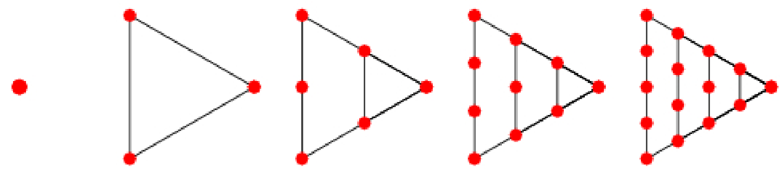
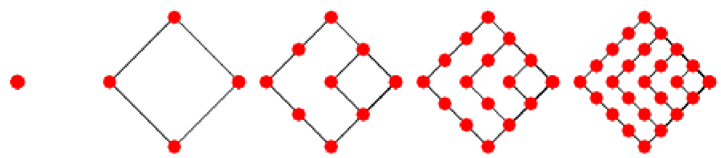
Stanford Math Circle: Sunday, May 9, 2010
Square-Triangular Numbers, Pell's Equation, and Continued Fractions

Recall that triangular numbers are numbers of the form $T_m = \dfrac{m(m+1)}{2}$. Geometrically, they're numbers that can be arranged in the shape of a triangle:



The n-th triangular number is formed using an outer triangle whose sides have n dots. Similarly, *square* numbers are numbers that can be arranged in the shape of a square:



The n-th square number is formed using an outer square whose sides have n dots. The n-th square number is $S_n = n^2$.

Example 1 Make a list of the first 10 triangular and square numbers. Are there any numbers in both lists?

In this chapter, we'll develop a method for finding all *square-triangular numbers* (i.e. numbers which are both square and triangular numbers). One of the major questions we'll be interested in answering is whether or not there are infinitely many square-triangular numbers. Since triangular numbers are of the form

$$T_m = \frac{m(m+1)}{2}$$

and square numbers are of the form

$$S_n = n^2,$$

square-triangular numbers are integer solutions of the equation

$$n^2 = \frac{m(m+1)}{2}.$$

Next, multiply both sides by 8 and show that we can rewrite the previous equation as

$$8n^2 = 4m^2 + 4m = (2m+1)^2 - 1.$$

This suggests the substitution

$$x = 2m + 1 \text{ and } y = 2n.$$

Make this substitution, and rearrange to obtain the equation

$$x^2 - 2y^2 = 1.$$

Solutions to this equation produce square-triangular numbers with

$$m = \frac{x-1}{2} \text{ and } n = \frac{y}{2}.$$

In other words, if (x, y) is a solution of the equation $x^2 - 2y^2 = 1$, then $N = n^2 = \left(\frac{y}{2}\right)^2$ is a square-triangular number. Geometrically, the square has $y/2$ dots on a side and the triangle has $(x-1)/2$ dots on the bottom row.

Example 2 Show that $(x, y) = (3, 2)$ and $(x, y) = (17, 12)$ are both solutions of $x^2 - 2y^2 = 1$. Then find the corresponding values of (m, n) and the resulting square-triangular numbers. Can you find any other solutions (perhaps using a calculator or computer)?

To find all square-triangular numbers, we need to find all solutions of

$$x^2 - 2y^2 = 1.$$

Observe that we can factor this equation as

$$x^2 - 2y^2 = (x + y\sqrt{2})(x - y\sqrt{2}).$$

For example, we can write the solution $(x, y) = (3, 2)$ as

$$1 = 3^2 - 2 \cdot 2^2 = (3 + 2\sqrt{2})(3 - 2\sqrt{2}).$$

Next, observe what happens if we square both sides of this equation:

$$\begin{aligned} 1 = 1^2 &= (3 + 2\sqrt{2})^2(3 - 2\sqrt{2})^2 \\ &= (17 + 12\sqrt{2})(17 - 12\sqrt{2}) \\ &= 17^2 - 2 \cdot 12^2 \end{aligned}$$

Thus, "squaring" the solution $(x, y) = (3, 2)$ produced the next solution $(x, y) = (17, 12)$! We call the solution $(3, 2)$ the **fundamental solution** of the equation.

