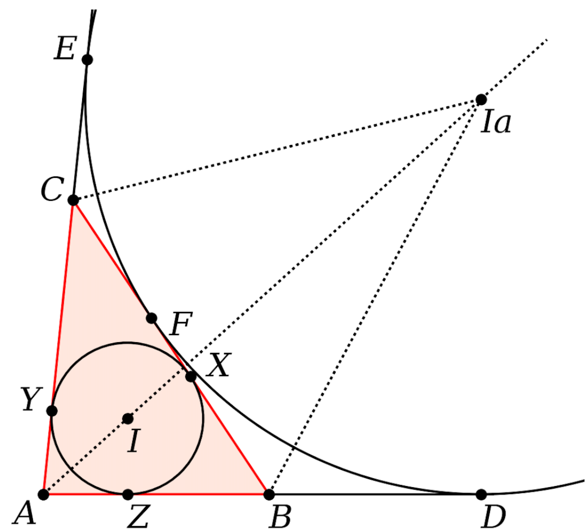
Excircles and Heron



Any triangle has an *incircle*, defined as the circle tangent to each of the three sides of the triangle. The incircle is on an *incenter* I.

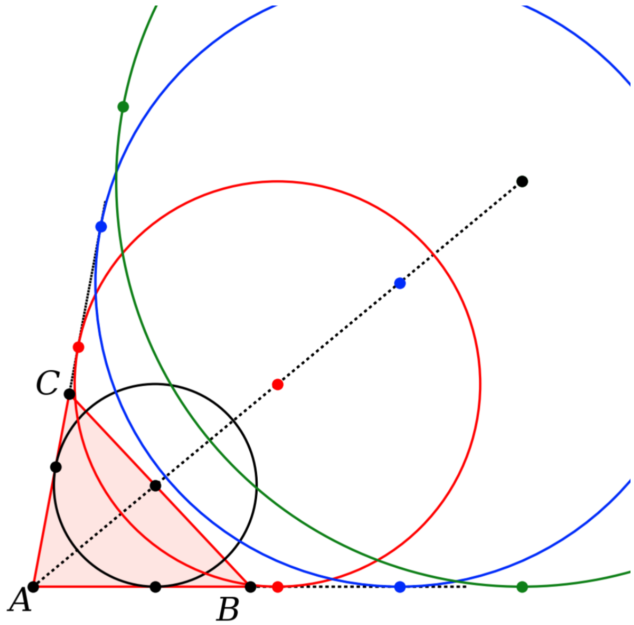
I lies on the angle bisectors of the three angles A, B and C. The points where the incircle is tangent to the sides are marked X, Y, and Z, so for example $IZ \perp AB$. The circle through X,Y and Z is tangent to each of the sides.

Proof. Find I as the intersection of two bisectors. Draw pairs of right triangles such as $\triangle IZA$ and $\triangle IYA$. These are congruent since they have two (thus three) angles the same and a shared side. So $IZ = IY = IX$. Draw the circle through X, Y and Z on center I. Since IZ et al are radii, $AB \perp IZ$ is tangent to the circle at I. $\square$

Each side of the triangle has a corresponding *excircle*. The excircle on center I_a is the circle tangent to three lines: side a as well as the extensions of sides AB and AC to D and E.

One idea for how to find the relevant points is to notice that D and E also tangents from A to the circle we are looking for. Hence $AD = AE$.

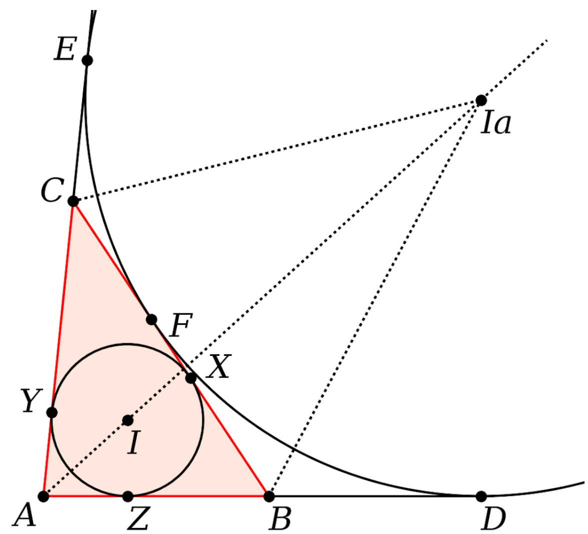
So for a candidate point D, find E the same distance from A on the extension of AC and then draw the perpendiculars to cross at I_a. It might help a little to know that $BD + CE$ is equal in length to side a. If the circle drawn on I_a with radius $r = AD = AE$ is tangent to side a, you're done.



