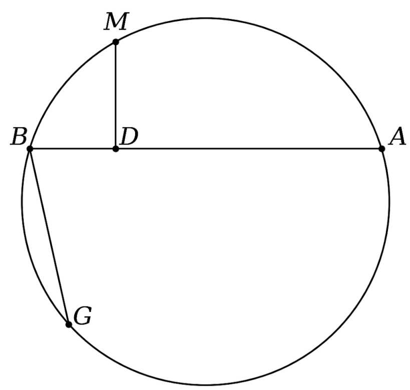
Broken Chord

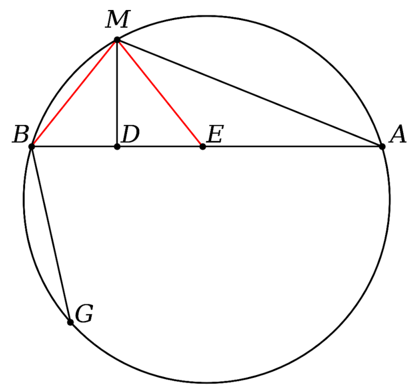
The theorem of the "broken chord" is ascribed to Archimedes, although his original work — the *Book of Circles* — has been lost. It was analyzed in proofs collected by the Arabic mathematician Al Biruni in his *Book on the Derivation of Chords in a Circle*. The theorem was not simply of academic interest, but related to the construction of tables of chords in the *Almagest* by Pappus.



Let A and G be any two points on a circle, and let M be equidistant from both, so that arc AM is equal to arc GM. It does not matter whether AMG is a major or minor arc.

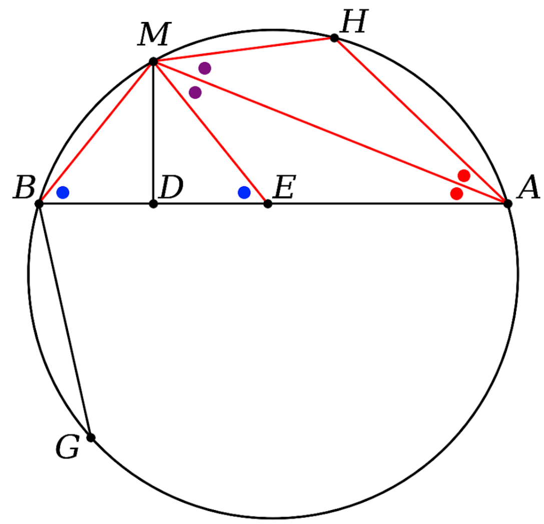
Let B be another point on the circle, lying between G and M. Drop the perpendicular from M to D on BC. The claim of the theorem is that $GB + BD = AD$.

Here we focus first on what are believed to be Archimedes' three proofs, and then continue with several more. A major motivation is that I have written a library in Python to draw planar geometrical figures. The advantage is that it is easier to keep a series of figures consistent with each other.



Starting from the figure above, we can, in addition, find point $E \mid BD = ED$. ($\mid$ means "such that"). We draw MB and ME. $\triangle MBD \cong \triangle MED$ by SAS, so $\triangle MBE$ is isosceles, with $MB = ME$. This triangle is used for several proofs.

first proof



As before, find E. The blue dots indicate equal base angles for an isosceles triangle.

Now find $H \mid MB = MH$. The red dots indicate angles equal by the well-known corollary of the inscribed angle theorem.