Example 3 Cube the solution $(x, y) = (3, 2)$, and show that you obtain the solution $(x, y) = (99, 70)$. Which square-triangular number does this produce? What about taking the fourth power?

Example 4 What is the solution of Pell's equation corresponding to $(3 + 2\sqrt{2})^{16}$? What is the corresponding square-triangular number?

Theorem 1 There are infinitely many square-triangular numbers.

Proof. For every positive integer k,

$$1 = 1^k = (3 + 2\sqrt{2})^k(3 - 2\sqrt{2})^k.$$

By raising $(3 + 2\sqrt{2})$ to higher and higher powers, we continue to find more and more solutions to the equation $x^2 - 2y^2 = 1$, and each new solution gives us a new square-triangular number. (Note: the technique that we have used is interesting from a number-theoretic point of view. In attempting to solve a question about *integers*, we've used irrational numbers!)

Thus, there are infinitely square-triangular numbers, but it's natural to ask at this point whether or not our procedure actually produces all of them.

Theorem 2 Square-Triangular Number Theorem.

(a) Every solution (x_k, y_k) in positive integers to the equation

$$x^2 - 2y^2 = 1$$

is of the form

$$x_k + y_k\sqrt{2} = (3 + 2\sqrt{2})^k, \quad k = 1, 2, 3, \ldots.$$

(b) Every square-triangular number $n^2 = \frac{1}{2}m(m+1)$ is given by

$$m = \frac{x_k - 1}{2} \text{ and } n = \frac{y_k}{2}.$$

Proof. We've already checked (b). We just need to check that if (u, v) is *any* solution of $x^2 - 2y^2 = 1$, then it is of the form

$$u + v\sqrt{2} = (3 + 2\sqrt{2})^k$$

for some k. To do this, we'll use the method of descent. First, note that $u \geq 3$, and if $u = 3$, then $v = 2$, so there's nothing to check. Next, suppose that $u > 3$, and try to show that there must be another solution (s, t) in positive integers such that

$$u + v\sqrt{2} = (3 + 2\sqrt{2})(s + t\sqrt{2}) \text{ with } s < u.$$

If $(s, t) = (3, 2)$, then we're done (i.e. (u, v) is of the correct form). If not, then try to find another solution (q, r) such that

$$s + t\sqrt{2} = (3 + 2\sqrt{2})(q + r\sqrt{2}) \text{ with } q < s.$$

If we can do this, then we have

$$u + v\sqrt{2} = (3 + 2\sqrt{2})^2(q + r\sqrt{2}),$$

so if $(q, r) = (3, 2)$, then we're done. If not, we'll apply the procedure again. Observe that this process can't go on forever, since each time we get a new solution, the value of "x" is smaller (e.g. $q < s < u$. Since these values are all positive integers, they cannot get smaller forever, so the process must end in a finite number of steps. Thus, we eventually must reach $(3, 2)$ as a solution, so eventually we're able to write $u + v\sqrt{2}$ as a power of $3 + 2\sqrt{2}$.

Thus, it remains to show that if we start with a solution (u, v) with $u > 3$, then we can find a solution (s, t) with the property

$$u + v\sqrt{2} = (3 + 2\sqrt{2})(s + t\sqrt{2}) \text{ with } s < u.$$

To do this, multiply out the right-hand side to obtain

$$u + v\sqrt{2} = (3s + 4t) + (2s + 3t)\sqrt{2}.$$

Thus, we need to solve

$$u = 3s + 4t \text{ and } v = 2s + 3t$$

for s and t. Show that the solution is

$$s = 3u - 4v \text{ and } t = -2u + 3v.$$

Now, there are three things left to check. We need to make sure that this (s, t) is really a solution of $x^2 - 2y^2 = 1$, that s and t are both positive, and that $s < u$. For the first, just check that