In the figure above, a circle tangent to the extensions of AB and AC has been drawn with its center on evenly spaced points along the bisector, starting from where the bisector crosses side a. The tangent points as well as the points where the circle itself crosses the bisector, are also evenly spaced. The ratio between these distances depends only on the angle at A.

The tangent point we are looking for on BC does *not* lie on the bisector, unless $AB = AC$ (and $\triangle ABC$ is isosceles). Intuitively, there is one position for the center in which the circle just "kisses" side a.

More practically, I_a can be found first, as the intersection of the bisectors of the external angles $\angle CBD$ and $\angle ECB$. Then find D and E as points on the extensions of AB and AC whose perpendiculars go through I_a. Last, draw the circle of radius $AD = AE$.



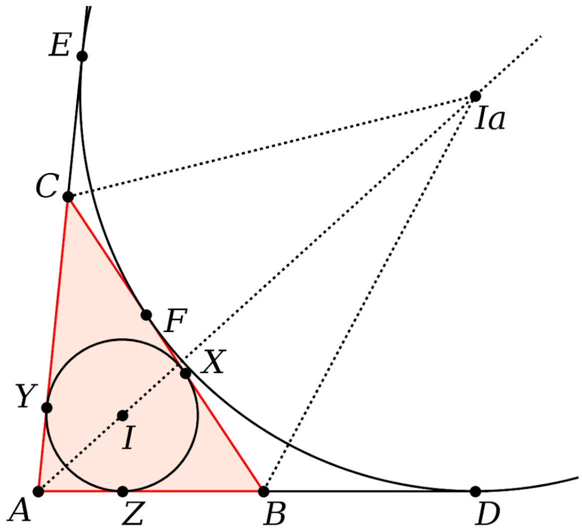
We can see that this will be correct, since $BD = BF$ as tangents from B and $CE = CF$ as tangents from C, so the bisectors of $\angle CBF$ and $\angle FCB$ will go through the center of the circle.

In what follows, we will first find a formula for the length of BF (and thus, BD, CE, and CF).

1

Let

$$AZ = AY = x \qquad BX = BZ = y \qquad CY = CX = z$$



Then let s be half the perimeter, called the *semiperimeter*:

$$2x + 2y + 2z = 2s$$

$$x + y + z = s$$

so

$$\begin{aligned} s &= AZ + BX + CX \\ &= AZ + a \end{aligned}$$

Re-arranging and generalizing

$$AZ = s - a \qquad BX = s - b \qquad CY = s - c$$

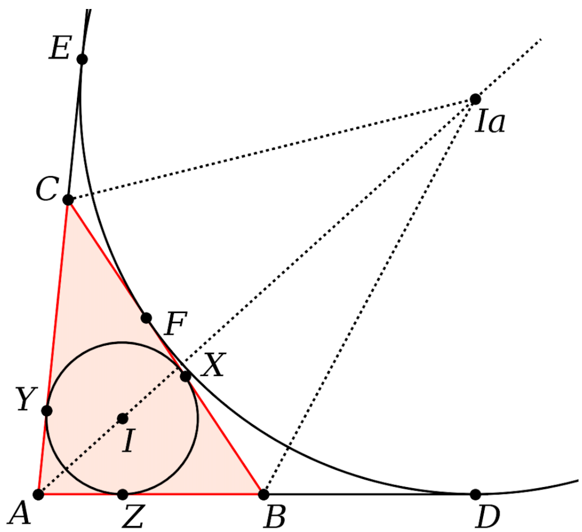
also

$$\begin{aligned} s - a &= (a + b + c)/2 - a \\ &= (-a + b + c)/2 \\ 2(s - a) &= -a + b + c \end{aligned}$$

