# Plane Geometry: Practice with Proofs

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# Introduction

The image below is a detail from a painting by Raphael entitled "School of Athens", which was used as the front cover of a wonderful book annotating the Heath translation of Euclid's *Elements*.

It took a genius to figure it out the first time, but it is within anyone's grasp to appreciate what they found. I imagine myself looking over Archimedes' shoulder as he explains the steps of a proof to me.

Most scientists I've met loved geometry in school, as I did. They like how visual it is, and they like clever simple proofs. Geometry should be fun!

A central feature of this book is the relentless use of proof. I emphasize the key insight for each, and have tried to make the proofs simple and as easy to follow as possible. You will notice that we frequently provide multiple proofs (using different methods) for the same theorem.

I express my sincere thanks to the authors of my favorite books, which are listed in the references and mentioned at various places in the text. Everything in here was appropriated from them in one way or another, and styled to my taste.

I offer my profound thanks also to Eugene Colosimo, S.J. He was among the best of a great group of teachers.

The books are provided as pdf files in that repository linked below. Most of the rest is the source files. There are several other books there as well, if you go up one level in Github. The books are, in order:

https://github.com/telliott99/geometry0

https://github.com/telliott99/geometry1

https://github.com/telliott99/geometry2

Note: the sites referred to by some urls in the text have disappeared from the web and the count of missing pages grows over time. I have left those URLs in, as a kind of protest, and because it is impractical to police them, but also because they may still be useful in connection with the wayback machine.

https://web.archive.org

# Part II

# Practice with proofs

# Chapter 2

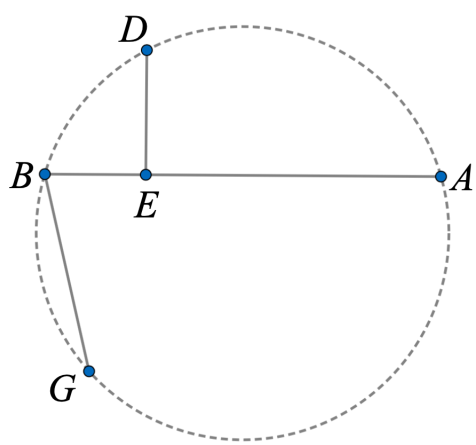
# Broken Chord

The theorem of the "broken chord" is ascribed to Archimedes, although his original work — the *Book of Circles* — has been lost.

It was analyzed in proofs collected by the Arabic mathematician Al Biruni in his *Book on the Derivation of Chords in a Circle*.

The theorem was not simply of academic interest, but related to the construction of tables of chords in the *Almagest* by Pappus (covered elsewhere).

Here is the general setup:




Let $A$ and $G$ be any two points on a circle, and let $D$ be equidistant from both, so that arc $AD$ is equal to arc $GD$. Let $B$ be another point on the circle, lying between

$G$ and $D$.

Drop the perpendicular from $D$ to $E$ on $BC$.

The claim of the theorem is that $GB + BE = AE$.

(I wondered about the choice of $G$ as one of the letters but then remembered that the Greek alphabet proceeds: $\alpha$, $\beta$, $\gamma \dots$ Later we will see $Z$ and $H$, and $T$ — zeta, eta, theta).

In this chapter we will look at a number of proofs including several ascribed to Archimedes. This topic is a great one for *elementary* geometry because the proofs are fairly easy and there are literally dozens of them. There is also the sense of connection with the ancients in working on a topic which got Archimedes' attention.

My original source for this problem was

https://www.uni-miskolc.hu/~matsefi/HMTM_2020/papers/HMTM_2020_Drakaki_Broken_chord.pdf

There, it is said that the book contains 22 proofs of the theorem, including 3 different ones due to Archimedes.

Later I found a link to Al Biruni's book, in Arabic:

https://tile.loc.gov/storage-services/service/gdc/gdcwdl/wd/l_/07/46/9/wdl_07469/wdl_07469.pdf

I cannot read Arabic, but the diagrams are reproduced, and using them I worked out a couple more proofs.

Even more recently, I came across a translation of this book, written in German by a historian named Heinrich Suter about 1910:

http://www.jphogendijk.nl/biruni/Suter-Chords.pdf

I was able to translate a part of that book into English. I don't read German either, but the scanned text of Suter is copyable, and Google Translate does a decent job. What makes it hard is that the OCR is so flaky — I'm not complaining — I think it's amazing that it works at all. But it took quite a bit of editing.

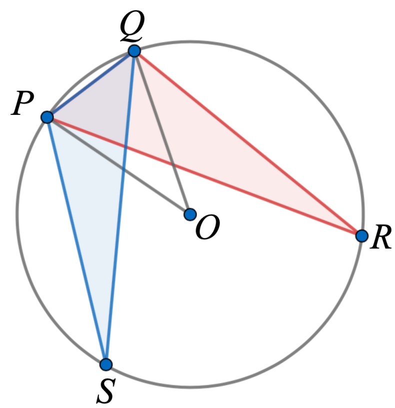
https://www.dropbox.com/scl/fi/zzuibok8apr6i2trwfb4r/

I count 23 examples (labeled $a$ through $x$, with no $j$), and four of those are Suter's own proofs, which leaves a total of 19 for Al Biruni. (Although some of these contain multiple approaches with the same diagram).

## inscribed angles

Before starting, let's recall the important corollary (Euclid III.21) of the inscribed angle theorem (Euclid III.20).

By definition, the central angle sweeping out a given arc is equal in measure to the length of the arc. Any peripheral angle subtended by the same arc is one-half that central angle. The result is that angles which lie on the same arc or are subtended by the same chord in the same circle, are equal.




$\angle R = \angle S$ (i.e., $\angle PRQ = \angle PSQ$)

We will also need the theorem that, in a given circle, equal arcs correspond to equal chords, and vice-versa, from **here**.

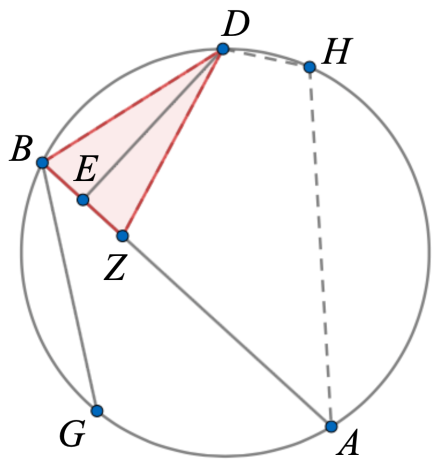
And we will also use several times the theorem that any point on the perpendicular bisector of a segment forms an isosceles triangle when connected with the endpoints of the segment.

## preliminary work

We are given arc $AD =$ arc $GD$.

Given $B$ lies on arc $GD$ (the minor arc, since $BD < GD <$ one half of the circle).

$DE \perp AB$.




Many constructions draw the isosceles triangle $DBZ$ as shown above. Note that $\angle DBZ$ cuts the arc $AD$ so it is equal to any other angle that might be drawn cutting $AD$ or $GD$.

$\angle DBZ = \angle DZB$, corresponding to the combined arcs $AD$ and $GD$, which leaves arc $AG$. It follows easily that $\angle BDZ = \angle ABG$.

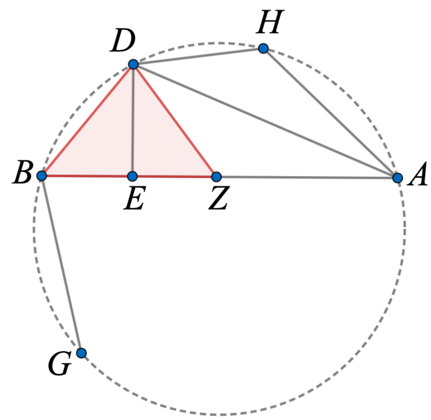
Several angles are equal to arc $AG + GD$. These include $\angle DZA$ (external to $\angle DBZ$ plus $\angle BDZ$), $\angle GBD$ (supplementary to arc $GD$) and $\angle AHD$ (supplementary to arc $AD$).

## first proof: Suter (a)

The first and second proofs are attributed to Archimedes.

*Proof.*

Draw the isosceles $\triangle DBZ$, with $\angle DBZ = \angle DZB$.




Now place $H$ such that $BD = HD$.

Subtracting equals, it follows that $GB = AH$. (Equally, we could find the latter first).

$\angle A$ is bisected since the two parts are subtended by $BD = HD$.

From the preliminaries, we have $\angle AZD = \angle AHD$

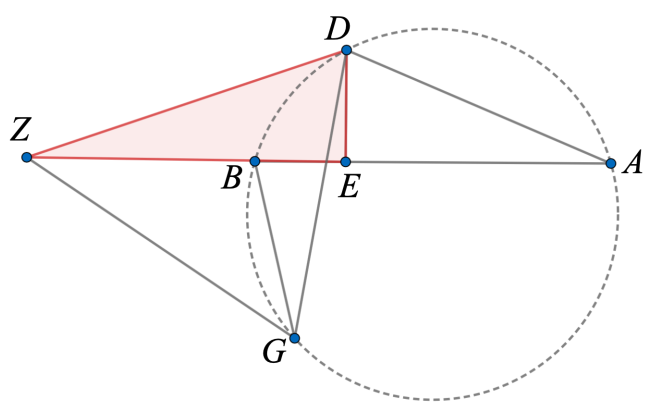
Thus $\triangle ZDA \cong \triangle HDA$ by ASA.

It follows that $AZ = AH = GB$.

Adding equals: $GB + BE = AZ + ZE = AE$.

$\square$

## second proof: Suter (c)



*Proof.*

Given arc $AD =$ arc $GD$, $B$ on arc $GD$, and $DE \perp AB$.

So $GD = AD$.

Extend $AEB$ to $Z$ such that $AE = ZE$.

$\triangle DZE \cong \triangle DAE$ by SAS.

So $\triangle DZA$ is isosceles and $DZ = AD = GD$.

Hence $\triangle DZG$ is isosceles with base angles equal.

Since $\triangle DZA$ is isosceles, $\angle DZE = \angle DAE$.

Since they correspond to the same arc, $\angle DAE = \angle DGB$.

Hence $\angle DZE = \angle DGB$.

Subtracting equals we have $\angle BZG = \angle BGZ$.

Thus $\triangle BZG$ is isosceles, and $ZB = BG$.

Adding equals: $BG + BE = ZB + BE = AE$.

$\square$

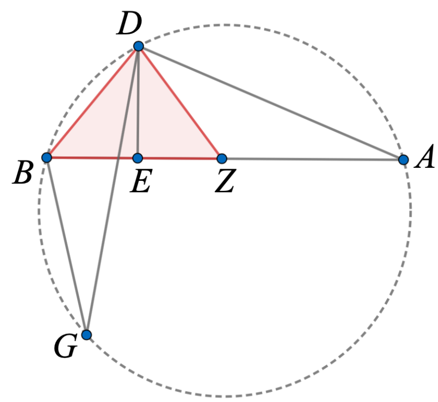
## third proof

This elegant proof is ascribed to Gregg Patruno, a student at Stuyvesant High School in New York (1980). I found it in

https://www.researchgate.net/publication/341579803_FROM_THE_THEOREM_OF_THE_BROKEN_CHORD_TO_THE_BEGINNING_OF_TRIGONOMETRY

[Drakaki attributes it to Patruno].

The diagram can be found in Al Biruni's book (p. 15), but, according to Suter's translation, that proof is subtly different. Patruno starts with $AZ = GB$, whereas Suter (m) starts with $DB = DZ$. We discuss this point at the end of the chapter.




*Proof.*

Find $Z$ such that $AZ = GB$.

$\angle G = \angle A$, as inscribed angles on the same arc.

We are given $GD = AD$.

Hence $\triangle BDG \cong \triangle ZDA$ by SAS.

So $BD = ZD$.

Hence $\triangle DBZ$ is isosceles.

It follows that $\triangle BDE \cong \triangle ZDE$ by HL.

So $BE = ZE$.

Adding equals:

$GB + BE = ZE + GB = ZE + AZ = AE$.

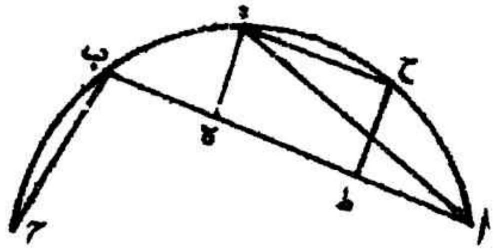
$\square$

## fourth proof: Suter (i)

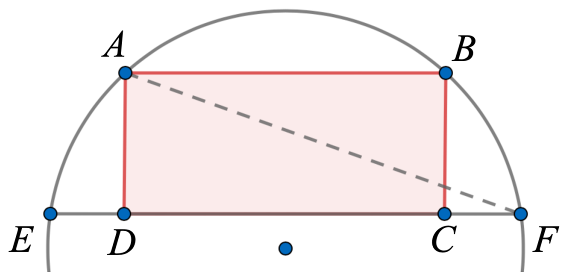
This one is attributed to El-Sidjzi (972), but may actually have been known to Apollonius.

https://www.researchgate.net/publication/341579803_FROM_THE_THEOREM_OF_THE_BROKEN_CHORD_TO_THE_BEGINNING_OF_TRIGONOMETRY

This figure from Al Biruni (below) suggests a neat line of attack.



Recall from our work on parallel chords in a circle, a result about the extensions from a rectangle in a circle. In the figure below, $DE = CF$.




*Proof.*

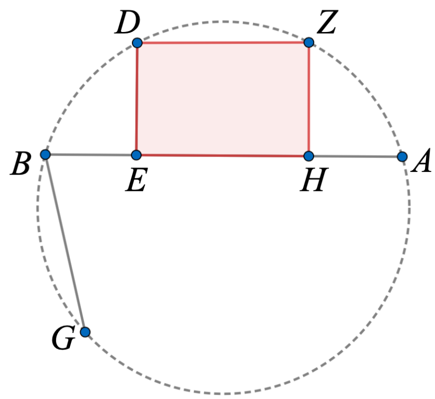
Let $AB$ and $EF$ be two parallel chords in a circle with unequal lengths. Draw the perpendiculars $AD$ and $BC$. Then $ABCD$ is a rectangle.

$\angle AFE = \angle BAF$ by alternate interior angles. So arc $AE =$ arc $BF$.

It follows easily that $\triangle ADE \cong \triangle BCF$ by HL.

□

So now




*Proof.*

As before, arc $GD =$ arc $AD$, and $DE \perp AEB$.

Erect the perpendicular to $DE$ at $D$ and find where it cuts the circle at $Z$. This always forms a polygon since $BD < AD$.

Erect the perpendicular $ZH$.

Since it has all right angles, $DZHE$ is a rectangle.

By our preliminary work, $\triangle DBE \cong \triangle ZAH$.

Thus arc $BD =$ arc $AZ$ and $BE = AH$.

Subtracting equals, arc $DZ =$ arc $GB$.

It follows that $GB = DZ = EH$.

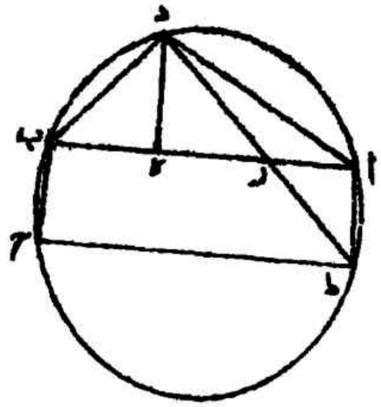
Adding equals:

$GB + BE = EH + AH = AE$.

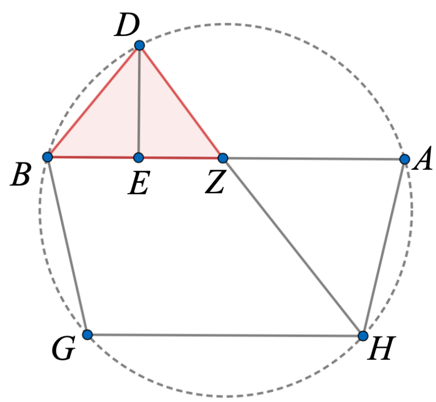
□

## fifth proof

Here is another diagram from Al Biruni's book in Arabic.



Re-rendered:



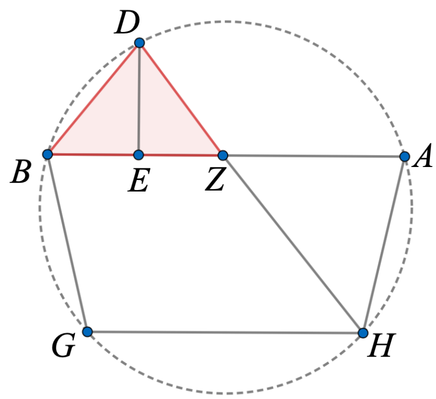

$\triangle DBZ$ is isosceles, as usual, and $DZ$ is extended to meet the circle at $H$. To make the discussion simpler, let us refer to the measure of $\angle DBZ$ as $\alpha$ and that of $\angle BDZ$ as $\beta$.

There are four other angles all equal to $\alpha$, $\angle DZB$ (by construction), $\angle AZH$ (vertical angles), $\angle AHZ$ (cuts the same arc $AD$), and $\angle GHD$ (cuts arc $GD =$ arc $AD$).

We also have the third angle in $\triangle DBZ$, namely $\angle BDZ = \beta$ which cuts arc $BGH$. This angle is equal to $\angle BAH$ (cuts the same arc).

It follows that the sum of all the angles at $A$ and $H$ is equal to 180, from which we obtain $AB \parallel GH$.

Parallel chords in a circle cut equal arcs and chords.



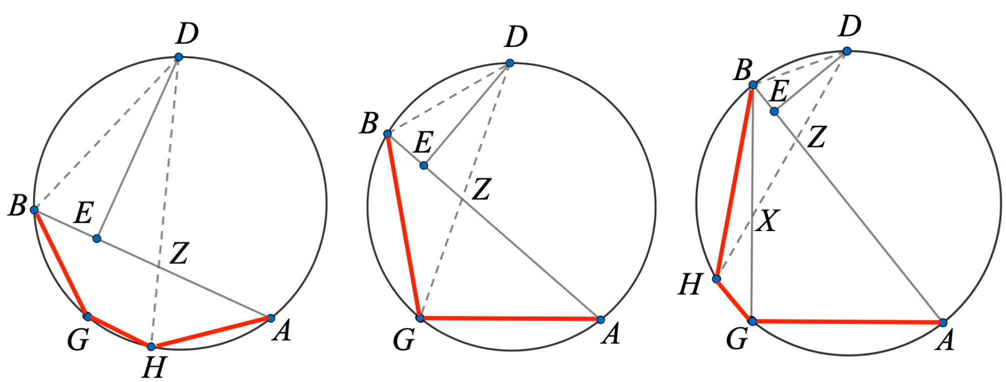

Thus $GB = AH$. And since $\triangle AZH$ is isosceles, $GB = AH = AZ$.

Adding equals:

$GB + BE = AZ + ZE = AE$.

□

However, this proof is incomplete, since for a given arrangement of $G$, $A$ and $D$, depending on the placement of $B$, one can draw $H$ in three different ways — coincident with $G$, or lying on either side of it.




The left panel is our original diagram. In the right panel, it appears that $GH \parallel AB$. Can the proof be rescued?

In a word, yes. There are various approaches. Here is a sketch of a simple proof.

*Proof.* (Sketch).

Draw the tangent at $D$ and connect $AD$ and $GD$.

Since the two other angles formed by the tangent cut the arcs $GD$ and $AD$, they are equal to $\angle DBZ = \alpha$.

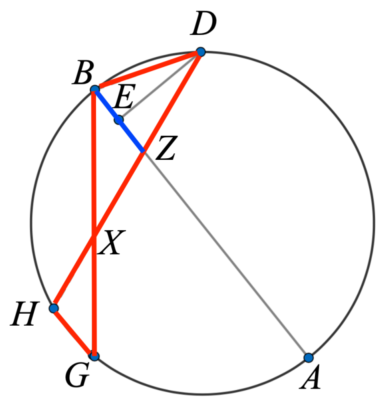
Then, $\angle ADG$ is equal to $\angle BDH$,

We obtain $AG = BH$ as arcs of equal angles, and thus $GH \parallel AB$ and then proceed as in the first case.

□

We might also proceed by showing that the obtuse angles at $X$, $Z$ and $B$ are all equal.

Since the acute angles at $X$ are equal, it follows that $\triangle BDX \sim \triangle BZX \sim \triangle GHX$.

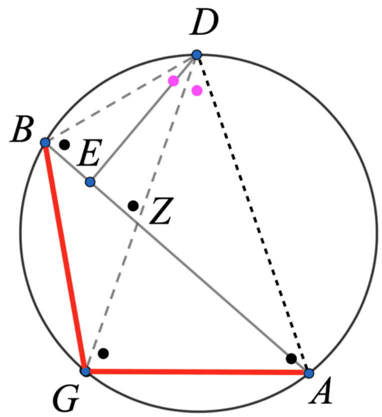



We still have to show that $AH = AZ$, but this follows since $\angle AHD$ cuts arc $AD$ so $\triangle AHZ$ is isosceles.

□

The **third case** is where the extension of $DZ$ terminates exactly on point $G$. $H$ coincides with $G$.

A simple proof is to complete $\triangle DGA$.




The base angles cut the arcs $AD$ and $GD$, so the triangle is similar to $\triangle BDZ$. In particular, the base angles are equal to $\alpha$, so they cut equal arcs. Thus $AG = BG$.

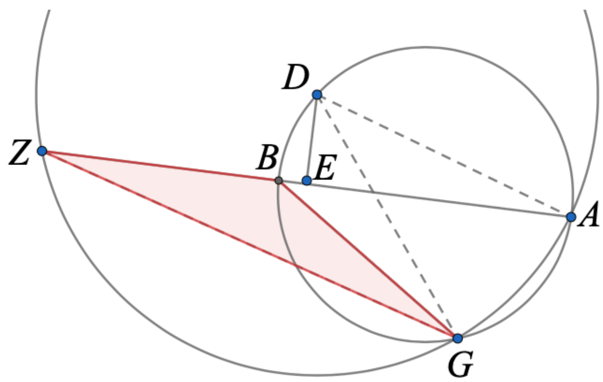
Since $\triangle AZG$ is also isosceles, we have $AG = AZ$ so $AZ = BG$, and the result follows easily.

□

## sixth proof: Suter(f)

*Proof.*

Draw the circle on center $D$ with radius $AD$. Extend $AB$ to $Z$. Draw $AD$ and $GD$.




$AZ$ is a chord of the circle on $D$. The perpendicular $DE$ goes through the center $D$, therefore it bisects $AZ$.

We have $AE = ZE = ZB + BE$.

Three angles intercept an arc between points $A$ and $G$, $\angle BZG$ on the big circle, $\angle ABG$ on the small circle, and $\angle ADG$ on both.

Since $\angle ADG$ is a central angle in the big circle on $D$, it is twice $\angle AZG$. But $\angle ADG = \angle ABG$.

Hence $\angle ABG$ is external to $\triangle BZG$ and twice the measure of $\angle BZG$. It follows that $\angle BZG = \angle BGZ$.

Thus $\triangle AZG$ is isosceles with $GB = ZB$.

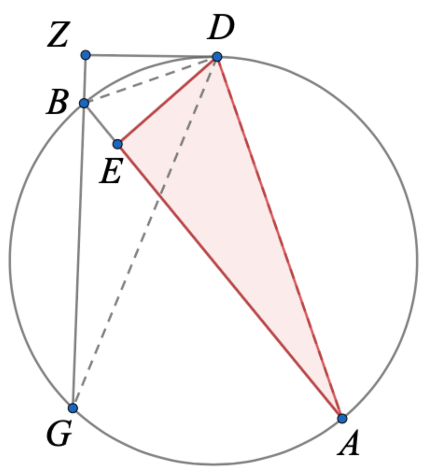
Substituting into what we had above: $AE = GB + BE$.

□

## seventh proof: Suter(p)

*Proof.*

Draw $GD$ and $AD$. Extend $GB$ to $Z$ and connect to $D$ such that $Z$ is a right angle.




$\angle ZGD = \angle BAD$ because they cut the same arc. We get the third angle by sum of angles and also $GD = AD$ so $\triangle GZD \cong \triangle AED$ by ASA.

$DZ = DE$ and then $\triangle DBZ \cong \triangle DBE$ by HL. So $BZ = BE$.

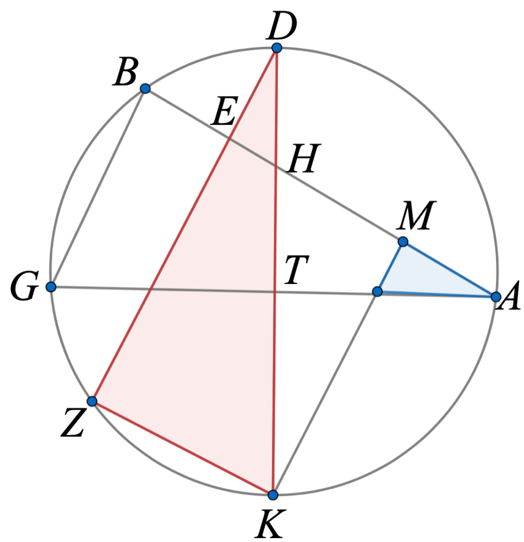
From the first congruence $GZ = AE$.

$GZ = GB + BZ = GB + BE$.

□

## eighth proof: Suter(n)

As before, $AD = GD$ and $DE \perp AB$.




*Proof.*

Extend $DE$ to cut the circle at $Z$.

Draw $AG$ and its perpendicular bisector $HTK$. It is a diameter of the circle and goes through $D$. Why?

Connect $ZK$ and then draw $MK \parallel DEZ$.

Since $DHTK$ is a diameter, $\angle Z$ is right. We're given that $\angle DEA$ is right.

Since $MK \parallel DEZ$, $EMKZ$ is a rectangle.

By similar right triangles, $\angle D = \angle HKM = \angle A$.

So $BG = ZK = EM$.

From rectangles in a circle we know that $BE = AM$.

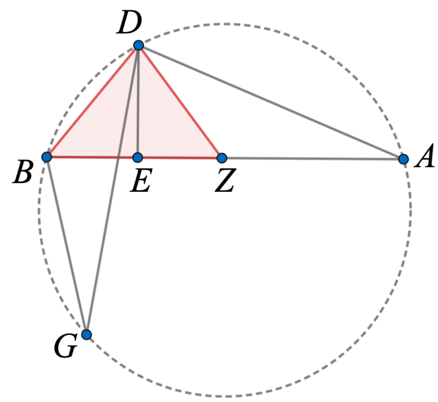
Add equals to equals to obtain the result.

□

That's enough proofs of this theorem. There are quite a few more. Each one follows naturally from an inspired diagram, and they are nearly all different.

## on the value of SSA

Something interesting happens with the third proof if you approach the premises slightly differently.




We draw $GD$ as before, but we forget to set $GB = AZ$ and instead put $BE = ZE$, as in some other proofs. Then what happens?

We have $SAS$ in the small right triangles so $\triangle DBE \cong \triangle DZE$, which means $DB = DZ$. We have $GD = AD$ as before and $\angle A = \angle G$.

We are tempted to compare $\triangle GBD$ with $\triangle AZD$. What we have is SSA, which — it's been drilled into our heads — is *not enough*, unless there is a right angle, in which case we call it hypotenuse-leg in a right triangle (HL).

But let's take a closer look. For SSA in the ambiguous case there are only two possibilities. We can see both of them in the figure!

Certainly $\angle GBD$ is obtuse. It is supplementary to an angle (not drawn) subtended by $GD$, which is certainly less than half of the circle (since it equals $AD$ and there is more than that in addition).

We also know something about $\angle AZD$. It is equal to the sum of two angles which add up to more than a right angle. It follows that $\angle AZD$ is obtuse.

Therefore, the angles are not only obtuse but clearly equal.

From Suter, here is what Archimedes says:

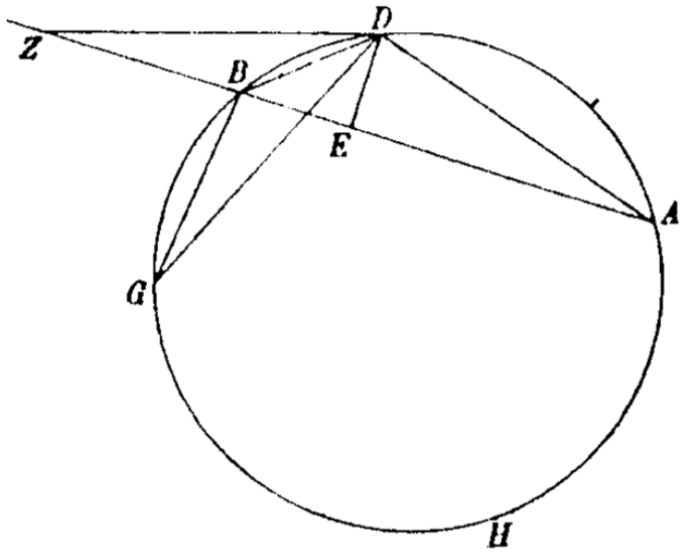


Fig. 1.

(re-phrasing): $GD = AD$ by the property of the median, $BD$ is shared, and $\angle G = \angle A$, so we have SSA.

Now, $Z$ has been drawn such that $ZE = AE$ which means $ZD = AD$, so we have actually another triangle sharing SSA with $\triangle GBD$.

Archimedes says that $\angle GBD$ corresponds to everything except arc $GD$, i.e. $DAHG$.

And $\angle DBZ$ is external to $\triangle ABD$, supplementary to $\angle DBA$ subtended by arc $AD$, so $\angle DBZ$ is corresponds to everything except arc $AD$, i.e. $AHGB$.

The missing parts are equal, so the angles are equal. Since we know two angles, we know three. Therefore, $\triangle ZBD \cong \triangle GBD$ by ASA.

## more

There is one additional proof of the broken chord theorem that I know about, beyond what is in Suter.

It is on the web attributed to someone named Bùi Quang Tuån. There is another proof from the same source of the **Pythagorean theorem** that I like even better.

`https://www.cut-the-knot.org/pythagoras/BrokenChordPythagoras.shtml`

Google also turns up a blog, but no biographical info.

https://artofproblemsolving.com/community/c1598

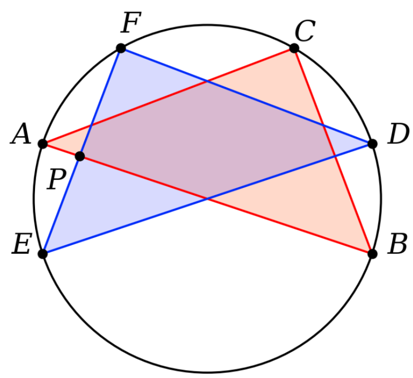
This proof is based on a rectangle, and I leave it to you to see how it relates to the fourth proof, above. I've written about it elsewhere.

Note: some additional material is here:

https://www.cut-the-knot.org/triangle/BrokenChordmpdlc.shtml

## ***Star of David*** proof of the Pythagorean theorem

https://www.cut-the-knot.org/pythagoras/PythStarOfDavid.shtml




Draw two congruent mirror-image right triangles in a circle, oriented so that $EF \perp AB$ (and $BC \perp DE$).

Note that $AB$ and $DE$ both pass through $O$, the center of the circle, because any right triangle inscribed in a circle has its hypotenuse as a diameter, by the converse of Thales' circle theorem.

$OA$ and $OD$ are perpendicular by construction and diagonals, so they are perpendicular bisectors. Thus

$$AE = AF = DC = DB$$

Arc $EC$ plus arc $CD$ is equal to $180°$.

So it is equal to the arc $EB$, which added to arc $DB$ gives 180.

Then $E$ is the median point between $B$ and $C$ and the perpendicular dropped from $E$ which meets $AB$ in a right angle, cuts $AB$ so

$$AP + AC = PB$$

$$AC = PB - AP$$

by the theorem of the broken chord. The two pieces of $AB$ are

$$AB = AP + PB$$

Putting this together, we have:

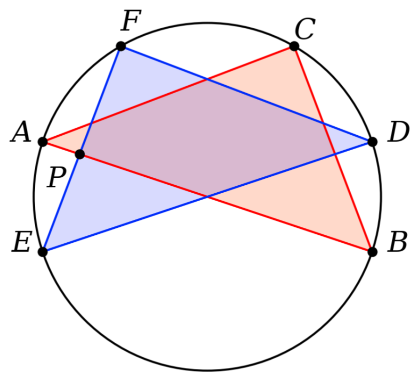
$$AB + AC = 2PB$$

and

$$AB - AC = 2AP$$

Hence

$$PB = \frac{1}{2}(AB + AC), \qquad AP = \frac{1}{2}(AB - AC)$$




By the theorem of crossed chords

$$\frac{AB + AC}{2} \cdot \frac{AB - AC}{2} = (\frac{EF}{2})^2$$

$$AB^2 - AC^2 = EF^2 = BC^2$$

The result follows.

□

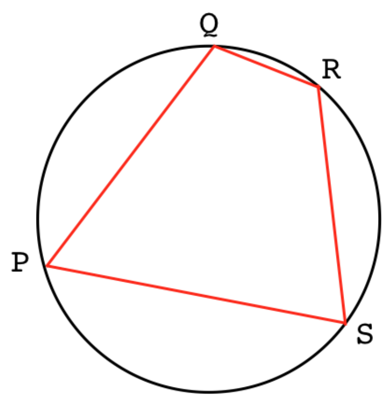
# Chapter 3

# Ptolemy

Ptolemy was a Greek astronomer and geographer who probably lived at Alexandria in the 2nd century AD (died c.168 AD). That is nearly 500 years after Euclid. (Ptolemy was a popular name for Egyptian pharaohs in earlier centuries).

Our Ptolemy is known for many works including his book the *Almagest*, and important to us here, for a theorem in plane geometry concerning cyclic quadrilaterals. These are 4-sided polygons with all four vertices on the same circle.

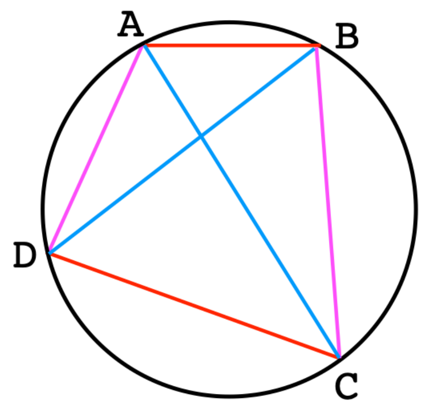
Recall that any triangle lies on a circle, so this is a restriction on the fourth vertex of the polygon.




Also recall the **quadrilateral supplementary theorem** (Euclid III.22):

- For *any* quadrilateral whose four vertices lie on a circle, the opposing angles are supplementary (they sum to 180°).

Now, draw the diagonals $AC$ and $BD$. Ptolemy's theorem says that if we take the products of the two pairs of opposing sides and add them, the result is equal to the product of the diagonals.

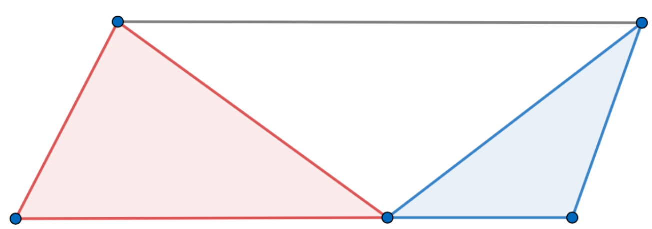



AB CD + BC AD = AC BD

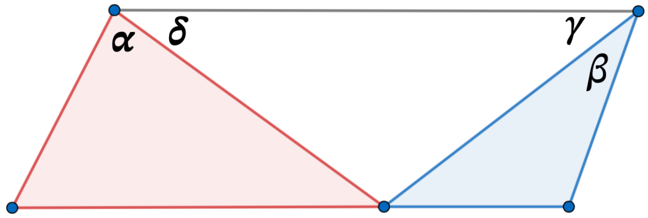
Notice that in the special case of a rectangle, the two diagonals are equal. Given the proof here, Ptolemy's result furnishes a trivial proof of the Pythagorean theorem (see later).

*Proof.*

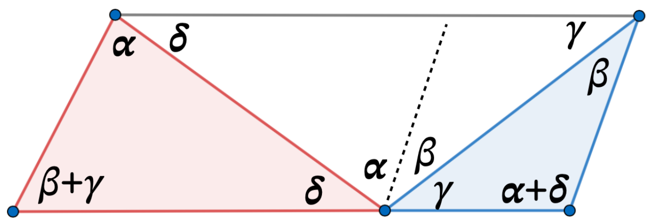
We're going to do a famous dissection of a cyclic quadrilateral as a proof of Ptolemy's theorem, but as a twist on the usual approach we'll do it in reverse, starting with a a parallelogram. I'm hoping this will make the whole thing clearer.



We have a parallelogram, and we've picked a point along the bottom edge and drawn lines to the opposing vertices. Let's label some angles.

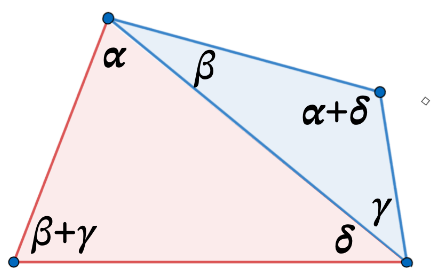



And then use the properties of parallels to get the rest of them labeled as well.

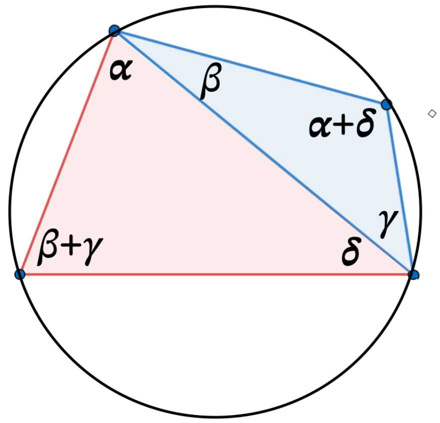



As a parallelogram, we have opposing angles equal, and adjacent angles summing to 180 as they must for parallel lines.

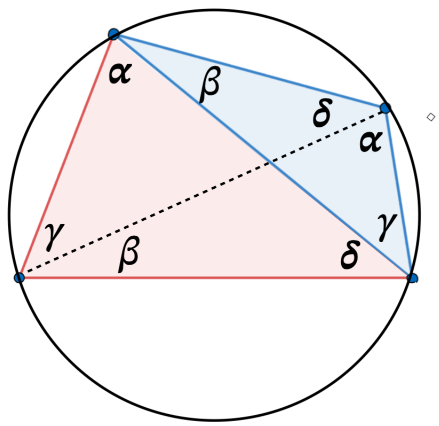
Now do the dissection. Cut out the red triangle and the blue triangle and join them as shown. In general, the scale will have to be adjusted so that they form a quadrilateral with that edge as the diagonal. We'll return to this point in a minute.




This is a cyclic quadrilateral, by the converse of the theorem on cyclic quadrilaterals, since opposing angles are supplementary.

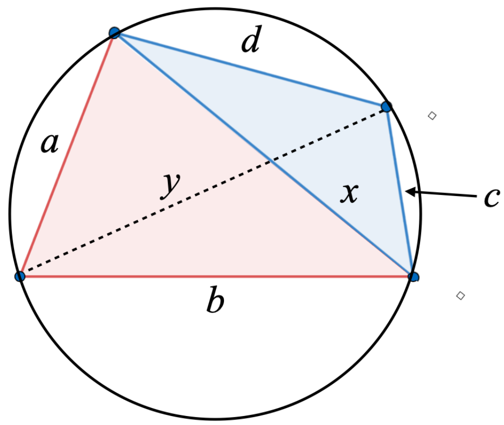



We can use the corollary of the inscribed angle theorem to draw the other diagonal and assign angles.

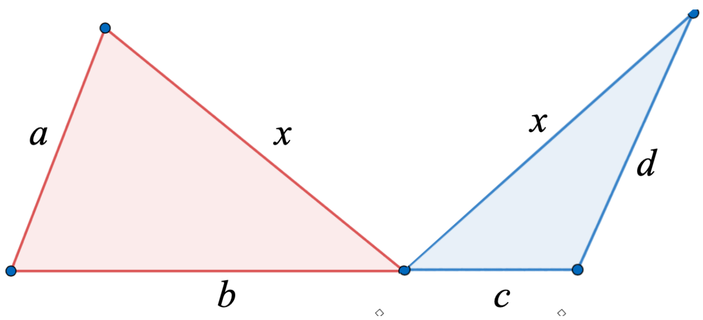



We're going switch our attention to the lengths of the sides and diagonals, but before we do, notice a very special triangle (partly red and partly blue) with angles $\gamma, \delta$, and $\alpha + \beta$.

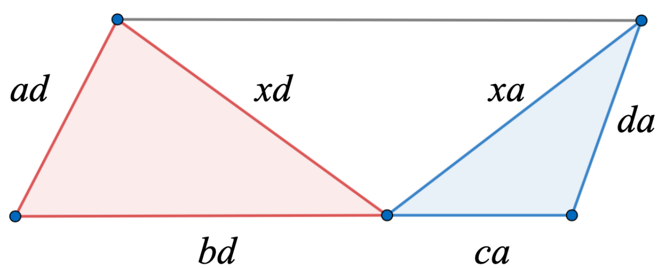
That triangle has sides $a, d$ and $y$.



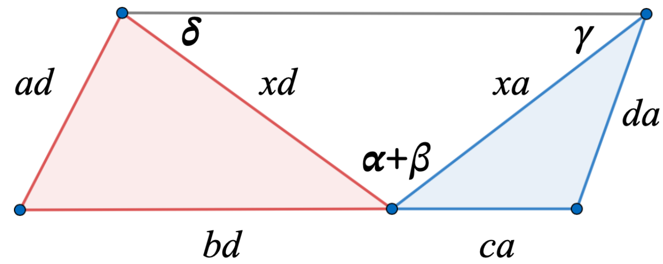

Let's reverse the process, dissecting the cyclic quadrilateral by cutting along the $x$ diagonal, and arranging the pieces so that the bases are collinear as in the original parallelogram.




In the general case, $a \neq d$. But we can scale the two sides to be equal. An easy way to do that is to cross multiply.

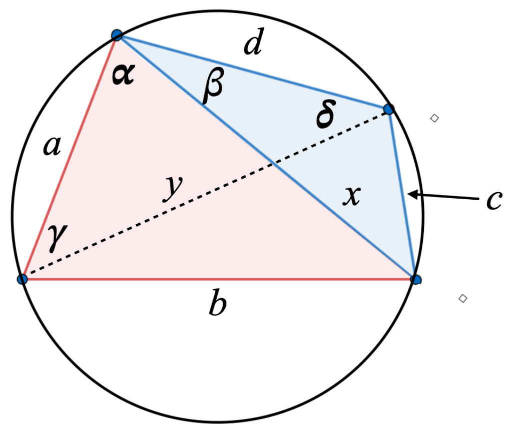



We have a parallelogram again. The angles are correct, and now the sides are equal. So are the top and bottom equal. We focus now on the white triangle.

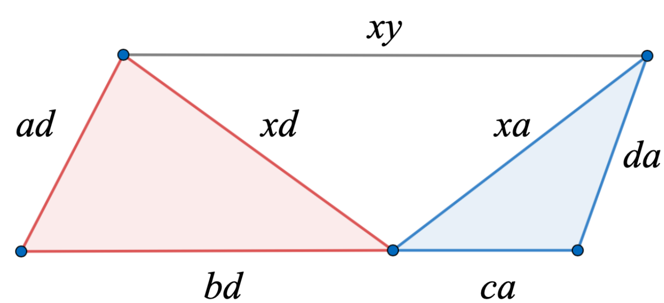



It has two sides with lengths $a$ and $d$, each scaled by a factor of $x$, and the two angles opposite those sides are $\delta$ and $\gamma$.

Remember the special triangle?




We can see it in the cyclic quadrilateral. The sides are $a, d, y$. In the parallelogram, that white triangle must be scaled by a factor of $x$, giving sides $xa, xd$ and $xy$.

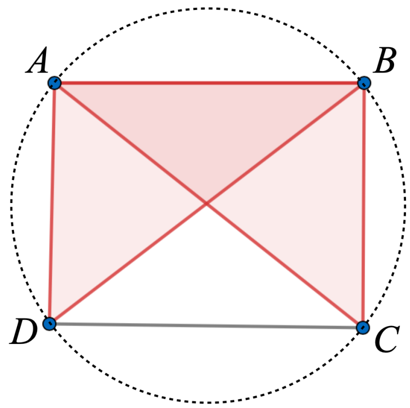



Since this is a parallelogram, the top and bottom are equal. Namely, $ac + bd = xy$.

This is Ptolemy's theorem.

□

## Pythagorean theorem from Ptolemy




*Proof.*

Consider an arbitrary rectangle, with all four vertices as right angles, and opposing sides equal.

The diagonals are equal, by our theorem on diagonals of a rectangle, and they bisect one another, by a general theorem about diagonals of a quadrilateral. Therefore, a circle can be drawn whose center is the point where they cross and that circle contains the four vertices.

We note that one can also start with a circle, then draw a diameter and any other point, forming a right triangle, by the converse of Thales' circle theorem. Draw the congruent triangle in the other hemisphere by SSS. We have opposite sides equal and at least one right angle, so the polygon is a rectangle.

Application of Ptolemy's theorem gives:

$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

but by equal sides in a rectangle and diameters in a circle:

$$AB^2 + BC^2 = AC^2$$

□

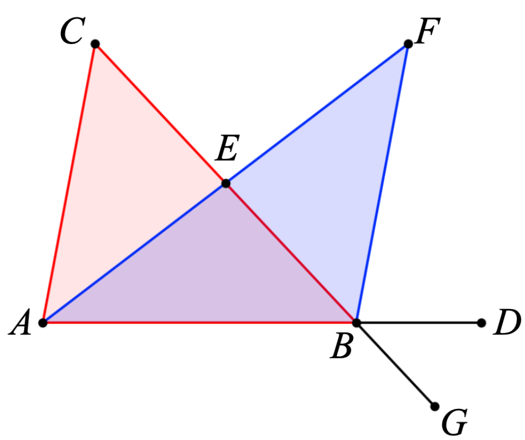
# Chapter 4

# Elements

In this chapter we will study some more *Propositions* from the first volume of Euclid's *Elements*. They are short, sweet and powerful.

### Euclid I.16

• In any triangle, if one of the sides is produced (extended), then the exterior angle is greater than either of the interior and opposite angles.




In $\triangle ABC$ extend side $AB$ to $D$. We are concerned with the exterior $\angle FBD$.

The claim is that $\angle FBD$ is greater (larger) than either of the interior angles $\angle ACB$ or $\angle CAB$.

*Proof.*

Find the midpoint of side $BC$ at $E$ so $BE = EC$ and then draw $AEF$ so that $AE = EF$ and join $BF$.

Note that $\angle EBD = \angle CBD$ is the external angle for $\triangle ABC$.

We have equal angles at $E$ by the vertical angle theorem.

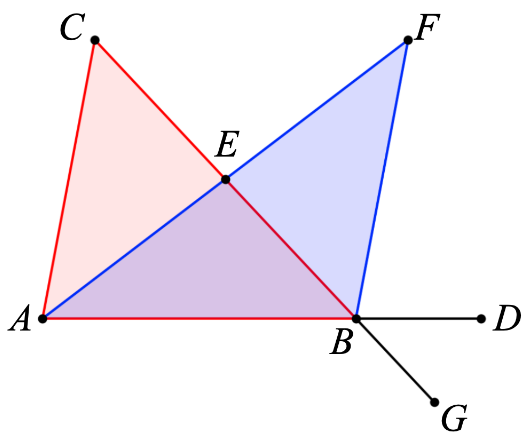
So by SAS the two smaller triangles $\triangle AEC$ and $\triangle BEF$ to the left and right are congruent.

Thus $\angle ACE = \angle FBE$. Since the whole is greater than the part, $\angle EBD > \angle FBE = \angle ACE$.

We can make a similar construction and proof for $\angle BAC$.

The exterior angle is greater than either of the interior and opposite angles.

□




One might ask why Euclid doesn't use supplementary angles to obtain a stronger proof. The answer seems to be that he is trying to go as far as he can without using the parallel postulate.

He will need that for the triangle sum theorem. That is Euclid I.32. And then that gives the stronger form of the external angle theorem.

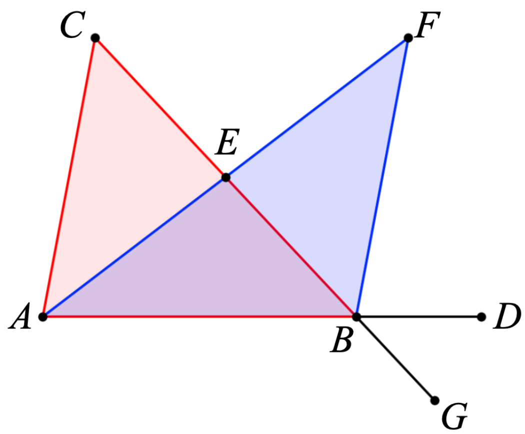
However, we did previously show a proof of the triangle sum theorem, that the sum of angles in any triangle is equal to two right angles.

And you were asked then to find the relationship between an external angle and the angles of the corresponding triangle. Here is our version of that proof.

### external angle theorem

- the external angle is equal to the sum of two internal angles

*Proof.*




As supplementary angles, $\angle ABC + \angle CBD =$ two right angles.

As the three angles of a triangle, $\angle ABC + \angle ACB + \angle CAB =$ two right angles as well.

But things equal to the same thing are equal to each other. Subtracting we obtain

$$\angle CBD = \angle ACB + \angle CAB$$

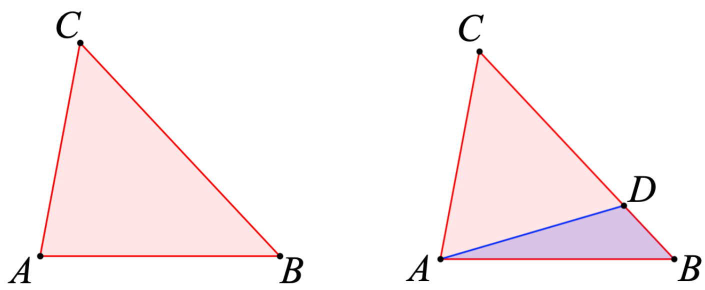
□

This relationship is obviously fundamental.

The next theorem is also extremely useful, and it follows from Euclid I.16.

## Euclid I.18: Longer side ⇒ larger angle

- Comparing two sides in any triangle, if one side is longer than the other, then the angle opposite is larger.




In $\triangle ABC$, suppose that $BC > AC$. We will show that $\angle A > \angle B$.

*Proof.*

Mark off $AD = AC$. By the isosceles triangle theorem, $\angle CAD = \angle CDA$.

Since the whole is greater than the part, $\angle A = \angle CAB > \angle CAD$.

But $\angle CDA$ is the external angle for $\triangle ABD$ so it is equal to $\angle BAD + \angle B$.

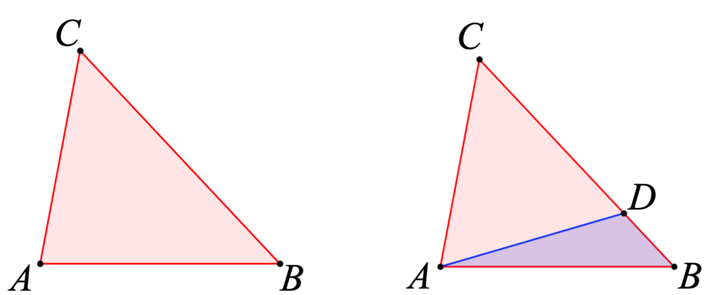
By transiitivity of inequality: $\angle A > \angle B$.

□

We get the converse almost for free.

## Euclid I.19: Larger angle ⇒ longer side

- Comparing two angles in any triangle, if one angle is larger than the other, then the side opposite is longer.




*Proof.*

We are given $\angle A > \angle B$ and claim that $BC > AC$.

We proceed by eliminating the other two possibilities. It cannot be that $BC = AC$ because then, by Euclid I.5, the angles would be equal but we are given that they are not.

Suppose that $AC > BC$. Then by Euclid I.18 just above, we have that $\angle B > \angle A$.

However, this also contradicts what was given. We must have $\angle A > \angle B$.

□

We have made use of what's called a trichotomy. There are only three possibilities:

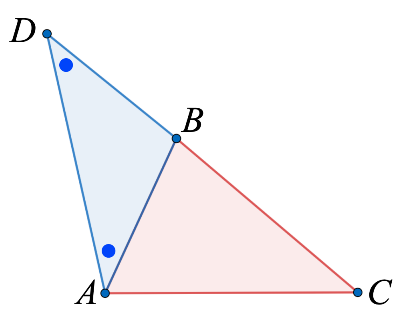
$$\angle A > \angle B, \qquad \angle A = \angle B, \qquad \angle A < \angle B$$

This applies to line segments and angles as well as many other things.

## Euclid I.20. Triangle inequality

- In any triangle, the longest side is smaller than the sum of the two shorter sides.

Clearly, if a given side is not the longest, so it is shorter than one of the others, then it must also be shorter than the sum of that other one plus the third. For this reason, we consider only the longest side.




Given that side $AC$ is the longest in $\triangle ABC$.

Extend side $BC$ so that $BD = AB$. By Euclid I.5, $\triangle ABD$ is isosceles, so the angles with blue dots are equal.

Then $\angle D$ is smaller than $\angle DAC$ and therefore, by Euclid I.19 just above, $AC$ is less than $DC$.

But $DC$ is equal to the sum of the two smaller sides of $\triangle ABC$. Hence

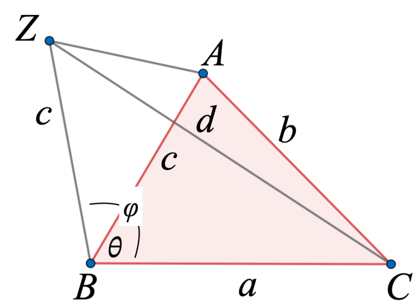
$$AC < AB + BC$$

$\square$

An equivalent statement is the famous "a straight line is the shortest distance between two points".

The triangle inequality has corresponding statements and proofs in other parts of mathematics including real and complex analysis.

## Euclid I.24. SAS inequality theorem




The next theorem is called the SAS inequality theorem, or informally, the "hinge" theorem.

- If two triangles have two pairs of sides which are the same length, the triangle with the larger included angle also has the larger third side.

We have $\triangle ABC$ with sides $a$ and $c$ flanking $\angle\theta$ and $\triangle ZBC$ with sides $a$ and $c$ flanking $\angle\phi$, and $\theta < \phi$. We claim that side $b$ in $\triangle ABC$ is smaller than side $d$ in $\triangle ZBC$.

Draw the two triangles nestled inside one another. Notice that $\triangle AZB$ is isosceles, so $\angle AZB = \angle ZAB$.

*Proof.*

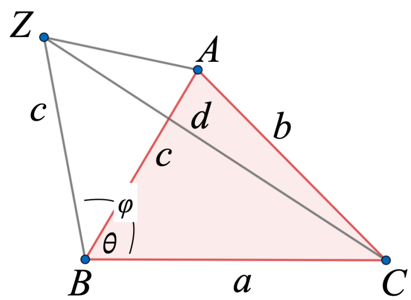
$$\angle AZC < \angle AZB \quad \text{and} \quad \angle ZAB < \angle ZAC$$

From above

$$\angle AZB = \angle ZAB$$

So

$$\angle AZC < \angle ZAC$$




By Euclid I.19, it follows that side $d$ opposite the larger angle $\angle ZAC$ is larger than side $b$ opposite the smaller angle $\angle AZC$ in $\triangle AZC$.

At the same time, in the two triangles $\triangle ABC$ and $\triangle ZBC$ the smaller angle $\theta$ is opposite the smaller side $b$, while the larger angle is $\phi$, opposite the longer side, $d$.

□

https://proofwiki.org/wiki/Hinge_Theorem

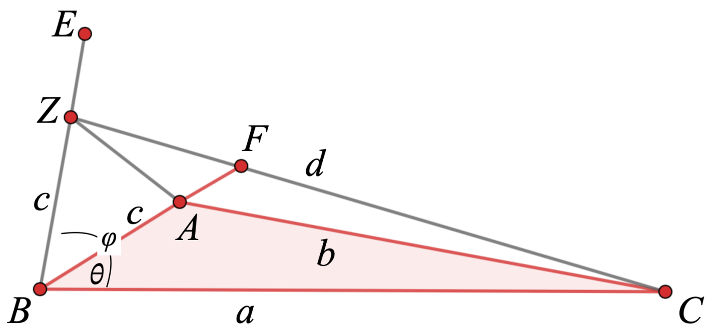
Note that Euclid I.24 says that where the included angle is larger, then the side opposite is larger. The converse is Euclid I.25.

*Proof.* Aiming for a contradiction, given side $d$ is smaller than side $b$, suppose that $\angle ABC < \angle ZBC$. The forward theorem gives $d > b$. This is a contradiction. □

Another subtlety is that it cannot be that $b = d$, by Euclid I.7, and also then the two triangles would be congruent by SSS, but the angles differ, so this is a contradiction.

One last point: this proof suffers from the same issue that Euclid I.7 does, namely that a different drawing can be made that makes the proof non-sensical. The solution is the same as well.

*Proof.*




Since $\triangle ABZ$ is isosceles, the base angles are also equal, namely $\angle AZE = \angle ZAF$. So then

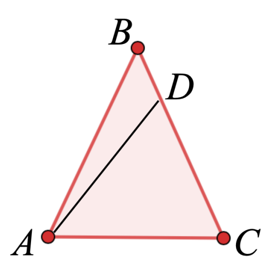
$$\angle AZF < \angle AZE = \angle ZAF < \angle ZAC$$

It follows that since the greater side is opposite the greater angle in $\triangle ACZ$, $d > b$.

$\square$

This is enough of the *Elements* to give us a good taste of the basics of Greek geometry of lines and triangles, and methods of proof.

## problem




Let $\triangle ABC$ be isosceles, and $D$ lie on $BC$. Using Euclid I.19, prove that $DC < DA$. Separately, use I.20 for the same proof.

Here is another idea. *Proof.* (Sketch). Let the perpendicular bisector of $AC$ be $BE$. Drop another perpendicular from $D$ to $AC$ at $F$. So $E$ is the midpoint and $F$ lies directly below $D$.

Clearly, $AF > AE > FC$. So

$$FC^2 < AF^2$$

$$FC^2 + DF^2 < AF^2 + DF^2$$

But by the Pythagorean theorem, the left-hand side is $DC^2$, while the right-hand side is $DA^2$. Taking side lengths as the positive square root, we have $DC < DA$. □

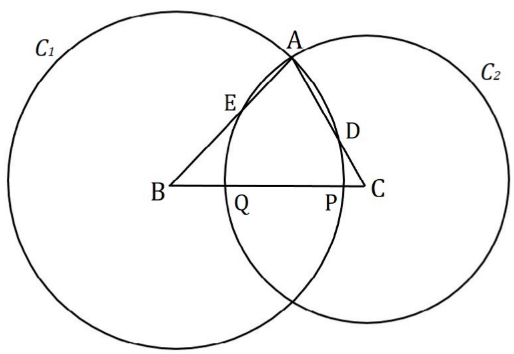
In fact, *every* point which lies on the same side of the perpendicular bisector of $AC$ as $C$, is closer to $C$ than to $A$.

## Triangle inequality by circles

Here is a modern proof of Euclid I.20.

*Proof.*

Consider $\triangle ABC$. Draw circle $\mathcal{C}_1$ on center $B$ with radius $AB$ and also circle $\mathcal{C}_2$ on center $C$ with radius $AC$. The third vertex $A$ lies on the intersection of the two circles.




If $BC$ is longer (greater) than the sum of the two radii, then the circles will not cross one another.

If $BC$ is equal to this sum, then they only touch, on the line $BC$. Thus, in order for there to be a triangle, $BC$ must be less than the sum of the diameters: $BC < AB + AC$.

□

https://arxiv.org/pdf/1803.01317

## angle of reflection

Suppose we shine a light at a mirror, or just look at the reflection of someone else, or even our own outstretched hand. The question arises, what determines the path of the light or the image as it travels to the eye? Heron of Alexandria discovered a proof of the answer in about 100 A.D.

The diagram shows two *possible* paths light might take, but there is only one path that it *does* take, shown in the right panel. Light takes the path where it reaches our eyes the fastest.

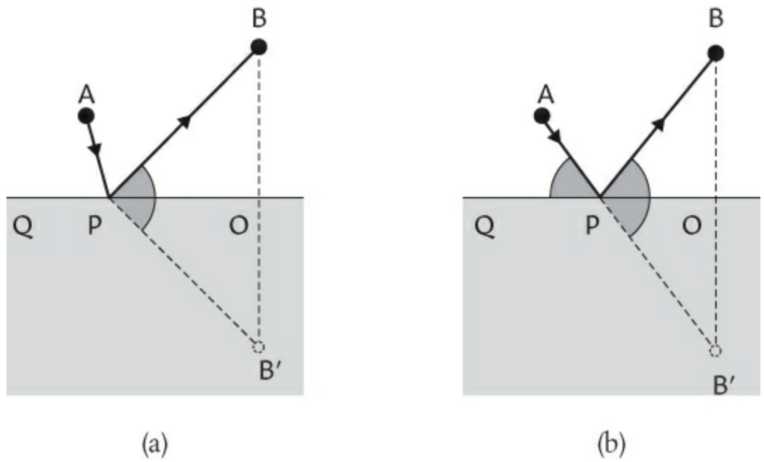



**Fig. 46** Finding the shortest path.

What is this path? What is the angle the ray of light makes with the mirror? This angle is called the angle of reflection.

Draw $\triangle POB'$, imagined to be on the other side of the mirror, with $B$ the same distance away from the mirror as $B'$, but on the other side. The two triangles $\triangle BPO$ and $\triangle B'PO$ are congruent by SAS.

Clearly, the shortest distance from $A$ to $B'$ is a straight line, by the triangle inequality.

So the result is that $\angle APQ$ equals $\angle B'PO$, which (by congruent triangles) equals $\angle BPO$. This is usually stated as "the angle of incidence is equal to the angle of reflection."

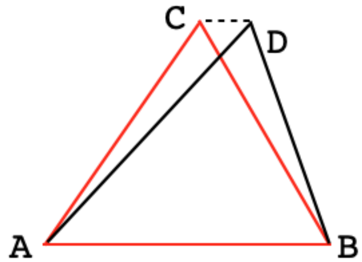
## smallest perimeter

Show that for two triangles with the same area, an isosceles triangle has the *smallest* perimeter.

*Proof.*

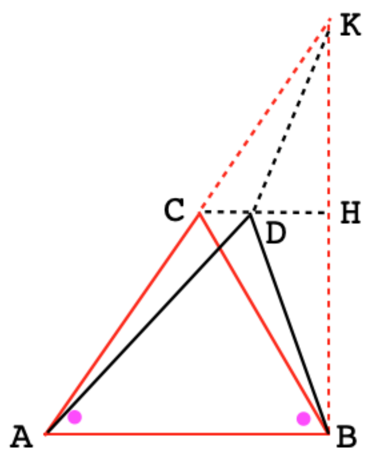
We suppose that the base of the isosceles triangle $\triangle ABC$ is equal to one of the sides of the other triangle $\triangle ABD$. If we would need to re-scale one side to have equality, we could then make a corresponding change in the altitude to that side to maintain equal area.

The equal area constraint means that points $C$ and $D$ lie along a horizontal line parallel to the common base $AB$.

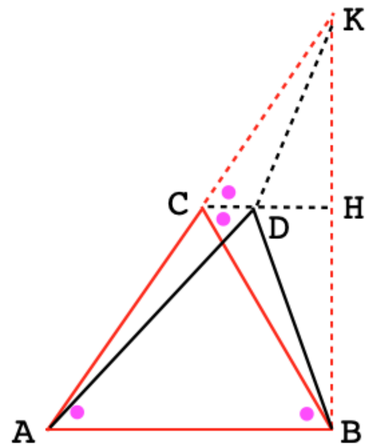



Draw a vertical from $B$ to meet the extension of $AC$ at $K$. Extend $CD$ to meet $KB$ at $H$ and also draw $DK$.

We're given that $AC = BC$ and so $\angle CAB = \angle CBA$ (magenta dots).




It follows that other angles are also equal to those two (by alternate interior angles).




$\angle BCH = \angle ABC$ and $\angle KCH = \angle CAB$.

Since $\angle CAB = \angle ABC$, it follows that $\angle BCH = \angle KCH$.

The angles at $H$ are right angles, since $BK$ is perpendicular to $AB$ and $CH$ is parallel to $AB$.

Therefore $\triangle CHK \cong \triangle BCH$ by ASA, so $BC = CK$.

Similar reasoning will give that $BD = DK$.

But now by the triangle inequality:

$$AC + CK < AD + DK$$

Substituting from above

$$AC + BC < AD + BD$$

Add $AB$ to both sides

$$AC + BC + AB < AD + BD + AB$$

The perimeter of $\triangle ABC$ is less than that of $\triangle ABD$.

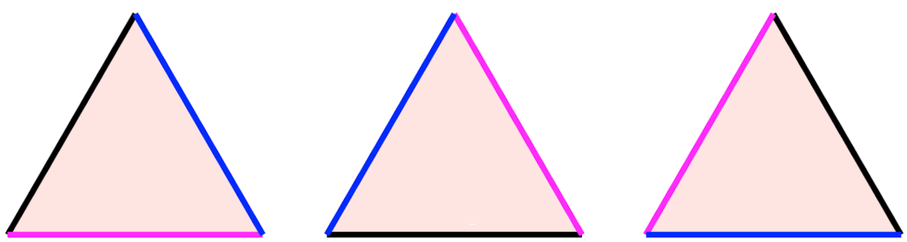
$\square$

# Chapter 5

# Equilateral triangles

## basic triangle

An equilateral triangle has all three sides the same length. By symmetry, as there is no reason to favor one vertex, the three vertices have the same angular measure, namely, $\pi/3$ (since the total is two right angles).



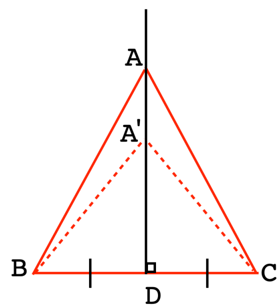
*Proof.*

The **isosceles triangle theorem** (Euclid I.5: equal sides $\rightarrow$ angles), which can be applied twice to give the result.

We also have **Euclid I.19**, which says that if one angle is larger than another in a triangle, then the corresponding side is also longer. So, assume the angles are not the same, then that implies the sides are not the same length either. Hence the angles are the same.

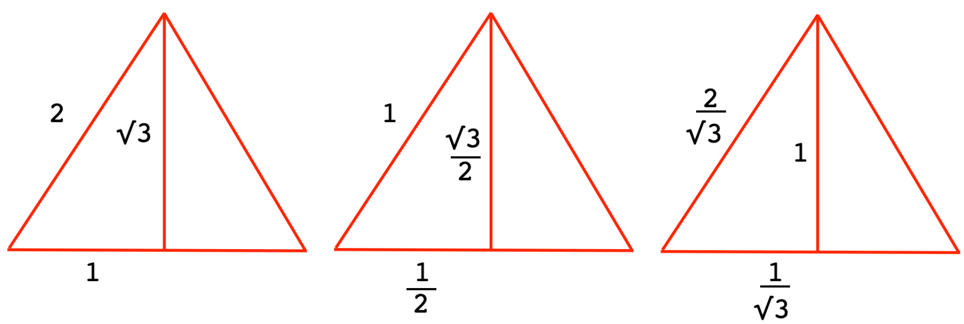
$\square$

An equilateral triangle is also isosceles (three times over), and as a result the altitude dropped from a vertex to the opposing base bisects both that base and the angle at the vertex.




## area

To calculate the area of the equilateral triangle, we could write a general formula, but it is usually easier to go back to basics and derive one.




In the left panel, the result of bisection is that, for a side of length 2, half the base is 1, and then Pythagoras tells us that the altitude is $\sqrt{3}$. Using the standard formula, the area of this triangle is also $\sqrt{3}$.

To re-scale the triangle, write whichever measurement is given. For example in the middle we're given a side length of 1 so that means that all lengths are multiplied by $1/2$.

On the right, we're given that the altitude is 1 so then we must multiply by $1/\sqrt{3}$. The logic is that to erase that $\sqrt{3}$ in the altitude, we must multiply by its inverse.

If you really insist on a general formula ...

Suppose the length of the side is $s$, then the ratio of the altitude to the side is

$$\frac{h}{s} = \frac{\sqrt{3}}{2}$$

Therefore twice the area of the triangle is

$$2A = \frac{\sqrt{3}}{2} \cdot s^2$$

We can check that against a general trigonometric formula for the area of any triangle. Twice the area is the product of two adjacent sides times the sine of the angle between. Here the angle is $\pi/3$ and its sine is $\sqrt{3}/2$ so we have $2A = \sqrt{3}/2 \cdot s^2$.

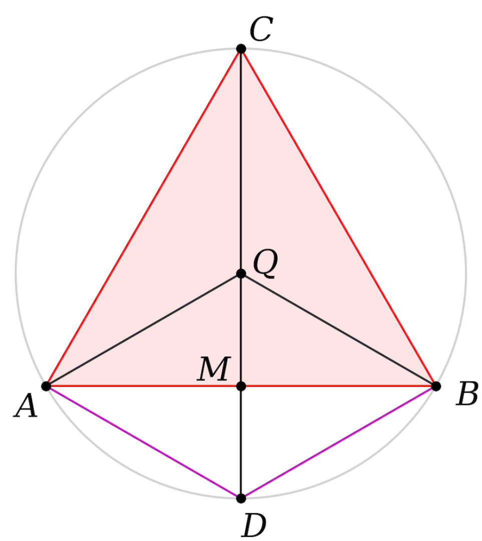
It can be quicker to rely on the fact that the area goes like the square of the side. For side length 2 we had $A = \sqrt{3}$. If we shrink the triangle to side length 1, the area goes down by a factor of four, to $A = \sqrt{3}/4$.

## circumscribed equilateral triangle

Here is a fun construction based on an equilateral triangle.

Any triangle fits into a unique circle (see the discussion of circles and bisectors of the chords). Draw the circumcircle for equilateral $\triangle ABC$.

We can prove a number of properties. First, draw the diameter from vertex $C$.



*Proof.*

(1) Given equal sides $AB = AC = BC$.

(2) We have three equal angles at $A$, $B$ and $C$ by two applications of isosceles triangles, or by inscribed angles on equal arcs.

(3) By Thales' theorem, $\angle CAD$ and $\angle CBD$ are both right.

(4) $\triangle CAD \cong \triangle CBD$ by HL.

(5) $\rightarrow AD = BD$.

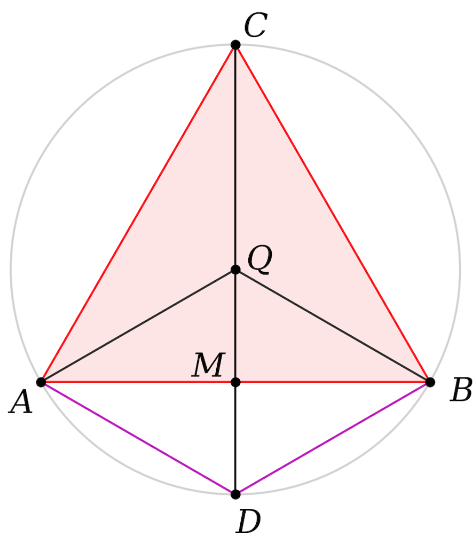
(6) $\rightarrow CD$ bisects $\angle C$ by inscribed angles on equal arcs.

(7) $AQ = DQ = BQ$ as radii of the circle.

(8) $\rightarrow \triangle QAD$ is isosceles.

(9) $\rightarrow \angle QAD = \angle QDA$.

(10) But $\angle QDA = \angle CBA$, so $\triangle QAD$ and $\triangle QBD$ are equilateral.




(11) $\rightarrow AD$ and $DB$ are also equal in length to radii.

(12) $\rightarrow AQBD$ is a parallelogram (two pairs of equal sides).

(13) $M$ bisects both $AB$ and $QD$ (crossed diagonals bisect each other).

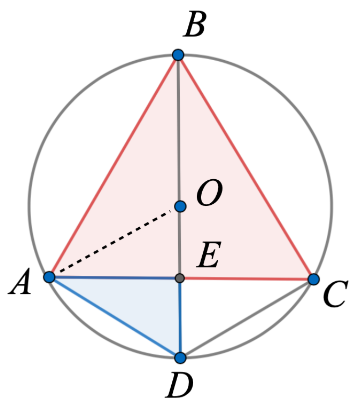
(Actually, this is a general result for altitudes as we show elsewhere).

(14) There are four congruent triangles with right angles at $M$ by SSS.

(15) $\rightarrow CQM$ is bisector, median and altitude of $\triangle ABC$ and $Q$ is incenter, centroid and orthocenter as well.

There are certainly other ways of proceeding. For example, since $\triangle AQC \cong \triangle BQC$, $\angle C$ is bisected and then $\triangle ACD \cong \triangle BCD$ and we have right angles at $D$.

□




*Proof.* (Alternate).

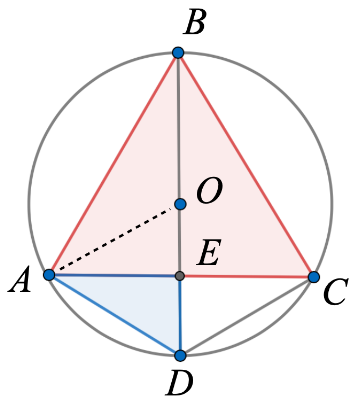
Let the side length be 2 and then $AE = CE = 1$. By crossed chords, $AE \cdot CE = 1 = BE \cdot DE$.