$s^2 - 2t^2 = 1$ (remember that $u^2 - 2v^2 = 1$ since (u, v) is a solution). Once we know that s and t are both positive, we can check that $s < u$ as follows:

$$\begin{aligned} s &= 3u - 4v \\ &= 3u - 4\left(\frac{1}{3}t + \frac{2}{3}u\right) \\ &= 3u - \frac{4}{3}t - \frac{8}{3}u \\ &= \frac{1}{3}u - \frac{4}{3}t \end{aligned}$$

So it remains to make sure that s and t are both positive. First, we'll check that s is positive:

$$\begin{aligned} u^2 &= 1 + 2v^2 > 2v^2 \\ u &> \sqrt{2}v \\ s &= 3u - 4v \\ &> 3\sqrt{2}v - 4v \\ &= (3\sqrt{2} - 4)v > 0 \end{aligned}$$

Finally, we'll check that t is positive:

$$\begin{aligned} u &> 3 \\ u^2 &> 9 \\ 9u^2 &> 9 + 8u^2 \\ 9u^2 - 9 &> 8u^2 \\ u^2 - 1 &> \frac{8}{9}u^2 \\ 2v^2 &> \frac{8}{9}u^2 \\ v &> \frac{2}{3}u \\ t &= -2u + 3v > -2u + 3\frac{2}{3}u = 0 \end{aligned}$$

This completes the descent proof.

More generally, any Diophantine equation of the form $x^2 - dy^2 = 1$, where d is a non-square positive integer is called a *Pell equation.* Pell's equation has an interesting history–its first recorded appearance is in the "Cattle problem of Archimedes" (287-212 BC), in a letter sent from Archimedes to Eratosthenes. In 1880, A. Amthor, a German mathematician, showed that the total number of cattle had to be a number with 206,545 digits, beginning with 7766. Over the next 85 years, an additional 40 digits were found, but it was not until 1965 at the University of Waterloo that a complete solution was found–it took over 7.5 hours of computation on an IBM 7040 computer. However, they didn't print out the solution, and the problem was solved a second time using a Cray-1 computer in 1981.

So, we know that if we can find one solution of a Pell equation, then we can find infinitely many. But how do we find the smallest (i.e. fundamental solution)? To answer this question, we'll investigate the relationship between *continued fractions* and Pell equations.

Example 5 A **continued fraction** is an expression of the form

$$4+\cfrac{1}{2+\frac{1}{7+\frac{1}{3}}}.$$

This is called the **continued fraction expansion** for the fraction $\dfrac{210}{47}$. Note that in the continued fraction expansion, all of the denominators are equal to 1. A more compact notation for this continued fraction expansion is $[4; 2, 7, 3]$.

Example 6 Find the continued fraction for $\dfrac{8}{5}$.

$$\begin{aligned} 8/5 &= 1+3/5 \\ &= 1+1/5/3 \\ &= 1+\cfrac{1}{1+\frac{2}{3}} \\ &= 1+\cfrac{1}{1+\frac{1}{1+\frac{1}{2}}} \end{aligned}$$

So $\dfrac{8}{5}=[1;1,1,2]$.

Example 7 Consider the decimal expansion of π:

$$\pi = 3.14159265358979323846264 33\ldots$$

Observe that we can write this as

$$\pi = 3+\text{something},$$

where the "something" is a number between 0 and 1. Next, observe that we can rewrite this as:

$$\begin{aligned} \pi &= 3+0.14159265358979323846264 33\ldots \\ &= 3+\cfrac{1}{\frac{1}{0.14159265358979323846264 33\ldots}} \\ &= 3+\cfrac{1}{7.06251330593104576930051\ldots} \\ &= 3+\cfrac{1}{7+0.06251330593104576930051\ldots} \\ &= 3+\cfrac{1}{7+\text{a little bit more}} \end{aligned}$$

The final equation above gives the fairly good approximation $\dfrac{22}{7}$ for π.

