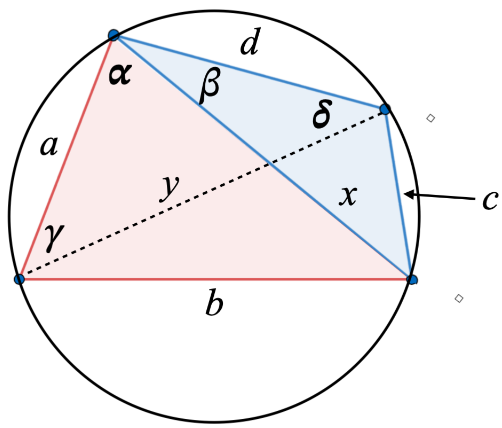


We have a parallelogram again. The angles are correct, and now the sides are equal. So are the top and bottom equal. We focus now on the white triangle.

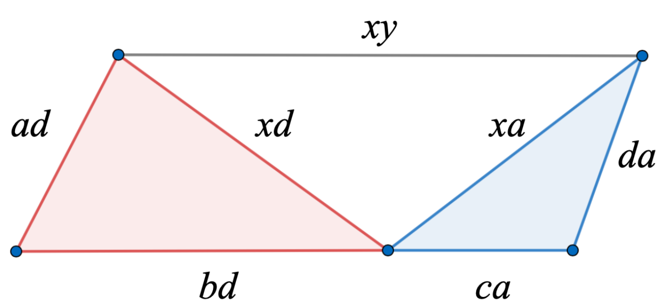


It has two sides with lengths a and d, each scaled by a factor of x, and the two angles opposite those sides are δ and γ.

Remember the special triangle?



We can see it in the cyclic quadrilateral. The sides are a, d, y. In the parallelogram, that white triangle must be scaled by a factor of x, giving sides xa, xd and xy.

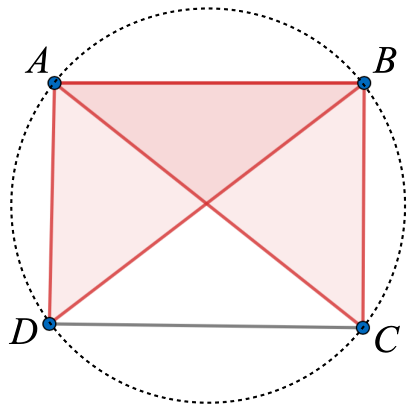


Since this is a parallelogram, the top and bottom are equal. Namely, $ac + bd = xy$.

This is Ptolemy's theorem.

□

Pythagorean theorem from Ptolemy



Proof.

Consider an arbitrary rectangle, with all four vertices as right angles, and opposing sides equal.

The diagonals are equal, by our theorem on diagonals of a rectangle, and they bisect one another, by a general theorem about diagonals of a quadrilateral. Therefore, a circle can be drawn whose center is the point where they cross and that circle contains the four vertices.

We note that one can also start with a circle, then draw a diameter and any other point, forming a right triangle, by the converse of Thales' circle theorem. Draw the congruent triangle in the other hemisphere by SSS. We have opposite sides equal and at least one right angle, so the polygon is a rectangle.

Application of Ptolemy's theorem gives:

$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

but by equal sides in a rectangle and diameters in a circle:

$$AB^2 + BC^2 = AC^2$$

□

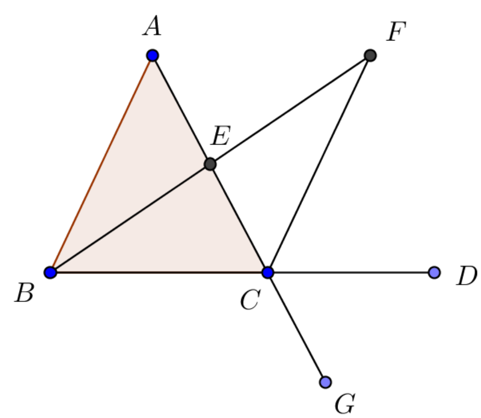
Chapter 29

Elements

In this chapter we will study some more *Propositions* from the first volume of Euclid's *Elements*. They are short, sweet and powerful.

Euclid I.16

• In any triangle, if one of the sides is produced (extended), then the exterior angle is greater than either of the interior and opposite angles.



In $\triangle ABC$ extend side BC to D. We are concerned with the exterior $\angle ACD$.

The claim is that $\angle ACD$ is greater (larger) than either of the interior angles $\angle ABC$ or $\angle BAC$.

Proof.

Find the midpoint of side AC at E so $AE = EC$ and then draw BEF so that $BE = EF$. Note that $\angle ECD$ is the same angle as $\angle ACD$.

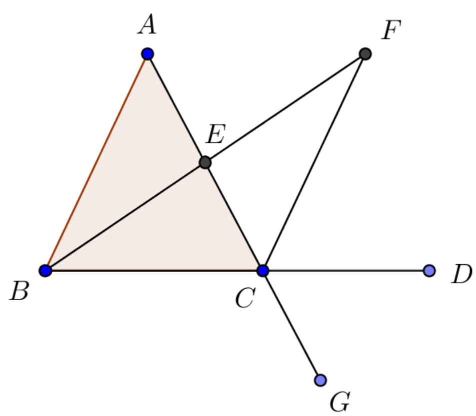
We have equal angles at E by the vertical angle theorem. So by SAS the two smaller triangles $\triangle BEA$ and $\triangle CEF$ to the left and right are congruent.

Thus $\angle BAE = \angle ECF$. Since the whole is greater than the part, $\angle ECD > \angle ECF = \angle BAE$.

We can make a similar construction and proof for $\angle ABC$.

The exterior angle is greater than either of the interior and opposite angles.

□



One might ask why Euclid doesn't use supplementary angles to obtain a stronger proof. The main reason is that he has not yet proved the triangle sum theorem. That is Euclid I.32.

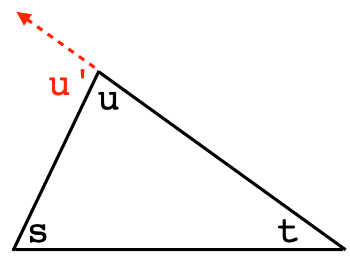
However, we did previously show a proof of the triangle sum theorem, that the sum of angles in any triangle is equal to two right angles.

And you were asked then to find the relationship between an external angle and the angles of the corresponding triangle. Here is our version of that proof.

external angle theorem

• the external angle is equal to the sum of two internal angles

Proof.



As supplementary angles, $u+u'=$ two right angles. As the three angles of a triangle, $s+t+u=$ two right angles as well.

But things equal to the same thing are equal to each other:

$$u+u'=s+t+u$$

Subtracting equals from equals:

$$u'=s+t$$

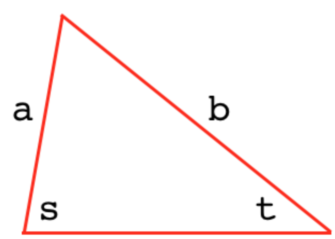
□

This relationship is fundamental.

The next theorem is also extremely useful, and it follows from Euclid I.16.

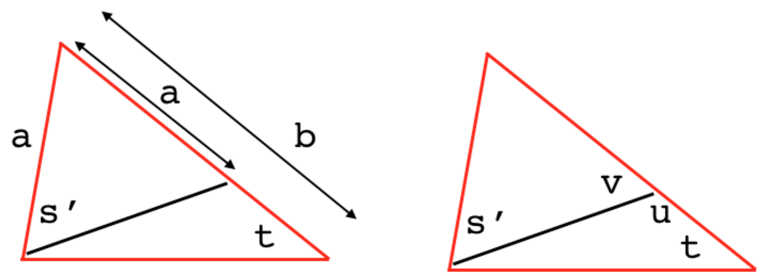
Euclid I.18

• Comparing two sides in any triangle, if one side is longer than the other, then the angle opposite is larger.



Proof.

Given that side $b > a$, mark off a on b.



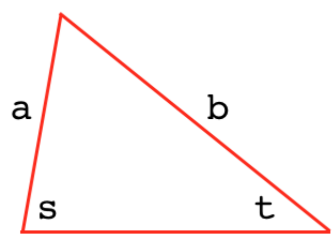
By the external angle theorem (I.16)

$$v > t$$

But $v = s'$ (by Euclid I.5) so

$$s' > t$$

And since $s > s'$



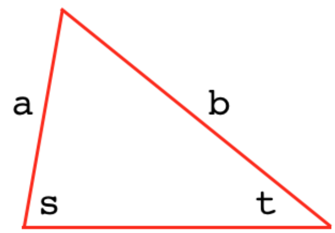
$$s > s' > t$$

□

We get the converse almost for free.

Euclid I.19

• Comparing two angles in any triangle, if one angle is larger than the other, then the side opposite is longer.



Proof.

We are given $s > t$ and claim that $b > a$.

We proceed by eliminating the other two possibilities. Rewrite the order of terms as $t < s$, with the claim $a < b$.

It cannot be that $a = b$ because then $s = t$ by isosceles $\triangle$ (Euclid I.5), but we are given $s > t$.

So then suppose $a > b$. By the previous proposition (Euclid I.18), we would have that $t > s$. But this is again contrary to what we were given.

Since the other two possibilities are eliminated, we must have $a < b$.

□

We have made use of what's called a trichotomy. There are only three possibilities:

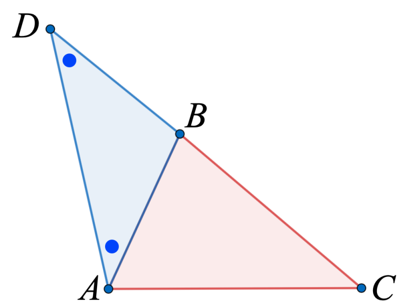
$$a < b, \qquad a = b, \qquad a > b$$

This applies to line segments and angles as well as many other things.

Euclid I.20. Triangle inequality

- In any triangle, the longest side is smaller than the sum of the two shorter sides.

Clearly, if a given side is not the longest, so it is shorter than one of the others, then it must also be shorter than the sum of that other one plus the third. For this reason, we consider only the longest side.



Given that side AC is the longest in $\triangle ABC$.

Extend side BC so that $BD = AB$. By Euclid I.5, $\triangle ABD$ is isosceles, so the angles with blue dots are equal.

Then $\angle D$ is smaller than $\angle DAC$ and therefore, by Euclid I.19 just above, AC is less than DC.

But DC is equal to the sum of the two smaller sides of $\triangle ABC$. Hence

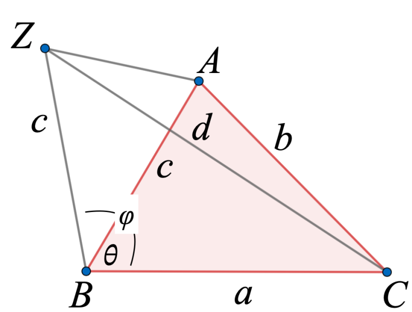
$$AC < AB + BC$$

□

An equivalent statement is the famous "a straight line is the shortest distance between two points".

The triangle inequality has corresponding statements and proofs in other parts of mathematics including real and complex analysis.

Euclid I.24. SAS inequality theorem



The next theorem is called the SAS inequality theorem, or informally, the "hinge" theorem.

- If two triangles have two pairs of sides which are the same length, the triangle with the larger included angle also has the larger third side.

We have $\triangle ABC$ with sides a and c flanking $\angle\theta$ and $\triangle ZBC$ with sides a and c flanking $\angle\phi$, and $\theta < \phi$. We claim that side b in $\triangle ABC$ is smaller than side d in $\triangle ZBC$.

Draw the two triangles nestled inside one another. Notice that $\triangle AZB$ is isosceles, so $\angle AZB = \angle ZAB$.

Proof.

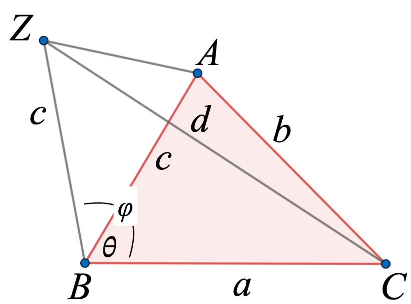
$$\angle AZC < \angle AZB \quad \text{and} \quad \angle ZAB < \angle ZAC$$

From above

$$\angle AZB = \angle ZAB$$

So

$$\angle AZC < \angle ZAC$$



By Euclid I.19, it follows that side d opposite the larger angle $\angle ZAC$ is larger than side b opposite the smaller angle $\angle AZC$ in $\triangle AZC$.

At the same time, in the two triangles $\triangle ABC$ and $\triangle ZBC$ the smaller angle θ is opposite the smaller side b, while the larger angle is ϕ, opposite the longer side, d.

□

https://proofwiki.org/wiki/Hinge_Theorem

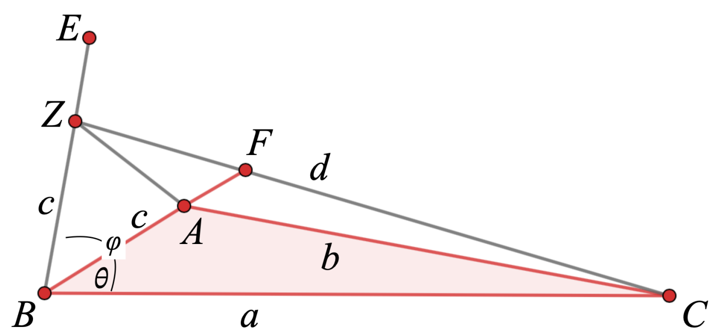
Note that Euclid I.24 says that where the included angle is larger, then the side opposite is larger. The converse is Euclid I.25.

Proof. Aiming for a contradiction, given side d is smaller than side b, suppose that $\angle ABC < \angle ZBC$. The forward theorem gives $d > b$. This is a contradiction. □

Another subtlety is that it cannot be that $b = d$, by Euclid I.7, and also then the two triangles would be congruent by SSS, but the angles differ, so this is a contradiction.

One last point: this proof suffers from the same issue that Euclid I.7 does, namely that a different drawing can be made that makes the proof non-sensical. The solution is the same as well.

Proof.



Since $\triangle ABZ$ is isosceles, the base angles are also equal, namely $\angle AZE = \angle ZAF$. So then

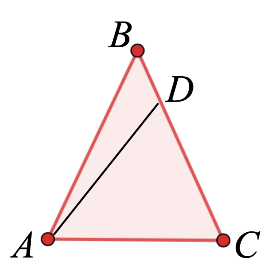
$$\angle AZF < \angle AZE = \angle ZAF < \angle ZAC$$

It follows that since the greater side is opposite the greater angle in $\triangle ACZ$, $d > b$.

□

This is enough of the *Elements* to give us a good taste of the basics of Greek geometry of lines and triangles, and methods of proof.

problem



Let $\triangle ABC$ be isosceles, and D lie on BC. Using Euclid I.19, prove that $DC < DA$. Separately, use I.20 for the same proof.

Here is another idea. *Proof.* (Sketch). Let the perpendicular bisector of AC be BE. Drop another perpendicular from D to AC at F. So E is the midpoint and F lies directly below D.

Clearly, $AF > AE > FC$. So

$$FC^2 < AF^2$$

$$FC^2 + DF^2 < AF^2 + DF^2$$

But by the Pythagorean theorem, the left-hand side is DC^2, while the right-hand side is DA^2. Taking side lengths as the positive square root, we have $DC < DA$. $\square$

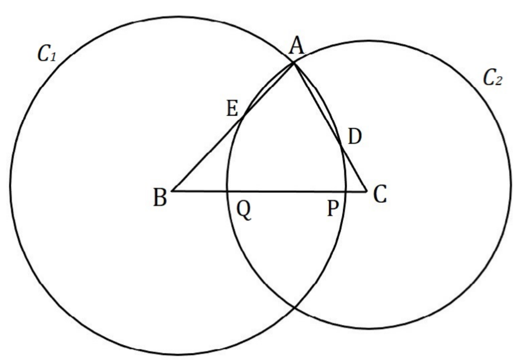
In fact, *every* point which lies on the same side of the perpendicular bisector of AC as C, is closer to C than to A.

Triangle inequality by circles

Here is a modern proof of Euclid I.20.

Proof.

Consider $\triangle ABC$. Draw circle $\mathcal{C}_1$ on center B with radius AB and also circle $\mathcal{C}_2$ on center C with radius AC. The third vertex A lies on the intersection of the two circles.



If BC is longer (greater) than the sum of the two radii, then the circles will not cross one another.

If BC is equal to this sum, then they only touch, on the line BC. Thus, in order for there to be a triangle, BC must be less than the sum of the diameters: $BC < AB + AC$.

□

https://arxiv.org/pdf/1803.01317

angle of reflection

Suppose we shine a light at a mirror, or just look at the reflection of someone else, or even our own outstretched hand. The question arises, what determines the path of the light or the image as it travels to the eye? Heron of Alexandria discovered a proof of the answer in about 100 A.D.

The diagram shows two *possible* paths light might take, but there is only one path that it *does* take, shown in the right panel. Light takes the path where it reaches our eyes the fastest.

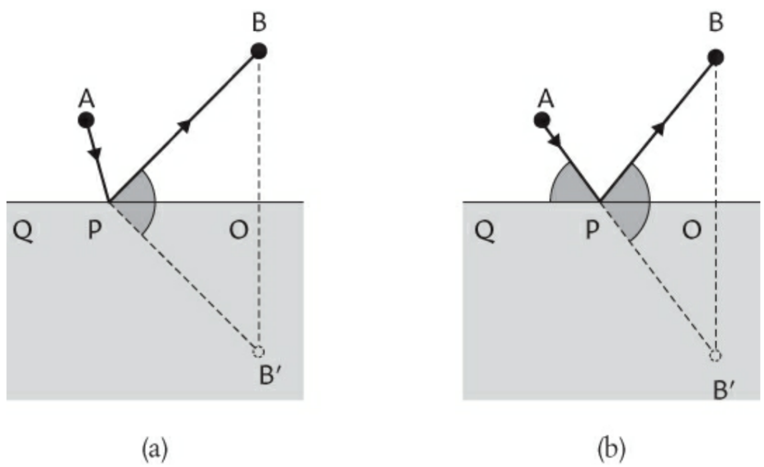


Fig. 46 Finding the shortest path.

What is this path? What is the angle the ray of light makes with the mirror? This angle is called the angle of reflection.

Draw $\triangle POB'$, imagined to be on the other side of the mirror, with B the same distance away from the mirror as B', but on the other side. The two triangles $\triangle BPO$ and $\triangle B'PO$ are congruent by SAS.

Clearly, the shortest distance from A to B' is a straight line, by the triangle inequality.

So the result is that $\angle APQ$ equals $\angle B'PO$, which (by congruent triangles) equals $\angle BPO$. This is usually stated as "the angle of incidence is equal to the angle of reflection."

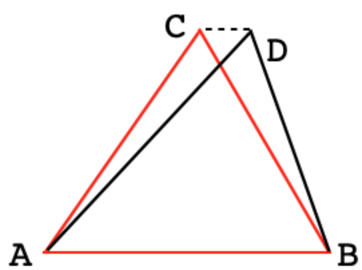
smallest perimeter

Show that for two triangles with the same area, an isosceles triangle has the *smallest* perimeter.

Proof.

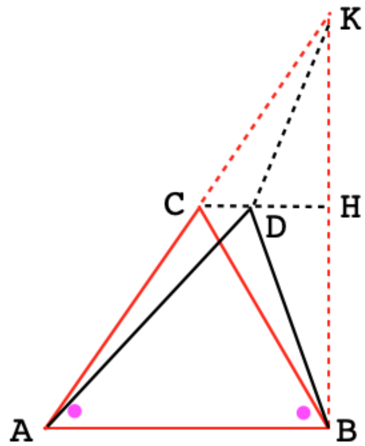
We suppose that the base of the isosceles triangle $\triangle ABC$ is equal to one of the sides of the other triangle $\triangle ABD$. If we would need to re-scale one side to have equality, we could then make a corresponding change in the altitude to that side to maintain equal area.

The equal area constraint means that points C and D lie along a horizontal line parallel to the common base AB.

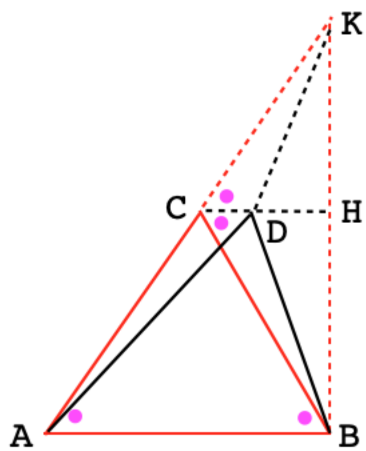


Draw a vertical from B to meet the extension of AC at K. Extend CD to meet KB at H and also draw DK.

We're given that $AC = BC$ and so $\angle CAB = \angle CBA$ (magenta dots).



It follows that other angles are also equal to those two (by alternate interior angles).



$\angle BCH = \angle ABC$ and $\angle KCH = \angle CAB$.

Since $\angle CAB = \angle ABC$, it follows that $\angle BCH = \angle KCH$.

The angles at H are right angles, since BK is perpendicular to AB and CH is parallel to AB.

Therefore $\triangle CHK \cong \triangle BCH$ by ASA, so $BC = CK$.

Similar reasoning will give that $BD = DK$.

But now by the triangle inequality:

$$AC + CK < AD + DK$$

Substituting from above

$$AC + BC < AD + BD$$

Add AB to both sides

$$AC + BC + AB < AD + BD + AB$$

The perimeter of $\triangle ABC$ is less than that of $\triangle ABD$.

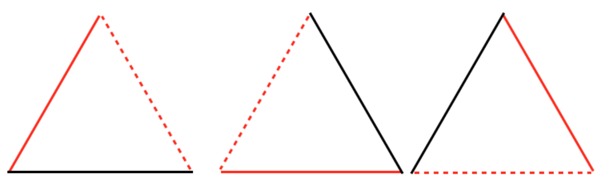
$\square$

Chapter 30

Equilateral triangles

basic triangle

An equilateral triangle has all three sides the same length. By symmetry, as there is no reason to favor one vertex, the three vertices have the same angular measure, namely, $\pi/3$ (since the total is two right angles).



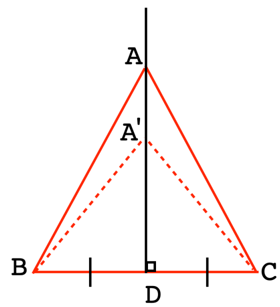
Proof.

The **isosceles triangle theorem** (Euclid I.5: equal sides $\rightarrow$ angles), which can be applied twice to give the result.

We also have **Euclid I.19**, which says that if one angle is larger than another in a triangle, then the corresponding side is also longer. So, assume the angles are not the same, then that implies the sides are not the same length either. Hence the angles are the same.

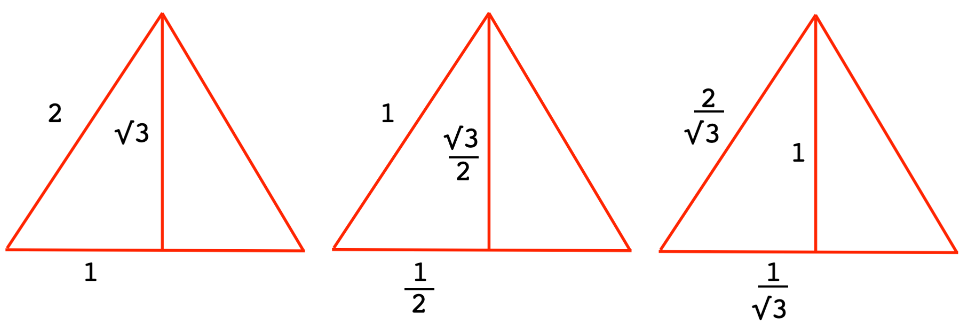
□

An equilateral triangle is also isosceles (three times over), and as a result the altitude dropped from a vertex to the opposing base bisects both that base and the angle at the vertex.



area

To calculate the area of the equilateral triangle, we could write a general formula, but it is usually easier to go back to basics and derive one.



In the left panel, the result of bisection is that, for a side of length 2, half the base is 1, and then Pythagoras tells us that the altitude is $\sqrt{3}$. Using the standard formula, the area of this triangle is also $\sqrt{3}$.

To re-scale the triangle, write whichever measurement is given. For example in the middle we're given a side length of 1 so that means that all lengths are multiplied by $1/2$.

On the right, we're given that the altitude is 1 so then we must multiply by $1/\sqrt{3}$. The logic is that to erase that $\sqrt{3}$ in the altitude, we must multiply by its inverse.

If you really insist on a general formula...

Suppose the length of the side is s, then the ratio of the altitude to the side is

$$\frac{h}{s} = \frac{\sqrt{3}}{2}$$

Therefore twice the area of the triangle is

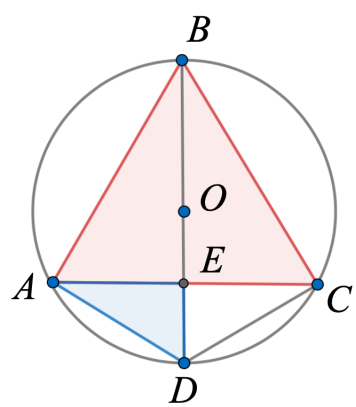
$$2A = \frac{\sqrt{3}}{2} \cdot s^2$$

We can check that against a general trigonometric formula for the area of any triangle. Twice the area is the product of two adjacent sides times the sine of the angle between. Here the angle is $\pi/3$ and its sine is $\sqrt{3}/2$ so we have $2A = \sqrt{3}/2 \cdot s^2$.

It can be quicker to rely on the fact that the area goes like the square of the side. For side length 2 we had $A = \sqrt{3}$. If we shrink the triangle to side length 1, the area goes down by a factor of four, to $A = \sqrt{3}/4$.

circumscribed equilateral triangle

Here is a fun construction based on an equilateral triangle.



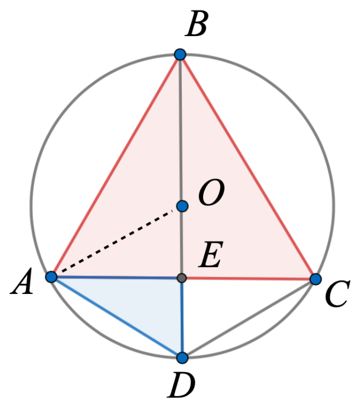
Any triangle fits into a unique circle (see the discussion of circles and bisectors of the chords). Draw the circumcircle for equilateral $\triangle ABC$.

Draw a radius to a vertex of the triangle, and extend it as the diameter BD, which crosses the opposing side at E.

We can prove a number of properties. A good place to start is that BD bisects $\angle B$.

Proof.

Connect AO and CO (only one is shown).



$\triangle AOB \cong \triangle COB$ by SSS.

It follows that $\angle B$ is bisected.

Since $AB = CB$, $\triangle ABE \cong \triangle CBE$ by SAS.

So the angles at E are all equal, and thus right angles.

□

The next property is that $\triangle OAD$ is also equilateral.

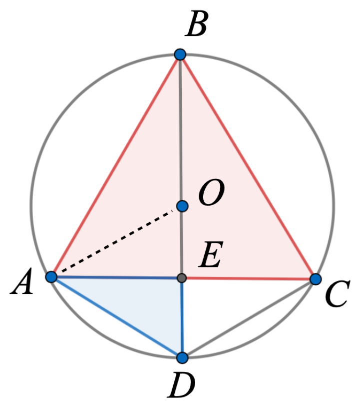
Proof.

$\angle AOD$ is the central angle for the half-angle at $\angle B$, hence $\angle AOD = \angle B$.

By inscribed angles, $\angle BDA = \angle C$, so $\triangle OAD$ is equi-angular, by sum of angles. It is therefore equilateral.

□

The three congruence classes of right triangles are all similar.



Proof.

$\triangle BAD$ is a right triangle by Thales' theorem.

From above, $\triangle BEA$ and $\triangle AED$ are both right as well.

The first two share β (the half-angle at B), while the third has $\angle ADB$ shared with $\triangle BAD$.

□

$AD = AO = CO = CD$.

Proof.

From above, $\triangle OAD$ is equilateral. Similarly, $\triangle OCD$ is equilateral.

It follows that $AOCD$ is a rhombus.

This would also follow by the converse of the diagonals theorem for parallelograms, since $AE = CE$.

□

Finally, $DE = OE =$ one-half of OB and one-third of BE.

Proof.

$\triangle BEA \sim \triangle BAD \sim \triangle AED$.

Then AD is one-half BD, but DE is one-half AD, hence 1/4 of BD.

Since OD is one-half of BD, DE is one-third of BE.

□

Proof. (Alternate).

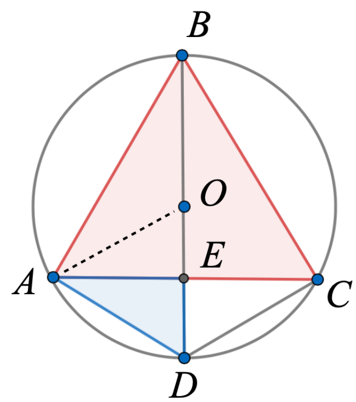
Let the side length be 2 and then $AE = CE = 1$. By crossed chords, $AE \cdot CE = 1 = BE \cdot DE$.

But by the Pythagorean theorem $BE = \sqrt{3}$ and then $DE = 1/\sqrt{3}$ and their ratio is

$$\frac{1}{\sqrt{3} \cdot \sqrt{3}} = \frac{1}{3}$$

□

Proof. (Alternate).



Let $AD = CD = d$ and r be the radius of the circle.

By Ptolemy's theorem:

$$sd + sd = 2rs$$

$$d = r$$

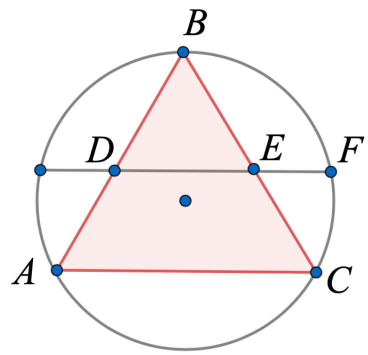
It follows that $AOCD$ is a parallelogram, the diagonals bisect one another, etc.

□

Thus, the altitude of the equilateral triangle is $3/4$ of the diameter of the circle that just encloses it. And the point where the altitudes meet in an equilateral triangle is $1/3$ of the way up from the base, since $r = 2a$.

problem

Here is a problem from Dr. Paul Yiu. Given that ABC is equilateral and that the points D and E are midpoints of the sides, find DF/DE.



We *could* calculate the length of the perpendicular from the center of the circle (O, not labeled) to DE and then use the Pythagorean theorem.

A simpler approach is to set the side length of the triangle to 2, so each half is 1, and since BDE is also equilateral, $DE = 1$ (or use the midline theorem).

Let DF be x. Then, by the theorem of crossed chords we have that

$$1 \cdot 1 = (x - 1) \cdot x$$

$$x^2 - x - 1 = 0$$

Solve using the quadratic equation:

$$x = \frac{1 + \sqrt{5}}{2} = \phi$$

This is the famous "golden" ratio.

Let's do it the hard way as well. If we keep the side length of the large triangle as 2, then the small triangle has side length 1, and the altitude is $\sqrt{3}/2$.

We need the radius. The altitude of the large triangle is $\sqrt{3}$ and the diameter is 4/3 of that, so the radius is 2/3 which makes $r = 2/\sqrt{3}$.

The distance from O to DE is the difference, and we will need the square of that:

$$\delta^2 = (\frac{\sqrt{3}}{2} - \frac{2}{\sqrt{3}})^2$$

$$= \frac{3}{4} + \frac{4}{3} - 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{2}{\sqrt{3}}$$
$$= \frac{3}{4} + \frac{4}{3} - 2$$

The distance from the midpoint of DE to F, squared, is:

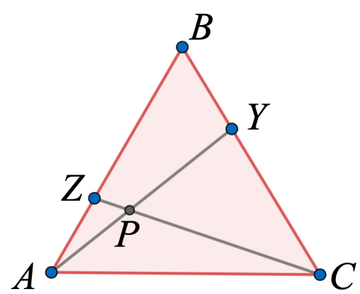
$$\frac{4}{3} - \delta^2 = \frac{4}{3} - (\frac{3}{4} + \frac{4}{3} - 2) = \frac{5}{4}$$

so the square root is $\sqrt{5}/2$ and then we must add $1/2$ to obtain DF. That matches what we had before.

problem

Here is a second problem from Dr. Yiu.

Crossing lines are drawn from the vertices in an equilateral triangle, such that the larger pair of angles at P are each $120°$.



We have that, since $\angle ZPY + \angle A =$ two right angles, the angles inside the quadrilateral at Z and Y are supplementary, so the opposing ones are equal.

A consequence of that is $\triangle AYB$ and $\triangle BZC$ have all three angles the same, but also share a side, hence they are congruent. Therefore $AY = BZ$ and $AZ = YC$.

By the same reasoning, $\angle AZC = \angle BYC$, which means that $\triangle AZC$ and $\triangle YCB$ share all three angles plus two sides, so they are congruent. Therefore $BY = ZC$.

There are a ridiculous number of similarities and equalities here. The acute angles at P are $60°$, which gives more similar triangles such as $\triangle PZB \sim \triangle AYB$ and $\triangle PYC \sim \triangle AZC$. I write the vertices flanking the short and medium sides in order to make it easy to write the ratios.

We are almost too wealthy. We are asked to show that

$$\frac{AY}{AZ} = \frac{PB}{PC}$$

Now, $\triangle BPC$ is not similar to anything else. But from the first similarity above ($\triangle PZB \sim \triangle AYB$) we get

$$PZ/ZB/BP = AY/YB/BA$$

so we can write

$$PB = AB \cdot ZB/YB$$

From the second similarity ($\triangle PYC \sim \triangle AZC$) we get

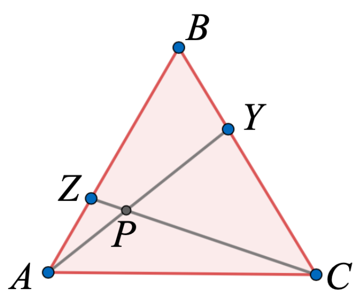
$$PY/YC/CP = AZ/ZC/AC$$

so we can write

$$PC = AC \cdot YC/ZC$$

Then we form the ratio (canceling $AB = AC$)

$$\frac{PB}{PC} = \frac{ZB}{YB} \cdot \frac{ZC}{YC}$$



Using the equalities $ZC = BY$, $AY = ZB$, $AZ = YC$:

$$\frac{PB}{PC} = \frac{ZB}{YC} = \frac{AY}{AZ}$$

I don't know if there is an easier solution, just happy to have found one.

Bertrand's paradox

Grinstead and Snell's wonderful *Introduction to Probability* has this problem (example 2.6). It's called Bertrand's paradox. We are asked to draw a chord of a unit circle randomly.

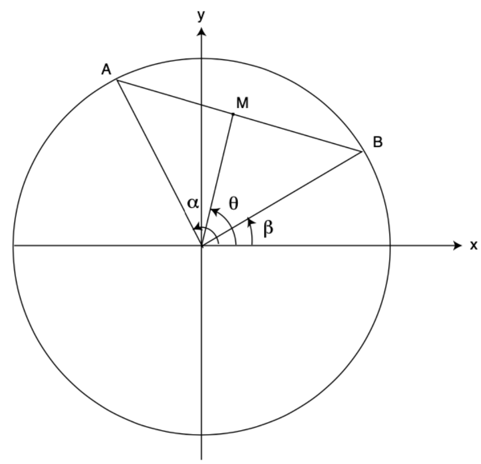


Figure 2.9: Random chord.

Here we might say, let's choose each of three angles α, β and θ randomly (uniform density) from $[0, 2\pi]$.

But there is no reason why the radius to B cannot lie along the x-axis, so there are really only two choices.

The question is posed: what is the probability that the length of this random chord is $> \sqrt{3}$.

However, there are several different approaches to parametrize the problem, and randomizing the different parameters leads to different results.

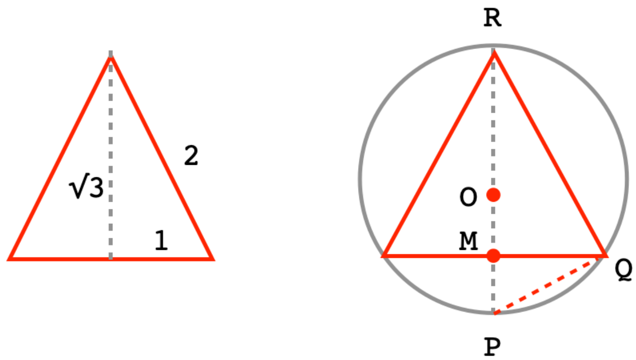
equilateral triangles

We briefly review some properties of equilateral triangles which we looked at earlier, with a slightly different take.

Drop an altitude and observe the ratio of side lengths. It is convenient to start with a side length of 2 for the original triangle, then in the bisected copies the sides are in the ratio 1-2-$\sqrt{3}$ (the 1 by bisection, and $\sqrt{3}$ by the Pythagorean theorem).

The angle at each vertex of the original equilateral triangle is $\pi/3$, so the new triangles have angles of $\pi/6$, both by bisection or because the altitude forms an angle of $\pi/2$ at the base, so the sum of angles theorem gives us the last angle.

In the right panel, the equilateral triangle is inscribed in a unit circle, so $OR = OP = OQ = 1$. We claim that the line segment OM has a length of $1/2$.



Proof.

$\angle PQR$ is a right angle, by Thales' circle theorem, and $\angle MRQ$ is shared, so $\triangle PQR$ is similar to $\triangle RMQ$. Therefore, $\angle RPQ = \angle MQR = \pi/3$.

Therefore the sides of $\triangle PQR$ are also in the ratio 1-2-$\sqrt{3}$, with $PQ/PR = 1/2$ and so $PQ = OP = OQ$. Thus, $\triangle OPQ$ is equilateral.

$QM \perp OP$ so MQ is the bisector of both $\angle PQO$ and the base OP.

Therefore, OM is one-half of OP and has a length of $1/2$.

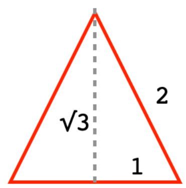
□

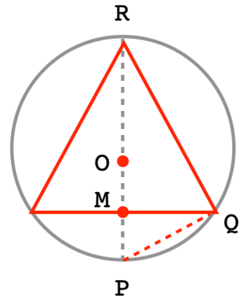
first parametrization

We have just shown that the altitude of the inscribed equilateral triangle in a unit circle has length $3/2$. This means that the ratio of the inscribed triangle to the standard one is $\sqrt{3}/2$.

And that means the side length of the inscribed equilateral triangle is $\sqrt{3}$.

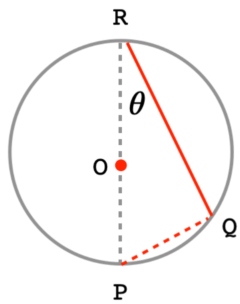
That explains the length chosen for the chord in this problem. We see that if M is chosen at random anywhere along OP, one-half of the time the chord formed will be larger than $\sqrt{3}$.

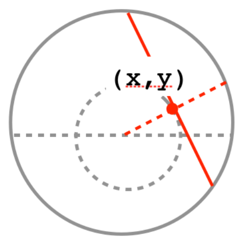




So the probability we are asked to give is just 1/2.

second parametrization





The second parametrization has the same triangle we just saw, $\triangle PQR$. The angle at vertex R is θ.

θ can lie in the interval $[0, \pi/2]$, and in the event that $\theta < \pi/6$, the chord length $RQ > \sqrt{3}$.

The probability that the chord is greater than $\sqrt{3}$ in length is 1/3, since $\pi/6$ is one-third of $\pi/2$.

third parametrization

Finally, we imagine picking two coordinates (x, y) at random from the interior of the circle. We place the midpoint of the chord M at (x, y).

If M is such that $r = \sqrt{x^2 + y^2} < 1/2$, then M will be closer to the center of the circle than 1/2 and so the chord length will be $> \sqrt{3}$.

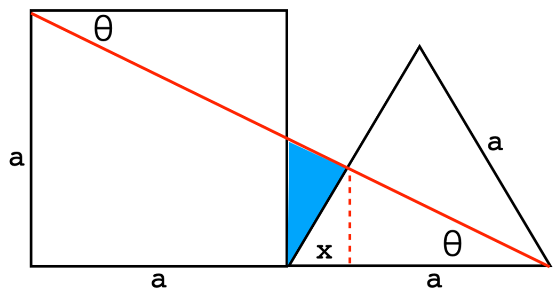
The number of points that have this property is proportional to the relative areas of the inside small circle, and the larger circle around is.

$$\frac{\pi(1/2)^2}{\pi} = \frac{1}{4}$$

We see that, depending on which parameter is randomized, we obtain a probability of 1/2, 1/3 or 1/4.

In Jaynes's words, the problem is not well-formed.

problem

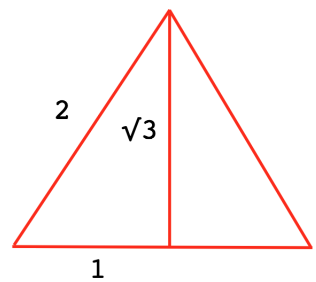


We have a square with side a and an equilateral triangle, also with side a. Their bases are colinear and a red line is drawn as shown. What is the area of the blue triangle?

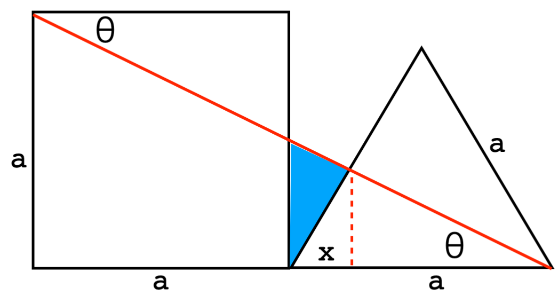
The angles marked θ are equal because the base and the top of the square are parallel, but the angle is *not* 30. The red line intersects the side of the square at half the height, so $a/2$.

We compute *twice* the area of the combined blue and part of the equilateral triangle as $a^2/2$ and that of the blue triangle as $xa/2$. But what is x?

We need a preliminary result. The Pythagorean theorem allows us to calculate that the altitude of an equilateral triangle is $\sqrt{3}$ times one-half the base.



Back to our problem. The triangle with x as its base does have an angle of 30 so its height h is $\sqrt{3} \cdot x$. We compute twice the area of the part of the equilateral triangle as $\sqrt{3}xa$.



Put the whole thing together as

$$2\mathcal{A} = \frac{a^2}{2} = \frac{xa}{2} + \sqrt{3}xa$$

$$a = x + 2\sqrt{3}x$$
$$x = \frac{a}{1 + 2\sqrt{3}}$$

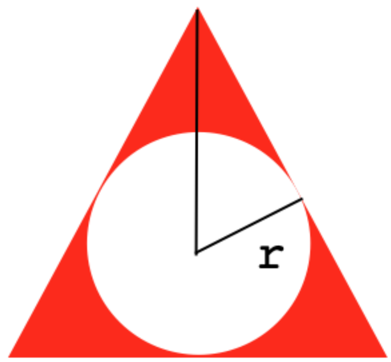
The final result is $1/2$ times $a/2$ times what we have above.

problem

Here is a problem from the entrance exam for Brown, 1901.

> **4. A regular triangle is circumscribed about a circle whose radius is 7. Find the area of that portion of the triangle which is outside of the circle.**

A regular triangle is what we would call an equilateral triangle. Applying the Pythagorean theorem to one-half of such a triangle yields 1-2-$\sqrt{3}$ for the sides, where 2 is the hypotenuse.



From this we deduce that the altitude is $3r$ and the side length is $2\sqrt{3}r$, so the triangle's area is

$$A = \frac{1}{2} \cdot 3 \cdot 2\sqrt{3} \cdot r^2$$

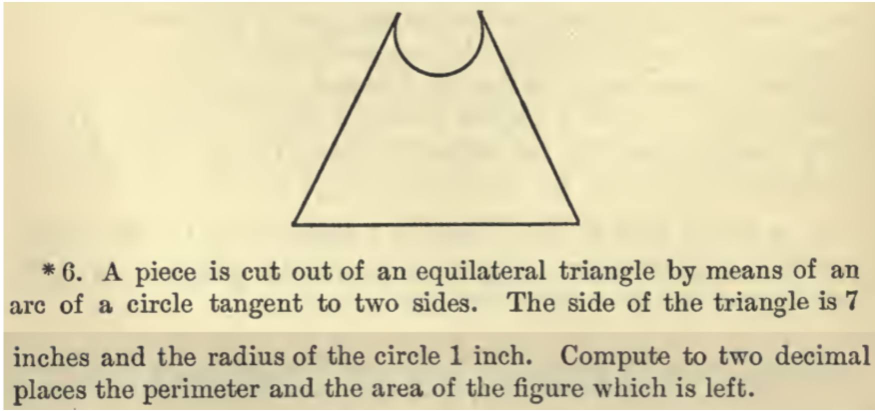
The red part is the difference between this and the circle's area.

$$\text{red} = (3\sqrt{3} - \pi)r^2$$

We're given a radius of 7 but that's arithmetic. We don't need that now.

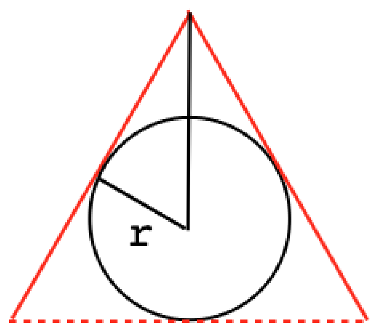
problem

Here is a problem from the entrance exam for Harvard, 1901.



*6. A piece is cut out of an equilateral triangle by means of an arc of a circle tangent to two sides. The side of the triangle is 7 inches and the radius of the circle 1 inch. Compute to two decimal places the perimeter and the area of the figure which is left.

The first thing to notice is that the circle is tangent to the sides. The tangent touches the circle at a single point, so the missing part of the diagram looks like this:



The whole angle at the top is $\pi/3$ so the half-angle is $\pi/6$. This right triangle has sides in the ratio 1-2-$\sqrt{3}$. The radius is given as 1 so the red base side of the triangle is $\sqrt{3}$.

The missing perimeter of the triangle is twice this or $2\sqrt{3}$. But the added perimeter from the circle is $2/3 \cdot 2\pi r = 4/3\pi$.

The area of the triangle is $1/2 \cdot \sqrt{3} \cdot 1 = \sqrt{3}/2$ and there are two copies so the total missing triangular area is $\sqrt{3}$. The added circular area is $2/3\pi$.

The original perimeter of the triangle was $7 \cdot 3 = 21$ and the area was $\sqrt{3}/4 \cdot s^2$.

We'll just set up the two calculations:

$$A = \frac{\sqrt{3}}{4} \cdot 7^2 - \sqrt{3} + 2/3 \cdot \pi$$

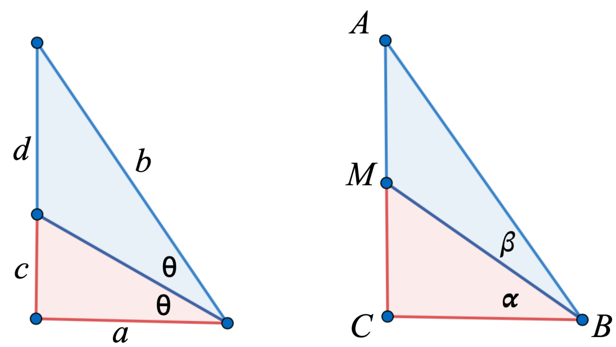
$$P = 7 \cdot 3 - 2\sqrt{3} + 4/3 \cdot \pi$$

It seems that the most efficient way to calculate this is to do $\sqrt{3} + 2/3 \cdot \pi$, then double it, then do the rest.

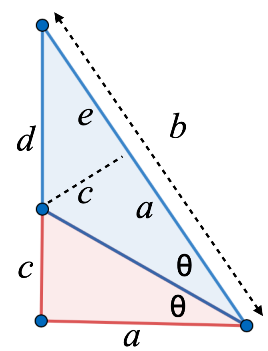
Chapter 31

Angle bisectors

Here's a classic problem: can we say anything about the ratio of sides when an internal line is drawn in a triangle? We first consider the case of a right triangle where the internal line is an angle bisector (left panel, below). We can prove an important theorem.



angle bisector theorem



The bisector forms two new internal triangles. In the upper one (blue), we draw a line perpendicular to side b which meets the bisector on the side opposite. This forms two triangles with two angles the same, θ and the right angle (thus three angles the same), and one side shared. So we have congruent right triangles by ASA.

Thus, the two sides labeled a are equal, as are the two sides labeled c.

The triangle with sides c, e, d is similar to the whole original triangle, because both are right triangles and the top vertex is shared. We have that the ratio of the short side to the hypotenuse is

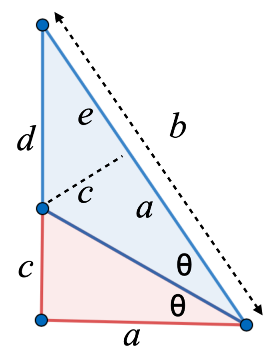
$$\frac{a}{b} = \frac{c}{d}$$

$$ad = bc$$

$$\frac{a}{c} = \frac{b}{d}$$

Tthe sides flanking the duplicated angle are in the same proportion as the parts of the base:

□



The result can be pushed a little further: Add 1 to both sides:

$$\frac{a+b}{b} = \frac{c+d}{d}$$
$$\frac{a+b}{c+d} = \frac{b}{d} = \frac{a}{c}$$

which may be a surprising result.

We took advantage of the fact that the large triangle was a right triangle. However, if you think about it, you should be able to see that the same result holds for an isosceles triangle. There, the two sides are equal, and if the top angle is bisected, so is the base. So the ratio of each side to its part of the base is also equal.

This might lead you to wonder whether the proof holds for a general triangle. Indeed, we will show later on that the sides and bases are in proportion for any triangle, if the angle is bisected.

In the formula from above

$$\frac{a+b}{c+d} = \frac{b}{d} = \frac{a}{c}$$
$$= \frac{a}{c+d} + \frac{b}{c+d}$$

we have a relationship between sides of the right triangle with angle 2θ and the right triangle with angle θ. Archimedes uses this in his method to place bounds on the value of π.

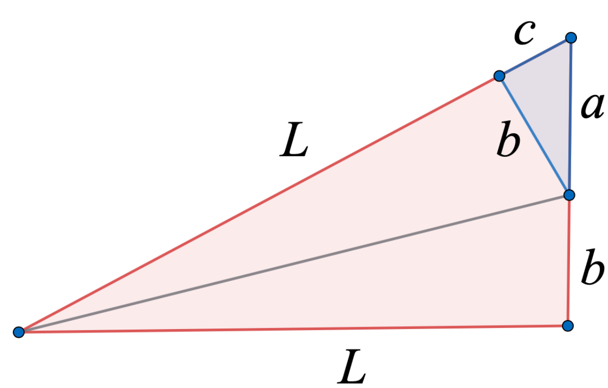
In trigonometry, the relationship is written:

$$\frac{a}{c} = \cot\theta = \frac{a}{c+d} + \frac{b}{c+d}$$
$$= \cot 2\theta + \csc 2\theta$$

angle bisector proof of Pythagoras

The same diagram (relabeled) gives an easy proof of the Pythagorean theorem (from Dunham's Problems).

Proof.



From similar triangles:

$$\frac{L}{a+b} = \frac{b}{c}$$
$$L = \frac{b(a+b)}{c}$$

Figure the area as twice the value. For the big triangle we have

$$2A = (a+b)L$$
$$= \frac{b(a+b)^2}{c}$$

From the three separate triangles:

$$2A = bc + 2bL$$
$$= bc + 2b^2\frac{a+b}{c}$$

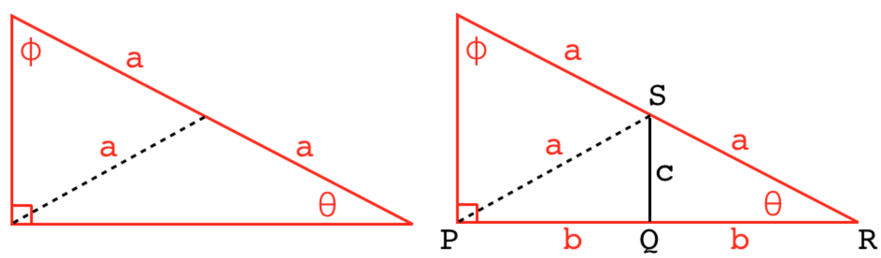
Equate them and do some algebra:

$$\frac{b(a+b)^2}{c} = bc + 2b^2\frac{a+b}{c}$$

$$(a+b)^2 = c^2 + 2b(a+b)$$
$$a^2 + 2ab + b^2 = c^2 + 2ab + 2b^2$$
$$a^2 = b^2 + c^2$$

□

midpoint theorem



In a right triangle, draw the line segment from the vertex that contains a right angle to the midpoint of the hypotenuse, separating it into two equal lengths a. We will show that the length of the bisector is also a.

We gave a proof of this earlier.

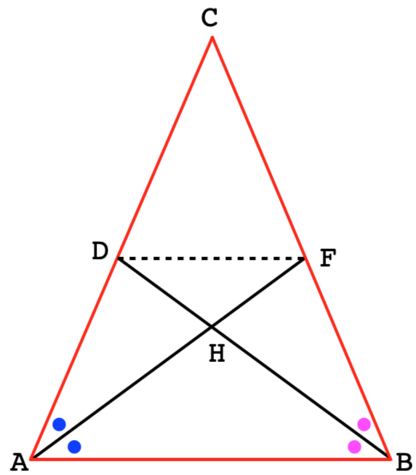
An elegant proof is based on Thales' circle theorem. Let the hypotenuse be the diameter of a circle.

Then, if the large triangle is a right triangle, the right angle lies on the circle and then PS is a radius.

Alternatively if the median is a radius of the circle, then P is on the circle. By the converse of Thales' theorem, it follows that the angle at P is a right angle.

isosceles angle bisector theorem

The next theorem involves angle bisectors in an isosceles triangle. It is easy in the forward direction, but the converse is very challenging, at least until you draw the right diagram. Then, as usual, it's not so bad.



We are given that $\triangle ABC$ is isosceles ($AC = BC$), and also that the angles at the base are both bisected.

We claim that the angle bisectors are equal in length: $AF = BD$.

Proof.

By the forward version of the isosceles triangle theorem, the entire $\angle A = \angle B$, so it follows that all four angles dotted blue and magenta are equal.

Then $\triangle ABD \cong \triangle BAF$ by ASA.

As a result, $AF = BD$ and $AD = BF$.

Furthermore, since the original triangle is isosceles and $AD = BF$, the smaller triangle $\triangle CDF$ is also isosceles, by subtraction. Alternatively, $\angle C$ is shared, and the long sides are equal so $\triangle CDB \cong \triangle CFA$ by AAS.

It follows that $CD = CF$ and $\triangle CDF$ is isosceles, and therefore, $\triangle CDF \sim \triangle ABC$ by AAA.

By alternate interior angles, $DF \parallel AB$.

$\square$

That's the easy part.

The converse theorem says that if we have angle bisectors and they are equal in length, then the triangle is isosceles. This is called the Steiner-Lehmus Theorem.

https://en.wikipedia.org/wiki/Steiner\OT1\textendashLehmus_theorem

We defer discussion of the **Steiner-Lehmus Theorem** to its own chapter.

angle bisector theorem revisited

Although we used right triangles in our first proof of the angle bisector theorem, that wasn't strictly necessary. The theorem is also true for the general case.

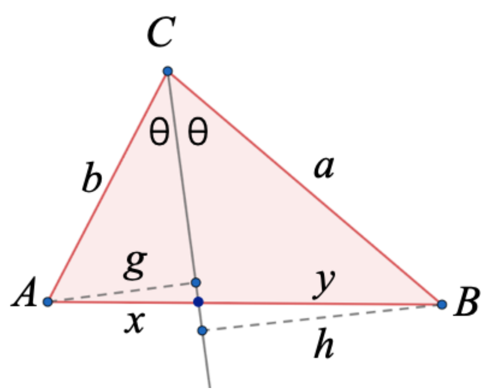
We discuss several different proofs. The first two draw right triangles involving the bisector, or part of it.

similar right triangles

Proof.