But by the Pythagorean theorem $BE = \sqrt{3}$ and then $DE = 1/\sqrt{3}$ and their ratio is

$$\frac{1}{\sqrt{3} \cdot \sqrt{3}} = \frac{1}{3}$$

□

*Proof.* (Alternate).




Let $AD = CD = d$ and $r$ be the radius of the circle.

By Ptolemy's theorem:

$$sd + sd = 2rs$$

$$d = r$$

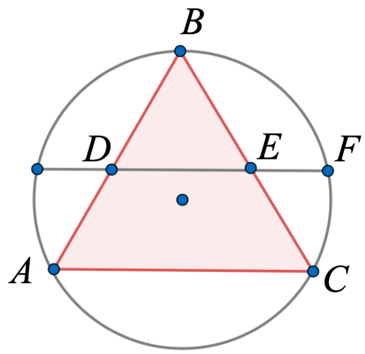
It follows that $AOCD$ is a parallelogram, the diagonals bisect one another, etc.

□

Thus, the altitude of the equilateral triangle is $3/4$ of the diameter of the circle that just encloses it. And the point where the altitudes meet in an equilateral triangle is $1/3$ of the way up from the base, since $r = 2a$.

## problem

Here is a problem from Dr. Paul Yiu. Given that $ABC$ is equilateral and that the points $D$ and $E$ are midpoints of the sides, find $DF/DE$.




We *could* calculate the length of the perpendicular from the center of the circle ($O$, not labeled) to $DE$ and then use the Pythagorean theorem.

A simpler approach is to set the side length of the triangle to 2, so each half is 1, and since $BDE$ is also equilateral, $DE = 1$ (or use the midline theorem).

Let $DF$ be $x$. Then, by the theorem of crossed chords we have that

$$1 \cdot 1 = (x - 1) \cdot x$$

$$x^2 - x - 1 = 0$$

Solve using the quadratic equation:

$$x = \frac{1 + \sqrt{5}}{2} = \phi$$

This is the famous "golden" ratio.

Let's do it the hard way as well. If we keep the side length of the large triangle as 2, then the small triangle has side length 1, and the altitude is $\sqrt{3}/2$.

We need the radius. The altitude of the large triangle is $\sqrt{3}$ and the diameter is $4/3$ of that, so the radius is $2/3$ which makes $r = 2/\sqrt{3}$.

The distance from $O$ to $DE$ is the difference, and we will need the square of that:

$$\delta^2 = (\frac{\sqrt{3}}{2} - \frac{2}{\sqrt{3}})^2$$

$$= \frac{3}{4} + \frac{4}{3} - 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{2}{\sqrt{3}}$$

$$= \frac{3}{4} + \frac{4}{3} - 2$$

The distance from the midpoint of $DE$ to $F$, squared, is:

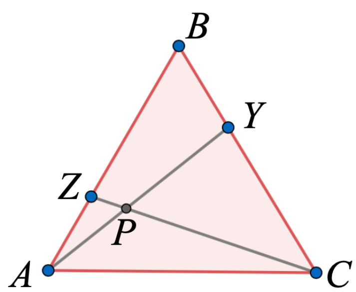
$$\frac{4}{3} - \delta^2 = \frac{4}{3} - (\frac{3}{4} + \frac{4}{3} - 2) = \frac{5}{4}$$

so the square root is $\sqrt{5}/2$ and then we must add $1/2$ to obtain $DF$. That matches what we had before.

## problem

Here is a second problem from Dr. Yiu.

Crossing lines are drawn from the vertices in an equilateral triangle, such that the larger pair of angles at $P$ are each 120°.




We have that, since $\angle ZPY + \angle A =$ two right angles, the angles inside the quadrilateral at $Z$ and $Y$ are supplementary, so the opposing ones are equal.

A consequence of that is $\triangle AYB$ and $\triangle BZC$ have all three angles the same, but also share a side, hence they are congruent. Therefore $AY = BZ$ and $AZ = YC$.

By the same reasoning, $\angle AZC = \angle BYC$, which means that $\triangle AZC$ and $\triangle YCB$ share all three angles plus two sides, so they are congruent. Therefore $BY = ZC$.

There are a ridiculous number of similarities and equalities here. The acute angles at $P$ are 60°, which gives more similar triangles such as $\triangle PZB \sim \triangle AYB$ and $\triangle PYC \sim \triangle AZC$. I write the vertices flanking the short and medium sides in order to make it easy to write the ratios.

We are almost too wealthy. We are asked to show that

$$\frac{AY}{AZ} = \frac{PB}{PC}$$

Now, $\triangle BPC$ is not similar to anything else. But from the first similarity above ($\triangle PZB \sim \triangle AYB$) we get

$$PZ/ZB/BP = AY/YB/BA$$

so we can write

$$PB = AB \cdot ZB/YB$$

From the second similarity ($\triangle PYC \sim \triangle AZC$) we get

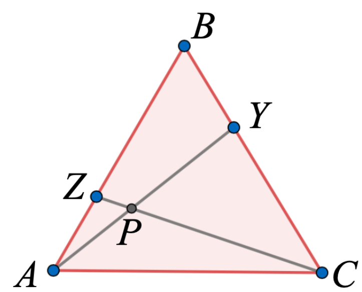
$$PY/YC/CP = AZ/ZC/AC$$

so we can write

$$PC = AC \cdot YC/ZC$$

Then we form the ratio (canceling $AB = AC$)

$$\frac{PB}{PC} = \frac{ZB}{YB} \cdot \frac{ZC}{YC}$$




Using the equalities $ZC = BY$, $AY = ZB$, $AZ = YC$:

$$\frac{PB}{PC} = \frac{ZB}{YC} = \frac{AY}{AZ}$$

I don't know if there is an easier solution, just happy to have found one.

## Bertrand's paradox

Grinstead and Snell's wonderful *Introduction to Probability* has this problem (example 2.6). It's called Bertrand's paradox. We are asked to draw a chord of a unit circle randomly.

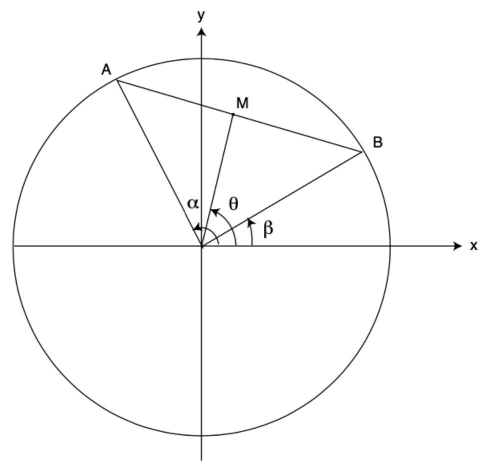



Figure 2.9: Random chord.

Here we might say, let's choose each of three angles $\alpha$, $\beta$ and $\theta$ randomly (uniform density) from $[0, 2\pi]$.

But there is no reason why the radius to $B$ cannot lie along the $x$-axis, so there are really only two choices.

The question is posed: what is the probability that the length of this random chord is $> \sqrt{3}$.

However, there are several different approaches to parametrize the problem, and randomizing the different parameters leads to different results.

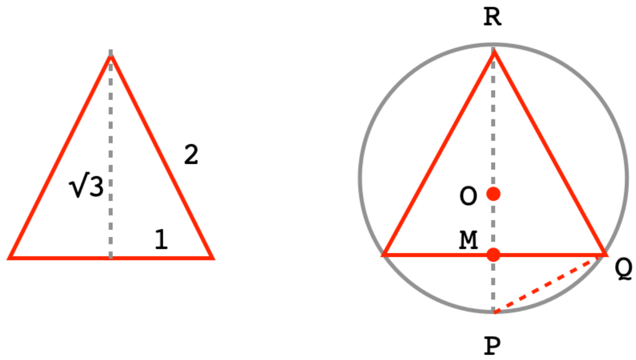
## equilateral triangles

We briefly review some properties of equilateral triangles which we looked at earlier, with a slightly different take.

Drop an altitude and observe the ratio of side lengths. It is convenient to start with a side length of 2 for the original triangle, then in the bisected copies the sides are in the ratio 1-2-$\sqrt{3}$ (the 1 by bisection, and $\sqrt{3}$ by the Pythagorean theorem).

The angle at each vertex of the original equilateral triangle is $\pi/3$, so the new triangles have angles of $\pi/6$, both by bisection or because the altitude forms an angle of $\pi/2$ at the base, so the sum of angles theorem gives us the last angle.

In the right panel, the equilateral triangle is inscribed in a unit circle, so $OR = OP = OQ = 1$. We claim that the line segment $OM$ has a length of $1/2$.




*Proof.*

$\angle PQR$ is a right angle, by Thales' circle theorem, and $\angle MRQ$ is shared, so $\triangle PQR$ is similar to $\triangle RMQ$. Therefore, $\angle RPQ = \angle MQR = \pi/3$.

Therefore the sides of $\triangle PQR$ are also in the ratio 1-2-$\sqrt{3}$, with $PQ/PR = 1/2$ and so $PQ = OP = OQ$. Thus, $\triangle OPQ$ is equilateral.

$QM \perp OP$ so $MQ$ is the bisector of both $\angle PQO$ and the base $OP$.

Therefore, $OM$ is one-half of $OP$ and has a length of $1/2$.

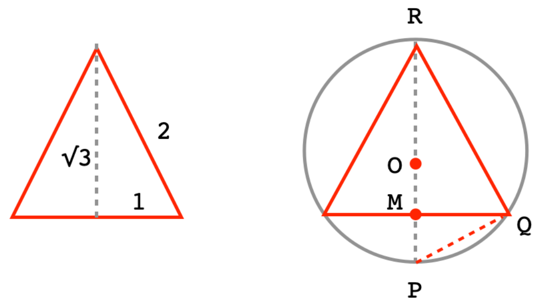
□

## first parametrization

We have just shown that the altitude of the inscribed equilateral triangle in a unit circle has length $3/2$. This means that the ratio of the inscribed triangle to the standard one is $\sqrt{3}/2$.

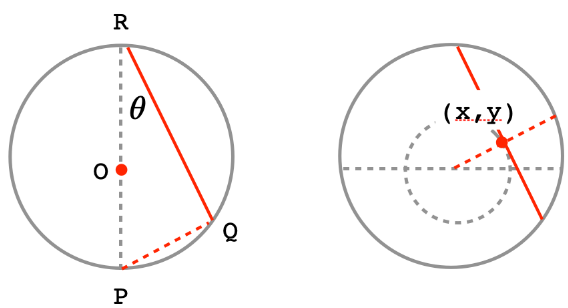
And that means the side length of the inscribed equilateral triangle is $\sqrt{3}$.

That explains the length chosen for the chord in this problem. We see that if $M$ is chosen at random anywhere along $OP$, one-half of the time the chord formed will be larger than $\sqrt{3}$.




So the probability we are asked to give is just $1/2$.

## second parametrization




The second parametrization has the same triangle we just saw, $\triangle PQR$. The angle at vertex $R$ is $\theta$.

$\theta$ can lie in the interval $[0, \pi/2]$, and in the event that $\theta < \pi/6$, the chord length $RQ > \sqrt{3}$.

The probability that the chord is greater than $\sqrt{3}$ in length is $1/3$, since $\pi/6$ is one-third of $\pi/2$.

## third parametrization

Finally, we imagine picking two coordinates $(x, y)$ at random from the interior of the circle. We place the midpoint of the chord $M$ at $(x, y)$.

If $M$ is such that $r = \sqrt{x^2 + y^2} < 1/2$, then $M$ will be closer to the center of the circle than $1/2$ and so the chord length will be $> \sqrt{3}$.

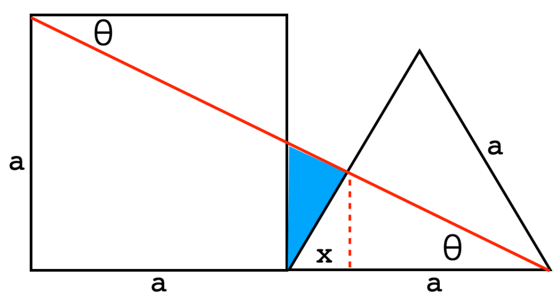
The number of points that have this property is proportional to the relative areas of the inside small circle, and the larger circle around is.

$$\frac{\pi(1/2)^2}{\pi} = \frac{1}{4}$$

We see that, depending on which parameter is randomized, we obtain a probability of 1/2, 1/3 or 1/4.

In Jaynes's words, the problem is not well-formed.

## problem

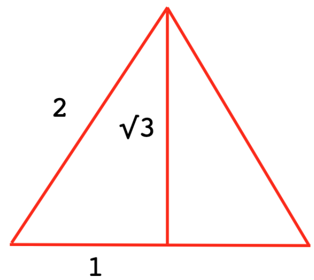



We have a square with side $a$ and an equilateral triangle, also with side $a$. Their bases are colinear and a red line is drawn as shown. What is the area of the blue triangle?

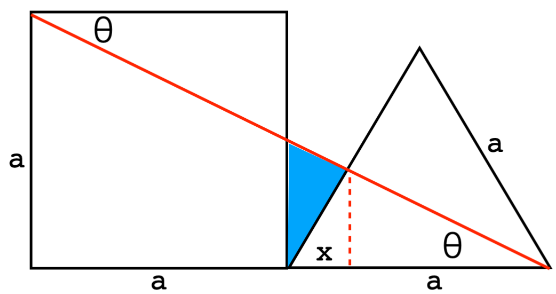
The angles marked $\theta$ are equal because the base and the top of the square are parallel, but the angle is *not* 30. The red line intersects the side of the square at half the height, so $a/2$.

We compute *twice* the area of the combined blue and part of the equilateral triangle as $a^2/2$ and that of the blue triangle as $xa/2$. But what is $x$?

We need a preliminary result. The Pythagorean theorem allows us to calculate that the altitude of an equilateral triangle is $\sqrt{3}$ times one-half the base.




Back to our problem. The triangle with $x$ as its base does have an angle of 30 so its height $h$ is $\sqrt{3} \cdot x$. We compute twice the area of the part of the equilateral triangle as $\sqrt{3}xa$.




Put the whole thing together as

$$2\mathcal{A} = \frac{a^2}{2} = \frac{xa}{2} + \sqrt{3}xa$$

$$a = x + 2\sqrt{3}x$$
$$x = \frac{a}{1 + 2\sqrt{3}}$$

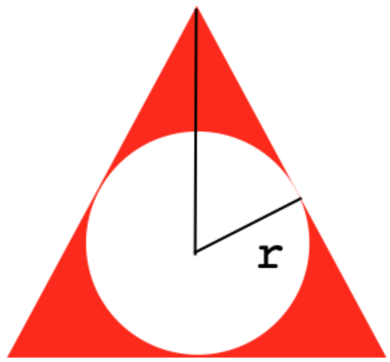
The final result is $1/2$ times $a/2$ times what we have above.

## problem

Here is a problem from the entrance exam for Brown, 1901.

**4. A regular triangle is circumscribed about a circle whose radius is 7. Find the area of that portion of the triangle which is outside of the circle.**

A regular triangle is what we would call an equilateral triangle. Applying the Pythagorean theorem to one-half of such a triangle yields 1-2-$\sqrt{3}$ for the sides, where 2 is the hypotenuse.




From this we deduce that the altitude is $3r$ and the side length is $2\sqrt{3}r$, so the triangle's area is

$$A = \frac{1}{2} \cdot 3 \cdot 2\sqrt{3} \cdot r^2$$

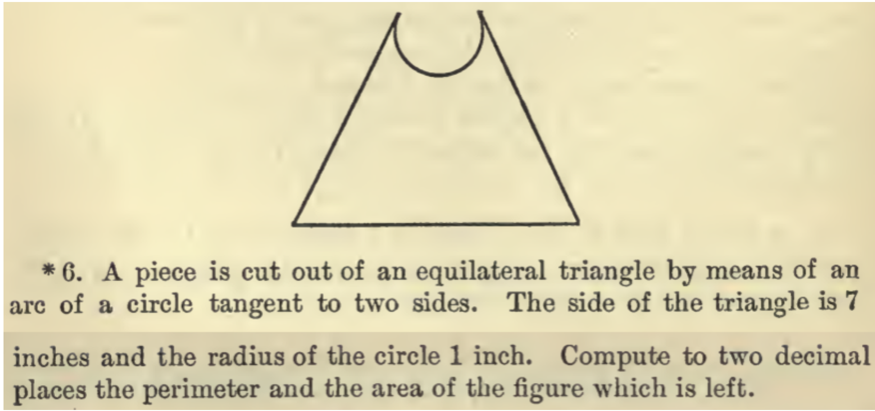
The red part is the difference between this and the circle's area.

$$\text{red} = (3\sqrt{3} - \pi)r^2$$

We're given a radius of 7 but that's arithmetic. We don't need that now.

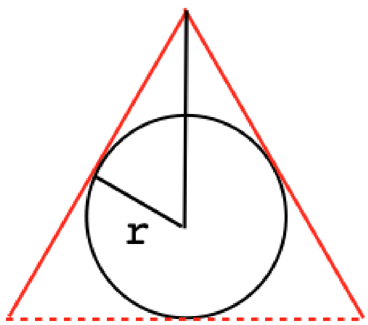
## problem

Here is a problem from the entrance exam for Harvard, 1901.



*6. A piece is cut out of an equilateral triangle by means of an arc of a circle tangent to two sides. The side of the triangle is 7 inches and the radius of the circle 1 inch. Compute to two decimal places the perimeter and the area of the figure which is left.

The first thing to notice is that the circle is tangent to the sides. The tangent touches the circle at a single point, so the missing part of the diagram looks like this:




The whole angle at the top is $\pi/3$ so the half-angle is $\pi/6$. This right triangle has sides in the ratio 1-2-$\sqrt{3}$. The radius is given as 1 so the red base side of the triangle is $\sqrt{3}$.

The missing perimeter of the triangle is twice this or $2\sqrt{3}$. But the added perimeter from the circle is $2/3 \cdot 2\pi r = 4/3\pi$.

The area of the triangle is $1/2 \cdot \sqrt{3} \cdot 1 = \sqrt{3}/2$ and there are two copies so the total missing triangular area is $\sqrt{3}$. The added circular area is $2/3\pi$.

The original perimeter of the triangle was $7 \cdot 3 = 21$ and the area was $\sqrt{3}/4 \cdot s^2$.

We'll just set up the two calculations:

$$A = \frac{\sqrt{3}}{4} \cdot 7^2 - \sqrt{3} + 2/3 \cdot \pi$$

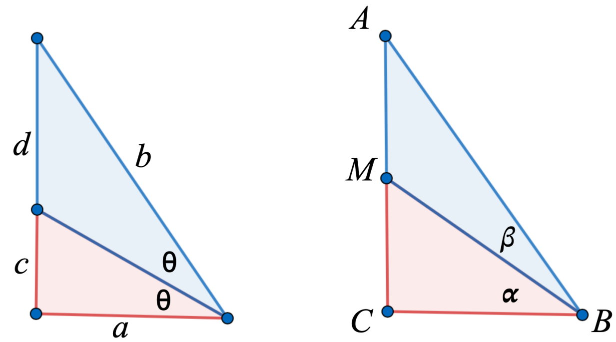
$$P = 7 \cdot 3 - 2\sqrt{3} + 4/3 \cdot \pi$$

It seems that the most efficient way to calculate this is to do $\sqrt{3} + 2/3 \cdot \pi$, then double it, then do the rest.

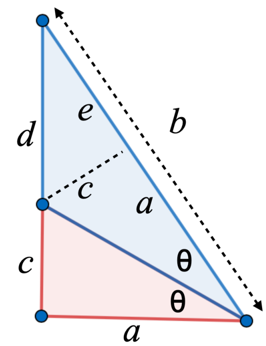
# Chapter 6

# Angle bisectors

Here's a classic problem: can we say anything about the ratio of sides when an internal line is drawn in a triangle? We first consider the case of a right triangle where the internal line is an angle bisector (left panel, below). We can prove an important theorem.



## angle bisector theorem




The bisector forms two new internal triangles. In the upper one (blue), we draw a line perpendicular to side $b$ which meets the bisector on the side opposite. This forms two triangles with two angles the same, $\theta$ and the right angle (thus three angles the same), and one side shared. So we have congruent right triangles by ASA.

Thus, the two sides labeled $a$ are equal, as are the two sides labeled $c$.

The triangle with sides $c, e, d$ is similar to the whole original triangle, because both are right triangles and the top vertex is shared. We have that the ratio of the short side to the hypotenuse is

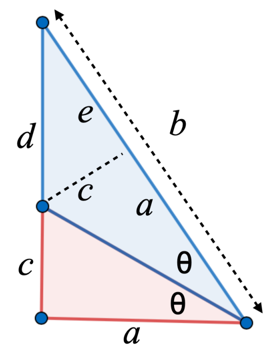
$$\frac{a}{b} = \frac{c}{d}$$

$$ad = bc$$

$$\frac{a}{c} = \frac{b}{d}$$

Tthe sides flanking the duplicated angle are in the same proportion as the parts of the base:

□




The result can be pushed a little further: Add 1 to both sides:

$$\frac{a+b}{b} = \frac{c+d}{d}$$
$$\frac{a+b}{c+d} = \frac{b}{d} = \frac{a}{c}$$

which may be a surprising result.

We took advantage of the fact that the large triangle was a right triangle. However, if you think about it, you should be able to see that the same result holds for an isosceles triangle. There, the two sides are equal, and if the top angle is bisected, so is the base. So the ratio of each side to its part of the base is also equal.

This might lead you to wonder whether the proof holds for a general triangle. Indeed, we will show later on that the sides and bases are in proportion for any triangle, if the angle is bisected.

In the formula from above

$$\frac{a+b}{c+d} = \frac{b}{d} = \frac{a}{c}$$
$$= \frac{a}{c+d} + \frac{b}{c+d}$$

we have a relationship between sides of the right triangle with angle $2\theta$ and the right triangle with angle $\theta$. Archimedes uses this in his method to place bounds on the value of $\pi$.

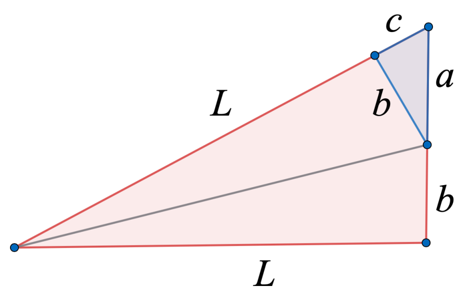
In trigonometry, the relationship is written:

$$\frac{a}{c} = \cot\theta = \frac{a}{c+d} + \frac{b}{c+d}$$
$$= \cot 2\theta + \csc 2\theta$$

## angle bisector proof of Pythagoras

The same diagram (relabeled) gives an easy proof of the Pythagorean theorem (from Dunham's Problems).

*Proof.*




From similar triangles:

$$\frac{L}{a+b} = \frac{b}{c}$$
$$L = \frac{b(a+b)}{c}$$

Figure the area as twice the value. For the big triangle we have

$$2A = (a+b)L$$
$$= \frac{b(a+b)^2}{c}$$

From the three separate triangles:

$$2A = bc + 2bL$$
$$= bc + 2b^2\frac{a+b}{c}$$

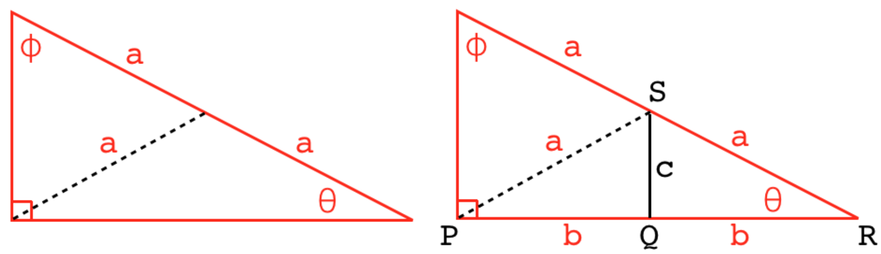
Equate them and do some algebra:

$$\frac{b(a+b)^2}{c} = bc + 2b^2\frac{a+b}{c}$$
$$(a+b)^2 = c^2 + 2b(a+b)$$

$$a^2 + 2ab + b^2 = c^2 + 2ab + 2b^2$$
$$a^2 = b^2 + c^2$$

□

## midpoint theorem




In a right triangle, draw the line segment from the vertex that contains a right angle to the midpoint of the hypotenuse, separating it into two equal lengths $a$. We will show that the length of the bisector is also $a$.

We gave a proof of this earlier.

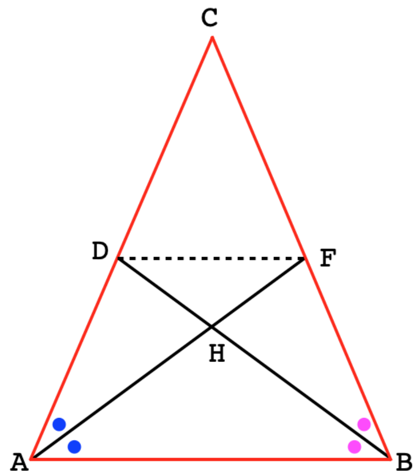
An elegant proof is based on Thales' circle theorem. Let the hypotenuse be the diameter of a circle.

Then, if the large triangle is a right triangle, the right angle lies on the circle and then $PS$ is a radius.

Alternatively if the median is a radius of the circle, then $P$ is on the circle. By the converse of Thales' theorem, it follows that the angle at $P$ is a right angle.

## isosceles angle bisector theorem

The next theorem involves angle bisectors in an isosceles triangle. It is easy in the forward direction, but the converse is very challenging, at least until you draw the right diagram. Then, as usual, it's not so bad.




We are given that $\triangle ABC$ is isosceles ($AC = BC$), and also that the angles at the base are both bisected.

We claim that the angle bisectors are equal in length: $AF = BD$.

*Proof.*

By the forward version of the isosceles triangle theorem, the entire $\angle A = \angle B$, so it follows that all four angles dotted blue and magenta are equal.

Then $\triangle ABD \cong \triangle BAF$ by ASA.

As a result, $AF = BD$ and $AD = BF$.

Furthermore, since the original triangle is isosceles and $AD = BF$, the smaller triangle $\triangle CDF$ is also isosceles, by subtraction. Alternatively, $\angle C$ is shared, and the long sides are equal so $\triangle CDB \cong \triangle CFA$ by AAS.

It follows that $CD = CF$ and $\triangle CDF$ is isosceles, and therefore, $\triangle CDF \sim \triangle ABC$ by AAA.

By alternate interior angles, $DF \parallel AB$.

$\square$

That's the easy part.

The converse theorem says that if we have angle bisectors and they are equal in length, then the triangle is isosceles. This is called the Steiner-Lehmus Theorem.

https://en.wikipedia.org/wiki/Steiner\OT1\textendashLehmus_theorem

We defer discussion of the **Steiner-Lehmus Theorem** to its own chapter.

## angle bisector theorem revisited

Although we used right triangles in our first proof of the angle bisector theorem, that wasn't strictly necessary. The theorem is also true for the general case.

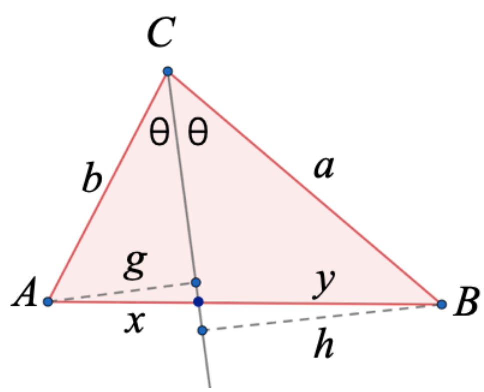
We discuss several different proofs. The first two draw right triangles involving the bisector, or part of it.

## similar right triangles

*Proof.*

In $\triangle ABC$ draw the bisector of $\angle C$ to join side $AB$, dividing it into lengths $x$ and $y$.

We draw verticals from $A$ and $B$ to the angle bisector.




This forms two pairs of similar right triangles. The first pair has complementary angle $\theta$

$$\frac{b}{a} = \frac{g}{h}$$

And the second pair have equal vertical angles at the bisector.

$$\frac{x}{y} = \frac{g}{h}$$

The result follows easily.

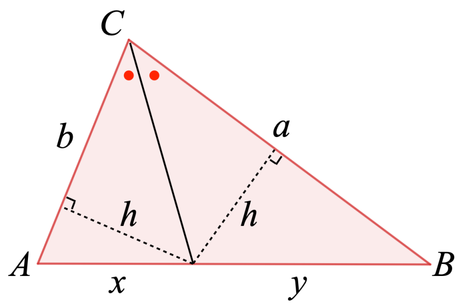
$$\frac{b}{a} = \frac{x}{y}$$

$\square$

The result we obtained previously is a special case where angle $A$ is a right angle, but the theorem is true generally.

Alternatively, use the notion of area. Draw verticals to the sides as shown.

We have $\triangle ABC$ with sides $a, b, c$ and the angle $C$ is bisected so the two angles marked with red dots are equal.




Start from the point where the bisector meets the side opposite $\angle C$, cutting $c$ into $x$ and $y$. Drop the perpendiculars to the sides $a$ and $b$.

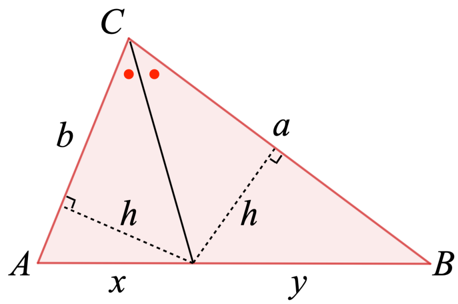
This forms two congruent right triangles, since the angles at the top are equal, and the hypotenuse is shared. The equal sides are marked as $h$.

We can compute the ratio of the areas of the left and right-hand sub-triangles in two ways.

The first way uses the area-ratio theorem.

$$\frac{A_L}{A_R} = \frac{x}{y}$$

The two triangles share a common altitude and the area is one-half the base times the altitude. Since the altitude is shared, it cancels, together with the factor of one-half.




The second approach is to use the sides $a$ and $b$ as the two bases. We have that the heights $h$ are equal. Hence

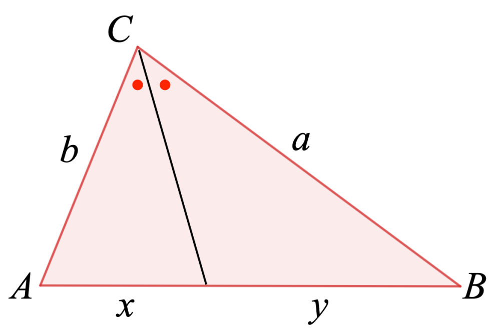
$$\frac{A_L}{A_R} = \frac{b}{a}$$

Equating the two results we have that

$$\frac{x}{y} = \frac{b}{a}$$

## similar triangles

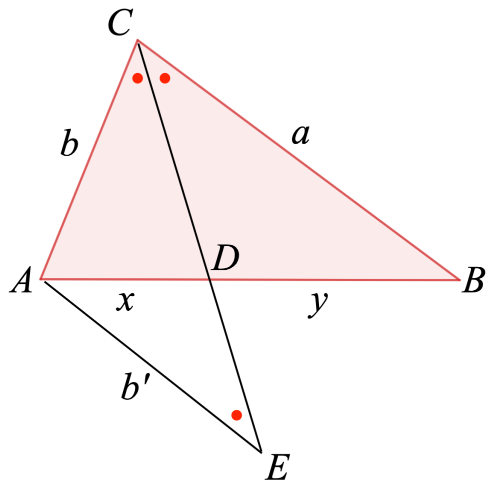
Here are two other proofs of the general angle bisector theorem by similar triangles. These involve constructions that either extend the bisector to a line parallel to one side, or extend one side to meet a line parallel to the bisector.




Given that the angle at $C$ is bisected. The claim is that $a/y = b/x$.

*Proof.*

Draw $AE$ parallel to $BC$ and extend the angle bisector to meet it at $E$. Then the angle at $E$ is equal to the half-angle, by alternate interior angles,




We have that $\triangle ACE$ is isosceles, so $b = b'$.

We also have two similar triangles with $\triangle AED \sim \triangle BCD$, since the angles at $D$ are equal by vertical angles.

Form the ratios of the sides opposite vertical angles to the sides opposite the angles marked with red dots:

$$\frac{a}{y} = \frac{b'}{x}$$

But since $b = b'$:

$$\frac{a}{y} = \frac{b}{x}$$

□

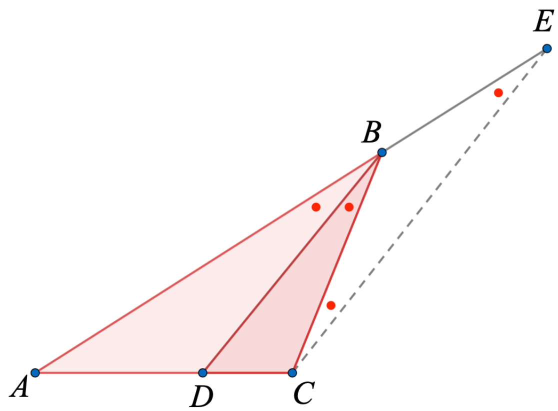
## Yiu proof

Here is another proof presented by Paul Yiu, together with a problem. It includes the bisector of the external angle as well.

The first part uses an extended line parallel to the bisector. (This is basically the same as Euclid VI.3).

In $\triangle ABC$ let $BD$ bisect $\angle ABC$. Then

$$\frac{AB}{AD} = \frac{BC}{CD}$$




*Proof.*

Draw $CE \parallel BD$ and extend $AB$ to meet it at $E$.

$\angle BEC = \angle BCE = \angle ABD = \angle CBD$. The last one is given and the others are by alternate interior angles.

$\triangle ABD \sim \triangle AEC$, since $\angle A$ is shared.

Thus

$$\frac{AB}{AD} = \frac{BE}{CD}$$

But $\triangle BCE$ is isosceles, with $BE = BC$.

It follows that

$$\frac{AB}{AD} = \frac{BC}{CD}$$

□

### converse

*Proof.* (Sketch).

Extend $AB$ so that $BE = BC$.

Follow the same steps as before, but in reverse order.

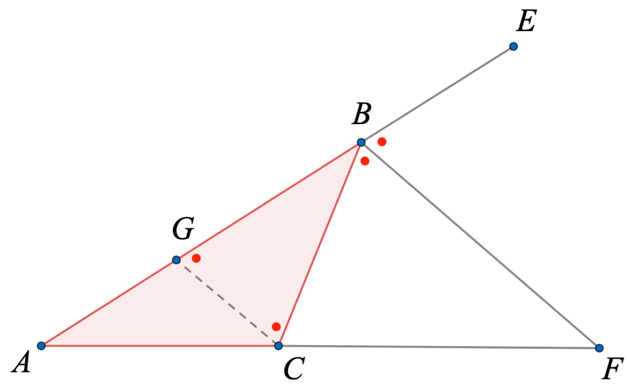
It follows that $\angle ABC$ is bisected by $BD$.

□

## external angle proof

Let $\angle CBE$ be an external angle for $\triangle ABC$, and $BF$ bisect it. Then

$$\frac{AB}{BC} = \frac{AF}{CF}$$




*Proof.*

Draw $GC \parallel BF$ (parallel to the bisector but *internal* to the triangle).

$\angle BGC = \angle BCG = \angle CBF = \angle EBF$ by alternate interior angles.

$\triangle ABF \sim \triangle AGC$.

Thus

$$\frac{AB}{AF} = \frac{BG}{CF}$$

But $\triangle BGC$ is isosceles, with $BG = BC$.

It follows that

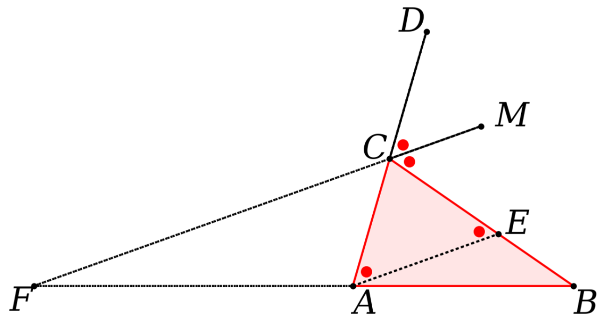
$$\frac{AB}{AF} = \frac{BC}{CF}$$

Rearranging

$$\frac{AB}{BC} = \frac{AF}{CF}$$

□

This approach works even when the bisectors do not reach the opposing side.




The bisector is $MC$, extended on the other side of $C$ to $F$.

$AE$ is drawn parallel to the extended bisector.

We have similar triangles with

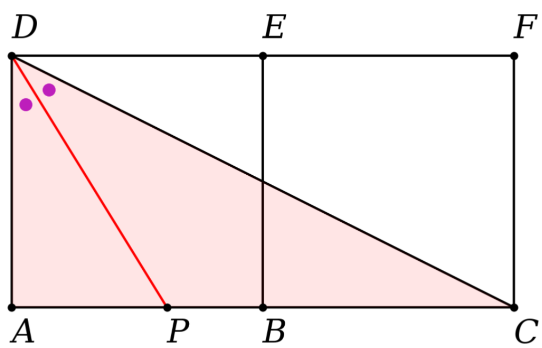
$$\frac{CE}{AF} = \frac{BC}{BF}$$

But from the isosceles triangle, $AC = CE$ so finally

$$\frac{AC}{AF} = \frac{BC}{BF}$$

$$\frac{BC}{AC} = \frac{BF}{AF}$$

Then to the problem: find the ratio $AP/PB$.




Let the squares have side length 1 so the long side of the rectangle is 2.

A simple application of the Pythagorean theorem gives $DC = \sqrt{5}$ and the angle bisector theorem says that $PC$ and $AP$ are in the same ratio as the flanking sides.

Let $AP = x$ and

$$\frac{PC}{AP} = \frac{2-x}{x} = \sqrt{5}$$

$$\frac{1}{x} = \frac{1+\sqrt{5}}{2}$$

The golden ratio! Do not solve for $x$ yet. $PB = 1 - x$ so the inverse of the desired ratio is

$$\frac{PB}{AP} = \frac{1-x}{x} = \frac{1}{x} - 1$$

$$= \frac{1+\sqrt{5}}{2} - 1 = \phi - 1$$

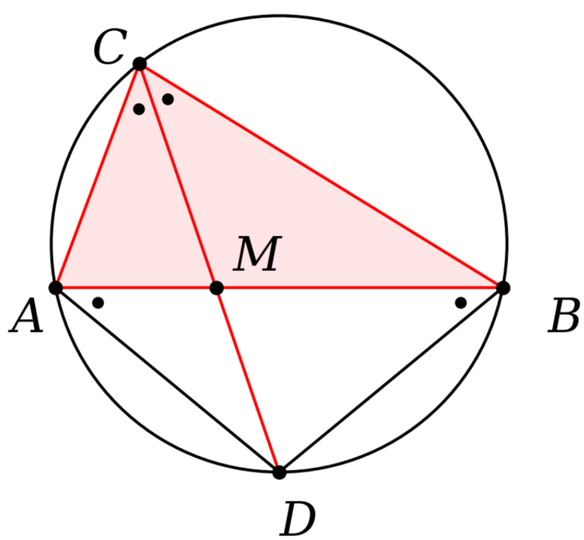
$$= \frac{\sqrt{5}-1}{2}$$

The inverse is

$$\frac{AP}{PB} = \frac{2}{\sqrt{5}-1}$$

## one more proof

Here is one more elegant proof of the basic theorem which depends on drawing the circumcircle.



Draw the bisector of $\angle C$ and extend it to meet the circumcircle at $D$. Inscribed angles gives the equal angles marked with black dots.

We discover similar triangles. $\triangle AMC \sim \triangle DMB$ and $\triangle AMD \sim \triangle CMB$. Also, $\triangle BDM$ is isosceles. The ratios are

$$\frac{AC}{AM} = \frac{BD}{MD} \qquad \frac{BC}{BM} = \frac{AD}{MD}$$

$$BD \cdot \frac{AM}{AC} = MD = AD \cdot \frac{BM}{BC}$$

but $AD = BD$, so

$$\frac{AM}{AC} = \frac{BM}{BC}$$

□

# Chapter 7

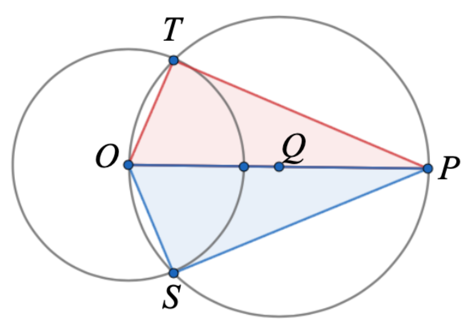
# Excircles and Heron

In this chapter we will look at excircles, which are kind of like incircles, only external to the triangle. These provide us with a nice approach to Heron's formula, which relates the side lengths and *semi-perimeter* $s$ to the area of the triangle as

$$\mathcal{A}^2 = s(s-a)(s-b)(s-c)$$

## tangents and incircles, again

First let's review how to draw the tangents from any point $P$ to a circle on center $O$. Note that $P$ must lie outside the circle. Euclid's method is to draw the line $OP$ and bisect it at $Q$. Then draw the circle on center $Q$ with radius $PQ = OQ$. Find where that circle intersects the first one at $S$ and $T$.

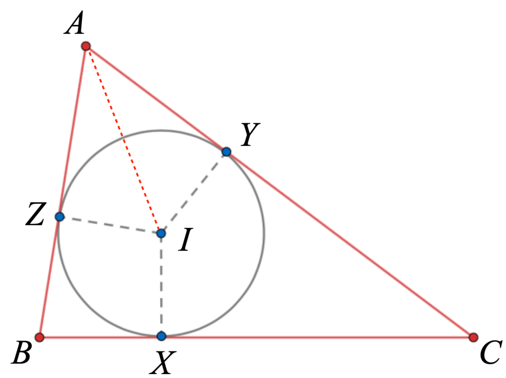



Since $OP$ is a diameter of circle $Q$, by Thales' theorem, $\angle PTO$ is right, so $PT \perp$ the radius $OT$ of circle $O$ and by definition then, $PT$ is a tangent to the circle on center $O$, through $T$. The other intersection at $S$ forms the other tangent $PS \perp OS$.

Since $\triangle OTP$ and $\triangle OSP$ are both right triangles and share the same hypotenuse, as well as bases equal to the radius of circle $O$, the two triangles are congruent by HL. Thus the two tangents from $P$ to circle $O$ are equal in length. In addition, the congruent triangles mean that the line $OP$ is the bisector of $\angle TPS$.

Conversely, if we bisect an angle and then draw perpendiculars from any point on the bisector, those triangles are congruent.

Recall that the *incircle* of a triangle is contained within and just touches (is tangent to) each of the three sides. It can be constructed by bisecting the angle at each vertex, and finding the point $I$ where the bisectors meet. $I$ is then the center of the incircle, and the radius is the length of the perpendicular to any side.




*Proof.* The proof just uses what we established above. Let the triangle be $\triangle ABC$ with side $a$ opposite vertex $A$, etc., as usual, and let $AI$ be the bisector of $\angle A$, $BI$ the bisector of $\angle B$, and so on. Drop perpendiculars from $I$ to the sides at $X$, $Y$ and $Z$.

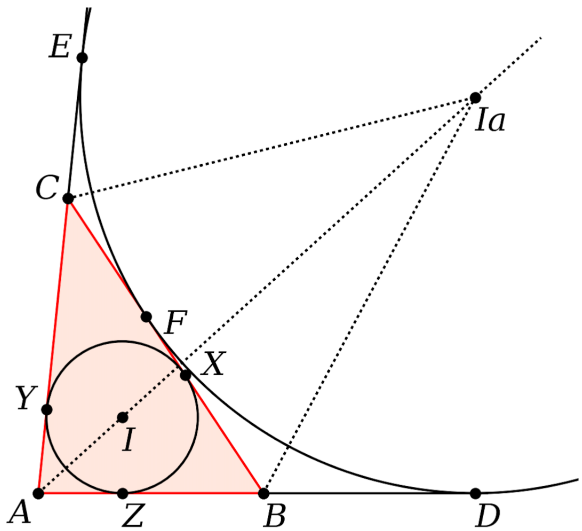
Above we showed that each pair of perpendiculars from an angle bisector forms two congruent right triangles, by hypotenuse-leg in a right triangle (HL), so, for example, $\triangle AIY \cong \triangle AIZ$. Thus, the two perpendiculars from any point on the bisector to the rays of the bisected angle are equal (**here**).

But we can do the same for any pair of sides. Therefore all three distances are equal, and we can then draw the circle, called the incircle, with that distance as the radius $r$. $IX = IY = IZ$.

□

We will often refer to the half-angles formed by bisection (not labeled here), named to remind us of the parent angle: $\alpha + \alpha = \angle A$ and $\angle\alpha = \angle IAZ$.

## excircles



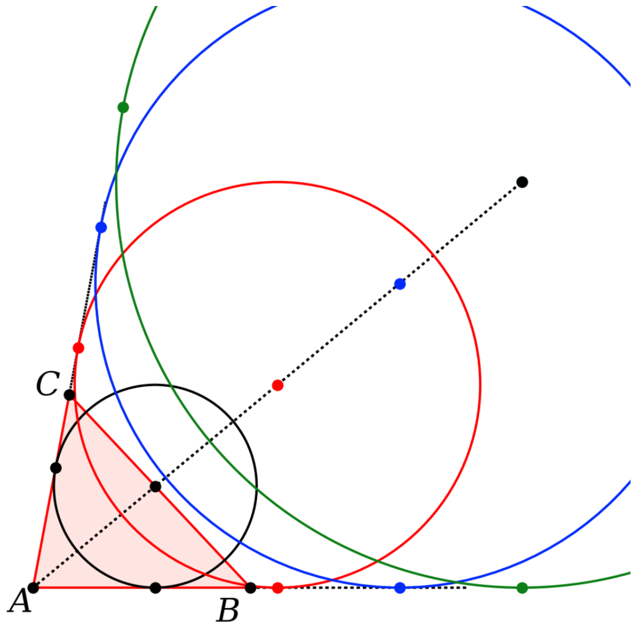

As just discussed, any triangle has an *incircle*, defined as the circle tangent to each of the three sides of the triangle. The incircle is on an *incenter* $I$.

$I$ lies on the angle bisectors of the three angles $A$, $B$ and $C$. The points where the incircle is tangent to the sides are marked $X$, $Y$, and $Z$, so for example $IZ \perp AB$. The circle through $X$,$Y$ and $Z$ is tangent to each of the sides.

Each side of the triangle has a corresponding *excircle*. The excircle on center $I_a$ is the circle tangent to three lines: side $a$ as well as the extensions of sides $AB$ and $AC$ to $D$ and $E$. Since there are infinitely many circles tangent to $AB$ and $AC$ after extension, it is puzzling how to find the correct, unique excircle.

One idea for how to find the relevant points is to notice that $D$ and $E$ also tangents from $A$ to the circle we are looking for. Hence $AD = AE$.

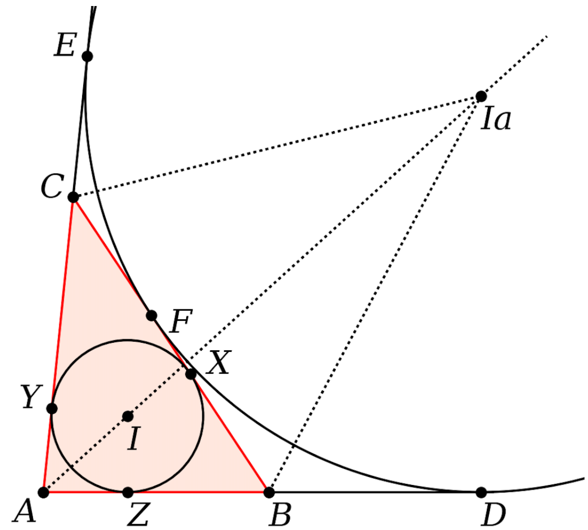
So for a candidate point $D$, find $E$ the same distance from $A$ on the extension of $AC$ and then draw the perpendiculars to cross at $I_a$. It might help a little to know that $BD + CE$ is equal in length to side $a$. If the circle drawn on $I_a$ with radius $r = AD = AE$ is tangent to side $a$, you're done.




In the figure above, a circle tangent to the extensions of $AB$ and $AC$ has been drawn with its center on evenly spaced points along the bisector, starting from where the bisector crosses side $a$. The tangent points as well as the points where the circle itself crosses the bisector, are also evenly spaced. The ratio between these distances depends only on the angle at $A$.

The tangent point we are looking for on $BC$ does *not* lie on the bisector, unless $AB = AC$ (and $\triangle ABC$ is isosceles). Intuitively, there is one position for the center in which the circle just "kisses" side $a$.

More practically, $I_a$ can be found first, as the intersection of the bisectors of the external angles $\angle CBD$ and $\angle ECB$. Then find $D$ and $E$ as points on the extensions of $AB$ and $AC$ whose perpendiculars go through $I_a$. Last, draw the circle of radius $AD = AE$ to find point $F$, tangent to side $a$



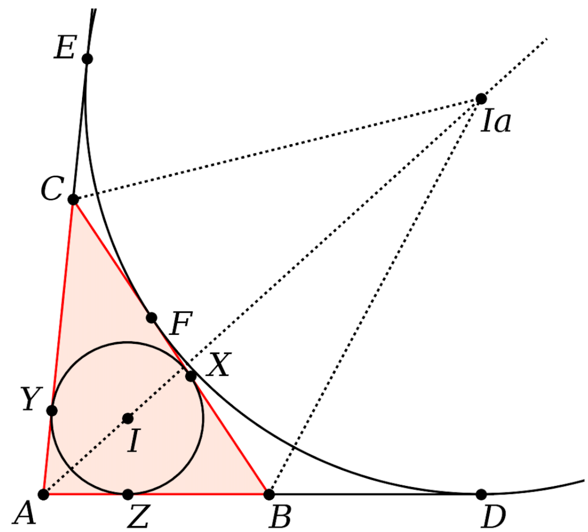

We can see that this will be correct, since $BD = BF$ as tangents from $B$ and $CE = CF$ as tangents from $C$, so the bisectors of $\angle CBF$ and $\angle FCB$ will go through the center of the circle.

In what follows, we will find a formula for the length of $BF$ (and thus, $BD$, $CE$, and $CF$).

## 1

Let

$$AZ = AY = x \qquad BX = BZ = y \qquad CY = CX = z$$




Then let $s$ be half the perimeter, called the *semiperimeter*:

$$2x + 2y + 2z = 2s$$

$$x + y + z = s$$

so

$$s = AZ + BX + CX$$
$$= AZ + a$$

Re-arranging and generalizing

$$AZ = s - a \qquad BX = s - b \qquad CY = s - c$$

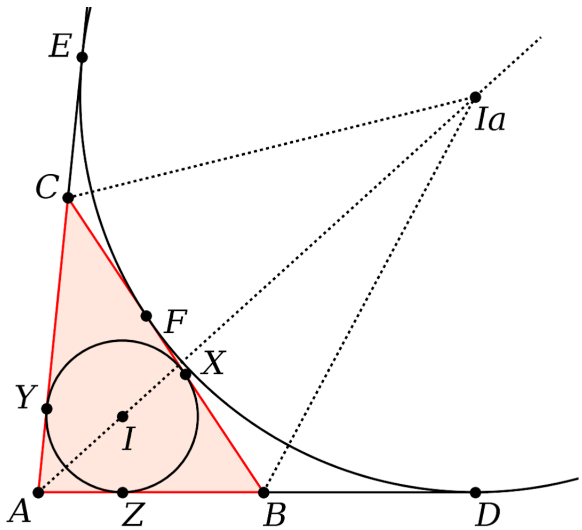
also

$$s - a = (a + b + c)/2 - a$$
$$= (-a + b + c)/2$$
$$2(s - a) = -a + b + c$$

## 2

Let $BD = BF = p$ and $CE = CF = q$. We see that together $p + q = a$ so

$$p = a - q$$




The two tangents from $A$ to the excircle are also equal. We have

$$AD = AE$$

$$(s - a) + (s - b) + p = (s - a) + (s - c) + q$$

$$p - b = q - c$$

Substituting

$$a - q - b = q - c$$

$$2q = a - b + c$$

$$= 2(s - b)$$

$$q = s - b$$

Similarly

$$p - b = a - p - c$$

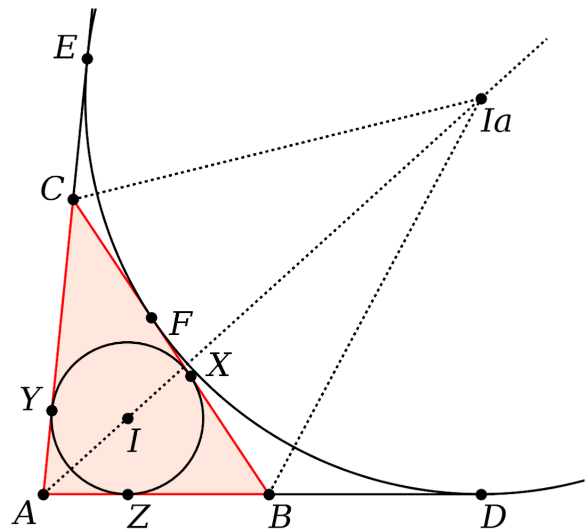
$$2p = a + b - c = 2(s - c)$$

$$p = s - c$$

Again, the two long tangents are equal:

$$AD = s - a + s - b + s - c$$

$$= 3s - 2s = s$$




The total length of the tangents $AD = AE$ is just $s$.

$X$ divides side $a$ into lengths $BX = s - b$ and $CX = s - c$.

Now we see that since (for example) $CE = CF = s - b$, the point $F$ (tangent to the excircle) divides the side $a$ into lengths $BF = s - c$ and $CF = s - b$.

## 3

Comparing the incircle and excircle, we can find two similar right triangles: $\triangle ADI_a$ and $\triangle AZI$. The relevant ratios are

$$\frac{R}{s} = \frac{r}{s - a}$$

$$rs = R(s - a)$$

where $R = I_aD$ is the radius of the excircle on side $a$.

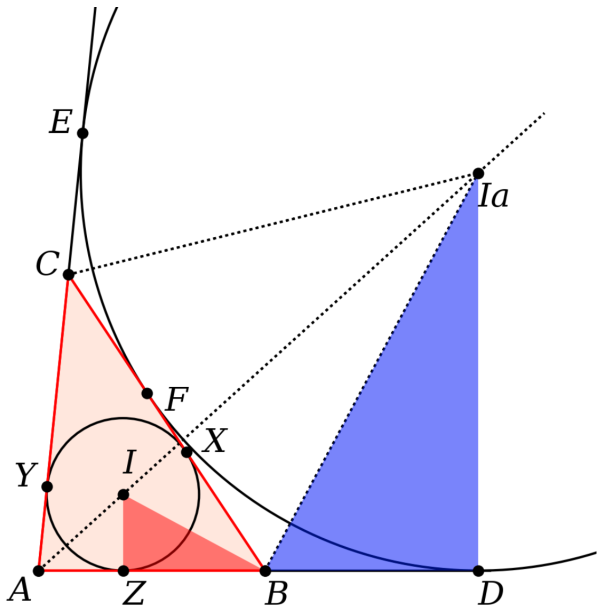
We have three pairs of congruent triangles so the total area of $\triangle ABC$ is

$$\mathcal{A} = rx + ry + rz = rs = R(s - a)$$

We might have used a subscript for $R$ such as $R_a$ so then

$$\mathcal{A} = R_a(s-a) = R_b(s-b) = R_c(s-c)$$

## 4




It is a property of the internal and external angle bisectors that the sum of the half-angles is a right angle, since the internal and external angles are in total two right angles.

So $\angle IBI_a$ is right. It follows that $\angle IBZ$ and $\angle I_aBD$ are complementary, and we know that they are both contained in right triangles.

Thus $\triangle IZB \sim \triangle BDI_a$ (marked red and blue in the figure). The relevant ratios are:

$$\frac{R}{s-c} = \frac{s-b}{r}$$

$$Rr = (s-b)(s-c)$$

## 5

We combine the last result from above with that from (3):

$$rs = R(s-a)$$

Multiplying the two equalities:

$$r^2 Rs = R(s-a)(s-b)(s-c)$$

$$r^2 s = (s-a)(s-b)(s-c)$$
$$r^2 s^2 = s(s-a)(s-b)(s-c)$$

Since $rs = \mathcal{A}$ we have, finally

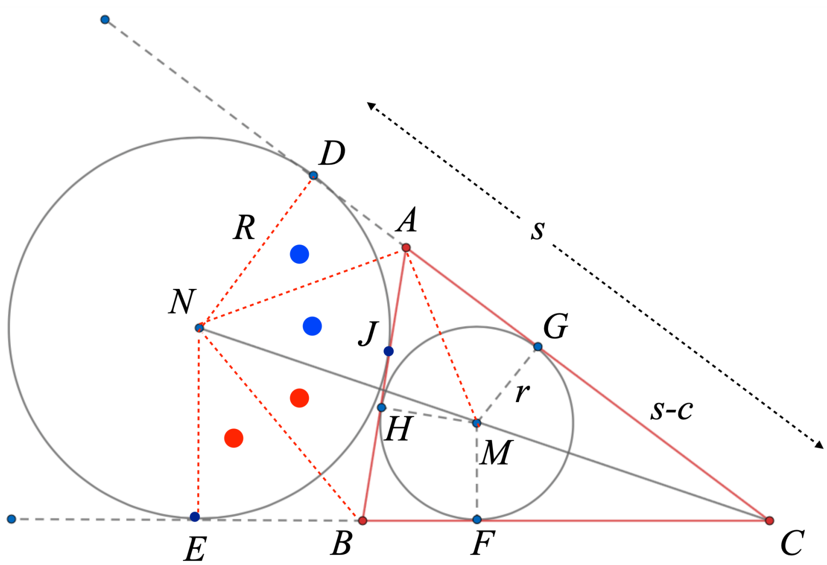
$$\mathcal{A}^2 = s(s-a)(s-b)(s-c)$$

□

This is Heron's theorem. The square of the area of the triangle is equal to the product on the right. As expected, the formula is symmetric in $a$, $b$ and $c$ and has the dimensions of the fourth power of a length.

The formula can also be obtained by computing areas in different ways. Here is an idea of the proof. (The notation in this figure is a bit different than we've been using). We leave it as an exercise to apply the idea to the diagrams from above.

*Proof.*



Using parentheses to signify area, we have that the polygon $ADNEB$ consists of two pairs of equal triangles, since $AN$ and $BN$ are bisectors, and there are two pairs of tangents.

Its area is twice that of $\triangle ANB$:

$$(ADNEB) = 2(\triangle ANB) = R_c \cdot c$$

The area of quadrilateral $CDNE$ is twice that of $\triangle CDN$:

$$(CDNE) = 2(\triangle CDN) = R_c \cdot s$$

The area of $\triangle ABC$ is simply the difference:

$$(\triangle ABC) = (CDNE) - (ADNEB)$$

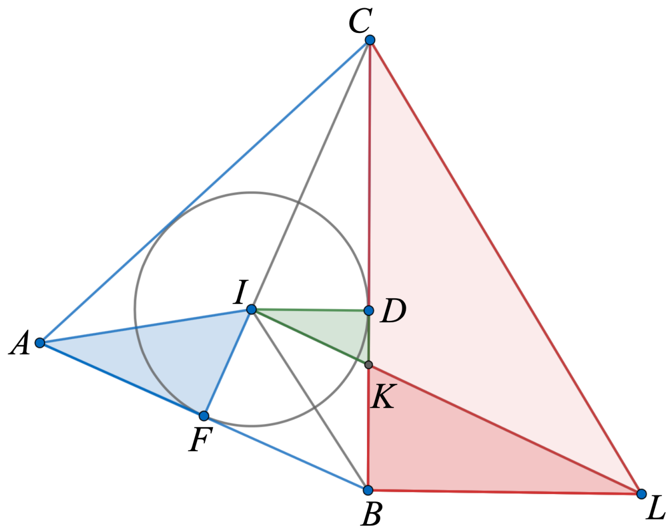
$$= R_c s - R_c c = R_c(s - c)$$

This is the same relationship we might obtain by determining the total length of the tangent $CD = s$ and then noting similar triangles $\triangle ADN \sim \triangle MGA$.

## Heron's proof

Lastly, we go through Heron's proof of the eponymous theorem. I obtained this from a web page written by Dr. Paul Yiu, which has disappeared. There is also a complete discussion in Dunham.

Let us start with a sketch of the proof.



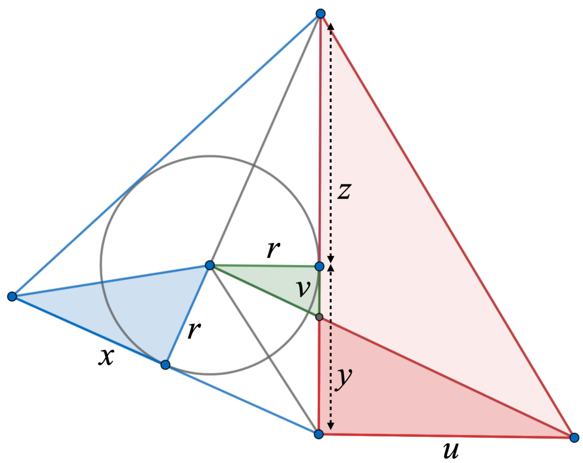

We have $\triangle ABC$ with its incircle on center $I$ and perpendiculars drawn to the tangent points $D$ and $F$ (note a slight change in notation from $X$ and $Z$).

Extend a line from $B$ perpendicular to $CB$, forming a right angle $\angle CBL$. Also extend a line from $I$ perpendicular to $CI$, forming a right angle $\angle CIL$. The two lines meet at $L$.

We we will be able to find two pairs of similar triangles: easily, the dark red $\triangle KBL \sim \triangle KDI$, and in a more complicated fashion, the whole red $\triangle LBC \sim \triangle IFA$, in blue. We first proceed assuming that these similarity relationships have been demonstrated.

I find it much more convenient to write the proof using single letters for the lengths, so these have been labeled as shown below.

The similar triangles give us two relationships:




$$\frac{u}{r} = \frac{y-v}{v}$$
$$\frac{u}{r} = \frac{y+z}{x}$$

Setting equals to equals:

$$\frac{y-v}{v} = \frac{y+z}{x}$$

After that it is just a matter of algebra. That is harder for Heron (actually the proof is probably from Archimedes), but relatively easy for us with our improved notation.

We simply add 1 to both sides. This is the step where Heron needs an extension $BT = x$ since $x + y + z$ must equal a straight line segment in his diagram (see Dunham's chapter on this).

$$\frac{y-v}{v} + \frac{v}{v} = \frac{y+z}{x} + \frac{x}{x}$$
$$\frac{y}{v} = \frac{x+y+z}{x}$$

The ratio of the whole semi-perimeter to $x$ ($AF$) is equal to the ratio of $BD$ to $KD$.

$$\frac{y}{v} = \frac{s}{x}$$
$$xy = vs$$

The last step is to involve $z$, and also somehow eliminate $v$.

We notice that in the right $\triangle CIK$, the radius of the incircle $r$ divides the hypotenuse into two lengths $z$ and $v$. By the standard proof of the geometric mean, we have

$$r^2 = vz$$

so

$$\frac{r^2}{z} = v = \frac{xy}{s}$$
$$r^2 s = xyz$$
$$(rs)^2 = xyzs$$

This is the famous formula in a simple form.

We proceed more formally to establish the similarity relations.

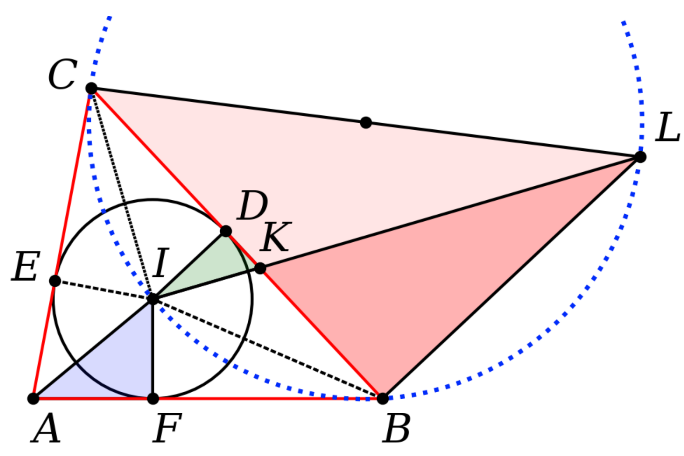
The triangle is $\triangle ABC$ with incircle radii $ID$, $IE$ and $IF$. The tangents are $AF = x, BD = y$, and $CD = z$.

There are two parts to the construction. Most important, $\angle CIL$ and $\angle CBL$ are both drawn as right angles.

*Proof.*

**part 1**: $\triangle CBL \sim \triangle AIF$.

By construction, $\triangle CBL$ and $\triangle CIL$ are both right triangles with the same hypotenuse $CL$.



By the converse of Thales' theorem, $B$ and $I$ lie on the same circle, with diameter $CL$. It follows that $BICL$ is a cyclic quadrilateral and therefore $\angle L$ is supplementary to $\angle BIC$.

But the latter is supplementary to $\beta + \gamma$ in $\triangle BIC$ (using our standard notation for the half-angles at $B$ and $C$). Hence $\angle L = \beta + \gamma$.

This is much easier to see if we draw the circle on diameter $CL$: the arc corresponding to $\angle L$ is divided between $\beta$ and $\gamma$.

We also have that the sum of the half-angles $\alpha + \beta + \gamma = 90$. Thus, $\angle AIF$, which is complementary to $\alpha$ in the right triangle $\triangle AIF$, is also equal to $\beta + \gamma$.

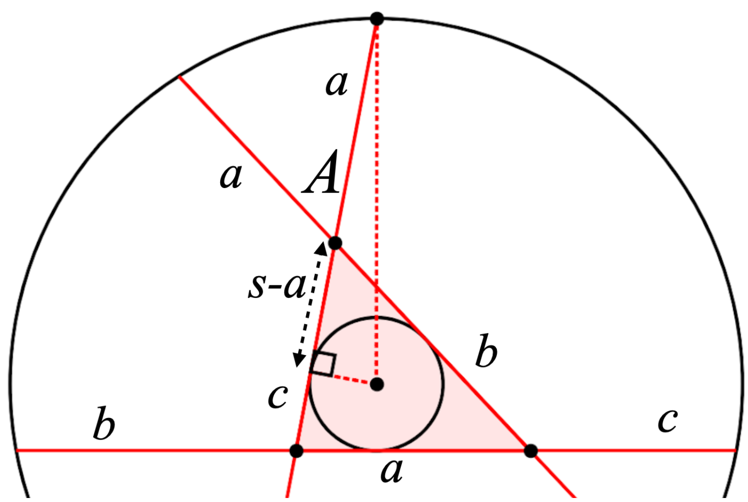
It follows that $\angle L = \angle AIF$, and then the two right triangles are similar: $\triangle CBL \sim \triangle AIF$.

**part 2** $\triangle LBK \sim \triangle IDK$

This is simple. They are both right triangles and share vertical angles at $K$.

$\square$

## One last construct




Extend the sides of $\triangle ABC$ a distance $a$ past vertex $A$ and so on.

Then one can draw a circle whose center is also the incenter of the triangle, that passes through the ends of all of the extended line segments.

*Proof.*

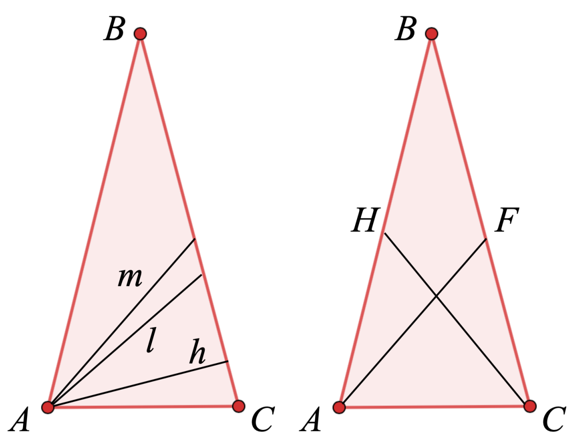
Draw the right triangle with one side equal to $r$, the radius of the incircle, and the other side extending part of side $c$ through vertex $A$, as shown. The second side has length $s - a + a = s$.

Therefore the hypotenuse of every such right triangle has the same length, and this is the radius of the large circle. The length of each chord is $2s$, the perimeter of $\triangle ABC$.

$\square$

# Chapter 8

# More isosceles




The following statements about isosceles triangles are equivalent:

1. two sides equal
2. base angles equal
3. medians ($m$) equal
4. angle bisectors ($l$) equal
5. altitudes ($h$) equal
6. the bisector, median and altitude at $b$ coincide

where in (1-5) the medians, bisectors and altitudes in question are those extending to the equal sides.

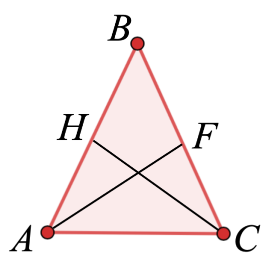
The figure shows (right panel) the case for the medians, but we will use the same labels for the other cases in the discussion below.

The question is then how to prove these one from the other, for example 1 $\Rightarrow$ 3 or 5 $\Rightarrow$ 2. Of course, we already have 1 $\Rightarrow$ 2 and 2 $\Rightarrow$ 1.

In what we might call the forward direction { 1,2 } $\Rightarrow$ { 3,4,5,6 }, these proofs are quite easy. I encourage you to try them before reading further.

The converse proofs are another matter.

## forward




1 $\Rightarrow$ 3

*Proof.* $AF$ and $CH$ are medians in $\triangle ABC$. Given $AB = CB$ and $\angle A = \angle C$, it follows that $AH = CF$ (by the definition of median) and thus by SAS we have $\triangle ACH \cong \triangle CAF$. Hence $AF = CH$. $\square$

1 $\Rightarrow$ 4

*Proof.* $AF$ and $CH$ are angle bisectors in $\triangle ABC$. Given $\angle A = \angle C$, the half-angles are also equal, so by ASA we have $\triangle ACH \cong \triangle CAF$. Hence $AF = CH$. $\square$

1 $\Rightarrow$ 5

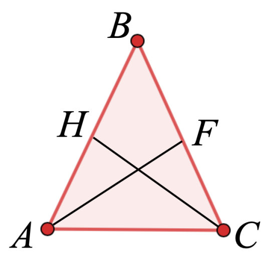
*Proof.* $AF$ and $CH$ are altitudes in $\triangle ABC$. Twice the area of $\triangle ABC$ may be computed as $AF \cdot CB = CH \cdot AB$. But $AB = CB$. Hence $AF = CH$. $\square$

## converse

One of the converse proofs is also easy.

$5 \Rightarrow 1$

*Proof.* $AF$ and $CH$ are altitudes in $\triangle ABC$. Twice the area of $\triangle ABC$ may be computed as $AF \cdot CB = CH \cdot AB$. But $AF = CH$. Hence $AB = CB$. $\square$




$4 \Rightarrow 1$

If the angle bisectors are equal, then the triangle is isosceles. This is a famous theorem, and there is a whole chapter about it here: **Steiner-Lehmus theorem**.

$3 \Rightarrow 1$

For this one, we look ahead to the **Law of cosines**.

*Proof.*

We have that the medians are equal, namely, $AF = CH$. Let them be equal to $m$. Using the law of cosines, compute the length $m$ squared of the side opposite $\angle B$ in two different triangles.

Let side $a = BC$ (the side opposite $\angle A$), and similarly side $c = AB$ (the side opposite $\angle C$).

So $BF = a/2$ and $BH = c/2$. Then

$$m^2 = a^2 + (c/2)^2 - 2a \cdot c/2 \cdot \cos B$$

$$m^2 = c^2 + (a/2)^2 - 2c \cdot a/2 \cdot \cos B$$

The last terms on the right-hand side are equal, namely

$$ac \cdot \cos B = ac \cdot \cos B$$

Setting the two expressions equal, it follows that

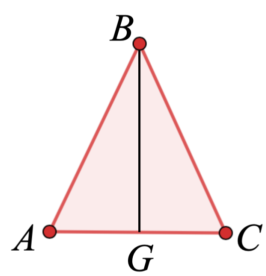
$$a^2 + (c/2)^2 = c^2 + (a/2)^2$$

$$\frac{3}{4}a^2 = \frac{3}{4}c^2$$
$$a^2 = c^2$$

Since $a$ and $c$ are both lengths, we may take the positive square roots for each, and thus $a = c$.

□

## median, bisector and altitude at $b$




Let $b$ be the side opposite $\angle B$ as usual.

**1 ⇒ 6**

We have that $AB = CB$ and $\angle A = \angle C$.

*Proof.* Let $BG$ be the median to $AC$ such that $AG = GC$ Then $\triangle ABG \cong \triangle CBG$ by SSS. Thus $\angle B$ is bisected and $\angle AGB = \angle CGB$ and both are right angles. □

*Proof.* Let $BG$ be the altitude to $AC$ such that $\angle AGB = \angle CGB$ and both are right angles. Then $\triangle ABG \cong \triangle CBG$ by HL. Thus $\angle B$ is bisected and $AG = GC$. □

*Proof.* Let $BG$ bisect $\angle B$. Then $\triangle ABG \cong \triangle CBG$ by SAS. Thus $\angle AGB = \angle CGB$ and both are right angles, and $AG = GC$. □

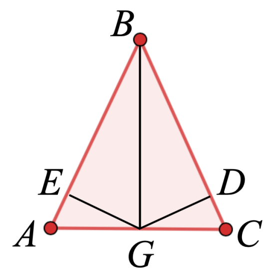
**6 ⇒ 1**

We will prove that if any two of these statements are both true, then $a = c$. In what follows, if we have $\triangle ABG \cong \triangle CBG$, then (1) and (2) follow immediately.

