Geometry by proof

Tom Elliott

January 25, 2022

Contents

Part I

Introduction

Chapter 1

Forward

This is a concise introduction to geometry, and weighs in at a little over a hundred pages, or it did before I added a couple new things. It has hand-drawn figures, in homage to Lockhart's *Measurement* and Acheson's *The Wonder Book of Geometry*.

And although there are some extra words in the first few chapters, effectively all that we do is prove theorems. By page 34 we have all the big ones. We are done with the basics of Euclidean geometry by page 120 or so.

However, the last third of the book has significantly more advanced material, so you may find it a bit challenging if you are really just beginning.

There aren't any problems, *per se*. I suggest that when you finish reading a chapter, go back and prove all the theorems again, without peeking. Or even better, try to do them before you even read the chapter!

My hope is that this book will bring pleasure to anyone who reads it, but especially those who write out the proofs themselves.

Chapter 2

Parallel

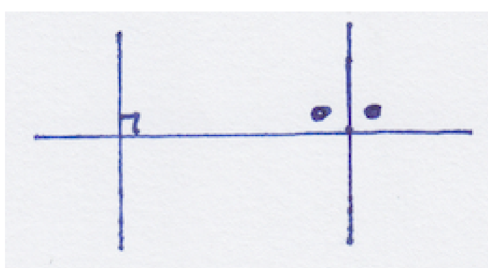
right angles

When two lines cross, they form four angles.

∘ If the angles adjacent to each other (on one side of any line) are equal, then we say that those angles are both right angles.

For a Greek mathematician, this is the definition of a right angle. If adjacent angles are equal, both are right angles. They also had an axiom about them: that *all right angles are equal*.

We say that two lines which cross to form right angles are *perpendicular*.



I'm sure you've been taught that the *measure* of a right angle is 90°. But the Greeks never used degrees. We will write 180 when we mean two right angles, and perhaps, 90 when we mean one, only because it is more concise.

In the figure above, on the left, a standard symbol for a right angle is shown: a small square. On the right, we have marked two equal angles with dots. This is much cleaner than the standard notation, which uses arcs and bars. It's even better when you can use colored circles.

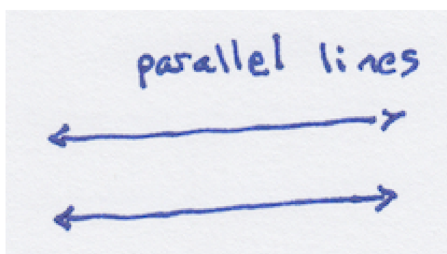
When one of the four angles formed by two lines crossing is a right angle, all four of them are right angles.

Each angle above is adjacent to two different angles, so both of those are equal right angles. And in turn they are both adjacent to the fourth angle, so it is a right angle as well.

parallel lines

∘ If two lines are drawn so that they never cross, they are parallel lines.

Arrowheads indicate that a line continues in both directions, forever.



Parallel lines never meet. They are the same distance apart, everywhere along their lengths.

This leads to the question of how to measure the distance between two lines. That's actually a tricky question so we will not try to answer it now.

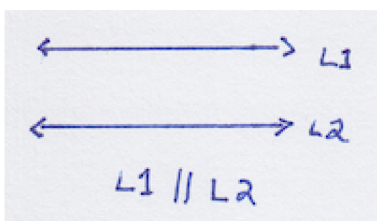
Let's just say that we choose a point on one line, then measure the distance to the second line along a perpendicular from the first. Move the first point over a bit and repeat the measurement. If it changes, the two lines are not parallel.

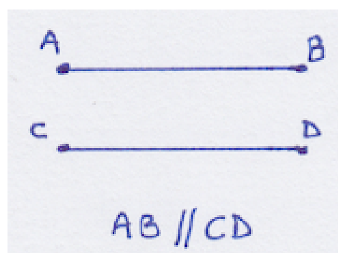
Suppose the second line is 1 foot from the first at given point, but further along in a particular direction, say 1 mile, we measure 11 inches instead.

Why would this be problem? Since a line goes forever, if we travel 12 miles that down the road rather than one, the two lines that we supposed were parallel, would meet. We must conclude that they are not parallel after all.

○ Two parallel lines never meet and they stay exactly the same distance apart, forever.

We can make a similar statement about line segments. Here $L1 \parallel L2$ and $AB \parallel CD$.





We mean that, if both line segments were extended to be lines, they would never cross.

An open circle (○), is used to mark an important statement, an assumption or *axiom* about the world. We will need a few of those, to get started.

We also assume *common notions*, such as that equals added to equals are equal, and that a thing is equal to itself. We gladly accept these as well, without much fanfare.

Other statements are *theorems*, marked with a $\bullet$, truths that we have deduced in a particular situation shown by a diagram. In the next chapter we get started on proving theorems. We will not stop until the end.

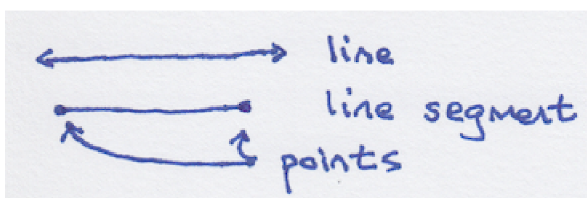
Chapter 3

Angles

lines and angles

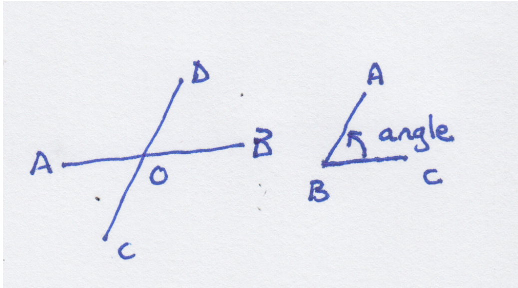
We start with line segments: a section of a line, which is delimited by two points. Many geometric shapes are built from them.

A line segment is "straight", which is defined as the "shortest distance between the two end points of the line segment.



Technically, if a line segment is extended (to infinity) in both directions it is called a line and should be represented using arrows on the ends of the part that we draw. We won't often do that.

Sometimes I may refer to a line segment as a line, as a shorthand, especially when a given statement is true for both lines and line segments.



On the left above, four angles are formed when two lines cross at the point O. An angle can also be constructed from line segments.

On the right, the angle ABC, written $\angle ABC$. has its *vertex* at the point B. If there is only one angle for a vertex, I would rather refer to it as $\angle B$ rather than $\angle ABC$. This is not standard usage, but I think it's easier to comprehend.

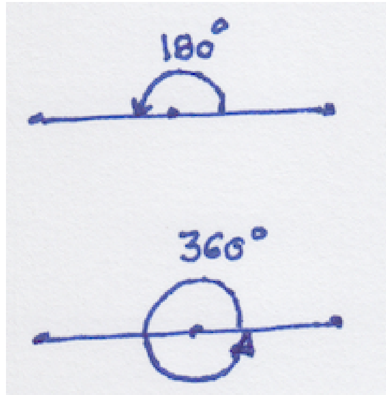
Angles have measure, a number which describes how "large" they are. In the figure above, $\angle AOD$ is larger or "greater" than $\angle DOB$.

The sum of all the angles around a given point is equal to four right angles. In a different system of measurement, it is 360° or just 360. We are not going to use the degrees symbol ° here. We say that 360 is equal to four right angles.

The choice of the number 360 is ancient. It's tempting to relate it to the length of the year (approximately 360 days). Probably it has as much to do with the fact that $2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20$ and so on are all factors of 360, so it can be divided up evenly in many ways.

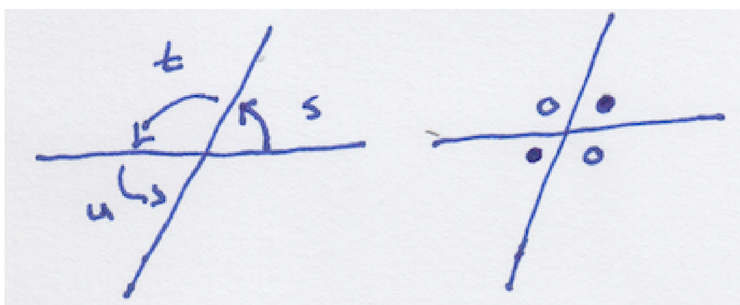
You can try to imagine slicing a pizza up into 360 equal slices. Then the measure of an angle could be determined by how many tiny slices we can fit between the two adjacent sides. This is basically what a protractor does.

To make it more practical, you could cut slices of size 30, 15 and 5 as well.



The sum of all the angles on one side of a line is half of 360, namely 180 or two right angles. Here, we have measured the angles starting from the line segment on the right of the central point.

Angles on one side of a line add up to 180 (two right angles) even when the angles are not right angles.



In the figure above, s and t are adjacent and lie above the horizontal line — they add up to 180. But t and u are both to the left of the (nearly) vertical line. So the sum $t + u$ is *also* equal to 180.

$$s + t = 180 = t + u$$

subtracting t from both sides:

$$s = u$$

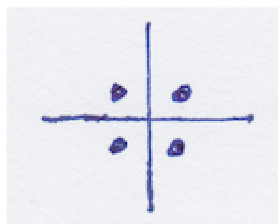
This leads to a theorem, a true statement that we have deduced, given certain assumptions. Most of geometry will consist of proving theorems based on a few assumptions, as well as theorems that have already been proved.

- s and u are *vertical* angles. Vertical angles are equal.

In the figure above, there is also a vertical angle corresponding to t, which is marked equal on the right by the open circle.

The angles s and t which add to 180 are called *supplementary* angles. The theorem we just proved is a consequence of the fact that

- Two angles which are supplementary to the same angle are equal to each other.

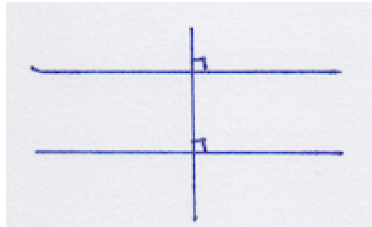


It follows that when two lines cross and one angle is a right angle, all four angles are equal and right angles.

Parallel postulate

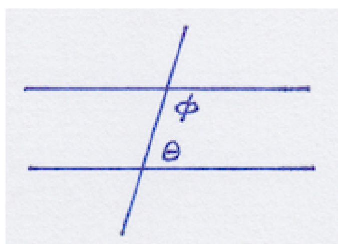
Let's return to parallel lines. The *parallel* postulate states that if a third line crosses two parallel lines, it forms the same angles with both.

Our first example is a right angle, and I think you'll agree that this idea makes intuitive sense. It is like the grid of streets in a standard city layout.



Suppose we had one line crossing two other lines and at the first crossing the angles were all equal, all right angles, but at the second line the angles were not. If the sum was less than 180 on one side the two lines would eventually meet on that side. These would not be parallel lines.

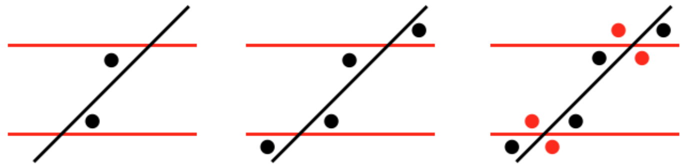
Here is the crucial idea behind the parallel postulate. Even if the angle formed is not a right angle, it will still be the case that the angles labeled θ and ϕ add up to 180 or two right angles.



- The internal angles on one side of a traversal of two parallel lines always add up to exactly two right angles.

There are altogether two sets of four equal angles in the figure above. Can you find them?

We combine what we've said up to now. Red and black add up to 180. In the left panel, the two angles marked with black dots are *alternate interior angles.*



• For a traversal of two parallel lines, alternate interior angles are equal.

The second statement is the converse of the first.

• For a traversal of two lines, if alternate interior angles are equal, the two lines are parallel.

These theorems are important in nearly every proof in elementary geometry: vertical angles, supplementary angles, and alternate interior angles of parallel lines.

A shorthand way of writing the alternate interior angles theorem is:

$$\text{alternate interior angles equal} \iff \parallel$$

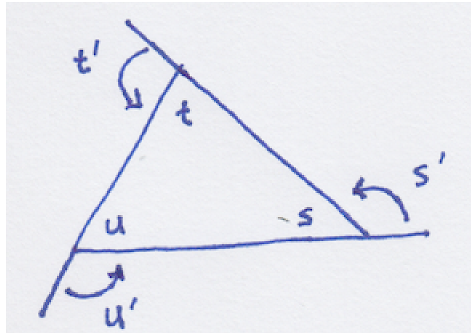
The symbol $\iff$ means *if and only if.*

triangle sum

• The sum of angles in any triangle is 180

You've probably seen this fact at some point, and we'll develop an elegant proof of it later. Here is a simple arithmetic proof.

Imagine that we walk along the sides of a triangle, turning at each vertex to follow the new side. It seems pretty clear that the sum of all three turns is 360, which we'll write as twice 180.



$$s' + t' + u' = 2 \cdot 180$$

By the supplementary angle theorem, we have three sets of angles, each pair of which adds up to 180, so the three pairs should sum to 3 times 180.

$$s + s' + t + t' + u + u' = 3 \cdot 180$$

If we subtract the first from the second:

$$s + t + u = 180$$

□

We put a little square at the end to show we're done with the proof.

reflection

The vertical angles theorem already gives us insight into problems. As an example, consider the path that light takes going from a source at A to your eye at B, reflected in a mirror.

A pastoral equivalent might be the shepherd who decides to water his flock at the river and then make it back to the barn at B by the shortest path. Light also takes the shortest path from A to B.

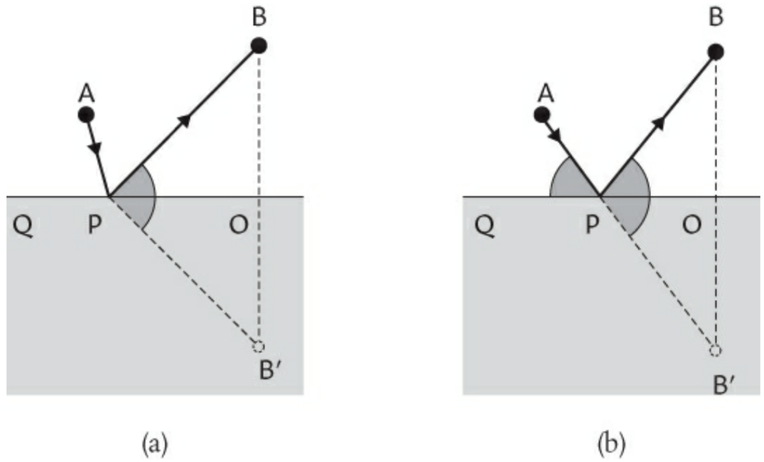


Fig. 46 Finding the shortest path.

The question is, what is the angle for the shortest path?

Let us change the problem and tackle it in reverse. Suppose we place a point B' on the other side of the mirror or across the river. Not worrying about the barrier, what is the shortest distance from A to B'. It's a straight line, by definition.

But then construct two triangles with the same sides that are mirror images. We will say more about our methods for proving that the two triangles are identical (allowing mirror images), in the next chapter.

Clearly the distance from A to B' is the same as the distance from A to the mirror and then back to B.

At the same time, $\angle\theta$ is equal to the dotted angle below it, and that angle is equal to the angle made by the light coming from A, by vertical angles.

We have the *law of reflection*, that for the shortest path, the angle of incidence is equal to the angle of reflection.

Chapter 4

Eratosthenes

The widely held theory, that the ancient world believed the earth to be flat, is just wrong. People with any level of sophistication not only knew the earth is roughly spherical but also knew its size within a few percent of the true value.

One likely basis is the false story that Columbus had trouble getting financing for his proposed trip to China because everyone thought he would fall off the edge of the earth. This was a tall tale invented by Washington Irving, who also made up several remarkable fables about George Washington.

The real reason the Italians and the Portuguese thought Columbus would fail is that they had a pretty good idea of the size of the spherical earth and thus of the distance to China, while the over-optimistic Columbus believed it was about half the true value. The prospective financiers knew that he was not able to carry the supplies necessary for a trip of this length.

Morris Kline (*Mathematics and the Physical World*) says that the error is due to geographers after Eratosthenes, who reduced the estimated circumference from 24,000 to 17,000 miles.

Eratosthenes

Views of the Greek philosophers on the earth and its sphericity are detailed here

https://www.iep.utm.edu/thales/#SH8d

Here is a partial quotation:

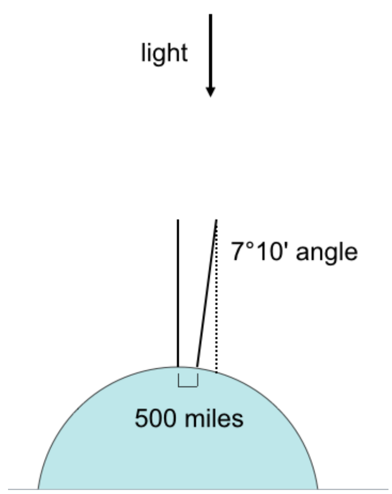
> There are several good reasons to accept that Thales envisaged the earth as spherical. Aristotle used these arguments to support his own view [...] . First is the fact that during a solar eclipse, the shadow caused by the interposition of the earth between the sun and the moon is always convex; therefore the earth must be spherical. In other words, if the earth were a flat disk, the shadow cast during an eclipse would be elliptical. Second, Thales, who is acknowledged as an observer of the heavens, would have observed that stars which are visible in a certain locality may not be visible further to the north or south, a phenomen[on] which could be explained within the understanding of a spherical earth.

https://en.wikipedia.org/wiki/Eratosthenes

Eratosthenes (ca. 276 - 195 BCE) measured the circumference of the earth from this observation: at high noon on June 21st there was no shadow seen at Syene, allegedly from a stick placed vertically in the ground. Some people say a deep well had the bottom illuminated at midday.

Syene is presently known as Aswan. It is on the Nile about 150 miles upstream of Luxor, which includes the famous site called the Valley of the Kings. At 24.1 degrees north latitude, Aswan or Syene is close enough to having the sun directly overhead on June 21. (The "Tropic of Cancer" is at 23 degrees, 26 minutes north).

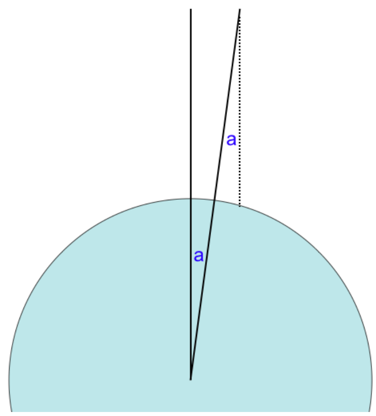
Alexandria was a famous center of learning of the ancient world, and Eratosthenes was hired by the pharaoh Ptolemy III to be the librarian in 245 BCE. Alexandria lies on the Mediterranean some 500 miles north of Syene, and anyone there who was looking could observe that at high noon on June 21st there *was a shadow*. This shadow Eratosthenes measured to be some 7 degrees and a bit (7 degrees and 10 minutes).



A full 360 degrees divided by 7 degrees and a bit is approximately 50.

So we can calculate on this basis that the circumference of the earth is about $50 \times 500 = 25000$ miles. That's pretty close to the correct value.

For this calculation, we assume that the sun's rays are effectively parallel (not a bad assumption given a distance of 93 million miles). Then we just use this:



an application of the alternate-interior-angles theorem.

It is curious how the distance from Alexandria to Syene was calculated.

Kline:

> Camel trains, which usually traveled 100 stadia a day, took 50 days to reach Syene. Hence the distance was 5000 stadia...It is believed that a stadium was 157 meters.

We obtain

$$157 \times 5000 \times 50 = 39,250 \text{ km}$$

The result is also sometimes given as 5000 stadia is about 500 miles, so that gives 25000 miles for the circumference, when the true value at the equator is 24902 miles (Maor). That's an error of 1 part in 250.

That's a much better estimate than a method that relies on camels really deserves.

Some people suspect that the conversion factor from stadia to meters might have been chosen so as to make Eratosthenes estimate look closer than it really was. According to this

http://www.geo.hunter.cuny.edu/~jochen/gtech201/lectures/lec6concepts/datums/determining%20the%20earths%20size.htm

The actual angular measurement corresponding to Syene v. Alexandria should have been $7^\circ\ 30'$ rather than $7^\circ\ 12'$. The estimate is thus under the true value by $432/450 \approx 1\%$. I can live with that.

Part II

Fundamental theorems

Chapter 5

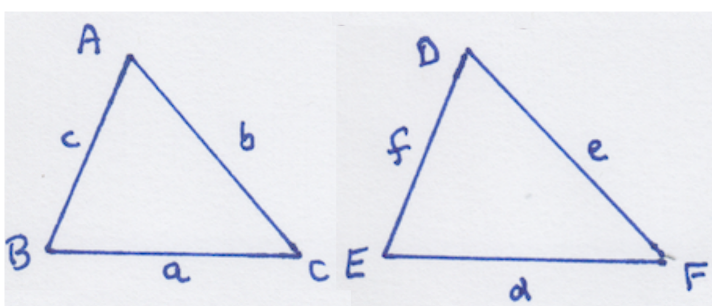
Congruence

A frequent task in geometry is to decide whether two shapes are congruent. Congruence is a fancy word for equal, or the same.

There is a special symbol, a modified equals sign, for congruence ($\cong$). For example:

$$\triangle ABC \cong \triangle DEF$$

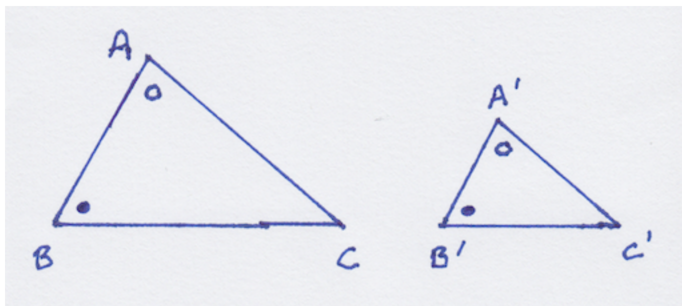
Every triangle (symbol: $\triangle$) is constructed from three line segments, called its *sides*. Hence, each triangle has three points where its sides meet, called vertices (singular, vertex), and an angle is formed there.



When two triangles are congruent, all 3 angles and all 3 sides are equal. $\angle A = \angle D$ and so on. Side a equals side d, and so on, as well.

Also, for congruence, equal sides must lie between the corresponding angles. But, this doesn't come up often, for the simple reason that keeping the angles the same but switching two sides usually means we don't have a triangle any more.

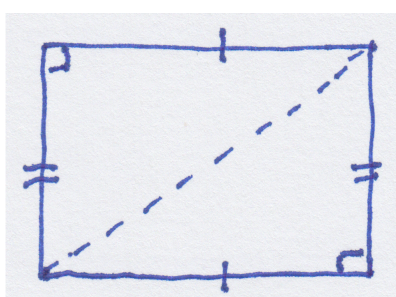
There is another type of relationship that we also care about, called similarity. Two triangles are similar if they have all the angles equal. The shape is the same but all the sides are scaled by some factor.



Because of the triangle sum theorem, we know that if any two angles are equal, then the third angle is equal as well. Similarity is an amazingly powerful concept and we'll have a lot to say about it later.

rectangles

A rectangle is defined as a four-sided figure (called a quadrilateral) with both pairs of opposite sides equal, and right angles at all four corners (only two are marked here). Bars mark the equal sides and dots or right angle symbols mark angles that are equal.



Think for a second what it would take to prove that two rectangles are congruent. Clearly, you do not need all those conditions to be met. Four right angles and one side will do it.

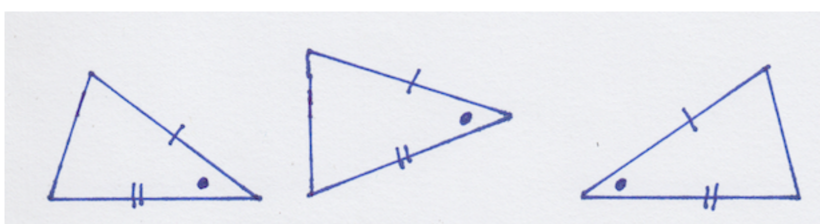
One pair of equal and opposite sides and any two right angles might be enough. How about two pairs of equal and opposite sides and one right angle?

Proving triangle congruence

In the figure above the diagonal splits the rectangle into two right triangles. Since we started with a rectangle, all four angles are right angles and the sides are equal as marked.

Our first test or method for proving congruence of triangles is called SAS, which stands for side-angle-side. Look again at the diagonal inside the rectangle. The long sides marked with a single cross-bar are equal, as are the other two sides. The angle between them is also equal.

We say that we *have* (i.e. we have shown or proved) SAS, and conclude that the two triangles are congruent.



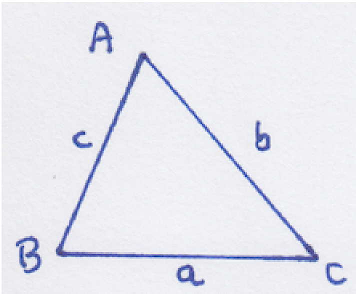
In the figure above, there are three triangles each with two sides marked as equal with bars, and the angle between them marked as equal with a filled blue dot. These triangles are all congruent, by SAS.

Notice that one on the right is a mirror image of the others, it has undergone a *reflection*. We consider two triangles that are mirror images to be congruent. Rotation, translation (movement from one place to another), and reflection are all allowed and two triangles can still be congruent.

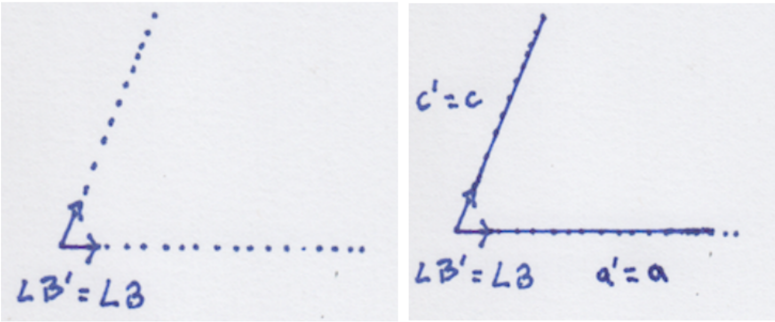
why SAS works

A good way to think about congruence tests is to ask how much information we need to draw a copy of an existing triangle so as to have the two be congruent.

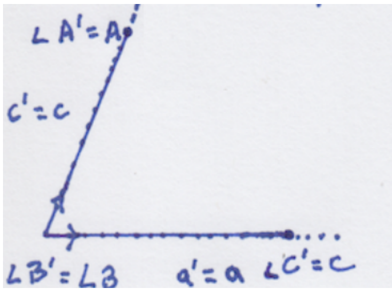
Suppose we're starting with $\triangle ABC$

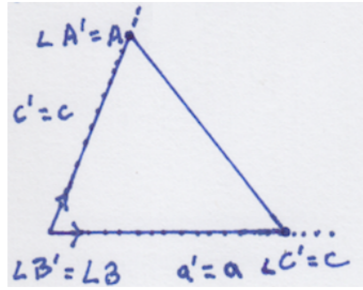


In the SAS method, think about first duplicating $\angle B$ so that $\angle B' = \angle B$. Draw the sides as dotted lines until we know how long they should be.



But that's exactly what the two side lengths gives us. Their lengths determine the location of the other two vertices of the triangle.