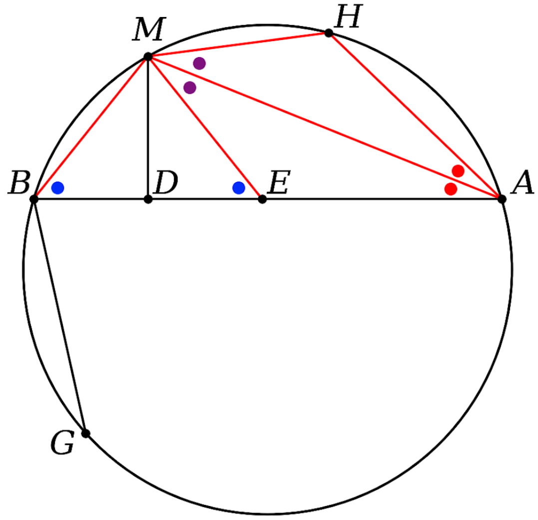
$\angle MBE$ is equal to the sum of red and $\angle HMA$ by the inscribed angle theorem.

$\angle BEM$ (equal to $\angle MBE$) is equal to red plus $\angle AME$ by the external angle theorem.

It follows that $\angle HMA = \angle EMA$. Hence the purple dots.

So $\triangle HMA \cong \triangle EMA$, by ASA.

Then $AE = AH$.



Subtract $MB = MH$ from $GM = AM$, leaving $AH = GB$. Then

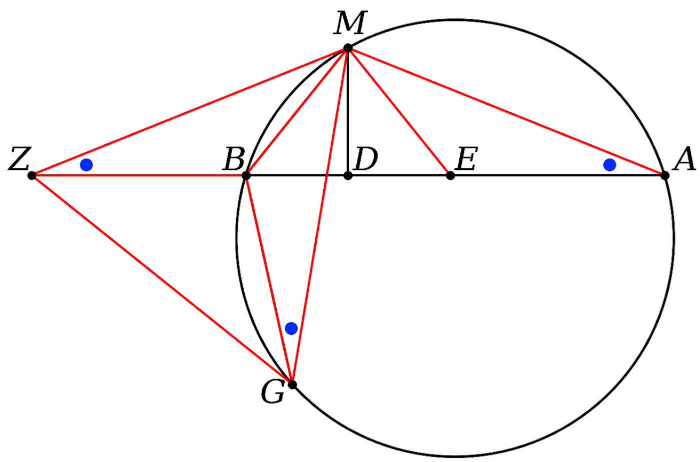
$$GB = AE$$

Adding equals

$$GB + BD = AE + ED = AD$$

$\square$

second proof



Extend AB to $Z \mid MZ = MA$.

Then $\triangle MZA$ is isosceles with $DZ = DA$ and equal base angles as blue dots. Draw GZ, GB and GM.

$\angle BGM = \angle MAB$ by the corollary of the inscribed angle theorem and then $\angle BGM = MZD$ because of the isosceles triangle.

Since $MG = MA = MZ$, $\triangle MZG$ is isosceles, and the whole $\angle Z$ is equal to the whole $\angle G$.

Subtracting the blue dots, $\triangle BZG$ is also isosceles.

It follows that

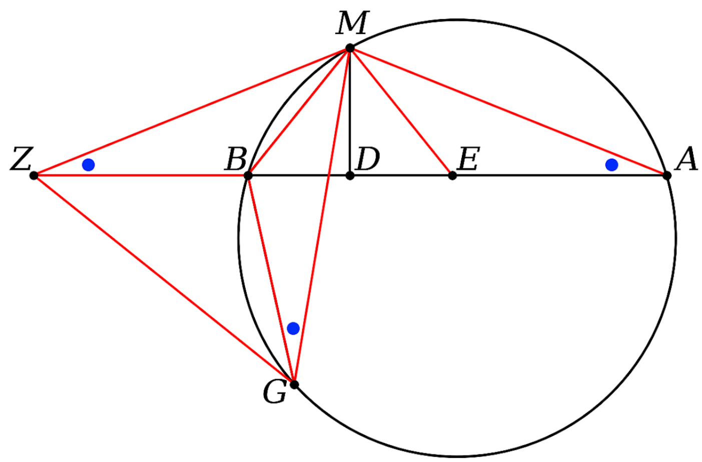
$$ZD = AD$$

$$ZB + BD = AD$$

$$GB + BD = AD$$

$\square$

proof 3



We use the same diagram as in the previous proof. We will show that $\triangle MBZ \cong \triangle MBG$.