*Proof.* If the bisector and the altitude at $b$ coincide, then $\triangle ABG \cong \triangle CBG$ by ASA.
□

*Proof.* If the median and the altitude at $b$ coincide: then $\triangle ABG \cong \triangle CBG$ by SAS.$\square$

Finally, suppose the bisector and the median at $b$ coincide. Draw perpendiculars from the midpoint of $b$ to each of sides $a$ and $c$.




*Proof.*

$BG$ bisects $\angle B$, $\angle GDB = \angle GEB$ and both are right angles, therefore $\angle BGE = \angle BGD$ by sum of angles. The hypotenuse $BG$ is shared. It follows that $\triangle BEG \cong \triangle BDG$ by ASA. Thus $GE = GD$.

Then (since $AG = GC$), $\triangle AEG \cong \triangle CDG$ by HL. It follows that $\angle A = \angle C$ and then by $2 \Rightarrow 1$, we have $a = c$.

$\square$

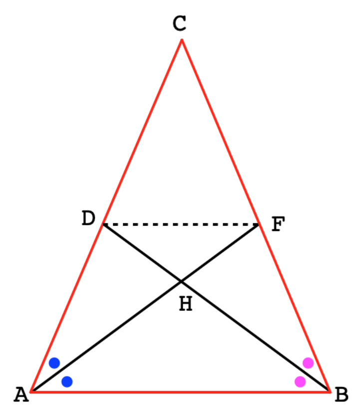
This material is covered in Byer (see **References**).

# Chapter 9

# Steiner Lehmus

This chapter discusses a theorem about angle bisectors in an isosceles triangle. We called the forward version the **isosceles bisector** theorem.

It is easy in the forward direction, but the converse is very challenging, at least until you draw the right diagram. Then, as usual, it's not so bad.




*Proof.* We are given that $\triangle ABC$ is isosceles ($AC = BC$), and also that the angles at the base are both bisected. It follows that the half-angles are also equal, and thus $\triangle CDB \cong \triangle CFA$ by ASA. So the angle bisectors are equal in length: $AF = BD$.

□

That's the easy part.

## Steiner-Lehmus Theorem

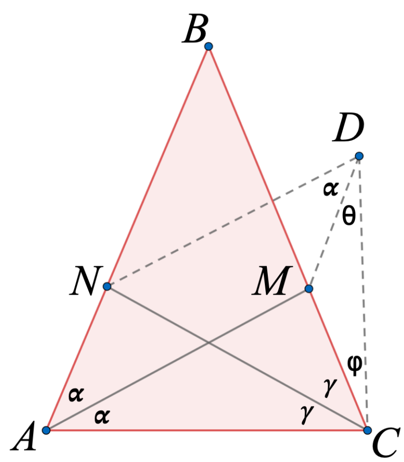
The converse theorem says that if we have angle bisectors and they are equal in length, then the triangle is isosceles.

https://en.wikipedia.org/wiki/Steiner\OT1\textendashLehmus_theorem

The problem is that, even though we can draw triangles with two sides equal, we don't know anything about the angles except for some vertical angles, which don't help.

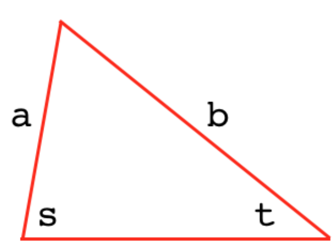
Here is an approach which I found on the web. It's a proof by contradiction.

https://www.algebra.com/algebra/homework/word/geometry/Angle-bisectors-in-an-isosceles-triangle.lesson




We claim that if $AM = CN$ and the angles are bisected, then $\alpha = \gamma$.

We rely on Euclid's propositions I.18 and I.19. In any triangle if one side is larger than another, then the angle opposite the longer side is greater (I.18) and conversely, if one angle is larger than another, then the side opposite is greater (I.19).



In the diagram above

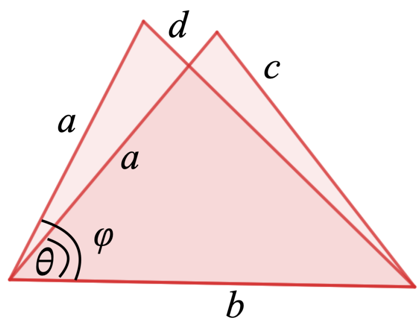
$$s > t \Rightarrow b > a$$

$$b > a \Rightarrow s > t$$

We proved both these theorems **here**.

The problem is that these are *within-triangle* results. We also need the following result.

If two triangles $\triangle ABC$ and $\triangle DEF$ have two pairs of sides equal, and the included angle is greater in one ($\phi > \theta$), then the side opposite $\phi$ also greater.




*Proof.*

Let $d$ be opposite $\phi$ and $c$ be opposite $\theta$. Use the law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$d^2 = a^2 + b^2 - 2ab\cos\phi$$

Then if $d > c$, so $d^2 > c^2$, and

$$a^2 + b^2 - 2ab\cos\phi > a^2 + b^2 - 2ab\cos\theta$$

$$-2ab\cos\phi > -2ab\cos\theta$$

$$\cos\phi < \cos\theta$$

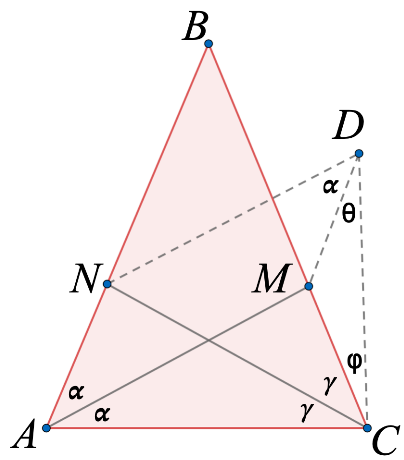
$$\phi > \theta$$

This chain of reasoning works just as well in reverse. So, $\phi > \theta \Rightarrow d^2 > c^2$, and then $d > c$. $\square$

This is known as the **hinge theorem**. We showed a different proof earlier. It is Euclid I.24.

□

Back to our problem. We claim that $\alpha = \gamma$ and the triangle is isosceles. We argue by contradiction.




*Proof.*

In $\triangle ABC$, let the base angles be bisected as shown.

Let the bisectors be equal: $AM = CN$.

Draw $ND \parallel AM$ and $MD \parallel ANB$.

So $ANDM$ is a parallelogram.

Thus $\angle NDM = \alpha$.

Aiming for a contradiction, suppose $\gamma > \alpha$.

By I.24, $AN > CM$.

So $DM > CM$.

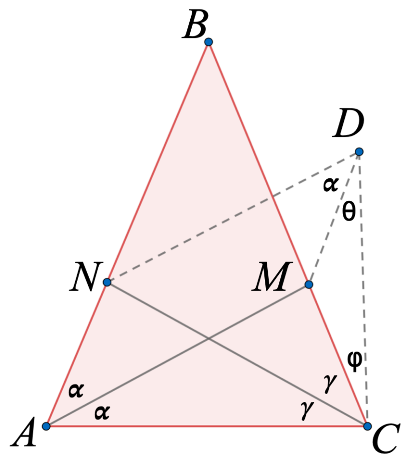
By I.18 $\phi > \theta$.

By addition of inequalities:

$$\gamma + \phi > \alpha + \theta$$

By I.19, $ND > CN$.

But $AM = ND$ so $AM > CN$.




This is a contradiction, we were given that $AM = CN$.

$\square$

Therefore, it cannot be that $\gamma > \alpha$.

The reverse supposition, that $\alpha < \gamma$, also leads to a contradiction by a symmetrical argument, substituting $<$ for $>$.

(Or draw the parallelogram on the other side of $\triangle ABC$ and use the same argument as previously).

Since $\alpha$ is neither greater than nor less than $\gamma$, we conclude that $\alpha = \gamma$. $\triangle ABC$ is therefore isosceles by I.6.

$\square$.

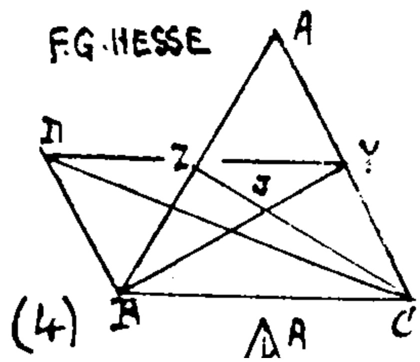
According to the internet, the Steiner-Lehmus theorem is famous for being difficult, for having many different proofs, and for some controversy over whether even one of the proofs is *direct* or not. By direct we mean, not using the technique of proof by contradiction or *reductio ad absurdum*.

I was lucky to find a (non-paywalled) review published on its centenary in 1942.

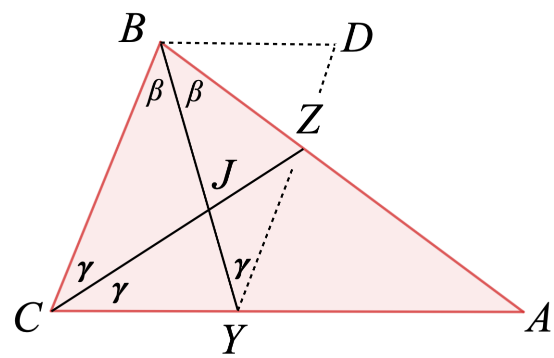
https://www.cambridge.org/core/services/aop-cambridge-core/content/view/7B625B08567935CAE06A0AC9430477C0/S0950184300000021a.pdf

It includes this proof.

## Hesse (1842)




The construction is to draw $YD = BC$ and $BD = BZ$. Later we will find that $CD$ divides $BDYC$ into two congruent triangles. Then it easily follows that $\beta = \gamma$.




*Proof.*

The equalities of the construction are $YD = BC$ and $BD = BZ$, and $BY = CZ$ is given.

(1) $\triangle DYB \cong \triangle BCZ$ by SSS.

Thus the corresponding angles of $\triangle DBY$ and $\triangle BZC$ are equal, namely:

$$\angle DYB = \angle BCZ = \gamma$$

$$\angle BDY = \angle ZBC = 2\beta$$

$$\angle DBY = \angle BZC = \angle A + \gamma$$

the last by sum of angles.

At this point I had some trouble with the details of the proof, so we may diverge from the source.

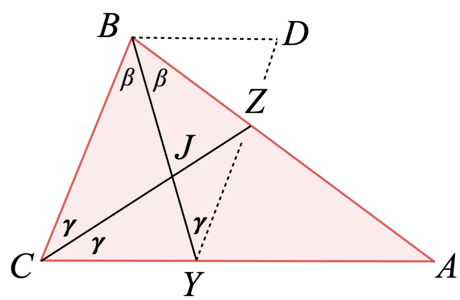
(2) $\angle BDY = 2\beta$

so by sum of angles in $\triangle DBY$ we have:

$$\angle ZBD = A - \beta + \gamma$$

Then

$$\angle CBD = A + \beta + \gamma = 90 + \alpha$$




Now all we need is to get the measure of $\angle CYD$

(3) By sum of angles $\angle BJC = \angle A + \beta + \gamma$ so by the external angle theorem:

$$\angle BYC = \angle A + \beta$$

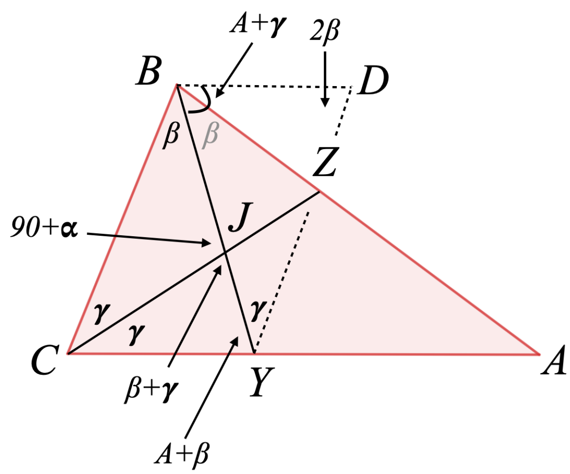
Then

$$\angle CYD = \angle A + \beta + \gamma$$

We have established that

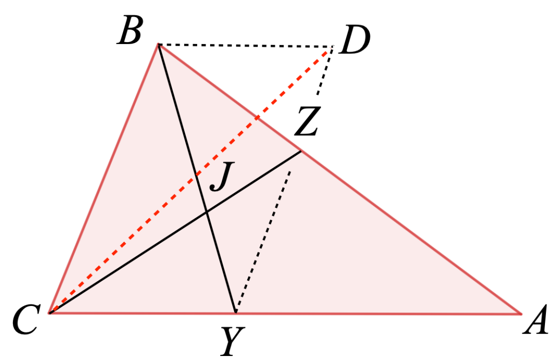
$$\angle CYD = \angle CBD$$

Crucially, they are not only equal but obtuse.




We don't necessarily have a parallelogram yet because we have only one pair of sides equal and one pair of opposing angles equal.

But we can draw $CD$.




(4) Comparing $\triangle CBD$ and $\triangle DYC$, we have that $CD$ is shared, and $DY = BC$ by construction. $\angle CYD = \angle CBD$.

We have SSA and in addition, the angles that we know in each, $\angle CYD$ and $\angle CBD$, are obtuse. It follows that $\triangle CBD \cong \triangle DYC$.

At this point we could just invoke the converse of the diagonal theorem for quadrilaterals.

(5) Instead Hesse says:

$$YC = BD = BZ$$

SSS then gives:

$$\triangle ZCB \cong \triangle YBC$$

$$\angle CBZ = \angle BCY$$

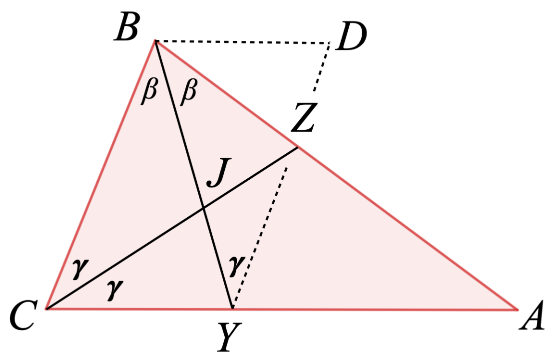
With base angles equal, all we need is I.6.

□

For more about SSA see **here**.

With the same diagram, I cooked up a proof by careful bookkeeping with the angles.

*Proof.* (Alternate).




Given $\angle B$ and $\angle C$ bisected into $2\beta$ and $2\gamma$.

Given $BY = CZ$.

Draw $BD = BZ$ and $DY = BC$. (Again, we do not claim $DZY$ collinear).

$\triangle BYD \cong \triangle ZCB$ by SSS.

So $\angle BDY = \angle ZBC = 2\beta$ and $\angle BYD = \angle ZCB = \gamma$.

At $J$, one vertical angle ($\angle BJC$) is $\angle A + \beta + \gamma$ by sum of angles.

$\angle BJC$ is external to $\triangle JCY$, hence $\angle CYJ = \angle A + \beta$.

Up to now, nothing has changed. Let $\angle DBZ = \theta$.

Summing angles in the quadrilateral $BDYC$:

$$\theta + 2\beta + 2\gamma + \angle A + \beta + \gamma + 2\beta = 360$$

$$\theta + \angle A + 5\beta + 3\gamma = 360$$

Since $\angle A + 2\beta + 2\gamma = 180$, subtracting

$$\theta + 3\beta + 1\gamma = 180$$

We also have that $\triangle BDZ$ is isosceles:

$$\theta + 4\beta = 180$$

Subtract again:

$$-\beta + \gamma = 0$$

$$\beta = \gamma$$

By I.6, $\triangle ABC$ is isosceles.

□

## afterward

There is some interesting discussion in Coxeter as well. According to what I can find on the web, most of the literature concerns the question of whether it is possible to provide a direct proof of the theorem. The algebraic proof, postponed for now to book II, has been cited as such.

However, that proof depends on Stewart's Theorem, which as we derive it there depends on the Law of Cosines, which depends in turn on the theorem of Pythagoras. And although there are several hundred proofs of Pythagoras most (all?) of them depend on the sum of angles and also on the parallel postulate, which explicitly depends on a proof by contradiction.

The question of a direct proof for Steiner-Lehmus is hard to answer conclusively. I have a write-up from John Conway claiming that it is impossible, but I don't really understand his argument. Unfortunately, nearly all the writing in mathematics journals is paywalled and very expensive.

# Part III

# Trigonometry

# Chapter 10

# Basic trigonometry

Here we take a look at fundamental ideas from trigonometry and begin to show why they are useful. Let's start by reviewing some previous work.

## similarity

A simple rectangular construction shows that right triangles with the same two complementary angles (*similar* right triangles), have equal ratios of sides.

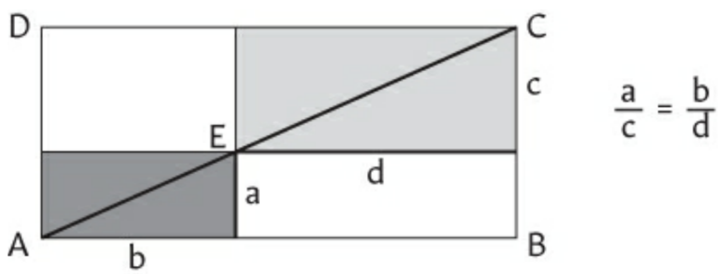



**Fig. 42** Area and similarity.

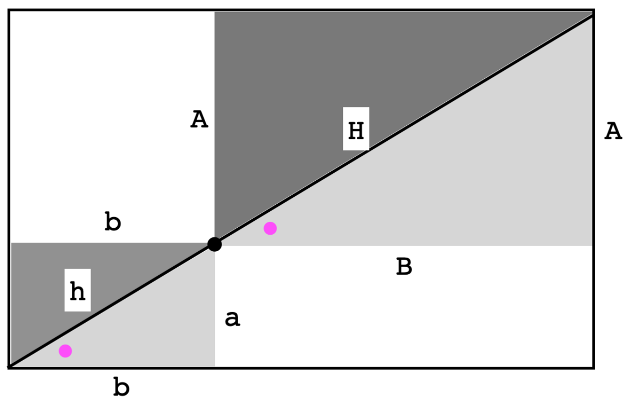
The proof is to use three sets of congruent triangles to show that the white areas are equal to each other which means that

$$bc = ad$$

Two simple rearrangements give different equalities

$$\frac{a}{b} = \frac{c}{d}, \qquad \frac{a}{c} = \frac{b}{d}$$

I want to switch notation for a second, to try to make a point. In the figure below, the two similar right triangles have sides $a, b$ and $h$, and then $A, B$ and $H$.




The equality is

$$Ab = aB$$

which can be rearranged to

$$\frac{a}{b} = \frac{A}{B}, \qquad \frac{a}{A} = \frac{b}{B}$$

The first one equates ratios of sides within a triangle, $a$ and $b$ from the first triangle, $A$ and $B$ from the second. The other one equates ratios of *corresponding* sides in different triangles. The sides of a triangle can be ordered by length, where one triangle has sides $a < b < h$ and a similar triangle has sides $A < B < H$.

Let $k$ be the constant of proportionality between triangles, with $A = ka$. Then

$$\frac{a}{b} = \frac{A}{B} = \frac{ka}{B}$$

$$B = \frac{ka}{a} \cdot b = kb$$

So our ratios imply that

$$\frac{A}{a} = k = \frac{B}{b}$$

The factor $k$ is a scaling factor which says how much bigger the second triangle is than the first.

This result is easily extended to the hypotenuse.

One way is to use the Pythagorean theorem. Let

$$a^2 + b^2 = h^2$$

and

$$A^2 + B^2 = H^2$$

But $A = ka$ and $B = kb$ so

$$\begin{aligned} H^2 &= A^2 + B^2 \\ &= (ka)^2 + (kb)^2 \\ &= k^2(a^2 + b^2) = k^2h^2 \\ H &= kh \end{aligned}$$

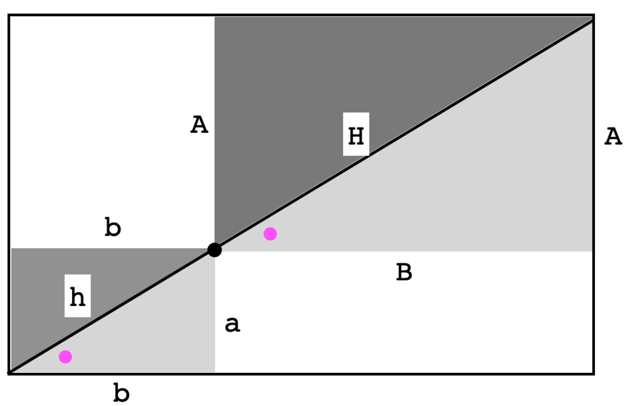
Hence

$$k = \frac{A}{a} = \frac{B}{b} = \frac{H}{h}$$

Now change focus back to comparisons within each triangle, namely

$$\frac{a}{h} = \frac{A}{H}$$

Consider the smaller angle, marked $\angle CAB$ in the original figure, and labeled by a magenta dot below



This angle is flanked by the hypotenuse $h$ and the side $b$. We will call $a$ the side *opposite* to $\angle CAB$ and $b$ the side *adjacent*.

Then the previous equation
$$\frac{a}{h} = \frac{A}{H}$$
says that the ratio of the opposite side to the hypotenuse is somehow independent of the scale of the triangle. It is the same for both.

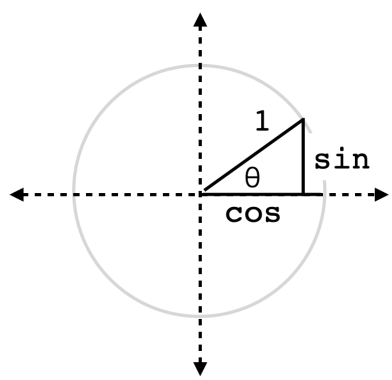
A similar thing happens with the *adjacent* sides $b$ and $B$. Their ratios to the hypotenuse are also equal.
$$\frac{b}{h} = \frac{B}{H}$$
a result which is easily obtained by manipulating what we had above.

These ratios between sides in right triangles with the same angles are somehow characteristic of the angles and independent of the side lengths.

We call the ratio opposite/hypotenuse the *sine* of the angle and the ratio adjacent/hypotenuse the *cosine* and write them like this

$$\sin \angle CAB = \frac{a}{h}, \qquad \cos \angle CAB = \frac{b}{h}$$

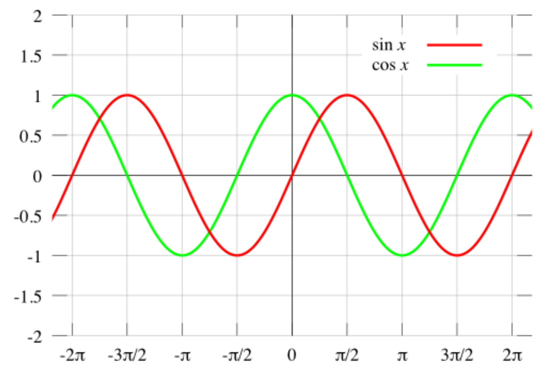
Often, we are working with central angles drawn in a unit circle, with radius 1.




Then the length of the side opposite the central angle is exactly the sine of the angle (since the hypotenuse has length 1), while the side adjacent is the cosine. This is the easiest picture to remember.

From a geometrical perspective, the importance of sine and cosine is that they incorporate the knowledge that these ratios are independent of the size of the triangle. You can think of them as ratios of sides to the hypotenuse for a central angle in a unit circle.

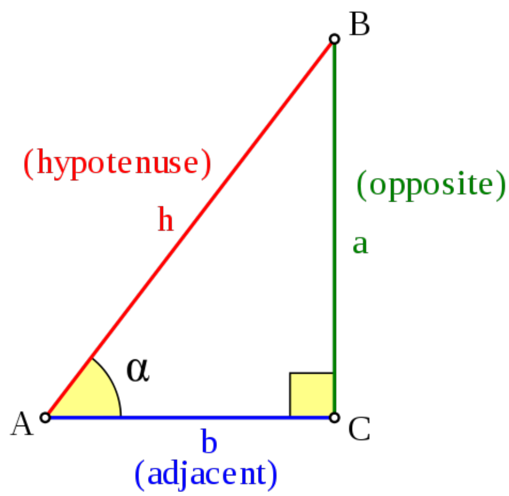
However, they are far more than that. In analytic geometry, we will see sine and cosine as *functions* of the angle $\theta$. The curve traced is a periodic one, which repeats as $\theta$ goes all the way around the circle. This becomes central in calculus, where we will deal often with periodic phenomena modeled using the sine and cosine functions.




That's getting ahead of ourselves.

## sine and cosine

To repeat:




In a right triangle, for angle $\alpha$, the sine of the angle is defined as the ratio between the side opposite and the hypotenuse. In the drawing

$$\sin\alpha = \frac{a}{h}$$

while the cosine is

$$\cos\alpha = \frac{b}{h}$$

From our work earlier, we know that any similar right triangle (with the same angles) has its sides in the same ratios. The sine and cosine are functions of the angle, but are independent of the size of the triangle.

Frequently, the hypotenuse is scaled to be equal to 1, so then the opposite side is equal to the sine, and the adjacent side to the cosine.

Switching our focus from $\alpha$ (the angle at $A$) to the angle at vertex $B$, swaps opposite to adjacent and vice-versa. This means that the cosine of an angle is the sine of its complementary angle, and vice-versa.

## radian measure

In this book, we will make an effort not to use degrees for angular measure. They aren't seen much after elementary school. The Greeks thought in terms of one or two right angles, or four right angles for an entire circle.

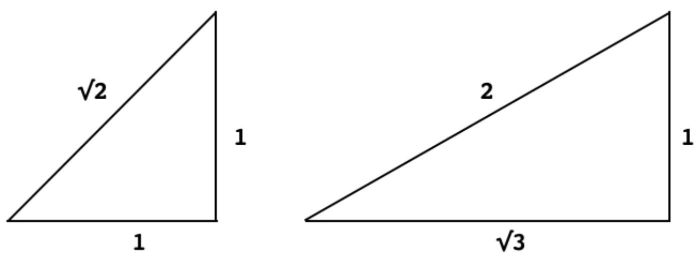
In more advanced mathematics we invariably use radians. Angles are measured in terms of the arc of the unit circle that they are *subtended by* or sweep out. Since the circumference of an entire circle is equal to $2\pi$, a right angle is one-quarter of that or $\pi/2$ radians.

Some other common angles: $\pi/6 = 30°$, $\pi/4 = 45°$, and $\pi/3 = 60°$.

## particular values

We can easily determine the values for these functions in three special cases.

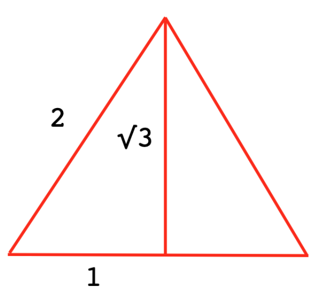
The first is the angle 45 degrees or $\pi/4$. Draw an isosceles right triangle with sides of length 1 (left panel).




Then the hypotenuse has length $\sqrt{2}$ (from Pythagoras) and the values are equal

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$$

If we start with an equilateral triangle (all angles equal to $60°$ and drop the angle bisector, we get two 30-60-90 triangles.




From the figure, we can read off that the sine of $30°$ is $1/2$ and the cosine is $\sqrt{3}/2$. The values for $60°$ are reversed since they are complementary angles.

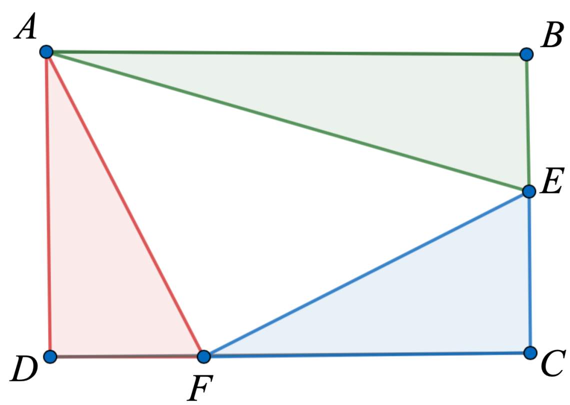
Hence

$$\sin \pi/6 = \frac{1}{2} = \cos \pi/3$$

$$\cos \pi/6 = \frac{\sqrt{3}}{2} = \sin \pi/3$$

$$\sin \pi/4 = \frac{1}{\sqrt{2}} = \cos \pi/4$$

The sine and cosine for other angles can be obtained by similar methods. Take two congruent 30-60-90 right triangles, such as $\triangle ADF$ and $\triangle FCE$, and orient them so that $DFC$ are collinear.




Extend $CE$ to make $AD = BC$ and $AD \parallel BC$. Draw $AB$ to complete rectangle $ABCD$.

$\triangle AEF$ is also a right triangle, by sum of angles, and it is isosceles, since $AF = EF$.

By sum of angles we have that $\angle BAE = 15°$. The lengths of the sides of $\triangle ABE$ are readily computed using the Pythagorean theorem. Form the ratios such as $BE/AE = \sin 15°$. We leave that as an exercise.

We will see later that the properties of the regular pentagon allow us to deduce the values for 18° (and 72°).

General formulas for the sum and difference of angles will be covered soon. These give (for example) $\sin 3°$ from the values above.

## extreme values

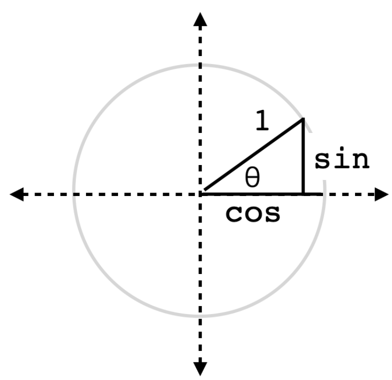
If you draw a very small angle, it will have a very short length for the vertical, $y = \sin\theta$, while the adjacent side becomes nearly equal to the radius. What happens when $\theta \to 0$? It turns out that

$$\begin{aligned} \sin 0 = 0, &\qquad \cos 0 = 1 \\ \sin \pi/2 = 1, &\qquad \cos \pi/2 = 0 \\ \sin \pi = 0, &\qquad \cos \pi = -1 \end{aligned}$$

We cannot prove any of this yet (we do not even have a *coordinate system*), but just take it on faith for now, and you will see why when we get farther along.

## a favorite trigonometric identity

Now that we know about the sine and cosine, we can look at what Pythagoras tells us about them:




As we said, in a unit circle, the sine and cosine of an angle are the sides of a right triangle with hypotenuse equal to 1. It follows from the Pythagorean theorem that for any angle $\theta$

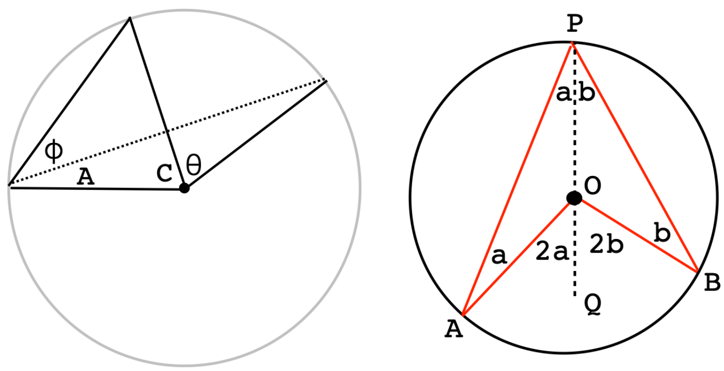
$$\begin{aligned} \sin^2\theta + \cos^2\theta = 1 \\ \sin\theta = \sqrt{1 - \cos^2\theta} \\ \cos\theta = \sqrt{1 - \sin^2\theta} \end{aligned}$$

This identity is fundamental since it provides a way of converting from sine to cosine or vice-versa.

It is traditional to write $\sin^2 \theta$ rather than $(\sin \theta)^2$, but they have the same meaning.

## inscribed angles

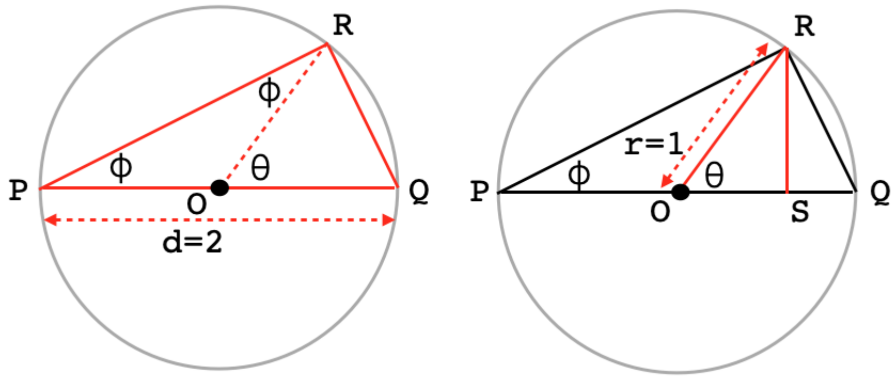
We also established previously the inscribed angle theorem. We showed two proofs that if the same arc on a circle subtends both a central angle $\theta$ and by a peripheral or inscribed angle $\phi$, then $\theta = 2\phi$.




The first proof covers the case where the inscribed angle's arc does not include the center of the circle, while in the second case it does. It follows that all inscribed angles with the same arc are equal.

But that means we can study the arc or chord corresponding to any angle (in a given circle) by drawing the angle with one arm as the diagonal of a circle, since the chord is the same no matter where we place the angle.

## inscribed and central angles




.

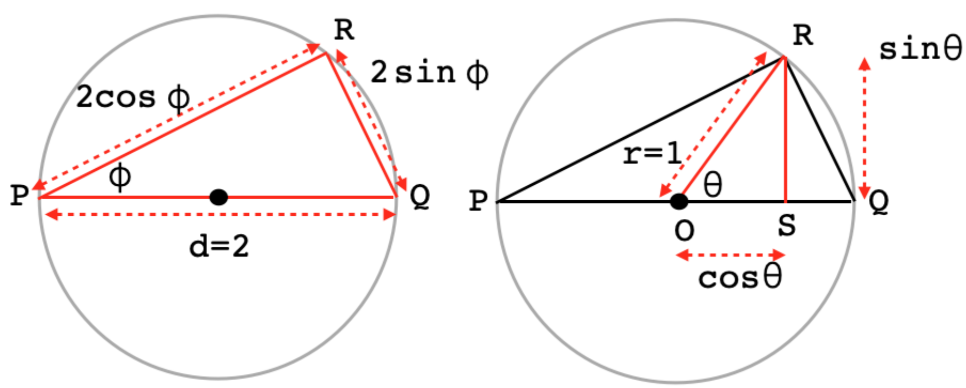
In the figure above, on the left we have drawn $\triangle PQR$ in a unit circle on center $O$ with $\angle\phi$ being the inscribed or peripheral angle and $\theta$ the corresponding central angle. Since $\triangle POR$ is isosceles, and $\theta$ is the external angle to that triangle, this is yet another demonstration that $\theta = 2\phi$.

In both panels we have scaled things so that the radius of the circle has length 1 and the diameter is twice that. One right angle is at $R$. Use our definitions of sine and cosine to see that

$$\sin\phi = \frac{RQ}{PQ} = \frac{RQ}{2}, \qquad \cos\phi = \frac{RP}{PQ} = \frac{RP}{2}$$

while

$$\sin\theta = \frac{RS}{OR} = RS, \qquad \cos\theta = \frac{OS}{OR} = OS$$



.

To repeat, for such a right triangle drawn in a unit circle, the chord $L$ subtends any peripheral angle $\phi$ is equal to

$$L = 2\sin\phi$$

and in general, if the diameter has length $d = 2r$, then

$$L = d\sin\phi = 2r\sin\phi$$

## history

The relationship between the chord $RQ$ as twice the sine of $\phi$ and $RS$ as the sine of $\theta$ and hence that of $2\phi$, was of great interest in geometry for centuries, partly for practical reasons. Chords were useful for astronomy, and astronomy was useful in turn for navigation.

Ptolemy constructed a *Table of Chords* containing values for arc lengths in 1 degree increments. The Greeks thought of values in terms of whole numbers or ratios of whole numbers, and since the smallest angle he dealt with was 1/360 of a circle, this provides a rationale for why 360 was chosen as the total number of degrees.

It also helps that 360 has so many integer factors:

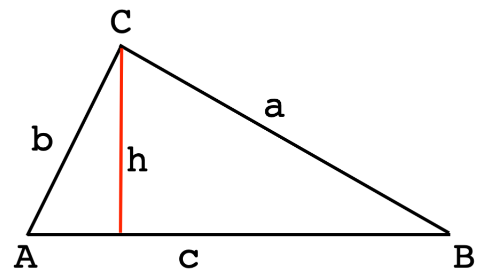
$1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18 \ldots.$

so we can talk about the 30 identical triangles that result from cutting a circle into 12 equal parts.

We will now spend some time looking at the "double angle" formula that connects the sine of an angle with that of twice the angle.

## area of any triangle

The altitude to any side of a triangle is equal to the length of the side of a triangle times the sine of the angle that side makes with the base.




$$\frac{h}{b} = \sin A$$
$$h = b \sin A$$

Therefore (if $\Delta$ is the area of the triangle):

$$2\Delta = hc = bc \sin A$$

One can equally well write

$$h = a \sin B$$
$$2\Delta = hc = ac \sin B$$

Twice the area of any triangle is the product of the two sides times the sine of the angle between them.

## problem

We usually compute the area of a parallelogram in terms of the sides. However, another formula for (twice) the area is:

$$2\Delta = d_1 d_2 \sin \theta$$

where $d_1$ and $d_2$ are the two diagonals and $\theta$ is either one of the central angles.

Derive this formula.

## law of sines

Since the area must be the same no matter how we compute it, this also leads to the equality

$$h = b \sin A = a \sin B$$
$$\frac{a}{b} = \frac{\sin A}{\sin B}$$

which by symmetry we extend to all three angles

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This simple formula is called the law of sines.

The constant ratio has an interesting value, which can be seen by going back to what we said above, namely

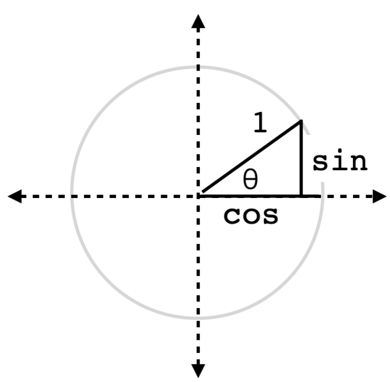
$$L = 2r \sin \phi$$

which, applied to this case, gives

$$\frac{a}{\sin A} = 2r$$

## other functions

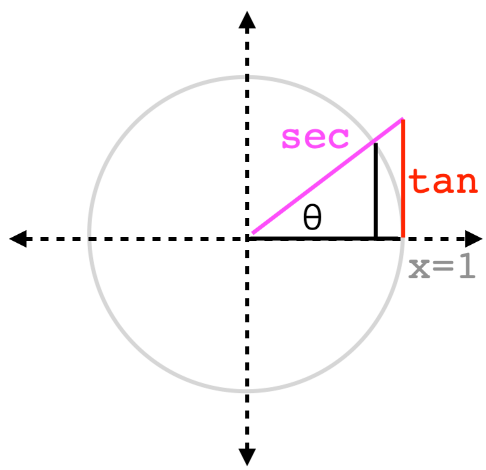
We'll just mention a few other common trig (short for trigonometric) functions that are constructed from sine and cosine, although they aren't used much in this book.



First, the tangent of the angle is defined as the opposite side divided by the adjacent side. In other words, it is equal to the sine divided by the cosine.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

We can see the tangent as a length. Extend the hypotenuse to make a similar triangle, where the adjacent side has length equal to the radius. Now, opposite over adjacent gives tangent, but adjacent is just 1.



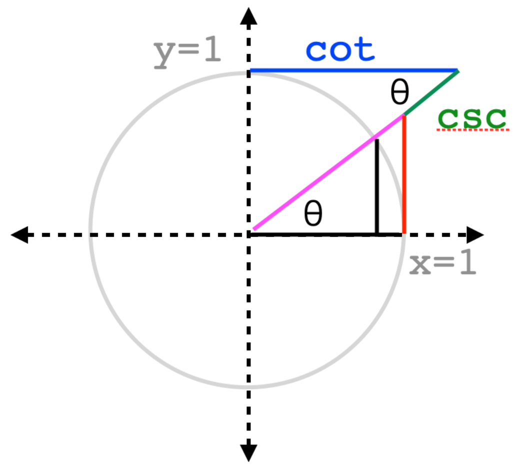

There are also three inverse functions: secant (inverse of the cosine), cosecant (inverse of the sine), and cotangent (inverse of the tangent). From the drawing above, we can get that

$$\frac{1}{\sec\theta} = \cos\theta$$

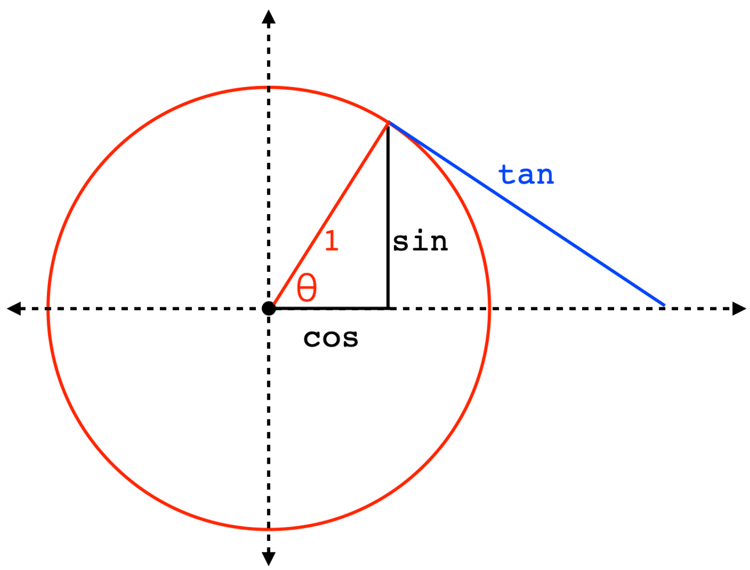
$$\sec\theta = \frac{1}{\cos\theta}$$

(We might also check by looking at the ratio $\tan\theta/\sec\theta = \sin\theta$).

We include figures with the cotangent and cosecant as well, but put off discussion of them for now.

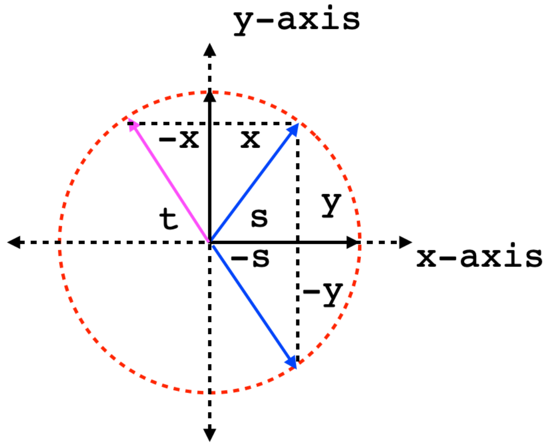



There are several other common representations of the six functions. We'll see more when we get to analytic geometry. Here is a common one where the tangent to the circle, perpendicular to the ray with $\angle\theta$ is equal to $\tan\theta$. See if you can work this out by using the complementary angle to $\theta$.




## signed angles

In analytic geometry we will introduce the idea of two axes in the plane, one for $x$-values and one for $y$-values. The origin will become $(0, 0)$, and values will have signs, i.e., $x$-values to the left of the origin will be minus some number, and $y$-values below the origin will be minus some number as well.




Without getting into all the gory details, I hope you can see that when, as here, $s$ and $t$ are supplementary angles, then

$$\sin s = \sin t$$

$$\cos s = -\cos t$$

while for $s$ and $-s$

$$\sin s = -\sin -s$$

$$\cos s = \cos -s$$

and as we said before, when $s$ and $t$ are complementary angles

$$\sin s = \cos t$$

$$\cos s = \sin t$$

This leads to an elementary proof of the **area-ratio theorem**.

*Proof.*

Let the a triangle be divided by a line from the upper vertex to the base such that the base is divided into two parts, $x$ and $y$. Call the dividing line $e$.

The angles at the base on either side of the intersection point are supplementary, hence they have the same sine. Let those angles be $\phi$ and $\phi'$.

Then twice the area of the left-hand triangle is

$$2A_L = xe \sin \phi$$

while twice the area of the other one is

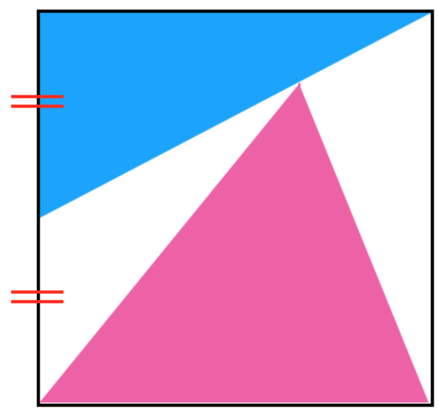
$$2A_R = ye \sin \phi' = ye \sin \phi$$

and the ratio of the areas is simply $x/y$.

□

## problem

Here is a nice area problem I saw on Twitter. (The attribution is "Paul Eigenmann, Stuttgart, 1967". Web search finds what looks like a great text, but in German).



Given that the side is bisected and the unshaded areas are equal. What is the area of the triangle shaded magenta?

*Solution.*

Let's say this is a unit square and let the top vertex of the magenta triangle be a distance $x$ from the left-hand side of the box and a distance $1-x$ from the right-hand side. Then the left-hand unshaded area is

$$\frac{1}{2} \cdot x \cdot \frac{1}{2}$$

and the right-hand shaded area is

$$\frac{1}{2} \cdot (1-x) \cdot 1$$

Set them equal:

$$\frac{x}{4} = \frac{1}{2} - \frac{x}{2}$$
$$x = 2 - 2x$$
$$x = \frac{2}{3}$$

To find the height of the magenta triangle using pure geometry, notice that a line drawn vertically from the vertex forms two similar triangles, therefore $y$ is a distance $1/3 \cdot 1/2 = 1/6$ down vertically from the upper boundary.

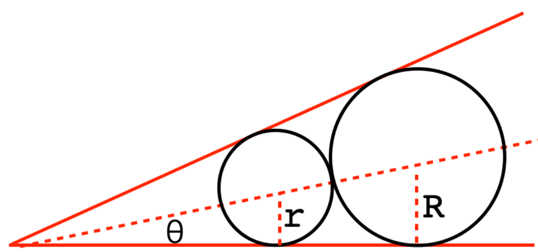
The height of the magenta triangle is therefore 5/6 and its area is one-half that, or 5/12 of the unit cube.

Using analytic geometry, by inspection of the figure, write the equation for the side of the upper triangle as

$$y = \frac{1}{2}x + \frac{1}{2}$$
$$y = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

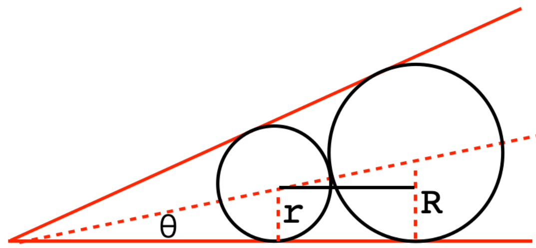
## double scoop problem

We have two lines tangent to two circles that just touch each other, the smaller one of radius $r$, and the larger of radius $R$.




There is a simple expression for the sine and cosine of $\theta$, the angle between the the two lines. Recall our introduction to trigonometry **here**.

The distance between the centers of the two circles is $r + R$. Draw a horizontal line through the center of the smaller circle.




We have constructed a right triangle, which is similar to the original one. It includes the angle $\theta$ and the hypotenuse is the distance between the two centers, $R+r$. The opposite side has length $R-r$ and so

$$\sin\theta = \frac{R-r}{R+r}$$

The adjacent side (the line segment colored black) has its squared length equal to

$$(R+r)^2 - (R-r)^2 = 4Rr$$

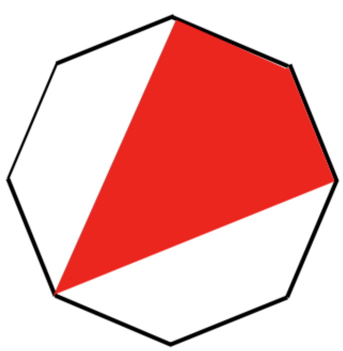
thus

$$\cos\theta = \frac{2\sqrt{Rr}}{R+r}$$

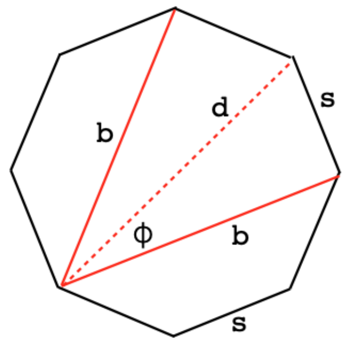
## David Bowie problem

Here's a problem from the web:


Vincent Pantal🍩ni @panlepan · Feb 25
The octagon that thinks it's David Bowie.
What fraction of this regular octagon is coloured in red ?
#GeometrySnacks #ShowYourWork #WFIS



One way to look at this is to imagine the octagon inscribed in a circle of diameter $d = 2r$ (below). We reason that the two triangles are right triangles with a shared hypotenuse.



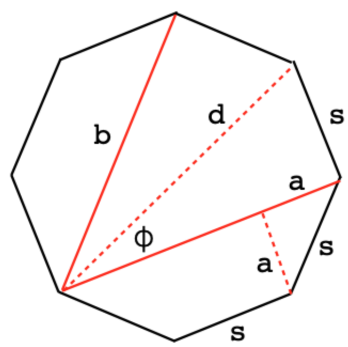

We can apply trigonometry to find that the whole area required is simply

$$A_{\text{ region}} = d^2 \sin\phi \cos\phi$$

But we aren't ready to do this kind of calculation yet.

However, it will turn out that the part shaded red is simply one-half of the whole.

Therefore, the three lines divide the octagon into four equal parts. That simple answer is a clue that something important makes the problem easy.




The triangle with small sides $a$ can be moved so one of the unshaded base parts becomes a rectangle with area

$$a(s + a) = sa + a^2$$

while the base of the triangle is $s + 2a$ so its area is

$$\frac{1}{2}s(s+2a) = \frac{1}{2}s^2 + sa$$

It's not obvious at first that these are equal, but working with it, they would be equal if we can show that $2a^2 = s^2$.

Of course we can do exactly that, because $a$ is the side of an isosceles right triangle with hypotenuse $s$. By the Pythagorean theorem, we have

$$a^2 + a^2 = s^2$$

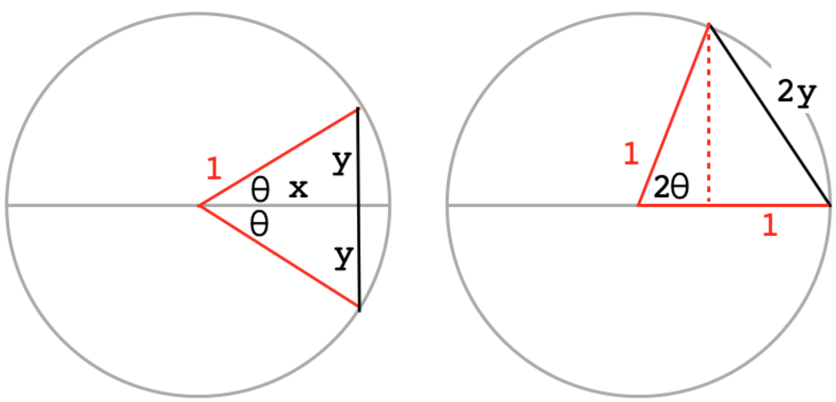
□

# Chapter 11

# Double angle

## double angle formulas

Draw a unit circle with central angle $\theta$ and then double the central angle below the horizontal diameter. Let $x$ be the base of the two smaller triangles, the side adjacent to angle $\theta$, and $y$ the side opposite.




.

Then by the basic definitions of trigonometry, we have that $x = \cos\theta$ and $y = \sin\theta$. The area of each small triangle is $xy/2$ and the area of the two combined is simply

$$A = xy = \sin\theta\cos\theta$$

Now, rotate the triangle counter-clockwise until the red side on the right lies along the diagonal. The dotted vertical line is clearly $\sin 2\theta$ and since the base is length 1, the area of that same triangle is now

$$A = \frac{1}{2} \cdot \sin 2\theta$$

Combining the results

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

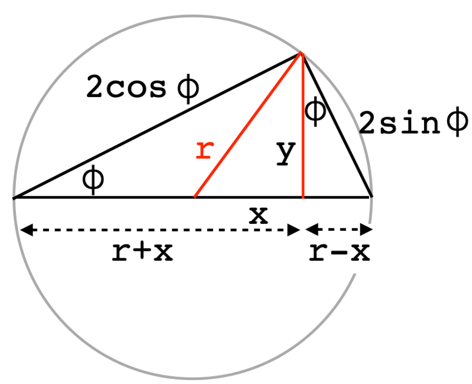
This is the first double angle formula, for sine. We claim that it still works even if the sum is greater than a right angle but keep things simple by not talking about it more.

## double angle for cosine

Consider this right triangle with an inscribed angle of $\phi$. Label the sides of a triangle containing the central angle $2\phi$ as $x$ and $y$ to simplify the figure. Let $r = 1$ so then

$$y = \sin 2\phi, \qquad x = \cos 2\phi$$

Whenever we drop an altitude in a right triangle, the smaller triangles are similar to the original. This accounts for the second $\angle\phi$ in the figure.




We note for later that the base of the altitude divides the base into $r + x$ and $r - x$.

Using the small triangle with angle $\phi$

$$\frac{y}{2\sin\phi} = \cos\phi$$

$$y = 2\sin\phi\cos\phi$$

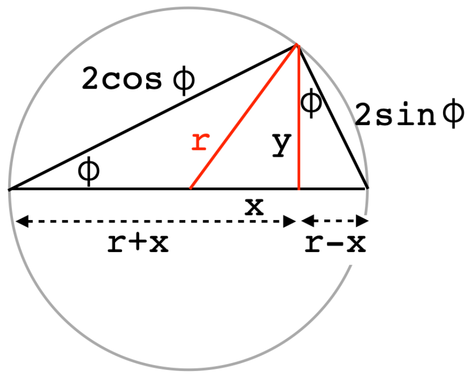
Since $y = \sin 2\phi$, this is just what we had before.

Along the base of the triangle extending from the left, we have (again, $r = 1$):

$$\frac{r+x}{2\cos\phi} = \frac{1+x}{2\cos\phi} = \cos\phi$$

$$\frac{1+x}{2} = \cos^2\phi$$

But (from the small triangle again)




$$\frac{r-x}{2\sin\phi} = \frac{1-x}{2\sin\phi} = \sin\phi$$

$$\frac{1-x}{2} = \sin^2\phi$$

And then

$$\frac{1+x}{2} - \frac{1-x}{2} = x$$

but $x = \cos 2\phi$ so

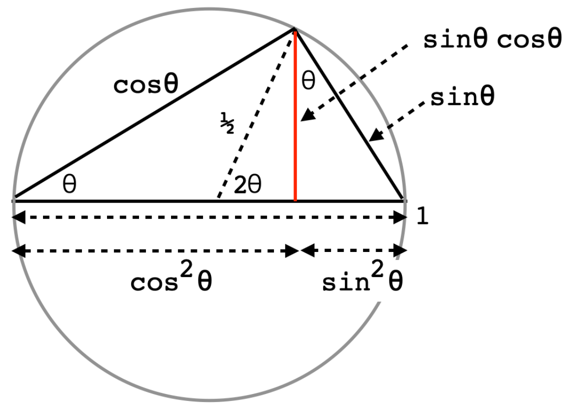
$$\cos 2\phi = \cos^2\phi - \sin^2\phi$$

Note also that

$$\cos^2\phi + \sin^2\phi = \frac{1+x}{2} + \frac{1-x}{2} = 1$$

which is true for any angle $\phi$, a restatement of the Pythagorean theorem.

Here is the same proof but scaled so that the diameter has unit length and we use sine and cosine labels rather than $x$ and $y$. The details are left to you.




## half-angle formulas

The half-angle formulas are easy to derive from the double angle cosine formula, because the sine and cosine terms are separated:

$$\cos 2A = \cos^2 A - \sin^2 A$$

Just use our favorite identity ($\sin^2 A + \cos^2 A = 1$) to obtain

$$\cos 2A = \cos^2 A - (1 - \cos^2 A)$$

or

$$\cos 2A = (1 - \sin^2 A) - \sin^2 A$$

We'll do the sine first

$$2\sin^2 A = 1 - \cos 2A$$

$$\sin A = \sqrt{\frac{1 - \cos 2A}{2}}$$

which is easily verified using the values for 30° and 60°.

On the other hand:

$$\cos 2A = \cos^2 A + \cos^2 A - 1$$

Solve for $\cos A$:

$$\cos A = \sqrt{\frac{1 + \cos 2A}{2}}$$

This is called the half-angle formula since $A$ is one-half of $2A$. That the sum of the squares is equal to 1 is also easily verified. We see that the square roots go away, the $\cos 2A$ terms will cancel and we end up with $1/2 + 1/2$.

## some algebra

You might be tempted to try to derive a formula for $\sin 2A$ from that for $\cos 2A$ (or vice-versa), using our favorite identity. Let us re-write the formulas substituting $s$ for $\sin A$ and $c$ for $\cos A$:

$$\sin 2A = 2sc, \qquad \cos 2A = c^2 - s^2$$

They have different forms: the first mixes $s$ and $c$, while the second has a difference of squares. If we were to take

$$(c+s)^2 = c^2 + 2sc + s^2$$

$$(c-s)^2 = c^2 - 2sc + s^2$$

$$(c+s)(c-s) = c^2 - s^2$$

There is some of what we want, but in the formula for $\cos 2A$ one term is positive and one negative. What's going on?

First, just verify what $\sin^2 2A + \cos^2 2A$ is equal to:

$$= 4s^2c^2 + c^4 - 2s^2c^2 + s^4$$

$$= c^4 + 2s^2c^2 + s^4$$

$$= (s^2 + c^2)^2 = 1^2 = 1$$

That checks out, and it contains a hint to the answer. We used $4s^2c^2 - 2s^2c^2$ to convert $-2s^2c^2$ to the positive $2s^2c^2$.

Remember, to do the conversion we square the sine or cosine, subtract it from 1 and then take the square root.

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

$$\cos\theta = \sqrt{1 - \sin^2\theta}$$

It is easier to see how this works using the first formula. So we start with the formula for cosine of $2A$ and plug in:

$$1 - (c^2 - s^2)^2 = 1 - (c^4 - 2s^2c^2 + s^4)$$

Add (and subtract) $2s^2c^2$ in the parentheses!

$$= 1 - (c^4 + 2s^2c^2 + s^4 - 4s^2c^2)$$

$$= 1 - \; [\; (c^2 + s^2)^2 - 4s^2c^2 \;]$$

$$= 1 - 1 + 4s^2c^2 = 4s^2c^2$$

and now the square root is easy and gives exactly what we need.

Then it becomes clear that we can add $(s^2 + c^2) - 1$ or $(s^2 + c^2)^2 - 1$ to expressions and if they will simplify, it helps us. Hence to obtain the formula for $\cos 2A$, starting with 1 minus sine squared:

$$1 - 4s^2c^2$$

$$= 1 - 4s^2c^2 + (s^2 + c^2)^2 - 1$$

$$= -4s^2c^2 + \; [\; s^4 + 2s^2c^2 - c^4 \;]$$

$$= s^4 - 2s^2c^2 + c^4)$$

$$= (s^2 - c^2)^2 = (c^2 - s^2)^2$$

remembering that $(x - y)^2 = (y - x)^2$.

Take the square root to finish.

## examples

Restating the double-angle formula for sine

$$\sin 2A = 2 \sin A \cos A$$

Plugging in

$$\sin 60^\circ = 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

and

$$\cos 2A = \cos^2 A - \sin^2 A$$

plugging in again

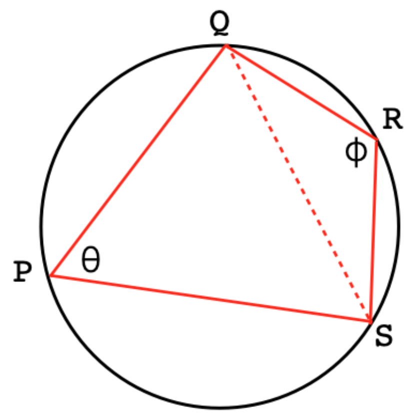
$$\cos 60^\circ = (\frac{\sqrt{3}}{2})^2 - (\frac{1}{2})^2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

They both look good.

We can also use the formulas to check supplementary angles (recall that $\sin 180 = 0$ and $\cos 180 = -1$):

$$\begin{aligned}\sin 180 - A &= \sin 180 \cos A - \sin A \cos 180 \\ &= 0 - (-\sin A) = \sin A \\ \cos 180 - A &= \cos 180 \cos A + \sin 180 \sin A \\ &= -\cos A + 0 = -\cos A\end{aligned}$$

Of course, we hardly need algebra for the result about the sine of the supplementary angle. Two inscribed angles with the supplementary arcs clearly have the same sine:



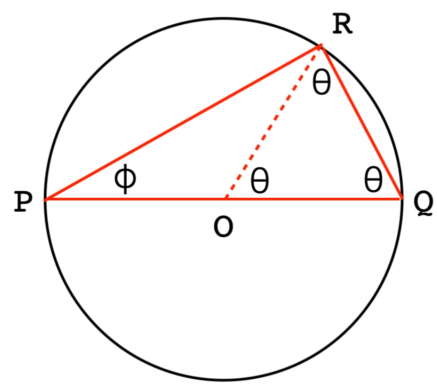

The total arc of the circle is $2\pi$, while the two angles $\theta$ and $\phi$ together take up that entire arc. As inscribed angles they sum to one-half or $\pi$. Hence $\theta$ and $\phi$ are supplementary.

But we previously had

$$L = d \sin \phi = 2r \sin \phi$$

where $L$ is the length of the dotted line, and if the circle is scaled appropriately, it *is* $\sin \phi$. But it is also $\sin \theta$.

## equilateral triangle




Here is another way of looking at 30° and 60°. Let the central angle $\theta$ be such that the triangle formed is equilateral.

Now the central angle is equal to both of the base angles, and the measure of all these angles is 60° or $\pi/3$.

Since $\triangle OQR$ is equilateral, the chord $QR$ is the same length as the radius. It follows that the sine of the inscribed angle $\phi$ is $r/2r = 1/2$. And of course the length of $PR$, which is 2 times the sine of 60°, is $\sqrt{3}$.

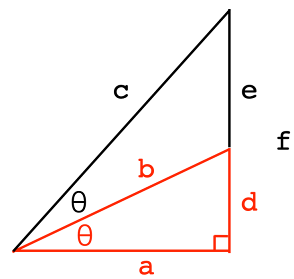
## Archimedes and pi

Later we will explore the math relating to Archimedes' method to approximate $\pi$, resulting in the bounds $3\ 10/71 < \pi < 3\ 1/7$.

The lower bound is about 3.1408 and the upper bound about 3.1429 while the true value is about 3.1416. (There is an interesting story relating to a fraction which is a much, much better estimate of $\pi$, namely 355/113, good to six digits).

As you probably know, the method involves calculating the length of the perimeter of circumscribed and inscribed regular polygons.

We can build on two basic results from the geometry of right triangles:




The first is that the sum of the cotangent and cosecant for the original angle are equal to the cotangent of the half-angle:

$$\frac{a}{f} + \frac{c}{f} = \frac{a}{d}$$

$$\cot 2\theta + \csc 2\theta = \cot \theta$$

This follows from the angle bisector theorem, $a/d = c/e$, with a bit of manipulation.

$$\frac{a}{c} = \frac{d}{e}$$

$$\frac{a}{c} + \frac{c}{c} = \frac{d}{e} + \frac{e}{e}$$
$$\frac{a+c}{d+e} = \frac{c}{e} = \frac{a}{d}$$

Adding components in the denominator of the first term:

$$\frac{a+c}{f} = \frac{a}{d}$$

Another way to get there is to use the double angle formulas. We'll abbreviate $\sin, \cos, \tan$ as $S, C, T$ and identify the values for the half-angle as lowercase. The formulas are:

$$S = 2sc$$

$$C = c^2 - s^2 = 2c^2 - 1$$

The cotangent of $2\theta$ is then (cosine/sine):

$$\frac{1}{T} = \frac{C}{S} = \frac{2c^2 - 1}{2sc}$$

$$= \frac{1}{t} - \frac{1}{S}$$

which rearranges to give the result

$$\frac{1}{t} = \frac{1}{T} + \frac{1}{S}$$

# Chapter 12

# Sum of angles by scaling

## sum of angles

In order to gain some practice thinking about sine and cosine, we will derive what are called the sum of angles formulas. These are really for the sum and difference of two angles:

$$\sin(s \pm\ t), \qquad \cos(s \pm\ t)$$

Previously we talked about formulas for the case where $s = t$, but now we want more general expressions. Here is the first, for the difference of angles $s$ and $t$

$$\cos s - t = \cos s \cos t + \sin s \sin t$$

By $\cos s - t$ we mean $\cos(s - t)$, but have left off the parentheses.

There are four formulas, and then some special examples. These are used a lot in calculus, not only for solving problems, but most important, in finding an expression for the derivatives of the sine and cosine functions.

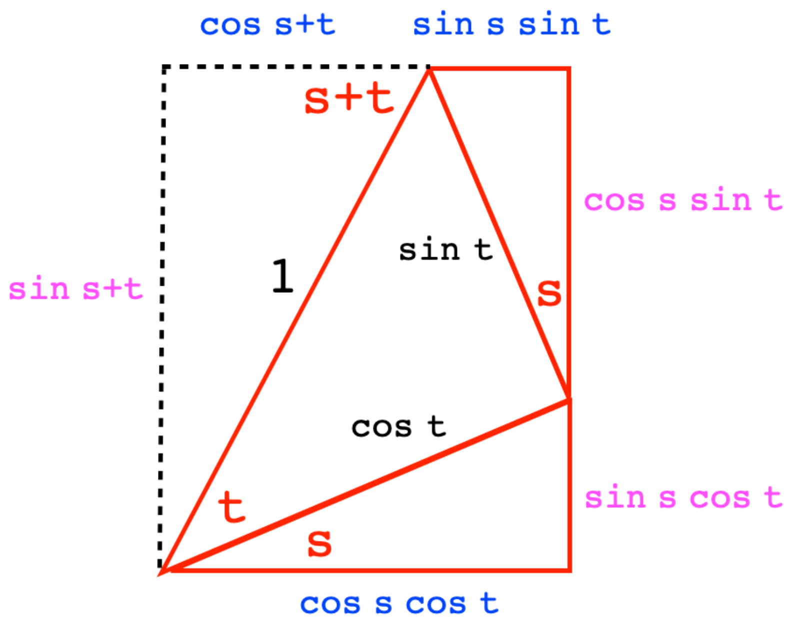
I've memorized only the single equation given above. Say "cos cos" and then recall the difference in sign, minus on the left, plus on the right.

I like this version because it can be checked easily. Just set $s = t$. Then $s-t$ becomes $s - s$ and we have

$$\cos s - s = \cos 0 = 1 = \cos^2 s + \sin^2 s$$

which is our favorite trigonometric identity and obviously correct.

This diagram shows where we're headed:



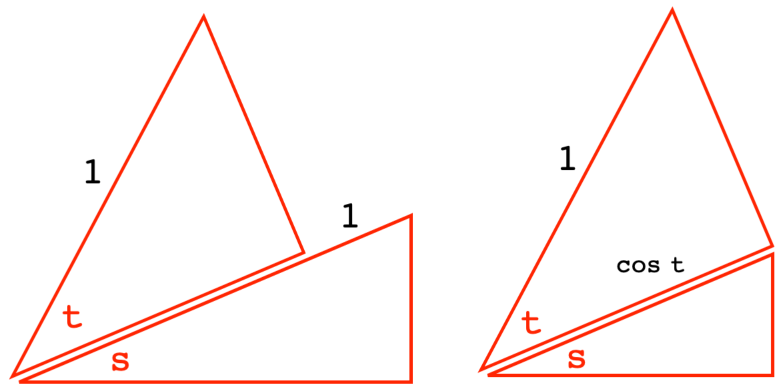

Since the figure is a rectangle, you can read the relevant equalities off the opposite sides.

$$\sin s + t = \sin s \cos t + \cos s \sin t$$

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

As so often, we begin with an inspired diagram. Let's build it up in stages.

On the left, below, we have stacked two right triangles. The one containing angle $t$ is rotated so that its base is parallel to the hypotenuse of the triangle containing angle $s$.




The crucial step is to re-scale the triangle on the bottom so that the parallel line segments are also equal in length (right panel).

By re-scaling, we change the length of the hypotenuse of the triangle containing angle $s$. Its hypotenuse is now the length $\cos t$.

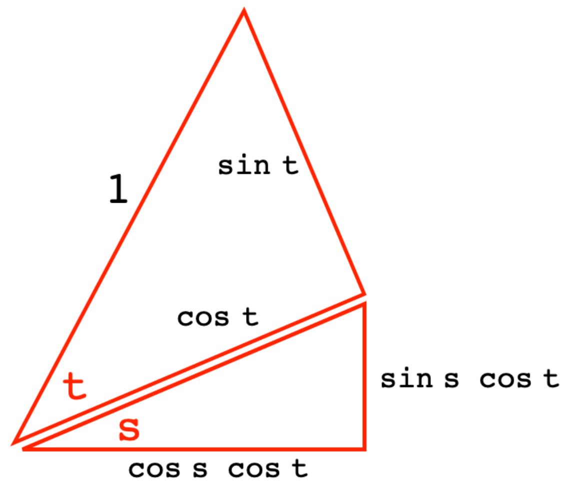
Recall that when the length of the hypotenuse was $h$, we divided the length of the adjacent side by the hypotenuse to get the cosine. There we had $b/h = \cos s$. Now we have

$$\cos s = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{adjacent}}{\cos t}$$

What should be the length of the base of the triangle with angle $s$? It must be $\cos s \cos t$, the product of cosines! Just multiply both sides above by $\cos t$.

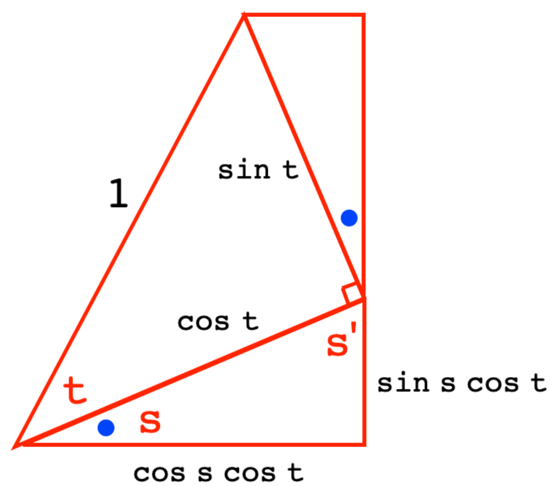
Or recognize that after dividing $\cos s \cos t$ by the hypotenuse, which is $\cos t$, we will then have what we want, $\cos s$.

$$\frac{\cos s \cos t}{\cos t} = \cos s$$




By the same reasoning, the opposite side in the triangle with angle $s$ is $\sin s \cos t$, so that after dividing by $\cos t$ we obtain the correct value, $\sin s$.

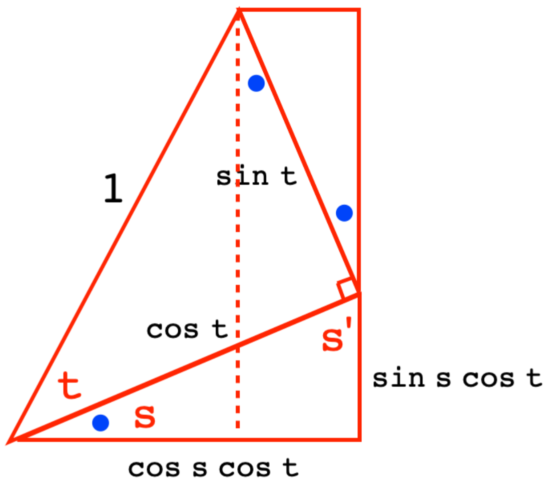
## a similar triangle

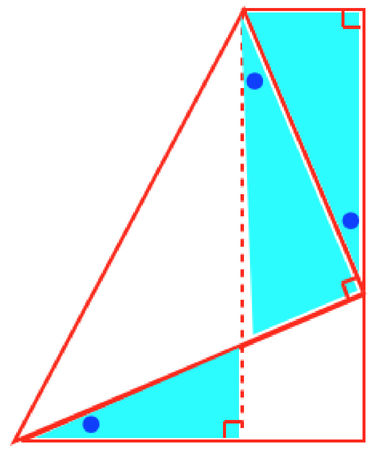



We add another right triangle to the figure

We claim that the angles labeled with blue dots are equal.

The easiest way to do that is algebraic. The angle that is complementary to $s$ in the bottom triangle is $s'$. Complementarity means that $s + s'$ is equal to a right angle.