In $\triangle ABC$ draw the bisector of $\angle C$ to join side AB, dividing it into lengths x and y.

We draw verticals from A and B to the angle bisector.



This forms two pairs of similar right triangles. The first pair has complementary angle θ

$$\frac{b}{a} = \frac{g}{h}$$

And the second pair have equal vertical angles at the bisector.

$$\frac{x}{y} = \frac{g}{h}$$

The result follows easily.

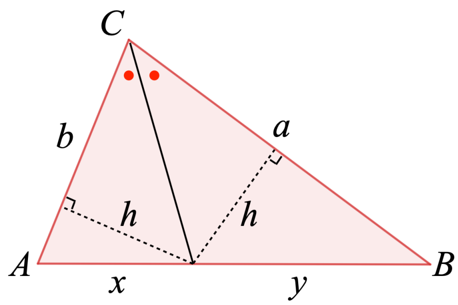
$$\frac{b}{a} = \frac{x}{y}$$

□

The result we obtained previously is a special case where angle A is a right angle, but the theorem is true generally.

Alternatively, use the notion of area. Draw verticals to the sides as shown.

We have $\triangle ABC$ with sides a, b, c and the angle C is bisected so the two angles marked with red dots are equal.



Start from the point where the bisector meets the side opposite $\angle C$, cutting c into x and y. Drop the perpendiculars to the sides a and b.

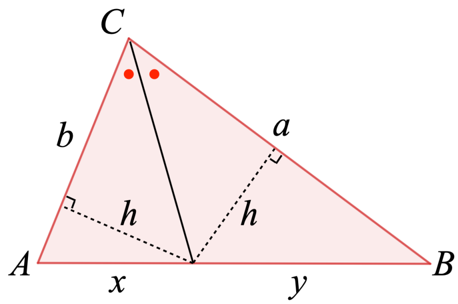
This forms two congruent right triangles, since the angles at the top are equal, and the hypotenuse is shared. The equal sides are marked as h.

We can compute the ratio of the areas of the left and right-hand sub-triangles in two ways.

The first way uses the area-ratio theorem.

$$\frac{A_L}{A_R} = \frac{x}{y}$$

The two triangles share a common altitude and the area is one-half the base times the altitude. Since the altitude is shared, it cancels, together with the factor of one-half.



The second approach is to use the sides a and b as the two bases. We have that the heights h are equal. Hence

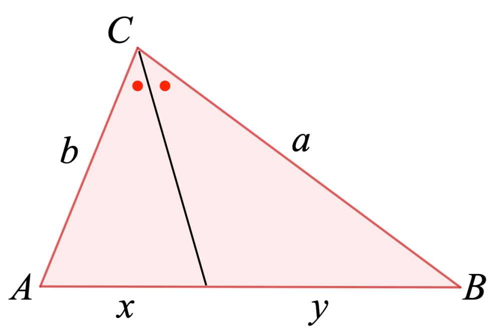
$$\frac{A_L}{A_R} = \frac{b}{a}$$

Equating the two results we have that

$$\frac{x}{y} = \frac{b}{a}$$

similar triangles

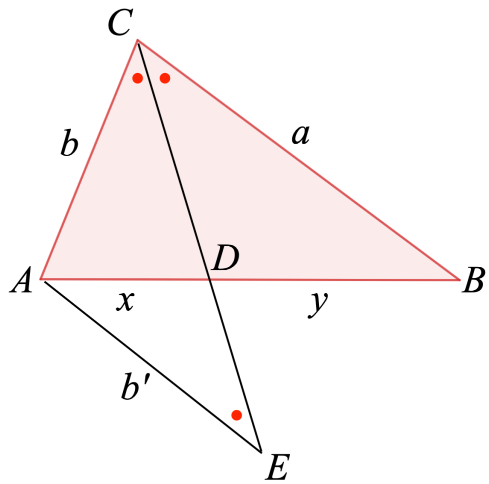
Here are two other proofs of the general angle bisector theorem by similar triangles. These involve constructions that either extend the bisector to a line parallel to one side, or extend one side to meet a line parallel to the bisector.



Given that the angle at C is bisected. The claim is that $a/y = b/x$.

Proof.

Draw AE parallel to BC and extend the angle bisector to meet it at E. Then the angle at E is equal to the half-angle, by alternate interior angles,



We have that $\triangle ACE$ is isosceles, so $b = b'$.

We also have two similar triangles with $\triangle AED \sim \triangle BCD$, since the angles at D are equal by vertical angles.

Form the ratios of the sides opposite vertical angles to the sides opposite the angles marked with red dots:

$$\frac{a}{y} = \frac{b'}{x}$$

But since $b = b'$:

$$\frac{a}{y} = \frac{b}{x}$$

□

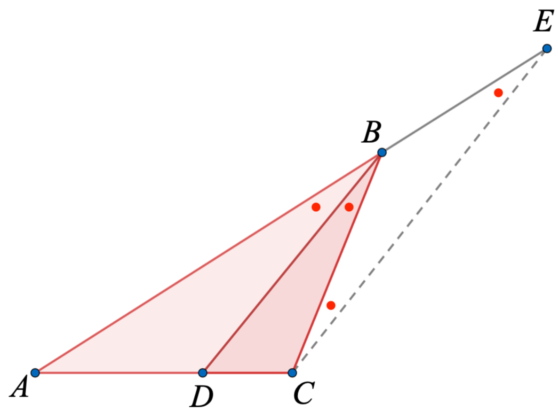
Yiu proof

Here is another proof presented by Paul Yiu, together with a problem. It includes the bisector of the external angle as well.

The first part uses an extended line parallel to the bisector. (This is basically the same as Euclid VI.3).

In $\triangle ABC$ let BD bisect $\angle ABC$. Then

$$\frac{AB}{AD} = \frac{BC}{CD}$$



Proof.

Draw $CE \parallel BD$ and extend AB to meet it at E.

$\angle BEC = \angle BCE = \angle ABD = \angle CBD$. The last one is given and the others are by alternate interior angles.

$\triangle ABD \sim \triangle AEC$, since $\angle A$ is shared.

Thus

$$\frac{AB}{AD} = \frac{BE}{CD}$$

But $\triangle BCE$ is isosceles, with $BE = BC$.

It follows that

$$\frac{AB}{AD} = \frac{BC}{CD}$$

□

converse

Proof. (Sketch).

Extend AB so that $BE = BC$.

Follow the same steps as before, but in reverse order.

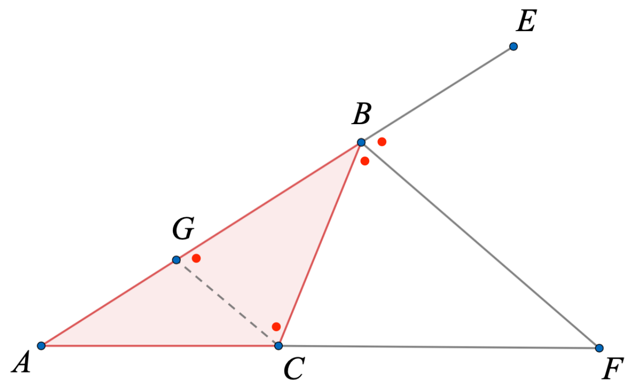
It follows that $\angle ABC$ is bisected by BD.

□

external angle proof

Let $\angle CBE$ be an external angle for $\triangle ABC$, and BF bisect it. Then

$$\frac{AB}{BC} = \frac{AF}{CF}$$



Proof.

Draw $GC \parallel BF$ (parallel to the bisector but *internal* to the triangle).

$\angle BGC = \angle BCG = \angle CBF = \angle EBF$ by alternate interior angles.

$\triangle ABF \sim \triangle AGC$.

Thus

$$\frac{AB}{AF} = \frac{BG}{CF}$$

But $\triangle BGC$ is isosceles, with $BG = BC$.

It follows that

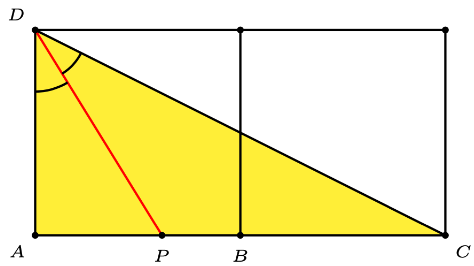
$$\frac{AB}{AF} = \frac{BC}{CF}$$

Rearranging

$$\frac{AB}{BC} = \frac{AF}{CF}$$

□

Then to the problem: find the ratio AP/PB.



Pythagoras gives $DC = \sqrt{5}$ and this theorem says that PC and AP are in the same ratio as the flanking sides.

Scale the problem so that $AD = 1$, $AC = 2$ and $AP = x$ and

$$\frac{PC}{AP} = \frac{2-x}{x} = \sqrt{5}$$

$$\frac{1}{x} = \frac{1+\sqrt{5}}{2}$$

Do not solve for x yet. $PB = 1 - x$ so the inverse of the desired ratio is

$$\frac{PB}{AP} = \frac{1-x}{x} = \frac{1}{x} - 1$$

$$= \frac{1+\sqrt{5}}{2} - 1 = \phi - 1$$

$$= \frac{\sqrt{5}-1}{2}$$

The inverse is

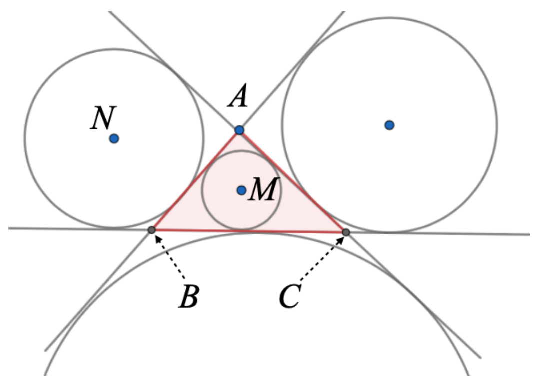
$$\frac{AP}{PB} = \frac{2}{\sqrt{5}-1}$$

Chapter 32

Excircles and Heron

This chapter is an introduction to *excircles*, which are so named by reference to *incircles*. The incircle is tangent to all three sides of a triangle and lies inside. An excircle is tangent to one side and also tangent to extensions of the other two sides.

In $\triangle ABC$, with opposite sides a, b and c, the incircle on center M is that circle to which all three sides are tangent.



The circle on center N is tangent to side c opposite $\angle C$, and also tangent to the extensions of sides a and b.

There is one excircle for each side of the original triangle. We show how to construct an excircle below.

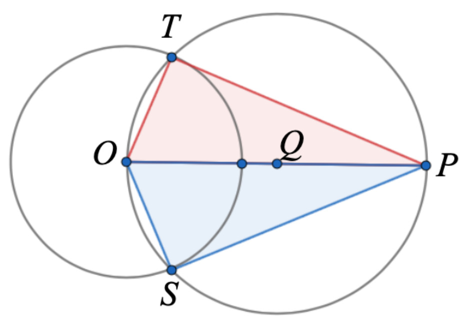
We can use this construction to simplify derivation of Heron's famous formula, which relates the area of any triangle to one-half of its perimeter, the *semi-perimeter* s, and the three side lengths:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

We develop a simple proof of Heron's theorem in this chapter. It utilizes similar triangles and a construction with an excircle. An algebraic excursion is delayed until **later**, when we also look at Brahmagupta. At the end of the chapter we also look at Heron's original proof.

tangents and incircles, again

First let's review how to draw the tangents from any point P to a circle on center O. Note that P must lie outside the circle. Draw the line OP and bisect it at Q. Then draw the circle on center Q with radius $PQ = OQ$. Find where that circle intersects the first one.

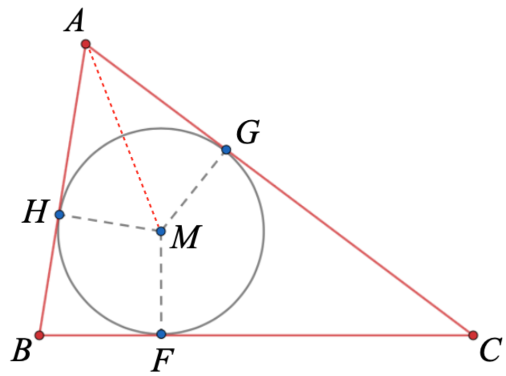


Since OP is a diameter of circle Q, by Thales' theorem, $\angle PTO$ is right, so $PT \perp$ the radius OT of circle O and by definition then, PT is a tangent to the circle on center O, through T. The other intersection at S forms the other tangent PS.

Since $\triangle OTP$ and $\triangle OSP$ are both right triangles and share the same hypotenuse, as well as bases equal to the radius of circle O, the two triangles are congruent by HL. Thus the two tangents from P to circle O are equal. In addition, the congruent triangles mean that the line OP is the bisector of $\angle TPS$.

Conversely, if we bisect an angle and then draw perpendiculars from any point on the bisector, the two triangles are congruent.

Now, the *incircle* of a triangle is contained within and just touches (is tangent to) each of the three sides. It can be constructed by bisecting the angle at each vertex, and finding the point M where the bisectors meet. M is then the center of the incircle.



Proof. The proof just uses what we established above. Let the triangle be $\triangle ABC$ with side a opposite vertex A, etc., as usual, and let AM be the bisector of $\angle A$, BM the bisector of $\angle B$, and so on. Drop perpendiculars from M to the sides at F, G and H.

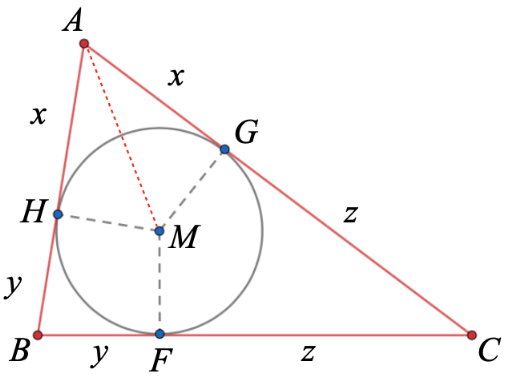
Above we showed that each pair of perpendiculars from an angle bisector forms two congruent right triangles, by hypotenuse-leg in a right triangle (HL), so, for example, $\triangle AMG \cong \triangle AMH$. Thus, the two perpendiculars from any point on the bisector to the rays of the bisected angle are equal (**here**).

But we can do the same for any pair of sides. Therefore all three distances are equal, and we can then draw the circle, called the incircle, with that distance as the radius r. $MF = MG = MH$.

$\square$

Note: the half-angles formed by bisection (not labeled here) are named for the parent angle: $\alpha + \alpha = \angle A$ and $\angle \alpha = \angle MAH$.

Before moving on to excircles, we do a little arithmetic on the incircle. Let the two congruent right triangles containing the half-angle at vertex A have base $AG = AH = x$, and the two at B base y and the two at C base z.



The perimeter is $p = 2x + 2y + 2z$, and so the *semi-perimeter* s is

$$s = \frac{a+b+c}{2} = x + y + z$$

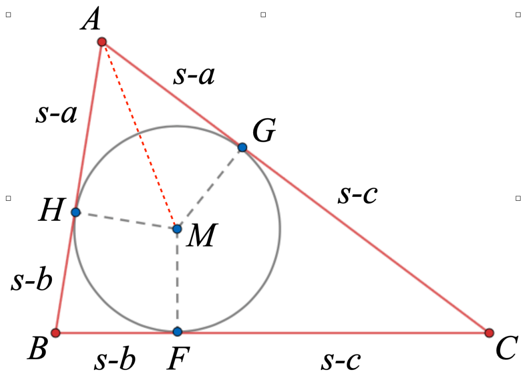
Side a lies opposite $\angle A$ (i.e. a is the same as BC) so $a = y + z$, using the equation above we can write

$$s = x + a$$

from which

$$x = s - a$$

and this explains the labels in the figure below.



We can check this. From the diagram:

$$(s-a) + (s-c) = 2s - a - c = b$$

In addition, the area of the whole triangle is

$$A = 2 \cdot (\frac{xr}{2} + \frac{yr}{2} + \frac{zr}{2})$$
$$= r(x + y + z) = rs$$

A commonly used notation for area is $\mathcal{A} = (ABC)$, so we can rewrite this as

$$\mathcal{A} = (ABC) = rs$$

We will see terms like $s - a$, $s - b$, etc. below. We note that

$$s - a = \frac{a + b + c}{2} - a = \frac{-a + b + c}{2}$$

Subtract side a from s on the left and change the sign on a in the formula for the semi-perimeter.

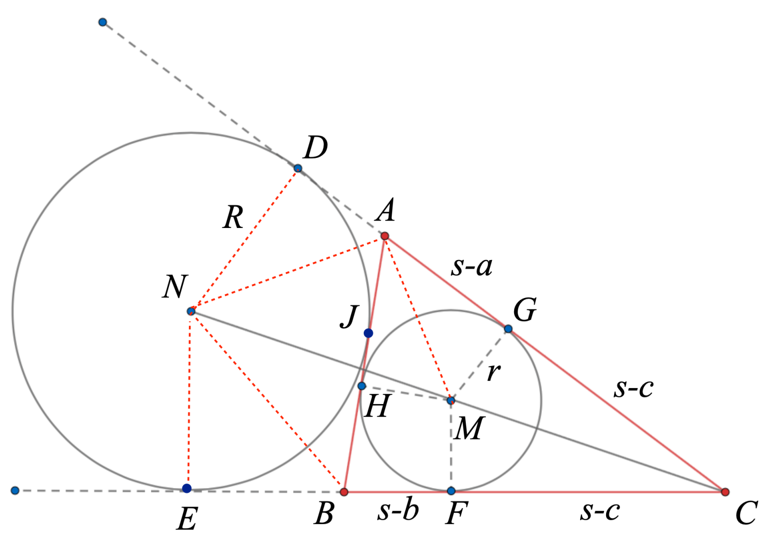
Another useful result that we will come back for later is

$$(s - a) + (s - b) + (s - c) = 3s - (a + b + c)$$
$$3s - 2s = s$$

problem

Show that the incircle for a $3 - 4 - 5$ right triangle has a radius equal to 1.

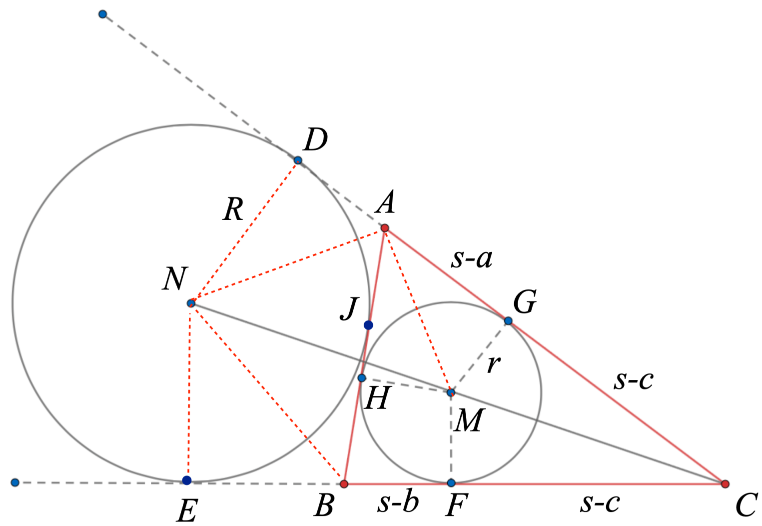
excircles



Any triangle has a single incircle and three excircles, one for each side of the triangle. The excircle on side c (opposite $\angle C$) is formed by extending sides a and b (i.e. BC and AC above) and then finding the circle that is tangent to those two extensions and also tangent to side c.

Let N be the center of the excircle on side c. N lies on the bisector of the original $\angle C$. This follows from the definition that CD and CE are tangent to this excircle.

Since AD and AJ are also tangents to the excircle from point A, if the angle supplementary to $\angle A$ (i.e. $\angle BAD$) is bisected, then the bisector also runs through N, so N can be constructed by finding the intersection of the bisectors of $\angle DAB$ and $\angle C$. The bisector of $\angle ABE$ goes through N as well.



Construct the perpendicular through N from the extension of AC (or of BC) meeting the extension at D (or E) to find the radius of the excircle R.

$R = ND = NE = NJ$. Since there is one excircle for each side, we may introduce a subscript like R_c.

We can see that $ND \perp CGD$ and also $MG \perp CGD$. It follows that $\triangle MGC \sim \triangle NDC$ which gives:

$$\frac{R_c}{CD} = \frac{r}{CG} = \frac{r}{s-c}$$

Rearranging

$$R_c \cdot (s-c) = r \cdot CD$$

It will turn out (below) that the length CD is equal to the semiperimeter s. So going back to our previous result, the product above is equal to the area of the triangle:

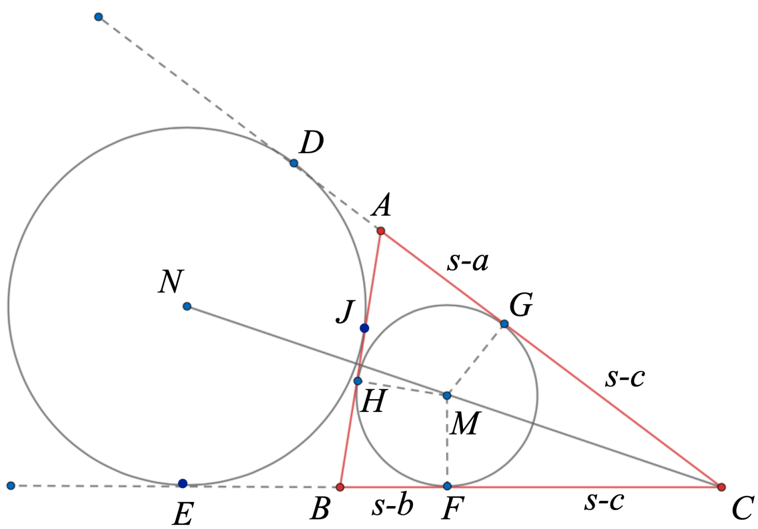
$$\mathcal{A}_{\triangle ABC} = rs = R_c \cdot (s - c)$$

This can be done for any side so

$$R_a \cdot (s-a) = R_b \cdot (s-b) = R_c \cdot (s-c)$$

tangent length

We can show that the length of the tangents $CD = CE$ is just s.



Proof.

Let $AD = AJ = x$ and $BE = BJ = y$.

We have two unknowns and two equations.

$$AJ + BJ = x + y = c$$

and since the two tangents from any point are equal:

$$x + (s-a) + (s-c) = y + (s-b) + (s-c)$$

$$x - a = y - b$$

$$= c - x - b$$

Rearranging

$$2x = a - b + c$$

Recall that (for example)

$$2(s - b) = a - b + c$$

Hence

$$2x = 2(s - b)$$

$$x = s - b$$

It follows that $y = s - a$ and

$$CD = CE = (s - a) + (s - b) + (s - c)$$

$$= s$$

Here is a slightly different and perhaps even simpler approach. Let the length of each tangent from C be t. Then

$$2t = CD + CE$$

$$= x + (s - a) + (s - c) + y + (s - b) + (s - c)$$

$$= c + (s - a) + (s - c) + (s - b) + (s - c)$$

$$= (s - a) + (s - c) + (s - b) + s$$

$$= 4s - 2s = 2s$$

Hence $t = s$ and the length of the entire tangent is just s.

□

To summarize, the lengths of all these tangents, 3 pairs to the excircle and 3 pairs to the incircle, are particularly simple.

The points of tangency J and H divide the side c into the same two lengths:

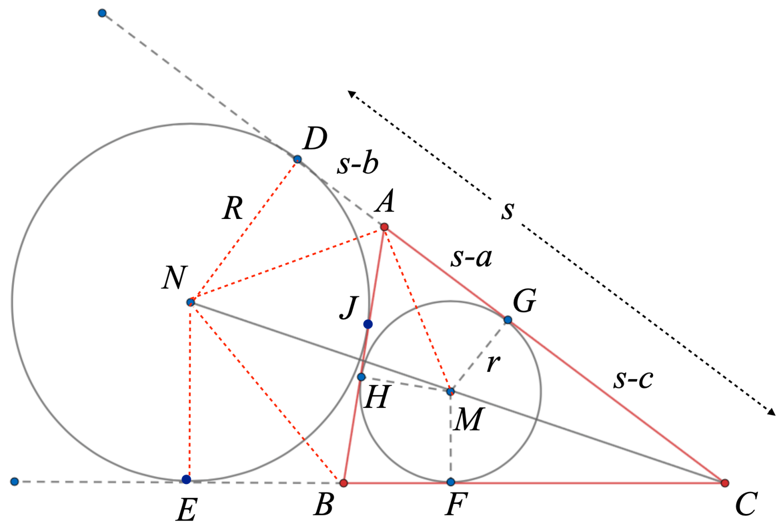
$$AD = AJ = BH = s - b$$

and

$$BE = BJ = AH = s - a$$

Heron's formula

Since $ND \perp CD$ and $MG \perp CG$, we have a pair of similar right triangles, namely one with sides R and s, and the other with sides r and $s - c$.



The proportion is:

$$\frac{r}{R} = \frac{s-c}{s}$$

which justifies what we wrote previously:

$$rs = R(s-c)$$

This leads to a beautiful, simple proof of Heron's formula.

The angle between two tangents from the same point is bisected by the line through the center of the circle. Thus $\angle DAB$ is bisected by AN and also $\angle BAC$ is bisected by AM.

It follows that $\angle NAM$ is a right angle.

So $\angle DAN$ is complementary to $\angle MAG$, since their sum is one right angle, and therefore, $\triangle ADN \sim \triangle MGA$.

The ratios of interest are:

$$\frac{s-b}{R} = \frac{r}{s-a}$$

$$rR = (s-a)(s-b)$$

But

$$rs = R(s-c)$$

Solve both equations for R and set them equal:

$$\frac{(s-a)(s-b)}{r} = \frac{rs}{s-c}$$

Thus

$$r^2 s = (s-a)(s-b)(s-c)$$

and

$$(rs)^2 = s(s-a)(s-b)(s-c)$$

But rs is the area of the triangle ABC.

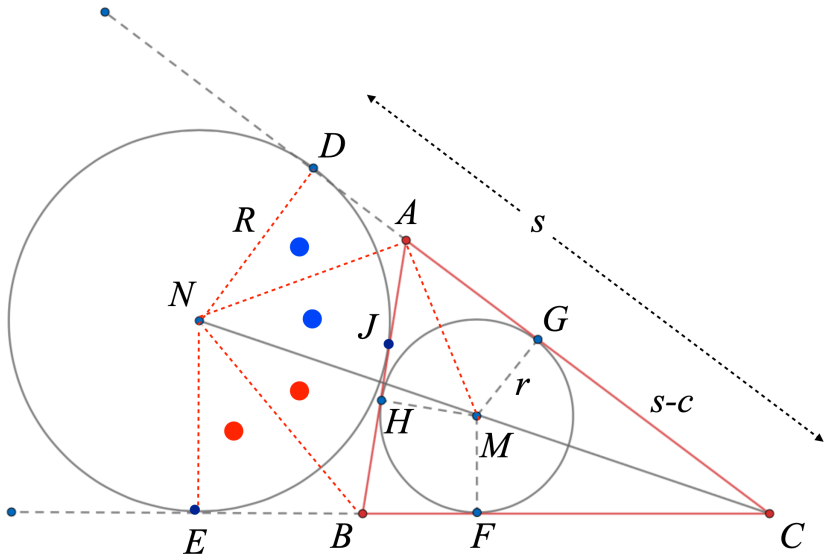
$$\mathcal{A}^2 = s(s-a)(s-b)(s-c)$$
$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

□

This is Heron's formula. Very elegant.

The formula can also be obtained by computing areas in different ways, as described in the link above. Here is an idea of the proof.

Proof.



Again using parentheses to signify area, we have that the polygon $ADNEB$ consists of two pairs of equal triangles, so that its area is twice that of $\triangle ANB$:

$$(ADNEB) = 2(\triangle ANB) = Rc$$

The area of quadrilateral $CDNE$ is twice that of $\triangle CDN$:

$$(CDNE) = 2(\triangle CDN) = Rs$$

The area of $\triangle ABC$ is simply the difference:

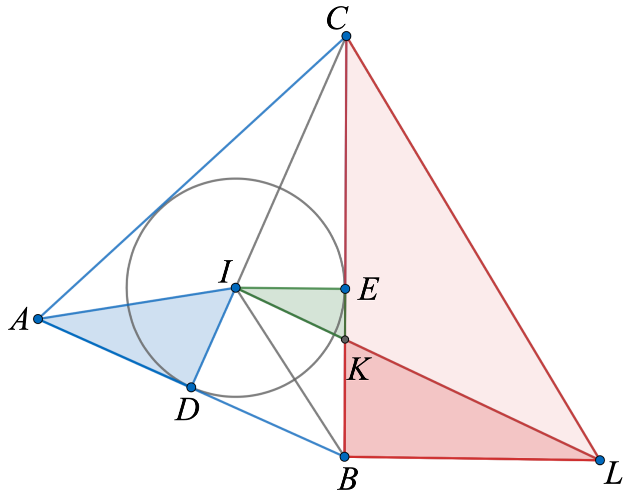
$$\begin{aligned}(\triangle ABC) &= (CDNE) - (ADNEB) \\ &= Rs - Rc = R(s - c)\end{aligned}$$

This is the same relationship we obtained by determining the total length of the tangent $CD = s$ and then noting similar triangles $\triangle ADN \sim \triangle MGA$.

Heron's proof

Lastly, we go through Heron's proof of the eponymous theorem. I obtained this from a web page written by Dr. Paul Yiu, which has disappeared. There is also a nice discussion in Dunham.

Let us start with a sketch of the proof.

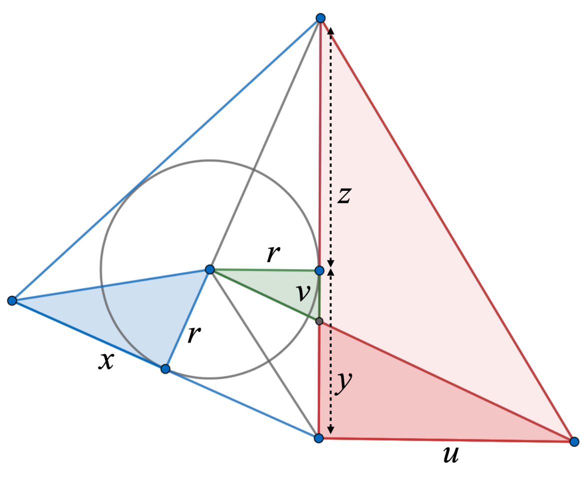


We have $\triangle ABC$ with its incircle on center I and perpendiculars drawn to the tangent points D and E. Extend a line from B perpendicular to CB, forming a right angle $\angle CBL$. Also extend a line from I perpendicular to CI, forming a right angle $\angle CIL$.

We we will be able to find two pairs of similar triangles: easily, the dark red $\triangle KBL \sim \triangle KEI$, and in a more complicated fashion, the whole red $\triangle LBC \sim \triangle IDA$, in blue. We first proceed assuming that has been done.

I find it much more convenient to write the proof using single letters for the lengths, so these have been labeled as shown below.

The similar triangles give us two relationships:



$$\frac{u}{r} = \frac{y - v}{v}$$
$$\frac{u}{r} = \frac{y + z}{x}$$

Setting equals to equals:

$$\frac{y - v}{v} = \frac{y + z}{x}$$

After that it is just a matter of algebra. That is harder for Heron (actually the proof is probably from Archimedes), but relatively easy for us with our improved notation.

We simply add 1 to both sides. This is the step where Heron needs to extension $BT = x$ since $x + y + z$ must equal a straight line segment in the diagram (see below).

$$\frac{y - v}{v} + \frac{v}{v} = \frac{y + z}{x} + \frac{x}{x}$$
$$\frac{y}{v} = \frac{x + y + z}{x}$$

The ratio of the whole semi-perimeter (CT below) to x (BT below) is equal to the ratio of BE to KE.

$$\frac{y}{v} = \frac{s}{x}$$
$$xy = vs$$

The last step is to involve z, and also somehow eliminate v.

We notice that in the right $\triangle CIK$, the radius of the incircle r divides the hypotenuse into two lengths z and v. By the standard proof of the geometric mean, we have

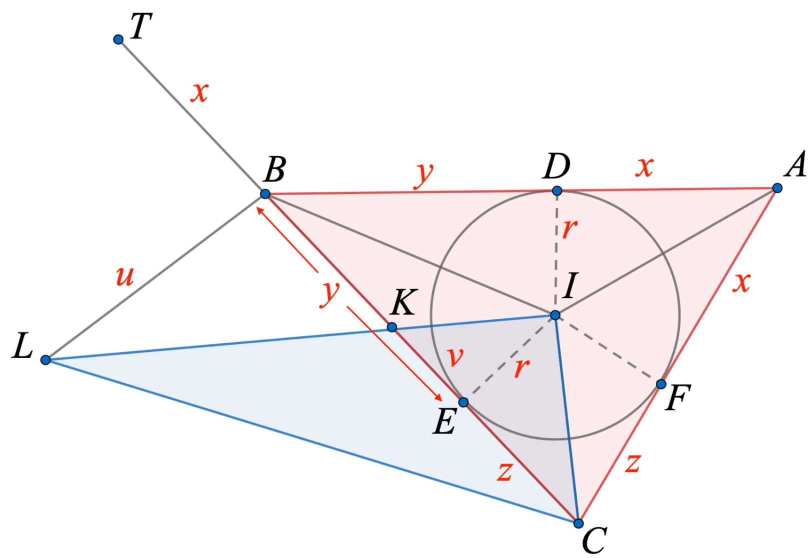
$$r^2 = vz$$

so

$$\frac{r^2}{z} = v = \frac{xy}{s}$$
$$r^2 s = xyz$$
$$(rs)^2 = xyzs$$

This is the famous formula in a simple form. Now we proceed more formally to establish the similarity relations.

The triangle is $\triangle ABC$ with incircle radii ID, IE and IF. The tangents are drawn as x, y, and z.



There are two parts to the construction. Most important, $\angle LIC$ and $\angle LBC$ are both drawn as right angles. Also, BT is drawn equal in length to $AD = x$, so that the whole length CT is equal to the semi-perimeter, $s = x + y + z$. We do not need this second part, but it makes Heron's task simpler.

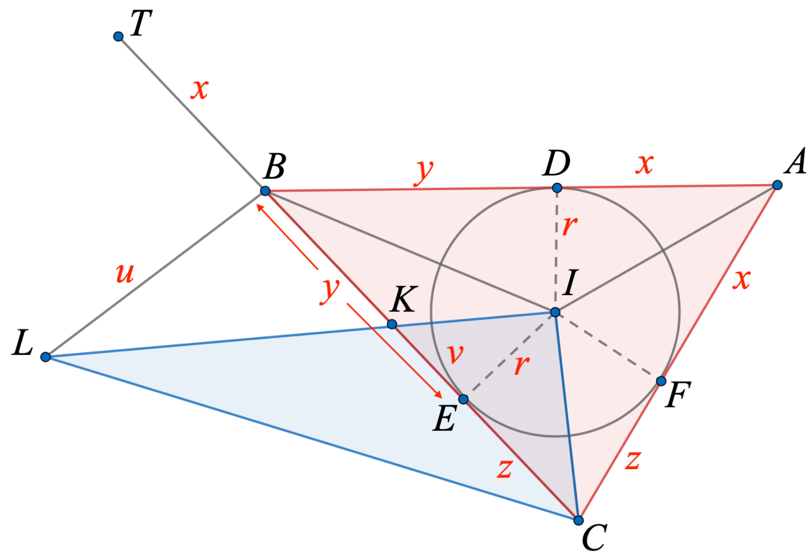
Proof.

part 1: $\triangle CBL \sim \triangle AID$.

By construction, $\triangle CBL$ and $\triangle CIL$ are both right triangles with the same hypotenuse CL.

By the converse of Thales' theorem, B and I lie on the same circle, with diameter CL. It follows that $BICL$ is a cyclic quadrilateral and therefore $\angle L$ is supplementary to $\angle BIC$.

But the latter is supplementary to $\beta + \gamma$ in $\triangle BIC$ (using our standard notation for the half-angles at B and C). Hence $\angle L = \beta + \gamma$.



(This is even easier to see if we draw the circle on diameter CL: the arc corresponding to $\angle L$ is divided between β and γ).

We also have that the sum of the half-angles $\alpha + \beta + \gamma = 90$. Thus, $\angle AID$, which is complementary to α in the right triangle $\triangle AID$, is also equal to $\beta + \gamma$.

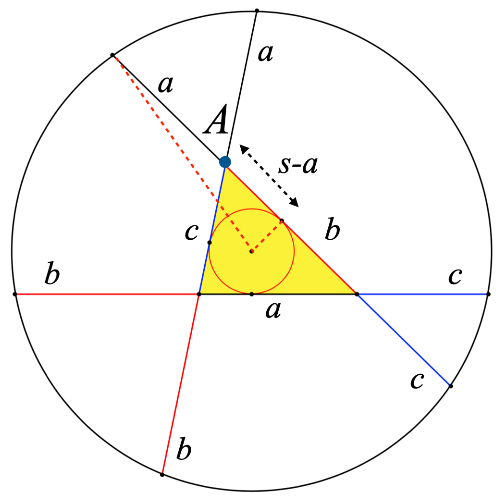
It follows that $\angle L = \angle AID$, and then the two right triangles are similar: $\triangle CBL \sim \triangle AID$.

part 2 $\triangle LBK \sim \triangle IEK$

This is easier. They are both right triangles and share vertical angles.

$\square$

One last construct



Extend the sides of $\triangle ABC$ a distance a past vertex A and so on. Then one can draw a circle whose center is also the incenter of the triangle, that passes through the ends of all of the extended line segments.

Proof.

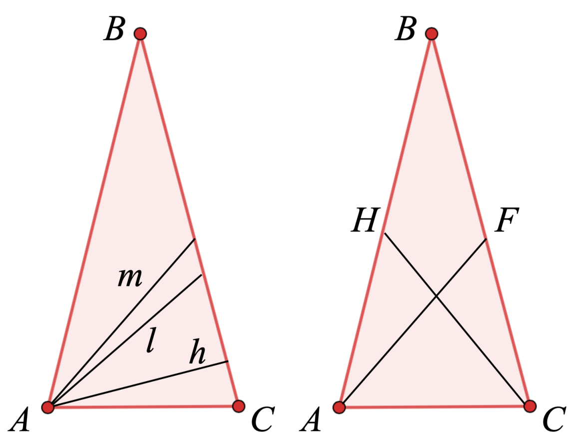
Draw the right triangle with one side equal to r, the radius of the incircle, and the other side extending side b through vertex A, as shown. The second side has length $s - a + a = s$.

Therefore the hypotenuse of every such right triangle is the same, and this is the radius of the large circle.

□

Chapter 33

More isosceles



The following statements about isosceles triangles are equivalent:

1. two sides equal
2. base angles equal
3. medians (m) equal
4. angle bisectors (l) equal
5. altitudes (h) equal
6. the bisector, median and altitude at b coincide

where in (1-5) the medians, bisectors and altitudes in question are those extending to the equal sides.

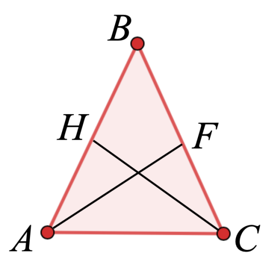
The figure shows (right panel) the case for the medians, but we will use the same labels for the other cases in the discussion below.

The question is then how to prove these one from the other, for example 1 $\Rightarrow$ 3 or 5 $\Rightarrow$ 2. Of course, we already have 1 $\Rightarrow$ 2 and 2 $\Rightarrow$ 1.

In what we might call the forward direction { 1,2 } $\Rightarrow$ { 3,4,5,6 }, these proofs are quite easy. I encourage you to try them before reading further.

The converse proofs are another matter.

forward



1 $\Rightarrow$ 3

Proof. AF and CH are medians in $\triangle ABC$. Given $AB = CB$ and $\angle A = \angle C$, it follows that $AH = CF$ (by the definition of median) and thus by SAS we have $\triangle ACH \cong \triangle CAF$. Hence $AF = CH$. $\square$

1 $\Rightarrow$ 4

Proof. AF and CH are angle bisectors in $\triangle ABC$. Given $\angle A = \angle C$, the half-angles are also equal, so by ASA we have $\triangle ACH \cong \triangle CAF$. Hence $AF = CH$. $\square$

1 $\Rightarrow$ 5

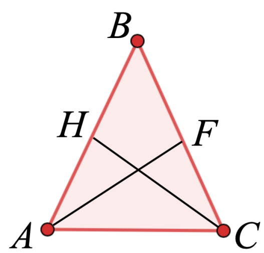
Proof. AF and CH are altitudes in $\triangle ABC$. Twice the area of $\triangle ABC$ may be computed as $AF \cdot CB = CH \cdot AB$. But $AB = CB$. Hence $AF = CH$. $\square$

converse

One of the converse proofs is also easy.

$5 \Rightarrow 1$

Proof. AF and CH are altitudes in $\triangle ABC$. Twice the area of $\triangle ABC$ may be computed as $AF \cdot CB = CH \cdot AB$. But $AF = CH$. Hence $AB = CB$. $\square$



$4 \Rightarrow 1$

If the angle bisectors are equal, then the triangle is isosceles. This is a famous theorem, and there is a whole chapter about it here: **Steiner-Lehmus theorem**.

$3 \Rightarrow 1$

For this one, we look ahead to the **Law of cosines**.

Proof.

We have that the medians are equal, namely, $AF = CH$. Let them be equal to m. Using the law of cosines, compute the length m squared of the side opposite $\angle B$ in two different triangles.

Let side $a = BC$ (the side opposite $\angle A$), and similarly side $c = AB$ (the side opposite $\angle C$).

So $BF = a/2$ and $BH = c/2$. Then

$$m^2 = a^2 + (c/2)^2 - 2a \cdot c/2 \cdot \cos B$$

$$m^2 = c^2 + (a/2)^2 - 2c \cdot a/2 \cdot \cos B$$

The last terms on the right-hand side are equal, namely

$$ac \cdot \cos B = ac \cdot \cos B$$

Setting the two expressions equal, it follows that

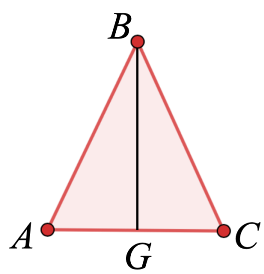
$$a^2 + (c/2)^2 = c^2 + (a/2)^2$$

$$\frac{3}{4}a^2 = \frac{3}{4}c^2$$
$$a^2 = c^2$$

Since a and c are both lengths, we may take the positive square roots for each, and thus $a = c$.

□

median, bisector and altitude at b



Let b be the side opposite $\angle B$ as usual.

1 ⇒ 6

We have that $AB = CB$ and $\angle A = \angle C$.

Proof. Let BG be the median to AC such that $AG = GC$ Then $\triangle ABG \cong \triangle CBG$ by SSS. Thus $\angle B$ is bisected and $\angle AGB = \angle CGB$ and both are right angles. □

Proof. Let BG be the altitude to AC such that $\angle AGB = \angle CGB$ and both are right angles. Then $\triangle ABG \cong \triangle CBG$ by HL. Thus $\angle B$ is bisected and $AG = GC$. □

Proof. Let BG bisect $\angle B$. Then $\triangle ABG \cong \triangle CBG$ by SAS. Thus $\angle AGB = \angle CGB$ and both are right angles, and $AG = GC$. □

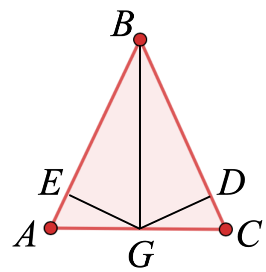
6 ⇒ 1

We will prove that if any two of these statements are both true, then $a = c$. In what follows, if we have $\triangle ABG \cong \triangle CBG$, then (1) and (2) follow immediately.

Proof. If the bisector and the altitude at b coincide, then $\triangle ABG \cong \triangle CBG$ by ASA. □

Proof. If the median and the altitude at b coincide: then $\triangle ABG \cong \triangle CBG$ by SAS.□

Finally, suppose the bisector and the median at b coincide. Draw perpendiculars from the midpoint of b to each of sides a and c.



Proof.

BG bisects $\angle B$, $\angle GDB = \angle GEB$ and both are right angles, therefore $\angle BGE = \angle BGD$ by sum of angles. The hypotenuse BG is shared. It follows that $\triangle BEG \cong \triangle BDG$ by ASA. Thus $GE = GD$.

Then (since $AG = GC$), $\triangle AEG \cong \triangle CDG$ by HL. It follows that $\angle A = \angle C$ and then by $2 \Rightarrow 1$, we have $a = c$.

□

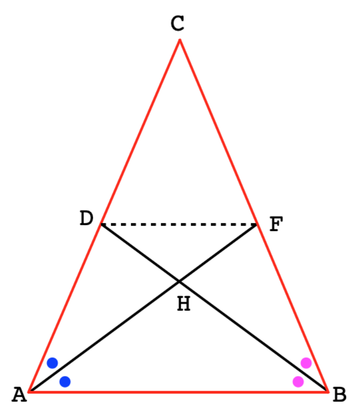
This material is covered in Byer (see **References**).

Chapter 34

Steiner Lehmus

This chapter discusses a theorem about angle bisectors in an isosceles triangle. We called the forward version the **isosceles bisector** theorem.

It is easy in the forward direction, but the converse is very challenging, at least until you draw the right diagram. Then, as usual, it's not so bad.



Proof. We are given that $\triangle ABC$ is isosceles ($AC = BC$), and also that the angles at the base are both bisected. It follows that the half-angles are also equal, and thus $\triangle CDB \cong \triangle CFA$ by ASA. So the angle bisectors are equal in length: $AF = BD$.

□

That's the easy part.

Steiner-Lehmus Theorem

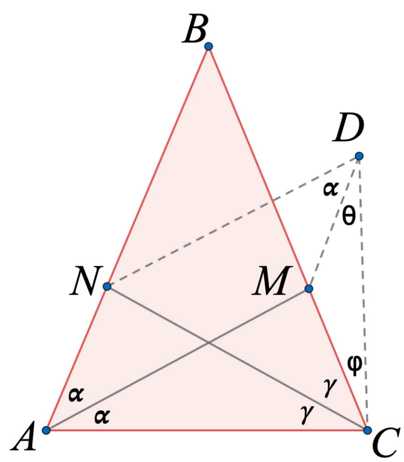
The converse theorem says that if we have angle bisectors and they are equal in length, then the triangle is isosceles.

https://en.wikipedia.org/wiki/Steiner\OT1\textendashLehmus_theorem

The problem is that, even though we can draw triangles with two sides equal, we don't know anything about the angles except for some vertical angles, which don't help.

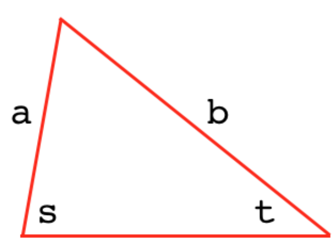
Here is an approach which I found on the web. It's a proof by contradiction.

https://www.algebra.com/algebra/homework/word/geometry/Angle-bisectors-in-an-isosceles-triangle.lesson



We claim that if $AM = CN$ and the angles are bisected, then $\alpha = \gamma$.

We rely on Euclid's propositions I.18 and I.19. In any triangle if one side is larger than another, then the angle opposite the longer side is greater (I.18) and conversely, if one angle is larger than another, then the side opposite is greater (I.19).



In the diagram above

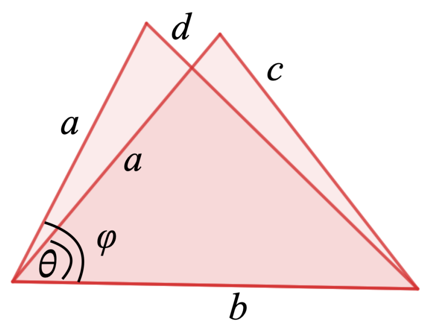
$$s > t \Rightarrow b > a$$

$$b > a \Rightarrow s > t$$

We proved both these theorems **here**.

The problem is that these are *within-triangle* results. We also need the following result.

If two triangles $\triangle ABC$ and $\triangle DEF$ have two pairs of sides equal, and the included angle is greater in one ($\phi > \theta$), then the side opposite ϕ also greater.



Proof.

Let d be opposite ϕ and c be opposite θ. Use the law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$d^2 = a^2 + b^2 - 2ab\cos\phi$$

Then if $d > c$, so $d^2 > c^2$, and

$$a^2 + b^2 - 2ab\cos\phi > a^2 + b^2 - 2ab\cos\theta$$

$$-2ab\cos\phi > -2ab\cos\theta$$

$$\cos\phi < \cos\theta$$

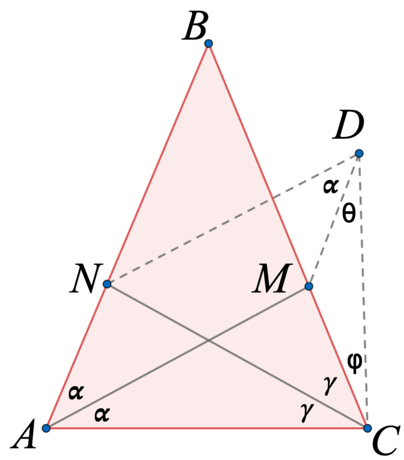
$$\phi > \theta$$

This chain of reasoning works just as well in reverse. So, $\phi > \theta \Rightarrow d^2 > c^2$, and then $d > c$. $\square$

This is known as the **hinge theorem**. We showed a different proof earlier. It is Euclid I.24.

□

Back to our problem. We claim that $\alpha = \gamma$ and the triangle is isosceles. We argue by contradiction.



Proof.

In $\triangle ABC$, let the base angles be bisected as shown.

Let the bisectors be equal: $AM = CN$.

Draw $ND \parallel AM$ and $MD \parallel ANB$.

So $ANDM$ is a parallelogram.

Thus $\angle NDM = \alpha$.

Aiming for a contradiction, suppose $\gamma > \alpha$.

By I.24, $AN > CM$.

So $DM > CM$.

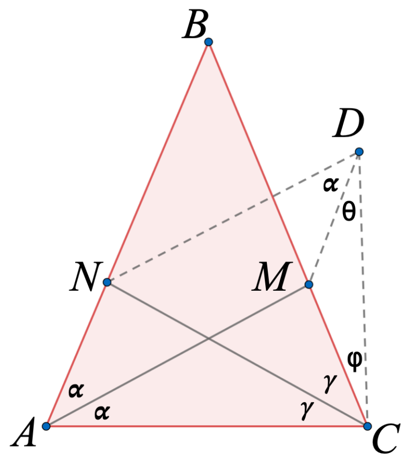
By I.18 $\phi > \theta$.

By addition of inequalities:

$$\gamma + \phi > \alpha + \theta$$

By I.19, $ND > CN$.

But $AM = ND$ so $AM > CN$.



This is a contradiction, we were given that $AM = CN$.

□

Therefore, it cannot be that $\gamma > \alpha$.

The reverse supposition, that $\alpha < \gamma$, also leads to a contradiction by a symmetrical argument, substituting $<$ for $>$.

(Or draw the parallelogram on the other side of $\triangle ABC$ and use the same argument as previously).

Since α is neither greater than nor less than γ, we conclude that $\alpha = \gamma$. $\triangle ABC$ is therefore isosceles by I.6.

□.

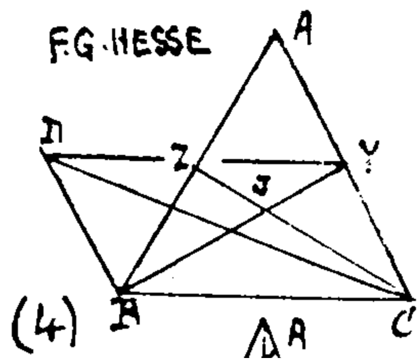
According to the internet, the Steiner-Lehmus theorem is famous for being difficult, for having many different proofs, and for some controversy over whether even one of the proofs is *direct* or not. By direct we mean, not using the technique of proof by contradiction or *reductio ad absurdum*.

I was lucky to find a (non-paywalled) review published on its centenary in 1942.

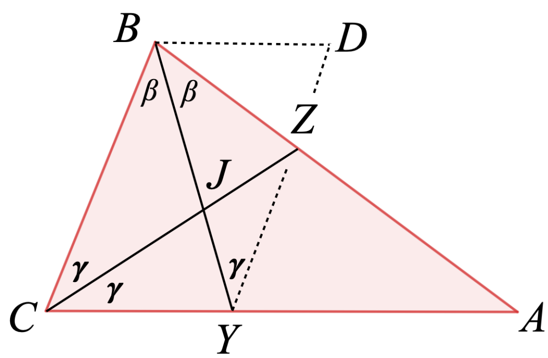
https://www.cambridge.org/core/services/aop-cambridge-core/content/view/7B625B08567935CAE06A0AC9430477C0/S0950184300000021a.pdf

It includes this proof.

Hesse (1842)



The construction is to draw $YD = BC$ and $BD = BZ$. Later we will find that CD divides $BDYC$ into two congruent triangles. Then it easily follows that $\beta = \gamma$.



Proof.

The equalities of the construction are $YD = BC$ and $BD = BZ$, and $BY = CZ$ is given.

(1) $\triangle DYB \cong \triangle BCZ$ by SSS.

Thus the corresponding angles of $\triangle DBY$ and $\triangle BZC$ are equal, namely:

$$\angle DYB = \angle BCZ = \gamma$$

$$\angle BDY = \angle ZBC = 2\beta$$

$$\angle DBY = \angle BZC = \angle A + \gamma$$

the last by sum of angles.

At this point I had some trouble with the details of the proof, so we may diverge from the source.

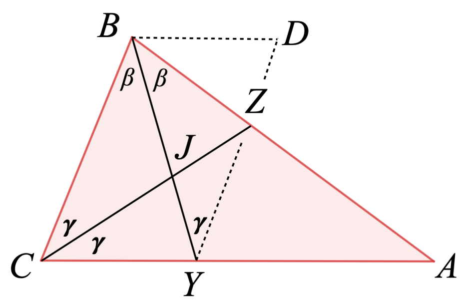
(2) $\angle BDY = 2\beta$

so by sum of angles in $\triangle DBY$ we have:

$$\angle ZBD = A - \beta + \gamma$$

Then

$$\angle CBD = A + \beta + \gamma = 90 + \alpha$$



Now all we need is to get the measure of $\angle CYD$

(3) By sum of angles $\angle BJC = \angle A + \beta + \gamma$ so by the external angle theorem:

$$\angle BYC = \angle A + \beta$$

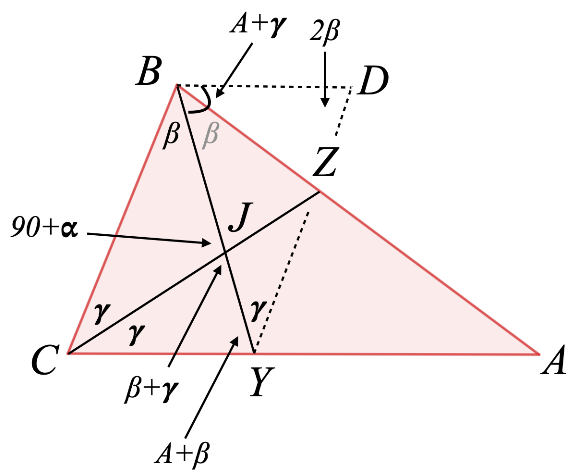
Then

$$\angle CYD = \angle A + \beta + \gamma$$

We have established that

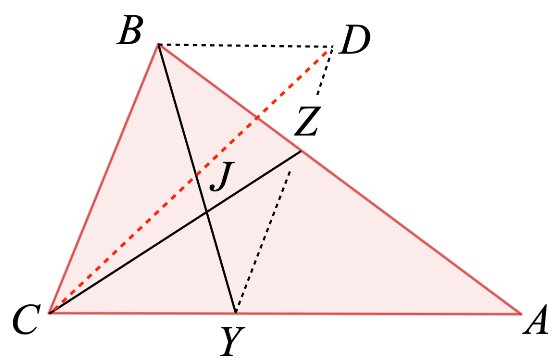
$$\angle CYD = \angle CBD$$

Crucially, they are not only equal but obtuse.