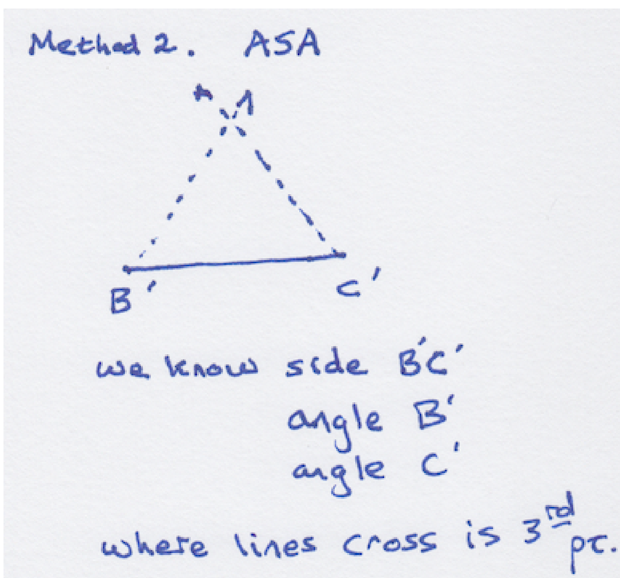


If you think about it you will see that switching the sides will also work. If we draw a length $a' = a$ on the side going up, and a length $c' = c$ on the horizontal, then we will still have SAS.

The resulting triangles are mirror images of each other. Such triangles, all 3 angles equal and all 3 sides equal, but flipped, are still congruent.

ASA

A second method for establishing congruence is called angle-side-angle.



The diagram above shows why ASA works to determine congruent △.

SSS $\to$ SAS

A third method for proving congruence is SSS. It is logically equivalent to SAS, but to prove this we need the isosceles triangles theorem from the next chapter.

right triangles

If two triangles are both right triangles, then if any two pairs of sides are equal, the two triangles are congruent. If both are base sides, this follows from SAS.

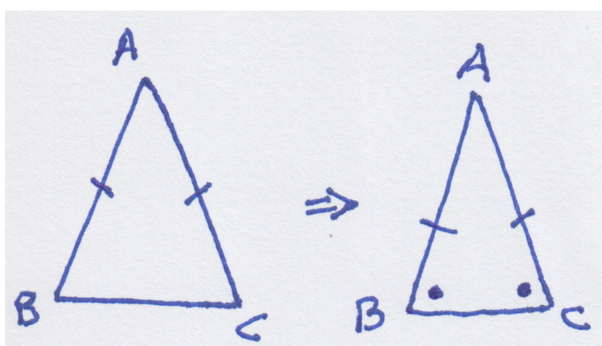
If one side is the hypotenuse, what is often called HL (hypotenuse-leg), then one can swing the hypotenuse against the other base to see why it works. Or just look forward to the Pythagorean theorem and say that if we know two sides in a right triangle, we know the third. So again we will have SAS.

Chapter 6

Isosceles triangles

We continue in this chapter with the subject of congruence and our second major theorem. The first one was the triangle sum theorem. In this chapter we will prove the isosceles triangle theorem.

• Given that two sides are equal in a triangle (it is isosceles), the base angles opposite those sides are also equal.



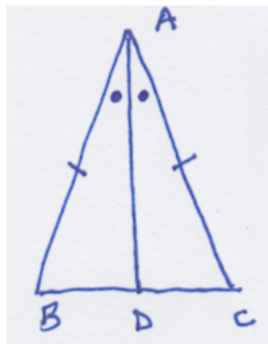
We're given that $AB = AC$. We claim that $\angle B = \angle C$.

I think we can agree, to begin with, that this seems rather obvious. After all, what reason would you give for saying that the two angles are different? Perhaps you are right-handed and have a bias toward the right angle being larger.

But then, imagine looking at a reflection of the triangle. We said that reflection is legal. Now right has become left, and the angle on the left side is the larger one. That makes no sense.

Proof.

A simple proof is to draw the angle bisector of the angle at A, the line that splits the angle in half. We'll give a method for doing that later.



Since $AB = AC$, and side AD is shared, we have two pairs of equal sides. The angle bisector construction makes $\angle BAD = \angle DAC$ at the top vertex.

Therefore, the two smaller triangles are congruent, by SAS.

Since $\triangle ADB \cong \triangle ACD$, it follows that the angles at B and C are equal.

It also follows that the base is cut into two equal pieces ($BD = DC$), and that the two angles at D are equal, which makes them both right angles.

□

converse of isosceles triangle theorem

The converse of the isosceles triangle theorem is also true.

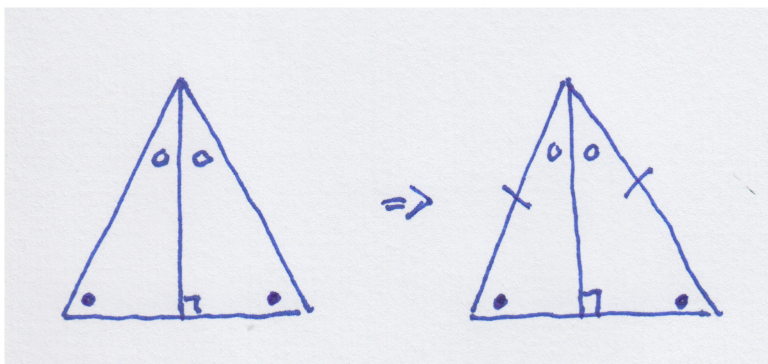
The forward version is that if two sides are equal in a triangle, then the sides opposite are equal. The converse is that if two angles are equal in a triangle, then the sides opposite are equal.

It isn't alway the case that the converse of a true theorem is true. We must prove (or at least try to prove) each case.

- Given two equal sides at the base of a triangle, the sides are also equal and the triangle is isosceles.

Proof.

Draw the angle bisector again. Now, we have two smaller triangles with two angles equal, since we are given that the base angles are equal.



In comparing these two smaller triangles, the third angle must also be equal, by the triangle sum theorem. This is the angle where the bisector meets the base. Since the pair of them are equal and also supplementary, they are both right angles.

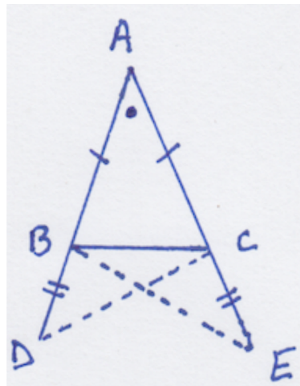
Then the shared side between the right angle and the one at the top marked with the open circle, gives ASA and therefore, the two smaller triangles are congruent.

It follows that the two sides of the original triangle are equal.

□

Euclid's proof

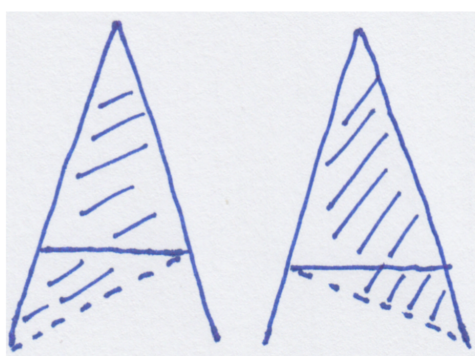
Let's approach the same problem a different way. We're going to look at two different triangles inside the same figure.



In the diagram above we have two sides of $\triangle ABC$ that have been extended as far as D and E. We know (we're *given*) that $AB = AC$ and $AD = AE$, and by subtraction of equals, we know that $BD = CE$.

The apex $\angle A$ is, of course, equal to itself, so it's marked with a dot as a reminder.

Now, look at the two triangles that have one side as a dotted line. I mean $\triangle ACD$ and $\triangle ABE$.



It's a little tricky because they overlap each other and are also mirror images. Be sure you trace out the correct three vertices for each one.

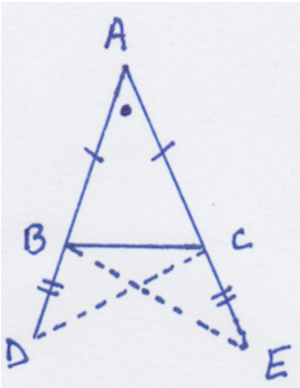
We were given

- $AB = AC$
- $AD = AE$
- $\angle A$ is certainly equal to itself.

By subtraction:

- $BD = CE$
- BC is certainly equal to itself

Since $\angle A$ is shared (it's exactly the same in both triangles), we have SAS. We conclude that the two $\triangle$ are congruent: $\triangle ACD \cong \triangle ABE$.

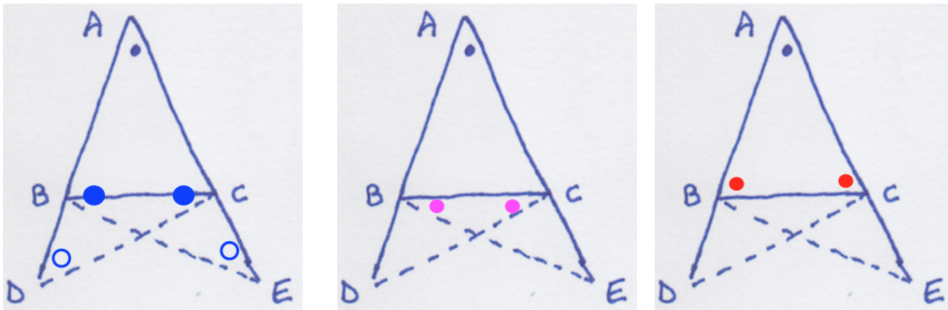


Now that we know they are congruent, it follows that there are other equalities:

- $BE = CD$
- $\angle D = \angle E$
- $\angle ABE = \angle ACD$

I have marked the first pair with larger dots because what we have shown so far is only that two angles, taken together, are equal. We must still find a way that the individual components are equal.

In the second step we will show that the angles marked with magenta dots are equal (below). Then subtraction will give the result we are after, that the red dotted angles are equal.



Again, the goal is to show that $\angle ABC = \angle ACB$. We already have equal angles that include those two, namely $\angle ABE = \angle ACD$. Now we need to show that $\angle CBE = \angle BCD$ (magenta dots).

We will show that $\triangle BCD \cong \triangle BCE$. We have that the sides flanking $\angle D$ and $\angle E$ are equal. Check above to see which statements apply. We also have that $\angle D = \angle E$.

We conclude that $\triangle BCD \cong \triangle BCE$ by SAS.

Therefore, by congruent triangles, $\angle CBE = \angle BCD$.

And finally, since $\angle ABE = \angle ACD$ (left panel) and $\angle CBE = \angle BCD$ (middle panel), subtract the second from the first to obtain the base angles.

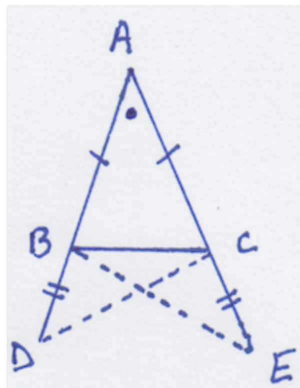
$$\angle ABC = \angle ACB$$

$\square$

This is a moderately complicated proof of the isosceles triangle theorem. It's more challenging than the first one, but Euclid proves it before dealing with angle bisection, so he can't use that idea here.

Euclid also proves the converse. He uses a method that we haven't seen yet, so let's not worry about it for now.

Historically the diagram is known as the "bridge of asses" (*pons asinorum* in Latin), because it looks something like a bridge, and asses (that is, dullards), proved incapable of being led over it.

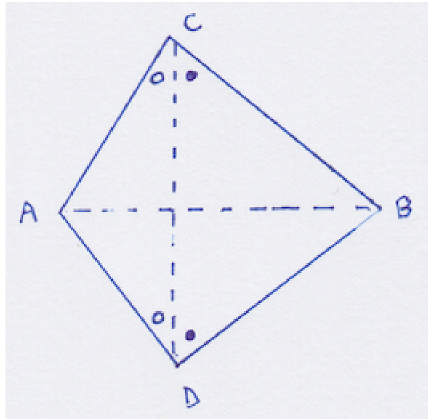


SSS → SAS

Continuing with a theorem from the last chapter, a third method for proving congruence is SSS.

Proof.

Suppose that $\triangle ABC$ and $\triangle ABD$ have all three sides equal. Then place the two triangles together to form a kite, two pairs of adjacent sides equal, and one side shared.



Draw CD. By the forward version of the isosceles triangle theorem, the angles marked with open dots are equal, as are the angles marked with filled dots. Therefore $\angle D = \angle C$.

We have SAS, so $\triangle ABC \cong \triangle ABD$.

$\square$

Chapter 7

Parallelograms

So far we have dealt mainly with triangles. Triangles are members of the first of two large categories of shapes in geometry, namely polygons, which have straight sides, and curves. Polygons are composed of line segments, and curves include circles as well as various conic sections like the ellipse, the parabola, and so on.

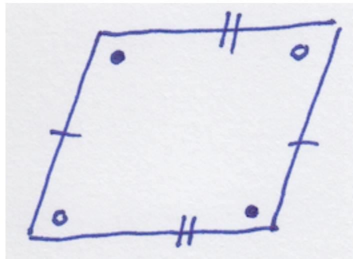
Triangles are polygons with three sides, of course. Quadrilaterals have four sides. They are in turn organized by whether one pair of opposing sides is equal, or possibly both pairs are equal, and whether the angles inside are right angles.

It is helpful to remember that any quadrilateral can be divided by a diagonal into two triangles. As a result, we have that the total of all the angles in such a polygon is equal to four right angles, or 360.

parallelogram

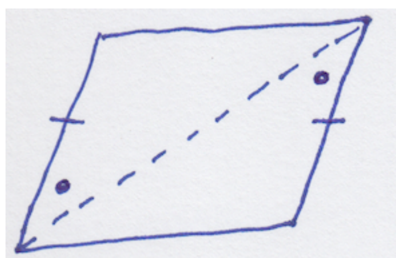
This chapter introduces the parallelogram. Its theorems provide practice in the methods of proving congruence for triangles.

A parallelogram is a quadrilateral with both pairs of opposing sides equal and parallel, and both pairs of opposite angles equal.



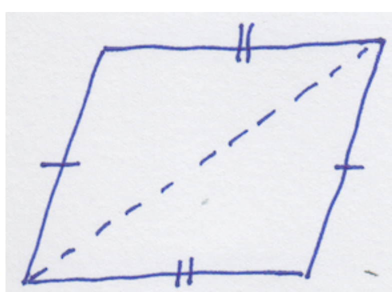
However, just as with congruent triangles, it is not necessary to show all these conditions are true before we can know that we have a parallelogram.

The first example is one pair of opposite sides equal and parallel. The marked angles are equal by alternate interior angles.



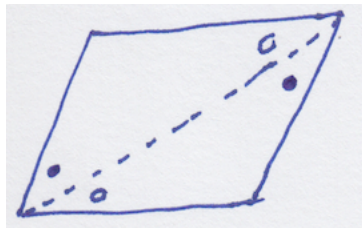
The two triangles formed by the diagonal are congruent by SAS. This gives the other properties.

The second example is both pairs of opposing sides equal.



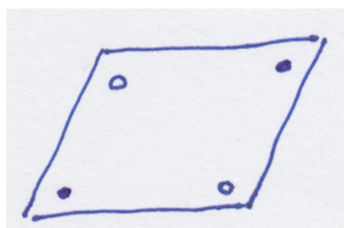
Here we have the third side shared, hence the two triangles formed by the diagonal are congruent by SSS. Again, this gives the other properties.

The third case is both pairs of opposing sides parallel.



This gives the indicated angle equalities, which means we have congruent triangles by ASA. This gives the rest of the equalities.

The fourth case is both pairs of opposing angles equal:



Any quadrilateral can be cut into two triangles, hence the sum of angles in a quadrilateral is four right angles. Since this is equal to two pairs of equal angles, the sum of one of each pair is equal to two right angles. Hence both pairs of opposing sides are parallel.

Therefore, we have the third case.

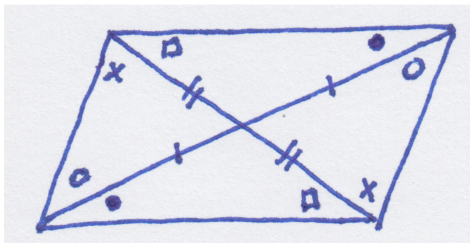
The fifth and last case is one pair of opposing sides equal and parallel (refer to the diagram for the third case and make the relevant substitution). When the diagonal is drawn we again have equal angles by alternate interior angles, and equal flanking sides, which from which it follows that the two triangles are congruent, by SAS.

The last case becomes useful later in dealing with the central theorem regarding similar triangles, due to Euclid.

midpoints of diagonals

The two diagonals of a parallelogram cross at their midpoints.

The angles marked equal are equal by alternate interior angles.



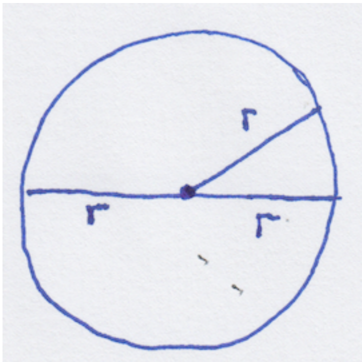
We have two sets of congruent triangles, by ASA (given that the opposing sides are equal).

Therefore, each diagonal is divided in half, even though one is longer than the other.

Chapter 8

Thales

circles



The easiest way to draw a circle is to just pick some point to be the center. Then draw all the points that lie a given distance away from the center. That distance is called the radius, r.

$\circ$ All the points on the circle are equidistant (the same distance) from the center.

That is not a theorem but an axiom, something that we choose to believe about the world.

A diameter is a straight line through the center, and since each half is a radius, the diameter $d = 2r$.

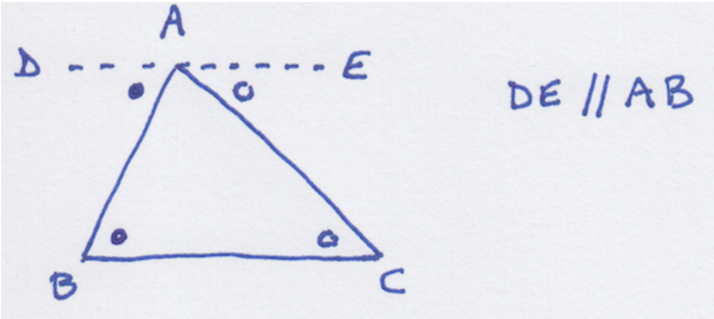
Thales (624-546 BC) was from a Greek town called Miletus on the coast of Asia Minor (modern Turkey). He lived long before Euclid — about 300 years before. Unfortunately, neither his writing nor an extensive history that was written later survives.

The following is one of three elementary but novel theorems for which he is thought to have developed proofs.

Triangle sum theorem

We did this one previously so we could use it before this, but Thales' proof is beautiful.

- The angle sum of a triangle is equal to two right angles.



Proof.

Draw a line segment through the top vertex of the triangle parallel to the base.

Now, use alternate interior angles. By the theorem, the two angles marked with filled circles are equal, as are the two angles marked with open circles.

The three angles under the top red line add up to two right angles, by the supplementary angle theorem. So the total measure of three angles in a triangle is equal to two right angles.

Again, we put a little box to show that the proof is complete.

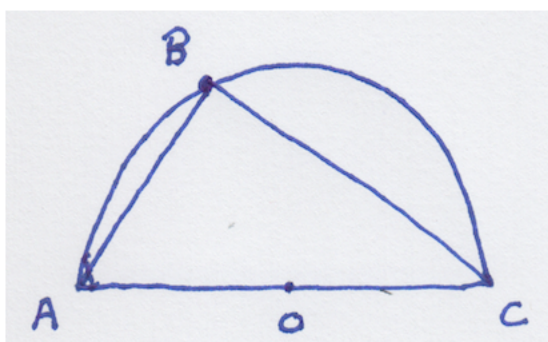
□

The triangle sum theorem is very closely related to the parallel postulate. There, we said that for a traversal of two parallel lines, which never meet, the sum of the internal angles is 180. The converse would be that if the two lines did meet, somewhere far away, then we would have a triangle.

Meeting at a third angle means that the sum of the internal angles must be less than 180, by the triangle sum theorem, since we must leave some (small) part of the 180 for the far distant angle.

Thales circle theorem

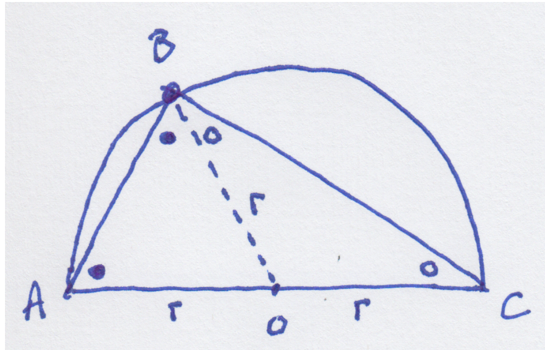
Draw a circle and its diameter AOC (actually, only a semi-circle is required, as we see here.



Pick any point on the periphery, B, and draw AB and BC. I claim that $\angle ABC$ is a right angle.

Proof.

Draw BO.



Since AO is on the diameter, it is a radius of the circle. OB is also a radius, so they are equal. Therefore $\triangle AOB$ is isosceles. By the isosceles triangle theorem, the angles marked with filled dots are equal.

But then the two base angles of $\triangle BOC$ are also equal, by the same reasoning.

The sum of the two angles at vertex B is one-half the sum of all the angles in the original triangle ABC. Therefore $\angle ABC$ is a right angle.

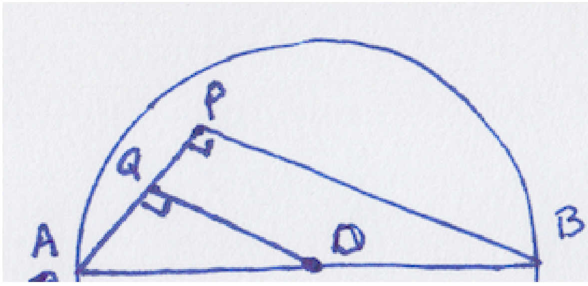
$\square$

Converse of Thales circle theorem

We showed above that given a diameter of the circle and any point on the circle (not on the diameter), the angle formed with the sides drawn from the diameter is a right angle.

We are now given that the angle at P is a right angle, and $\triangle APB$ has AB as a diagonal of the circle centered at O with radius AO.

We will prove that P must be on the circle.



Proof.

Draw the perpendicular at Q so that it goes through the center of the circle at O. By similar triangles (three angles equal), $\triangle APB \sim \triangle AQO$ and since AO is one-half the diameter, the ratio of sides is 2:1.

Using that ratio, $AQ = QP$. (Note: this step uses similarity and ratios of sides, which we won't prove for a few more chapters).

But this means that $\triangle AQO \cong \triangle POQ$, so $AO = OP$. Thus, $\triangle AOP$ is isosceles.

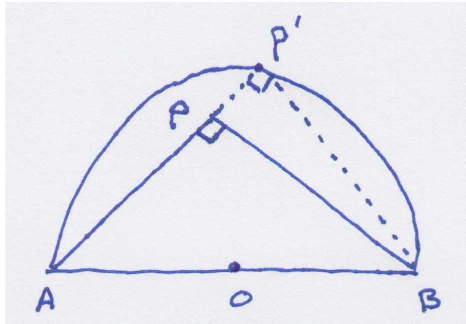
But AO is a radius of the circle, so OP is also. Therefore, P must lie on the circle.

□

We will prove the previous theorem again, in order to give a first proof by contradiction.

In this method, we assume something that is the opposite of what we think may be true, and want to prove. Then follow the logic until we reach a contradiction. This logical dead end implies that the original assumption was incorrect, so the converse of the original statement is true.

We have the same situation but a slightly different diagram. If $\angle APB$ is a right angle and AB is a diameter, then P is on the circle.



Proof.

By contradiction. Assume that $\angle APB$ is a right angle, but P lies inside the circle.

Draw the continuation of AP to form AP' with P' on the circle. Then, by the forward version of the theorem, $\angle AP'B$ is a right angle.

We assumed that $\angle APB$ is also a right angle. Therefore, by the parallel postulate, PB is parallel to $P'B$.

But these two line segments meet at B so they are not parallel. This is a contradiction.

Therefore our assumption that APB lies inside the circle was mistaken.

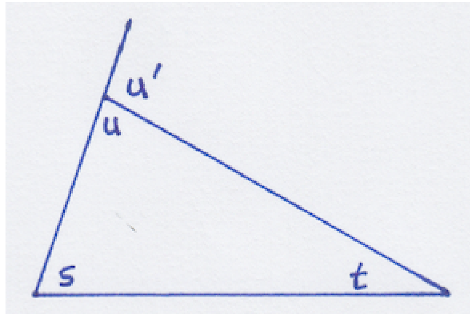
A similar argument shows that P is not outside the circle, either. If it does not lie either outside or inside the circle, it must lie on the circle.

□

Since the converse is true, we know that *any* right triangle can be embedded in a semicircle.

exterior angle theorem

The form of the exterior angle theorem that we will use most often says that the angle u' is equal to $s + t$.



Proof.

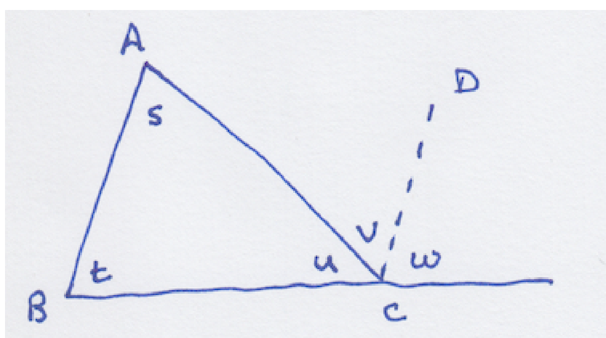
u' is supplementary to u, but the sum $s+t$ is supplementary also. Therefore u' is equal to $s+t$.

□

A longer proof gives some more detail on the components of the exterior angle, although honestly, it is really just a repeat of the triangle sum theorem with the alternate interior angles theorem added in.

Proof

Draw a triangle and extend one side. Also draw a line segment at that vertex parallel to the opposite side.



The exterior angle at vertex C is the sum of the two angles $v+w$. These are equal to the opposing angles s and t because

$$s + t + u = 180 = u + v + w$$

$$s + t = v + w$$

Also $s = v$, by alternate interior angles, so $t = w$, by subtraction.

The exterior angle at any vertex is equal to the sum of the two internal angles on the side facing that vertex.

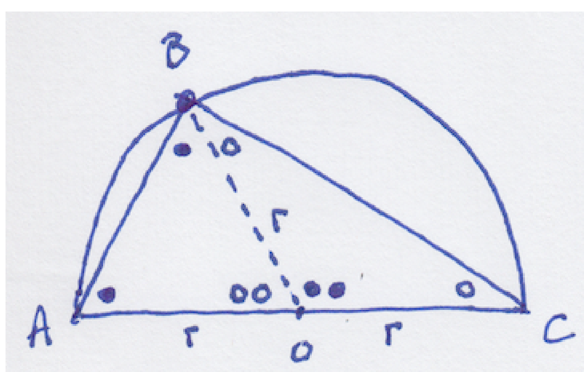
$\square$

Technically, the exterior angle theorem refers to Euclid's proof of a weaker result, which was that the exterior angle is greater than either one of the two internal angles on the side facing that vertex.