But the second blue dotted angle plus $s'$ is also equal to a right angle. The reason is that, when added to another right angle, it makes two right angles or a straight line. Using the language of radian measure

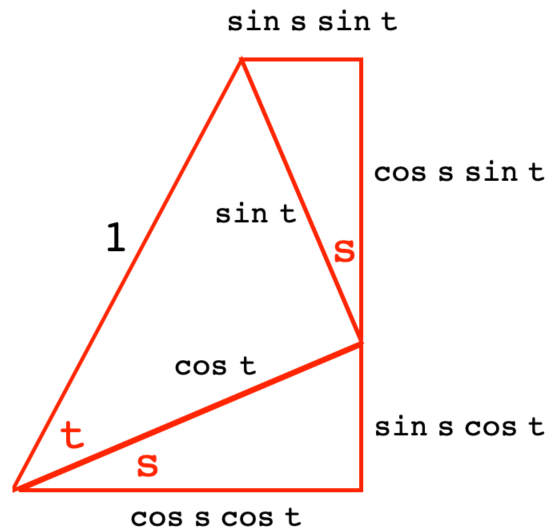
$$s + s' = \frac{\pi}{2}$$

$$s' + \frac{\pi}{2} + \text{blue dot} = \pi$$

$$s' + \text{blue dot} = \frac{\pi}{2}$$

$$\text{blue dot} = s$$

So we add the correct label, and then play the same trick as before.



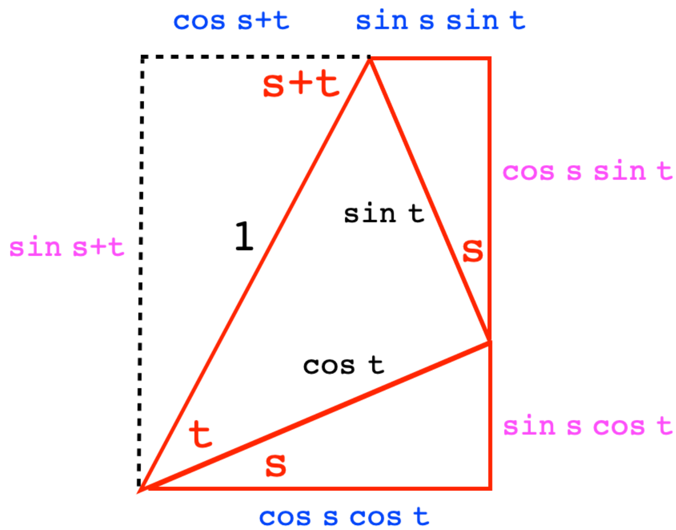

The difference is that now, the length of the hypotenuse of the upper triangle containing angle $s$ is $\sin t$. The two similar triangles are scaled differently from one another.

We obtain $\sin s \sin t$ and $\cos s \sin t$ for the sides of the upper triangle, so that after dividing by the hypotenuse, which is $\sin t$, we obtain the correct values for the sine and cosine of angle $s$.

## finale

Why have we gone to the trouble of doing all this?

The two angles, $s$ and $t$, taken together, are equal to the angle at the top of the figure labeled, naturally, $s + t$.




We fill in lengths for the dotted lines of the fourth right triangle in the figure below.

This forms a rectangle (all four corners are right angles). Therefore, opposite sides are equal.

We just write down the formula by reading off the figure:

$$\sin s + t = \sin s \cos t + \cos s \sin t$$

and

$$\cos s + t + \sin s \sin t = \cos s \cos t$$

which can be rearranged to give

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

These are the sum of angles formulas.

## change signs

For $\cos s - t$, flip the sign on the second term.

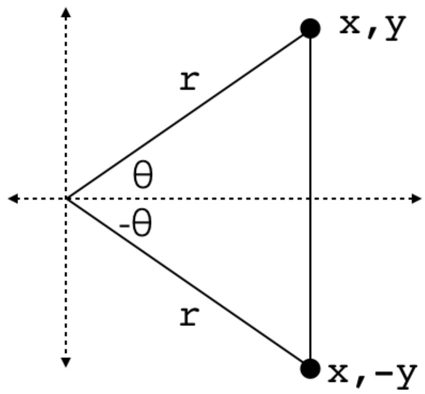
$$\cos s - t = \cos s \cos t + \sin s \sin t$$

That's because

$$\cos -\theta = \cos \theta$$

$$\sin -\theta = -\sin\theta$$

To show this we need to invoke analytic geometry (there's no such thing as a negative angle in classical geometry).




The diagram shows the reason:

$$\cos\theta = x/r = \cos -\theta$$

while

$$\sin\theta = y/r = -(\sin -\theta) = -(-y/r)$$

Substitute $-\sin\theta$ for $\sin -\theta$ and $\cos\theta$ for $\cos -\theta$:

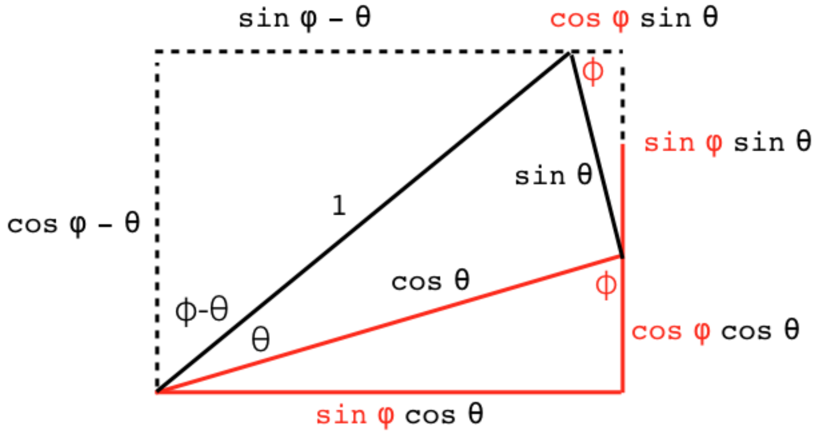
$$\cos s - t = \cos s \cos -t + \sin s \sin -t$$

$$= \cos s \cos t - \sin s \sin t$$

and

$$\sin s - t = \sin s \cos t - \cos s \sin t$$

It's kind of overkill, but still worth noting that a simple change to the figure we had above will give the difference formulas:




We've changed symbols to $\theta$ and $\phi$ for the complementary angles.

We can justify the label $\phi - \theta$ for the angle at the lower left in various ways, for example, by adding up the three angles at that corner:

$$(\phi - \theta) + \theta + (90 - \phi) = 90$$

Switch the labels appropriately (it's easy since this $\phi$ is the complement of the old one).

Read the result:

$$\sin \phi - \theta = \sin \phi \cos \theta - \cos \phi \sin \theta$$

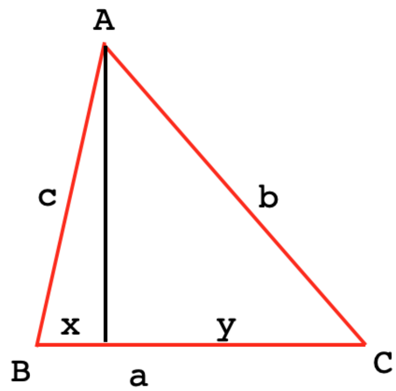
$$\cos \phi - \theta = \cos \phi \cos \theta + \sin \phi \sin \theta$$

## alternative derivation

There are many derivations of the sum of angles formulas. Here is an algebraic one based on the **law of sines**.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Start with this triangle




$$a = x + y = c \cos B + b \cos C$$

From the law of sines: $\sin A = (a/b) \sin B$. Substituting for $a$:

$$\sin A = \frac{c \cos B + b \cos C}{b} \sin B$$

$$= \frac{c}{b} \sin B \cos B + \sin B \cos C$$

Again from the law of sines: $\sin B = (b/c) \sin C$, so

$$\sin A = \sin C \cos B + \sin B \cos C$$

But since $A$ and $B + C$ are supplementary, their sines are equal, thus

$$\sin B + C = \sin C \cos B + \sin B \cos C$$

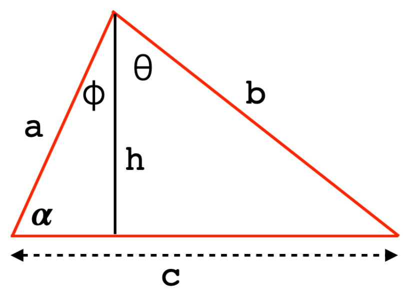
□

The proof for an obtuse angle is left as an exercise.

## general formula for sine

Here is a related, simple proof of the formula for sine.

Consider the following triangle, where the angle at one vertex is divided by the altitude to the opposing side, forming angles $\theta$ and $\phi$. We are interested in finding the sine of the two angles added together.




But first we must introduce a standard formula for the area of a triangle, illustrating it by the angle $\alpha$. From our previous work we know that twice the area of the triangle is $hc$, but $h$ is also part of a formula for sine, namely $h/a = \sin\alpha$, which can be rearranged to give $h = a\sin\alpha$. In other words

$$2A = hc = ac\sin\alpha$$

In general, twice the area of any triangle is the product of two sides times the sine of the angle between. So in this case, we also have that

$$2A = ab\sin(\phi + \theta)$$

Now we just calculate the area of the two smaller triangles and add them together. We have

$$2A = ha\sin\phi + hb\sin\theta$$

I'm going to rearrange this slightly

$$2A = a\sin\phi \cdot h + b\sin\theta \cdot h$$

For these angles, $h$ is connected to a trig function, but this time it's the cosine.

$$h = a\cos\phi = b\cos\theta$$

Substituting two times into the previous equation, we obtain

$$2A = a\sin\phi \cdot b\cos\theta + b\sin\theta \cdot a\cos\phi$$

and equate it to the first result

$$ab\sin(\phi + \theta) = a\sin\phi \cdot b\cos\theta + b\sin\theta \cdot a\cos\phi$$

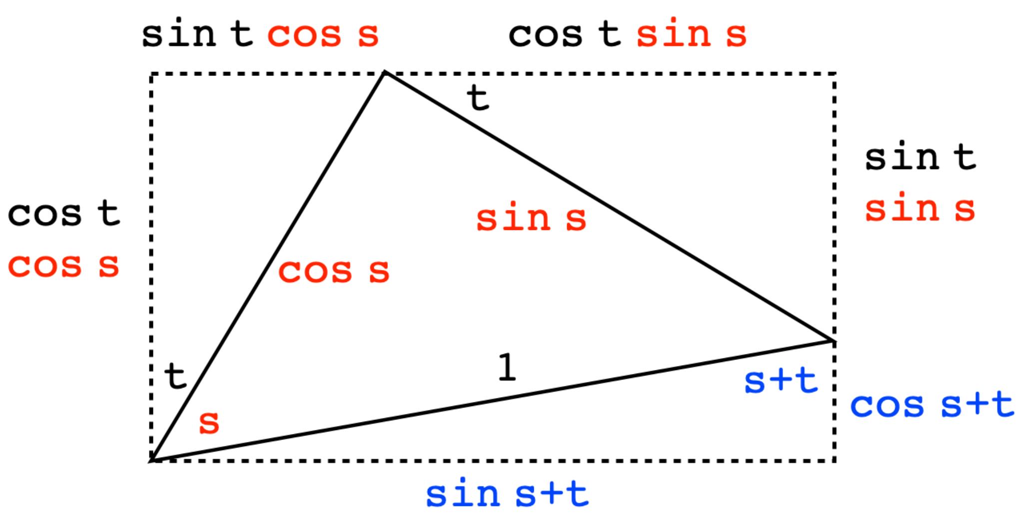
We can cancel $ab$ from all three terms:

$$\sin(\phi + \theta) = \sin\phi\cos\theta + \sin\theta\cos\phi$$

This is the general form for the sine of the sum of two angles. It is apparent that it reduces to the previous double angle formula in the case where $\theta = \phi$.

## alternate version

Just to be complete, here is an alternative version I found that may be even simpler.




## yet another proof of Pythagoras's Theorem

Consider the right triangle in a unit circle with opposite side $\sin\theta$ and adjacent side $\cos\theta$. We will prove that

$$1 = \cos^2\theta + \sin^2\theta$$

*Proof.*

We have the formula above:

$$\cos A + B = \cos A\cos B - \sin A\sin B$$

Let $A = \theta$ and $B = -\theta$.

$$\cos\theta + -\theta = \cos\theta\cos -\theta - \sin\theta\sin -\theta$$

For the left-hand side, we have

$$\cos\theta + -\theta = \cos 0 = 1$$

And for the right-hand side we use the fact that cosine is an odd function ($\cos -s = \cos s$). Sine is an even function ($\sin s = -\sin -s$) so that gives

$$\cos\theta\cos\theta - \sin\theta(-\sin\theta)$$

$$= \cos^2\theta + \sin^2\theta$$

Bringing back the left-hand side

$$1 = \cos^2\theta + \sin^2\theta$$

□

# Chapter 13

# Sum of angles by Ptolemy

## general sum of angles

Previously, we saw the double angle formulas

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

We'd like to find a general formula for the sine of two angles added together. $\sin A + B = ?$

Of course, if $A = B$ then the formula must reduce to what we have above, so it seems there are two reasonable possibilities if we keep things simple:

$$\sin A + B \stackrel{?}{=} \sin A\cos A + \sin B\cos B$$

$$\sin A + B = \sin A\cos B + \sin B\cos A$$

Without explaining it at present, we will just assume it is known that $\sin 0 = 0$ and $\cos 0 = 1$. It is easy to see that the sine and cosine approach these values for very small angles but a proper explanation will have to wait for analytic geometry.

If you start with any angle $A$, set $B = 0$, and then use the first formula it gives $\sin A + 0 = \sin A\cos A + 0 \cdot 1$ which means $\sin A = \sin A\cos A$ which makes no sense. Indeed, we will now derive the second formula as the correct result.

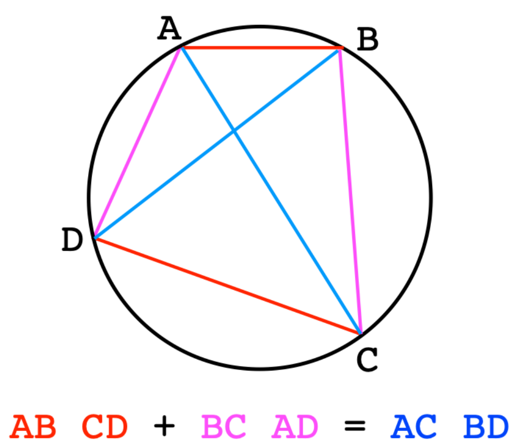
If we start with the cosine formula there seems to be only one simple candidate, namely

$$\cos A + B = \cos A \cos B - \sin A \sin B$$

We use Ptolemy's theorem, which we proved previously using the properties of parallelograms. In fact, a big reason for including this section now is to illustrate the power of this theorem, which is normally thought to be difficult, but since we found an easy proof we are in great shape.

Ptolemy's theorem says that if a four-sided figure, a quadrilateral, has all of its vertices on a circle, then we can form the two products of opposing sides and add them together to obtain the product of the diagonals.

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$




Our first proof of this is **here**. And we'll revisit the topic later.

Notice that if the quadrilateral is a rectangle, then the diagonals of the rectangle are also diagonals of the circle (**here**), so this theorem gives a simple proof of the Pythagorean theorem.

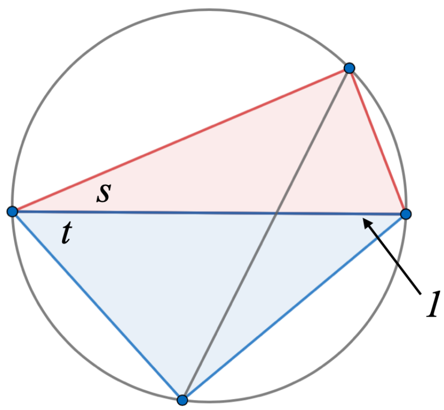
## sum of angles

Ptolemy's theorem can be used to give direct proofs of the sum (and difference) of angles formulas for both sine and cosine. It's a fun exercise because the results come easily from inspired diagrams with slight changes between them.

It helps that we have an idea about what we want.

We need to say a bit more before we start. Each diagram contains a diameter of the circle, scaled so its length is 1.

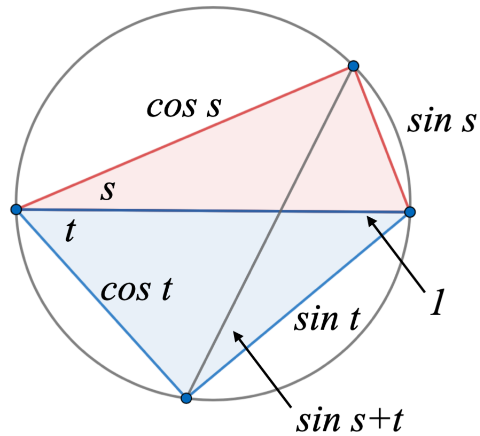
In two cases that length is also a diagonal of the quadrilateral, and in the other two it is a side. In the first case, the product with the other diagonal of the quadrilateral is just equal to whatever value that second diagonal has. Here is the first figure:




There are two right angles on the circle with sides extending to the ends of the diameter. This makes the setup a natural for sine and cosine formulas.

Second, a peripheral or inscribed angle $\theta$ is related to its chord length by $L = 2R \sin \theta$. We saw this important result previously (**here**). Since we have $2R = 1$, this simplifies to $L = \sin \theta$, where in this case $\theta = s + t$.

We start with the sum of sines. The idea is that $\sin s + t$ should be the second diagonal.




From there the formula basically writes itself.

$$\sin s + t = \sin s \cos t + \sin t \cos s$$

Our analysis of what was likely to be the form of the final result turns out to be correct.

## sine of the difference

Algebraically, the difference of sines is easily derived using the fact that cosine is an even function, $\cos(x) = \cos(-x)$ while sine is odd, so $\sin(x) = -\sin(-x)$.

Thus, substituting $-t$ for $t$ changes the "sign" of the second term but not the first.

$$\sin s + (-t) = \sin s \cos(-t) + \sin(-t) \cos s$$

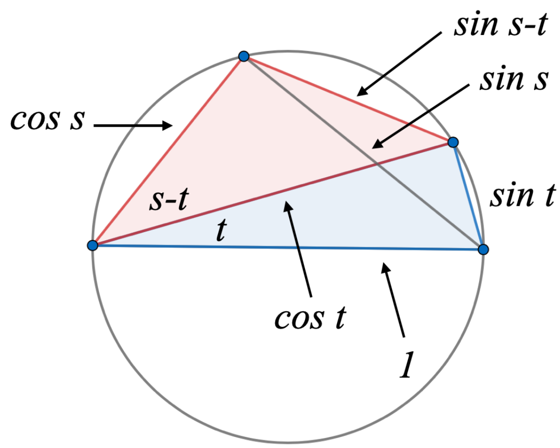
$$= \sin s \cos t - \sin t \cos s$$

However, we can use a diagram (kind of) like the one above to do this formula too. See:

https://www.cut-the-knot.org/proofs/sine_cosine.shtml

The trick is that, because of the minus sign on the right-hand side in the final formula, we want $\sin s - t$ to be one of the sides. So the diagonal of the circle must be opposite, and also be a side of the quadrilateral.

Now, $s$ is the whole angle, i.e. s - t + t = s. So one diagonal is $\sin s$ and the other is $\cos t$.




And again, the formula writes itself:

$$\sin(s-t) + \sin t \cos s = \sin s \cos t$$

which rearranges to give

$$\sin s - t = \sin s \cos t - \sin t \cos s$$

## cosine of the sum

For the cosine formulas, we'll need to relate them to the sine of another angle. Recall that if

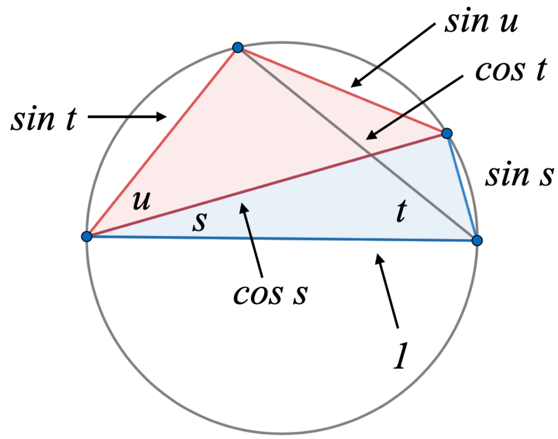
$$s + t + u = 90$$

then

$$\cos(s+t) = \sin u = \sin 90 - (s+t)$$

So for the first cosine formula we add an additional angle $u$, and use the fact that $s+t$ is complementary to $u$.

We make $\sin u = \cos s + t$ one of the sides, because we know the formula has a minus sign in it.




$$\sin s \sin t + \cos(s+t) = \cos s \cos t$$

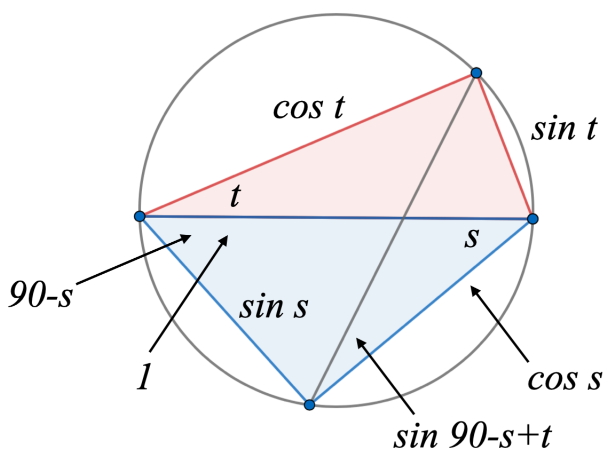
$$\cos s + t = \cos s \cos t - \sin s \sin t$$

### cosine of the difference

Last is the difference formula for cosine. Again, we can derive this formula easily by using the fact that cosine is even and sine is odd.

Knowing where we're headed, we want opposing sides to be both sine or both cosine, and we have them as sides, because they add in the final formula.

So then, somehow, the diagonal must be $\cos s - t$.



The complementary angle to $s$ is $90 - s$. Adding it to the adjacent angle $t$ we have that the diagonal is $\sin(90 - s + t)$. But that is

$$\cos 90 - (90 - s + t) = \cos s - t$$

which is just what we need.

$$\cos s - t = \cos s \cos t + \sin s \sin t$$

Perhaps you may object that possibly it coulbe be that $s < t$ so $t - s < 0$. But we can also use the angle on the opposite side and write:

$$\sin s + (90 - t) = \cos 90 - \ [\ (90 - t) + s\ ]$$

$$= \cos t - s$$

Again, this works because cosine is an even function so $\cos s - t = \cos t - s$, or if you prefer you can use the fact that the sine of an angle is equal to the the sine of its supplementary angle, which is obvious from the diagram.

$$\sin s + (90 - t) = \sin 180 - \ [\ s + (90 - t)\ ] = \sin t + (90 - s)$$

# Chapter 14

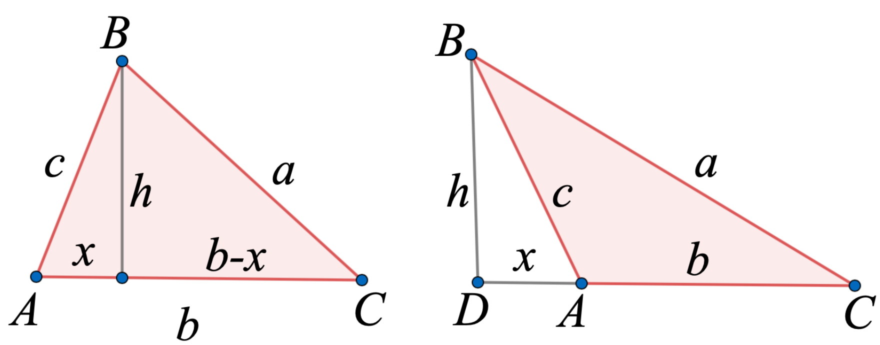
# Law of Cosines

This theorem is used extensively in geometry. It relates the side lengths of any triangle to the cosine of an angle. For example, if the side opposite $\angle A$ is $a$, then $a^2 = b^2 + c^2 - 2bc \cos A$.

Rewritten appropriately, it can give any side in terms of the other two.

Consider $\triangle ABC$ in two versions. In one (left panel), $\angle A$ is acute, and in the other (right panel) $\angle A$ is obtuse. Note that the supplementary $\angle BAD$ on the right is equal to $\angle A$ from the acute case. In particular, it has the same cosine, $x/h$.

Let the sides opposite be $a, b, c$, as usual.




For the acute case, we have

$$a^2 = h^2 + (b - x)^2$$
$$c^2 = h^2 + x^2$$

Subtracting:

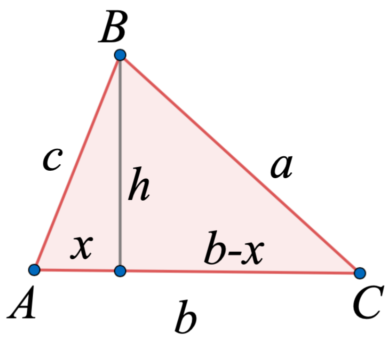
$$a^2 - c^2 = (b - x)^2 - x^2 = b^2 - 2xb$$

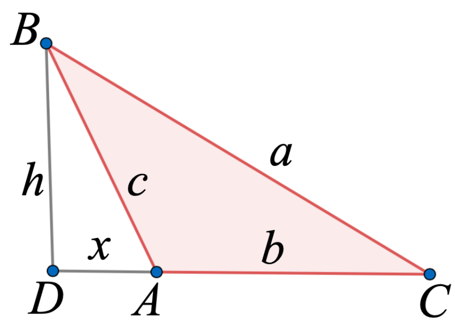
Rearranging

$$a^2 = b^2 + c^2 - 2xb$$

$x/c$ is the cosine of $\angle A$ so $x = c \cos A$ so then finally

$$a^2 = b^2 + c^2 - 2bc \cos A$$




For the obtuse case the initial arithmetic has a change of sign. ($a$ is still the side opposite $\angle A$, but it is obviously bigger now).

$$a^2 = h^2 + (b + x)^2$$

$$c^2 = h^2 + x^2$$

Subtracting as before:

$$a^2 - c^2 = (b + x)^2 - x^2 = b^2 + 2xb$$

Rearranging

$$a^2 = b^2 + c^2 + 2xb$$

There is a change of sign for the last term.

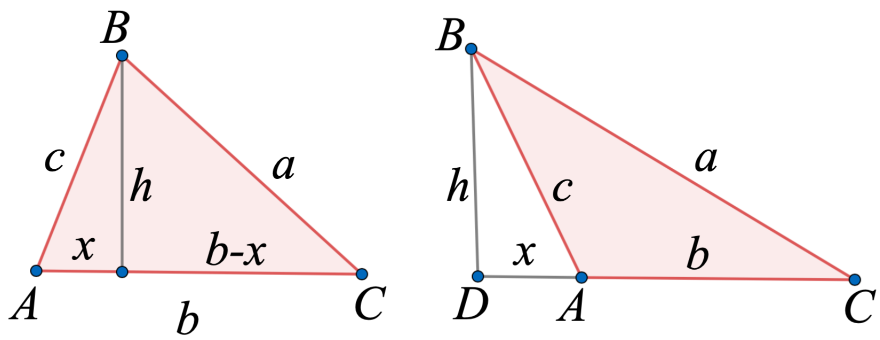
However, there is another difference for the obtuse case. Now the angle *inside* the triangle is supplementary to the angle whose cosine is $x/c$.

Recall the sum of angles formula for cosine. Let $\theta$ and $\phi$ be supplementary so $\theta + \phi = \pi$. Then

$$\cos \theta = \cos \pi - \phi$$

$$= \cos \pi \cos \phi + \sin \pi \sin \phi = - \cos \phi$$

Thus, when we think about $\angle A$ as the angle at vertex $A$ *inside* the triangle




for the obtuse case we have

$$\cos A = -\frac{x}{c} \qquad -c\cos A = x$$

giving the same ormula for both cases:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

where $\angle A$ is $\angle BAC$ for the acute case and $\angle B'AC$ for the obtuse case.

This is the law of cosines. We compute the length of one side $a$ in terms of the two other sides $b$ and $c$ and the angle between them, $\angle A$. This proof is identical to Euclid Book II.12 and II.13, except for the last step.

Using the cosine in the formula is just a form of shorthand for the ratio $x/c$ and gets rid of that pesky term $x$.

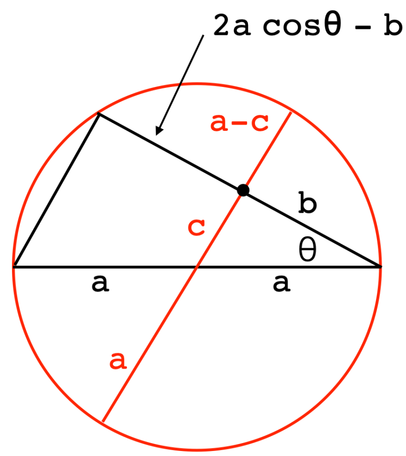
This is Pythagorean theorem with an added correction factor, $-2bc\cos A$, that depends on the angle opposite the hypotenuse and which disappears when that angle is a right angle, since the cosine is zero.

The factor is negative for an acute angle, which makes sense, since the smaller angle squeezes the hypotenuse to be smaller as well, while it is positive for an obtuse angle, giving a longer side opposite the greater angle.

## alternative proof

Here is an alternative derivation based on the products of parts of two secants, for the special case of a right triangle.

Draw a right triangle on one diameter in a circle of radius $a$. Draw a second diameter such that it crosses the base of the right triangle at a right angle, forming a smaller, similar right triangle.




The smaller triangle has sides $a, b$ and $c$. The lengths of the other parts are easy to compute. Now multiply

$$(a + c)(a - c) = b(2a \cos \theta - b)$$

$$a^2 - c^2 = 2ab \cos \theta - b^2$$

The result follows immediately.

$$a^2 - c^2 = 2ab \cos \theta - b^2$$

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

□

## algebraic proof

In $\triangle ABC$ drop the altitude from vertex $A$ to side $a$ opposite then

$$a = b \cos C + c \cos B$$

In the same way:

$$b = a \cos C + c \cos A$$

$$c = a\cos B + b\cos A$$

Multiply the first by $a$

$$a^2 = ab\cos C + ac\cos B$$

In the same way

$$b^2 = ab\cos C + bc\cos A$$

$$c^2 = ac\cos B + bc\cos A$$

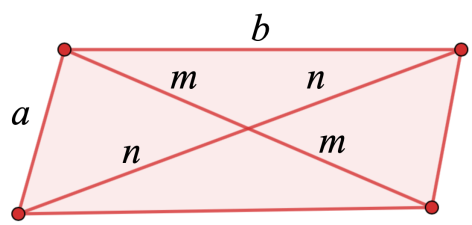
Subtract the first and second from the third:

$$c^2 - a^2 - b^2 = -2ab\cos C$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

## parallelogram sides

The law of cosines leads to an interesting relationship between the sides and diagonals of a parallelogram. Recall that the diagonals divide the figure into two congruent triangles. They also bisect one another.




*Proof.*

For convenience we label the half-diagonals as $m$ and $n$. Applying the theorem twice we have

$$a^2 = m^2 + n^2 - 2mn\cos\theta$$

$$b^2 = m^2 + n^2 + 2mn\cos\theta$$

The sign change on the last term arises because the angles at the center are supplementary, so their cosines are negatives. By addition:

$$a^2 + b^2 = 2(m^2 + n^2)$$

But if $e$ and $f$ are the diagonals then $m^2 = e^2/4$ and $n^2 = f^2/4$ so

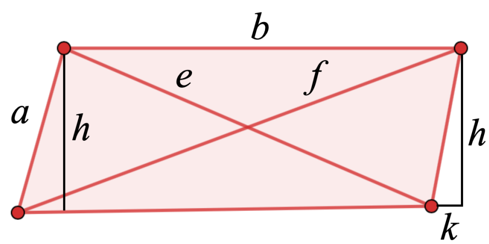
$$2(a^2 + b^2) = e^2 + f^2$$

The sum of the squares of all four sides is equal to the sum of the squares of the diagonals.

Of course this is correct for a rectangle (by Pythagoras's theorem), but it remains true when the shape leans to one side.

□

Al alternative proof (Byer) is to apply Pythagoras directly




*Proof.*

$$\begin{aligned} a^2 &= h^2 + k^2 \\ e^2 &= h^2 + (b-k)^2 \\ f^2 &= h^2 + (b+k)^2 \\ e^2 + f^2 &= 2h^2 + 2b^2 + 2k^2 \\ &= 2(a^2 + b^2) \end{aligned}$$

□

## philosophy

Trigonometry is not just about problems like finding the measure of the angle complementary to 23° as 67°.

Instead, trigonometry uses formulas like the sum of angles, and especially, the law of cosines, to solve problems in calculus.

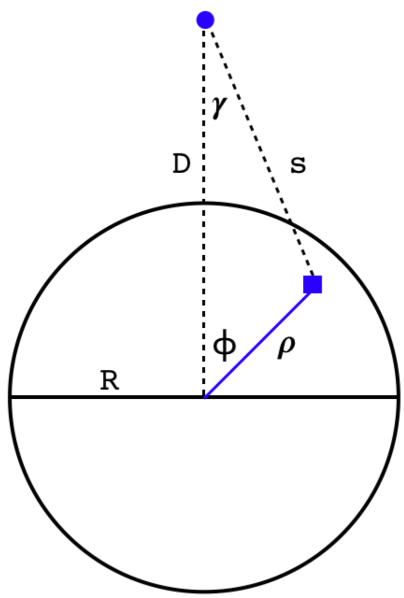
One of the most famous applications came when Newton derived Kepler's laws about the orbits of the planets. Originally, to do that he made the approximation that the mass of the earth acts *as if* it were concentrated at a single point corresponding to the center of the earth, and likewise for the sun.

However, for a rigorous demonstration he needed to prove that this approximation is correct. We do not have the tools yet to see how he did that, but here are two equations from my write-up:

$$\rho^2 = D^2 + s^2 - 2Ds\cos\gamma$$

$$\cos\gamma = \frac{D^2 + s^2 - \rho^2}{2Ds}$$

and the relevant diagram:




You can probably recognize the law of cosines at work.

Trigonometry is hugely important in math and science. Although it has (simple) applications for activities like surveying that is not at all what it is about.

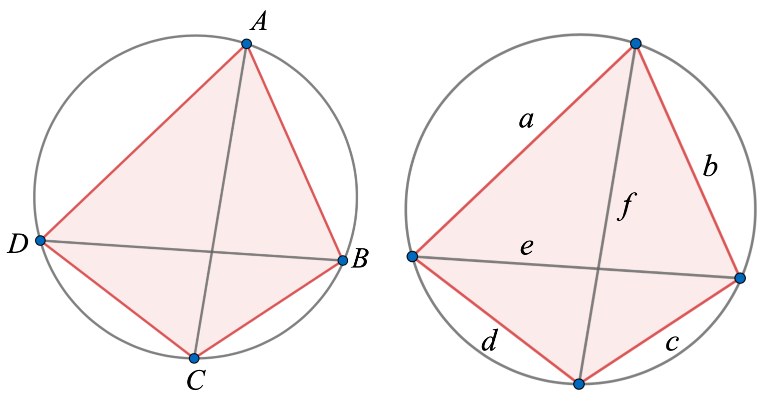
# Part IV

# Applications of similarity

# Chapter 15

# Ptolemy revisited

In a previous **chapter** we introduced Ptolemy's theorem.

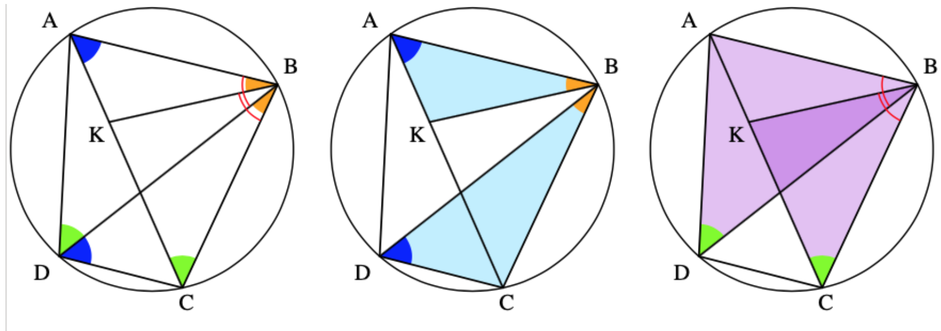



In the left panel

$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

or somewhat more compactly in the right panel: $ac + bd = ef$.

Here we provide two more proofs of this theorem, as examples of wonderful proofs, and then explore some consequences.

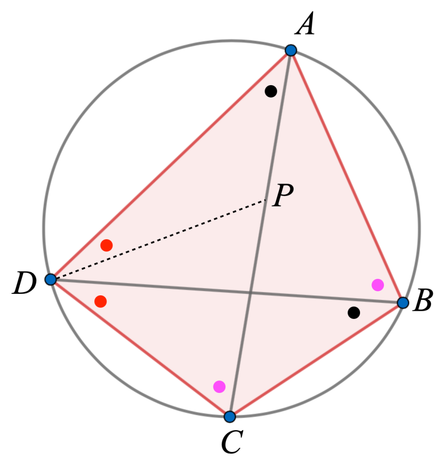



Above is a graphic from wikipedia that shows where we're going in the first proof. We will form two sets of similar triangles and use our knowledge about corresponding ratios.

https://en.wikipedia.org/wiki/Ptolemy%27s_theorem

## Ptolemy's theorem from similar triangles

*Proof.*




Find $P$ on $AC$ such that $\angle ADP = \angle CDB$ (red dots).

Since $ABCD$ is a cyclic quadrilateral, we can find other equal angles (black and magenta dots).

We write the vertices in the same order as the equal angles.

$\triangle ADP \sim \triangle BDC$.

So

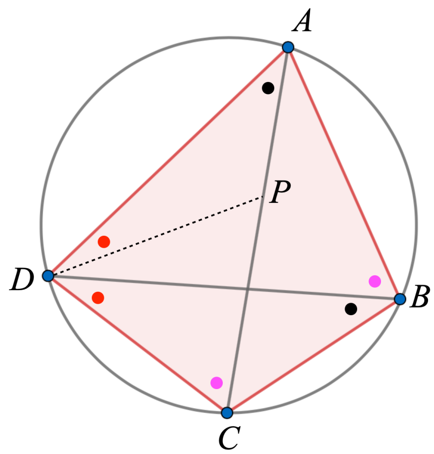
$$\frac{AD}{BD} = \frac{DP}{CD} = \frac{AP}{BC}$$
$$AD \cdot BC = AP \cdot BD$$

Since $\angle PDB$ is shared, $\angle ADB = \angle PDC$.

$\triangle PDC \sim \triangle ADB$.

So

$$\frac{PD}{AD} = \frac{CD}{BD} = \frac{PC}{AB}$$
$$AB \cdot CD = PC \cdot BD$$




Adding

$$AB \cdot CD + AD \cdot BC = PC \cdot BD + AP \cdot BD$$
$$= AC \cdot BD$$

□

This is Ptolemy's theorem.

Yiu also proves the converse theorem.

The dots in this proof make it clear which two pairs of triangles are similar, and I've taken care to list the vertices of the triangles in the same order as the sides in each pair of similar triangles, from smallest to largest.

For an alternate notation see **here**.

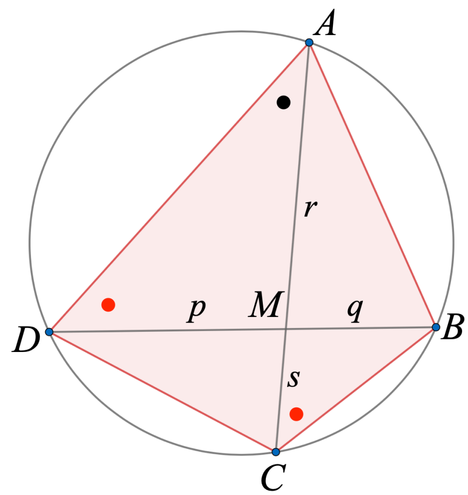
## proof by switching sides

(adapted from wikipedia).

https://en.wikipedia.org/wiki/Ptolemy%27s_theorem

In the proof below, we denote the area of polygons such as $ABCD$ as $(ABCD)$.

*Proof.*




We add up (twice) the areas of the component triangles:

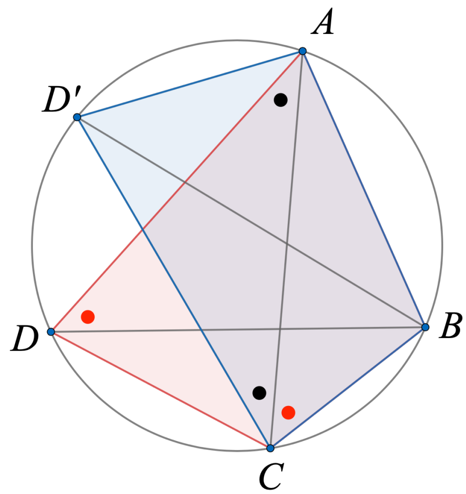
$$\begin{aligned} 2(AMB) &= qr \cdot \sin AMB \\ 2(BMC) &= qs \cdot \sin BMC \\ 2(CMD) &= ps \cdot \sin AMB \\ 2(AMD) &= pr \cdot \sin BMC \end{aligned}$$

But $\sin AMB = \sin BMC$. Hence twice the area of $(ABCD)$ is

$$\begin{aligned} 2(ABCD) &= (qr + qs + pr + ps) \cdot \sin AMB \\ &= (q + p)(r + s) \cdot \sin AMB \\ &= AC \cdot BD \cdot \sin AMB \end{aligned}$$

We're on to something. Now, the great idea.

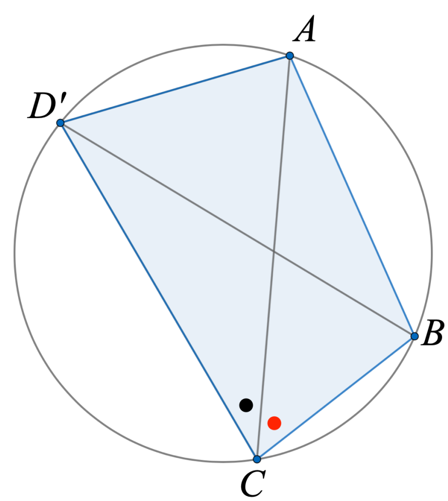
Move the point $D$ to $D'$ such that $AD' = CD$ and $AD = CD'$.




When we do this, note two things. First, $\angle CAD = \angle ACD'$ (black dots), because the arcs they intercept are equal, $AD' = CD$.

Second, $\triangle ADC \cong \triangle ACD'$ by SSS. Therefore the area hasn't changed: $(ABCD) = (ABCD')$.

Now compute (twice) the area of two component triangles.



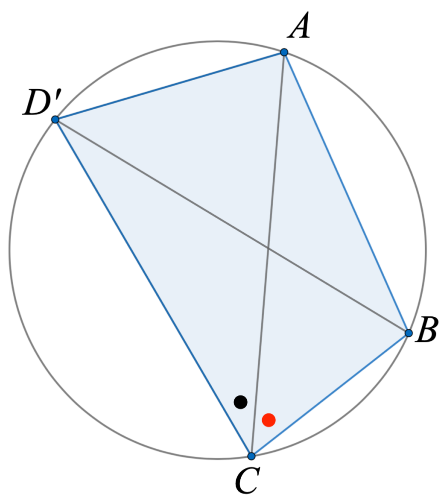

$$
\begin{aligned}
2(ABD') &= AB \cdot AD' \cdot \sin \angle BAD' \\
&= AB \cdot AD' \cdot \sin \angle BCD'
\end{aligned}
$$

$$2(CBD') = BC \cdot CD' \cdot \sin \angle BCD'$$

since the two angles are supplementary.

But $\angle BCD' = \angle ADC + \angle CAD$. (Check out the dots).

So $\angle BCD' = \angle AMB$. Thus, their sines are equal. We have




$$2(ABD') + 2(CBD') = 2(ABCD)$$

$$[\ AB \cdot AD' + BC \cdot CD'\ ]\ \sin \angle BCD' = AC \cdot BD \cdot \sin AMB$$

$$AB \cdot AD' + BC \cdot CD' = AC \cdot BD$$

Finally, since $AD = CD'$ and $CD = AD'$:

$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

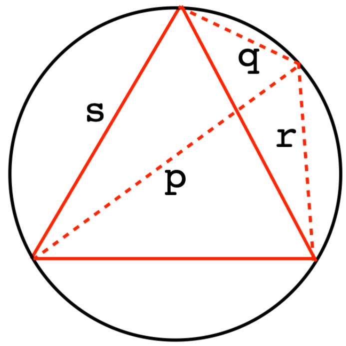
$\square$

This is Ptolemy's theorem.

**corollaries**

Here are just a few of the results that follow from this remarkable theorem.

## equilateral triangle



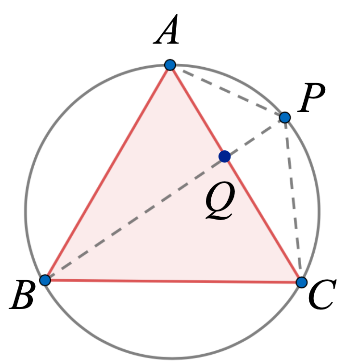

Inscribe an equilateral triangle in a circle and pick any point on the circle.

$$ps = qs + rs$$

$$p = q + r$$

We proved this earlier, without using Ptolemy's theorem, as Van Schooten's theorem.

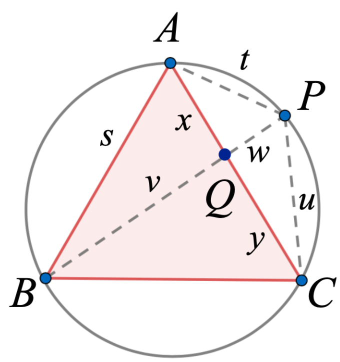
Here's a different problem from basically the same diagram (Coxeter).




$$\frac{1}{PA} + \frac{1}{PC} = \frac{1}{PQ}$$

Let us re-label with $PA = t$, $PC = u$ and $PQ = w$. We will prove that:

$$\frac{1}{t} + \frac{1}{u} = \frac{1}{w}$$




*Proof.*

We have similar triangles:

$$\triangle AQB \sim \triangle PQC, \qquad \triangle AQP \sim \triangle BQC$$

Which gives the ratios

$$\frac{v}{y} = \frac{s}{u} = \frac{x}{w}, \qquad \frac{w}{y} = \frac{t}{s} = \frac{x}{v}$$

Thus:

$$\frac{s}{u} = \frac{x}{w}, \qquad \frac{s}{t} = \frac{y}{w}$$

Adding

$$\frac{s}{u} + \frac{s}{t} = \frac{x+y}{w} = \frac{s}{w}$$
$$\frac{1}{t} + \frac{1}{u} = \frac{1}{w}$$

□

## Pythagorean theorem

Let the quadrilateral be a rectangle. The the sum of squares of opposing sides is

$$a^2 + b^2$$

Triangles made by opposing diagonals are congruent, so the diagonals are equal in length. The diagonal is the hypotenuse, hence

$$a^2 + b^2 = c^2$$

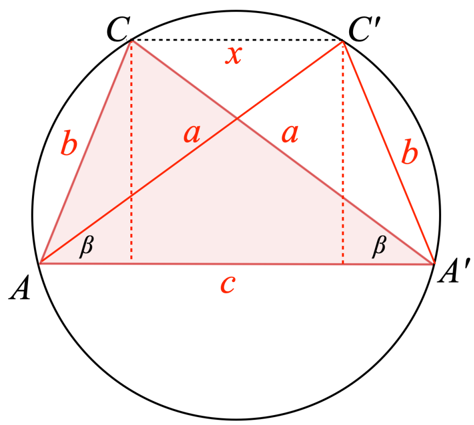
We saw this proof previously (**here**).

## Law of Cosines

Draw $\triangle ABC$ (suppress the $B$ label) and then draw another triangle congruent with it, with a shared base, and all four points in a circle, forming a cyclic quadrilateral.

Relying on previous work with a rectangle in a circle (**here**), we know this construction is possible.

The points are $A, A', C, C'$. $\beta$ marks the original $\angle B$, but will not be used.




We need an expression for $x$. We have that the base of the altitude from $C$ to side $c$ is a distance from $A$ equal to $(c-x)/2$. It follows that

$$\frac{c-x}{2} \div b = \cos A$$

$$c - x = 2b\cos A$$

$$x = c - 2b\cos A$$

Now, apply Ptolemy's Theorem. We have:

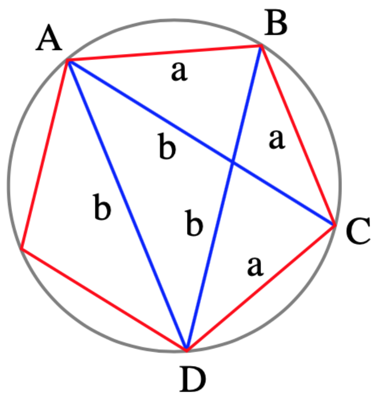
$$a^2 = b^2 + cx$$

$$= b^2 + c(c - 2b\cos A)$$

$$= b^2 + c^2 - 2bc\cos A$$

$\square$

## golden mean in the pentagon




Take four vertices of the regular pentagon and draw two diagonals. From the theorem, we have

$$b \cdot b = a \cdot a + a \cdot b$$
$$\frac{b^2}{a^2} = 1 + \frac{b}{a}$$

Rather than use the quadratic equation, rearrange and add 1/4 to both sides to "complete the square":

$$\frac{b^2}{a^2} - \frac{b}{a} + \frac{1}{2^2} = 1 + \frac{1}{2^2}$$

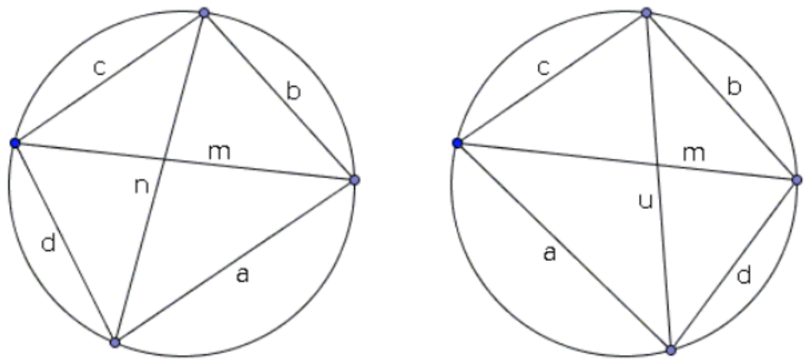
So

$$(\frac{b}{a} - \frac{1}{2})^2 = \frac{5}{4}$$
$$\frac{b}{a} - \frac{1}{2} = \pm\ \frac{\sqrt{5}}{2}$$
$$\frac{b}{a} = \frac{1 \pm \sqrt{5}}{2}$$

This ratio $b/a$ is known as $\phi$, the golden mean.

## diagonals

Let us look at something like what we used for the proof of Ptolemy's theorem in the beginning.

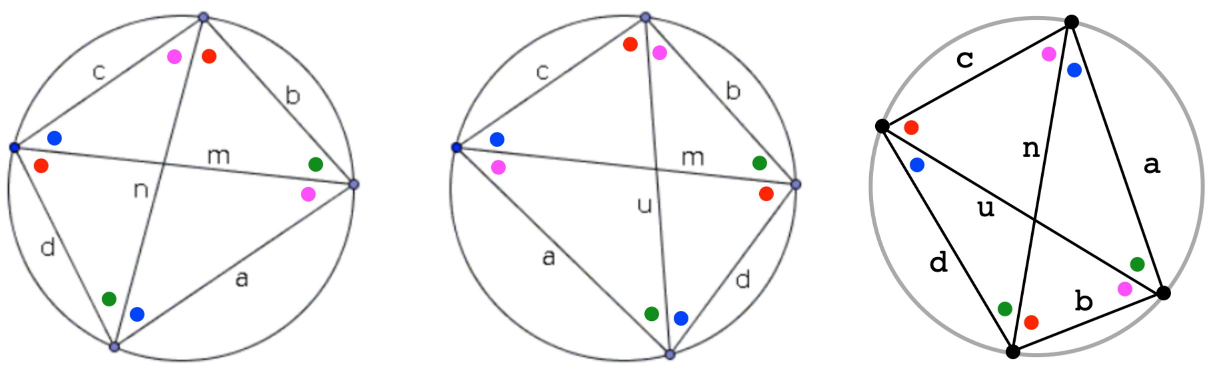



$$nm = ac + bd$$

We move one of the points, exchanging sides $a$ and $d$. Then, one of the diameters, $n$, changes length to $u$.

$$mu = ab + cd$$

If, instead, we exchange sides $a$ and $b$, the old $m$ changes to $u$. Why?




Mark the peripheral angles with equal arcs ($abcd$: red, blue, green, magenta).

The triangle with sides $b$ and $d$ in the middle, and magenta plus red for the vertex angle, is congruent to one in the right panel. So their long sides are equal, both have length $u$.

Thus,

$$nu = ad + bc$$

We get a formula for the square of the diagonal:

$$m^2 = \frac{(mu)(nm)}{nu} = \frac{(ab+cd)(ac+bd)}{(ad+bc)}$$

There is a similar formula for $n^2$. These formulas are sometimes attributed to Brahmagupta. This beautiful proof is due to Paramesvara (14th century).

https://www.cut-the-knot.org/proofs/PtolemyDiagonals.shtml

The ratio is

$$\frac{m}{n} = \frac{ab+cd}{ad+bc}$$

which is referred to as Ptolemy's second theorem.

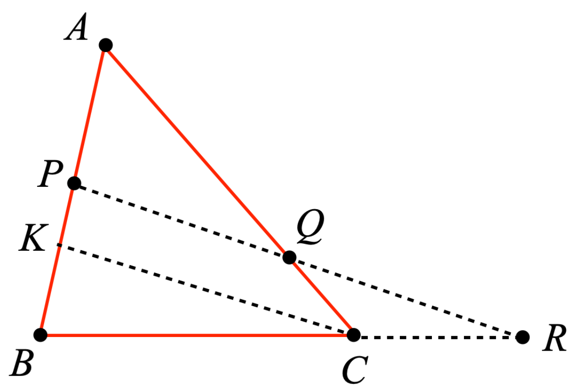
# Chapter 16

# Menelaus's theorem

In this chapter we establish a simple but very valuable result called **Menelaus's theorem**, due to Menelaus of Alexandria (the geometer, not the mythological figure). He lived about 100 CE. It is supposed that Menelaus grew up in Alexandria but is known to have lived in Rome.

## Menelaus's theorem

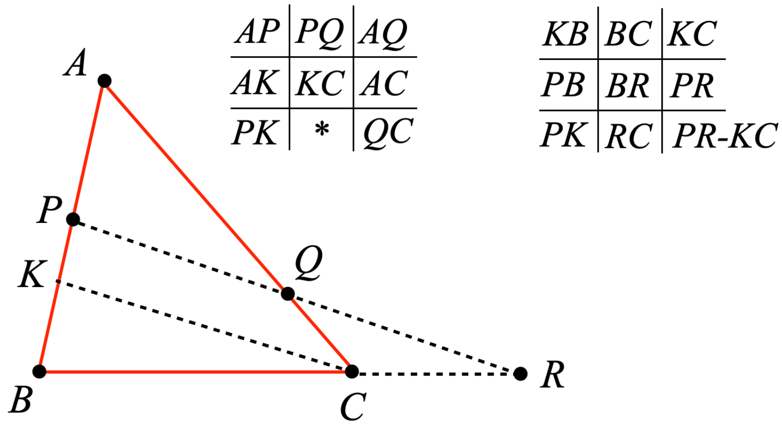
Consider $\triangle ABC$ and a line that goes through the triangle (not through a vertex and not along a side), called a transversal. There is one side, which when it is extended, meets the transversal. Here the transversal $PQR$ meets $BC$ at $R$.




*Proof.*

Draw a line parallel to the transversal, $CK$. We have two pairs of similar triangles.

The first is $\triangle APQ \sim \triangle AKC$, and the second is $\triangle KBC \sim \triangle PBR$. I will write the ratios as shown in the figure. I call these ratio boxes.




From the first set:

$$\frac{AP}{AQ} = \frac{PK}{QC}$$

And from the second set:

$$\frac{PB}{BR} = \frac{PK}{RC}$$

Isolate $PK$ and equate:

$$\frac{AP}{AQ} \cdot QC = \frac{PB}{BR} \cdot RC$$

$$\frac{AP}{PB} \cdot \frac{BR}{RC} \cdot \frac{QC}{AQ} = 1$$

□

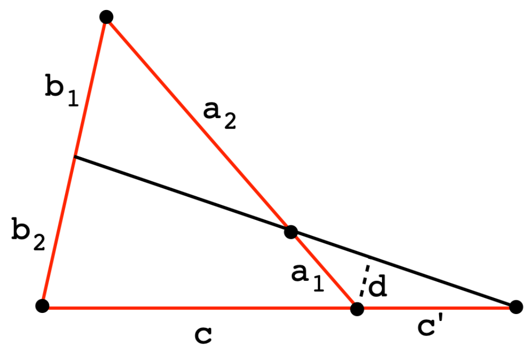
We will show several different proofs, for practice.

In the figure below, subscripts indicate the parts of a side, for example, $a_1$ and $a_2$ are the components of side $a$.

$c + c'$ is the length of the whole side *plus the extension*.

We will show that the product of three ratios is equal to 1:

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c + c'}{c'} = 1$$




*Proof.*

Draw the dotted line segment parallel to $b$ and label it $d$. We have two pairs of similar triangles. The first pair has side ratios

$$\frac{d}{a_1} = \frac{b_1}{a_2}$$

while the second has

$$\frac{d}{c'} = \frac{b_2}{c + c'}$$

Combining the two results:

$$d = \frac{a_1 b_1}{a_2} = \frac{b_2 c'}{c + c'}$$

So

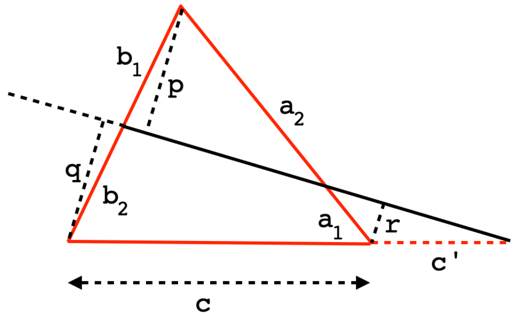
$$\frac{a_1 \cdot b_1 \cdot c + c'}{a_2 \cdot b_2 \cdot c'} = 1$$

□

## Einstein's proof of Menelaus

There is a story that Einstein disliked the proofs of Menelaus's theorem based on similar triangles.

He said they were ugly, and didn't involve the vertices symmetrically. I read that Einstein's proof starts by dropping altitudes to the transversal, and then uses similar triangles. Here's what I came up with.




*Proof.*

The right triangle with $c+c'$ as hypotenuse is similar to the one with $c'$ as hypotenuse. This gives

$$\frac{q}{r} = \frac{c+c'}{c'}$$

The right triangles with $a_1$ and $a_2$ as hypotenuse are similar. We form the ratio of altitudes:

$$\frac{r}{p} = \frac{a_1}{a_2}$$

Finally, the right triangles with $b_1$ and $b_2$ as hypotenuse are similar. We form the ratio of altitudes again:

$$\frac{p}{q} = \frac{b_1}{b_2}$$

Multiply the left-hand sides all together to obtain 1.

$$\frac{q}{r} \cdot \frac{r}{p} \cdot \frac{p}{q} = 1$$

Therefore, the product of the right-hand sides is also 1:

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c+c'}{c'} = 1$$

□

According to

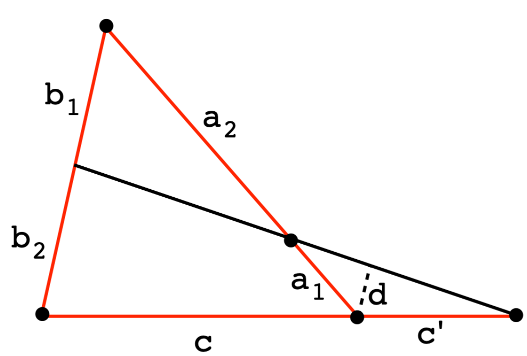
https://www.cut-the-knot.org/Generalization/MenelausByEinstein.shtml

> ... [in] correspondence between Albert Einstein and a friend of his, Max Wertheimer. In the first letter, Einstein apparently continues a discussion on elegance of mathematical proofs. A proof may require introduction of additional elements, like line AP in the first of the cited proofs. In Einstein's opinion, "... we are completely satisfied only if we feel of each intermediate concept that it has to do with the proposition to be proved."
>
> As an illustration of his viewpoint, Einstein gives two proofs of the same proposition - one ugly, the other elegant. Curiously, the proposition he proves is that of the Menelaus theorem, and the proof ugly in his view is the first of the cited proofs. He writes, "Although the first proof is somewhat simpler, it is not satisfying. For it uses an auxiliary line which has nothing to do with the content of the proposition to be proved, and the proof favors, for no reason, the vertex A, although the proposition is symmetrical in relation to A, B, and C. The second proof, however, is symmetrical, and can be read off directly from the figure."

## converse

We've drawn $\triangle ABC$ as an acute triangle, with the transversal crossing through points on two sides and on the extension of the third. There are other possibilities. However, if we restrict ourselves to this case, then the converse can be stated as

Given that the product of ratios is equal to 1, the three points are collinear.




*Proof.*

An easy proof draws on the forward theorem. Assume that the product of ratios is 1.

Suppose the points are not colinear, so the case with product 1 has $c''$ and the case with colinear points has $c'$. We are given that

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c + c''}{c''} = 1$$

But, by the forward theorem

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c + c'}{c'} = 1$$

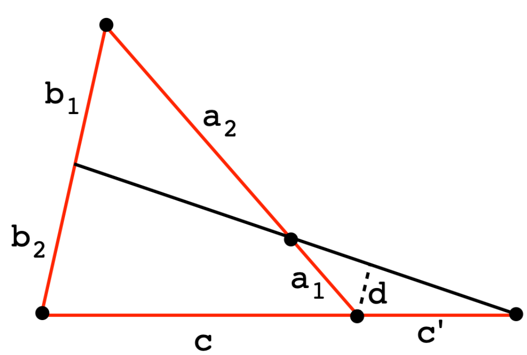
We have

$$\frac{c + c''}{c''} = \frac{c + c'}{c'}$$
$$\frac{c}{c''} = \frac{c}{c'}$$
$$\frac{1}{c''} = \frac{1}{c'}$$

We conclude that $c' = c''$ Hence, if the product is equal to 1, the points lie on the same line.

□

## more about Menelaus



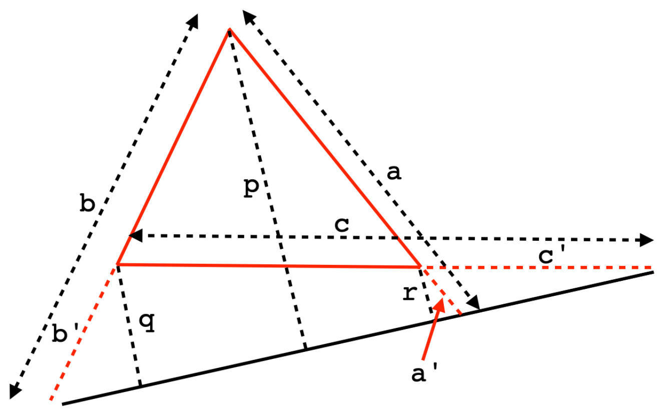
We showed that the product of three ratios is equal to 1:

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c + c'}{c'} = 1$$

One way to remember the terms is that, going around in any direction, we take the first length we run into and put it into the numerator. It doesn't matter which direction you go, as long as you take $c'$ first when going clockwise and $c + c'$ first when going counter-clockwise.

Take the extension first if you run into it first.

We also need to prove the theorem for an alternative setup. Even though the black line in the figure below does not go through the triangle, it is technically a transversal.




Notice that we are re-defining $a$, $b$ and $c$.

$$\frac{r}{p} \cdot \frac{p}{q} \cdot \frac{q}{r} = \frac{a'}{a} \cdot \frac{b}{b'} \cdot \frac{c}{c'}$$

Once again, we can remember this as, take the first length we run into, and put it on top.

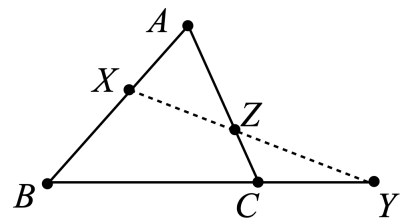
Menelaus is a preliminary lemma to an easy proof of Ceva's theorem, which becomes important in thinking about the concurrency of lines in a triangle, like those which bisect the opposing sides and meet at the centroid. Ceva's theorem gives a condition under which three lines like this are concurrent, or meet at a point.

Menelaus's theorem can be seen as in some sense as similar to Ceva, because it provides conditions under which three points can be proved collinear, or on one line.

It will come in handy in another problem, called Pappus's Theorem.

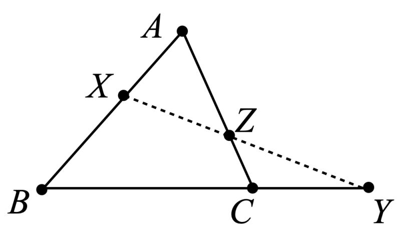
## alternative definition

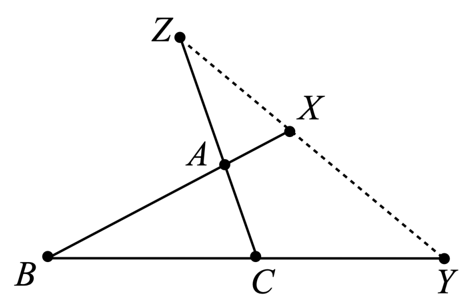
There is a somewhat more sophisticated version of Menelaus's Theorem which views the path around a triangle as consisting of *directed* (that is, *signed*) line segments.




The transversal is $XZY$, which meets the extension of $BC$ at $Y$. Since $Y$ does not lie between $B$ and $C$, the line segments $BY$ and $YC$ point in opposite directions. The ratio $BY : YC$ thereby acquires a minus sign.

Every transversal has this property. Here (on the right) is a transversal that does not go through the triangle at all.




The path around the triangle has three parts:

$A$ to $X$ to $B \Rightarrow AX : XB$

$B$ to $Y$ to $C \Rightarrow BY : YC$

$C$ to $Z$ to $A \Rightarrow CZ : ZA$

Each ratio has a minus sign so the total product also has a minus sign. Menelaus's Theorem says:

$$\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZA} = -1$$

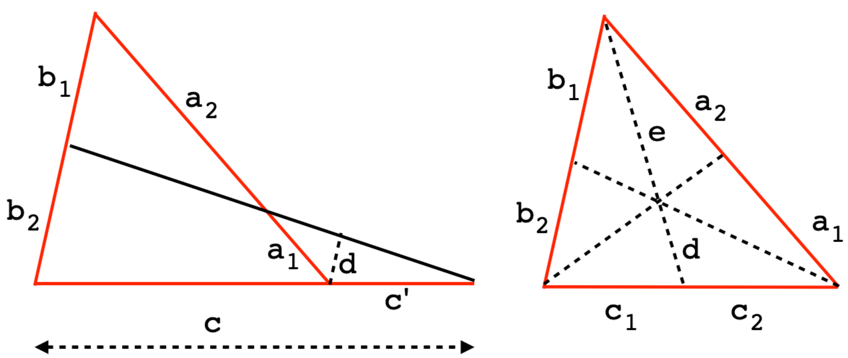
# Chapter 17

# Ceva's Theorem

## Ceva's theorem

We previously showed a proof of Menelaus' theorem. Let us continue with Ceva's theorem (right panel, below). We will show that if the *three lines are concurrent* (they all cross at the same point), then

$$\frac{a_1 \cdot b_1 \cdot c_1}{a_2 \cdot b_2 \cdot c_2} = 1$$




We ignore here the signed or directed line segments of the original, which give minus 1 as the product above. These would cancel at the next step.

*Proof.*

The left panel is from the proof of Menelaus' theorem. In the right panel, consider the left half-triangle with a side composed of $e + d$. Apply Menelaus' theorem once.

$$\frac{d}{e} \cdot \frac{b_1}{b_2} \cdot \frac{c}{c_2} = 1$$

For the right half-triangle, apply Menelaus again but go clockwise from the middle:

$$\frac{d}{e} \cdot \frac{a_2}{a_1} \cdot \frac{c}{c_1} = 1$$

Combine the two results:

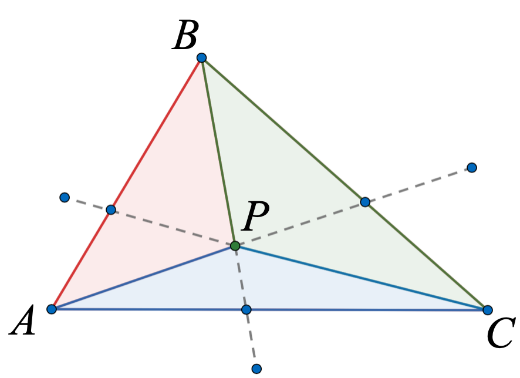
$$\frac{b_1}{b_2} \cdot \frac{c}{c_2} = \cdot \frac{a_2}{a_1} \cdot \frac{c}{c_1}$$

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1$$

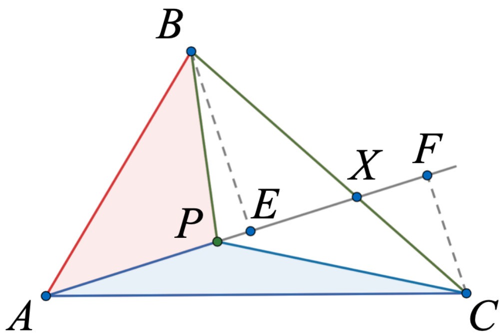
□

### proof based on area

Draw $\triangle ABC$ and let $P$ be a point somewhere inside. Extend lines from each vertex through $P$ to reach the opposite side.




Consider two of the smaller triangles, say, $\triangle APB$ and $\triangle APC$.




If we think of $AP$ as the base for both triangles, then $BE$ is the altitude of the first one and $CF$ is the altitude of the second. Since they have the same base, the ratio of areas is in the proportion:

$$\frac{\mathcal{A}_{APB}}{\mathcal{A}_{APC}} = \frac{BE}{CF}$$

But now consider $\triangle BEX$ and $\triangle CFX$. They are both right triangles with shared vertical angles. Thus

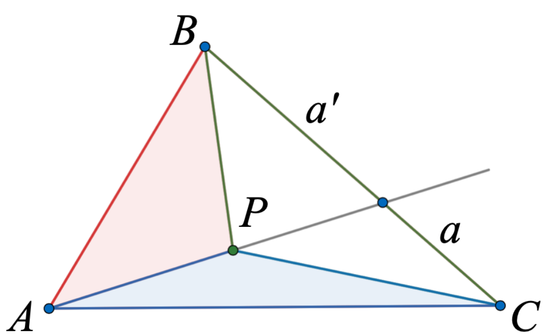
$$\triangle BEX \sim \triangle CFX$$

As similar triangles, the sides are in proportion

$$\frac{BX}{CX} = \frac{BE}{CF} = \frac{\mathcal{A}_{APB}}{\mathcal{A}_{APC}}$$

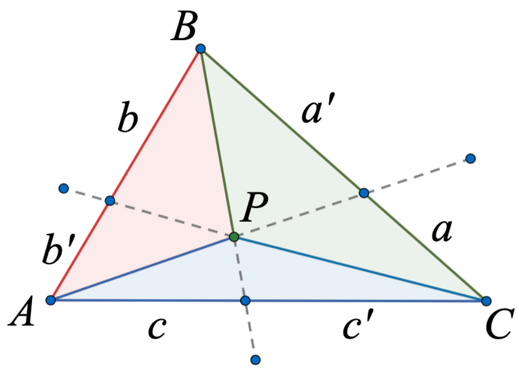
Reversing the order of the triangles and using a simpler notation for the parts of side $a$:

$$\frac{a}{a'} = \frac{\mathcal{A}_{APC}}{\mathcal{A}_{APB}}$$



This result is completely general. $P$ could be anywhere, and we can compare any two of the three triangles.

$$\frac{b}{b'} = \frac{\mathcal{A}_{BPC}}{\mathcal{A}_{APC}}, \qquad \frac{c}{c'} = \frac{\mathcal{A}_{APB}}{\mathcal{A}_{BPC}}$$




Multiplying:

$$\frac{\mathcal{A}_{APC}}{\mathcal{A}_{APB}} \cdot \frac{\mathcal{A}_{BPC}}{\mathcal{A}_{APC}} \cdot \frac{\mathcal{A}_{APB}}{\mathcal{A}_{BPC}} = \frac{a}{a'} \cdot \frac{b}{b'} \cdot \frac{c}{c'}$$

Since the left-hand side cancels, we obtain:

$$\frac{a}{a'} \cdot \frac{b}{b'} \cdot \frac{c}{c'} = 1$$

□

This is Ceva's theorem.

The proof also works in reverse, We change notation slightly, since we will use the prime symbol for another purpose.

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1 \iff 3 \text{ lines cross at point P}$$

*Proof.*

Everything is the same as before, except we suppose that the ratio of the parts of $a$ has to be slightly different in order to obtain 1 when we multiply everything together.