Our construction gives $MA = MZ = MG$ and side MB is shared so in comparing $\triangle MBZ$ and $\triangle MBG$ we have SSA (side-side-angle).

As every student knows, this is not a sufficient criterion for congruence. However, once it is satisfied, if the two triangles are both acute, or both obtuse, *then they are congruent*.

$\angle ZBM$ is external to $\triangle MBD$ so it is equal to a right angle plus something more. Therefore $\angle ZBM$ is obtuse.

$\angle MBG$ corresponds to the supplement of arc MG. But $MG = MA$ and together they are less than the whole circle. It follows that $\angle MBG$ is also obtuse.

So then $\triangle ZBM \cong \triangle MBG$, which gives $ZB = GB$.

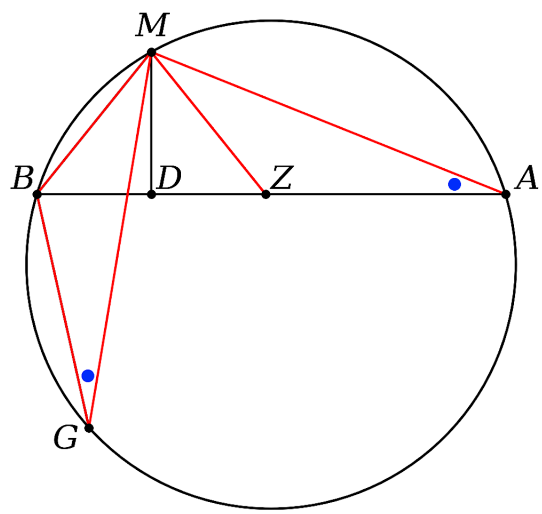
By construction

$$ZB + BD = AD$$

$$GB + BD = AD$$

□

proof 4



Find $Z \mid AZ = BG$.

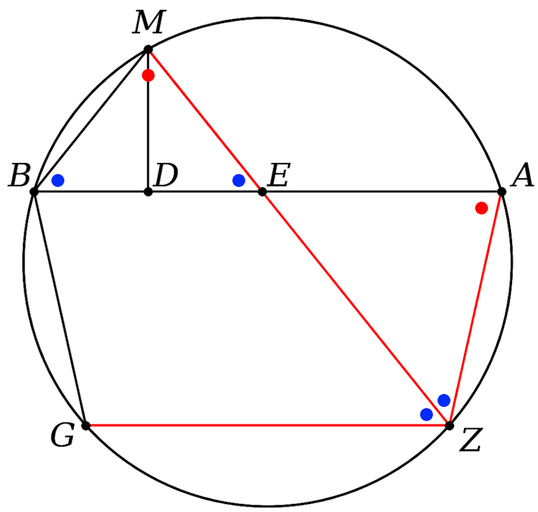
We use the point E of the isosceles triangle, but obtain it indirectly, since it turns out that Z and E are the *same point*.

Given the equal angles marked with blue dots (by the corollary of the inscribed angle theorem), we have $AM = GM$ and $AZ = GB$ and the angle in between, so $\triangle BGM \cong \triangle ZAM$ by SAS.

It follows that $MZ = MB$. (i.e. Z is the same point as E). The proof follows easily in the same way as before.

□

proof 5



Find E as part of the isosceles $\triangle MBE$ as before and then extend MB to meet the circle at Z. $\angle GZA$ is bisected, since both halves cut equal arcs, and this is the same arc cut by $\angle MBE$.

We also have vertical angles at E ($\angle AEZ$ is not yet marked).

It follows that $\triangle AEZ$ is isosceles, with $AE = AZ$. The whole angle at M is equal to $\angle A$ because they cut equal arcs, and also by sum of angles.