We don't necessarily have a parallelogram yet because we have only otne pair of sides equal and one pair of opposing angles equal.

But we can draw CD.



(4) Comparing $\triangle CBD$ and $\triangle DYC$, we have that CD is shared, and $DY = BC$ by construction. $\angle CYD = \angle CBD$.

We have SSA and in addition, the angles that we know in each, $\angle CYD$ and $\angle CBD$, are obtuse. It follows that $\triangle CBD \cong \triangle DYC$.

At this point we could just invoke the converse of the diagonal theorem for quadrilaterals.

(5) Instead Hesse says:

$$YC = BD = BZ$$

SSS then gives:

$$\triangle ZCB \cong \triangle YBC$$

$$\angle CBZ = \angle BCY$$

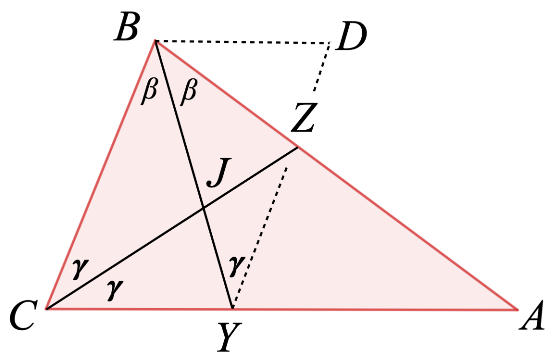
With base angles equal, all we need is I.6.

□

For more about SSA see **here**.

With the same diagram, I cooked up a proof by careful bookkeeping with the angles.

Proof. (Alternate).



Given $\angle B$ and $\angle C$ bisected into 2β and 2γ.

Given $BY = CZ$.

Draw $BD = BZ$ and $DY = BC$. (Again, we do not claim DZY collinear).

$\triangle BYD \cong \triangle ZCB$ by SSS.

So $\angle BDY = \angle ZBC = 2\beta$ and $\angle BYD = \angle ZCB = \gamma$.

At J, one vertical angle ($\angle BJC$) is $\angle A + \beta + \gamma$ by sum of angles.

$\angle BJC$ is external to $\triangle JCY$, hence $\angle CYJ = \angle A + \beta$.

Up to now, nothing has changed. Let $\angle DBZ = \theta$.

Summing angles in the quadrilateral $BDYC$:

$$\theta + 2\beta + 2\gamma + \angle A + \beta + \gamma + 2\beta = 360$$

$$\theta + \angle A + 5\beta + 3\gamma = 360$$

Since $\angle A + 2\beta + 2\gamma = 180$, subtracting

$$\theta + 3\beta + 1\gamma = 180$$

We also have that $\triangle BDZ$ is isosceles:

$$\theta + 4\beta = 180$$

Subtract again:

$$-\beta + \gamma = 0$$

$$\beta = \gamma$$

By I.6, $\triangle ABC$ is isosceles.

$\square$

afterward

There is some interesting discussion in Coxeter as well. According to what I can find on the web, most of the literature concerns the question of whether it is possible to provide a direct proof of the theorem. The algebraic proof, postponed for now to book II, has been cited as such.

However, that proof depends on Stewart's Theorem, which as we derive it there depends on the Law of Cosines, which depends in turn on the theorem of Pythagoras. And although there are several hundred proofs of Pythagoras most (all?) of them depend on the sum of angles and also on the parallel postulate, which explicitly depends on a proof by contradiction.

The question of a direct proof for Steiner-Lehmus is hard to answer conclusively. I have a write-up from John Conway claiming that it is impossible, but I don't really understand his argument. Unfortunately, nearly all the writing in mathematics journals is paywalled and very expensive.

Part VII

Trigonometry

Chapter 35

Basic trigonometry

Here we take a look at fundamental ideas from trigonometry and begin to show why they are useful. Let's start by reviewing some previous work.

similarity

A simple rectangular construction shows that right triangles with the same two complementary angles (*similar* right triangles), have equal ratios of sides.

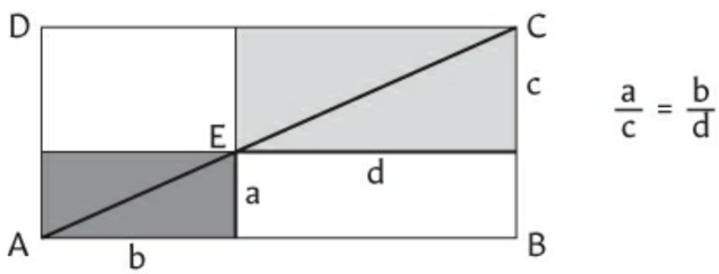


Fig. 42 Area and similarity.

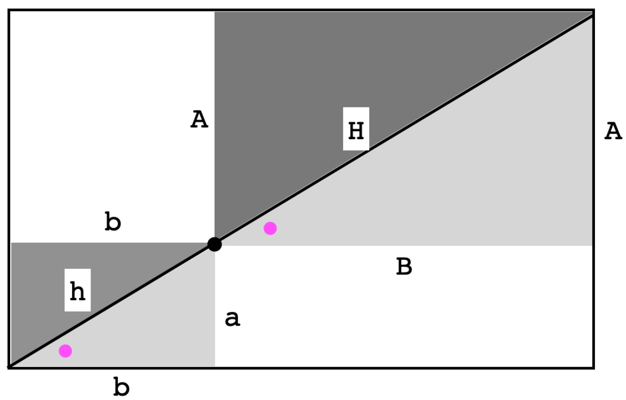
The proof is to use three sets of congruent triangles to show that the white areas are equal to each other which means that

$$bc = ad$$

Two simple rearrangements give different equalities

$$\frac{a}{b} = \frac{c}{d}, \qquad \frac{a}{c} = \frac{b}{d}$$

I want to switch notation for a second, to try to make a point. In the figure below, the two similar right triangles have sides a, b and h, and then A, B and H.



The equality is

$$Ab = aB$$

which can be rearranged to

$$\frac{a}{b} = \frac{A}{B}, \qquad \frac{a}{A} = \frac{b}{B}$$

The first one equates ratios of sides within a triangle, a and b from the first triangle, A and B from the second. The other one equates ratios of *corresponding* sides in different triangles. The sides of a triangle can be ordered by length, where one triangle has sides $a < b < h$ and a similar triangle has sides $A < B < H$.

Let k be the constant of proportionality between triangles, with $A = ka$. Then

$$\frac{a}{b} = \frac{A}{B} = \frac{ka}{B}$$

$$B = \frac{ka}{a} \cdot b = kb$$

So our ratios imply that

$$\frac{A}{a} = k = \frac{B}{b}$$

The factor k is a scaling factor which says how much bigger the second triangle is than the first.

This result is easily extended to the hypotenuse.

One way is to use the Pythagorean theorem. Let

$$a^2 + b^2 = h^2$$

and

$$A^2 + B^2 = H^2$$

But $A = ka$ and $B = kb$ so

$$H^2 = A^2 + B^2$$
$$= (ka)^2 + (kb)^2$$
$$= k^2(a^2 + b^2) = k^2h^2$$
$$H = kh$$

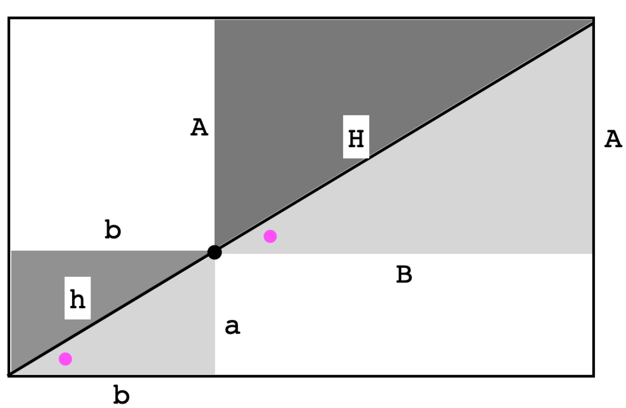
Hence

$$k = \frac{A}{a} = \frac{B}{b} = \frac{H}{h}$$

Now change focus back to comparisons within each triangle, namely

$$\frac{a}{h} = \frac{A}{H}$$

Consider the smaller angle, marked $\angle CAB$ in the original figure, and labeled by a magenta dot below



This angle is flanked by the hypotenuse h and the side b. We will call a the side *opposite* to $\angle CAB$ and b the side *adjacent*.

Then the previous equation

$$\frac{a}{h} = \frac{A}{H}$$

says that the ratio of the opposite side to the hypotenuse is somehow independent of the scale of the triangle. It is the same for both.

A similar thing happens with the *adjacent* sides b and B. Their ratios to the hypotenuse are also equal.

$$\frac{b}{h} = \frac{B}{H}$$

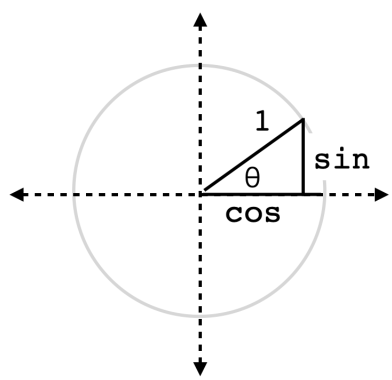
a result which is easily obtained by manipulating what we had above.

These ratios between sides in right triangles with the same angles are somehow characteristic of the angles and independent of the side lengths.

We call the ratio opposite/hypotenuse the *sine* of the angle and the ratio adjacent/hypotenuse the *cosine* and write them like this

$$\sin \angle CAB = \frac{a}{h}, \qquad \cos \angle CAB = \frac{b}{h}$$

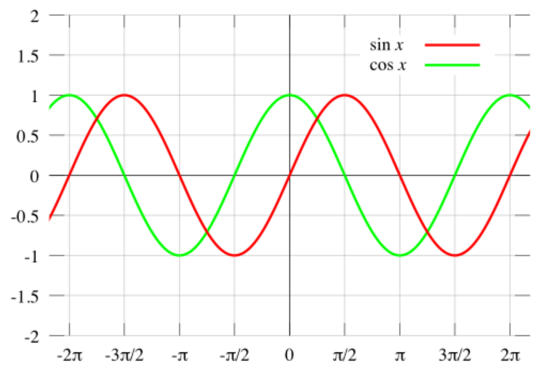
Often, we are working with central angles drawn in a unit circle, with radius 1.



Then the length of the side opposite the central angle is exactly the sine of the angle (since the hypotenuse has length 1), while the side adjacent is the cosine. This is the easiest picture to remember.

From a geometrical perspective, the importance of sine and cosine is that they incorporate the knowledge that these ratios are independent of the size of the triangle. You can think of them as ratios of sides to the hypotenuse for a central angle in a unit circle.

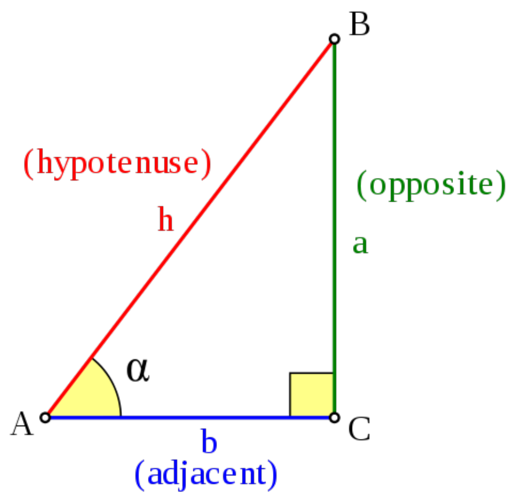
However, they are far more than that. In analytic geometry, we will see sine and cosine as *functions* of the angle θ. The curve traced is a periodic one, which repeats as θ goes all the way around the circle. This becomes central in calculus, where we will deal often with periodic phenomena modeled using the sine and cosine functions.



That's getting ahead of ourselves.

sine and cosine

To repeat:



In a right triangle, for angle α, the sine of the angle is defined as the ratio between the side opposite and the hypotenuse. In the drawing

$$\sin\alpha = \frac{a}{h}$$

while the cosine is

$$\cos\alpha = \frac{b}{h}$$

From our work earlier, we know that any similar right triangle (with the same angles) has its sides in the same ratios. The sine and cosine are functions of the angle, but are independent of the size of the triangle.

Frequently, the hypotenuse is scaled to be equal to 1, so then the opposite side is equal to the sine, and the adjacent side to the cosine.

Switching our focus from α (the angle at A) to the angle at vertex B, swaps opposite to adjacent and vice-versa. This means that the cosine of an angle is the sine of its complementary angle, and vice-versa.

radian measure

In this book, we will make an effort not to use degrees for angular measure. They aren't seen much after elementary school. The Greeks thought in terms of one or two right angles, or four right angles for an entire circle.

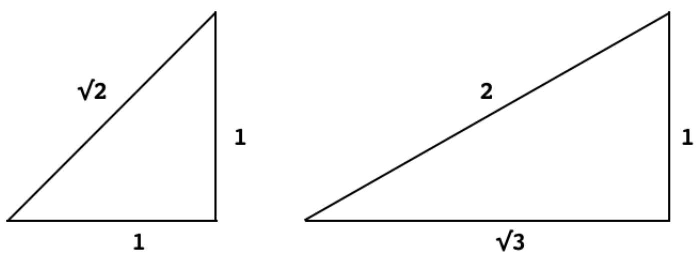
In more advanced mathematics we invariably use radians. Angles are measured in terms of the arc of the unit circle that they are *subtended by* or sweep out. Since the circumference of an entire circle is equal to 2π, a right angle is one-quarter of that or $\pi/2$ radians.

Some other common angles: $\pi/6 = 30°$, $\pi/4 = 45°$, and $\pi/3 = 60°$.

particular values

We can easily determine the values for these functions in three special cases.

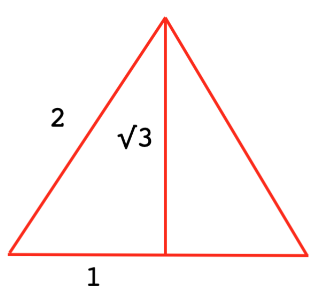
The first is the angle 45 degrees or $\pi/4$. Draw an isosceles right triangle with sides of length 1 (left panel).



Then the hypotenuse has length $\sqrt{2}$ (from Pythagoras) and the values are equal

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$$

If we start with an equilateral triangle (all angles equal to 60° and drop the angle bisector, we get two 30-60-90 triangles.

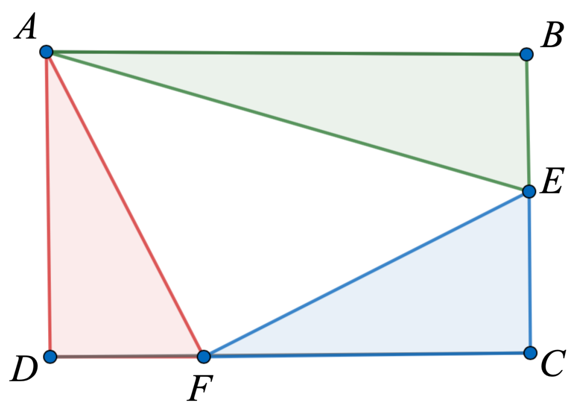


From the figure, we can read off that the sine of 30° is $1/2$ and the cosine is $\sqrt{3}/2$. The values for 60° are reversed since they are complementary angles.

Hence

$$\sin \pi/6 = \frac{1}{2} = \cos \pi/3$$
$$\cos \pi/6 = \frac{\sqrt{3}}{2} = \sin \pi/3$$
$$\sin \pi/4 = \frac{1}{\sqrt{2}} = \cos \pi/4$$

The sine and cosine for other angles can be obtained by similar methods. Take two congruent 30-60-90 right triangles, such as $\triangle ADF$ and $\triangle FCE$, and orient them so that DFC are collinear.



Extend CE to make $AD = BC$ and $AD \parallel BC$. Draw AB to complete rectangle $ABCD$.

$\triangle AEF$ is also a right triangle, by sum of angles, and it is isosceles, since $AF = EF$.

By sum of angles we have that $\angle BAE = 15°$. The lengths of the sides of $\triangle ABE$ are readily computed using the Pythagorean theorem. Form the ratios such as $BE/AE = \sin 15°$. We leave that as an exercise.

We will see later that the properties of the regular polygon allow us to deduce the values for 18° (and 72°).

General formulas for the sum and difference of angles will be covered soon. These give (for example) $\sin 3°$ from the values above.

extreme values

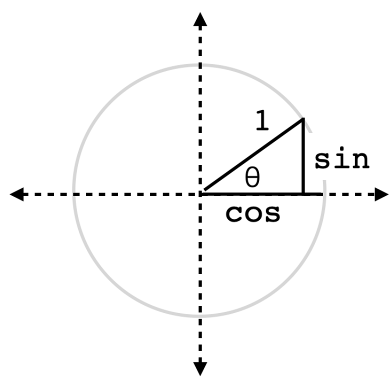
If you draw a very small angle, it will have a very short length for the vertical, $y = \sin\theta$, while the adjacent side becomes nearly equal to the radius. What happens when $\theta \to 0$? It turns out that

$$\begin{aligned} \sin 0 = 0, &\qquad \cos 0 = 1 \\ \sin \pi/2 = 1, &\qquad \cos \pi/2 = 0 \\ \sin \pi = 0, &\qquad \cos \pi = -1 \end{aligned}$$

We cannot prove any of this yet (we do not even have a *coordinate system*), but just take it on faith for now, and you will see why when we get farther along.

a favorite trigonometric identity

Now that we know about the sine and cosine, we can look at what Pythagoras tells us about them:



As we said, in a unit circle, the sine and cosine of an angle are the sides of a right triangle with hypotenuse equal to 1. It follows from the Pythagorean theorem that for any angle θ

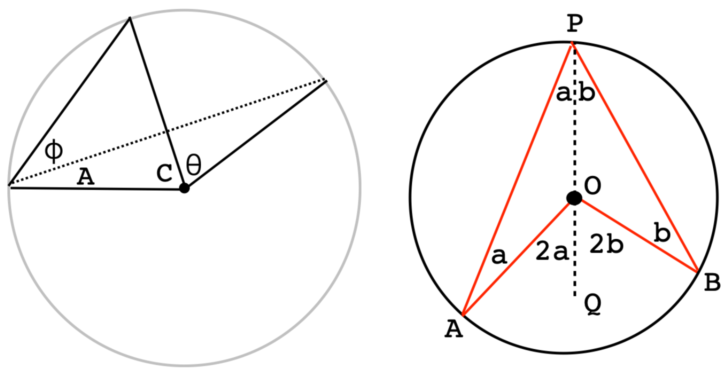
$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin\theta = \sqrt{1 - \cos^2\theta}$$
$$\cos\theta = \sqrt{1 - \sin^2\theta}$$

This identity is fundamental since it provides a way of converting from sine to cosine or vice-versa.

It is traditional to write $\sin^2\theta$ rather than $(\sin\theta)^2$, but they have the same meaning.

inscribed angles

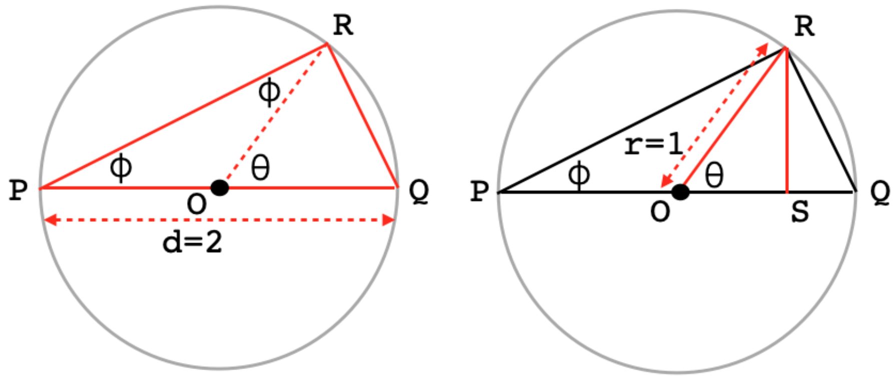
We also established previously the inscribed angle theorem. We showed two proofs that if the same arc on a circle subtends both a central angle θ and by a peripheral or inscribed angle ϕ, then $\theta = 2\phi$.



The first proof covers the case where the inscribed angle's arc does not include the center of the circle, while in the second case it does. It follows that all inscribed angles with the same arc are equal.

But that means we can study the arc or chord corresponding to any angle (in a given circle) by drawing the angle with one arm as the diagonal of a circle, since the chord is the same no matter where we place the angle.

inscribed and central angles



.

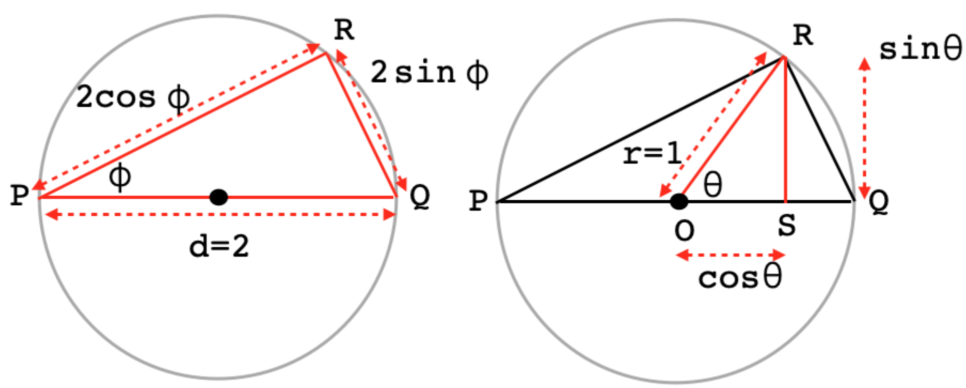
In the figure above, on the left we have drawn $\triangle PQR$ in a unit circle on center O with $\angle\phi$ being the inscribed or peripheral angle and θ the corresponding central angle. Since $\triangle POR$ is isosceles, and θ is the external angle to that triangle, this is yet another demonstration that $\theta = 2\phi$.

In both panels we have scaled things so that the radius of the circle has length 1 and the diameter is twice that. One right angle is at R. Use our definitions of sine and cosine to see that

$$\sin\phi = \frac{RQ}{PQ} = \frac{RQ}{2}, \qquad \cos\phi = \frac{RP}{PQ} = \frac{RP}{2}$$

while

$$\sin\theta = \frac{RS}{OR} = RS, \qquad \cos\theta = \frac{OS}{OR} = OS$$



.

To repeat, for such a right triangle drawn in a unit circle, the chord L subtends any peripheral angle ϕ is equal to

$$L = 2 \sin \phi$$

and in general, if the diameter has length $d = 2r$, then

$$L = d \sin \phi = 2r \sin \phi$$

history

The relationship between the chord RQ as twice the sine of ϕ and RS as the sine of θ and hence that of 2ϕ, was of great interest in geometry for centuries, partly for practical reasons. Chords were useful for astronomy, and astronomy was useful in turn for navigation.

Ptolemy constructed a *Table of Chords* containing values for arc lengths in 1 degree increments. The Greeks thought of values in terms of whole numbers or ratios of whole numbers, and since the smallest angle he dealt with was 1/360 of a circle, this provides a rationale for why 360 was chosen as the total number of degrees.

It also helps that 360 has so many integer factors:

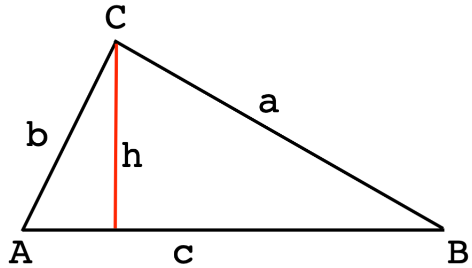
$1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18 \ldots.$

so we can talk about the 30 identical triangles that result from cutting a circle into 12 equal parts.

We will now spend some time looking at the "double angle" formula that connects the sine of an angle with that of twice the angle.

area of any triangle

The altitude to any side of a triangle is equal to the length of the side of a triangle times the sine of the angle that side makes with the base.



$$\frac{h}{b} = \sin A$$
$$h = b \sin A$$

Therefore (if Δ is the area of the triangle):

$$2\Delta = hc = bc \sin A$$

One can equally well write

$$h = a \sin B$$
$$2\Delta = hc = ac \sin B$$

Twice the area of any triangle is the product of the two sides times the sine of the angle between them.

problem

We usually compute the area of a parallelogram in terms of the sides. However, another formula for (twice) the area is:

$$2\Delta = d_1 d_2 \sin \theta$$

where d_1 and d_2 are the two diagonals and θ is either one of the central angles.

Derive this formula.

law of sines

Since the area must be the same no matter how we compute it, this also leads to the equality

$$h = b \sin A = a \sin B$$
$$\frac{a}{b} = \frac{\sin A}{\sin B}$$

which by symmetry we extend to all three angles

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This simple formula is called the law of sines.

The constant ratio has an interesting value, which can be seen by going back to what we said above, namely

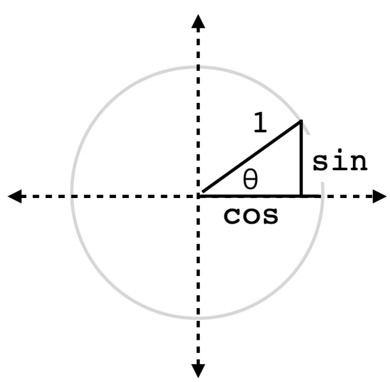
$$L = 2r \sin \phi$$

which, applied to this case, gives

$$\frac{a}{\sin A} = 2r$$

other functions

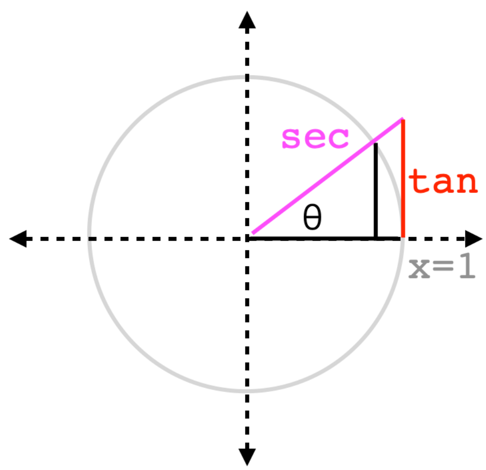
We'll just mention a few other common trig (short for trigonometric) functions that are constructed from sine and cosine, although they aren't used much in this book.



First, the tangent of the angle is defined as the opposite side divided by the adjacent side. In other words, it is equal to the sine divided by the cosine.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

We can see the tangent as a length. Extend the hypotenuse to make a similar triangle, where the adjacent side has length equal to the radius. Now, opposite over adjacent gives tangent, but adjacent is just 1.



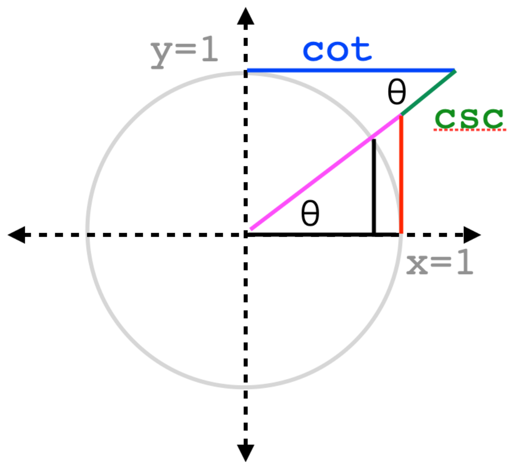
There are also three inverse functions: secant (inverse of the cosine), cosecant (inverse of the sine), and cotangent (inverse of the tangent). From the drawing above, we can get that

$$\frac{1}{\sec\theta} = \cos\theta$$

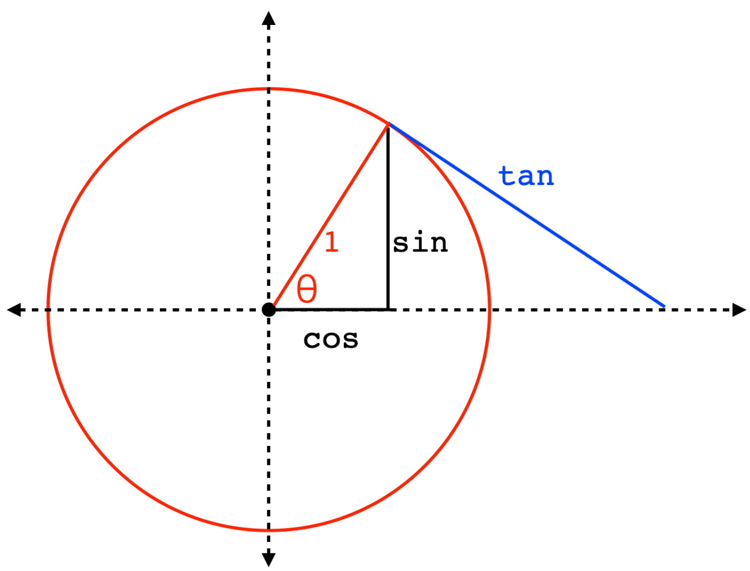
$$\sec\theta = \frac{1}{\cos\theta}$$

(We might also check by looking at the ratio $\tan\theta/\sec\theta = \sin\theta$).

We include figures with the cotangent and cosecant as well, but put off discussion of them for now.

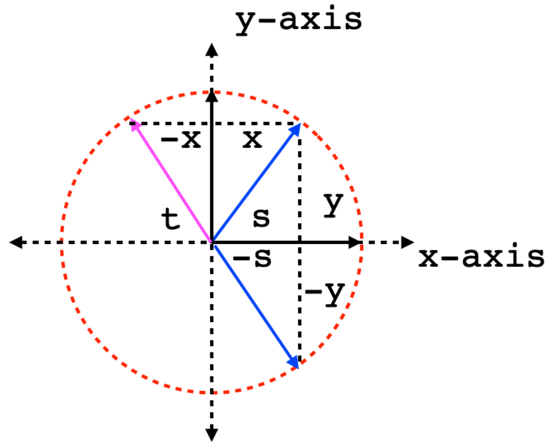


There are several other common representations of the six functions. We'll see more when we get to analytic geometry. Here is a common one where the tangent to the circle, perpendicular to the ray with $\angle\theta$ is equal to $\tan\theta$. See if you can work this out by using the complementary angle to θ.



signed angles

In analytic geometry we will introduce the idea of two axes in the plane, one for x-values and one for y-values. The origin will become $(0, 0)$, and values will have signs, i.e., x-values to the left of the origin will be minus some number, and y-values below the origin will be minus some number as well.



Without getting into all the gory details, I hope you can see that when, as here, s and t are supplementary angles, then

$$\sin s = \sin t$$

$$\cos s = -\cos t$$

while for s and $-s$

$$\sin s = -\sin -s$$

$$\cos s = \cos -s$$

and as we said before, when s and t are complementary angles

$$\sin s = \cos t$$

$$\cos s = \sin t$$

This leads to an elementary proof of the **area-ratio theorem**.

Proof.

Let the a triangle be divided by a line from the upper vertex to the base such that the base is divided into two parts, x and y. Call the dividing line e.

The angles at the base on either side of the intersection point are supplementary, hence they have the same sine. Let those angles be ϕ and ϕ'.

Then twice the area of the left-hand triangle is

$$2A_L = xe \sin \phi$$

while twice the area of the other one is

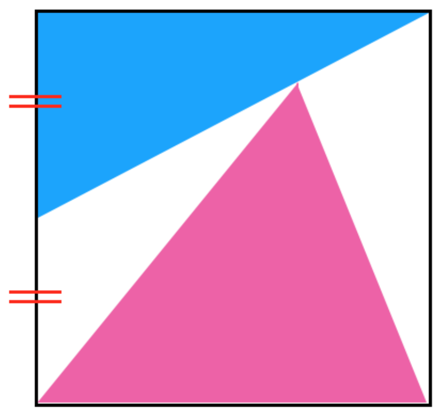
$$2A_R = ye \sin \phi' = ye \sin \phi$$

and the ratio of the areas is simply x/y.

$\square$

problem

Here is a nice area problem I saw on Twitter. (The attribution is "Paul Eigenmann, Stuttgart, 1967". Web search finds what looks like a great text, but in German).



Given that the side is bisected and the unshaded areas are equal. What is the area of the triangle shaded magenta?

Solution.

Let's say this is a unit square and let the top vertex of the magenta triangle be a distance x from the left-hand side of the box and a distance $1-x$ from the right-hand side. Then the left-hand unshaded area is

$$\frac{1}{2} \cdot x \cdot \frac{1}{2}$$

and the right-hand shaded area is

$$\frac{1}{2} \cdot (1-x) \cdot 1$$

Set them equal:

$$\frac{x}{4} = \frac{1}{2} - \frac{x}{2}$$
$$x = 2 - 2x$$
$$x = \frac{2}{3}$$

To find the height of the magenta triangle using pure geometry, notice that a line drawn vertically from the vertex forms two similar triangles, therefore y is a distance $1/3 \cdot 1/2 = 1/6$ down vertically from the upper boundary.

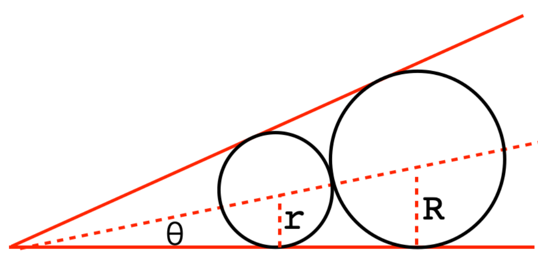
The height of the magenta triangle is therefore $5/6$ and its area is one-half that, or $5/12$ of the unit cube.

Using analytic geometry, by inspection of the figure, write the equation for the side of the upper triangle as

$$y = \frac{1}{2}x + \frac{1}{2}$$
$$y = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

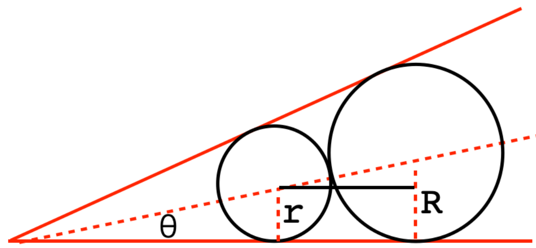
double scoop problem

We have two lines tangent to two circles that just touch each other, the smaller one of radius r, and the larger of radius R.



There is a simple expression for the sine and cosine of θ, the angle between the the two lines. Recall our introduction to trigonometry **here**.

The distance between the centers of the two circles is $r + R$. Draw a horizontal line through the center of the smaller circle.



We have constructed a right triangle, which is similar to the original one. It includes the angle θ and the hypotenuse is the distance between the two centers, $R + r$. The opposite side has length $R - r$ and so

$$\sin\theta = \frac{R-r}{R+r}$$

The adjacent side (the line segment colored black) has its squared length equal to

$$(R+r)^2 - (R-r)^2 = 4Rr$$

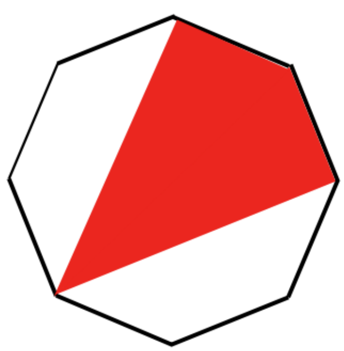
thus

$$\cos\theta = \frac{2\sqrt{Rr}}{R+r}$$

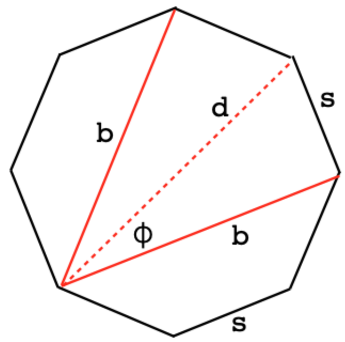
David Bowie problem

Here's a problem from the web:

Vincent Pantal🍩ni @panlepan · Feb 25
The octagon that thinks it's David Bowie.
What fraction of this regular octagon is coloured in red ?
#GeometrySnacks #ShowYourWork #WFIS



One way to look at this is to imagine the octagon inscribed in a circle of diameter $d = 2r$ (below). We reason that the two triangles are right triangles with a shared hypotenuse.



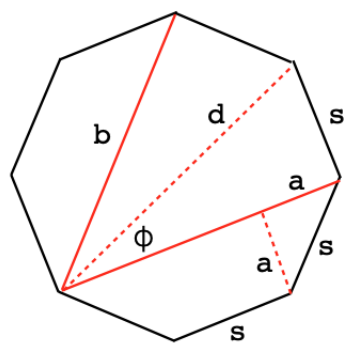
We can apply trigonometry to find that the whole area required is simply

$$A_{\text{ region}} = d^2 \sin\phi \cos\phi$$

But we aren't ready to do this kind of calculation yet.

However, it will turn out that the part shaded red is simply one-half of the whole.

Therefore, the three lines divide the octagon into four equal parts. That simple answer is a clue that something important makes the problem easy.



The triangle with small sides a can be moved so one of the unshaded base parts becomes a rectangle with area

$$a(s + a) = sa + a^2$$

while the base of the triangle is $s + 2a$ so its area is

$$\frac{1}{2}s(s + 2a) = \frac{1}{2}s^2 + sa$$

It's not obvious at first that these are equal, but working with it, they would be equal if we can show that $2a^2 = s^2$.

Of course we can do exactly that, because a is the side of an isosceles right triangle with hypotenuse s. By the Pythagorean theorem, we have

$$a^2 + a^2 = s^2$$

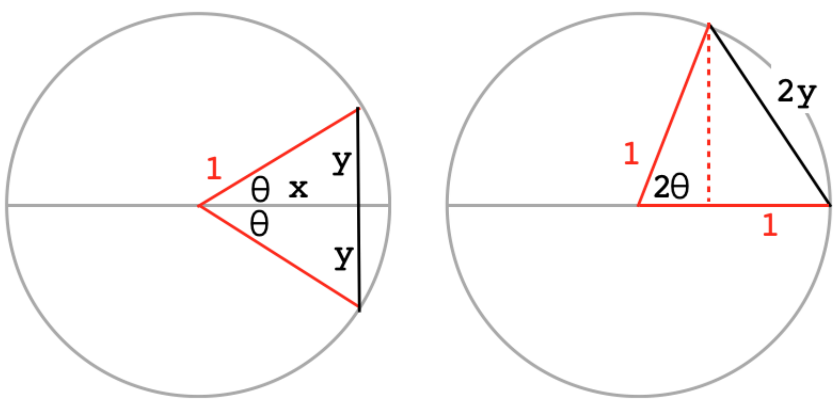
□

Chapter 36

Double angle

double angle formulas

Draw a unit circle with central angle θ and then double the central angle below the horizontal diameter. Let x be the base of the two smaller triangles, the side adjacent to angle θ, and y the side opposite.



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Then by the basic definitions of trigonometry, we have that $x = \cos\theta$ and $y = \sin\theta$. The area of each small triangle is $xy/2$ and the area of the two combined is simply

$$A = xy = \sin\theta\cos\theta$$

Now, rotate the triangle counter-clockwise until the red side on the right lies along the diagonal. The dotted vertical line is clearly $\sin 2\theta$ and since the base is length 1, the area of that same triangle is now

$$A = \frac{1}{2} \cdot \sin 2\theta$$

Combining the results

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

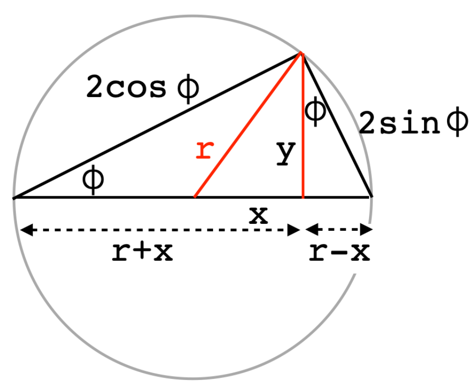
This is the first double angle formula, for sine. We claim that it still works even if the sum is greater than a right angle but keep things simple by not talking about it more.

double angle for cosine

Consider this right triangle with an inscribed angle of ϕ. Label the sides of a triangle containing the central angle 2ϕ as x and y to simplify the figure. Let $r = 1$ so then

$$y = \sin 2\phi, \qquad x = \cos 2\phi$$

Whenever we drop an altitude in a right triangle, the smaller triangles are similar to the original. This accounts for the second $\angle\phi$ in the figure.



We note for later that the base of the altitude divides the base into $r + x$ and $r - x$.

Using the small triangle with angle ϕ

$$\frac{y}{2\sin\phi} = \cos\phi$$

$$y = 2\sin\phi\cos\phi$$

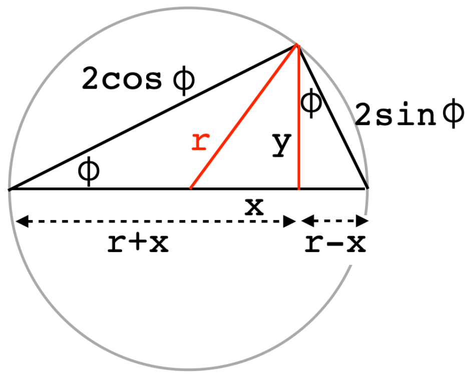
Since $y = \sin 2\phi$, this is just what we had before.

Along the base of the triangle extending from the left, we have (again, $r = 1$):

$$\frac{r + x}{2\cos\phi} = \frac{1 + x}{2\cos\phi} = \cos\phi$$

$$\frac{1 + x}{2} = \cos^2\phi$$

But (from the small triangle again)



$$\frac{r - x}{2\sin\phi} = \frac{1 - x}{2\sin\phi} = \sin\phi$$

$$\frac{1 - x}{2} = \sin^2\phi$$

And then

$$\frac{1 + x}{2} - \frac{1 - x}{2} = x$$

but $x = \cos 2\phi$ so

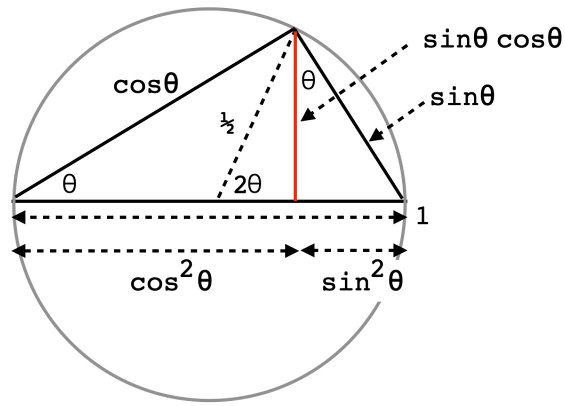
$$\cos 2\phi = \cos^2\phi - \sin^2\phi$$

Note also that

$$\cos^2\phi + \sin^2\phi = \frac{1 + x}{2} + \frac{1 - x}{2} = 1$$

which is true for any angle ϕ, a restatement of the Pythagorean theorem.

Here is the same proof but scaled so that the diameter has unit length and we use sine and cosine labels rather than x and y. The details are left to you.



half-angle formulas

The half-angle formulas are easy to derive from the double angle cosine formula, because the sine and cosine terms are separated:

$$\cos 2A = \cos^2 A - \sin^2 A$$

Just use our favorite identity ($\sin^2 A + \cos^2 A = 1$) to obtain

$$\cos 2A = \cos^2 A - (1 - \cos^2 A)$$

or

$$\cos 2A = (1 - \sin^2 A) - \sin^2 A$$

We'll do the sine first

$$2\sin^2 A = 1 - \cos 2A$$

$$\sin A = \sqrt{\frac{1 - \cos 2A}{2}}$$

which is easily verified using the values for 30° and 60°.

On the other hand:

$$\cos 2A = \cos^2 A + \cos^2 A - 1$$

Solve for $\cos A$:

$$\cos A = \sqrt{\frac{1 + \cos 2A}{2}}$$

This is called the half-angle formula since A is one-half of $2A$. That the sum of the squares is equal to 1 is also easily verified. We see that the square roots go away, the $\cos 2A$ terms will cancel and we end up with $1/2 + 1/2$.

some algebra

You might be tempted to try to derive a formula for $\sin 2A$ from that for $\cos 2A$ (or vice-versa), using our favorite identity. Let us re-write the formulas substituting s for $\sin A$ and c for $\cos A$:

$$\sin 2A = 2sc, \qquad \cos 2A = c^2 - s^2$$

They have different forms: the first mixes s and c, while the second has a difference of squares. If we were to take

$$(c+s)^2 = c^2 + 2sc + s^2$$
$$(c-s)^2 = c^2 - 2sc + s^2$$
$$(c+s)(c-s) = c^2 - s^2$$

There is some of what we want, but in the formula for $\cos 2A$ one term is positive and one negative. What's going on?

First, just verify what $\sin^2 2A + \cos^2 2A$ is equal to:

$$= 4s^2c^2 + c^4 - 2s^2c^2 + s^4$$
$$= c^4 + 2s^2c^2 + s^4$$
$$= (s^2 + c^2)^2 = 1^2 = 1$$

That checks out, and it contains a hint to the answer. We used $4s^2c^2 - 2s^2c^2$ to convert $-2s^2c^2$ to the positive $2s^2c^2$.

Remember, to do the conversion we square the sine or cosine, subtract it from 1 and then take the square root.

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$
$$\cos\theta = \sqrt{1 - \sin^2\theta}$$

It is easier to see how this works using the first formula. So we start with the formula for cosine of $2A$ and plug in:

$$1 - (c^2 - s^2)^2 = 1 - (c^4 - 2s^2c^2 + s^4)$$

Add (and subtract) $2s^2c^2$ in the parentheses!

$$= 1 - (c^4 + 2s^2c^2 + s^4 - 4s^2c^2)$$

$$= 1 - \ [\ (c^2 + s^2)^2 - 4s^2c^2\ ]$$

$$= 1 - 1 + 4s^2c^2 = 4s^2c^2$$

and now the square root is easy and gives exactly what we need.

Then it becomes clear that we can add $(s^2 + c^2) - 1$ or $(s^2 + c^2)^2 - 1$ to expressions and if they will simplify, it helps us. Hence to obtain the formula for $\cos 2A$, starting with 1 minus sine squared:

$$1 - 4s^2c^2$$

$$= 1 - 4s^2c^2 + (s^2 + c^2)^2 - 1$$

$$= -4s^2c^2 + \ [\ s^4 + 2s^2c^2 - c^4\ ]$$

$$= s^4 - 2s^2c^2 + c^4)$$

$$= (s^2 - c^2)^2 = (c^2 - s^2)^2$$

remembering that $(x - y)^2 = (y - x)^2$.

Take the square root to finish.

examples

Restating the double-angle formula for sine

$$\sin 2A = 2 \sin A \cos A$$

Plugging in

$$\sin 60^\circ = 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

and

$$\cos 2A = \cos^2 A - \sin^2 A$$

plugging in again

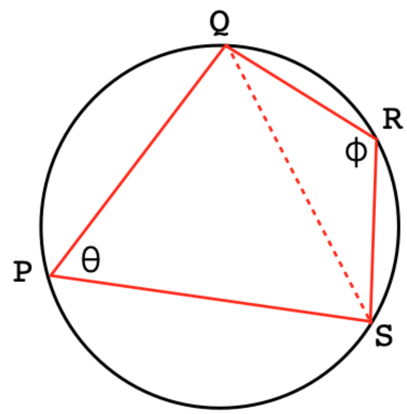
$$\cos 60^\circ = (\frac{\sqrt{3}}{2})^2 - (\frac{1}{2})^2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

They both look good.

We can also use the formulas to check supplementary angles (recall that $\sin 180 = 0$ and $\cos 180 = -1$):

$$
\begin{aligned}
\sin 180 - A &= \sin 180 \cos A - \sin A \cos 180 \\
&= 0 - (-\sin A) = \sin A \\
\cos 180 - A &= \cos 180 \cos A + \sin 180 \sin A \\
&= -\cos A + 0 = -\cos A
\end{aligned}
$$

Of course, we hardly need algebra for the result about the sine of the supplementary angle. Two inscribed angles with the supplementary arcs clearly have the same sine:



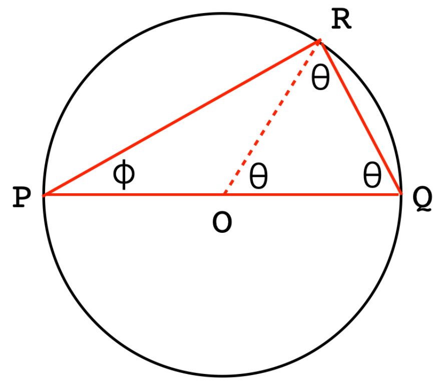
The total arc of the circle is 2π, while the two angles θ and ϕ together take up that entire arc. As inscribed angles they sum to one-half or π. Hence θ and ϕ are supplementary.

But we previously had

$$L = d \sin \phi = 2r \sin \phi$$

where L is the length of the dotted line, and if the circle is scaled appropriately, it *is* $\sin \phi$. But it is also $\sin \theta$.

equilateral triangle



Here is another way of looking at 30° and 60°. Let the central angle θ be such that the triangle formed is equilateral.

Now the central angle is equal to both of the base angles, and the measure of all these angles is 60° or $\pi/3$.

Since $\triangle OQR$ is equilateral, the chord QR is the same length as the radius. It follows that the sine of the inscribed angle ϕ is $r/2r = 1/2$. And of course the length of PR, which is 2 times the sine of 60°, is $\sqrt{3}$.

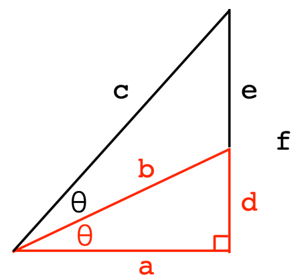
Archimedes and pi

Later we will explore the math relating to Archimedes' method to approximate π, resulting in the bounds $3\ 10/71 < \pi < 3\ 1/7$.

The lower bound is about 3.1408 and the upper bound about 3.1429 while the true value is about 3.1416. (There is an interesting story relating to a fraction which is a much, much better estimate of π, namely 355/113, good to six digits).

As you probably know, the method involves calculating the length of the perimeter of circumscribed and inscribed regular polygons.

We can build on two basic results from the geometry of right triangles:



The first is that the sum of the cotangent and cosecant for the original angle are equal to the cotangent of the half-angle:

$$\frac{a}{f} + \frac{c}{f} = \frac{a}{d}$$

$$\cot 2\theta + \csc 2\theta = \cot \theta$$

This follows from the angle bisector theorem, $a/d = c/e$, with a bit of manipulation.

$$\frac{a}{c} = \frac{d}{e}$$

$$\frac{a}{c} + \frac{c}{c} = \frac{d}{e} + \frac{e}{e}$$
$$\frac{a+c}{d+e} = \frac{c}{e} = \frac{a}{d}$$

Adding components in the denominator of the first term:

$$\frac{a+c}{f} = \frac{a}{d}$$

Another way to get there is to use the double angle formulas. We'll abbreviate $\sin, \cos, \tan$ as S, C, T and identify the values for the half-angle as lowercase. The formulas are:

$$S = 2sc$$

$$C = c^2 - s^2 = 2c^2 - 1$$

The cotangent of 2θ is then (cosine/sine):

$$\frac{1}{T} = \frac{C}{S} = \frac{2c^2 - 1}{2sc}$$

$$= \frac{1}{t} - \frac{1}{S}$$

which rearranges to give the result

$$\frac{1}{t} = \frac{1}{T} + \frac{1}{S}$$

Chapter 37

Sum of angles by scaling

sum of angles

In order to gain some practice thinking about sine and cosine, we will derive what are called the sum of angles formulas. These are really for the sum and difference of two angles:

$$\sin(s \pm\ t), \qquad \cos(s \pm\ t)$$

Previously we talked about formulas for the case where $s = t$, but now we want more general expressions. Here is the first, for the difference of angles s and t

$$\cos s - t = \cos s \cos t + \sin s \sin t$$

By $\cos s - t$ we mean $\cos(s - t)$, but have left off the parentheses.

There are four formulas, and then some special examples. These are used a lot in calculus, not only for solving problems, but most important, in finding an expression for the derivatives of the sine and cosine functions.

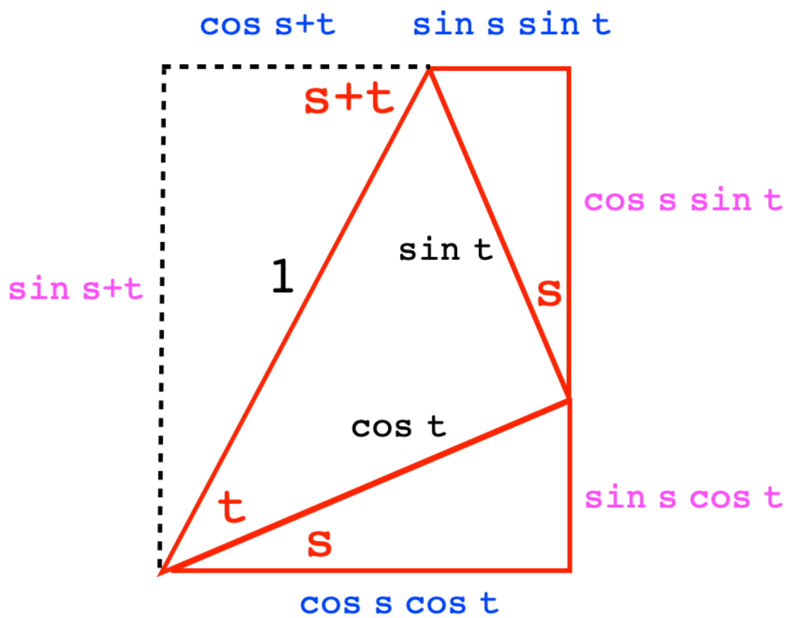
I've memorized only the single equation given above. Say "cos cos" and then recall the difference in sign, minus on the left, plus on the right.

I like this version because it can be checked easily. Just set $s = t$. Then $s - t$ becomes $s - s$ and we have

$$\cos s - s = \cos 0 = 1 = \cos^2 s + \sin^2 s$$

which is our favorite trigonometric identity and obviously correct.

This diagram shows where we're headed:



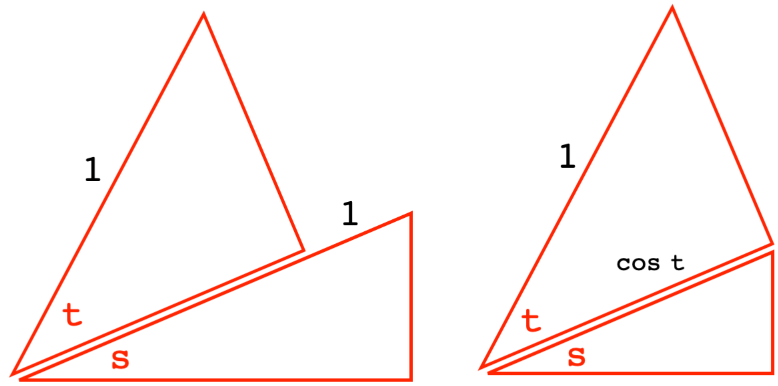
Since the figure is a rectangle, you can read the relevant equalities off the opposite sides.

$$\sin s + t = \sin s \cos t + \cos s \sin t$$

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

As so often, we begin with an inspired diagram. Let's build it up in stages.

On the left, below, we have stacked two right triangles. The one containing angle t is rotated so that its base is parallel to the hypotenuse of the triangle containing angle s.



The crucial step is to re-scale the triangle on the bottom so that the parallel line segments are also equal in length (right panel).

By re-scaling, we change the length of the hypotenuse of the triangle containing angle s. Its hypotenuse is now the length $\cos t$.

