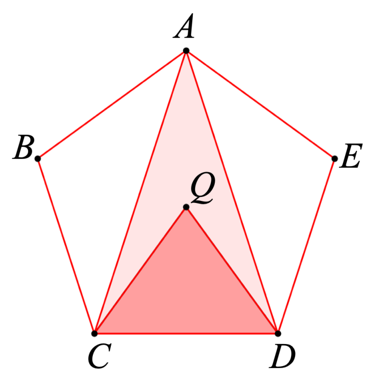
Pentagon

By the standard definition, a pentagon is a polygon having five equal sides. However, the pentagon can also be defined as being formed by five evenly spaced points on a circle, with five equal central angles that total four right angles.



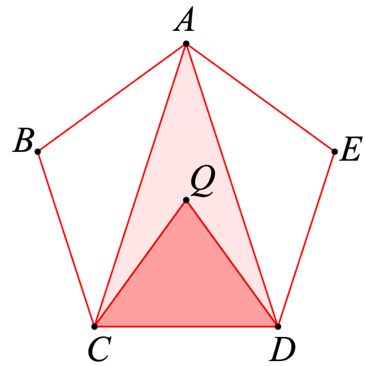
Either way, we will need more insight if we wish to construct a pentagon.

Since there are five triangles like $\triangle QCD$, the central angles are each $1/5$ of the whole circle, or $2/5$ of a triangle, 72° in modern notation. By the inscribed angle theorem, $\angle CAD$ with the same endpoints has one-half the measure, $1/5$ of a triangle.

We claim that $\triangle ACD$ is isosceles, thus, the base angles are twice the vertex angle, in the overall ratio 1:2:2.

Proof.

Since the five triangles like $\triangle QCD$ have the same central angle and the same arms, they are all congruent by SAS. It follows that the total angles at the vertices $A, B \ldots$, such as $\angle BCD$ are all equal and twice the base angles of $\triangle QCD$ such as $\angle QCD$.



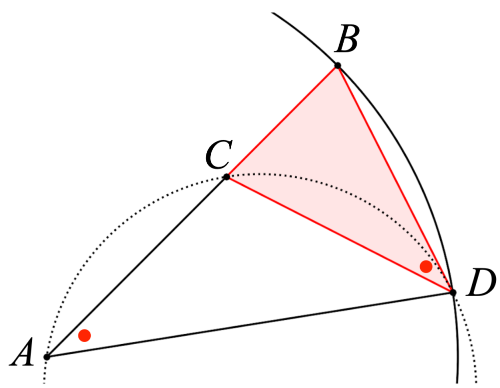
Then $\triangle ABC \cong \triangle AED$ by SAS. By subtraction, the base $\angle ACD = \angle ADC$ and $AC = AD$ by I.6.

$\square$

As a result, we have 4/5 of two right angles to be divided equally for the base angles of the isosceles $\triangle ACD$.

IV.10

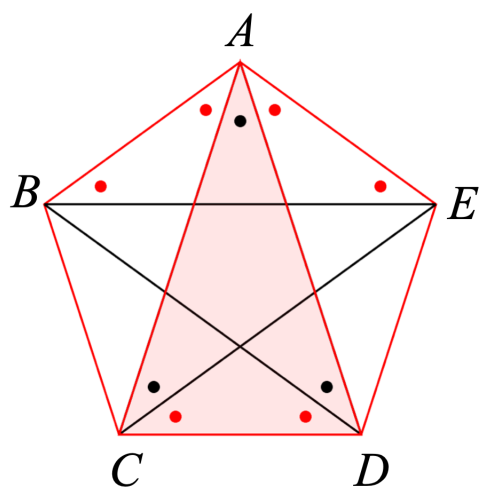
Euclid shows how to construct such a triangle in IV.10.



Briefly, point C is placed so that $AB/AC = AC/BC$; D is placed on the circle on center A with radius AB, such that the length $BD = AC$. We can use the converse of the tangent-secant theorem to show that BD is tangent to the circle containing ACD, and then $\angle BDC = \angle BAD$.