2

Let $BD = BF = p$ and $CE = CF = q$. We see that together $p+q = a$ so

$$p = a - q$$



The two tangents from A to the excircle are also equal. We have

$$AD = AE$$

$$(s-a) + (s-b) + p = (s-a) + (s-c) + q$$

$$p - b = q - c$$

Substituting

$$a - q - b = q - c$$

$$2q = a - b + c$$

$$= 2(s-b)$$

$$q = s - b$$

Similarly

$$p - b = a - p - c$$

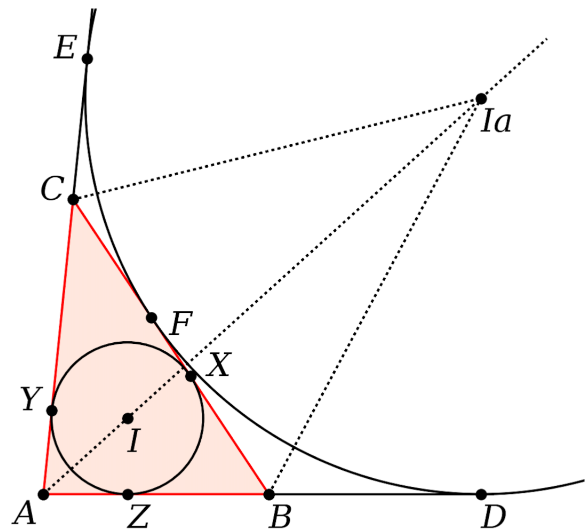
$$2p = a + b - c = 2(s - c)$$

$$p = s - c$$

Again, the two long tangents are equal:

$$AD = s - a + s - b + s - c$$

$$= 3s - 2s = s$$



The total length of the tangents $AD = AE$ is just s.

Also, the point X divides the side a into lengths $BX = s - b$ and $CX = s - c$. Now we see that since (for example) $CE = CF = s - b$, the point F (tangent to the excircle) divides the side a into lengths $BF = s - c$ and $CF = s - b$.

3

Comparing the incircle and excircle, we can find two similar right triangles: $\triangle ADI_a$ and $\triangle AZI$. The relevant ratios are

$$\frac{R}{s} = \frac{r}{s - a}$$

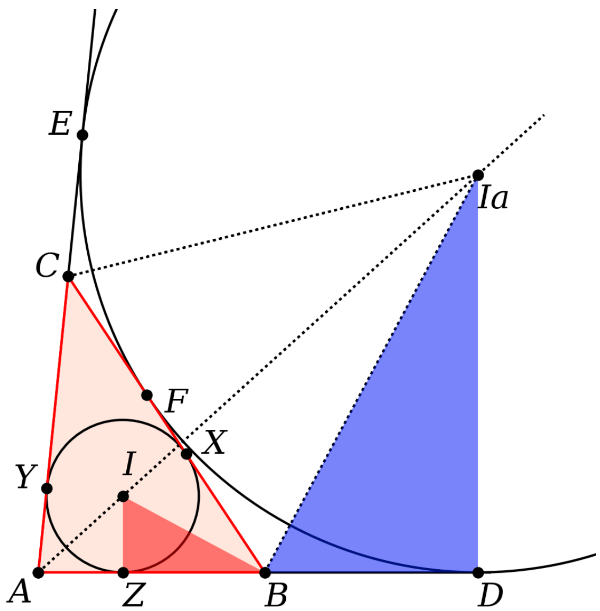
$$rs = R(s-a)$$

where $R = I_aD$ is the radius of the excircle on side a.

We have three pairs of congruent triangles so the total area of $\triangle ABC$ is

$$\mathcal{A} = rx + ry + rz = rs$$

4



It is a property of the internal and external angle bisectors that the sum of the half-angles is a right angle, since the internal and external angles are in total two right angles.

So $\angle IBI_a$ is right. It follows that $\angle IBZ$ and $\angle I_aBD$ are complementary.

Thus $\triangle IZB \sim \triangle BDI_a$ (marked red and blue in the figure). The relevant ratios are:

$$\frac{R}{s-c} = \frac{s-b}{r}$$

$$Rr = (s-b)(s-c)$$

5

We combine the last result from above with that from (3):

$$rs = R(s-a)$$

Multiplying

$$r^2Rs = R(s-a)(s-b)(s-c)$$
$$r^2s = (s-a)(s-b)(s-c)$$
$$r^2s^2 = s(s-a)(s-b)(s-c)$$

Since $rs = \mathcal{A}$ we have, finally

$$\mathcal{A}^2 = s(s-a)(s-b)(s-c)$$

□

This is Heron's theorem. The square of the area of the triangle is equal to the product on the right. As expected, the formula is symmetric in a, b and c and has the dimensions of the fourth power of a length.