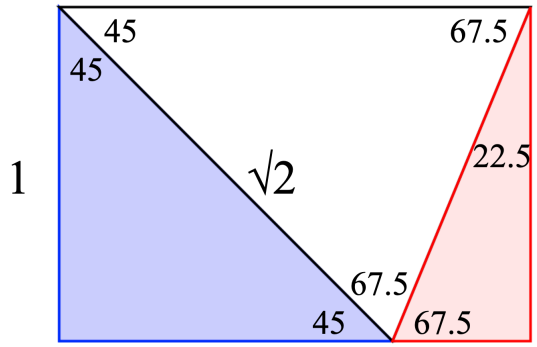
Proof Without Words: trigonometry

Here is a proof without words for $\tan 22.5^\circ = \sqrt{2} - 1$:



And a derivation. The standard double-angle formulas:

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

When $A = 22.5^\circ$, the left-hand sides are equal so

$$2 \sin A \cos A = \cos^2 A - \sin^2 A$$

$$2 \tan A = 1 - \tan^2 A$$

Let $t = \tan A$

$$2t = 1 - t^2$$

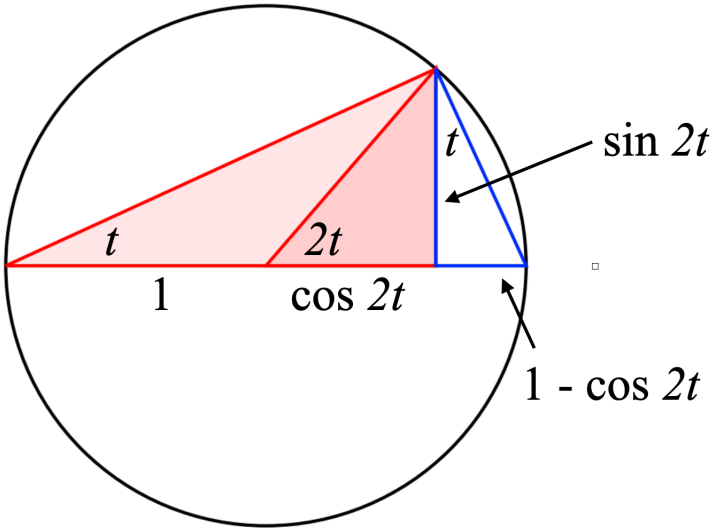
$$t^2 + 2t - 1 = 0$$

$$t = \frac{-2 \pm \sqrt{4+4}}{2}$$
$$= -1 \pm \sqrt{2}$$

We take the positive root

$$\tan 22.5^\circ = \sqrt{2} - 1$$

And here is another proof without words for the general case:



$$\tan t = \frac{\sin 2t}{1 + \cos 2t} = \frac{1 - \cos 2t}{\sin 2t}$$

When $2t = 45^\circ$, then the sine and cosine are both $1/\sqrt{2}$. Multiply on top and bottom to obtain $\tan t = \sqrt{2} - 1$.