Of course, this would mean that the line from $A$ would no longer go through $P$. Suppose the new line gives $a_1'$ and $a_2'$ as the components of the side $a$.

$$a_1 \neq a_1', \qquad a_2 \neq a_2'$$

But, by the forward theorem we have that the line which *does* go through $P$ divides the side into $a_1$ and $a_2$ such that

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1$$

We conclude that

$$\frac{a_1}{a_2} = \frac{a_1'}{a_2'}$$

Since they are parts of the same length, the components are individually equal. That is, if the whole side is $a$ then $a_2 = a - a_1$ and $a_2' = a - a_1'$ so

$$\frac{a_1}{a - a_1} = \frac{a_1'}{a - a_1'}$$

$$a_1 a - a_1 a_1' = a_1' a - a_1 a_1'$$

$$a_1 = a_1'$$

□

## centroid

In the general case, the crossing lines are called cevians. In the case of medians, they divide the sides opposite in half, and then the central point is called the centroid. Recall that we had

$$\frac{a}{a'} = \frac{\mathcal{A}_{APC}}{\mathcal{A}_{APB}}$$

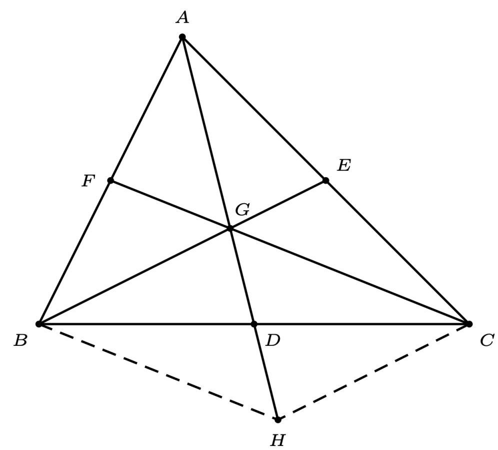
$$\frac{b}{b'} = \frac{\mathcal{A}_{BPC}}{\mathcal{A}_{APC}}$$

But if $a = a'$ and $b = b'$, then $\mathcal{A}_{APC} = \mathcal{A}_{APB} = \mathcal{A}_{BPC}$.

For a physical object, the point $P$ would be the *center of mass*.

## Centroid

Consider the $\triangle ABC$. Draw the medians $BE$ and $CF$ (bisectors of the sides). Extend $AG$ through the intersection of the two medians at $G$ to $D$ and then finally to $H$, where $CH$ is drawn parallel to $BGE$. We profess not to know anything about $BH$ yet.



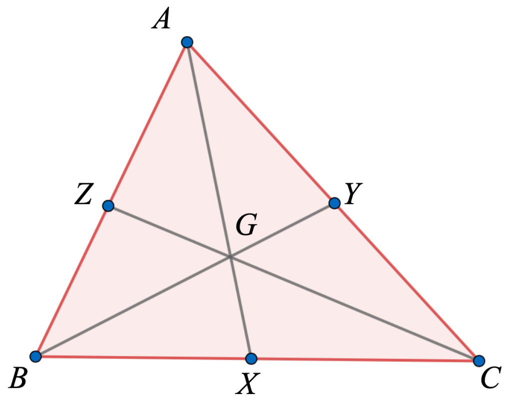

Since $CH \parallel BGE$ and $AE = EC$, we have that $\triangle AGE \sim \triangle AHC$ with ratio 2 by the midpoint theorem (or what we've called SAS similarity). Thus, $AG = GH$.

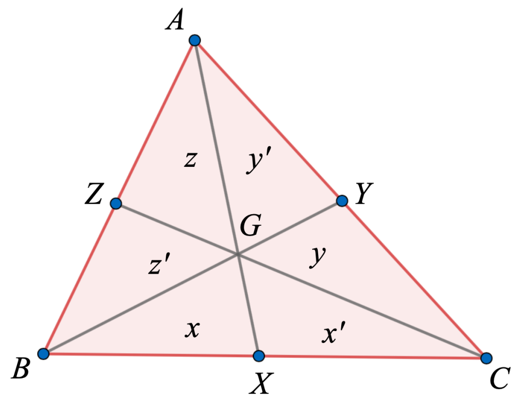
But we also have $AF = FB$, which means that $\triangle AFG \sim \triangle ABH$ with ratio 2. This gives $BH \parallel FGC$.

Therefore, $BGCH$ has two pairs of opposing sides parallel so it is a parallelogram. The diagonals cross at their midpoints, which means that $AD$ is a median ($BC$ is bisected at $D$).

In addition, $GD = DH$. Thus, $GD$ is one-half of $GH$, and $GH$ is equal to $AG$, so $GD$ is one-quarter of $AH$ and one-third of $AD$. $G$ lies on all three medians, and is called the *centroid* of the triangle. For a physical triangle it would be the center of mass.

## Centroid alternate proof






The medians divide any triangle into six equal small triangles.

*Proof.*

$(\triangle GBX) = (\triangle GCX)$ by the **area-ratio theorem**. Hence the two small triangles on any one side have equal area: $x = x'$, $y = y'$, $z = z'$.

Also by the same theorem, $(\triangle ABX) = (\triangle ACX)$. So by subtraction $(\triangle AGB) = (\triangle AGC)$: $y + y' = z + z'$. Thus $y = z$. But this is true for any side. Hence all six triangles are equal in area.

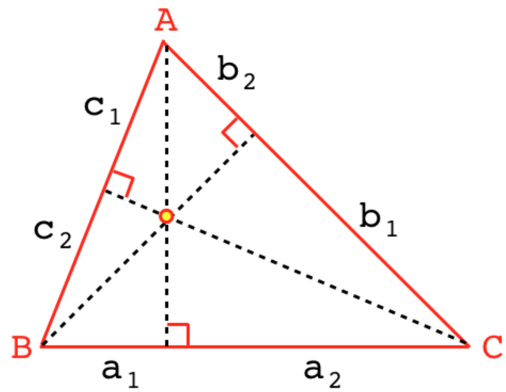
□

Consider $\triangle GBX$ and $\triangle ABX$ on base $XGA$. They have the same altitude, from the base to the vertex at $B$. The ratio of areas is $1 : 3$. So that must also be the ratio $XG : XA$.

The centroid lies one-third of the way up the median from the side.

## orthocenter

Consider this triangle in which we have drawn the altitudes to each side. We claim that they cross at a single point, called the orthocenter.




Let the angles be $A, B, C$ as labeled, and the sides opposite be $a, b, c$, subdivided as shown.

Then $\angle A$ is part of two right triangles, and by similar triangles we have that

$$\frac{c_1}{b} = \frac{b_2}{c} \quad \rightarrow \quad \frac{b_2}{c_1} = \frac{c}{b}$$

Similarly for $\angle B$

$$\frac{a_1}{c} = \frac{c_2}{a} \quad \rightarrow \quad \frac{c_2}{a_1} = \frac{a}{c}$$

And $\angle C$

$$\frac{b_1}{a} = \frac{a_2}{b} \quad \rightarrow \quad \frac{a_2}{b_1} = \frac{b}{a}$$

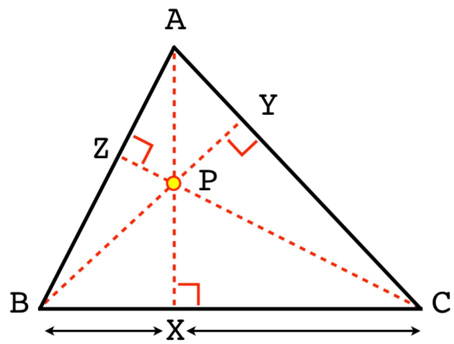
The product of the three right-hand sides above is $c/b \cdot a/c \cdot b/a = 1$. Therefore the product of the left-hand sides is also 1:

$$\frac{b_2}{c_1} \cdot \frac{c_2}{a_1} \cdot \frac{a_2}{b_1} = 1$$

Invert and re-order the terms

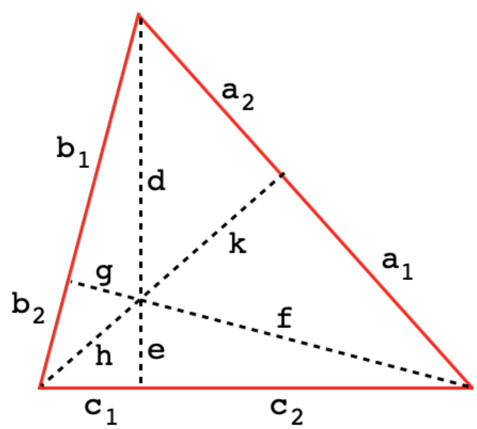
$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1$$

$\square$




Since we have satisfied Ceva's condition, the 3 altitudes all cross at a single point. That point is the orthocenter, and this is a proof that it exists.

## another approach to the orthocenter




There is a different set of similar triangles one can use for the orthocenter. The triangle with sides $a_2$, $d$ and $k$ is similar to the triangle with sides $c_1$, $h$, and $e$. For one pair, we obtain

$$\frac{e}{k} = \frac{c_1}{a_2}$$

You should be able to use the other two pairs to construct Ceva's ratio equal to

$$\frac{e}{k} \cdot \frac{k}{g} \cdot \frac{g}{e}$$

which is, of course, equal to 1.

Another easily derived relationship is that

$$\frac{e}{k} = \frac{h}{d}$$

so

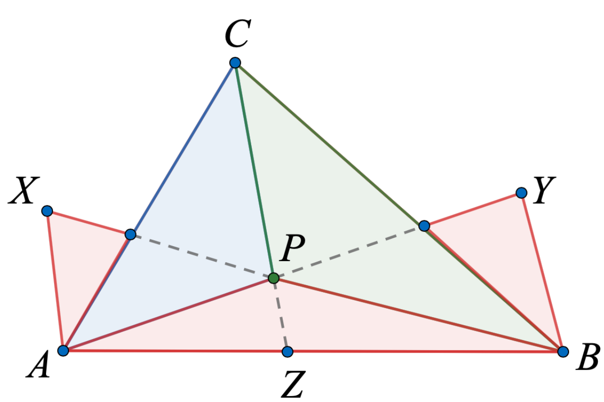
$$de = hk$$

Going around the triangle we will get

$$de = fg = hk$$

This occurs because the position of $P$ with respect to each altitude is the same fraction of the whole.

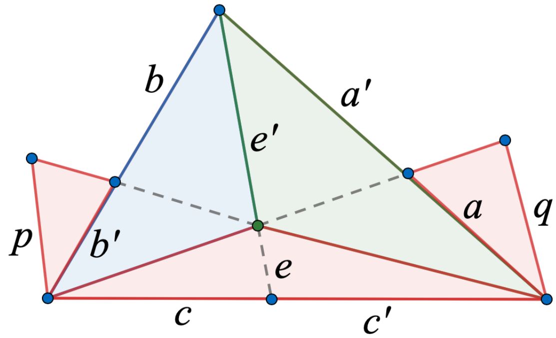
## another proof of Ceva's theorem




I found an alternative approach in a geometry textbook by Jurgensen *et al.* It's a bit weird because this is the only mention of the theorem in the book. I like the proof, which is only hinted at, because we enhance the original diagram, and it provides exercise using similar triangles.

Draw $AX$ and $BY$ parallel to $CPZ$.

I found the algebra to be a little cleaner with different notation, using lowercase letters for the lengths.




*Proof.*

Parallel lines give similar triangles with the ratios:

$$\frac{a}{a'} = \frac{q}{e'}, \qquad \frac{b'}{b} = \frac{p}{e'}$$

which combine to give

$$\frac{q}{p} = \frac{a}{a'} \cdot \frac{b}{b'}$$

We also have

$$\frac{c+c'}{c} = \frac{q}{e}, \qquad \frac{c+c'}{c'} = \frac{p}{e}$$

combined

$$\frac{c'}{c} = \frac{q}{p}$$

Equating the two results

$$\frac{c'}{c} = \frac{a}{a'} \cdot \frac{b}{b'}$$

which rearranges to give

$$\frac{a}{a'} \cdot \frac{b}{b'} \cdot \frac{c}{c'} = 1$$

□

## Viviani

Here's a problem from Acheson that looks challenging, but yields easily to the right perspective.

**Viviani's theorem**

This states that for any internal point P of an *equilateral* triangle the sum of the perpendicular distances from the sides is a constant, independent of the position of P (Fig. 145).

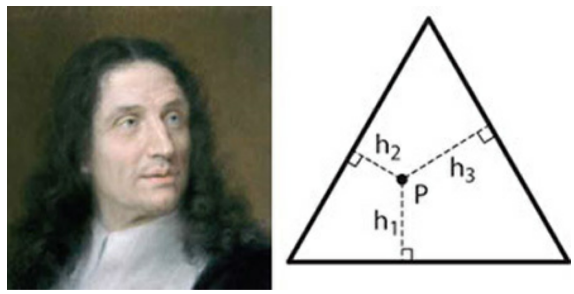
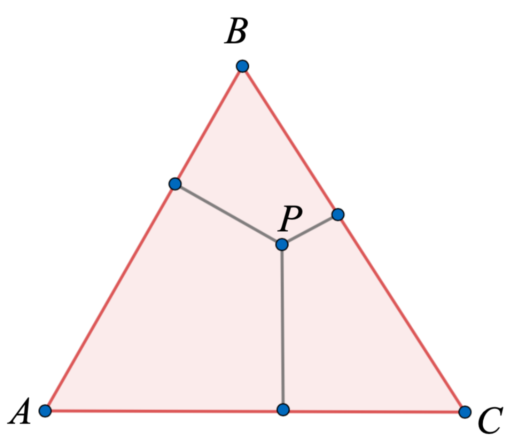



**Fig. 145** Viviani's theorem: in an equilateral triangle, $h_1 + h_2 + h_3 = \text{constant.}$

Rather than letting $P$ be a special point, it can be anywhere inside the triangle. $P$ is the opposite of a special point, it is completely general. However, we are also given that the triangle is equilateral.

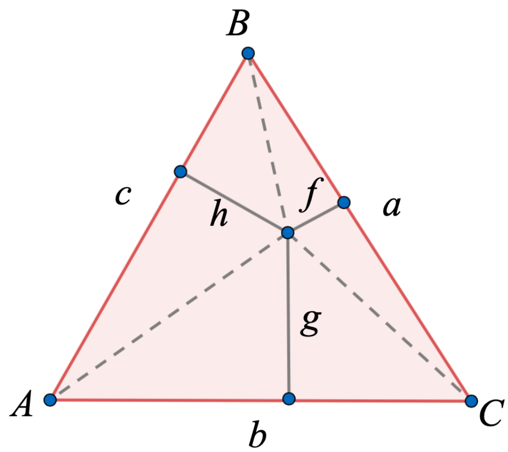
Let $\triangle ABC$ be equilateral, and let $P$ be an arbitrary point internal.




Draw the lines perpendicular to each of the sides from $P$. The sum of the three lengths is the same no matter where $P$ is chosen inside $\triangle ABC$.

*Proof.*

Draw the lines connecting the three vertices with $P$.




$\mathcal{A}_{ABC}$ is the sum of the areas of the three small triangles:

$$2\mathcal{A}_{ABC} = af + bg + ch$$

Let $a = b = c = s$

$$2\mathcal{A}_{ABC} = s(f + g + h)$$

We also know that the altitude of an equilateral triangle is in the ratio to the side as $\sqrt{3}/2$ so twice the area is

$$2\mathcal{A}_{ABC} = \frac{\sqrt{3}}{2}\, s \cdot s$$

It follows that

$$f + g + h = \frac{\sqrt{3}}{2}\, s$$

which is a constant for any given triangle, independent of the position of $P$ inside this equilateral triangle.

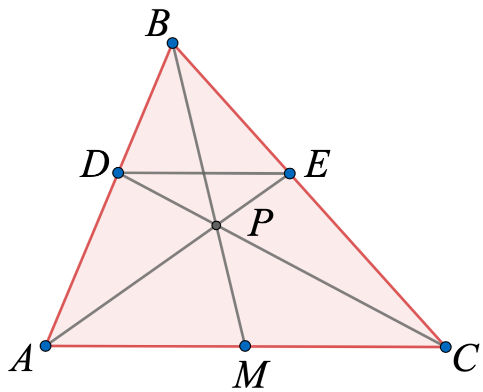
## problem (Posamentier 1.18)

In $\triangle ABC$, draw $DE \parallel AC$.

Let $BM$ be the median to side $b$.

Draw $AE$ and $CD$.

Then, the two lines $AE$ and $CD$ are concurrent with $BM$ at a point, $P$.




*Proof.*

By the converse of Ceva's theorem, the lines will be concurrent if

$$\frac{AM}{MC} \cdot \frac{CE}{EB} \cdot \frac{BD}{DA} = 1$$

Since $AM = MC$, the first term is just 1. Then, it remains to show that

$$\frac{CE}{EB} \cdot \frac{BD}{DA} = 1$$

We have $\triangle ABC \sim \triangle DBE$ so

$$\frac{BD}{AD} = \frac{BE}{CE}$$

Substituting, we satisfy the condition for concurrence.

$$\frac{CE}{BE} \cdot \frac{BE}{CE} = 1$$

$\square$

# Chapter 18

# Pythagoras by area

## Einstein

Late in life, Albert Einstein wrote that he had two experiences as a youth that influenced him tremendously. The first was when he received a compass as a child and was intrigued by the question of what "force" made the needle turn to the north.

The second was that, while studying geometry at about age 12, he had developed on his own a proof of the Pythagorean theorem. He said the achievement had a profound effect on him, illustrating the power of pure thought.

Strogatz says that the proof relied in a simple way on symmetry.

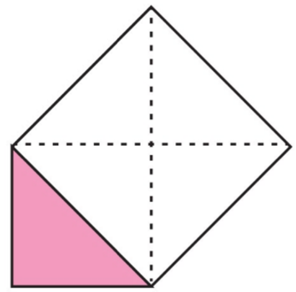
https://www.newyorker.com/tech/annals-of-technology/einsteins-first-proof-pythagorean-theorem

## areas in proportion

We will prove the part that Strogatz says was obvious to Einstein from symmetry, which he may have assumed. The idea is that for two similar right triangles, the area of each triangle is in a constant ratio to the product on any two sides, including the square on any one side.

So the area is proportional to the hypotenuse squared.

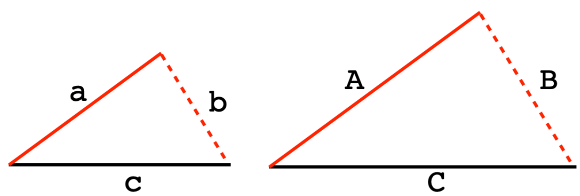
This is easy to see for an isosceles right triangle where we can calculate simply that $k = 1/4$.



But it is true for similar right triangles in general.

*Proof.*

We assume that the sides of similar triangles are in proportion. This fundamental proof has been given elsewhere.




$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = k$$

For right triangles, twice the area is the product of the two sides, hence the ratio of the areas is

$$\frac{ab}{AB} = \frac{a}{A} \cdot \frac{b}{B} = k^2$$

But $k = c/C$ so

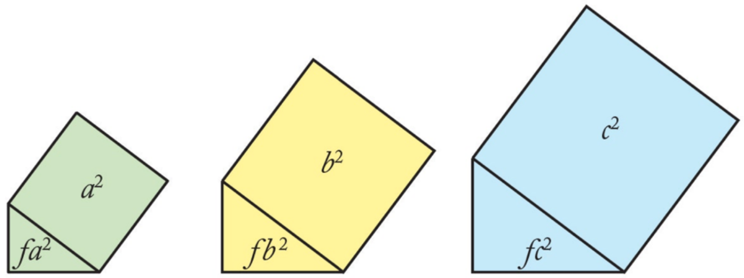
$$\frac{ab}{AB} = \frac{c^2}{C^2}$$

which can be rearranged to give

$$\frac{ab}{c^2} = \frac{AB}{C^2}$$

We conclude that the area of each right triangle is in a constant ratio to the square on any side, including the hypotenuse.

□




## Einstein's proof

Given this symmetry, here is what is thought to be Einstein's proof of the Pythagorean theorem.

*Proof.*

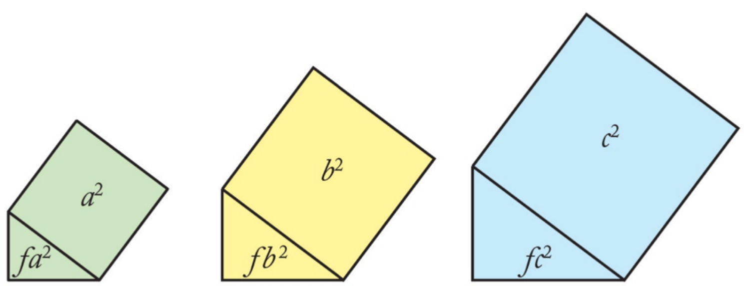
Let the constant of proportionality between area and hypotenuse squared be $f$.

Form two smaller triangles with hypotenuse of lengths $a$ and $b$. These are the two triangles formed by dropping an altitude to the hypotenuse in the original right triangle.

The two smaller triangles have areas $fa^2$ and $fb^2$ but they add to give the larger one so:

$$fa^2 + fb^2 = fc^2$$

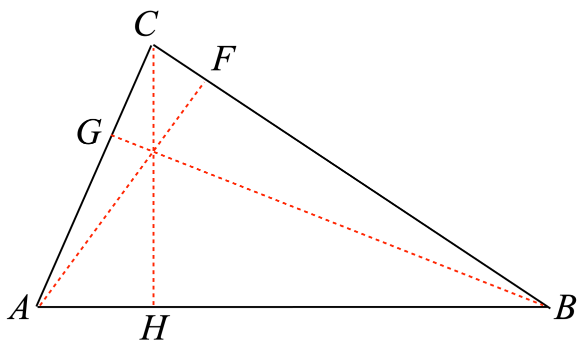
Divide by $f$ and we're done.




□

We have done the arithmetic to prove this relationship, but an appeal to symmetry abstracts away the underlying arithmetic of ratios. A true believer would simply write the last equation and then divide by $f$.

## extension of Pythagoras




Draw the altitudes in a triangle such as $\triangle ABC$. We can form pairs of similar triangles by sharing one of the vertex angles. For example $\triangle AGB \sim \triangle AHC$ and $\triangle BFA \sim \triangle BHC$. Form one ratio for each pair:

$$\frac{AG}{AB} = \frac{AH}{AC} \qquad \frac{BF}{AB} = \frac{BH}{BC}$$

Be careful to pick the side *not* opposite the shared angle.

Cross multiply:

$$AG \cdot AC = AB \cdot AH$$

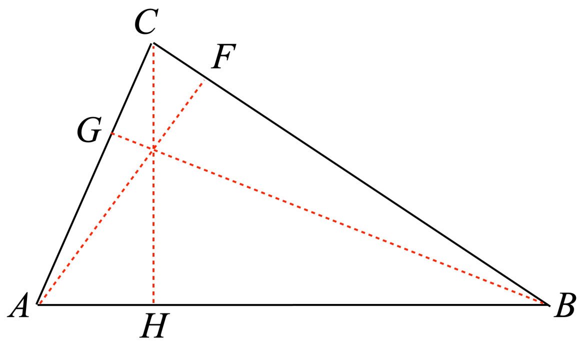
$$BF \cdot BC = AB \cdot BH$$

and add:

$$AG \cdot AC + BF \cdot BC = AB \cdot (AH + BH)$$

A small rearrangement gives a general extension of the Pythagorean Theorem.

$$AG \cdot AC + BF \cdot BC = AB \cdot AB$$




Now let the angle at vertex $C$ become a right angle. $AG \to AC$ and $BF \to BC$ so

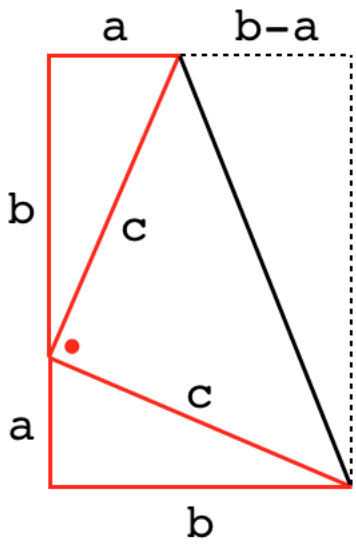
$$AC \cdot AC + BC \cdot BC = AB \cdot AB$$

□

It would be worth thinking about whether this proof extends to the case of an obtuse triangle. We leave that as an exercise.

## Garfield's proof

Here is a proof by a future President of the United States, James A. Garfield. (He was a congressman at the time).




*Proof.*

Draw a right triangle with sides $a, b$ and $c$, and a second, rotated copy as shown. The angles opposite sides $a$ and $b$ are complementary angles. So the angle marked with a dot is a right angle, and the triangle with sides labeled $c$ is a right triangle.

The area of the entire quadrilateral is the product of the left side $(a + b)$ and the *average* of $a$ and $b$ (top and bottom). This can be seen intuitively.

The halfway point of the solid red line has horizontal dimension $(a + b)/2$. Hence

$$A = (a + b) \cdot \frac{1}{2}(a + b)$$

If you're worried about that argument, just subtract the area of the triangle with two dotted sides from the quadrilateral that includes it:

$$A = (a + b)b - \frac{(a + b)(b - a)}{2}$$

$$= (a + b)(b - \frac{b}{2} + \frac{a}{2})$$

$$= (a + b) \cdot \frac{1}{2}(a + b)$$

which is just what we said.

So now:

$$A = \frac{a^2}{2} + ab + \frac{b^2}{2}$$

But we can also calculate the area of the quadrilateral as the sum of the three triangles:

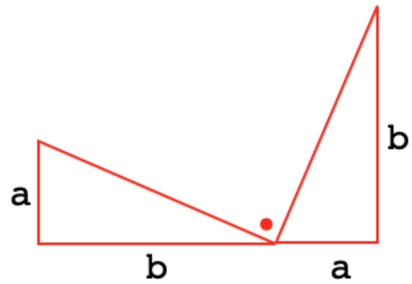
$$A = \frac{ab}{2} + \frac{ab}{2} + \frac{c^2}{2}$$

Equate the two and the result follows almost immediately.

□

## product of slopes

Let's take Garfield's basic figure and turn it sideways.




As we said, since the triangles are right triangles, the angle marked with a red dot is also a right angle.

Later, in analytical geometry, we will define the slope of a line as *rise over run*. So, for example, the slope of the hypotenuse of the right-hand triangle is $b/a$.

In a similar way, the slope of the hypotenuse of the left-hand triangle is $-a/b$. We think of the "run" of a line as going from left to right. This line heads down as we go to the right, hence the minus sign.

So then the product of slopes is

$$\frac{-a}{b} \cdot \frac{b}{a} = -1$$

This is a proof that the product of the slopes of two line segments that meet at a right angle is equal to $-1$.

## lunes

Anything else that goes like the square of the side has the same relationship:

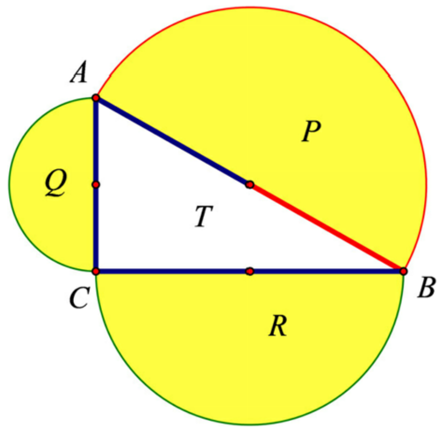



Figure 5.12.

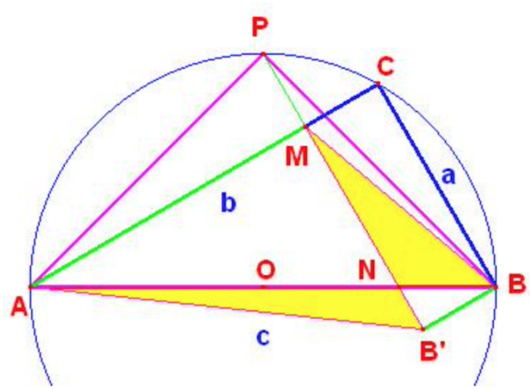
The areas add: $Q + R = P$.

These semi-circular areas are called lunes.

## fancy proof

Tuån extended the broken chord theorem of Archimedes to a proof of the Pythagorean theorem in a very clever way.

Here is a diagram.




We start with two right triangles ($AB$ is a diameter of the circle). One of the triangles, $\triangle APB$, is isosceles.

The sides of $\triangle ABC$ are labeled as $a, b$ and $c$, opposite the corresponding vertices. Side $BC$ has length $a$.

$PM$ is drawn perpendicular to $AC$. By the broken chord theorem,

$$AM = MC + BC$$

Twice that is

$$AM + MC + BC = AC + BC = b + a$$

so

$$AM = \frac{b+a}{2}$$

while

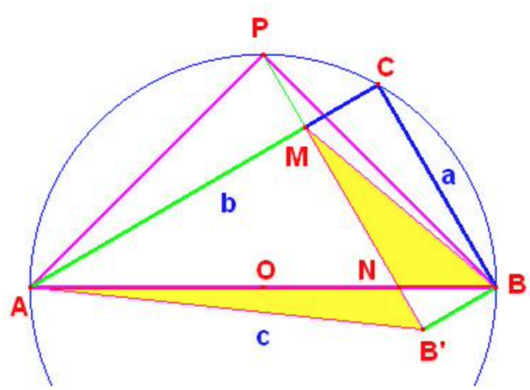
$$MC = AM - BC$$
$$= \frac{b+a}{2} - a = \frac{b-a}{2}$$

$PM$ is extended to meet the diagonal at $N$ and past it to $B'$. $B'$ is chosen so that $B'BCM$ is a rectangle. Thus side $MB'$ is equal to $BC$ and so to $a$.

We make two preliminary claims. The first is that $\triangle PMC$ is a right *isosceles* triangle. Let us accept that provisionally.

$$PM = MC = \frac{b-a}{2}$$

The second is that the areas of the two triangles shaded yellow are equal.




We reason as follows. Add $\triangle NB'B$ to both.

$\triangle AB'B$ and $\triangle MB'B$ have the same base $BB'$, and the opposing vertices $A$ and $M$ both lie on $AC$, which is parallel to the base $B'B$. Hence the two triangles have the same altitude, namely, $a$, so they have the same area.

By subtraction of $(\triangle B'BN)$ we obtain:

$$(\triangle AB'N) = (\triangle MNB)$$

Triangle area is indicated by the parentheses.

Now we find the area of $\triangle APB$ in two different ways.

The first is $c^2/4$, since it is one-quarter of a square with sides $c$.

The second way is as the sum of smaller triangles:

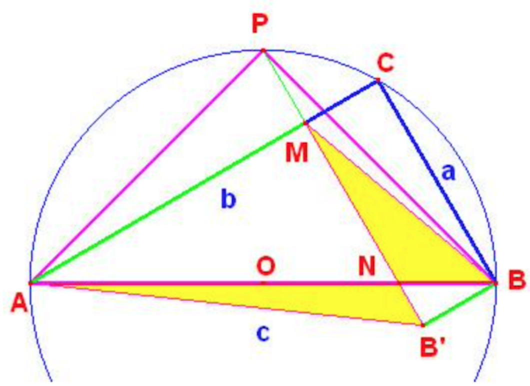
$$(\triangle APM) + (\triangle PMB) + (\triangle AMN) + (\triangle MNB)$$

Since $(\triangle AB'N) = (\triangle MNB)$:

$$(\triangle APM) + (\triangle PMB) + (\triangle AMN) + (\triangle AB'N)$$

And since $(\triangle AMN) + (\triangle AB'N) = (\triangle AMB')$:

$$(\triangle APM) + (\triangle PMB) + (\triangle AMB')$$




So then the areas are

$$(\triangle APM) = \frac{1}{2} \cdot AM \cdot MC = \frac{1}{2} \cdot \frac{b+a}{2} \cdot \frac{b-a}{2}$$

$$(\triangle PMB) = \frac{1}{2} \cdot PM \cdot MC = \frac{1}{2} \cdot \frac{(b-a)}{2} \cdot \frac{(b-a)}{2}$$

$$(\triangle AMB') = \frac{1}{2} \cdot AM \cdot MB' = \frac{1}{2} \cdot a \cdot \frac{b+a}{2}$$

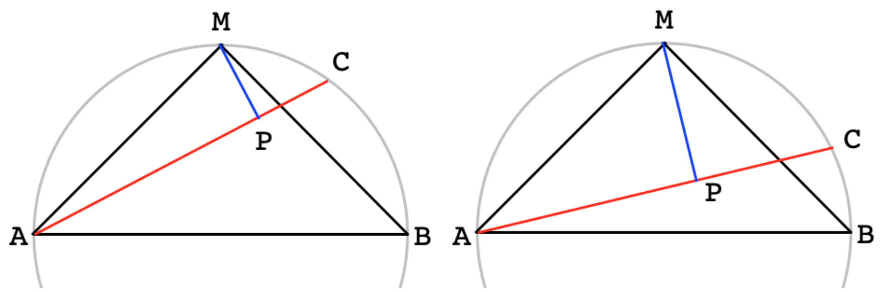
We compute 8 times the sum, so as not to have to deal with fractions:

$$(b^2 - a^2) + (b^2 - 2ab + a^2) + (2ab + 2a^2) = 2b^2 + 2a^2$$

Previously, we calculated the area as $c^2/4$, and 8 times that is $2c^2$. The result follows immediately.

## last step

However, the proof is not yet complete. We must show that $\triangle PMC$ is isosceles. Simplifying the figure:

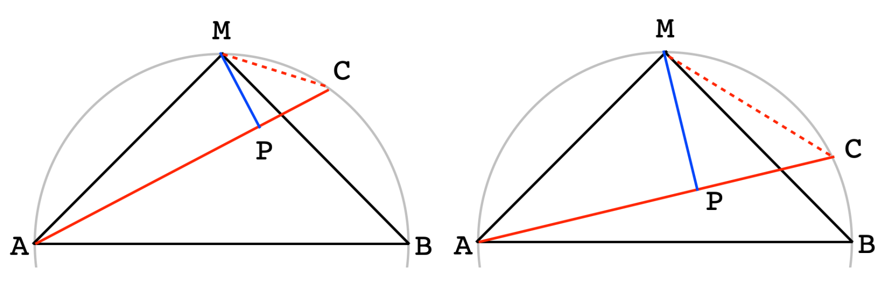



Let $AB$ be a diameter of the circle and $\triangle AMB$ isosceles. Let $C$ be any point on the perimeter, with $MP \perp AC$. Then, we claim that $MP = PC$.

It seems reasonable. If $C \to B$, the $P$ becomes the origin and the statement is true, while if $C \to M$, both vanish. I spent some time fooling around with similar triangles before insight came.

*Proof.*

Connect the two vertices by drawing $MC$.




Clearly $\angle MCA$ is one-half of a right angle since it intercepts the same arc as $\angle ABM$, by the inscribed angle theorem. Since $\angle MPC$ is right, it follows that $\triangle PMC$ is isosceles (by complementary angles) and so $MP = PC$ (by the converse of the isosceles triangle theorem).

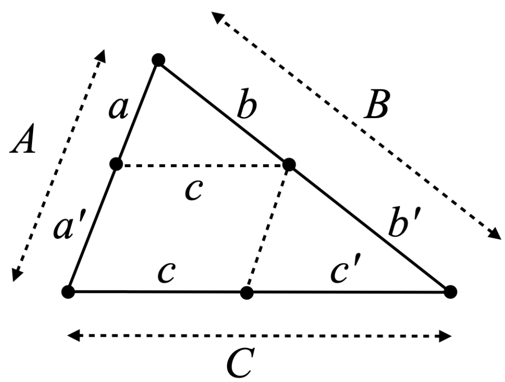
$\square$

Notice how we incorporate the information that $\triangle AMB$ is isosceles.

# Chapter 19

# Ratio Boxes

In this short chapter, we will look at a device which, at least for now, I'm going to call *ratio boxes*. Here's the idea:




| $a$ | $b$ | $c$ |
|---|---|---|
| $A$ | $B$ | $C$ |
| $a'$ | $b'$ | $c'$ |

We start with two similar triangles, $\triangle abc$ nested inside $\triangle ABC$.

One part of the definition is that the sides have equal ratios. For example, $a : A = b : B$, also written as

$$\frac{a}{A} = \frac{b}{B}$$

Now, you can look at the figure and take away that result, easily enough. But sometimes you need more than one relationship, and if the vertices have the labels, then it's more complicated.

You may notice that there are two similar triangles but nine entries in the box. The reason is the following:

$$\frac{a}{b} = \frac{A}{B} = \frac{a+a'}{b+b'}$$

$$\frac{a+a'}{a} = \frac{b+b'}{b}$$

$$\frac{a'}{a} = \frac{b'}{b}$$

$$\frac{a'}{b'} = \frac{a}{b} = \frac{A}{B}$$

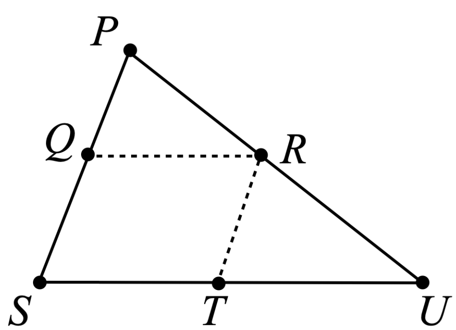
I like to write the sides that are in proportion as shown in the rectangular grid.

| | | |
|---|---|---|
| $a$ | $b$ | $c$ |
| $A$ | $B$ | $C$ |
| $a'$ | $b'$ | $c'$ |

Orient it as you like. I usually proceed from the shortest side to the longest side. Sometimes that is hard to see so we start matching sides with the angle opposite.

But however, you do it, there are nine sides or differences of sides. Here the ratios *within* triangles go across, and the ones *between* triangles go down.

The next example uses the traditional Greek notation (well, with Roman letters).




| | | |
|---|---|---|
| $PQ$ | $PR$ | $QR$ |
| $PS$ | $PU$ | $SU$ |
| $QS$ | $RU$ | $TU$ |

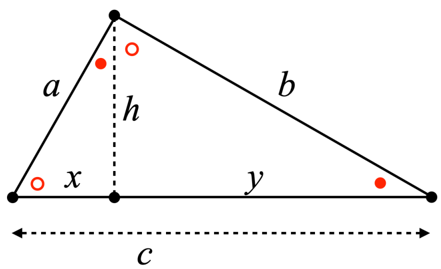
This is exactly the same as before, except the sides are labeled by the flanking vertices. It is fairly easy to go through a figure and make such a box.

The value is this: any four entries that are in the shape of a rectangle form a valid ratio. For example: $PQ : PS = QR : SU$. You can also see immediately that $PQ \cdot SU = PS \cdot QR$.

| | | |
|---|---|---|
| $PQ$ | $PR$ | $QR$ |
| $PS$ | $PU$ | $SU$ |
| $QS$ | $RU$ | $TU$ |

If you have three picked you know the fourth. If you have two picked, and they are not related horizontally or vertically (like $PQ$ and $SU$), then you obtain the other two partners immediately.

## right triangles




| | | |
|---|---|---|
| $x$ | $h$ | $a$ |
| $h$ | $y$ | $b$ |
| $a$ | $b$ | $c$ |

As you know, if you draw the (one) altitude in a right triangle, it forms two smaller similar triangles. The reason is complementary angles, as shown by the red dots marking equal angles.

Say we want a proof of Pythagoras's Theorem. We need $a^2$ and $b^2$. They jump right out of the box:

$$a^2 = xc$$

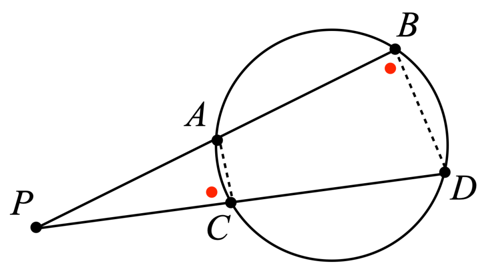
$$b^2 = yc$$

$$a^2 + b^2 = xc + yc = c^2$$

$\square$

We also see, without ever referring back to the figure, that $h^2 = xy$. $h$ is the *geometric mean* of $x$ and $y$.

## secants




| | | |
|---|---|---|
| $AC$ | $BD$ | * |
| $PC$ | $PB$ | $AB$ |
| $PA$ | $PD$ | $CD$ |

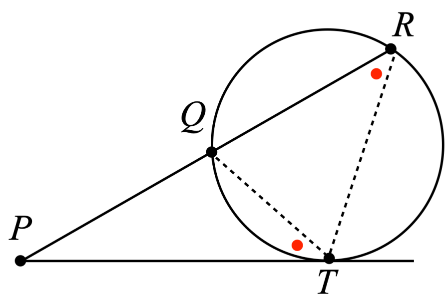
$PAB$ and $PCD$ are secants of this circle. The red dots show equal angles. The reason is that $\angle B$ and $\angle ACD$ are supplementary, because they are opposite angles of a *cyclic quadrilateral*. Together they correspond to a complete arc of the circle.

But $\angle PCA$ and $\angle ACD$ are supplementary too. So the marked angles are equal. And since two pairs of angles are equal, we have similar triangles: $\triangle PCA \sim \triangle PBD$. I try to remember to write the vertices in the order of similarity, so for example: $PC : CA = PB : BD$.

Go through each triangle and write the sides that have equal angles opposite. The famous result is: $PA \cdot PB = PC \cdot PD$.

I have left out the entry in the starred box, because it looks funny. It is $BD - AC$.

## tangent-secant




| | |
|---|---|
| $PQ$ | $PT$ |
| $QT$ | $RT$ |
| $PT$ | $PR$ |

Here we have a secant $PQR$ and a tangent, $PT$. The angle that the tangent makes with a chord is easily shown to be equal to any inscribed angle subtended by that chord. That accounts for the red dots.

Again we have similar triangles, this time, one is nested inside another, so there are only six terms in the box. Here I followed the angles, first red then $\angle P$.

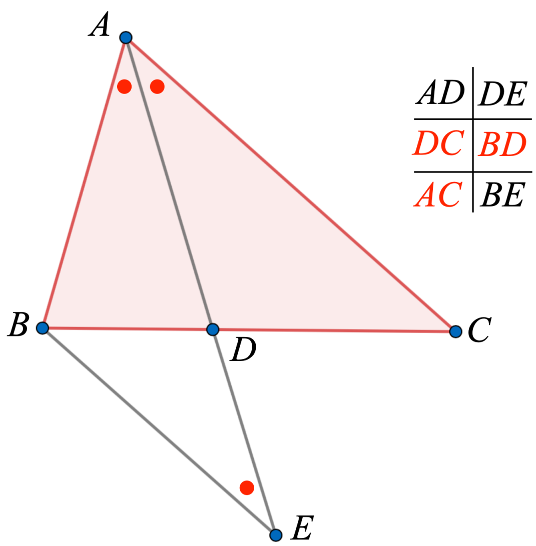
The famous result is the Secant Tangent Theorem (or Tangent Secant Theorem): $PQ \cdot PR = PT^2$.

It writes itself.

## Angle Bisector Theorem

We revisit this theorem, from **here**. We are given that the angle at $A$ ($\angle BAC$), is bisected.

So we draw a line segment $BE$ parallel to side $AC$ and extend the bisector to meet it.




Because of the parallel lines, $\angle BED$ gets a red dot as well. And that means $\triangle BED \sim \triangle DAC$, which gives the ratio box in the figure. Here we want to be careful to match the sides with equal angles opposite.

Highlight the segments we think might be interesting. Finally, notice that $\triangle ABE$ is isosceles, so $AB = BE$. We obtain:

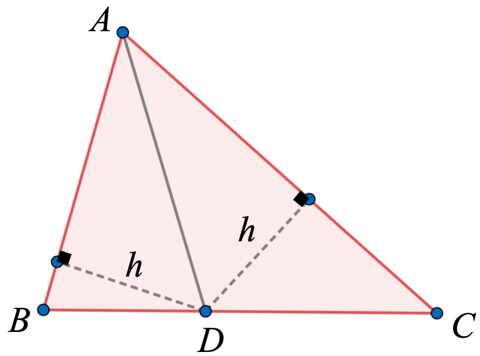
$$\frac{AC}{DC} = \frac{BE}{BD} = \frac{AB}{BD}$$

□

The sides and divisions of the base are in equal proportion. This can also be written as

$$\frac{AB}{AC} = \frac{BD}{DC}$$

This one can also be done by areas (next figure). $\triangle ABD$ and $\triangle ACD$ have the same altitude $h$. So the areas are in the ratio $AB : AC$.

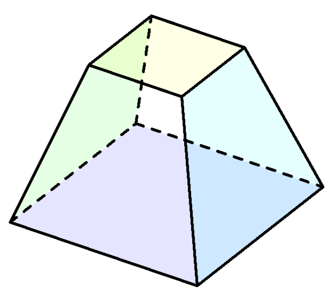



But looked at another way, they have bases $BD$ and $DC$ on the same parallel, so they have the same height drawn to vertex $A$. Hence, the areas are in the ratio $BD : DC$. Thus

$$\frac{AB}{AC} = \frac{BD}{DC}$$

## frustum

A frustum is a truncated pyramid.



It's known, and we discuss elsewhere a proof, that the volume of a cone or pyramid is

$$V = \frac{1}{3} \cdot hA$$

where $h$ is the height and $A$ is the area of the base. We'll make things slightly easier by writing

$$3V = hs^2$$

with $s$ being the side length, for a square pyramid.

A frustum has the top chopped off. We will be given the side length at the bottom $a$, and the side length on the top $b$, as well as the height of the frustum $h$.

We need to compute the height of the whole thing $H$, then find the height of the missing top part as $H - h$. We'll use those two to get the volume of the frustum by subtraction.

Let $H$ be the entire height, and then by similar triangles we have

$$\frac{H}{a} = \frac{H-h}{b} = \frac{h}{a-b}$$

This is our ratio box for the problem. The height of the small missing piece is

$$H - h = \frac{b}{a}H$$

Then the volume we seek uses the difference:

$$\begin{aligned} 3V &= a^2H - b^2(H-h) \\ &= a^2H - b^2H\ \frac{b}{a} \\ &= (a^2 - \frac{b^3}{a})H \end{aligned}$$

But now we need $H$ in terms of $h$, since that's what we're given, so going back to the ratios we have:

$$H = \frac{a}{a-b}h$$

which gives

$$\begin{aligned} 3V &= (a^2 - \frac{b^3}{a})\frac{a}{a-b}\ h \\ &= \frac{a^3 - b^3}{a-b}\ h \end{aligned}$$

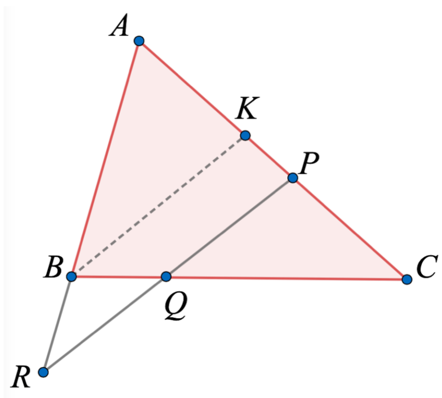
And if we fool around for a while, we may discover that the numerator can be factored as

$$(a^2 + ab + b^2)(a - b) = a^3 - b^3$$

which gives, finally

$$3V = h(a^2 + ab + b^2)$$

## Menelaus's Theorem




| | | |
|---|---|---|
| AB | AR | BR |
| BK | RP | RP-BK |
| KA | PA | PK |

| | | |
|---|---|---|
| CP | CK | PK |
| PQ | KB | KB-PQ |
| QC | BC | BQ |

The next example is one where the method really proves its worth. We prove Menelaus's theorem (we will look at it in more detail later **here**).

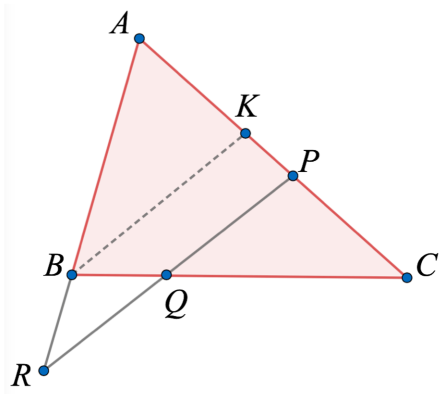
In $\triangle ABC$ we have the transversal $RQP$. Draw an internal line segment $BK$ parallel to the transversal.

Here, it is usual to label vertices rather than sides.

For the moment, each vertex is listed in order it is encountered going around a triangle, starting at the same vertex each time (since we have parallel bases).

The values in the right-hand columns of the two ratio boxes are all subtractions, but for one there is no labeled segment to equate to the result.

The segments needed for the proof are shown in red.




| AB | AR | BR |
|---|---|---|
| BK | RP | RP-BK |
| KA | PA | PK |

| CP | CK | PK |
|---|---|---|
| PQ | KB | KB-PQ |
| QC | BC | BQ |

It writes itself! Since

$$\frac{PK}{BR} = \frac{PA}{AR}$$

$$PK = BR \cdot \frac{PA}{AR}$$

Similarly

$$\frac{PK}{CP} = \frac{BQ}{QC}$$

$$PK = CP \cdot \frac{BQ}{QC}$$

Combining the two results:

$$CP \cdot \frac{BQ}{QC} = PK = BR \cdot \frac{PA}{AR}$$

$$\frac{BQ}{QC} \cdot \frac{CP}{PA} \cdot \frac{AR}{BR} = 1$$

□

Finally, it is usual to view each segment as having a direction or sign. Namely

$$BQ \Rightarrow QC \Rightarrow CP \Rightarrow PA \Rightarrow AR \Rightarrow RB$$

Since $RB$ and $BR$ are in opposite directions, $RB = -BR$. Substitute $RB$ for $BR$ and adjust the sign of the result accordingly.

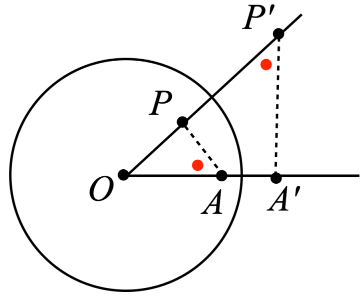
$$\frac{BQ}{QC} \cdot \frac{CP}{PA} \cdot \frac{AR}{RB} = -1$$

## inversion

Some good examples involve a topic called *Inversive Geometry*. Given a circle of radius $r$, any point (except the origin), like $A$, can be transformed into its *image under an inverse transformation*, resulting in A'. Draw the line from $O$ through $A$ and calculate the length of $OA'$ by this rule:

$$OA \cdot OA' = r^2$$

Since $P$ and $P'$ are related by the same transformation, we have $OA{\cdot}OA' = OP{\cdot}OP'$. Since $\angle O$ is shared, we again have similar triangles: $\triangle OAP \sim \triangle OP'A'$.




| $AP$ | $A'P'$ |
|---|---|
| $OP$ | $OA'$ |
| $OA$ | $OP'$ |

You can read the rule right off the box.

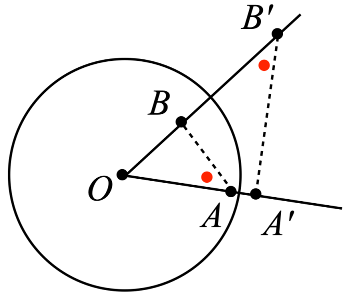
And if $OAA' \perp A'P'$ then the small $\triangle OAP$ is a right triangle. By the converse of Thales' Theorem, we can draw a circle with diameter $OA$ and $P$ will lie on the circle.

This is true for *any* point on $A'P'$. Say $Q'$ is on $A'P'$, then if $Q$ is the image of $Q'$, $Q$ lies on the same circle, the one with radius $OA$.

As a result, the *image* of any point on the line $A'P'$ lies on the circle with radius $OA$. We say that the image of the line is the circle (and it goes through $O$).

The transformation is an *involution*, so the converse is also true: the image of a circle through $O$ is a line not through $O$.

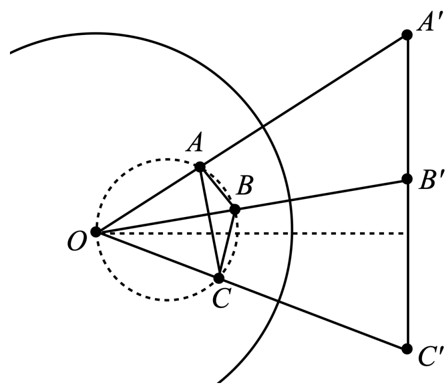
A second example from inversion is more general.




| | |
|---|---|
| $AB$ | $A'B'$ |
| $OB$ | $OA'$ |
| $OA$ | $OB'$ |

You can read the rule right off the box, again. What this means is that if we take any two points $A$ and $B$ and their images $A'$ and $B'$, we get similar triangles.

## Ptolemy's theorem




We will prove a famous theorem. All we need are some boxes and the previous result.

| | |
|---|---|
| $AB$ | $A'B'$ |
| $OB$ | $OA'$ |
| $OA$ | $OB'$ |

| | |
|---|---|
| $AC$ | $A'C'$ |
| $OC$ | $OA'$ |
| $OA$ | $OC'$ |

| | |
|---|---|
| $BC$ | $B'C'$ |
| $OC$ | $OB'$ |
| $OB$ | $OC'$ |

I didn't even have to think about it. I just copied the box from before, and substituted $C$ for $B$ in the middle, then $B$ for $A$ on the right.

We see that that the transformed circle is a line with $A'B' + B'C' = A'C'$. We can find expressions for those lengths in our boxes. We know that eventually we will want things like $AB$, $BC$ and $AC$, as well as $OB$, etc.

I get

$$A'B' = \frac{AB \cdot OA'}{OB} \qquad A'C' = \frac{AC \cdot OA'}{OC} \qquad B'C' = \frac{BC \cdot OB'}{OC}$$

Again, these may be obtained mechanically, by straight substitution. The one for $A'C'$ first, by substituting $C'$ for $B'$ in the one on the left. Wash, rinse, repeat.

Substitute and then clear the denominator:

$$AB \cdot OA' \cdot OC + BC \cdot OB' \cdot OB = AC \cdot OA' \cdot OB$$

Divide by $OA'$:

$$AB \cdot OC + BC \cdot \frac{OB'}{OA'} \cdot OB = AC \cdot OB$$

Can you find the four entries we need in the first box on the left and complete the proof?

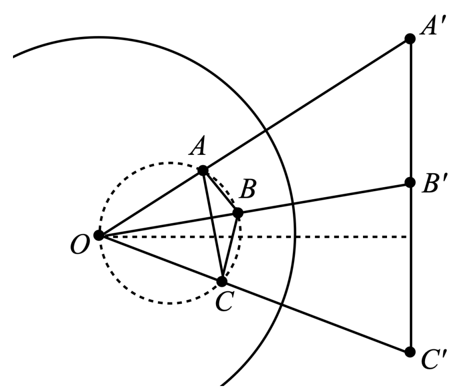
| $AB$ | $A'B'$ |
|---|---|
| $OB$ | $OA'$ |
| $OA$ | $OB'$ |

| $AC$ | $A'C'$ |
|---|---|
| $OC$ | $OA'$ |
| $OA$ | $OC'$ |

| $BC$ | $B'C'$ |
|---|---|
| $OC$ | $OB'$ |
| $OB$ | $OC'$ |

The final result is

$$AB \cdot OC + BC \cdot OA = AC \cdot OB$$




We have four vertices on a circle, another cyclic quadrilateral. Take the product of opposing sides, add them, and obtain the product of the two diagonals.

This is Ptolemy's Theorem.

$$AB \cdot OC + BC \cdot OA = AC \cdot OB$$

□

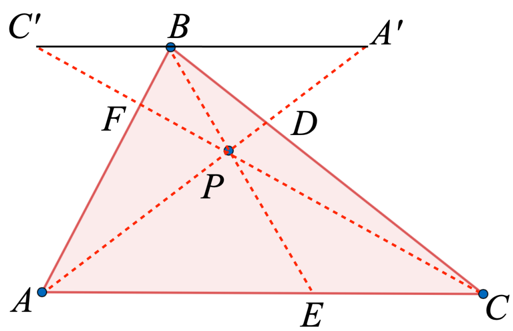
## Ceva's Theorem by parallel lines

Here is another proof of **Ceva's theorem**.

*Proof.*

Consider $\triangle ABC$. Draw $AD$, $BE$ and $CF$ concurrent at $P$. Draw a line through $B$ parallel to $AEC$ and extend $APDA'$ and $CPFC'$.

We have five pairs of similar triangles.




Three with vertical angles at $P$:

$$(1)\ \triangle APC \sim \triangle A'PC' \Rightarrow \frac{AP}{A'P} = \frac{PC}{PC'} = \frac{CA}{C'A'}$$

$$(2)\ \triangle APE \sim \triangle A'PB \Rightarrow \frac{AP}{A'P} = \frac{PE}{PB} = \frac{EA}{BA'}$$

$$(3)\ \triangle CPE \sim \triangle C'PB \Rightarrow \frac{CP}{C'P} = \frac{PE}{PB} = \frac{EC}{BC'}$$

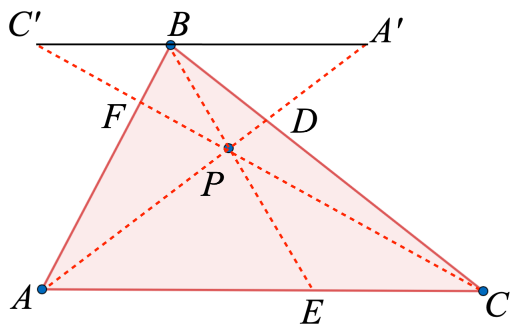
And two more with vertical angles at $D$ and $F$:

$$(4)\ \triangle ADC \sim \triangle A'DB \Rightarrow \frac{AD}{A'D} = \frac{DC}{DB} = \frac{CA}{BA'}$$

$$(5)\ \triangle AFC \sim \triangle BFC' \Rightarrow \frac{AF}{BF} = \frac{FC}{FC'} = \frac{CA}{C'B}$$

It is helpful to remember that when similar triangles are formed by vertical angles between two parallel lines, corresponding sides either lie across from each other, or are reflected through the point with the vertical angles. Thus $A'B$ corresponds to $AE$, and $PB$ to $PE$.

We label the triangles so that corresponding vertices match. Also, do not worry yet about the order in each line segment so for example, $AB = BA$.




If the triangles are labeled carefully it is easy to make the ratio boxes:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $AP$ | $A'P$ | $AP$ | $A'P$ | $CP$ | $C'P$ | $AD$ | $A'D$ | $AF$ | $BF$ |
| $PC$ | $PC'$ | $PE$ | $PB$ | $PE$ | $PB$ | $DC$ | $DB$ | $FC$ | $FC'$ |
| $CA$ | $C'A'$ | $EA$ | $BA'$ | $EC$ | $BC'$ | $CA$ | $BA'$ | $CA$ | $C'B$ |
| (1) | | (2) | | (3) | | (4) | | (5) | |

Now look for the ratios we need namely:

$$\frac{CD}{DB} = \frac{CA}{BA'} \qquad \frac{BF}{FA} = \frac{C'B}{CA}$$

We are encouraged by the fact that $CA$ cancels in the product.

$$\frac{CD}{DB} \cdot \frac{BF}{FA} = \frac{C'B}{BA'}$$

The third ratio is $AE/EC$. $AE$ and $EC$ do not occur in the same box (they do not lie in similar triangles), but in (2) and (3) we find:

$$AE = A'B \cdot \frac{PE}{PB} \qquad EC = BC' \cdot \frac{PE}{PB}$$

so the ratio is just

$$\frac{AE}{EC} = \frac{A'B}{BC'}$$

which is exactly what we need to cancel!

$$\frac{CD}{DB} \cdot \frac{BF}{FA} \cdot \frac{AE}{EC} = \frac{C'B}{BA'} \cdot \frac{A'B}{BC'} = 1$$

□

# Part V

# Special stuff

# Chapter 20

# Euler Theorem

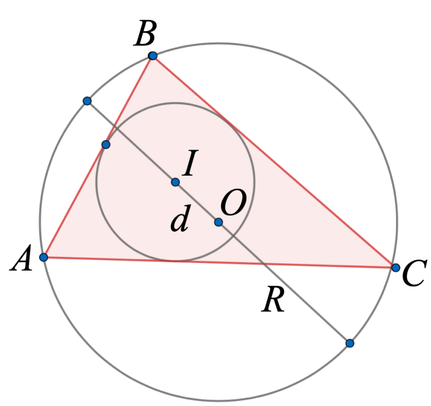
In this chapter we talk about the famous Euler Theorem (the one in geometry, there are others).

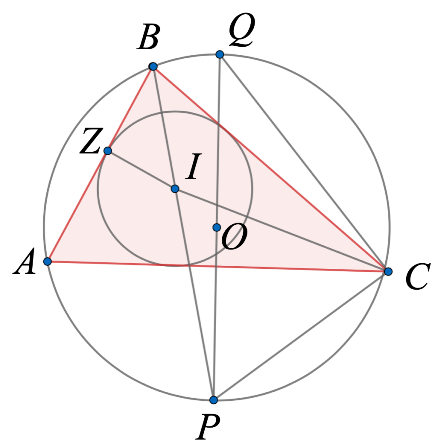
Consider $\triangle ABC$ with its circumcircle on center $O$ with radius $R$, and its incircle on center $I$ with radius $r$.

Let the distance between $O$ and $I$ be $d$.

Then the distance from I to the outer circle is $R + d$ in one direction and $R - d$ in the other direction. The product of parts of the chord is

$$(R + d)(R - d) = R^2 - d^2$$



We consider another chord drawn through $I$, namely the bisector of $\angle B$, extended to meet the circumcircle at $P$.

Let the bisected angles be $2\beta = \angle B$ and $2\gamma = \angle C$, as usual.

Draw the diameter $POQ$. By Thales' theorem, $\triangle PQC$ is right.

Let $IZ \perp BZA$, so $\triangle IBZ$ is right.

By inscribed angles, $\angle PQC = \angle PBC = \beta$.

Thus, the two right triangles are similar, since they both contain $\beta$.

Form the ratios:

$$\frac{IZ}{IB} = \frac{PC}{PQ}$$

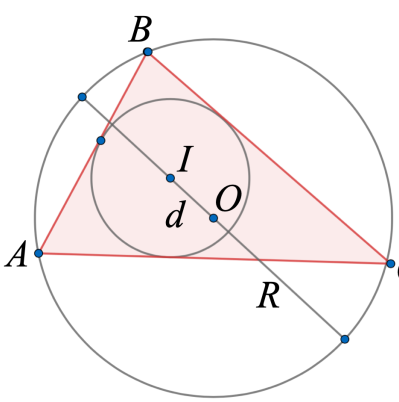
Substituting

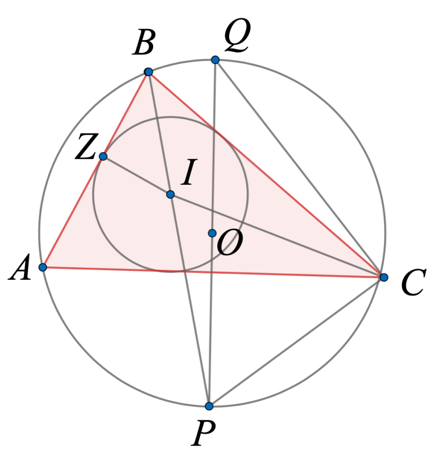
$$\frac{r}{IB} = \frac{PC}{2R}$$

$$2rR = PC \cdot IB$$

If we can show that $PC = PI$, we will have that the right-hand side of the previous equation is equal to $PI \cdot IB = R^2 - d^2$, by crossed chords.

We consider $\triangle PCI$.




$\angle PCI$ is equal to $\gamma + \beta$ by inscribed angles.

$\angle CPI$ is equal to $\angle A$.

By sum of angles, $\angle PIC$ is also equal to $\gamma + \beta$.

With equal base angles, by I.6 $\triangle PCI$ is isosceles.

Thus $PI = PC$, and the main result follows:

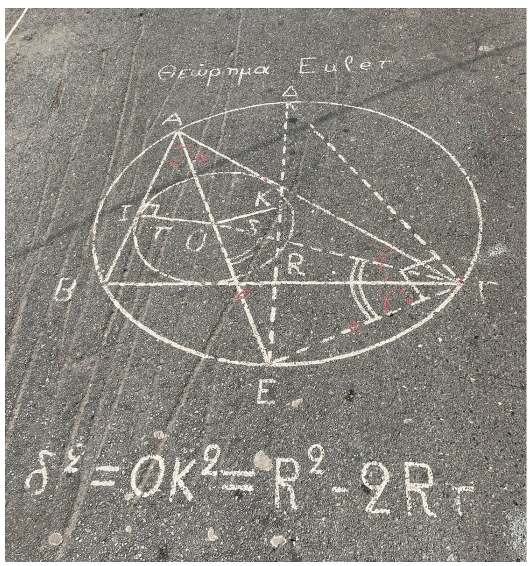
$$2rR = R^2 - d^2$$

$$= (R+d)(R-d)$$

$$\frac{1}{r} = \frac{2R}{(R+d)(R-d)}$$

$$= \frac{1}{R+d} + \frac{1}{R-d}$$

Drawn on a small street in Athens.



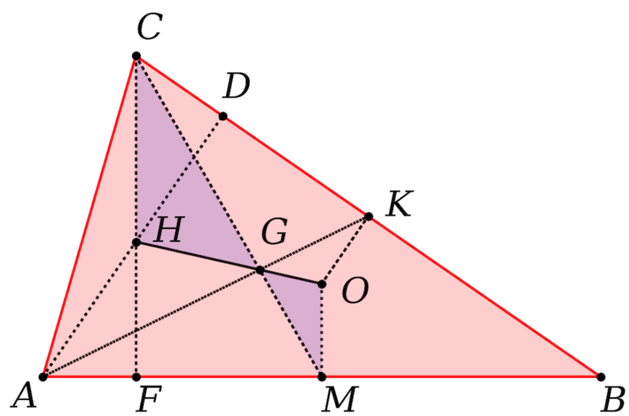

That is indeed (part of) our proof.

# Chapter 21

# Euler Line

In this chapter we talk about the famous Euler Line of a given triangle:

https://en.wikipedia.org/wiki/Euler_line

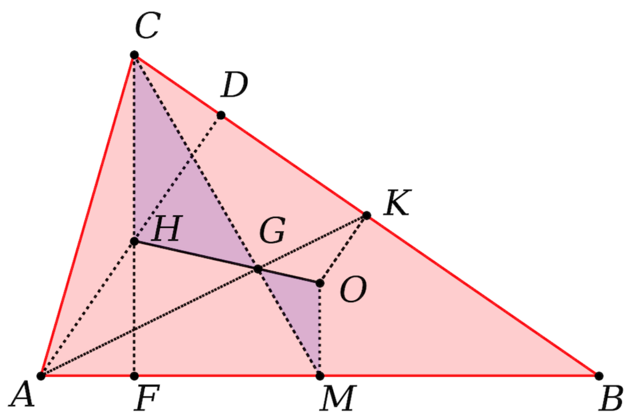



Recall that the circumcenter is the center of the circle that contains all three vertices, and it can be found by erecting the perpendicular bisectors of the sides. Side $a$ opposite $\angle A$ is bisected at $K$, while side $c$ opposite $\angle C$ is bisected at $M$. The perpendicular bisectors meet at $O$.

The orthocenter $H$ is the point where altitudes drawn from the three vertices of a triangle cross. Here $AD$ meets $CF$ at $H$.

The centroid $G$ is the point where the medians cross, before they bisect the opposite sides. $CM$ is one of the medians of $\triangle ABC$.

We will assume that these points actually exist (that the three altitudes or three medians are concurrent). That proof is coming later.




Suppose we find $O$ as described above, draw the median $CM$, find $G$ arithmetically, and then extend $OG$ to find $H$ such that $HG = 2OG$.

We will show that $H$ is the orthocenter of $\triangle ABC$, that it actually lies on $CM$.

We also need a couple of preliminary results. We will prove that the centroid divides the medians by the $2/3 - 1/3$ rule: $CG = 2MG$. Let's do this below as a lemma.

The second is SAS similarity. In the discussion of similarity for a general or arbitrary triangle, we showed that if two triangles have two pairs of sides in the same proportion, and the angle between is also equal, then they are similar triangles.

*Lemma*

We depend on the area-ratio theorem. If two triangles have one vertex shared and the bases lie on the same line, then their areas are in proportion as the lengths of the bases. The reason is that they have the same altitude, since for a given line and a point not on the line, only one perpendicular can be drawn from the point to the line.

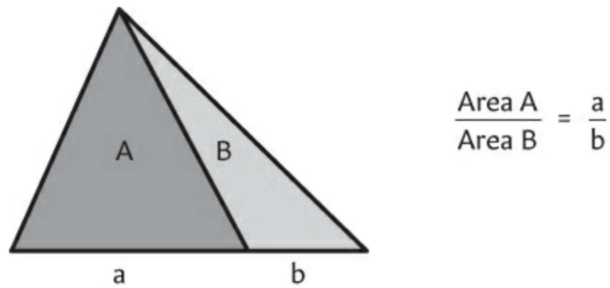
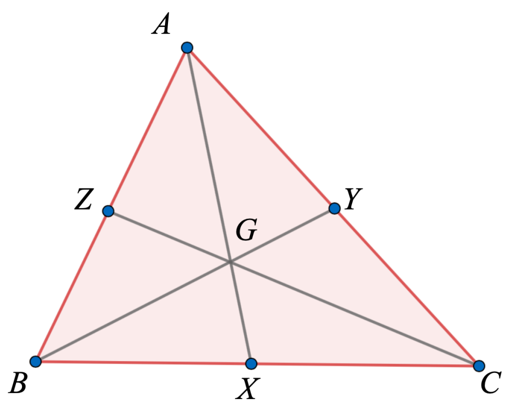
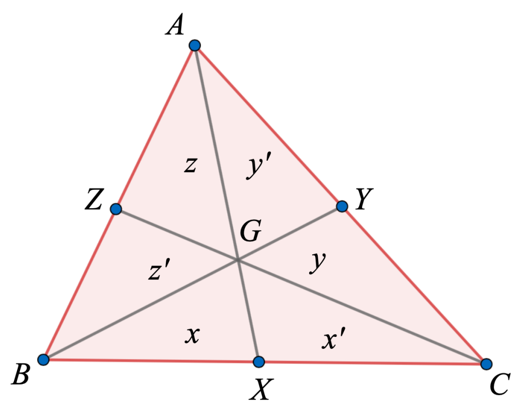



**Fig. 120** An area-ratio theorem.

So now we consider an arbitrary triangle, divided by its three medians. We will show that each of the small triangles is equal in area.




Compare $\triangle GBX$ and $\triangle GCX$ to find that they are equal in area, since $GX = CX$. Thus area $x$ = area $x'$. and so on. Similarly, compare $\triangle ABX$ and $\triangle ACX$ to find that they are equal in area. It follows that:

$$x + z + z' = x' + y + y'$$

which reduces to $z = y$. The same argument will give $x = y$ and $x = z$.

Finally, we use what is technically the converse of the area-ratio theorem. We have that the area of $\triangle ABG$ is twice that of $\triangle AYG$. Since they have the same altitude, it follows that the bases are in proportion: $BG = 2YG$. Again, this can be extended to any median by the same argument.

□

## SAS for similar triangles

• If two triangles share an equal angle, and the two sets of flanking sides are both in proportion, then the two triangles are similar.

We proved this theorem **here**.

Here is a simple algebraic proof.

*Proof.*

Suppose the small triangle has sides $a, b, c$ and the larger triangle has sides $A, B, C$ with $A/a = B/b = k$. Suppose that $\angle\gamma$ flanked by $a$ and $b$ is also flanked by $A$ and $B$.

Use the law of cosines. We have

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$C^2 = A^2 + B^2 - 2AB\cos\gamma$$

Since $A = ka$ and $B = kb$

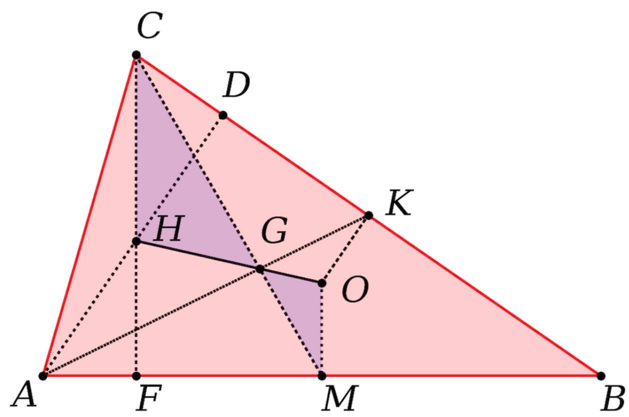
$$\begin{aligned} C^2 &= (ka)^2 + (kb)^2 - 2(k^2ab)\cos\gamma \\ &= k^2c^2 \end{aligned}$$

Hence $C/c = k$.

□

*Proof.*

Returning to the main problem, we justify the placement of $G$ by noting that $CG = 2MG$. We found $H$ by $HG = 2OG$. Finally, we have vertical angles at $G$. It follows that $\triangle CGH \sim \triangle MGO$ with sides in proportion $2:1$.




It follows that $\angle CHG = \angle MOG$.

Thus, by alternate interior angles, we find that $OM \parallel CHF$, which means that since $OM \perp AFMB$, so is $CH$ when extended $\perp AFMB$.

But again, there is only one perpendicular to be drawn from $C$ through $ABMB$ at $F$, thus, $CF$ *is* the altitude to the vertex $C$ in $\triangle ABC$.

$\square$

Two more quick points. First, in an equilateral triangle, the three points $O$, $G$ and $H$ coincide — they are the same point. Second, there is much more to the Euler line, including the center of the *nine point circle*, which we discuss elsewhere.

# Chapter 22

# Special circles

## triangle area and radii for incircle and circumcircle

Acheson gives formulas that connect the area of a triangle (he uses the symbol $\Delta$ for the area), and the radius of either the incircle or the circumcircle.