His proof has been discussed *a lot*, mainly because it is closely related to the parallel postulate. We don't need to get off track by doing that here. Proposition 32 of book 1 is equivalent to what we've done here.

beyond Thales

Let us go back to Thales theorem about the right angle on a diameter of the circle.



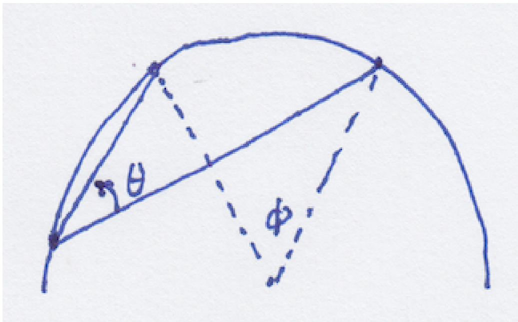
Fill in the measures of the central angles at O, using the exterior angle theorem from above.

We notice something interesting: the angle at vertex A is exactly one-half the central $\angle BOC$, and they cut off (subtend) the same arc on the circle.

This is a special case, because both A and O lie on the diameter. But we begin to wonder whether it might be true in general.

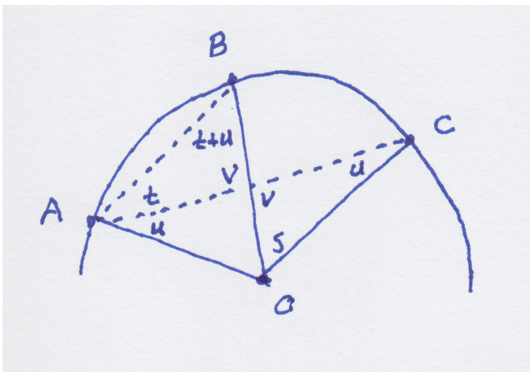
inscribed angle theorem

- Any angle inscribed in a circle is one-half the measure of the central angle that cuts off the same arc. In the diagram, $2\theta = \phi$.



Proof.

Label a few more angles. Central angle s and peripheral t ($\angle BAC$) subtend the same arc of the circle.



In isosceles $\triangle AOB$, the base angles of measure $t + u$ are equal.

In isosceles $\triangle AOC$, the base angles labeled u are equal. Furthermore, the angles labeled v are vertical angles.

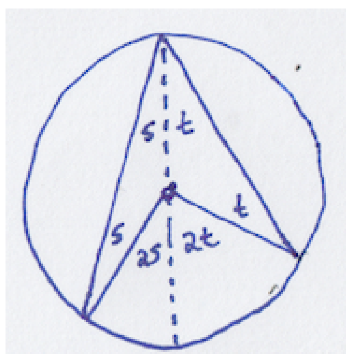
The triangle sum theorem says that

$$t + t + u + v = s + u + v$$

$$2t = s$$

$\square$.

This proof is simple but not complete because the drawing assumes that the center of the triangle O is not contained within the angle formed by BAC. A more general proof by a similar construction is possible, but instead we introduce a different proof.



$s = s$ and $t = t$ by the isosceles triangle theorem. Then the exterior angle theorem gives the central angles, and the sum gives our basic inscribed angle theorem.

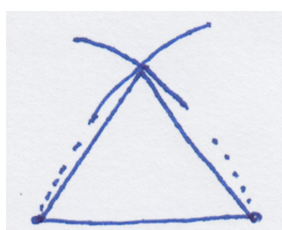
Chapter 9

Constructs

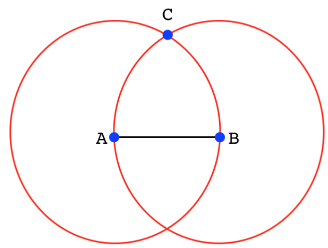
equilateral triangle

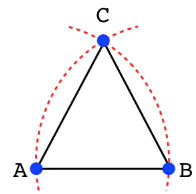
The first figure shows how to construct an equilateral triangle with straight-edge and compass. (A straight-edge is a ruler with no rules, no marked divisions).

Pick a base of the desired length. Then using one end of the base as the center of a circle, draw the circle of radius equal to the length of the base. (i.e. set the compass to the base length).



We're only showing the part of the arc that matters. Repeat using the other point. Where the arcs cross, we have a point that is the same distance from each end of the original base, and that distance is equal to the base, so together with the ends of the base, this point forms one of the vertices of an equilateral triangle.





perpendicular bisector

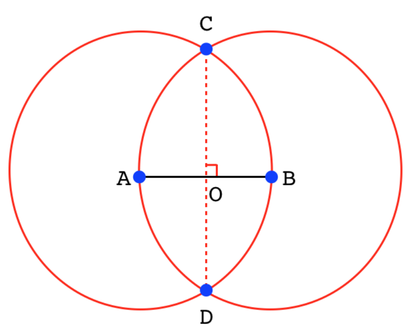
There are three different cases:

∘ (1) A perpendicular bisector at a point on a line.

∘ (2) A perpendicular bisector through a line, midway between two other points.

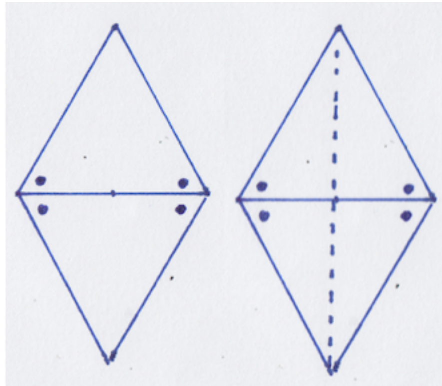
∘ (3) A perpendicular through a line and an external point.

Let us solve the second one first. Draw two circles to form two equilateral triangles above and below the line.



Draw the line CD connecting the apex of the first triangle with that of the second triangle below. I claim that the angles at O are all right angles and AB is bisected.

Proof.



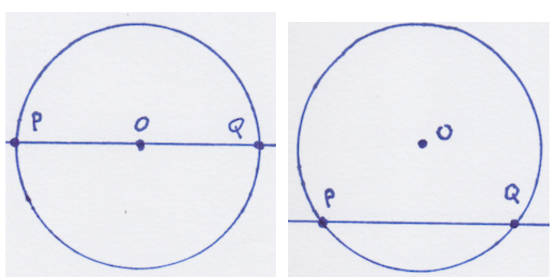
In the left panel, we have two triangles congruent by ASA (or SSS or SAS, take your pick). All five lines in the figure are equal length, and all angles are equal as well (the top and bottom aren't marked).

The two triangles formed by the dotted line (right panel) are congruent by SAS, so the top and bottom angles are bisected. Therefore, all four small triangles are congruent, by ASA.

It follows from the congruent triangles that $\angle AOC = \angle BOC$ so they are both right angles, and then all the angles at O are right angles and AB is bisected at O.

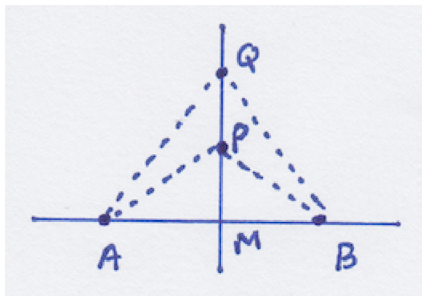
$\square$

The other two cases can be converted into the second. For the first, draw a circle around the given point, smaller than the radius to be used for the equilateral triangles. That circle will cross the line at two points equidistant from the given point. Proceed with case 2.



For the third case, draw a circle around the external point, to find two points on the line that are equidistant from the given point and so form an isosceles triangle with it. Proceed with case 2 as before.

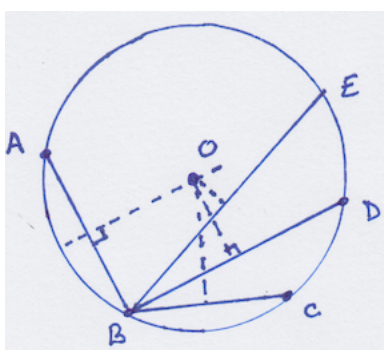
Points on the perpendicular bisector are equidistant from the points A and B. Examples: P has $PA = PB$ and Q has $QA = QB$. This is true of every point on the vertical line (figure below).



circle containing any three points

Next, for any three points, we can draw a circle that contains those points. Consider first the points A, B and C in the figure below. Draw AB and its perpendicular bisector. Draw BC and its perpendicular bisector. The point where they cross is O, the center of the circle containing all three points.

Proof. All the points on the first bisector are equidistant from A and B, and on the second, equidistant from B and C. Hence $AO = BO = CO$.

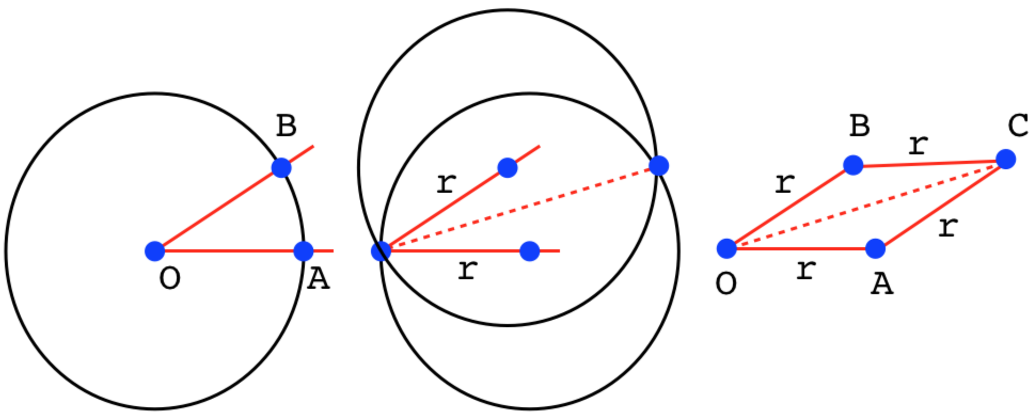


The point where the bisectors meet is equidistant from all three points. Therefore it can be used as the center for a circle that contains all three of them.

For any other point on the circle, such as D or E, the perpendicular bisector has the same property. Hence they all go through point O.

angle bisection

Next, we will bisect an angle. A method is shown here:



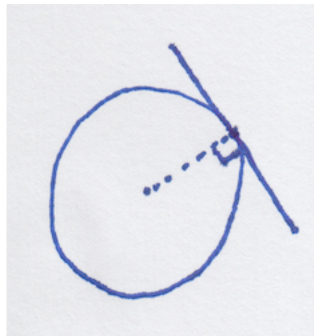
Suppose we wish to bisect the angle at vertex O. Mark off $OA = OB$. Draw two circles with the same radius, whose arcs are shown meeting at C. We have that $AC = BC$.

By the isosceles triangle theorem, $\angle OBA = \angle OAB$ and $\angle ABC = \angle BAC$ so their sums, namely the total $\angle OBC = \angle OAC$. Now we have SAS.

Therefore, $\triangle OBC \cong \triangle OAC$. So the two angles at O, $\angle BOC$ and $\angle AOC$, are equal.

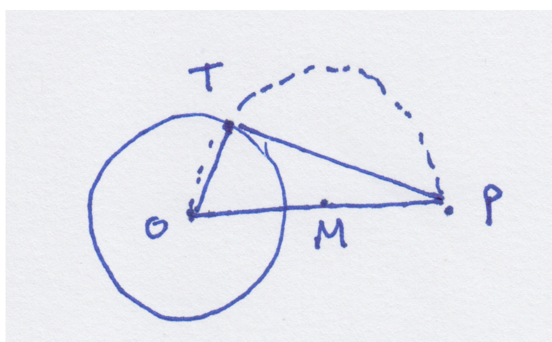
tangent

At each point where the radius intersects the circle, we can draw the tangent to the circle at that point.



One way to do this is to extend the radius the same length again and draw its perpendicular bisector.

On the other hand, if we want the tangent to go through a particular exterior point P, connect the center of the circle O to that point, then bisect the resulting length at M.



A circle drawn with length OM and its center at the bisection point will form a right angle at the point T where it intersects the circle. This follows from Thales circle theorem.

Part III

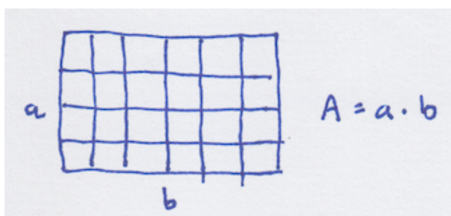
Area

Chapter 10

Area

Area is defined by squares of whatever unit size we choose, whether it's square miles (miles^2) or nanometers (nm^2) or something in between, like inches (in^2), or centimeters (cm^2).

In the simplest cases, the area to be computed is that of a rectangle or a square, and the lengths of the sides are an even multiple of our unit length.



In the example above $a = 4$ and $b = 6$ and the area is

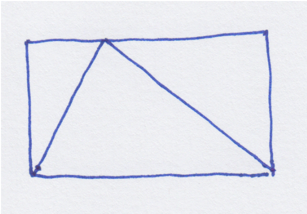
$$A = a \cdot b = 4 \cdot 6 = 24$$

We are interested in finding the areas of different triangles.

In a famous essay, Lockhart says that it is a mistake (an error in teaching) to just give the formula to begin with.

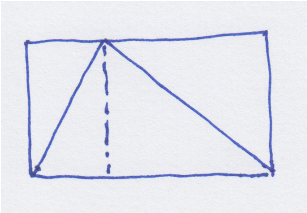
https://www.maa.org/sites/default/files/pdf/devlin/LockhartsLament.pdf

Instead, what we should do to appreciate the art of geometry is to draw a triangle in a box, like this



and then wonder, or guess, what the area is inside the triangle compared to the surrounding box.

The "aha" moment comes when we draw a vertical down, and see that the two smaller triangles are exactly one-half of their surrounding boxes.



Aha!

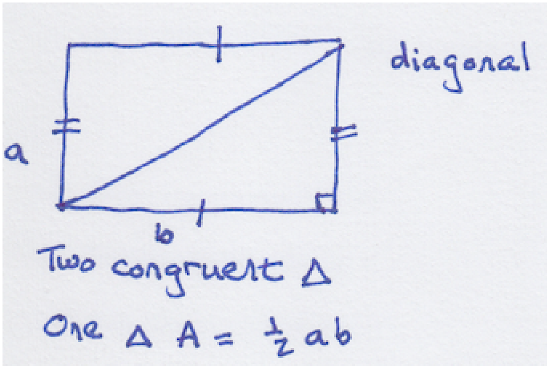
This is characteristic of many proofs, as you may notice. A line is extended or a new line drawn at a point and parallel to some other line. The drawing we start with gets enhanced, and then you suddenly see the way forward.

So how do we know that the diagonal cuts a rectangle into two equal parts? One answer: symmetry. There is no reason to favor one piece over the other. Or use the properties of rectangles in the following way.

systematic approach

Any rectangle can be divided by drawing its diagonal. The result is two right triangles, and these two triangles are congruent.

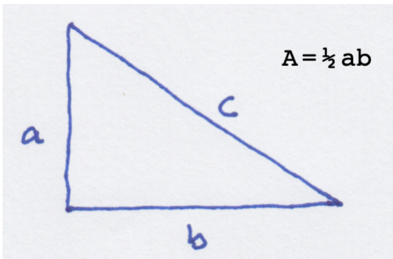
In the drawing, all four vertices are right angles (this is a rectangle), but only one is marked as such.



Since the area of the whole is $a \cdot b$ and we have produced two identical halves, the area of each right triangle is

$$A_{\triangle} = \frac{ab}{2}$$

This is true for *any* right triangle.

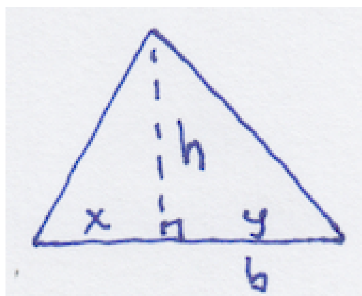


The longest side in a right triangle is called the hypotenuse. The product of the other two sides, times one-half, is the area of any right triangle.

It is mildly annoying to keep track of that factor of one-half all the time. A solution, which we will follow for the rest of this chapter, is to talk about *twice* the area of the triangle, and say that

$$2A_{\triangle} = ab$$

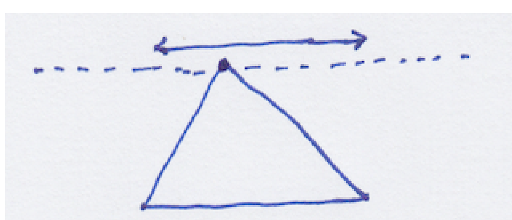
Now, suppose we do not have a right triangle, but a line drawn from the top vertex down to the base (called the *altitude*), perpendicular to it, and that line stays inside the triangle.



Then the altitude h divides the triangle into two smaller right triangles. The total area is the sum of those smaller triangles.

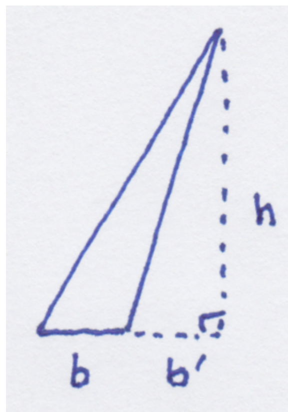
$$2A = hx + hy = h(x + y) = hb$$

In the diagram below, if we would move the top vertex back and forth along a line parallel to the base, the height would not change, and neither would the area of the triangle.



• Triangles with the same base length and a vertex anywhere on a line parallel to the base, have equal areas.

This approach still works for an obtuse triangle. In that case, if the largest angle is one of the base angles, then a line dropped perpendicular to the base, meets the base outside of the triangle. An easy way to see that the formula still works is to compute the areas of the two right triangles in the figure.



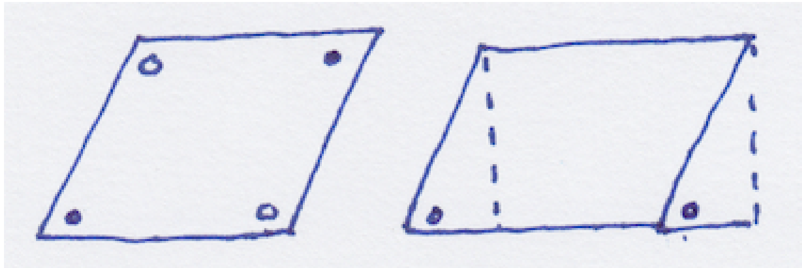
The base of the triangle is b and the short dotted line which extends b to meet the vertical line is b'. Twice the area of the large right triangle is $h(b + b')$ and twice the area of the small dotted right triangle is $2A = ab'$. The area of the original triangle is the difference.

$$2A = a(b + b') - ab' = ab$$

a different view

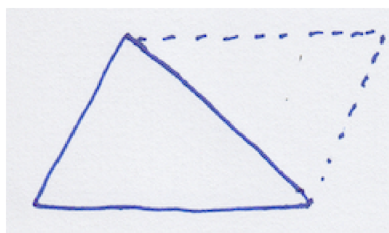
Another way to think about the area of triangles is to start with a parallelogram, which is a quadrilateral (four-sided figure) with both pairs of opposite sides parallel.

As a consequence the opposing sides and angles are equal. Angles that flank one another add up to 180.



The area of a parallelogram is the base times the length of a perpendicular from one vertex to the base, or an extension of the base. You can see that this works by cutting off a small triangle on one end and attaching it at the other.

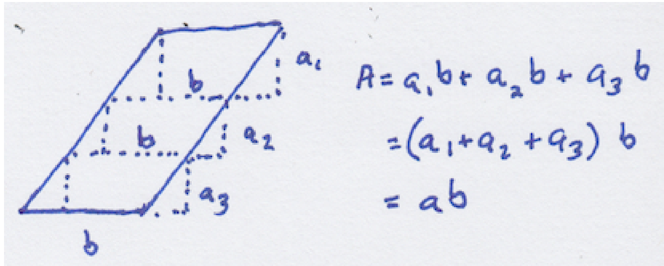
But any triangle can be converted into a parallelogram, by drawing a congruent triangle, turning it around, and pasting them together, as shown.



So the area is the base times the vertical height, times one-half since there are two copies of the original triangle.

For an obtuse triangle, the parallelogram may lean over so far that you can't easily turn it into a rectangle.

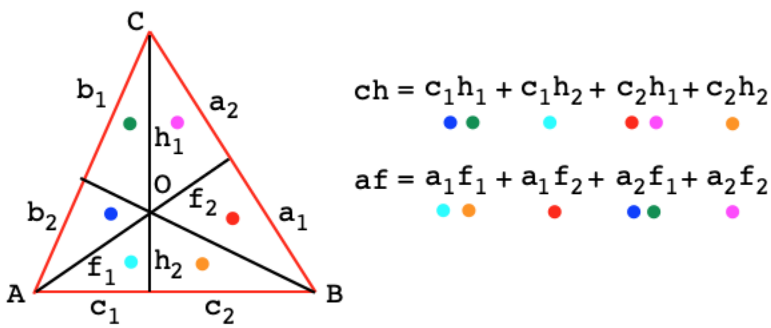
However, this difficulty can be solved by dividing the parallelogram into sections horizontally. Each individual section can be converted to a rectangle, and the areas added up for the pieces.



note

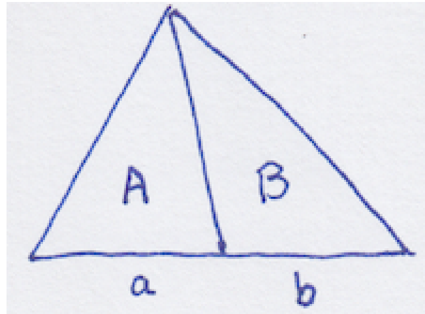
The area of a triangle must be the same no matter which side we choose as the base.

We won't go through the proof in detail, but by cutting the triangle up into smaller ones, we can show that $af = ch$ (and b times its altitude is also equal, as well).



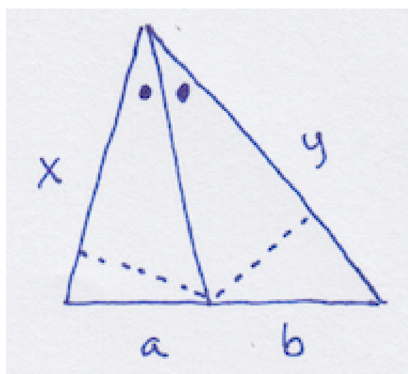
area-ratio theorem

A triangle is divided into two smaller ones by drawing a line from the top vertex to some arbitrary point along the base. Since both triangles have the same altitude (a perpendicular from the apex), twice the areas are in the same ratio as the bases.



$$\frac{(2A)}{(2B)} = \frac{ha}{hb} = \frac{a}{b}$$

Now we take an arbitrary triangle and draw the angle bisector at the top, splitting the base into a and b. The right triangles that are formed by the dotted line are congruent. We have two angles (thus three angles) and the shared side.



So let the dotted line be h and then

$$\frac{(2A)}{(2B)} = \frac{(A)}{(B)} = \frac{hx}{hy} = \frac{x}{y}$$

But the area-ratio theorem also applies, giving

$$\frac{(2A)}{(2B)} = \frac{(A)}{(B)} = \frac{a}{b} = \frac{x}{y}$$

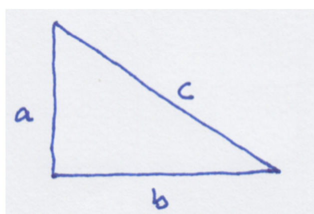
which also means

$$\frac{x}{a} = \frac{y}{b}$$

Chapter 11

Pythagoras

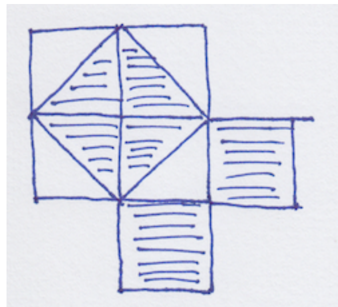
We're going to develop several proofs of the Pythagorean theorem. This theorem says that in any right triangle, the sum of the squares on the shorter sides is equal to the square on the hypotenuse.



$$a^2 + b^2 = c^2$$

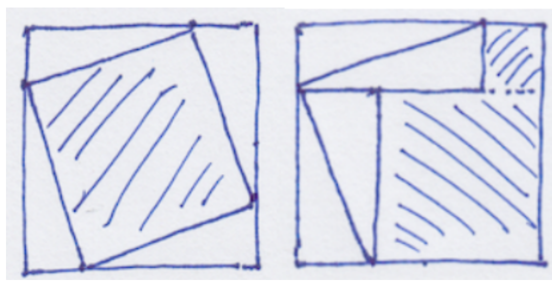
The Greeks thought of it as a strictly geometric theorem, in terms of area. We usually think of this result algebraically, for example using the integers $a = 3$, $b = 4$ and $c = 5$ to solve the equation.

Surely, first there were proofs without words. We start with an isosceles right triangle.



The square on the diagonal is obviously equal to the two small squares put together. This is a proof of the special case of the theorem when the two small sides are equal.

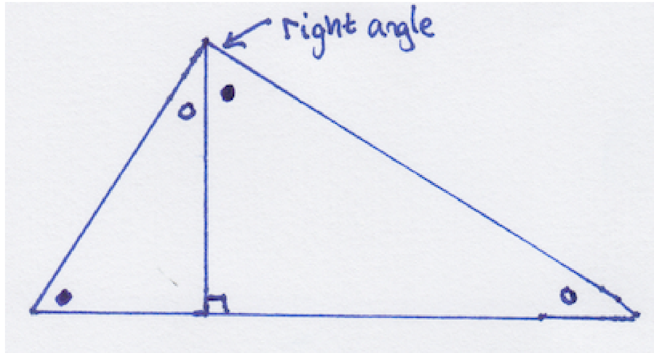
But now, just tilt the inner square.



Four copies of the triangle take up the same space in both diagrams. What's left is clearly the square of the hypotenuse on the left, and the sum of squares on the other sides, on the right.

proof from similar triangles

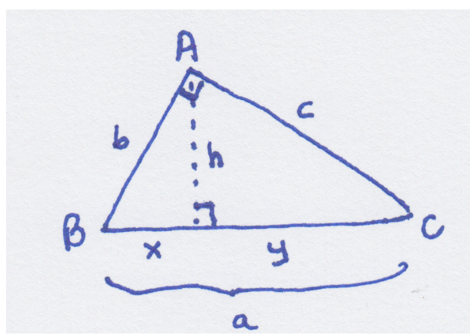
We need a preliminary result.



If we start with any right triangle, we know that the two angles which are not right angles, are complementary. They must add up to 90, or one right angle, by the sum of angles theorem.

So if we draw the altitude, which is perpendicular to the base and forms a right angle with it, then the two new triangles have angles that are also complementary to the originals, and therefore equal to other angles in the figure, as shown.

In the figure below, we have done the same construction. $\angle A$ is a right angle. The altitude or height h is drawn vertically, perpendicular to the base.



Our proof is a simple consequence of the relationships of similar triangles (all angles equal), namely, that corresponding sides are in the same ratio. We postpone the demonstration of this until later, so for the moment, I hope you'll just believe me.

Proof.