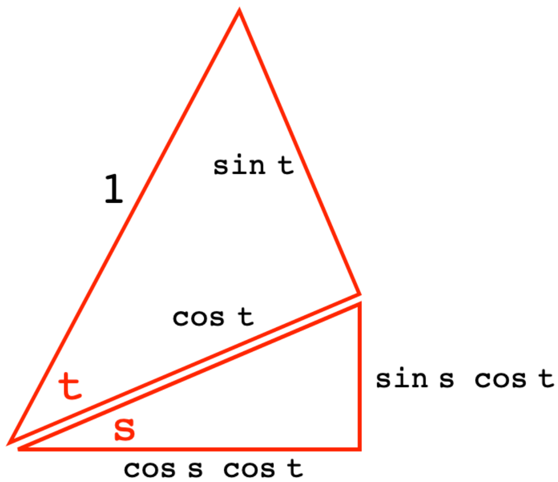
Recall that when the length of the hypotenuse was h, we divided the length of the adjacent side by the hypotenuse to get the cosine. There we had $b/h = \cos s$. Now we have

$$\cos s = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{adjacent}}{\cos t}$$

What should be the length of the base of the triangle with angle s? It must be $\cos s \cos t$, the product of cosines! Just multiply both sides above by $\cos t$.

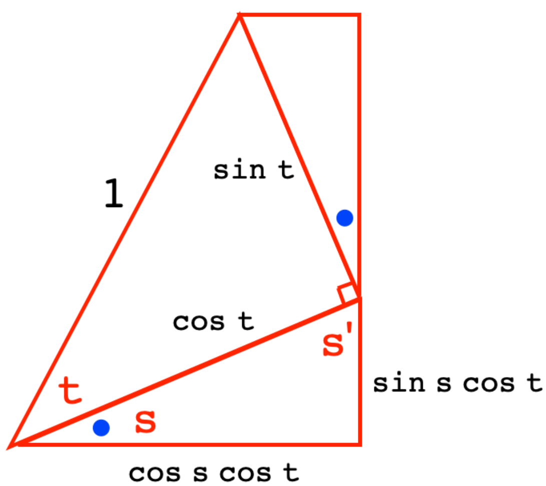
Or recognize that after dividing $\cos s \cos t$ by the hypotenuse, which is $\cos t$, we will then have what we want, $\cos s$.

$$\frac{\cos s \cos t}{\cos t} = \cos s$$



By the same reasoning, the opposite side in the triangle with angle s is $\sin s \cos t$, so that after dividing by $\cos t$ we obtain the correct value, $\sin s$.

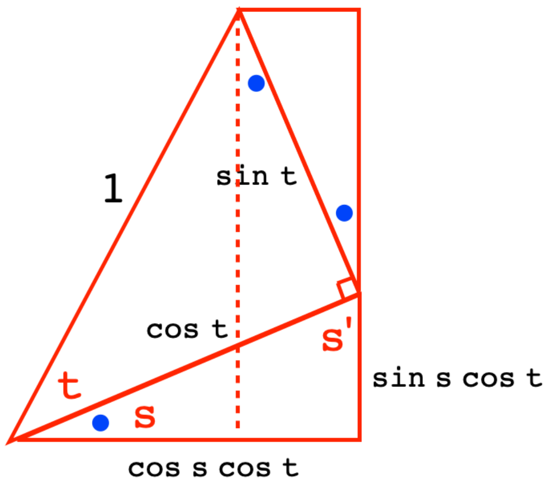
a similar triangle

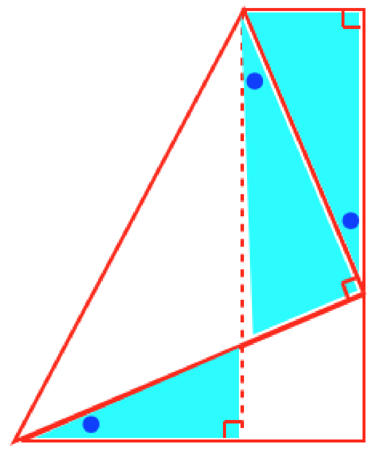


We add another right triangle to the figure

We claim that the angles labeled with blue dots are equal.

The easiest way to do that is algebraic. The angle that is complementary to s in the bottom triangle is s'. Complementarity means that $s + s'$ is equal to a right angle.





But the second blue dotted angle plus s' is also equal to a right angle. The reason is that, when added to another right angle, it makes two right angles or a straight line. Using the language of radian measure

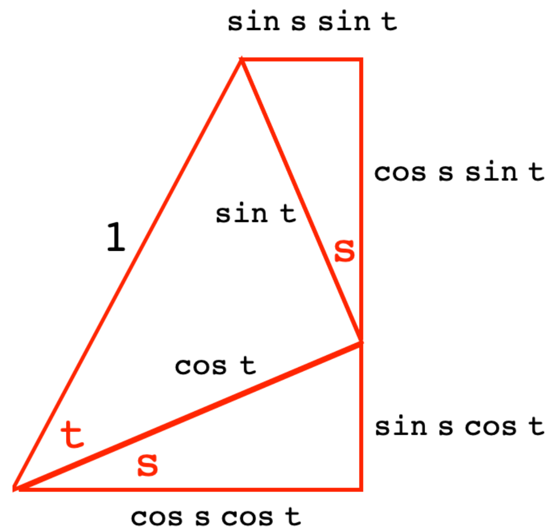
$$s + s' = \frac{\pi}{2}$$

$$s' + \frac{\pi}{2} + \text{blue dot} = \pi$$

$$s' + \text{blue dot} = \frac{\pi}{2}$$

$$\text{blue dot} = s$$

So we add the correct label, and then play the same trick as before.



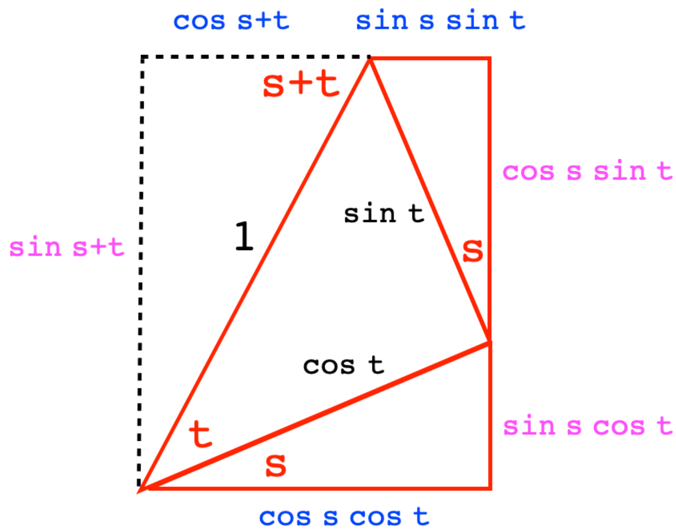
The difference is that now, the length of the hypotenuse of the upper triangle containing angle s is $\sin t$. The two similar triangles are scaled differently from one another.

We obtain $\sin s \sin t$ and $\cos s \sin t$ for the sides of the upper triangle, so that after dividing by the hypotenuse, which is $\sin t$, we obtain the correct values for the sine and cosine of angle s.

finale

Why have we gone to the trouble of doing all this?

The two angles, s and t, taken together, are equal to the angle at the top of the figure labeled, naturally, $s + t$.



We fill in lengths for the dotted lines of the fourth right triangle in the figure below.

This forms a rectangle (all four corners are right angles). Therefore, opposite sides are equal.

We just write down the formula by reading off the figure:

$$\sin s + t = \sin s \cos t + \cos s \sin t$$

and

$$\cos s + t + \sin s \sin t = \cos s \cos t$$

which can be rearranged to give

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

These are the sum of angles formulas.

change signs

For $\cos s - t$, flip the sign on the second term.

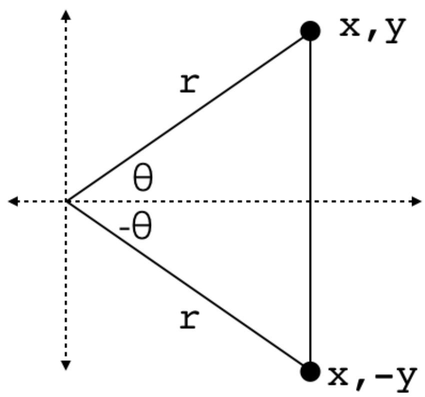
$$\cos s - t = \cos s \cos t + \sin s \sin t$$

That's because

$$\cos -\theta = \cos \theta$$

$$\sin -\theta = -\sin\theta$$

To show this we need to invoke analytic geometry (there's no such thing as a negative angle in classical geometry).



The diagram shows the reason:

$$\cos\theta = x/r = \cos -\theta$$

while

$$\sin\theta = y/r = -(\sin -\theta) = -(-y/r)$$

Substitute $-\sin\theta$ for $\sin -\theta$ and $\cos\theta$ for $\cos -\theta$:

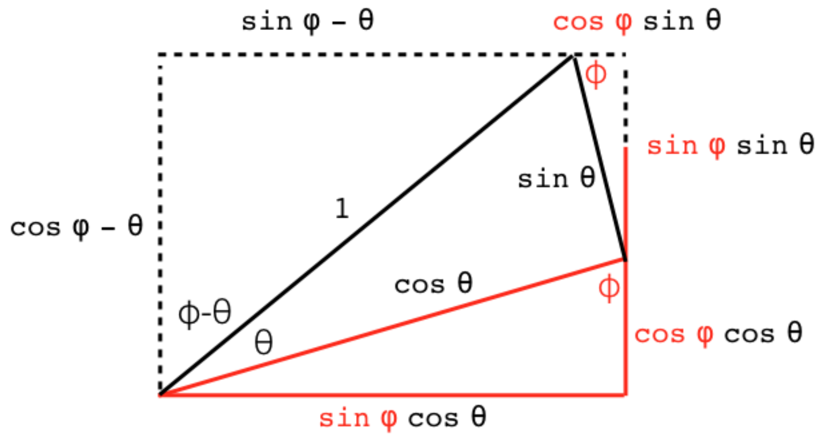
$$\cos s - t = \cos s \cos -t + \sin s \sin -t$$

$$= \cos s \cos t - \sin s \sin t$$

and

$$\sin s - t = \sin s \cos t - \cos s \sin t$$

It's kind of overkill, but still worth noting that a simple change to the figure we had above will give the difference formulas:



We've changed symbols to θ and ϕ for the complementary angles.

We can justify the label $\phi - \theta$ for the angle at the lower left in various ways, for example, by adding up the three angles at that corner:

$$(\phi - \theta) + \theta + (90 - \phi) = 90$$

Switch the labels appropriately (it's easy since this ϕ is the complement of the old one).

Read the result:

$$\sin \phi - \theta = \sin \phi \cos \theta - \cos \phi \sin \theta$$

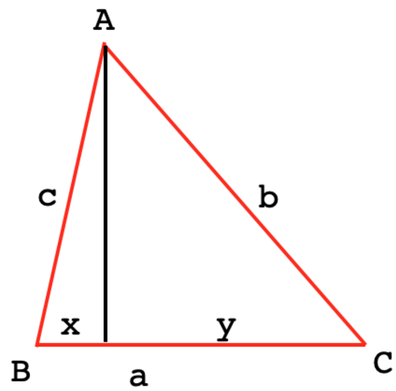
$$\cos \phi - \theta = \cos \phi \cos \theta + \sin \phi \sin \theta$$

alternative derivation

There are many derivations of the sum of angles formulas. Here is an algebraic one based on the **law of sines**.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Start with this triangle



$$a = x + y = c\cos B + b\cos C$$

From the law of sines: $\sin A = (a/b)\sin B$. Substituting for a:

$$\sin A = \frac{c\cos B + b\cos C}{b}\sin B$$

$$= \frac{c}{b}\sin B\cos B + \sin B\cos C$$

Again from the law of sines: $\sin B = (b/c)\sin C$, so

$$\sin A = \sin C\cos B + \sin B\cos C$$

But since A and $B + C$ are supplementary, their sines are equal, thus

$$\sin B + C = \sin C\cos B + \sin B\cos C$$

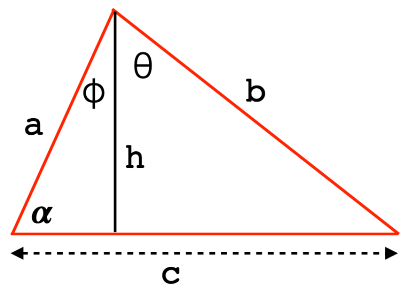
$\square$

The proof for an obtuse angle is left as an exercise.

general formula for sine

Here is a related, simple proof of the formula for sine.

Consider the following triangle, where the angle at one vertex is divided by the altitude to the opposing side, forming angles θ and ϕ. We are interested in finding the sine of the two angles added together.



But first we must introduce a standard formula for the area of a triangle, illustrating it by the angle α. From our previous work we know that twice the area of the triangle is hc, but h is also part of a formula for sine, namely $h/a = \sin\alpha$, which can be rearranged to give $h = a\sin\alpha$. In other words

$$2A = hc = ac\sin\alpha$$

In general, twice the area of any triangle is the product of two sides times the sine of the angle between. So in this case, we also have that

$$2A = ab\sin(\phi + \theta)$$

Now we just calculate the area of the two smaller triangles and add them together. We have

$$2A = ha\sin\phi + hb\sin\theta$$

I'm going to rearrange this slightly

$$2A = a\sin\phi \cdot h + b\sin\theta \cdot h$$

For these angles, h is connected to a trig function, but this time it's the cosine.

$$h = a\cos\phi = b\cos\theta$$

Substituting two times into the previous equation, we obtain

$$2A = a\sin\phi \cdot b\cos\theta + b\sin\theta \cdot a\cos\phi$$

and equate it to the first result

$$ab\sin(\phi + \theta) = a\sin\phi \cdot b\cos\theta + b\sin\theta \cdot a\cos\phi$$

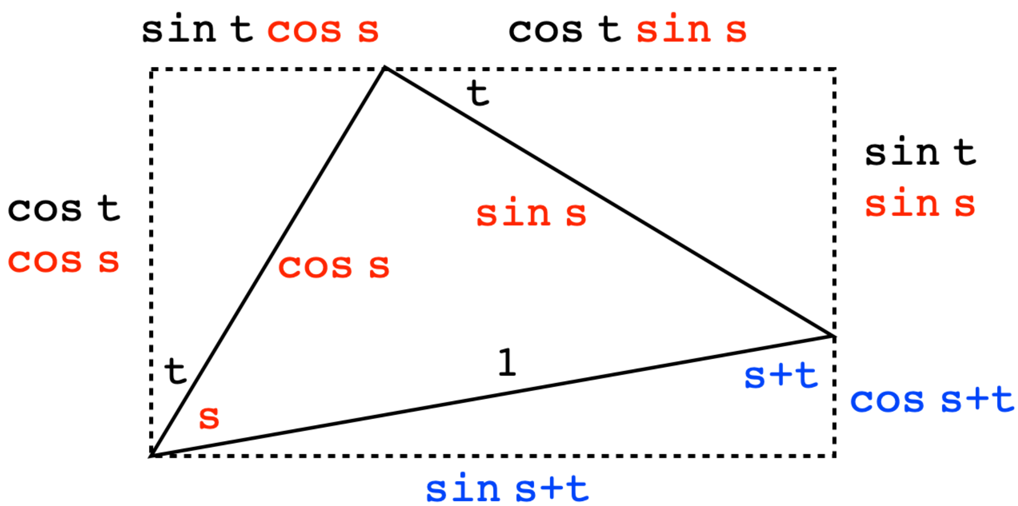
We can cancel ab from all three terms:

$$\sin(\phi + \theta) = \sin\phi\cos\theta + \sin\theta\cos\phi$$

This is the general form for the sine of the sum of two angles. It is apparent that it reduces to the previous double angle formula in the case where $\theta = \phi$.

alternate version

Just to be complete, here is an alternative version I found that may be even simpler.



yet another proof of Pythagoras's Theorem

Consider the right triangle in a unit circle with opposite side $\sin\theta$ and adjacent side $\cos\theta$. We will prove that

$$1 = \cos^2\theta + \sin^2\theta$$

Proof.

We have the formula above:

$$\cos A + B = \cos A\cos B - \sin A\sin B$$

Let $A = \theta$ and $B = -\theta$.

$$\cos \theta + -\theta = \cos \theta \cos -\theta - \sin \theta \sin -\theta$$

For the left-hand side, we have

$$\cos \theta + -\theta = \cos 0 = 1$$

And for the right-hand side we use the fact that cosine is an odd function ($\cos -s = \cos s$). Sine is an even function ($\sin s = -\sin -s$) so that gives

$$\cos \theta \cos \theta - \sin \theta (- \sin \theta)$$

$$= \cos^2 \theta + \sin^2 \theta$$

Bringing back the left-hand side

$$1 = \cos^2 \theta + \sin^2 \theta$$

$\square$

Chapter 38

Sum of angles by Ptolemy

general sum of angles

Previously, we saw the double angle formulas

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

We'd like to find a general formula for the sine of two angles added together. $\sin A + B = ?$

Of course, if $A = B$ then the formula must reduce to what we have above, so it seems there are two reasonable possibilities if we keep things simple:

$$\sin A + B \stackrel{?}{=} \sin A \cos A + \sin B \cos B$$

$$\sin A + B = \sin A \cos B + \sin B \cos A$$

Without explaining it at present, we will just assume it is known that $\sin 0 = 0$ and $\cos 0 = 1$. It is easy to see that the sine and cosine approach these values for very small angles but a proper explanation will have to wait for analytic geometry.

If you start with any angle A, set $B = 0$, and then use the first formula it gives $\sin A + 0 = \sin A \cos A + 0 \cdot 1$ which means $\sin A = \sin A \cos A$ which makes no sense. Indeed, we will now derive the second formula as the correct result.

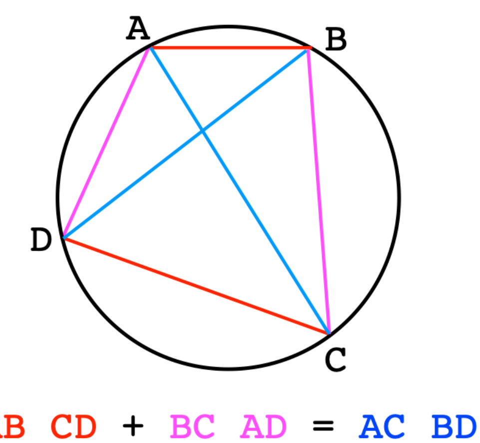
If we start with the cosine formula there seems to be only one simple candidate, namely

$$\cos A + B = \cos A \cos B - \sin A \sin B$$

We use Ptolemy's theorem, which we proved previously using the properties of parallelograms. In fact, a big reason for including this section now is to illustrate the power of this theorem, which is normally thought to be difficult, but since we found an easy proof we are in great shape.

Ptolemy's theorem says that if a four-sided figure, a quadrilateral, has all of its vertices on a circle, then we can form the two products of opposing sides and add them together to obtain the product of the diagonals.

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$



Our first proof of this is **here**. And we'll revisit the topic later.

Notice that if the quadrilateral is a rectangle, then the diagonals of the rectangle are also diagonals of the circle (**here**), so this theorem gives a simple proof of the Pythagorean theorem.

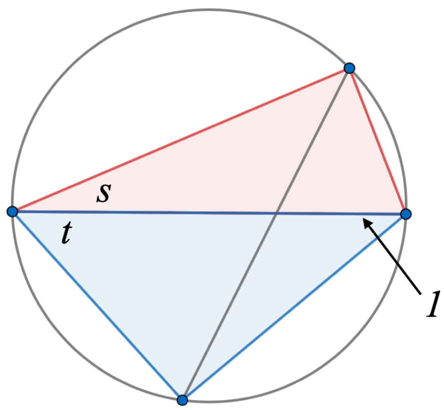
sum of angles

Ptolemy's theorem can be used to give direct proofs of the sum (and difference) of angles formulas for both sine and cosine. It's a fun exercise because the results come easily from inspired diagrams with slight changes between them.

It helps that we have an idea about what we want.

We need to say a bit more before we start. Each diagram contains a diameter of the circle, scaled so its length is 1.

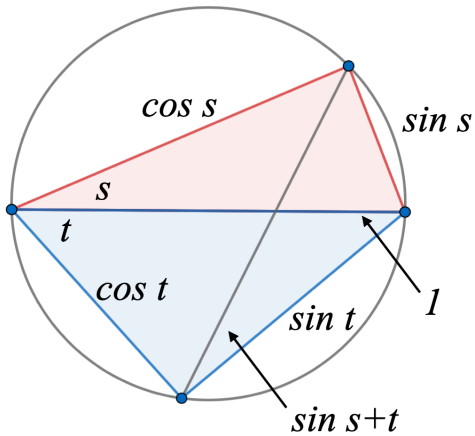
In two cases that length is also a diagonal of the quadrilateral, and in the other two it is a side. In the first case, the product with the other diagonal of the quadrilateral is just equal to whatever value that second diagonal has. Here is the first figure:



There are two right angles on the circle with sides extending to the ends of the diameter. This makes the setup a natural for sine and cosine formulas.

Second, a peripheral or inscribed angle θ is related to its chord length by $L = 2R\sin\theta$. We saw this important result previously (**here**). Since we have $2R = 1$, this simplifies to $L = \sin\theta$, where in this case $\theta = s + t$.

We start with the sum of sines. The idea is that $\sin s + t$ should be the second diagonal.



From there the formula basically writes itself.

$$\sin s + t = \sin s \cos t + \sin t \cos s$$

Our analysis of what was likely to be the form of the final result turns out to be correct.

sine of the difference

Algebraically, the difference of sines is easily derived using the fact that cosine is an even function, $\cos(x) = \cos(-x)$ while sine is odd, so $\sin(x) = -\sin(-x)$.

Thus, substituting $-t$ for t changes the "sign" of the second term but not the first.

$$\sin s + (-t) = \sin s \cos(-t) + \sin(-t) \cos s$$

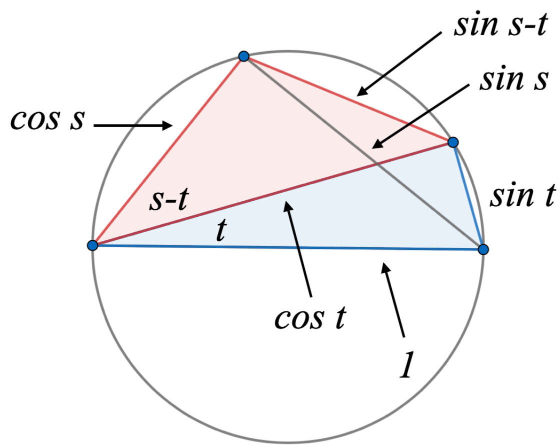
$$= \sin s \cos t - \sin t \cos s$$

However, we can use a diagram (kind of) like the one above to do this formula too. See:

https://www.cut-the-knot.org/proofs/sine_cosine.shtml

The trick is that, because of the minus sign on the right-hand side in the final formula, we want $\sin s - t$ to be one of the sides. So the diagonal of the circle must be opposite, and also be a side of the quadrilateral.

Now, s is the whole angle, i.e. s - t + t = s. So one diagonal is $\sin s$ and the other is $\cos t$.



And again, the formula writes itself:

$$\sin(s - t) + \sin t \cos s = \sin s \cos t$$

which rearranges to give

$$\sin s - t = \sin s \cos t - \sin t \cos s$$

cosine of the sum

For the cosine formulas, we'll need to relate them to the sine of another angle. Recall that if

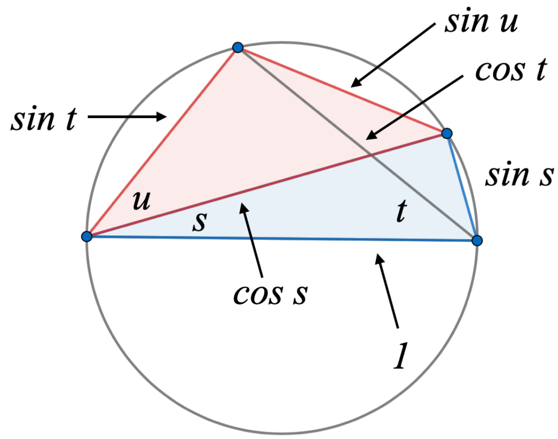
$$s + t + u = 90$$

then

$$\cos(s + t) = \sin u = \sin 90 - (s + t)$$

So for the first cosine formula we add an additional angle u, and use the fact that $s + t$ is complementary to u.

We make $\sin u = \cos s + t$ one of the sides, because we know the formula has a minus sign in it.



$$\sin s \sin t + \cos(s + t) = \cos s \cos t$$

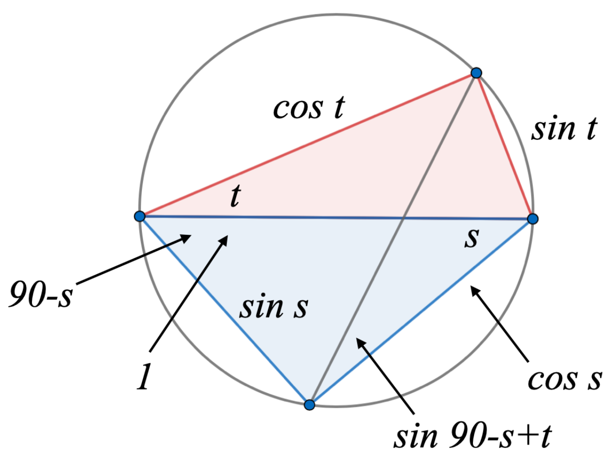
$$\cos s + t = \cos s \cos t - \sin s \sin t$$

cosine of the difference

Last is the difference formula for cosine. Again, we can derive this formula easily by using the fact that cosine is even and sine is odd.

Knowing where we're headed, we want opposing sides to be both sine or both cosine, and we have them as sides, because they add in the final formula.

So then, somehow, the diagonal must be $\cos s - t$.



The complementary angle to s is $90 - s$. Adding it to the adjacent angle t we have that the diagonal is $\sin(90 - s + t)$. But that is

$$\cos 90 - (90 - s + t) = \cos s - t$$

which is just what we need.

$$\cos s - t = \cos s \cos t + \sin s \sin t$$

Perhaps you may object that possibly it coulbe be that $s < t$ so $t - s < 0$. But we can also use the angle on the opposite side and write:

$$\sin s + (90 - t) = \cos 90 - \ [\ (90 - t) + s\ ]$$

$$= \cos t - s$$

Again, this works because cosine is an even function so $\cos s - t = \cos t - s$, or if you prefer you can use the fact that the sine of an angle is equal to the the sine of its supplementary angle, which is obvious from the diagram.

$$\sin s + (90 - t) = \sin 180 - \ [\ s + (90 - t)\ ] = \sin t + (90 - s)$$

Chapter 39

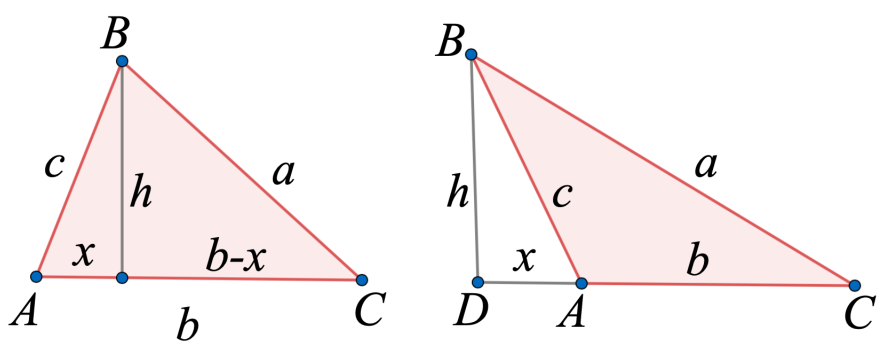
Law of Cosines

This theorem is used extensively in geometry. It relates the side lengths of any triangle to the cosine of an angle. For example, if the side opposite $\angle A$ is a, then $a^2 = b^2 + c^2 - 2bc \cos A$.

Rewritten appropriately, it can give any side in terms of the other two.

Consider $\triangle ABC$ in two versions. In one (left panel), $\angle A$ is acute, and in the other (right panel) $\angle A$ is obtuse. Note that the supplementary $\angle BAD$ on the right is equal to $\angle A$ from the acute case. In particular, it has the same cosine, x/h.

Let the sides opposite be a, b, c, as usual.



For the acute case, we have

$$a^2 = h^2 + (b - x)^2$$
$$c^2 = h^2 + x^2$$

Subtracting:

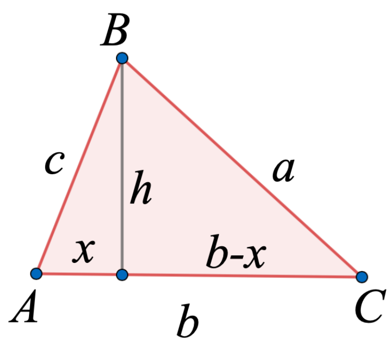
$$a^2 - c^2 = (b-x)^2 - x^2 = b^2 - 2xb$$

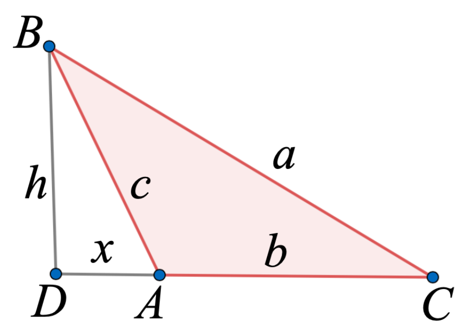
Rearranging

$$a^2 = b^2 + c^2 - 2xb$$

x/c is the cosine of $\angle A$ so $x = c\cos A$ so then finally

$$a^2 = b^2 + c^2 - 2bc\cos A$$





For the obtuse case the initial arithmetic has a change of sign. (a is still the side opposite $\angle A$, but it is obviously bigger now).

$$a^2 = h^2 + (b+x)^2$$

$$c^2 = h^2 + x^2$$

Subtracting as before:

$$a^2 - c^2 = (b+x)^2 - x^2 = b^2 + 2xb$$

Rearranging

$$a^2 = b^2 + c^2 + 2xb$$

There is a change of sign for the last term.

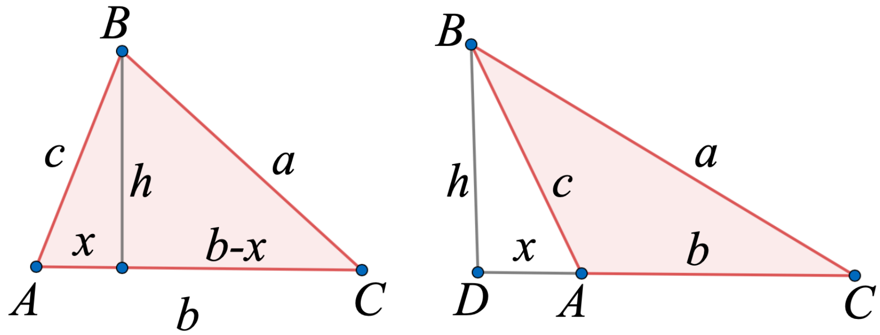
However, there is another difference for the obtuse case. Now the angle *inside* the triangle is supplementary to the angle whose cosine is x/c.

Recall the sum of angles formula for cosine. Let θ and ϕ be supplementary so $\theta + \phi = \pi$. Then

$$\cos\theta = \cos\pi - \phi$$

$$= \cos\pi\cos\phi + \sin\pi\sin\phi = -\cos\phi$$

Thus, when we think about $\angle A$ as the angle at vertex A *inside* the triangle



for the obtuse case we have

$$\cos A = -\frac{x}{c} \qquad\qquad -c\cos A = x$$

giving the same ormula for both cases:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

where $\angle A$ is $\angle BAC$ for the acute case and $\angle B'AC$ for the obtuse case.

This is the law of cosines. We compute the length of one side a in terms of the two other sides b and c and the angle between them, $\angle A$. This proof is identical to Euclid Book II.12 and II.13, except for the last step.

Using the cosine in the formula is just a form of shorthand for the ratio x/c and gets rid of that pesky term x.

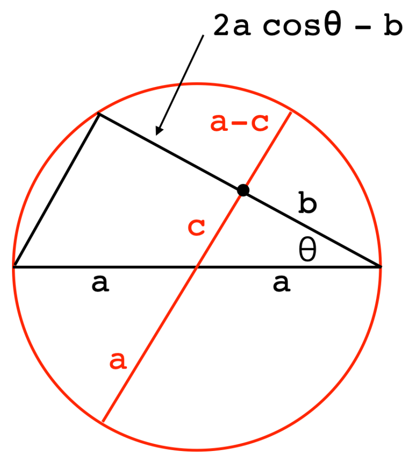
This is Pythagorean theorem with an added correction factor, $-2bc\cos A$, that depends on the angle opposite the hypotenuse and which disappears when that angle is a right angle, since the cosine is zero.

The factor is negative for an acute angle, which makes sense, since the smaller angle squeezes the hypotenuse to be smaller as well, while it is positive for an obtuse angle, giving a longer side opposite the greater angle.

alternative proof

Here is an alternative derivation based on the products of parts of two secants, for the special case of a right triangle.

Draw a right triangle on one diameter in a circle of radius a. Draw a second diameter such that it crosses the base of the right triangle at a right angle, forming a smaller, similar right triangle.



The smaller triangle has sides a, b and c. The lengths of the other parts are easy to compute. Now multiply

$$(a+c)(a-c) = b(2a\cos\theta - b)$$

$$a^2 - c^2 = 2ab\cos\theta - b^2$$

The result follows immediately.

$$a^2 - c^2 = 2ab\cos\theta - b^2$$

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

□

algebraic proof

In $\triangle ABC$ drop the altitude from vertex A to side a opposite then

$$a = b\cos C + c\cos B$$

In the same way:

$$b = a\cos C + c\cos A$$

$$c = a \cos B + b \cos A$$

Multiply the first by a

$$a^2 = ab \cos C + ac \cos B$$

In the same way

$$b^2 = ab \cos C + bc \cos A$$

$$c^2 = ac \cos B + bc \cos A$$

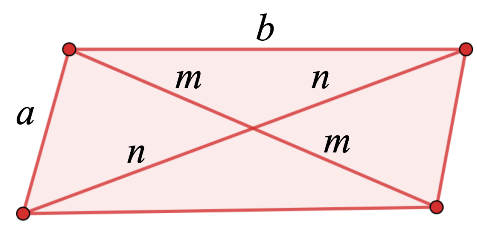
Subtract the first and second from the third:

$$c^2 - a^2 - b^2 = -2ab \cos C$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

parallelogram sides

The law of cosines leads to an interesting relationship between the sides and diagonals of a parallelogram. Recall that the diagonals divide the figure into two congruent triangles. They also bisect one another.



Proof.

For convenience we label the half-diagonals as m and n. Applying the theorem twice we have

$$a^2 = m^2 + n^2 - 2mn \cos \theta$$

$$b^2 = m^2 + n^2 + 2mn \cos \theta$$

The sign change on the last term arises because the angles at the center are supplementary, so their cosines are negatives. By addition:

$$a^2 + b^2 = 2(m^2 + n^2)$$

But if e and f are the diagonals then $m^2 = e^2/4$ and $n^2 = f^2/4$ so

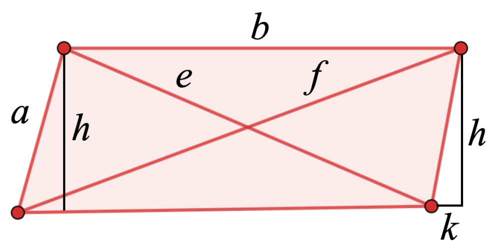
$$2(a^2 + b^2) = e^2 + f^2$$

The sum of the squares of all four sides is equal to the sum of the squares of the diagonals.

Of course this is correct for a rectangle (by Pythagoras's theorem), but it remains true when the shape leans to one side.

□

Al alternative proof (Byer) is to apply Pythagoras directly



Proof.

$$\begin{aligned}
a^2 &= h^2 + k^2 \\
e^2 &= h^2 + (b-k)^2 \\
f^2 &= h^2 + (b+k)^2 \\
e^2 + f^2 &= 2h^2 + 2b^2 + 2k^2 \\
&= 2(a^2 + b^2)
\end{aligned}$$

□

philosophy

Trigonometry is not just about problems like finding the measure of the angle complementary to 23° as 67°.

Instead, trigonometry uses formulas like the sum of angles, and especially, the law of cosines, to solve problems in calculus.

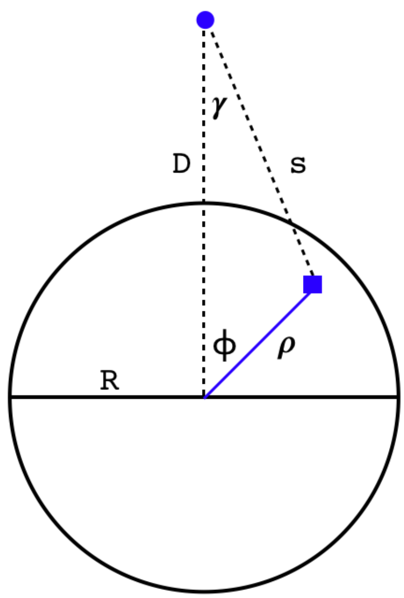
One of the most famous applications came when Newton derived Kepler's laws about the orbits of the planets. Originally, to do that he made the approximation that the mass of the earth acts *as if* it were concentrated at a single point corresponding to the center of the earth, and likewise for the sun.

However, for a rigorous demonstration he needed to prove that this approximation is correct. We do not have the tools yet to see how he did that, but here are two equations from my write-up:

$$\rho^2 = D^2 + s^2 - 2Ds\cos\gamma$$

$$\cos\gamma = \frac{D^2 + s^2 - \rho^2}{2Ds}$$

and the relevant diagram:



You can probably recognize the law of cosines at work.

Trigonometry is hugely important in math and science. Although it has (simple) applications for activities like surveying that is not at all what it is about.

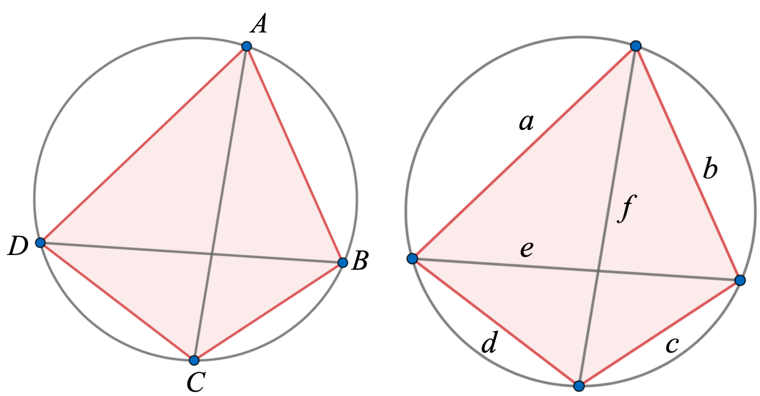
Part VIII

Applications of similarity

Chapter 40

Ptolemy revisited

In a previous **chapter** we introduced Ptolemy's theorem.

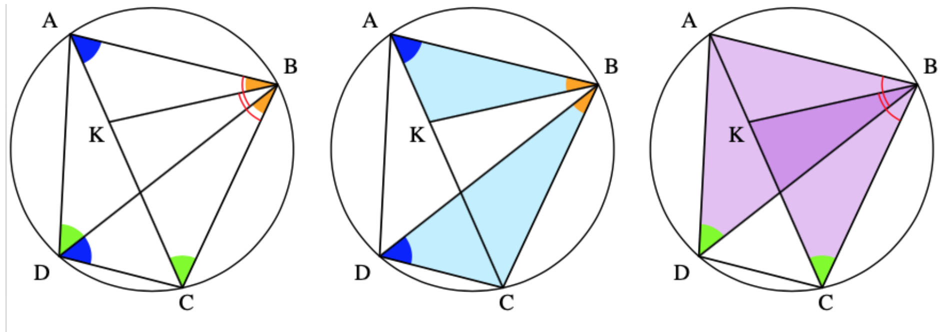


In the left panel

$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

or somewhat more compactly in the right panel: $ac + bd = ef$.

Here we provide two more proofs of this theorem, as examples of wonderful proofs, and then explore some consequences.

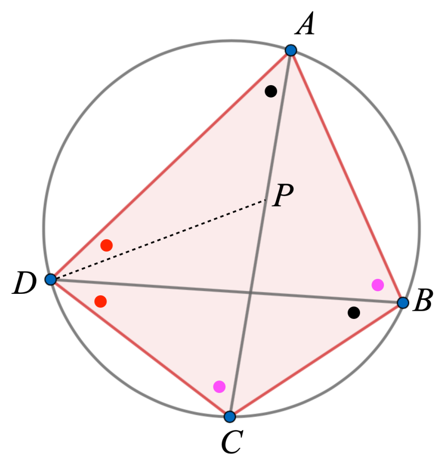


Above is a graphic from wikipedia that shows where we're going in the first proof. We will form two sets of similar triangles and use our knowledge about corresponding ratios.

https://en.wikipedia.org/wiki/Ptolemy%27s_theorem

Ptolemy's theorem from similar triangles

Proof.



Find P on AC such that $\angle ADP = \angle CDB$ (red dots).

Since $ABCD$ is a cyclic quadrilateral, we can find other equal angles (black and magenta dots).

We write the vertices in the same order as the equal angles.

$\triangle ADP \sim \triangle BDC$.

So

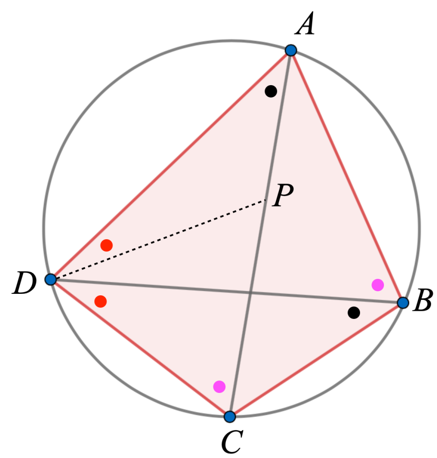
$$\frac{AD}{BD} = \frac{DP}{CD} = \frac{AP}{BC}$$
$$AD \cdot BC = AP \cdot BD$$

Since $\angle PDB$ is shared, $\angle ADB = \angle PDC$.

$\triangle PDC \sim \triangle ADB$.

So

$$\frac{PD}{AD} = \frac{CD}{BD} = \frac{PC}{AB}$$
$$AB \cdot CD = PC \cdot BD$$



Adding

$$AB \cdot CD + AD \cdot BC = PC \cdot BD + AP \cdot BD$$
$$= AC \cdot BD$$

□

This is Ptolemy's theorem.

Yiu also proves the converse theorem.

The dots in this proof make it clear which two pairs of triangles are similar, and I've taken care to list the vertices of the triangles in the same order as the sides in each pair of similar triangles, from smallest to largest.

For an alternate notation see **here**.

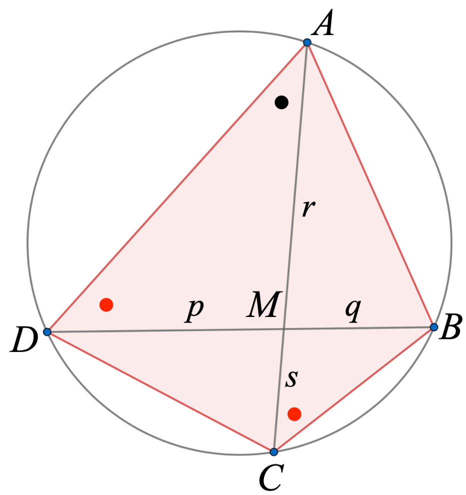
proof by switching sides

(adapted from wikipedia).

https://en.wikipedia.org/wiki/Ptolemy%27s_theorem

In the proof below, we denote the area of polygons such as $ABCD$ as $(ABCD)$.

Proof.



We add up (twice) the areas of the component triangles:

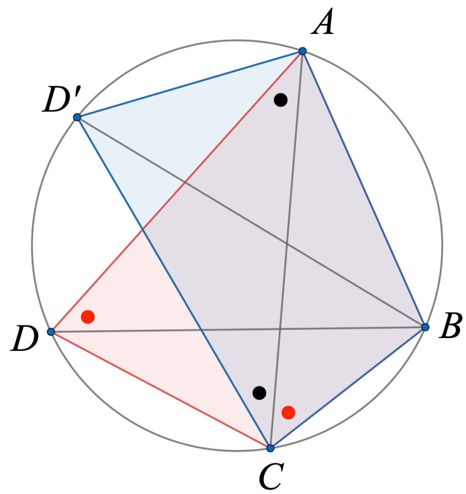
$$
\begin{aligned}
2(AMB) &= qr \cdot \sin AMB \\
2(BMC) &= qs \cdot \sin BMC \\
2(CMD) &= ps \cdot \sin AMB \\
2(AMD) &= pr \cdot \sin BMC
\end{aligned}
$$

But $\sin AMB = \sin BMC$. Hence twice the area of $(ABCD)$ is

$$
\begin{aligned}
2(ABCD) &= (qr + qs + pr + ps) \cdot \sin AMB \\
&= (q + p)(r + s) \cdot \sin AMB \\
&= AC \cdot BD \cdot \sin AMB
\end{aligned}
$$

We're on to something. Now, the great idea.

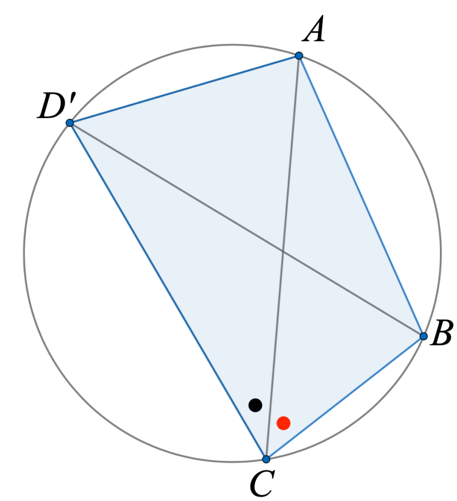
Move the point D to D' such that $AD' = CD$ and $AD = CD'$.



When we do this, note two things. First, $\angle CAD = \angle ACD'$ (black dots), because the arcs they intercept are equal, $AD' = CD$.

Second, $\triangle ADC \cong \triangle ACD'$ by SSS. Therefore the area hasn't changed: $(ABCD) = (ABCD')$.

Now compute (twice) the area of two component triangles.



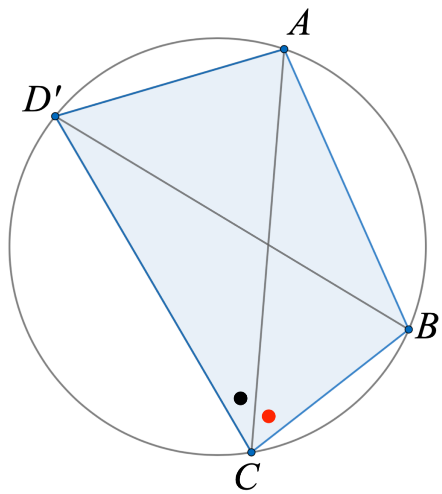
$$\begin{aligned} 2(ABD') &= AB \cdot AD' \cdot \sin \angle BAD' \\ &= AB \cdot AD' \cdot \sin \angle BCD' \end{aligned}$$

$$2(CBD') = BC \cdot CD' \cdot \sin \angle BCD'$$

since the two angles are supplementary.

But $\angle BCD' = \angle ADC + \angle CAD$. (Check out the dots).

So $\angle BCD' = \angle AMB$. Thus, their sines are equal. We have



$$2(ABD') + 2(CBD') = 2(ABCD)$$

$$[\ AB \cdot AD' + BC \cdot CD'\ ]\ \sin \angle BCD' = AC \cdot BD \cdot \sin AMB$$

$$AB \cdot AD' + BC \cdot CD' = AC \cdot BD$$

Finally, since $AD = CD'$ and $CD = AD'$:

$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

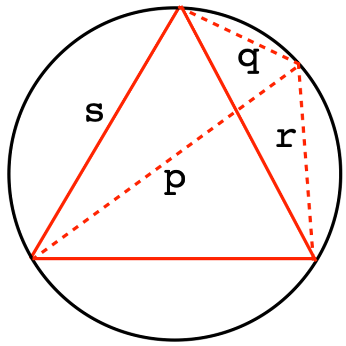
$\square$

This is Ptolemy's theorem.

corollaries

Here are just a few of the results that follow from this remarkable theorem.

equilateral triangle



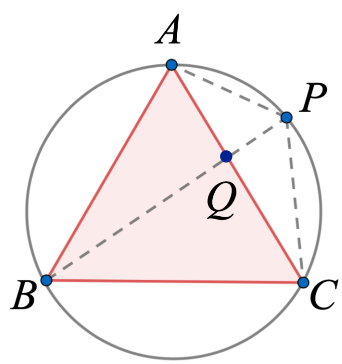
Inscribe an equilateral triangle in a circle and pick any point on the circle.

$$ps = qs + rs$$

$$p = q + r$$

We proved this earlier, without using Ptolemy's theorem, as Van Schooten's theorem.

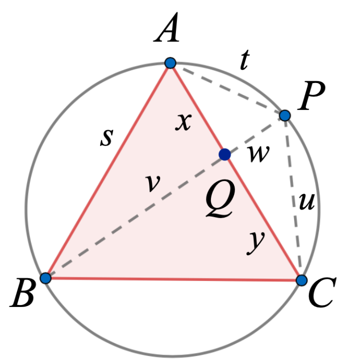
Here's a different problem from basically the same diagram (Coxeter).



$$\frac{1}{PA} + \frac{1}{PC} = \frac{1}{PQ}$$

Let us re-label with $PA = t$, $PC = u$ and $PQ = w$. We will prove that:

$$\frac{1}{t} + \frac{1}{u} = \frac{1}{w}$$



Proof.

We have similar triangles:

$$\triangle AQB \sim \triangle PQC, \qquad \triangle AQP \sim \triangle BQC$$

Which gives the ratios

$$\frac{v}{y} = \frac{s}{u} = \frac{x}{w}, \qquad \frac{w}{y} = \frac{t}{s} = \frac{x}{v}$$

Thus:

$$\frac{s}{u} = \frac{x}{w}, \qquad \frac{s}{t} = \frac{y}{w}$$

Adding

$$\frac{s}{u} + \frac{s}{t} = \frac{x+y}{w} = \frac{s}{w}$$
$$\frac{1}{t} + \frac{1}{u} = \frac{1}{w}$$

□

Pythagorean theorem

Let the quadrilateral be a rectangle. The the sum of squares of opposing sides is

$$a^2 + b^2$$

Triangles made by opposing diagonals are congruent, so the diagonals are equal in length. The diagonal is the hypotenuse, hence

$$a^2 + b^2 = c^2$$

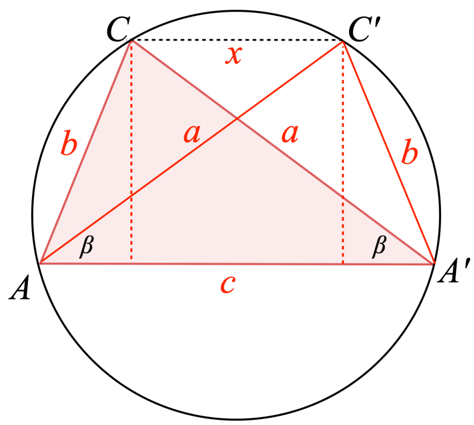
We saw this proof previously (**here**).

Law of Cosines

Draw $\triangle ABC$ (suppress the B label) and then draw another triangle congruent with it, with a shared base, and all four points in a circle, forming a cyclic quadrilateral.

Relying on previous work with a rectangle in a circle (**here**), we know this construction is possible.

The points are A, A', C, C'. β marks the original $\angle B$, but will not be used.



We need an expression for x. We have that the base of the altitude from C to side c is a distance from A equal to $(c-x)/2$. It follows that

$$\frac{c-x}{2} \div b = \cos A$$

$$c - x = 2b \cos A$$

$$x = c - 2b \cos A$$

Now, apply Ptolemy's Theorem. We have:

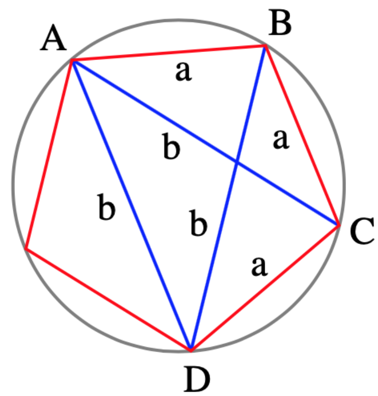
$$a^2 = b^2 + cx$$

$$= b^2 + c(c - 2b \cos A)$$

$$= b^2 + c^2 - 2bc \cos A$$

$\square$

golden mean in the pentagon



Take four vertices of the regular pentagon and draw two diagonals. From the theorem, we have

$$b \cdot b = a \cdot a + a \cdot b$$
$$\frac{b^2}{a^2} = 1 + \frac{b}{a}$$

Rather than use the quadratic equation, rearrange and add 1/4 to both sides to "complete the square":

$$\frac{b^2}{a^2} - \frac{b}{a} + \frac{1}{2^2} = 1 + \frac{1}{2^2}$$

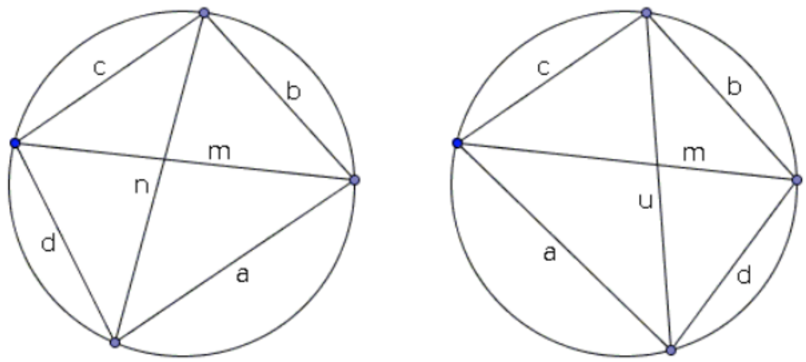
So

$$(\frac{b}{a} - \frac{1}{2})^2 = \frac{5}{4}$$
$$\frac{b}{a} - \frac{1}{2} = \pm\ \frac{\sqrt{5}}{2}$$
$$\frac{b}{a} = \frac{1 \pm \sqrt{5}}{2}$$

This ratio b/a is known as ϕ, the golden mean.

diagonals

Let us look at something like what we used for the proof of Ptolemy's theorem in the beginning.

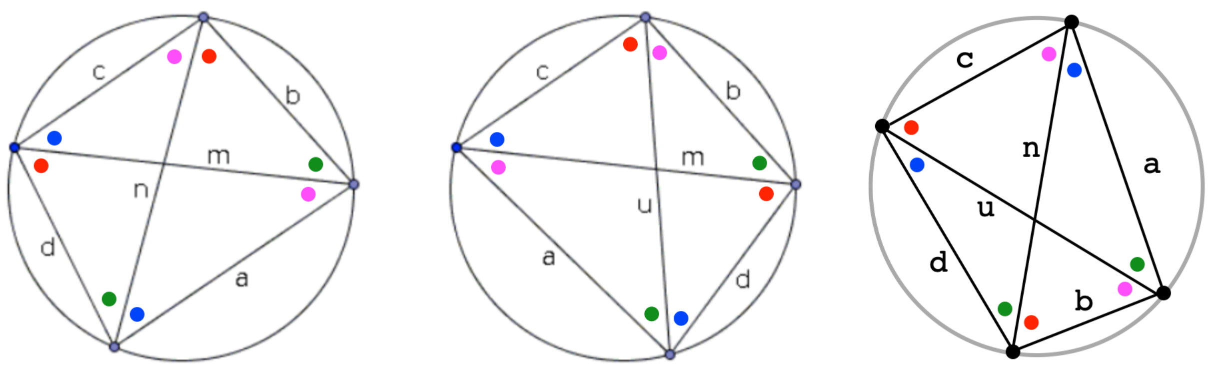


$$nm = ac + bd$$

We move one of the points, exchanging sides a and d. Then, one of the diameters, n, changes length to u.

$$mu = ab + cd$$

If, instead, we exchange sides a and b, the old m changes to u. Why?