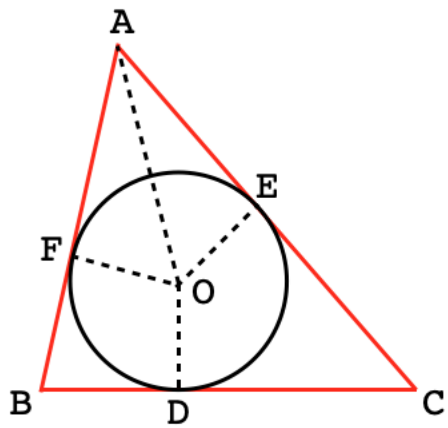
The first one is just a matter of algebra, but the second is gorgeous. It is really worth it to try to solve before you look at the answer. So, write down the answer, close the book and then try! Once again, an inspired diagram is everything.

$$r = \frac{2\Delta}{a+b+c}$$

$$R = \frac{abc}{4\Delta}$$

Let $r$ be the radius of the incircle and $a$ be the length of the base opposite vertex $A$.

Then the area of $\triangle BOC$ is equal to one-half $r \cdot a$.




So the area of the whole triangle is equal to one-half $r \cdot (a+b+c)$.

$$2\Delta = r \cdot (a+b+c)$$
$$r = \frac{2\Delta}{a+b+c}$$

Define the *semi-perimeter* $s$ as

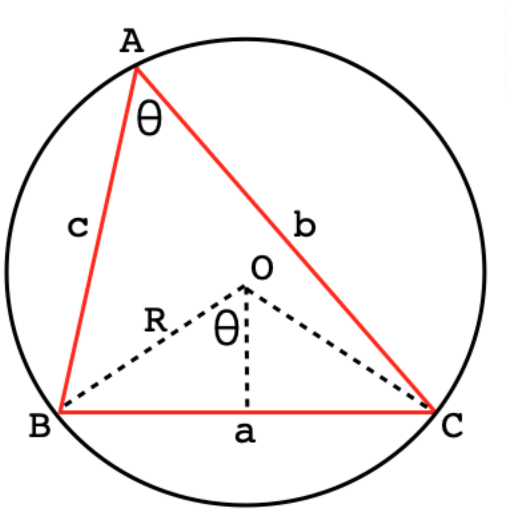
$$s = \frac{a+b+c}{2}$$

and then

$$r = \frac{\Delta}{s}$$
$$\Delta = rs$$

That's an interesting parallel, that this formula is so similar to that for the area of the circle. Here we have the radius of the incircle times the one-half the perimeter of the triangle. Of course, for a circle, we have the radius times one-half its perimeter as well.

For the second problem, the radius of the circumcircle, the key insight is in the diagram below. After that it's easy.




*Proof.*

The altitude to side $b$ (not shown) has length $c \sin \theta$. So the area of the triangle is

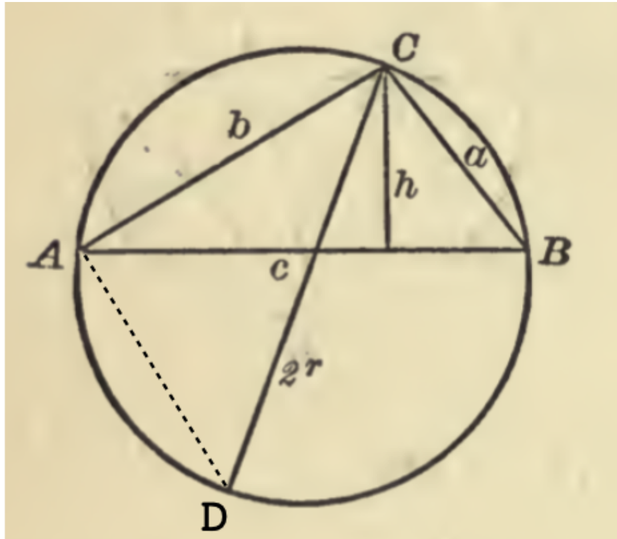
$$\Delta = \frac{1}{2} bc \sin \theta$$

But $\sin \theta = a/2$ divided by $R$ so

$$\Delta = \frac{1}{2} bc \cdot \frac{a}{2R}$$
$$\Delta = \frac{abc}{4R}$$

$\square$

Here is an alternate proof from Hopkins.




*Proof.* (Alternate).

As a preliminary matter, note that $\triangle ABC$ is any triangle, and the circle is its circumcircle, with radius $r$. Then the extension of the radius to $D$ forms a right triangle $\triangle ACD$. Since $\angle B$ and $\angle D$ cut off the same arc of the circle, they are equal.

Therefore, $\triangle ACD$ is similar to the triangle formed by the altitude $h$ and including side $a$. By similar triangles:

$$\frac{h}{a} = \frac{b}{2r}$$

$$h = \frac{ab}{2r}$$

Twice the area of the triangle is

$$2\Delta = ch$$

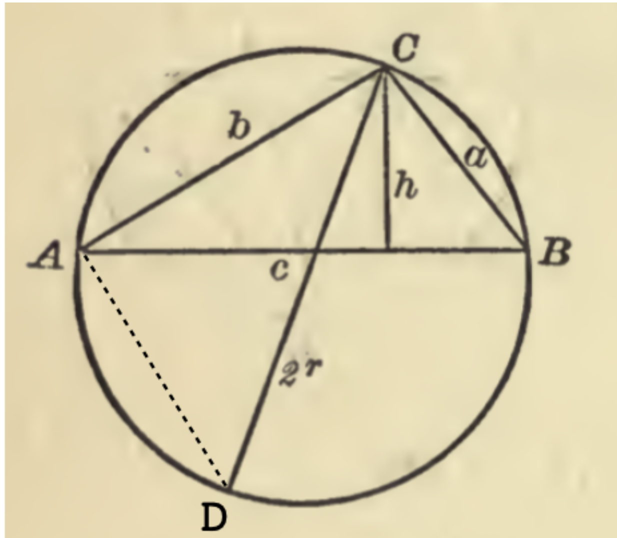
$$\Delta = \frac{abc}{4r}$$

□

Hopkins also notes that this result can be expressed purely in terms of the side lengths by using **Heron's formula** (which we introduced above and will say more about soon):

$$A^2 = s \cdot (s-a) \cdot (s-b) \cdot (s-c)$$

(where $s = (a+b+c)/2$).




So

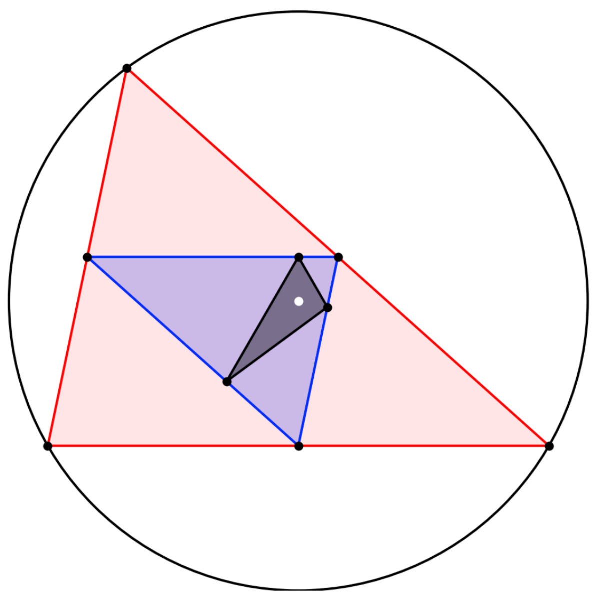
$$r = \frac{abc}{4\sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)}}$$

You should be able to show that

$$r = \frac{abc}{\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}}$$

# Chapter 23

# Triangles in triangles



In the figure above, the outer red triangle lies on its circumcircle, where the circumcenter is the white point at the center.

The midpoints of the sides of the outer triangle are joined to form a second triangle, in blue. We will show that the perpendicular bisectors of the sides of the outer triangle are the altitudes of the blue triangle. The proof is due to Gauss.

Even more interesting, if the feet of the altitudes of the blue triangle are joined to make a third triangle, the angle bisectors of that triangle meet at an incenter, I. The incenter of the dark gray third triangle is the orthocenter of the blue triangle and also the circumcenter of the original red triangle.

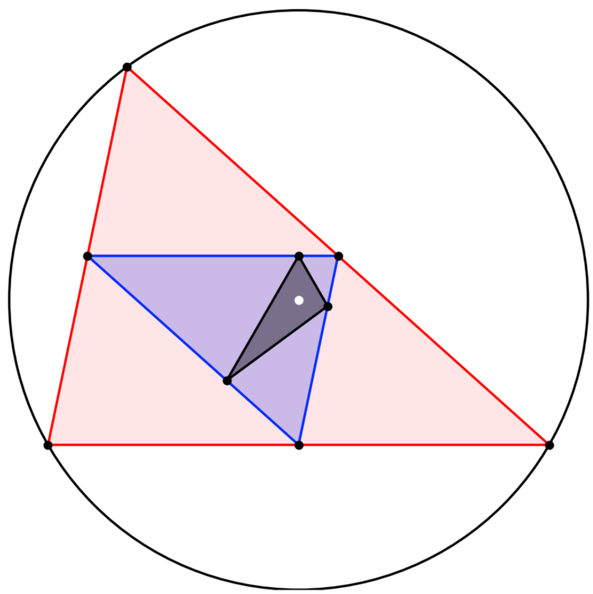
## Gauss and altitude

*Proof.*

Recall that we can find the circumcenter of any triangle by erecting perpendicular bisectors at the midpoints of the sides.

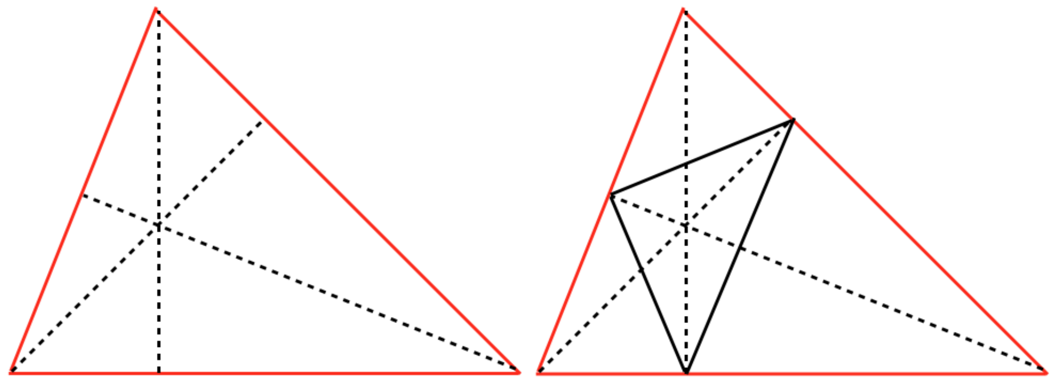
But those midpoints are the vertices of the blue triangle. And in drawing the triangle on the midpoints, by the midpoint theorem the resulting triangle has sides parallel to the original. So the perpendicular bisectors of the first triangle are perpendicular to the sides of the second, and also go through its vertices. Thus, they are altitudes of the second triangle.

□



## orthocenter and incenter

We have proven previously that the three altitudes meet at a single point, the orthocenter. The proofs include one from **Newton**, and the previous one (from **Gauss**).



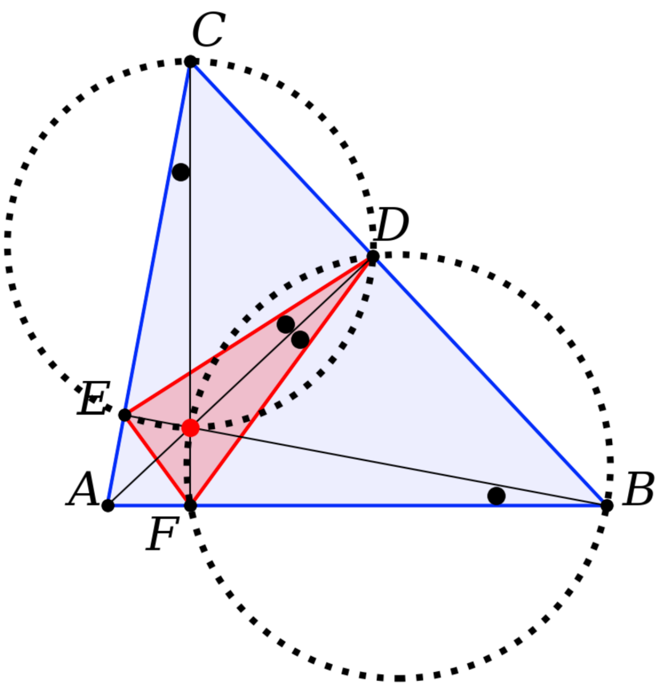
Above we have drawn the altitudes (left panel) and then also connected the points where the altitudes meet the sides at right angles. We will prove that the dotted lines are the bisectors of the angles at the vertices of the small inset triangle.

In other words, the incenter of the small triangle is the same point as the orthocenter of the bigger one.

*Proof.*

The key to the proof is to recognize that we can use a part of an altitude as the diameter of a circle. Draw the circle that has for its diameter the line segment connecting the orthocenter $H$ and vertex $B$ of the large blue triangle.

In this diagram we have changed the color of the inner triangle to be red, and also, the orthocenter of the blue triangle, where the altitudes cross, is not labeled because the figure is already too busy. It is, however, marked with a red dot.




Now consider the parts of the other two altitudes that terminate at $D$ and $F$ I claim that these two points lie on the same circle.

The reason is that, each one individually, taken together with the first two points, forms a right triangle. By the converse of Thales theorem, they must lie on the circle.

The same thing can be done with $CH$ as the diameter of a different circle, through $D$ and $E$.

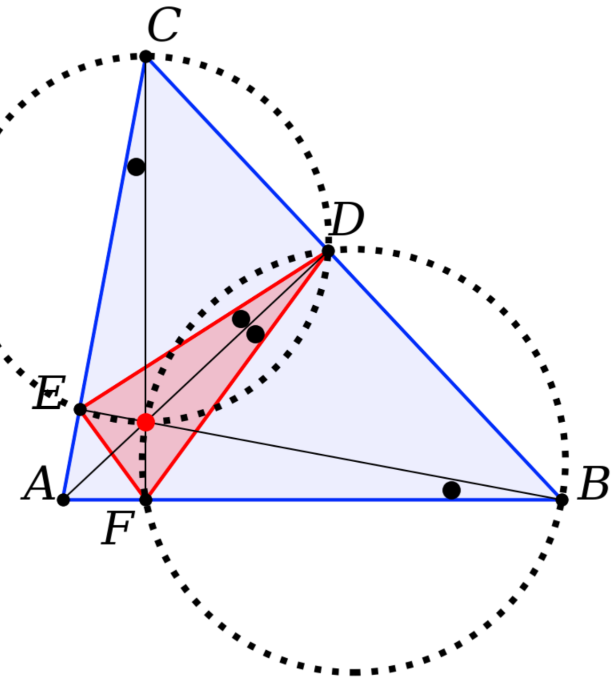
Now we can use the **theorem** about arcs that subtend an angle on the perimeter of the circle.

$\angle ADF = \angle EBA$ for this reason, while $\angle ACF = \angle ADE$ as well.

But then we notice that $\triangle HEC$ and $\triangle HFB$ are right triangles that share vertical angles, so they are similar. In particular $\angle ACF = \angle EBA$.

Thus, all four angles marked with black dots are equal. Therefore, the angle at vertex $D$ in $\triangle EDF$ is bisected by the altitude of the outer triangle $AD$.

Exactly the same logic will show that the angles at the other two vertices of $\triangle DEF$ are also bisected by the altitudes of $\triangle ABC$.




We conclude that the orthocenter $H$ of the blue triangle is the incenter $I$ of the red triangle.

$\square$

# Chapter 24

# Nine point circle

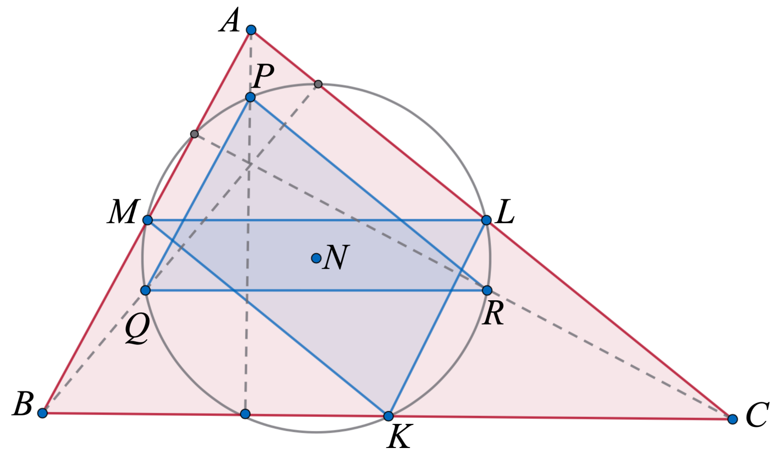
We've seen that the perpendicular bisectors of the sides and, separately, the angle bisectors of triangles, converge on (are concurrent with) points that are the centers of circles with interesting properties. These circles contain either the vertices of the triangle (circumcircle) or have the sides as tangents to the circle (incircle).

Now we investigate the altitudes and their point of convergence, the orthocenter. It turns out there is a special circle involving the altitudes, but it does not have the orthocenter as its center.

Quite surprisingly, there are nine points on the circle. Three of these are midpoints of the sides. This means that all four categories of special points of a triangle are connected to circles of one kind or another.

The circle that goes through the midpoints of the sides also goes through the points where the altitudes of a triangle meet the sides, as well as the midpoints of that part of each altitude lying between the orthocenter and the corresponding vertex.

### one approach




*Proof.*

Here is a fairly simple way to approach the nine point circle.

Draw $\triangle ABC$, then find the midpoints of the sides to form $\triangle KLM$. By the midpoint theorem, we have four congruent smaller triangles.

One of them is inverted: $\triangle KLM$. Draw the circumcircle of $\triangle KLM$. This is the nine point circle.

Now draw another triangle, congruent to $\triangle KLM$, with its vertices lying on the same circle, such that the corresponding sides are parallel. That is $\triangle PQR$.

The easiest way to do this is to draw a rectangle such as $MLRQ$ with $ML$, the base of $\triangle KLM$ equal to $QR$, the base of $\triangle PQR$.

There are actually three such rectangles on the circle, one for each side. These are $MLRQ$, $PRKM$ and $MPLK$.

Opposing vertices are the endpoints of diameters of the circle: $MR$, $LQ$, and $PK$.

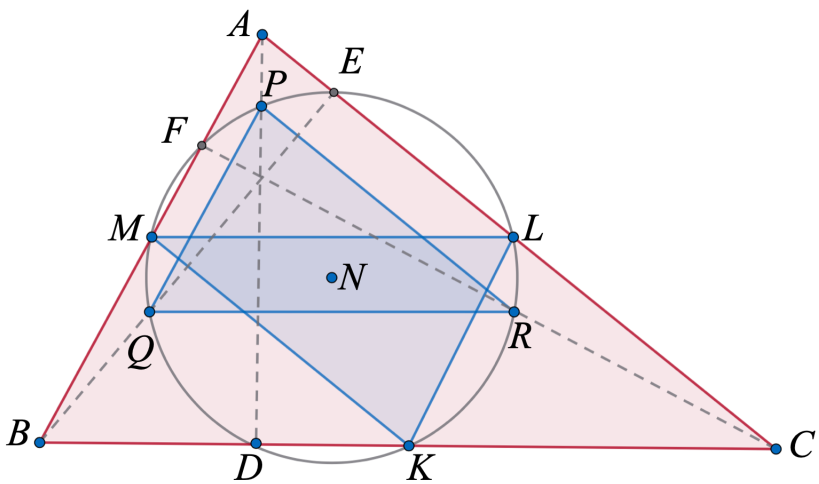
Comparing $\triangle KLM$ and $\triangle PQR$, there has been a rotation around the center $N$.

So then the altitude from $A$ is perpendicular not only to $BC$ but also to $QR$ and to $ML$. Let that altitude meet the base at $D$. The trick is to show that $AD$ goes through $P$.

Note that $\triangle PQR \cong \triangle AML$.

The bases of these two are displaced by movement parallel to $MQ$, i.e. vertically, since $MLRQ$ is a rectangle and $APQM$ is a parallelogram.

It follows that $AP \parallel MQ$ and perpendicular to $ML$, $QR$ and $BC$ so it coincides with $AD$.




Now since $PK$ is a diameter, it follows that $\angle KDP$ is right. The same follows for $\angle LEQ$ and $\angle RFM$.

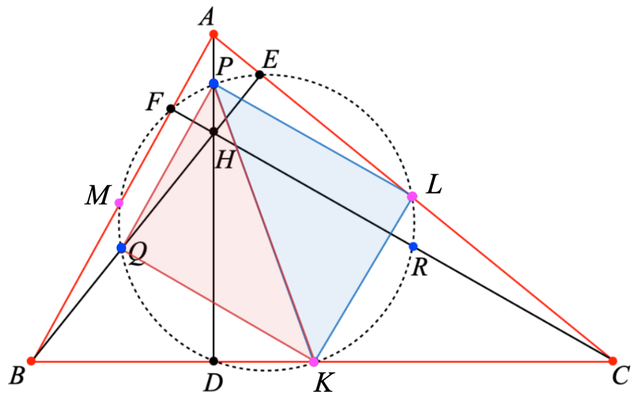
Thus, each of the three altitudes of $\triangle ABC$ has its foot lying on the nine point circle.

$\square$

The last point would be to show that $AP$ is one-half the distance from $A$ to the orthocenter. We'll see how to do this in the next part..

It's challenging to draw the circle.

In the next figure, some of the measurements may look a little off, but the logic will show that the circle indeed contains the nine points cited.




*Proof.*

In $\triangle ABC$ draw the altitudes $AD$, $BE$ and $CF$. Bisect the sides at midpoints $K$, $L$ and $M$. Mark the orthocenter at $H$ and bisect $AH$ at $P$, $BH$ at $Q$ and $CH$ at $R$.

Because of the six bisections, we can find a number of similar triangles to show that certain lines in the diagram are parallel, and then involving the altitudes, find certain right angles.

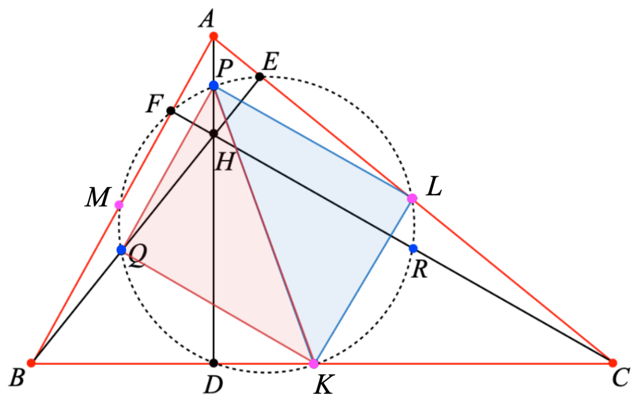
For example, in $\triangle AHC$ we have that $P$ and $L$ are midpoints of two sides, so it follows that $\triangle APL \sim \triangle AHC$, and then $PL \parallel FHC$.

Looking at $\triangle BHC$, by the same logic we can show that $QK \parallel FHC$. But this implies $PL \parallel QK$.

In exactly the same way, we can show that $PQ \parallel AB$ and $LK \parallel AB$ so $PQ \parallel LK$. Thus, $PLKQ$ is a parallelogram.

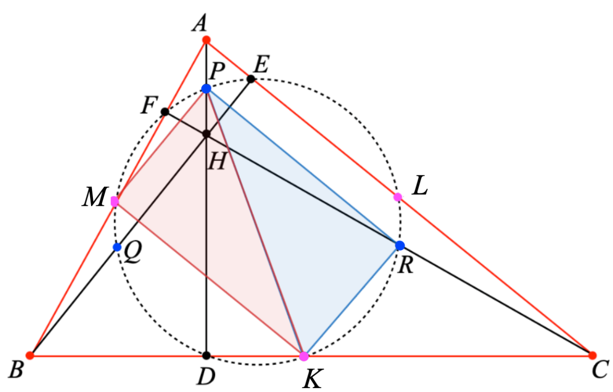
Since $AB \perp CF$, it follows that $LK \perp PL$. Thus, $\angle PLK$ is right. It follows that $PLKQ$ is a rectangle.

Next, draw the circle on diameter $PK$ (the center is not shown, but can be found by bisecting $PK$). $\angle PLK$ and $\angle PQK$ are both right, which means that points $L$ and $Q$ lie on the circle with diameter $PK$. And since $\angle ADK$ is right, $D$ also lies on the circle. That's five of the nine points.

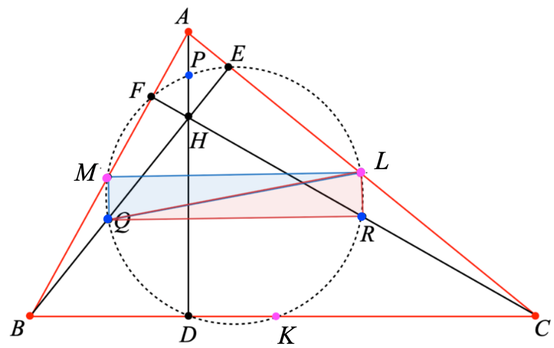



Since $Q$ and $L$ lie on the circle and $\angle QPL$ and $\angle QKL$ are right, it follows that $QL$ is also a diameter of the same circle. Alternatively, note that the center of the circle is at the bisector of $PK$, which since we have a rectangle, is also the bisector of $QL$ and $PK = QL$.

Since $QEL$ is right, $E$ also lies on this circle.




A similar series of steps will show that $PRKM$ is a rectangle, which is enough to establish that $M$ and $R$ are on the circle, and that $MR$ is actually a diameter of the circle. Since $\angle MFR$ is right, then $F$ also lies on the circle.




We do not need it, but there is a third rectangle, namely, $MLRQ$.

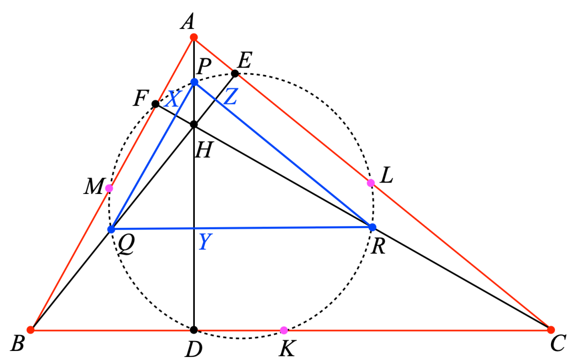
□

## radius

It turns out that the radius of the nine-point circle is $1/2$ the radius of the circumcircle of the same triangle.

*Proof.*

Draw $\triangle PQR$.




The nine-point circle *is* the circumcircle of $\triangle PQR$. We know a formula that connects the area $\mathcal{A}$ of a triangle with its side lengths and the radius of the circumcircle $R$:

$$R = \frac{abc}{4\mathcal{A}}$$

Let $\triangle ABC$ have side lengths $a$, $b$ and $c$ and area $\mathcal{A}$. Let $\triangle PQR$ have side lengths $a'$,

$b'$ and $c'$ and area $\mathcal{A}'$. This triangle with vertices halfway along the sides of $\triangle ABC$ is called its medial triangle.

We can use similar triangles to show that each side and altitude in $\triangle ABC$ is twice that of $\triangle PQR$.

For example, we will show that $AB = 2PQ$.

Label the point $X$ where $HF$ cuts $PQ$. $PXQ \parallel AB$, so $PXQ \perp CHF$, and $\triangle AFH \sim \triangle PXH$ with a ratio of 2. It follows that $2PX = AF$ and $2XH = FH$.

The latter equality gives us $\triangle BFH \sim \triangle QXH$, again with a ratio of 2. Thus $2XQ = FB$ and $2QH = BH$. The result follows by addition. In a similar way we can show that $2HZ = HE$ so by addition the altitude $BE$ is also scaled by a factor of 2.

This means that the area is scaled by a factor of 4. The ratio of radii is:

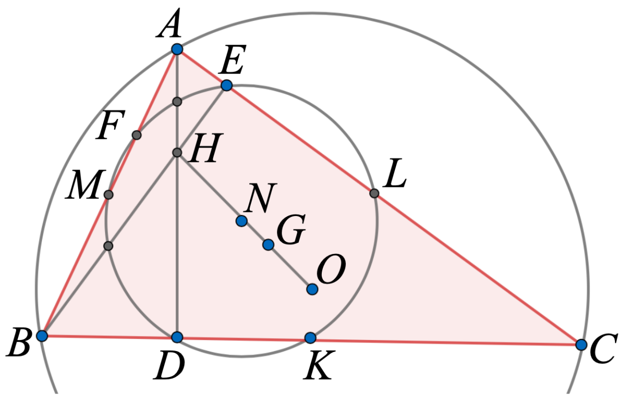
$$\frac{abc}{a'b'c'} \cdot \frac{\mathcal{A}'}{\mathcal{A}}$$

The first term is 8 and the second term is 1/4 so the result is just a factor of 2.

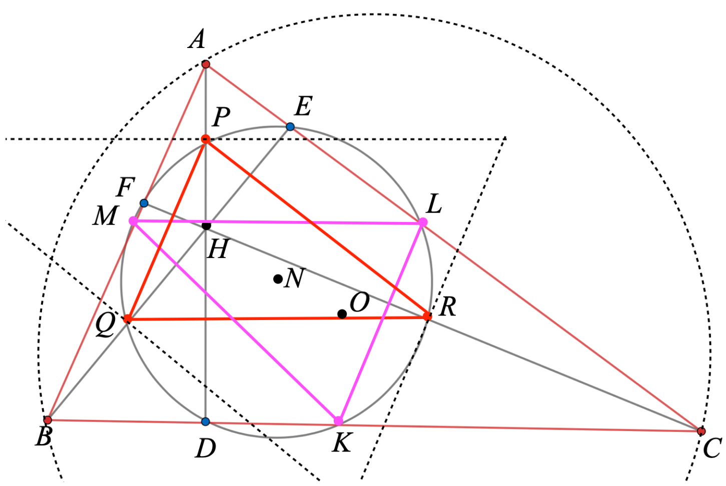
## center of nine point circle

I find from reading Coxeter that the center of the nine point circle bisects the Euler line.

Recall that the centroid $G$ lies one-third of the way along the Euler line from the circumcenter $O$ to the orthocenter $H$. Then $N$ bisects $OH$ and $G$ lies one-sixth of that length away from $N$, on the side toward $O$.




*Proof.*




We established above that the side lengths of $\triangle PQR$ are exactly half those of $\triangle ABC$.

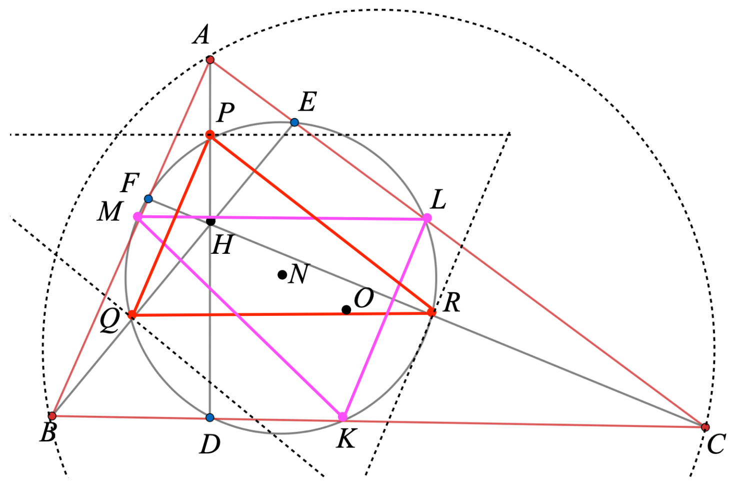
But the sides of $\triangle KLM$ are also exactly half those of $\triangle ABC$.

So the two smaller triangles are congruent. They share the same circumcircle on center $N$, since all the vertices lie on the circle, which is the nine point circle for the parent $\triangle ABC$.

Since $ML \parallel BC$ (and so on), the two triangles are related by rotation through a half-turn on center $N$.

Find the circumcenter of $\triangle ABC$ by dropping perpendiculars from points $K$, $L$ and $M$ to meet at $O$. The same lines are the altitudes of $\triangle KLM$, since for example,

$ML \parallel QR \parallel BC$.




The circumcenter of $\triangle ABC$, point $O$, is the same point as the orthocenter of $\triangle KLM$. Rotation about $N$ converts $H$ into $O$ and vice-versa. It follows that the distances are the same: $HN = ON$.

$\square$

https://en.wikipedia.org/wiki/Nine-point_circle

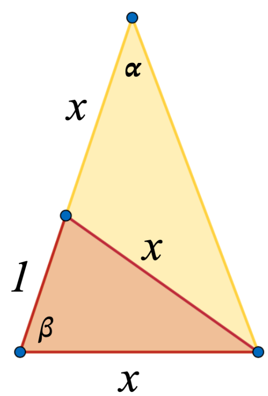
# Part VI

# Polygons

# Chapter 25

# Pentagon

Before we look at the pentagon, let's start with three isosceles triangles, one tall and skinny and one short and squat, both nestled inside another tall skinny one.




Since the base angles of the tall triangles are equal, they are similar. Scale the base of the brown one to be equal to 1, then label the other side as $x$:

$$\frac{x}{1} = \frac{1+x}{x}$$

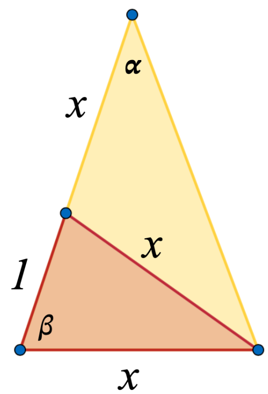
$$x^2 = 1 + x$$

This is the famous golden ratio, what the Greeks called the mean proportion, and is often labeled as $\phi$:

$$\phi^2 = 1 + \phi$$

We've seen this occasionally elsewhere, and it shows up repeatedly in consideration of the pentagon. The other thing we notice is the value of the two angles. We have:

$$\beta + \beta + \alpha = \pi$$




From the lower right hand vertex:

$$\beta = 2\alpha$$

It follows that $\alpha = \pi/5$ or $36°$, and $\beta$ is twice that.

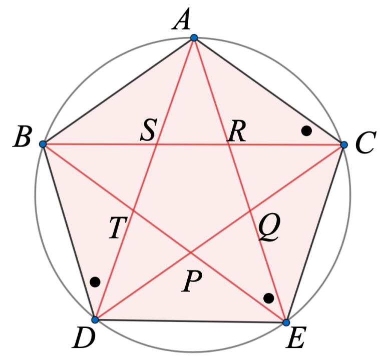
## angles in the pentagon

A pentagon has 5 sides, and a regular pentagon has all five sides equal. In this chapter we explore some of its properties.

The first question that arises is how to draw one. We will look at two constructions later, a very quick modern one, and also a classic due to Euclid.

For the moment, we assume that this has been done. This is not a big deal, since most constructions start with a circle.

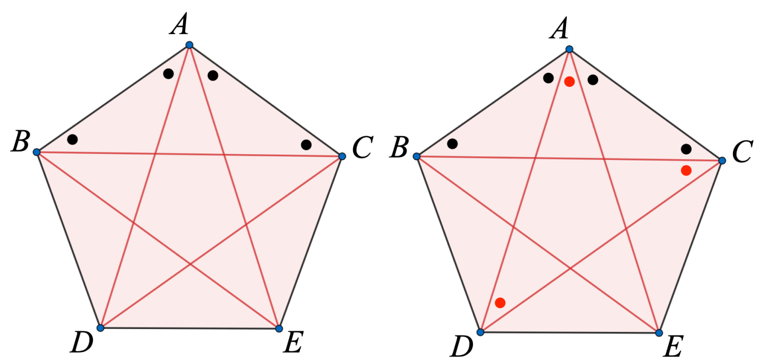
Here is a regular pentagon circumscribed by a circle.




The inscribed angle theorem makes the derivation of equal angles easy.. In the figure above, the three angles marked with black (B) dots are all equal because each is subtended by chord $AB$.

Because all the sides are equal, $AB = AC$ and so on, it follows that all three of the angles at any one vertex are equal. $\angle DAE$ is subtended by $DE$ but $DE = CE$, so $\angle DAE = \angle CAE$.

In fact, all of the vertices are also equal, since central triangles (not drawn) are congruent by SSS. We could also appeal to the five-fold rotational symmetry.




Symmetry also gives us the equal central angles labeled with red (R) dots in the right panel. Then by sum of angles we have:

$$2B + 3R = 4B + 1R$$

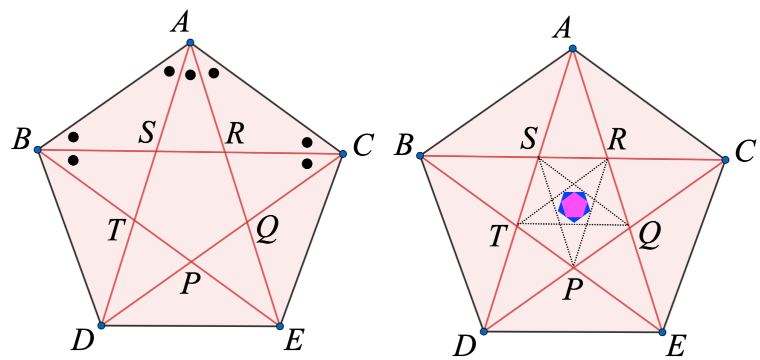
which quickly leads to $B = R$, as we have already concluded.

Each vertex consists of three copies of the same angle, each one of those small angles is 1/5 of two right angles or 36°. They add up to 104° at the vertices.

All five chords are equal as well since, for example, $\triangle ABC \cong \triangle ACE$ by SAS. It follows that $\triangle ADE$ is isosceles.

## parallelogram

Again, each small angle at any vertex has the same measure. We have marked only some of them in the figure below (left).




In $ACPB$, adjacent vertices have five copies of the small angles, which add up to two right angles. Thus, the figure is a parallelogram. Moreover, it is a rhombus, because adjacent sides are equal, such as $AB = AC$. Altogether, there are 5 such rhombi in the figure.

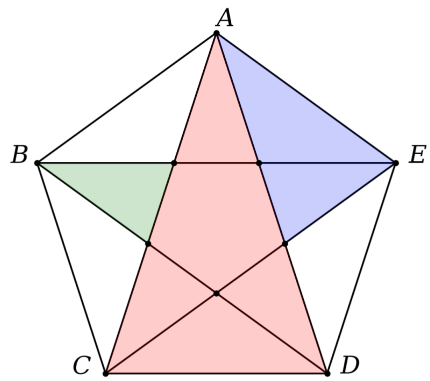
Because of the parallelograms, $BC \parallel DE$ and so on. It follows that $\triangle ADE \sim \triangle ASR$.

There are two classes of similar isosceles triangle in the figure: tall skinny ones (36-72-72), and short fat ones (108-36-36). The tall skinny ones come in three sizes, so for example

$$\triangle ASR \cong \triangle BTS$$
$$\triangle ABT \cong \triangle DSB$$
$$\triangle ADE \cong \triangle EAB$$

Each of these triangles contains the same angles, so they are similar, and they are all isosceles.

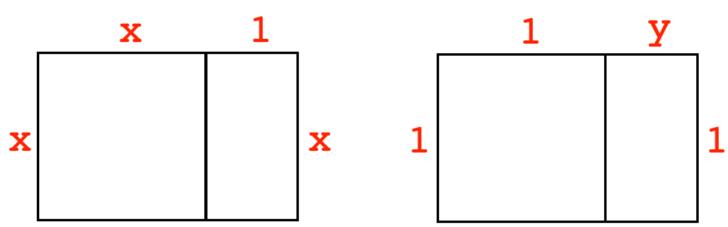
Here are three examples of the tall skinny triangle:




By sum of angles, the central $PQRST$ is a pentagon, and by symmetry, it is a regular pentagon. One can repeat the process of drawing the diagonals and generate new pentagons inside pentagons, forever.

When we work through the similar triangles using relationships between lengths (like those from the rhombus sides equal to pentagon sides), we'll see something very interesting. But I think it may be useful to stop here and preview the answer, so we can get the arithmetic straight.

## the golden ratio




We draw a square and then extend two parallel sides to make a large rectangle and a small one at the same time. We don't want just any rectangles, but require that they be similar: they should have the same ratio of the long side to the short side.

We can conveniently model this in two ways. In the first, the square has side length $x$ and the extension is 1, while in the second, the square is scaled to have side length 1 and the extension is $y$. These will give inverses.

We choose the first method. Hence similarity gives:

$$\frac{x+1}{x} = \frac{x}{1}$$

$$x + 1 = x^2$$

$$x^2 - x - 1 = 0$$

The solutions are:

$$x = \frac{1 \pm \sqrt{5}}{2}$$

Since $\sqrt{5} > 1$ (also $> 2$ !) the minus branch gives $x < 0$. We choose the positive branch:

$$x = \frac{1 + \sqrt{5}}{2}$$

This is called $\phi$, the famous golden mean or ratio. It has a value of about 1.65.

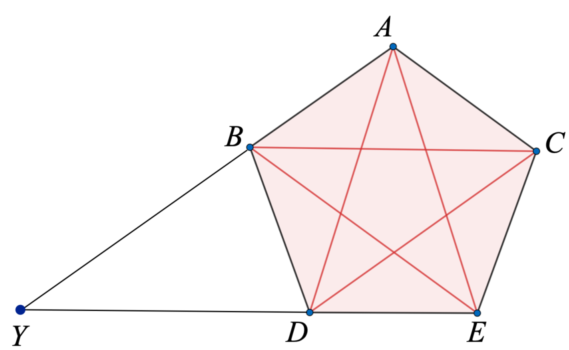
We can check that $\phi$ really does solve the equation:

$$\phi^2 = \frac{1+\sqrt{5}}{2} \cdot \frac{1+\sqrt{5}}{2}$$

$$= \frac{1}{4}(1 + 2\sqrt{5} + 5)$$

$$= 1 + \frac{2 + 2\sqrt{5}}{4} = 1 + \phi$$

To return to our problem:



Extend $AB$ and $ED$ to meet at $Y$.

Since the external angles to $\triangle YBD$ are equal, by I.6 $\triangle YBD$ is isosceles.

Since $\triangle ADE$ has the same base angles and equal base $BD = DE$, $\triangle ADE \cong \triangle YDB$ by ASA.

Let $\triangle YDB$ have a ratio of the side length to the base of $x$.

Since $BD \parallel AE$, we have $\triangle YDB \sim \triangle YEA$. Remembering that $DY = AD$, we can construct the ratios:

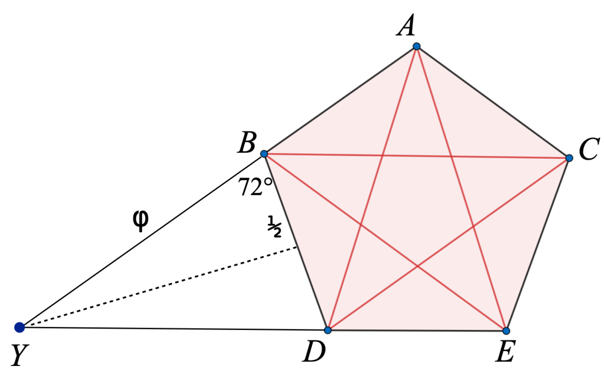
$$\frac{x}{1} = \frac{x+1}{x}$$
$$x^2 = x + 1$$

Hence $x$ is really $\phi$.

The equation below is the one to remember, with $\phi$ substituted for $x$:

$$\phi^2 = 1 + \phi$$

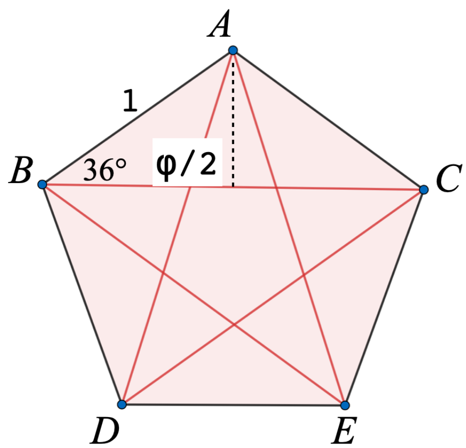
There are some other values we will need that can be seen in this figure:




$\angle YBD = 72°$, since it has two copies of the fundamental angle. Its cosine is one-half of $BD$, that is, $1/2$, divided by $\phi$ or

$$\cos 72° = \frac{1}{2\phi}$$

We also will need:




$\angle ABC = 36°$. Its cosine is one-half of $BC$, $\phi/2$, divided by 1, or just

$$\cos 36° = \frac{\phi}{2}$$

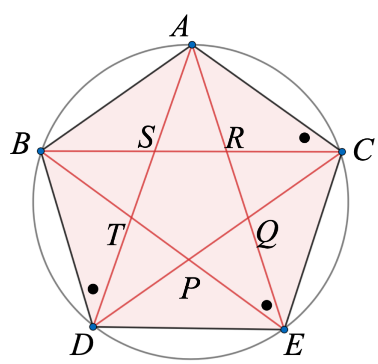
We will come back for both of these results.

### more

There are occurrences of $\phi$ all over the regular pentagon (see the first reference below).

One can also get the golden ratio, $\phi$, from the short squat triangles.

$$BC = \phi, \qquad AB = AC = 1$$



We find that $BS = \phi - 1$. Hence we compare $\triangle ABS \sim \triangle BCA$. We form the ratio of the longer side to the shorter one:

$$\frac{1}{x-1} = \frac{x}{1}$$

$$x^2 - x = 1$$

$$x^2 = 1 + x$$

Hence this $x$ is also $\phi$.

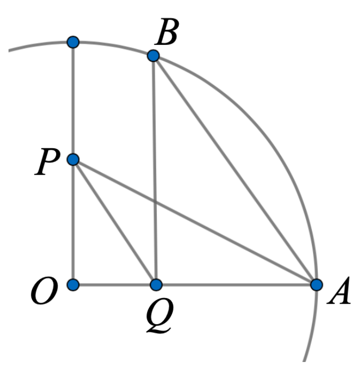
## construction of circumscribed pentagon

Joyce and Bogolmony both give this construction, due to Richmond.

https://mathcs.clarku.edu/~djoyce/java/elements/bookIV/propIV11.html

https://www.cut-the-knot.org/pythagoras/RichmondPentagon.shtml

Wikipedia gives a rearranged version of the same thing.




In a circle on center $O$, draw the radius $OA$ and make $\angle AOP$ a right angle, with the second radius bisected at $P$.

Draw $AP$. Now bisect the angle $APO$ to find $Q$ on $OA$.

Finally, draw the perpendicular to $OA$ at $Q$ and find where it cuts the circle at $B$. $AB$ is one side of the pentagon circumscribed by this circle.

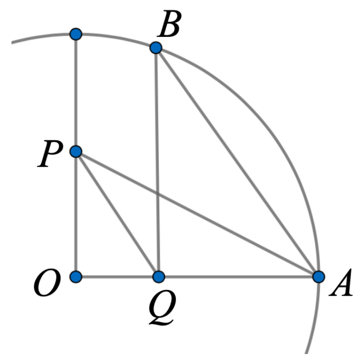
*Proof.* We first show that $AQ/OQ = \sqrt{5}$.

Let the radius equal 2. Then $OP = 1$ and the Pythagorean theorem gives $AP = \sqrt{5}$.

From the bisector theorem

$$\frac{OQ}{AQ} = \frac{OP}{AP} = \frac{1}{\sqrt{5}}$$

The result follows easily.




We also know that $OQ + AQ = 2$ so

$$OQ + \sqrt{5} \cdot OQ = 2$$

$$(1 + \sqrt{5}) \cdot OQ = 2$$

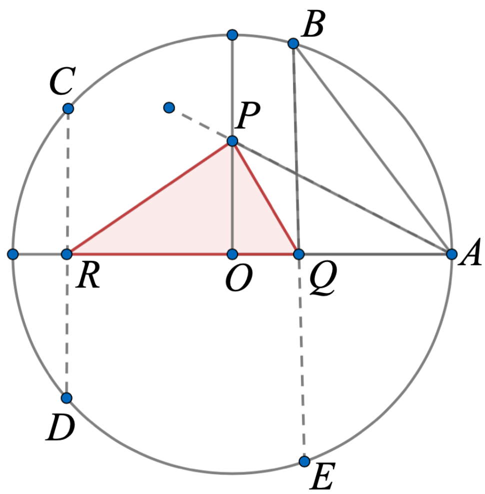
$$\frac{1}{OQ} = \phi$$

In other words, the ratio $OP : OQ = \phi$.

$\triangle BOQ$ has adjacent side $OQ = 1/\phi$ and the hypotenuse is 2 so the cosine of $\angle BOQ$ is $1/2\phi$.

But this is the cosine of 72°, which means that $\angle BOQ = \angle BOA = 72°$, and corresponds to one-fifth of the complete circle. It is the central angle of a pentagon.

Here is a figure (redrawn from Bogolmony) which shows how Richmond's approach extends to finding all of the vertices of the regular pentagon (Bogolmony cites Conway and Guy).




$APO$ is bisected by $PQ$. $PR$ is drawn as the bisector of the external angle to $\angle APO$ (i.e. supplementary). Two adjacent bisectors of supplementary angles together form a right angle.

Thus $PR$ is perpendicular to $PQ$ at $P$ and forms $\triangle RPQ$ as a right triangle. $\triangle RPQ \sim \triangle ROP \sim \triangle POQ$.

An easy consequence is that by similar triangles we have that

$$\frac{OP}{OR} = \frac{OQ}{OP}$$

$$OP^2 = OR \cdot OQ$$

$$1 = OR \cdot \frac{1}{\phi}$$

$$OR = \phi$$

But $OR$ is the adjacent side in the right triangle $COR$, with hypotenuse equal to 2. So the cosine of $\angle COR$ is $\phi/2$.

We showed above that the angle whose cosine is $\phi/2$ is $36°$.

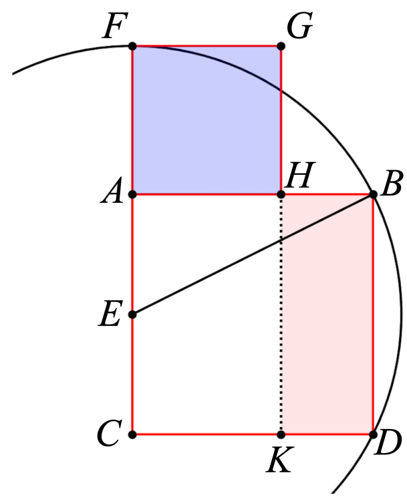
$\angle COD$ is twice $\angle COR$ or $72°$, which is the correct measure for the central angle of one sector of the regular pentagon.

## Euclid's construction

Euclid uses several steps to construct a pentagon circumscribed by a circle. The critical chain of dependencies is II.11 ⇒ IV.10 ⇒ IV.11. We first look at II.11.

## Euclid II.11

> To cut the line $AB$ at $H$ such that the rectangle contained by the whole and one of the segments is equal to the square on the other segment.




The idea is to find $H$ such that

$$AB \cdot HB = AH^2$$

In other words, find $H$ such that

$$\frac{AB}{AH} = \frac{AH}{HB} = \phi$$

We will prove that the area of rectangle $HBDK$, abbreviated $HD$, equals $FH$, the square on $AH$.

*Geometric Proof.*

Draw the square on $AB$ so $AB = BD$.

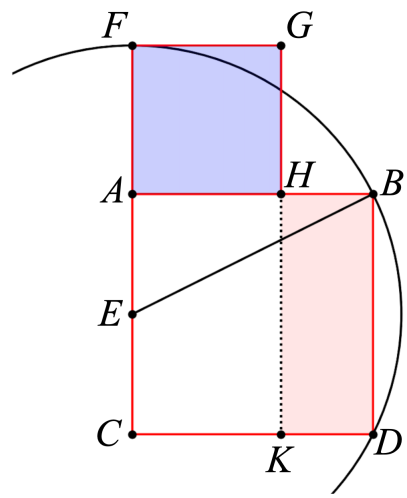
Bisect $AC$ at $E$. Draw $BE$.

Extend $EA$ so that $EF = BE$.

Draw the square on $AF$. $H$ is the desired point.

I claim $HB \cdot AB = AH^2$.

$$\frac{AH}{HB} = \frac{AB}{AH}$$




In other words, the golden ratio or mean.

By II.6

$$CF \cdot FA = EF^2 - AE^2$$

Since $EF = BE$ and by I.47

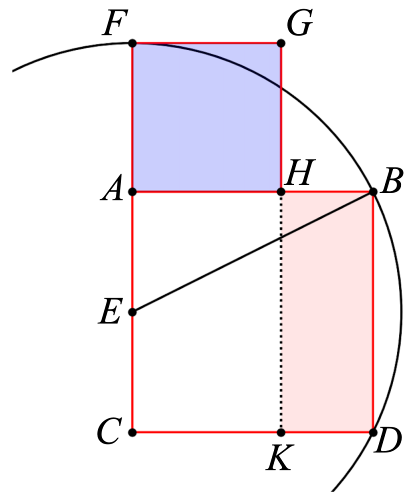
$$CF \cdot FA = BE^2 - AE^2 = AB^2$$

The left-hand side is $FK$, and the right-hand side is $AD$.

Subtract the shared area $AK$ from each. We obtain:

$$FH = AH^2 = HD = AB \cdot HB$$

□




Algebraically, let $AB = x, AH = y, HB = x - y$, and then

$$(x - y) \cdot x = y^2$$

$$x^2 - xy = y^2$$
$$x^2 = xy + y^2$$

Scale so that $y = 1 = AH$ then

$$x^2 = x + 1$$

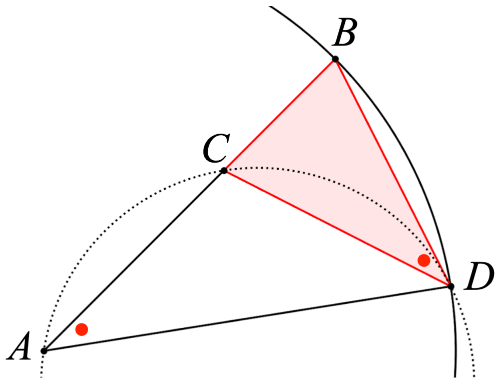
As advertised, $x = \phi = AB$, the golden ratio.

## Euclid IV.10 and IV.11

Next, Euclid uses II.10 to construct an isosceles triangle that has its base angles twice the vertex. What is below does not follow word for word, but it's close.

Draw the circle $\mathcal{O}$ on center $A$ with arbitrary radius $AB$.

By the construction of II.11, cut $AB$ at $C$ so that $AB \cdot BC = AC^2$.




Now find $D$ on $\mathcal{O}$ such that BD = AC. So

$$AB \cdot BC = AC^2 = BD^2$$

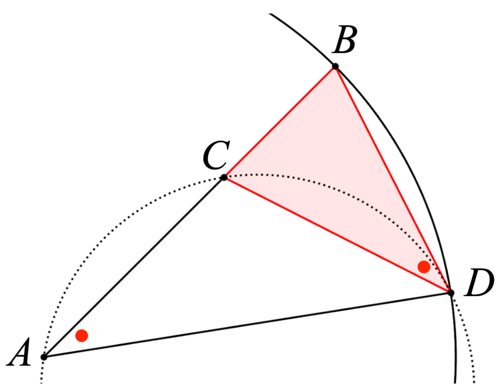
As radii of $\mathcal{O}$, $AB = AD$ and $\triangle ABD$ is isosceles.

Join $CD$ and $AD$. Draw the circle $\mathcal{Q}$ containing points $A$,$C$,$D$.

A crucial step is that that by the converse of the tangent-secant theorem, the relationship above means that $BD$ is tangent to $\mathcal{Q}$ at $D$.

Therefore, since they are subtended by the same arc:

$$\angle A = \angle BDC$$




Since they have equal apex angle and share the base angle at $B$, $\triangle ABD \sim \triangle ADB$.

Thus $\triangle DBC$ is also isosceles, with $CD = BD = AC$. But this means that $\triangle ACD$ is also isosceles.

It follows that $\angle CDA = \angle CAD = \angle BDC$, and $CD$ bisects $\angle D$.

Finally the base $\angle DCB$ is equal to $\angle A$ plus $\angle CDA$ by the external angle theorem, and since these are equal, $\angle B = \angle DCB$ is twice $\angle A$.

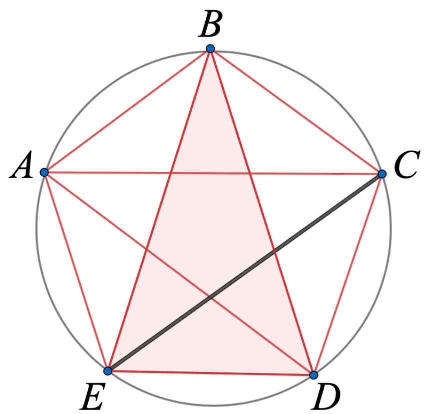
$\square$

Since $\triangle ABD$ has central $\angle A$ equal to one-fifth of a right triangle, it is one-tenth of the complete circle.

Therefore, the side $BD$ forms one side of a regular decagon inscribed in $\mathcal{O}$.

Following Bogolmony, we use this shortcut: connecting alternate vertices yields a regular pentagon.. This completes the construction.

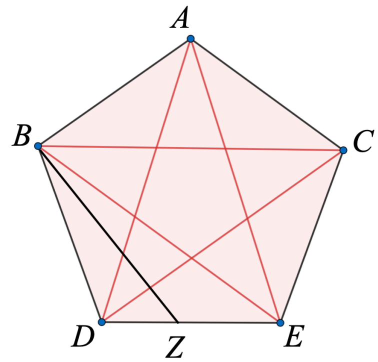
Euclid IV.11 inscribes a triangle with the same angles as above into a circle ($\triangle BED$, below).




Bisect $\angle BED$ and extend the bisector to meet the circle at $C$. Do the same with $\angle BDE$ to find $A$. This completes Euclid's construction.

## another bisection

Draw the perpendicular at any vertex, say $BZ \perp BA$, so that $\angle ABZ$ is a right angle.



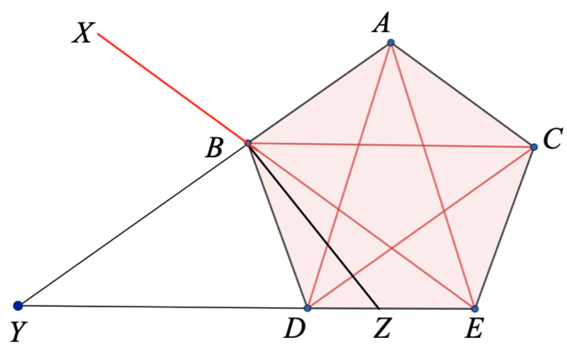

$\angle DBE$ is bisected, since the total angle at $B$ is 108°, so by subtraction $\angle DBZ = 18°$, which is one half of $\angle DBE = 36°$.

(Of course, the Greeks would reason in terms of fractions of a right angle).

We use the bisector theorem again:

$$\frac{EZ}{DZ} = \frac{BE}{BD} = \frac{1+x}{x} = x = \phi$$

We can extend adjacent sides in the previous figure. Since $\angle ABZ$ is a right angle and $\angle DBZ$ is one-half of the bisected internal $\angle DBE$ (in $\triangle BDE$), it follows that $\angle DBY$ is the bisector of the external $\angle DBX$.




By the external bisector theorem:

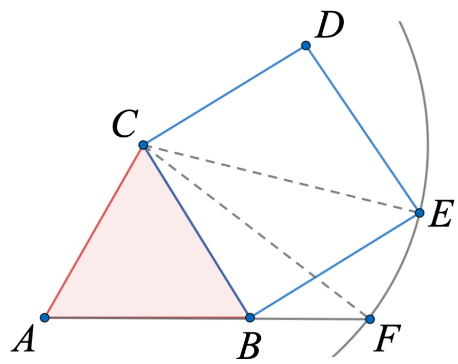
$$\frac{EY}{DY} = \frac{BE}{BD} = \phi$$

When two adjacent sides of a regular pentagon are extended, then the length of the side plus the extension is in proportion to the side length as $\phi$.

This is also evident from the fact that $BD \parallel AE$, so $\triangle BYD$ is similar to $\triangle AYE$.

## problem

https://www.cut-the-knot.org/do_you_know/GoldenRatio.shtml




Let $\triangle ABC$ be equilateral and $BCDE$ be a square. Construct the circle on center $C$ with radius $CE$ and find where it cuts the extension of side $AB$ at $F$. Prove that $AB : BF = \phi$.

*Proof.*

Scale the triangle so that $AC = 1$

Then $CE = \sqrt{2} = CF$.

We have $\cos A = \sin 30^\circ = \frac{1}{2}$

Use the Law of Cosines:

$$CF^2 = AC^2 + AF^2 - 2AC \cdot AF \cos A$$

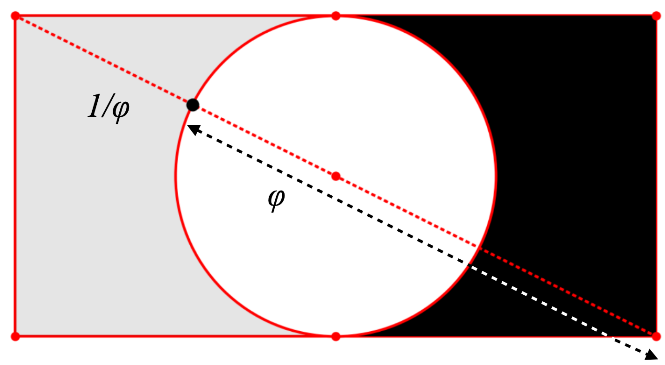
$$2 = 1 + AF^2 - AF$$

$$AF = \phi$$

The result follows easily.

□

## problem




Here's another construction in the collection curated by Bogolmony, from John Arioni. Let the diameter of the circle be 1, so the rectangle has sides of length 1 and 2 and the diameter is $\sqrt{5}$.

Let the distance from the corner of the rectangle to the first edge of the circle be $x$. We have that

$$x + 1 + x = \sqrt{5}$$

Then

$$x = \frac{\sqrt{5} - 1}{2} = \phi - 1$$

But then the distance from the corner to the *other* edge of the circle is just $1 + x = \phi$.

Finally, recall that

$$\phi^2 = \phi + 1$$

$$\phi = 1 + \frac{1}{\phi}$$

So $\phi - 1 = 1/\phi$.

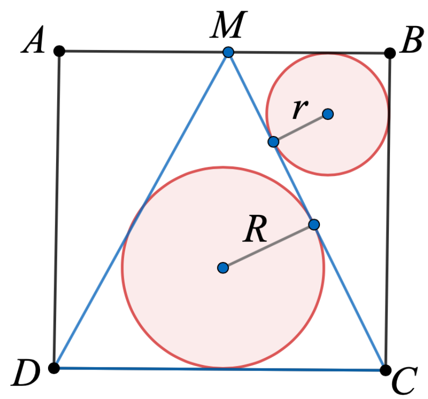
□

## $\phi$ in the square

Let's finish the chapter with one last example I found on Bogolmony's site.

https://www.cut-the-knot.org/do_you_know/GoldenRatioInSquare.shtml

It was "contributed by Ercole Suppa (Italy) at the Peru Geometrico facebook group."




One side of a square is bisected at $M$ and then the two incircles are drawn as shown. Remarkably, the radii are in the ratio $R/r = \phi$.

*Proof.*

Let the square $AC$ have sides of length 2, so $MB = 1$ and $MC = \sqrt{5}$. We find the semi-perimeters of the two triangles:

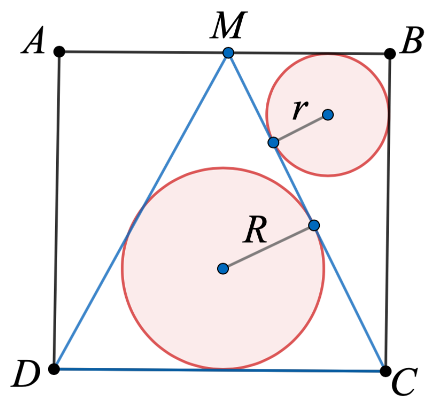
$$s_{\triangle MBC} = \frac{1}{2} \cdot (3 + \sqrt{5})$$

$$s_{\triangle MDC} = \frac{1}{2} \cdot (2 + 2\sqrt{5}) = 1 + \sqrt{5}$$

Then we find the areas as

$$\mathcal{A}_{\triangle MBC} = rs_{\triangle MBC} = \frac{1}{2}\ r\ (3 + \sqrt{5})$$

$$\mathcal{A}_{\triangle MDC} = Rs_{\triangle MDC} = R\ (1 + \sqrt{5})$$




Since the base of one triangle is twice the other, we have

$$\mathcal{A}_{\triangle MDC} = 2A_{\triangle MBC} = r(3+\sqrt{5})$$

Then

$$R(1+\sqrt{5}) = r(3+\sqrt{5})$$

$$\frac{R}{r} = \frac{3+\sqrt{5}}{1+\sqrt{5}}$$

which certainly doesn't look like $\phi$, although it does have $\sqrt{5}$.

We could clear the denominator, multiplying by $(1-\sqrt{5})/(1-\sqrt{5})$, but instead let's just play around:

$$\phi = \frac{1+\sqrt{5}}{2}$$

$$2\phi = 1+\sqrt{5}$$

$$2\phi + 2 = 3+\sqrt{5}$$

Hence we have that

$$\frac{R}{r} = \frac{2\phi+2}{2\phi}$$

$$= \frac{\phi+1}{\phi}$$

Recalling that $\phi + 1 = \phi^2$, it follows that

$$\frac{R}{r} = \phi$$

□

That's amazing!

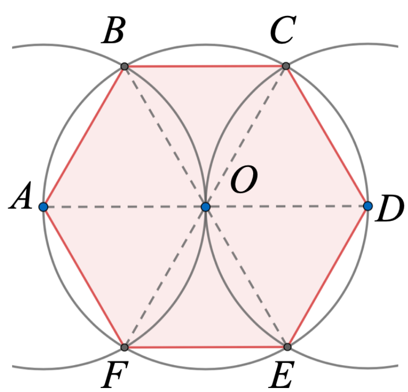
# Chapter 26

# Hexagon

## angles in the hexagon

A hexagon has 6 sides, and a regular hexagon has all six sides equal. In this chapter we explore some of its properties.

Euclid, IV.15 has a nice construction.



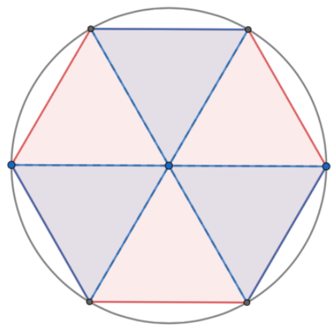

Draw a circle on center $O$, and draw a diameter $AD$. Then draw circles on centers $A$ and $D$ with the same radius, so $AO = DO$.

First, $OA$, $OB$, $OC$, $OD$, $OE$ and $OF$ are all radii of the original circle.

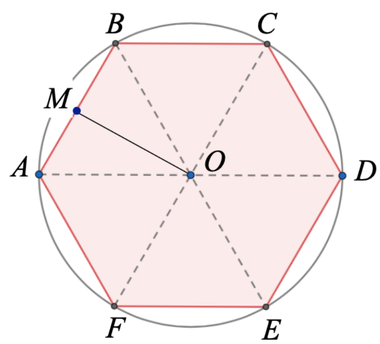
Second, the segments $AB = AF = AO$, and $DC = DE = DO$. It follows that $\triangle AOB$ is equilateral and so is $\triangle COD$.

Finally, the base angle of $\triangle BOC$, $\angle BOC$, is $60°$, and the triangle is isosceles, so it is also equilateral.

As a result, we have six equilateral triangles, and the polygon $ABCDEF$ is a regular hexagon.



Let the side length (and radius) be 1. Then Pythagoras's theorem gives us that the apothem $OM$ has a length of $\sqrt{3}/2$.




So twice the area of each component triangle is

$$2\mathcal{A}_{\triangle AOB} = \frac{\sqrt{3}}{2} \cdot 1$$

The total area of the hexagon is three times that or

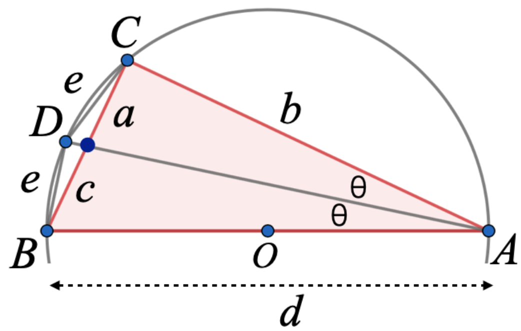
$$\mathcal{A}_{hex} = \frac{3\sqrt{3}}{2}$$

This approximates the area of the unit circle ($\pi$), but it is not a very good approximation.

Similarly, the total perimeter is 6, which approximates the total perimeter of the unit circle ($2\pi$), but it is not especially close yet either.

## lower bound for $\pi$

Let $\triangle ABC$ be a right triangle inscribed in a semicircle, and the angle at $A$ is bisected.




The idea is that the side of $\triangle ABC$ that is a chord of the circle, $BC = a + c$, approximates the perimeter of the circle.

Let $\triangle OBC$ (the arms are not drawn) be an equilateral triangle. By the inscribed angle theorem, $2\theta = 30°$, one-half of $\angle BOC$. The total perimeter is 6 times that.

Now imagine that we are able to find the length $e$. $e$ is clearly much closer to the actual circle, so 6 times $2e$ would be a better approximation to the perimeter.

Using this idea, with both inscribed and circumscribed hexagons, Archimedes came up with both upper and lower bounds, namely, $223/71 < \pi < 22/7$.

We begin with a result that comes from the bisector theorem.

$$\frac{a}{c} = \frac{b}{d}$$

$$\frac{a}{c} + 1 = \frac{b}{d} + 1$$

$$\frac{a+c}{c} = \frac{b+d}{d}$$

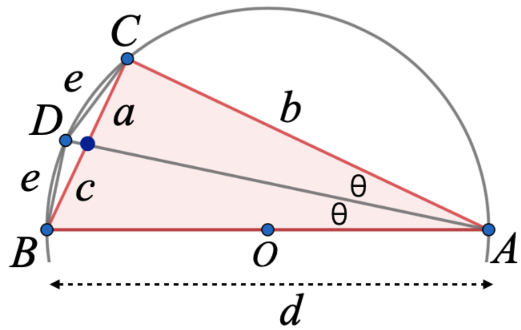
$$\frac{a+c}{b+d} = \frac{c}{d} = \frac{a}{b}$$

$$\frac{b+d}{a+c} = \frac{b}{a}$$

$$\frac{b}{a+c} + \frac{d}{a+c} = \frac{b}{a}$$

Using trigonometric functions, we can write this as

$$\cot 2\theta + \csc 2\theta = \cot \theta$$




If we're doing multiple rounds we would need Pythagoras's theorem

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \cot^2 \theta = \frac{1}{\sin^2 \theta} = \csc^2 \theta$$

First add the cotangent and cosecant of the double angle to find the cotangent, then use the above procedure to get the cosecant. Do the same thing again, as many times as you wish.

## computation

We start with a $30 - 60 - 90$ triangle ($2\theta = 30°$).

The next thing to decide is the scale. If $d = 1$ then the perimeter equals $\pi$; alternatively with $d = 2$ then the perimeter is $2\pi$.

Let $d = 2$.

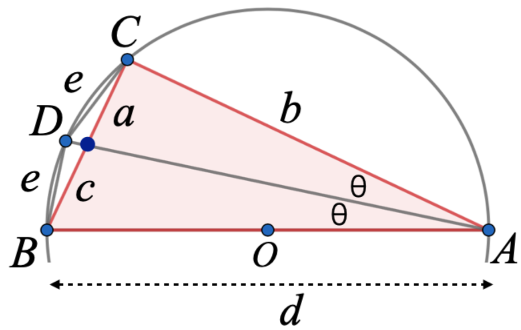
$$BC = a + c = 1$$

$$b = \sqrt{3}$$

The whole computation has to be done with rational numbers, because that's how the Greeks did things.

We need a rational approximation for $\sqrt{3}$. The value Archimedes used was $\sqrt{3} \approx 1351/780$. (I have written elsewhere about how he may have come up with that).

The ratio to the true value is $\approx 1.0000003$. It is over, but we will use the value as the inverse, so that's actually under, appropriate for a lower limit.




$$\csc 2\theta = 2$$

$$\cot 2\theta = \sqrt{3} = \frac{1351}{780}$$

$$\cot\theta = \frac{1560 + 1351}{780} = \frac{2911}{780}$$

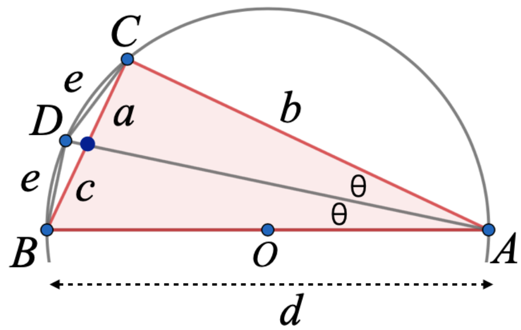
$$\csc^2\theta = 1 + \frac{8473921}{608400} = \frac{9082321}{608400}$$

Archimedes approximates this as

$$\csc\theta = \frac{AB}{BD} = \frac{3013 - 3/4}{780} = \frac{2}{e}$$

Note that $3013 - 3/4$ is slightly larger than the true value of the square root, but this error makes our estimate *lower*, since it shows up as the inverse, below. So the estimate is valid as a lower bound.

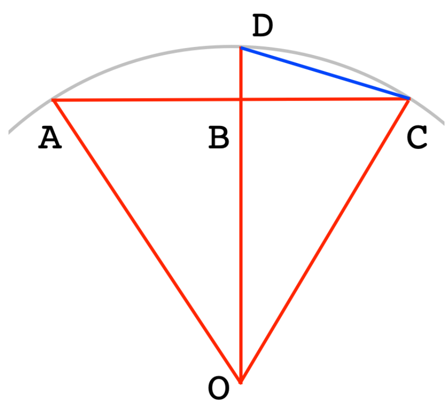
$e$ is twice $\approx 0.25881$. There are 12 copies of $e$ in the perimeter but that's for $2\pi$




To get $\pi$, we do $12\cdot \approx 0.25881 = \approx 3.10577$.