The small triangles are similar to the original. As a result, *the ratio of the hypotenuse to the short side* is equal for each, meaning that comparing the left side to the whole we get

$$\frac{b}{x} = \frac{a}{b}$$

Similarly, the ratio of the hypotenuse to the *long* side of the triangle on the right is equal to the same ratio in the original triangle:

$$\frac{c}{y} = \frac{a}{c}$$

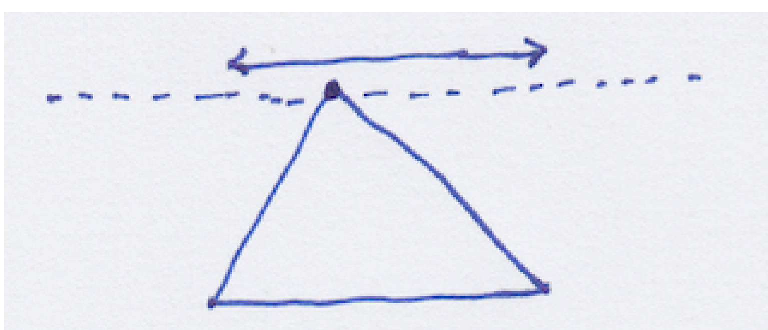
Rearranging, we have $b^2 = ax$ and $c^2 = ay$ so

$$b^2 + c^2 = a(x + y) = a^2$$

$\square$.

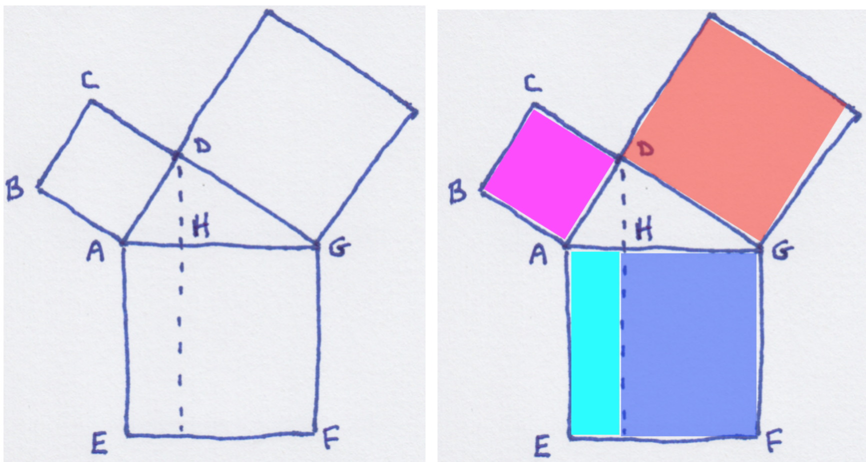
second proof

The second proof, due to Euclid, is a little more complicated. We need a preliminary result which we proved in the chapter on area:



For any triangle, if we draw a line parallel to the base, then the apex (top vertex) can be moved anywhere along that line, and the area of the resulting triangle will be the same.

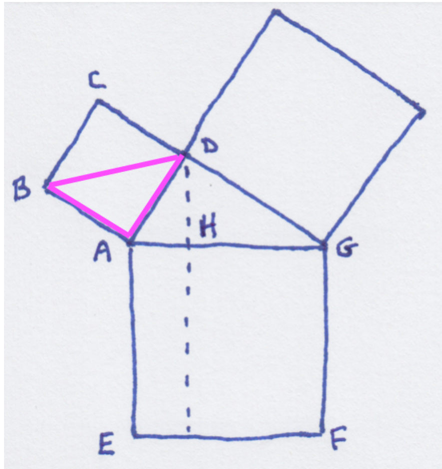
We have drawn the squares on each side of a right triangle ($\triangle DAG$). We draw a vertical down from D through H all the way to the bottom, parallel to AE.

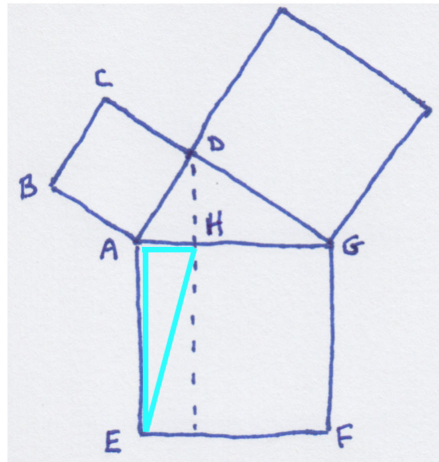


We will prove that the part of the large square that is cut off by the vertical extended from DH (cyan rectangle) is equal to the entire small square (magenta).

We'll achieve that by proving that the areas of triangles formed by drawing the diagonal in each quadrilateral are equal.

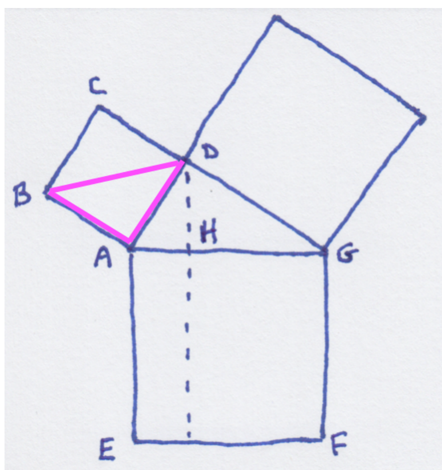
To prove. $\triangle ABD \cong \triangle AEH$.

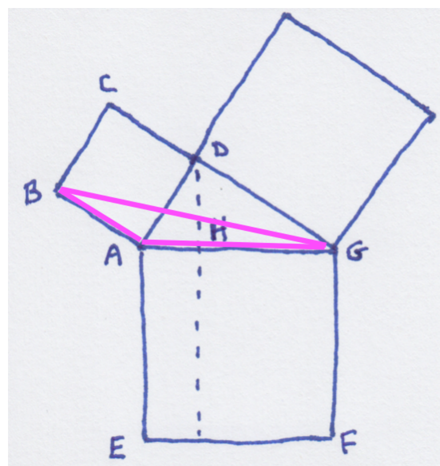




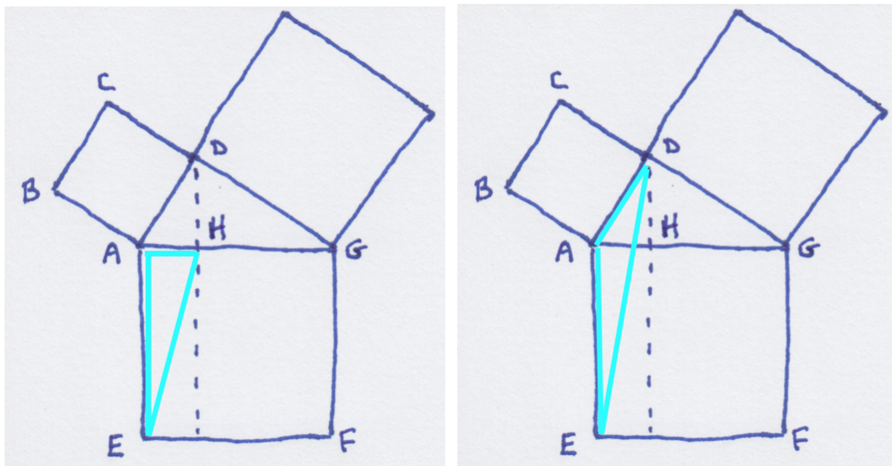
Proof.

For the first one, compare $\triangle ABD$ and $\triangle ABG$.



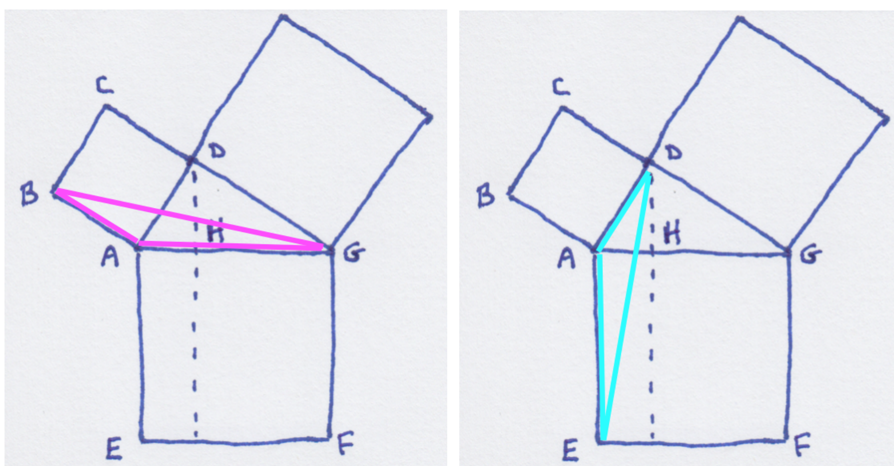


I say that the areas of the two triangles outlined in magenta are equal. The reason is that they have the same base AB, and the vertex moves from D to G along a line CDG that is parallel to the base AB.

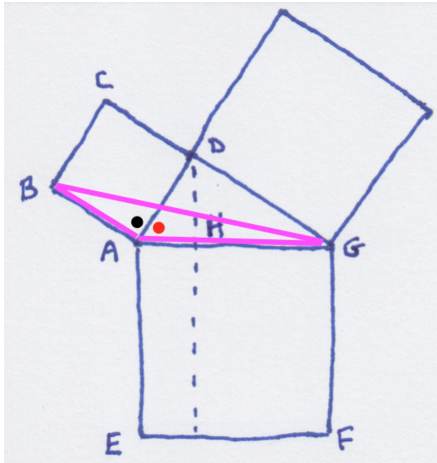


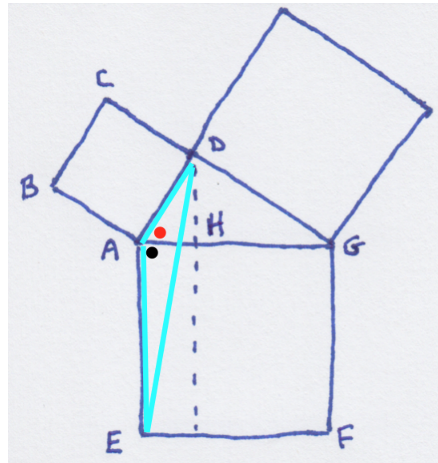
In the same way, I claim that the areas of the two triangles outlined in cyan are equal, because they have the same base AE and the vertex moves from H to D along the line DH, parallel to the base AE.

Now we will prove that $\triangle ABG \cong \triangle ADE$.



First, they share the side $AB = AD$. They also share the side $AG = AE$. If we can show that the central angles are equal, we will have SAS.



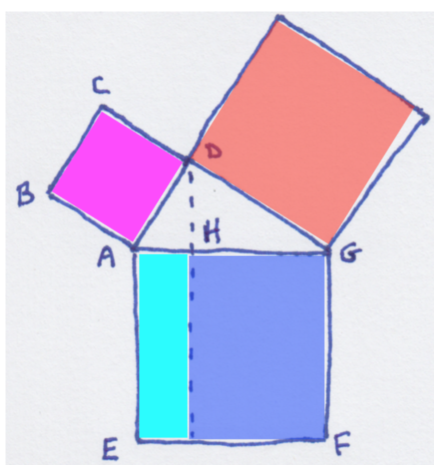


For the magenta triangle on the left, the central angle is one right angle (black dot) plus $\angle DAH$ (red dot).

For the cyan triangle, the central angle is one right angle (black again) plus the same angle as before, $\angle DAH$. Since each angle is the sum of a right angle and the same angle $\angle DAH$, they are equal.

$$\angle BAG = \angle DAE$$

We have SAS. Since each of the two triangles, $\triangle BAG$ and $\triangle ADE$, has equal area, so do the original triangles – $\triangle BAD$ and $\triangle AEH$.



And since the magenta square and cyan rectangle are twice the triangles referenced above, they are also equal in area.

We could repeat the argument for the other part of the figure, but we just appeal to symmetry.

$\square$.

Chapter 12

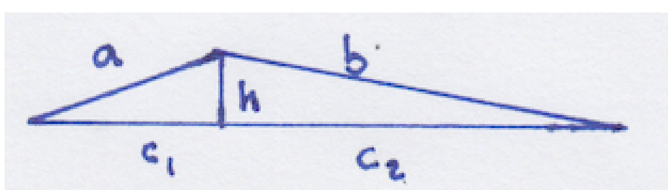
Triangle inequality

There is a famous inequality in geometry, the *triangle inequality*. It probably has more application in the theory of real numbers and complex analysis but we will use it for something important later.

The theorem says that in any triangle, the length of the longest side is always less than the sum of the other two.

It seems fairly obvious and one informal proof is a single statement: a straight line is the shortest distance between two points.

A slightly longer informal proof is to say that in any triangle we can drop a vertical line (an altitude) to the longest side.



Then, by the Pythagorean theorem, the two parts of c are clearly shorter than the sum of the two other sides since

$$c_1^2 + h^2 = a^2, \qquad c_2^2 + h^2 = b^2$$

Simple algebra (and a positive square root, for a length) gives $c_1 < a$ and $c_2 < b$ so

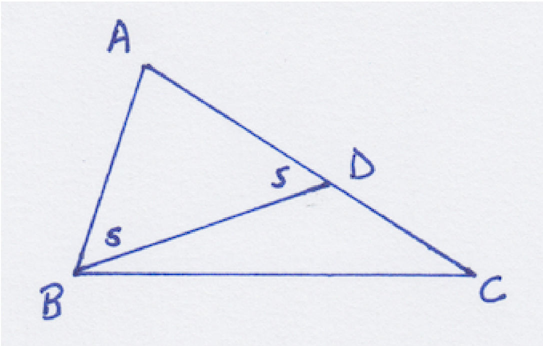
$$c = c_1 + c_2 < a + b$$

Euclid proves this theorem in book 1 of *Elements*. We need a preliminary theorem, called a *lemma*.

• In any triangle, if one side is the longest, then the angle opposite is the greatest.

Proof.

Given that side $AC > AB$, mark off the latter on AC, forming an isosceles triangle.



By the exterior angle theorem, $s > \angle C$. But $\angle B > s$ (the whole is greater than any part). It follows that $\angle B > \angle C$.

$\square$

The converse is also true:

Proof

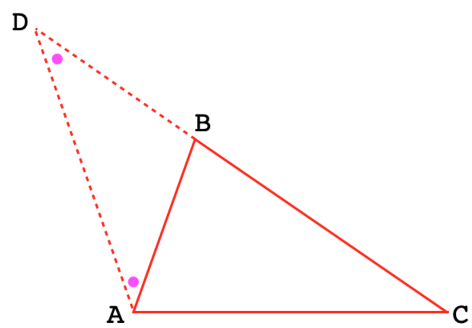
Given that $\angle B > \angle C$ above. It cannot be that $AB = AC$, because then the angles would be equal. Suppose that $AB > AC$. But then $\angle C > \angle B$ by the forward theorem. But that contradicts what we're given. Hence $AC > AB$.

□

Triangle inequality

In any triangle, the longest side is smaller than the sum of the two shorter sides. Clearly, if a given side is not the longest, i.e. it is shorter than one of the others, then it must also be shorter than the sum of that other one plus the third. For this reason, we consider only the longest side.

Proof.



Given that side AC is the longest in $\triangle ABC$.

Extend side BC so that $BD = AB$. By the isosceles triangle theorem, since $\triangle ABD$ is isosceles, $\angle D = \angle DAB$ (marked with magenta dots).

Then $\angle D$ is smaller than $\angle DAC$ and therefore, by the previous lemma, AC is less than DC. But DC is equal to the sum of the two smaller sides of $\triangle ABC$. Hence

$$AC < AB + BC$$

□

hypotenuse

We said above that if one side is longer, then the angle opposite is larger. The converse is also true but we will skip the general case. However, we prove the specific case of a right triangle.

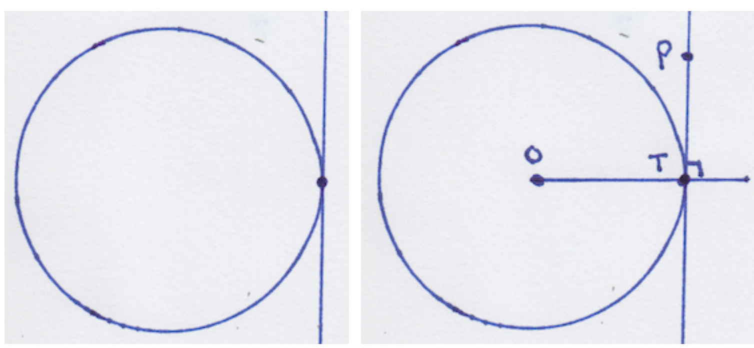
Proof.

In any right triangle, the right angle is the largest angle. The reason is that the two base angles (flanking the hypotenuse) add up to be one right angle, and they must both be non-zero. Hence the largest must be smaller than a right angle. It follows that the hypotenuse is the longest side in a right triangle.

□

tangent to a circle

The simple definition of the tangent is that it touches the circle at only one point, and as a consequence, it forms a right angle there.



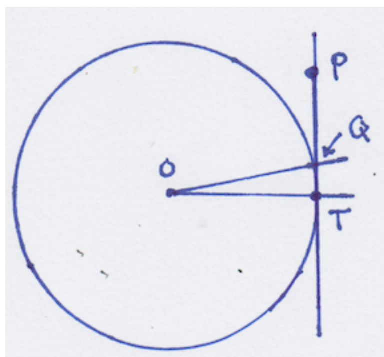
Proof.

Suppose that the angle between the tangent and the radius at T is not a right angle. Then, there must be some *other* point on the tangent line which does form a right angle when connected to the center. There is always one such point, because we have a construction that gives the perpendicular (bisector) from a line to an external point.

But that would make the radius to the point of tangency the hypotenuse of a right triangle. The hypotenuse is the longest side in a right triangle, so the new point must be *inside* the circle. This is a contradiction, since the tangent touches the circle in only one point.

□

Given that the tangent forms a right angle at its single point of contact, the result about the hypotenuse above (our lemma), says that any other line drawn from the center of the circle to the tangent line would be a hypotenuse and so necessarily longer than the radius to the tangent. Such a point must be outside the circle.



Chapter 13

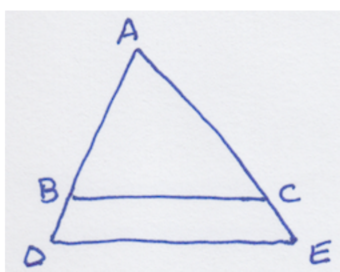
Similarity

equal angles $\rightarrow$ parallel

When two triangles are *similar*, they have the same shape but are scaled differently. By the same shape, we mean they have the same three angles.

Because the angles are the same, we can draw the two triangles nestled inside one another, picking any of the three angles as the common vertex.

In the figure, $\triangle ABC \sim \triangle ADE$ ($\sim$ is the symbol for similarity).

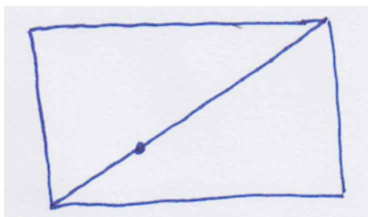


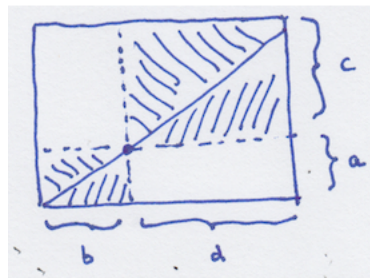
And because the angles are the same, say, $\angle ABC = \angle D$, it follows that $BC \parallel DE$ by the converse of alternate interior angles.

similar sides in a right triangle

There are other properties that go along with similarity. We will prove that for similar right triangles, all angles equal implies equal ratios of sides. Our approach is from Acheson and is based on an observation about area.

Draw a rectangle and one diagonal and pick a point on the diagonal





All of the right triangles in the figure are similar. The ones have a shared length of the diagonal are congruent. This part is easily proved with the alternate interior angles theorem, then use complementary angles in a right triangle, and finish with vertical angles.

By changing the height of the figure, we can obtain any two complementary angles we wish. And by changing the placement of the central point we can get any ratio.

Finally, the two shaded rectangles are bisected by the diagonal. This is a basic property of rectangles; we have congruent $\triangle$ by SSS). So both pairs of congruent triangles have equal area.

But the big triangles formed by the diagonal are also congruent and also have equal area.

Therefore, we just subtract equal areas to find that the two unshaded rectangles above and below the diagonal are equal in area. The one on top has area bc and the one below has area ad. We have

$$bc = ad$$
$$\frac{a}{c} = \frac{b}{d}$$

A bit of algebra gives:

$$\frac{a}{c} + \frac{c}{c} = \frac{b}{d} + \frac{d}{d}$$
$$\frac{a+c}{c} = \frac{b+d}{d}$$
$$\frac{a+c}{b+d} = \frac{c}{d} = \frac{a}{b}$$

We have added one to each side before the rearrangement, maintaining equality.

hypotenuse

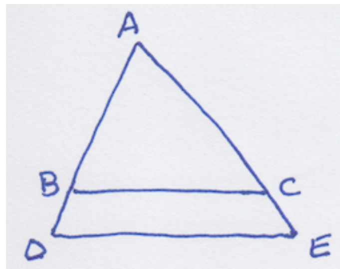
It is natural to ask, what about the hypotenuse? There is an algebraic proof based on the Pythagorean theorem. However, our easiest proof of the Pythagorean theorem is based on similarity. It would be a circular (and logically invalid) argument!

Luckily, we have Euclid's proof of the Pythagorean theorem, which uses SAS, and this would get us out of the trap. So, we could prove that similar right triangles have equal ratios of hypotenuse as well as sides, and then extend that proof to all triangles by dissecting them into right triangles.

But we have two terrific proofs from Euclid so why not just go ahead to the general case.

□

all similar triangles



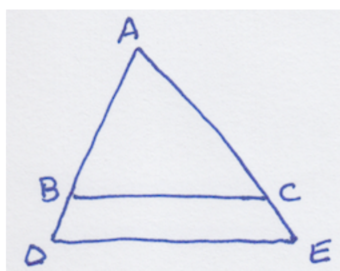
Above, $BC \parallel DE$ (BC is parallel to DE). In addition, we have equal ratios of sides.

$$\frac{AB}{AD} = \frac{BC}{DE} = \frac{AC}{AE}$$

and

$$\frac{AB}{BD} = \frac{BC}{DE - BC} = \frac{AC}{CE}$$

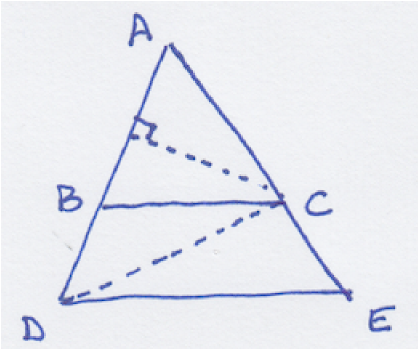
The second set of ratios is easily derived from the first exactly as we did before.



parallel $\rightarrow$ equal ratios

Given: $BC \parallel DE$. We claim that the ratios given above follow.

Notice that C is a vertex of both $\triangle ABC$ and $\triangle BCD$. Therefore, the altitude from C to ABD is the same for both triangles.

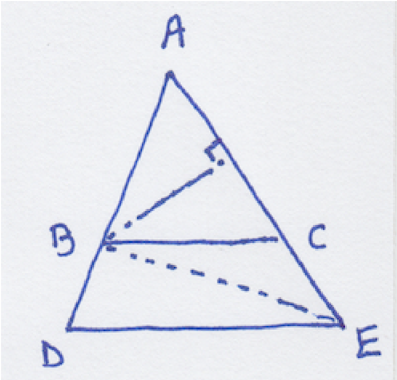


If we define the symbol (ABC) to mean area, then

$$\frac{(ABC)}{(BCD)} = \frac{AB}{BD}$$

The areas of the two triangles are in the same ratio as the lengths of the bases, since the altitudes are the same.

A symmetric construction



shows that

$$\frac{(ABC)}{(BCE)} = \frac{AC}{CE}$$

However, $(BCD) = (BCE)$. The two triangles have the same base, BC, and their altitudes are the same because the respective vertices at D and E lie on the same line parallel to BC.

Therefore the first two expressions from above are equal.

$$\frac{(ABC)}{(BCD)} = \frac{AB}{BD}$$

$$\frac{(ABC)}{(BCE)} = \frac{AC}{CE}$$

Since $(BCD) = (BCE)$, the left-and sides are identical, therefore, so are the right-hand sides.

$$\frac{AB}{BD} = \frac{AC}{CE}$$

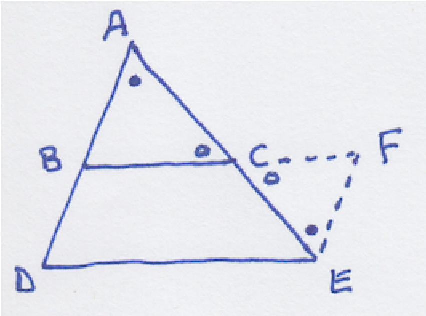
□

equal ratios → parallel

Given that $\triangle ABC$ has equal ratios to $\triangle ADE$.

$$\frac{AB}{BD} = \frac{AC}{CE} = \frac{BC}{DE}$$

We claim that $BC \parallel DE$.



Just to be clear, we are allowed to use the previous theorem to help us prove its converse. Given all 3 angles equal, it follows that equal ratios of the sides obtains. We will then use equal ratios to prove the parallel property.

Proof.

First, draw EF extending up from E, with $EF \parallel ABD$.

Alternate interior angles gives us that $\angle A$ is equal to $\angle CEF$, marked with filled dots. Another angle equality is at vertex C, by vertical angles. Since two angles are equal, all three are, and this means that all three sides are in equal ratios by the previous theorem.

Since $\triangle ABC \sim \triangle CEF$

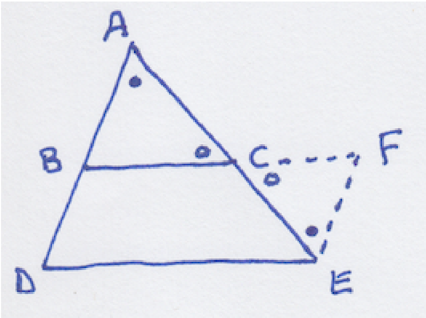
$$\frac{AB}{EF} = \frac{AC}{CE} = \frac{BC}{CF}$$

but we were given that

$$\frac{AC}{CE} = \frac{AB}{BD}$$

so

$$BD = EF$$



We have $BD = EF$ and we started with $BD \parallel EF$. We also have

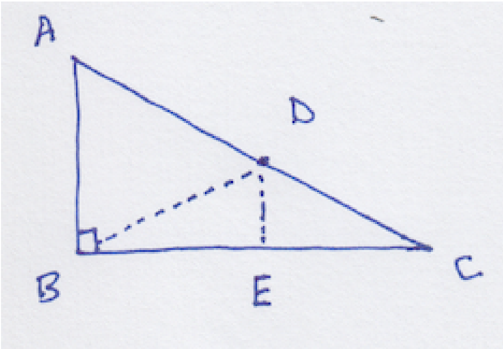
$$\frac{BC}{BF} = \frac{AC}{AE} = \frac{BC}{DE}$$

So $BF = DE$ and it follows that $BCED$ is a parallelogram. Therefore $BCF \parallel DE$.

$\square$

median theorem

Draw a line segment from the right angle of a right △ to the hypotenuse, such that it is the median, that is $AD = DC$. Prove that BD is also equal to AD and DC.



Proof.

Here's a standard approach using similar triangles. Drop the perpendicular from D to E. By similar triangles, since $AD = DC$, $BE = EC$. So $\triangle BDE \cong \triangle DCE$.