Mark the peripheral angles with equal arcs ($abcd$: red, blue, green, magenta).

The triangle with sides b and d in the middle, and magenta plus red for the vertex angle, is congruent to one in the right panel. So their long sides are equal, both have length u.

Thus,

$$nu = ad + bc$$

We get a formula for the square of the diagonal:

$$m^2 = \frac{(mu)(nm)}{nu} = \frac{(ab+cd)(ac+bd)}{(ad+bc)}$$

There is a similar formula for n^2. These formulas are sometimes attributed to Brahmagupta. This beautiful proof is due to Paramesvara (14th century).

https://www.cut-the-knot.org/proofs/PtolemyDiagonals.shtml

The ratio is

$$\frac{m}{n} = \frac{ab+cd}{ad+bc}$$

which is referred to as Ptolemy's second theorem.

Chapter 41

Euler Theorem

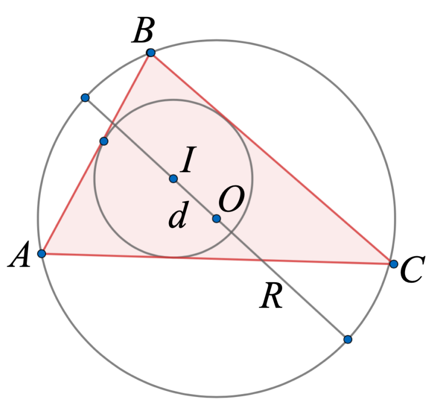
In this chapter we talk about the famous Euler Theorem (the one in geometry, there are others).

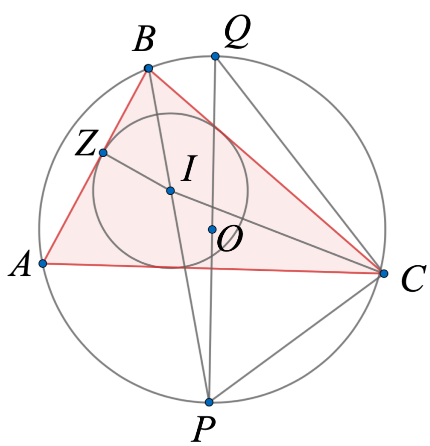
Consider $\triangle ABC$ with its circumcircle on center O with radius R, and its incircle on center I with radius r.

Let the distance between O and I be d.

Then the distance from I to the outer circle is $R + d$ in one direction and $R - d$ in the other direction. The product of parts of the chord is

$$(R + d)(R - d) = R^2 - d^2$$





We consider another chord drawn through I, namely the bisector of $\angle B$, extended to meet the circumcircle at P.

Let the bisected angles be $2\beta = \angle B$ and $2\gamma = \angle C$, as usual.

Draw the diameter POQ. By Thales' theorem, $\triangle PQC$ is right.

Let $IZ \perp BZA$, so $\triangle IBZ$ is right.

By inscribed angles, $\angle PQC = \angle PBC = \beta$.

Thus, the two right triangles are similar, since they both contain β.

Form the ratios:

$$\frac{IZ}{IB} = \frac{PC}{PQ}$$

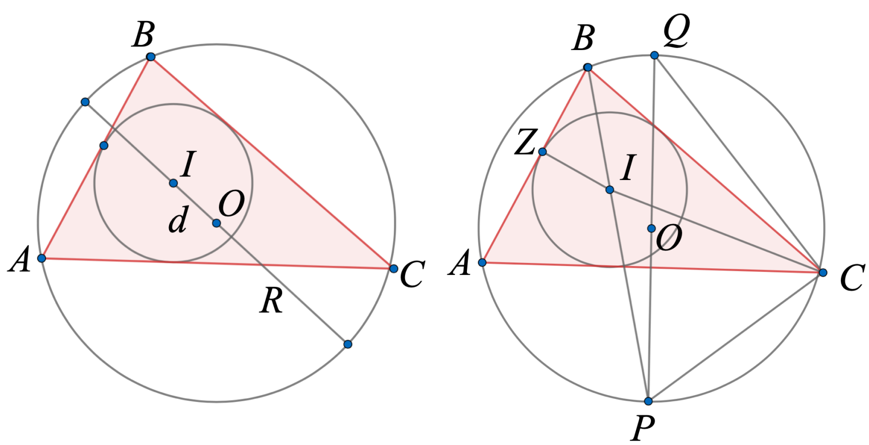
Substituting

$$\frac{r}{IB} = \frac{PC}{2R}$$

$$2rR = PC \cdot IB$$

If we can show that $PC = PI$, we will have that the right-hand side of the previous equation is equal to $PI \cdot IB = R^2 - d^2$, by crossed chords.

We consider $\triangle PCI$.



$\angle PCI$ is equal to $\gamma + \beta$ by inscribed angles.

$\angle CPI$ is equal to $\angle A$.

By sum of angles, $\angle PIC$ is also equal to $\gamma + \beta$.

With equal base angles, by I.6 $\triangle PCI$ is isosceles.

Thus $PI = PC$, and the main result follows:

$$2rR = R^2 - d^2$$

$$= (R + d)(R - d)$$

$$\frac{1}{r} = \frac{2R}{(R + d)(R - d)}$$

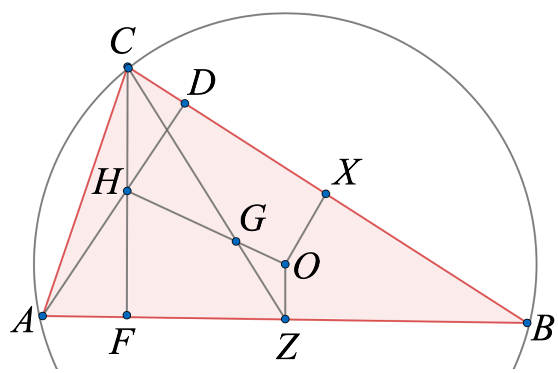
$$= \frac{1}{R + d} + \frac{1}{R - d}$$

Chapter 42

Euler Line

In this chapter we talk about the famous Euler Line of a given triangle:

https://en.wikipedia.org/wiki/Euler_line



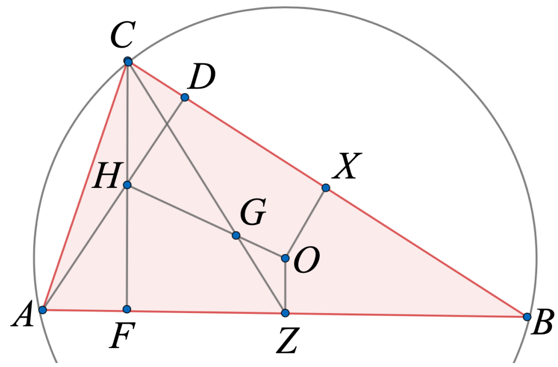
The proof is quite easy (once you know how).

Recall that the circumcenter is the center of the circle that contains all three vertices, and it can be found by erecting the perpendicular bisectors of the sides. Side a opposite $\angle A$ is bisected at X, while side c opposite $\angle C$ is bisected at Z. The perpendicular bisectors OX and OZ meet at O.

The orthocenter H is the point where altitudes drawn from the three vertices of a triangle cross. Here AD meets CF at H.

The centroid G is the point where the medians cross, before they bisect the opposite sides. CZ is one of the medians of $\triangle ABC$.

We will assume that these points actually exist (that the three altitudes or three medians are concurrent). That proof is coming later.



Suppose we find O as described above, draw the median CZ, find G arithmetically, and then extend OG to find H such that $HG = 2OG$.

We will show that H is the orthocenter of $\triangle ABC$, that it actually lies on CF.

We also need a couple of preliminary results. We will prove that the centroid divides the medians by the $2/3 - 1/3$ rule: $CG = 2ZG$. Let's do this below as a lemma.

The second is SAS similarity. In the discussion of similarity for a general or arbitrary triangle, we showed that if two triangles have two pairs of sides in the same proportion, and the angle between is also equal, then they are similar triangles.

Lemma

We depend on the area-ratio theorem. If two triangles have one vertex shared and the bases lie on the same line, then their areas are in proportion as the lengths of the bases. The reason is that they have the same altitude, since for a given line and a point not on the line, only one perpendicular can be drawn from the point to the line.

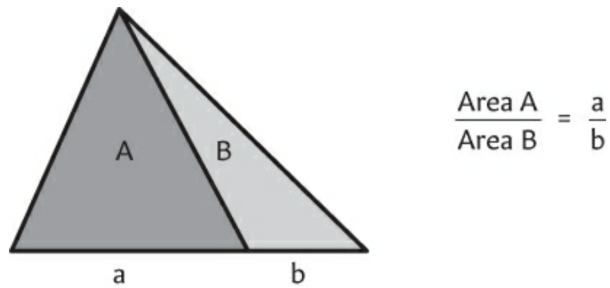
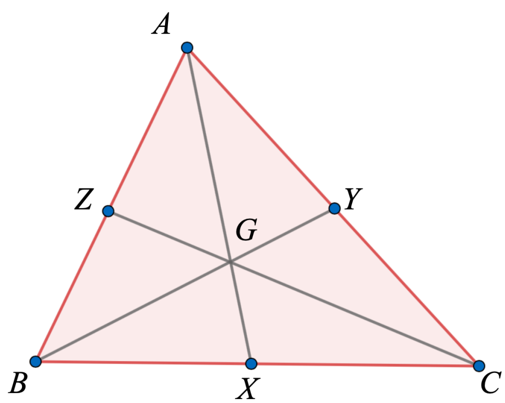
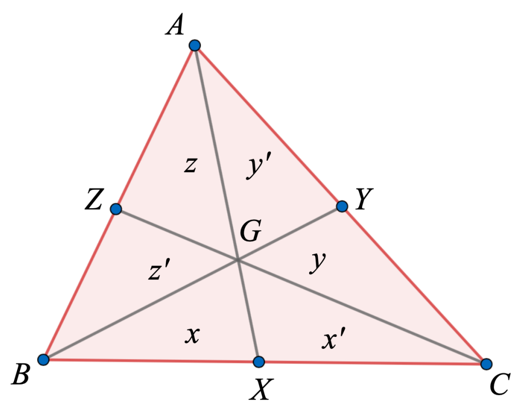


Fig. 120 An area-ratio theorem.

So now we consider an arbitrary triangle, divided by its three medians. We will show that each of the small triangles is equal in area.





Compare $\triangle GBX$ and $\triangle GCX$ to find that they are equal in area, since $GX = CX$. Thus area x = area x'. and so on. Similarly, compare $\triangle ABX$ and $\triangle ACX$ to find that they are equal in area. It follows that:

$$x + z + z' = x' + y + y'$$

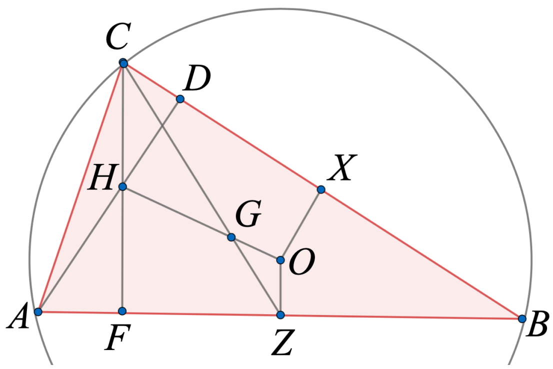
which reduces to $z = y$. The same argument will give $x = y$ and $x = z$.

Finally, we use what is technically the converse of the area-ratio theorem. We have that the area of $\triangle ABG$ is twice that of $\triangle AYG$. Since they have the same altitude, it follows that the bases are in proportion: $BG = 2YG$. Again, this can be extended to any median by the same argument.

□

Proof.

Returning to the main problem, we justify the placement of G by noting that $CG = 2ZG$. We found H by $HG = 2OG$. Finally, we have vertical angles at G. It follows that $\triangle CGH \sim \triangle ZGO$ with sides in proportion $2:1$.



It follows that $\angle CHG = \angle ZOG$.

Thus, by alternate interior angles, we find that $OZ \parallel CHF$, which means that since $OZ \perp AFZB$, so is CH when extended $\perp AFZB$.

But again, there is only one perpendicular to be drawn from C through $ABZB$ at F, thus, CF *is* the altitude to the vertex C in $\triangle ABC$.

$\square$

Two more quick points. First, in an equilateral triangle, the three points O, G and H coincide — they are the same point. Second, there is much more to the Euler line, including the center of the *nine point circle*, which we discuss in the other volume.

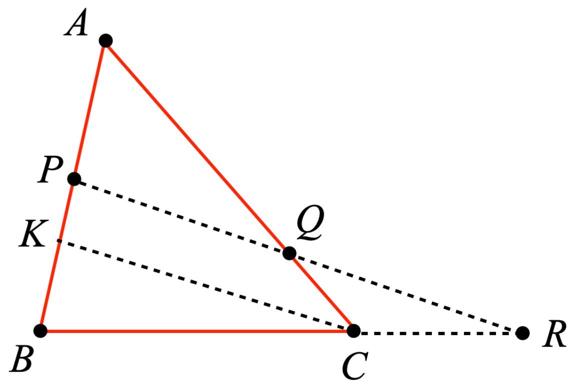
Chapter 43

Menelaus's theorem

In this chapter we establish a simple but very valuable result called **Menelaus's theorem**, due to Menelaus of Alexandria (the geometer, not the mythological figure). He lived about 100 CE. It is supposed that Menelaus grew up in Alexandria but is known to have lived in Rome.

Menelaus's theorem

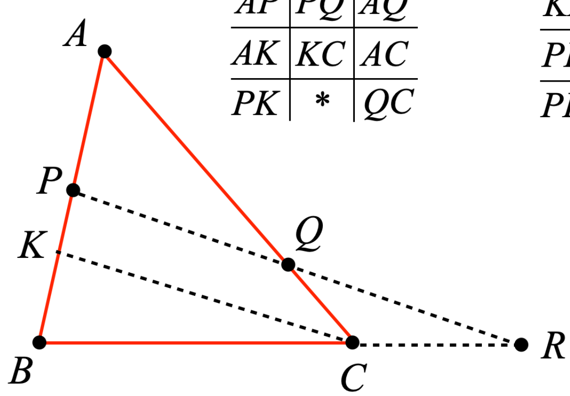
Consider $\triangle ABC$ and a line that goes through the triangle (not through a vertex and not along a side), called a transversal. There is one side, which when it is extended, meets the transversal. Here the transversal PQR meets BC at R.



Proof.

Draw a line parallel to the transversal, CK. We have two pairs of similar triangles.

The first is $\triangle APQ \sim \triangle AKC$, and the second is $\triangle KBC \sim \triangle PBR$. I will write the ratios as shown in the figure. I call these ratio boxes.



AP	PQ	AQ
AK	KC	AC
PK	*	QC

KB	BC	KC
PB	BR	PR
PK	RC	PR-KC

From the first set:

$$\frac{AP}{AQ} = \frac{PK}{QC}$$

And from the second set:

$$\frac{PB}{BR} = \frac{PK}{RC}$$

Isolate PK and equate:

$$\frac{AP}{AQ} \cdot QC = \frac{PB}{BR} \cdot RC$$

$$\frac{AP}{PB} \cdot \frac{BR}{RC} \cdot \frac{QC}{AQ} = 1$$

□

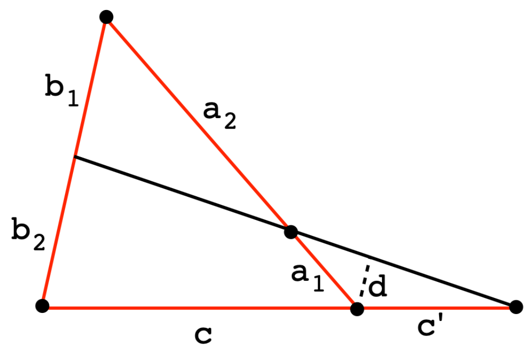
We will show several different proofs, for practice.

In the figure below, subscripts indicate the parts of a side, for example, a_1 and a_2 are the components of side a.

$c + c'$ is the length of the whole side *plus the extension*.

We will show that the product of three ratios is equal to 1:

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c + c'}{c'} = 1$$



Proof.

Draw the dotted line segment parallel to b and label it d. We have two pairs of similar triangles. The first pair has side ratios

$$\frac{d}{a_1} = \frac{b_1}{a_2}$$

while the second has

$$\frac{d}{c'} = \frac{b_2}{c + c'}$$

Combining the two results:

$$d = \frac{a_1 b_1}{a_2} = \frac{b_2 c'}{c + c'}$$

So

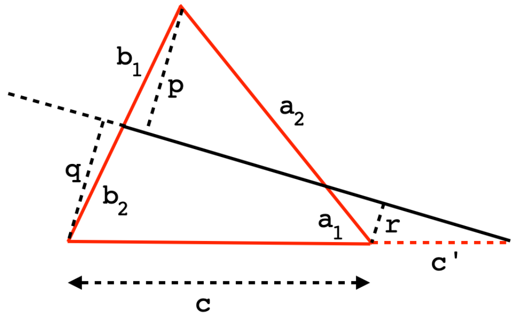
$$\frac{a_1 \cdot b_1 \cdot c + c'}{a_2 \cdot b_2 \cdot c'} = 1$$

□

Einstein's proof of Menelaus

There is a story that Einstein disliked the proofs of Menelaus's theorem based on similar triangles.

He said they were ugly, and didn't involve the vertices symmetrically. I read that Einstein's proof starts by dropping altitudes to the transversal, and then uses similar triangles. Here's what I came up with.



Proof.

The right triangle with $c+c'$ as hypotenuse is similar to the one with c' as hypotenuse. This gives

$$\frac{q}{r} = \frac{c+c'}{c'}$$

The right triangles with a_1 and a_2 as hypotenuse are similar. We form the ratio of altitudes:

$$\frac{r}{p} = \frac{a_1}{a_2}$$

Finally, the right triangles with b_1 and b_2 as hypotenuse are similar. We form the ratio of altitudes again:

$$\frac{p}{q} = \frac{b_1}{b_2}$$

Multiply the left-hand sides all together to obtain 1.

$$\frac{q}{r} \cdot \frac{r}{p} \cdot \frac{p}{q} = 1$$

Therefore, the product of the right-hand sides is also 1:

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c+c'}{c'} = 1$$

□

According to

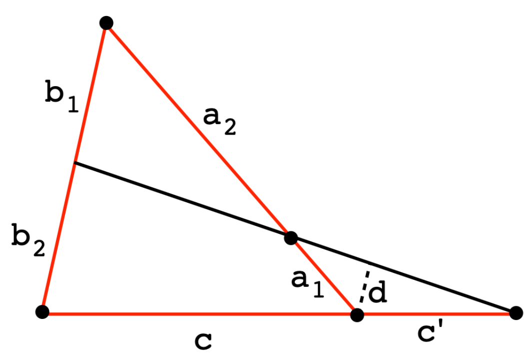
https://www.cut-the-knot.org/Generalization/MenelausByEinstein.shtml

> ... [in] correspondence between Albert Einstein and a friend of his, Max Wertheimer. In the first letter, Einstein apparently continues a discussion on elegance of mathematical proofs. A proof may require introduction of additional elements, like line AP in the first of the cited proofs. In Einstein's opinion, "... we are completely satisfied only if we feel of each intermediate concept that it has to do with the proposition to be proved."
>
> As an illustration of his viewpoint, Einstein gives two proofs of the same proposition - one ugly, the other elegant. Curiously, the proposition he proves is that of the Menelaus theorem, and the proof ugly in his view is the first of the cited proofs. He writes, "Although the first proof is somewhat simpler, it is not satisfying. For it uses an auxiliary line which has nothing to do with the content of the proposition to be proved, and the proof favors, for no reason, the vertex A, although the proposition is symmetrical in relation to A, B, and C. The second proof, however, is symmetrical, and can be read off directly from the figure."

converse

We've drawn $\triangle ABC$ as an acute triangle, with the transversal crossing through points on two sides and on the extension of the third. There are other possibilities. However, if we restrict ourselves to this case, then the converse can be stated as

Given that the product of ratios is equal to 1, the three points are collinear.



Proof.

An easy proof draws on the forward theorem. Assume that the product of ratios is 1.

Suppose the points are not colinear, so the case with product 1 has c'' and the case with colinear points has c'. We are given that

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c+c''}{c''} = 1$$

But, by the forward theorem

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c+c'}{c'} = 1$$

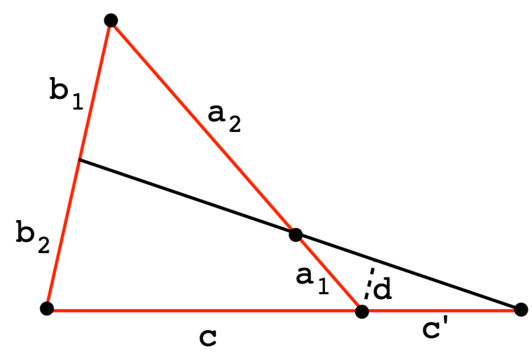
We have

$$\frac{c+c''}{c''} = \frac{c+c'}{c'}$$
$$\frac{c}{c''} = \frac{c}{c'}$$
$$\frac{1}{c''} = \frac{1}{c'}$$

We conclude that $c' = c''$ Hence, if the product is equal to 1, the points lie on the same line.

□

more about Menelaus



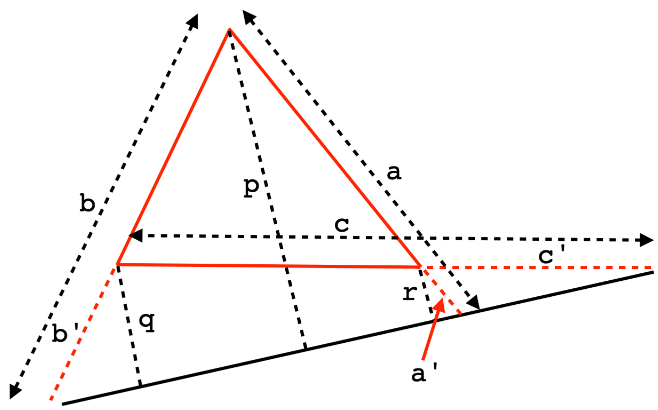
We showed that the product of three ratios is equal to 1:

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c+c'}{c'} = 1$$

One way to remember the terms is that, going around in any direction, we take the first length we run into and put it into the numerator. It doesn't matter which direction you go, as long as you take c' first when going clockwise and $c + c'$ first when going counter-clockwise.

Take the extension first if you run into it first.

We also need to prove the theorem for an alternative setup. Even though the black line in the figure below does not go through the triangle, it is technically a transversal.



Notice that we are re-defining a, b and c.

$$\frac{r}{p} \cdot \frac{p}{q} \cdot \frac{q}{r} = \frac{a'}{a} \cdot \frac{b}{b'} \cdot \frac{c}{c'}$$

Once again, we can remember this as, take the first length we run into, and put it on top.

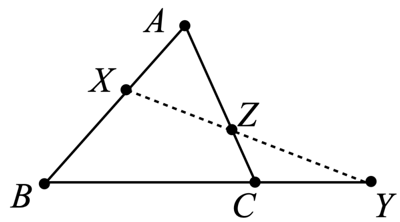
Menelaus is a preliminary lemma to an easy proof of Ceva's theorem, which becomes important in thinking about the concurrency of lines in a triangle, like those which bisect the opposing sides and meet at the centroid. Ceva's theorem gives a condition under which three lines like this are concurrent, or meet at a point.

Menelaus's theorem can be seen as in some sense as similar to Ceva, because it provides conditions under which three points can be proved collinear, or on one line.

It will come in handy in another problem, called Pappus's Theorem.

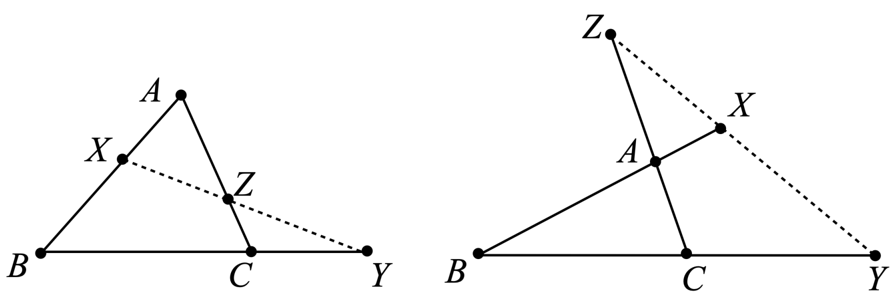
alternative definition

There is a somewhat more sophisticated version of Menelaus's Theorem which views the path around a triangle as consisting of *directed* (that is, *signed*) line segments.



The transversal is XZY, which meets the extension of BC at Y. Since Y does not lie between B and C, the line segments BY and YC point in opposite directions. The ratio $BY : YC$ thereby acquires a minus sign.

Every transversal has this property. Here (on the right) is a transversal that does not go through the triangle at all.



The path around the triangle has three parts:

A to X to $B \Rightarrow AX : XB$

B to Y to $C \Rightarrow BY : YC$

C to Z to $A \Rightarrow CZ : ZA$

Each ratio has a minus sign so the total product also has a minus sign. Menelaus's Theorem says:

$$\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZA} = -1$$

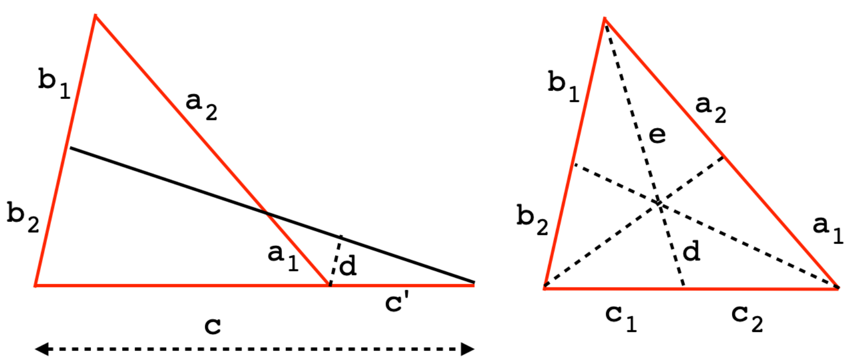
Chapter 44

Ceva's Theorem

Ceva's theorem

We previously showed a proof of Menelaus' theorem. Let us continue with Ceva's theorem (right panel, below). We will show that if the *three lines are concurrent* (they all cross at the same point), then

$$\frac{a_1 \cdot b_1 \cdot c_1}{a_2 \cdot b_2 \cdot c_2} = 1$$



We ignore here the signed or directed line segments of the original, which give minus 1 as the product above. These would cancel at the next step.

Proof.

The left panel is from the proof of Menelaus' theorem. In the right panel, consider the left half-triangle with a side composed of $e + d$. Apply Menelaus' theorem once.

$$\frac{d}{e} \cdot \frac{b_1}{b_2} \cdot \frac{c}{c_2} = 1$$

For the right half-triangle, apply Menelaus again but go clockwise from the middle:

$$\frac{d}{e} \cdot \frac{a_2}{a_1} \cdot \frac{c}{c_1} = 1$$

Combine the two results:

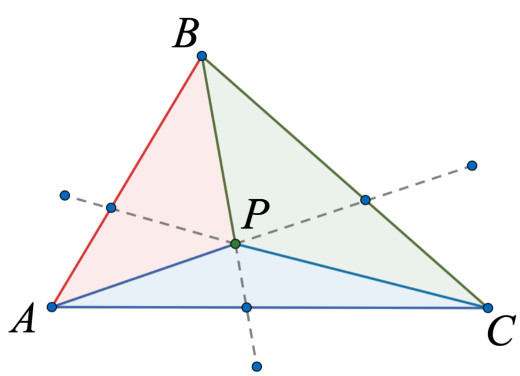
$$\frac{b_1}{b_2} \cdot \frac{c}{c_2} = \cdot \frac{a_2}{a_1} \cdot \frac{c}{c_1}$$

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1$$

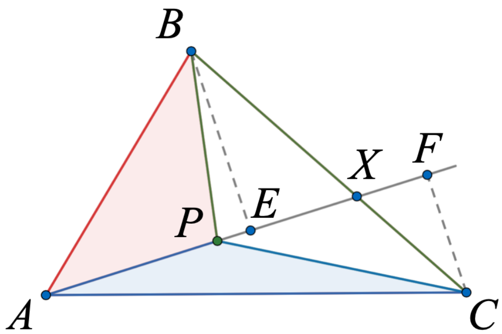
□

proof based on area

Draw $\triangle ABC$ and let P be a point somewhere inside. Extend lines from each vertex through P to reach the opposite side.



Consider two of the smaller triangles, say, $\triangle APB$ and $\triangle APC$.



If we think of AP as the base for both triangles, then BE is the altitude of the first one and CF is the altitude of the second. Since they have the same base, the ratio of areas is in the proportion:

$$\frac{\mathcal{A}_{APB}}{\mathcal{A}_{APC}} = \frac{BE}{CF}$$

But now consider $\triangle BEX$ and $\triangle CFX$. They are both right triangles with shared vertical angles. Thus

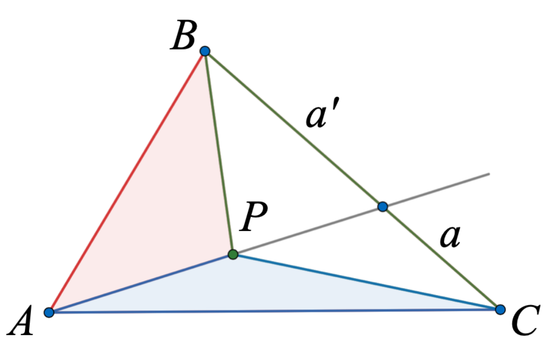
$$\triangle BEX \sim \triangle CFX$$

As similar triangles, the sides are in proportion

$$\frac{BX}{CX} = \frac{BE}{CF} = \frac{\mathcal{A}_{APB}}{\mathcal{A}_{APC}}$$

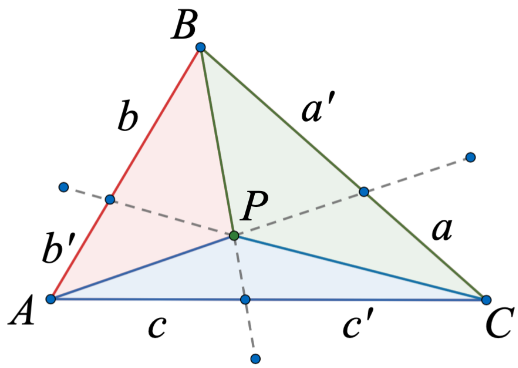
Reversing the order of the triangles and using a simpler notation for the parts of side a:

$$\frac{a}{a'} = \frac{\mathcal{A}_{APC}}{\mathcal{A}_{APB}}$$



This result is completely general. P could be anywhere, and we can compare any two of the three triangles.

$$\frac{b}{b'} = \frac{\mathcal{A}_{BPC}}{\mathcal{A}_{APC}}, \qquad \frac{c}{c'} = \frac{\mathcal{A}_{APB}}{\mathcal{A}_{BPC}}$$



Multiplying:

$$\frac{\mathcal{A}_{APC}}{\mathcal{A}_{APB}} \cdot \frac{\mathcal{A}_{BPC}}{\mathcal{A}_{APC}} \cdot \frac{\mathcal{A}_{APB}}{\mathcal{A}_{BPC}} = \frac{a}{a'} \cdot \frac{b}{b'} \cdot \frac{c}{c'}$$

Since the left-hand side cancels, we obtain:

$$\frac{a}{a'} \cdot \frac{b}{b'} \cdot \frac{c}{c'} = 1$$

□

This is Ceva's theorem.

The proof also works in reverse, We change notation slightly, since we will use the prime symbol for another purpose.

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1 \iff \text{3 lines cross at point P}$$

Proof.

Everything is the same as before, except we suppose that the ratio of the parts of a has to be slightly different in order to obtain 1 when we multiply everything together.

Of course, this would mean that the line from A would no longer go through P. Suppose the new line gives a_1' and a_2' as the components of the side a.

$$a_1 \neq a_1', \qquad a_2 \neq a_2'$$

But, by the forward theorem we have that the line which *does* go through P divides the side into a_1 and a_2 such that

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1$$

We conclude that

$$\frac{a_1}{a_2} = \frac{a_1'}{a_2'}$$

Since they are parts of the same length, the components are individually equal. That is, if the whole side is a then $a_2 = a - a_1$ and $a_2' = a - a_1'$ so

$$\frac{a_1}{a - a_1} = \frac{a_1'}{a - a_1'}$$

$$a_1 a - a_1 a_1' = a_1' a - a_1 a_1'$$

$$a_1 = a_1'$$

□

centroid

In the general case, the crossing lines are called cevians. In the case of medians, they divide the sides opposite in half, and then the central point is called the centroid. Recall that we had

$$\frac{a}{a'} = \frac{\mathcal{A}_{APC}}{\mathcal{A}_{APB}}$$

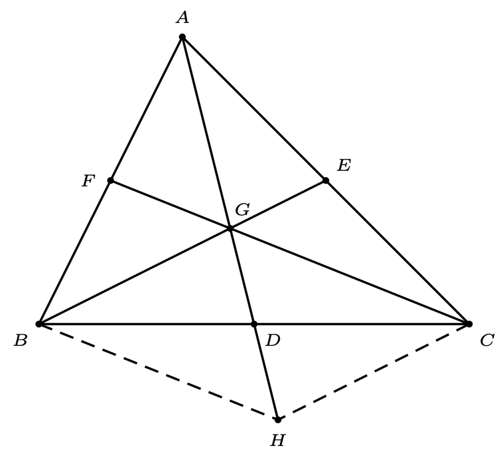
$$\frac{b}{b'} = \frac{\mathcal{A}_{BPC}}{\mathcal{A}_{APC}}$$

But if $a = a'$ and $b = b'$, then $\mathcal{A}_{APC} = \mathcal{A}_{APB} = \mathcal{A}_{BPC}$.

For a physical object, the point P would be the *center of mass*.

Centroid

Consider the $\triangle ABC$. Draw the medians BE and CF (bisectors of the sides). Extend AG through the intersection of the two medians at G to D and then finally to H, where CH is drawn parallel to BGE. We profess not to know anything about BH yet.



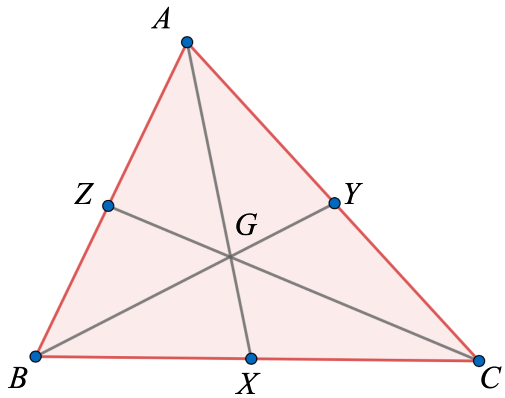
Since $CH \parallel BGE$ and $AE = EC$, we have that $\triangle AGE \sim \triangle AHC$ with ratio 2 by the midpoint theorem (or what we've called SAS similarity). Thus, $AG = GH$.

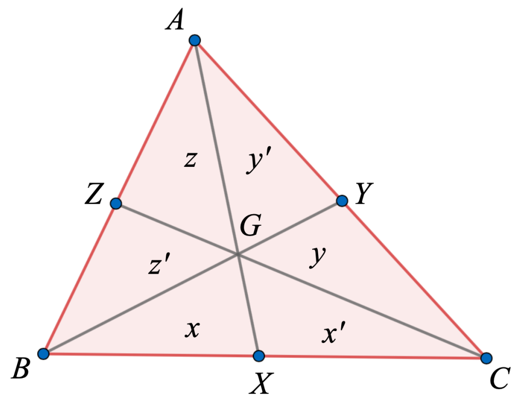
But we also have $AF = FB$, which means that $\triangle AFG \sim \triangle ABH$ with ratio 2. This gives $BH \parallel FGC$.

Therefore, $BGCH$ has two pairs of opposing sides parallel so it is a parallelogram. The diagonals cross at their midpoints, which means that AD is a median (BC is bisected at D).

In addition, $GD = DH$. Thus, GD is one-half of GH, and GH is equal to AG, so GD is one-quarter of AH and one-third of AD. G lies on all three medians, and is called the *centroid* of the triangle. For a physical triangle it would be the center of mass.

Centroid alternate proof





The medians divide any triangle into six equal small triangles.

Proof.

$(\triangle GBX) = (\triangle GCX)$ by the **area-ratio theorem**. Hence the two small triangles on any one side have equal area: $x = x'$, $y = y'$, $z = z'$.

Also by the same theorem, $(\triangle ABX) = (\triangle ACX)$. So by subtraction $(\triangle AGB) = (\triangle AGC)$: $y + y' = z + z'$. Thus $y = z$. But this is true for any side. Hence all six triangles are equal in area.

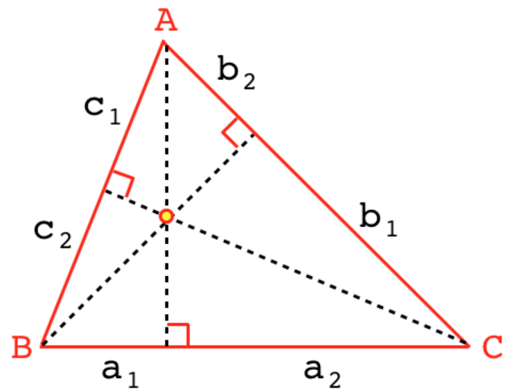
□

Consider $\triangle GBX$ and $\triangle ABX$ on base XGA. They have the same altitude, from the base to the vertex at B. The ratio of areas is $1 : 3$. So that must also be the ratio $XG : XA$.

The centroid lies one-third of the way up the median from the side.

orthocenter

Consider this triangle in which we have drawn the altitudes to each side. We claim that they cross at a single point, called the orthocenter.



Let the angles be A, B, C as labeled, and the sides opposite be a, b, c, subdivided as shown.

Then $\angle A$ is part of two right triangles, and by similar triangles we have that

$$\frac{c_1}{b} = \frac{b_2}{c} \quad \rightarrow \quad \frac{b_2}{c_1} = \frac{c}{b}$$

Similarly for $\angle B$

$$\frac{a_1}{c} = \frac{c_2}{a} \quad \rightarrow \quad \frac{c_2}{a_1} = \frac{a}{c}$$

And $\angle C$

$$\frac{b_1}{a} = \frac{a_2}{b} \quad \rightarrow \quad \frac{a_2}{b_1} = \frac{b}{a}$$

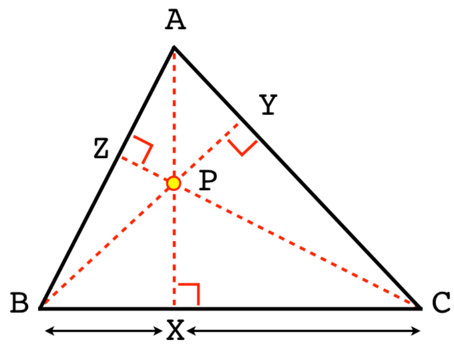
The product of the three right-hand sides above is $c/b \cdot a/c \cdot b/a = 1$. Therefore the product of the left-hand sides is also 1:

$$\frac{b_2}{c_1} \cdot \frac{c_2}{a_1} \cdot \frac{a_2}{b_1} = 1$$

Invert and re-order the terms

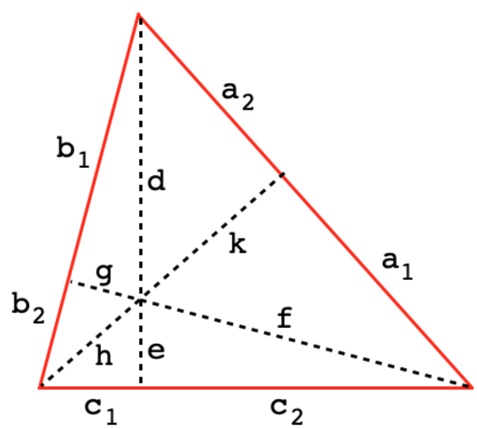
$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1$$

□



Since we have satisfied Ceva's condition, the 3 altitudes all cross at a single point. That point is the orthocenter, and this is a proof that it exists.

another approach to the orthocenter



There is a different set of similar triangles one can use for the orthocenter. The triangle with sides a_2, d and k is similar to the triangle with sides c_1, h, and e. For one pair, we obtain

$$\frac{e}{k} = \frac{c_1}{a_2}$$

You should be able to use the other two pairs to construct Ceva's ratio equal to

$$\frac{e}{k} \cdot \frac{k}{g} \cdot \frac{g}{e}$$

which is, of course, equal to 1.

Another easily derived relationship is that

$$\frac{e}{k} = \frac{h}{d}$$

so

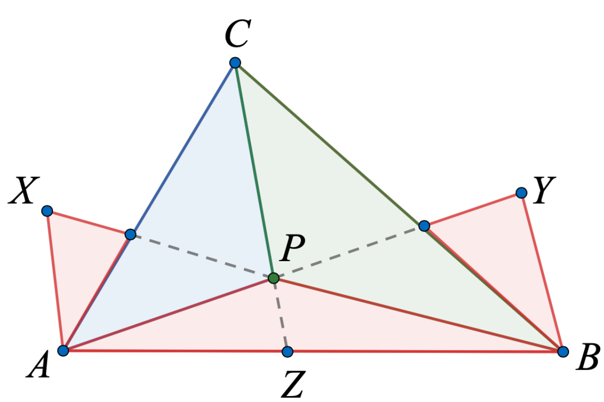
$$de = hk$$

Going around the triangle we will get

$$de = fg = hk$$

This occurs because the position of P with respect to each altitude is the same fraction of the whole.

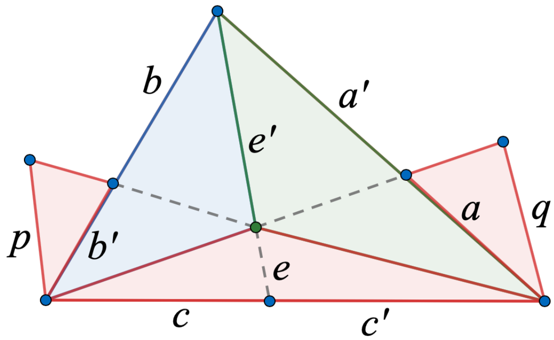
another proof of Ceva's theorem



I found an alternative approach in a geometry textbook by Jurgensen *et al.* It's a bit weird because this is the only mention of the theorem in the book. I like the proof, which is only hinted at, because we enhance the original diagram, and it provides exercise using similar triangles.

Draw AX and BY parallel to CPZ.

I found the algebra to be a little cleaner with different notation, using lowercase letters for the lengths.



Proof.

Parallel lines give similar triangles with the ratios:

$$\frac{a}{a'} = \frac{q}{e'}, \qquad \frac{b'}{b} = \frac{p}{e'}$$

which combine to give

$$\frac{q}{p} = \frac{a}{a'} \cdot \frac{b}{b'}$$

We also have

$$\frac{c+c'}{c} = \frac{q}{e}, \qquad \frac{c+c'}{c'} = \frac{p}{e}$$

combined

$$\frac{c'}{c} = \frac{q}{p}$$

Equating the two results

$$\frac{c'}{c} = \frac{a}{a'} \cdot \frac{b}{b'}$$

which rearranges to give

$$\frac{a}{a'} \cdot \frac{b}{b'} \cdot \frac{c}{c'} = 1$$

□

Viviani

Here's a problem from Acheson that looks challenging, but yields easily to the right perspective.

Viviani's theorem

This states that for any internal point P of an *equilateral* triangle the sum of the perpendicular distances from the sides is a constant, independent of the position of P (Fig. 145).

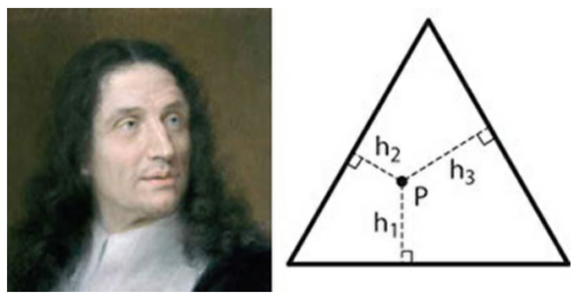
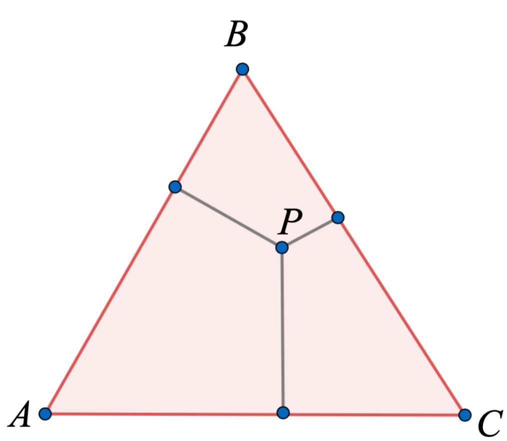


Fig. 145 Viviani's theorem: in an equilateral triangle, $h_1 + h_2 + h_3 = \text{constant}$.

Rather than letting P be a special point, it can be anywhere inside the triangle. P is the opposite of a special point, it is completely general. However, we are also given that the triangle is equilateral.

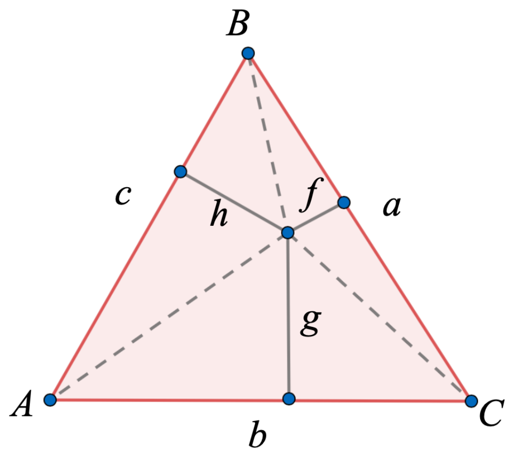
Let $\triangle ABC$ be equilateral, and let P be an arbitrary point internal.



Draw the lines perpendicular to each of the sides from P. The sum of the three lengths is the same no matter where P is chosen inside $\triangle ABC$.

Proof.

Draw the lines connecting the three vertices with P.



$\mathcal{A}_{ABC}$ is the sum of the areas of the three small triangles:

$$2\mathcal{A}_{ABC} = af + bg + ch$$

Let $a = b = c = s$

$$2\mathcal{A}_{ABC} = s(f + g + h)$$

We also know that the altitude of an equilateral triangle is in the ratio to the side as $\sqrt{3}/2$ so twice the area is

$$2\mathcal{A}_{ABC} = \frac{\sqrt{3}}{2}\, s \cdot s$$

It follows that

$$f + g + h = \frac{\sqrt{3}}{2}\, s$$

which is a constant for any given triangle, independent of the position of P inside this equilateral triangle.

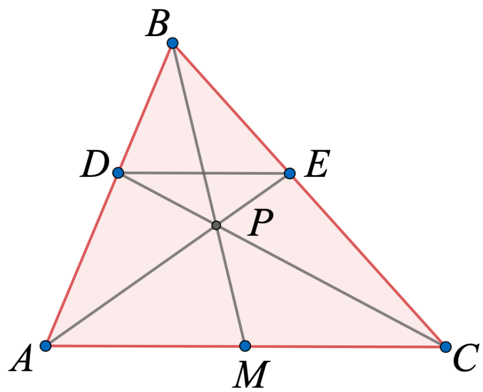
problem (Posamentier 1.18)

In $\triangle ABC$, draw $DE \parallel AC$.

Let BM be the median to side b.

Draw AE and CD.

Then, the two lines AE and CD are concurrent with BM at a point, P.



Proof.

By the converse of Ceva's theorem, the lines will be concurrent if

$$\frac{AM}{MC} \cdot \frac{CE}{EB} \cdot \frac{BD}{DA} = 1$$

Since $AM = MC$, the first term is just 1. Then, it remains to show that

$$\frac{CE}{EB} \cdot \frac{BD}{DA} = 1$$

We have $\triangle ABC \sim \triangle DBE$ so

$$\frac{BD}{AD} = \frac{BE}{CE}$$

Substituting, we satisfy the condition for concurrence.

$$\frac{CE}{BE} \cdot \frac{BE}{CE} = 1$$

□

Chapter 45

Pythagoras by area

Einstein

Late in life, Albert Einstein wrote that he had two experiences as a youth that influenced him tremendously. The first was when he received a compass as a child and was intrigued by the question of what "force" made the needle turn to the north.

The second was that, while studying geometry at about age 12, he had developed on his own a proof of the Pythagorean theorem. He said the achievement had a profound effect on him, illustrating the power of pure thought.

Strogatz says that the proof relied in a simple way on symmetry.

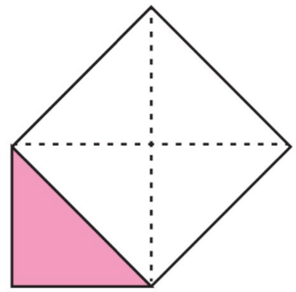
https://www.newyorker.com/tech/annals-of-technology/einsteins-first-proof-pythagorean-theorem

areas in proportion

We will prove the part that Strogatz says was obvious to Einstein from symmetry, which he may have assumed. The idea is that for two similar right triangles, the area of each triangle is in a constant ratio to the product on any two sides, including the square on any one side.

So the area is proportional to the hypotenuse squared.

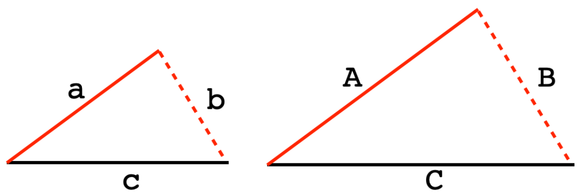
This is easy to see for an isosceles right triangle where we can calculate simply that $k = 1/4$.



But it is true for similar right triangles in general.

Proof.

We assume that the sides of similar triangles are in proportion. This fundamental proof has been given elsewhere.



$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = k$$

For right triangles, twice the area is the product of the two sides, hence the ratio of the areas is

$$\frac{ab}{AB} = \frac{a}{A} \cdot \frac{b}{B} = k^2$$

But $k = c/C$ so

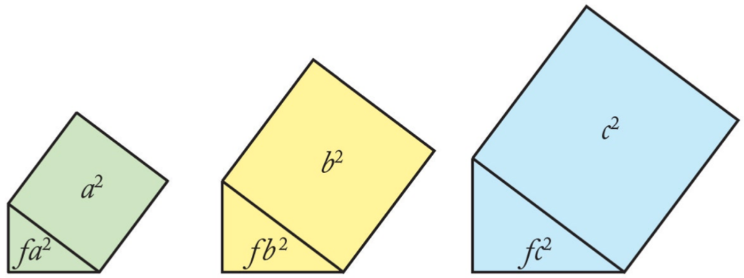
$$\frac{ab}{AB} = \frac{c^2}{C^2}$$

which can be rearranged to give

$$\frac{ab}{c^2} = \frac{AB}{C^2}$$

We conclude that the area of each right triangle is in a constant ratio to the square on any side, including the hypotenuse.

□



Einstein's proof

Given this symmetry, here is what is thought to be Einstein's proof of the Pythagorean theorem.

Proof.

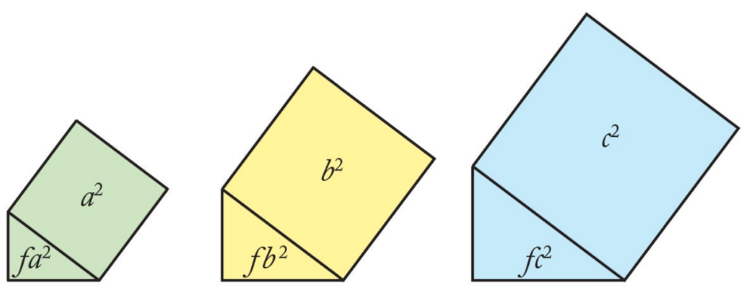
Let the constant of proportionality between area and hypotenuse squared be f.

Form two smaller triangles with hypotenuse of lengths a and b. These are the two triangles formed by dropping an altitude to the hypotenuse in the original right triangle.

The two smaller triangles have areas fa^2 and fb^2 but they add to give the larger one so:

$$fa^2 + fb^2 = fc^2$$

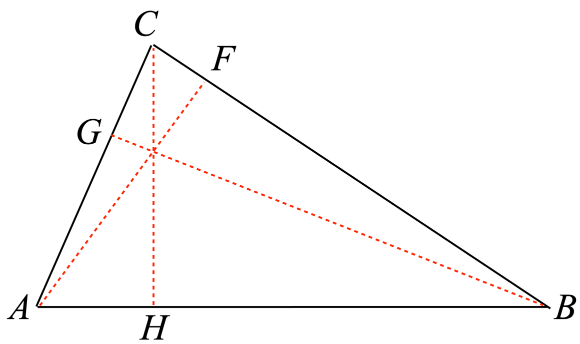
Divide by f and we're done.



□

We have done the arithmetic to prove this relationship, but an appeal to symmetry abstracts away the underlying arithmetic of ratios. A true believer would simply write the last equation and then divide by f.

extension of Pythagoras



Draw the altitudes in a triangle such as $\triangle ABC$. We can form pairs of similar triangles by sharing one of the vertex angles. For example $\triangle AGB \sim \triangle AHC$ and $\triangle BFA \sim \triangle BHC$. Form one ratio for each pair:

$$\frac{AG}{AB} = \frac{AH}{AC} \qquad \frac{BF}{AB} = \frac{BH}{BC}$$

Be careful to pick the side *not* opposite the shared angle.

Cross multiply:

$$AG \cdot AC = AB \cdot AH$$

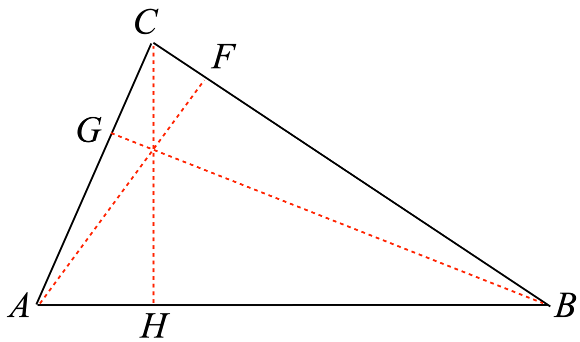
$$BF \cdot BC = AB \cdot BH$$

and add:

$$AG \cdot AC + BF \cdot BC = AB \cdot (AH + BH)$$

A small rearrangement gives a general extension of the Pythagorean Theorem.

$$AG \cdot AC + BF \cdot BC = AB \cdot AB$$



Now let the angle at vertex C become a right angle. $AG \to AC$ and $BF \to BC$ so

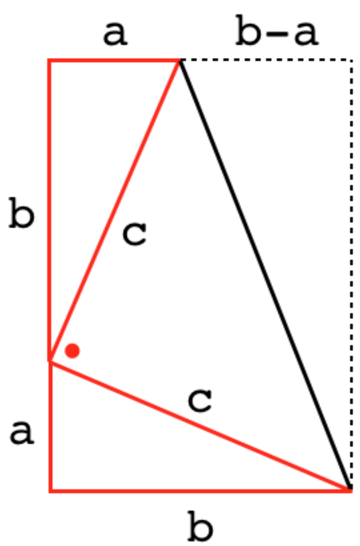
$$AC \cdot AC + BC \cdot BC = AB \cdot AB$$

□

It would be worth thinking about whether this proof extends to the case of an obtuse triangle. We leave that as an exercise.

Garfield's proof

Here is a proof by a future President of the United States, James A. Garfield. (He was a congressman at the time).



Proof.