## Liu Hui

Another approach is rely entirely on the Pythagorean theorem. Let $AC$ be the chord for an $n$-gon.




Then the distance from $B$ to the circle is

$$BD = OC - \sqrt{OC^2 - BC^2}$$

$$\begin{aligned} BD^2 &= OC^2 - 2\sqrt{OC^2 - BC^2} + OC^2 - BC^2 \\ &= 2OC^2 - 2\sqrt{OC^2 - BC^2} - BC^2 \end{aligned}$$

$CD$ is the chord for the $2n$-gon and its squared length is

$$\begin{aligned} CD^2 &= BD^2 + BC^2 \\ &= BD^2 + 2OC^2 - 2\sqrt{OC^2 - BC^2} \end{aligned}$$

Each round of angle-halving involves three squares (one is used twice), two square roots and some arithmetic.

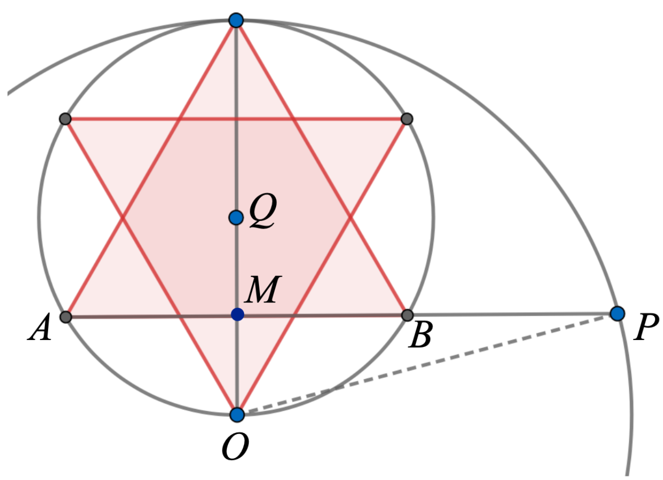
This is not as elegant as Archimedes' method, which needs only a single square and one square root, but Liu Hui was persistent.

He was persistent enough to recognize that 355/113 is a much better approximation to $\pi$ than either of Archimedes' values or even Ptolemy's 377/120.

355/113 has a *much* smaller error than any other rational approximation until the integer components get very large. Likely, Ptolemy missed that because he did not know the true value to enough accuracy.

## $\phi$ in the hexagon

Suppose we inscribe a hexagon into a circle on center $Q$. The problem comes with a big hint since a hexagram (aka Star of David) has been drawn using the vertices of the hexagon.




Now draw the circle on center $O$ whose half-radius is $OQ$, and extend one of the sides of the hexagram, $AB$ to meet the larger circle at $P$.

Show that $AP/AB = \phi$.

*Proof.*

By our work with equilateral triangles we know that $OM$ is one-quarter of $OP$.

So then let $OM = 1$ and $OP = 4$ and

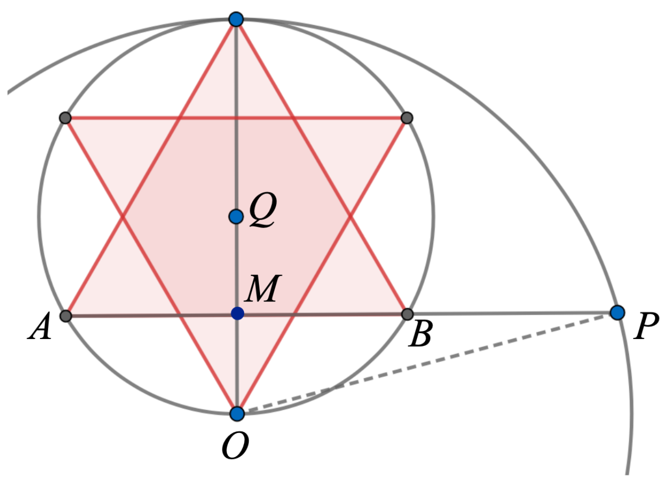
$$PM^2 = OP^2 - OM^2 = 15$$

$$PM = \sqrt{15}$$

$$MB = \sqrt{3}$$
$$\frac{PM}{MB} = \sqrt{5}$$
$$\frac{PM}{AB} = \frac{\sqrt{5}}{2}$$
$$\frac{AM}{AB} = \frac{1}{2}$$

$AP/AB$ is the sum of the last two terms, which is just $\phi$.

□



# Part VII

# Vectors without coordinates

# Chapter 27

# Simple vectors

## length and direction

In this chapter and the next we use vectors to solve several problems in plane geometry. Some of these are difficult to solve by other methods.

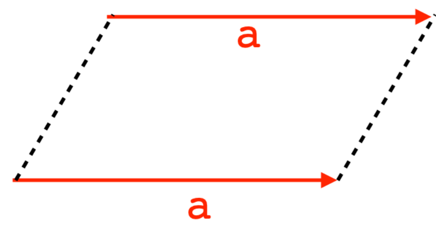
Your basic vector is a mathematical construct that represents both a length and a direction in the plane (2D), or in space (3D). It can be helpful to think of them as arrows extending from a tail up to the head.

A vector might represent an object's position, or its velocity, or the direction and strength of the electric field at some location.

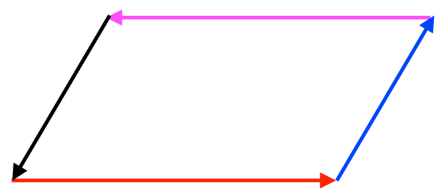
Normally, this idea is only introduced after defining a coordinate system, such as the orthogonal $x$- and $y$-axes in the plane. Using a coordinate system, the length and direction can be computed as the difference in coordinates between head and tail. If the difference in the $x$-coordinate is $\Delta x = x' - x$ and the difference in the $y$-coordinate is $\Delta y = y' - y$, then $\mathbf{v} = \langle \Delta x, \Delta y \rangle$.

But it is possible to get very interesting results even without a coordinate system, as we will see. Hence the motivation for including this chapter in a book on plane geometry.

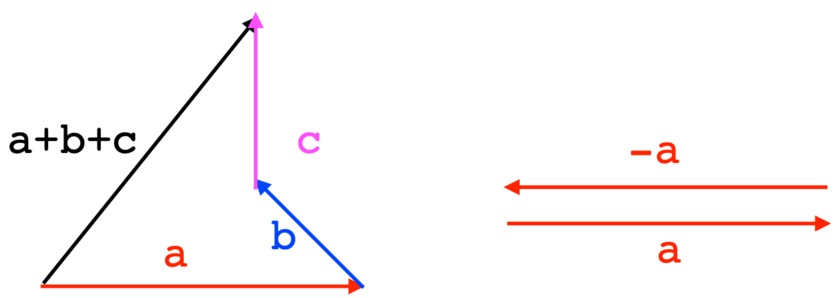
In physical applications it is probably more usual to think of the vector as fixed in space, whereas for mathematical ones, it may be moveable. For example, suppose a vector in the plane is duplicated and the copy moved anywhere else (without changing the orientation). When the ends are connected to form a quadrilateral, the result is always a parallelogram.




We will look at polygons, and use vectors to represent the lengths of sides, and also their relative orientations. For this purpose, we'll put vectors together head to tail to form the figure.



∘ Vector *addition* is defined as placing the tail of the second vector coincident with the head of the first.

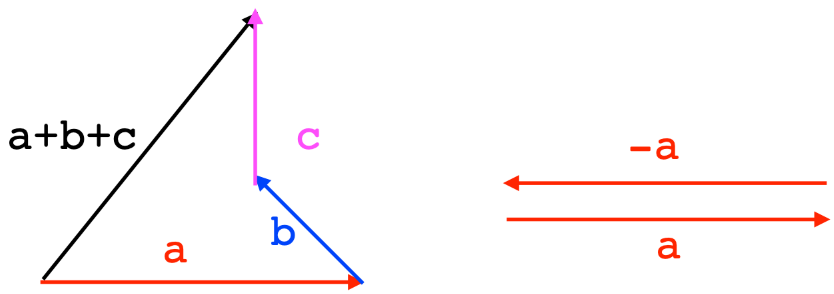



Addition of vectors is like a path that you might walk along, where the final vector connects the point where your journey started and the one where you end up. A bird might fly straight along $\mathbf{a} + \mathbf{b} + \mathbf{c}$ — "as the crow flies" — while you meander through some medieval town, puzzling at your map.

∘ The sum of the vectors comprising a closed polygon is zero.

Walking all the way around a polygon gets you back to where you started.

Applying this new idea to the parallelogram from before, we see that opposing sides are the negatives of each other and add to zero.



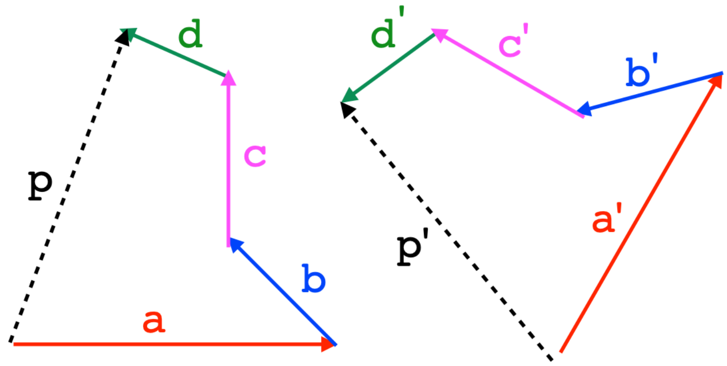

In the figure above, if we defined $\mathbf{d}$ to join the head of $\mathbf{c}$ with the tail of $\mathbf{a}$, closing the quadrilateral, then

$$\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0$$

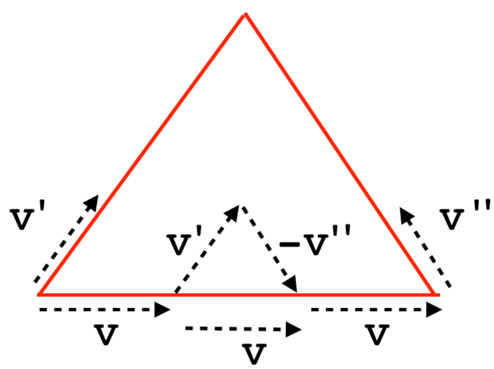
○ The negative of a vector is defined to be $-\mathbf{a} + \mathbf{a} = \mathbf{0}$.

If you walk backward along the same path you'll end up where you started. And if we want to know whether $\mathbf{u} = \mathbf{v}$, we can ask whether $-\mathbf{u} + \mathbf{v} = 0$.

○ A transformation such as rotation can be carried out by rotating each component vector of a figure, one after the other. If a vector $\mathbf{p} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$ is transformed by the rotation denoted by $'$ then $\mathbf{p}' = \mathbf{a}' + \mathbf{b}' + \mathbf{c}' + \mathbf{d}'$.



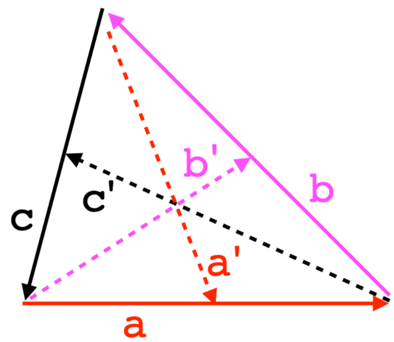

In the figure below we have an equilateral triangle. $\mathbf{v}$ is defined as 1/3 of the side length, and its orientation the same as the bottom side, from left to right. The operation of rotation through an angle is represented by a prime. Here, prime is a 60 degree rotation counter-clockwise. In another problem it will be defined to be a rotation of 90 degrees.




Using this figure for the definition, you should be able to see that $\mathbf{v}'''$ (*triple*-prime) is equal to $-\mathbf{v}$.

## vector sums

In an arbitrary triangle, draw the three vectors that connect the vertices with the sides opposite. These vectors are called medians in geometry, and they cross at a unique point called the centroid. For now, we just label them with the same letter as the side they meet, adding a prime.




To prove: the sum of the three primed vectors is zero.

As we said above, the sum of the paths for a closed polygon is zero:

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$$

Scaling by a constant $k$ doesn't change the result:

$$k\mathbf{a} + k\mathbf{b} + k\mathbf{c} = 0$$

Following closed paths, we can write the following equations:

$$\mathbf{a}' + \mathbf{a}/2 + \mathbf{b} = 0$$

$$\mathbf{b}' + \mathbf{b}/2 + \mathbf{c} = 0$$

$$\mathbf{c}' + \mathbf{c}/2 + \mathbf{a} = 0$$

Adding these equations, we find that taking the third term from each gives zero. Similarly, taking the second term from each also gives zero, replacing $k = 1/2$. So finally

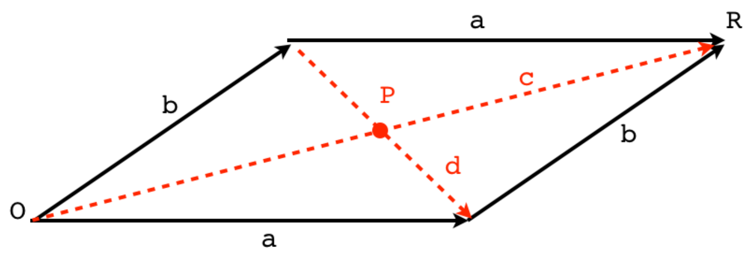
$$\mathbf{a}' + \mathbf{b}' + \mathbf{c}' = 0$$

□

## parallelogram

To prove: the two diagonals of a parallelogram cross at their midpoints.

Suppose $\mathbf{c}$ and $\mathbf{d}$ are diagonals as shown below.




Clearly, $\mathbf{a} + \mathbf{b} = \mathbf{c}$.

Finding $\mathbf{d}$ takes some thought. $\mathbf{d}$ is that vector, which when added to $\mathbf{b}$, gives $\mathbf{a}$.

$$\mathbf{b} + \mathbf{d} = \mathbf{a}$$

So

$$\mathbf{d} = \mathbf{a} - \mathbf{b}$$

Follow $\mathbf{b} + \mathbf{d}/2$ to $P$. That is

$$\mathbf{b} + \frac{\mathbf{a} - \mathbf{b}}{2} = \frac{\mathbf{a} + \mathbf{b}}{2}$$

But that is one-half of $\mathbf{c}$.

Similarly

$$\frac{\mathbf{c}}{2} + \frac{\mathbf{d}}{2} = \frac{\mathbf{a} + \mathbf{b}}{2} + \frac{\mathbf{a} - \mathbf{b}}{2} = \mathbf{a}$$

□

## Varignon

Varignon's theorem concerns a general quadrilateral, with four vertices whose positions can be anywhere.

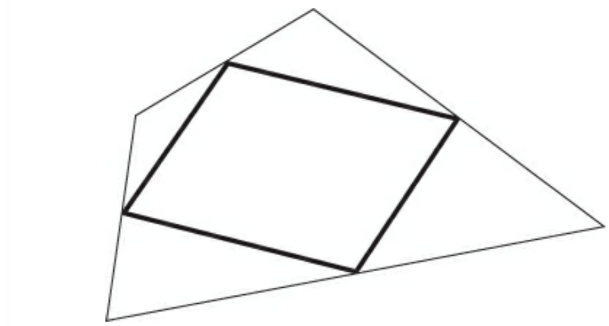


**Fig. 50** Varignon's theorem.

The theorem states that when the *midpoints* of the sides of the quadrilateral are connected, the result is a parallelogram.

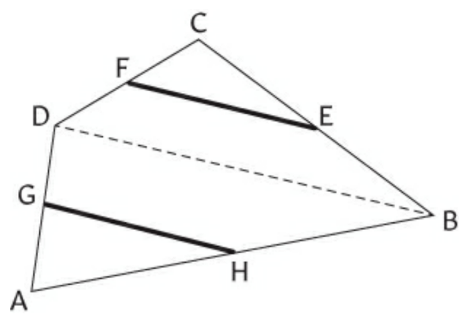


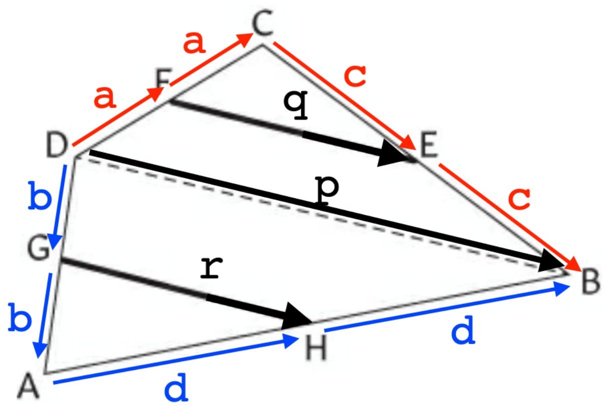

**Fig. 51** Proof of Varignon's theorem.

$EFGH$ is a parallelogram. With the midpoint theorem from similar triangles, the proof is trivial.

$$GH \parallel DB, \qquad FE \parallel DB \quad \Rightarrow \quad GH \parallel FE$$

and $GH = FE = DB/2$.

Let's suppose we don't have the midpoint theorem and apply vectors to this problem.




We use the fact that $E, F, G$, and $H$ are midpoints to draw the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, as well as $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$.

For the lower $\triangle ABD$, we get from point $D$ to point $B$ by one of three paths: $\mathbf{p}$, $\mathbf{b}+\mathbf{r}+\mathbf{d}$ or $2\mathbf{b}+2\mathbf{d}$. Since these all go from point $D$ to point $B$, they must be *equal*.

We have two additional paths that are also equal to the starting three: $\mathbf{a}+\mathbf{q}+\mathbf{c}$ or $2\mathbf{a}+2\mathbf{c}$.

Equate

$$2\mathbf{b}+2\mathbf{d} = 2\mathbf{a}+2\mathbf{c}$$

$$\mathbf{b}+\mathbf{r}+\mathbf{d} = \mathbf{a}+\mathbf{q}+\mathbf{c}$$

Subtract one-half of the first equation from the second to obtain

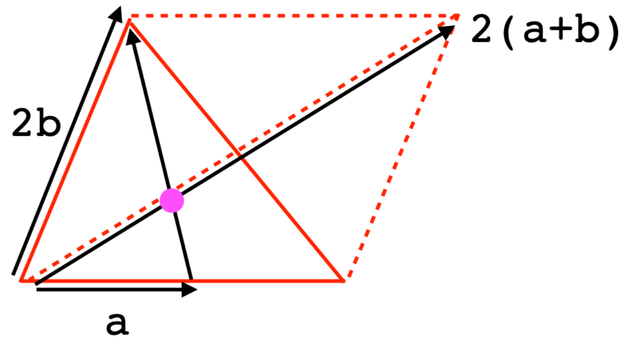
$$\mathbf{r} = \mathbf{q}$$

The opposing sides of the new figure, $FE$ and $GH$, are equal *vectors*. That means $FE$ and $GH$ point in the same direction and are the same length. It follows that $EFGH$ is a parallelogram.

□

## Ceva

Ceva's theorem says that if we draw a line from each vertex of a triangle to the midpoint of the opposite side, the lines cross at a single point called the centroid, and that point is 1/3 of the distance from the side and 2/3 of the distance from the vertex.



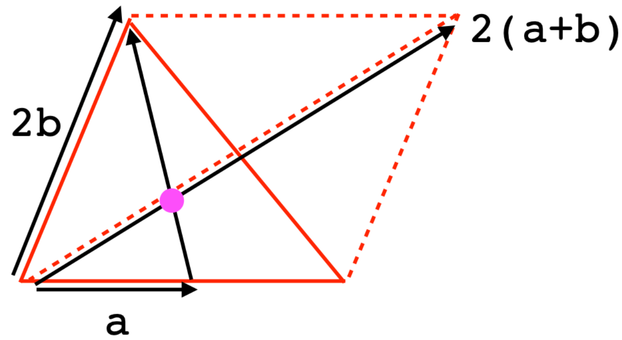

In this diagram the point in magenta is the centroid of the red triangle; as we said, it is 1/3 of the way up from the base.

We can derive this result using vectors as follows. The origin is at the bottom left. The base of the triangle is $2\mathbf{a}$, so $\mathbf{a}$ extends to the midpoint of the side. The second side is $2\mathbf{b}$.

The vector going up from the base, starting from the "head" of $\mathbf{a}$, and passing through the centroid, has the vector formula $2\mathbf{b} - \mathbf{a}$. To find this, I simply ask, what is the vector that when added to $\mathbf{a}$, gives $2\mathbf{b}$?

When a rotated version of the triangle is added (dotted lines), the diagonal passes through the midpoint of the third side, by a standard property of parallelograms. In vector notation, that diagonal is $2(\mathbf{a}+\mathbf{b})$, or $2\mathbf{a}+2\mathbf{b}$, so the diagonal to the midpoint is just $\mathbf{a}+\mathbf{b}$.

So then we say: there is no reason to think that one side of the triangle is any closer to the centroid than the other two. We expect then that the ratio of the distance from the side to the centroid compared to the length of the whole median, is the same for all three medians. Let us call that ratio $r$.




We compare two paths to the centroid, they must be equal:

$$\mathbf{a} + r(2\mathbf{b} - \mathbf{a}) = (1 - r)(\mathbf{a} + \mathbf{b})$$

$$\mathbf{a} + 2r\mathbf{b} - r\mathbf{a} = \mathbf{a} + \mathbf{b} - r\mathbf{a} - r\mathbf{b}$$

$$2r\mathbf{b} = \mathbf{b} - r\mathbf{b}$$

$$2r = 1 - r, \qquad r = \frac{1}{3}$$

The same result can be obtained using as one of the paths **2b** and back down to the centroid.

$$\mathbf{a} + r(2\mathbf{b} - \mathbf{a}) = \mathbf{b} + r(2\mathbf{a} - \mathbf{b})$$

$$\mathbf{a} + 2r\mathbf{b} - r\mathbf{a} = \mathbf{b} + 2r\mathbf{a} - r\mathbf{b}$$

$$\mathbf{a} + 3r\mathbf{b} = \mathbf{b} + 3r\mathbf{a}$$

$$3r = 1, \qquad r = \frac{1}{3}$$

□

Those are simple, pretty results. We look at two spectacular applications in the next chapter.

# Chapter 28

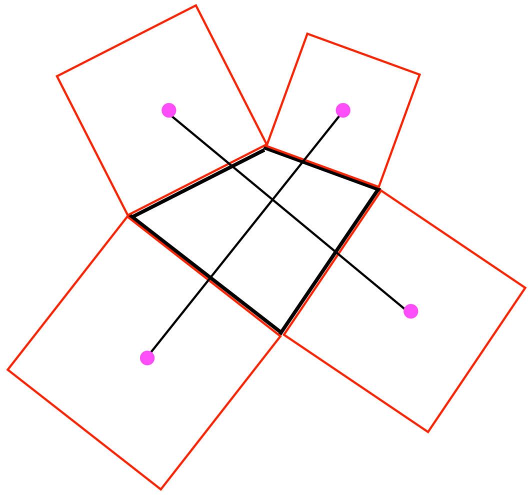
# Napoleon's theorem

We continue looking at vectors, without coordinates, and apply them to two difficult problems.

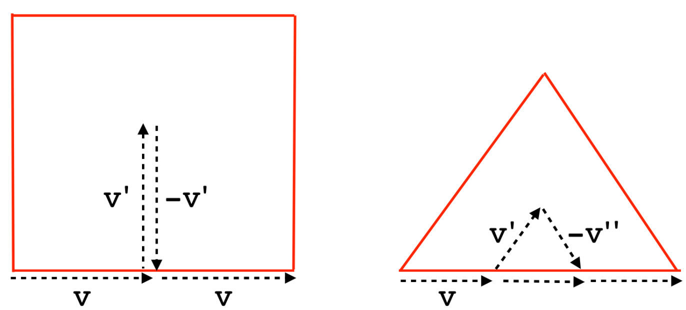
## sides of a quadrilateral

We choose any four points in the plane to form a generalized quadrilateral, so the vertices could be anything, and the vectors corresponding to the sides could be anything as well.

Draw a square on each side of the quadrilateral and connect the centers of opposing squares, as shown in the diagram below. The resulting vectors will be perpendicular and equal in length, no matter which points are chosen initially.

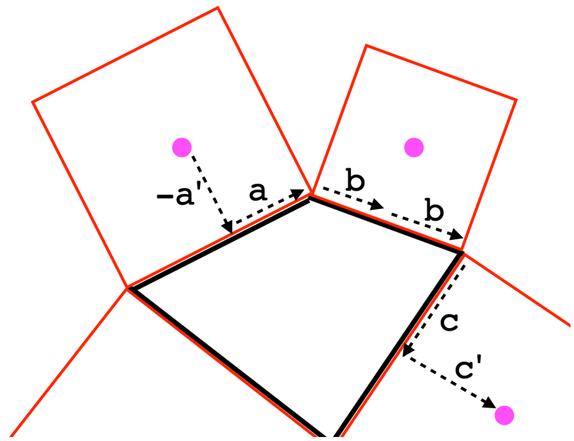


We are going to reason about these paths using vectors but not coordinates. For this first problem, we are concerned with squares. The path from vertex to center and back to the next vertex is shown in the left panel, below.




$\mathbf{v}'$ is defined to be $\mathbf{v}$ transformed by a quarter-turn counter-clockwise. $-\mathbf{v}'$ is $\mathbf{v}'$, but in the opposite direction.

Since the sides can be any length, rather than $\mathbf{v}$, we will use vectors $\mathbf{a}$ through $\mathbf{d}$ for the four different sides.




So the first path, from center to center is

$$\mathbf{p} = -\mathbf{a}' + \mathbf{a} + \mathbf{b} + \mathbf{b} + \mathbf{c} + \mathbf{c}'$$

And the second is

$$\mathbf{q} = -\mathbf{b}' + \mathbf{b} + \mathbf{c} + \mathbf{c} + \mathbf{d} + \mathbf{d}'$$

Notice that the first path goes roughly left-to-right, while the second goes top-to-bottom.

In order to check whether the two paths are at right angles, we should rotate the first path by one-quarter turn counter-clockwise. If the theorem is correct, $\mathbf{p}'$ should run in the exact opposite direction as $\mathbf{q}$, with the same length. They should add to give zero.

First path, rotated:

$$\mathbf{p}' = -\mathbf{a}'' + \mathbf{a}' + \mathbf{b}' + \mathbf{b}' + \mathbf{c}' + \mathbf{c}''$$

Each component has gained a prime ($'$). That's all it takes to rotate a vector path, just rotate each component.

Now, $-\mathbf{a}'' = \mathbf{a}$ and $\mathbf{c}'' = -\mathbf{c}$ (rotating 180 is the same as minus). Also, it makes no difference in which order you do the operations, whether first minus and then prime, or vice-versa.

Added together

$$\mathbf{p}' + \mathbf{q} = \mathbf{a} + \mathbf{a}' + \mathbf{b}' + \mathbf{b}' + \mathbf{c}' - \mathbf{c} - \mathbf{b}' + \mathbf{b} + \mathbf{c} + \mathbf{c} + \mathbf{d} + \mathbf{d}'$$

The question is, do these add up to zero? If so, we can conclude that the original two vector paths were orthogonal and the same length.

We can knock out the full path of the quadrilateral, as we end up where we started so that sum should be zero.

We subtract $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=0$, leaving:

$$\mathbf{p}'+\mathbf{q}=\mathbf{a}'+\mathbf{b}'+\mathbf{b}'+\mathbf{c}'+-\mathbf{c}+-\mathbf{b}'+\mathbf{c}+\mathbf{d}'$$

Similarly $\mathbf{a}'+\mathbf{b}'+\mathbf{c}'+\mathbf{d}'=0$. Subtract again:

$$\mathbf{p}'+\mathbf{q}=\mathbf{b}'-\mathbf{c}-\mathbf{b}'+\mathbf{c}$$

What's left is zero, so the whole thing is zero.

We conclude that the vectors which connect opposing centers are orthogonal and the same length.

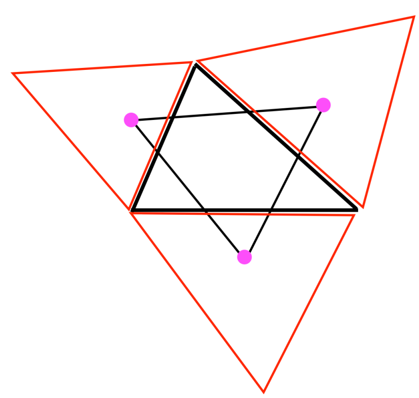
□

That's pretty amazing. The theorem is called van Aubel's theorem (1878). I learned about this proof and the theorem here:

https://www.youtube.com/c/MathyJaphy/videos

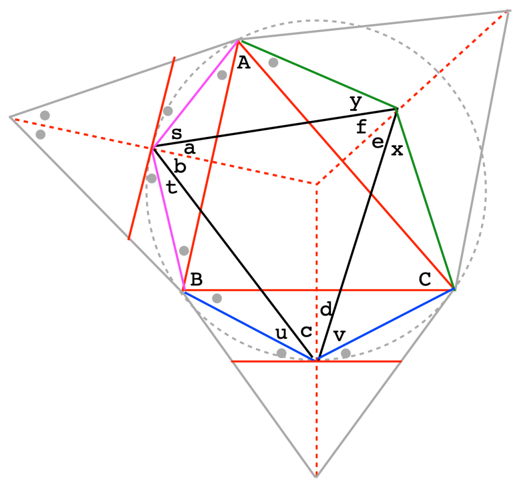
Among other things, the animation is terrific.

## sides of a triangle

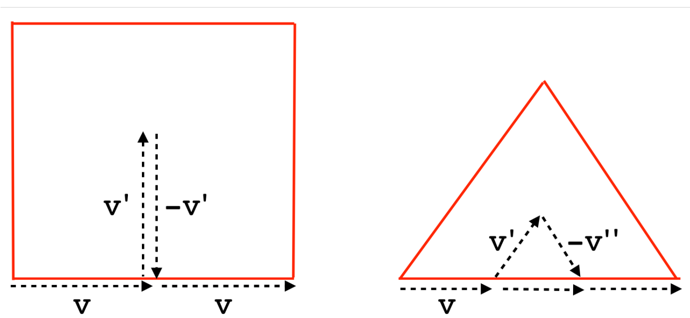


This problem is similar to the last one. On the three sides of any triangle, we construct equilateral triangles. Find the "center" of each triangle, and connect them. The result is an equilateral triangle.

I saw this problem in another book and worked on it for quite a while (days) without getting anywhere.




The vector approach provides a simple answer. The right panel, below, shows the idea. First, we need to find the center of an equilateral triangle. Luckily, all the centers — centroid, circumcenter, orthocenter and incenter — are the same point for an equilateral triangle.



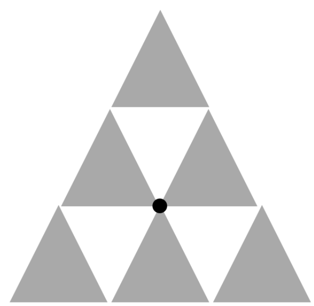

That point lies on a central line of the triangle and is one-third of the way up from the base.

To get there, we will use rotations that are 60 degrees, and define those as the transformation called prime. As shown, $\mathbf{v}'$ is a 60 degree rotation counter-clockwise from any starting vector $\mathbf{v}$.

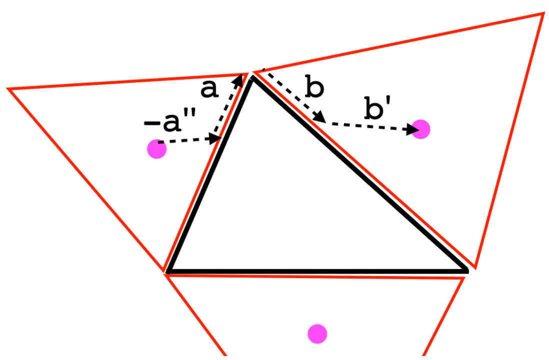
A clever idea is to go to the centroid by addition of one-third of a side plus the same vector, primed. The path along one edge is as shown above.

For any triangle, the centroid is the average of the three vertices. (This gives a different simple proof for $r$ in the centroid problem, using coordinates).

From one vertex, take one-third of a step toward each of the other two vertices. Here is a visual proof without words.



Using this approach, we construct two paths as shown. One is from the center of the triangle with sides constructed from $\mathbf{a}$ to the one with sides $\mathbf{b}$.




$$\mathbf{p} = -\mathbf{a}'' + \mathbf{a} + \mathbf{b} + \mathbf{b}'$$

The other is (by symmetry)

$$\mathbf{q} = -\mathbf{b}'' + \mathbf{b} + \mathbf{c} + \mathbf{c}'$$

The big question is then: how do we decide when two vectors form the sides of an equilateral triangle? One set of criteria is that the two sides should be the same length, and flank an angle of 60 degrees.

If you look at the graphic, you'll see that **p** points toward the vertex of interest, while **q** points away from it (vectors are head to tail). Therefore, if we turn the first vector **p** counter-clockwise, it should point in the exact opposite direction from **q**, and then the sum $\mathbf{p}' + \mathbf{q}$ will be $\mathbf{0}$. Let's see.

Starting with

$$\mathbf{p} = -\mathbf{a}'' + \mathbf{a} + \mathbf{b} + \mathbf{b}'$$

After the rotation, each one has acquired a prime. The first component, $-\mathbf{a}''$, turns into $-\mathbf{a}''' = \mathbf{a}$.

$$\mathbf{p}' = \mathbf{a} + \mathbf{a}' + \mathbf{b}' + \mathbf{b}''$$

Add together:

$$\mathbf{p}' + \mathbf{q} = \mathbf{a} + \mathbf{a}' + \mathbf{b}' + \mathbf{b}'' - \mathbf{b}'' + \mathbf{b} + \mathbf{c} + \mathbf{c}'$$

Now, $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ and similarly for the primed versions, so those terms all drop out, leaving $\mathbf{b}'' - \mathbf{b}''$, which is of course, zero!

We have proven that the vectors corresponding to two sides of our construct are the same length, and have an angle of 60 degrees between them. Thus, they form an equilateral triangle.

□

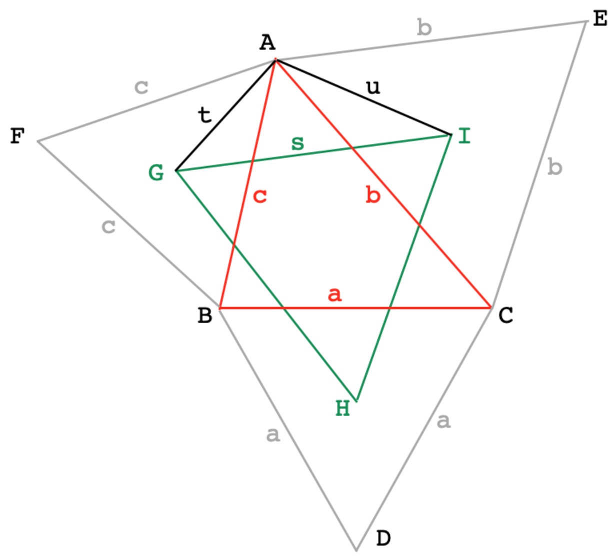
This theorem is called Napoleon's theorem. It seems likely that the association with Napoleon is apocryphal.

https://en.wikipedia.org/wiki/Napoleon%27s_theorem

## algebraic proof

To simplify the notation, let the angle at vertex $A$ be $A$, the side opposite be $a$ with length $a$, and so on.

The medians of the equilateral triangles erected on the sides will be $H$, $I$ and $G$, while $s$, $t$ and $u$ are sides of $\triangle AGI$ with vertices being two of the medians and vertex $A$.




We use the law of cosines, several times. First, recall that the line from the centroid of an equilateral triangle to any vertex bisects the angle at the vertex. So side $u$ bisects $\angle CAE$ and side $t$ bisects $\angle BAF$, and the angle between $t$ and $u$ is $A + 60$.

Then, by the law of cosines

$$s^2 = t^2 + u^2 - 2tu\cos(A + 60)$$

A second key point is that we can obtain sides $t$ and $u$ in terms of the original sides $c$ and $b$.

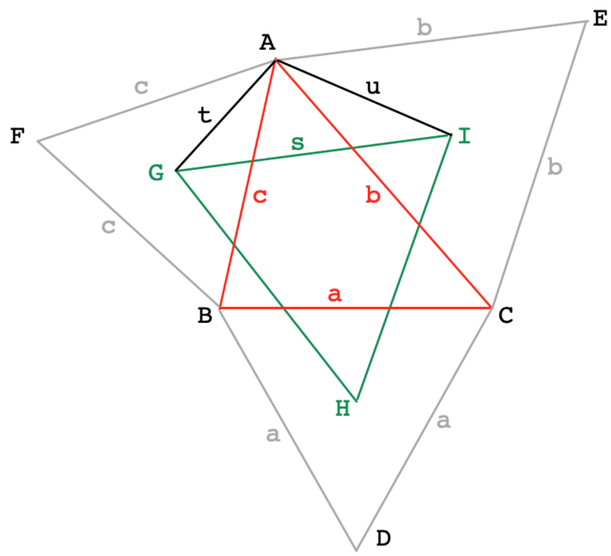
The median of an equilateral triangle forms a triangle with sides in the ratio 1-2-$\sqrt{3}$. The triangle with hypotenuse $u$ is similar to this triangle, so the ratio of the long side to the hypotenuse is:

$$\frac{b/2}{u} = \frac{\sqrt{3}}{2}, \qquad b = \sqrt{3}u, \qquad b^2 = 3u^2$$

Also, $c^2 = 3t^2$.

Going back to the first equation and multiplying by 3 we obtain:

$$3s^2 = c^2 + b^2 - 2bc\cos(A + 60)$$




This could be done for any of the other two sides, by symmetry. For example, we can relate $HI$ to sides $a$ and $b$:

$$3HI^2 = a^2 + b^2 - 2ab\cos(C + 60)$$

We can connect these two expressions through the line segment $BE$.

Looking toward vertex $A$ and employing the law of cosines again, we have:

$$BE^2 = b^2 + c^2 - 2bc\cos(A + 60)$$

Looking instead toward vertex $C$ we have

$$BE^2 = a^2 + b^2 - 2ab\cos(C + 60)$$

These two expressions are equal, and therefore the things we found equal to them previously are also equal. Namely:

$$3s^2 = 3HI^2$$

$$s = HI$$

The same thing could be done for either of the other pairs of sides.

$$s = GI = GH = HI$$

$\triangle GHI$ is equilateral.

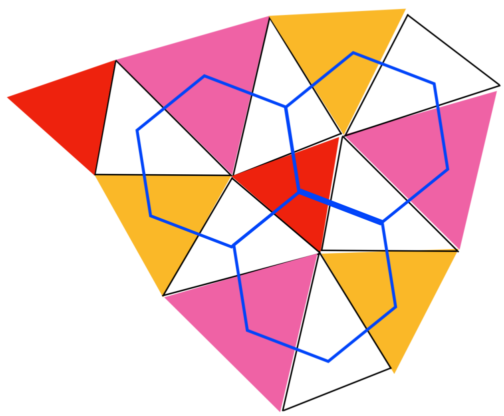
$\square$

We just note in passing that since $AD^2 = a^2 + b^2 - 2ab\cos(C + 60)$, and $BE^2$ is equal to the same expression, we have that $AD = BE$ and by extension, both are equal to $CF$.

Also, connect each vertex of the original triangle with that of the equilateral triangle opposite: forming $AH, BI$ and $CG$. These lines cross at *Fermat's point*.

https://www.cut-the-knot.org/proofs/napoleon.shtml

Apparently, there is a tiling pattern in the figure. This graphic isn't perfect (my poor skills with the program), but I think you get the idea.



There is a hexagonal pattern that contains the centroids of the equilateral triangles, and a three-fold rotational symmetry around each of them.

# Part VIII

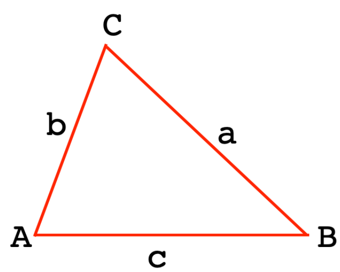
# Later geometry

# Chapter 29

# Heron and Brahmagupta

Heron (or Hero) of Alexandria lived in the first century AD. He was primarily an engineer, but is also remembered for Heron's Formula, which can be used to compute the area of a triangle from the lengths of its sides. It is a simple formula that does not explicitly include the altitude $h$ or the components of side $c$.

Heron's formula was later found to be a special case of a similar formula for quadrilaterals, discovered by Brahmagupta, which we'll study later. Here's a standard triangle with vertex $A$ opposite side $a$ and so on




Let $s$ be one-half the perimeter, called the semi-perimeter:

$$s = \frac{1}{2}(a + b + c)$$

$$2s = a + b + c$$

then Heron says that the area, call it $\Delta$, is

$$\Delta = \sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)}$$

$$\Delta^2 = s \cdot (s-a) \cdot (s-b) \cdot (s-c)$$

## simple derivation

We use a bit of trigonometry. The triangle shown above has area $\Delta$ in terms of angle $B$:

$$\Delta = \frac{1}{2} ac \sin B$$

$$16\Delta^2 = (2ac)^2 \sin^2 B$$

$$= (2ac)^2 \; [\; 1 - \cos^2 B \;]$$

From the law of cosines, substitute for $\cos B$:

$$16\Delta^2 = (2ac)^2 \; [\; 1 - (\frac{a^2 + c^2 - b^2}{2ac})^2 \;]$$

$$= (2ac)^2 - (a^2 + c^2 - b^2)^2$$

Some algebra with differences of squares follows:

$$16\Delta^2 = (2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)$$

$$= \; [\; (a+c)^2 - b^2 \;] \; [\; (b^2 - (a-c)^2 \;]$$

$$= (a+c+b)(a+c-b)(b+a-c)(b-a+c)$$

Let $2s = a + b + c$. Then

$$16\Delta^2 = 2s \cdot 2(s-a) \cdot 2(s-b) \cdot 2(s-c)$$

$$\Delta^2 = s(s-a)(s-b)(s-c)$$

□

We now explore a justification for why each term is present in the equation. To begin, note that the equation is symmetrical in $a, b$ and $c$. This is expected, since there is no reason to distinguish among the sides.

## Levi

Mark Levi has a short proof of Heron's formula, linked on this page:

https://www.marklevimath.com/sinews

The url I have for this quote no longer points to the correct document, but I still like it:

> The area-squared is obviously a symmetric and homogeneous polynomial of degree 4 in a, b, c, divisible by (a + b - c)(a + c - b)(b + c - a), since degenerate triangles have zero area.
>
> Hence the area-squared divided by (a + b - c)(a + c - b)(b + c - a) is a symmetric and homogeneous polynomial of degree 1 in a, b, c, and so is (a + b + c) times some constant that must be 1 by considering, say, the 90, 45, 45 triangle.

Let's just play with the formula. Take what is under the square root above:

$$s \cdot (s - a) \cdot (s - b) \cdot (s - c)$$

Multiply each term by 2

$$\begin{gathered} 2s \cdot (2s - 2a) \cdot (2s - 2b) \cdot (2s - 2c) \\ = (a + b + c)(b + c - a)(a + c - b)(a + b - c) \end{gathered}$$

According to the formula above, $16A^2$, and hence the area itself, will be zero when

∘ $a + b + c = 0$

that is, when the sum of all three sides is equal to zero. Since lengths are always positive, this means that $a = b = c = 0$, or

∘ one of the other terms is zero, e.g. $a + b - c = 0$.

that is, when one side length is equal to the sum of the other two.

These are all "degenerate" triangles, where the shape has collapsed either to a point (the first case) or to a line segment.

The factor of 16 may be deduced from an example, e.g., an equilateral triangle with unit sides, altitude equal to $\sqrt{3}/2$ and area of $\sqrt{3}/4$.

Suppose we do not know the factor, so let it be $k$ (rather than 16):

$$k \cdot (\frac{\sqrt{3}}{4})^2 = (a+b+c)(b+c-a)(a+c-b)(a+b-c)$$

$$= 3 \cdot 1 \cdot 1 \cdot 1 = 3$$

Clearly, $k = 4^2 = 16$.

Or, for an isosceles right triangle with sides 1, 1, $\sqrt{2}$:

$$k \cdot (\frac{1}{2})^2 = (2+\sqrt{2})(\sqrt{2})(\sqrt{2})(2-\sqrt{2})$$

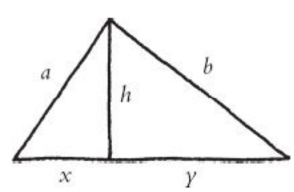
$$= (4-2)(2) = 4$$

The proof starts with the deduction that the area squared is "a polynomial of degree 4 in $a, b, c$", and he works through why that is so. It makes sense, since area is itself the product of two lengths, each of which must be proportional somehow to the lengths of the sides.

## Lockhart

Here is a long-winded algebraic proof, from Lockhart, but there is a point! It does not depend explicitly on trigonometry. Instead, it hides what is in effect, a derivation of the law of cosines.

*Proof.*




Side $c$ is split into $x$ and $y$. We can write three equations:

$$x^2 + h^2 = a^2$$

$$y^2 + h^2 = b^2$$

$$x + y = c$$

Our objective is an equation that contains only $a$, $b$ and $c$. From the first two:

$$a^2 - b^2 = x^2 - y^2$$

and from the third:

$$y^2 = c^2 - 2xc + x^2$$

so

$$\begin{aligned} a^2 - b^2 &= x^2 - c^2 + 2xc - x^2 \\ &= 2xc - c^2 \end{aligned}$$

then

$$a^2 + c^2 - b^2 = 2xc$$

Finally a slight rearrangement:

$$x = \frac{c^2 + a^2 - b^2}{2c} = \frac{c}{2} + \frac{a^2 - b^2}{2c}$$

This says that to find the point where $c$ is divided into $x$ and $y$, we move from the center $c/2$ a distance of $(a^2 - b^2)/2c$.

The corresponding equation for $y$ is

$$y = \frac{c}{2} - \frac{a^2 - b^2}{2c}$$

which is easily checked by adding together the final two equations, obtaining $x+y = c$.

For the area, we will need $h$ somehow. It is easier to use $h^2$.

$$\begin{aligned} h^2 &= a^2 - x^2 \\ &= a^2 - \frac{(c^2 + a^2 - b^2)^2}{(2c)^2} \end{aligned}$$

The area squared is

$$\begin{aligned} \Delta^2 &= \frac{1}{4}c^2h^2 \\ &= \frac{1}{4}c^2a^2 - \frac{1}{4}c^2\frac{(c^2 + a^2 - b^2)^2}{(2c)^2} \end{aligned}$$

Lockhart:

> the algebraic form of this measurement is aesthetically unacceptable. First of all, it is not symmetrical; second, it's hideous. I simply refuse to believe that something as natural as the area of a triangle should depend on the sides in such an absurd way. It must be possible to rewrite this ridiculous expression...

Here's a start:

$$16\Delta^2 = (2ac)^2 - (c^2 + a^2 - b^2)^2$$

This is much better. It is still problematic, in that $a$ and $c$ do not appear symmetric with $b$.

However, we immediately notice that it is a difference of squares. First

$$16\Delta^2 = \ [\ 2ac + (c^2 + a^2 - b^2)\ ]\ [\ 2ac - (c^2 + a^2 - b^2)\ ]$$

And that has within it two squares, namely $(a+c)^2$ in the first term on the right-hand side, and $(a - c)^2$ in the second.

$$16\Delta^2 = \ [\ (a + c)^2 - b^2)\ ]\ [\ b^2 - (a - c)^2\ ]$$

A second difference of squares. Thus

$$16\Delta^2 = (a + c + b)(a + c - b)(b + a - c)(b - a + c)$$

At this point, we introduce the semi-perimeter $2s = a + b + c$ and then obtain after several steps

$$\Delta = \sqrt{s \cdot (s - a)(s - b)(s - c)}$$

□

And that is symmetric in each of the three sides, as we hope and expect.

## check

As a simple example, if we have a right triangle with sides 3,4,5, then the area is one-half of 3 times 4 = 6. The semi-perimeter is s

$$s = \frac{(3 + 4 + 5)}{2} = \frac{12}{2} = 6$$

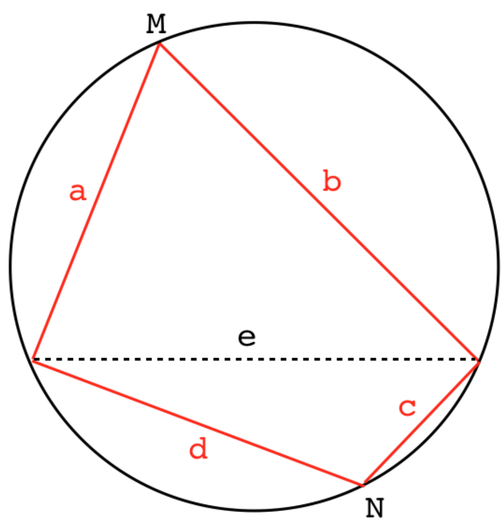
We have

$$\Delta = \sqrt{6(6-5)(6-4)(6-3)} = \sqrt{6(1)(2)(3)} = 6$$

## Brahmagupta

Brahmagupta was an Indian mathematician who lived in the 7th century AD in a region of India called Bhinmal, which is in Rajastan. He completed the square to obtain the quadratic equation, and did many other amazing things in trigonometry and arithmetic, as well as this example from geometry.

We consider a quadrilateral inscribed into a circle. This is a special case, where the fourth point fits into the same circle determined by any three of the points.




We will prove that the area of this quadrilateral is given by Brahmagupta's formula.

$$A = \sqrt{(s-a)\cdot(s-b)\cdot(s-c)\cdot(s-d)}$$
$$A^2 = (s-a)\cdot(s-b)\cdot(s-c)\cdot(s-d)$$

Heron's formula is thus a special case where $d = 0$.

$$A = \sqrt{s\cdot(s-a)\cdot(s-b)\cdot(s-c)}$$

## preliminary

We need two preliminary results. If $M$ and $N$ are supplementary angles, then

$$\sin M = \sin N, \qquad \cos M = -\cos N$$

Supplementary angles have mirror image symmetry across the $y$-axis. This becomes obvious if you plot them.

Then, draw the line connecting the two opposing vertices which are not $M$ and $N$. Using the law of cosines we can write two equal expressions for $e^2$, namely:

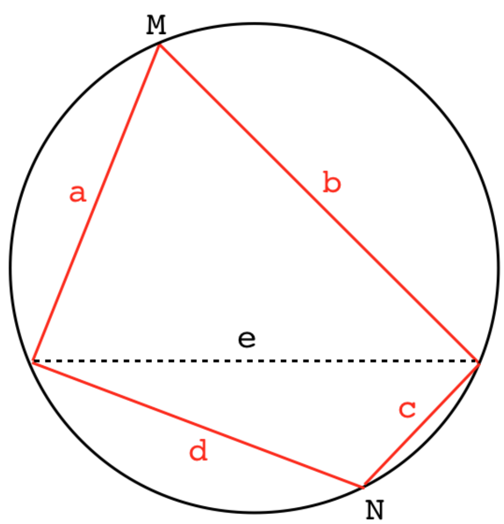
$$e^2 = a^2 + b^2 - 2ab\cos M$$

$$e^2 = c^2 + d^2 - 2cd\cos N = c^2 + d^2 + 2cd\cos M$$

Equating the two and grouping terms:

$$a^2 + b^2 - c^2 - d^2 = 2(ab + cd)\cos M$$

Look at the diagram again.




The triangle above the dotted line has area $(1/2)\ ab\sin M$ and similarly for the one below so the total area is

$$A_1 = \frac{1}{2}ab\sin M$$

$$A_2 = \frac{1}{2} cd \sin N = \frac{1}{2} cd \sin M$$

Adding, the total area is:

$$A = \frac{1}{2}(ab + cd) \sin M$$

$$4A = 2(ab + cd) \sin M$$

## algebra

Square the two main equations so far:

$$(a^2 + b^2 - c^2 - d^2)^2 = \ [\ 2(ab + cd)\ ]^2 \cos^2 M$$

$$16A^2 = \ [\ 2(ab + cd)\ ]^2 \sin^2 M$$

and add

$$16A^2 + (a^2 + b^2 - c^2 - d^2)^2 = \ [\ 2(ab + cd)\ ]^2$$

Rearrange

$$16A^2 = \ [\ 2(ab + cd)\ ]^2 - (a^2 + b^2 - c^2 - d^2)^2$$

As before, we proceed to factor two differences of squares.

First:

$$16A^2 = \ [\ 2(ab + cd) + (a^2 + b^2 - c^2 - d^2)\ ]\ [\ \ 2(ab + cd) - (a^2 + b^2 - c^2 - d^2)\ ]$$

$$= \ [\ (a + b)^2 - (c - d)^2\ ]\ \ [\ (c + d)^2 - (a - b)^2\ ]$$

Second

$$= (a + b + (c - d))(a + b - (c - d))\ (c + d + (a - b))(c + d - (a - b))\ ]$$

$$= (a + b + c - d)(a + b - c + d)(c + d + a - b)(c + d - a + b)$$

If the semi-perimeter is $s$ then

$$2s = a + b + c + d$$

So we have

$$16A^2 = (2s - 2d)(2s - 2c)(2s - 2b)(2s - 2a)$$

$$A^2 = (s-d)(s-c)(s-b)(s-a)$$

So lastly

$$A^2 = (s-a)(s-b)(s-c)(s-d)$$
$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

In comparing the two proofs, it's clear that the latter proof draws on the ideas of (i) using the semi-perimeter and (ii) difference of squares, which are in Heron's proof. The main new ideas are trigonometric: the law of cosines and the cancellation of $\sin^2 x + \cos^2 x$.

We might see this by rewriting the proof of Heron's formula in the style of the Brahmagupta proof. (Note: Brahmagupta does not give proofs or say how he obtained his results).

But we don't have to! That's essentially what the proof from Twitter does, above. First, the law of cosines. Let $\alpha$ be the angle opposite side $a$:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$(b^2 + c^2 - a^2)^2 = (2bc)^2\cos^2\alpha$$

And the area is

$$A = (1/2)bc\sin\alpha$$
$$4A = 2bc\sin\alpha$$
$$16A^2 = (2bc)^2\sin^2\alpha$$

Adding

$$16A^2 + (b^2 + c^2 - a^2)^2 = (2bc)^2$$
$$16A^2 = (2bc)^2 - (b^2 + c^2 - a^2)^2$$

The rest is exactly as before, it's just a matter of two differences of squares.

# Chapter 30

# The Almagest

*The Almagest* is a treatise written by Ptolemy in the second century AD which laid out a geocentric (earth-centered) picture of the universe. Among other things, it relies on his calculation of a table of lengths of chords formed by angles between 0 and 180 degrees in very small increments.

Chords are related to the sine function, as we've seen. The ratio of the chord length to the diameter is equal to the sine of the half-angle.

So far, the only two angles for which we've calculated the sine and cosine are those which are either one-half or one-third of a right angle. We get another one (two-thirds of a right angle) by the properties of complementary angles.

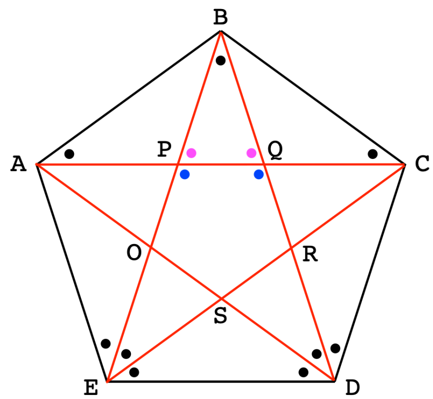
Here, we start by extending our calculation to angles that are one-fifth or two-fifths of a right angle. These come naturally from the study of the pentagon.

## regular pentagon

The regular pentagon has five sides of equal length. We looked at its properties **here**.

The first step was to show that each vertex angle is divided into three equal parts by the internal chords of the pentagon.

Let the measure of that angle be $t$. Each one has a measure of $t = 1/5 \cdot \pi$ or $36^\circ$. (From now on we will suppress the degree symbol in angle measurements, so the angle is just 36). Five of them add up to one triangle or 180. Only some of the equal angles are marked in this diagram.




Another major result is that the figure contains five parallelograms (actually, each is a rhombus, with four equal sides). Each of the five internal chords is the base of one of five congruent triangles containing two sides and a vertex, so they are all equal in length.

Since entire chords are equal and the sides of the parallelograms are all equal, each of the small parts remaining, such as $BP$, are equal to their counterparts.

The small, central pentagon is a regular pentagon, so all of its vertex angles are equal, and their measure is $3t$. Therefore the supplementary angles marked in magenta are equal and their measure is $2t$. Thus, $\triangle BPQ$ is isosceles and is similar to $\triangle BED$ by SAS similarity.

Let the side length of the small, central pentagon (e.g. $PQ$) be equal to 1 and the length $BP$ be equal to $x$, so the ratio of the long side to the base in $\triangle BPQ$ is $x$.

$BE$ is equal to $2x + 1$ and $DE = AQ$ is equal to $x + 1$ so by equal ratios in similar triangles

$$x = \frac{2x+1}{x+1}$$

$$x^2 + x = 2x + 1$$

$$x^2 - x - 1 = 0$$

We will re-label $x$ as $\phi$. The solution to this quadratic equation is $\phi = (1 + \sqrt{5})/2$. The second solution is negative, so we will not worry about it here, where we are concerned with ratios of lengths.

In any isosceles triangle with vertex angle 36, the ratio of either of the equal sides to the base is $\phi$.

## algebra with $\phi$

We first verify that $\phi$ solves the equation

$$\phi^2 = \frac{1}{4}(6+2\sqrt{5}) = \frac{1}{2}(3+\sqrt{5})$$

$$= 1 + \frac{1+\sqrt{5}}{2} = \phi + 1$$

So far we have

$$\phi^2 - \phi - 1 = 0$$

and

$$\phi^2 = \phi + 1$$

$$\phi^3 = \phi^2 + \phi = 2\phi + 1$$

$$\phi^4 = \phi^3 + \phi^2 = 3\phi + 2$$

If we continue, we'll get the Fibonacci sequence. Divide by $\phi$ and rearrange:

$$\phi = \frac{1}{\phi} + 1$$

$$\frac{1}{\phi} = \phi - 1$$

And finally

$$\phi = \phi^2 - 1 = (\phi+1)(\phi-1)$$

$$\phi = \frac{1}{\phi - 1}$$

$$(\phi+1)(\phi-1) = \frac{1}{\phi-1}$$

Some expressions involving $\phi$ I have only been able to figure out by working backward from the answer. We'll see examples in a bit.

## a little trigonometry

We have that the ratio of $BE$ to $ED$ is equal to $\phi$, so the inverse ratio is $1/\phi$.

One-half of that is the sine of one-half of $t$, $36/2 = 18$.

$$\sin 18 = \frac{1}{2\phi}$$

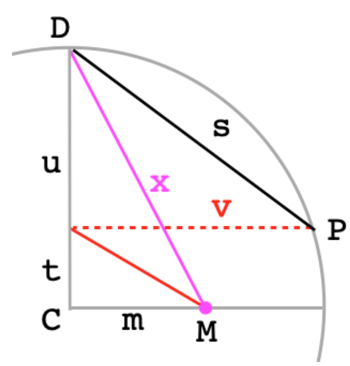
If the pentagon is inscribed in a circle then the length of the side $s$ is the chord formed by an inscribed angle of 36 or $s = 2r \sin 36$.

The double angle formulas allow us to get to $\sin 36$ and so find $s$ in terms of $r$, but we also have some constructions that work. We look at one of those now.

## construction 1

Wikipedia gives two methods to construct a regular pentagon. I have redrawn their first figure.

Inscribe the pentagon in a unit circle ($CD = 1$). Draw perpendicular radii (or diameters) and divide the right horizontal radius in half at $M$, so length $m = 1/2$.




Draw $DM$. Bisect $\angle CMD$.

Extend the bisector to the vertical diagonal $CD$. Finally, draw the horizontal to intersect the circle at $P$. We label the lengths with single letters to make the algebra more intuitive.

We claim that $DP$ or $s$ is one side of a regular inscribed pentagon. We will first verify that the construction gives the correct side length.

## calculation

Let the length of $DM$ (magenta line) be $x$. By the Pythagorean theorem

$$x^2 = m^2 + 1^2 = (\frac{1}{2})^2 + 1^2 = \frac{5}{4}$$

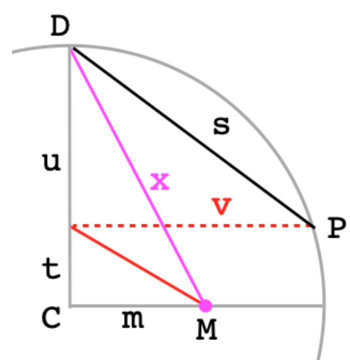
$$x = \frac{\sqrt{5}}{2}$$

Next we invoke the angle bisector theorem:

$$\frac{t}{m} = \frac{u+t}{x+m}$$

$$\frac{t}{1/2} = \frac{1}{\sqrt{5}/2 + 1/2}$$

$$t = \frac{1}{1+\sqrt{5}} = \frac{1}{2\phi}$$




We note in passing that $t$ has the value of the sine of 18. Next:

$$v^2 = 1^2 - t^2$$

We haven't drawn the hypotenuse for the above triangle, but its base is the dotted red line. Then

$$u^2 = (1-t)^2 = 1 - 2t + t^2$$

and

$$s^2 = u^2 + v^2$$

$$= 2 - 2t = 2 - \frac{1}{\phi}$$

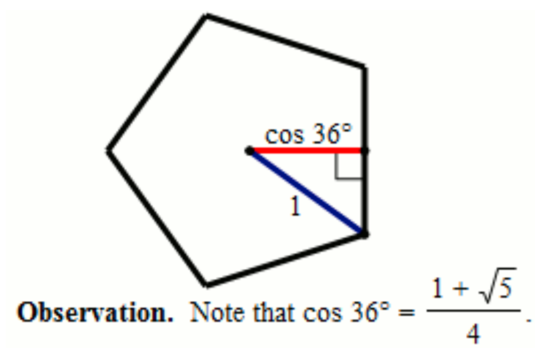
Since $1/\phi = \phi - 1$, the right-hand side is $2 - (\phi - 1) = 3 - \phi$ and the result is finally:

$$s = \sqrt{3 - \phi}$$

Now consider the $\angle CDP$, that is one-half of the vertex angle of a pentagon, namely, one-half of 108 or 54. The cosine of that angle is side $s$ divided by the diameter, or $s/2$.

The same value is also the sine of the complementary angle, 36. I get that sine 36 is equal to $\sqrt{3 - \phi}/2$. We will use trigonometry to derive the same result.

Another way to look at this is to actually draw the whole pentagon inscribed into a unit circle, and then draw its apothem, the line that splits the side in half. Since there are five sides, the central angle is 72 and one-half that is 36. So the red line is the cosine of 36 and the sine of 36 is one-half the side length.




The figure claims that

$$\cos 36 = \frac{1 + \sqrt{5}}{4}$$

It's a simple challenge to derive it directly:

$$\cos^2 36 = 1 - (\frac{\sqrt{3 - \phi}}{2})^2$$

$$= 1 - \frac{3 - \phi}{4}$$

$$= \frac{1 + \phi}{4} = \frac{\phi^2}{4}$$

$$\cos 36 = \frac{\phi}{2}$$

We can also check by squaring both results and adding:

$$(\frac{\sqrt{3-\phi}}{2})^2 + (\frac{\phi}{2})^2 = \frac{1}{4}(3 - \phi + \phi^2)$$

but $\phi^2 - \phi = 1$ so the result is just 1, naturally.

Writing what we have so far all in one place:

- $\sin 18 = 1/2\phi = (\phi - 1)/2$
- $\cos 18 = \sqrt{2+\phi}/2$ (see below)
- $\sin 36 = \sqrt{3-\phi}/2$
- $\cos 36 = \phi/2$

Recall the half-angle formula for cosine:

$$\cos A = \sqrt{\frac{1 + \cos 2A}{2}}$$

so

$$\cos^2 18 = \frac{1 + \phi/2}{2}$$

$$\cos 18 = \sqrt{\frac{2+\phi}{4}} = \frac{\sqrt{2+\phi}}{2}$$

We can also check this using our favorite identity, as follows:

$$(\frac{1}{2\phi})^2 + \frac{1 + \phi/2}{2}$$

$$= \frac{1}{4}(\frac{1}{\phi^2} + 2 + \phi)$$

Go back to the section on algebra:

$$\frac{1}{\phi^2} = (\phi - 1)^2$$

$$= \phi^2 - 2\phi + 1 = 2 - \phi$$

Substituting into what's in the parentheses above immediately yields the correct answer, 1.

We can also confirm that

$$\sin 36 = 2 \sin 18 \cos 18$$
$$= 2 \cdot \frac{1}{2\phi} \cdot \frac{\sqrt{2+\phi}}{2} = \frac{\sqrt{2+\phi}}{2\phi}$$

This isn't close to what we had, but algebra with $\phi$ can get pretty weird. Let's just go backwards from the answer.

$$\frac{\sqrt{2+\phi}}{2\phi} = \frac{\sqrt{3-\phi}}{2}$$

squaring

$$\begin{aligned} 2 + \phi &= (3-\phi)\phi^2 \\ &= 3\phi^2 - \phi^3 \\ &= 3(\phi+1) - (2\phi+1) \\ &= 2 + \phi \end{aligned}$$

which checks.

Let's do one more:

$$\begin{aligned} \sin 54 = \sin 36 + 18 &= \sin 36 \cos 18 + \sin 18 \cos 36 \\ &= \frac{\sqrt{3-\phi}}{2} \cdot \frac{\sqrt{2+\phi}}{2} + \frac{1}{2\phi} \cdot \frac{\phi}{2} \\ &= \frac{1}{4}\ (\sqrt{(3-\phi)(2+\phi)} + 1) \\ &= \frac{1}{4}\ (\sqrt{6+\phi-\phi^2} + 1) \end{aligned}$$

It looks like a bit of a mess. But we know the answer must be simple because $\sin 54 = \cos 36 = \phi/2$. In other words, the expression in parentheses must be equal to $2\phi$.

$$2\phi \stackrel{?}{=} \sqrt{6+\phi-\phi^2} + 1$$

But $\phi^2 = 1 + \phi$ so the right-hand side is

$$\sqrt{6+\phi-1-\phi} + 1 = \sqrt{5} + 1$$

which is, indeed, $2\phi$!

## trigonometry

So now we actually have all the pieces and might follow Ptolemy's method.

I would rather digress to show a bit of new trigonometry. It will connect to another place in the book where we talk about de Moivre's theorem.

And there is a fancy trick that's specific to 18 degrees. Later we'll adjust to 36.

First, the new general result: the formula for the cosine of three times the angle.

## cos 3A

We use the standard angle sum formula in a new version:

$$\cos 3A = \cos 2A \cos A - \sin 2A \sin A$$

so then a second application of the formula gives

$$\cos 3A = (\cos^2 A - \sin^2 A) \cos A - (2 \sin A \cos A) \sin A$$
$$= (2 \cos^2 A - 1) \cos A - 2 \cos A(1 - \cos^2 A)$$

Just count up the terms. We have

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

□

This result can also be obtained by **de Moivre's theorem**:

$$\cos nx + i \sin nx = (\cos x + i \sin x)^n$$

We have

$$\cos 3x + i \sin 3x = (\cos x + i \sin x)^3$$
$$= \cos^3 x + 3i \cos^2 x \sin x + 3i^2 \cos x \sin^2 x + i^3 \sin^3 x$$

We only need the real part, which is

$$\cos 3x = \cos^3 x - 3 \cos x \sin^2 x$$
$$= \cos^3 x - 3 \cos x(1 - \cos^2 x)$$

which simplifies to the same result.

## sine of 18 degrees

Let $A = 18$. Then

$$5A = 90$$

$$2A = 90 - 3A$$

$$\sin 2A = \cos 3A$$

Plug in the previous result

$$\sin 2A = 4\cos^3 A - 3\cos A$$

$$2\sin A\cos A - 4\cos^3 A + 3\cos A = 0$$

Each term contains one copy of $\cos A$, and since that is non-zero, we simply multiply by $1/\cos A$ on both sides, giving

$$2\sin A - 4\cos^2 A + 3 = 0$$

$$2\sin A - 4(1 - \sin^2 A) + 3 = 0$$

$$4\sin^2 A + 2\sin A - 1 = 0$$

Now we have a quadratic in $\sin A$. The roots are

$$\sin A = \frac{-2 \pm \sqrt{4 + 16}}{8}$$

$$= \frac{-2 \pm 2\sqrt{5}}{8}$$

We take the positive root because $\sin A > 0$ in the first quadrant.

$$\sin 18 = \frac{-1 + \sqrt{5}}{4}$$

$$= \frac{1}{2} \cdot \frac{\sqrt{5} - 1}{2}$$

This is almost $\phi$.

$$\phi - 1 = \frac{\sqrt{5} - 1}{2}$$

so

$$\sin 18 = \frac{\phi - 1}{2}$$

and since we showed earlier that

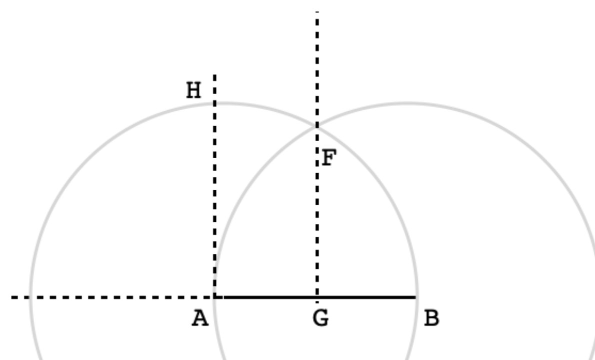
$$\frac{1}{\phi} = \phi - 1$$

the previous result can be re-written as

$$\sin 18 = \frac{1}{2\phi}$$

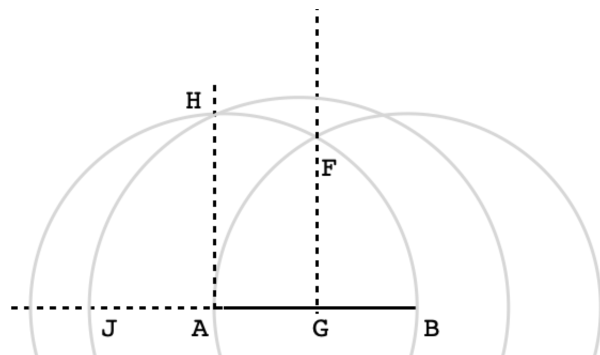
## construction 2, starting from a side length

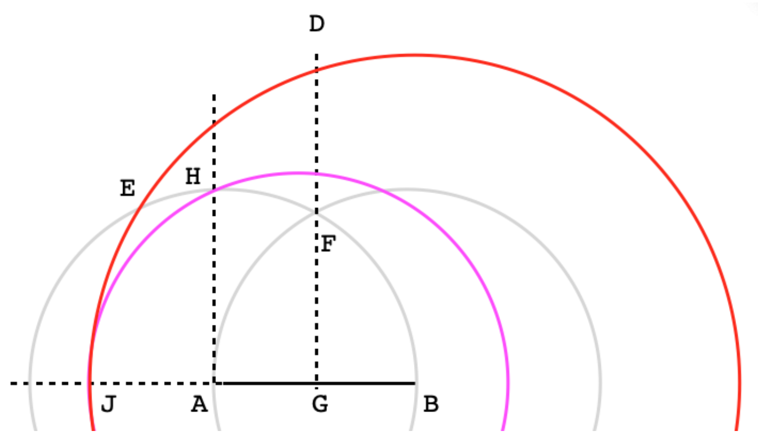
Wikipedia gives another method for constructing a regular pentagon, this time starting from a given side length, $AB$. The first part is straightforward.




Draw two congruent circles with a radius of length $AB$ centered at $A$ and $B$. Construct the perpendicular bisector $FG$ (not shown). Construct the perpendicular from $A$ to meet the circle centered on $A$, at $H$.

We will construct two more circles. The first one is centered at $G$ with radius $GH$, intercepting the extension of $AB$ at $J$.




The second one is centered at $B$ with radius $BJ$. The intersection with the previous circle (at $E$) and with the perpendicular at $D$ are vertices of the pentagon.

The last vertex, at $C$, can be constructed by laying off arcs of the distance $DE$ from $D$ and $B$, or by repeating the construction with a large circle centered on $A$.