So then it also follows that $AB \parallel ZG$. Parallel chords in a circle cut equal arcs. (We leave this to the reader). Thus

$$GB = AZ = AE$$

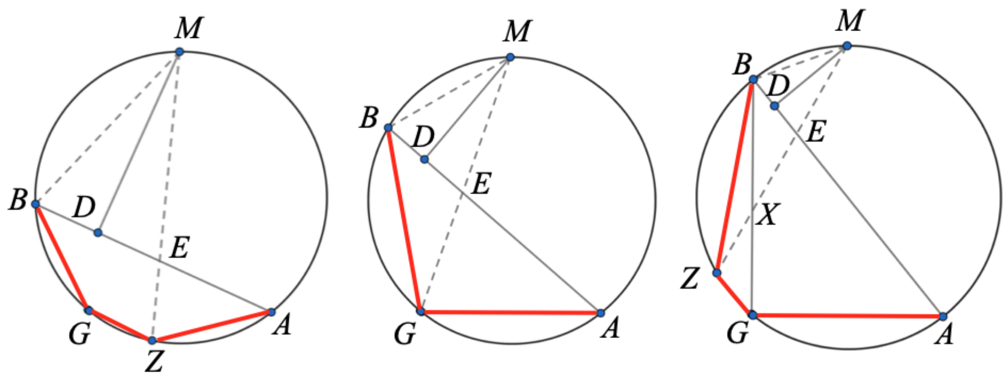
Adding equals

$$GB + BD = AE + ED = AD$$

$\square$

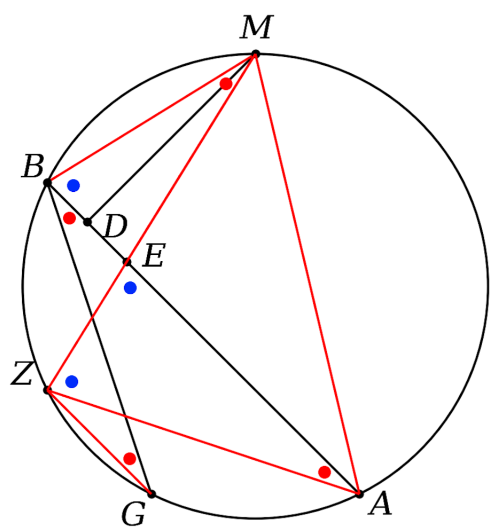
However, this proof is incomplete, since for a given arrangement of G,

A and D, depending on the placement of B, one can draw H in three different ways — coincident with G, or lying on either side of it.



The left panel is similar to our original diagram. Some of the points are labeled differently. But the important thing is that A and G are closer together, and B is fairly close to M.

As a result, in the right panel, it appears that $GH \parallel AB$. Can the proof be rescued? We redraw the standard figure with Z CW from G.



Three of the angles marked with red dots are on arc BZ.

The fourth one, $\angle ABG$, cuts arc AG, which means that it is supplementary to two copies of $\angle MBE$ each cutting the arc AM. So they are all equal.

The angles marked blue either cut the same arc or are equal by vertical angles and the isosceles $\triangle MBE$.

It follows that $GZ \parallel AB$ so $BZ = AG$ and thus $\triangle ZGB \cong \triangle GZA$.

Then $AZ = GB$ and since $\triangle AEZ$ is isosceles

$$AZ = AE = GB$$

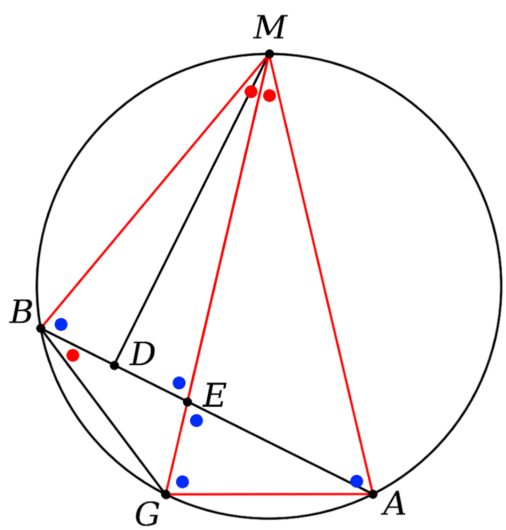
Adding equals

$$AE + DE = GB + BD$$

$$AD = GB + BD$$

$\square$

The **third case** is where the extension of DZ terminates exactly on point G. Z coincides with G.