It follows that $BD = DC$, $\triangle BDC$ is isosceles, and so is $\triangle ABD$.

□

Even better, use the converse of Thales' theorem.

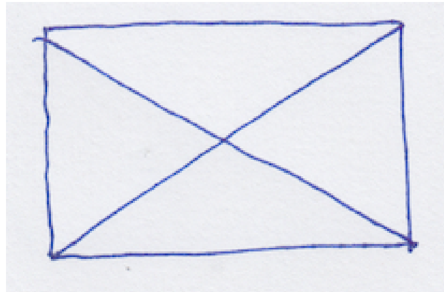
Proof.

Since $\angle ABC$ is a right angle, we can draw it in a triangle with AC as the diameter. We are given that $AD = DC$.

Since B is on the circle and D is the center of the circle, BD is also a radius.

□

Still better, a proof without words.



With words: draw a second right $\triangle$ to form a rectangle. The diameters of any rectangle cross at the midpoints of each.

Proof.

The triangles on top and bottom are congruent (SSS). Therefore the two diameters are bisected.

□

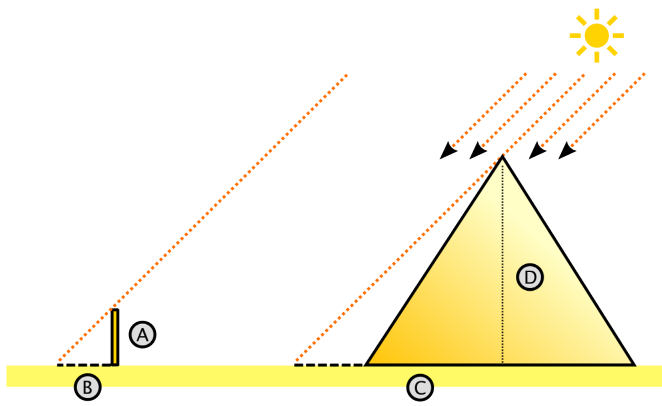
pyramid height

Thales was from Miletus and he lived around 600 BC. He is believed to have traveled extensively and was likely of Phoenician heritage. As you probably know, the Phoenicians were famous sailors who founded many settlements around the Mediterranean.

They competed with the mainland Greeks and later with the Romans for colonies, and their major city, Carthage, was destroyed much later by the Romans, in the third Punic War. Hannibal rode his famous elephants over the Alps in the second Punic war.

During his travels, Thales went to Egypt, home to the great pyramids at Giza, which were already ancient then. They had been built about 2560 BC (dated by reference to Egyptian kings) and were already 2000 years old at that time!

The story is that Thales asked the Egyptian priests about the height of the Great Pyramid of Cheops, and they would not tell him. So he set about measuring it himself. He used similar triangles. I'm sure he wrote down his answer, but I'm not aware that it survives. The current height is 480 feet.



Plutarch says this:

> the king finds much to admire in you, and in particular he was immensely pleased with your method of measuring the pyramid, because, without making any ado or asking for any instrument, you simply set your walking stick upright at the edge of the shadow which the pyramid cast, and, two triangles being formed by the intercepting of the sun's rays, you demonstrated that the height of the pyramid bore the same relation to the length of the stick as the one shadow to the other.

So the actual method is even cleverer than the diagram shows.

Part IV

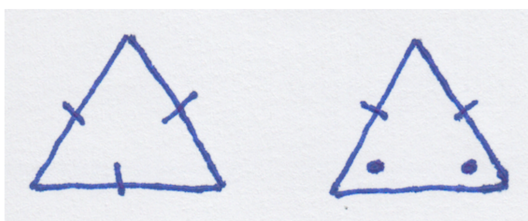
Other polygons

Chapter 14

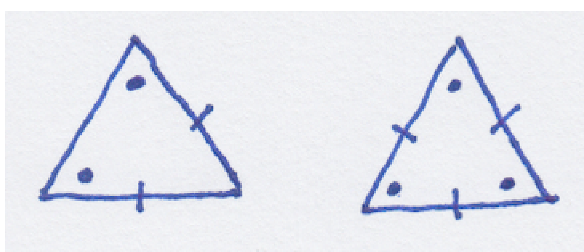
Equilateral triangles

We've seen a lot about isosceles triangles already, triangles with two sides equal. Now we look at equilateral triangles, with three sides equal. That's what "equilateral" means.

We can use the forward version of the isosceles triangle theorem once.

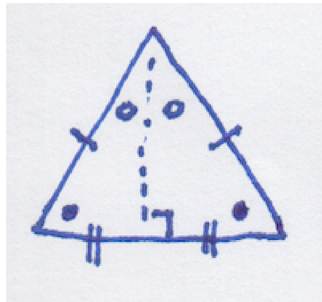


And then use it again.



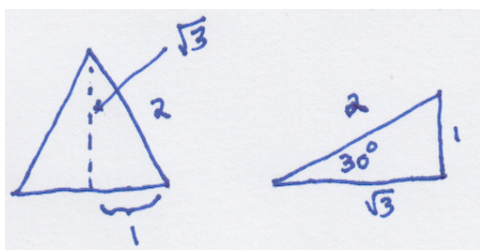
We see that an equilateral triangle has all three angles equal as well. Thus each angle is $180 \div 3 = 60$.

Looking back at our proof of the isosceles triangle theorem,



The angle at the top is bisected, so each half is 30. That matches the base of the altitude, which is a right angle. We have two 30-60-90 right triangles and the two smaller angles in each of those small triangles add up to one right angle. The base is also bisected.

Let's look at some ratios of lengths. It is most convenient to let one-half of the base have length 1, so then the side is 2.



The third side comes from the Pythagorean theorem:

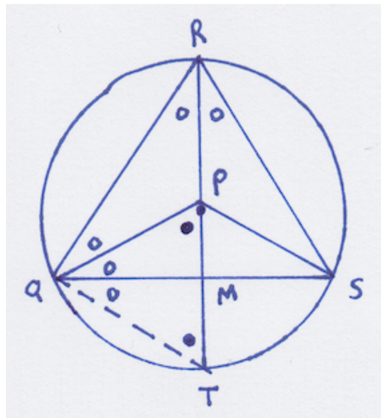
$$2^2 = 1^2 + h^2$$

$$h = \sqrt{3}$$

inscribed triangle

We want to draw an equilateral triangle with each of its vertices in the same circle, called a circumcircle. Start by drawing a diameter of the circle, vertically.

Anticipating the next result, to draw the base of the triangle, divide the radius from the center to the bottom of the circle in half ($PM = MT$), then draw a horizontal line QS there. Draw QR and SR to the apex at the top.

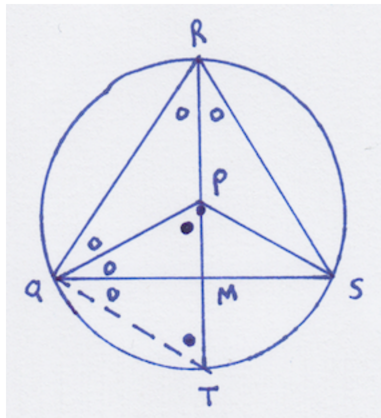


QS is called a "chord" of the circle. Since we've drawn it perpendicular to the radius PT, the resulting $\triangle PMQ \cong \triangle PMS$ by hypotenuse-leg in a right triangle. The extension of the radius to R forms a shared side for two triangles $\triangle RMQ$ and $\triangle RMS$. Since RM is shared and $QM = MS$, and the angles at M are right angles, these two triangles are also congruent.

Therefore, the diameter bisects the angle at vertex R, each half marked with an open dot. The angle $\angle RQP$ is equal by the isosceles triangle theorem. $\angle RQP = \angle PQM$ for the same reason as the angle at R is bisected, because QP is a diameter of the circle drawn through the vertex at Q.

Lastly, the two angles $\angle RQP$ and $\angle PQM$ together form one whole angle of an equilateral triangle, which is two-thirds of a right angle, but $\angle RQT$ is a right angle, by Thales' theorem. So $\angle MQT$ is equal to the others.

This construction also forms a smaller $\triangle PQT$. We can use algebra to show that it is equilateral. The right angle contains three of the dotted angles, so $\angle PQT$ contains two, and since $\triangle QPT$ contains two radii of the circle $\angle PQT = \angle QTP$. We get the third angle from the triangle sum theorem.

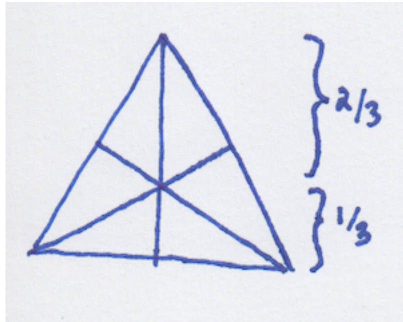


Another way to see that is to use the inscribed angle theorem and show that $\angle QTP = \angle QSR$, because they are both peripheral angles that cut off the same arc of the circle.

We wonder about the distance from the base of the triangle to the center, PM. It's the same as the distance as MT, by construction. So MT is one-half the radius.

Hence the distance RM is one whole radius plus a half. So the distance from the base to the center at P divided by the whole altitude is

$$\frac{1/2}{3/2} = \frac{1}{3}$$



The place where lines like this meet has various names, depending on how the lines are chosen. Since we're talking about altitudes of the triangle, the point is called the orthocenter.

Another way to calculate MT is to use similar triangles. That length is one-half the base of $\triangle QPT$, so it is to its altitude (1/2 of RS, which is $1/2 \cdot 2 = 1$) as the ratio $1/\sqrt{3}$ in the large triangle.

$$MT = \frac{1}{\sqrt{3}}$$

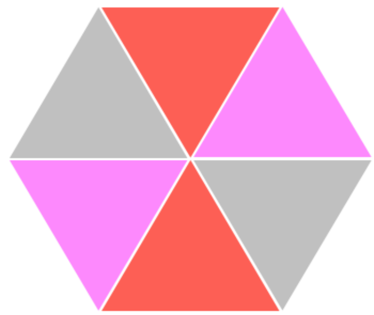
Hence the ratio of MT to the altitude of the large triangle is

$$\frac{1/\sqrt{3}}{\sqrt{3}} = \frac{1}{3}$$

as we said.

hexagon

A hexagon can be composed of six equilateral triangles. One reason is that the whole central angle of 360 is divided into six equal parts, so each triangle has central angle of 60, but it is also isosceles, so the other angles are 60 as well..



We notice that walking along the perimeter, at each turn we go 60 to the left, meaning that the inside angle is 120 or twice 60.

There is a famous theorem, usually proved by induction, that the sum of internal angles in any polygon is equal to 180 times the number of sides greater than 3, plus 180. Here, that number is 3, giving 180(3) + 180 = 720.

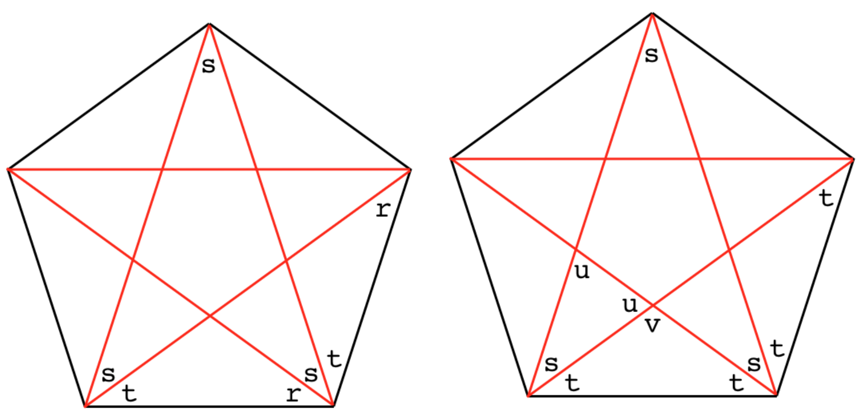
You can easily confirm that 6(180) = 720. It's worth thinking about why that theorem makes sense. Hint: it has to do with the fact that a 4-sided figure can be cut into 2 triangles, a 5-sided figure into 3, and a 6-sided figure into 4.

Chapter 15

Pentagon

In this chapter we explore some properties of a regular pentagon. A pentagon has 5 sides, and a regular polygon has all sides equal. The regular pentagon has five-fold rotational symmetry.

Draw all of the internal chords of the figure and label a few angles. By rotational symmetry each of the five vertices of the pentagon has the same three components, the central one labeled s, and two flanking angles r and t (left panel).



First, we observe that the triangle formed with two equal sides at the lower right, has equal base angles by the isosceles triangle theorem.

Therefore $r = t$ (right panel).

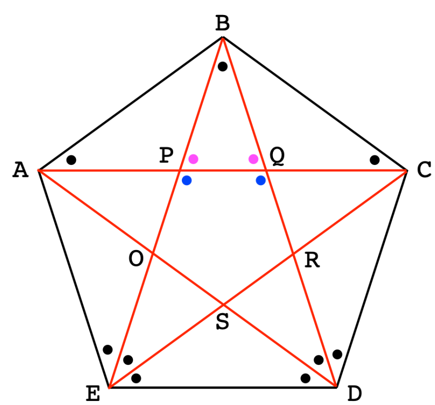
Next, we count up all the angles in two triangles. In the tall skinny triangle we have $3s + 2t$ while in the short, squat one on the right we have $4t + s$. But these are equal, by the triangle sum theorem:

$$3s + 2t = 4t + s$$

I'm sure you can complete the proof. $s = t$. All three angles at each vertex are equal. We also have that s is one-fifth of 180, so $s = 36$.

The central angle subtending each face (not shown) is $2s = 72$. We can get that by the inscribed angle theorem, since we know s, or simply from the five-fold symmetry.

Continuing:

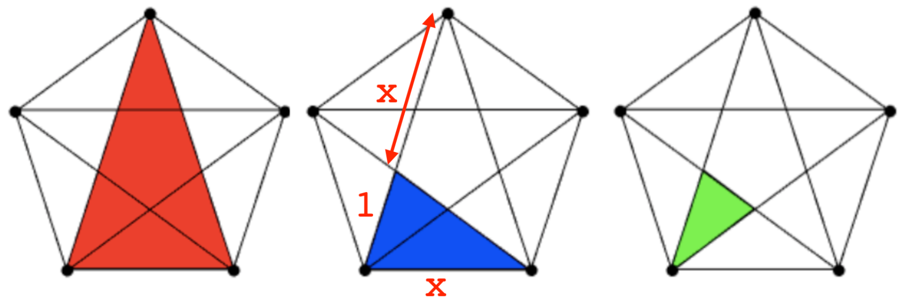


By counting the angles in a triangle again, we conclude that the angles marked with pink dots (e.g. $\angle BPQ$) are equal to two copies of s. We conclude that $AC \parallel ED$ by the converse of alternate interior angles.

The chord above the base that looks like it's parallel *is* parallel. The parallelograms inside the figure like $PCED$ are actually rhombi, with all sides equal, since opposite sides of a parallelogram are equal.

All this means that we have two sets of similar triangles, as well as a small pentagon at the center. $\triangle BPQ \sim \triangle BED$ and $\triangle ABE \sim \triangle ABP$.

Here are three examples of the tall skinny type:



Suppose we scale things so the base of the skinny blue triangle is 1 and the side of the pentagon is x. Then, for the tall skinny red triangle the ratio of sides is $(x+1)/x$ and for the blue one it's $x/1$. This gives

$$\frac{x+1}{x} = x$$

$$x^2 = x + 1$$

We won't solve this equation here, but just give the result.

x is usually called ϕ, the famous *golden mean* or golden ratio:

$$\phi = \frac{1}{2}(1+\sqrt{5})$$

We should check that ϕ really does solve the equation:

$$\phi^2 = \frac{1}{4}(1+\sqrt{5})^2$$

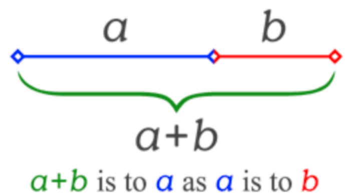
$$= \frac{1}{4}(1+2\sqrt{5}+5)$$

$$= \frac{1}{4}(6+2\sqrt{5})$$

$$= 1+\phi$$

golden mean

One way to introduce the golden ratio is to take an arbitrary length and pick a point on it, dividing the whole into one part of length a and the other of length b. Suppose a is greater than b.

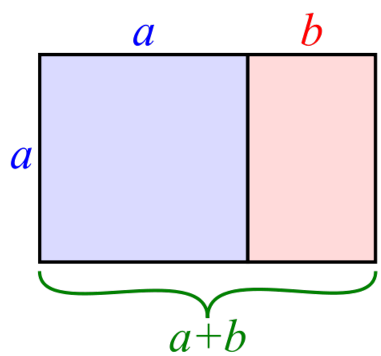


Then, the definition of the golden ratio $a : b$ is

$$\frac{a+b}{a} = \frac{a}{b}$$

The ratio of the whole length to a is the same as the ratio of a to b.

We can do a similar trick with rectangles.



Draw a square with sides a, and extend it a distance b to form a rectangle of side length $a + b$. We want the proportions of the two rectangles in the figure to be the equal, and this gives the same equation as before.

Going back to that equation, since the length is arbitrary, we can scale it so $b = 1$.

$$a = \frac{a+1}{a}$$
$$a^2 = a + 1$$

This is exactly what we had above.

construction

It is quite a challenge to construct a pentagon freehand. Some very interesting methods using ruler and compass are given in wikipedia

https://en.wikipedia.org/wiki/Pentagon

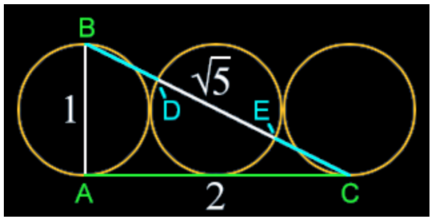
Here is a third one.

https://www.goldennumber.net/phi-formula-geometry

We first construct the length

$$\phi = \frac{1}{2}(1 + \sqrt{5})$$

Draw 3 identical circles of diameter 1 next to each other. It might help to first construct a rectangle with sides of length 2 and 1/2 and subdivide the length into two parts, in order to place the centers and bases in the right orientation.

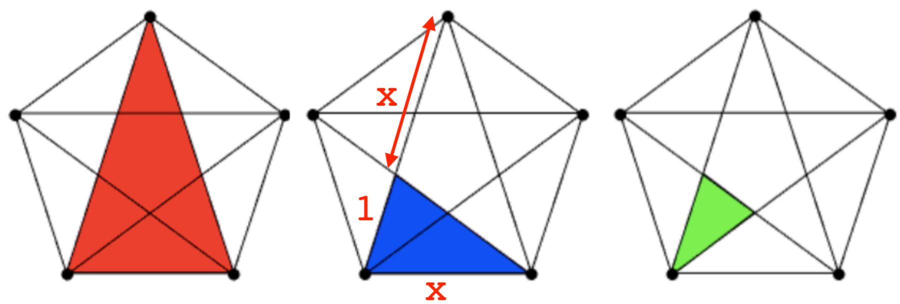


Find the base point at C and connect BC. The Pythagorean theorem gives us that $BC = \sqrt{5}$. So from B to the center of the middle circle is one-half that, and extending along the line to the point E adds another $1/2$. We have

$$BE = \frac{\sqrt{5}}{2} + \frac{1}{2}$$

which, with a slight rearrangement, can be seen to be equal to ϕ.

So now we have the length ϕ, based on a given length 1. Go back to the tall skinny red triangle:



To duplicate this, mark off a length 1 for the base. Then find the intersection of two arcs of length ϕ, putting one pin of the compass on each endpoint of the base. This fixes the top vertex.

Finally, mark off arcs of length 1 from the top and the sides of the base to place the other two vertices.

Brief aside on ϕ and the Fibonacci sequence

$\phi \approx 1.618$.

The Fibonacci sequence is defined as $F_n = F_{n-1} + F_{n-2}$, starting with $1, 1$ as the first two numbers.

The first ten numbers in the sequence are:

`1 1 2 3 5 8 13 21 34 55 ...`

As the numbers in the Fibonacci sequence get larger, the ratio F_n/F_{n-1} becomes a better and better approximation to ϕ. Binet's formula turns this around and uses ϕ to calculate the nth Fibonacci number.

Here is some Python code to give large values of F_n and calculate the ratio F_n/F_{n-1}:

```
>>> phi = (1 + 5**0.5)/2
>>> phi
1.618033988749895
>>>
>>> def f(n):
...     a,b = 1,1
...     for i in range(n):
...         a,b = a+b, a
...     return a,b,a/b
...
>>> f(10)
(144, 89, 1.6179775280898876)
>>> f(30)
(2178309, 1346269, 1.6180339887496482)
>>>
```

(This code relies on the fact that the value of a is cached for the tuple assignment a,b = a+b, a). It should be clear that F_n/F_{n-1} becomes a better and better approximation to ϕ as it gets larger.

irrationality

ϕ is an *irrational* number, a concept that gave the Greeks fits.

This means that it cannot be written as the ratio of two integers, and what amounts to the same thing, an accurate decimal representation goes on forever.

One way to see this is to construct the continued fraction representation of ϕ.

$$\phi^2 = \phi + 1$$

Dividing by ϕ we get

$$\phi = 1 + \frac{1}{\phi}$$

The trick here is to see that the *entire* right-hand side of the equation above is equal to ϕ so that whole thing can be substituted for the denominator of the second term on that side.

$$\phi = 1 + \frac{1}{1 + \frac{1}{\phi}}$$

This goes on forever.

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}$$

Those dots mean that to evaluate this expression, at some point we must decide to neglect the remaining terms. If we take a finite number of terms, there will always be some error between what we calculate and the true value of ϕ.

Going back up the chain, we start with by ignoring the dots to get 1 and then repeatedly add one, invert, add one, invert, etc.

```
1 + 1 = 2
1/2 + 1 = 3/2
2/3 + 1 = 5/3
3/5 + 1 = 8/5
5/8 + 1 = 13/8
```

Can you see the Fibonacci sequence developing?

Part V

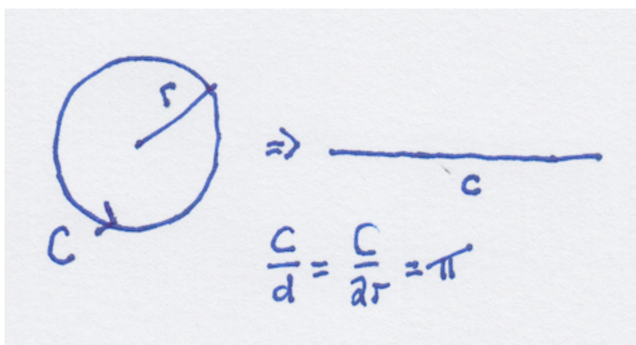
Circles

Chapter 16

Circles

Pi

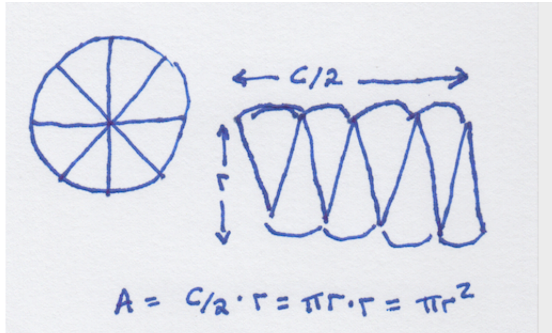
You undoubtedly know that the ratio of the circumference of a circle to its diameter is equal to the special value, π.



Since the diameter is twice the radius, we have that $C = 2\pi r$. This is true no matter how big or how small we make the circle.

It is amazing that the same constant π is involved in the area of a circle. The formula for area is

$$A = \pi r^2$$



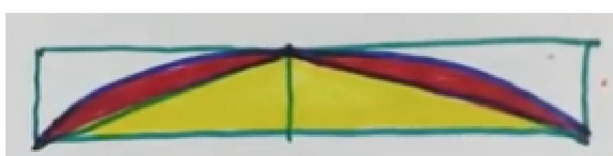
Above is a figure which makes this result seem reasonable.

Cut a circle into an even number of equal-sized slices and arrange them as shown on the right. The height of the resulting rectangle-like figure is r and the length of the combined top and bottom is C, so the width is $C/2 = \pi r$. Using the rule for rectangles the area is

$$C = \frac{C}{2} \cdot r = \pi r \cdot r = \pi r^2$$

One simple thing to improve the estimate is to cut one of the end slices in half and add it to the other end. That makes the end pieces vertical. The other improvement is to cut the pizza into more slices. This will have the effect of making the curvy edges straighter.

Imagine if we just cut off the curved edges and ignored them. The error will be significant with 8 pieces, but if the number of pieces is doubled, the error is more than halved. Here's how it looks.



The error (underestimate for the area) for the whole sector is the yellow plus the red, that for the half-sector is just the red alone. The latter is clearly less than half of the former, since there is less red than yellow.

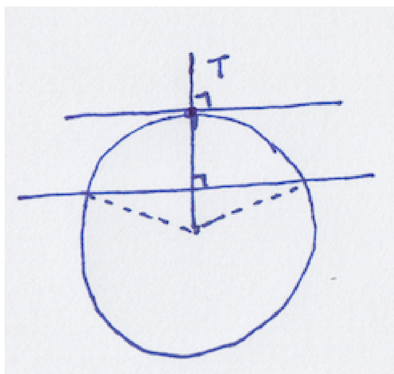
Perhaps it is more like 1/3. So the total error obviously decreases as we make the curvy edges straighter by cutting the pizza into more slices.

https://www.youtube.com/watch?v=R1HUtt2oo7A

There is a subtle argument to make sure that we get all the way to the actual value. But it looks persuasive, and in fact, it works.

one chord

Draw any radius in a circle and the tangent where it meets the circle. Then draw a chord that is parallel.

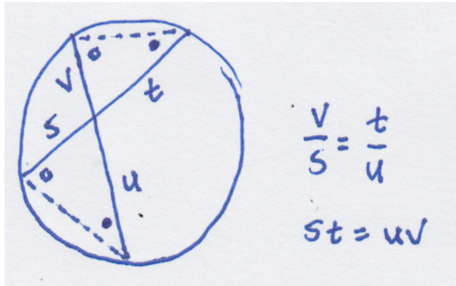


The radius cuts the chord in a right angle (because of the parallel construction). This means that the two small triangles with parts of the chord and radii for sides are congruent (hypotenuse and a side plus a right angle). So the chord is bisected.

Conversely, if we are given that the chord is bisected, then the two triangles are congruent by SSS. So the angle where the radius cuts the chord is a right angle, which makes the chord parallel to the tangent drawn at the point where the radius meets the circle.

two chords

Draw any two straight lines (called chords) in a circle.



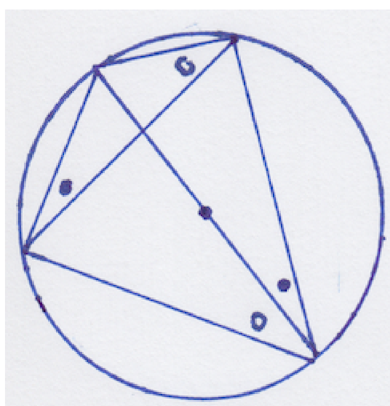
Recall the inscribed angle theorem. The marked angles are peripheral angles that cut off the same arc from the circle. By the theorem, they are equal, as marked.

This means that the two triangles are similar. Ratios of similar sides give the result.