## why this works

It is claimed that

$$\frac{BJ}{AB} = \frac{AB}{AJ} = \phi$$

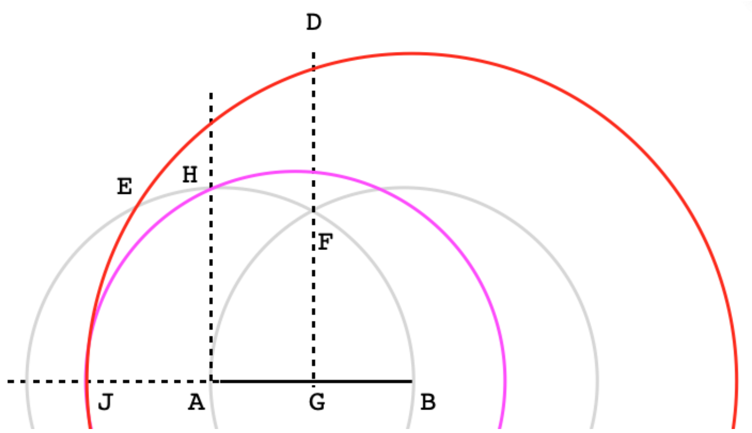
Let's see.

$$AB = 1, \qquad AG = \frac{1}{2}$$

$$AH = AB = 1$$

$$GJ = GH = \sqrt{1^2 + (\frac{1}{2})^2} = \frac{\sqrt{5}}{2}$$

$$BJ = GJ + \frac{1}{2} = \phi$$

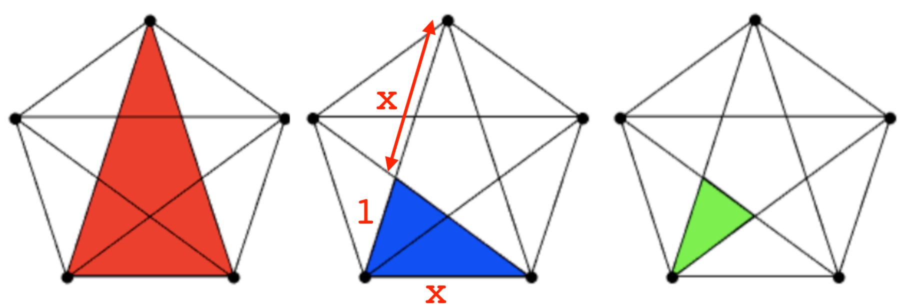



and

$$AJ = BJ - 1 = \phi - 1$$

$$\frac{AB}{AJ} = \frac{1}{\phi - 1} = \frac{\phi}{\phi^2 - \phi} = \phi$$

So we can confirm that the ratios are correct. Now we just need to connect the lengths in the diagram to sides of triangles in our view of the pentagon with internal chords.




The top vertex is easy.

$$AD = BD = BJ = \phi$$

while $AB = 1$ so the ratio is $\phi$, which matches the red triangle.

For vertex $E$, we must show that $BE = \phi$. But $BE = BJ = \phi$.

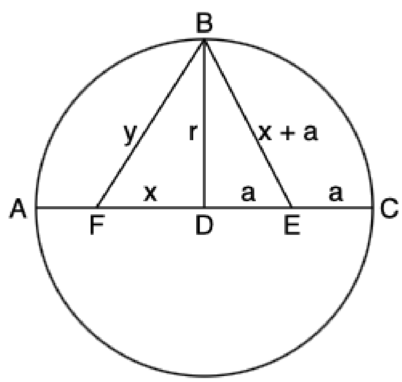
We verify that $D$ is placed correctly, as follows. $D$ lies on the perpendicular bisector of $AB$ and is also a distance $\phi$ away from both $A$ and $B$, forming an isosceles triangle with long sides $\phi$ and short side 1.

We could try to verify that $D$ is placed correctly by proving that the base $DE = 1$ to complete the triangle. But this seems difficult, and it's easy to show that $\triangle ABE$ has two sides of length 1 and one, $BE$, of length $\phi$. Since this triangle is congruent to the short fat ones we find in a pentagon of side length 1, we're done.

$\square$

## construction 3

The third construction I have is said to be contained in Ptolemy's book *The Almagest* (link below).




Once again, we divide the radius in half: $a = r/2$. The hypotenuse $BE$ is marked out so that $EF = BE$. Then the second hypotenuse, $BF$ is claimed to be the side of a pentagon inscribed in a unit circle and $BE$ is the side of the decagon.

If we let $r = 1$ and $a = 1/2$ then

$$x + a = \sqrt{(\frac{1}{2})^2 + 1)} = \frac{\sqrt{5}}{2}$$

so

$$x = \frac{\sqrt{5}}{2} - \frac{1}{2} = \phi - 1$$

and then $BF$ or

$$y = \sqrt{1^2 + (\phi - 1)^2}$$
$$= \sqrt{1 + \phi^2 - 2\phi + 1}$$

but $\phi^2 - \phi = 1$ so we have

$$y = \sqrt{3 - \phi}$$

$BF = y$ is supposed to be the side of a pentagon and that matches what we had before. $BE = x$ is supposed to be the side of a decagon. We check that by recalling that the side of the pentagon is twice the sine of 36, which matches.

So the side of the decagon is twice the sine of 18 which is simply $1/\phi$. We must show that this is equal to $\phi - 1$. But we did this already, back near the beginning.

$$\phi^2 = 1 + \phi$$

$$\phi = \frac{1}{\phi} + 1$$

And that's it.

## more about Ptolemy

This completes the first part of Ptolemy's *The Almagest*, described in the link below.

https://hypertextbook.com/eworld/chords/#table1

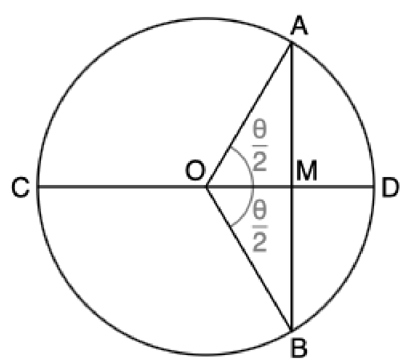



Figure 1

This figure is just to remind us that the discussion in the link describes values for the chord $AB$ of a central angle $\theta$, or inscribed (peripheral) angle $\theta/2$. In modern language, we would say that the chord length $AB = 2r\sin\theta/2$ or

$$\frac{AB}{CD} = \sin\theta/2$$

When Ptolemy associates a value to a particular angle $\theta$ we should find the sine of one-half $\theta$ for comparison.

Also, Ptolemy deals in ratios and his units are degrees, minutes and seconds. In particular, the diameter $CD$ is also divided into degrees, namely, 120°.

Table 1: Chords of the special angles

| angle | crd |
|---|---|
| 36° | 37°4'55" |
| 60° | 60° |
| 72° | 70°32'3" |
| 90° | 84°51'10" |
| 108° | 97°4'56" |
| 120° | 103°55'23" |
| 144° | 114°7'37" |
| 180° | 120° |

On the first line, the angle is given as 36° and from what we said, we need to look for the sine of 18°, which we have as

$$\frac{\phi - 1}{2} = \frac{\sqrt{5} - 1}{4} = 0.3090$$

This result must be converted to degrees for comparison. I don't want to do arithmetic in degrees, etc., so I multiply $120 \times 3600 = 432000$. The diameter is divided into that many parts.

What we have is 0.309017 times that or 133495.34. I divide by 3600 and the whole part is 37, the modulus is 295. Dividing by 60 I then get 4 with 55 and a bit left over. That's a match.

The second part of *The Almagest* involves what we know as the sum of angles formulas. We actually used Ptolemy's theorem and his derivation of those relationships as our preferred proofs of them (**here**).

He uses the sum of arcs (i.e. angles), difference of arcs and half-arc formulas to fill out a table for every angle up to $180°$ in increments of $3/4°$.

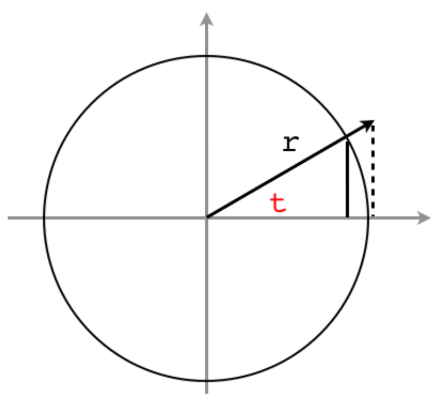
So he went from $36 \to 18$ and $30 \to 15$ and then $18 - 15 = 3$ and half 3 is 1.5 and half again is $3/4$. That's a lot of work.

## approximation

The last part of the book involves an approximation for the sine of very small angles. This allows not just small values to be calculated, but also fine-grained interpolation for the table as a whole.

The basic idea is that as $\theta$ gets small, the sine becomes approximately equal to the arc traced by the angle. For a hemisphere the arc is $\pi$ and the chord is the diameter, which I am calling 2, for a unit circle. For one-quarter circle the arc is $\pi/2$ and the chord is $\sqrt{2}$. So the first ratio is $\pi$ divided by 2 and the second is $\pi$ divided by $2\sqrt{2}$. Clearly, the denominator is increasing.

To put it another way




For $0 < t < \pi/2$, we have $\sin t < t < \tan t$, but as $t \to 0$ they all become equal.

If you already know about power series, you know (or can look up) that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

so

$$\sin x \approx x$$

when $x$ is sufficiently small.

The justification used by Ptolemy involves a relationship called Aristarchus' inequality, which says that for $\alpha < \beta$:

$$\frac{\sin\alpha}{\sin\beta} > \frac{\alpha}{\beta}, \qquad \frac{\sin\alpha}{\alpha} > \frac{\sin\beta}{\beta}$$

If $\alpha$ is smaller than $\beta$, $\sin\alpha$ gets proportionately closer to $\alpha$ than $\sin\beta$ is to $\beta$.

$$\sin\alpha \cdot \frac{\beta}{\alpha} > \sin\beta$$

An aid to memory: when the fraction multiplying the sine or chord is greater than one, then we need the greater than symbol on that side of the inequality.

Suppose that $\angle\alpha$ is one-half $\angle\beta$. Then the chord corresponding to $\angle\alpha$ or the sine of $\alpha$ is *more than* one-half the corresponding value for $\angle\beta$.

## lemma and proof

I want to give a proof of Aristarchus' inequality. Rather than use the one he or Ptolemy would use, we'll do something more modern.

It's based on a proof in wikipedia, and this gives a chance to introduce a trigonometric identity we've not seen until now.

https://en.wikipedia.org/wiki/Aristarchus%27s_inequality

Let's start with a lemma. Recall the formulas for the sine of the sum and difference of two angles:

$$\sin x + y = \sin x \cos y + \sin y \cos x$$

$$\sin x - y = \sin x \cos y - \sin y \cos x$$

Adding them makes the second term disappear and gives

$$\sin(x+y) + \sin(x-y) = 2\sin x \cos y$$

Now, the neat idea. We do something close to what we want and then go back and fix it. For starters, let

$$A = x + y, \qquad B = x - y$$

Then

$$A + B = 2x, \qquad A - B = 2y$$

so

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

We actually want something slightly different. We want the sine term to have $B - A$. So go back and instead let

$$-A = x - y, \qquad B = x + y$$

Then

$$\begin{aligned} B - A &= B + (-A) \\ &= x + y + (x - y) = 2x \\ x &= (B - A)/2 \end{aligned}$$

and

$$\begin{aligned} B + A &= B - (-A) \\ &= x + y - (x - y) = 2y \\ y &= (B + A)/2 \end{aligned}$$

and

$$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$$

becomes

$$\sin B + \sin -A = 2 \sin \frac{B-A}{2} \cos \frac{B+A}{2}$$

Finally, since $\sin -\theta = -\sin\theta$

$$\sin B - \sin A = 2 \sin \frac{B-A}{2} \cos \frac{B+A}{2}$$

We proceed to the main part of the proof.

*Proof.*

Consider two angles and let $A < B$, $A$ is smaller than $B$. We are to prove that

$$\frac{\sin B}{B} < \frac{\sin A}{A}$$

The meaning of this is that, as we said above, if $A$ is smaller than $B$ then proportionally, $\sin A$ is closer to $A$ than $\sin B$ is to $B$.

$$\sin A > \frac{A}{B} \cdot \sin B$$

The first step is an algebraic trick that we've seen several times. First, rearrange what we are to prove like this:

$$\frac{\sin B}{\sin A} < \frac{B}{A}$$

Subtracting 1 from both sides doesn't change the inequality

$$\frac{\sin B}{\sin A} - \frac{\sin A}{\sin A} < \frac{B}{A} - \frac{A}{A}$$

$$\frac{\sin B - \sin A}{\sin A} < \frac{B - A}{A}$$

$$\frac{\sin B - \sin A}{B - A} < \frac{\sin A}{A}$$

This statement is equivalent to what we need to prove. If we can show that this is true, we'll be done.

We proceed by finding a value in between the two expressions.

It turns out that $\cos A$ will work.

$$\frac{\sin B - \sin A}{B - A} < \cos A < \frac{\sin A}{A}$$

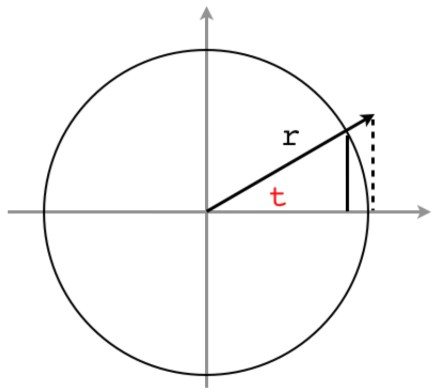
The second part is easy

$$\cos A < \frac{\sin A}{A}$$

This is just

$$A < \frac{\sin A}{\cos A} = \tan A$$

which can be shown by considering areas. Compare the triangle with the dotted line with the area of the sector in the figure below.




The complete result is $\sin A < A < \tan A$. (Recall that the area of the whole unit circle is $\pi$, while the entire angle is $2\pi$, so the area of a sector swept out be angle $\theta$ is $A = \theta/2$.)

Now we look more closely at the left-hand side.

$$\frac{\sin B - \sin A}{B - A} < \cos A$$

First, from our lemma, the numerator is

$$\sin B - \sin A = 2 \sin \frac{B - A}{2} \cos \frac{B + A}{2}$$

We will make a simple substitution that make this term larger. Then we'll show that the resulting inequality really is correct.

Since we've made the left-hand side *bigger* and the relationship still holds, we know that the original expression is also valid.

$B > A$ so the difference $B - A$ is positive but smaller than $B$.

For a small positive angle, the sine of the angle is smaller than the angle itself so we can replace $\sin(B - A)/2$ by $(B - A)/2$. This makes the whole left-hand side larger.

$$2 \cdot \frac{B - A}{2} \cos \frac{A + B}{2}$$

We notice that we can cancel the original denominator (which was $B - A$), as well as that factor of 2. As a result, the whole inequality is reduced to

$$\cos \frac{B + A}{2} < \cos A$$

But the average of two angles $(B+A)/2$ is larger than the smaller angle $A$:

$$A < B$$

$$2A < B + A$$
$$A < \frac{B+A}{2}$$

And, as the angle gets larger, the cosine gets smaller. Hence, $\cos(B+A)/2$ is smaller than the cosine of $A$.

In other words, the inequality is correct. Since our manipulation made the numerator bigger and the inequality definitely holds, the original expression is valid.

$$\frac{\sin B}{B} < \frac{\sin A}{A}$$

□

## examples

The example from the book uses known values for the *chord* of an angle of $3/4°$ as well as 1-1/2°. As we said, the chord corresponds to the sine of the half-angle.

The values are given in degrees, minutes and seconds of arc. Let $s$ and $t$ be those values:

$$s = 0° \ 47' \ 08'' = crd \ 3/4°$$

$$t = 1° \ 34' \ 15'' = crd \ 1.5°$$

It is immediately apparent that $2s > t$ but not by much, only one second of arc. It should not be surprising that we end up with a linear approximation.

The diameter is divided into 432000 parts (120 degrees, 60 minutes, 60 seconds). A calculator (and some fiddling) gives the following value for the sine of $3/4°$ in parts of 432000: 5654.705.

A similar calculation for the sine of $3/8°$ gives: 2827.413. The whole part is exactly one-half, while the fractional part is slightly larger than half for the smaller angle.

Doing the modular arithmetic, I get 47′ 07" for the first and 1° 34′ 14" for the second. Our source says they "didn't believe in rounding up, ever", as we do today for fractions larger than one half. I suppose the error may come from the previous calculations.

In any event, let us estimate the value for the chord $c$ of $1^\circ$ as Ptolemy would. The angle $\theta = 1^\circ$ is $4/3 \cdot s$ so the calculation goes:

$$c < \frac{4}{3} \cdot 0^\circ\ 47'\ 08'' = 1^\circ\ 2'\ 50''$$

On the other hand, $\theta = 1^\circ$ is $2/3 \cdot t$ so

$$c > \frac{2}{3} \cdot 1^\circ\ 34'\ 15'' = 1^\circ\ 2'\ 50''$$

(Remember, the greater than symbol goes with the fraction larger than one).

Clearly, we have a good approximation for $c$. The results seem to be exactly the same, though they cannot really be. It just appears so because of round-off error. If we do the same calculation with our modern precise decimals we get

$$c < \frac{4}{3} \cdot 2827.413 = 3769.884$$

$$c > \frac{2}{3} \cdot 5654.705 = 3769.803$$

The true value is something like $1^\circ\ 2'\ 49''$ plus 800 and some thousandths, which I cannot help but note is smaller than the reported value by (truncated to 49") by one second.

According to the source:

> The remainder of the Almagest consists of astronomical calculations: the position of the sun, moon, and planets at various times relative to the fixed stars. The Table of Chords played an important role in their compilation.

# Chapter 31

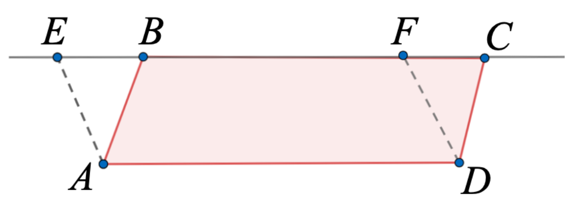
# Pappus's proof of Pythagoras

Pappus came up with a beautiful theorem which includes the Pythagorean theorem as an extension. First, we need a simple lemma about parallelograms.

*Lemma.*

Given two parallel lines: $AD \parallel EBFC$.

If two parallelograms $AEFD$ and $ABCD$ have opposite sides on the two lines, and those two segments are of equal length, then they have equal areas.



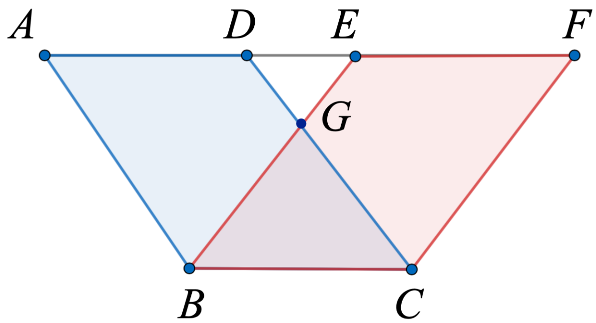

Here is a special case where they have the same base $AD$.

We proved earlier that the perpendicular between two parallel lines is of equal length no matter where it is drawn.

So the two parallelograms are each composed of pairs of triangles, which, having the same base and the same altitude, also have equal area.

□

One might also just invoke Euclid I.35.

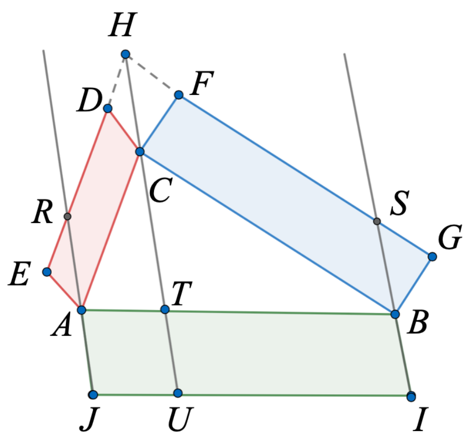



## Pappus's parallelogram theorem

In the figure below, on the two sides of any $\triangle ABC$ draw any parallelograms $ACDE$ and $BCFG$. Extend the two new sides to meet at $H$.

Draw $HC$ and its extension such that it cuts $AB$ at $T$ and then let $HC = TU$.

Draw $AJ \parallel HCTU \parallel BI$.




*Proof.*

The new parallelogram with $AC$ as one side and $RH$ the other, is equal, by our lemma. $(ACDE) = (ACHR)$.

Since $RAJ \parallel HCTU$ and $RA = HC = TU = AJ$, $(ATUJ) = (ACHR)$ for the same reason.

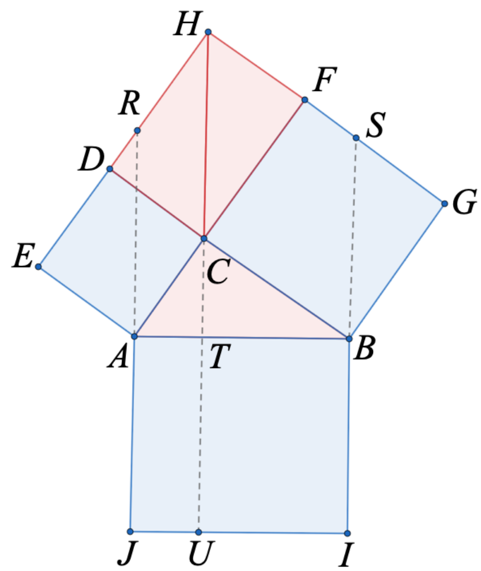
So $(ATUJ) = (ACDE)$.

We use the same argument for $(BCFG) = (TBIU)$ on the right.

Then add the two results: $(ACDE) + (BCFG) = (ABIJ)$.

$\square$.

Let $\angle ACB$ be a right angle, and let the parallelograms be squares.




$\triangle DHC$ and $\triangle FHC$ are right triangles and they are congruent to $\triangle ABC$ by SAS.

So $\angle CHF = \angle CAT$ and we also have vertical angles, so $\triangle HFC \sim \triangle CTA$.

It follows that $\angle HFC = \angle CTA$ and both are right angles.

Further, $CH = TU = AB$, so $ABIJ$ is the square on $AB$.

It is easy to see that $(AEDC) = (ARHC) = (ATUJ)$, and the rest follows.

Thus Pappus' theorem becomes the Pythagorean theorem as a special case.

[ Reference: George F. Simmons, *Calculus Gems*. ]

# Part IX

# Addendum

# Chapter 32

# Author's notes

A central feature of this book is the relentless use of proof. I emphasize the key insight for each, and have tried to make the proofs simple and as easy to follow as possible.

This volume is distinguished from most other texts, since they maintain that a proper proof should be watertight, with each step carefully justified and following closely from the one before. I don't deny that rigor has its proper place in math education, but I also think that this rigidity obscures the core insights. Our purpose here is to view beauty clearly.

We prefer instead to be like the famous mountaineer Ueli Steck. Reach the summit quickly and emphasize the key steps. You should be able to fill in the details if there are loose ends.

Multiple proofs for important theorems are sometimes given, because proof is our stock in trade, and different approaches shed light on how proofs may be found and developed.

Another distinguishing feature is a set of simple proofs based on scaling of triangles. This happens for the Pythagorean theorem and for Ptolemy's theorem, as well as the sum of angles theorems and then later, a fairly sophisticated theorem of Euler's.

Recently I came across a fantastic book by Acheson, called *The Wonder Book of Geometry*. I helped myself to some of his examples, and now have more than a dozen. Please go find Acheson, and buy it. It's truly magical. In fact, all of his books are wonderful!

A saying attributed to Manaechmus, speaking to Alexander the Great, is that "there is no royal road to geometry". Others write that this was actually Euclid, speaking to Ptolemy I of Egypt. Since the two sources lived some 700 years after the fact, it is difficult to know.

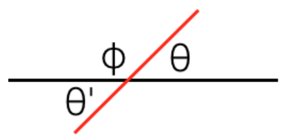
Practically, this means that in learning mathematics you must follow the argument with pencil and paper and work out each step yourself, to your own satisfaction. That is the only way of really learning, and at heart, a principal reason why I wrote this book.

Having read a chapter, see if you can prove the theorems yourself, without looking at the text.

There are a few problems listed in the later chapters, perhaps thirty or more altogether. Most of them have worked out solutions. It is highly recommended that you attempt each problem yourself before reading my answer. Since the crucial point is often to draw an inspired diagram, you must stop reading as soon as the problem is stated!

# Chapter 33

# Angular measure




We can see that obviously

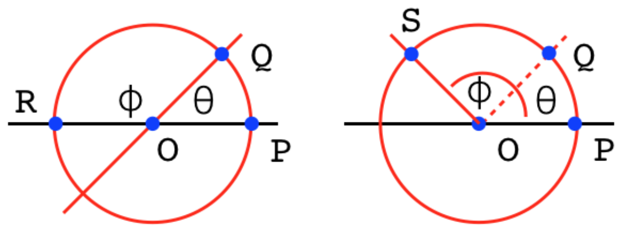
$$\phi > \theta$$

Writing the statement $\phi > \theta$ is easy, but for this to make sense (what if they are almost equal?), we must have some way of taking the measure of an angle. We can't just rely on a picture.

Our answer is to construct a circle around the central point.

∘ The measure of the angle is the distance along the circumference between the points where the lines cross the circle.

If that distance along the edge is larger for $\phi$ than for $\theta$, then $\phi > \theta$. In the left panel, the arc between $Q$ and $R$ (call it arc $QR$) is larger than arc $PQ$.



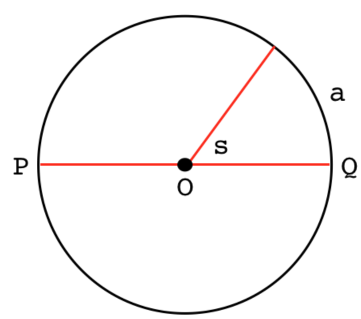

If we lay off the arcs starting from the same point $P$, the arc $PS$ is longer than the arc $PQ$ (right panel).

We don't need to actually measure the arc itself to do this. Measuring curved lengths is a bit tricky to do.

Instead we can use a standard compass to lay off the linear distance from $P$ to $S$ and compare that with the distance from $P$ to $Q$. Since the distance from $P$ to $S$ is more than from $P$ to $Q$, $\phi > \theta$.

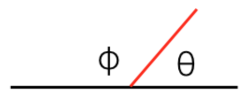
This doesn't work very well as $S$ approaches the diameter through $O$ and $P$, but we could lay off the distance in two (or more than two) parts and that would be fine. For example, take a unit circle, draw a diameter and the perpendicular bisector to form four right angles. Then bisect each right angle and bisect again. That would divide the circle into 16 arcs of equal length.

No matter how the measurement is to be performed, we *define* the measure of an angle $s$ to be equal to the arc $a$ it sweeps out, or is subtended by, in a circle of radius 1, a *unit* circle. Angles are not lengths, but numerically, the measure of the angle is the measure of the arc.



## right angles

• the sum of all the angles on one side of a line (at a given point), is equal to two right angles.




This is simply a matter of symmetry and subtraction. If the sum of all the angles at a point is equal to four right angles, then the sum of the angles on each side of a line through that point is equal to two right angles.

The convention that there are 360° total in a circle dates to the time of the Babylonians (c. 2400 B.C.).

In degrees, a right angle is 90° and two supplementary angles measure 180°.

There is nothing particularly special about using 90° as the measure of a right angle, or 360° for one whole turn. Well, here is one thing: there are *approximately* 360 days in a year, which marks the sun's track across the sky. Another idea is that 360 is special because it has so many factors, which makes it possible to divide up a circle evenly in $2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20 \ldots 180, 360$ parts.

Euclid just talks about angle measurements in terms of right angles. For example, that the sum of two supplementary angles is equal to two right angles.

In his book, *Measurement*, Lockhart adopts the convention that a whole turn is equal to 1.

## radians

We'll just mention here that one whole turn can be defined using a different unit of measure as $2\pi$ *radians*, and that convention turns out to be quite important for calculus.

It is based on two ideas: the first, from above, that angular measure is numerically equal to arc length, and second, that in a unit circle, the circumference or total of the arc length is equal to $2\pi$.

In calculus, all angles will be in radians. One big reason for that comes in working out the *derivative* of sine and cosine. There, it will be important to consider the following expression:

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

The notation on the left asks us to consider what happens as $\theta$ gets close to the value 0, and the rest of it states that the ratio $\sin \theta / \theta$ is equal to 1. As a result, the derivative of $\sin \theta$ is simply $\cos \theta$.

Well, if you're working in degrees *that's not true.* There's an awkward constant of proportionality. So we work in radians.

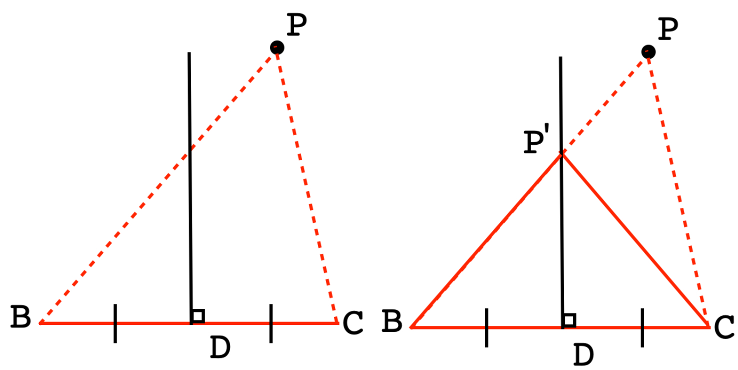
It also makes plots prettier, since the sine of $x$ goes from 0 to 1 as $x$ goes from 0 to $\pi/2 \approx 1.57$. The sine of 1 degree is effectively zero, so that wouldn't look so nice.

# Chapter 34

# Additional proofs

## perpendicular bisector (converse)

The argument in this section is a little complicated.




Suppose a given point $P$ does not lie on the vertical bisector. We claim that any such point cannot be equidistant from $B$ and $C$.

*Proof.*

We will use an argument by contradiction. We assume the opposite of the statement we want to prove, and show that it leads to a contradiction and so cannot be correct.

Now, suppose that $P$ *is* equidistant, with $PB = PC$. By the forward theorem we have that $\angle PBC = \angle PCB$.

Draw the perpendicular bisector and find point $P'$ on both $PB$ and the perpendicular bisector.

By the forward theorem we have that $\angle P'BC = \angle P'CB$. But $\angle P'BC$ and $\angle PBC$ are the same angle.

Therefore

$$\angle PBC = \angle P'CB = \angle PCB$$

But clearly $\angle PCB > \angle P'CB$.

This is a contradiction. Therefore, $PB \neq PC$.

$\square$

If we raise the perpendicular bisector from a line segment, *every* point on the bisector is equidistant from the ends of the line segment, and when the points are connected, forms an isosceles triangle.

No other point not on the perpendicular bisector can be equidistant from the two endpoints.

*Proof.* (Alternate).

There is also a much easier proof that relies on **Euclid I.7**, which says there cannot be two points on the same side of $BC$ above which have the same distance to $B$ and $C$.

Yet that is exactly what this situation calls for. $P$ is claimed *not* to be on the bisector, with $PB = PC$. Yet by the forward theorem, there must be another point $Q$ *on* the bisector with $QB = QC$. By **Euclid I.7**, this is impossible. $\square$.

## Euclid I.6

We proved the converse of I.5 early in the book based on angle bisection. Later, we gave another proof by contradiction.

- **isosceles triangle theorem** (Euclid $I$.6: angles $\rightarrow$ sides)

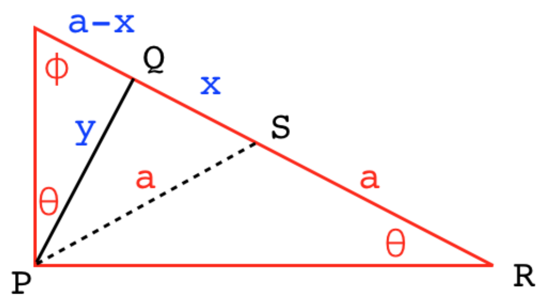
## tangents of similar triangles

Here is a mixed geometric/algebraic proof of the Pythagorean theorem.

*Proof.*

Let $S$ be the midpoint of the hypotenuse in a right triangle, and draw $PS$ connecting the midpoint to the vertex at $P$. If the hypotenuse has length $2a$, then the length of $PS$ is $a$, by the **midpoint theorem** that we talked about previously.

[Quick proof: inscribe the original triangle in a circle, with the hypotenuse of the right triangle as the diameter. The length $a$ is the radius of the circle centered at $S$ that contains the three vertices of the original right triangle.]




Now, draw the altitude from the right angle at $P$ to the hypotenuse $PQ$. Suppose $QS = x$, then the length $a$ is divided into $x$ and $a - x$ as shown.

The angle $\phi$ and the angle at vertex $R$ labeled $\theta$ are complementary angles in a right triangle, they add up to one right angle. Therefore both angles labeled $\theta$ are equal.

So we can form the equal ratios of sides:

$$\frac{a-x}{y} = \frac{y}{a+x}$$
$$(a-x)(a+x) = y^2$$
$$x^2 + y^2 = a^2$$

□

The second line from the last:

$$(a-x)(a+x) = y^2$$

is a statement of our theorem that the altitude of a right triangle is the geometric mean of the two sections of the base.

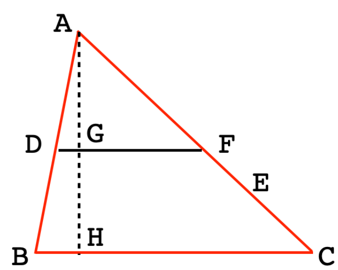
# similar right triangles

## altitudes in proportion

Note that, as well as the sides, the altitudes are also in proportion with the same ratio.

One way to see this is to drop an altitude and then consider the two similar right triangles on one side of it. The altitudes are sides of this triangle.

For similar triangles, where the sides are in proportion $k$, the areas are in proportion $k^2$. The reason is that the altitudes are in the same proportion, namely $k$.




Given that $DF \parallel BC$, so $\triangle ADF \sim \triangle ABC$.

Drop the altitude $AGH$.

Now we see that $\triangle ADG \sim \triangle ABH$ and $\triangle AGF \sim \triangle AHC$, with the same proportionality constant, *since they share the sides* $AG$ and $AH$.

Suppose that $AD/AB = DF/BC = k$. Then $AG/AH$ also is equal to $k$.

The ratio of areas is then

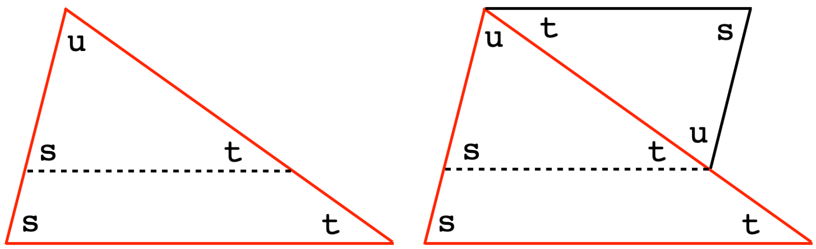
$$\frac{\Delta_{ADF}}{\Delta_{ABC}} = \frac{\frac{1}{2}DF \cdot AG}{\frac{1}{2}BC \cdot AH} = k^2$$

## all triangles

Any triangle can be decomposed into two right triangles.

We previously proved the equal angles for two similar right triangles implies equal ratios of sides. Now we combine the results for the two sub-triangles, both right triangles, and we will have the result for the general case.

We use a flipped and rotated copy of the smaller triangle. Start with two triangles similar because the angles are the same (left panel).



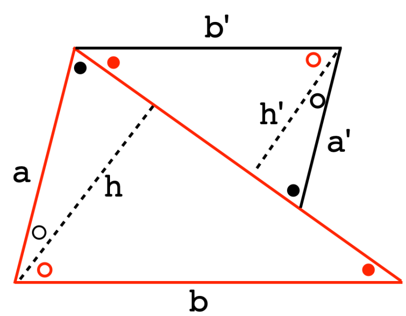

Make a copy of the smaller triangle and rotate it and then attach at the top (forming a parallelogram). The original small triangle and the flipped version are congruent by our construction.

From congruent triangles we get the angle equalities in the right panel.

We also have two pairs of parallel sides, either by alternate interior angles or because $s + t + u$ is equal to two right angles.

Now, draw the two altitudes, label the sides, and suppress the labels for the angles but just mark them with colored circles.

The angles marked with filled dots are equal (black and black, and red and red) by parallel sides as we just said, and the open dots with the same colors are equal because they are complementary angles in a right triangle.




Thus, we have two different pairs of similar right triangles.

$$\frac{a}{h} = \frac{a'}{h'}$$

$$\frac{b}{h} = \frac{b'}{h'}$$

So then

$$\frac{h}{h'} = \frac{a}{a'} = \frac{b}{b'}$$

Corresponding sides of the two original triangles have equal ratios of sides. (It is important that we extended the equal ratios result to the hypotenuse for this to work).

But there is nothing special about this pair of sides, we could have chosen any other pair, either $a$ and $c$ or $b$ and $c$, and have the same result.

Therefore if any two triangles have three angles the same, the side lengths are all in the same proportion.

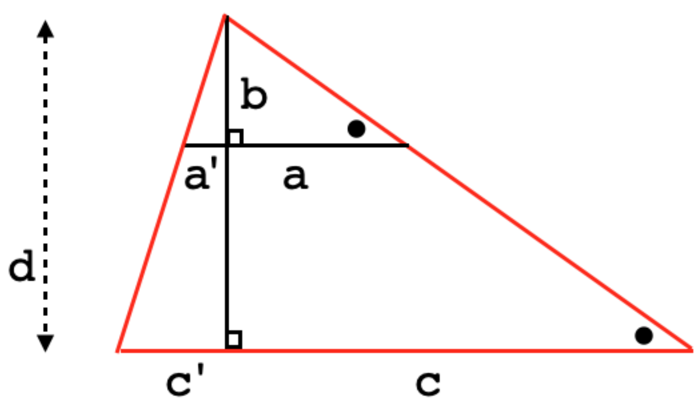
□

## similar triangles

Euclid VI.2 gives us all we need. However, let's take a moment to look at a different proof.

We showed previously that all the statements about similarity that we made in this chapter apply to similar right triangles.

But we can cut any triangle into two right triangles.




If these are right triangles, then the two on the right are similar as well as the two on the left, using complementary angles.

We have

$$\frac{a}{c} = \frac{b}{d}$$

but also

$$\frac{b}{d} = \frac{a'}{c'}$$

So then

$$\frac{a}{c} = \frac{a'}{c'}$$

and then

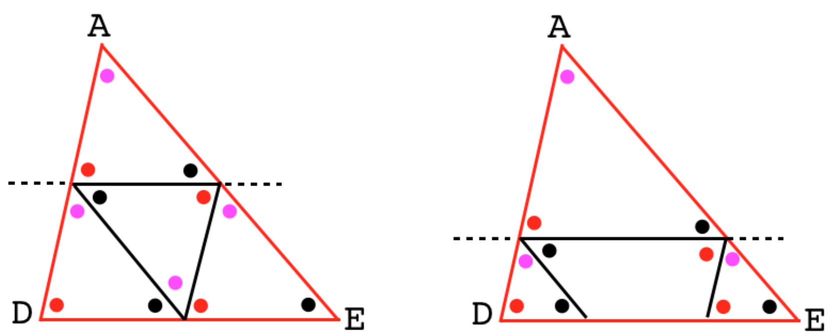
$$\frac{a}{a'} = \frac{c}{c'}$$

$$\frac{a+a'}{a'} = \frac{c+c'}{c'}$$

$$\frac{a+a'}{c+c'} = \frac{a'}{c'} = \frac{a}{c} = \frac{b}{d}$$

□

## AAA similarity theorem

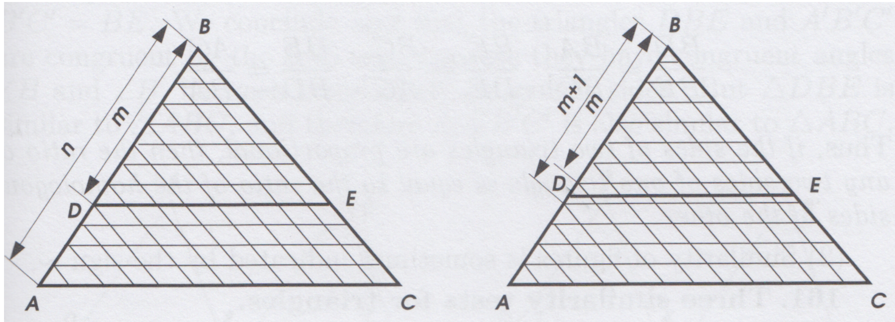



On the left is the easy case where $AB = BD$.

We will show that the sides are in proportion even when that proportion is not 1 : 2, as on the right.

Note: we proved this theorem for right triangles already, based on an idea I found in Acheson's book, and combined it with an extension to all triangles.

If the horizontal bisector is parallel to the base, then the triangles are similar. We will have AAA. This is true regardless of which side of the large triangle we choose to be the base.




His notation is different than what we used above, drawing $\triangle BDE$ smaller than $\triangle BAC$. We follow Kiselev for this section.

There are two cases. The first is when the lengths of $BA$ and $BD$ are commensurable.

Two lengths are commensurable when there is some small length $\ell$ that we can define as one unit, such that for integers $m$ and $n$, $BD = m\ell = m$ and $BA = n\ell = n$.

Divide the side as shown. Draw lines parallel to $AC$ and also those parallel to $BC$.

Then $BE$ and $BC$ will be divided into congruent parts, numbering $m$ and $n$ for each, respectively. The same thing happens on the bottom. It is clear that

$$\frac{m}{n} = \frac{BD}{BA} = \frac{DE}{AC} = \frac{BE}{BC}$$

The second, harder, case is shown in the right panel above.

$BD$ and $BA$ are not commensurate and there is some small remainder when dividing the first into the second. Put another way, if $BA = n\ell$, then there are two integers $m$ and $m + 1$ such that

$$m\ell < BD < (m+1)\ell$$

But if $\ell$ is small, with $n$ and $m$ large,

$$\frac{m}{n} \approx \frac{BD}{BA}, \qquad \frac{m}{n} \approx \frac{DE}{AC}, \qquad \frac{m}{n} \approx \frac{BE}{BC}$$

Crucially, by choosing the unit length $\ell$ smaller and smaller, and thus $n$ being larger and larger, we can make the remainder $BD - m\ell$ **as small as we like**.

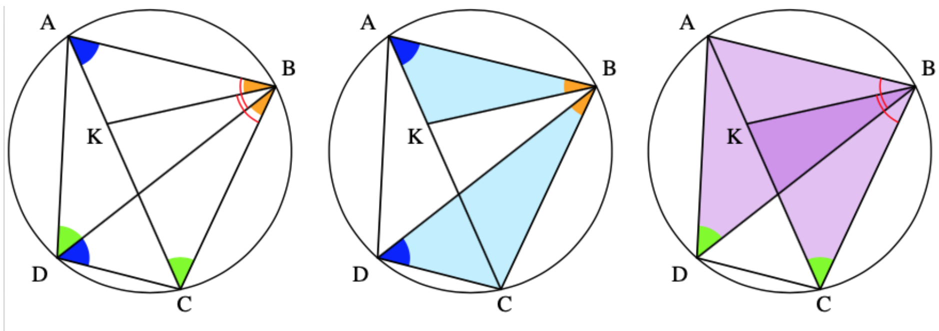
As $n$ gets very large we approach equality:

$$\frac{m}{n} = \frac{BD}{BA} = \frac{DE}{AC} = \frac{BE}{BC}$$

for the second case as well.

In calculus we say that, in the limit, as $n \to \infty$, they become equal. If this seems strange, wait for the discussion of the limit concept, in calculus.

## Ptolemy's by similar triangles




Above is a graphic from wikipedia that shows where we're going in the first proof. We will form two sets of similar triangles and use our knowledge about corresponding ratios.

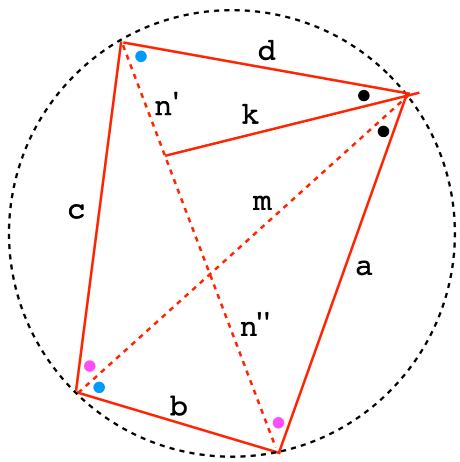
https://en.wikipedia.org/wiki/Ptolemy%27s_theorem

*Proof.*

It is often easier to adopt a modern notation, designating side lengths by single letters. In this problem, we have sides $a, b, c, d$ and diagonals $m$ and $n$.

The angles marked with magenta dots are equal as peripheral angles subtended by the same arc, and the same with the blue dots.

The key insight is to draw the line segment marked $k$, separating $n$ into two parts, $n'$ and $n''$. The line is drawn so that the angles marked with black dots are equal.




Other pairs of angles are equal by the inscribed angle theorem (blue and magenta).

The first pair of similar triangles has one vertex with the black dotted angles, and a second vertex with blue dots. Taking the opposite sides in the same order, we have the ratios:

$$\frac{n'}{b} = \frac{k}{a} = \frac{d}{m}$$

The second pair of similar triangles contain a vertex consisting of the black dotted angle *plus* the central angle between the two black dots. Refer to the wikipedia figure if this sounds confusing. There these angles are marked with red arcs.

This pair of triangles also has a second vertex with magenta dots. Taking the opposite sides in the same order we have

$$\frac{n''}{c} = \frac{k}{d} = \frac{a}{m}$$

The second trick is to pick the right relationships to manipulate. We know we don't want $k$ in the answer, (and we do want all of $abcd$ plus $mn'$ and $n''$), so choose from the first:

$$\frac{n'}{b} = \frac{d}{m} \quad \Rightarrow \quad bd = mn'$$

and from the second:

$$\frac{n''}{c} = \frac{a}{m} \quad \Rightarrow \quad ac = mn''$$

Simply add the two equations

$$ac + bd = m(n' + n'') = mn$$

□

## difference of sines

The theorem of the broken chord can be used to derive the formula for the sine of the difference of two angles.

We first recall fundamental result about chords.

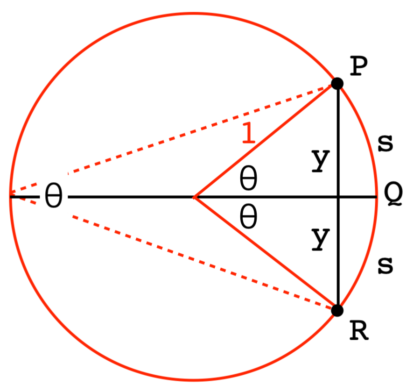
*Lemma.*

Consider any chord of a circle. Place another point on the circle in the larger arc to form a triangle.. All such triangles have the same peripheral angle, by the inscribed angle theorem.

Then choose the vertex for the peripheral angle such that it is bisected by a diameter of the circle and form the central angle corresponding to the same chord.

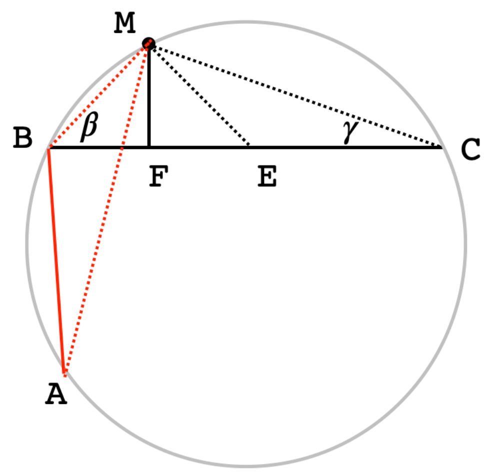
One-half of that central angle, equal to the peripheral angle, has a sine that is equal to one-half the chord.

In other words, the chord corresponding to any peripheral angle is twice the sine of that angle.




We now derive the formula for the sine of a difference of angles.

*Proof.*




Consider $\angle MBF$ at vertex $B$, let us call that angle $\beta$ for convenience.

Then the lemma says that

$$MC = 2\sin\beta$$

Similarly, let us label the angle at vertex $C$ as $\gamma$. Then

$$BM = 2\sin\gamma$$

Now, we can also use the right triangle to find

$$\cos\beta = \frac{BF}{BM}$$

so

$$\begin{aligned} BF &= BM\cos\beta \\ &= 2\sin\gamma\cos\beta \end{aligned}$$

And

$$\cos\gamma = \frac{FC}{MC}$$

so

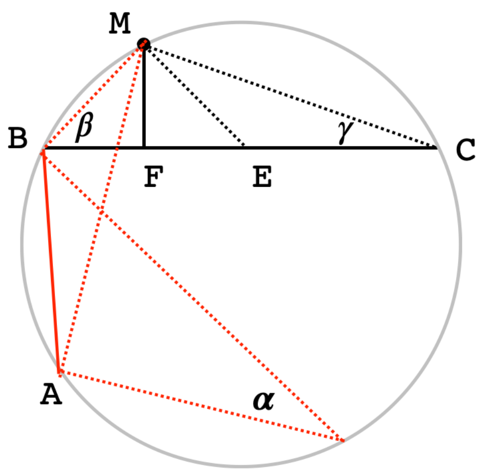
$$\begin{aligned} FC &= MC\cos\gamma \\ &= 2\sin\beta\cos\gamma \end{aligned}$$

which starts to look familiar.

By the theorem of the broken chord:

$$FC - BF = AB$$

$$AB = 2\sin\beta\cos\gamma - 2\sin\gamma\cos\beta$$




What can we do with $AB$? We also know the sum of arcs, namely

$$\text{arc } AB + \text{arc } BM = \text{arc } MC$$

which means that if $\alpha$ is any peripheral angle subtended by arc $AB$:

$$\alpha + \gamma = \beta$$

$$\alpha = \beta - \gamma$$

and by the lemma

$$AB = 2\sin\alpha = 2\sin(\beta - \gamma)$$

So finally,

$$2\sin(\beta - \gamma) = 2\sin\beta\cos\gamma - 2\sin\gamma\cos\beta$$
$$\sin(\beta - \gamma) = \sin\beta\cos\gamma - \sin\gamma\cos\beta$$

□

Sine is an odd function ($f(x) = -f(-x)$), so $\sin -x = -\sin x$.

Cosine is even so $\cos x = \cos -x$. Thus,

$$\sin(\beta + \gamma) = \sin\beta\cos\gamma + \sin\gamma\cos\beta$$

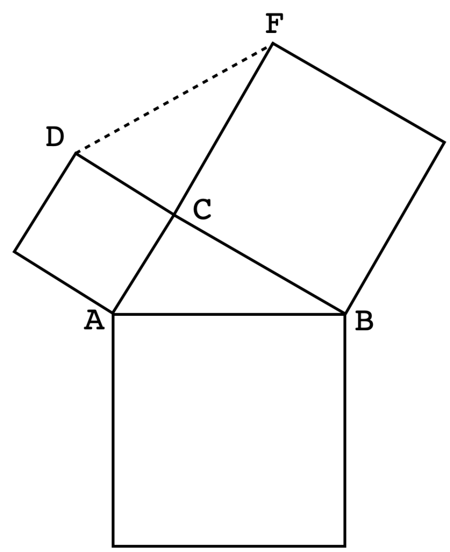
# Chapter 35

# Two final proofs

## Condit proof

Here is a proof by a student named Ann Condit, who was a sophomore in high school at the time. It is one that Euclid would approve of.

We start by drawing the right $\triangle ABC$ and the squares on each side, as with Euclid $I$.47.

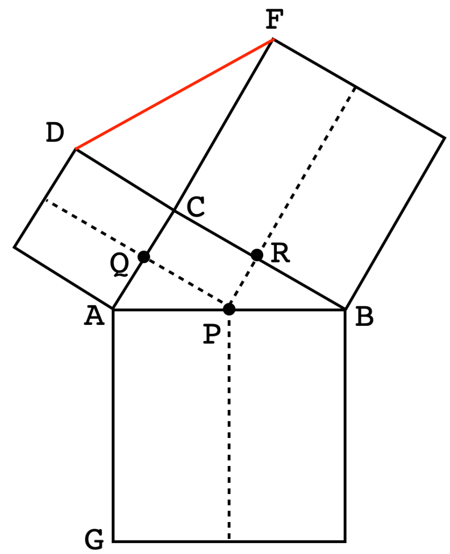



Next, join two nearby points on the two smaller squares to form $DF$ as shown.

(1) $\triangle CDF$ is congruent to the original triangle $ABC$, by SAS.

We number our conclusions to keep track of everything, because the proof is a bit longer than many.

Next, find the midpoints of each side of $\triangle ABC$ at $P$, $Q$ and $R$.



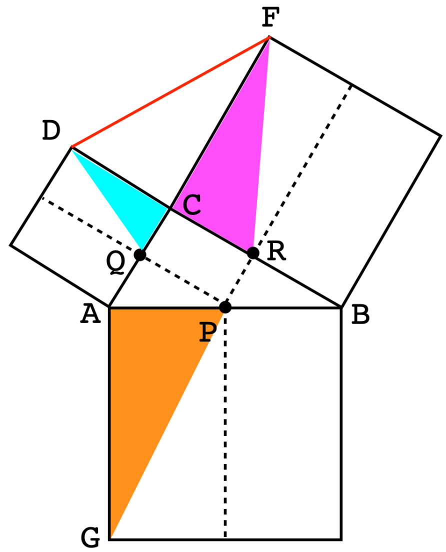

The line segment joining the midpoints of opposite sides forms a smaller triangle similar to the larger one. $\triangle BPR$ and $\triangle APQ$ are both similar to $\triangle ABC$ and congruent to each other. Therefore

(2) $PQ$ and $PR$ form right angles with the sides $AC$ and $CB$, respectively.

(3) Thus, $CRPQ$ is a rectangle (four right angles), so opposing sides are equal.

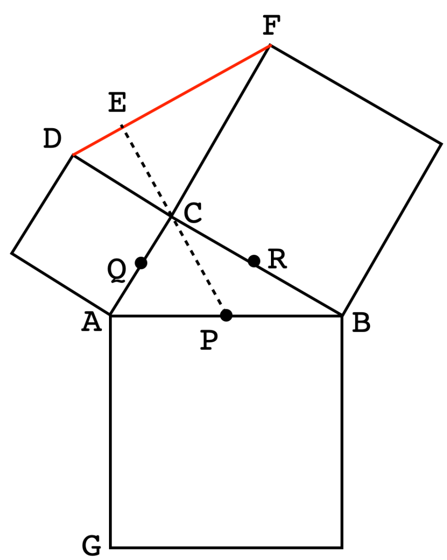
We will erase the dotted lines to reduce clutter, but remember the essential property that $P$, $Q$ and $R$ are midpoints; as well as that $QP$ and $CR$ are parallel sides of a rectangle and equal in length.

We will show that the area of $\triangle APG$ is the sum of the area of $\triangle DQC$ plus that of $\triangle FRC$.




Since each triangle is one-fourth the area of its respective square, this proves that the sum the two smaller squares is equal in area to the largest square. This is the theorem of Pythagoras.

We need one more line segment.




Draw $PC$ and extend it to the opposite side $DF$ ending at $E$.

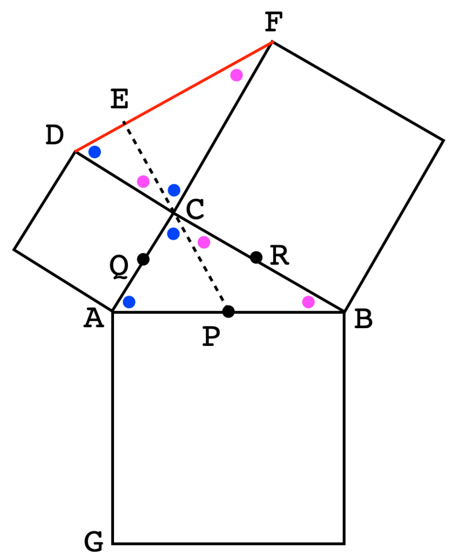
## main idea of the proof

We claim that $PC$ extended to $CE$, is perpendicular to $DF$.

Recall that when the side of the hypotenuse is bisected in a right triangle, as we did by drawing $CP$, the two resulting smaller triangles are both isosceles.

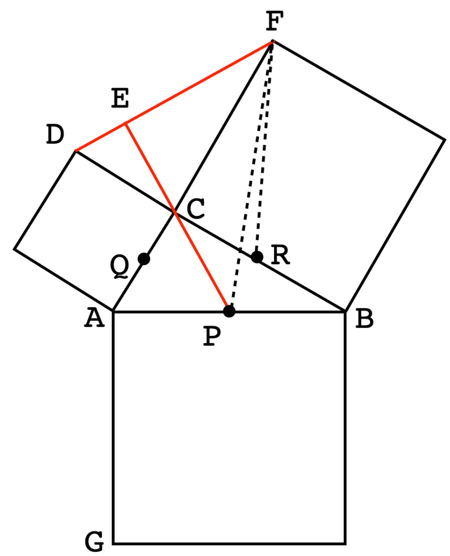
(4) Because of this, $\angle ACP$ is equal to the angle at the vertex $A$ ($\angle CAB$) of the original $\triangle ABC$. $\angle ACP$ is also equal to $\angle EDC$, by congruent triangles.

(5) $\angle PCB$ is equal to the angle at vertex $B$ ($\angle CBA$) as well as to $\angle EFC$, for the same reason.




(6) Therefore $\triangle CDE$ is similar to $\triangle CEF$, which means that the angles at $E$ are identical and so are right angles.

This is the key step of the proof. Now we just look at the areas of different triangles.




First, the area of $\triangle FRC$ (one-quarter of the square of $BC$) is equal to the area of $\triangle FCP$, because they share the base $CF$, and $QP$ is parallel to $CR$ and equal in length to it (opposite sides of a rectangle).

But the area of $\triangle FCP$ is equal to $PC \cdot EF$, because $EF$ is the altitude of $\triangle FCP$ (since $\angle FEP$ is a right angle).

By symmetry, the one-quarter of the small square's area is equal to that of $PC \cdot DE$, so together they equal

$$PC \cdot DE + PC \cdot EF = PC \cdot DF$$

But we have that $AP = CP$ and $AG = AB = DF$, so they have the same base and the same height and therefore the same area.

Thus, the two triangles together have the same area as the triangle with base $AG$ and height $AP$.

$\square$

## yet another proof

Recently (April, 2023) I came across an article by Keith McNulty, referenced on Hacker News. It's an analysis of a very recent proof of the Pythagorean theorem by two teenagers from New Orleans named Calcea Johnson and Ne'Kiya Jackson.

https://keith-mcnulty.medium.com/heres-how-two-new-orleans-teenagers-found-a-new-proof-of-the-pythagorean-theorem-b4f6e7e9ea2d

I don't have a reference to an original publication, the article simply refers to a presentation at a meeting.

McNulty claims that the proof involves trigonometry and that this is novel — "[it] might make a few established mathematicians eat their words...because their proof uses trigonometry."

Trigonometry contains two sorts of theorems, those which explicitly depend on the Pythagorean theorem, and those which do not, including some which use similar triangles. (We hedge the statement because of the complication that similar triangles can be used to prove the theorem).

The law of cosines is an example of a theorem that depends on Pythagoras, while the law of sines does not. Naturally, one could not use anything which depends on the Pythagorean theorem to prove the theorem, that would be circular logic.

What this proof mainly does is to use similar triangles, plus the law of sines. Therefore I think McNulty's claim is wrong.

It's still worth taking a look. The really novel part is the use of infinite series. We're going to use a geometric series:

$$S = 1 + r + r^2 + \cdots + r^n + r^{n+1} + \dots$$

The sum of the first $n$ terms of such a series is

$$S_n = 1 + r + r^2 + \cdots + r^n$$

Provided $|r| < 1$, as $n$ gets large, the terms beyond $r^n$ become negligible, so $S_n$ gets closer and closer to the true sum, $S$. Now,

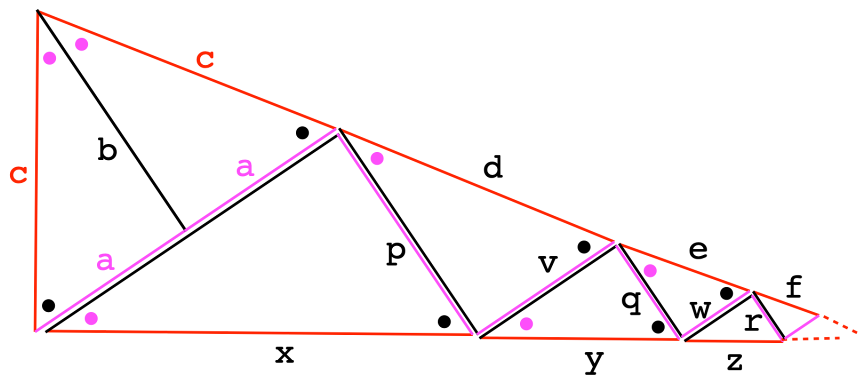
$$(1 - r) \cdot S_n = 1 - r^{n+1}$$

a classic "telescoping" series, and if $r^{n+1}$ and subsequent terms are small enough then

$$S \approx S_n \approx \frac{1}{1 - r}$$

## construction

The idea of the new proof is that a right triangle with angle $\theta$ (magenta dot) can be combined with scaled versions of itself to give a different right triangle with angle $2\theta$, as shown in the figure.




All the smaller triangles are similar. We will work out the lengths of these pieces in terms of the original triangle's sides.

Let the secant ($\sec\theta$) be $S = c/b$ and the tangent $T = a/b$. Then the small triangles have sides:

$$\begin{aligned} x &= 2aS, & p &= 2aT \\ d = pS &= 2aST, & v &= pT = 2aT^2 \\ y = vS &= 2aST^2, & q &= vT = 2aT^3 \\ e = qS &= 2aST^3, & w &= qT = 2aT^4 \\ z = wS &= 2aST^4, & r &= wT = 2aT^5 \end{aligned}$$

The last value to check is

$$f = rS = 2aST^5$$

Let us call the top side of the whole large right triangle $C$ and the bottom side $A$. We have that

$$A = x + y + z + \dots$$

$$C = c + d + e + f \dots$$

Looking at the above results, A and C (minus the first term, $c$) are geometric series with the same ratio, $T^2$.

$$x = 2aS, \quad y = 2aST^2, \quad z = 2aST^4$$
$$d = 2aST, \quad e = 2aST^3, \quad f = 2aST^5$$

As we said, the sum of a geometric series with ratio $r$ and initial term 1 is $1/(1-r)$, provided that $|r| < 1$. Here $T = a/b$ If $b > a$ then $T > 1$ and then $1/T^2 < 1$, so the series converges. If $a > b$ then just swap $a$ and $b$. We deal with the case $a = b$ at the end.

The sum for initial term $k$ is

$$S = k \cdot \frac{1}{1-r}$$

This series has ratio $T^2$ so its sum is

$$A = 2aS \cdot \frac{1}{1-T^2}$$

Recalling that $S = c/b$ and $T = a/b$ so

$$aS = ac/b = cT$$

and then

$$A = 2cT \cdot \frac{1}{1-T^2}$$

The other series is

$$d + e + f + \cdots = 2aST \cdot \frac{1}{1-T^2}$$

Using again the relation $aS = ac/b = cT$ we obtain

$$d + e + f + \cdots = \frac{2cT^2}{1-T^2}$$

The whole of the top side $C$ is $c + d + e + f + \ldots$

$$C = c + \frac{2cT^2}{1-T^2}$$

$$= c\ (1 + \frac{2T^2}{1-T^2}) = c \cdot \frac{1+T^2}{1-T^2}$$

We can finally form the ratio $A/C$, canceling the factor of $1-T^2$.

$$\frac{A}{C} = \frac{2cT}{c(1+T^2)}$$

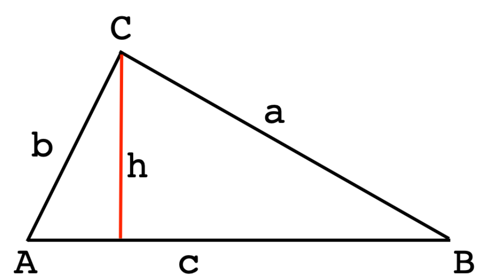
$$= \frac{2a/b}{1+a^2/b^2}$$

$$= \frac{2ab}{a^2+b^2}$$

The ratio $A/C$ is also the sine of the double angle $2\theta$.

## law of sines

In any triangle, drop the altitude from vertex $C$ to the opposite side.




$$\sin A = \frac{h}{b}$$

$$\sin B = \frac{h}{a}$$

So

$$b \sin A = a \sin B$$

The law of sines follows:

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

Apply the law of sines to the isosceles (double) triangle with angle $2\theta$ and sides $c$ and $2a$:

$$\frac{\sin 2\theta}{2a} = \frac{\sin(90-\theta)}{c} = \frac{\cos\theta}{c} = \frac{b}{c^2}$$

$$\sin 2\theta = \frac{2ab}{c^2}$$

Substituting the ratio of $A/C$ for the left-hand side:

$$\frac{2ab}{a^2+b^2} = \frac{2ab}{c^2}$$

The Pythagorean theorem immediately follows.

Finally, we must handle the case where $a = b$. Then the small triangles are all isosceles as well. So $\theta = 45°$ and $2\theta = 90°$ and a square is formed with sides $c$.

The ratio of opposing sides $C/A = c/c = 1 = \sin 2\theta$.

The law of sines gives

$$\frac{\sin 2\theta}{2a} = \frac{\sin\theta}{c}$$

Since $\sin 2\theta = 1$ and $\sin\theta = a/c$

$$\begin{aligned} \frac{1}{2a} &= \frac{a/c}{c} \\ c^2 &= 2a^2 \\ &= a^2 + b^2 \end{aligned}$$

$\square$

# Chapter 36

# Resources

I have given very few problems, and solved nearly all of those, but you will need to do a lot of problems to really learn the material.

Start with Simmons. He summarizes Algebra, Geometry, Analytic Geometry and Trigonometry in a little more than one hundred pages. The book can easily be found used, and I highly recommend it.

David Acheson's books are wonderful, including *The Wonder Book of Geometry*.

Another resource that I enthusiastically recommend is a website with a highly annotated version of Euclid's *Elements*.

https://mathcs.clarku.edu/~djoyce/elements/elements.html

You owe it to yourself to work your way through the first book, at least.

Here is an old textbook by Hopkins that I found online:

https://archive.org/details/inductiveplanege00hopkrich

As a unique feature it contains more than a dozen entrance examinations for major colleges from the years around 1900.

Two examples from Dartmouth:

## DARTMOUTH — 1900.

1. Prove that the three perpendicular bisectors of the sides of any triangle meet in a common point.

2. Circumscribe a circle about a given triangle.

3. Construct a circle having its center in a given line, and passing through two given points.

4. Prove, algebraically, that the square of the side opposite the obtuse angle of an obtuse-angled triangle is equal to the sum of the squares of the other sides plus the product of one of these sides and the projection of the other side upon it.

5. If the side of an equilateral triangle is $a$, find its area.

6. Prove that the area of a circle is equal to half the product of its radius and circumference.

7. The radius of a circle is 2; find the area of the regular inscribed dodecagon.

8. The radius of a circle is $R$; what is the radius of a concentric circle which divides it into two equivalent parts?

## DARTMOUTH — 1901.

1. Prove that the diagonals of a parallelogram bisect each other. When are they equal?

2. How many degrees in one angle of a regular decagon? of a regular dodecagon? What is the largest number of degrees possible in one angle of a regular polygon?

3. One angle between two chords intersecting within a circle is 50°; its intercepted arc is 10°. How many degrees in the arc intercepted by its vertical angle?

4. Prove that the bisector of an angle of a triangle divides the opposite side into segments proportional to the sides of the angle.

5. The diagonals of a rhombus are 9 and 12. Find its perimeter and area.

6. In a circle whose radius is $R$, show that the area of the inscribed square is $2R^2$, and of the inscribed regular dodecagon is $3R^2$.

7. Draw the tangents to a circle from a point outside the circle, and prove that they are equal.

# Chapter 37

# List of theorems

The geometry book has recently been split into parts. If a proof is listed here but the link is broken, it's very likely to be in a different volume. I have left the broken ones in for reference.

## proofs of the Pythagorean theorem

- **Pythagorean Thm: Condit**
- **Pythagorean Thm: Tuan**
- **Pythagorean Thm: Quorra corollary**
- **Pythagorean Thm: converse**

## proofs from Euclid

- **construct an equilateral triangle** (Euclid I.1)
- **side angle side (SAS)** (Euclid I.4)
- **isosceles triangle theorem** (Euclid I.5: equal sides → angles)
- **isosceles triangle theorem converse** (Euclid I.6: equal angles → sides)
- **Preliminary to SSS** (Euclid I.7)
- **angle bisection** (Euclid I.9)
- **perpendicular bisector** (Euclid I.10)
- **perpendicular through a point** (Euclid I.11)
- **perpendicular to a point** (Euclid I.12)
- **external angle inequality** (Euclid I.16)
- **longer side → larger angle** (Euclid I.18)
- **larger angle → longer side** (Euclid I.19)
- **triangle inequality** (Euclid I.20)
- **hinge theorem** (Euclid I.24)
- **ASA for congruence** (Euclid I.26)
- **line parallel to another line** (Euclid I.31)
- **parallelogram area** (Euclid I.35)
- **parallelogram complements equal** (Euclid I.43)
- **Pythagorean theorem** (Euclid I.47)
- $(x+y)(x-y) = x^2 - y^2$ (Euclid II.5)

- $(2x+y)y+x^2=(x+y)^2$ (Euclid II.6)
- **Law of cosines, obtuse case** (Euclid II.12)
- **Law of cosines, acute case** (Euclid II.13)
- **square of a rectangle** (Euclid II.14)
- **find circle center** (Euclid III.1)
- **find circle center** (Euclid III.12)
- **inscribed angle theorem** (Euclid III.20)
- **same arc → equal angles** (Euclid III.21)
- **quadrilateral supplementary theorem** (Euclid III.22)
- **equal angles, on same circle** (Euclid III.26 converse)
- **Thales' theorem** (Euclid III.31)
- **tangent-chord theorem** (Euclid III.32)
- **crossed chord theorem** (Euclid III.35)
- **tangent-secant theorem** (Euclid III.36)
- **similarity: AAA → equal ratios** (Euclid VI.2)
- **equal divisions of a line segment** (Euclid VI.9)

## proofs of other theorems

- **alternate interior angles**
- **angle bisector theorem** (right triangle)
- **angle bisector theorem** (general)
- **ASA for congruence**
- **area ratio theorem**
- **bisector equidistant from sides**
- **equidistant from sides → bisector**
- **Centroid is one-third of cevian**

- **Ceva's theorem**
- **Ceva's theorem** (Menelaus)
- **Ceva's theorem** (by area)
- **Ceva's theorem** (alternate proof)
- **Ceva's theorem by parallel lines**
- **crossed chord theorem** (product of lengths)
- **complementary angles**
- **cyclic quadrilateral** (opposing angles are supplementary)
- **cyclic quadrilateral** (converse)
- **diameter divides circle in half**
- **diameters form a rectangle**
- **equal arcs** $\iff$ **equal chords**
- **equal angles** $\iff$ **equal arcs**
- **eyeball theorem**
- **excircle theorems**
- **extended altitude theorem**
- **external angle theorem**
- **extraordinary property of the circle**
- **Heron's formula by excircles**
- **Heron's formula, Heron's proof**
- **hypotenuse-leg in a right triangle (HL)**
- **hypotenuse longest side in a triangle**
- **incenter** (incenter: angle bisectors meet at a point)
- **inscribed angle theorem** (on a circle is one-half central angle)
- **inscribed angles converse**
- **isosceles triangle theorem** (sides $\rightarrow$ angles)

- **isosceles triangle theorem** (angles $\rightarrow$ sides)
- **Law of cosines**
- **Law of cosines, algebraic proof**
- **Law of cosines, Ptolemy**
- **Menelaus' theorem**
- **midline theorem**
- **midpoint theorem** (right triangle)
- **orthocenter exists** (Newton)
- **Pappus parallelogram theorem**
- **parallelogram theorems**
- **special parallelogram theorem** (one pair of sides)
- **circumcenter** (perpendicular bisectors of a chord is diameter)
- **circumcenter** (perpendicular bisectors meet at a point)
- **Ptolemy's theorem, by cutting**
- **Ptolemy's theorem, similar triangles**
- **Ptolemy's theorem, switch sides**
- **Ptolemy's theorem, by inversion**
- **rectangle in a circle**
- **right angle is largest in a triangle**
- **tangent-secant theorem**
- **shortest distance from a point to a line**
- **supplementary angles**
- **similar right triangles**
- **midline theorem** (similar triangles)
- **similar triangles** (ratio of sides)
- **similar triangles** (right triangle composition)

- **AAA similarity theorem** (Kiselev)
- **SAS for congruence**
- **SAS inequality, hinge theorem**
- **SAS to establish similarity**
- **SSS implies SAS**
- **supplementary angle theorem**
- **supplementary angles equal to two right angles**
- **tangent theorem** (right angle → touches one point)
- **tangent-chord theorem**
- **tangent construction**
- **tangent theorem** (touches one point → right angle)
- **tangent-secant theorem**
- **Thales' circle theorem** (right angle in a semi-circle)
- **Thales circle theorem: converse**
- **triangular area**
- **triangle inequality** (triangle inequality)
- **sum of angles**
- **triangles are similar if two angles equal**
- **Varignon's theorem**
- **vertical angle theorem**

# Chapter 38

# References


- Acheson. *The Wonder Book of Geometry.*
- Byer, Lazebnik and Smeltzer. *Methods for Euclidean Geometry.*
- Coxeter and Greitzer. *Geometry Revisited.*
- Densmore. *Euclid's Elements* (annotated Heath translation).
- Dunham. *Journey Through Genius.*
- Dunham. *The Mathematical Universe.*
- Gelfand and Saul. *Trigonometry.*
- Hopkins. *Inductive Plane Geometry.*
- Jacobs. *Geometry: Seeing, Doing, Understanding.*
- Kiselev. *Geometry, Book I. Planimetry.*
- Lehocsky and Rusczyk. *the Art of Problem Solving, Volume 1: the Basics.*
- Lockhart. *Measurement.*
- Nelsen. *Proofs Without Words.*
- Posamentier. *The Joy of Geometry.*
- Simmons. *Precalculus Mathematics in a Nutshell.*
- Surowski. *Advanced High School Math*

https://www.math.ksu.edu/~dbski/index.htm