Draw a right triangle with sides a, b and c, and a second, rotated copy as shown. The angles opposite sides a and b are complementary angles. So the angle marked with a dot is a right angle, and the triangle with sides labeled c is a right triangle.

The area of the entire quadrilateral is the product of the left side $(a + b)$ and the *average* of a and b (top and bottom). This can be seen intuitively.

The halfway point of the solid red line has horizontal dimension $(a + b)/2$. Hence

$$A = (a + b) \cdot \frac{1}{2}(a + b)$$

If you're worried about that argument, just subtract the area of the triangle with two dotted sides from the quadrilateral that includes it:

$$A = (a + b)b - \frac{(a + b)(b - a)}{2}$$

$$= (a + b)(b - \frac{b}{2} + \frac{a}{2})$$

$$= (a + b) \cdot \frac{1}{2}(a + b)$$

which is just what we said.

So now:

$$A = \frac{a^2}{2} + ab + \frac{b^2}{2}$$

But we can also calculate the area of the quadrilateral as the sum of the three triangles:

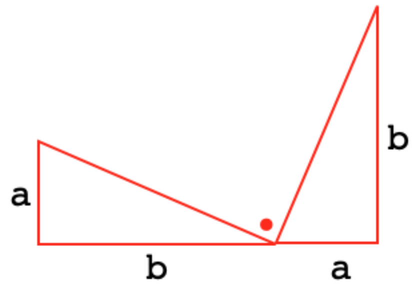
$$A = \frac{ab}{2} + \frac{ab}{2} + \frac{c^2}{2}$$

Equate the two and the result follows almost immediately.

□

product of slopes

Let's take Garfield's basic figure and turn it sideways.



As we said, since the triangles are right triangles, the angle marked with a red dot is also a right angle.

Later, in analytical geometry, we will define the slope of a line as *rise over run*. So, for example, the slope of the hypotenuse of the right-hand triangle is b/a.

In a similar way, the slope of the hypotenuse of the left-hand triangle is $-a/b$. We think of the "run" of a line as going from left to right. This line heads down as we go to the right, hence the minus sign.

So then the product of slopes is

$$\frac{-a}{b} \cdot \frac{b}{a} = -1$$

This is a proof that the product of the slopes of two line segments that meet at a right angle is equal to -1.

lunes

Anything else that goes like the square of the side has the same relationship:

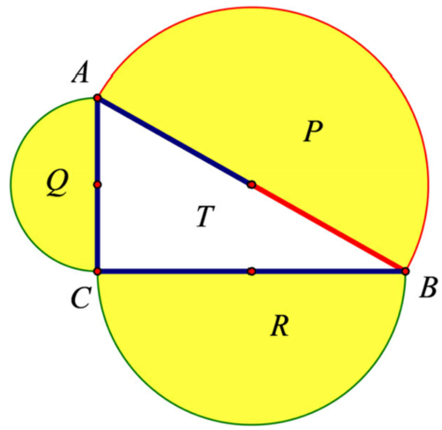


Figure 5.12.

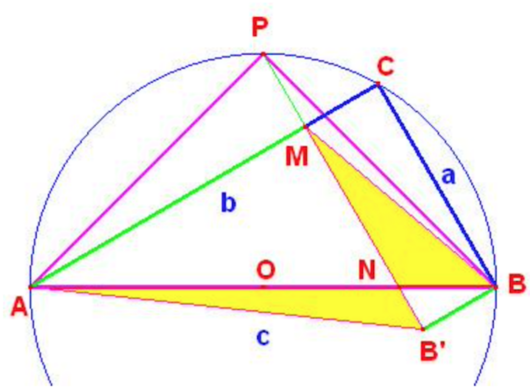
The areas add: $Q + R = P$.

These semi-circular areas are called lunes.

fancy proof

Tuån extended the broken chord theorem of Archimedes to a proof of the Pythagorean theorem in a very clever way.

Here is a diagram.



We start with two right triangles (AB is a diameter of the circle). One of the triangles, $\triangle APB$, is isosceles.

The sides of $\triangle ABC$ are labeled as a, b and c, opposite the corresponding vertices. Side BC has length a.

PM is drawn perpendicular to AC. By the broken chord theorem,

$$AM = MC + BC$$

Twice that is

$$AM + MC + BC = AC + BC = b + a$$

so

$$AM = \frac{b+a}{2}$$

while

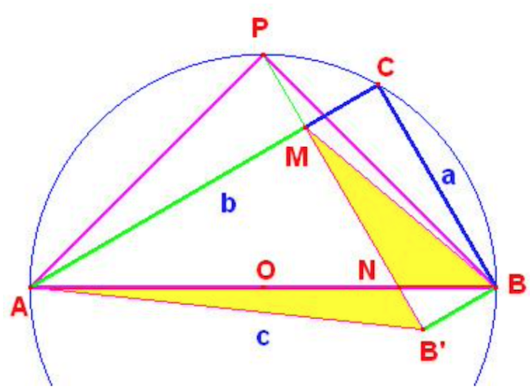
$$MC = AM - BC$$
$$= \frac{b+a}{2} - a = \frac{b-a}{2}$$

PM is extended to meet the diagonal at N and past it to B'. B' is chosen so that $B'BCM$ is a rectangle. Thus side MB' is equal to BC and so to a.

We make two preliminary claims. The first is that $\triangle PMC$ is a right *isosceles* triangle. Let us accept that provisionally.

$$PM = MC = \frac{b-a}{2}$$

The second is that the areas of the two triangles shaded yellow are equal.



We reason as follows. Add $\triangle NB'B$ to both.

$\triangle AB'B$ and $\triangle MB'B$ have the same base BB', and the opposing vertices A and M both lie on AC, which is parallel to the base $B'B$. Hence the two triangles have the same altitude, namely, a, so they have the same area.

By subtraction of $(\triangle B'BN)$ we obtain:

$$(\triangle AB'N) = (\triangle MNB)$$

Triangle area is indicated by the parentheses.

Now we find the area of $\triangle APB$ in two different ways.

The first is $c^2/4$, since it is one-quarter of a square with sides c.

The second way is as the sum of smaller triangles:

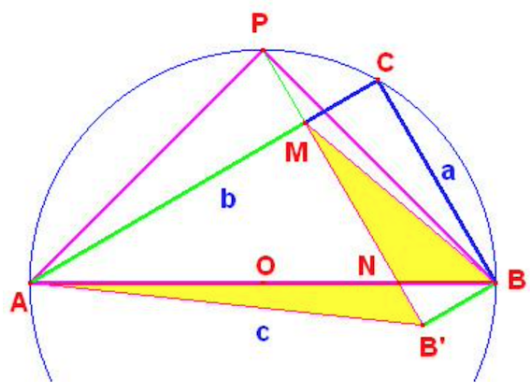
$$(\triangle APM) + (\triangle PMB) + (\triangle AMN) + (\triangle MNB)$$

Since $(\triangle AB'N) = (\triangle MNB)$:

$$(\triangle APM) + (\triangle PMB) + (\triangle AMN) + (\triangle AB'N)$$

And since $(\triangle AMN) + (\triangle AB'N) = (\triangle AMB')$:

$$(\triangle APM) + (\triangle PMB) + (\triangle AMB')$$



So then the areas are

$$(\triangle APM) = \frac{1}{2} \cdot AM \cdot MC = \frac{1}{2} \cdot \frac{b+a}{2} \cdot \frac{b-a}{2}$$

$$(\triangle PMB) = \frac{1}{2} \cdot PM \cdot MC = \frac{1}{2} \cdot \frac{(b-a)}{2} \cdot \frac{(b-a)}{2}$$

$$(\triangle AMB') = \frac{1}{2} \cdot AM \cdot MB' = \frac{1}{2} \cdot a \cdot \frac{b+a}{2}$$

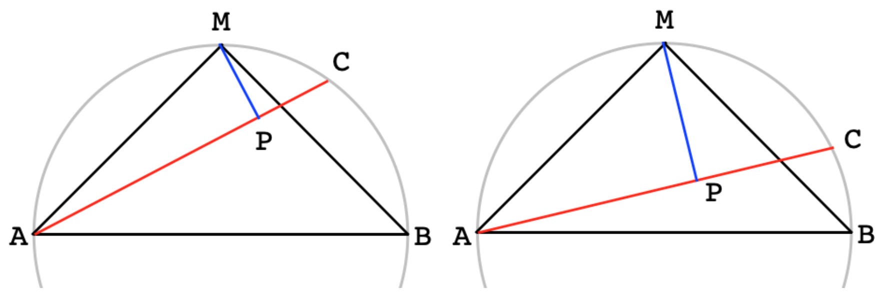
We compute 8 times the sum, so as not to have to deal with fractions:

$$(b^2 - a^2) + (b^2 - 2ab + a^2) + (2ab + 2a^2) = 2b^2 + 2a^2$$

Previously, we calculated the area as $c^2/4$, and 8 times that is $2c^2$. The result follows immediately.

last step

However, the proof is not yet complete. We must show that $\triangle PMC$ is isosceles. Simplifying the figure:

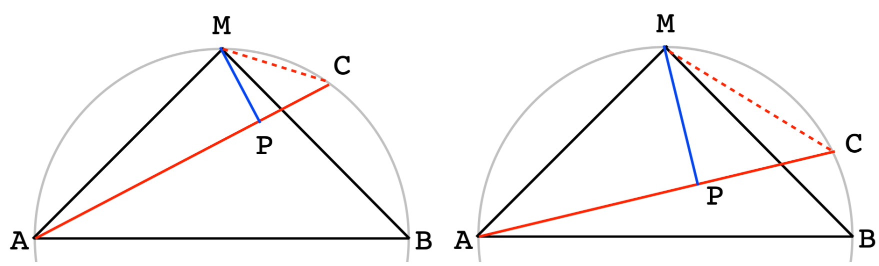


Let AB be a diameter of the circle and $\triangle AMB$ isosceles. Let C be any point on the perimeter, with $MP \perp AC$. Then, we claim that $MP = PC$.

It seems reasonable. If $C \to B$, the P becomes the origin and the statement is true, while if $C \to M$, both vanish. I spent some time fooling around with similar triangles before insight came.

Proof.

Connect the two vertices by drawing MC.



Clearly $\angle MCA$ is one-half of a right angle since it intercepts the same arc as $\angle ABM$, by the inscribed angle theorem. Since $\angle MPC$ is right, it follows that $\triangle PMC$ is isosceles (by complementary angles) and so $MP = PC$ (by the converse of the isosceles triangle theorem).

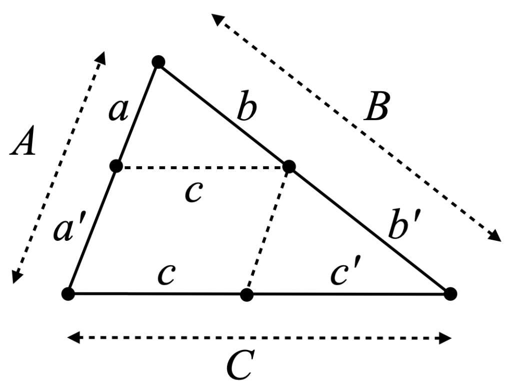
□

Notice how we incorporate the information that $\triangle AMB$ is isosceles.

Chapter 46

Ratio Boxes

In this short chapter, we will look at a device which, at least for now, I'm going to call *ratio boxes*. Here's the idea:



a	b	c
A	B	C
a'	b'	c'

We start with two similar triangles, $\triangle abc$ nested inside $\triangle ABC$.

One part of the definition is that the sides have equal ratios. For example, $a : A = b : B$, also written as

$$\frac{a}{A} = \frac{b}{B}$$

Now, you can look at the figure and take away that result, easily enough. But sometimes you need more than one relationship, and if the vertices have the labels, then it's more complicated.

You may notice that there are two similar triangles but nine entries in the box. The reason is the following:

$$\frac{a}{b} = \frac{A}{B} = \frac{a+a'}{b+b'}$$
$$\frac{a+a'}{a} = \frac{b+b'}{b}$$
$$\frac{a'}{a} = \frac{b'}{b}$$
$$\frac{a'}{b'} = \frac{a}{b} = \frac{A}{B}$$

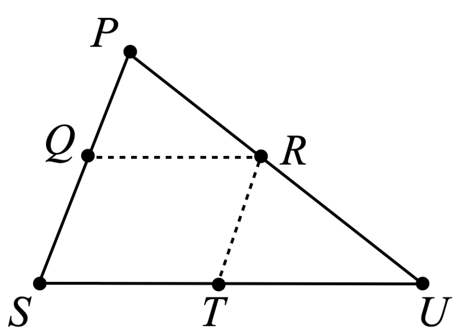
I like to write the sides that are in proportion as shown in the rectangular grid.

a	b	c
A	B	C
a'	b'	c'

Orient it as you like. I usually proceed from the shortest side to the longest side. Sometimes that is hard to see so we start matching sides with the angle opposite.

But however, you do it, there are nine sides or differences of sides. Here the ratios *within* triangles go across, and the ones *between* triangles go down.

The next example uses the traditional Greek notation (well, with Roman letters).



PQ	PR	QR
PS	PU	SU
QS	RU	TU

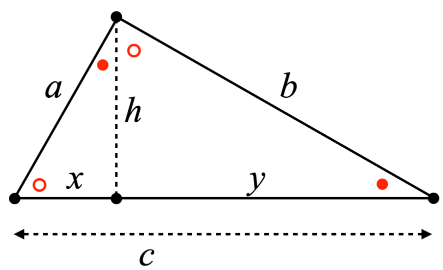
This is exactly the same as before, except the sides are labeled by the flanking vertices. It is fairly easy to go through a figure and make such a box.

The value is this: any four entries that are in the shape of a rectangle form a valid ratio. For example: $PQ : PS = QR : SU$. You can also see immediately that $PQ \cdot SU = PS \cdot QR$.

PQ	PR	QR
PS	PU	SU
QS	RU	TU

If you have three picked you know the fourth. If you have two picked, and they are not related horizontally or vertically (like PQ and SU), then you obtain the other two partners immediately.

right triangles



x	h	a
h	y	b
a	b	c

As you know, if you draw the (one) altitude in a right triangle, it forms two smaller similar triangles. The reason is complementary angles, as shown by the red dots marking equal angles.

Say we want a proof of Pythagoras's Theorem. We need a^2 and b^2. They jump right out of the box:

$$a^2 = xc$$

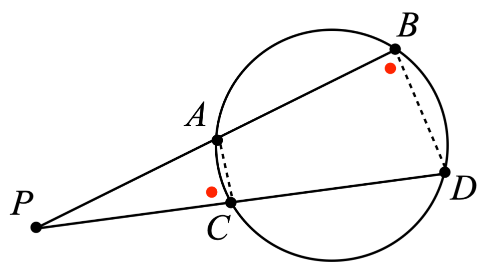
$$b^2 = yc$$

$$a^2 + b^2 = xc + yc = c^2$$

$\square$

We also see, without ever referring back to the figure, that $h^2 = xy$. h is the *geometric mean* of x and y.

secants



AC	BD	*
PC	PB	AB
PA	PD	CD

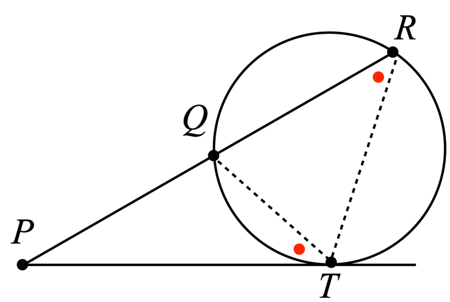
PAB and PCD are secants of this circle. The red dots show equal angles. The reason is that $\angle B$ and $\angle ACD$ are supplementary, because they are opposite angles of a *cyclic quadrilateral*. Together they correspond to a complete arc of the circle.

But $\angle PCA$ and $\angle ACD$ are supplementary too. So the marked angles are equal. And since two pairs of angles are equal, we have similar triangles: $\triangle PCA \sim \triangle PBD$. I try to remember to write the vertices in the order of similarity, so for example: $PC : CA = PB : BD$.

Go through each triangle and write the sides that have equal angles opposite. The famous result is: $PA \cdot PB = PC \cdot PD$.

I have left out the entry in the starred box, because it looks funny. It is $BD - AC$.

tangent-secant



PQ	PT
QT	RT
PT	PR

Here we have a secant PQR and a tangent, PT. The angle that the tangent makes with a chord is easily shown to be equal to any inscribed angle subtended by that chord. That accounts for the red dots.

Again we have similar triangles, this time, one is nested inside another, so there are only six terms in the box. Here I followed the angles, first red then $\angle P$.

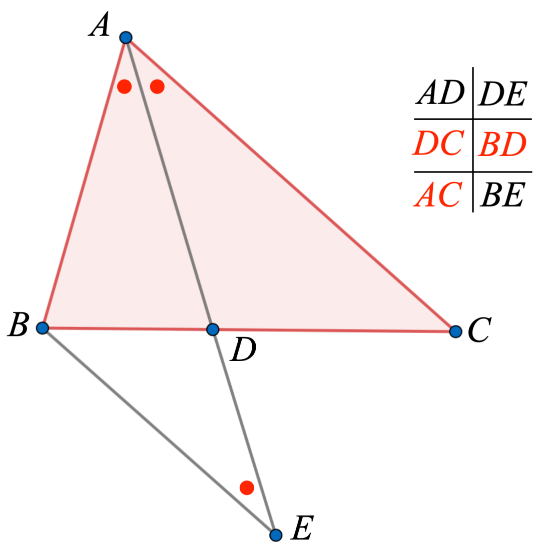
The famous result is the Secant Tangent Theorem (or Tangent Secant Theorem): $PQ \cdot PR = PT^2$.

It writes itself.

Angle Bisector Theorem

We revisit this theorem, from **here**. We are given that the angle at A ($\angle BAC$), is bisected.

So we draw a line segment BE parallel to side AC and extend the bisector to meet it.



Because of the parallel lines, $\angle BED$ gets a red dot as well. And that means $\triangle BED \sim \triangle DAC$, which gives the ratio box in the figure. Here we want to be careful to match the sides with equal angles opposite.

Highlight the segments we think might be interesting. Finally, notice that $\triangle ABE$ is isosceles, so $AB = BE$. We obtain:

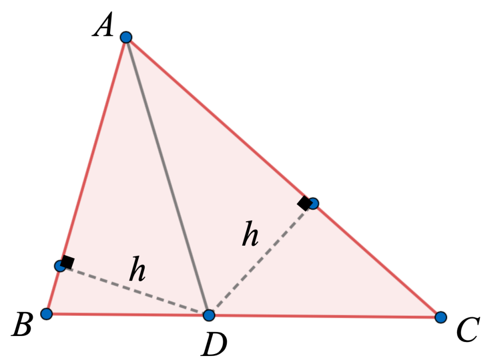
$$\frac{AC}{DC} = \frac{BE}{BD} = \frac{AB}{BD}$$

□

The sides and divisions of the base are in equal proportion. This can also be written as

$$\frac{AB}{AC} = \frac{BD}{DC}$$

This one can also be done by areas (next figure). $\triangle ABD$ and $\triangle ACD$ have the same altitude h. So the areas are in the ratio $AB : AC$.

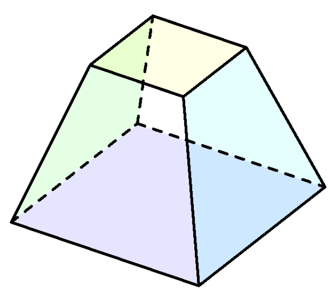


But looked at another way, they have bases BD and DC on the same parallel, so they have the same height drawn to vertex A. Hence, the areas are in the ratio $BD : DC$. Thus

$$\frac{AB}{AC} = \frac{BD}{DC}$$

frustum

A frustum is a truncated pyramid.



It's known, and we discuss elsewhere a proof, that the volume of a cone or pyramid is

$$V = \frac{1}{3} \cdot hA$$

where h is the height and A is the area of the base. We'll make things slightly easier by writing

$$3V = hs^2$$

with s being the side length, for a square pyramid.

A frustum has the top chopped off. We will be given the side length at the bottom a, and the side length on the top b, as well as the height of the frustum h.

We need to compute the height of the whole thing H, then find the height of the missing top part as $H - h$. We'll use those two to get the volume of the frustum by subtraction.

Let H be the entire height, and then by similar triangles we have

$$\frac{H}{a} = \frac{H-h}{b} = \frac{h}{a-b}$$

This is our ratio box for the problem. The height of the small missing piece is

$$H - h = \frac{b}{a}H$$

Then the volume we seek uses the difference:

$$3V = a^2H - b^2(H-h)$$

$$= a^2H - b^2H\ \frac{b}{a}$$

$$= (a^2 - \frac{b^3}{a})H$$

But now we need H in terms of h, since that's what we're given, so going back to the ratios we have:

$$H = \frac{a}{a-b}h$$

which gives

$$3V = (a^2 - \frac{b^3}{a})\frac{a}{a-b}\ h$$

$$= \frac{a^3 - b^3}{a-b}\ h$$

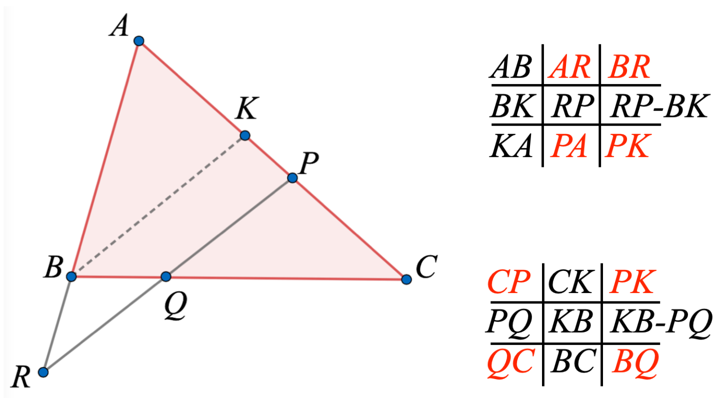
And if we fool around for a while, we may discover that the numerator can be factored as

$$(a^2 + ab + b^2)(a - b) = a^3 - b^3$$

which gives, finally

$$3V = h(a^2 + ab + b^2)$$

Menelaus's Theorem



The next example is one where the method really proves its worth. We prove Menelaus's theorem (we will look at it in more detail later **here**).

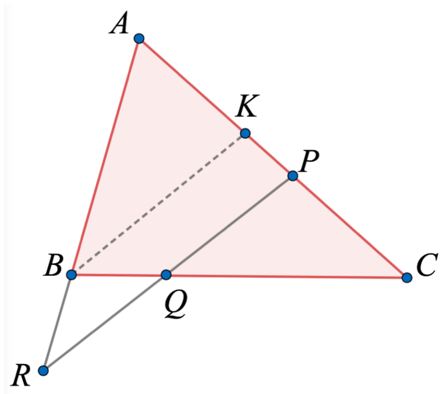
In $\triangle ABC$ we have the transversal RQP. Draw an internal line segment BK parallel to the transversal.

Here, it is usual to label vertices rather than sides.

For the moment, each vertex is listed in order it is encountered going around a triangle, starting at the same vertex each time (since we have parallel bases).

The values in the right-hand columns of the two ratio boxes are all subtractions, but for one there is no labeled segment to equate to the result.

The segments needed for the proof are shown in red.



AB	AR	BR
BK	RP	RP-BK
KA	PA	PK

CP	CK	PK
PQ	KB	KB-PQ
QC	BC	BQ

It writes itself! Since

$$\frac{PK}{BR} = \frac{PA}{AR}$$
$$PK = BR \cdot \frac{PA}{AR}$$

Similarly

$$\frac{PK}{CP} = \frac{BQ}{QC}$$
$$PK = CP \cdot \frac{BQ}{QC}$$

Combining the two results:

$$CP \cdot \frac{BQ}{QC} = PK = BR \cdot \frac{PA}{AR}$$
$$\frac{BQ}{QC} \cdot \frac{CP}{PA} \cdot \frac{AR}{BR} = 1$$

□

Finally, it is usual to view each segment as having a direction or sign. Namely

$$BQ \Rightarrow QC \Rightarrow CP \Rightarrow PA \Rightarrow AR \Rightarrow RB$$

Since RB and BR are in opposite directions, $RB = -BR$. Substitute RB for BR and adjust the sign of the result accordingly.

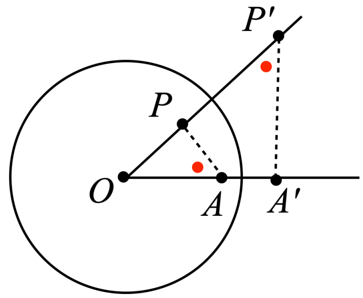
$$\frac{BQ}{QC} \cdot \frac{CP}{PA} \cdot \frac{AR}{RB} = -1$$

inversion

Some good examples involve a topic called *Inversive Geometry*. Given a circle of radius r, any point (except the origin), like A, can be transformed into its *image under an inverse transformation*, resulting in A'. Draw the line from O through A and calculate the length of OA' by this rule:

$$OA \cdot OA' = r^2$$

Since P and P' are related by the same transformation, we have $OA{\cdot}OA' = OP{\cdot}OP'$. Since $\angle O$ is shared, we again have similar triangles: $\triangle OAP \sim \triangle OP'A'$.



AP	$A'P'$
OP	OA'
OA	OP'

You can read the rule right off the box.

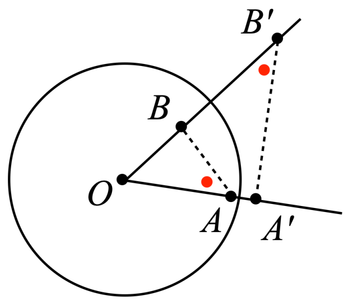
And if $OAA' \perp A'P'$ then the small $\triangle OAP$ is a right triangle. By the converse of Thales' Theorem, we can draw a circle with diameter OA and P will lie on the circle.

This is true for *any* point on $A'P'$. Say Q' is on $A'P'$, then if Q is the image of Q', Q lies on the same circle, the one with radius OA.

As a result, the *image* of any point on the line $A'P'$ lies on the circle with radius OA. We say that the image of the line is the circle (and it goes through O).

The transformation is an *involution*, so the converse is also true: the image of a circle through O is a line not through O.

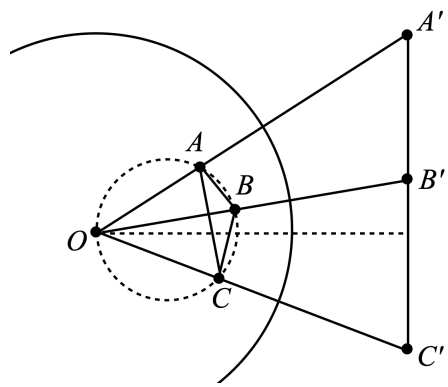
A second example from inversion is more general.



AB	$A'B'$
OB	OA'
OA	OB'

You can read the rule right off the box, again. What this means is that if we take any two points A and B and their images A' and B', we get similar triangles.

Ptolemy's theorem



We will prove a famous theorem. All we need are some boxes and the previous result.

AB	$A'B'$
OB	OA'
OA	OB'

AC	$A'C'$
OC	OA'
OA	OC'

BC	$B'C'$
OC	OB'
OB	OC'

I didn't even have to think about it. I just copied the box from before, and substituted C for B in the middle, then B for A on the right.

We see that that the transformed circle is a line with $A'B' + B'C' = A'C'$. We can find expressions for those lengths in our boxes. We know that eventually we will want things like AB, BC and AC, as well as OB, etc.

I get

$$A'B' = \frac{AB \cdot OA'}{OB} \qquad A'C' = \frac{AC \cdot OA'}{OC} \qquad B'C' = \frac{BC \cdot OB'}{OC}$$

Again, these may be obtained mechanically, by straight substitution. The one for $A'C'$ first, by substituting C' for B' in the one on the left. Wash, rinse, repeat.

Substitute and then clear the denominator:

$$AB \cdot OA' \cdot OC + BC \cdot OB' \cdot OB = AC \cdot OA' \cdot OB$$

Divide by OA':

$$AB \cdot OC + BC \cdot \frac{OB'}{OA'} \cdot OB = AC \cdot OB$$

Can you find the four entries we need in the first box on the left and complete the proof?

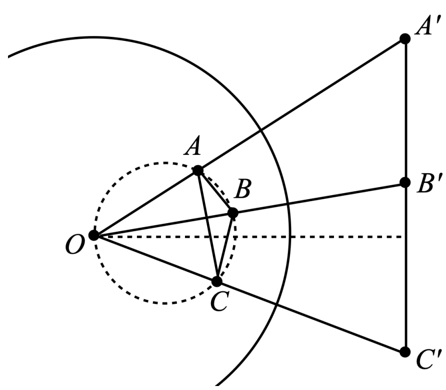
AB	$A'B'$
OB	OA'
OA	OB'

AC	$A'C'$
OC	OA'
OA	OC'

BC	$B'C'$
OC	OB'
OB	OC'

The final result is

$$AB \cdot OC + BC \cdot OA = AC \cdot OB$$



We have four vertices on a circle, another cyclic quadrilateral. Take the product of opposing sides, add them, and obtain the product of the two diagonals.

This is Ptolemy's Theorem.

$$AB \cdot OC + BC \cdot OA = AC \cdot OB$$

□

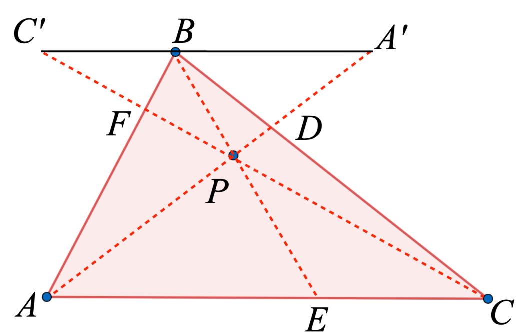
Ceva's Theorem by parallel lines

Here is another proof of **Ceva's theorem**.

Proof.

Consider $\triangle ABC$. Draw AD, BE and CF concurrent at P. Draw a line through B parallel to AEC and extend $APDA'$ and $CPFC'$.

We have five pairs of similar triangles.



Three with vertical angles at P:

$$(1)\ \triangle APC \sim \triangle A'PC' \Rightarrow \frac{AP}{A'P} = \frac{PC}{PC'} = \frac{CA}{C'A'}$$

$$(2)\ \triangle APE \sim \triangle A'PB \Rightarrow \frac{AP}{A'P} = \frac{PE}{PB} = \frac{EA}{BA'}$$

$$(3)\ \triangle CPE \sim \triangle C'PB \Rightarrow \frac{CP}{C'P} = \frac{PE}{PB} = \frac{EC}{BC'}$$

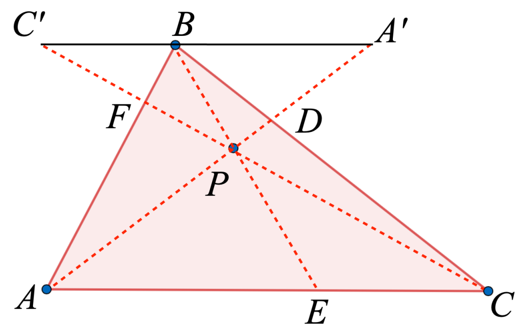
And two more with vertical angles at D and F:

$$(4)\ \triangle ADC \sim \triangle A'DB \Rightarrow \frac{AD}{A'D} = \frac{DC}{DB} = \frac{CA}{BA'}$$

$$(5)\ \triangle AFC \sim \triangle BFC' \Rightarrow \frac{AF}{BF} = \frac{FC}{FC'} = \frac{CA}{C'B}$$

It is helpful to remember that when similar triangles are formed by vertical angles between two parallel lines, corresponding sides either lie across from each other, or are reflected through the point with the vertical angles. Thus $A'B$ corresponds to AE, and PB to PE.

We label the triangles so that corresponding vertices match. Also, do not worry yet about the order in each line segment so for example, $AB = BA$.



If the triangles are labeled carefully it is easy to make the ratio boxes:

AP	$A'P$
PC	PC'
CA	$C'A'$

(1)

AP	$A'P$
PE	PB
EA	BA'

(2)

CP	$C'P$
PE	PB
EC	BC'

(3)

AD	$A'D$
DC	DB
CA	BA'

(4)

AF	BF
FC	FC'
CA	$C'B$

(5)

Now look for the ratios we need namely:

$$\frac{CD}{DB} = \frac{CA}{BA'} \qquad \frac{BF}{FA} = \frac{C'B}{CA}$$

We are encouraged by the fact that CA cancels in the product.

$$\frac{CD}{DB} \cdot \frac{BF}{FA} = \frac{C'B}{BA'}$$

The third ratio is AE/EC. AE and EC do not occur in the same box (they do not lie in similar triangles), but in (2) and (3) we find:

$$AE = A'B \cdot \frac{PE}{PB} \qquad EC = BC' \cdot \frac{PE}{PB}$$

so the ratio is just

$$\frac{AE}{EC} = \frac{A'B}{BC'}$$

which is exactly what we need to cancel!

$$\frac{CD}{DB} \cdot \frac{BF}{FA} \cdot \frac{AE}{EC} = \frac{C'B}{BA'} \cdot \frac{A'B}{BC'} = 1$$

$\square$

Part IX

Special stuff

Chapter 47

Special circles

triangle area and radii for incircle and circumcircle

Acheson gives formulas that connect the area of a triangle (he uses the symbol Δ for the area), and the radius of either the incircle or the circumcircle.

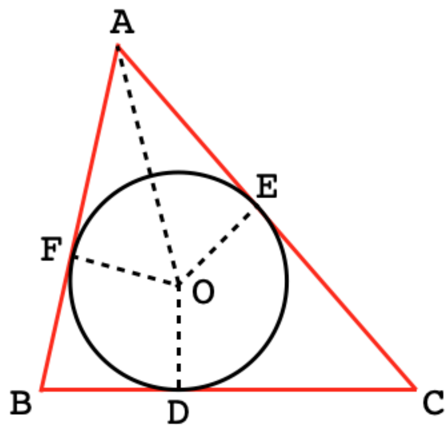
The first one is just a matter of algebra, but the second is gorgeous. It is really worth it to try to solve before you look at the answer. So, write down the answer, close the book and then try! Once again, an inspired diagram is everything.

$$r = \frac{2\Delta}{a+b+c}$$

$$R = \frac{abc}{4\Delta}$$

Let r be the radius of the incircle and a be the length of the base opposite vertex A.

Then the area of $\triangle BOC$ is equal to one-half $r \cdot a$.



So the area of the whole triangle is equal to one-half $r \cdot (a + b + c)$.

$$2\Delta = r \cdot (a + b + c)$$
$$r = \frac{2\Delta}{a + b + c}$$

Define the *semi-perimeter* s as

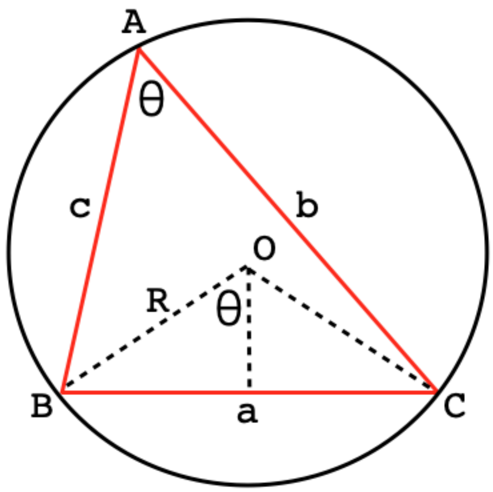
$$s = \frac{a + b + c}{2}$$

and then

$$r = \frac{\Delta}{s}$$
$$\Delta = rs$$

That's an interesting parallel, that this formula is so similar to that for the area of the circle. Here we have the radius of the incircle times the one-half the perimeter of the triangle. Of course, for a circle, we have the radius times one-half its perimeter as well.

For the second problem, the radius of the circumcircle, the key insight is in the diagram below. After that it's easy.



Proof.

The altitude to side b (not shown) has length $c \sin\theta$. So the area of the triangle is

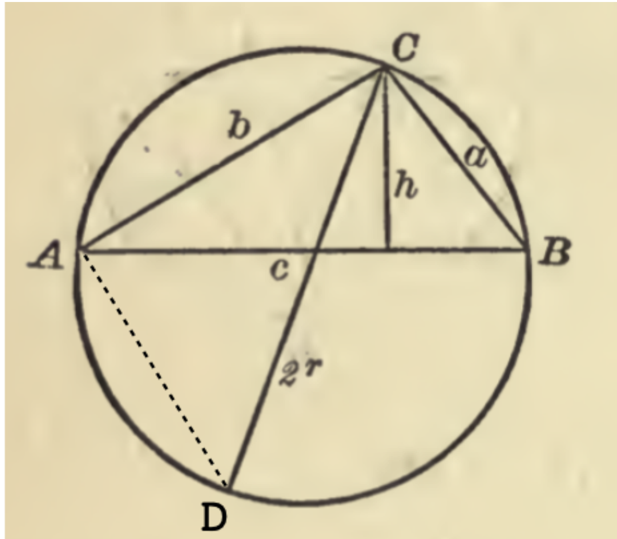
$$\Delta = \frac{1}{2} bc \sin\theta$$

But $\sin\theta = a/2$ divided by R so

$$\Delta = \frac{1}{2} bc \cdot \frac{a}{2R}$$
$$\Delta = \frac{abc}{4R}$$

□

Here is an alternate proof from Hopkins.



Proof. (Alternate).

As a preliminary matter, note that $\triangle ABC$ is any triangle, and the circle is its circumcircle, with radius r. Then the extension of the radius to D forms a right triangle $\triangle ACD$. Since $\angle B$ and $\angle D$ cut off the same arc of the circle, they are equal.

Therefore, $\triangle ACD$ is similar to the triangle formed by the altitude h and including side a. By similar triangles:

$$\frac{h}{a} = \frac{b}{2r}$$

$$h = \frac{ab}{2r}$$

Twice the area of the triangle is

$$2\Delta = ch$$

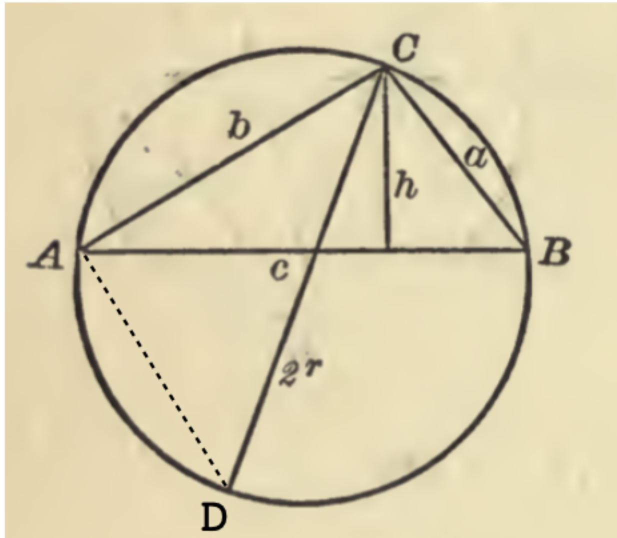
$$\Delta = \frac{abc}{4r}$$

□

Hopkins also notes that this result can be expressed purely in terms of the side lengths by using **Heron's formula** (which we introduced above and will say more about soon):

$$A^2 = s \cdot (s-a) \cdot (s-b) \cdot (s-c)$$

(where $s = (a + b + c)/2$).



So

$$r = \frac{abc}{4\sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)}}$$

You should be able to show that

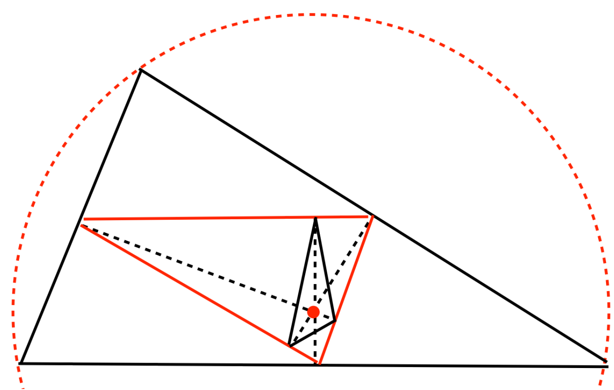
$$r = \frac{abc}{\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}}$$

Chapter 48

Triangles in triangles

In the figure below, the outer black triangle lies on its circumcircle, where the circumcenter is the red point at the center.

The midpoints of the sides of the black outer triangle are joined to form a second triangle, in red.



We will show that the perpendicular bisectors of the sides of the outer triangle are the altitudes of the red triangle.

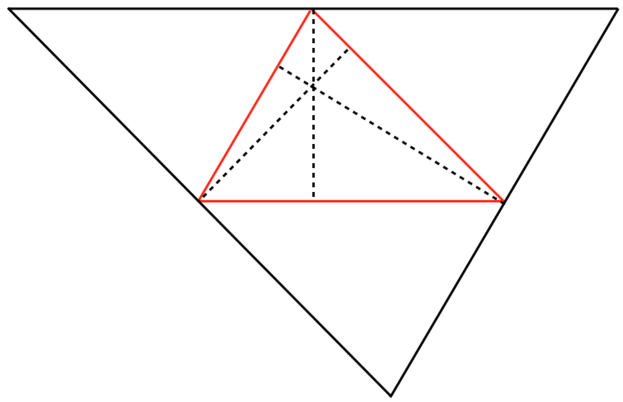
Furthermore, if the points where the altitudes meet the sides of the red triangle are joined to form a third inner triangle, those same altitudes are the angle bisectors of the inner triangle.

So for this arrangement, we have circumcenter concurrent (at the same point) with the orthocenter, which in turn is concurrent with the incenter.

Gauss and altitude

We will prove that the altitudes of a triangle are the perpendicular bisectors of a particular triangle which encloses it. The proof is due to Gauss.

Proof.



Draw the outer triangle, in black. Connect the midpoints of the sides to form an inner triangle, in red. Also draw the perpendicular bisectors of the outer triangle.

By the **midpoint theorem**, each side of the outer triangle is parallel to one side of the inner triangle and equal to twice its length. Since we have opposing sides equal and parallel, this gives three parallelograms in the figure.

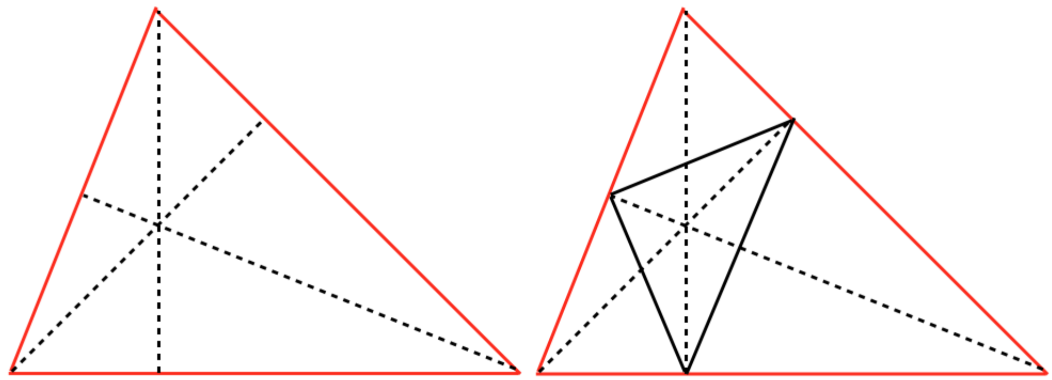
The perpendicular bisector of each black side (dotted line) is the altitude of the paired red side, because it starts from a vertex of the red triangle and meets the base at a right angle due to the parallel sides.

This shows that the circumcenter of the enclosing triangle is the orthocenter of the smaller, enclosed triangle. Since the circumcenter exists and is a single point, so is the orthocenter.

□

orthocenter

We have shown previously that the three altitudes meet at a single point, the orthocenter. The proofs include one from **Newton**, and the previous one (from **Gauss**).



Above we have drawn the altitudes (left panel) and then also connected the points where the altitudes meet the sides at right angles. We will prove that the dotted lines are the bisectors of the angles at the vertices of the small inset triangle.

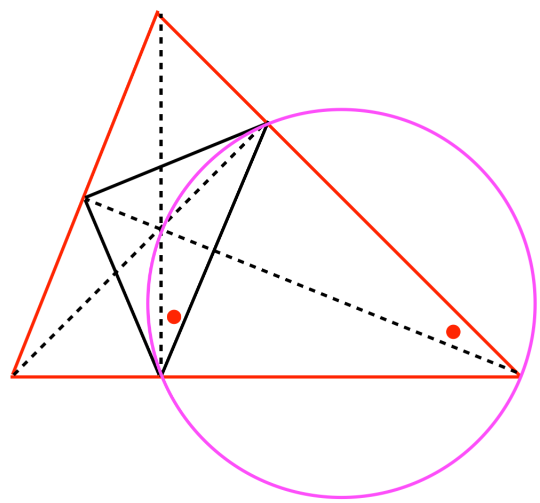
In other words, the incenter of the small triangle is the same point as the orthocenter of the bigger one.

Proof.

The key to the proof is to recognize that we can use a part of an altitude as the diameter of a circle. Draw the circle that has for its diameter the line segment connecting the orthocenter and one vertex of the large triangle.

Now consider the parts of the other two altitudes that terminate in right angles at the sides of the red triangle. I claim that these two points lie on the same circle.

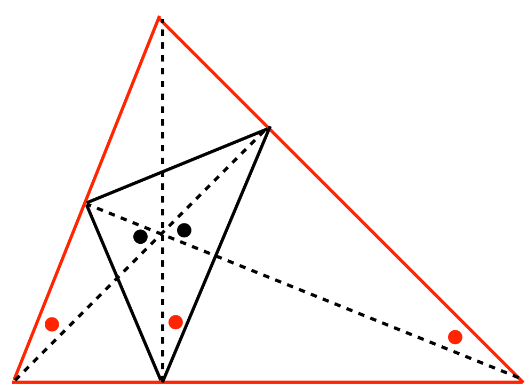
The reason is that, each one individually, taken together with the first two points, forms a right triangle. By the converse of Thales theorem, they must lie on the circle.



(The included side of the inner triangle is *not necessarily* perpendicular to the diameter, the first one looks so because the original triangle is nearly isosceles — see below).

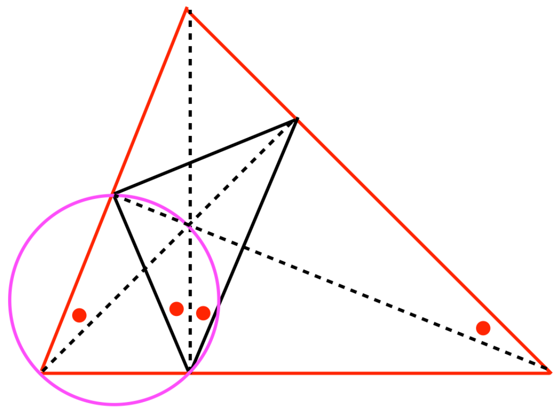
Now we can use the **theorem** about arcs that subtend an angle on the perimeter of the circle. The two angles on the circle marked with red dots correspond to the same arc of the magenta circle, so they are equal.

For the next step, we use vertical angles (marked with black dots) to show that two triangles are similar, since they also contain right angles. Therefore, we can mark a third angle as equal to the others with a red dot.



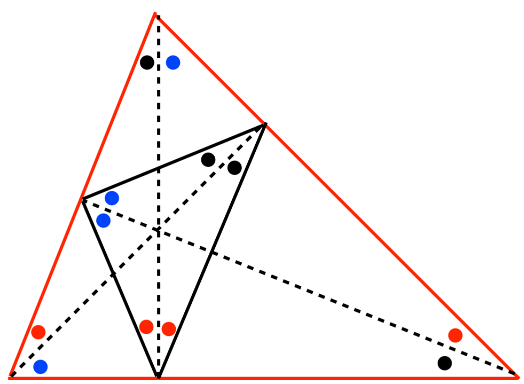
Finally we draw a circle for a different vertex. Now it is obvious that the solid black line is not necessarily perpendicular to the altitude.

Using the arc theorem, we find another equal angle, for a total of four angles marked with red dots. We see that the one vertex of the inner triangle is bisected into two equal angles marked with red dots.



But the same thing can be done for the two other vertices of the inner triangle. The pattern of the angles is the same.

With all the dots filled in:



This shows that the dotted lines are angle bisectors for the small triangle.

Thus, the orthocenter of the large triangle and the incenter of the triangle inscribed between the points where altitudes meet the base, are the same point.

□

Chapter 49

Nine point circle

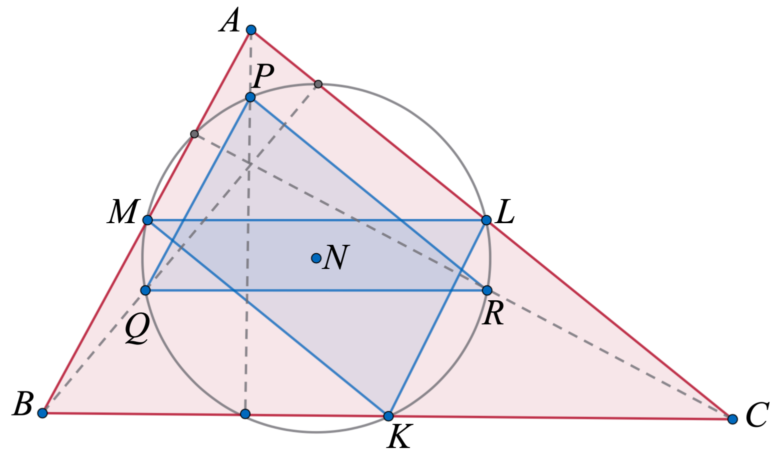
We've seen that the perpendicular bisectors of the sides and, separately, the angle bisectors of triangles, converge on (are concurrent with) points that are the centers of circles with interesting properties. These circles contain either the vertices of the triangle (circumcircle) or have the sides as tangents to the circle (incircle).

Now we investigate the altitudes and their point of convergence, the orthocenter. It turns out there is a special circle involving the altitudes, but it does not have the orthocenter as its center.

Quite surprisingly, there are nine points on the circle. Three of these are midpoints of the sides. This means that all four categories of special points of a triangle are connected to circles of one kind or another.

The circle that goes through the midpoints of the sides also goes through the points where the altitudes of a triangle meet the sides, as well as the midpoints of that part of each altitude lying between the orthocenter and the corresponding vertex.

one approach



Proof.

Here is a fairly simple way to approach the nine point circle.

Draw $\triangle ABC$, then find the midpoints of the sides to form $\triangle KLM$. By the midpoint theorem, we have four congruent smaller triangles.

One of them is inverted: $\triangle KLM$. Draw the circumcircle of $\triangle KLM$. This is the nine point circle.

Now draw another triangle, congruent to $\triangle KLM$, with its vertices lying on the same circle, such that the corresponding sides are parallel. That is $\triangle PQR$.

The easiest way to do this is to draw a rectangle such as $MLRQ$ with ML, the base of $\triangle KLM$ equal to QR, the base of $\triangle PQR$.

There are actually three such rectangles on the circle, one for each side. These are $MLRQ$, $PRKM$ and $MPLK$.

Opposing vertices are the endpoints of diameters of the circle: MR, LQ, and PK.

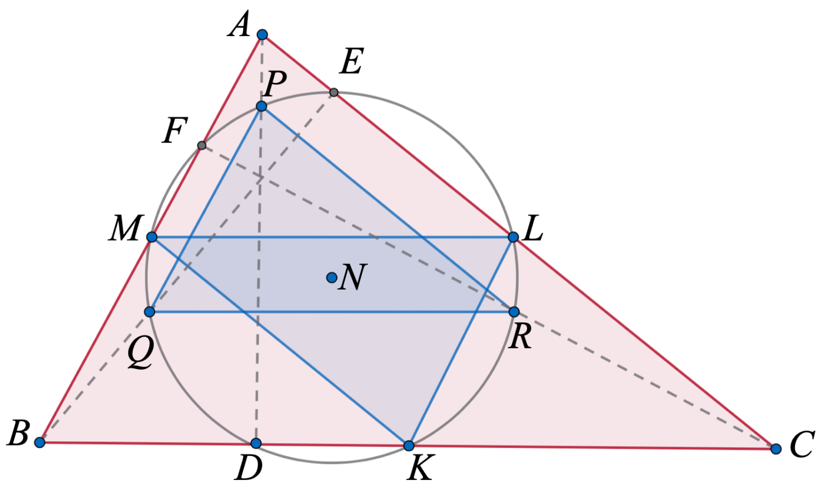
Comparing $\triangle KLM$ and $\triangle PQR$, there has been a rotation around the center N.

So then the altitude from A is perpendicular not only to BC but also to QR and to ML. Let that altitude meet the base at D. The trick is to show that AD goes through P.

Note that $\triangle PQR \cong \triangle AML$.

The bases of these two are displaced by movement parallel to MQ, i.e. vertically, since $MLRQ$ is a rectangle and $APQM$ is a parallelogram.

It follows that $AP \parallel MQ$ and perpendicular to ML, QR and BC so it coincides with AD.



Now since PK is a diameter, it follows that $\angle KDP$ is right. The same follows for $\angle LEQ$ and $\angle RFM$.

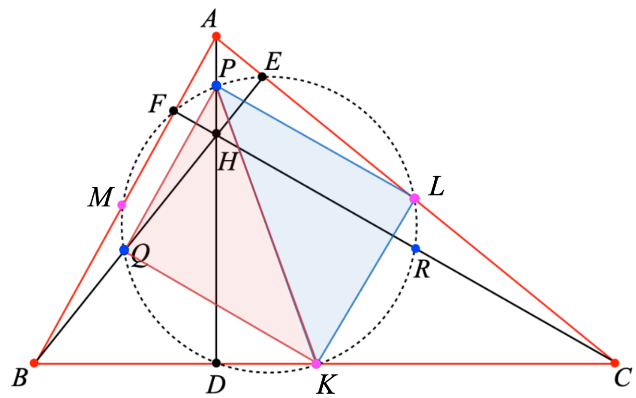
Thus, each of the three altitudes of $\triangle ABC$ has its foot lying on the nine point circle.

$\square$

The last point would be to show that AP is one-half the distance from A to the orthocenter. We'll see how to do this in the next part..

It's challenging to draw the circle.

In the next figure, some of the measurements may look a little off, but the logic will show that the circle indeed contains the nine points cited.



Proof.

In $\triangle ABC$ draw the altitudes AD, BE and CF. Bisect the sides at midpoints K, L and M. Mark the orthocenter at H and bisect AH at P, BH at Q and CH at R.

Because of the six bisections, we can find a number of similar triangles to show that certain lines in the diagram are parallel, and then involving the altitudes, find certain right angles.

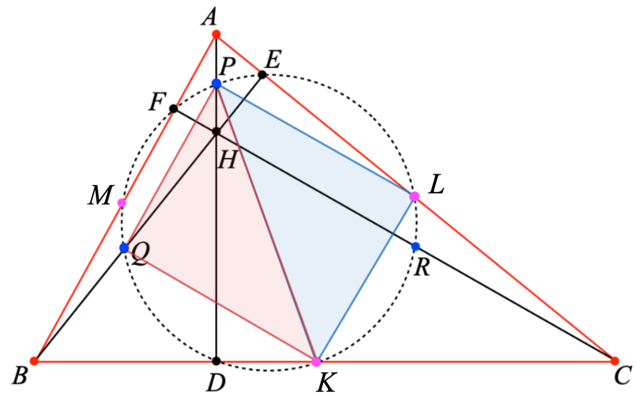
For example, in $\triangle AHC$ we have that P and L are midpoints of two sides, so it follows that $\triangle APL \sim \triangle AHC$, and then $PL \parallel FHC$.

Looking at $\triangle BHC$, by the same logic we can show that $QK \parallel FHC$. But this implies $PL \parallel QK$.

In exactly the same way, we can show that $PQ \parallel AB$ and $LK \parallel AB$ so $PQ \parallel LK$. Thus, $PLKQ$ is a parallelogram.