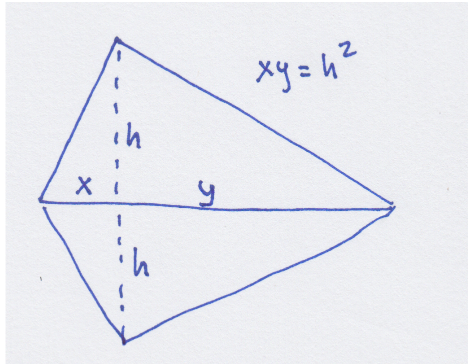
$$\frac{v}{s} = \frac{t}{u}$$
$$st = uv$$

This is true for any two arcs of a circle.

Suppose we have two right angles inscribed into a circle along the same diameter.



Below is a special case, where the dotted line forms right angles where it crosses the diameter.



Then, we have peripheral angles that subtend the same arc, so they are equal, and that gives similar right triangles. We form the ratios: $y/h = h/x$.

Alternatively, the result from above about chords means that

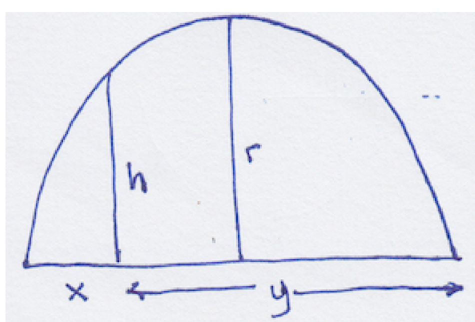
$$xy = h^2$$

Rearranging

$$\frac{x}{h} = \frac{h}{y}$$

which means that the two smaller triangles are similar. A third way we know this is from complementary angles. We used it when we proved the Pythagorean theorem.

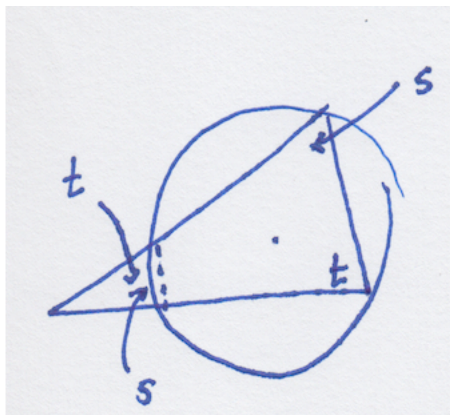
In the construction, we have the relationship $h = \sqrt{xy}$. In statistics this is called the geometric mean of x and y.



But we also have that $x + y = d = 2r$. Hence $r = (x + y)/2$. r is the arithmetic mean of x and y. Imagine making x larger and y smaller or vice-versa.

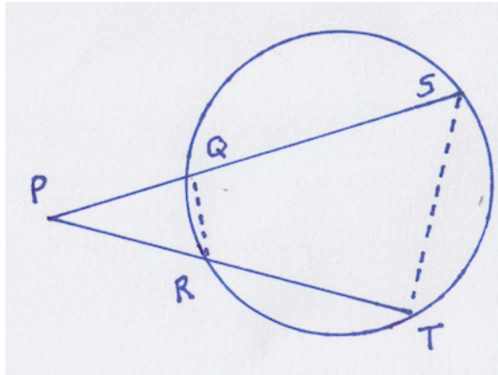
The diagram shows that the arithmetic mean is always greater than the geometric mean, except when $x = y$. Then, both are equal to r.

In the figure below, pick any two points in a circle and one exterior point and draw the triangle. Also connect the points where the two sides we just drew intersect the circle, to form a smaller triangle.



We have four points in a circle, forming a special four-sided figure called a cyclic quadrilateral. In this figure, the opposing angles are supplementary, they add to 180. This means the two angles s are supplementary to the same angle. The same is true of t.

Therefore we have two similar triangles. So then we form the ratios of corresponding parts.



$$\frac{PR}{PS} = \frac{PQ}{PT}$$
$$PR \cdot PT = PQ \cdot PS$$

Finally, imagine that we slide the ends of the chord so that S and Q move toward each other, and finally meet at the tangent point. Call the new point S'.

PS' is a tangent to the circle and

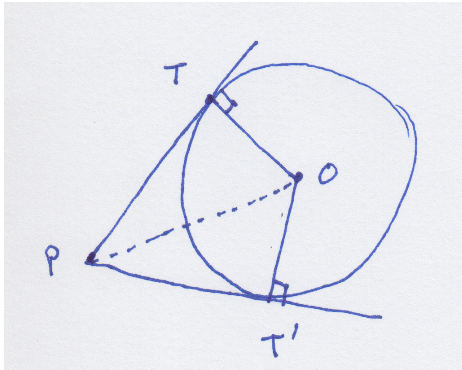
$$PS'^2 = PR \cdot PT$$

Do the same thing on the other side, forming PT'. Now

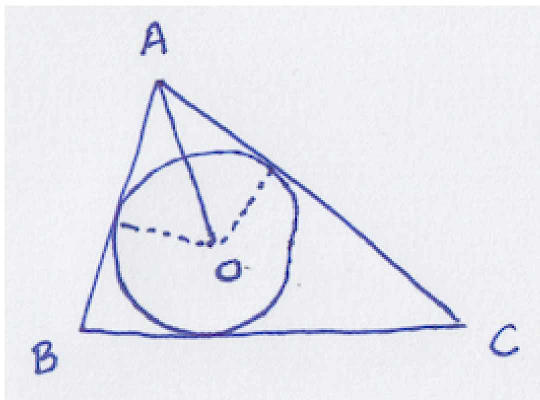
$$PS'^2 = PT'^2$$

$$PS' = PT'$$

The two tangents drawn to a circle from any external point are equal. Draw the two tangents, plus a line from the point to the center. Form two congruent triangles (by side-hypotenuse in a right $\triangle$). So $PT = PT'$.



There is a circle that is just inscribed into a triangle, where each side is a tangent and just touches the circle. It is called the incircle.



Draw AO to form two triangles. They are right triangles, because the sides are tangent. Since they share a side and have a second side that is a radius, they are congruent. Therefore AO bisects $\angle A$.

Also, the distance from A to the points of tangency is equal on both sides. The tangents drawn from any exterior point to a circle are equal. Call this distance x. Then the combined area of these two triangles is just rx. The total area of the triangle is $r(x + y + z)$ where $x + y + z$ is equal to the semi-perimeter of the triangle (one-half).

But the perimeter is the sum of the sides $a + b + c$. So

$$\triangle = r(x + y + z) = rs$$

where

$$s = \frac{(a+b+c)}{2}$$

Chapter 17

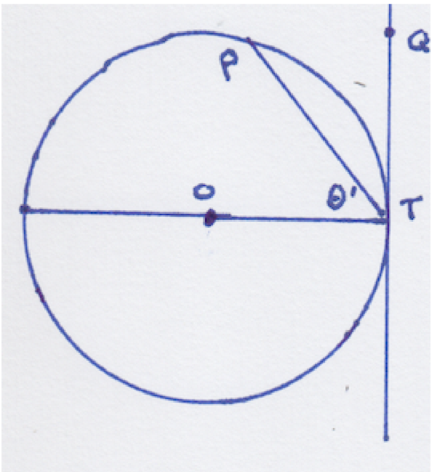
Arcs

arc of a tangent

We've talked about angles and arcs in the context of the inscribed angle theorem, previously.

What about the arc formed by an angle containing a tangent line?

We have a circle centered on O with $OT \perp QT$, the tangent line.

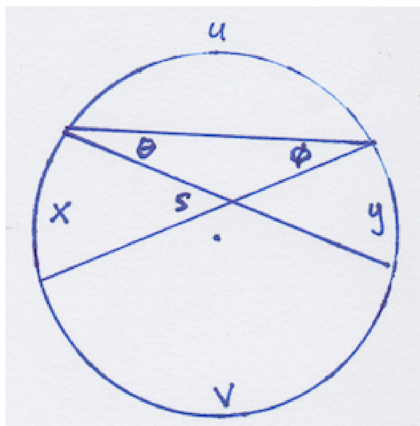


We want to find the angle formed with a chord of the circle like PT. Let that angle be θ and then, because space is a bit tight, label its complementary angle θ'.

θ' subtends the arc from P counter-clockwise to where the diameter meets the circle on the left. But $\angle OTQ$ is a right angle, $\theta + \theta' = 90$. It's clear that θ subtends the arc PT, as expected. Twice θ will be equal to the length of the arc PT.

crossed chords

Two chords cross in a circle. We already know a theorem about the parts of the chords, but this question is about angles.



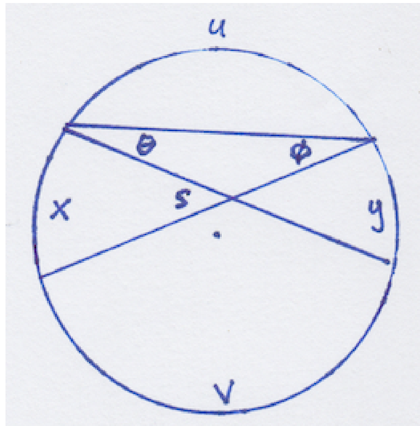
One of the angles where the two chords cross is s. What is θ in terms of the arcs x and y? We might guess. If the arcs crossed at the center then s would be exactly equal to either of the two equal arcs x and y, so $2s = x + y$.

And if s were on the left or right periphery then one of x and y would be zero and $2s$ would be equal to the other one. Again, $2s = x + y$.

So we guess that $s = (x + y)/2$, s is the *average* of x and y.

Proof.

Draw the chord connecting the points on the circle delimiting one of the other arcs, for example, v. Then, s is the exterior angle for this triangle, so it is equal to the two new angles formed by the chord.



We have:

$$s = \theta + \phi$$

but $2\theta = x$ and $2\phi = y$ so

$$x + y = 2\theta + 2\phi = 2s$$

□

One can also work with x or y and use the triangle sum theorem. Draw the chord corresponding to arc x. (Not shown). Then

$$u/2 + v/2 + s = 180$$

$$u + v + 2s = 360$$

But the sum of the arcs is equal to that:

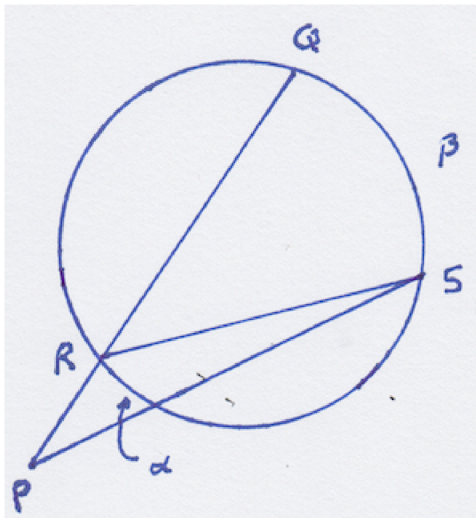
$$u + v + x + y = 360$$

Hence

$$2s = x + y$$

Chords from an external point

Two chords are drawn extending out to the same external point, P. We already know a theorem about the segments of these secant lines, but this question is about arcs and angles.



What can we say about the angle P in terms of the arcs α and β?

We might guess here as well. If P were on the circle then $2P$ would just be β. Since P is smaller than it would be if on the circle, we'll need to subtract something from β. On the other hand if P were very far away from the circle then P would be very small, and also α would be very nearly equal to β. Let's see.

Draw RS. $\angle QRS$ subtends arc β so $2\angle QRS = \beta$.

Since $\angle QRS$ is the exterior angle for $\triangle PRS$, we know that $\angle QRS = P + S$ so

$$\beta = 2\angle QRS = 2P + 2S$$

Finally, we have that $2S = \alpha$.

$$\beta = 2P + \alpha$$

$$P = \frac{\beta - \alpha}{2}$$

We subtract the arc that the angle subtends going in the "wrong" direction.

Part VI

More

Chapter 18

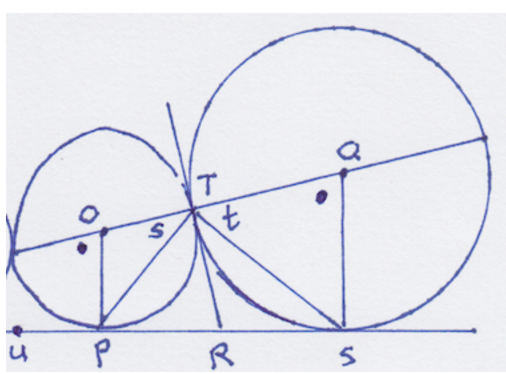
Broken chord

We look at a few problems involving circles, culminating in the broken chord theorem of Archimedes.

a tangent problem

Two circles of different sizes are shown below with a base line tangent to both, namely $UPRS$.

TR is drawn tangent to both circles, meeting them at point T, on the line joining their centers O and Q. Show that $\angle PTS$ is a right angle.



Solution.

Note first that the blue dotted angles, one at O, supplementary to $\angle POT$, and the other one at Q, namely $\angle TQS$, are equal, by similar triangles.

$\triangle OPT$ is isosceles (two radii of the circle on center O), so $2s$ is equal to the blue dotted angle by the exterior angle theorem. $\triangle TQS$ is also isosceles, for the same reason, so $2t$ is supplementary to the blue dotted angle by the angle sum theorem.

$$2s + 2t = 180$$

We conclude that $s + t$ is a right angle, so $\angle PTS$ is also, by supplementary angles.

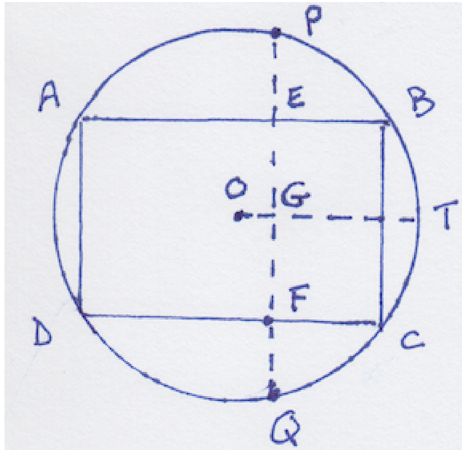
There are other approaches, e.g. the quadrilaterals $POTR$ and $RTQS$ are similar with two right angles and one copy each of the blue dotted angle and its supplement.

$\square$

rectangle extensions

In the figure below, $ABCD$ is a rectangle inscribed in a circle. PQ is drawn parallel to BC, forming rectangle $EBCF$.

$OT \perp BC$. Show that $PE = FQ$.



Solution.

$\triangle OBC$ is isosceles. Since $OT \perp BC$ and goes through the vertex O of $\triangle OBC$, the base of the isosceles triangle BC is bisected by OT. Refer to the standard proof of the isosceles triangle theorem.

Then, $PQ \parallel BC$. But $\triangle OPQ$ is also isosceles, so PQ is also bisected by OT, which means that $PG = GQ$.

Since $EBCF$ is a rectangle and OT is the perpendicular bisector of one side BC, it is also the perpendicular bisector of the other side, EF. So $EG = GF$.

$$PG = GQ$$
$$BT = TC = EG = GF$$

so

$$PG - EG = GQ - GF$$

The left-hand side is PE and the right-hand side is FQ. Hence

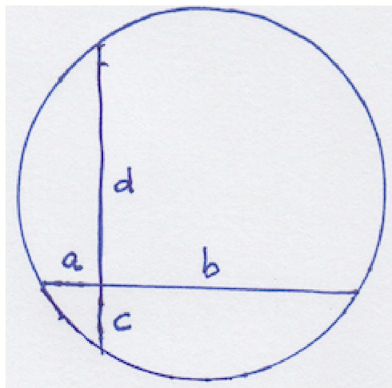
$$PE = FQ$$

□

extraordinary property of the circle

Two perpendicular chords are drawn AB and EF. They may cross anywhere in the circle. The pieces are a, b, c and d. The radius of the circle is R.

Show that $a^2 + b^2 + c^2 + d^2 = 4R^2$.

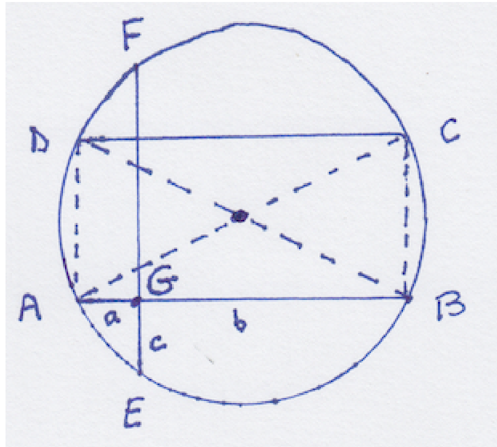


We draw the rectangle $ABCD$ and its diagonals AC and BD (below).

These are also diagonals of the circle, by the converse of Thales theorem. Given that $\angle DAB$ is a right angle, it follows that BD is a diagonal.

Two diagonals of a circle cross at a single point, and they both go through the center, so that's where they must cross. (Alternatively, use congruent triangles).

The length of the rectangle is $a + b$. The length of FG is d. The parts of FE that lie outside the rectangle are both equal, that is, c. This is the previous result. As a result, we have that the width of the rectangle is $d - c$.



Apply the Pythagorean theorem

$$(a + b)^2 + (d - c)^2 = (2R)^2$$

$$a^2 + 2ab + b^2 + c^2 - 2cd + d^2 = 4R^2$$

As the arms of chords that cross in a circle, we showed previously that $ab = cd$. With the cancelation, we have

$$a^2 + b^2 + c^2 + d^2 = 4R^2$$

□

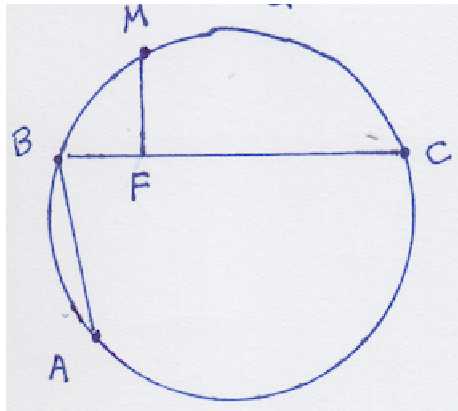
the broken chord, I

The theorem of the "broken chord" is ascribed to Archimedes, although his original work has been lost. It was analyzed by the Arabic mathematician Al'Biruni in his *Book on the Derivation of Chords in a Circle.*

Pick two points on a circle, A and C and let M lie midway between them. If the chords AM and MC were drawn, they would be equal. Now pick B to one side of M and draw AB and BC.

Drop the perpendicular from M to BC at F.

I claim that $AB + BF = FC$.

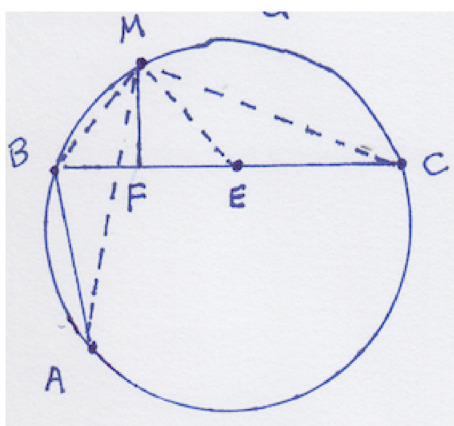


Proof.

The clever idea is to place point E such that $AB = EC$ (below).

Now, if we can show that $\triangle BME$ is isosceles, then MF bisects the base and we are done.

By the property of the midpoint, we know that $AM = MC$. Having drawn those chords, we notice that $\angle A$ and $\angle C$ subtend the same arc of the circle, so they are equal. The other flanking side is AB which is equal to EC by our construction.



Thus, we have two congruent triangles by SAS. $\triangle ABM \cong \triangle ECM$.

Since they are corresponding parts of congruent triangles, $BM = ME$.

That means $\triangle BME$ is isosceles. Since MF is the perpendicular to base BE, it bisects it, by the properties of isosceles triangles.

We have that $BF = FE$, and $AB = EC$, so their sums are equal as well. $AB + BF = FE + EC = FC$.

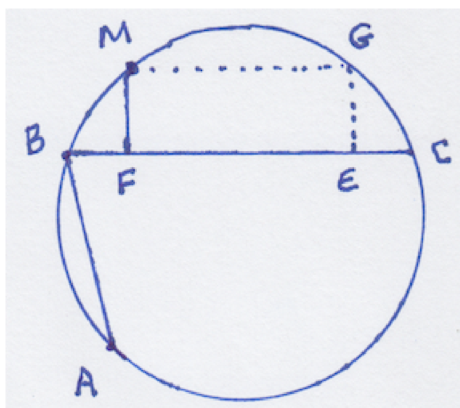
$\square$

the broken chord, II

There is another neat proof of this theorem.

Proof.

Draw the rectangle $MGEF$. By the work on parallel chords earlier in this chapter, we know that $BF = EC$.



Hence we have two small congruent triangles $\triangle BFM \cong \triangle GEC$ by SAS, and so it follows that $BM = GC$.

Chords of equal length in a given circle have the same arc and vice-versa. Furthermore, arc lengths are additive, though chord lengths are not (See below).

Hence the arcs BM and GC are equal as are AM and MC, so by subtraction $AB = MG$. So the chords corresponding to those arcs, AB and MG are also equal.

But MG and FE are opposite sides in a rectangle. It follows that $AB = FE$.

By addition, $AB + BF = FE + EC$.

□

The first proof duplicated the length AB with C as the endpoint, while this proof duplicates the length BF with C as the endpoint.

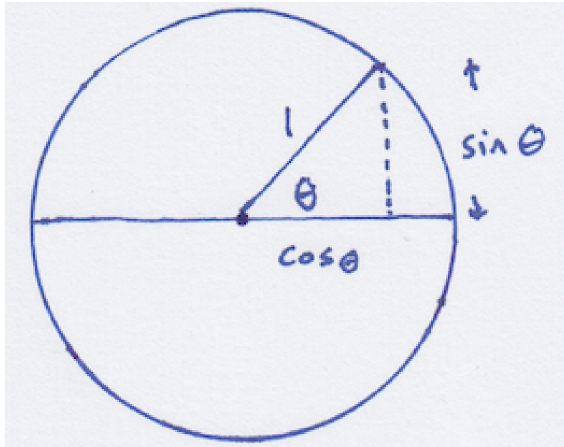
arcs and chords

We said that chords of equal length in a given circle have the same arc and vice-versa. We haven't stated this relationship explicitly so far.

The measure of an angle θ is the length of arc s that it subtends *in a unit circle*. Hence θ gives us s and s gives us θ, and of course, any peripheral angle subtending the same arc as a central angle is just one-half that angle.

There is a difference between arcs and chords, however. Arcs add (as do angles), but chords do not. A simple reason is the triangle inequality.

Anticipating the chapter on basic trigonometry, we consider the chord corresponding to a central angle of 2θ in a unit circle, and introduce the sine of that angle θ as one-half of the chord. This also corresponds to one-half the chord subtended by a peripheral angle of θ.

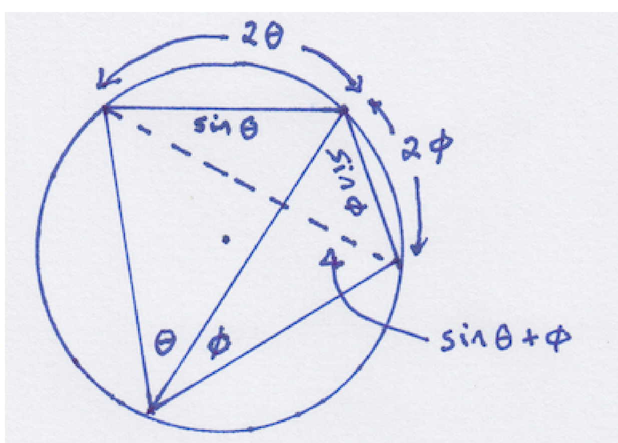


Both sine and its inverse are functions. The sine returns a unique value in the range $[0,1]$ for any input in the range $0-90$ degrees (For radian measure, see the chapter on basic trigonometry). Both functions are increasing everywhere on their intervals. This means that any angle corresponds to a unique chord, and any chord corresponds to a unique angle.

The arc determines the angle determines the chord, and vice-versa.

It will turn out that to add chords, we must adjust the lengths. We will show later that

$$\sin\theta + \phi = \sin\theta\cos\phi + \sin\phi\cos\theta$$



Each of the smaller chords is shrunk by an amount equal to the cosine of the same angle. This shrinkage is larger (as a percentage) as the measure of the angle is greater, since the cosine gets closer to 0.

The result that we need for this chapter is that if two arcs of a circle are equal, then so are the chords of those arcs, and vice-versa.

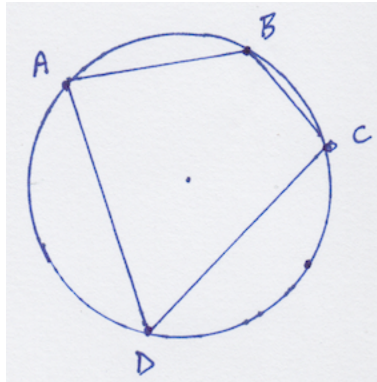
Chapter 19

Ptolemy

Ptolemy was a Greek astronomer and geographer who probably lived at Alexandria in the 2nd century AD (died c. 168 AD). That is nearly 500 years after Euclid. Ptolemy was a popular name for Egyptian pharaohs in earlier centuries.

Our Ptolemy is known for many works including his book the *Almagest*, and important to us, for a theorem in plane geometry concerning cyclic quadrilaterals. These are 4-sided polygons all of whose vertices lie on a circle. Recall that any triangle lies on a circle, so this is a restriction on the fourth vertex of the polygon.

A cyclic quadrilateral is a four-sided polygon whose vertices all lie in a circle. Consider $ABCD$.



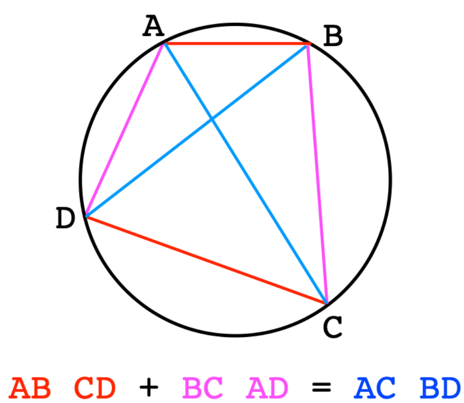
In this figure $\angle A$ and $\angle C$ are supplementary. The reason is that together, their arcs sweep out the entire circle. The sum of all four peripheral angles is

$$\angle A + \angle B + \angle C + \angle D = 360$$

Of course we knew that, it is true for any four-sided polygon.

□

Draw the diagonals AC and BD.



Ptolemy's theorem says that for a cyclic quadrilateral, the product of opposing sides, summed, is equal to the product of the diagonals:

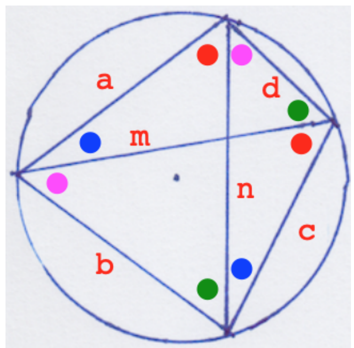
$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

Proof.