A nice proof is to complete $\triangle MGA$. Since $GM = AM$, the triangle is isosceles, with equal base angles cutting arcs equal to that of $\angle MBE$. It follows that MG bisects $\angle AMB$.

So

$$\angle EMB = \angle EMA = \angle EBG$$

Then $\triangle ABG$ is isosceles, with $BG = AG$ and $\triangle AEG$ is isosceles as well. It follows that

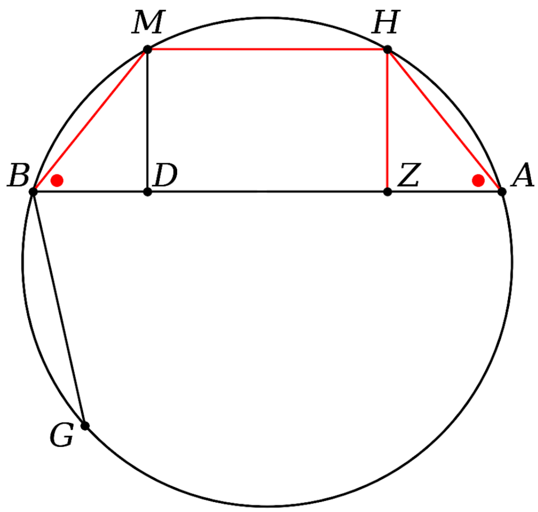
$$AG = AE = BG$$

Adding equals

$$BG + BD = AE + ED = AD$$

$\square$

proof 6



Draw the rectangle $MHZD$, by drawing $MH \perp MD$ to meet the circle at H, and then $HZ \parallel MD$.

It is easy to show that the red dotted angles are equal (draw AM and BH), and then we have two congruent right triangles in $\triangle MBD$ and $\triangle HAZ$.

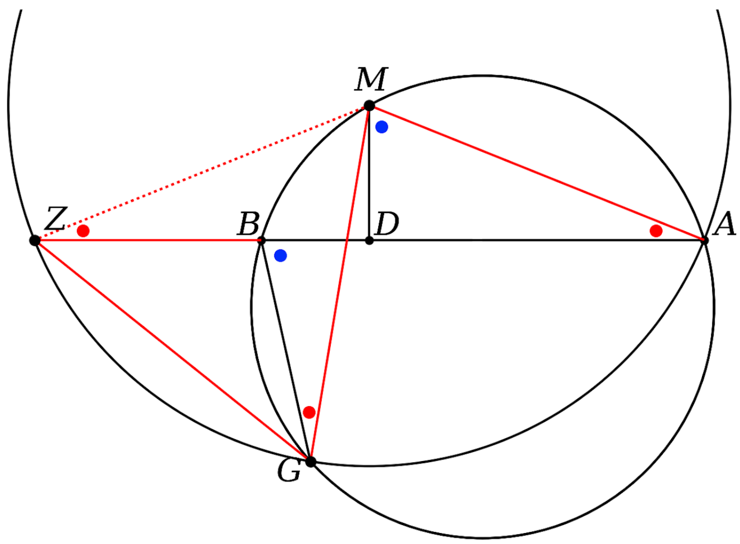
So $BD = AZ$.

Given arc $MG = MA$, and $MB = AH$, so find $MH = BG$.

$$GB + BD = MH + AZ = DZ + AZ = AD$$

$\square$

proof 7



Draw the circle on center M with radius $MA = MG$. Extend AB to meet the circle at Z.