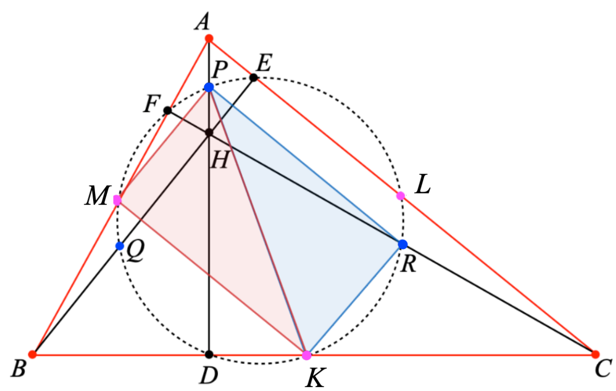
Since $AB \perp CF$, it follows that $LK \perp PL$. Thus, $\angle PLK$ is right. It follows that $PLKQ$ is a rectangle.

Next, draw the circle on diameter PK (the center is not shown, but can be found by bisecting PK). $\angle PLK$ and $\angle PQK$ are both right, which means that points L and Q lie on the circle with diameter PK. And since $\angle ADK$ is right, D also lies on the circle. That's five of the nine points.

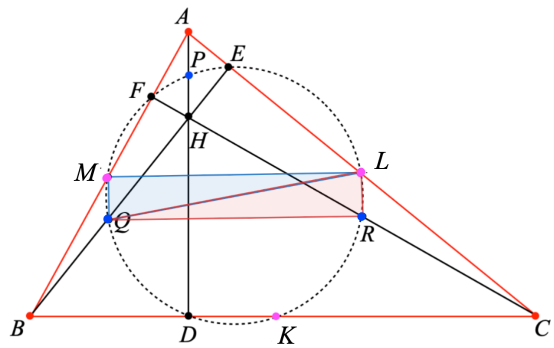


Since Q and L lie on the circle and $\angle QPL$ and $\angle QKL$ are right, it follows that QL is also a diameter of the same circle. Alternatively, note that the center of the circle is at the bisector of PK, which since we have a rectangle, is also the bisector of QL and $PK = QL$.

Since QEL is right, E also lies on this circle.



A similar series of steps will show that $PRKM$ is a rectangle, which is enough to establish that M and R are on the circle, and that MR is actually a diameter of the circle. Since $\angle MFR$ is right, then F also lies on the circle.



We do not need it, but there is a third rectangle, namely, $MLRQ$.

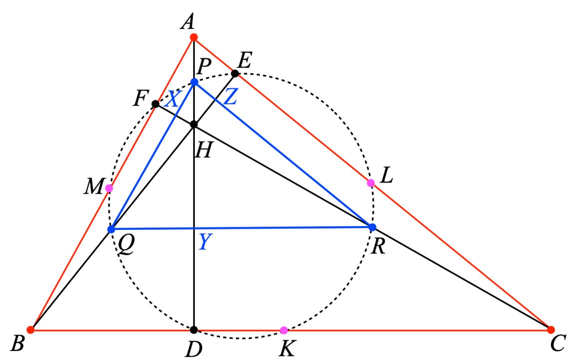
□

radius

It turns out that the radius of the nine-point circle is $1/2$ the radius of the circumcircle of the same triangle.

Proof.

Draw $\triangle PQR$.



The nine-point circle *is* the circumcircle of $\triangle PQR$. We know a formula that connects the area $\mathcal{A}$ of a triangle with its side lengths and the radius of the circumcircle R:

$$R = \frac{abc}{4\mathcal{A}}$$

Let $\triangle ABC$ have side lengths a, b and c and area $\mathcal{A}$. Let $\triangle PQR$ have side lengths a',

b' and c' and area $\mathcal{A}'$. This triangle with vertices halfway along the sides of $\triangle ABC$ is called its medial triangle.

We can use similar triangles to show that each side and altitude in $\triangle ABC$ is twice that of $\triangle PQR$.

For example, we will show that $AB = 2PQ$.

Label the point X where HF cuts PQ. $PXQ \parallel AB$, so $PXQ \perp CHF$, and $\triangle AFH \sim \triangle PXH$ with a ratio of 2. It follows that $2PX = AF$ and $2XH = FH$.

The latter equality gives us $\triangle BFH \sim \triangle QXH$, again with a ratio of 2. Thus $2XQ = FB$ and $2QH = BH$. The result follows by addition. In a similar way we can show that $2HZ = HE$ so by addition the altitude BE is also scaled by a factor of 2.

This means that the area is scaled by a factor of 4. The ratio of radii is:

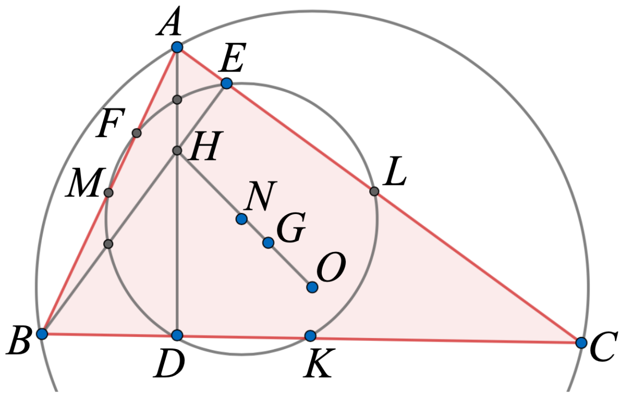
$$\frac{abc}{a'b'c'} \cdot \frac{\mathcal{A}'}{\mathcal{A}}$$

The first term is 8 and the second term is 1/4 so the result is just a factor of 2.

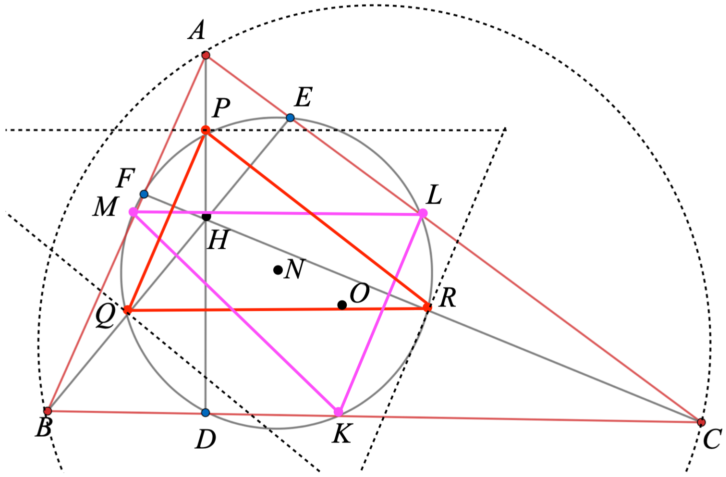
center of nine point circle

I find from reading Coxeter that the center of the nine point circle bisects the Euler line.

Recall that the centroid G lies one-third of the way along the Euler line from the circumcenter O to the orthocenter H. Then N bisects OH and G lies one-sixth of that length away from N, on the side toward O.



Proof.



We established above that the side lengths of $\triangle PQR$ are exactly half those of $\triangle ABC$.

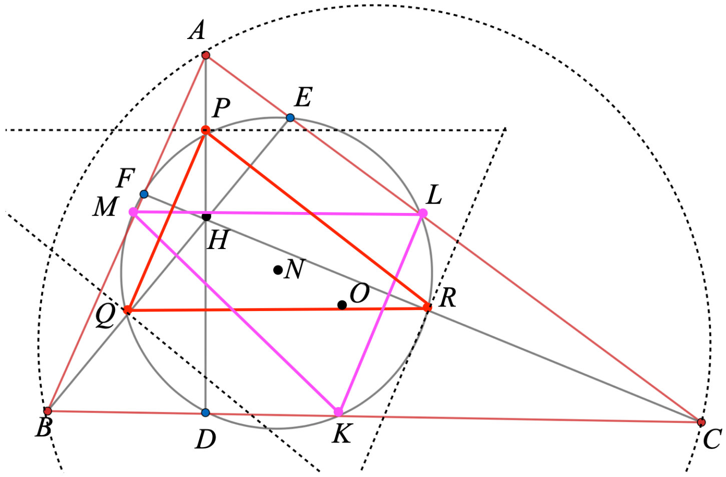
But the sides of $\triangle KLM$ are also exactly half those of $\triangle ABC$.

So the two smaller triangles are congruent. They share the same circumcircle on center N, since all the vertices lie on the circle, which is the nine point circle for the parent $\triangle ABC$.

Since $ML \parallel BC$ (and so on), the two triangles are related by rotation through a half-turn on center N.

Find the circumcenter of $\triangle ABC$ by dropping perpendiculars from points K, L and M to meet at O. The same lines are the altitudes of $\triangle KLM$, since for example,

$ML \parallel QR \parallel BC$.



The circumcenter of $\triangle ABC$, point O, is the same point as the orthocenter of $\triangle KLM$. Rotation about N converts H into O and vice-versa. It follows that the distances are the same: $HN = ON$.

$\square$

https://en.wikipedia.org/wiki/Nine-point_circle

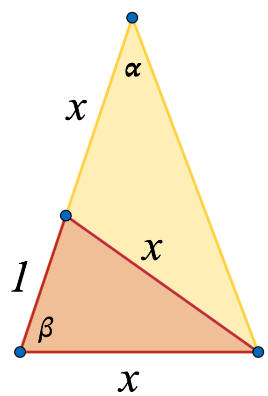
Part X

Polygons

Chapter 50

Pentagon

Before we look at the pentagon, let's start with three isosceles triangles, one tall and skinny and one short and squat, both nestled inside another tall skinny one.



Since the base angles of the tall triangles are equal, they are similar. Scale the base of the brown one to be equal to 1, then label the other side as x:

$$\frac{x}{1} = \frac{1+x}{x}$$

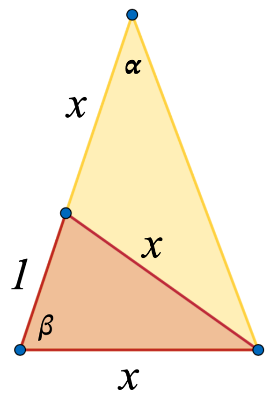
$$x^2 = 1 + x$$

This is the famous golden ratio, what the Greeks called the mean proportion, and is often labeled as ϕ:

$$\phi^2 = 1 + \phi$$

We've seen this occasionally elsewhere, and it shows up repeatedly in consideration of the pentagon. The other thing we notice is the value of the two angles. We have:

$$\beta + \beta + \alpha = \pi$$



From the lower right hand vertex:

$$\beta = 2\alpha$$

It follows that $\alpha = \pi/5$ or 36°, and β is twice that.

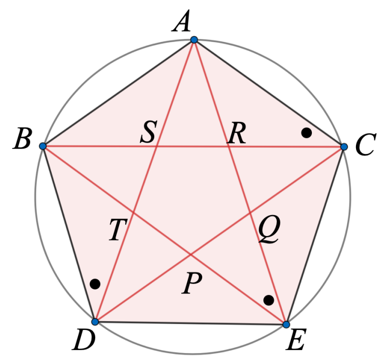
angles in the pentagon

A pentagon has 5 sides, and a regular pentagon has all five sides equal. In this chapter we explore some of its properties.

The first question that arises is how to draw one. We will look at two constructions later, a very quick modern one, and also a classic due to Euclid.

For the moment, we assume that this has been done. This is not a big deal, since most constructions start with a circle.

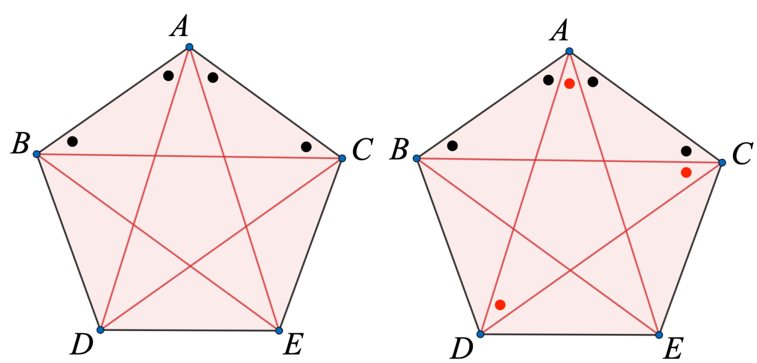
Here is a regular pentagon circumscribed by a circle.



The inscribed angle theorem makes the derivation of equal angles easy.. In the figure above, the three angles marked with black (B) dots are all equal because each is subtended by chord AB.

Because all the sides are equal, $AB = AC$ and so on, it follows that all three of the angles at any one vertex are equal. $\angle DAE$ is subtended by DE but $DE = CE$, so $\angle DAE = \angle CAE$.

In fact, all of the vertices are also equal, since central triangles (not drawn) are congruent by SSS. We could also appeal to the five-fold rotational symmetry.



Symmetry also gives us the equal central angles labeled with red (R) dots in the right panel. Then by sum of angles we have:

$$2B + 3R = 4B + 1R$$

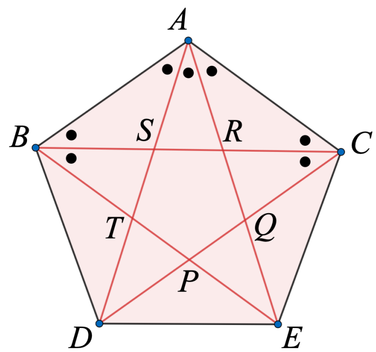
which quickly leads to $B = R$, as we have already concluded.

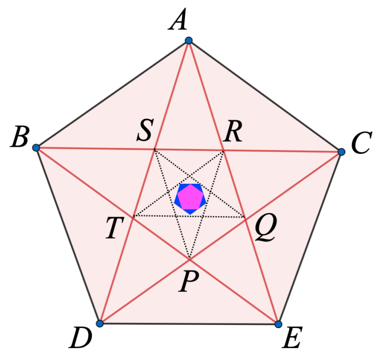
Each vertex consists of three copies of the same angle, each one of those small angles is 1/5 of two right angles or 36°. They add up to 104° at the vertices.

All five chords are equal as well since, for example, $\triangle ABC \cong \triangle ACE$ by SAS. It follows that $\triangle ADE$ is isosceles.

parallelogram

Again, each small angle at any vertex has the same measure. We have marked only some of them in the figure below (left).





In $ACPB$, adjacent vertices have five copies of the small angles, which add up to two right angles. Thus, the figure is a parallelogram. Moreover, it is a rhombus, because adjacent sides are equal, such as $AB = AC$. Altogether, there are 5 such rhombi in the figure.

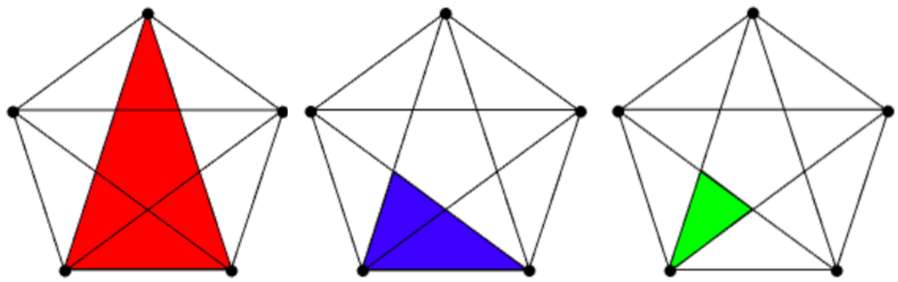
Because of the parallelograms, $BC \parallel DE$ and so on. It follows that $\triangle ADE \sim \triangle ASR$.

There are two classes of similar isosceles triangle in the figure: tall skinny ones (36-72-72), and short fat ones (108-36-36). The tall skinny ones come in three sizes, so for example

$$\triangle ASR \cong \triangle BTS$$
$$\triangle ABT \cong \triangle DSB$$
$$\triangle ADE \cong \triangle EAB$$

Each of these triangles contains the same angles, so they are similar, and they are all isosceles.

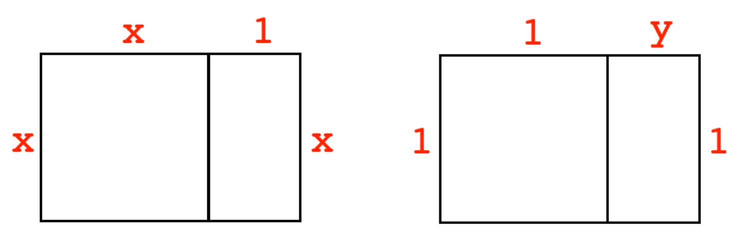
Here are three examples of the tall skinny triangle:



By sum of angles, the central $PQRST$ is a pentagon, and by symmetry, it is a regular pentagon. One can repeat the process of drawing the diagonals and generate new pentagons inside pentagons, forever.

When we work through the similar triangles using relationships between lengths (like those from the rhombus sides equal to pentagon sides), we'll see something very interesting. But I think it may be useful to stop here and preview the answer, so we can get the arithmetic straight.

the golden ratio



We draw a square and then extend two parallel sides to make a large rectangle and a small one at the same time. We don't want just any rectangles, but require that they be similar: they should have the same ratio of the long side to the short side.

We can conveniently model this in two ways. In the first, the square has side length x and the extension is 1, while in the second, the square is scaled to have side length 1 and the extension is y. These will give inverses.

We choose the first method. Hence similarity gives:

$$\frac{x+1}{x} = \frac{x}{1}$$

$$x + 1 = x^2$$
$$x^2 - x - 1 = 0$$

The solutions are:

$$x = \frac{1 \pm \sqrt{5}}{2}$$

Since $\sqrt{5} > 1$ (also > 2 !) the minus branch gives $x < 0$. We choose the positive branch:

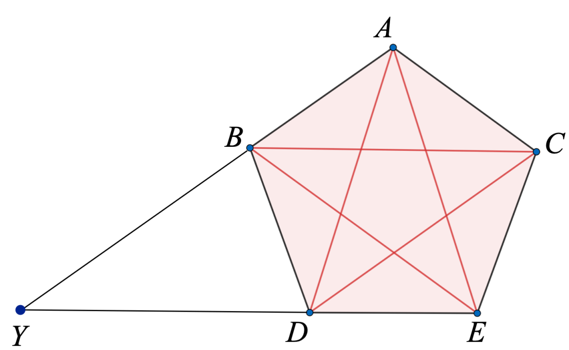
$$x = \frac{1 + \sqrt{5}}{2}$$

This is called ϕ, the famous golden mean or ratio. It has a value of about 1.65.

We can check that ϕ really does solve the equation:

$$\phi^2 = \frac{1 + \sqrt{5}}{2} \cdot \frac{1 + \sqrt{5}}{2}$$
$$= \frac{1}{4}(1 + 2\sqrt{5} + 5)$$
$$= 1 + \frac{2 + 2\sqrt{5}}{4} = 1 + \phi$$

To return to our problem:



Extend AB and ED to meet at Y.

Since the external angles to $\triangle YBD$ are equal, by I.6 $\triangle YBD$ is isosceles.

Since $\triangle ADE$ has the same base angles and equal base $BD = DE$, $\triangle ADE \cong \triangle YDB$ by ASA.

Let $\triangle YDB$ have a ratio of the side length to the base of x.

Since $BD \parallel AE$, we have $\triangle YDB \sim \triangle YEA$. Remembering that $DY = AD$, we can construct the ratios:

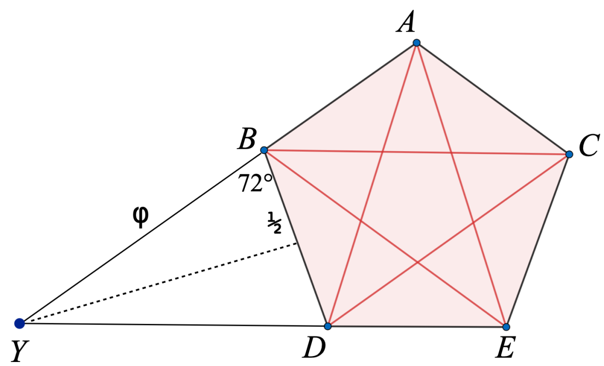
$$\frac{x}{1} = \frac{x+1}{x}$$
$$x^2 = x + 1$$

Hence x is really ϕ.

The equation below is the one to remember, with ϕ substituted for x:

$$\phi^2 = 1 + \phi$$

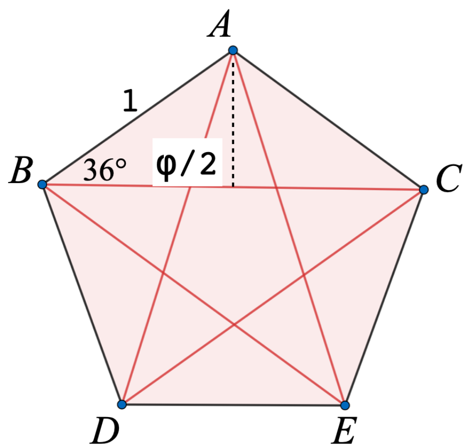
There are some other values we will need that can be seen in this figure:



$\angle YBD = 72°$, since it has two copies of the fundamental angle. Its cosine is one-half of BD, that is, $1/2$, divided by ϕ or

$$\cos 72° = \frac{1}{2\phi}$$

We also will need:



$\angle ABC = 36°$. Its cosine is one-half of BC, $\phi/2$, divided by 1, or just

$$\cos 36° = \frac{\phi}{2}$$

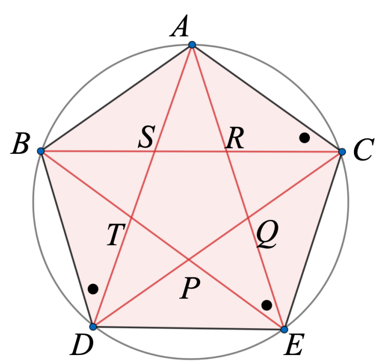
We will come back for both of these results.

more

There are occurrences of ϕ all over the regular pentagon (see the first reference below).

One can also get the golden ratio, ϕ, from the short squat triangles.

$$BC = \phi, \qquad AB = AC = 1$$



We find that $BS = \phi - 1$. Hence we compare $\triangle ABS \sim \triangle BCA$. We form the ratio of the longer side to the shorter one:

$$\frac{1}{x-1} = \frac{x}{1}$$

$$x^2 - x = 1$$

$$x^2 = 1 + x$$

Hence this x is also ϕ.

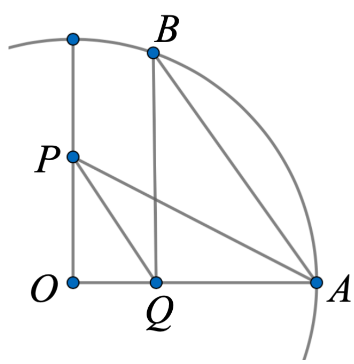
construction of circumscribed pentagon

Joyce and Bogolmony both give this construction, due to Richmond.

https://mathcs.clarku.edu/~djoyce/java/elements/bookIV/propIV11.html

https://www.cut-the-knot.org/pythagoras/RichmondPentagon.shtml

Wikipedia gives a rearranged version of the same thing.



In a circle on center O, draw the radius OA and make $\angle AOP$ a right angle, with the second radius bisected at P.

Draw AP. Now bisect the angle APO to find Q on OA.

Finally, draw the perpendicular to OA at Q and find where it cuts the circle at B. AB is one side of the pentagon circumscribed by this circle.

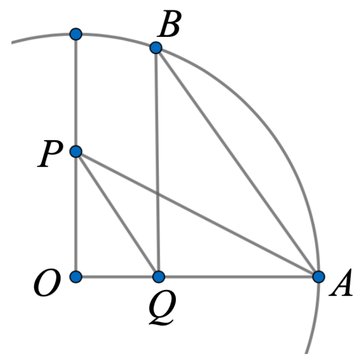
Proof. We first show that $AQ/OQ = \sqrt{5}$.

Let the radius equal 2. Then $OP = 1$ and the Pythagorean theorem gives $AP = \sqrt{5}$.

From the bisector theorem

$$\frac{OQ}{AQ} = \frac{OP}{AP} = \frac{1}{\sqrt{5}}$$

The result follows easily.



We also know that $OQ + AQ = 2$ so

$$OQ + \sqrt{5} \cdot OQ = 2$$

$$(1 + \sqrt{5}) \cdot OQ = 2$$

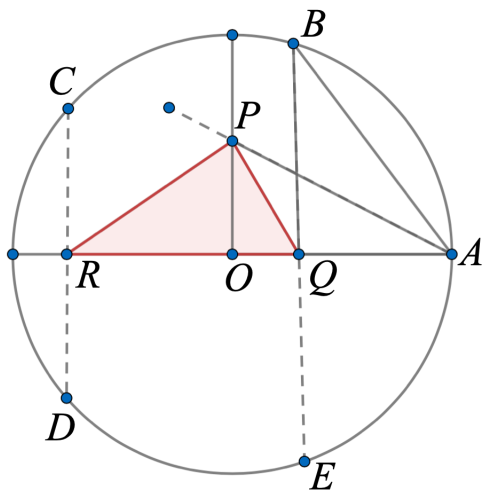
$$\frac{1}{OQ} = \phi$$

In other words, the ratio $OP : OQ = \phi$.

$\triangle BOQ$ has adjacent side $OQ = 1/\phi$ and the hypotenuse is 2 so the cosine of $\angle BOQ$ is $1/2\phi$.

But this is the cosine of 72°, which means that $\angle BOQ = \angle BOA = 72°$, and corresponds to one-fifth of the complete circle. It is the central angle of a pentagon.

Here is a figure (redrawn from Bogolmony) which shows how Richmond's approach extends to finding all of the vertices of the regular pentagon (Bogolmony cites Conway and Guy).



APO is bisected by PQ. PR is drawn as the bisector of the external angle to $\angle APO$ (i.e. supplementary). Two adjacent bisectors of supplementary angles together form a right angle.

Thus PR is perpendicular to PQ at P and forms $\triangle RPQ$ as a right triangle. $\triangle RPQ \sim \triangle ROP \sim \triangle POQ$.

An easy consequence is that by similar triangles we have that

$$\frac{OP}{OR} = \frac{OQ}{OP}$$

$$OP^2 = OR \cdot OQ$$

$$1 = OR \cdot \frac{1}{\phi}$$

$$OR = \phi$$

But OR is the adjacent side in the right triangle COR, with hypotenuse equal to 2. So the cosine of $\angle COR$ is $\phi/2$.

We showed above that the angle whose cosine is $\phi/2$ is 36°.

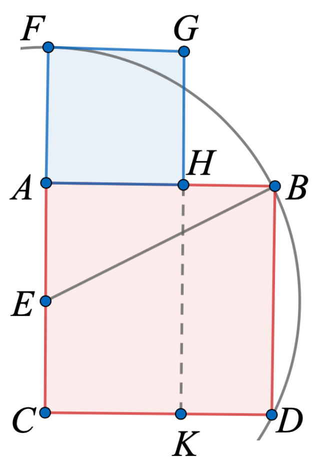
$\angle COD$ is twice $\angle COR$ or 72°, which is the correct measure for the central angle of one sector of the regular pentagon.

Euclid's construction

Euclid uses several steps to construct a pentagon circumscribed by a circle. The critical chain of dependencies is II.11 $\Rightarrow$ IV.10 $\Rightarrow$ IV.11. We first look at II.11.

II.11

> To cut the line AB at H such that the rectangle contained by the whole and one of the segments is equal to the square on the other segment.



The idea is to find H such that

$$AB \cdot HB = AH^2$$

In other words, find H such that

$$\frac{AB}{AH} = \frac{AH}{HB} = \phi$$

We will prove that the area of rectangle $HBDK$, abbreviated HD, equals FH, the square on AH.

Geometric Proof.

Draw the square on AB so $AB = BD$.

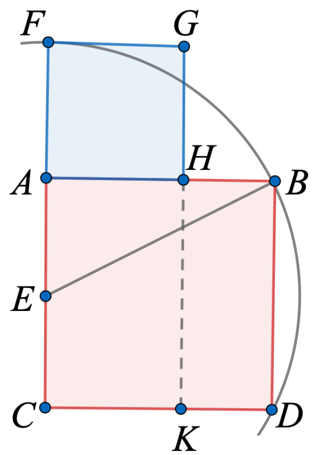
Bisect AC at E. Draw BE.

Extend EA so that $EF = BE$.

Draw the square on AF. H is the desired point.

I claim $HB \cdot AB = AH^2$.

$$\frac{AH}{HB} = \frac{AB}{AH}$$



In other words, the golden ratio or mean.

By II.6

$$CF \cdot FA = EF^2 - AE^2$$

Since $EF = BE$ and by I.47

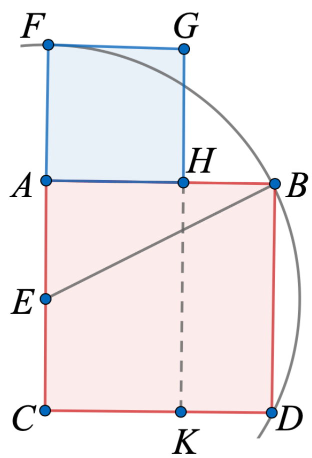
$$CF \cdot FA = BE^2 - AE^2 = AB^2$$

The left-hand side is FK, and the right-hand side is AD.

Subtract the shared area AK from each. We obtain:

$$FH = AH^2 = HD = AB \cdot HB$$

$\square$



Algebraically, let $AB = x, AH = y, HB = x - y$, and then

$$(x - y) \cdot x = y^2$$

$$x^2 - xy = y^2$$

$$x^2 = xy + y^2$$

Scale so that $y = 1 = AH$ then

$$x^2 = x + 1$$

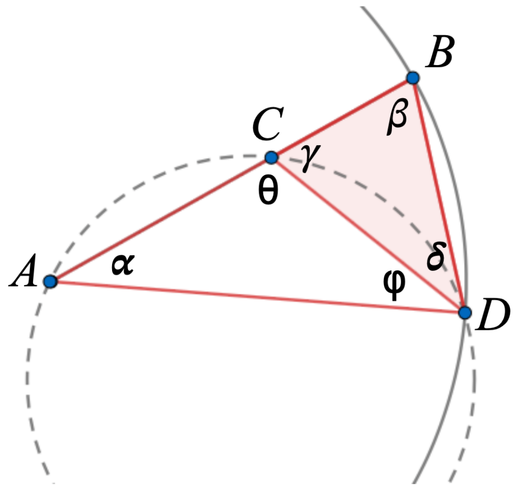
As advertised, $x = \phi = AB$, the golden ratio.

IV.10

Next, Euclid uses II.10 to construct an isosceles triangle that has its base angles twice the vertex. What is below does not follow word for word, but it's close.

Draw the circle $\mathcal{O}$ on center A with arbitrary radius AB.

By the construction of II.11, cut AB at C so that $AB \cdot BC = AC^2$.



Now find D on $\mathcal{O}$ such that

$$(1)\ BD = AC$$

So

$$(2)\ AB \cdot BC = AC^2 = BD^2$$

As radii of $\mathcal{O}$, $AB = AD$ and $\triangle ABD$ is isosceles, with

$$(3)\ \beta = \angle BDA = \phi + \delta$$

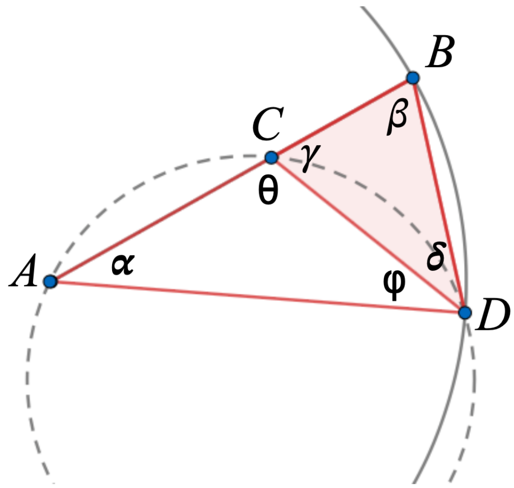
Join CD and AD.

Draw the circle $\mathcal{Q}$ containing points A,C,D.

A crucial step is that that by the converse of the tangent-secant theorem, (2) means that BD is tangent to $\mathcal{Q}$ at D.

Therefore, since they are subtended by the same arc:

$$(4)\ \delta = \alpha$$



Adding equals:

$$\angle BDA = \delta + \phi = \alpha + \phi$$

As the external angle to $\triangle CAD$:

$$(5)\ \gamma = \alpha + \phi = \angle BDA$$

By (3) $\beta = \angle BDA$, and by (5) $\gamma = \angle BDA$, hence

$$\beta = \gamma$$

Thus $\triangle DBC$ is also isosceles.

[Once we have (4), we can get here even quicker. $\triangle ABD$ has apex angle α and base angles β. $\triangle DBC$ has the equal apex angle $\delta = \alpha$ and one base angle known (β). It follows that all three angles are shared, so $\triangle ABD \sim \triangle DBC$.]

Furthermore, $\triangle DBC$ and $\triangle ABD$ are equi-angular (similar, although we won't need that).

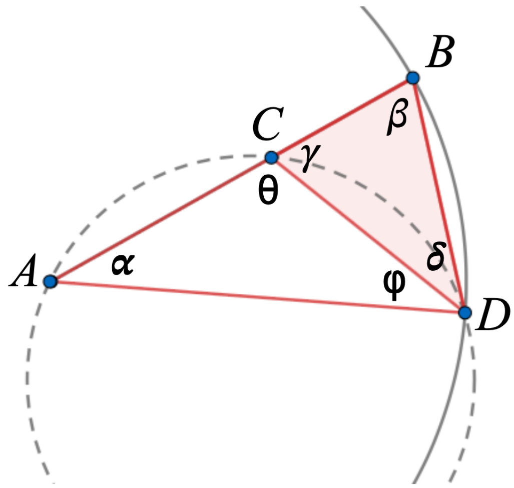
It follows that $BD = CD$ but by (2) $BD = AC$ so $CD = AC$ and thus $\triangle CAD$ is isosceles and then

$$\alpha = \phi$$

By (4) $\delta = \alpha$ so

$$\delta = \phi$$

$\angle BDA$ is bisected by CD.



In summary, we have

$$\alpha = \delta = \phi$$

and three isosceles triangles. So

$$2\alpha = \angle BDA = \beta = \gamma$$

In the isosceles $\triangle ABD$ each of the equal base angles is equal to β and is twice the vertex angle α.

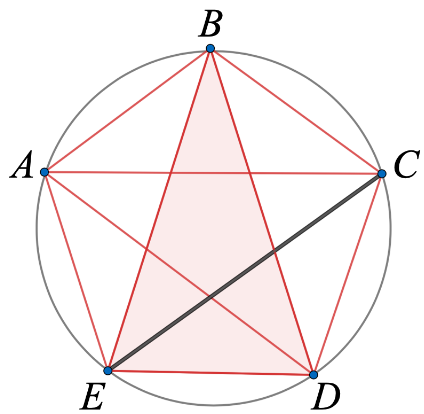
$\square$

Since $\triangle ABD$ has central angle α equal to one-fifth of a right triangle, it is one-tenth of the complete circle.

Therefore, the side BD forms one side of a regular decagon inscribed in $\mathcal{O}$.

Following Bogolmony, we use this shortcut: connecting alternate vertices yields a regular pentagon.. This completes the construction.

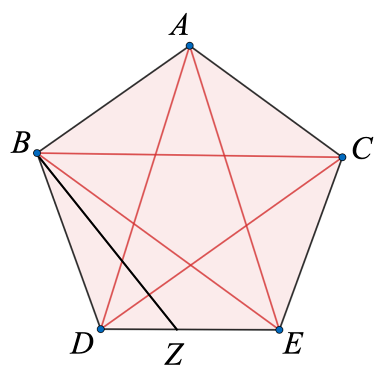
Euclid IV.11 inscribes a triangle with the same angles as above into a circle ($\triangle BED$, below).



Bisect $\angle BED$ and extend the bisector to meet the circle at C. Do the same with $\angle BDE$ to find A. This completes Euclid's construction.

another bisection

Draw the perpendicular at any vertex, say $BZ \perp BA$, so that $\angle ABZ$ is a right angle.

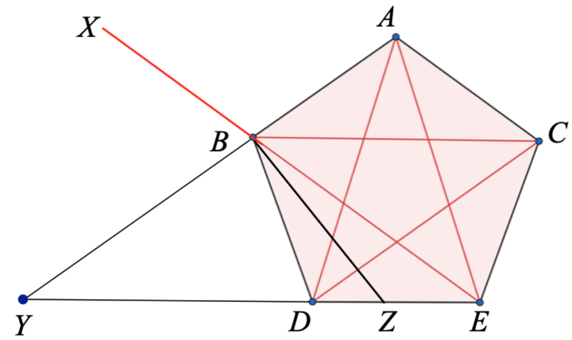


$\angle DBE$ is bisected, since the total angle at B is 108°, so by subtraction $\angle DBZ = 18°$, which is one half of $\angle DBE = 36°$.

We use the bisector theorem again:

$$\frac{EZ}{DZ} = \frac{BE}{BD} = \frac{1+x}{x} = x = \phi$$

We can extend adjacent sides in the previous figure. Since $\angle ABZ$ is a right angle and $\angle DBZ$ is one-half of the bisected internal $\angle DBE$ (in $\triangle BDE$), it follows that $\angle DBY$ is the bisector of the external $\angle DBX$.



By the external bisector theorem:

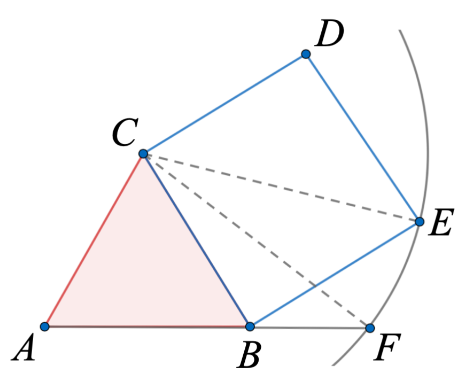
$$\frac{EY}{DY} = \frac{BE}{BD} = \phi$$

When two adjacent sides of a regular pentagon are extended, then the length of the side plus the extension is in proportion to the side length as ϕ.

This is also evident from the fact that $BD \parallel AE$, so $\triangle BYD$ is similar to $\triangle AYE$.

problem

https://www.cut-the-knot.org/do_you_know/GoldenRatio.shtml



Let $\triangle ABC$ be equilateral and $BCDE$ be a square. Construct the circle on center C with radius CE and find where it cuts the extension of side AB at F. Prove that $AB : BF = \phi$.

Proof.

Scale the triangle so that $AC = 1$

Then $CE = \sqrt{2} = CF$.

We have $\cos A = \sin 30° = \frac{1}{2}$

Use the Law of Cosines:

$$CF^2 = AC^2 + AF^2 - 2AC \cdot AF \cos A$$

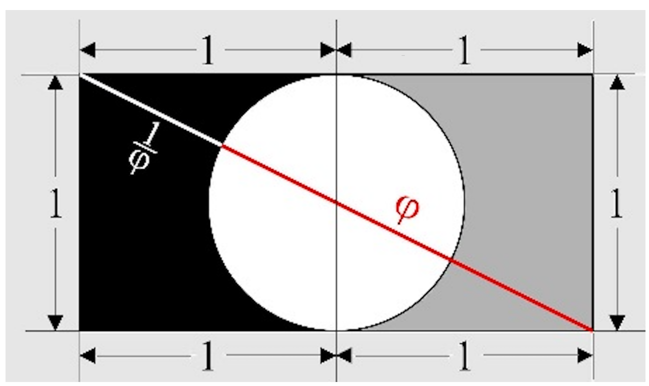
$$2 = 1 + AF^2 - AF$$

$$AF = \phi$$

The result follows easily.

□

problem



Here's another construction in the collection curated by Bogolmony, from John Arioni.

The distance from the center of the circle to the corner of the rectangle is just $\sqrt{5}/2$. So ϕ is quite literally constructed by adding $1/2$ (the radius of the circle) to give the length of the red line.

It remains to show that

$$\phi + \frac{1}{\phi} = \sqrt{5}$$

Start from the usual expression for ϕ and solve for $\sqrt{5}$. Equate the result to the left-hand side of what is above.

$$\phi + \frac{1}{\phi} = 2\phi - 1$$

Multiply by ϕ:

$$\phi^2 + 1 = 2\phi^2 - \phi$$
$$1 = \phi^2 - \phi$$

That looks correct. To obtain the proof, reverse these steps.

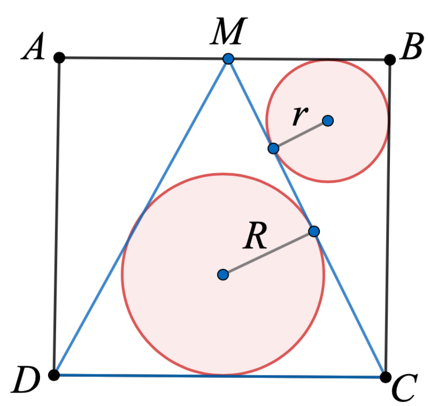
□

ϕ in the square

Let's finish the chapter with one last example I found on Bogolmony's site.

https://www.cut-the-knot.org/do_you_know/GoldenRatioInSquare.shtml

It was "contributed by Ercole Suppa (Italy) at the Peru Geometrico facebook group."



One side of a square is bisected at M and then the two incircles are drawn as shown. Remarkably, the radii are in the ratio $R/r = \phi$.

Proof.

Let the square AC have sides of length 2, so $MB = 1$ and $MC = \sqrt{5}$. We find the semi-perimeters of the two triangles:

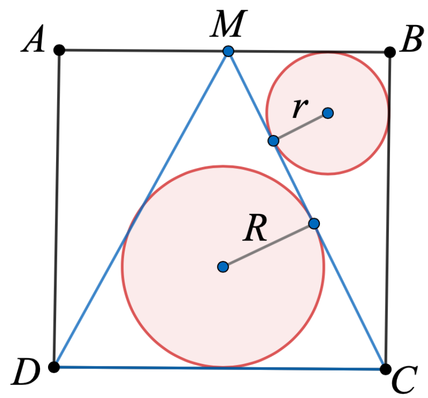
$$s_{\triangle MBC} = \frac{1}{2} \cdot (3 + \sqrt{5})$$

$$s_{\triangle MDC} = \frac{1}{2} \cdot (2 + 2\sqrt{5}) = 1 + \sqrt{5}$$

Then we find the areas as

$$\mathcal{A}_{\triangle MBC} = rs_{\triangle MBC} = \frac{1}{2}\, r\ (3 + \sqrt{5})$$

$$\mathcal{A}_{\triangle MDC} = Rs_{\triangle MDC} = R\ (1 + \sqrt{5})$$



Since the base of one triangle is twice the other, we have

$$\mathcal{A}_{\triangle MDC} = 2A_{\triangle MBC} = r(3 + \sqrt{5})$$

Then

$$R(1 + \sqrt{5}) = r(3 + \sqrt{5})$$

$$\frac{R}{r} = \frac{3 + \sqrt{5}}{1 + \sqrt{5}}$$

which certainly doesn't look like ϕ, although it does have $\sqrt{5}$.

We could clear the denominator, multiplying by $(1 - \sqrt{5})/(1 - \sqrt{5})$, but instead let's just play around:

$$\phi = \frac{1 + \sqrt{5}}{2}$$

$$2\phi = 1 + \sqrt{5}$$

$$2\phi + 2 = 3 + \sqrt{5}$$

Hence we have that

$$\begin{aligned} \frac{R}{r} &= \frac{2\phi + 2}{2\phi} \\ &= \frac{\phi + 1}{\phi} \end{aligned}$$

Recalling that $\phi + 1 = \phi^2$, it follows that

$$\frac{R}{r} = \phi$$

□

That's amazing!

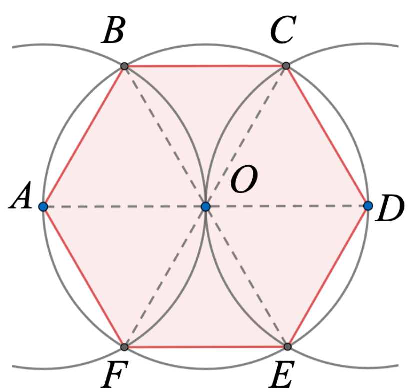
Chapter 51

Hexagon

angles in the hexagon

A hexagon has 6 sides, and a regular hexagon has all six sides equal. In this chapter we explore some of its properties.

Euclid, IV.15 has a nice construction.



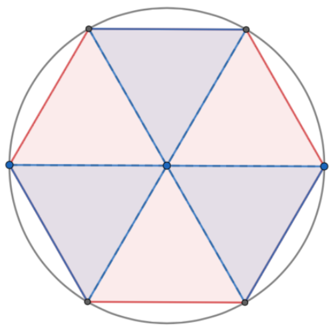
Draw a circle on center O, and draw a diameter AD. Then draw circles on centers A and D with the same radius, so $AO = DO$.

First, OA, OB, OC, OD, OE and OF are all radii of the original circle.

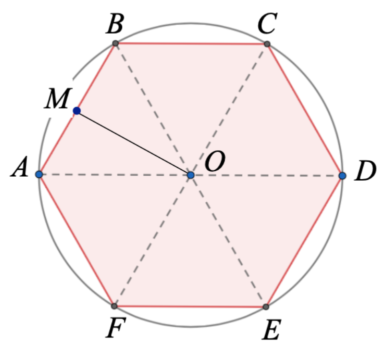
Second, the segments $AB = AF = AO$, and $DC = DE = DO$. It follows that $\triangle AOB$ is equilateral and so is $\triangle COD$.

Finally, the base angle of $\triangle BOC$, $\angle BOC$, is $60°$, and the triangle is isosceles, so it is also equilateral.

As a result, we have six equilateral triangles, and the polygon $ABCDEF$ is a regular hexagon.



Let the side length (and radius) be 1. Then Pythagoras's theorem gives us that the apothem OM has a length of $\sqrt{3}/2$.



So twice the area of each component triangle is

$$2\mathcal{A}_{\triangle AOB} = \frac{\sqrt{3}}{2} \cdot 1$$

The total area of the hexagon is three times that or

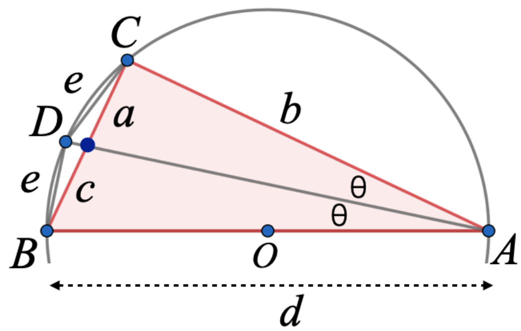
$$\mathcal{A}_{hex} = \frac{3\sqrt{3}}{2}$$

This approximates the area of the unit circle (π), but it is not a very good approximation.

Similarly, the total perimeter is 6, which approximates the total perimeter of the unit circle (2π), but it is not especially close yet either.

lower bound for π

Let $\triangle ABC$ be a right triangle inscribed in a semicircle, and the angle at A is bisected.



The idea is that the side of $\triangle ABC$ that is a chord of the circle, $BC = a + c$, approximates the perimeter of the circle.

Let $\triangle OBC$ (the arms are not drawn) be an equilateral triangle. By the inscribed angle theorem, $2\theta = 30°$, one-half of $\angle BOC$. The total perimeter is 6 times that.

Now imagine that we are able to find the length e. e is clearly much closer to the actual circle, so 6 times $2e$ would be a better approximation to the perimeter.

Using this idea, with both inscribed and circumscribed hexagons, Archimedes came up with both upper and lower bounds, namely, $223/71 < \pi < 22/7$.

We begin with a result that comes from the bisector theorem.

$$\frac{a}{c} = \frac{b}{d}$$

$$\frac{a}{c} + 1 = \frac{b}{d} + 1$$

$$\frac{a+c}{c} = \frac{b+d}{d}$$

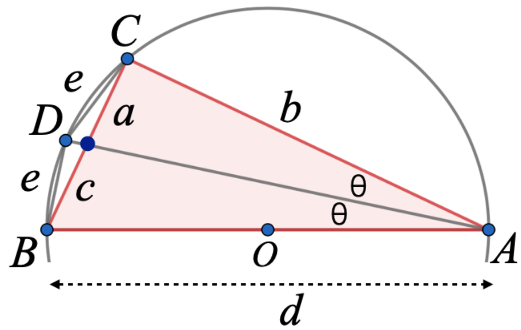
$$\frac{a+c}{b+d} = \frac{c}{d} = \frac{a}{b}$$

$$\frac{b+d}{a+c} = \frac{b}{a}$$

$$\frac{b}{a+c} + \frac{d}{a+c} = \frac{b}{a}$$

Using trigonometric functions, we can write this as

$$\cot 2\theta + \csc 2\theta = \cot \theta$$



If we're doing multiple rounds we would need Pythagoras's theorem

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \cot^2 \theta = \frac{1}{\sin^2 \theta} = \csc^2 \theta$$

First add the cotangent and cosecant of the double angle to find the cotangent, then use the above procedure to get the cosecant. Do the same thing again, as many times as you wish.

computation

We start with a $30 - 60 - 90$ triangle ($2\theta = 30°$).

The next thing to decide is the scale. If $d = 1$ then the perimeter equals π; alternatively with $d = 2$ then the perimeter is 2π.

Let $d = 2$.

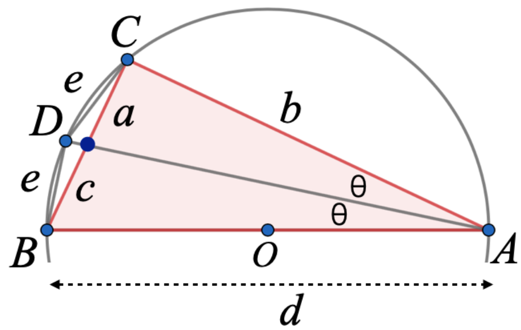
$$BC = a + c = 1$$

$$b = \sqrt{3}$$

The whole computation has to be done with rational numbers, because that's how the Greeks did things.

We need a rational approximation for $\sqrt{3}$. The value Archimedes used was $\sqrt{3} \approx 1351/780$. (I have written elsewhere about how he may have come up with that).

The ratio to the true value is ≈ 1.0000003. It is over, but we will use the value as the inverse, so that's actually under, appropriate for a lower limit.



$$\csc 2\theta = 2$$

$$\cot 2\theta = \sqrt{3} = \frac{1351}{780}$$

$$\cot \theta = \frac{1560 + 1351}{780} = \frac{2911}{780}$$

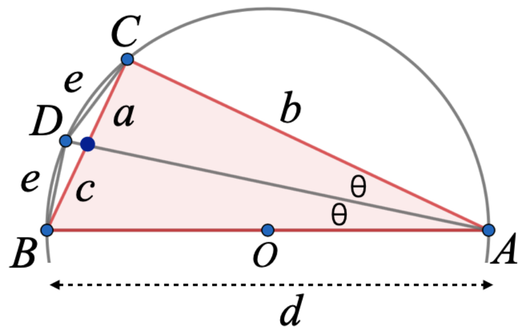
$$\csc^2 \theta = 1 + \frac{8473921}{608400} = \frac{9082321}{608400}$$

Archimedes approximates this as

$$\csc \theta = \frac{AB}{BD} = \frac{3013 - 3/4}{780} = \frac{2}{e}$$

Note that $3013 - 3/4$ is slightly larger than the true value of the square root, but this error makes our estimate *lower*, since it shows up as the inverse, below. So the estimate is valid as a lower bound.

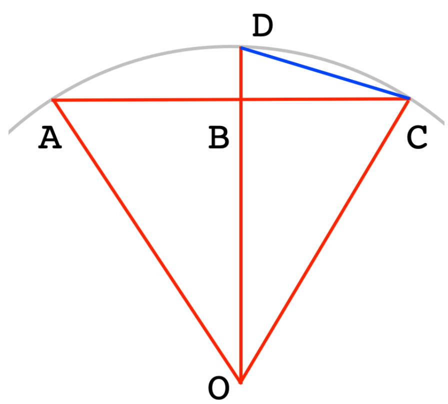
e is twice ≈ 0.25881. There are 12 copies of e in the perimeter but that's for 2π



To get π, we do $12\cdot \approx 0.25881 = \approx 3.10577$.

Liu Hui

Another approach is rely entirely on the Pythagorean theorem. Let AC be the chord for an n-gon.



Then the distance from B to the circle is

$$BD = OC - \sqrt{OC^2 - BC^2}$$

$$\begin{aligned} BD^2 &= OC^2 - 2\sqrt{OC^2 - BC^2} + OC^2 - BC^2 \\ &= 2OC^2 - 2\sqrt{OC^2 - BC^2} - BC^2 \end{aligned}$$

CD is the chord for the $2n$-gon and its squared length is

$$\begin{aligned} CD^2 &= BD^2 + BC^2 \\ &= BD^2 + 2OC^2 - 2\sqrt{OC^2 - BC^2} \end{aligned}$$

Each round of angle-halving involves three squares (one is used twice), two square roots and some arithmetic.

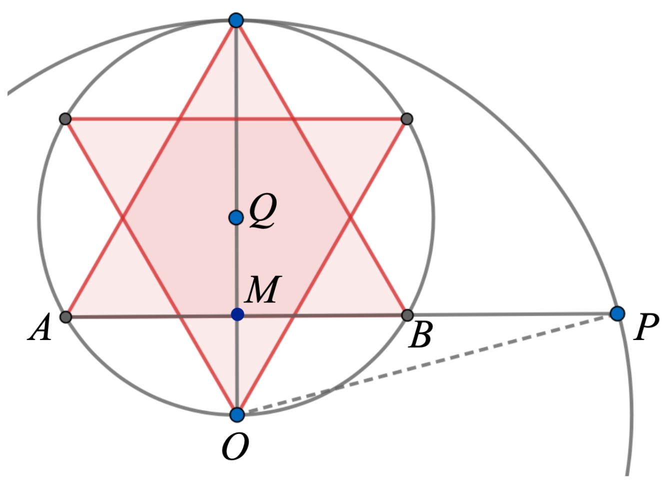
This is not as elegant as Archimedes' method, which needs only a single square and one square root, but Liu Hui was persistent.

He was persistent enough to recognize that 355/113 is a much better approximation to π than either of Archimedes' values or even Ptolemy's 377/120.

355/113 has a *much* smaller error than any other rational approximation until the integer components get very large. Likely, Ptolemy missed that because he did not know the true value to enough accuracy.

ϕ in the hexagon

Suppose we inscribe a hexagon into a circle on center Q. The problem comes with a big hint since a hexagram (aka Star of David) has been drawn using the vertices of the hexagon.



Now draw the circle on center O whose half-radius is OQ, and extend one of the sides of the hexagram, AB to meet the larger circle at P.

Show that $AP/AB = \phi$.

Proof.

By our work with equilateral triangles we know that OM is one-quarter of OP.

So then let $OM = 1$ and $OP = 4$ and

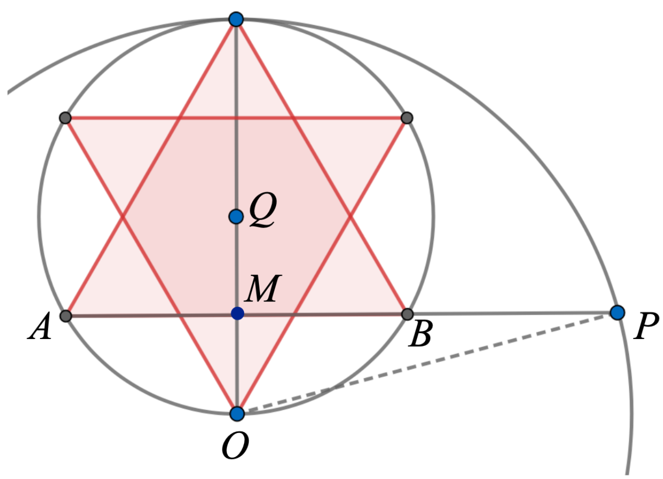
$$PM^2 = OP^2 - OM^2 = 15$$

$$PM = \sqrt{15}$$

$$MB = \sqrt{3}$$
$$\frac{PM}{MB} = \sqrt{5}$$
$$\frac{PM}{AB} = \frac{\sqrt{5}}{2}$$
$$\frac{AM}{AB} = \frac{1}{2}$$

AP/AB is the sum of the last two terms, which is just ϕ.