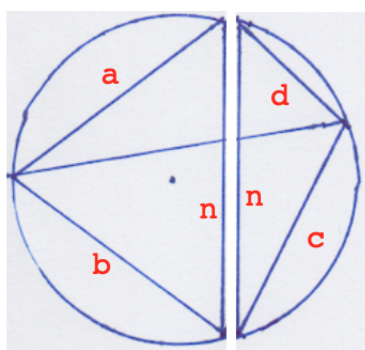
I found a great proof without words here:

https://www.cut-the-knot.org/proofs/PtolemyTheoremPWW.shtml

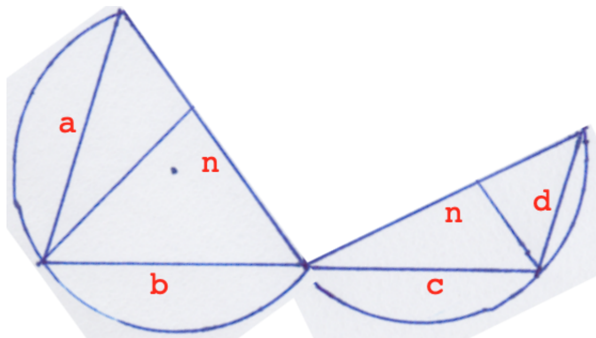
We'll actually use some words in an attempt to see how one might have developed this proof. We have switched to use single letters for the sides, and the angles are labeled with colored dots. Equal angles come from the inscribed angle theorem.



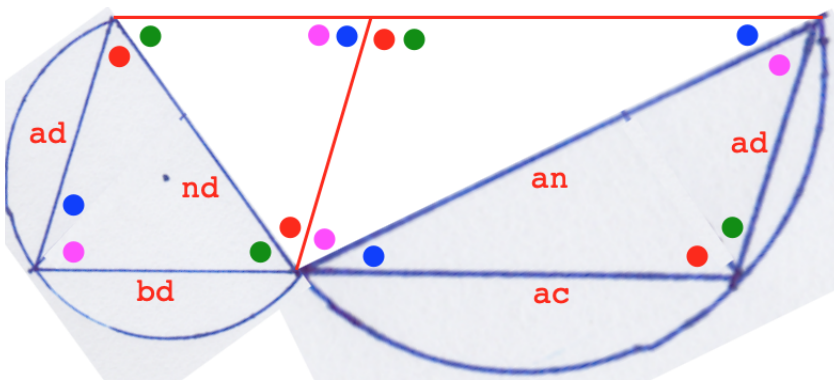
Now, the great idea is to construct a parallelogram. We begin by cutting the figure in half along one of the diagonals, say n. This makes two pieces from the other diagonal m, which we will ignore for the moment.



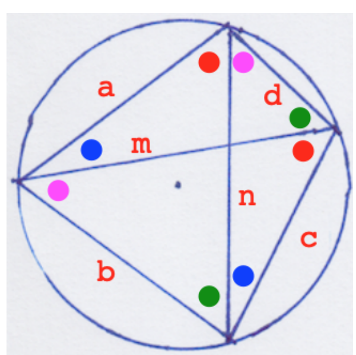
Next, pick any two sides and align them horizontally.



At this point, we need to scale the figures to make a parallelogram. Algebraically, we do this by multiplying. Each of the three sides on the left is scaled by d, and each of three on the right by a.



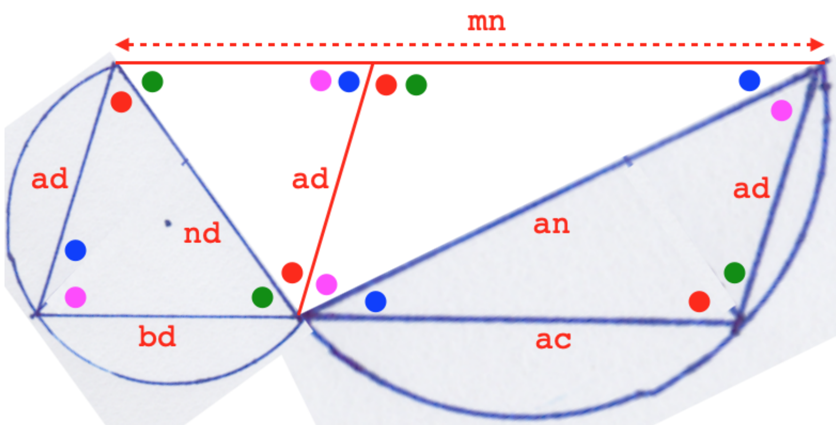
We can also fill in the angles by referring back to the original diagram. We have a parallelogram because the two short sides are equal and also parallel.



The angles in the new part of the parallelogram are determined by using the alternate interior angles theorem. Draw a central red line to help with placing the angles.

At this point, we realize that we have a matching triangle in the original figure. It is the one with sides a, d and m. The red and green dotted angles are switched with respect to the central angle composed of red and magenta (the large triangle is a reflected version), but that's no problem.

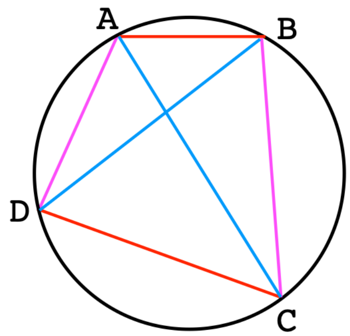
It just needs to be scaled by n to fit.



As opposing sides in a parallelogram, the top and bottom must be equal in length. Therefore

$$mn = ac + bd$$

□

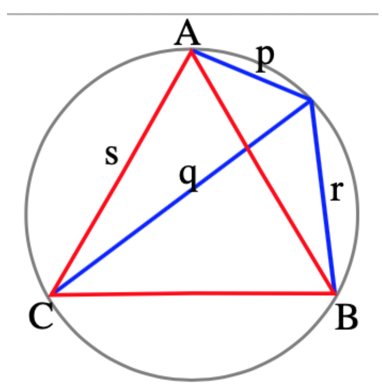


AB CD + BC AD = AC BD

corollaries

Here are just a few of the results that follow from this remarkable theorem.

equilateral triangle



Inscribe an equilateral triangle in a circle and pick any point on the circle.

$$qs = ps + rs$$

$$q = p + r$$

Pythagorean theorem

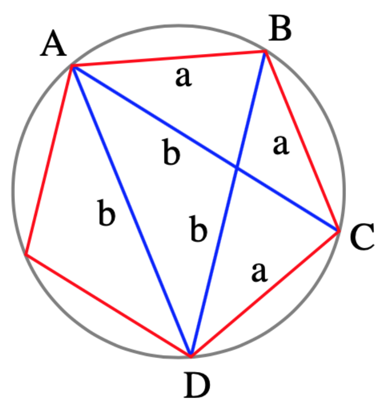
Let the quadrilateral be a rectangle. The the sum of squares of opposing sides is

$$a^2 + b^2$$

Triangles made by opposing diagonals are congruent, so the diagonals are equal in length. The diagonal is the hypotenuse, hence

$$a^2 + b^2 = h^2$$

golden mean in the pentagon



Take four vertices of the regular pentagon and draw two diagonals. From the theorem, we have

$$b \cdot b = a \cdot a + a \cdot b$$

$$\frac{b^2}{a^2} = 1 + \frac{b}{a}$$

Rather than use the quadratic equation, rearrange and add 1/4 to both sides to "complete the square":

$$\frac{b^2}{a^2} - \frac{b}{a} + \frac{1}{2^2} = 1 + \frac{1}{2^2}$$

So

$$(\frac{b}{a} - \frac{1}{2})^2 = \frac{5}{4}$$

$$\frac{b}{a} - \frac{1}{2} = \pm \frac{\sqrt{5}}{2}$$

$$\frac{b}{a} = \frac{1 \pm \sqrt{5}}{2}$$

This ratio b/a is known as ϕ, the golden mean.

We'll see one more result when we get to trigonometry. All of the sum of angles formulas easily follow from Ptolemy's theorem.

Chapter 20

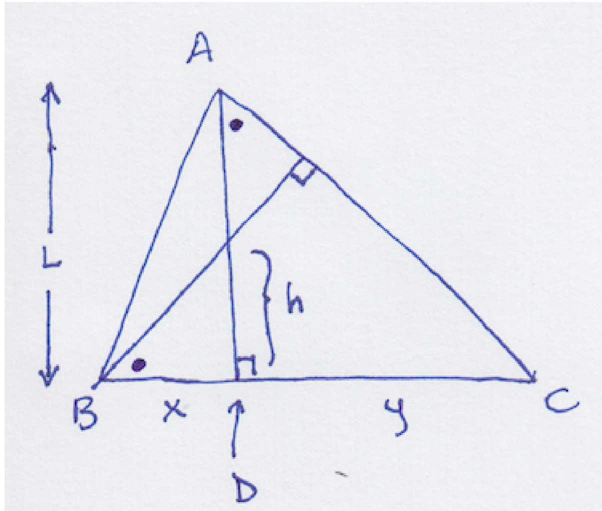
Concurrence

When we draw a line from each vertex in a triangle to the side opposite, those lines have different names depending on how we pick the end-point. Some possibilities are:

- to form the perpendicular (altitude)
- to bisect the side (median)
- to bisect the angle where we start

These three lines always seem to cross at the same point. Here, we look at the conditions under which this happens. Start with the altitude.

In $\triangle ABC$ draw two of the altitudes, from A and B.



The altitude from the vertex A meets the base in a right angle at D. This altitude divides the base into lengths x and y.

Now draw a second altitude from vertex B to the side opposite.

What is the height h above the base where the two lines cross?

The two small triangles formed by the junction of the altitudes are similar. They share vertical angles and both are right triangles, so the angles marked with a dot are equal.

But the large $\triangle ACD$ is also a right triangle and contains the same dotted angle so it is similar to the first two.

The ratio of the base lengths for the lower small triangle is h/x, and that for $\triangle ACD$ is y/L. (h and y go on top because they are opposite the dotted angle).

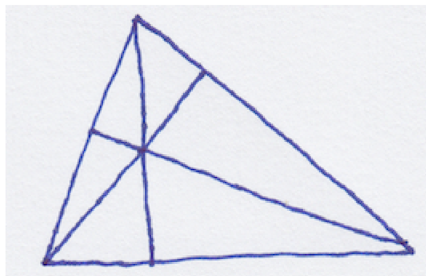
As similar triangles

$$\frac{h}{x} = \frac{y}{L}$$

$$h = \frac{xy}{L}$$

The formula is really interesting. It is symmetrical in x and y and it does not contain any term related to the length of side AB.

Therefore, if we draw the third altitude to side AB and then calculate the height for it above the intersection with AD, we will get the same answer.



And this means that the three altitudes cross at a single point. Newton published this proof about 1680. This is a proof of existence for the orthocenter, and of concurrence, for the altitudes.

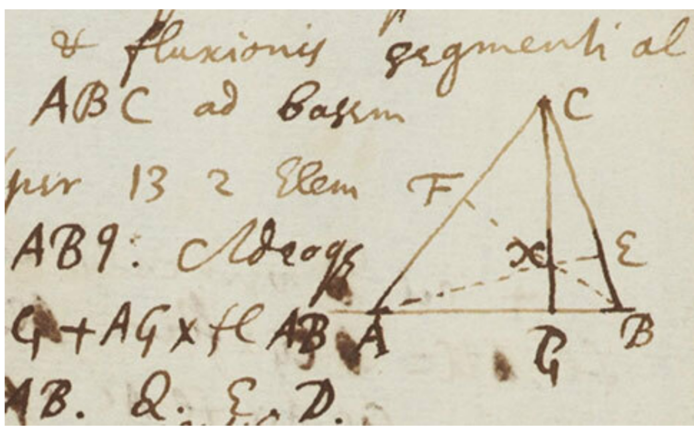


Fig. 129 The diagram for Newton's proof, from his *Geometria curvilinea and Fluxions*, Ms Add. 3963, p54r.

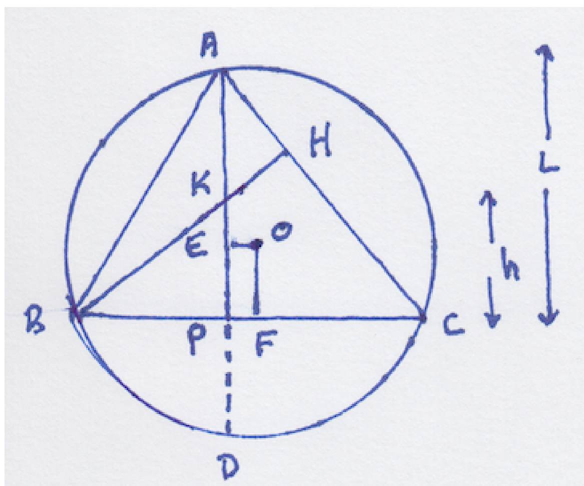
One other idea about the orthocenter. We can draw a circle around any triangle. If we do that, and apply the theorem on chords that cross in a circle, we can draw an extension of any altitude to the circle beneath it. Call that extension g. The crossed chords theorems says that the product of the chord segments are equal:

$$xy = gL$$

Rearranging what we had above gives

$$xy = hL$$

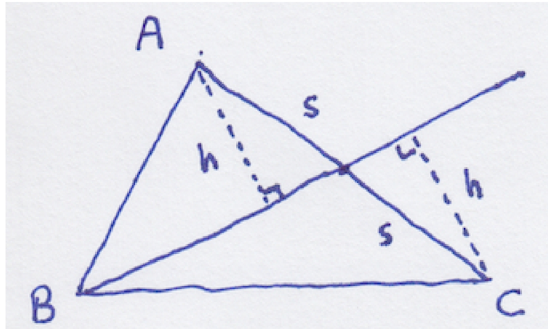
So $g = h$. The extension of the altitude to meet the circumcircle is equal to the height of the orthocenter above the side to which the altitude has been drawn.



$$PD = PK$$

median

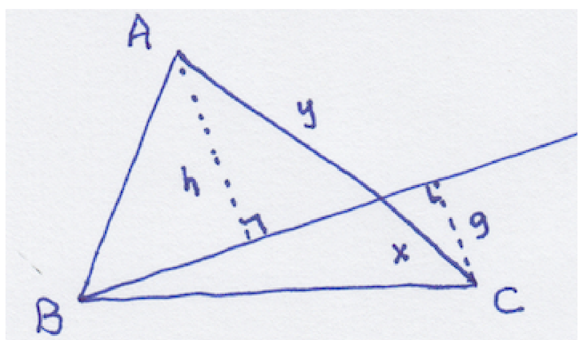
Next, we look at a median. The line from vertex B is drawn to bisect the opposite side.



The altitudes from A and C to the median are drawn. This forms two small triangles that are congruent, since $s = s$ and we have two (three) angles equal. Therefore, the altitudes are the same length.

We conclude that the median divides the original triangle into two smaller triangles with the shared base and equal altitudes, so they have the same area.

Now we think about the general case.



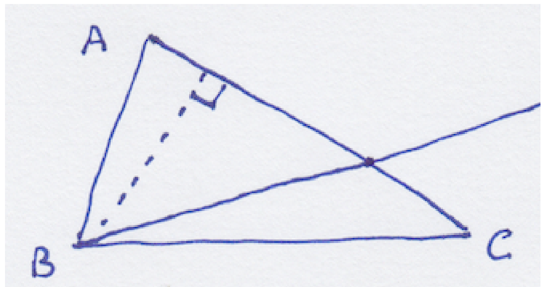
Here, $x \neq y$. But we still have two similar triangles formed by the altitudes g and h By similar triangles the altitudes are in the same ratio as the bases:

$$\frac{g}{h} = \frac{x}{y}$$

Again the base is shared. Therefore the areas of the two triangles formed by the solid line have their areas in the same ratio as the bases

$x : y$.

Here is a diagram for another proof for the same conclusion.

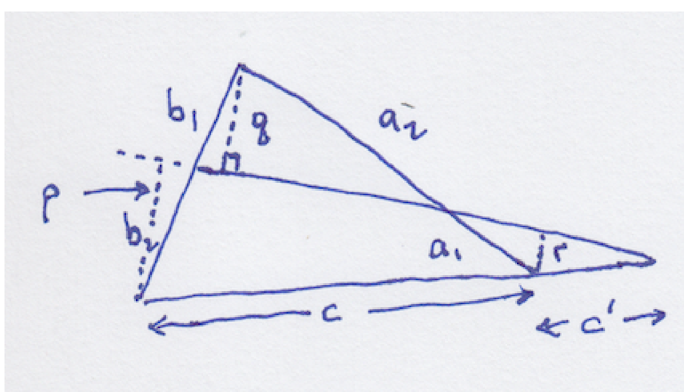


Menelaus' theorem

We come now to a theorem by Menelaus. We are headed for Ceva's theorem. I have heard that theorem is too hard to be taught in high school, but that's because they generally use a complicated proof. Menelaus' theorem makes it easy, and Menelaus is itself easy, so we're going to do them both.

There are multiple possibilities for a proof of Menelaus' theorem, we will look at one Einstein valued for its symmetry.

In a triangle, draw the *traversal*, the solid line in the middle, that meets the extended base on the right.



The sides are divided into lengths a_1 and a_2, and b_1 and b_2. The extended base is labeled with a prime to emphasize the extension. Three altitudes are drawn as well.

From similar triangles we can write

$$\frac{r}{p} = \frac{c'}{c+c'}, \qquad \frac{p}{q} = \frac{b_2}{b_1}, \qquad \frac{q}{r} = \frac{a_2}{a_1}$$

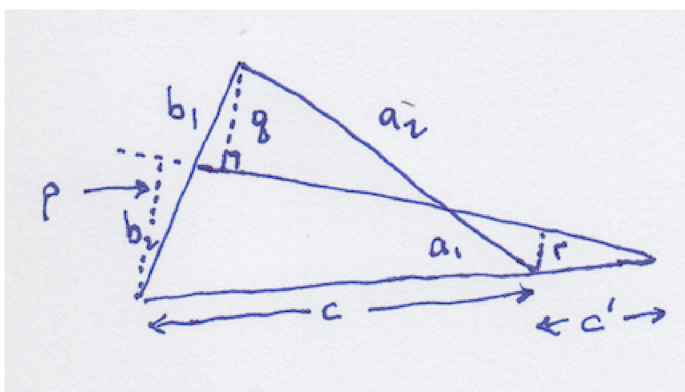
Since the left-hand sides, multiplied together are equal to 1, so is the product of the right-hand sides.

$$1 = \frac{c'}{c+c'} \cdot \frac{b_2}{b_1} \cdot \frac{a_2}{a_1}$$

which we can rewrite as

$$1 = \frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c+c'}{c'}$$

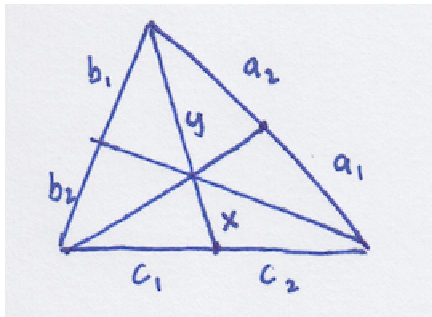
□



Going around the circle from whichever vertex is convenient, we take the lengths in the order that we first encounter them, remembering that c and c' are not like the other two.

Ceva's theorem

Finally, we come to Ceva's theorem.



Draw three lines crossing at any point P in the interior of the triangle.

Let us apply Menelaus' theorem to the two half triangles formed by the line drawn from the apex to the base. For the left we have that

$$1 = \frac{x}{y} \cdot \frac{b_1}{b_2} \cdot \frac{c_1 + c_2}{c_2}$$

The other one is taken in the clockwise direction (or form the mirror image):

$$1 = \frac{x}{y} \cdot \frac{a_2}{a_1} \cdot \frac{c_1 + c_2}{c_1}$$

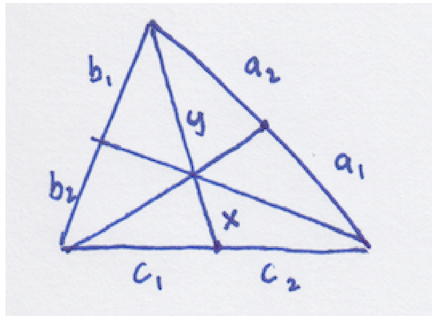
Equating the two expressions and canceling x/y and $c_1 + c_2$ we obtain

$$\frac{b_1}{b_2} \cdot \frac{1}{c_2} = \frac{a_2}{a_1} \cdot \frac{1}{c_1}$$

which can be rearranged to give:

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1$$

If the lines meet at the center, then the proportions of the parts of the sides multiply to give 1.



The converse theorem is also true. Concurrence of lines follows from the fact that the product of proportions is equal to 1. We will skip the proof for this part, but it's relatively easy to construct a proof by contradiction.

□

placement of the centroid

Did you notice? We had

$$1 = \frac{x}{y} \cdot \frac{b_1}{b_2} \cdot \frac{c_1 + c_2}{c_2}$$

For the case of the centroid, the middle term is 1 and the right-hand term is 2 so

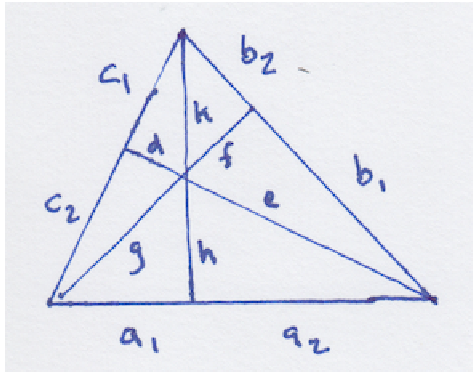
$$1 = \frac{x}{y} \cdot 2$$

$$y = 2x$$

The centroid is one-third of the distance up from the base.

back to the orthocenter

Here is a second proof of existence for the orthocenter, using Ceva's theorem directly.



For each vertex, find the two right triangles that contain the entire angle. These triangles are similar, so we can form the same ratios. Start at the top and work clockwise:

$$\frac{c_1}{b} = \frac{b_2}{c}, \qquad \frac{b_1}{a} = \frac{a_2}{b}, \qquad \frac{a_1}{c} = \frac{c_2}{a}$$

Notice the symmetry! Rearranging

$$\frac{c_1}{b_2} = \frac{b}{c}, \qquad \frac{b_1}{a_2} = \frac{a}{b}, \qquad \frac{a_1}{c_2} = \frac{c}{a}$$

The product of the three fractions on the right-hand side is 1, which is equal to the product of the three fractions from the left-hand side:

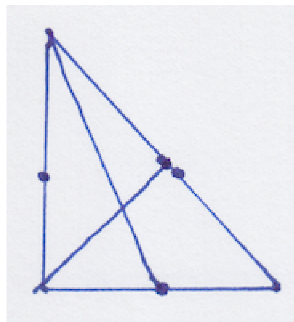
$$\frac{c_1}{b_2} \cdot \frac{b_1}{a_2} \cdot \frac{a_1}{c_2} = 1$$

□

Euler line

One final thought about concurrence. There cannot be any general ratio that describes the placement of the orthocenter, where altitudes cross. The reason is that in a right triangle, that ratio is zero because they cross at the right angle.

Let's look a little closer at an isosceles right triangle. If you recall, we showed that the median drawn to the hypotenuse is itself a radius of the circumcircle. So the half-way point of the hypotenuse is the circumcenter.



If we draw a line from the orthocenter, at the right angle, to the circumcenter, this *is* a median of the triangle. So the centroid, where the medians cross, lies on a line between the circumcenter and the orthocenter.

It is not difficult to show that the centroid's distance from the orthocenter is twice its distance from the median, but we will leave that to you.

Another special case is the equilateral triangle, where the altitude and the median are the same line for each vertex, or the isosceles triangles, where they are the same line for one vertex, and both go through the circumcenter.

Euler showed that both of these results hold for all triangles. The centroid, where the medians cross, lies on a line between the circumcenter and the orthocenter. The distance from the centroid to the orthocenter is twice the distance to the circumcenter.

We'll save that as something for the future.

Chapter 21

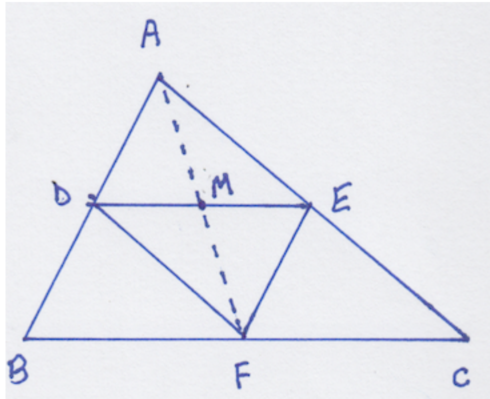
Triangle relations

There are a number of interesting examples of triangles drawn inside triangles using points placed by medians or altitudes and so on. We look at three.

centroid

The first is the median. Midpoints are marked for $\triangle ABC$ and one median is drawn (dotted line).

The midpoint criterion means that, for example, $AD = DB$ and $AE = EC$. Since $\angle A$ is shared, $\triangle ADE \sim \triangle ABC$, with a ratio of sides of $1:2$.



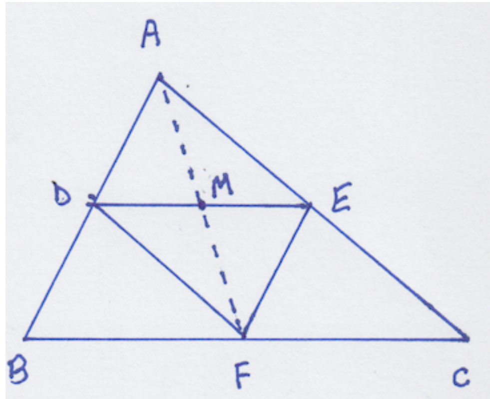
Symmetry or similar logic will show that $\triangle BDF$ and $\triangle EFC$ have exactly the same relationship with $\triangle ABC$. Since the side ratios to the large triangle are all equal, we conclude that the three small triangles are not just similar but congruent.

Another way to do this is to use the midpoint relationships to show that $DE \parallel BC$. Alternate interior angles and triangle sum of angles will give the result.

Note that $DEFB$ is a parallelogram. It has two pairs of opposite sides equal and the shared diagonal DF means we have SSS so opposing angles are equal and we have a parallelogram. Therefore $\triangle DEF$ is congruent to the other three.

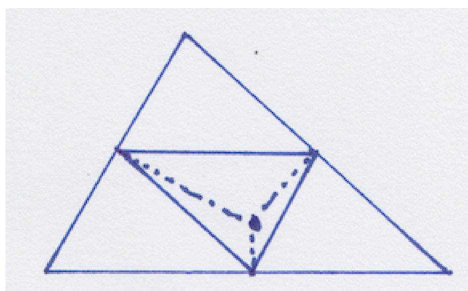
But MF is a median of $\triangle DEF$. We can draw another small triangle inside $\triangle DEF$ and it will have the same medians as $\triangle ABC$ and $\triangle DEF$.

Therefore, the centroid exists, and we could make an algebraic argument for why it lies one-third of the way from F up along the median, below M, but we rely on our previous demonstration instead.



circumcenter and orthocenter

In a triangle we draw the perpendicular bisectors of the sides.



They meet at the circumcenter, which we know must exist, since having drawn two bisectors and finding where they meet, we know the circumcenter.

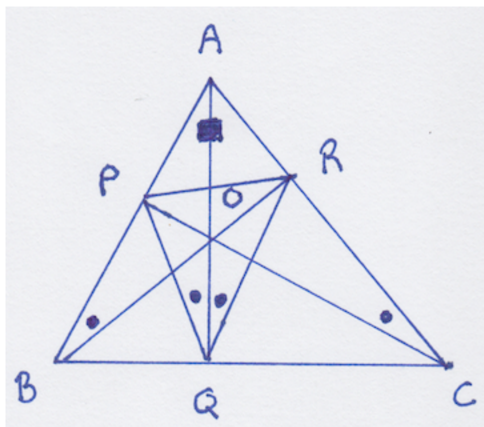
But this is the same arrangement we had in the previous diagram, we are just drawing the dotted lines in a different direction. Hence we have the same three parallelograms and the same congruent triangles.