$\angle BGM = \angle A$ by inscribed angles.

Since $MZ = MA$, $\triangle MZA$ is isosceles, with $\angle MZD = \angle A$. This explains the angles marked with red dots.

The angles marked with blue dots are equal by inscribed angles.

Since $MZ = MG$, $\triangle MZG$ is isosceles, with $\angle MZG = \angle MGZ$. $\angle BZG = \angle BGZ$ by subtraction.

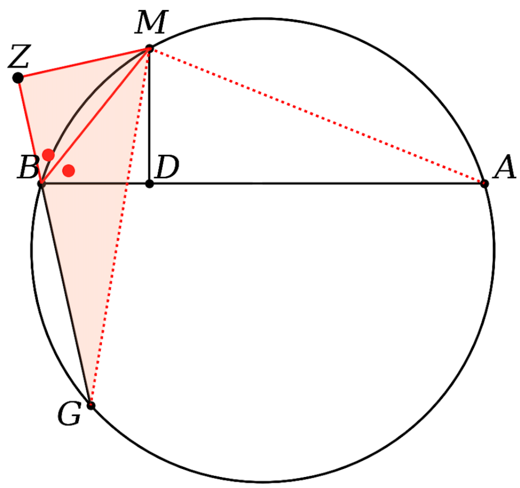
Hence, $\triangle ZBG$ is isosceles and $BG = BZ$.

Given $MD \perp AB$. AZ is a chord of the circle and MD is perpendicular through the circle center, so AZ is bisected at D.

$AD = ZD = ZB + BD = GB + BD$

$\square$

proof 8



Extend GB to $Z \mid MZ \perp GZ$.

$\angle MBG$ is supplementary to $\angle MBZ$. $\angle MBG$ lies on arc MAG. Hence, $\angle MBZ$ corresponds to arc MG.

But MG =MA and $\angle MBD$ lies on arc MA, hence the red dots for equal angles.

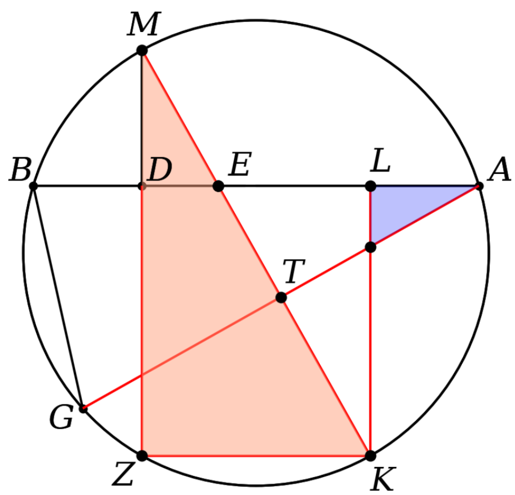
$\triangle MBZ$ and $\triangle MBD$ are right triangles, sharing a second angle and one side, so they are congruent by ASA. $BZ = BD$.

Now consider $\triangle MZG$ and $\triangle MDA$. Both are right and also $\angle A = \angle G$. Further, $MG = MA$. So they are congruent with

$$AD = GZ = GB + BZ = GB + BD$$

$\square$

proof 9



Extend MD to the circle at Z. Draw MK as a diameter of the circle.

Draw AG. Find where MK crosses AG at T.

Also draw $LK \perp AB$. So $MZ \parallel LK$.

The angle at Z is right, by Thales circle theorem, and $ZK \parallel AB$.

Thus $DLKZ$ is a rectangle with $ZK = DL$.

MK traverses parallel lines so $\angle M = \angle LKM$

$\triangle LEK$ is similar to the blue triangle, hence $\angle A = \angle LKM =< M$

But $AL = BD$ because $DLKZ$ is a rectangle in a circle. It follows that

As arcs of equal angles, $BG = ZK = DL$.

$$BG + BD = DL + AL = AD$$

$\square$