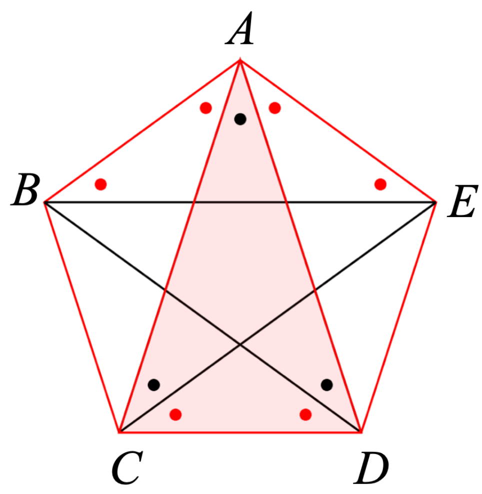
Then since $\triangle ACD$ is isosceles, $\angle ADC = \angle DAC$, so $\angle ADB$ is bisected. But $\triangle ABD$ is isosceles, so the two equal base angles are each twice the entral angle at A.

continuation



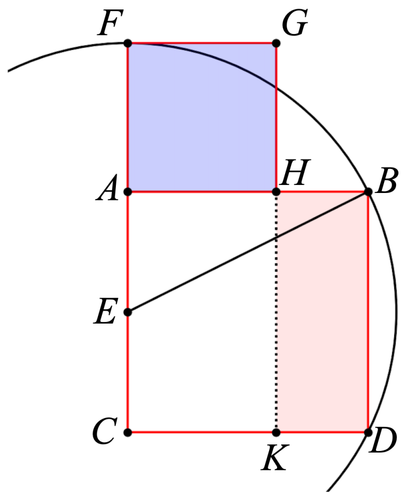
Going back to the first diagram, we can show that $\angle CAD$ is one-third the total angle at the vertex A. Since the flanking angles are equal, together AC and AD form three equal angles at vertex A.

Proof. We might count fractions of a right angle. Another approach is to use the triangle sum theorem. We showed that the base angles of $\triangle BAC$ are equal. Let them equal x (red dots). Let the angles like $\angle CAD$ have measure y (black dots). Then in $\triangle ABE$ we have $4x + y$ and in $\triangle ACD$ we have $3x + 2y$. Since these sums are equal, it follows that $x = y$. $\square$



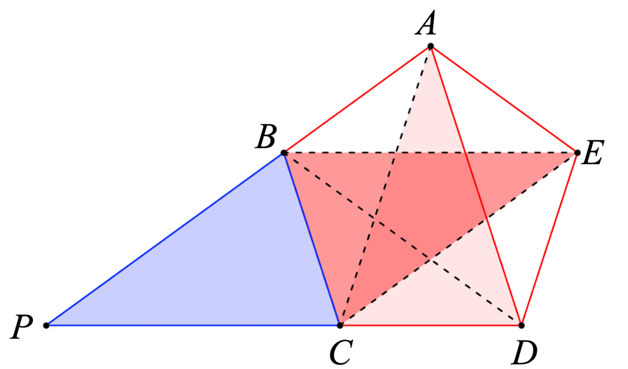
As a result, we have that the base angles of $\triangle ACD$ have the same measure as the central angle QCD in the first figure. We also have that $\angle A$ plus $\angle ABD$ is equal to two right angles, so $AE \parallel BD$, $AB \parallel CE$, $ABXD$ is a rhombus (parallelogram), and the inner figure is a regular pentagon.

In $\triangle ACD$, the ratio of the long sides to the base is the same as the golden ratio or mean, usually called ϕ, as defined by Euclid's construct above.



Go back to the construction of the golden ratio in EII.11. If AE has length 1, then $BE = EF$ has length $\sqrt{5}$. $AH = AF$ has length $\sqrt{5}-1$. $AB/AH = \phi = 2/(\sqrt{5}-1)$. Rationalizing the denominator we have

$$\phi = \frac{2}{\sqrt{5}-1} \cdot \frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{\sqrt{5}+1}{2}$$



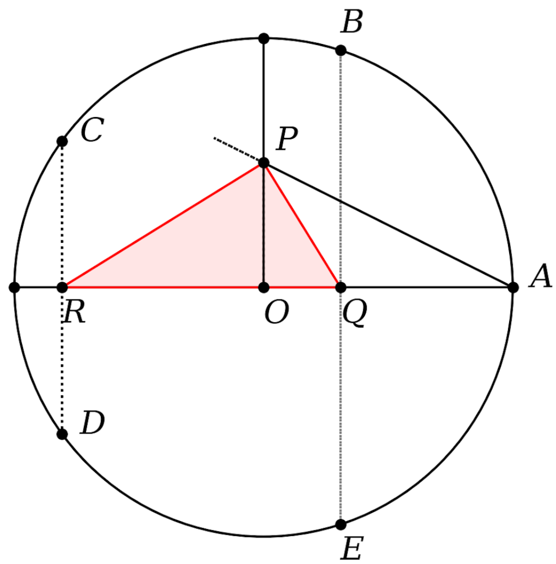
In the figure above $\triangle PBC \cong \triangle ACD$. *Proof.* $\triangle PAB$ is similar to both, and they have equal bases. $\square$

Then the ratio of one-half of BC to PB is $1/2$ divided by ϕ or $1/2\phi$. This is the cosine of $\angle PCB = \angle ACD = 72°$, the same as the central

$\angle CQD$.