Now, if we repeat this process, we obtain:

$$\begin{aligned} 0.0625133059310457693005 1\ldots &= \frac{1}{\frac{1}{0.0625133059310457693005 1\ldots}} \\ &= 15.9965944066857198889230 60 \\ &= 15 + 0.9965944066857198889230 60 \end{aligned}$$

Thus, we have the following representation of π:

$$\pi \quad = \quad 3 + \frac{1}{7 + \frac{1}{15 + 0.9965944066857198889230 60}}$$

The bottom level of this fraction is 15.996594406685719888923060, which is very close to 16. If we replace it with 16, we get a rational number that is very close to π:

$$3 + \frac{1}{7 + \frac{1}{16}} = \frac{355}{113} = 3.1415929203539823008849557\ldots$$

The fraction $\dfrac{355}{113}$ agrees with π to six decimal places.

Continue this process, at each stage flipping the decimal that is left over and then separating off the whole integer part, to obtain a four-layer fraction representation of π. Use your final representation to get a rational number approximation for π, and compare with the known decimal approximation of π to see how accurate your approximation is.

Using our more compact notation, we can express the continued fraction expansion of π as Using this notation, our continued fraction expansion of π can be written as

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \ldots].$$

Definition 1 The **n-th convergent to** α is the rational number

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_n}}}}$$

obtained by using the terms up to a_n in the continued fraction expansion of α.

Example 8 For the continued fraction expansion of $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \ldots]$, the first few convergents are:

$$\begin{aligned} \frac{p_0}{q_0} &= 3 \\ \frac{p_1}{q_1} &= 3 + \frac{1}{7} = \frac{22}{7} = 3.142857143 \\ \frac{p_2}{q_2} &= 3 + \frac{1}{7 + \frac{1}{15}} = \frac{333}{106} = 3.141509434 \end{aligned}$$

Consider again the Pell equation with $d = 2$:

$$x^2 - 2y^2 = 1.$$

Start by finding the continued fraction expansion of $\sqrt{2} = 1.414213562...$:

$$\sqrt{2} = [1; 2, 2, 2, 2, 2, 2, 2, \ldots].$$

The first few convergents are:

$$\begin{aligned}
\frac{p_0}{q_0} &= 1 \\
\frac{p_1}{q_1} &= 1 + \frac{1}{2} = \frac{3}{2} \\
\frac{p_2}{q_2} &= 1 + \frac{1}{2 + \frac{1}{2}} = \frac{7}{5}
\end{aligned}$$

What do you notice? The fundamental solution of $x^2 - 2y^2 = 1$ is $(3, 2)$, which is one of our convergents!

Theorem 3 Continued Fractions and Fundamental Solutions of Pell Equations. Consider the Pell equation $x^2 - dy^2 = 1$. Let $\frac{h_i}{k_i}$, $i = 0, 1, \ldots$ denote the sequence of convergents to the continued fraction expansion for $\sqrt{d}$. Then the fundamental solution (x_1, y_1) of the Pell equation satisfies $x_1 = h_i$ and $y_1 = k_i$ for some i.

Example 9 Consider the Pell equation $x^2 - 3y^2 = 1$. Find the continued fraction expansion of $\sqrt{3} = 1.7320508075688...$, and use it to find the fundamental solution of the Pell equation.

$$\begin{aligned}
\sqrt{3} &= [1; 1, 2, 1, 2, 1, 2, 1, 2, \ldots] \\
\frac{p_0}{q_0} &= 1 \\
\frac{p_1}{q_1} &= 1 + \frac{1}{1} = \frac{2}{1} \\
\frac{p_2}{q_2} &= 1 + \frac{1}{1 + \frac{1}{2}} = \frac{5}{3} = 1.6666...
\end{aligned}$$

Example 10 Consider the Pell equation $x^2 - 7y^2 = 1$. Find the continued fraction expansion of $\sqrt{7} = 2.6457513110645907...$, and use it to find the fundamental solution of the Pell equation.

$$\begin{aligned}
\sqrt{7} &= [2; 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, \ldots] \\
\frac{p_0}{q_0} &= 2 \\
\frac{p_1}{q_1} &= 2 + \frac{1}{1} = \frac{3}{1} \\
\frac{p_2}{q_2} &= 2 + \frac{1}{1 + \frac{1}{1}} = \frac{5}{2} = 2.5 \\
\frac{p_3}{q_3} &= 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = \frac{8}{3} = 2.6666...
\end{aligned}$$