□



Part XI

Addendum

Chapter 52

Author's notes

A central feature of this book is the relentless use of proof. I emphasize the key insight for each, and have tried to make the proofs simple and as easy to follow as possible.

This volume is distinguished from most other texts, since they maintain that a proper proof should be watertight, with each step carefully justified and following closely from the one before. I don't deny that rigor has its proper place in math education, but I also think that this rigidity obscures the core insights. Our purpose here is to view beauty clearly.

We prefer instead to be like the famous mountaineer Ueli Steck. Reach the summit quickly and emphasize the key steps. You should be able to fill in the details if there are loose ends.

Multiple proofs for important theorems are sometimes given, because proof is our stock in trade, and different approaches shed light on how proofs may be found and developed.

Another distinguishing feature is a set of simple proofs based on scaling of triangles. This happens for the Pythagorean theorem and for Ptolemy's theorem, as well as the sum of angles theorems and then later, a fairly sophisticated theorem of Euler's.

Recently I came across a fantastic book by Acheson, called *The Wonder Book of Geometry*. I helped myself to some of his examples, and now have more than a dozen. Please go find Acheson, and buy it. It's truly magical. In fact, all of his books are wonderful!

A saying attributed to Manaechmus, speaking to Alexander the Great, is that “there is no royal road to geometry”. Others write that this was actually Euclid, speaking to Ptolemy I of Egypt. Since the two sources lived some 700 years after the fact, it is difficult to know.

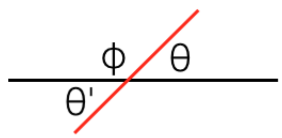
Practically, this means that in learning mathematics you must follow the argument with pencil and paper and work out each step yourself, to your own satisfaction. That is the only way of really learning, and at heart, a principal reason why I wrote this book.

Having read a chapter, see if you can prove the theorems yourself, without looking at the text.

There are a few problems listed in the later chapters, perhaps thirty or more altogether. Most of them have worked out solutions. It is highly recommended that you attempt each problem yourself before reading my answer. Since the crucial point is often to draw an inspired diagram, you must stop reading as soon as the problem is stated!

Chapter 53

Angular measure



We can see that obviously

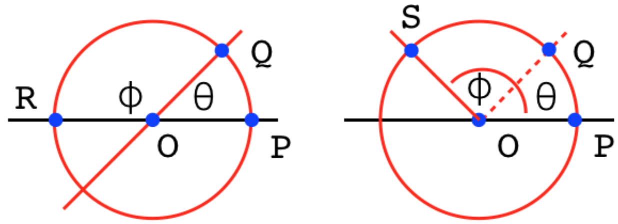
$$\phi > \theta$$

Writing the statement $\phi > \theta$ is easy, but for this to make sense (what if they are almost equal?), we must have some way of taking the measure of an angle. We can't just rely on a picture.

Our answer is to construct a circle around the central point.

∘ The measure of the angle is the distance along the circumference between the points where the lines cross the circle.

If that distance along the edge is larger for ϕ than for θ, then $\phi > \theta$. In the left panel, the arc between Q and R (call it arc QR) is larger than arc PQ.



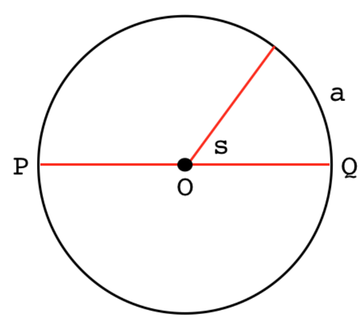
If we lay off the arcs starting from the same point P, the arc PS is longer than the arc PQ (right panel).

We don't need to actually measure the arc itself to do this. Measuring curved lengths is a bit tricky to do.

Instead we can use a standard compass to lay off the linear distance from P to S and compare that with the distance from P to Q. Since the distance from P to S is more than from P to Q, $\phi > \theta$.

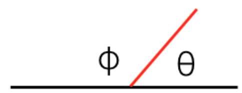
This doesn't work very well as S approaches the diameter through O and P, but we could lay off the distance in two (or more than two) parts and that would be fine. For example, take a unit circle, draw a diameter and the perpendicular bisector to form four right angles. Then bisect each right angle and bisect again. That would divide the circle into 16 arcs of equal length.

No matter how the measurement is to be performed, we *define* the measure of an angle s to be equal to the arc a it sweeps out, or is subtended by, in a circle of radius 1, a *unit* circle. Angles are not lengths, but numerically, the measure of the angle is the measure of the arc.



right angles

• the sum of all the angles on one side of a line (at a given point), is equal to two right angles.



This is simply a matter of symmetry and subtraction. If the sum of all the angles at a point is equal to four right angles, then the sum of the angles on each side of a line through that point is equal to two right angles.

The convention that there are 360° total in a circle dates to the time of the Babylonians (c. 2400 B.C.).

In degrees, a right angle is 90° and two supplementary angles measure 180°.

There is nothing particularly special about using 90° as the measure of a right angle, or 360° for one whole turn. Well, here is one thing: there are *approximately* 360 days in a year, which marks the sun's track across the sky. Another idea is that 360 is special because it has so many factors, which makes it possible to divide up a circle evenly in $2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20 \dots 180, 360$ parts.

Euclid just talks about angle measurements in terms of right angles. For example, that the sum of two supplementary angles is equal to two right angles.

In his book, *Measurement*, Lockhart adopts the convention that a whole turn is equal to 1.

radians

We'll just mention here that one whole turn can be defined using a different unit of measure as 2π *radians*, and that convention turns out to be quite important for calculus.

It is based on two ideas: the first, from above, that angular measure is numerically equal to arc length, and second, that in a unit circle, the circumference or total of the arc length is equal to 2π.

In calculus, all angles will be in radians. One big reason for that comes in working out the *derivative* of sine and cosine. There, it will be important to consider the following expression:

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

The notation on the left asks us to consider what happens as θ gets close to the value 0, and the rest of it states that the ratio $\sin \theta / \theta$ is equal to 1. As a result, the derivative of $\sin \theta$ is simply $\cos \theta$.

Well, if you're working in degrees *that's not true.* There's an awkward constant of proportionality. So we work in radians.

It also makes plots prettier, since the sine of x goes from 0 to 1 as x goes from 0 to $\pi/2 \approx 1.57$. The sine of 1 degree is effectively zero, so that wouldn't look so nice.

Chapter 54

Additional constructions

collapsible compass

We note briefly that there is a restriction in Euclid's *Elements* to a *collapsible* compass, one which loses its setting when lifted from the page. That means that generally, you wouldn't be able to draw two circles of the same radius on different centers.

We get around that restriction by drawing the circles on Q and R with the same radius QR.

We will call a compass that is able to hold its setting, a *standard* compass. Within the first few pages of that book, it is shown how to use a collapsible compass to carry out the very construction we said we couldn't do, namely, construct two circles on Q and R with equal radius and that radius not equal to QR or QP.

Also, see the video at the url:

https://www.mathopenref.com/constperpextpoint.html

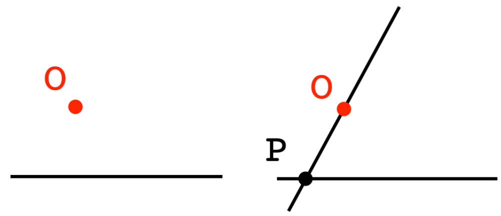
We have skipped that part here.

Euclid I.31: construct a line parallel to another line

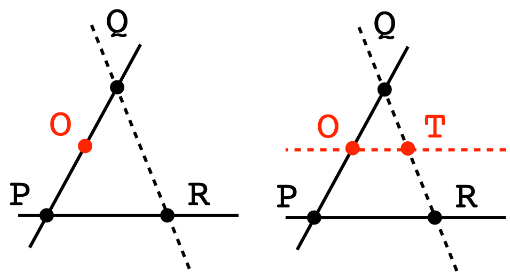
Suppose we are asked to construct a line parallel to a line or line segment, through a given point. We remain true to the Greek ideal, that dividers should not come off the paper.

Proof.

First, pick some point on the line segment P, and draw a line segment through OP.



Find Q on the second line such that $QP > OQ$.



Now draw the circle with center Q and radius QP and, at the intersection with the first line, R. Draw the line QR.

Finally, draw the circle with center Q and radius OQ, and at the intersection of the circle with the last line, find T.

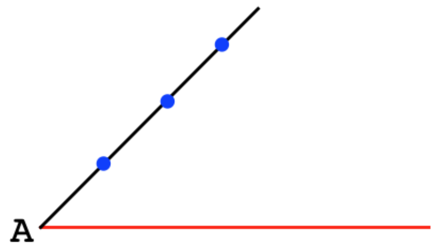
We have that $OQ = QT$ and $QP = QR$. Therefore the base angles of $\triangle QOT$ and $\triangle QPR$ are equal and therefore the triangles are similar by AAA.

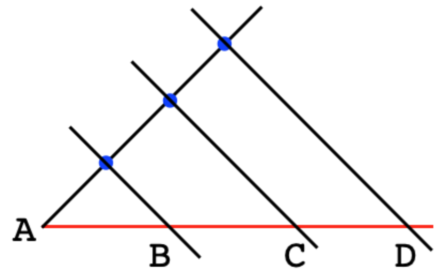
Therefore the bases PR and OT are parallel, by the converse of the alternate interior angles theorem.

□

Euclid VI.9: division of a line segment into parts

We wish to divide a general segment (in red, below) into an even number of pieces. Suppose that number is three.





Using one end of the target segment, draw any other line, and mark off on that line segments of equal length, using a compass. (Even with a collapsible compass, this can be done sequentially by moving the fixed point).

Then, erect the perpendicular bisector of the black line at each point and extend the bisector to the target red line.

We will have that $AB = BC = CD$. Furthermore, since $AB = BC$, $AB = \frac{1}{2}(AB + BC)$ so $AC = 2AB$.

$\square$

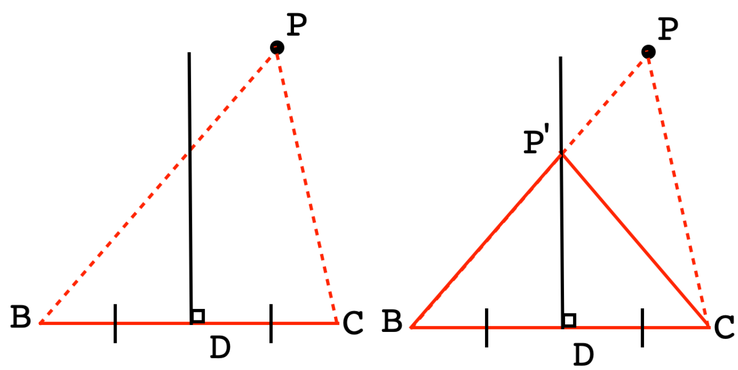
This construction uses properties of *similar* triangles that we have not explained yet. Since the angle at A is shared and the angle with the black line is always the same, all the triangles have the same shape and their sides are in proportion. We have fixed that proportion as an integer: 1, 2 or 3.

Chapter 55

Additional proofs

perpendicular bisector (converse)

The argument in this section is a little complicated.



Suppose a given point P does not lie on the vertical bisector. We claim that any such point cannot be equidistant from B and C.

Proof.

We will use an argument by contradiction. We assume the opposite of the statement we want to prove, and show that it leads to a contradiction and so cannot be correct.

Now, suppose that P *is* equidistant, with $PB = PC$. By the forward theorem we have that $\angle PBC = \angle PCB$.

Draw the perpendicular bisector and find point P' on both PB and the perpendicular bisector.

By the forward theorem we have that $\angle P'BC = \angle P'CB$. But $\angle P'BC$ and $\angle PBC$ are the same angle.

Therefore

$$\angle PBC = \angle P'CB = \angle PCB$$

But clearly $\angle PCB > \angle P'CB$.

This is a contradiction. Therefore, $PB \neq PC$.

$\square$

If we raise the perpendicular bisector from a line segment, *every* point on the bisector is equidistant from the ends of the line segment, and when the points are connected, forms an isosceles triangle.

No other point not on the perpendicular bisector can be equidistant from the two endpoints.

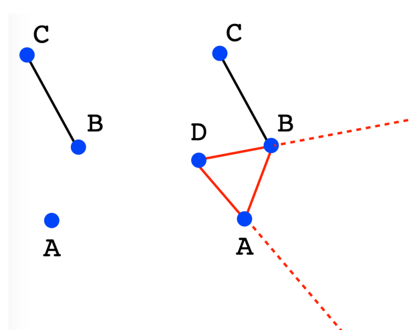
Proof. (Alternate).

There is also a much easier proof that relies on **Euclid I.7**, which says there cannot be two points on the same side of BC above which have the same distance to B and C.

Yet that is exactly what this situation calls for. P is claimed *not* to be on the bisector, with $PB = PC$. Yet by the forward theorem, there must be another point Q *on* the bisector with $QB = QC$. By **Euclid I.7**, this is impossible. $\square$.

Euclid I.2: transfer a length

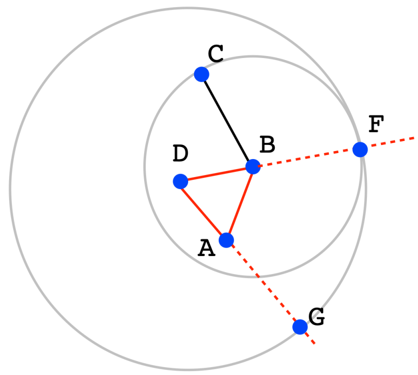
To construct a line segment on a point, equal to a given line segment.



Since this immediately follows the first construction, it seems likely we will need an equilateral triangle. The other thing we know how to do is to draw circles.

We notice that we can construct a circle of radius BC on center B. Where that intersects the extension of DB at F we have a length equal to one side of the $\triangle$ plus BC.

So then draw a circle on center D of radius DF.



The intersection of that circle with the extension of DA marks off a length equal to one side of the $\triangle$ plus BC.

So $AG = BF = BC$, as required.

If it is desired to draw a line segment of length BC in some other direction from A, we just need another circle, centered at A. That is Proposition 3.

In sum then, we can mark off any length from one line segment onto another, even with just a collapsing compass.

Euclid I.6

We proved the converse of I.5 early in the book based on angle bisection. Later, we gave another proof by contradiction.

- **isosceles triangle theorem** (Euclid I.6: angles $\rightarrow$ sides)

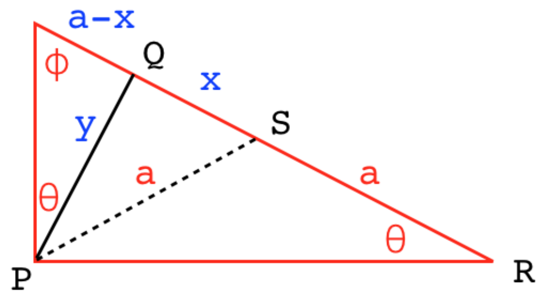
tangents of similar triangles

Here is a mixed geometric/algebraic proof of the Pythagorean theorem.

Proof.

Let S be the midpoint of the hypotenuse in a right triangle, and draw PS connecting the midpoint to the vertex at P. If the hypotenuse has length $2a$, then the length of PS is a, by the **midpoint theorem** that we talked about previously.

[Quick proof: inscribe the original triangle in a circle, with the hypotenuse of the right triangle as the diameter. The length a is the radius of the circle centered at S that contains the three vertices of the original right triangle.]



Now, draw the altitude from the right angle at P to the hypotenuse PQ. Suppose $QS = x$, then the length a is divided into x and $a - x$ as shown.

The angle ϕ and the angle at vertex R labeled θ are complementary angles in a right triangle, they add up to one right angle. Therefore both angles labeled θ are equal.

So we can form the equal ratios of sides:

$$\frac{a-x}{y} = \frac{y}{a+x}$$
$$(a-x)(a+x) = y^2$$
$$x^2 + y^2 = a^2$$

□

The second line from the last:

$$(a-x)(a+x) = y^2$$

is a statement of our theorem that the altitude of a right triangle is the geometric mean of the two sections of the base.

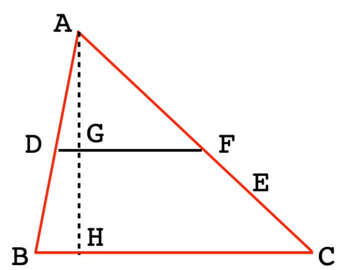
similar right triangles

altitudes in proportion

Note that, as well as the sides, the altitudes are also in proportion with the same ratio.

One way to see this is to drop an altitude and then consider the two similar right triangles on one side of it. The altitudes are sides of this triangle.

For similar triangles, where the sides are in proportion k, the areas are in proportion k^2. The reason is that the altitudes are in the same proportion, namely k.



Given that $DF \parallel BC$, so $\triangle ADF \sim \triangle ABC$.

Drop the altitude AGH.

Now we see that $\triangle ADG \sim \triangle ABH$ and $\triangle AGF \sim \triangle AHC$, with the same proportionality constant, *since they share the sides* AG and AH.

Suppose that $AD/AB = DF/BC = k$. Then AG/AH also is equal to k.

The ratio of areas is then

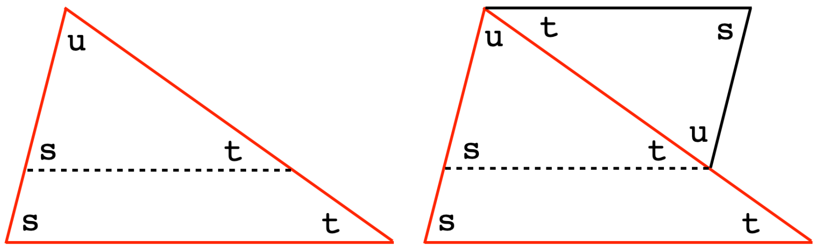
$$\frac{\Delta_{ADF}}{\Delta_{ABC}} = \frac{\frac{1}{2}DF \cdot AG}{\frac{1}{2}BC \cdot AH} = k^2$$

all triangles

Any triangle can be decomposed into two right triangles.

We previously proved the equal angles for two similar right triangles implies equal ratios of sides. Now we combine the results for the two sub-triangles, both right triangles, and we will have the result for the general case.

We use a flipped and rotated copy of the smaller triangle. Start with two triangles similar because the angles are the same (left panel).



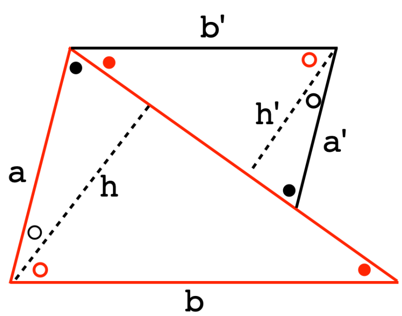
Make a copy of the smaller triangle and rotate it and then attach at the top (forming a parallelogram). The original small triangle and the flipped version are congruent by our construction.

From congruent triangles we get the angle equalities in the right panel.

We also have two pairs of parallel sides, either by alternate interior angles or because $s + t + u$ is equal to two right angles.

Now, draw the two altitudes, label the sides, and suppress the labels for the angles but just mark them with colored circles.

The angles marked with filled dots are equal (black and black, and red and red) by parallel sides as we just said, and the open dots with the same colors are equal because they are complementary angles in a right triangle.



Thus, we have two different pairs of similar right triangles.

$$\frac{a}{h} = \frac{a'}{h'}$$

$$\frac{b}{h} = \frac{b'}{h'}$$

So then

$$\frac{h}{h'} = \frac{a}{a'} = \frac{b}{b'}$$

Corresponding sides of the two original triangles have equal ratios of sides. (It is important that we extended the equal ratios result to the hypotenuse for this to work).

But there is nothing special about this pair of sides, we could have chosen any other pair, either a and c or b and c, and have the same result.

Therefore if any two triangles have three angles the same, the side lengths are all in the same proportion.

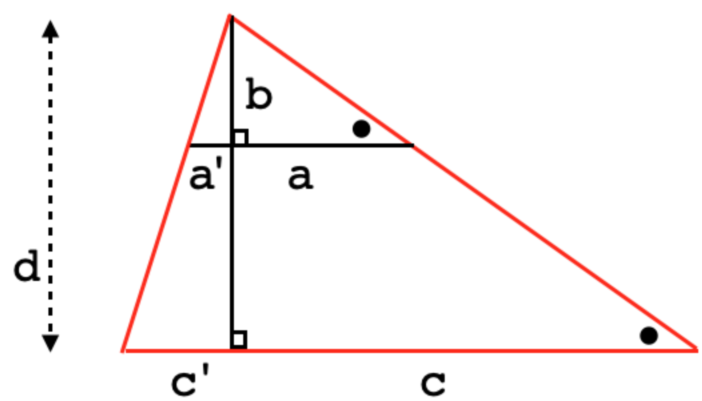
$\square$

similar triangles

Euclid VI.2 gives us all we need. However, let's take a moment to look at a different proof.

We showed previously that all the statements about similarity that we made in this chapter apply to similar right triangles.

But we can cut any triangle into two right triangles.



If these are right triangles, then the two on the right are similar as well as the two on the left, using complementary angles.

We have

$$\frac{a}{c} = \frac{b}{d}$$

but also

$$\frac{b}{d} = \frac{a'}{c'}$$

So then

$$\frac{a}{c} = \frac{a'}{c'}$$

and then

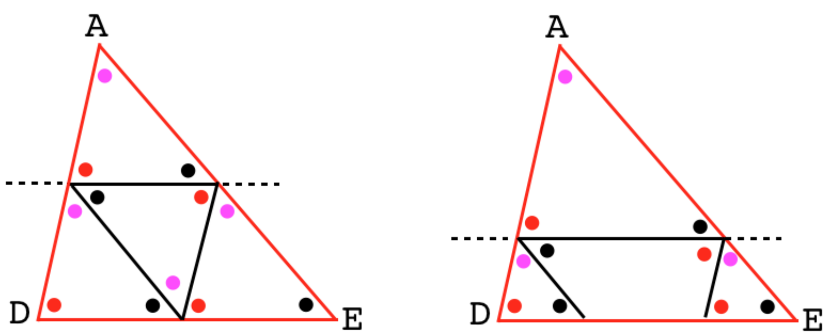
$$\frac{a}{a'} = \frac{c}{c'}$$

$$\frac{a+a'}{a'} = \frac{c+c'}{c'}$$

$$\frac{a+a'}{c+c'} = \frac{a'}{c'} = \frac{a}{c} = \frac{b}{d}$$

□

AAA similarity theorem

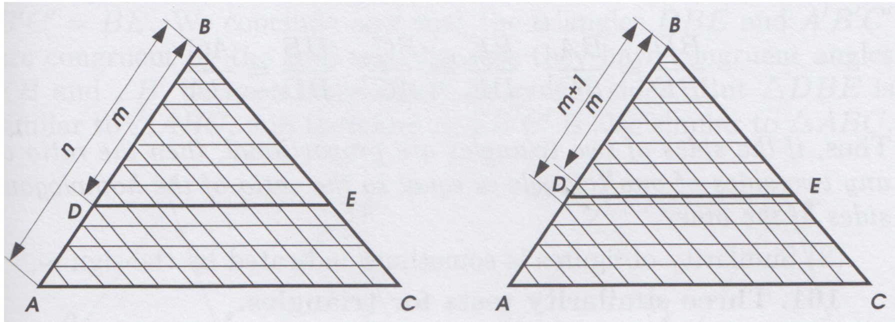


On the left is the easy case where $AB = BD$.

We will show that the sides are in proportion even when that proportion is not $1:2$, as on the right.

Note: we proved this theorem for right triangles already, based on an idea I found in Acheson's book, and combined it with an extension to all triangles.

If the horizontal bisector is parallel to the base, then the triangles are similar. We will have AAA. This is true regardless of which side of the large triangle we choose to be the base.



His notation is different than what we used above, drawing $\triangle BDE$ smaller than $\triangle BAC$. We follow Kiselev for this section.

There are two cases. The first is when the lengths of BA and BD are commensurable.

Two lengths are commensurable when there is some small length ℓ that we can define as one unit, such that for integers m and n, $BD = m\ell = m$ and $BA = n\ell = n$.

Divide the side as shown. Draw lines parallel to AC and also those parallel to BC.

Then BE and BC will be divided into congruent parts, numbering m and n for each, respectively. The same thing happens on the bottom. It is clear that

$$\frac{m}{n} = \frac{BD}{BA} = \frac{DE}{AC} = \frac{BE}{BC}$$

The second, harder, case is shown in the right panel above.

BD and BA are not commensurate and there is some small remainder when dividing the first into the second. Put another way, if $BA = n\ell$, then there are two integers m and $m + 1$ such that

$$m\ell < BD < (m + 1)\ell$$

But if ℓ is small, with n and m large,

$$\frac{m}{n} \approx \frac{BD}{BA}, \qquad \frac{m}{n} \approx \frac{DE}{AC}, \qquad \frac{m}{n} \approx \frac{BE}{BC}$$

Crucially, by choosing the unit length ℓ smaller and smaller, and thus n being larger and larger, we can make the remainder $BD - m\ell$ **as small as we like**.

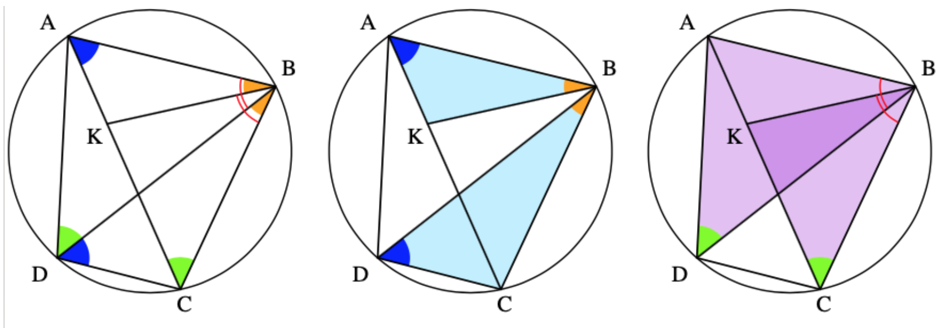
As n gets very large we approach equality:

$$\frac{m}{n} = \frac{BD}{BA} = \frac{DE}{AC} = \frac{BE}{BC}$$

for the second case as well.

In calculus we say that, in the limit, as $n \to \infty$, they become equal. If this seems strange, wait for the discussion of the limit concept, in calculus.

Ptolemy's by similar triangles



Above is a graphic from wikipedia that shows where we're going in the first proof. We will form two sets of similar triangles and use our knowledge about corresponding ratios.

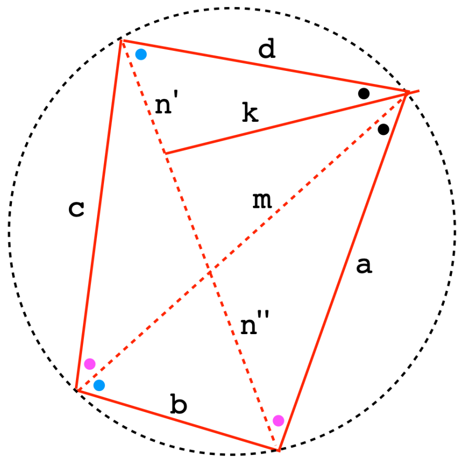
https://en.wikipedia.org/wiki/Ptolemy%27s_theorem

Proof.

It is often easier to adopt a modern notation, designating side lengths by single letters. In this problem, we have sides a, b, c, d and diagonals m and n.

The angles marked with magenta dots are equal as peripheral angles subtended by the same arc, and the same with the blue dots.

The key insight is to draw the line segment marked k, separating n into two parts, n' and n''. The line is drawn so that the angles marked with black dots are equal.



Other pairs of angles are equal by the inscribed angle theorem (blue and magenta).

The first pair of similar triangles has one vertex with the black dotted angles, and a second vertex with blue dots. Taking the opposite sides in the same order, we have the ratios:

$$\frac{n'}{b} = \frac{k}{a} = \frac{d}{m}$$

The second pair of similar triangles contain a vertex consisting of the black dotted angle *plus* the central angle between the two black dots. Refer to the wikipedia figure if this sounds confusing. There these angles are marked with red arcs.

This pair of triangles also has a second vertex with magenta dots. Taking the opposite sides in the same order we have

$$\frac{n''}{c} = \frac{k}{d} = \frac{a}{m}$$

The second trick is to pick the right relationships to manipulate. We know we don't want k in the answer, (and we do want all of $abcd$ plus mn' and n''), so choose from the first:

$$\frac{n'}{b} = \frac{d}{m} \quad \Rightarrow \quad bd = mn'$$

and from the second:

$$\frac{n''}{c} = \frac{a}{m} \quad \Rightarrow \quad ac = mn''$$

Simply add the two equations

$$ac + bd = m(n' + n'') = mn$$

□

difference of sines

The theorem of the broken chord can be used to derive the formula for the sine of the difference of two angles.

We first recall fundamental result about chords.

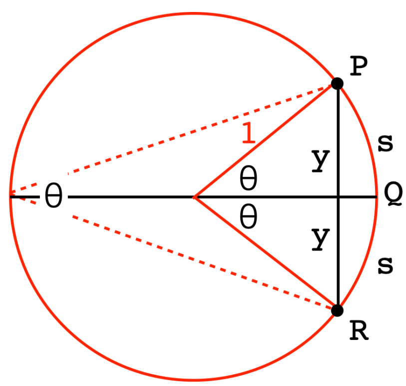
Lemma.

Consider any chord of a circle. Place another point on the circle in the larger arc to form a triangle.. All such triangles have the same peripheral angle, by the inscribed angle theorem.

Then choose the vertex for the peripheral angle such that it is bisected by a diameter of the circle and form the central angle corresponding to the same chord.

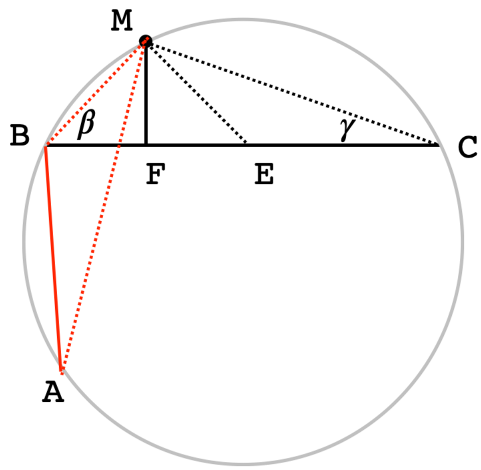
One-half of that central angle, equal to the peripheral angle, has a sine that is equal to one-half the chord.

In other words, the chord corresponding to any peripheral angle is twice the sine of that angle.



We now derive the formula for the sine of a difference of angles.

Proof.



Consider $\angle MBF$ at vertex B, let us call that angle β for convenience.

Then the lemma says that

$$MC = 2 \sin \beta$$

Similarly, let us label the angle at vertex C as γ. Then

$$BM = 2 \sin \gamma$$

Now, we can also use the right triangle to find

$$\cos \beta = \frac{BF}{BM}$$

so

$$\begin{aligned} BF &= BM \cos \beta \\ &= 2 \sin \gamma \cos \beta \end{aligned}$$

And

$$\cos \gamma = \frac{FC}{MC}$$

so

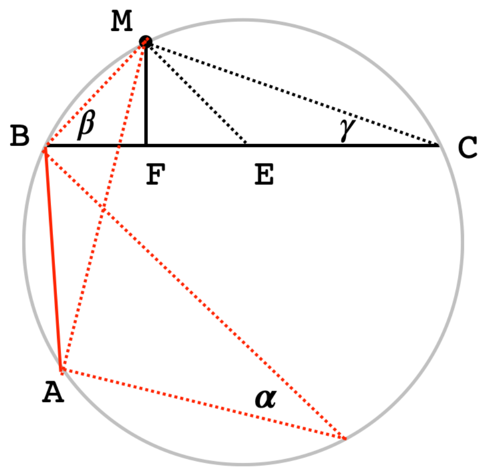
$$\begin{aligned} FC &= MC \cos \gamma \\ &= 2 \sin \beta \cos \gamma \end{aligned}$$

which starts to look familiar.

By the theorem of the broken chord:

$$FC - BF = AB$$

$$AB = 2\sin\beta\cos\gamma - 2\sin\gamma\cos\beta$$



What can we do with AB? We also know the sum of arcs, namely

$$\text{arc } AB + \text{arc } BM = \text{arc } MC$$

which means that if α is any peripheral angle subtended by arc AB:

$$\alpha + \gamma = \beta$$

$$\alpha = \beta - \gamma$$

and by the lemma

$$AB = 2\sin\alpha = 2\sin(\beta - \gamma)$$

So finally,

$$2\sin(\beta - \gamma) = 2\sin\beta\cos\gamma - 2\sin\gamma\cos\beta$$
$$\sin(\beta - \gamma) = \sin\beta\cos\gamma - \sin\gamma\cos\beta$$

□

Sine is an odd function ($f(x) = -f(-x)$), so $\sin -x = -\sin x$.

Cosine is even so $\cos x = \cos -x$. Thus,

$$\sin(\beta + \gamma) = \sin\beta\cos\gamma + \sin\gamma\cos\beta$$

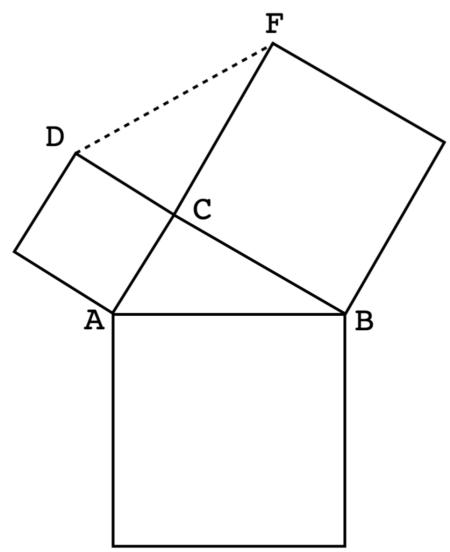
Chapter 56

Two final proofs

Condit proof

Here is a proof by a student named Ann Condit, who was a sophomore in high school at the time. It is one that Euclid would approve of.

We start by drawing the right $\triangle ABC$ and the squares on each side, as with Euclid I.47.

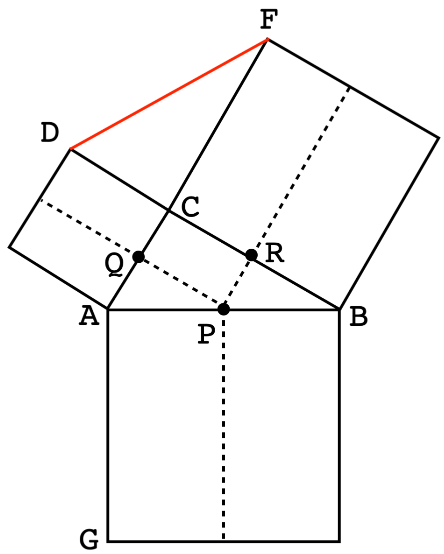


Next, join two nearby points on the two smaller squares to form DF as shown.

(1) $\triangle CDF$ is congruent to the original triangle ABC, by SAS.

We number our conclusions to keep track of everything, because the proof is a bit longer than many.

Next, find the midpoints of each side of $\triangle ABC$ at P, Q and R.



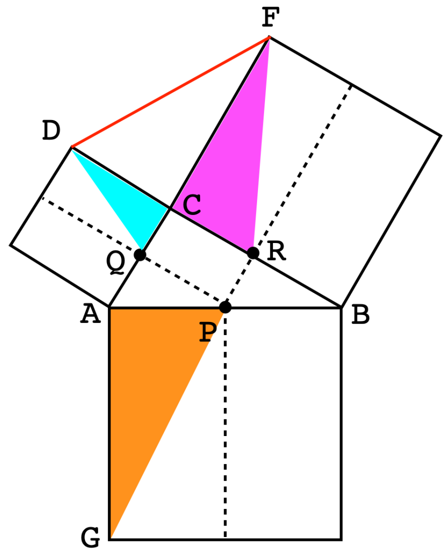
The line segment joining the midpoints of opposite sides forms a smaller triangle similar to the larger one. $\triangle BPR$ and $\triangle APQ$ are both similar to $\triangle ABC$ and congruent to each other. Therefore

(2) PQ and PR form right angles with the sides AC and CB, respectively.

(3) Thus, $CRPQ$ is a rectangle (four right angles), so opposing sides are equal.

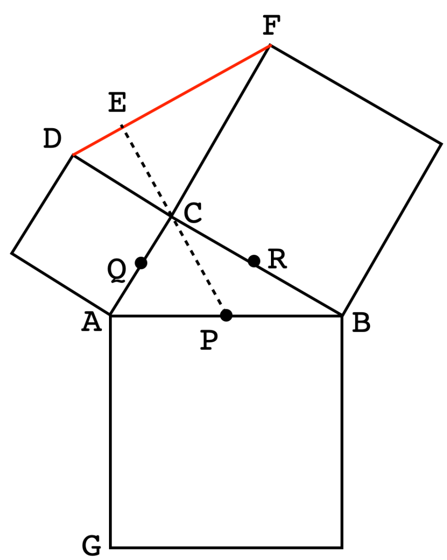
We will erase the dotted lines to reduce clutter, but remember the essential property that P, Q and R are midpoints; as well as that QP and CR are parallel sides of a rectangle and equal in length.

We will show that the area of $\triangle APG$ is the sum of the area of $\triangle DQC$ plus that of $\triangle FRC$.



Since each triangle is one-fourth the area of its respective square, this proves that the sum the two smaller squares is equal in area to the largest square. This is the theorem of Pythagoras.

We need one more line segment.



Draw PC and extend it to the opposite side DF ending at E.

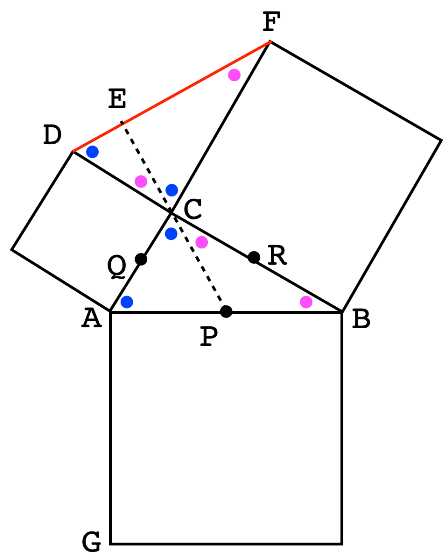
main idea of the proof

We claim that PC extended to CE, is perpendicular to DF.

Recall that when the side of the hypotenuse is bisected in a right triangle, as we did by drawing CP, the two resulting smaller triangles are both isosceles.

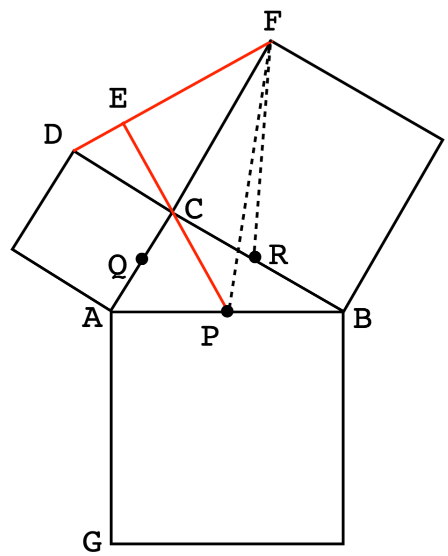
(4) Because of this, $\angle ACP$ is equal to the angle at the vertex A ($\angle CAB$) of the original $\triangle ABC$. $\angle ACP$ is also equal to $\angle EDC$, by congruent triangles.

(5) $\angle PCB$ is equal to the angle at vertex B ($\angle CBA$) as well as to $\angle EFC$, for the same reason.



(6) Therefore $\triangle CDE$ is similar to $\triangle CEF$, which means that the angles at E are identical and so are right angles.

This is the key step of the proof. Now we just look at the areas of different triangles.



First, the area of $\triangle FRC$ (one-quarter of the square of BC) is equal to the area of $\triangle FCP$, because they share the base CF, and QP is parallel to CR and equal in length to it (opposite sides of a rectangle).

But the area of $\triangle FCP$ is equal to $PC \cdot EF$, because EF is the altitude of $\triangle FCP$ (since $\angle FEP$ is a right angle).

By symmetry, the one-quarter of the small square's area is equal to that of $PC \cdot DE$, so together they equal

$$PC \cdot DE + PC \cdot EF = PC \cdot DF$$

But we have that $AP = CP$ and $AG = AB = DF$, so they have the same base and the same height and therefore the same area.

Thus, the two triangles together have the same area as the triangle with base AG and height AP.

□

yet another proof

Recently (April, 2023) I came across an article by Keith McNulty, referenced on Hacker News. It's an analysis of a very recent proof of the Pythagorean theorem by two teenagers from New Orleans named Calcea Johnson and Ne'Kiya Jackson.

https://keith-mcnulty.medium.com/heres-how-two-new-orleans-teenagers-found-a-new-proof-of-the-pythagorean-theorem-b4f6e7e9ea2d

I don't have a reference to an original publication, the article simply refers to a presentation at a meeting.

McNulty claims that the proof involves trigonometry and that this is novel — "[it] might make a few established mathematicians eat their words...because their proof uses trigonometry."

Trigonometry contains two sorts of theorems, those which explicitly depend on the Pythagorean theorem, and those which do not, including some which use similar triangles. (We hedge the statement because of the complication that similar triangles can be used to prove the theorem).

The law of cosines is an example of a theorem that depends on Pythagoras, while the law of sines does not. Naturally, one could not use anything which depends on the Pythagorean theorem to prove the theorem, that would be circular logic.

What this proof mainly does is to use similar triangles, plus the law of sines. Therefore I think McNulty's claim is wrong.

It's still worth taking a look. The really novel part is the use of infinite series. We're going to use a geometric series:

$$S = 1 + r + r^2 + \cdots + r^n + r^{n+1} + \ldots$$

The sum of the first n terms of such a series is

$$S_n = 1 + r + r^2 + \cdots + r^n$$

Provided $|r| < 1$, as n gets large, the terms beyond r^n become negligible, so S_n gets closer and closer to the true sum, S. Now,

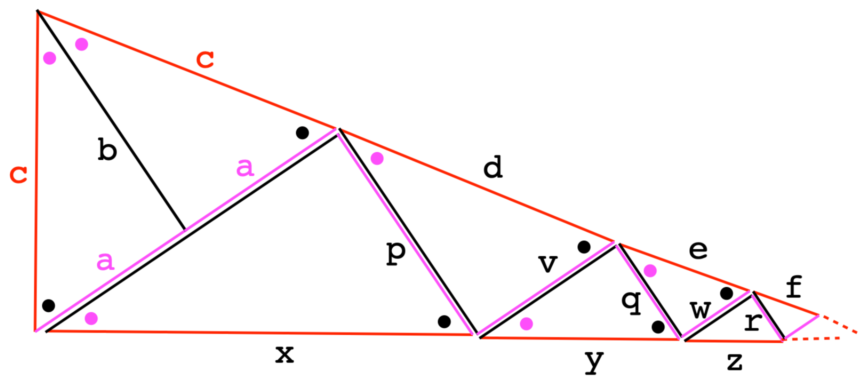
$$(1 - r) \cdot S_n = 1 - r^{n+1}$$

a classic "telescoping" series, and if r^{n+1} and subsequent terms are small enough then

$$S \approx S_n \approx \frac{1}{1 - r}$$

construction

The idea of the new proof is that a right triangle with angle θ (magenta dot) can be combined with scaled versions of itself to give a different right triangle with angle 2θ, as shown in the figure.



All the smaller triangles are similar. We will work out the lengths of these pieces in terms of the original triangle's sides.

Let the secant ($\sec\theta$) be $S = c/b$ and the tangent $T = a/b$. Then the small triangles have sides:

$$\begin{aligned} x &= 2aS, & p &= 2aT \\ d &= pS = 2aST, & v &= pT = 2aT^2 \\ y &= vS = 2aST^2, & q &= vT = 2aT^3 \\ e &= qS = 2aST^3, & w &= qT = 2aT^4 \\ z &= wS = 2aST^4, & r &= wT = 2aT^5 \end{aligned}$$

The last value to check is

$$f = rS = 2aST^5$$

Let us call the top side of the whole large right triangle C and the bottom side A. We have that

$$A = x + y + z + \dots$$

$$C = c + d + e + f \dots$$

Looking at the above results, A and C (minus the first term, c) are geometric series with the same ratio, T^2.

$$x = 2aS, \quad y = 2aST^2, \quad z = 2aST^4$$
$$d = 2aST, \quad e = 2aST^3, \quad f = 2aST^5$$

As we said, the sum of a geometric series with ratio r and initial term 1 is $1/(1-r)$, provided that $|r| < 1$. Here $T = a/b$ If $b > a$ then $T > 1$ and then $1/T^2 < 1$, so the series converges. If $a > b$ then just swap a and b. We deal with the case $a = b$ at the end.

The sum for initial term k is

$$S = k \cdot \frac{1}{1-r}$$

This series has ratio T^2 so its sum is

$$A = 2aS \cdot \frac{1}{1-T^2}$$

Recalling that $S = c/b$ and $T = a/b$ so

$$aS = ac/b = cT$$

and then

$$A = 2cT \cdot \frac{1}{1-T^2}$$

The other series is

$$d + e + f + \cdots = 2aST \cdot \frac{1}{1-T^2}$$

Using again the relation $aS = ac/b = cT$ we obtain

$$d + e + f + \cdots = \frac{2cT^2}{1-T^2}$$

The whole of the top side C is $c + d + e + f + \ldots$

$$C = c + \frac{2cT^2}{1-T^2}$$

$$= c\ (1 + \frac{2T^2}{1-T^2}) = c \cdot \frac{1+T^2}{1-T^2}$$

We can finally form the ratio A/C, canceling the factor of $1 - T^2$.

$$\frac{A}{C} = \frac{2cT}{c(1+T^2)}$$

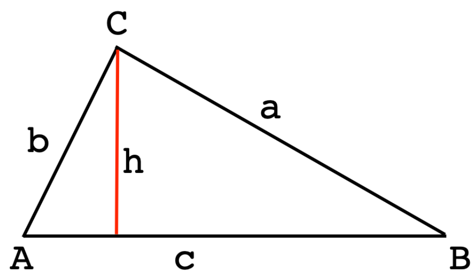
$$= \frac{2a/b}{1 + a^2/b^2}$$

$$= \frac{2ab}{a^2 + b^2}$$

The ratio A/C is also the sine of the double angle 2θ.

law of sines

In any triangle, drop the altitude from vertex C to the opposite side.



$$\sin A = \frac{h}{b}$$

$$\sin B = \frac{h}{a}$$

So

$$b \sin A = a \sin B$$

The law of sines follows:

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

Apply the law of sines to the isosceles (double) triangle with angle 2θ and sides c and $2a$:

$$\frac{\sin 2\theta}{2a} = \frac{\sin(90-\theta)}{c} = \frac{\cos\theta}{c} = \frac{b}{c^2}$$
$$\sin 2\theta = \frac{2ab}{c^2}$$

Substituting the ratio of A/C for the left-hand side:

$$\frac{2ab}{a^2+b^2} = \frac{2ab}{c^2}$$

The Pythagorean theorem immediately follows.

Finally, we must handle the case where $a = b$. Then the small triangles are all isosceles as well. So $\theta = 45°$ and $2\theta = 90°$ and a square is formed with sides c.

The ratio of opposing sides $C/A = c/c = 1 = \sin 2\theta$.

The law of sines gives

$$\frac{\sin 2\theta}{2a} = \frac{\sin\theta}{c}$$

Since $\sin 2\theta = 1$ and $\sin\theta = a/c$

$$\frac{1}{2a} = \frac{a/c}{c}$$
$$c^2 = 2a^2$$
$$= a^2 + b^2$$

□

Chapter 57

Resources

I have given very few problems, and solved nearly all of those, but you will need to do a lot of problems to really learn the material.

Start with Simmons. He summarizes Algebra, Geometry, Analytic Geometry and Trigonometry in a little more than one hundred pages. The book can easily be found used, and I highly recommend it.

David Acheson's books are wonderful, including *The Wonder Book of Geometry*.

Another resource that I enthusiastically recommend is a website with a highly annotated version of Euclid's *Elements*.

https://mathcs.clarku.edu/~djoyce/elements/elements.html

You owe it to yourself to work your way through the first book, at least.

Here is an old textbook by Hopkins that I found online:

https://archive.org/details/inductiveplanege00hopkrich

As a unique feature it contains more than a dozen entrance examinations for major colleges from the years around 1900.

Two examples from Dartmouth:

DARTMOUTH — 1900.

1. Prove that the three perpendicular bisectors of the sides of any triangle meet in a common point.

2. Circumscribe a circle about a given triangle.

3. Construct a circle having its center in a given line, and passing through two given points.

4. Prove, algebraically, that the square of the side opposite the obtuse angle of an obtuse-angled triangle is equal to the sum of the squares of the other sides plus the product of one of these sides and the projection of the other side upon it.

5. If the side of an equilateral triangle is a, find its area.

6. Prove that the area of a circle is equal to half the product of its radius and circumference.

7. The radius of a circle is 2; find the area of the regular inscribed dodecagon.

8. The radius of a circle is R; what is the radius of a concentric circle which divides it into two equivalent parts?

DARTMOUTH — 1901.

1. Prove that the diagonals of a parallelogram bisect each other. When are they equal?

2. How many degrees in one angle of a regular decagon? of a regular dodecagon? What is the largest number of degrees possible in one angle of a regular polygon?

3. One angle between two chords intersecting within a circle is 50°; its intercepted arc is 10°. How many degrees in the arc intercepted by its vertical angle?

4. Prove that the bisector of an angle of a triangle divides the opposite side into segments proportional to the sides of the angle.

5. The diagonals of a rhombus are 9 and 12. Find its perimeter and area.

6. In a circle whose radius is R, show that the area of the inscribed square is $2 R^2$, and of the inscribed regular dodecagon is $3 R^2$.

7. Draw the tangents to a circle from a point outside the circle, and prove that they are equal.

Chapter 58

List of theorems

proofs of the Pythagorean theorem

- Pythagorean Thm: similar triangles
- Pythagorean Thm: Euclid I.47
- Pythagorean Thm: scaled triangles
- Pythagorean Thm: sum of angles
- Pythagorean Thm: area
- Pythagorean Thm: Garfield
- Pythagorean Thm: Pappus
- Pythagorean Thm: crossed chords
- Pythagorean Thm: incircle
- Pythagorean Thm: angle bisector
- Pythagorean Thm: Star of David, Anderson
- Pythagorean Thm: Ptolemy
- Pythagorean Thm: Condit
- Pythagorean Thm: Tuan
- Pythagorean Thm: Quorra corollary

- **Pythagorean Thm: converse**

proofs from Euclid

- **construct an equilateral triangle** (Euclid I.1)
- **side angle side (SAS)** (Euclid I.4)
- **isosceles triangle theorem** (Euclid I.5: equal sides → angles)
- **isosceles triangle theorem converse** (Euclid I.6: equal angles → sides)
- **Preliminary to SSS** (Euclid I.7)
- **angle bisection** (Euclid I.9)
- **perpendicular bisector** (Euclid I.10)
- **perpendicular through a point** (Euclid I.11)
- **perpendicular to a point** (Euclid I.12)
- **external angle inequality** (Euclid I.16)
- **longer side → larger angle** (Euclid I.18)
- **larger angle → longer side** (Euclid I.19)
- **triangle inequality** (Euclid I.20)
- **hinge theorem** (Euclid I.24)
- **ASA for congruence** (Euclid I.26)
- **line parallel to another line** (Euclid I.31)
- **parallelogram area** (Euclid I.35)
- **parallelogram complements equal** (Euclid I.43)
- **Pythagorean theorem** (Euclid I.47)
- $(x+y)(x-y) = x^2 - y^2$ (Euclid II.5)
- $(2x+y)y + x^2 = (x+y)^2$ (Euclid II.6)
- **Law of cosines, obtuse case** (Euclid II.12)
- **Law of cosines, acute case** (Euclid II.13)

- **square of a rectangle** (Euclid II.14)
- **find circle center** (Euclid III.1)
- **find circle center** (Euclid III.12)
- **inscribed angle theorem** (Euclid III.20)
- **same arc → equal angles** (Euclid III.21)
- **quadrilateral supplementary theorem** (Euclid III.22)
- **equal angles, on same circle** (Euclid III.26 converse)
- **Thales' theorem** (Euclid III.31)
- **tangent-chord theorem** (Euclid III.32)
- **crossed chord theorem** (Euclid III.35)
- **tangent-secant theorem** (Euclid III.36)
- **similarity: AAA → equal ratios** (Euclid VI.2)
- **equal divisions of a line segment** (Euclid VI.9)

proofs of other theorems

- **alternate interior angles**
- **angle bisector theorem** (right triangle)
- **angle bisector theorem** (general)
- **ASA for congruence**
- **area ratio theorem**
- **bisector equidistant from sides**
- **equidistant from sides → bisector**
- **Centroid is one-third of cevian**
- **Ceva's theorem**
- **Ceva's theorem** (Menelaus)
- **Ceva's theorem** (by area)

- **Ceva's theorem** (alternate proof)
- **Ceva's theorem by parallel lines**
- **crossed chord theorem** (product of lengths)
- **complementary angles**
- **cyclic quadrilateral** (opposing angles are supplementary)
- **cyclic quadrilateral** (converse)
- **diameter divides circle in half**
- **diameters form a rectangle**
- **equal arcs $\iff$ equal chords**
- **equal angles $\iff$ equal arcs**
- **eyeball theorem**
- **excircle theorems**
- **extended altitude theorem**
- **external angle theorem**
- **extraordinary property of the circle**
- **Heron's formula by excircles**
- **Heron's formula, Heron's proof**
- **hypotenuse-leg in a right triangle (HL)**
- **hypotenuse longest side in a triangle**
- **incenter** (incenter: angle bisectors meet at a point)
- **inscribed angle theorem** (on a circle is one-half central angle)
- **inscribed angles converse**
- **isosceles triangle theorem** (sides $\rightarrow$ angles)
- **isosceles triangle theorem** (angles $\rightarrow$ sides)
- **Law of cosines**
- **Law of cosines, algebraic proof**

- **Law of cosines, Ptolemy**
- **Menelaus' theorem**
- **midline theorem**
- **midpoint theorem** (right triangle)
- **orthocenter exists** (Newton)
- **Pappus parallelogram theorem**
- **parallelogram theorems**
- **special parallelogram theorem** (one pair of sides)
- **circumcenter** (perpendicular bisectors of a chord is diameter)
- **circumcenter** (perpendicular bisectors meet at a point)
- **Ptolemy's theorem, by cutting**
- **Ptolemy's theorem, similar triangles**
- **Ptolemy's theorem, switch sides**
- **Ptolemy's theorem, by inversion**
- **rectangle in a circle**
- **right angle is largest in a triangle**
- **tangent-secant theorem**
- **shortest distance from a point to a line**
- **supplementary angles**
- **similar right triangles**
- **midline theorem** (similar triangles)
- **similar triangles** (ratio of sides)
- **similar triangles** (right triangle composition)
- **AAA similarity theorem** (Kiselev)
- **SAS for congruence**
- **SAS inequality, hinge theorem**

- **SAS to establish similarity**
- **SSS implies SAS**
- **supplementary angle theorem**
- **supplementary angles equal to two right angles**
- **tangent theorem** (right angle → touches one point)
- **tangent-chord theorem**
- **tangent construction**
- **tangent theorem** (touches one point → right angle)
- **tangent-secant theorem**
- **Thales' circle theorem** (right angle in a semi-circle)
- **Thales circle theorem: converse**
- **triangular area**
- **triangle inequality** (triangle inequality)
- **sum of angles**
- **triangles are similar if two angles equal**
- **Varignon's theorem**
- **vertical angle theorem**

Chapter 59

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