We also have that the length of BE is $\phi/2$, but this same length divided by the unit side length gives the cosine of $\angle ABE = 36°$.

We can use these results to verify the correctness of the next method for construction of a pentagon.



Let the radius AO be 2, bisected at P, so $AP = \sqrt{5}$. Bisect $\angle OPA$, then $AQ/OQ = \sqrt{5}$.

So $AQ + OQ = (1 + \sqrt{5}) \cdot OQ = 2$. Thus $OQ = 1/\phi$.

$OQ/OB = 1/2\phi$ and so $\angle BOA$ is equal to the central angle in a pentagon constructed on this circle.

Finally, draw the right $\triangle QPR$. $OP^2 = 1$ and this is equal to $OQ \cdot OR$. Thus OR equals ϕ, which gives $\phi/2$ as the cosine of $\angle COR$ which is $36°$.

We have the following values for cosine:

$$\cos 36 = \frac{\phi}{2} \qquad \cos 72 = \frac{1}{2\phi}$$

We can get more values from the Pythagorean theorem, the definition ($\phi^2 = \phi + 1$), and from the double angle formula. For example:

$$\sin^2 36 = 1 - \frac{\phi^2}{4} = \frac{4 - \phi - 1}{4}$$

$$\sin 36 = \frac{\sqrt{3-\phi}}{2}$$

and

$$\sin 72 = 2 \sin 36 \cos 36 = 2\frac{\sqrt{3-\phi}}{2} \cdot \frac{\phi}{2}$$

$$\sin^2 72 = \frac{(3-\phi)(\phi+1)}{4} = \frac{3 + 2\phi - (\phi + 1)}{4}$$

$$= \frac{\phi + 2}{4}$$

$$\sin 72 = \frac{\sqrt{2+\phi}}{2}$$

which can be checked

$$\frac{2+\phi}{4} + \frac{1}{4\phi^2} = \frac{2\phi^2 + \phi^3 + 1}{4\phi^2}$$

The numerator is

$$2\phi + 2 + (\phi + 1)\phi + 1 = 4\phi + 4$$

which is equal to the denominator. We satisfy the identity $\sin^2\theta + \cos^2\theta = 1$.

Of course, now that we know sine and cosine of 36° and 72°, we also know them for 18° and 54°.

ratio of sines

We can also say something about the ratio of two sines. Recall that the ratio of the chord of a regular pentagon to the side is equal to ϕ.

Considered as part of the circumcircle, the side is a chord as well, of the circumcircle. Using the standard result that the chord is the diameter times the sine of the angle, we have that

$$\frac{\sin 72^\circ}{\sin 36^\circ} = \frac{\phi}{1} = \phi$$

Plugging in the values from our work above, we must have that

$$\phi = \frac{\sqrt{2+\phi}}{\sqrt{3-\phi}}$$

which seems hard to believe. Let us work backward. Square and rearrange:

$$\phi^2 \cdot (3-\phi) = (2+\phi)$$

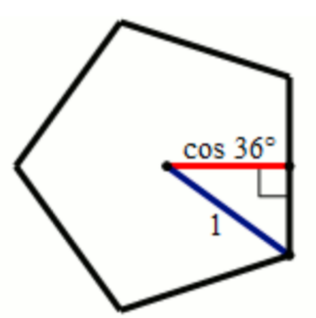
Working with the left-hand side:

$$(\phi+1)(3-\phi) = 2\phi + 3 - \phi - 1 = 2 + \phi$$

which is correct.

radius

Another important question is the radius of the circle that circumscribes the pentagon. Returning to the central angle, observe that



If the side length is s then the half side is $s/2$ and $s/2r$ is the sine of the central angle, $\sin 36°$ so the ratio is

$$\frac{s}{r} = 2\sin 36 = \sqrt{3-\phi} = \sqrt{\frac{5-\sqrt{5}}{2}}$$