Since the horizontal in the middle is parallel to the base, the perpendicular bisector of the base meets that horizontal in a right angle. It *is* the altitude of the small inset triangle. Since the circumcenter certainly exists, this is a proof that the orthocenter also exists.

altitudes and angle bisectors

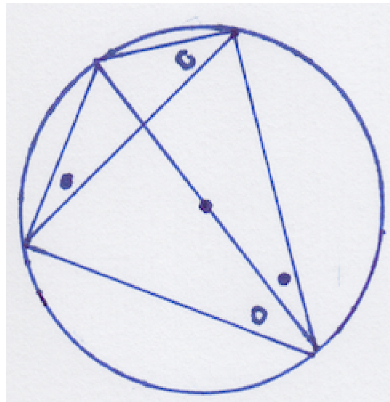
In the first two examples the proofs related to concurrence. When we said that a point, whether the centroid or the orthocenter *exists*, we mean that the three medians or altitudes are concurrent at that point.

In the previous example we showed that the triangle drawn to the points perpendicular bisectors hit the sides, has those same lines for its altitudes.



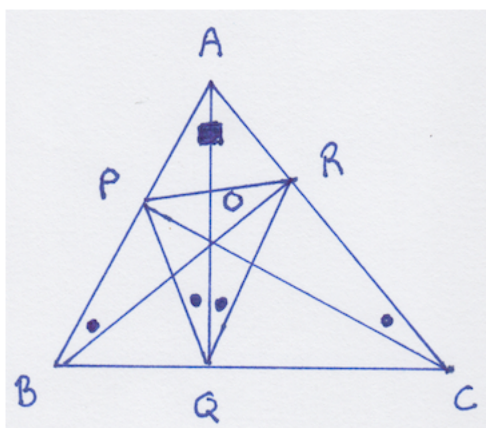
Here, we will show that when a triangle is drawn using the points where the altitudes hit the sides, those same lines bisect the angles of the new triangle. Draw the altitudes in $\triangle ABC$ and join the points where the right angles are formed, along the bases, to give $\triangle PQR$. The key is that the altitudes connect as right angles and they go through the orthocenter, which we have *already shown to exist.*

Quadrilateral $ORCQ$ has two right angles at opposing vertices. If we make OC the diameter of a circle, then R and Q will have right angles. This is a consequence of the inscribed angle theorem.



Here's an example. The twisted kite (two short sides not equal, two long sides not equal either) is not symmetric. Nevertheless, the angles marked with blue dots, as well as the pair marked with open circles, are equal.

So, in our original diagram, $\angle RCO$ and $\angle OQR$, are inscribed angles that subtend the same arc, so they are equal. That accounts for two of the filled dots.



Either symmetry, or an equivalent circle construction for quadrilateral $OQBP$ will show that $\angle PBO$ and $\angle PQO$ sweep out the same arc, so they are equal as well.

But now consider $\triangle ABR$ and $\triangle APC$. Both are right triangles that contain $\angle PAR$. Therefore, the complementary angles in these two triangles are equal. These are the two angles that are components of $\angle B$ and $\angle C$. All four blue dotted angles are equal.

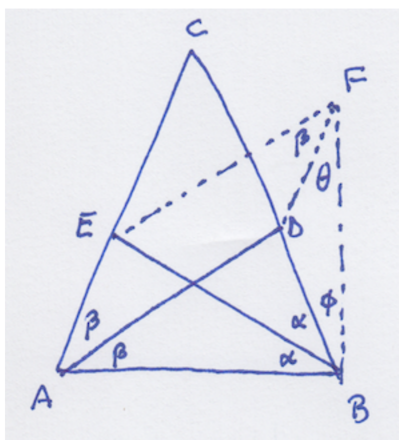
We conclude that $\angle PQR$ is bisected, which by symmetry, means that all the angles of $\triangle PQR$ are bisected.

O is both the incenter of the $\triangle PRQ$, the center of its incircle, and also the orthocenter of the original $\triangle ABC$. Since the incenter certainly exists, this is yet another proof that the orthocenter exists.

We have proceeded first from circumcenter to orthocenter, and onward to incenter.

challenge

This is our last problem in pure geometry. See if you can prove it before reading further.

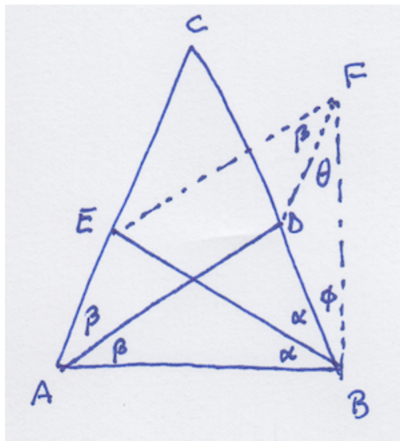


Given that $\angle A$ and $\angle B$ are bisected. If we also know that $\angle A = \angle B$ (isosceles $\triangle$), then it follows that the lengths of the bisectors are equal: $AD = BE$. This follows from ASA and $\triangle ABE \cong \triangle ADB$.

What about the converse? Given $AD = BE$, show that the triangle is isosceles. Draw the parallelogram $AEFD$.

Proof.

By contradiction. Suppose $\alpha > \beta$. By the theorem of greater side $\iff$ greater angle, $AE > BD$. But $AE = DF$ (parallelogram), so $DF > BD$. By the converse theorem, $\phi > \theta$.



Adding inequalities gives $\alpha + \phi > \beta + \theta$. By the forward theorem, $EF > BE$. Since $AD = EF$, $AD > BE$. But this is a contradiction.

Assuming that $\beta > \alpha$ gives a similar contradiction by the symmetric construction. Hence $\alpha = \beta$ and $2\alpha = 2\beta$. $\square$

Part VII

Trigonometry

Chapter 22

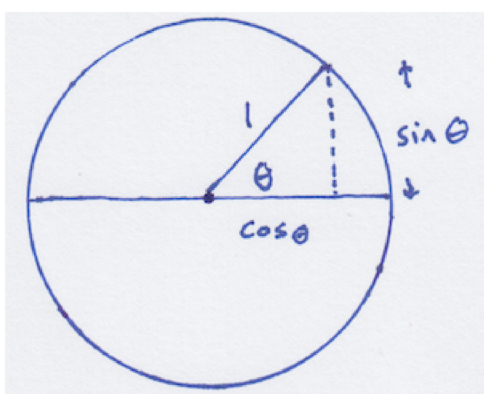
Basic trig

definitions

The sine and cosine are really more a convenience than a necessity in basic geometry, they allow us to use one symbol for the ratio of two sides. It is only when we come to consider them as functions, and see their utility for modeling periodic phenomena, that they because extremely important.

Here we just scratch the surface.

I will define them in a slightly different way than usual, and then show the connection:

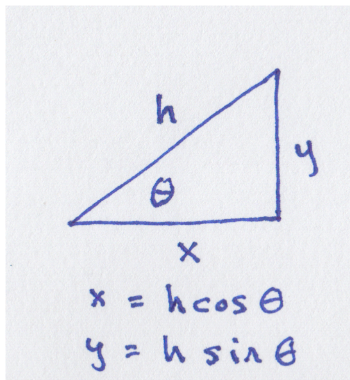


If you draw a right triangle in a unit circle with angle θ as the central angle, then the sides of the triangle are $\cos\theta$ and $\sin\theta$.

If the triangle and circle are resized so that the length of the radius is h, then the x part of the triangle (the side adjacent to the angle) becomes $h\cos\theta$, and the y part of the triangle (the side opposite the angle) becomes $h\sin\theta$.

$$y = h\sin\theta$$

$$x = h\cos\theta$$



Rearranging slightly:

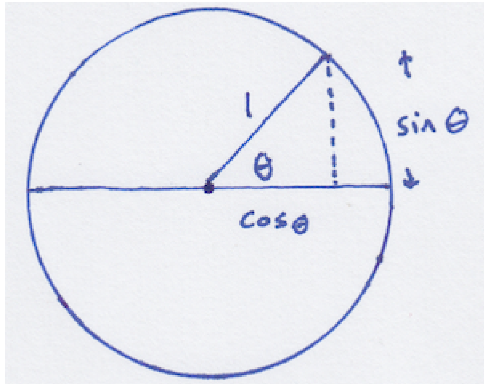
$$\frac{y}{h} = \sin\theta$$

$$\frac{x}{h} = \cos\theta$$

This is the normal presentation. People recite sine is *opposite over hypotenuse* and cosine is *adjacent over hypotenuse*, but there seems to be endless confusion over this, so I am starting here from a different, but equivalent, viewpoint.

The Pythagorean theorem can be restated as

$$\sin^2\theta + \cos^2\theta = 1$$



It is an unusual but standard notation that $\sin^2 x$ is written for $(\sin x)^2$ and the same for cosine.

radian measure

In talking about the measure of an angle, we have largely avoided the degree system that you are undoubtedly familiar with, which divides the circle into 360°. We just follow the Greeks, who thought only in terms of right angles.

In trigonometry we will move to the radian system of measure. We *define* the measure of an angle θ to be equal to the arc s it sweeps out, or subtends, in a circle of radius 1, a *unit* circle. Angles are not lengths, but numerically, the measure of the angle is the measure of the arc.

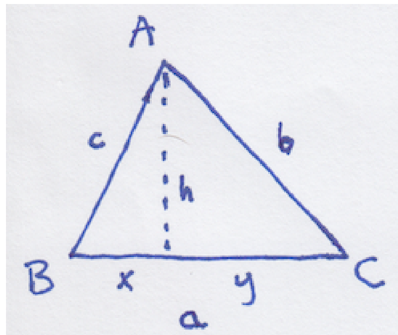
This means that the whole circle corresponds to 2π radians, and a right angle to $\pi/2$. One big reason for that comes in working out the *derivative* of sine and cosine. We use the symbol d/dx for the derivative and say that

$$\frac{d}{d\theta}\ \sin\theta = \cos\theta$$

If we wrote this using the degree system, this would not be a true statement without adding an awkward constant of proportionality.

laws of sines and cosines

Next we look at 2 powerful relationships that are easily derived from these definitions.



The first law is the law of sines.

$$h = c \sin B, \qquad h = b \sin C$$

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

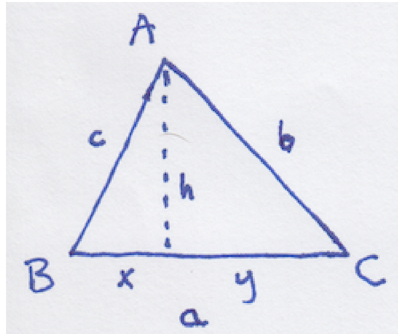
By symmetry

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The second law is the law of cosines. Again we use h (or rather, h^2) to connect two equal expressions. We also have $y = a - x$.

$$c^2 - x^2 = h^2 = b^2 - (a - x)^2$$

$$a^2 + c^2 = b^2 + 2ax$$



But $x = c\cos B$ so

$$b^2 = a^2 + c^2 - 2ac\cos B$$

Generally, for any angle we can rewrite the formula in terms of the square of the opposite side.

special values

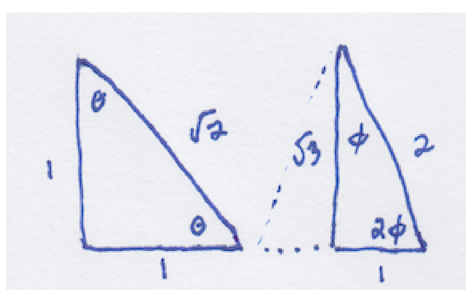
Since θ is duplicated in a 45-45-90 right triangle, we have

$$\sin 45 = \cos 45 = \frac{1}{\sqrt{2}}$$

Cutting an equilateral triangle in half, if the original side is 2, we obtain

$$\sin 30 = \frac{1}{2}$$

$$\cos 30 = \frac{1}{\sqrt{3}}$$



The complementary angle switches adjacent and opposite sides (see above), so

$$\sin 30 = \cos 60$$

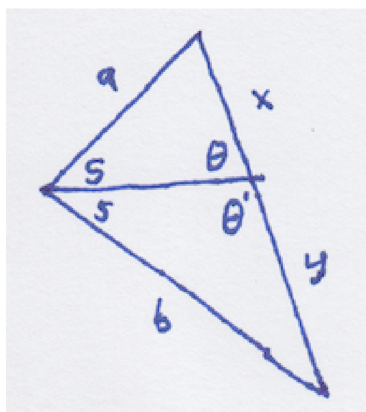
$$\cos 30 = \sin 60$$

Another useful tidbit is that if θ' is supplementary to θ ($\theta + \theta' = 180$), then $\sin\theta = \sin\theta'$.

ratios of sides

In the chapter on area, we showed a different proof of the following theorem. Here we rely on the law of sines.

The angle on the left is bisected. I claim that a is to x as b is to y.



We have that

$$\frac{x}{\sin s} = \frac{a}{\sin\theta}, \qquad \frac{y}{\sin s} = \frac{b}{\sin\theta'}$$

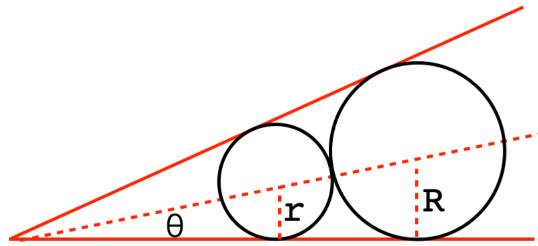
But θ and θ' are supplementary so their sines are equal. Therefore

$$\frac{x}{a} = \frac{\sin s}{\sin\theta} = \frac{y}{b}$$

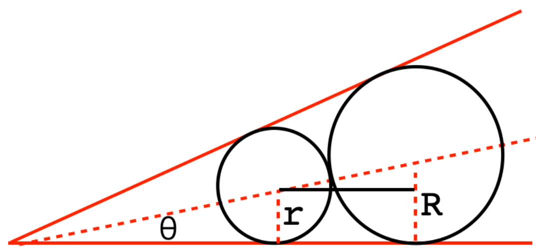
□

double scoop problem

We have two lines tangent to two circles that just touch each other, the smaller one of radius r, and the larger of radius R.



There is a simple expression for the sine and cosine of θ, the angle between the base and the line through both centers. The distance between the centers of the two circles is $r + R$. Draw a horizontal line through the center of the smaller circle.



We have constructed a right triangle, which is similar to the original one. It includes the angle θ and the hypotenuse is the distance between the two centers, $R + r$. The opposite side has length $R - r$ and so

$$\sin\theta = \frac{R - r}{R + r}$$

The adjacent side (the line segment colored black) has its squared length equal to

$$(R + r)^2 - (R - r)^2 = 4Rr$$

thus

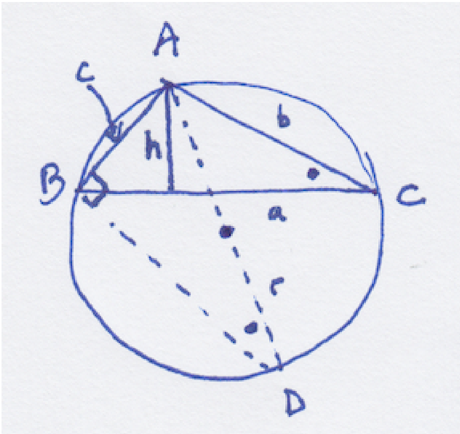
$$\cos\theta = \frac{2\sqrt{Rr}}{R+r}$$

You can confirm that the identity $\sin^2\theta + \cos^2\theta = 1$ works here.

area

The radius of the circumcircle of the triangle can be expressed in terms of the side lengths.

Draw $\angle D$ on a diameter. By equal arcs subtended $\angle D = \angle C$. And by Thales' circle theorem, the $\angle ABD$ is a right angle.



As a result $\triangle ABD \sim$ the right triangle formed with one vertex at C and third at A. h is the side opposite $\angle C$.

We have that the ratios of the short side to the hypotenuse are:

$$\frac{h}{b} = \frac{c}{2r}$$

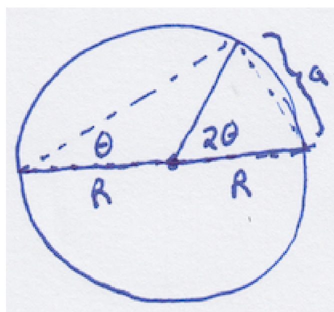
$$r = \frac{bc}{2h}$$

We also have that twice the area of $\triangle ABC$, written $2(ABC)$ is equal to ha. Substituting for h:

$$r = \frac{abc}{4(ABC)}$$

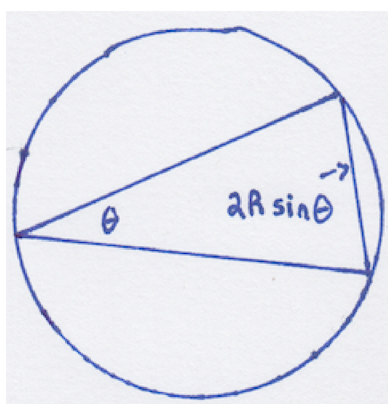
another method

If we go back to the first example of the inscribed angle theorem, we have that the central angle is twice the peripheral angle subtending the same arc.



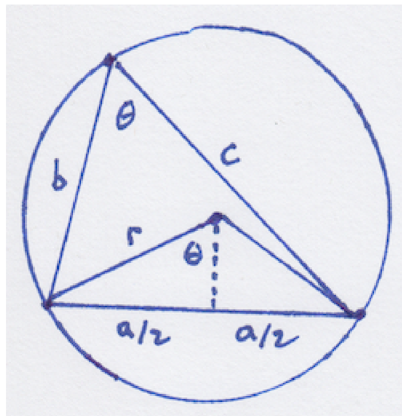
$$\frac{a}{2R} = \sin\theta$$

But θ can move anywhere on the circle



$$a = 2R \sin\theta$$

As before, except we switched r for R



The altitude to side c is $b \sin\theta$, so the area of the triangle (symbolized by $\triangle$) is

$$\triangle = \frac{bc \sin\theta}{2}$$

We substitute for $\sin\theta = a/2r$:

$$\triangle = \frac{abc}{4r}$$

which is rearranged to

$$r = \frac{abc}{4\triangle}$$

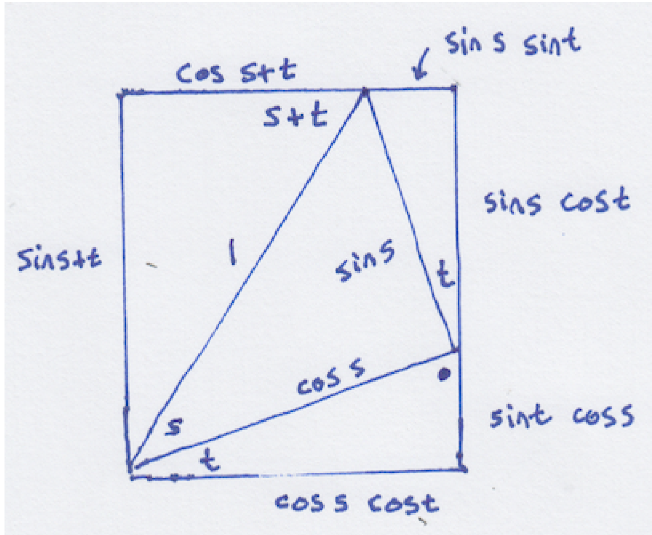
Chapter 23

Sum of angles

This topic is a bit advanced, but important. We want to know how to compute the sine and cosine for the sum and difference of two angles.

I have seen numerous ways of deriving the formulas, some easy and some complicated. But the following is my favorite method and it requires no calculation at all.

The trick here is to stand one triangle on top of the other. The scaling means the triangle on the bottom has a hypotenuse of $\cos s$. The other piece of the puzzle is that the small triangle with angle t is known because it is complementary to the complementary angle of t.



For angle t, the sine is $\sin t \cos s$ since when it is divided by the hypotenuse $\cos s$ it gives the desired result. You can just read off both addition formulas.

$$\sin s + t = \sin s \cos t + \sin t \cos s$$

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

To get the subtraction formulas, use the fact that $-\sin t = \sin -t$ but $\cos t = \cos -t$. Cosine is an even function, and sine an odd one. For example

$$\begin{aligned} \cos s - t &= \cos s \cos(-t) - \sin s \sin(-t) \\ &= \cos s \cos t + \sin s \sin t \end{aligned}$$

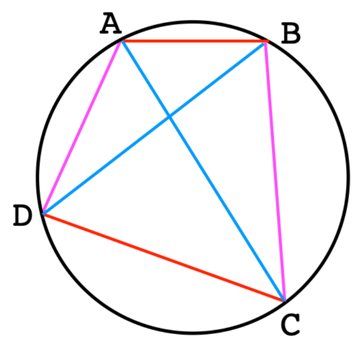
This is the one to remember if you need to do so. It's easy to check since if $s = t$, the left-hand side is the cosine of zero which is 1, and the right-hand side is our favorite trig identity.

□

I don't usually write parentheses in these formulas, but for the rest of the chapter we're going to be doing some algebra and I think it'll make things less confusing.

proof from Ptolemy

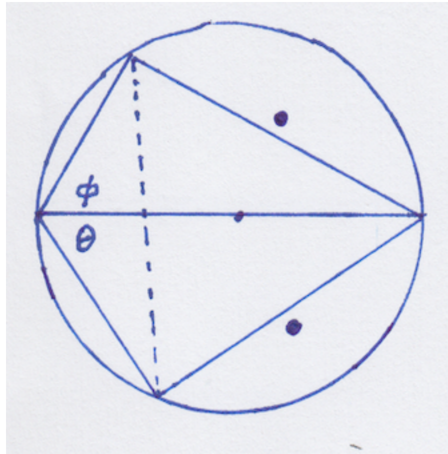
We recall Ptolemy's theorem: for a cyclic quadrilateral (all 4 vertices on one circle), the sum of the products of the opposing sides is equal to the product of the diagonals.



AB CD + BC AD = AC BD

Draw the diagram corresponding to a special case, where one of the diagonals is a diameter of the circle, scaled so that the length of the diameter is 1 (below).

Then, draw any two points on the circle (split by the diameter), which give two right triangles, by Thales theorem. With this scaling, the dotted line is exactly $\sin(\theta + \phi)$, since the factor of $2R$ is equal to 1.



So the product of the diagonals is exactly the same.

What about the sides? Rather than label them, I've just marked two sides with dots, these correspond to the sines of the two angles. We can read the formula for the sum of sines off the diagram, it is:

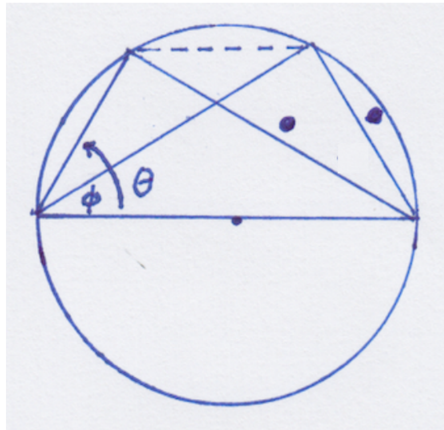
$$\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$$

□

Notice that both products on the right-hand side are the sides of the quadrilateral so they add in the formula.

The other two formulas which have a minus sign will be set up differently, with the dotted line and diagonal as sides of the quadrilateral.

In the next figure, the dotted line is $\sin\theta - \phi$. It is multiplied by the diameter, which is 1, as before.



The product of the other pair of sides of the quadrilateral is $\sin\phi\cos\theta$. The product of diagonals is $\sin\theta\cos\phi$, giving

$$\sin\phi\cos\theta + \sin(\theta-\phi) = \sin\theta\cos\phi$$

Rearranged

$$\sin(\theta-\phi) = \sin\theta\cos\phi - \sin\phi\cos\theta$$

□

See if you can show that the other angle that has the dotted line as its chord in the figure above, is also equal to $\theta - \phi$.

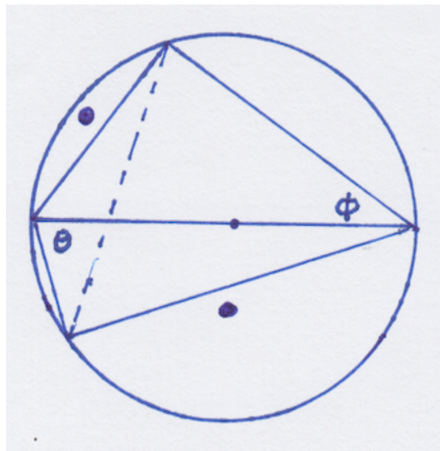
It is pretty straightforward to set these up and play around with the angles and sides to massage them into the formulas (since we already know the answer).

There is one more little trick. So far the dotted line has been the sine of an angle, but now we need it to be the cosine of an angle. We obtain that by the following manipulation. If s and t are complementary angles then

$$\sin s = \cos t = \cos(90-s)$$

Next, we use a diagram where the dotted line is a diagonal of the quadrilateral. That will lead to a formula in which the other terms are summed, which is the formula for the difference of cosines.

Let's see.



The dotted line is the sine of θ plus something, namely, the complement of ϕ, so that's $\sin(\theta + 90 - \phi)$. Remembering what we said about switching to cosine, we take the complement:

$$\cos\,[\,90 - (\theta + 90 - \phi)\,] \;= \cos(\phi - \theta)$$

The terms from the sides are easy

$$\cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\phi - \theta)$$

Rearranged:

$$\cos(\phi - \theta) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

The left-hand side is symmetrical in θ and ϕ, but the other side is not. However, notice that we could just as easily have written $\sin(\phi + 90 - \theta)$ at the beginning. Then we would have $\cos(\theta - \phi)$ now. So

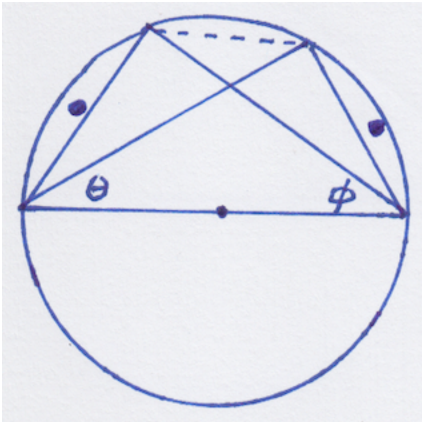
$$\cos(\phi - \theta) = \cos\theta - \phi = \cos\theta\cos\phi + \sin\theta\sin\phi$$

and that's right! Cosine is an even function:

$$f(x) = f(-x) \quad \text{and} \quad -(\theta - \phi) = \phi - \theta$$

□

There is one more. This formula has a minus sign in it so it must have the dotted line as one of the sides of the quadrilateral.



The product of diagonals is $\cos\theta\cos\phi$ and one pair of sides is $\sin\theta\sin\phi$. Since the formula is

$$\sin\theta\sin\phi + \cos(\theta+\phi) = \cos\theta\cos\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

somehow, the dotted line *must* be $\cos(\theta+\phi)$. I leave the last step to you.

Don't forget the square at the end of the proof.